7.3: Line Integrals

The significance of the distinction between exact and inexact differential expressions comes into focus when we use the differential, df, to find how the quantity, f, changes when the system passes from the state defined by (x_1,y_1) to the state defined by (x_2,y_2). We suppose that the system undergoes this change along some continuous path in the xy-plane. We can specify such a path as a function, $c=g(x,y)$, where c is a constant, or as $y=h(x)$. Whether the differential is exact or inexact, we can sum up increments of change, Δf, along short segments of the path to find the change in f between (x_1,y_1) and (x_2,y_2). Let (x_i,y_i) and $(x_i+\Delta x, y_i+\Delta y)$ be two neighboring points on the curve $c=g(x,y)$. As the system traverses $c=g(x,y)$ between these points, the change in f is

$$\Delta f \approx M(x_i,y_i)\,\Delta x + N(x_i,y_i)\,\Delta y$$

If we sum up such increments of Δf, along the curve $c=g(x,y)$, from (x_1,y_1) to (x_2,y_2), the sum approximates the change in f along this path. In the limit that all of the incremental Δx and Δy become arbitrarily small, the approximation becomes exact. The limit of this sum is called the ***line integral*** of df along the path $c=g(x,y)$, between (x_1,y_1) and (x_2,y_2).

Whether df is exact or inexact, the line integral of df is defined along any continuous path in the xy-plane. If the path is $c=g(x,y)$ and it connects the points (x_1,y_1) and (x_2,y_2) in the xy-plane, we designate the value of the line integral as

$$\Delta f=\int_g df=\int_{c=g(x_1,y_1)}^{c=g(x_2,y_2)} df$$

(any differential expression)

However, if df is exact, we know that $\Delta f=f(x_2,y_2)-f(x_1,y_1)$. In this case, the line integral of df along curve $c=g(x,y)$ between these points has the value

$$\Delta f=f(x_2,y_2)-f(x_1,y_1)=\int_{c=g(x_1,y_1)}^{c=g(x_2,y_2)} df$$

(for exact differential df)

Because the value of the line integral depends only on the values of $f(x,y)$ at the end points of the integration path, the line integral of the total differential, df, is independent of the path, $c=g(x,y)$. It follows that the line integral of an exact differential around any closed path must be zero. A circle in the middle of the integral sign is often used to indicate that the line integral is being taken around a closed path. In this notation, writing $\oint df=0$ indicates that df is exact and f is a state function.

In concept, the evaluation of line integrals is straightforward. Since the path of integration is a line, the integrand involves only one dimension. A line integral can always be expressed using a single variable of integration. Three approaches to the evaluation of line integrals are noteworthy.

If we are free to choose an arbitrary path, we can choose the two-segment path $(x_1,y_1)\rightarrow(x_2,y_1)\rightarrow(x_2,y_2)$. Along the first segment, y is constant at y_1, so we can evaluate the change in f as

$$\Delta f_I=\int_{x_1}^{x_2} M(x,y_1)\,dx$$

Along the second segment, x is constant at x_2, so we can evaluate the change in f as

$$\Delta f_{II}=\int_{y_1}^{y_2} N(x_2,y)\,dy$$

Then $\Delta f=\Delta f_I+\Delta f_{II}$.

If the path, $c=g(x,y)$, is readily solved for y as a function of x, say $y=h(x)$, substitution converts the differential expression into a function of only x:

$$df=M(x,h(x))\,dx+N(x,h(x))\left(\frac{dh}{dx}\right)dx$$

Integration of this expression from x_1 to x_2 gives Δf.

The path, $c = g(x, y)$, can always be expressed as a parametric function of a dummy variable, t. That is, we can always find functions $x = x(t)$ and $y = y(t)$ such that $c = g(x(t), y(t)) = g(t)$, $x_1 = x(t_1)$, $y_1 = y(t_1)$, $x_2 = x(t_2)$, and $y_2 = y(t_2)$. Then substitution converts the differential expression into a function of t:

$$df = M(x(t), y(t))\,dt + N(x(t), y(t))\,dt$$

Integration of this expression from t_1 to t_2 gives Δf.

While the line integral of an exact differential between two points is independent of the path of integration, this not the case for an inexact differential. For an inexact differential, the integral between two points depends on the path of integration. To illustrate these ideas, let us consider some examples. These examples illustrate methods for finding the integral of a differential along a particular path. They illustrate also the path-independence of the integral of an exact differential and the path-dependence of the integral of an inexact differential.

Example 7.3.1: An exact Differential

We begin by considering the function

$$f(x, y) = xy^2$$

for which $df = y^2 dx + 2xy\,dy$. Since $f(x, y)$ exists, df must be exact. Let us integrate df between the points $(1,\ 1)$ and $(2,\ 2)$ along four different paths, sketched in Figure 2, that we denote as paths a, b, c, and d.

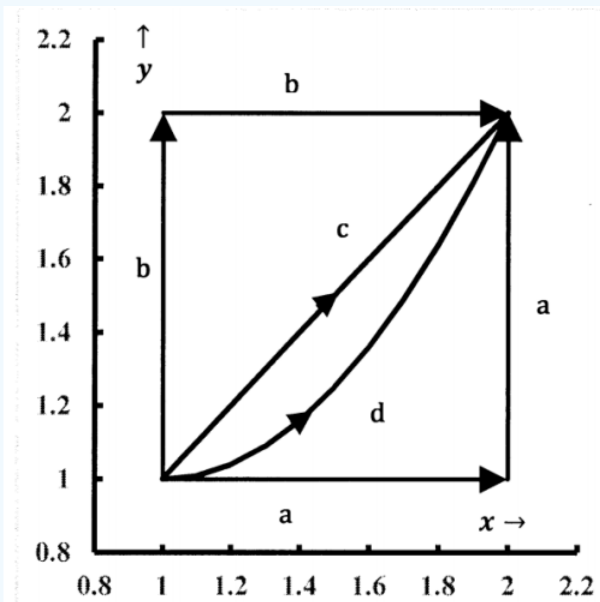

Figure 2. Paths a, b, c, and d.

- **Path a** has two linear segments. The first segment is the portion of the line $y = 1$ from $x = 1$ to $x = 2$. Along this segment, $dy = 0$. The second segment is portion of the line $x = 2$ from $y = 1$ to $y = 2$. Along the second segment, $dx = 0$.
- **Path b** has two linear segments also. The first segment is the portion of the line $x = 1$ from $y = 1$ to $y = 2$. Along the first segment, $dx = 0$. The second segment is portion of the line $y = 2$ from $x = 1$ to $x = 2$. Along the second segment, $dy = 0$.
- **Path c** is the line $y = x$, from $x = 1$ to $x = 2$, and for which $dy = dx$.
- **Path d** is the line $y = x^2 - 2x + 2$, which we can express in parametric form as $y = t^2 + 1$ and $x = t + 1$. At $(1,\ 1)$, $t = 0$. At $(2,\ 2)$, $t = 1$. Also, $dx = dt$ and $dy = 2t\ dt$.

The integrals along these paths are

- **Path a:**

$$\begin{aligned}\int_a df &= \int_{x=1}^{x=2} 1^2 dx + \int_{y=1}^{y=2} (2)\,(2)\,y\,dy \\ &= x|_1^2 + 2y^2|_1^2 \\ &= 7\end{aligned}$$

- **Path b:**

$$\begin{aligned}\int_b df &= \int_{x=1}^{x=2} 2^2 dx + \int_{y=1}^{y=2} (2)\,(1)\,y\,dy \\ &= 4x|_1^2 + y^2|_1^2 \\ &= 7\end{aligned}$$

- **Path c:**

$$\begin{aligned}\int_c df &= \int_{x=1}^{x=2} 3x^2\,dx \\ &= x^3|_1^2 \\ &= 7\end{aligned}$$

- **Path d:**

$$\begin{aligned}\int_d df &= \int_{t=0}^{t=1} \left\{(t^2+1)^2 + 2(t+1)(t^2+1)(2t)\right\} dt \\ &= \int_{t=0}^{t=1} \left\{5t^4 + 4t^3 + 6t^2 + 4t + 1\right\} dt \\ &= t^5 + t^4 + 2t^3 + 2t^2 + t\Big|_{\,0}^{\,1} = 7\end{aligned}$$

The integrals along all four paths are the same. The value is 7, which, as required, is the difference $f\,(2,2) - f\,(1,1) = 7$.

Example 7.3.2: An inexact Differential

Now, let us consider the differential expression

$$dh = y\,dx + 2xy\,dy.$$

This expression has the form of a total differential, but we will see that there is no function, $h\,(x,y)$, for which this expression is the total differential. That is, dh is an inexact differential. If we integrate dh over the same four paths, we find

- **Path a:**

$$\begin{aligned}\int_a dh &= \int_{x=1}^{x=2} (1)\,dx + \int_{y=1}^{y=2} (2)(2)y\,dy \\ &= x]_1^2 + 2y^2|_1^2 \\ &= 7\end{aligned}$$

- **Path b:**

$$\begin{aligned}\int_b dh &= \int_{x=1}^{x=2} (2)\,dx + \int_{y=1}^{y=2} (2)\,(1)\,y\,dy \\ &= 2x|_1^2 + y^2|_1^2 \\ &= 5\end{aligned}$$

- **Path c:**

$$\begin{aligned}\int_c dh &= \int_{x=1}^{x=2} \left(x+2x^2\right)\, dx \\ &= \left[\frac{x^2}{2}+\frac{2x^3}{3}\right]_1^2 \\ &= 6\frac{1}{6}\end{aligned}$$

- **Path d:**

$$\begin{aligned}\int_d dh &= \int_{t=0}^{t=1} \left\{(t^2+1)+2(t+1)(t^2+1)(2t)\right\} dt \\ &= \int_{t=0}^{t=1} \left\{4t^4+4t^3+5t^2+4t+1\right\} dt \\ &= \left[4t^5/5+t^4+5t^3/3+2t^2+t\right]_0^1 \\ &= 6\frac{7}{15}\end{aligned}$$

For $dh\left(x,y\right)$, the value of the integral depends on the path of integration, confirming that $dh\left(x,y\right)$ is an **inexact differential**: Since the value of the integral depends on path, there can be no $h\left(x,y\right)$ for which

$$\Delta h = h\left(x_2,y_2\right)-h\left(x_1,y_1\right)=\int_{(x_1,y_1)}^{(x_2,y_2)} dh$$

That is, $h\left(x_2,y_2\right)-h\left(x_1,y_1\right)$ cannot have four different values.