# 7.4: Exact Differentials and State Functions

Now, let us consider the general case of a continuous function $f(x,y)$, for which the exact differential is

$$df = f_x(x,y)\,dx + f_y(x,y)\,dy.$$

We want to integrate the exact differential over very short paths like paths a and b in Section 7.3. Let us evaluate the integral between $(x_0, y_0)$ and $(x_0+\Delta x, y_0+\Delta y)$ over the paths a* and b* sketched in Figure 3.

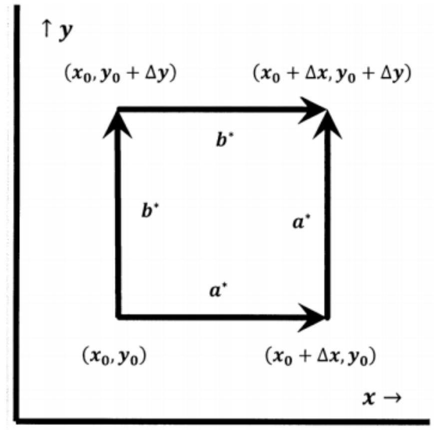


Figure 3. Alternative paths from $(x_0,\ y_0)$ to $(x_0+\Delta x,\ y_0+\Delta y\ )$

- **Path a*** has two linear segments. The first segment is the portion of the line $y = y_0$ as $x$ goes from $x_0$ to $x_0+\Delta x$. Along the first segment $\Delta y = 0$. The second segment is the portion of the line $x = x_0 + \Delta x$ as $y$ goes from $y_0$ to $y_0+\Delta y$. Along the second segment, $\Delta x = 0$.
- **Path b*** has two linear segments also. The first segment is the portion of the line $x = x_0$ as $y$ goes from $y_0$ to $y_0+\Delta y$. Along the first segment, $\Delta x = 0$. The second segment is the portion of the line $y = y_0 + \Delta y$ as $x$ goes from $x_0$ to $x_0+\Delta x$. Along the second segment, $\Delta y = 0$.

Along path a*, we have

$$\Delta_{a^*} f = f_x(x_0, y_0)\,\Delta x + f_y(x_0+\Delta x, y_0)\,\Delta y$$

Along path b*,

$$\Delta_{b^*} f = f_x(x_0, y_0+\Delta y)\,\Delta x + f_y(x_0, y_0)\,\Delta y$$

In the limit as $\Delta x$ and $\Delta y$ become arbitrarily small, we must have $\Delta_{a^*} f = \Delta_{b^*} f$, so that

$$f_x(x_0, y_0)\,\Delta x + f_y(x_0+\Delta x, y_0)\,\Delta y = f_x(x_0, y_0+\Delta y)\,\Delta x + f_y(x_0, y_0)\,\Delta y$$

Rearranging this equation so that terms in $f_x$ are on one side and terms in $f_y$ are on the other side, dividing both sides by $\Delta x \Delta y$, and taking the limit as $\Delta x \to 0$ and $\Delta y \to 0$, we have

$$\lim_{\Delta x \to 0}\left[\frac{f_y(x_0+\Delta x, y_0) - f_y(x_0, y_0)}{\Delta x}\right] = \lim_{\Delta y \to 0}\left[\frac{f_x(x_0, y_0+\Delta y) - f_x(x_0, y_0)}{\Delta y}\right]$$

These limits are the partial derivative of $f_y(x_0, y_0)$ with respect to $x$ and of $f_x(x_0, y_0)$ with respect to $y$. That is

$$\left[\frac{\partial}{\partial x} f_y(x_0, y_0)\right]_y = \left[\frac{\partial}{\partial x}\left(\frac{\partial f(x_0, y_0)}{\partial y}\right)\right]_y = \frac{\partial^2 f(x_0, y_0)}{\partial y \partial x}$$

and

$$\left[\frac{\partial}{\partial y} f_x(x_0, y_0)\right]_x = \left[\frac{\partial}{\partial y}\left(\frac{\partial f(x_0, y_0)}{\partial x}\right)\right]_x = \frac{\partial^2 f(x_0, y_0)}{\partial x \partial y}$$

This shows that, if $f(x,y)$ is a continuous function of $x$ and $y$ whose partial derivatives exist, then

$$\frac{\partial^2 f\left(x_0, y_0\right)}{\partial y \partial x}=\frac{\partial^2 f\left(x_0, y_0\right)}{\partial x \partial y}$$

The mixed second partial derivative of $f\left(x, y\right)$ is independent of the order of differentiation. We also write these second partial derivatives as $f_{xy}\left(x_0, y_0\right)$ and $f_{yx}\left(x_0, y_0\right)$.

To summarize these points, if $f\left(x, y\right)$ is a continuous function of $x$ and $y$, all of the following are true:

1. $f\left(x, y\right)$ represents a surface in a three-dimensional space.
2. $f\left(x, y\right)$ is a state function.
3. The total differential is

$$df=\left(\partial f/\partial x\right)_y dx+\left(\partial f/\partial y\right)_x dy.$$

4. The total differential is exact.
5. The line integral of $df$ between two points is independent of the path of integration.
6. The line integral of $df$ around any closed path is zero: $\oint df=0$.
7. The mixed second-partial derivatives are equal; that is,

$$\frac{\partial^2 f}{\partial y \partial x}=\frac{\partial^2 f}{\partial x \partial y}$$