6.4: Dual basis

The dual basis is a basis associated to the basis of a vector space. In three-dimensional space, it is isomorphous to the basis of the reciprocal lattice. It is mathematically defined as follows:

Given a basis of n vectors $\mathbf{e_i}$ spanning the direct space E^n, and a vector $\mathbf{x} = x^i\, \mathbf{e_i}$, let us consider the n quantities defined by the scalar products of $\mathbf{x}$ with the basis vectors, $\mathbf{e_i}$:

$x_i = \mathbf{x} \,.\, \mathbf{e_i} = x^j\, \mathbf{e_j} \,.\, \mathbf{e_i} = x^j\, g_{ji}$,

where the g_{ji} 's are the doubly covariant components of the metric tensor.

By solving these equations in terms of x^j, one gets:

$x^j = x_i\, g^{ij}$

where the matrix of the g^{ij} 's is inverse of that of the g_{ij} 's ($g^{ik} g_{jk} = \delta^{\mathrm{i}}{}_{\mathrm{j}}$). The development of vector $\mathbf{x}$ with respect to basis vectors $\mathbf{e_i}$ can now also be written:

$\mathbf{x} = x^i\, \mathbf{e_i} = x_i\, g^{ij}\, \mathbf{e_j}$

The set of n vectors $\mathbf{e^i} = g^{ij}\, \mathbf{e_j}$ that span the space E^n forms a basis since vector $\mathbf{x}$ can be written:

$\mathbf{x} = x_i\, \mathbf{e^i}$

This basis is the *dual basis* and the n quantities x_i defined above are the coordinates of $\mathbf{x}$ with respect to the dual basis. In a similar way one can express the direct basis vectors in terms of the dual basis vectors:

$\mathbf{e_i} = g_{ij}\, \mathbf{e^j}$

The scalar products of the basis vectors of the dual and direct bases are:

$g^i{}_j = \mathbf{e^i} \,.\, \mathbf{e_j} = g^{ik}\, \mathbf{e_k} \,.\, \mathbf{e_j} = g^{ik} g_{jk} = \delta^{\mathrm{i}}{}_{\mathrm{j}}$.

One has therefore, since the matrices g^{ik} and g_{ij} are inverse:

$g^i{}_j = \mathbf{e^i} \,.\, \mathbf{e_j} = \delta^{\mathrm{i}}{}_{\mathrm{j}}$.

These relations show that the dual basis vectors satisfy the definition conditions of the reciprocal vectors. In a three-dimensional space the dual basis and the basis of reciprocal space are identical.