24.2: Heisenberg's Uncertainty Principle

Let's now revisit the simple case of a free particle. As we saw in chapter 20, the wave function that solved the TISEq:

$$\psi(x) = A\exp(\pm ikx), \tag{24.2.1}$$

is the equation of a plane wave along the x direction. This result is in agreement with the de Broglie hypothesis, which says that every object in the universe is a wave. If this wave function describes a particle with mass (such as an electron), freely moving along one spatial direction x, it would be reasonable to ask the question: where is the particle located? Analyzing Equation 24.2.1, however, it is not possible to answer this question since $\psi(x)$ is delocalized in space from $x=-\infty$ to $x=+\infty$.[1] In other words, the particle position is extremely uncertain because it could be essentially anywhere along the wave.

Thus for a free particle, the particle side of the wave-particle duality seems completely lost. We can, however, bring it back into the picture by writing the wave function as a sum of many plane waves, called a *wave packet*:

$$\psi(x) \propto \sum_n A_n \exp\left(\frac{ip_n x}{\hbar}\right), \tag{24.2.2}$$

where A_n represents the relative contribution of the mode p_n to the overall total. We are allowed to write the wave function this way because each individual plane wave is a solution of the TISEq, and as we already saw in chapter 22 and several other places, the sum of each individual solution is also a solution. An interesting consequence of writing the wave function as a wave packet is that when we sum different waves, they interfere with each other, and they might localize in some region of space. Thus for a wave function written as in Equation 24.2.2, the wave packet can become more localized. We may also make this procedure a step further to the continuum limit, where the wave function goes from a sum to an integral over all possible modes:

$$\psi(x) = \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty}\varphi(p)\cdot\exp\left(\frac{ipx}{\hbar}\right)dp, \tag{24.2.3}$$

where $\varphi(p)$ represents the amplitude of these modes and is called the wave function in momentum space. In mathematical terms, we say that $\varphi(p)$ is the Fourier transform of $\psi(x)$ and that x and p are conjugate variables. Adding together all of these plane waves comes at a cost; namely, the momentum has become less precise since it becomes a mixture of waves of many different momenta.

One way to quantify the precision of the position and momentum is the standard deviation, σ. Since $|\psi(x)|^2$ is a probability density function for position, we calculate its standard deviation. The precision of the position is improved—i.e., reduced σ_x—by using many plane waves, thereby weakening the precision of the momentum—i.e., increased σ_p. Another way of stating this is that σ_x and σ_p have an inverse relationship (once we know one with absolute precision, the other becomes completely unknown). This fact was discovered by Werner Heisenberg and is now called the **Heisenberg's uncertainty principle**. The mathematical treatment of this procedure results in the simple formula:

$$\sigma_x\sigma_p \geq \frac{\hbar}{2}. \tag{24.2.4}$$

The uncertainty principle can be extended to any couple of conjugated variables, including, for example, energy and time, angular momentum components along perpendicular directions, spin components along perpendicular directions, etc. It is also easy to show that conjugate variables in quantum mechanics correspond to non-commuting operators.[2]

1. The time-dependent picture does not help us either, but since it is a little more complicated to work with the TDSEq, we are not showing it here.
2. Therefore, a simpler way of finding if two variables are subject to the uncertainty principle is to check if their corresponding operators commute.