4.4: The Car Engine and the Carnot Cycle

When civilization began, people survived by the work of their hands. Farming meant pushing a plow, although the discovery of the mule made this easier. Later, the industrial revolution demonstrated the utility of machines for work. Unlike other people and mules, machines never tire and can be turned off and on at will, and their food is super-cheap. As this fuel is oil.

Machines burn fuel to do work, meaning $\partial q \rightarrow \partial w$, and this direction of energy flow has profound implications for engine efficiency. We must consider how to practically power a machine. When you were a baby your first interaction with a machine was likely a wind-up toy, perhaps a car. Compressing and releasing the toy's spring propelled the miniature vehicle forward, until the spring got sprung. Although this mechanism works just fine for a child's toy, would a company design a real car around the same principle? Such a vehicle wouldn't propel anyone very far before having to be wound up again, which would be clumsy at best (and the spring would have to be huge!). Overall, we reject the use of a spring-driven vehicle and prefer a car with a *reciprocating engine*.

A reciprocating engine uses heat to create work in a cyclical process (this is where the end state is the same as at the beginning). An example is a modern 4-stroke piston engine shown in Figure 4.3. A mixture of petroleum and air is drawn inside, compressed, and then ignited by a sparkplug. The resulting expansion pushes the piston downward in the power stroke, which turns the car's wheels that are connected by the crankshaft. Afterwards, the spinning crankshaft pushes the piston back into the original state, which expels the resultant CO_2 gas and water vapor out into the tailpipe in the process. More fuel and air are drawn in and burned as the cycle is repeated. The net work is negative, and also recall that expansion work is energy *out of the system*. Another, easier way to think of this is to ask yourself, would you rather be pushed forward or pulled (dragged) behind a car? Hopefully you said pushed forward, which means you want the reciprocating engine to produce a negative amount of work.

The idea of a reciprocating process was first realized in steam engines, which were becoming ubiquitous in 1830 when Sadi Carnot was in the French military. At that time Carnot was busy shooting cannon balls at things; this was his job in the military which made him wonder how heat is converted into work. Carnot imagined the simple system shown in Figure 4.4, whereby negative work is created by expanding a gas in cylinder body by placing it on a heat source. Afterwards, the system is returned to its initial volume by moving it onto a cold block. This simple model allowed Carnot to derive the theoretical efficiency of converting heat to work, which can be used to demonstrate that cars need both pistons and tailpipes and explains why planes fly at high altitudes. Carnot also introduced the concept of entropy (although he didn't use that word), and as a result Carnot is called the father of Thermodynamics. Carnot died soon after of cholera in 1832; I'm sure his cat was sad.

Carnot expanded on the simple model described above by developing an idealized cycle to represent a reciprocating engine shown in Figure 4.5. The piston initially has a small volume (V_1) and experiences an input of heat (Δq_1). The resulting high temperature and pressure ($T_{hot},\ P_1$) of gas expands the piston to create some negative, "pushing" type isothermal work (Δw_1). The system lands at volume V_2 and pressure P_2, still at T_{hot}. Note that the relevant thermodynamic quantities are defined in the figure. The next step is to lower the temperature, for which Carnot imagined using an adiabatic expansion to generate small bit more negative work (Δw_2) to take the system to volume V_3 and pressure P_3. The adiabatic expansion cools the cylinder to a lower temperature, T_{cold}. To compress the piston back to the original state, Carnot simply reversed the first two steps using an isothermal reversible compression that releases heat followed by an adiabatic compression to return it to $T_{hot},\ P_1$, and V_1.

4.4.1 Efficiency of the Carnot Cycle

The Carnot cycle is clever in how it forms a perfect loop. Here we perform a thermodynamic analysis to understand the implications of a cyclical hot expansion followed by a cold compression. First, we add up the total work performed by the cycle, which is:

$$\Delta w_{tot} = \Delta w_1 + \Delta w_2 + \Delta w_3 + \Delta w_4$$

where $\Delta w_1 = -nRT_{hot} \cdot \ln\left(\frac{V_2}{V_1}\right)$;$Deltaw_2 = C_V \cdot (T_{cold} - T_{hot})$; $\Delta w_3 = -nRT_{cold} \cdot \ln\left(\frac{V_4}{V_3}\right)$; and $\Delta w_4 = C_V \cdot (T_{hot} - T_{cold})$. Obviously $\Delta w_2 = -\Delta w_4$, so these cancel out and the total work is now composed of just two terms:

$$\Delta w_{tot} = \Delta w_1 + \Delta w_3 = -nRT_{hot} \cdot \ln\left(\frac{V_2}{V_1}\right) - nRT_{cold} \cdot \ln\left(\frac{V_4}{V_3}\right)$$

This can be simplified if we can demonstrate a relationship between $\ln\left(\frac{V_2}{V_1}\right)$ and $\ln\left(\frac{V_4}{V_3}\right)$, which we will do using the adiabatic expansion and compression steps that connect state 2 to 3 and state 4 to the initial state 1, respectively. Specifically, the adiabatic equation of state $\left(\frac{T_f}{T_i}\right)^{\frac{C_V}{nR}} = \frac{V_i}{V_f}$ stipulates that, in the 2^{nd} step:

$$\left(\frac{T_{cold}}{T_{hot}}\right)^{\frac{C_V}{nR}} = \frac{V_2}{V_3}$$

and in the 4^{th} step: $\left(\frac{T_{hot}}{T_{cold}}\right)^{\frac{C_V}{nR}} = \frac{V_4}{V_1}$ which can be inverted into:

$$\left(\frac{T_{cold}}{T_{hot}}\right)^{\frac{C_V}{nR}} = \frac{V_1}{V_4}$$

which means: $\frac{V_2}{V_3} = \frac{V_1}{V_4}$ and can be rearranged to: $\frac{V_2}{V_1} = \frac{V_3}{V_4}$. Taking the natural log of both sides:

$$\ln\left(\frac{V_2}{V_1}\right) = \ln\left(\frac{V_3}{V_4}\right) = -\ln\left(\frac{V_4}{V_3}\right)$$

since $-\ln\left(\frac{a}{b}\right) = \ln\left(\frac{b}{a}\right)$. This allows us to substitute: $+nRT_{cold} \cdot \ln\left(\frac{V_2}{V_1}\right)$ for: $-nRT_{cold} \cdot \ln\left(\frac{V_4}{V_3}\right)$ in the equation for total work:

$$\Delta w_{tot} = -nRT_{hot} \cdot \ln\left(\frac{V_2}{V_1}\right) - nRT_{cold} \cdot \ln\left(\frac{V_4}{V_3}\right)$$

$$= -nRT_{hot} \cdot \ln\left(\frac{V_2}{V_1}\right) + nRT_{cold} \cdot \ln\left(\frac{V_2}{V_1}\right) = -nR \cdot \ln\left(\frac{V_2}{V_1}\right)(T_{hot} - T_{cold})$$

Since $V_2 > V_1$ and $T_{hot} > T_{cold}$, clearly Δw_{tot} is a negative quantity. Perhaps this seemed to be a large amount of effort to demonstrate a point than can be proven graphically as shown in Figure 4.6 (left), which is an exaggeration of the Carnot cycle. Here, the graph of P vs. V clearly has more area under the hot expansion compared to the cold compression, which is consistent with the generation of net negative work.

Next we determine the efficiency of the Carnot cycle, although we must make some decisions on what "efficiency" means. Since modern engines turn heat into work, it seems that we should determine the ratio of the total work divided by the heat: $\frac{\Delta w_{tot}}{\Delta q_{tot}}$, where the total heat is the sum of the 1^{st} and 3^{rd} steps. However, this isn't quite right. To understand, note that in the 1^{st} expansion step we add the heat energy which takes action on our part plus fuel (which means \[s!). In contrast, the 3^{rd} compression step loses heat which doesn't take any effort from us, the engine user. In an actual motor vehicle the 3^{rd} step is when the exhaust gas in the engine is opened to the tailpipe, which dissipates the leftover heat all on its own. Thus, efficiency should be the total work divided by just the heat added in the 1^{st} step (Δq_1). As that the total work is also negative, so we are going to define the efficiency as the absolute value of the work to heat ratio:

$$Efficiency = \left|\frac{\Delta w_{tot}}{\Delta q_1}\right| \tag{4.4.1}$$

Using the equations derived previously the efficiency is:

$$\left|\frac{\Delta w_{tot}}{\Delta q_1}\right| = \left|\frac{-nR \cdot \ln\left(\frac{V_2}{V_1}\right)(T_{hot} - T_{cold})}{nR \cdot \ln\left(\frac{V_2}{V_1}\right)(T_{hot})}\right| = \frac{T_{hot} - T_{cold}}{T_{hot}} = 1 - \frac{T_{cold}}{T_{hot}}$$

(If you are following along but have lost track of a minus sign in the numerator, that is the effect of the absolute value). A very important result of this relationship is that the Carnot cycle cannot be 100% efficient unless $\frac{T_{cold}}{T_{hot}} \sim 0$. This is a bit of a bummer, and it also shows us that those wacky Youtube videos of perpetually motion contraptions are impossible. So how can we make a machine at least more efficient, to save money and use less fuel? We try to make the ratio $\frac{T_{cold}}{T_{hot}}$ as low as possible. For example, we can increase the temperature in the engines, unless it gets so hot that it melts. As for the cold sink, do you see now why airplanes what fly at $\sim$30 K feet where the atmospheric temperature is quite low (-55 $^\circ$C)? Overall, one of the values of such thermodynamic analyses as presented here is that our fundamental understandings of science and nature help us manipulate our surroundings to be of maximum benefit to us and the environment.

Equation 4.4.1 reveals more about the inner workings of reciprocating engines by the fact that the efficiency is 100% if the cold block (i.e. car tailpipe) is at 0 K. How can a cold tailpipe affect mechanical motion such that all the heat energy is converted into work? Recall that all the work is performed by a gas. Using the perfect gas law: $PV = nRT_{cold}$ with $T_{cold} = 0\ K$ in cycle steps 3 and 4 means that the exterior pressure in those compressions is 0 Pa. In other words, the system is compressing against a vacuum to return to the original state, and since $\partial w = -P_{ext}\partial V$, this isn't any work at all! This situation is depicted in Figure 4.6, which makes it clear that the lack of perfect efficiency of the Carnot cycle is due to the loss of energy when performing compression work to return the system back to the original state.

A relevant practice problem is provided in Example Problem 4.1. Here you can see that the input of energy into an adiabatic reversible device results in negative work out of the system. Consistency with the second law is demonstrated by the fact that there is no change in total entropy (in or out the system!). This makes the work out the most possible, which is nonetheless less than the input energy of heat. Hence, machine efficiency is finite. Upon closer analysis the efficiency of the transition $\left(\left|\frac{\Delta w_{tot}}{\Delta q_1}\right| = \frac{32.1\ J}{100\ J}\right)$ appears greater than allowed from our analysis of the Carnot cycle: $\left(1 - \frac{T_{cold}}{T_{hot}}\right)$; however, the example problem doesn't follow a full Carnot cycle because we didn't make the piston transform back into its original state. If we had, work would have been lost in the compression.

4.4.2 Entropy of the Carnot Cycle

While we have demonstrated that a Carnot cycle cannot be 100% efficient, at least not practically, we now ask the question: so what? Just like I'm not concerned about the efficiency of a wind-up spring toy, perhaps the Carnot cycle describes a mechanical system that is not of interest. To show why this isn't the case, we must calculate the total change in system entropy of the Carnot cycle by summing up the changes. As steps 2 & 4 are adiabatic and reversible, there is no change in entropy at all (see Table 4.1 on pg.); thus $\Delta S_{2+4} = 0\ J/K$. As for steps 1 & 3, as these are isothermal reversible transitions:

$$\Delta S_1 + \Delta S_3 = nR \cdot \ln\left(\frac{V_2}{V_1}\right) + nR \cdot \ln\left(\frac{V_4}{V_3}\right)$$

While we can't make much headway with this, note that we previously determined that $\ln\left(\frac{V_2}{V_1}\right) = -\ln\left(\frac{V_4}{V_3}\right)$ using the adiabatic equation of state. Thus:

$$\Delta S_1 + \Delta S_3 = -nR \cdot \ln\left(\frac{V_4}{V_3}\right) + nR \cdot \ln\left(\frac{V_4}{V_3}\right) = 0\ J/K$$

Consequently, the entropy change of the system throughout the cycle is $\Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 = 0\ J/K$. Now we can consider the exterior changes in entropy. First, $\Delta S_{ext,2} + \Delta S_{ext,4} = 0\ J/K$ since these steps are adiabatic, and likewise $\Delta S_{ext,1} + \Delta S_{ext,3} = 0\ J/K$ because of the reversibility of those steps. As a result, the total $\Delta S_{ext} = 0\ J/K$ and thus $\Delta S_{tot} = 0\ J/K$. This means that the Carnot cycle is fully reversible. As reversible systems provide the most exterior (negative) work as we established in Sec. 2.2, it must be true that an engine that operates by the Carnot cycle is the most efficient engine that could ever exist. As a result, we can now stipulate that, for any machine a person could ever build, that efficiency is less than $\left(1 - \frac{T_{cold}}{T_{hot}}\right)$; this is the most important result of thermodynamics.

This page titled 4.4: The Car Engine and the Carnot Cycle is shared under a CC BY-NC 4.0 license and was authored, remixed, and/or curated by Preston Snee.