

18.2: The Time-Dependent Schrödinger Equation

Unfortunately, the analogy with the classical wave equation that allowed us to obtain the TISEq in the previous section cannot be extended to the time domain by considering the equation that involves the partial first derivative with respect to time. Schrödinger himself presented his time-independent equation first, and then went back and postulated the more general time-dependent equation. We are following here the same strategy and just give the time-independent variable as a postulate. The single-particle time-dependent Schrödinger equation is:

$$i\hbar \frac{\partial \psi(x, t)}{\partial t} = -\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x, t)}{\partial x^2} + V(x)\psi(x, t) \quad (18.2.1)$$

where $V \in \mathbb{R}^n$ represents the potential energy of the system. Obviously, the time-dependent equation can be used to derive the time-independent equation. If we write the wavefunction as a product of spatial and temporal terms, $\psi(x, t) = \psi(x)f(t)$, then Equation 18.2.1 becomes:

$$\psi(x)i\hbar \frac{df(t)}{dt} = f(t) \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x), \quad (18.2.2)$$

which can be rearranged to:

$$\frac{i\hbar}{f(t)} \frac{df(t)}{dt} = \frac{1}{\psi(x)} \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x). \quad (18.2.3)$$

Since the left-hand side of Equation 18.2.3 is a function of t only and the right hand side is a function of x only, the two sides must equal a constant. If we tentatively designate this constant E (since the right-hand side clearly must have the dimensions of energy), then we extract two ordinary differential equations, namely:

$$\frac{1}{f(t)} \frac{df(t)}{dt} = -\frac{iE}{\hbar} \quad (18.2.4)$$

and:

$$-\frac{\hbar^2}{2m} \frac{\partial^2 \psi(x)}{\partial x^2} + V(x)\psi(x) = E\psi(x). \quad (18.2.5)$$

The latter equation is the TISEq. The former equation is easily solved to yield

$$f(t) = e^{-iEt/\hbar} \quad (18.2.6)$$

The solutions of Equation 18.2.6, $f(t)$, are purely oscillatory, since $f(t)$ never changes in magnitude. Thus if:

$$\psi(x, t) = \psi(x) \exp\left(\frac{-iEt}{\hbar}\right), \quad (18.2.7)$$

then the total wave function $\psi(x, t)$ differs from $\psi(x)$ only by a phase factor of constant magnitude. There are some interesting consequences of this. First of all, the quantity $|\psi(x, t)|^2$ is time independent, as we can easily show:

$$|\psi(x, t)|^2 = \psi^*(x, t)\psi(x, t) = \psi^*(x) \exp\left(\frac{iEt}{\hbar}\right) \psi(x) \exp\left(\frac{-iEt}{\hbar}\right) = \psi^*(x)\psi(x). \quad (18.2.8)$$

Wave functions of the form of Equation 18.2.7 are called stationary states. The state $\psi(x, t)$ is “stationary,” but the particle it describes is not! Of course Equation 18.2.6 represents only a particular solution to the time-dependent Schrödinger equation. The general solution is much more complicated, and the factorization of the temporal part is often not possible:¹

$$\psi(\mathbf{r}, t) = \sum_i c_i e^{-iE_i t/\hbar} \psi_i(\mathbf{r})$$

1. This sections was adapted in part from Prof. C. David Sherrill's A Brief Review of Elementary Quantum Chemistry Notes available [here](#).

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