7.6: The Chain Rule and the Divide-through Rule

If we have $f\left(x,y\right)$ while x and y are functions of another variable, u, the chain rule states that

$$\frac{df}{du}=\left(\frac{\partial f}{\partial x}\right)_y\frac{dx}{du}+\left(\frac{\partial f}{\partial y}\right)_x\frac{dy}{du}$$

If x and y are functions of variables u and v; that is, $x=x\left(u,v\right)$ and $y=y\left(u,v\right)$, the chain rule for partial derivatives is

$$\left(\frac{\partial f}{\partial u}\right)_v=\left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial u}\right)_v+\left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial u}\right)_v$$

A useful mnemonic recognizes that these equations can be generated from the total differential by "dividing through" by du. We must specify that the "new" partial derivatives are taken with v held constant. This is sometimes called the ***divide-through rule***.

The divide-through rule is a reliable expedient for generating new relationships among partial derivatives. As a further example, dividing by dx and specifying that any other variable is to be held constant produces a valid equation. Letting w be the variable held constant, we obtain

$$\begin{aligned}\left(\frac{\partial f}{\partial x}\right)_w&=\left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial x}\right)_w+\left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_w\\&=\left(\frac{\partial f}{\partial x}\right)_y+\left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial x}\right)_w\end{aligned}$$

where we recognize that $\left(\partial x/\partial x\right)_w=1$. The result is just the chain rule for $\left(\partial f/\partial x\right)_w$ when $f=f\left(x,y\right)$ and $y=y\left(x,w\right)$; that is, when $f=f\left(x,y\left(x,w\right)\right)$.

If we require that $f\left(x,y\right)$ remain constant while x and y vary, we can use the divide-though rule to obtain another useful relationship from the total differential. If $f\left(x,y\right)$ is constant, $df\left(x,y\right)=0$. This can only be true if there is a relationship between x and y. To find this relationship we use the divide-through rule to find $\left(\partial f/\partial y\right)_f$ when $f=f\left(x\left(y\right),y\right)$. Dividing

$$df=\left(\frac{\partial f}{\partial x}\right)_y dx+\left(\frac{\partial f}{\partial y}\right)_x dy$$

by dy, and stipulating that f is constant, we find

$$\left(\frac{\partial f}{\partial y}\right)_f=\left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_f+\left(\frac{\partial f}{\partial y}\right)_x\left(\frac{\partial y}{\partial y}\right)_f$$

Since $\left(\partial f/\partial y\right)_f=0$ and $\left(\partial y/\partial y\right)_f=1$, we have

$$\left(\frac{\partial f}{\partial y}\right)_x=-\left(\frac{\partial f}{\partial x}\right)_y\left(\frac{\partial x}{\partial y}\right)_f$$

In Chapter 10, we find that the divide-through rule is a convenient way to generate thermodynamic relationships.