

## 6.5: Combining the One-dimensional Probability Density Functions

In [Section 4.4](#), we derive the probability density function for one Cartesian component of the velocity of a gas molecule. The probability density functions for the other two Cartesian components are the same function. For  $\vec{v} = (v_x, v_y, v_z)$ , we have  $v^2 = v_x^2 + v_y^2 + v_z^2$ , and

$$\begin{aligned}\frac{df_x(v_x)}{dv_x} &= \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left(\frac{-\lambda v_x^2}{2}\right) \\ \frac{df_y(v_y)}{dv_y} &= \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left(\frac{-\lambda v_y^2}{2}\right) \\ \frac{df_z(v_z)}{dv_z} &= \left(\frac{\lambda}{2\pi}\right)^{1/2} \exp\left(\frac{-\lambda v_z^2}{2}\right)\end{aligned}$$

We now want to derive the three-dimensional probability density function from these relationships. Given these probability density functions for the Cartesian components of  $\vec{v}$ , we can find the probability density function in spherical coordinates

$$\begin{aligned}&\left(\frac{df_x(v_x)}{dv_x}\right) \left(\frac{df_y(v_y)}{dv_y}\right) \left(\frac{df_z(v_z)}{dv_z}\right) \\ &= \left(\frac{\lambda}{2\pi}\right)^{3/2} \exp\left(\frac{-\lambda v_x^2}{2}\right) \exp\left(\frac{-\lambda v_y^2}{2}\right) \exp\left(\frac{-\lambda v_z^2}{2}\right) \\ &= \left(\frac{\lambda}{2\pi}\right)^{3/2} \exp\left(\frac{-\lambda v^2}{2}\right) \\ &= \rho(v, \theta, \varphi)\end{aligned}$$

Since the differential volume element in spherical coordinates is  $v^2 \sin\theta \, dv d\theta d\varphi$ , the probability that a molecule has a velocity vector whose magnitude lies between  $v$  and  $v + dv$ , while its  $\theta$ -component lies between  $\theta$  and  $\theta + d\theta$ , and its  $\varphi$ -component lies between  $\varphi$  and  $\varphi + d\varphi$  becomes

$$\begin{aligned}&\left(\frac{df_v(v)}{dv}\right) \left(\frac{df_\theta(\theta)}{d\theta}\right) \left(\frac{df_\varphi(\varphi)}{d\varphi}\right) dv d\theta d\varphi \\ &= \rho(v, \theta, \varphi) v^2 \sin\theta \, dv d\theta d\varphi \\ &= \left(\frac{\lambda}{2\pi}\right)^{3/2} v^2 \exp\left(\frac{-\lambda v^2}{2}\right) \sin\theta \, dv d\theta d\varphi\end{aligned}$$

(We found the same result in [Section 4.3](#), of course.) We can find the probability-density function for the scalar velocity by eliminating the dependence on the angular components. To do this, we need only sum up, at a given value of  $v$ , the contributions from all possible values of  $\theta$  and  $\varphi$ , recalling that  $0 \leq \theta < \pi$  and  $0 \leq \varphi < 2\pi$ . This sum is just

$$\begin{aligned}\frac{df_v(v)}{dv} \int_{\theta=0}^{\pi} \left(\frac{df_\theta(\theta)}{d\theta}\right) d\theta \int_{\varphi=0}^{2\pi} \left(\frac{df_\varphi(\varphi)}{d\varphi}\right) d\varphi &= \\ &= \left(\frac{\lambda}{2\pi}\right)^{3/2} v^2 \exp\left(\frac{-\lambda v^2}{2}\right) \int_{\theta=0}^{\pi} \sin\theta \, d\theta \int_{\varphi=0}^{2\pi} d\varphi\end{aligned}$$

Since  $\int_{\theta=0}^{\pi} \left(\frac{df_\theta(\theta)}{d\theta}\right) d\theta = \int_{\varphi=0}^{2\pi} \left(\frac{df_\varphi(\varphi)}{d\varphi}\right) d\varphi = 1$ ,  $\int_0^\pi \sin\theta \, d\theta = 2$ , and  $\int_0^{2\pi} d\varphi = 2\pi$ , we again obtain the Maxwell-Boltzmann probability-density function for the scalar velocity:

$$\frac{df_v(v)}{dv} = 4\pi \left(\frac{\lambda}{2\pi}\right)^{3/2} v^2 \exp\left(\frac{-\lambda v^2}{2}\right)$$

Unlike the distribution function for the Cartesian components of velocity, the Maxwell-Boltzmann distribution for scalar velocities is not a normal distribution. Possible speeds lie in the interval  $0 \leq v < \infty$ . Because of the  $v^2$  term, the Maxwell-Boltzmann equation is asymmetric; it has a pronounced tail at high velocities.

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