13.5: Problems

Problem 13.5.1

Finish the problem of Example $13.1.1$ and obtain y and z.

Problem 13.5.2

Use determinants to solve the equations:

A)

$$\begin{gathered} x+y+z=6 \\ x+2y+3z=14 \\ x+4y+9z=36 \end{gathered}$$

B)

$$\begin{gathered} x+iy-z=0 \\ ix+y+z=0 \\ x+2y-iz=1 \end{gathered}$$

Problem 13.5.3

Show that a 3×3 determinant that contains zeros above the principal diagonal is the product of the diagonal elements.

$$D=\begin{vmatrix} a & 0 & 0 \\ b & c & 0 \\ d & e & f \end{vmatrix}=acf$$

Problem 13.5.4

Prove that

$$D=\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \\ 3 & 4 & 3 \end{vmatrix}=0$$

using the properties of determinants (that is, without calculating the determinant!). Clearly state the properties you use in each step.

Exercise 13.5.5

In previous lectures, we discussed how to perform double and triple integrals in different coordinate systems. For instance, we learned that the area elements and volume elements are:

2D:

Cartesian: $dA=dx.\,dy$
Polar: $dA=r.\,dr.\,d\theta$

3D:

Cartesian: $dV=dx.\,dy.\,dz$
Spherical: $dV=r^2.\sin\theta dr.\,d\theta d\phi$

In general, for any coordinate system, we can express the area (or volume) element in a new coordinate system using the Jacobian (J). For example, in polar coordinates in two dimensions:

$$dA = dx.\,dy = J.\,dr.\,d\theta$$

where the Jacobian is defined as:

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}$$

a) Calculate the Jacobian in two-dimensional polar coordinates and show that $dA = r.\,dr.\,d\theta$.

In spherical coordinates,

$$dV = dx.\,dy.\,dz = J.\,dr.\,d\theta.\,d\phi$$

where

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix}$$

b) Calculate the Jacobian in three-dimensional spherical coordinates and show that

$$dV = r^2.\sin\theta dr.\,d\theta d\phi$$