22.3: Spin Operators

The mathematics of quantum mechanics tell us that $\hat{S}_z$ and $\hat{S}_x$ do not commute. When two operators do not commute, the two measurable quantities that are associated with them cannot be known at the same time.

In 3-dimensional space there are three directions that are orthogonal to each other $\{x, y, z\}$. Thus, we can define a third spin projection operator along the y direction, $\hat{S}_y$, corresponding to a new set of Stern-Gerlach experiments where the second magnet is oriented along a direction that is orthogonal to the two that we consider in the previous section. The total spin operator, $\hat{S}^2$, can then be constructed similarly to the total angular momentum operator of Equation 22.3.5, as:

$$\begin{aligned} \hat{S}^2 &= \hat{S}\cdot\hat{S} = \left(\mathbf{i}\hat{S}_x + \mathbf{j}\hat{S}_y + \mathbf{k}\hat{S}_z\right)\cdot\left(\mathbf{i}\hat{S}_x + \mathbf{j}\hat{S}_y + \mathbf{k}\hat{S}_z\right) \\ &= \hat{S}_x^2 + \hat{S}_y^2 + \hat{S}_z^2, \end{aligned} \tag{22.3.1}$$

with $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ the unitary vectors in three-dimensional space.

Wolfgang Pauli explicitly derived the relationships between all three spin projection operators. Assuming the magnetic field along the z axis, Pauli's relations can be written using simple equations involving the two possible eigenstates $\phi_\uparrow$ and $\phi_\downarrow$:

$$\begin{aligned} \hat{S}_x\phi_\uparrow &= \frac{\hbar}{2}\phi_\downarrow \qquad & \hat{S}_y\phi_\uparrow &= \frac{\hbar}{2}i\phi_\downarrow \qquad & \hat{S}_z\phi_\uparrow &= \frac{\hbar}{2}\phi_\uparrow \\ \hat{S}_x\phi_\downarrow &= \frac{\hbar}{2}\phi_\uparrow \qquad & \hat{S}_y\phi_\downarrow &= -\frac{\hbar}{2}i\phi_\uparrow \qquad & \hat{S}_z\phi_\downarrow &= -\frac{\hbar}{2}\phi_\downarrow, \end{aligned} \tag{22.3.2}$$

where i is the imaginary unit ($i^2 = -1$). In other words, for $\hat{S}_z$ we have eigenvalue equations, while the remaining components have the effect of permuting state $\phi_\uparrow$ with state $\phi_\downarrow$ after multiplication by suitable constants. We can use these equations, together with Equation 23.1.7, to calculate the commutator for each couple of spin projector operators:

$$\begin{aligned} \left[\hat{S}_x, \hat{S}_y\right] &= i\hat{S}_z \\ \left[\hat{S}_y, \hat{S}_z\right] &= i\hat{S}_x \\ \left[\hat{S}_z, \hat{S}_x\right] &= i\hat{S}_y, \end{aligned} \tag{22.3.3}$$

which prove that the three projection operators do not commute with each other.

Example 22.3.1

Proof of Commutator Between Spin Projection Operators.

Solution

The equations in 22.3.3 can be proved by writing the full eigenvalue equation and solving it using the definition of commutator, Equation 23.1.7, in conjunction with Pauli's relation, Equations 22.3.2. For example, for the first couple:

$$\begin{aligned} \left[\hat{S}_x, \hat{S}_y\right]\phi_\uparrow &= \hat{S}_x\hat{S}_y\phi_\uparrow - \hat{S}_y\hat{S}_x\phi_\uparrow \\ &= \hat{S}_x\left(\frac{\hbar}{2}i\phi_\downarrow\right) - \hat{S}_y\left(\frac{\hbar}{2}\phi_\downarrow\right) \\ &= \frac{\hbar}{2}i\left(\frac{\hbar}{2}i\phi_\downarrow\right) - \frac{\hbar}{2}\left(\frac{\hbar}{2}\phi_\downarrow\right) \\ &= \left(\frac{\hbar}{4} + \frac{\hbar}{4}\right)i\phi_\uparrow \\ &= \frac{\hbar}{2}i\phi_\uparrow \\ &= i\hat{S}_z\phi_\uparrow \end{aligned} \tag{22.3.4}$$