ELEMENTARY COLLEGE GEOMETRY

Henry Africk
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Elementary College Geometry

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This text was compiled on 02/01/2024

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About this Book

This text is intended for a brief introductory course in plane geometry. It covers the topics from elementary geometry that are most likely to be required for more advanced mathematics courses. The only prerequisite is a semester of algebra.

The emphasis is on applying basic geometric principles to the numerical solution of problems. For this purpose the number of theorems and definitions is kept small. Proofs are short and intuitive, mostly in the style of those found in a typical trigonometry or precalculus text. There is little attempt to teach theorem-proving or formal methods of reasoning. However the topics are ordered so that they may be taught deductively.

The problems are arranged in pairs so that just the odd-numbered or just the even-numbered can be assigned. For assistance, the student may refer to a large number of completely worked-out examples. Most problems are presented in diagram form so that the difficulty of translating words into pictures is avoided. Many problems require the solution of algebraic equations in a geometric context. These are included to reinforce the student's algebraic and numerical skills, A few of the exercises involve the application of geometry to simple practical problems. These serve primarily to convince the student that what he or she is studying is useful. Historical notes are added where appropriate to give the student a greater appreciation of the subject.

This book is suitable for a course of about 45 semester hours. A shorter course may be devised by skipping proofs, avoiding the more complicated problems and omitting less crucial topics.

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Preface

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The emphasis is on applying basic geometric principles to the numerical solution of problems. For this purpose the number of theorems and definitions is kept small, Proofs are short and intuitive, mostly in the style of those found in a typical trigonometry or precalculus text. There is little attempt to teach theorem-proving or formal methods of reasoning, However the topics are ordered so that they may be taught deductively.

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I would like to thank my colleagues at New York City Technical College who have contributed, directly or indirectly, to the development o: this work. In particular, I would like to acknowledge the influence of L. Chosid, M. Graber, S, Katoni, F. Parisi and E. Stern.

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CHAPTER OVERVIEW

1: Lines, Angles, and Triangles

Thumbnail: Angles A and B are adjacent. (Public Domain; Limaner via Wikipedia)

1.1: Lines

Geometry (from Greek words meaning earth-measure) originally developed as a means of surveying land areas, In its simplest form, it is a study of figures that can be drawn on a perfectly smooth flat surface, or **plane**. It is this **plane geometry** which we will study in this bock and which serves as a foundation for trigonometry, solid and analytic geometry, and calculus.

The simplest figures that can be drawn on a plane are the point and the line. By a line we will always mean a **straight line**. Through two distinct points one and only one (straight) line can be drawn. The line through points A and B will be denoted by $\overleftrightarrow{AB}$ (Figure $1.1.1$). The arrows indicate that the line extends indefinitely in each direction, The **line segment** from A to B consists of A, B and that part of $\overleftrightarrow{AB}$ between A and B, It is denoted by AB (some textbooks use the notation $\overline{AB}$ for line segment). The ray $\overrightarrow{AB}$ is the part of $\overleftrightarrow{AB}$ which begins at A and extends indefinitely in the direction of B.

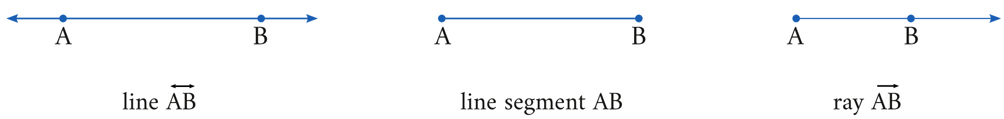


Figure $1.1.1$: Line $\overleftrightarrow{AB}$, line segment $\overline{AB}$, and ray $\overrightarrow{AB}$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

We assume everyone is familiar with the notion of length of a line segment and how it can be measured in inches, or feet, or meters, etc, The distance between two points A and B is the same as the length of AB.

Two line segments are equal if they have the same length, e.g., in Figure 1.1.2, $AB = CD$,

Figure $1.1.2$: $AB = CD$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

We often indicate two line segments are equal by marking them in the same way, e.g., in Figure $1.1.3$, $AB = CD$ and $EF = GH$.

Figure $1.1.3$: $AB = CD$ and $EF = GH$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Example $1.1.1$

Find x if $AB = CD$:

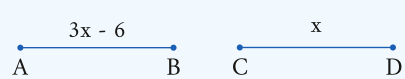


Figure $1.1.E1$: (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Solution

$$\begin{aligned} AB &= CD \\ 3x - 6 &= x \\ 3x - x &= 6 \\ 2x &= 6 \\ x &= 3 \end{aligned}$$

Check:

AB	CD
3x - 6	x
3(3) - 6	3
9 - 6	
3	

Answer: $x = 3$.

Notice that in Example 1.1.1 we have not indicated the unit of measurement. Strictly speaking, we should specify that $AB = 3x - 6$ inches (or feet or meters) and that $BC = x$ inches. However since the answer would still be $x = 3$ we will usually omit this information to save space.

We say that B is the midpoint of AC if B is A point on AC and $AB = BC$ (Figure 1.1.4).

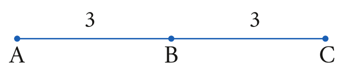


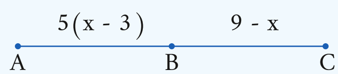
Figure 1.1.4: B is the midpoint of AC. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

Example 1.1.2

Find x and AC if B is the midpoint of AC and $AB = 5(x - 3)$ and $BC = 9 - x$,

Solution

We first draw a picture to help visualize the given information:



Since B is a midpoint,

$$\begin{aligned} AB &= BC \\ 5(x-3) &= 9-x \\ 5x-15 &= 9-x \\ 5x+x &= 9+15 \\ 6x &= 24 \\ x &= 4 \end{aligned}$$

Check:

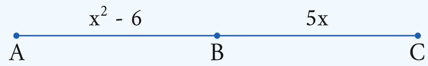
AB	BC
5(x - 3)	9 - x
5(4 - 3)	9 - 4
5(1)	5
5	

We obtain $AC = AB + BC = 5 + 5 = 10$.

Answer: $x = 4$, $AC = 10$.

Example 1.1.3

Find AB if B is the midpoint of AC:



Solution

$$\begin{aligned} AB &= BC \\ x^2-6 &= 5x \\ x^2-5x-6 &= 0 \\ (x-6)(x+1) &= 0 \end{aligned}$$

$$\begin{aligned} x-6 &= 0 & x+1 &= 0 \\ x &= 6 & x &= -1 \end{aligned}$$

If $x=6$ then $AB = x^2-6 = 6^2-6 = 36-6 = 30$.

If $x=-1$ then $AB = (-1)^2-6 = 1-6 = -5$.

We reject the answer $x=-1$ and $AB=-5$ because the length of a line segment is always positive. Therefore $x=6$ and $AB=30$.

Check:

AB	= BC
x^2 - 6	5x
6^2 - 6	5(6)
36 - 6	30
30	

Answer: $AB=30$.

Three points are collinear if they lie on the same line.

Figure 1.1.5: A, B, and C are collinear $AB=5$, $BC=3$, and $AC=8$ (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

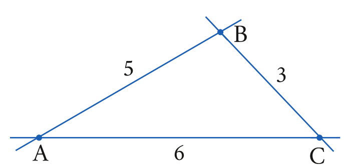
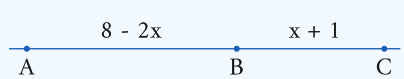


Figure 1.1.6: A, B, and C are not collinear. $AB=5$, $BC=3$, $AC=6$. (CC BY-NC 4.0; Ümit Kaya via LibreTexts)

A, B, and C are collinear if and only if $AB+BC=AC$.

Example 1.1.4

If A, B, and C are collinear and $AC=7$, find x:



Solution

$$\begin{aligned} AB + BC &= AC \\ 8 - 2x + x + 1 &= 7 \\ 9 - x &= 7 \\ 2 &= x \end{aligned}$$

Check:

AB + BC	=	AC
8 - 2x + x + 1		7
8 - 2(2) + 2 + 1		
8 - 4 + 3		
4 + 3		
7		

Answer: $x = 2$

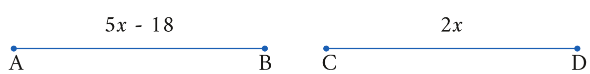
Historical Note

Geometry originated in the solution of practical problems, The architectural remains of Babylon, Egypt, and other ancient civilizations show a knowledge of simple geometric relationships, The digging of canals, erection of buildings, and the laying out of cities required computations of lengths, areas, and volumes, Surveying is said to have developed in Egypt so that tracts of land could be relocated after the annual overflow of the Nile, Geometry was also utilized by ancient civilizations in their astronomical observations and the construction of their calendars.

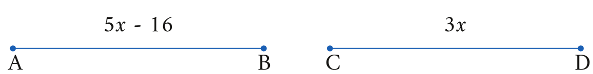
The Greeks transformed the practical geometry of the Babylonians and Egyptians into an organized body of knowledge. Thales (c, 636 - c. 546 B,C.), one of the "seven wise men" of antiquity, is credited with being the first to obtain geometrical results by logical reasoning, instead of just by intuition and experiment. Pythagoras (c. 582 - c. 507 B,C.) continued the work of Thales, He founded the Pythagorean school, a mystical society devoted to the unified study of philosophy, mathematics, and science, About 300 B,C., Euclid, a Greek teacher of mathematics at the university at Alexandria, wrote a systematic exposition of elementary geometry called the **Elements**, In his **Elements**, Euclid used a few simple principles, called *axioms* or *postulates*, to derive most of the mathematics known at the time, For over 2000 years, Euclid's Elements has been accepted as the standard textbook of geometry and is the basis for most other elementary texts, including this one.

Problems

1. Find x if $AB = CD$:



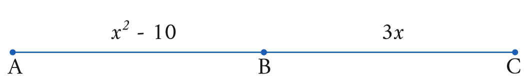
2. Find x if $AB = CD$:



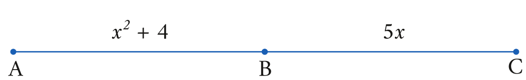
3. Find x and AC if B is the midpoint of AC and $AB = 3(x-5)$ and $BC = x + 3$.

4. Find x and AC if B is the midpoint of AC and $AB = 2x + 9$ and $BC = 5(x-9)$,

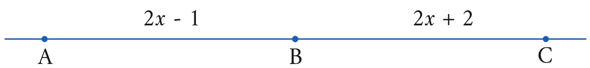
5. Find AB if B is the midpoint of AC:



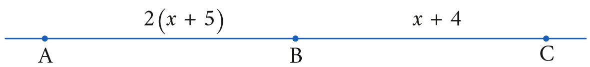
6, Find AB if B is the midpoint of AC:



7. If A, B, and C are collinear and $AC = 13$ find x:



8. If A, B, and C are collinear and $AC = 26$ find x:



1.2: Angles

An *angle* is the figure formed by two rays with a common end point, The two rays are called the sides of the angle and the common end point is called the *vertex* of the angle, The symbol for angle is $\angle$

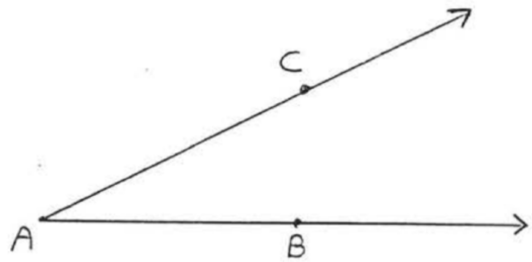


Figure $1.2.1$: Angle BAC has vertex A and sides $\overrightarrow{AB}$ and $\overrightarrow{AC}$

The angle in Figure $1.2.1$ has vertex A and sides AB and AC, It is denoted by $\angle BAC$ or $\angle CAB$ or simply $\angle A$. When three letters are used, the middle letter is always the vertex, In Figure $1.2.2$ we would not use the notation $\angle A$ as an abbreviation for $\angle BAC$ because it could also mean $\angle CAD$ or $\angle BAD$, We could however use the simpler name $\angle x$ for $\angle BAC$ if "x" is marked in as shown,

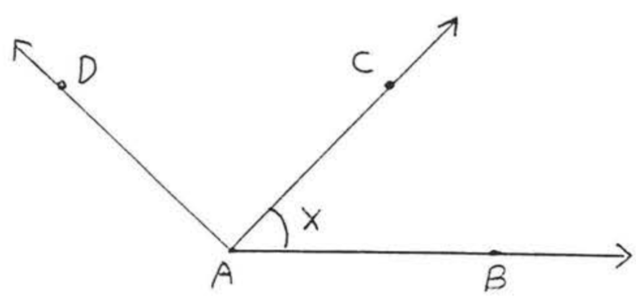


Figure $1.2.2$: $\angle BAC$ may also be denoted by $\angle x$.

Angles can be measured with an instrument called a *protractor*. The unit of measurement is called a *degree* and the symbol for degree is $^\circ$.

To measure an angle, place the center of the protractor (often marked with a cross or a small circle) on the vertex of the angle, Position the protractor so that one side of the angle cuts across 0, at the beginning of the scale, and so that the other side cuts across a point further up on the scale, We use either the upper scale or the lower scale, whichever is more convenient, For example, in Figure $1.2.3$, one side of $\angle BAC$ crosses 0 on the lower scale and the other side crosses 50 on the lower scale. The measure of $\angle BAC$ is therefore 50° and we write $\angle BAC$ = 50°.

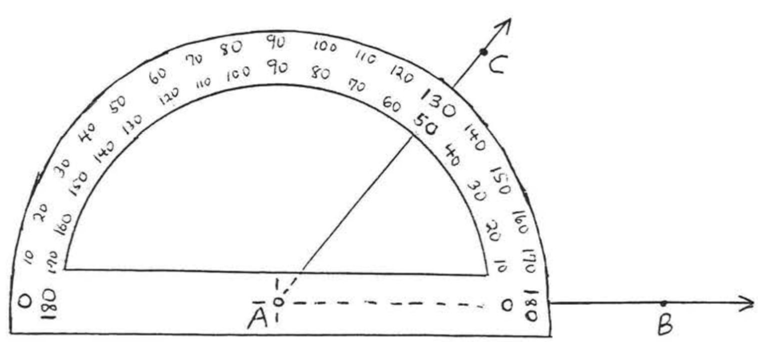


Figure $1.2.3$: The protractor shows $\angle BAC = 50^\circ$

In Figure 1.2.4, side $\overrightarrow{AD}$ of $\angle DAC$ crosses 0 on the upper scale. Therefore we look on the upper scale for the point at which $\overrightarrow{AC}$ crosses and conclude that $\angle DAC = 130^\circ$.

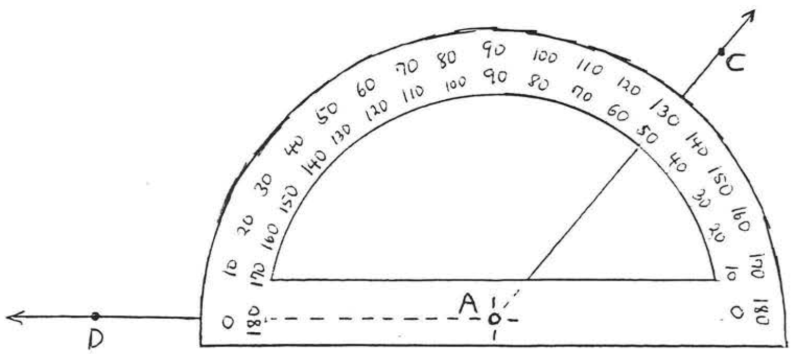


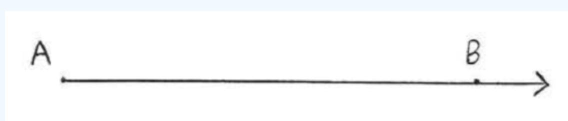
Figure 1.2.4: $\angle DAC = 130^\circ$.

✓ Example 1.2.1

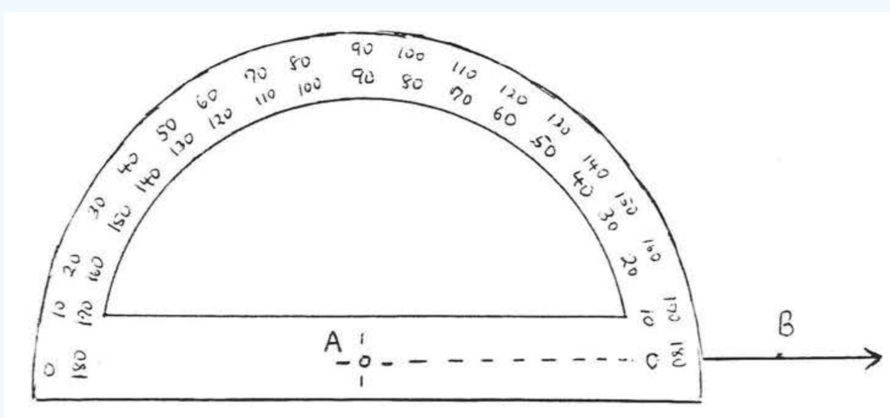
Draw an angle of 40° and label it $\angle BAC$.

Solution

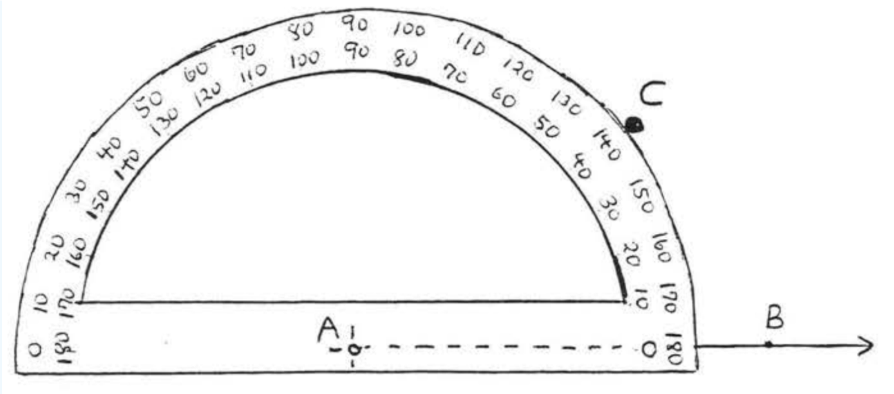
Draw ray $\overrightarrow{AB}$ using a straight edge:



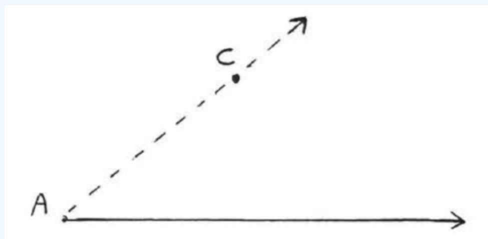
Place the protractor so that its center coincides with A and $\overrightarrow{AB}$ crosses the scale at 0:



Mark the place on the protractor corresponding to 40°. Label this point C:



Connect A with C:



Two angles are said to be equal if they have the same measure in degrees. We often indicate two angles are equal by marking them in the same way. In Figure $1.2.5$, $\angle A = \angle B$.

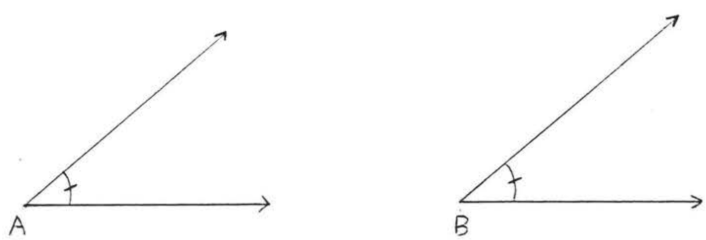


Figure $1.2.5$: Equal angles.

An angle bisector is a ray which divides an angle into two equal angles. In Figure $1.2.6$, $\overrightarrow{AC}$ is an angle bisector of $\angle BAD$. We also say $\overrightarrow{AC}$ bisects $\angle BAD$.

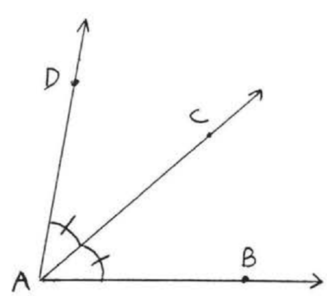
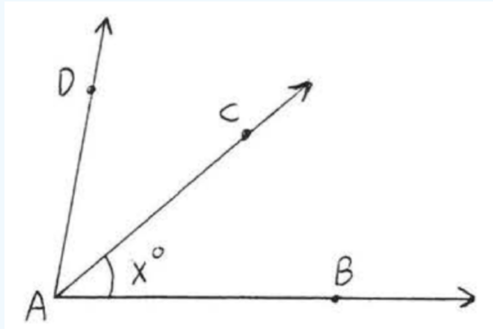


Figure $1.2.6$: $\overrightarrow{AC}$ bisects $\angle BAD$.

Example 1.2.2

Find x if $\overrightarrow{AC}$ bisects $\angle BAD$ and $\angle BAD = 80^\circ$:



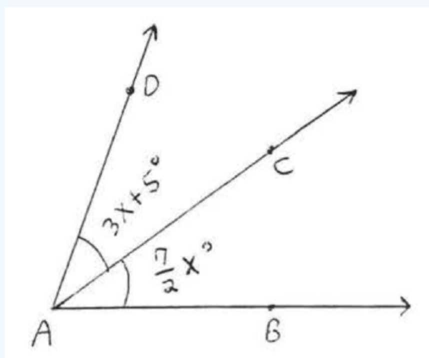
Solution

$x^\circ = \frac{1}{2}\angle BAD = \frac{1}{2}(80^\circ) = 40^\circ$

Answer: $x = 40$.

Example 1.2.3

Find x if $\overrightarrow{AC}$ bisects $\angle BAD$:



Solution

$$\begin{aligned} \angle BAC &= \angle CAD \\ \frac{7}{2}x &= 3x+5 \\ (2)\frac{7}{2}x &= (2)(3x+5) \\ 7x &= 6x+10 \\ 7x-6x &= 10 \\ x &= 10 \end{aligned} \tag{1.2.1}$$

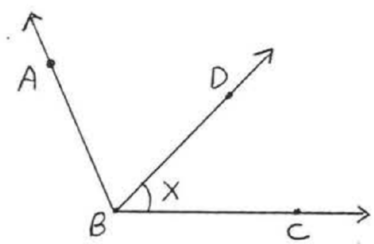
Check:

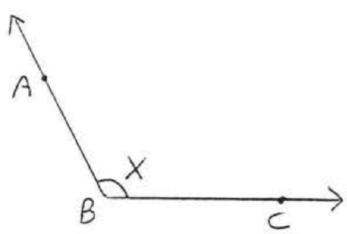
$\angle BAC$	= $\angle CAD$
$\frac{7}{2}x^\circ$	$3x + 5^\circ$
$\frac{7}{2}(10)^\circ$	$3(10) + 5^\circ$
35°	$30 + 5^\circ$
	35°

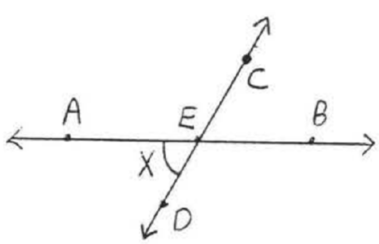
Answer: $x = 10$.

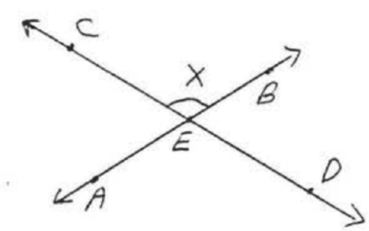
Problems

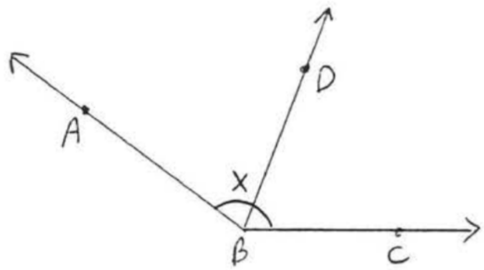
1 - 6. For each figure, give another name for $\angle x$:

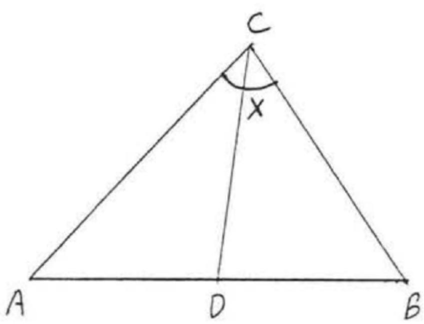
1. 

2. 

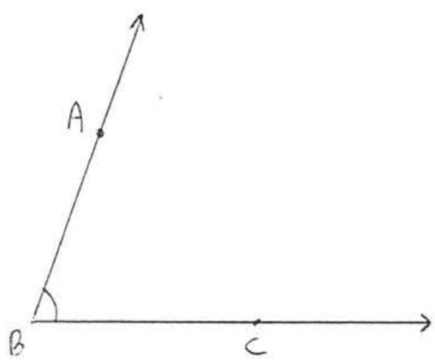
3. 

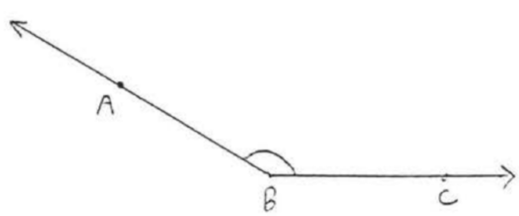
4. 

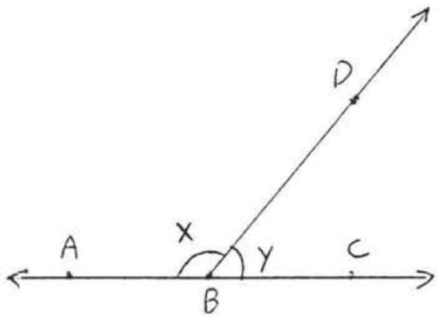
5. 

6. 

7 - 16, Measure each of the indicated angles:

7. 

8. 



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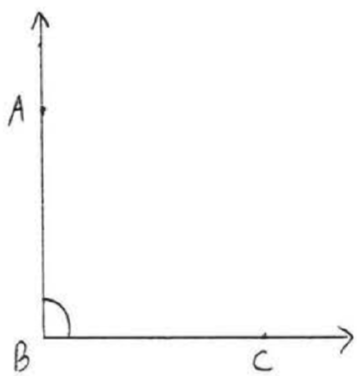

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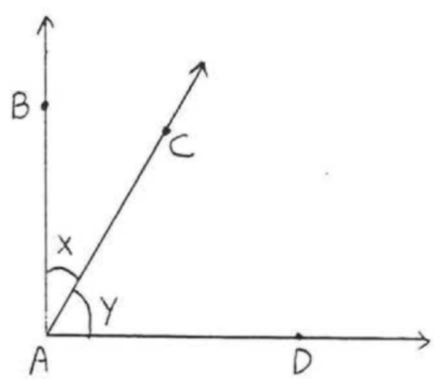

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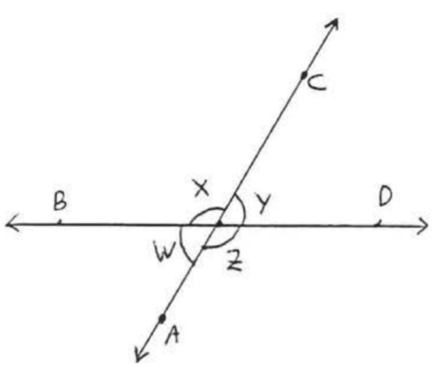

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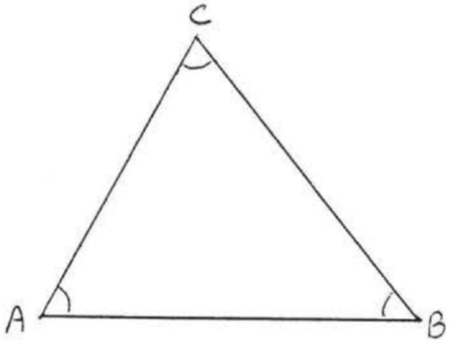

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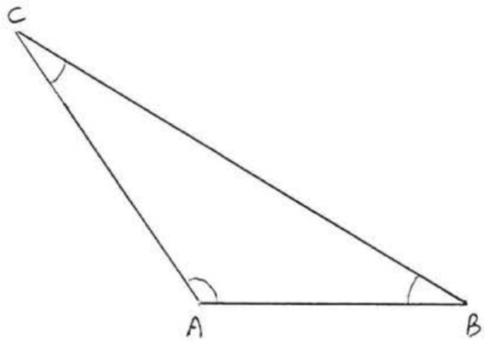
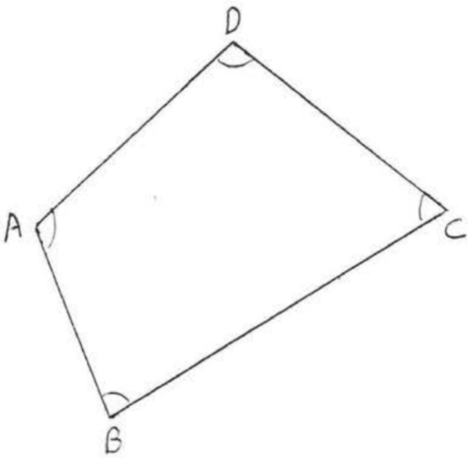


14.



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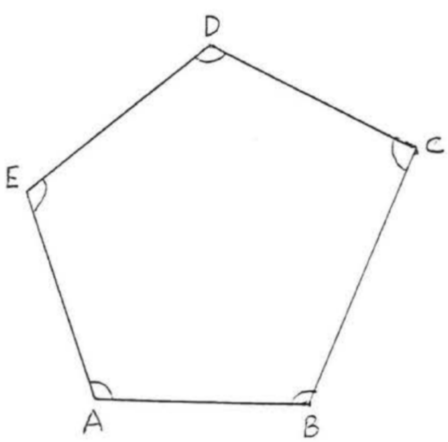


16.

17 - 24. Draw and label each angle:

17. $\angle BAC = 30^\circ$

18. $\angle BAC = 40^\circ$

19. $\angle ABC = 45^\circ$

20. $\angle EFG = 60^\circ$

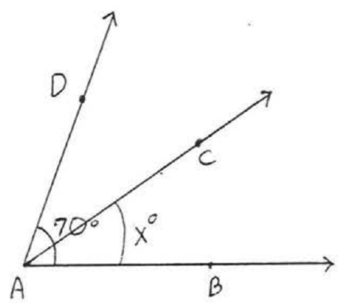
21. $\angle RST = 72^\circ$

22. $\angle XYZ = 90^\circ$

23. $\angle PQR = 135^\circ$

24. $\angle JKL = 164^\circ$

25 - 28. Find x if $\overrightarrow{AC}$ bisects $\angle BAD$:



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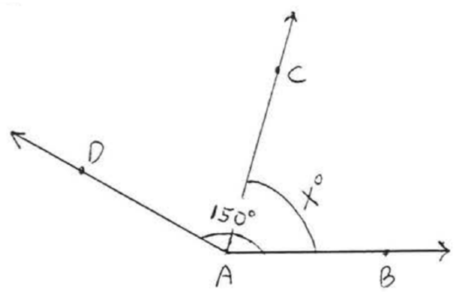
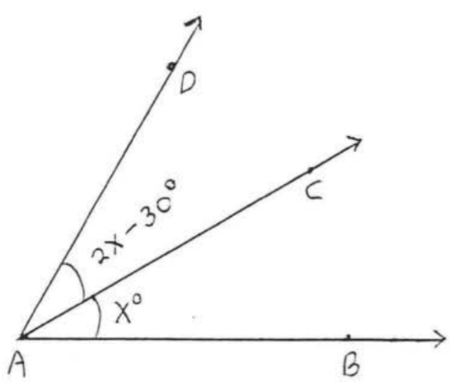


26.



27.

28.