12.2: Biased Random Walk

The diffusion with drift equation can be obtained from a biased random walk problem. To illustrate, we extend the earlier description of a walker on a 1D lattice that can step left or right by an amount distance Δx for every time interval Δt. However, in this case there is unequal probability of stepping right (+) or left (–) during $\Delta t : P_+ \neq P_-$. Probabilistically speaking, the change in position for a given time interval can be expressed as

$$\begin{aligned} \langle x(t+\Delta t)\rangle &= \langle x(t) + \Delta x P_+ - \Delta x P_-\rangle \\ &= \langle x(t)\rangle + \Delta x (P_+ - P_-) \end{aligned} \tag{12.2.1}$$

We see that the average position of random walkers depends on the difference in left and right stepping rates. To help link stepping with time, we define rate constants for stepping left or right,

$$k_\pm = \frac{P_\pm}{\Delta t} \tag{12.2.2}$$

with $k_+ \neq k_-$. Then Equation 12.2.1 can be written as

$$\begin{aligned} \langle x(t+\Delta t)\rangle &= \langle x(t)\rangle + (k_+ - k_-)\Delta t \Delta x \\ &= \langle x(t)\rangle + v_x \Delta t \end{aligned} \tag{12.2.3}$$

where the drift velocity is related to the difference in hopping rates

$$v_x = (k_+ - k_-)\Delta x$$

Expressing Equation 12.2.3 as the result of many steps says that the mean of the position distribution behaves like traditional linear motion: $\langle x(t)\rangle = x_0 + v_x t$.

What about the variance in the distribution? Calculating the mean-square value of x from Equation 12.2.1 gives

$$\begin{aligned} \langle x^2(t+\Delta t)\rangle &= \langle x^2(t) \pm 2\Delta x \Delta t k_\pm x(t) + (k_+ + k_-)^2 \Delta x^2 \Delta t^2\rangle \\ &= \langle x^2(t)\rangle + 2 v_x \Delta t \langle x(t)\rangle + (k_+ + k_-)\Delta x^2 \Delta t \end{aligned} \tag{12.2.4}$$

where we used $(k_+ + k_-)\Delta t = 1$.

Using this to calculate the variance in x:

$$\sigma^2(t) = (k_+ + k_-)\,\Delta\, x^2 t \tag{12.2.5}$$

and then comparing with $\langle x^2\rangle^{1/2} = 2Dt$, leads to the conclusion that the breadth of the distribution σ spreads as it would in the absence of a drift velocity, and the diffusion coefficient for this biased random walk is given by

$$D = \frac{1}{2}(k_+ + k_-)\Delta x^2$$

When the left and right stepping rates are the same, we recover our earlier result $2D = \Delta x^2/\Delta t$.