19.1: The Free Particles

By definition, the particle does not feel any external force, therefore $V(x)=0$, and the TISEq is written simply:

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2}=E\psi(x). \tag{19.1.1}$$

This equation can be rearranged to:

$$\frac{d^2\psi}{dx^2}=-\frac{2mE}{\hbar^2}\psi(x), \tag{19.1.2}$$

which corresponds to a mathematical problem where the second derivative of a function should be equal to a constant, $-\frac{2mE}{\hbar^2}$ multiplied by the function itself. Such a problem is easily solved by the function:

$$\psi(x)=A\exp(\pm ikx). \tag{19.1.3}$$

The first and second derivatives of this function are:

$$\begin{aligned} \frac{d\psi(x)}{dx} &= \pm ikA\exp(\pm ikx)=\pm ik\psi(x) \\ \frac{d^2\psi(x)}{dx^2} &= \mp k^2A\exp(\pm ikx)=-(\pm k^2)\psi(x). \end{aligned} \tag{19.1.4}$$

Comparing the second derivative in Equation 19.1.4 with Equation 19.1.2, we immediately see that if we set:

$$k^2=\frac{2mE}{\hbar^2}, \tag{19.1.5}$$

we solve the original differential equation. Considering de Broglie’s equation, Equation 17.5.4, we can replace $k=\frac{p}{\hbar}$, to obtain:

$$E=\frac{k^2\hbar^2}{2m}=\frac{p^2}{2m}, \tag{19.1.6}$$

which is exactly the classical value of the kinetic energy of a free particle moving in one direction of space. Since the function in Equation 19.1.3 solves the Schrödinger equation for the free particle, it is called an *eigenfunction* (or *eigenstate*) of the TISEq. The energy result of Equation 19.1.6 is called *eigenvalue* of the TISEq. Notice that, since k is continuous in the eigenfunction, the energy eigenvalue is also continuous (i.e., all values of E are acceptable).