# 7.6: Appendix - Review of Free Electromagnetic Field

Here we review the derivation of the vector potential for the plane wave in free space. We begin with Maxwell's equations (SI):

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{7.6.1}$$

$$\overline{\nabla} \cdot \overline{E} = \rho/\varepsilon_0 \tag{7.6.2}$$

$$\overline{\nabla} \times \overline{E} = -\frac{\partial \overline{B}}{\partial t} \tag{7.6.3}$$

$$\overline{\nabla} \times \overline{B} = \mu_0 \overline{J} + \varepsilon_0 \mu_0 \frac{\partial \overline{E}}{\partial t} \tag{7.6.4}$$

Here the variables are: $\overline{E}$, electric field; $\overline{B}$, magnetic field; $\overline{J}$, current density; $\rho$, charge density; $\mathcal{E}_0$, electrical permittivity; $\mu_0$, magnetic permittivity. We are interested in describing $\overline{E}$ and $\overline{B}$ in terms of a vector and scalar potential, $\overline{A}$ and $\varphi$.

Next, let's review some basic properties of vectors and scalars. Generally, vector field $\overline{F}$ assigns a vector to each point in space. The divergence of the field

$$\overline{\nabla} \cdot \overline{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \tag{7.6.5}$$

is a scalar. For a scalar field $\phi$, the gradient

$$\nabla \phi = \frac{\partial \phi}{\partial x}\hat{x} + \frac{\partial \phi}{\partial y}\hat{y} + \frac{\partial \phi}{\partial z}\hat{z} \tag{7.6.6}$$

is a vector for the rate of change at one point in space. Here

$$\hat{x}^2 + \hat{y}^2 + \hat{z}^2 = \hat{r}^2 \tag{7.6.7}$$

are unit vectors. Also, the curl

$$\overline{\nabla} \times \overline{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{7.6.8}$$

is a vector whose $x$, $y$, and $z$ components are the circulation of the field about that component. Some useful identities from vector calculus that we will use are

$$\overline{\nabla} \cdot (\overline{\nabla} \times \overline{F}) = 0 \tag{7.6.9}$$

$$\nabla \times (\nabla \phi) = 0 \tag{7.6.10}$$

$$\nabla \times (\overline{\nabla} \times \overline{F}) = \overline{\nabla}(\overline{\nabla} \cdot \overline{F}) - \overline{\nabla}^2 \overline{F} \tag{7.6.11}$$

## Gauge Transforms

We now introduce a vector potential $\overline{A}(\bar{r}, t)$ and a scalar potential $\varphi(\bar{r}, t)$, which we will relate to $\overline{E}$ and $\overline{B}$. Since

$$\overline{\nabla} \cdot \overline{B} = 0 \tag{7.6.12}$$

and

$$\overline{\nabla}(\overline{\nabla} \times \overline{A}) = 0, \tag{7.6.13}$$

we can immediately relate the vector potential and magnetic field

$$\overline{B} = \overline{\nabla} \times \overline{A} \tag{7.6.14}$$

Inserting this into Equation 7.6.3 and rewriting, we can relate the electric field and vector potential:

$$\overline{\nabla}\times\left[\overline{E}+\frac{\partial\overline{A}}{\partial t}\right]=0 \tag{7.6.15}$$

Comparing Equations 7.6.15 and 7.6.10 allows us to state that a scalar product exists with

$$\overline{E}=\frac{\partial\overline{A}}{\partial t}-\nabla\varphi \tag{7.6.16}$$

So summarizing our results, we see that the potentials $\overline{A}$ and $\varphi$ determine the fields $\overline{B}$ and $\overline{E}$:

$$\overline{B}(\overline{r},t)=\overline{\nabla}\times\overline{A}(\overline{r},t) \tag{7.6.17}$$

$$\overline{E}(\overline{r},t)=-\overline{\nabla}\varphi(\overline{r},t)-\frac{\partial}{\partial t}\overline{A}(\overline{r},t) \tag{7.6.18}$$

We are interested in determining the classical wave equation for $\overline{A}$ and $\varphi$. Using Equation 7.6.17, differentiating Equation 7.6.18, and substituting into Equation 7.6.4, we obtain

$$\overline{\nabla}\times(\overline{\nabla}\times\overline{A})+\varepsilon_0\mu_0\left(\frac{\partial^2\overline{A}}{\partial t^2}+\overline{\nabla}\frac{\partial\varphi}{\partial t}\right)=\mu_0\overline{J} \tag{7.6.19}$$

Using Equation 7.6.11,

$$\left[-\overline{\nabla}^2\overline{A}+\varepsilon_0\mu_0\frac{\partial^2\overline{A}}{\partial t^2}\right]+\overline{\nabla}\left(\overline{\nabla}\cdot\overline{A}+\varepsilon_0\mu_0\frac{\partial\varphi}{\partial t}\right)=\overline{\mu}_0\overline{J} \tag{7.6.20}$$

From Equation 7.6.16, we have

$$\overline{\nabla}\cdot\overline{E}=-\frac{\partial\overline{\nabla}\cdot\overline{A}}{\partial t}-\overline{\nabla}^2\varphi \tag{7.6.21}$$

and using Equation 7.6.2,

$$\frac{-\partial\overline{V}\cdot\overline{A}}{\partial t}-\overline{\nabla}^2\varphi=\rho/\varepsilon_0 \tag{7.6.22}$$

Notice from Equations 7.6.17 and 7.6.18 that we only need to specify four field components ($A_x, A_y, A_z, \varphi$ to determine all six $\bar{E}$ and $\bar{B}$ components. But $\bar{E}$ and $\bar{B}$ do not uniquely determine $\bar{A}$ and $\varphi$. So we can construct $\bar{A}$ and $\varphi$ in any number of ways without changing $\bar{E}$ and $\bar{B}$. Notice that if we change $\bar{A}$ by adding $\bar{\nabla}\chi$ where $\chi$ is any function of $\bar{r}$ and $t$ this will not change $\bar{B}$ $\quad(\nabla\times(\nabla\cdot B)=0)$ . It will change $E$ by $\left(-\frac{\partial}{\partial t}\bar{\nabla}\chi\right)$, but we can change $\varphi$ to $\varphi'=\varphi-(\partial\chi/\partial t)$ . Then $\bar{E}$ and $\bar{B}$ will both be unchanged. This property of changing representation (gauge) without changing $\bar{E}$ and $\bar{B}$ is **gauge invariance**. We can define a gauge transformation with

$$\bar{A}'(\bar{r},t)=\bar{A}(\bar{r},t)+\bar{\nabla}\cdot\chi(\bar{r},t) \tag{7.6.23}$$

$$\varphi'(\bar{r},t)=\varphi(\bar{r},t)-\frac{\partial}{\partial t}\chi(\bar{r},t) \tag{7.6.24}$$

Up to this point, $A'\mathrm{and}\varphi'$ are undetermined. Let's choose a $\chi$ such that:

$$\overline{\nabla}\cdot\overline{A}+\varepsilon_0\mu_0\frac{\partial\varphi}{\partial t}=0 \tag{7.6.25}$$

which is known as the **Lorentz condition**. Then from Equation 7.6.19:

$$-\nabla^2\overline{A}+\varepsilon_0\mu_0\frac{\partial^2\overline{A}}{\partial t^2}=\mu_0\overline{J} \tag{7.6.26}$$

The right hand side of this equation can be set to zero when no currents are present. From Equation 7.6.22, we have:

$$\varepsilon_0\mu_0\frac{\partial^2\varphi}{\partial t^2}-\nabla^2\varphi=\frac{\rho}{\varepsilon_0} \tag{7.6.27}$$

Equations 7.6.26 and 7.6.27 are wave equations for $\overline{A}$ and $\varphi$. Within the Lorentz gauge, we can still arbitrarily add another $\chi$; it must only satisfy Equation 7.6.25. If we substitute Equations 7.6.23 and 7.6.24 into Equation 7.6.27, we see

$$\nabla^2\chi-\varepsilon_0\mu_0\frac{\partial^2\chi}{\partial t^2}=0 \tag{7.6.28}$$

So we can make further choices/constraints on $\bar{A}\text{and}\varphi$ as long as it obeys Equation 7.6.28. We now choose $\varphi=0$, the **Coulomb gauge**, and from Equation 7.6.25 we see

$$\overline{\nabla}\cdot\overline{A}=0 \tag{7.6.29}$$

So the wave equation for our vector potential when the field is far currents ($J=0$) is

$$-\overline{\nabla}^2\overline{A}+\varepsilon_0\mu_0\frac{\partial^2\overline{A}}{\partial t^2}=0 \tag{7.6.30}$$

The solutions to this equation are plane waves:

$$\overline{A}=\overline{A}_0\sin(\omega t-\bar{k}\cdot\bar{r}+\alpha) \tag{7.6.31}$$

where $\alpha$ is a phase. $k$ is the wave vector which points along the direction of propagation and has a magnitude

$$k^2=\omega^2\mu_0\varepsilon_0=\omega^2/c^2 \tag{7.6.32}$$

Since $\overline{\nabla}\cdot\overline{A}=0$ (Equation 7.6.29), then

$$-\bar{k}\cdot\overline{A}_0\cos(\omega t-\bar{k}\cdot\bar{r}+\alpha)=0 \tag{7.6.33}$$

and

$$\bar{k}\cdot\overline{A}_0=0 \tag{7.6.34}$$

So the direction of the vector potential is perpendicular to the direction of wave propagation ($\bar{k}\perp\overline{A_0}$). From Equations 7.6.17 and 7.6.18, we see that for $\varphi=0$:

$$\overline{E}=-\frac{\partial\overline{A}}{\partial t} \tag{7.6.35}$$

$$=-\omega\overline{A}_0\cos(\omega t-\bar{k}\cdot\bar{r}+\alpha) \tag{7.6.36}$$

$$\overline{B}=\overline{\nabla}\times\overline{A} \tag{7.6.37}$$

$$=-\left(\bar{k}\times\overline{A}_0\right)\cos(\omega t-\bar{k}\cdot\bar{r}+\alpha) \tag{7.6.38}$$

Here the electric field is parallel with the vector potential, and the magnetic field is perpendicular to the electric field and the direction of propagation ($\bar{k}\perp\overline{E}\perp\overline{B}$). The Poynting vector describing the direction of energy propagation is

$$\overline{S}=\varepsilon_0c^2(\overline{E}\times\overline{B}) \tag{7.6.39}$$

and its average value, the intensity, is

$$I=\langle S\rangle=\frac{1}{2}\varepsilon_0cE_0^2. \tag{7.6.40}$$

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