

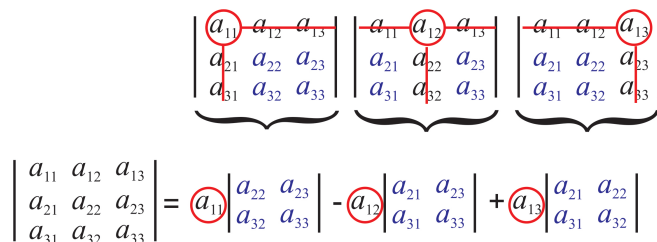
## 13.2: Calculation of a $3 \times 3$ determinant

There are several techniques to calculate determinants, but if this topic is new to you, expanding along the first row is the easiest (although maybe not the most computationally efficient) way of doing it.

A  $3 \times 3$  matrix is calculated as:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \quad (13.2.1)$$

Notice that we multiply each entry in the first row by the determinant formed by what is left after deleting the corresponding row and column. In addition, notice that we alternate signs. Formally, the sign that corresponds to a particular entry  $a_{ij}$  is  $(-1)^{i+j}$ , but if we use the first row we will always start with a “+” and alternate signs afterwards. A schematic of the procedure is shown in Figure 13.2.1.



$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

Figure 13.2.1: A  $3 \times 3$  determinant (CC BY-NC-SA; Marcia Levitus)

We can use the same idea to calculate a determinant of any size. For example, for a  $4 \times 4$  determinant:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} = a_{11} \begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} & a_{24} \\ a_{31} & a_{33} & a_{34} \\ a_{41} & a_{43} & a_{44} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} & a_{24} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix} - a_{14} \begin{vmatrix} a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{vmatrix} \quad (13.2.2)$$

The  $3 \times 3$  determinants are then calculated using Equation 13.2.1.

### ✓ Example 13.2.1

Find  $x$  in the following system of equations:

$$2x + 3y + 8z = 0$$

$$x - \frac{1}{2}y - 3z = \frac{1}{2}$$

$$-x - y - z = \frac{1}{2}$$

#### Solution

We can calculate the  $x$  as:

$$x = \frac{D_1}{D}$$

where

$$D = \begin{vmatrix} 2 & 3 & 8 \\ 1 & -1/2 & -3 \\ -1 & -1 & -1 \end{vmatrix}$$

and

$$D_1 = \begin{vmatrix} 0 & 3 & 8 \\ 1/2 & -1/2 & -3 \\ 1/2 & -1 & -1 \end{vmatrix}$$

$D$  is a  $3 \times 3$  determinant  $D$  and can be expanded using equation 13.2.1:

$$D = 2 \begin{vmatrix} -1/2 & -3 \\ -1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 1 & -3 \\ -1 & -1 \end{vmatrix} + 8 \begin{vmatrix} 1 & -1/2 \\ -1 & -1 \end{vmatrix}$$

$$= 2 \times (-5/2) - (3) \times (-4) + 8 \times (-3/2) = -5$$

The determinant  $D_1$  is similarly expanded:

$$D_1 = 0 \begin{vmatrix} -1/2 & -3 \\ -1 & -1 \end{vmatrix} - 3 \begin{vmatrix} 1/2 & -3 \\ 1/2 & -1 \end{vmatrix} + 8 \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & -1 \end{vmatrix}$$

$$= 0 \times (-5/2) - (3) \times (1) + 8 \times (-1/4) = -5$$

So,

$$x = \frac{D_1}{D} = 1$$

To practice, finish this problem and obtain  $y$  and  $z$  (Problem 13.1).

### ✓ Example 13.2.2

Show that a  $3 \times 3$  determinant that contains zeros below the principal diagonal (top left to bottom right) is the product of the diagonal elements.

#### Solution

We are asked to prove that

$$D = \begin{vmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{vmatrix} = adf$$

Because we have two zeros in the first column, it makes more sense to calculate the determinant by expanding along the first column instead of the first row. Yet, if you feel uncomfortable doing this at this point we can expand along the first row as we just learned:

$$D = a \times \begin{vmatrix} d & e \\ 0 & f \end{vmatrix} - b \times \begin{vmatrix} 0 & e \\ 0 & f \end{vmatrix} + c \times \begin{vmatrix} 0 & d \\ 0 & 0 \end{vmatrix} = a \times d \times f$$

The conclusion is true in any dimension.

Need help? The links below contain solved examples. The determinant of a 3x3 matrix: <http://tinyurl.com/n2a3uxw>

External links:

- Finding the determinant of a 3x3 matrix: <http://patrickjmt.com/finding-the-determinant-of-a-3-x-3-matrix/>
- Solving a system of 3 linear equations: <http://patrickjmt.com/cramers-rule-to-solve-a-system-of-3-linear-equations-example-1/>

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