# 19.5: Problems

1. Leland got a train set for Christmas. It came with seven rail cars. (We say that all seven cars are “distinguishable.”) Four of the rail cars are box cars and three are tank cars. If we distinguish between permutations in which the box cars are coupled (lined up) differently but not between permutations in which tank cars are coupled differently, how many ways can the seven cars be coupled so that all of the tank cars are together? What are they? What formula can we use to compute this number?

(Hint: We can represent one of the possibilities as $b_1b_2b_3b_4T$. This is one of the possibilities in which the first four cars behind the engine are all box cars. There are $4!$ such possibilities; that is, there are $4!$ possible permutations for placing the four box cars.)

2. If we don’t care about the order in which the box cars are coupled, and we don’t care about the order in which the tank cars are coupled, how many ways can the rail cars in problem 1 be coupled so that all of the tank cars are together? What are they? What formula can we use to compute this number?

3. If we distinguish between permutations in which either the box cars or the tank cars in problem 1 are ordered differently, how many ways can the rail cars be coupled so that all of the tank cars are together? What formula can we use to compute this number?

4. How many ways can all seven rail cars in problem 1 be coupled if the tank cars need not be together?

5. If, as in the previous problem, we distinguish between permutations in which any of the rail cars are ordered differently, how many ways can the rail cars be coupled so that not all of the tank cars are together?

6. If we distinguish between box cars and tank cars, but we do not distinguish one box car from another box car, and we do not distinguish one tank car from another tank car, how many ways can the rail cars in problem 1 be coupled?

7. If Leland gets five flat cars for his birthday, he will have four box cars, three tank cars and five flat cars. How many ways will Leland be able to couple (permute) these twelve rail cars?

8. If we distinguish between box cars and tank cars, between box cars and flat cars, and between tank cars and flat cars, but we do not distinguish one box car from another box car, and we do not distinguish one tank car from another tank car, and we do not distinguish one flat car from another flat car, how many ways can the rail cars in problem seven be coupled? What formula can we use to compute this number?

9. We are given four distinguishable marbles, labeled $A--D$, and two cups, labeled $1$ and $2$. We want to explore the number of ways we can put two marbles in cup $1$ and two marbles in cup $2$. This is the number of combinations, $C\left(2,2\right)$, for the population set $N_1=2$, $N_2=2$.

(a) One combination is $\left[AB\right]_1\left[CD\right]_2$. Find the remaining combinations. What is $C\left(2,2\right)$?

(b) There are four permutations for the combination given in (a): $\left[AB\right]_1\left[CD\right]_2$; $\left[BA\right]_1\left[CD\right]_2$; $\left[AB\right]_1\left[DC\right]_2$; $\left[BA\right]_1\left[DC\right]_2$. Find all of the permutations for each of the remaining combinations.

(c) How many permutations are there for each combination?

(d) Write down all of the possible permutations of marbles $A--D$. Show that there is a one-to-one correspondence with the permutations in (b).

(e) Show that the total number of permutations is equal to the number of combinations times the number of permutations possible for each combination.

10. We are given seven distinguishable marbles, labeled $A--G$, and two cups, labeled $1$ and $2$. We want to find the number of ways we can put three marbles in cup $1$ and four marbles in cup $2$. That is, we seek $C\left(3,4\right)$, the number of combinations in which $N_1=3$ and $N_2=4$. $\left[ABC\right]_1\left[DEFG\right]_2$ is one such combination.

(a) How many different ways can these marbles be placed in different orders without exchanging any marbles between cup $1$ and cup $2$? (This is the number of permutations associated with this combination.)

(b) Find a different combination with $N_1=3$ and $N_2=4$.

(c) How many permutations are possible for the marbles in (b)? How many permutations are possible for any combination with $N_1=3$ and $N_2=4$?

(d) If $C\left(3,4\right)$ is the number of combinations in which $N_1=3$ and $N_2=4$, and if $P$ is the number of permutations for each such combination, what is the total number of permutations possible for 7 marbles?

(e) How else can one express the number of permutations possible for 7 marbles?

(f) Equate your conclusions in (d) and (e). Find $C\left(3,4\right)$.

11.

(a) Calculate the probabilities of 0, 1, 2, 3, and 4 heads in a series of four tosses of an unbiased coin. The event of 2 heads is $20\%$ of these five events. Note particularly the probability of the event: 2 heads in 4 tosses.

(b) Calculate the probabilities of 0, 1, 2, 3,…, 8, and 9 heads in a series of nine tosses of an unbiased coin. The events of 4 heads and 5 heads comprise $20\%$ of these ten cases. Calculate the probability of 4 heads or 5 heads; i.e., the probability of being in the middle $20\%$ of the possible events.

(c) Calculate the probabilities of 0, 1, 2, 3,…, 13, and 14 heads in a series of fourteen tosses of an unbiased coin. The events of 6 heads, 7 heads, and 8 heads comprise 20% of these fifteen cases. Calculate the probability of 6, 7, or 8 heads; i.e., the probability of being in the middle $20\%$ of the possible events.

(d) What happens to the probabilities for the middle $20\%$ of possible events as the number of tosses becomes very large? How does this relate to the fraction heads in a series of tosses when the total number of tosses becomes very large?

12. Let the value of the outcome heads be one and the value of the outcome tails be zero. Let the "score" from a particular simultaneous toss of $n$ coins be

$$\mathrm{score}=1\times \left(\frac{number\ of\ heads}{number\ of\ coins}\right)\ +0\times \left(\frac{number\ of\ tails}{number\ of\ coins}\right)$$

Let us refer to the distribution of scores from tosses of $n$ coins as the "$S_n$ distribution."

(a) The $S_1$ distribution comprises two outcomes: $\{$1 head, 0 tail$\}$ and $\{$0 head, 1 tail$\}$.

What is the mean of the $S_1$ distribution?

(b) What is the variance of the $S_1$ distribution?

(c) What is the mean of the $S_n$ distribution?

(d) What is the variance of the $S_n$ distribution?

13. Fifty unbiased coins are tossed simultaneously.

(a) Calculate the probability of 25 heads and 25 tails.

(b) Calculate the probability of 23 heads and 27 tails.

(c) Calculate the probability of 3 heads and 47 tails.

(d) Calculate the ratio of your results for parts (a) and (b).

(e) Calculate the ratio of your results for parts (a) and (c).

14. For $N=3,\ 6$ and 10, calculate

(a) The exact value of $N!$

(b) The value of $N!$ according to the approximation

$$N!\approx N^N{\left(2\pi N\right)}^{1/2}\mathrm{exp}\left(-N\right)\mathrm{exp}\left(\frac{1}{12N}\right)$$

(c) The value of N! according to the approximation

$$N!\approx N^N{\left(2\pi N\right)}^{1/2}\mathrm{exp}\left(-N\right)$$

(d) The value of N! according to the approximation

$$N!\approx N^N\mathrm{exp}\left(-N\right)$$

(e) The ratio of the value in (b) to the corresponding value in (a).

(f) The ratio of the value in (c) to the corresponding value in (a).

(g) The ratio of the value in (d) to the corresponding value in (a).

(h) Comment.

15. Find , $d\ \ln N!/dN$ using each of the approximations

$$N! \approx N^N (2\pi N)^{1/2} \exp(-N) \exp\left(\frac{1}{12N}\right) \approx N^N (2\pi N)^{1/2} \exp(-N) \approx N^N \exp(-N)$$

How do the resulting approximations for $d\ \ln N!/dN$ compare to one another as $N$ becomes very large?

16. There are three energy levels available to any one molecule in a crystal of the substance. Consider a crystal containing $1000$ molecules. These molecules are distinguishable because each occupies a unique site in the crystalline lattice. How many combinations (microstates) are associated with the population set $N_1 = 800$, $N_2 = 150$, $N_3 = 50$?

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