10.5: Appendix - Jacobians

Appendix: Jacobians

A.1. 3D to 1D

A Jacobian is a mathematical entity used to switch partials inside an integral. For the case of velocity in three dimensions being converted into spherical coordinates, where v is the net velocity and akin to a sphere's radius:

\[\partial v_x\cdot \partial v_y\cdot \partial v_z=\left|\frac{\partial (v_x,v_y,v_z)}{\partial (v,\theta ,\phi)}\right|\cdot \partial v\cdot \partial \phi \cdot \partial \theta =v^2\cdot

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sin\ (phi)\cdot \partial v\cdot \partial \phi \cdot \partial \theta \nonumber \]

To evaluate the above, we should have some idea how to convert v_x v_y v_z into spherical coordinates to begin with. It's just a lesson in geometry:

\[v_x=v\cdot

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sin\ (phi)\cdot

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (theta)v_y=v\cdot

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)\cdot

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (theta)v_z=v\cdot

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (phi) \nonumber \]

where $v=\sqrt{v_x^2+v_y^2+v_z^2}$. The Jacobian $\left|\frac{\partial(v_x,v_y,v_z)}{\partial(v,\theta,\phi)}\right|$ is the absolute values of the determinant of the following matrix:

\[\left|

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detleft[\frac{∂v_x}{\partial v}frac{∂v_x}{\partial \theta }frac{∂v_x}{\partial \varphi }frac{∂v_y}{\partial v}frac{∂v_y}{\partial \theta }frac{∂v_y}{\partial \varphi }frac{∂v_z}{\partial v}frac{∂v_z}{\partial \theta }frac{∂v_z}{\partial \varphi }right]right| \nonumber \]

In our coordinate system the absolute value of the determinant of the matrix is:

\[\left|

UndefinedNameError: reference to undefined name 'det' (click for details)

detleft[

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (theta)\ -v

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (theta)\ v

UndefinedNameError: reference to undefined name 'cos' (click for details) cos\ (phi)

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (theta)\

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (theta)\ v

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (theta)\ v

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (phi)

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (theta)\

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (phi)\ 0\ -v

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)right]right| \nonumber \]

\[=\left|-v^2(phi)(theta)-v^2

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sin\ (phi)(phi)(theta)+0-v^2(phi)(theta)-0-v^2

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)(phi)(theta)\right|=\left|-v^2(phi)((theta)+(theta))-v^2

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)(phi)((theta)+(theta))\right|=\left|-v^2(phi)-v^2

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)(phi)\right|=\left|-v^2

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)((phi)+(phi))\right|=\left|-v^2

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sin\ (phi)\right| \nonumber \]

Since we take the absolute value of this result (the negative sign goes away), the final answer is: \(\partial v_x\cdot \partial v_y\cdot \partial v_z=v^2\cdot

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)\cdot \partial v\cdot \partial \phi \cdot \partial \theta\)

When we apply the Jacobian to the Maxwell-Boltzmann formula we see that (bold emphasis added):

\[{\left(\frac{m}{2\cdot \pi \cdot k_B\cdot T}\right)}^{\frac{3}{2}}\cdot e^{\frac{-m\cdot v^2}{2\cdot k_B\cdot T}}\cdot \partial v_x\cdot \partial v_y\cdot \partial v_z={\left(\frac{m}{2\cdot \pi \cdot k_B\cdot T}\right)}^{\frac{3}{2}}\cdot e^{\frac{-m\cdot v^2}{2\cdot k_B\cdot T}}\cdot v^2\cdot

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sin\ (phi)\cdot \partial v\cdot \partial \phi \cdot \partial \theta \nonumber \]

Now we can integrate out the angles:

\[{\left(\frac{m}{2\cdot \pi \cdot k_B\cdot T}\right)}^{\frac{3}{2}}\cdot e^{\frac{-m\cdot v^2}{2\cdot k_B\cdot T}}\cdot v^2\cdot

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sin\ (phi)\cdot \partial v\cdot \int^{2\pi }_0{}

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sin\ (phi)\cdot \partial \phi \cdot \int^{\pi }_0{}\partial \theta \nonumber \]

which leaves us with just a constant:

\[\int^{\pi }_0{}

UndefinedNameError: reference to undefined name 'sin' (click for details)

sin\ (phi)\cdot \partial \phi \cdot \int^{2\pi }_0{}\partial \theta ={-

UndefinedNameError: reference to undefined name 'cos' (click for details)

cos\ (phi)]}^{\pi }_0\cdot 2\pi =4\pi \nonumber \]

leaving us with:

$$4\pi \cdot v^2 \cdot \left(\frac{m}{2\cdot\pi\cdot k_B\cdot T}\right)^{\frac{3}{2}} \cdot e^{\frac{-m\cdot v^2}{2\cdot k_B\cdot T}} \cdot \partial v$$

This is the velocity form of the Maxwell-Boltzmann equation.

A.2.

$\partial v_a \cdot \partial v_b \rightarrow \partial G \cdot \partial V_{rel}$ **:** The following Jacobian was used in the calculation of relative velocity:

$$\partial v_a \cdot \partial v_b = \left|\frac{\partial(v_a, v_b)}{\partial(G, V_{rel})}\right| \cdot \partial G \cdot \partial V_{rel}$$

The determinant is:

\left|\frac{\partial (v_a,v_b)}{\partial (G,V_{rel})}\right|=\left|det\left[\frac{\partial v_a}{\partial V_{rel}}frac{\partial v_a}{\partial G}frac{\partial v_b}{\partial V_{rel}}frac{\partial v_b}{\partial G}right]\right| \nonumb

Given that $v_a = G + \frac{m_b}{m_a+m_b}V_{rel}$ and $v_b = G - \frac{m_a}{m_a+m_b}V_{rel}$, we can fill out the matrix:

\left|det\left[\frac{m_b}{m_a+m_b}\ 1\ -\frac{m_a}{m_a+m_b}\ 1right]\right|=\frac{m_b}{m_a+m_b}+\frac{m_a}{m_a+m_b}=\frac{m_a+m_b}{m_a+m_b}=1 \nonumber

Done!