9.6: Exact and Inexact Differentials (Summary)

To summarize, given a function $f(x,y)$, its total differential df is, by definition:

$$df = \left(\frac{\partial f}{\partial x}\right)_y dx + \left(\frac{\partial f}{\partial y}\right)_x dy$$

Given an arbitrary differential

$$df = M(x,y)dx + N(x,y)dy$$

where M and N are functions of x and y, the differential is **exact** if it is the total differential of a function $f(x,y)$. To test for exactness we compare the partial derivative of $M(x,y)$ with respect to y, and the partial derivative of $N(x,y)$ with respect to x:

$$\left(\frac{\partial M(x,y)}{\partial y}\right)_x \stackrel{?}{=} \left(\frac{\partial N(x,y)}{\partial x}\right)_y$$

If the derivatives are identical, we conclude that the differential df is exact, and therefore it is the total differential of a function $f(x,y)$. To find the function, we notice that for an exact differential:

$$M(x,y) = \left(\frac{\partial f}{\partial x}\right)_y$$

$$N(x,y) = \left(\frac{\partial f}{\partial y}\right)_x$$

We can then find the function by partial integration:

$$f(x,y) = \int df = \int M(x,y)dx \text{ (at constant } y)$$

$$f(x,y) = \int df = \int N(x,y)dy \text{ (at constant } x)$$

It is important to keep in mind that the integration constant in the first case will be an arbitrary function of y, and in the second case an arbitrary function of x.

For an exact differential, the line integral does not depend on the path, but only on the initial and final points. Furthermore, because the differential is exact, it is the total differential of a state function $f(x,y)$. This means that the integral of df along any path is simply the function f evaluated at the final state minus the function f evaluated at the initial state:

$$\int_c M(x,y)dx + N(x,y)dy = \int_c df = \Delta f = f(x_2,y_2) - f(x_1,y_1)$$

where c represents the path that starts at the point (x_1,y_1) and ends at the point (x_2,y_2).

If the initial and the final states are identical, for an exact differential:

$$\oint df = 0$$

For an inexact differential, $\int_c df$ will in general depend on the path c.