7.3: Orthogonal Expansions

> **Note**
>
> As stated in Section 7.2, the coefficients of 7.3.7 are defined as so:
>
> $$a_0=\frac{1}{L}\int_{-L}^{L}f(x)dx \tag{7.3.1}$$
>
> $$a_n=\frac{1}{L}\int_{-L}^{L}f(x)\cos\left(\frac{n\pi x}{L}\right)dx \tag{7.3.2}$$
>
> $$b_n=\frac{1}{L}\int_{-L}^{L}f(x)\sin\left(\frac{n\pi x}{L}\right)dx \tag{7.3.3}$$

The idea of expressing functions as a linear combination of the functions of a given basis set is more general than what we just saw. The sines and cosines are not the only functions we can use, although they are a particular good choice for periodic functions. There is a fundamental theorem in function theory that states that we can construct any function using a **complete set** of orthonormal functions.

The term orthonormal means that each function in the set is normalized, and that all functions of the set are mutually orthogonal. For a function in one dimension, the normalization condition is:

$$\int_{-\infty}^{\infty}|f(x)|^2\,dx=1 \tag{7.3.4}$$

Two functions $f(x)$ and $g(x)$ are said to be orthogonal if:

$$\int_{-\infty}^{\infty}f(x)g^*(x)\,dx=0 \tag{7.3.5}$$

The idea that you can construct a function with a linear combination of orthonormal functions is analogous to the idea of constructing a vector in three dimensions by combining the vectors $\overrightarrow{v_1}=\{(1,0,0)\},\overrightarrow{v_2}=\{(0,1,0)\},\overrightarrow{v_3}=\{(0,0,1)\}$, which as we all know are mutually orthogonal, and have unit length.

The basis set we use to construct a Fourier series is

$$\{1,\sin(\frac{\pi}{L}x),\cos(\frac{\pi}{L}x),\sin(2\frac{\pi}{L}x),\cos(2\frac{\pi}{L}x),\sin(3\frac{\pi}{L}x),\cos(3\frac{\pi}{L}x)\ldots\}$$

We will prove that these functions are mutually orthogonal in the interval $[0,2L]$ (one period).

For example, let's prove that $\sin(n\frac{\pi}{L}x)$ and 1 are orthogonal:

$$\int sin\left(\frac{n\pi x}{L}\right)dx=-\frac{L}{n\pi}cos\left(\frac{n\pi x}{L}\right)$$

$$\int_0^{2L}sin\left(\frac{n\pi x}{L}\right)dx=-\frac{L}{n\pi}cos(2n\pi)+\frac{L}{n\pi}cos(0)=\frac{L}{n\pi}(1-cos(2n\pi))=0$$

We can also prove that any $\sin(nx)$ is orthogonal to any $\cos(nx)$:

$$\int sin\left(\frac{n\pi x}{L}\right)\cos\left(\frac{n\pi x}{L}\right)dx=-\frac{L}{4n\pi}\cos\left(\frac{2n\pi x}{L}\right)$$

$$\int_0^{2L}\sin\left(\frac{n\pi x}{L}\right)cos\left(\frac{n\pi x}{L}\right)dx=-\frac{L}{4n\pi}\cos(4n\pi)+\frac{L}{4n\pi}\cos(0)=0$$

Following the same procedure, we can also prove that

$$\int\sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right)dx=0\ n\neq m$$

$$\int \cos\left(\frac{n\pi x}{L}\right)\cos\left(\frac{m\pi x}{L}\right)dx = 0 \; n \neq m$$

The functions used in a Fourier series are mutually orthogonal. Are they normalized?

$$\int_0^{2L} \sin^2\left(\frac{n\pi x}{L}\right)dx = L$$

$$\int_0^{2L} \cos^2\left(\frac{n\pi x}{L}\right)dx = L$$

$$\int_0^{2L} 1^2 \; dx = 2L$$

They are not! The functions $1/2L$, $\frac{1}{L}\sin\left(\frac{\pi}{L}x\right)$ and $\frac{1}{L}\cos\left(\frac{\pi}{L}x\right)$ are normalized, so we may argue that our orthonormal set should be:

$$\left\{\frac{1}{2L}, \frac{1}{L}\sin\left(\frac{\pi}{L}x\right), \frac{1}{L}\cos\left(\frac{\pi}{L}x\right), \frac{1}{L}\sin\left(2\frac{\pi}{L}x\right), \frac{1}{L}\cos\left(2\frac{\pi}{L}x\right), \ldots\right\}$$

and the series should be written as:

$$f(x) = c_0\frac{1}{2L} + \frac{1}{L}\sum_{n=1}^{\infty} c_n \cos\left(\frac{n\pi x}{L}\right) + \frac{1}{L}\sum_{n=1}^{\infty} d_n \sin\left(\frac{n\pi x}{L}\right) \tag{7.3.6}$$

where we used the letters c and d to distinguish these coefficients from the ones defined in Equations 7.3.1, 7.3.2 and 7.3.3.

However if we compare this expression to Equation 7.3.7:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n sin\left(\frac{n\pi x}{L}\right) \tag{7.3.7}$$

we see that it is just a matter of how we define the coefficients. The coefficients in Equation 7.3.7 equal the coefficients in Equation 7.3.6 divided by L. In other words, the coefficients in Equation 7.3.7 already contain the constant L (look at Equations 7.3.1, 7.3.2 and 7.3.3), so we can write the sines and cosines without writing the factor $1/L$ every single time.

In conclusion, the set

$$\left\{1, \sin\left(\frac{\pi}{L}x\right), \cos\left(\frac{\pi}{L}x\right), \sin\left(2\frac{\pi}{L}x\right), \cos\left(2\frac{\pi}{L}x\right), \sin\left(3\frac{\pi}{L}x\right), \cos\left(3\frac{\pi}{L}x\right)\ldots\right\}$$

is not strictly orthonormal the way it is written, but it is once we include the constant L in the coefficients. Therefore, the cosines and sines form a complete set that allows us to express any other function using a linear combination of its members.

There are other orthonormal sets that are used in quantum mechanics to express a variety of functions. Just remember that we can construct any function using a complete set of orthonormal functions.

> *We can construct any function using a complete set of orthonormal functions.*