# 12.1: Diffusion with Drift

If diffusion occurs within a moving fluid, the time-dependent concentration profiles will be influenced by the local velocity of the fluid, or drift velocity $v_x$. The net advective flux density for the concentration passing through an area per unit time is then

$$J_{adv} = v_x C \tag{12.1.1}$$

So that the total flux according to eq. (12.1) is

$$J = -D\frac{\partial C}{\partial x} + v_x C \tag{12.1.2}$$

Now using the continuity expression $\partial C/\partial t = -\partial J/\partial x$, and assuming a constant drift velocity the diffusion coefficient is[1]

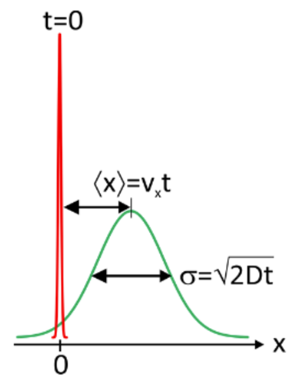
$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - v_x\frac{\partial C}{\partial x} \tag{12.1.3}$$

This equation is the same as the normal diffusion equation in the inertial frame of reference. If we shift to a frame moving at $v_x$, we can define the relative displacement

$$\overline{x} = x - v_x t$$

Remember, C is a function of x and t, and expressing eq. (12.1.2) in terms of $\overline{x}$ via the chain rule, we find that we can recast it as the simple diffusion equation:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial \overline{x}^2}$$




Then the solution for diffusion from a point source becomes

$$C(\overline{x}, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-\overline{x}^2/4Dt}$$

$$C(x, t) = \frac{1}{\sqrt{4\pi Dt}} e^{-(x - v_x t)^2/4Dt}$$

So the peak of the distribution moves as $\langle x\rangle = v_x t$ and the width grows as $\sigma = [\langle x^2\rangle - \langle x\rangle^2]^{1/2} = (2Dt)^{1/2}$.

Let's consider the relative magnitude of the diffusive and drift velocity contributions to the motion of a protein in water. A typical diffusion constant is $10^{-6}$ $cm^2/s$, meaning that the root mean square displacement in a one microsecond time period is 14 nm. If we compare this with the typical velocity of blood in capillaries, v = 0.3 mm/s, in the same microsecond the same protein is pushed $\langle x\rangle$ = 0.3 nm. For this example, diffusion dominates the transport process on the nanometer scale, however, with the increase of time scale and transport distance, the drift term will grow in significance due to the $t^{1/2}$ scaling of diffusive transport.

## Péclet Number

The Péclet number $P_e$ is a unitless number used in continuum hydrodynamics to characterize the relative importance of diffusive transport and advective transport processes. Language note:

- Convection: internal currents within fluid
- Advection: mass transport due to convection

We characterize this with a ratio of the rates or equivalently the characteristic time scale for transport with these processes:

$$P_e = \frac{\text{adjective flux}(J_{adv})}{\text{diffusive flux}(J_{diff})} \approx \frac{\text{diffusion timescale}(t_{diff})}{\text{advection timescale}(t_{adv})}$$

Limits

- $P_e \ll 1$ Diffusion dominated. In this limit, diffusive transport spreads the concentration profile symmetrically about the maximum as illustrated above.

- $P_e \gg 1$ Flow dominated. Effectively no spreading to concentration; it is just carried along with the flow.

If we define a characteristic transport length d and the flow velocity v, then

$$t_{adv} \approx \frac{d}{v}$$

Given a diffusion constant D, the diffusive time-scale is taken to be

$$t_{diff} \approx \frac{d^2}{D}$$

So That

$$P_e = \frac{vd}{D}$$

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1. In three dimensions: $\mathbf{J}(\mathbf{r}, t) = -D\overline{\overline{\Delta}}C(\mathbf{r}, t) + \mathbf{v}C(\mathbf{r}, t)$ and $\dot{C} = \nabla \cdot (D\overline{\overline{\nabla}}C) - \nabla \cdot (\mathbf{v}C)$ .