9.11: A Third Statement of the Second Law

Let us consider another frequently cited alternative statement of the second law, which, for easy reference, we call the **temperature-based** statement of the second law:

The temperature-based statement of the second law

The spontaneous transfer of heat from a colder body to a warmer body is impossible.

In the discussion below, we refer to this statement as proposition TSL. By "body", we simply mean any system or object. By the "spontaneous transfer of heat," we mean that the transfer of heat energy can be initiated by bringing the two bodies into contact with one another or by enabling the transmission of radiant energy between them. The surroundings do no work and exchange no heat with either reservoir; there is no change of any sort in the surroundings.

We can show that the entropy-based statement and the temperature-based statement of the second law are equivalent: Given the definition of entropy, one implies the other.

Let us begin by showing that the entropy-based statement implies the temperature-based statement of the second law. That is, we prove $\mathrm{SL} \Rightarrow \mathrm{TSL}$. To do so, we prove $\sim \mathrm{TSL} \Rightarrow \sim \mathrm{SL}$. That is, we assume that spontaneous transfer of heat from a colder to a warmer body is possible and show that this leads to a contradiction of the entropy-based statement of the second law. Let the quantity of heat received by the warmer body be $dq_{warmer} > 0$, and let the temperatures of the warmer and colder bodies be T_{warmer} and T_{colder}, respectively. We have $T_{warmer} - T_{colder} > 0$. The colder body receives heat

$dq_{colder} = -dq_{warmer} < 0$. We make the heat increment so small that there is no significant change in the temperature of either body. No other changes occur. The two bodies are the only portions of the universe that are affected. Let the entropy changes for the warmer and colder bodies be dS_{colder} and dS_{warmer}, respectively.

To find dS_{colder} and dS_{warmer} we must find a reversible path to effect the same changes. This is straightforward. We can effect identically the same change in the warmer body by transferring heat, $q_{warmer} > 0$, to it through contact with some third body, whose temperature is infinitesimally greater than T_{warmer}. This process is reversible, and the entropy change is $dS_{warmer} = dq_{warmer}/T_{warmer}$. Similarly, the entropy change for the colder body is $dS_{colder} = dq_{colder}/T_{colder} = -dq_{warmer}/T_{colder}$. It follows that

$$\begin{aligned} dS_{universe} &= dS_{warmer} + dS_{colder} \\ &= \frac{dq_{warmer}}{T_{warmer}} - \frac{dq_{warmer}}{T_{colder}} \\ &= -dq_{warmer}\left(\frac{T_{warmer} - T_{colder}}{T_{warmer}T_{colder}}\right) \\ &< 0 \end{aligned}$$

However, if $dS_{universe} < 0$ for a spontaneous process, the second law (SL) must be false. We have shown that a violation of the temperature-based statement implies a violation of the entropy-based statement of the second law: $\sim \mathrm{TSL} \Rightarrow \sim \mathrm{SL}$, so that $\mathrm{SL} \Rightarrow \mathrm{TSL}$.

It is equally easy to show that the temperature-based statement implies the entropy-based statement of the second law. To do so, we assume that the entropy-based statement is false and show that this implies that the temperature-based statement must be false. By the arguments above, the entropy change that the universe experiences during the exchange of the heat increment is

$$dS_{universe} = -dq_{warmer}\left(\frac{T_{warmer} - T_{colder}}{T_{warmer}T_{colder}}\right)$$

If the entropy-based statement of the second law is false, then $dS_{universe} < 0$. It follows that $dq_{warmer} > 0$; that is, the spontaneous process transfers heat from the colder to the warmer body. This contradicts the temperature-based statement. That is, $\sim \mathrm{SL} \Rightarrow \sim \mathrm{TSL}$, so that $\mathrm{TSL} \Rightarrow \mathrm{SL}$.