19.3: Distribution of Results for Multiple Trials with Many Possible Outcomes

It is now easy to extend our results to multiple trials with any number of outcomes. Let the outcomes be A, B, C, …., Z, for which the probabilities in a single trial are P_A, P_B, P_C,…P_Z. We again want to write an equation for the total probability after n trials. We let n_A, n_B, n_C,…n_Z be the number of A, B, C,…, Z outcomes exhibited in $n_A+n_B+n_C+\ldots+n_Z=n$ trials. If we do not care about the order in which the outcomes are obtained, the probability of n_A, n_B, n_C,…, n_Z outcomes in n trials is

$$C\left(n_A,n_B,n_C,\ldots,n_Z\right)P^{n_A}_AP^{n_B}_BP^{n_C}_C\ldots P^{n_Z}_Z$$

and the total probability sum is

$$1=\ \left(P_A+P_B+P_C+\dots+P_Z\right)^{\ n}=\sum_{n_I}{C\left(n_A,n_B,n_C,\ldots,n_Z\right)P^{n_A}_AP^{n_B}_BP^{n_C}_C\ldots P^{n_Z}_Z}$$

where the summation is to be carried out over all combinations of integer values for n_A, n_B, n_C,…, n_Z consistent with $n_A+n_B+n_C+\ldots+n_Z=n$.

Let one of the terms for n_A A-outcomes, n_B B-outcomes, n_C C-outcomes, …, n_Z Z-outcomes, be

$$\left(P_{A,a}P_{A,b}\dots P_{A,f}\right)\left(P_{B,g}P_{B,h}\dots P_{B,m}\right)\times \left(P_{C,p}P_{C,q}\dots P_{C,t}\right)\dots \left(P_{Z,u}P_{Z,v}\dots P_{Z,z}\right)$$

where there are n_A indices in the set $\{a,\ b,\ \dots ,\ f\}$, n_B indices in the set $\{g,\ h,\ \dots ,\ m\}$, n_C indices in the set $\{p,\ q,\ \dots ,\ t\}$, …, and n_Z indices in the set $\{u,\ v,\ \dots ,\ z\}$. There are $n_A!$ ways to order the A-outcomes, $n_B!$ ways to order the B-outcomes, $n_C!$ ways to order the C-outcomes, …, and $n_Z!$ ways to order the Z-outcomes. So, there are $n_A!n_B!n_C!\dots n_Z!$ ways to order n_A A-outcomes, n_B B-outcomes, n_C C-outcomes, …, and n_Z Z-outcomes. The same is true for any other distinguishable combination; for every distinguishable combination belonging to the population set $\{n_A,\ n_B,\ n_C,…,\ n_Z\}$ there are $n_A!n_B!n_C!\dots n_Z!$ indistinguishable permutations. Again, we can express this result as the general relationship:

total number of permutations = (number of distinguishable combinations) $\times$ (number of indistinguishable permutations for each distinguishable combination)

so that

$$n!=n_A!n_B!n_C!\dots n_Z!C\left(n_A,n_B,n_C,\dots ,n_Z\right)$$

and

$$C\left(n_A,n_B,n_C,\dots ,n_Z\right)=\frac{n!}{n_A!n_B!n_C!\dots n_Z!}$$

Equivalently, we can construct a sum, T, in which we add up all of the $n!$ permutations of $P_{A,a}$ factors for n_A A-outcomes, $P_{B,b}$ factors for n_B B-outcomes, $P_{C,c}$ factors for n_C C-outcomes, …, and $P_{Z,z}$ factors for n_Z Z-outcomes. The value of each term in T will be $P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$. So we have

$$T=n!P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$$

T will contain all $C\left(n_A,n_B,n_C,\dots ,n_Z\right)$ of the $P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$-valued products (distinguishable combinations) that are a part of the total-probability sum. Moreover, T will also include all of the $n_A!n_B!n_C!\dots n_Z!$ indistinguishable permutations of each of these $P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$-valued products. Then we also have

$$T=n_A!n_B!n_C!\dots n_Z!C\left(n_A,n_B,n_C,\dots ,n_Z\right)$$
$$\times P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$$

Equating these two expressions for T gives us the number of $P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$-valued products

$$n!P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z=n_A!n_B!n_C!\dots n_Z!$$
$$\times C\left(n_A,n_B,n_C,\dots ,n_Z\right)P^{n_A}_AP^{n_B}_BP^{n_C}_C\dots P^{n_Z}_Z$$

and hence,

$$C\left(n_A, n_B, n_C, \dots, n_Z\right) = \frac{n!}{n_A! n_B! n_C! \dots n_Z!}$$

In the special case that $P_A = P_B = P_C = \cdots = P_Z$, all of the products $P_A^{n_A} P_B^{n_B} P_C^{n_C} \dots P_Z^{n_Z}$ have the same value. Then, the probability of any set of outcomes, $\{n_A, n_B, n_C, \dots, n_Z\}$, is proportional to $C\left(n_A, n_B, n_C, \dots, n_Z\right)$.