1.2: Quantity Calculus

This section gives examples of how we may manipulate physical quantities by the rules of algebra. The method is called *quantity calculus*, although a better term might be "quantity algebra."

Quantity calculus is based on the concept that a physical quantity, unless it is dimensionless, has a value equal to the product of a *numerical value* (a pure number) and one or more *units*:

$$\text{physical quantity} = \text{numerical value} \times \text{units} \tag{1.2.1}$$

(If the quantity is dimensionless, it is equal to a pure number without units.) The physical property may be denoted by a symbol, but the symbol does *not* imply a particular choice of units. For instance, this e-book uses the symbol ρ for density, but ρ can be expressed in any units having the dimensions of mass divided by volume.

A simple example illustrates the use of quantity calculus. We may express the density of water at $25\,^\circ\mathrm{C}$ to four significant digits in SI base units by the equation

$$\rho = 9.970 \times 10^2\ \mathrm{kg\ m^{-3}} \tag{1.2.2}$$

and in different density units by the equation

$$\rho = 0.9970\ \mathrm{g\ cm^{-3}} \tag{1.2.3}$$

We may divide both sides of the last equation by 1 g cm^{-3} to obtain a new equation

$$\rho/\mathrm{g\ cm^{-3}} = 0.9970 \tag{1.2.4}$$

Now the pure number 0.9970 appearing in this equation is the number of grams in one cubic centimeter of water, so we may call the ratio $\rho/\mathrm{g\ cm^{-3}}$ "the number of grams per cubic centimeter." By the same reasoning, $\rho/\mathrm{kg\ m^{-3}}$ is the number of kilograms per cubic meter. In general, a physical quantity divided by particular units for the physical quantity is a pure number representing the number of those units.

> Just as it would be incorrect to call ρ "the number of grams per cubic centimeter," because that would refer to a particular choice of units for ρ, the common practice of calling n "the number of moles" is also strictly speaking not correct. It is actually the ratio n/mol that is the number of moles.

In a table, the ratio $\rho/\mathrm{g\ cm^{-3}}$ makes a convenient heading for a column of density values because the column can then show pure numbers. Likewise, it is convenient to use $\rho/\mathrm{g\ cm^{-3}}$ as the label of a graph axis and to show pure numbers at the grid marks of the axis. You will see many examples of this usage in the tables and figures in this e-book.

A major advantage of using SI base units and SI derived units is that they are *coherent*. That is, values of a physical quantity expressed in different combinations of these units have the same numerical value.

For example, suppose we wish to evaluate the pressure of a gas according to the ideal gas equation

$$p = \frac{nRT}{V} \tag{1.2.5}$$
(ideal gas)

This is the first equation that, like many others to follow, shows *conditions of validity* in parentheses immediately below the equation number at the right. Thus, Eq. 1.2.5 is valid for an ideal gas. In this equation, p, n, T, and V are the symbols for the physical quantities pressure, amount (amount of substance), thermodynamic temperature, and volume, respectively, and R is the gas constant.

The calculation of p for 5.000 moles of an ideal gas at a temperature of 298.15 kelvins, in a volume of 4.000 cubic meters, is

$$p = \frac{(5.000\ \mathrm{mol})(8.3145\ \mathrm{J\ K^{-1}\ mol^{-1}})(298.15\ \mathrm{K})}{4.000\ \mathrm{m^3}} = 3.099 \times 10^3\ \mathrm{J\ m^{-3}} \tag{1.2.6}$$

The mole and kelvin units cancel, and we are left with units of J m^{-3}, a combination of an SI derived unit (the joule) and an SI base unit (the meter). The units J m^{-3} must have dimensions of pressure, but are not commonly used to express pressure.

To convert J m^{-3} to the SI derived unit of pressure, the pascal (Pa), we can use the following relations from Table 1.2:

$$1\,\mathrm{J} = 1\,\mathrm{N\,m} \qquad 1\,\mathrm{Pa} = 1\,\mathrm{N\,m^{-2}} \tag{1.2.7}$$

When we divide both sides of the first relation by $1\,\mathrm{J}$ and divide both sides of the second relation by $1\,\mathrm{N\,m^{-2}}$, we obtain the two new relations

$$1 = (1\,\mathrm{N\,m/J}) \qquad (1\,\mathrm{Pa/N\,m^{-2}}) = 1 \tag{1.2.8}$$

The ratios in parentheses are *conversion factors*. When a physical quantity is multiplied by a conversion factor that, like these, is equal to the pure number 1, the physical quantity changes its units but not its value. When we multiply Eq. 1.2.6 by both of these conversion factors, all units cancel except Pa:

$$p \ = (3.099 \times 10^3\,\mathrm{J\,m^{-3}}) \times (1\,\mathrm{N\,m/J}) \times (1\,\mathrm{Pa/N\,m^{-2}}) \ = 3.099 \times 10^3\,\mathrm{Pa} \tag{1.2.9}$$

This example illustrates the fact that to calculate a physical quantity, we can simply enter into a calculator numerical values expressed in SI units, and the result is the numerical value of the calculated quantity expressed in SI units. In other words, as long as we use only SI base units and SI derived units (without prefixes), *all conversion factors are unity*.

Of course we do not have to limit the calculation to SI units. Suppose we wish to express the calculated pressure in torrs, a non-SI unit. In this case, using a conversion factor obtained from the definition of the torr in Table 1.3, the calculation becomes

$$p \ = (3.099 \times 10^3\,\mathrm{Pa}) \times (760\,\mathrm{Torr}/101,325\,\mathrm{Pa}) = 23.24\,\mathrm{Torr} \tag{1.2.10}$$