4.4: The Probability-density Function for Gas Velocities in One Dimension

In Section 4.3, we find a differential equation in the function $\rho_x(v_x)$. Unlike the velocity, which takes values from zero to infinity, the x-component, v_x, takes values from minus infinity to plus infinity. The probability density at an infinite velocity, in either direction, is necessarily zero. Therefore, we cannot evaluate the integral of $d\rho_x(v_x)/\rho_x(v_x)$ from $v_x=-\infty$ to an arbitrary velocity, v_x. However, we know from Maxwell's assumption that the probability density for v_x must be independent of whether the molecule is traveling in the direction of the positive x-axis or the negative x-axis. That is, $\rho_x(v_x)$ must be an even function; the probability density function must be symmetric around $v_x=0$; $\rho_x(v_x)=\rho_x(-v_x)$. Hence, we can express $\rho_x(v_x)$ relative to its fixed value, $\rho_x(0)$, at $v_x=0$. We integrate $d\rho_x(v_x)/\rho_x(v_x)$ from $\rho_x(0)$ to $\rho_x(v_x)$ as v_x goes from zero to an arbitrary velocity, v_x, to find

$$\int_{\rho_x(0)}^{\rho_x(v_x)} \frac{d\rho_x(v_x)}{\rho_x(v_x)} = -\lambda \int_0^{v_x} v_x dv_x$$

or

$$\rho_x(v_x) = \frac{df_x(v_x)}{dv_x} = \rho_x(0) \exp\left(\frac{-\lambda v_x^2}{2}\right)$$

The value of $\rho_x(0)$ must be such as to make the integral of $\rho_x(v_x)$ over all possible values of v_x, \(-\infty < v_x <\infty \), equal to unity. That is, we must have

$$\begin{aligned} 1 &= \int_{-\infty}^{\infty} \rho_x(v_x)\, dv_x \\ &= \int_{-\infty}^{\infty} \frac{df_x(v_x)}{dv_x} dv_x \\ &= \rho_x(0) \int_{-\infty}^{\infty} \exp\left(\frac{-\lambda v_x^2}{2}\right) dv_x \\ &= \rho_x(0)\sqrt{\frac{2\pi}{\lambda}} \end{aligned}$$

where we use the definite integral $\int_{-\infty}^{\infty} \exp(-ax^2)\, dx = \sqrt{\pi/a}$. (See Appendix D.) It follows that $\rho_x(0) = (\lambda/2\pi)^{1/2}$. The one-dimensional probability-density function becomes

$$\begin{aligned} \rho_x(v_x) &= \frac{df_x(v_x)}{dv_x} \\ &= \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left(\frac{-\lambda v_x^2}{2}\right) \end{aligned}$$

Note that this is the normal distribution with $\mu=0$ and $\sigma^2=\lambda^{-1}$. So λ^{-1} is the variance of the normal one-dimensional probability-density function. As noted above, in Section 4.6 we find that $\lambda = m/kT$.