3.4: Two Basic Rules of Probability

When calculating probability, there are two rules to consider when determining if two events are independent or dependent and if they are mutually exclusive or not.

The Multiplication Rule

If A and B are two events defined on a **sample space**, then $P(A\cap B)=P(B)*P(A|B)$. We can think of the intersection symbol as substituting for the word "and".

This rule may also be written as: $P(A|B)=\frac{P(A\cap B)}{P(B)}$

This equation is read as the probability of A given B equals the probability of A and B divided by the probability of B.

If A and B are **independent**, then $P(A|B)=P(A)$. Then $P(A\cap B)=P(A|B)*P(B)$ becomes $P(A\cap B)=P(A)*(B)$ because the $P(A|B)=P(A)$ if A and B are independent.

One easy way to remember the multiplication rule is that the word "and" means that the event has to satisfy two conditions. For example the name drawn from the class roster is to be both a female and a sophomore. It is harder to satisfy two conditions than only one and of course when we multiply fractions the result is always smaller. This reflects the increasing difficulty of satisfying two conditions.

The Addition Rule

If A and B are defined on a sample space, then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$. We can think of the union symbol substituting for the word "or". The reason we subtract the intersection of A and B is to keep from double counting elements that are in both A and B.

If A and B are **mutually exclusive**, then $P(A\cap B)=0$. Then $P(A\cup B)=P(A)+P(B)-P(A\cap B)$ becomes $P(A\cup B)=P(A)+P(B)$.

Example 3.4.1

Klaus is trying to choose where to go on vacation. His two choices are A = New Zealand and B = Alaska.

- Klaus can only afford one vacation. The probability that he chooses A is $P(A)=0.6$ and the probability that he chooses B is $P(B)=0.35$.
- $P(A\cap B)=0$ because Klaus can only afford to take one vacation.
- Therefore, the probability that he chooses either New Zealand or Alaska is $P(A\cup B)=P(A)+P(B)=0.6+0.35=0.95$. Note that the probability that he does not choose to go anywhere on vacation must be 0.05.

Example 3.4.2

Carlos plays college soccer. He makes a goal 65% of the time he shoots. Carlos is going to attempt two goals in a row in the next game. A = the event Carlos is successful on his first attempt. $P(A)=0.65$. B = the event Carlos is successful on his second attempt. $P(A)=0.65$. Carlos tends to shoot in streaks. The probability that he makes the second goal | that he made the first goal is 0.90.

a. What is the probability that he makes both goals?
b. What is the probability that Carlos makes either the first goal or the second goal?
c. Are A and B independent?
d. Are A and B mutually exclusive?

Answer

a. The problem is asking you to find $P(A\cap B)=P(B\cap A)$. Since $P(B|A)=0.90$, $P(B\cap A)=P(B|A)*P(A)=(0.90)(0.65)=0.585$.

Carlos makes the first and second goals with probability 0.585.

b. The problem is asking you to find $P(A \cup B)$.

$P(A \cup B) = P(A) + P(B) - P(A \cap B) = 0.65 + 0.65 - 0.585 = 0.715$

Carlos makes either the first goal or the second goal with probability 0.715.

c. No, they are not, because $P(B \cap A) = 0.585$.

$P(B) * P(A) = (0.65)(0.65) = 0.423$

0.423 ≠ 0.585 = $P(B \cap A)$

So, $P(B \cap A)$ is **not** equal to $P(B) * P(A)$.

d. No, they are not because $P(A \cap B) = 0.585$.

To be mutually exclusive, $P(A \cap B)$ must equal zero.

Exercise 3.4.2

Helen plays basketball. For free throws, she makes the shot 75% of the time. Helen must now attempt two free throws. C = the event that Helen makes the first shot. $P(C) = 0.75$. D = the event Helen makes the second shot. $P(D) = 0.75$. The probability that Helen makes the second free throw given that she made the first is 0.85. What is the probability that Helen makes both free throws?

Example 3.4.3

A community swim team has 150 members. Seventy-five of the members are advanced swimmers. Forty-seven of the members are intermediate swimmers. The remainder are novice swimmers. Forty of the advanced swimmers practice four times a week. Thirty of the intermediate swimmers practice four times a week. Ten of the novice swimmers practice four times a week. Suppose one member of the swim team is chosen randomly.

a. What is the probability that the member is a novice swimmer?
b. What is the probability that the member practices four times a week?
c. What is the probability that the member is an advanced swimmer and practices four times a week?
d. What is the probability that a member is an advanced swimmer and an intermediate swimmer? Are being an advanced swimmer and an intermediate swimmer mutually exclusive? Why or why not?
e. Are being a novice swimmer and practicing four times a week independent events? Why or why not?

Answer

a. $\frac{28}{150}$
b. $\frac{80}{150}$
c. $\frac{40}{150}$
d. $P(Advanced \cap Intermediate) = 0$, so these are mutually exclusive events. A swimmer cannot be an advanced swimmer and an intermediate swimmer at the same time.
e. No, these are not independent events.
$P(Novice \cap PracticesFourTimesPerWeek) = 0.0667$
$P(Novice) * P(PracticesFourTimesPerWeek) = 0.0996$
0.0667 ≠ 0.0996

Exercise 3.4.3

A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is taking a gap year?

Example 3.4.4

Felicity attends Modesto JC in Modesto, CA. The probability that Felicity enrolls in a math class is 0.2 and the probability that she enrolls in a speech class is 0.65. The probability that she enrolls in a math class | that she enrolls in speech class is 0.25.

Let M = math class, S = speech class, $M|S$ = math class given speech class.

a. What is the probability that Felicity enrolls in math and speech?
   Find $P(M \cap S) = P(M|S) * P(S)$.
b. What is the probability that Felicity enrolls in math or speech classes?
   Find $P(M \cup S) = P(M) + P(S) - P(M \cap S)$.
c. Are M and S independent? Is $P(M|S) = P(M)$?
d. Are M and S mutually exclusive? Is $P(M \cap S) = 0$?

Answer

a. 0.1625, b. 0.6875, c. No, d. No

Exercise 3.4.4

A student goes to the library. Let events B = the student checks out a book and D = the student check out a DVD. Suppose that $P(B) = 0.40$, $P(D) = 0.30$ and $P(D|B) = 0.5$.

a. Find $P(B \cap D)$.
b. Find $P(B \cup D)$.

Example 3.4.5

Studies show that about one woman in seven (approximately 14.3%) who live to be 90 will develop breast cancer. Suppose that of those women who develop breast cancer, a test is negative 2% of the time. Also suppose that in the general population of women, the test for breast cancer is negative about 85% of the time. Let B = woman develops breast cancer and let N = tests negative. Suppose one woman is selected at random.

a. What is the probability that the woman develops breast cancer? What is the probability that woman tests negative?
b. Given that the woman has breast cancer, what is the probability that she tests negative?
c. What is the probability that the woman has breast cancer AND tests negative?
d. What is the probability that the woman has breast cancer or tests negative?
e. Are having breast cancer and testing negative independent events?
f. Are having breast cancer and testing negative mutually exclusive?

Answer

a. $P(B) = 0.143$; $P(N) = 0.85$

b. $P(N|B) = 0.02$

c. $P(B \cap N) = P(B) * P(N|B) = (0.143)(0.02) = 0.0029$

d. $P(B \cup N) = P(B) + P(N) - P(B \cap N) = 0.143 + 0.85 - 0.0029 = 0.9901$

e. No. $P(N) = 0.85$; $P(N|B) = 0.02$. So, $P(N|B)$ does not equal $P(N)$.

f. No. $P(B \cap N) = 0.0029$. For B and N to be mutually exclusive, $P(B \cap N)$ must be zero.

Exercise 3.4.5

A school has 200 seniors of whom 140 will be going to college next year. Forty will be going directly to work. The remainder are taking a gap year. Fifty of the seniors going to college play sports. Thirty of the seniors going directly to work play sports. Five of the seniors taking a gap year play sports. What is the probability that a senior is going to college and plays sports?

Example $3.4.6$

Refer to the information in Example $3.4.5$. P = tests positive.

a. Given that a woman develops breast cancer, what is the probability that she tests positive. Find $P(P|B) = 1 - P(N|B)$.
b. What is the probability that a woman develops breast cancer and tests positive. Find $P(B \cap P) = P(B) * P(P|B)$.
c. What is the probability that a woman does not develop breast cancer. Find $P(B') = 1 - P(B)$.
d. What is the probability that a woman tests positive for breast cancer. Find $P(P) = 1 - P(N)$.

Answer

a. 0.98; b. 0.1401; c. 0.857; d. 0.15

Exercise $3.4.6$

A student goes to the library. Let events B = the student checks out a book and D = the student checks out a DVD. Suppose that $P(B) = 0.40$, $P(D) = 0.30$ and $P(D|B) = 0.5$.

1. Find $P(B')$.
2. Find $P(D \cap B)$.
3. Find $P(B|D)$.
4. Find $P(D \cap B')$.
5. Find $P(D|B')$.