12.3: Born interpretation

Back in the mid 1920's there was some confusion as to the purpose of the wave equation ψ. While it can deliver a numerical value for energy (quite useful), some questioned if they have any intrinsic meaning. For example, my Aunt Mary's dog only turns right when walking, hence an equation for the angle of his turns is $-|\theta|$. This implies that he tries to turn left (positive θ) an unseen force causes him to reverse (the negative of the absolute value). What do we call this doggie force? Is it fundamental, like gravity or electromagnetism, and can we measure this force acting on other dogs? *What about cats?*

In reality, the dog had surgery on his left paw and that is why he only turns right, a fact that isn't captured by $-|\theta|$. Hence, we shouldn't over interpret an equation that describes him. Are we doing the same thing with wave equations?

Max Born was the first to state that wave equations have substantial meaning, which is to say that they represent probability distributions. Specifically, if you square the wavefunction to make sure it is always positive as shown in Figure 12.2, it represents the probability that you can find the particle at some point in space (probability distributions were discussed at length in Chapter 10). The fact that the wavefunction squared is a probability distribution requires that it be "normalized", which means:

$$\int_{lower\ limit}^{upper\ limit} |\psi|^2 \partial\tau = \int_{lower\ limit}^{upper\ limit} \psi^*\psi\partial\tau = 1.0$$

where ψ^* is the complex conjugate of the wavefunction, which needs to be used because most wavefunctions are complex (i.e. they have $i=\sqrt{-1}$ in them). There is a substantial amount to unpack from this normalization equation. First, we didn't specify the limits because they depend on what is being represented and how "big" the wave equation is allowed to be. For example, if we are using quantum mechanics to describe a particle trapped in a box of length L, then the lower limit would likely be x=0 and the upper x=L. Also note the partial $\partial\tau$ in the integral. This is a symbol that is generic for the dimensionality of the wave equation. Thus far, we have been dealing with a wave in the x direction, so $\partial\tau=\partial x$. If we were trying to solve a quantum mechanical problem for a particle in three dimensions, then $\partial\tau=\partial x\partial y\partial z$, and of course that means that normalization integral is actually a triple integral. If we were working in radial coordinates then $\partial\tau=r^2 sin(\theta)\partial r\partial\phi\partial\theta$, where $r^2 sin(\theta)$ is the Jacobian that property accounts for the volume. If there is no angular dependence to a problem that involves radius, then $\partial\tau=4\pi r^2\partial r$. Last, you should know that we are going to have to use complex mathematics to work quantum mechanical problems. If you are not familiar, there is a short description of most of what you need to know on the next page; more can be found on the "internet". While this may seem like more to learn (and it is), the value is that complex mathematics makes solving quantum mechanical problems much easier.

12.3.1 Normalization. Let's take a look back at what it means for a wave equation to be normalized. Generally, when we determine that a wavefunction is something like: $\psi=cos(kx)$, for example, it is unlikely to be normalized. As a result, we have to make it normalized. To do so you multiply ψ by a normalization constant (N) as:

$$\psi_{norm}=N\psi=\frac{1}{\sqrt{\int|\psi|^2\partial\tau}}\psi$$

As a result:

$$\int\psi_{norm}^2\partial\tau=\frac{\int\psi^*\psi\partial\tau}{\sqrt{\int|\psi|^2\partial\tau}\sqrt{\int|\psi|^2\partial\tau}}=\frac{\int\psi^*\psi\partial\tau}{\int\psi^*\psi\partial\tau}=1$$

and clearly $N=\frac{1}{\sqrt{\int\psi^2\partial\tau}}$. It is often the case that we first figure out what kind of function (sine, cosine etc.) is the solution to the wave equation, and then normalize it after the fact. Sometimes we don't need to normalize the wave equation to answer problems, but it is a good practice. In fact, we will generally assume that wave equations have been properly normalized in our further discussions. It is interesting to note that the requirement for normalization means that not any function can be a wavefunction; in fact there are a few restrictions on solutions as discussed below.

12.3.2 Wave equation restrictions. Since the absolute value, i.e. the square of the wave equation, must be related to probability there are some restrictions on what wave equations can and cannot do as shown in Figure 12.3. First, they cannot be 0 everywhere. This is sort of silly, since $\psi=0$ doesn't leave much room for solving any problems. Second, they must be continuous. Otherwise, there are basically two probabilities for a particle to be found at a certain point in space- what kind of nonsense is that? Third, the wavefunctions must be smooth, which means that the derivative cannot approach ∞ at any point. As you will see later, if the derivative did so then the particle would have more kinetic energy that the Universe holds. Last, wavefunctions cannot be divergent, which means that they can be integrated to a finite value. If not, then the wavefunction could not be normalized, which would not be consistent with the rules of probability distributions.

One of the tricks of quantum mechanics is to use these restrictions to solve problems. Generally, the most relevant are the smooth and continuous stipulation at some sort of boundary. Often that boundary takes the form of a sudden change in the potential energy at a point in space. Another observation is that these boundary conditions mean that a solution for the wave equation can't be found for any energy, rather, often discrete energy values. This is the source of the "quantum" in quantum mechanics, and the solutions are likely to look like standing waves discussed earlier.