15.5: Matrix Inversion

The inverse of a square matrix $\mathbf{A}$, sometimes called a reciprocal matrix, is a matrix $\mathbf{A}^{-1}$ such that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the identity matrix.

It is easy to obtain $\mathbf{A}^{-1}$ in the case of a 2×2 matrix:

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix};\ \mathbf{A}^{-1} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & h \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ae + bg = 1 \tag{15.5.1}$$

$$af + bh = 0 \tag{15.5.2}$$

$$ce + dg = 0 \tag{15.5.3}$$

$$cf + dh = 1 \tag{15.5.4}$$

From Equations 15.5.1 and 15.5.3: $g = (1 - ae)/b = -ce/d \rightarrow ae = cbe/d + 1 \rightarrow e(a - cb/d) = 1 \rightarrow e(ad - cb) = d \rightarrow e = d/(ad - cb)$. You can obtain expressions for f, g and h in a similar way to obtain:

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Notice that the term $(ad - bc)$ is the determinant of $\mathbf{A}$, and therefore $\mathbf{A}^{-1}$ exists only if $|\mathbf{A}| \neq 0$. In other words, the inverse of a singular matrix is not defined.

If you think about a square matrix as an operator, the inverse "undoes" what the original matrix does. For example, the matrix $\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}$, when applied to a vector (x, y), gives $(-2x, y)$:

$$\begin{pmatrix} -2 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -2x \\ y \end{pmatrix}$$

The inverse of $\mathbf{A}$, when applied to $(-2x, y)$, gives back the original vector, (x, y):

$$\mathbf{A}^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \rightarrow \mathbf{A}^{-1} = -\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}$$

$$-\frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -2 \end{pmatrix}\begin{pmatrix} -2x \\ y \end{pmatrix} = \begin{pmatrix} x \\ y \end{pmatrix}$$

It is of course possible to calculate the inverse of matrices of higher dimensions, but in this course you will not be required to do so by hand.