9.2: Exact and Inexact Differentials

So far, we discussed how to calculate the total differential of a function. If you are given a function of more than one variable, you can calculate its total differential using the definition of a total differential of a function u: ($du = \left(\frac{\partial u}{\partial x_1}\right)_{x_2 \ldots x_n} dx_1 + \left(\frac{\partial u}{\partial x_2}\right)_{x_1, x_3 \ldots x_n} dx_2 + \ldots + \left(\frac{\partial u}{\partial x_n}\right)_{x_1 \ldots x_{n-1}} dx_n$). You will have one term for each independent variable. What if we are given a differential (e.g.

$$dz = (9x^2 + 6xy + y^2)dx + (3x^2 + 2xy)dy$$

see Example 9.1) and we are asked to calculate the function whose total differential is dz? This is basically working Example 9.1 backwards: we know the differential, and we are looking for the function. Things are a little bit more complicated than this, because not all differentials are the total differentials of a function. For example, from the example above we know that

$$dz = (9x^2 + 6xy + y^2)dx + (3x^2 + 2xy)dy$$

is the total differential of

$$z(x,y) = 3x^3 + 3yx^2 + xy^2.$$

However, the differential $dz = xydx + x^2dy$ is **not** the total differential of any function $z(x,y)$. You can write down every single function $z(x,y)$ in this planet, calculate their total differentials, and you will never see $dz = xydx + x^2dy$ in your list.

Therefore, the question we are addressing is the following: given a differential, 1) is it the total differential of any function? 2) if it is, which function?

To illustrate the question, let's say we are given the differential below (notice that I switched to P, V, and T, which are variables you will encounter often in thermodynamics):

$$dP = \frac{RT}{V-b}dT + \left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right]dV \tag{9.2.1}$$

The question is whether this is the total differential of a function $P = P(T,V)$ (we are told that a and b are constants, and we already know that R is a constant). By definition of total differential, if the function exists, its total differential will be:

$$dP = \left(\frac{\partial P}{\partial T}\right)_V dT + \left(\frac{\partial P}{\partial V}\right)_T dV \tag{9.2.2}$$

Comparing Equation 9.2.1 and 9.2.2, if the function exists, its derivatives will have to be:

$$\left(\frac{\partial P}{\partial T}\right)_V = \frac{RT}{V-b} \tag{9.2.3}$$

$$\left(\frac{\partial P}{\partial V}\right)_T = \left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2}\right] \tag{9.2.4}$$

If we find a function $P = P(T,V)$ that satisfies these equations at the same time, we know that Equation 9.2.1 will be its total differential.

From Equation 9.2.3, we can calculate P by integrating with respect to T at constant V:

$$\int dP = \int \frac{RT}{V-b}dT \rightarrow P = \frac{R}{V-b}\frac{T^2}{2} + f(V) \tag{9.2.5}$$

where we included an integration constant ($f(V)$) that can be any function of V (we are integrating at constant V).

In order to get an expression for $P(T,V)$, we need to find out $f(V)$ so we can complete the right side of Equation 9.2.5. To do that, we are going to take the derivative of P (Equation 9.2.5 with respect to V, and compare with Equation 9.2.4:

$$\left(\frac{\partial P}{\partial V}\right)_T = -\frac{RT^2}{2(V-b)^2} + \frac{df(V)}{dV} \tag{9.2.6}$$

Looking at Equation 9.2.4 and 9.2.6, we see that the two expressions do not match, regardless of which function we chose for $f(V)$. This means that Equation 9.2.1 does not represent the total differential of any function $P(V,T)$. We call these differentials **inexact differentials**. If a differential is the total differential of a function, we will call the differential **exact**.

What we did so far is correct, but it is not the easiest way to test whether a differential is exact or inexact. There is, in fact, a very easy way to test for exactness. We'll derive the procedure below, but in the future we can use it without deriving it each time.

Given the differential $dz = f_1(x,y)dx + f_2(x,y)dy$, the differential is exact if

$$\left(\frac{\partial f_1(x,y)}{\partial y}\right)_x = \left(\frac{\partial f_2(x,y)}{\partial x}\right)_y \tag{9.2.7}$$

If Equation 9.2.7 does not hold, the differential is inexact. For instance, if $dz = (9x^2+6xy+y^2)dx + (3x^2+2xy)dy$, the functions f_1 and f_2 are $f_1 = 9x^2+6xy+y^2$ and $f_2 = 3x^2+2xy$. To test this differential, we perform the partial derivatives

$$\left(\frac{\partial f_1(x,y)}{\partial y}\right)_x = 6x+2y$$

and

$$\left(\frac{\partial f_2(x,y)}{\partial x}\right)_y = 6x+2y$$

The two derivatives are the same, and therefore the differential is said to be exact.

Let's prove why the test of Equation 9.2.7 works. Let's consider a differential of the form $dz = f_1(x,y)dx + f_2(x,y)dy$. If the differential is exact, it is the total differential of a function $z(x,y)$, and therefore:

$$dz = f_1(x,y)dx + f_2(x,y)dy = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy \tag{9.2.8}$$

We know that the mixed partial derivatives of a function are independent of the order they are computed:

$$\left(\frac{\partial^2 z}{\partial y \partial x}\right) = \left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

From Equation 9.2.8,

$$f_1(x,y) = \left(\frac{\partial z}{\partial x}\right)_y \rightarrow \left(\frac{\partial f_1(x,y)}{\partial y}\right)_x = \left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

$$f_2(x,y) = \left(\frac{\partial z}{\partial y}\right)_x \rightarrow \left(\frac{\partial f_2(x,y)}{\partial x}\right)_y = \left(\frac{\partial^2 z}{\partial y \partial x}\right)$$

Because the mixed partial derivatives are the same, for an exact differential:

$$\left(\frac{\partial f_1(x,y)}{\partial y}\right)_x = \left(\frac{\partial f_2(x,y)}{\partial x}\right)_y$$

This equation is true only for an exact differential because we derived it by assuming that the function $z = z(x,y)$ exists, so its mixed partial derivatives are the same. We can use this relationship to test whether a differential is exact or inexact. If the equality of Equation 9.2.7 holds, the differential is exact. If it does not hold, it is inexact.

Example 9.2.1

Test whether the following differential is exact or inexact:

$$dz = \frac{1}{x^2}dx - \frac{y}{x^3}dy$$

Solution

To test whether dz is exact or inexact, we compare the following derivatives

$$\left(\frac{\partial(1/x^2)}{\partial y}\right)_x \overset{?}{=} \left(\frac{\partial(y/x^3)}{\partial x}\right)_y$$

$$\left(\frac{\partial(1/x^2)}{\partial y}\right)_x = 0$$

$$\left(\frac{\partial(y/x^3)}{\partial x}\right)_y = -3yx^{-4}$$

We conclude that dz is inexact, and therefore there is no function $z(x,y)$ whose total differential is dz.

Example 9.2.2

Determine whether the following differential is exact or inexact. If it is exact, determine $z = z(x,y)$.

$$dz = (2x+y)dx + (x+y)dy$$

Solution

To test whether dz is exact or inexact, we compare the following derivatives

$$\left(\frac{\partial(2x+y)}{\partial y}\right)_x \overset{?}{=} \left(\frac{\partial(x+y)}{\partial x}\right)_y$$

If this equality holds, the differential is exact.

$$\left(\frac{\partial(2x+y)}{\partial y}\right)_x = 1$$

$$\left(\frac{\partial(x+y)}{\partial x}\right)_y = 1$$

Therefore, the differential is exact. Because it is exact, it is the total differential of a function $z(x,y)$. The total differential of $z(x,y)$ is, by definition,

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Comparing this expression to the differential $dz = (2x+y)dx + (x+y)dy$:

$$\left(\frac{\partial z}{\partial x}\right)_y = (2x+y)$$

$$\left(\frac{\partial z}{\partial y}\right)_x = (x+y) \tag{9.2.9}$$

To find $z(x,y)$, we can integrate the first expression partially with respect to x keeping y constant:

$$\int dz = z = \int (2x+y)dx = x^2 + xy + f(y)$$

So far we have

$$z = x^2 + xy + f(y) \tag{9.2.10}$$

so we need to find the function $f(y)$ to complete the expression above and finish the problem. To do that, we'll take the derivative of z with respect to y, and compare with Equation 9.2.9. The derivative of Equation 9.2.10 is:

$$\left(\frac{\partial z}{\partial y}\right)_x = x + \frac{df(y)}{dy}$$

comparing with Equation 9.2.9 we notice that $\frac{df(y)}{dy} = y$, and integrating, we obtain $f(x) = y^2/2 + c$

Therefore, $dz = (2x + y)dx + (x + y)dy$ is the total differential of $z = x^2 + xy + y^2/2 + c$.

We can check our result by working the problem in the opposite direction. If we are given $z = x^2 + xy + y^2/2 + c$ and we are asked to calculate its total differential, we would apply the definition:

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

and because

$$\left(\frac{\partial z}{\partial x}\right)_y = y + 2x$$

and

$$+\left(\frac{\partial z}{\partial y}\right)_x = y + x$$

we would write $dz = (2x + y)dx + (x + y)dy$, which is the differential we were given in the problem.

Check two extra solved examples in this video: http://tinyurl.com/kq4qecu