7.2: The Total Differential

If $f\left(x,y\right)$ is a continuous function of the variables x and y, we can think of $f\left(x,y\right)$ as a surface in a three-dimensional space. $f\left(x,y\right)$ is the height of the surface above the xy-plane at the point $\left(x,y\right)$ in the plane. If we consider points $\left(x_1,y_1\right)$ and $\left(x_2,y_2\right)$ in the xy-plane, the vertical separation between the corresponding points on the surface, $f\left(x_1,y_1\right)$ and $f\left(x_2,y_2\right)$, is

$$\Delta f=f\left(x_2,y_2\right)-f\left(x_1,y_1\right)$$

We can add $f\left(x_1,y_2\right)-f\left(x_1,y_2\right)$ to Δf without changing its value. Then

$$\Delta f=\left[f\left(x_2,y_2\right)-f\left(x_1,y_2\right)\right]+\left[f\left(x_1,y_2\right)-f\left(x_1,y_1\right)\right]$$

If we consider a small change, such that $x_2=x_1+\Delta x$ and $y_2=y_1+\Delta y$, we have

$$\Delta f=\frac{\left[f\left(x_1+\Delta x,y_1+\Delta y_1\right)-f\left(x_1,y_1+\Delta y_1\right)\right]\Delta x}{\Delta x}+\frac{\left[f\left(x_1,y_1+\Delta y_1\right)-f\left(x_1,y_1\right)\right]\Delta y}{\Delta y}$$

Letting $df=\lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\Delta f$, we have

$$\begin{aligned} df &= \lim_{\substack{\Delta x\to 0\\ \Delta y\to 0}}\left\{\frac{\left[f\left(x_1+\Delta x,y_1+\Delta y_1\right)-f\left(x_1,y_1+\Delta y_1\right)\right]\Delta x}{\Delta x}\right\}+\lim_{\Delta y\to 0}\left\{\frac{\left[f\left(x_1,y_1+\Delta y_1\right)-f\left(x_1,y_1\right)\right]\Delta y}{\Delta y}\right\} \\ &= \lim_{\Delta y\to 0}\left\{\left(\frac{\partial f\left(x_1,y_1+\Delta y\right)}{\partial x}\right)_y dx\right\}+\left(\frac{\partial f\left(x_1,y_1\right)}{\partial y}\right)_x dy=\left(\frac{\partial f\left(x_1,y_1\right)}{\partial x}\right)_y dx+\left(\frac{\partial f\left(x_1,y_1\right)}{\partial y}\right)_x dy \end{aligned}$$

We call df the ***total differential*** of the function $f\left(x,y\right)$:

$$df=\left(\frac{\partial f}{\partial x}\right)_y dx+\left(\frac{\partial f}{\partial y}\right)_x dy$$

where df is the amount by which $f\left(x,y\right)$ changes when x changes by an arbitrarily small increment, dx, and y changes by an arbitrarily small increment, dy. We use the notation

$$f_x\left(x,y\right)=\left(\frac{\partial f}{\partial x}\right)_y$$

and

$$f_y\left(x,y\right)=\left(\frac{\partial f}{\partial y}\right)_x$$

to represent the partial derivatives more compactly. In this notation, $df=f_x\left(x,y\right)dx+f_y\left(x,y\right)dy$. We indicate the partial derivative with respect to x with y held constant at the particular value $y=y_0$ by writing $f_x\left(x,y_0\right)$.

We can also write the total differential of $f\left(x,y\right)$ as

$$df=M\left(x,y\right)dx+N\left(x,y\right)dy \tag{7.2.1}$$

in which case $M\left(x,y\right)$ and $N\left(x,y\right)$ are merely new names for $\left(\partial f/\partial x\right)_y$ and $\left(\partial f/\partial y\right)_x$, respectively. To express the fact that there exists a function, $f\left(x,y\right)$, such that $M\left(x,y\right)=\left(\partial f/\partial x\right)_y$ and $N\left(x,y\right)=\left(\partial f/\partial y\right)_x$, we say that df is an exact differential.

Inexact Differentials

It is important to recognize that a differential expression in Equation 7.2.1, may not be exact. In our efforts to model physical systems, **we encounter differential expressions that have this form, but for which there is no function**, $\boldsymbol{f\left(x,y\right)}$, such that $M\left(x,y\right)=\left(\partial f/\partial x\right)_y$ and $N\left(x,y\right)=\left(\partial f/\partial y\right)_x$. We call a differential expression, $df\left(x,y\right)$, for which there is no corresponding function, $f\left(x,y\right)$, an **inexact differential**. Heat and work are important examples. We will develop differential expressions that describe the amount of heat, dq, and work, dw, exchanged between a system and its surroundings. We will find that these differential expressions are not necessarily exact. (We develop examples in Section 7.17 to Section 7.20.) It follows that heat and work are not state functions.

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