1.1: Some Newtonian Physics

Consider the definition of **acceleration** (a) as the first time-derivative of **velocity** (v) and the second time-derivative of **position** (x).

$$a = \frac{dv}{dt} = \frac{d^2x}{dt^2}$$

Newton's second law states that **force** (F) is the product of **mass** (m) and acceleration.

$$\begin{aligned} F &= ma \\ &= m\frac{dv}{dt} \\ &= m\frac{d^2x}{dt^2} \end{aligned}$$

Since **momentum** (p) is related to velocity and mass thought the definition

$$p = mv$$

(and mass is invariant to time) the following must hold.

$$\frac{dp}{dt} = \frac{d\,(mv)}{dt} m\frac{dv}{dt} = ma = F$$

Now consider **potential energy** (U) – which is also related to force through the first derivative with respect to position.

$$F = -\frac{dU}{dx}$$

This indicates that the following equation must hold for any particle that can be described by Newtonian motion.

$$-\frac{dU}{dx} = \frac{dp}{dt}$$

The classical **Hamiltonian** (H) is the sum of **kinetic energy** (T) and **potential energy** (U). And as it turns out, the kinetic energy can be expressed in terms of momentum.

$$T = \frac{mv^2}{2} = \frac{p^2}{2m}$$

So the Hamiltonian function, which gives the sum of the kinetic and potential energies is given by

$$H = \frac{p^2}{2m} + U$$

The time-rate-of-change of the total energy can be found from the first derivative of H with respect to t.

$$\begin{aligned} \frac{d}{dt}H &= \frac{d}{dt}\left(\frac{p^2}{2m} + U\right) \\ &= \frac{1}{2m} \cdot 2p \cdot \frac{dp}{dt} + \frac{dU}{dt} \\ &= \frac{2mv}{2m} \cdot \frac{dp}{dt} + \frac{dU}{dx}\frac{dx}{dt} \\ &= \frac{dx}{dt}\left(\frac{dp}{dt} + \frac{dU}{dx}\right) \end{aligned}$$

And since

$$-\frac{dU}{dx} = \frac{dp}{dt}$$

it follows that

$$\begin{aligned}\frac{d}{dt}H &= \frac{dx}{dt}\left(-\frac{dU}{dx}+\frac{dU}{dx}\right)\\ &= \frac{dx}{dt}(0)\\ &= 0\end{aligned}$$

This indicates that the total energy of a system that follows Newtonian physics does not change in time. Another way to state this is that energy is conserved, or that total energy is a “constant of the motion”. This is also a mathematical proof that the sum of potential and kinetic energy must be conserved in all processes, since this sum cannot change in time.

Many discussions in this text will rely on derivations such as above in order to make specific points about the nature of matter. Keep in mind that the important points are the conclusions as well as the pathway to relating the conclusions to the initial parameters of the problem. The more you can focus on these aspects, rather than getting bogged down in the specifics of the math, the more sense quantum mechanics will make to you.