17.3: Hamiltonian Mechanics

A third way of obtaining the equation of motion is Hamiltonian mechanics, which uses the generalized momentum in place of velocity as a coordinate. The generalized momentum is defined in terms of the Lagrangian and the coordinates $(q, \dot{q})$:

$$p=\frac{\partial L}{\partial \dot{q}}. \tag{17.3.1}$$

The Hamiltonian is defined from the Lagrangian by applying a Legendre transformation as:[1]

$$H(p, q)=p\dot{q}-L(q, \dot{q}), \tag{17.3.2}$$

The Lagrangian equation of motion becomes a pair of equations known as the Hamiltonian system of equations:

$$\begin{aligned}
\dot{p} &= \frac{dp}{dt} = -\frac{\partial H}{\partial q} \\
\dot{q} &= \frac{dq}{dt} = +\frac{\partial H}{\partial p},
\end{aligned} \tag{17.3.3}$$

where $H=H(q, p, t)$ is the Hamiltonian of the system, which often corresponds to its total energy. For a closed system, it is the sum of the kinetic and potential energy in the system:

$$H=K+V. \tag{17.3.4}$$

Notice the difference between the Hamiltonian, Equation 17.3.4, and the Lagrangian, Equation 18.2.1. In Newtonian mechanics, the time evolution is obtained by computing the total force being exerted on each particle of the system, and from Newton's second law, the time evolutions of both position and velocity are computed. In contrast, in Hamiltonian mechanics, the time evolution is obtained by computing the Hamiltonian of the system in the generalized momenta and inserting it into Hamilton's equations. This approach is equivalent to the one used in Lagrangian mechanics, since the Hamiltonian is the Legendre transform of the Lagrangian. The main motivation to use Hamiltonian mechanics instead of Lagrangian mechanics comes from the more simple description of complex dynamic systems.

Example 17.3.1

Let's apply the Hamiltonian mechanics formulas to the same problem in the previous examples.

Solution

Using Equation 17.3.2, the Hamiltonian can be written as:

$$H=m\dot{q}\dot{q}-\frac{1}{2}m\dot{q}^2+mGq=\frac{1}{2}m\dot{q}^2+mGq. \tag{17.3.5}$$

Since the Hamiltonian really depends on position and momentum, we need to get this in terms of q and p, with $p=m\dot{q}$ for the momentum. This is not always the case, since it depends on the choice of coordinate system. For a trivial coordinate system for our simple 1-dimensional problem, we have:

$$H=\frac{p^2}{2m}+mGq,$$

from which we can use eqs. 17.3.3 to get:

$$\begin{aligned}
\dot{q} &= \frac{\partial H}{\partial p} = \frac{p}{m} \\
\dot{p} &= -\frac{\partial H}{\partial q} = -mG.
\end{aligned}$$

These equations represent a major diffference of the Hamiltonian method, since we describe the system using two first-order differential equations, rather than one second-order differential equation. In order to get the equation of motion, we need to take the derivative of $\dot{q}$:

$$\ddot{q} = \frac{d}{dt}\left(\frac{p}{m}\right) = \frac{\dot{p}}{m},$$

and then replacing the definition of $\dot{p}$ obtained above, we get:

$$\ddot{q} = \frac{-mG}{m} = -G$$

which—once again—is the same result obtained for the two previous cases. Integrating this twice, we get the familiar equation of motion for our problem.

1. We have already encountered Legendre transform in The Live Textbook of Physical Chemistry 1 when transforming from the thermodynamic energy to any of the other thermodynamic potentials.