14.5: Angular Momentum Operators

Angular momentum is defined as:

$$L\rightharpoonup = r\rightharpoonup \times p\rightharpoonup$$

where $r\rightharpoonup$ is the position of the particle, $p\rightharpoonup$ is momentum and reveals the direction of motion, and $L\rightharpoonup$ is the angular momentum vector. As everything is a vector the angular momentum can be decomposed into x, y and z components, i.e. $L\rightharpoonup = \hat{L}_x + \hat{L}_y + \hat{L}_z$ (this will be important later). While understanding angular momentum can be intimidating, if you look at Figure 14.10 you see that it is basically the axel of a wheel as defined by the rotation of a particle. As shown in the figure the only nuance is that the right-hand rule dictates whether the vector is up or down depending on the direction that the particle is rotating. The relationship $L\rightharpoonup = r\rightharpoonup \times p\rightharpoonup$ was part of your learning of classical mechanics; likely you learned about it Physics I or even in high school. To create a quantum mechanical operator for angular momentum, we simply insert $\hat{r}$ as the position operator and $\hat{p} = \frac{\hbar}{i}\left\{\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right\}$ as the momentum operator into $r \times p$. The cross product simplifies into relationships for angular momentum as projected into x, y and z as:

$\hat{L}_x = \frac{\hbar}{i}\left\{y \cdot \frac{\partial}{\partial z} - z \cdot \frac{\partial}{\partial y}\right\}$, $\hat{L}_y = \frac{\hbar}{i}\left\{z \cdot \frac{\partial}{\partial x} - x \cdot \frac{\partial}{\partial z}\right\}$, and $\hat{L}_z = \frac{\hbar}{i}\left\{x \cdot \frac{\partial}{\partial y} - y \cdot \frac{\partial}{\partial x}\right\}$

Next, we insert x, y, and z in spherical coordinates. You may recall we already showed how to covert $\frac{\partial}{\partial x}$ etc.: $\frac{\partial}{\partial x} = \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial x}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta}$ and likewise: $\frac{\partial}{\partial y} = \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \phi}{\partial y}\frac{\partial}{\partial \phi} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta}$. These simplify to:

\[\frac{\partial }{\partial x}=sin\left(\theta \right)

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cosleft(\phi \right)\frac{\partial }{\partial r}\ +\frac{cos\left(\theta \right)cos\left(\phi \right)}{r}\frac{\partial }{\partial \theta }-\frac{sin\left(\phi \right)}{r\cdot sin\left(\theta \right)}\frac{\partial }{\partial \phi } \nonumber \]

and:

\[\frac{\partial }{\partial y}=sin\left(\theta \right)

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sinleft(\phi \right)\frac{\partial }{\partial r}\ +\frac{cos\left(\theta \right)sin\left(\phi \right)}{r}\frac{\partial }{\partial \theta }+\frac{cos\left(\phi \right)}{r\cdot sin\left(\theta \right)}\frac{\partial }{\partial \phi } \nonumber \]

Hence the angular momentum in the z direction is: $\hat{L}_z = \frac{\hbar}{i}\left\{x \cdot \frac{\partial}{\partial y} - y \cdot \frac{\partial}{\partial x}\right\} = \frac{\hbar}{i}\bullet$

\[\{r\cdot sin\left(\theta \right)cos\left(\phi \right)\cdot sin\left(\theta \right)

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sinleft(\phi \right)\frac{\partial }{\partial r}\ +r\cdot sin\left(\theta \right)cos\left(\phi \right)\cdot \frac{cos\left(\theta \right)sin\left(\phi \right)}{r}\frac{\partial }{\partial \theta }+r\cdot sin\left(\theta \right)cos\left(\phi \right)\cdot \frac{cos\left(\phi \right)}{r\cdot sin\left(\theta \right)}\frac{\partial }{\partial \phi }\ -r\cdot sin\left(\theta \right)sin\left(\phi \right)\cdot sin\left(\theta \right)

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cosleft(\phi \right)\frac{\partial }{\partial r}\ -r\cdot sin\left(\theta \right)sin\left(\phi \right)\cdot \frac{cos\left(\theta \right)

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cosleft(\phi \right)\ }{r}\frac{\partial }{\partial \theta }+r\cdot sin\left(\theta \right)sin\left(\phi \right)\cdot \frac{sin\left(\phi \right)}{r\cdot sin\left(\theta \right)}\frac{\partial }{\partial \phi }\}\ \nonumber \]

While this looks overwhelming, note how the first two terms on each line cancel, leaving:

$$\hat{L}_z = \frac{\hbar}{i}\cdot\left\{r\cdot sin\left(\theta\right)cos\left(\phi\right)\cdot\frac{cos\left(\phi\right)}{r\cdot sin\left(\theta\right)} + r\cdot sin\left(\theta\right)sin\left(\phi\right)\cdot\frac{sin\left(\phi\right)}{r\cdot sin\left(\theta\right)}\right\}\frac{\partial}{\partial\phi} =$$

$$\frac{\hbar}{i}\cdot\left\{cos^2\left(\phi\right)+sin^2\left(\phi\right)\right\}\frac{\partial}{\partial\phi} = \frac{\hbar}{i}\frac{\partial}{\partial\phi}$$

Very simple! Along the same lines you find that:

\[{\hat{L}}_x=\frac\hslash{i}\left(-

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cotleft(\theta \right)\

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cosleft(\phi \right)frac{\partial }{\partial \phi }-

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sinleft(\phi \right)frac{\partial }{\partial \theta }\right) \nonumber \]

and:

\[{\hat{L}}_y=\frac\hslash{i}\left(-

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cotleft(\theta \right)\

UndefinedNameError: reference to undefined name 'sin' (click for details)

sinleft(\phi \right)frac{\partial }{\partial \phi }+

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cosleft(\phi \right)frac{\partial }{\partial \theta }\right) \nonumber \]

The angular kinetic energy can be derived from:

$$\frac{\hat{L}^2}{2I} = \frac{\left(\hat{L}_x+\hat{L}_y+\hat{L}_z\right)^2}{2I} = \frac{-\hbar^2}{2I}\left(\frac{1}{sin^2\left(\theta\right)}\frac{\partial^2}{\partial\phi^2} + \frac{1}{sin(\theta)}\frac{\partial}{\partial\theta}sin(\theta)\frac{\partial}{\partial\theta}\right)$$

which is exactly the same as shown before.

Overall is it unfortunate that all these operators are a bit complicated, but they are important because they reveal the existence of another uncertainty principle. You may recall from Ch. 13 section that, for two operators $\hat{A}$ and $\hat{B}$ if the following is true: $\left[\hat{A},\hat{B}\right]=\hat{A}\cdot\hat{B}-\hat{B}\cdot\hat{A}\neq 0$

(the operators don't commute), then the two operators don't share the same eigenvectors. In other words, if Φ is an eigenvector of $\hat{A}$, then $\hat{A}\Phi=\omega\Phi$ where ω is just a constant such as $\hbar$ or 0. However, the eigenvalue equation for won't work for the other operator: $\hat{B}\Phi\neq\omega\Phi$. Hence, whatever observable that $\hat{B}$ describes, you can't know what that is for the state described by Φ. For example, in the previous chapter it was shown that $[\hat{x},\hat{p}]=-i\hbar$ for operators $\hat{x}$ and $\hat{p}$, so you can't know where something is and where it is going fully at the same time.

Returning to our rotational wavefunctions (the spherical harmonics), these have to be eigenfunctions of the $\hat{L}^2$ operator because they were solved to be the wavefunctions of rotational kinetic energy which is just $\hat{L}^2$ times a constant. Since it is always true that an operator commutes with its square, i.e. $\left[\hat{L}^2,\hat{L}\right]=\hat{L}^2\hat{L}-\hat{L}\hat{L}^2=\hat{L}^3-\hat{L}^3=0$, then the spherical harmonics are all eigenfunctions of $\hat{L}$. If you apply $\hat{L}_z$, you see that all of the spherical harmonics are eigenfunctions of $\hat{L}_z$ too. Since $\hat{L}$ and $\hat{L}_z$ share a common set of eigenfunctions, then it must be true that $\left[\hat{L},\hat{L}_z\right]=0$. However, using the operators above you can show that: $\left[\hat{L},\hat{L}_x\right]\neq 0$, $\left[\hat{L},\hat{L}_y\right]\neq 0$, $\left[\hat{L}_z,\hat{L}_x\right]\neq 0$, and finally $\left[\hat{L}_z,\hat{L}_y\right]\neq 0$. Hence, when describing rotation we can know the total angular momentum $\langle\hat{L}\rangle$ (the length of the "axel" in Figure 14.9) and the projection onto the z-axis $\langle\hat{L}_z\rangle$ (the tilt of the angular momentum vector), but we don't know the projection of angular momentum onto the x or y axes, i.e. $\langle\hat{L}_x\rangle$ and $\langle\hat{L}_y\rangle$. This is represented in Figure 14.11, where the motion of the quantum particle is displaced from the origin of its angular momentum vector so you can more easily see the rotation. What it means for $\langle\hat{L}_x\rangle$ and $\langle\hat{L}_y\rangle$ to be unknown is that the ϕ angle of the angular momentum vector is not defined. As a result, the angular momentum vector $\langle\hat{L}\rangle$ can point anywhere in the x-y plane when $m_l=0$. If $m_l=1$, the angular momentum vector can be anywhere on the surface of a cone as shown to the right of Figure 14.11. The cone representation on the right also includes $m_l=-1$; examples for $l=\frac{1}{2}$ and $l=2$ are also provided in Figure 14.12 where again we emphasize that the L vectors lie on the surface of the cones, yet they describe rotation about those vectors like a wheel about its axel.