19.3: The Harmonic Oscillator

We now consider a particle subject to a restoring force $F = -kx$, as might arise for a mass-spring system obeying Hooke's Law. The potential is then:

$$V(x) = -\int_{-\infty}^{\infty} (-kx)dx = V_0 + \frac{1}{2}kx^2. \tag{19.3.1}$$

If we choose the energy scale such that $V_0 = 0$ then: $V(x) = \frac{1}{2}kx^2$, and the TISEq looks:

$$-\frac{\hbar^2}{2\mu}\frac{d^2\psi}{dx^2} + \frac{1}{2}kx^2\psi(x) = E\psi(x) \tag{19.3.2}$$

After some effort, the eigenfunctions are:

$$\psi_n(x) = N_n H_n(\alpha^{1/2}x)e^{-\alpha x^2/2} \quad n = 0, 1, 2, \ldots, \infty, \tag{19.3.3}$$

where H_n is the Hermite polynomial of degree n, and α and N_n are defined by

$$\alpha = \sqrt{\frac{k\mu}{\hbar^2}} \qquad N_n = \frac{1}{\sqrt{2^n n!}}\left(\frac{\alpha}{\pi}\right)^{1/4}. \tag{19.3.4}$$

The eigenvalues are:

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right), \tag{19.3.5}$$

with $\omega = \sqrt{k/\mu}$. Notice how, once again, the eigenfunctions and eigenvalues are not continuous. In this case, however, the first eigenvalue corresponds to $n = 0$, but because of the $\frac{1}{2}$ factor in Equation 19.3.5, the lowest energy state is, once again, not zero. In other words, the two masses of a quantum harmonic oscillator are always in motion. The frequencies at which they vibrate do not form a continuous spectrum. That is, the vibration frequency cannot take any value that we can think of, but only those given by Equation 19.3.5. The lowest possible energy (the ZPE) will be $E_0 = \frac{1}{2}\hbar\omega$.