7.5: Determining Whether an Expression is an Exact Differential

Since exact differentials have these important characteristics, it is valuable to know whether a given differential expression is exact or not. That is, given a differential expression of the form

$$df = M(x,y)\,dx + N(x,y)\,dy, \tag{7.5.1}$$

we would like to be able to determine whether df is exact or inexact. It turns out that there is a simple test for exactness:

test for exactness

The differential in the form of Equation 7.5.1 is exact **if and only if**

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{7.5.2}$$

That is, this condition is necessary and sufficient for the existence of a function, $f(x,y)$, for which $M(x,y) = f_x(x,y)$ and $N(x,y) = f_y(x,y)$.

In §4 we demonstrate that the condition is necessary. Now we want to show that it is sufficient. That is, we want to demonstrate: If Equation 7.5.2 hold, then there exists a $f(x,y)$, such that $M(x,y) = f_x(x,y)$ and $N(x,y) = f_y(x,y)$. To do this, we show how to find a function, $f(x,y)$, that satisfies the given differential relationship. If we integrate $M(x,y)$ with respect to x, we have

$$f(x,y) = \int M(x,y)\,dx + h(y)$$

where $h(y)$ is a function only of y; it is the arbitrary constant in the integration with respect to x, which we carry out with y held constant.

To complete the proof, we must find a function $h(y)$ such that this $f(x,y)$ satisfies the conditions:

$$M(x,y) = f_x(x,y) \Leftrightarrow \tag{7.5.3}$$

$$= \frac{\partial}{\partial x}\left[\int M(x,y)\,dx + h(y)\right] \tag{7.5.4}$$

$$N(x,y) = f_y(x,y) \Leftrightarrow \tag{7.5.5}$$

$$= \frac{\partial}{\partial y}\left[\int M(x,y)\,dx + h(y)\right] \tag{7.5.6}$$

The validity of condition in Equation 7.5.3 follows immediately from the facts that the order of differentiation and integration can be interchanged for a continuous function and that $h(y)$ is a function only of y, so that $\partial h/\partial x = 0$.

To find $h(y)$ such that condition in Equation 7.5.5 is satisfied, we observe that

$$\frac{\partial}{\partial y}\left[\int M(x,y)\,dx + h(y)\right] = \int \left(\frac{\partial M(x,y)}{\partial y}\right) dx + \frac{dh(y)}{dy}$$

But since

$$\frac{\partial M(x,y)}{\partial y} = \frac{\partial N(x,y)}{\partial x}$$

this becomes

$$\frac{\partial}{\partial y}\left[\int M(x,y)\,dx + h(y)\right] = \int \left(\frac{\partial N(x,y)}{\partial x}\right) dx + \frac{dh(y)}{d} y = N(x,y) + \frac{dh(y)}{dy}$$

Hence, condition in Equation 7.5.5 is satisfied if and only if $dh(y)/dy = 0$, so that $h(y)$ is simply an arbitrary constant.