19.2: Distribution of Results for Multiple Trials with Three Possible Outcomes

Let us extend the ideas we have developed for binomial probabilities to the case where there are three possible outcomes for any given trial. To be specific, suppose we have a coin-sized object in the shape of a truncated right-circular cone, whose circular faces are parallel to each other. The circular faces have different diameters. When we toss such an object, allowing it to land on a smooth hard surface, it can wind up resting on the big circular face ($\boldsymbol{H}$eads), the small circular face ($\boldsymbol{T}$ails), or on the conical surface ($\boldsymbol{C}$one-side). Let the probabilities of these outcomes in a single toss be P_H, P_T, and P_C, respectively. In general, we expect these probabilities to be different from one another; although, of course, we require $1=(P_H+P_T+P_C)$.

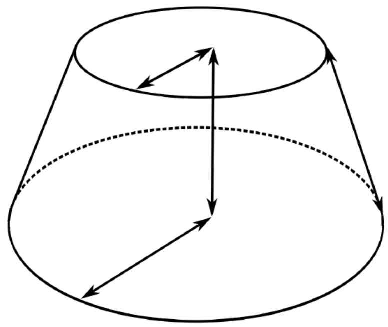

Figure 1: A truncated right-circular cone (CC BY-SA Unported; Dirk Hünniger via Wikipedia).

Following our development for the binomial case, we want to write an equation for the total probability sum after n tosses. Let n_H, n_T, and n_C be the number of H, T, and C outcomes exhibited in $n_H+n_T+n_C=n$ trials. We let the probability coefficients be $C\left(n_H,n_T,n_C\right)$. The probability of n_H, n_T, n_C outcomes in n trials is

$$C\left(n_H,n_T,n_C\right)P^{n_H}_HP^{n_T}_TP^{n_C}_C$$

and the total probability is

$$1={\left(P_H+P_T+P_C\right)}^n=\sum_{n_H,n_T,n_C}{C\left(n_H,n_T,n_C\right)P^{n_H}_HP^{n_T}_TP^{n_C}_C}$$

where the summation is to be carried out over all combinations of integer values for n_H, n_T, and n_C, consistent with $n_H+n_T+n_C=n$.

To find $C\left(n_H,n_T,n_C\right)$, we proceed as before. We suppose that one of the terms with n_H heads, n_T tails, and n_C cone-sides is

$$\left(P_{H,a}P_{H,b}\dots P_{H,f}\right)\left(P_{T,g}P_{T,h}\dots P_{T,m}\right)\left(P_{C,p}P_{C,q}\dots P_{C,z}\right)$$

where there are n_H indices in the set $\{a,\ b,\ \dots ,\ f\}$, n_T indices in the set $\{g,\ h,\ \dots ,\ m\}$, and n_C indices in the set {p, q,..., z }. There are $n_H!$ ways to order the heads outcomes, $n_T!$ ways to order the tails outcomes, and $n_C!$ ways to order the cone-sides outcomes. So, there are $n_H!n_T!n_C!$ possible ways to order n_H heads, n_T tails, and n_C cone-sides. There will also be $n_H!n_T!n_C!$ indistinguishable permutations of any combination (particular assignment) of n_H heads, n_T tails, and n_C cone-sides. There are $n!$ possible permutations of n probability factors and $C\left(n_H,n_T,n_C\right)$ distinguishable combinations with n_H heads, n_T tails, and n_C cone-sides. As before, we have

total number of permutations = (number of distinguishable combinations) $\times$ (number of indistinguishable permutations for each distinguishable combination)

so that

$$n!=n_H!n_T!n_C!C\left(n_H,n_T,n_C\right)$$

and hence,

$$C\left(n_H,n_T,n_C\right)=\frac{n!}{n_H!n_T!n_C!}$$

Equivalently, we can construct a sum of terms, S, in which the terms are all of the $n!$ permutations of $P_{H,r}$ factors for n_H heads, $P_{T,s}$ factors for n_T tails, and $P_{C,t}$ factors for n_C cone-sides. The value of each term in S will be $P^{n_H}_HP^{n_T}_TP^{n_C}_C$. Thus, we have

$$S=n!P_H^{n_H}P_T^{n_T}P_C^{n_C}$$

S will contain all $C\left(n_H,n_T,n_C\right)$ of the distinguishable combinations n_H heads, n_T tails, and n_C cone-sides outcomes that give rise to $P_H^{n_H}P_T^{n_T}P_C^{n_C}$-valued terms. Moreover, S will also include all of the $n_H!n_T!n_C!$ indistinguishable permutations of each of these $P_H^{n_H}P_T^{n_T}P_C^{n_C}$-valued terms, and we also have

$$S=n_H!n_T!n_C!C\left(n_H,n_T,n_C\right)P_H^{n_H}P_T^{n_T}P_C^{n_C}$$

Equating these two expressions for S gives us the number of $P_H^{n_H}P_T^{n_T}P_C^{n_C}$-valued terms in the total-probability product, $C\left(n_H,n_T,n_C\right)$. That is,

$$S=n!P_H^{n_H}P_T^{n_T}P_C^{n_C}=n_H!n_T!n_C!C\left(n_H,n_T,n_C\right)P_H^{n_H}P_T^{n_T}P_C^{n_C}$$

and, again,

$$C\left(n_H,n_T,n_C\right)=\frac{n!}{n_H!n_T!n_C!}$$

In the special case that $P_H=P_T=P_C=1/3$, all of the products $P_H^{n_H}P_T^{n_T}P_C^{n_C}$ will have the value $(1/3)^n$. Then the probability of any set of outcomes, $\{n_H,n_T,n_C,\}$, is proportional to $C\left(n_H,n_T,n_C\right)$ with the proportionality constant $(1/3)^n$.