7.1: Introduction to Fourier Series

In Chapter 3 we learned that a function $f(x)$ can be expressed as a series in powers of x as long as $f(x)$ and all its derivatives are finite at $x=0$. We then extended this idea to powers of $x-h$, and called these series "Taylor series". If $h=0$, the functions that form the basis set are the powers of $x: x^0, x^1, x^2 \ldots$, and in the more general case of $h \neq 0$, the basis functions are $(x-h)^0, (x-h)^1, (x-h)^2 \ldots$

The powers of x or $(x-h)$ are not the only choice of basis functions to expand a function in terms of a series. In fact, if we want to produce a series which will converge rapidly, so that we can truncate if after only a few terms, it is a good idea to choose basis functions that have as much as possible in common with the function to be represented. If we want to represent a periodic function, it is useful to use a basis set containing functions that are periodic themselves. For example, consider the following set of functions: $\sin(nx),\ n=1,2,\ldots,\infty$:

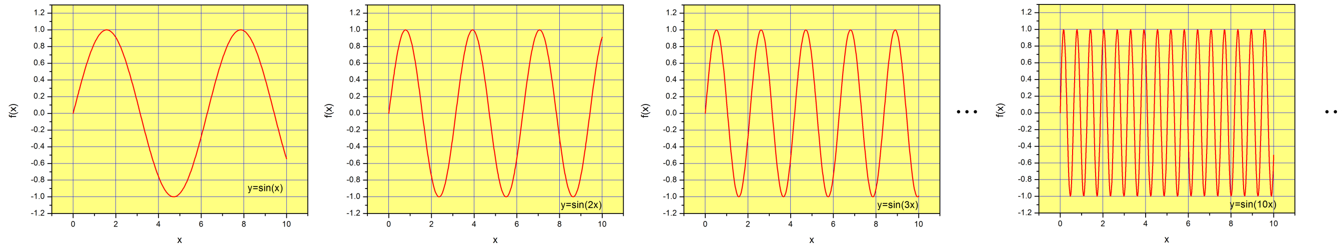

Figure 7.1.1: Some examples of the family of funtions $\sin(nx)$. From left to right: $\sin(x), \sin(2x), \sin(3x)$ and $\sin(10x)$ (CC BY-NC-SA; Marcia Levitus)

We can mix a finite number of these functions to produce a periodic function like the one shown in the left panel of Figure 7.1.2, or an infinite number of functions to produce a periodic function like the one shown on the right. Notice that an infinite number of sine functions creates a function with straight lines! We will see that we can create all kinds of periodic functions by just changing the coefficients (i.e. the numbers multiplying each sine function).

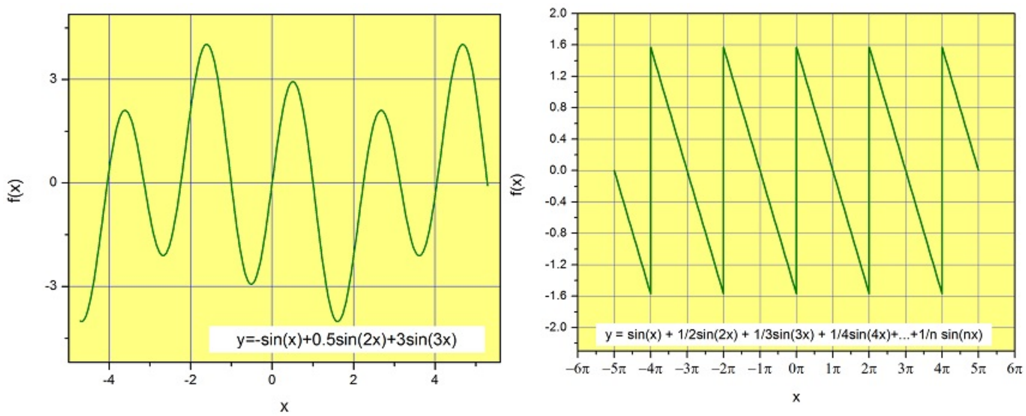

Figure 7.1.2: Examples of periodic functions that are linear combinations of $sin(nx)$ (CC BY-NC-SA; Marcia Levitus)

So far everything sounds fine, but we have a problem. The functions $\sin nx$ are all odd, and therefore any linear combination will produce an odd periodic function. We might need to represent an even function, or a function that is neither odd nor even. This tells us that we need to expand our basis set to include even functions, and I hope you will agree the obvious choice are the cosine functions $\cos(nx)$.

Below are two examples of even periodic functions that are produced by mixing a finite (left) or infinite (right) number of cosine functions. Notice that both are even functions.

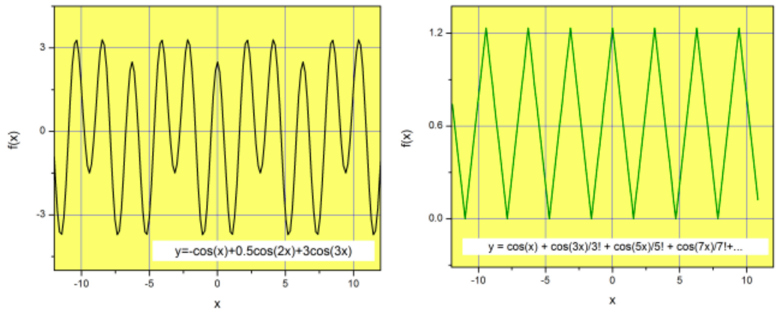

Figure $7.1.3$: Examples of periodic functions that are linear combinations of $\cos(nx)$ functions (CC BY-NC-SA; Marcia Levitus)

Before moving on, we need to review a few concepts. First, since we will be dealing with periodic functions, we need to define the period of a function. As we saw in Section 1.4, a function $f(x)$ is said to be periodic with period P if $f(x) = f(x+P)$. For example, the period of the function of Figure $7.1.4$ is 2π.

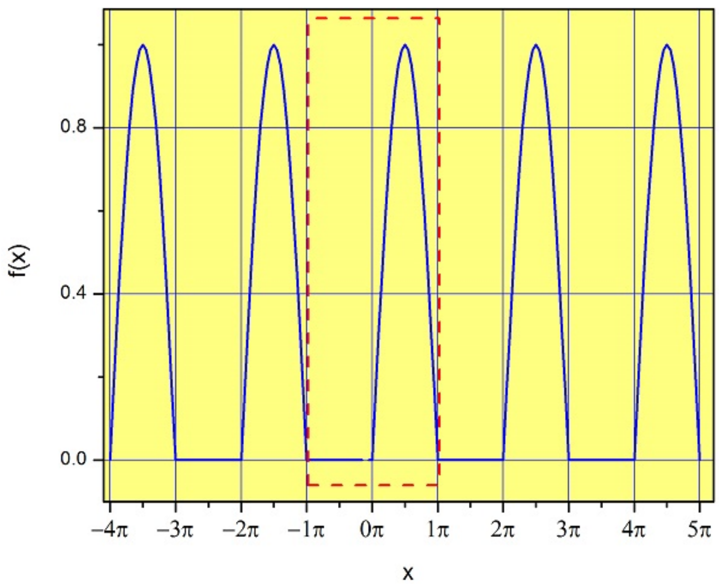

Figure $7.1.4$: A periodic function with period $P = 2\pi$ (CC BY-NC-SA; Marcia Levitus)

How do we write the equation for this periodic function? We just need to specify the equation of the function between $-P/2$ and $P/2$. This range is shown in a red dotted line in Figure $7.1.4$, and as you can see, it has the width of a period, and it is centered around $x = 0$. If we have this information, we just need to extend the function to the left and to the right to create the periodic function:

$$f(x)=\begin{cases}\sin x & 0\le x\le\pi\\ 0 & -\pi\le x\le 0\end{cases}$$

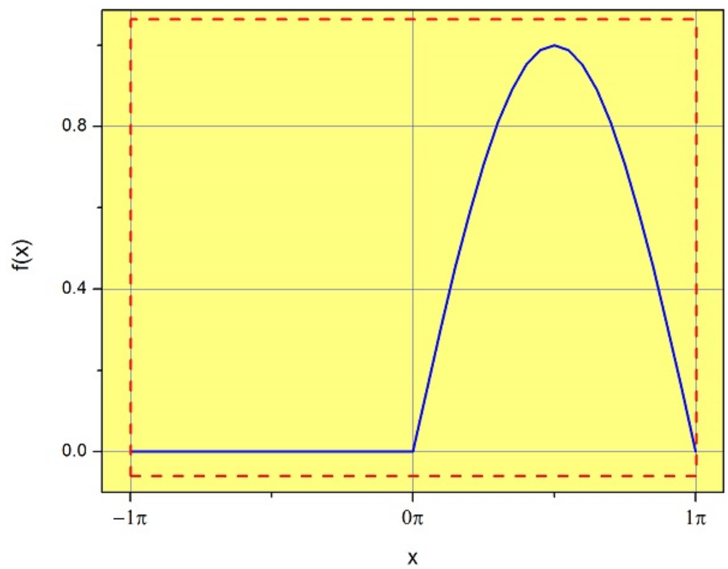

Figure 7.1.4 (CC BY-NC-SA; Marcia Levitus)