11.1: The Black body Radiator

We will begin with exploring the marvelous complexity of the wonder of nature known as the lightbulb. There are some technicalities that must be employed, which is why this derivation is generally referred to as the "black body radiator" problem. This means that we are describing an object that is hot, and self-contained like an empty box. The interior is perfectly black, causes any photons that come into existence to be re-absorbed. The fact that light photons, which have an energy $h\nu = \frac{hc}{\lambda}$ (where h is Planck's constant 6.626×10^{-34} J·s,nu is the frequency of light, c is the speed of light and$lambda$ is the wavelength), do not escape results in the system maintaining thermal equilibrium. This is sensible because, if the photons got out, the box would cool. Nonetheless, we do have to drill a small hole into the side to see what is going on, which means that we measure the number and wavelengths of all the photons inside of it as shown in Figure 11.1A. The best way to measure emission is to record the spectrum, which would vary with the equilibrium temperature as shown in Figure 11.1.B. What is interesting is how the intensity at first rises with decreasing wavelength (increasing energy), maximizes and then rapidly falls off. Also, the spectrum of a ∼5000K black body is nearly identical to the sun. This gives us a hint about the first step of our approach, which is that we must think about what kinds of wavelengths of photons fit inside the black body box to begin with.

11.1.1 Wavelength

Shown in Figure 11.2A are some representations about how light might fit inside the box of length L, which serves to dictate the wavelength of the box. These are the first four "allowed" wavelengths in the x-direction, and note how Figure 11.2.B shows that there are three directions since the Universe is three dimensional. Looking back to Figure 11.2.A we see that the first allowed photon has an infinite wavelength. While this seems odd, it is ok due to a sort of technicality because it has no energy and thus doesn't actually exist. It "counts" because there can be non-zero wavelengths components in the y- or z- directions; this will be more clear later on. Next, we see that the next allowed wavelength has \lambdaup=2L, then L, followed by 2L/3, which clearly reveals the empirical relationship:

$$\lambda = \frac{2L}{n_x},\ n_x = 1,\ 2,\ 3\ldots$$

where n_x is called the "mode number", which represents the #nodes-1 of the confined radiation. As you can see, increasing the mode number shortens the wavelength and raises the energy of the photon. There are mode numbers in the y- and z- directions, which can form a set $\{\mathrm{n}_x, \mathrm{n}_y, \mathrm{n}_z\}$ sort of like a vector. For example, we can describe the highest energy photon in Figure 11.2A as $\{3{,}0{,}0\}$ since there are no components of the wavelength in the y- and z- directions. There can be, as you can have mode number sets such as $\{3{,}1{,}1\}$ etc.

Now that we have a relationship that defines the wavelengths, we can determine a mathematical function to represent them, which is proposed to be:

$$E\left(x,y,z\right) = sin\left(\frac{n_x\pi}{L}\cdot x\right)\cdot sin\left(\frac{n_y\pi}{L}\cdot y\right)\cdot sin\left(\frac{n_z\pi}{L}\cdot z\right)$$

where E is the electric field of the photons. You can verify that these are the proper representation for the waves that have corresponding mode numbers such as those shown in Figure 112.

At this point in your academic career you should be aware that a photon is an oscillating electric and magnetic fields. The ability of light's electric component to perturb objects and impart force is significantly greater than the magnetic field, so we usually don't need to describe the magnetic properties of light. Regardless, the fact that electromagnetism has entered the discussion means that we now need to examine whether our relationship conforms to Maxwell's Equations, which are laws that govern all electromagnetism phenomena including light itself. As it applies here, what happens is that when we apply Gauss's Law to the equation, we can derive , which is known as the wave equation.

$$\left\{\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z}\right\} E\left(x,y,z,t\right) = \frac{-1}{c^2}\frac{\partial^2}{\partial t^2}E\left(x,y,z,t\right)$$

When we insert our relationship we find:

$$E(y,z,t)\frac{\partial^2}{\partial x^2}sin\left(\frac{n_x\pi}{L}\cdot x\right)+E(x,z,t)\frac{\partial^2}{\partial y^2}sin\left(\frac{n_y\pi}{L}\cdot y\right)+E(x,y,t)\frac{\partial^2}{\partial z}sin\left(\frac{n_z\pi}{L}\cdot z\right)$$
$$=E(x,y,z)\frac{1}{c^2}\frac{\partial^2}{\partial t^2}sin\left(\frac{2\pi c}{\lambda}\cdot t\right)$$

We can evaluate terms such as:

$$E(y,z,t)\frac{\partial^2}{\partial x^2}sin\left(\frac{n_x\pi}{L}\cdot x\right)=-\frac{n_x^2\pi^2}{L^2}E(x,y,z,t)$$

Which makes:

$$E(x,y,z,t)\left\{\frac{n_x^2\pi^2}{L^2}+\frac{n_y^2\pi^2}{L^2}+\frac{n_z^2\pi^2}{L^2}\right\}=E(x,y,z,t)\frac{4\pi^2}{\lambda^2}$$

The equation for the electric field can be divided out on the left and right sides leaving:

$$\frac{\pi^2}{L^2}\left(n_x^2+n_y^2+n_z^2\right)=\frac{4\pi^2}{\lambda^2}$$

If we simply treat the mode set like a vector, which has a net value:

$$n^2=n_x^2+n_y^2+n_z^2$$

then the above simplifies to:

$$\lambda^2=\frac{4L^2}{n^2}\ orlambda=\frac{2L}{n}$$

One important aspect of this result is that it demonstrates that mathematical models can tell us more than what the model is.

11.1.2 Mode Degeneracy

We now can relate the wavelengths that fit inside the box to the net mode number n. These values are discrete, since n is composed of a combination of whole numbers as shown in Table 11.1. To visualize, we can think of the spectrum of emission as being composed of posts that can accommodate a stack disks; each post is positioned to represent an allowed wavelength, and the disks are photons that reside at those wavelengths. Technically, each post can hold one disk, and each disk represents two photons since light comes right and left circularly polarized form. This idea is illustrated in Figure 11.3. However, there is a slight complication. Take for example that for the net mode number of $n=1$ corresponding to a wavelength of $\lambda=2L$. There are actually three sets of $\{n_x, n_y, n_z\}$ that given $n=1$; they are \left\{1,0,0\right\},left\{0,1,0\right\} and $\{0,0,1\}$. Likewise a mode number of $n=\sqrt{2}$ can also come about from three sets, \left\{1,0,1\right\},left\{1,1,0\right\} and $\{0,1,1\}$. However, $n=\sqrt{3}$ can only come about from $\{1,1,1\}$, and there is no combination of whole mode numbers that provide $n=\sqrt{7}$.

$n^2 = n_x^2 + n_y^2 + n_z^2$	$\{n_x,n_y,n_z\}$	Number of sets (degeneracy)
1	{1,0,0},{0,1,0},{0,0,1}	3
2	{1,1,0},{0,1,1},{1,0,1}	3
3	{1,1,1}	1
4	{2,0,0},{0,2,0},{0,0,2}	3
5	{0,1,2},{0,2,1},{1,0,2},{2,0,1},{1,2,0},{2,1,0}	6
6	{1,1,2},{1,2,1},{2,1,1}	3
7	**No examples**	-
8	{2,2,0},{0,2,2},{2,0,2}	3

$n^2 = n_x^2 + n_y^2 + n_z^2$	$\{n_x, n_y, n_z\}$	Number of sets (degeneracy)
9	{1,2,2},{2,1,2},{2,2,1},{0,0,3},{0,3,0},{3,0,0}	6
10	{0,1,3},{0,3,1},{1,0,3},{3,0,1},{1,3,0},{3,1,0}	6
⋮	⋮	⋮
10,800	{60,60,60},{20,76,68},…	28
⋮	⋮	⋮
24,300	{90,90,90},...	91

Table 11.1. How net mode numbers n^2 can be composed of multiple whole number $\{n_x, n_y, n_z\}$ sets.

The reason that the emission spectrum initially rises with decreasing wavelength as seen in Figure 11.1B is because of how the number of sets that yield the same net mode number is generally increasing as seen in Table 11.1 and Figure 11.3. Since the net mode number corresponds to a specific wavelength and thus energy of light, the number of sets is the degeneracy of the energy state. Now our purpose here is to generate a relationship between the net mode number and the degeneracy.

Unfortunately, as can be seen in Table 11.1 there isn't a simple formula that can take n or n^2 as an input and generate the degeneracy as the output. This can be visualized in 2D as shown in Figure 11.4, where we represent n as the radius of a circle on a graph of n_y vs. n_x. Each red cross represents distinct mode set, but we can see that a semi-circle of radius n doesn't cross many of them. In fact, it doesn't seem clear that there is a simple formula that relates the net mode number n to the number of nearby modes. However, as shown in Figure 11.4B, any particular point $\{n_x, n_y\}$ is offset from every other point by $\Delta n_x = \pm 1$ and/or $\Delta n_y = \pm 1$. As a result, we can say that each mode number occupies an area on the graph of $1^2 = 1$. Since one point occupies an area of 1, we now have a way to count up mode sets as a function of the net mode number n. This is easy because n looks just like a radius when plotted against n_x and n_y as in Figure 11.4C, and we can define an area associated with n using a quarter disk with a thickness of ∂n. Hence the degeneracy for the 2D system can be calculated via the area of a quarter disk, which is the circumference times the thickness:

$$2D\ degeneracy(n) = \left(\frac{1}{4}\right) 2\pi n \cdot \partial n$$

Of course, we live in three dimensions. Since $1^3 = 1$, we can imagine that the 3D degeneracy can be calculated by volume. In fact, if the number of degeneracies in 2D is 1/4th the area of a disk, then for 3D the degeneracy is 1/8th the volume of a shell:

$$3D\ degeneracy(n) = \left(\frac{1}{8}\right) 4\pi n^2 \cdot \partial n$$

where the shell volume is the surface area of a sphere ($4\pi n^2$) times the shell's thickness ∂n. To make further progress we have to remove the mode number n and insert \lambdaup, since spectrometers report on wavelength. Earlier, when we applied Maxwell's equations to the equation for the electric field of light we found that: $\lambda^2 = \frac{4L^2}{n^2}$ and therefore $n^2 = \frac{4L^2}{\lambda^2}$. Making this substitution into the above reveals:

$$3D\ degeneracy(n) = \left(\frac{1}{8}\right) 4\pi \frac{4L^2}{\lambda^2} cdot\partial n = \frac{2\pi L^2}{\lambda^2} cdot\partial n$$

Two more problems; we have to multiply the above by 2 to represent the fact that light has both left and right circularly polarized forms. Also, we have to convert ∂n to $\partial\lambda$ using a Jacobian: $\partial n \rightarrow \left|\frac{\partial n}{\partial \lambda}\right| \partial\lambda$:

\[3D\ degeneracy(\lambda)=2\times \frac{2\pi L^2}

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{\partial \lambda }\right|\partial \lambda =\frac{8\pi L^3}{{\lambda }^4}partial \lambda \nonumber \]

This is the analytical result plotted in Figure 11.3. Ultimately, we see that shorter wavelength photons "fit" into the black box better and thus have more degeneracies, which increases the light output from a bulb as seen in Figure 11.1.B. However, if the wavelength becomes too short then the intensity of light drops off, which is to say that there appears to be some sort of high energy cutoff. This suggests that the Boltzmann equation plays a role, since nature doesn't partition out energy into things if that energy is greater than $k_B T$.

11.1.3 Total Energy

Now that we have an expression for the degeneracy of light modes in the black body radiator, we can now calculate the total energy from a lightbulb using:

$$Total\ Energy = \sum_{\lambda} degeneracy(\lambda) \cdot \langle energy\,(\lambda)\rangle$$

Here we see that we need to calculate the average energy of the lightbulb's mode as a function of the mode's wavelength, $\langle energy\,(\lambda)\rangle$. We are already well aware that the energy of a photon is $\frac{h\cdot c}{\lambda}$, and with this we can apply the Boltzmann equation to calculate the average. A subtlety is revealed at this point- how is the energy of the photon dialed up or down to conform to the Boltzmann equation? Afterall, to calculate the energy at a defined wavelength using $\frac{h\cdot c}{\lambda}$, everything is a constant (h is the Plank constant, 6.626$\times 10^{-34}$ J•s, and c is the speed of light)! It turns out that the Boltzmann formula is satisfied by varying the *number* of photons, to which we now use in the equation:

$$P\left(\frac{j\cdot h\cdot c}{\lambda}\right) = \frac{e^{\frac{-Energy}{kT}}}{\sum e^{\frac{-Energy}{kT}}} = \frac{e^{\frac{-j\cdot h\cdot c}{\lambda\cdot k_B T}}}{\sum_{i=0}^{\infty} e^{\frac{-i\cdot h\cdot c}{\lambda\cdot k_B T}}}$$

where $P\left(\frac{j\cdot h\cdot c}{\lambda}\right)$ is the probability density for having j photons of wavelength λ given temperature T. First thing we do is to solve the normalizer (i.e. the denominator), $\sum_{i=0}^{\infty} e^{\frac{-i\cdot h\cdot c}{\lambda\cdot k_B T}}$. The summation reflects the fact that Nature can dictate that there are as little as 0 photons and as many as ∞. To solve this part, we can use the following identity:

$$\sum_{i=0}^{\infty} a^i = \frac{1}{(1-a)}; i = 0, 1, 2, 3\ldots$$

This works because the number of photons is discrete, i.e. there can only be whole numbers (0, 1, 2, 3, …) of them. Applying the identity means: $a = e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}$, which makes:

$$\sum_{i=0}^{\infty} e^{\frac{-i\cdot h\cdot c}{\lambda\cdot k_B T}} = \frac{1}{\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)}$$

Now we can try to deal with the average energy via:

$$\langle energy\rangle = \sum energy\cdot Boltzmann\ distribution = \sum_{j=0}^{\infty}\left(\frac{j\cdot h\cdot c}{\lambda}\right)\cdot\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right) e^{\frac{-j\cdot h\cdot c}{\lambda\cdot k_B T}}$$

where: $\left(\frac{j\cdot h\cdot c}{\lambda}\right)$ is the energy of a j number of \lambdaup−wavelength photons and: $\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right) e^{\frac{-j\cdot h\cdot c}{\lambda\cdot k_B T}}$ is the normalized Boltzmann distribution. To solve the expression above we use the following identity:

$$\sum_{i=0}^{\infty} i\cdot a^i = \frac{a}{(1-a)^2};\ i = 0, 1, 2, 3\ldots$$

Applying the identity above to $\sum_{j=0}^{\infty}\left(\frac{j\cdot h\cdot c}{\lambda}\right)\cdot e^{\frac{-j\cdot h\cdot c}{\lambda\cdot k_B T}}\cdot\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)$, we need to identify the constants: $\frac{h\cdot c}{\lambda}\cdot\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)$ and: $a = e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}$. Thus, the average energy is:

$$\sum_{j=0}^{\infty}\left(\frac{j\cdot h\cdot c}{\lambda}\right)\cdot e^{\frac{-j\cdot h\cdot c}{\lambda\cdot k_B T}}\cdot\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)=\left(\frac{h\cdot c}{\lambda}\right)\cdot e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\cdot\frac{\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)}{\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)^2}$$

This is:

$$\langle energy\rangle=\left(\frac{h\cdot c}{\lambda}\right)\cdot\frac{e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}}{\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)}$$

Now we can simplify further by this neat trick of multiplying the top and bottom by $e^{\frac{h\cdot c}{\lambda\cdot k_B T}}$:

$$\left(\frac{h\cdot c}{\lambda}\right)\cdot\frac{e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}}{\left(1-e^{\frac{-h\cdot c}{\lambda\cdot k_B T}}\right)}\cdot\frac{e^{\frac{h\cdot c}{\lambda\cdot k_B T}}}{e^{\frac{h\cdot c}{\lambda\cdot k_B T}}}=\left(\frac{h\cdot c}{\lambda}\right)\cdot\frac{e^0}{\left(e^{\frac{h\cdot c}{\lambda\cdot k_B T}}-e^0\right)}=\frac{h\cdot c}{\lambda\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k_B T}}-1\right)}$$

Done! The average energy of a \lambdaup-wavelength photon is:

$$\langle energy\,(\lambda)\rangle=\frac{h\cdot c}{\lambda\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k_B T}}-1\right)}$$

This result is plotted in Figure 11.5, where we can see that the average energy rises with increasing wavelength. This is sensible, because if the energy of a photon is on the order of $k_B T$ (or less), Nature allows you to have more of those photons.

We can now finally solve for the total energy output of a lightbulb and calculate the spectrum at the same time. Shown in Figure 11.5 is a representation of the average energy as a stack of discs that correlates to the number of photons. The modes and degeneracies of the blackbody were previously presented as a series of posts to stack the photon discs on. Now we see the reason that a lightbulb's spectrum rises and falls with increasing wavelength- Nature provides the energy to create more photons as the energy per photon drops with increasing wavelength. However, the photons must also reside in blackbody radiator modes, which decrease with increasing wavelength. The result is a rise and fall of intensity, which mimics the spectra shown in Figure 11.1.

Now to create a mathematical representation of the same. First, we return to our original expression for the total energy:

$$Total\ Energy=\sum_{\lambda} degeneracy(\lambda)\cdot\langle energy\,(\lambda)\rangle$$

Technically, the above result is dependent on the volume of the lightbulb, as a bigger bulb produces more energy. We should instead present the result as the energy density, the total energy divided by the volume, which can be applied to any sized black box radiator:

$$Energy\ Density=\frac{Total\ Energy}{Volume}=\frac{1}{L^3}\sum_{\lambda} degeneracy(\lambda)\cdot\langle energy(\lambda)\rangle$$

We insert the relationships derived above, and then we can approximate the wavelengths as varying continuously. This allows us to change the sum to an integral; we also do some algebraic cleaning:

$$Energy\ Density=\left(\frac{1}{L^3}\right)\int_0^{\infty}\left(\frac{8\pi L^3}{\lambda^4}\right)\cdot\frac{h\cdot c}{\lambda\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k_B T}}-1\right)}\cdot\partial\lambda=\int_0^{\infty}\frac{8\pi\cdot h\cdot c}{\lambda^5\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k_B T}}-1\right)}\cdot\partial\lambda$$

where the expression:

$$\frac{8\pi\cdot h\cdot c}{\lambda^5\cdot\left(e^{\frac{h\cdot c}{\lambda\cdot k_B T}}-1\right)}$$

is known as the Planck distribution and is exactly what is plotted in Figure 11.1B. Last, we evaluate the integral above, which unfortunately is rather difficult and requires a bit of sophistication to derive. Regardless, the result is:

$$\frac{8\pi^5 (k_B T)^4}{15 (h \cdot c)^4}$$

and is known as the **Stefan–Boltzmann law**.