3.6: Continuous Distribution Functions - the Envelope Function is the Derivative of the Area

When we can represent the envelope curve as a continuous function, the envelope curve is the derivative of the cumulative probability distribution function: The cumulative distribution function is $f\left(u\right)$; the envelope function is $df\left(u\right)/du$. The envelope function is a ***probability density***, and we will refer to the envelope function, $df\left(u\right)/du$, as the ***probability density function***. The probability density function is the derivative, with respect to the random variable, of the cumulative distribution function. This is an immediate consequence of the fundamental theorem of calculus.

If $H\left(u\right)$ is the anti-derivative of a function $h\left(u\right)$, we have $dH\left(u\right)/du=h\left(u\right)$, and the fundamental theorem of calculus asserts that the area under $h\left(u\right)$, from $u=a$ to $u=b$ is

$$\begin{aligned}\int_a^b h(u)du &= \int_a^b \left(\frac{dH(u)}{du}\right) du \\ &= H(b)-H(a)\end{aligned}$$

In the present instance, $H(u)=f(u)$, so that

$$\int_a^b \left(\frac{df(u)}{du}\right) du = f(b)-f(a)$$

and

$$h(u)=\frac{df(u)}{du}$$

The envelope function, $h(u)$, and $df(u)$ are the same function.

This point is also apparent if we consider the incremental change in the area, dA, under a histogram as the variable increases from u to $u+du$. If we let the envelope function be $h(u)$, we have

$$dA=h(u)du$$

or

$$h(u)=\frac{dA}{du}$$

That is, the envelope function is the derivative of the area with respect to the random variable, u. The area is $f(u)$, so the envelope function is $h(u)=df(u)/du$.

Calling the envelope curve the probability density function emphasizes that it is analogous to a function that expresses the density of matter. That is, for an incremental change in u, the incremental change in probability is

$$\Delta(probability)=\frac{df}{du}\Delta u$$

analogous to the incremental change in mass accompanying an incremental change in volume

$$\Delta(mass)=density\times\Delta(volume)$$

where

$$density=\frac{d\left(mass\right)}{d\left(volume\right)}.$$

In this analogy, we suppose that mass is distributed in space with a density that varies from point to point in the space. The mass enclosed in any particular volume is given by the integral of the density function over the volume enclosed; that is,

$$mass=\int_V \left(density\right)dV.$$

Conversely, the density at any given point is the limit, as the enclosing volume shrinks to zero, of the enclosed mass divided by the magnitude of the enclosing volume.

Similarly, for any value of the random variable, the probability density is the limit, as an interval spanning the value of the random variable shrinks to zero, of the probability that the random variable is in the interval, divided by the magnitude of the interval.
