1.9: Transformation matrices

Matrices can be used to map one set of coordinates or functions onto another set. Matrices used for this purpose are called *transformation matrices*. In group theory, we can use transformation matrices to carry out the various symmetry operations considered at the beginning of the course. As a simple example, we will investigate the matrices we would use to carry out some of these symmetry operations on a vector (x, y).

The identity Operation

The identity operation leaves the vector unchanged, and as you may already suspect, the appropriate matrix is the identity matrix.

$$(x, y)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = (x, y) \tag{1.9.1}$$

Reflection in a plane

The simplest example of a reflection matrix corresponds to reflecting the vector (x, y) in either the x or y axes. Reflection in the x axis maps y to $-y$, while reflection in the y axis maps x to $-x$. The appropriate matrix is very like the identity matrix but with a change in sign for the appropriate element. Reflection in the x axis transforms the vector (x, y) to $(x, -y)$, and the appropriate matrix is

$$(x, y)\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = (x, -y) \tag{1.9.2}$$

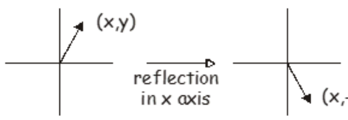

Figure 1.9.1: Reflection across the x-axis

Reflection in the y axis transforms the vector (x, y) to $(-x, y)$, and the appropriate matrix is

$$(x, y)\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = (-x, y) \tag{1.9.3}$$

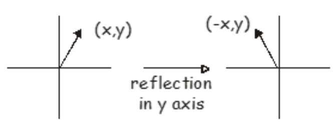

Figure 1.9.2: Reflection across the y-axis

More generally, matrices can be used to represent reflections in any plane (or line in 2D). For example, reflection in the 45° axis shown below maps

(x, y) onto $(-y, -x)$.

$$(x, y)\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = (-y, -x) \tag{1.9.4}$$

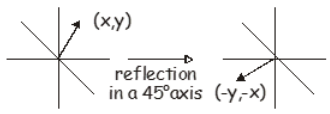

Figure 1.9.3: Reflection across the axis that is rotated 45° with with respect to x-axis.

Rotation about an Axis

In two dimensions, the appropriate matrix to represent rotation by an angle θ about the origin is

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{1.9.5}$$

In three dimensions, rotations about the x, y and z axes acting on a vector (x, y, z) are represented by the following matrices.

$$R_x(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{pmatrix} \tag{1.9.6}$$

$$R_y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{1.9.7}$$

$$R_z(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{1.9.8}$$