11.5: Bell's Inequality

Bohm's work on the EPR paradox reawakened an interest in the topic. One physicist who took a particular interest in the topic was John S. Bell. Bell proposed a mathematical model that could in fact distinguish between local hidden variable theories and quantum theory [16].

Consider a set of things U which can be subdivided into three overlapping subsets, A, B and C. Bell's theorem states: the number of members of A that are not a member of B plus all members of B that are not a member of C must be greater than or equal to the number of members in the subset of A that are not also in subset B.

To show this, let's first settle on some notation. We'll call the number of items that are in subset A, but not in subset B by the symbol $N(A_+B_-)$ and the number of items in subset B but not in subset C by $N(B_+C_-)$. Etcetera. This notation coupled with the use of some Venn diagrams, the concept of the inequality should become clear.

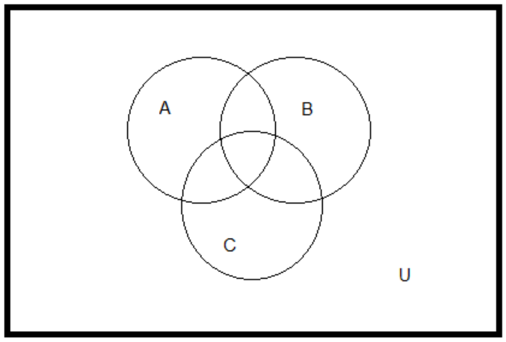

Figure 11.5.1

It should be clear that $N(A_+B_-)$ can be easily shown to be given by the number of items in subset A, not in subset B and in subset C, plus the number in A, not in B and not in C.

$$N(A_+B_-) = N(A_+B_-C_+) + N(A_+B_-C_-)$$

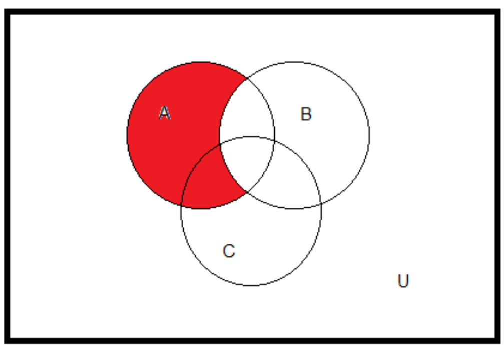

Figure 11.5.2

Similar sums can be derived for $N(B_+C_-)$ and $N(A_+C_-)$

$$N(B_+C_-) = N(A_+B_+C_-) + N(A_-B_+C_-)$$

$$N(A_+C_-) = N(A_+B_+C_-) + N(A_+B_-C_-)$$

Shown below is the sum for $N(B_+C_-)$.

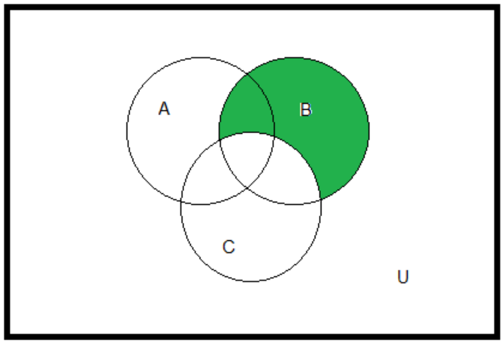

Figure 11.5.3

Adding the terms for $N(A_+B_-)$ and $N(B_+C_-)$ gives

$$N(A_+B_-) + N(B_+C_-) = N(A_+B_-C_+) + N(A_+B_-C_-) + N(A_+B_+C_-) + N(A_-B_+C_-)$$

This can be simplified by grouping the terms for $N(A_+B_+C_-)$ and $N(A_+B_-C_-)$ and recognizing that their sum gives $N(A_+C_-)$.

$$N(A_+B_-) + N(B_+C_-) = N(A_+B_-C_+) + N(A_-B_+C_-) + N(A_+C_-)$$

So long as neither $N(A_+B_-C_+)$ nor $N(A_-B_+C_-)$ are negative (which they can not be) then we arrive at CityplaceBell's inequality:

$$N(A_+B_-) + N(B_+C_-) \geq N(A_+C_-)$$

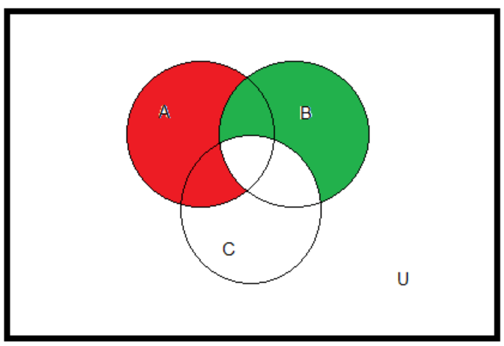

Figure 11.5.4

Employing the Stern-Gerlach results to Test Bell's Inequality

On the face of it, place CityBell's result does not seem that extraordinary. In fact, it almost seems trivial. However, it is only trivial when the results of tests that would place an object into group A, B or C are not correlated. When the results are correlated, the result becomes a bit perplexing.

Consider the dissociation of a pion (also called a π meson), which is a subatomic particle with zero spin and zero charge. It can decompose into a positron and an electron (to conserve charge), each traveling in opposite directions (such that momentum is conserved.) The spins will also be entangled in such a way as to conserve angular momentum.

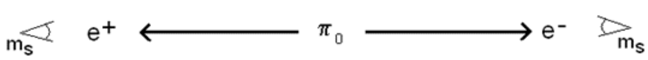

Figure 11.5.5

In fact, the spin state of the electron/positron pair will be given by the familiar singlet spin function:

$$\Psi = \frac{1}{\sqrt{2}}(\alpha_+\beta_- - \beta_+\alpha_-)$$

This suggests that if the positron (subscript +) is detected in the α spin state, the electron (subscript -) will necessarily be forced into the β spin state. The wavefunction allows for equal probability that the positron will be detected in the α spin state or the β spin state, but detection in either state forces an immediate collapse of the wavefunction for the electron. This is the "spooky action at a distance" that Einstein so vehemently rejected in the EPR paper [14]. Einstein also insisted that the spin state of the positron was a "real" property that existed with a definite value for the entire transit of the positron from the decay event to the detector. And quantum mechanics, in Einstein's view, was incomplete in that it could not predict the "realness" of that spin state. If Einstein's view was correct, then correlated measurements of the two spin states would have to satisfy CityplaceBell's inequality.

With the results of the Stern-Gerlach experiments, we can actually determine exactly what quantum mechanics will predict. To do this, we will set up our detectors to detect the spin to the dissociated fragments, but we will rotate the detectors relative to one another. In a laboratory-fixed coordinate system, we will set detector A at 0° rotation, B at 30° and C and 60°. What we want to know is the probability that if one detector measures its particle to be in spin state α that the other will measure its particle to be in spin state β. That probability will be related to the angle of rotation of the second detector relative to the first. According to the Stern-Gerlach result, the probability is given by $\frac{1}{2}\sin(\theta_2-\theta_1)$, where θ_2 and θ_1 are the angles of the second and first detectors in the pair respectively.

So if we define $P(A_+B_-)$ as the probability that detector A detects an $\alpha spin$ and detector B fails to detect a β spin, we can construct the following table based on three specific experimental configurations:

Experiment	θ_1	θ_2	Case	$\theta_1-\theta_2$	$\frac{1}{2}sin^2(\Delta\theta)$
1	0°	30°	$P(A_+B_-)$	30°	0.125
2	30°	60°	$P(B_+C_-)$	30°	0.125
3	0°	60°	$P(A_+C_-)$	60°	0.375

After collecting data from a very large set of measurements using these configurations, we will have can compare the experimental distribution of outcomes to what is predicted by quantum mechanics, and thus conclude if it is possible to have a locality variable that predetermines our outcomes, or if the measurements are purely probabilistic. If the locality variable exists, then Bell's Inequality must hold [17].

$$P(A_+B_-)+P(B_+C_-)\geq P(A_+C_-)$$

However, if Quantum Mechanics allows for a locality variable to redetermine the measured outcomes of the three experiments, then the following must be true:

$$0.125+0.125\geq 0.375$$

Except that it simply isn't true. (In fact, it isn't even true for extremely large values of the sum 0.125 + 0.125.) The above set of experiments was proposed by Alain Aspect in 1976 [17], and results published in 1982 [18]. And while the results were criticized due to the "detection loophole", results of similar experiments being conducted up to 2015 [20] confirmed Aspect's results. Alain Aspect shared the 2022 Nobel Prize in Physics with John Clauser and Anton Zeillinger "for experiments with entangled photons, establishing the violation of Bell inequalities and pioneering quantum information science". [21]

Since Aspect's result was derived completely independent of any theory of hidden variables, it should be clear that the result is incompatible with any such theory. In fact, the result shows that one must divorce oneself from any ideas of local realism for quantum mechanical particles. One simply must conclude that it is the observation that creates the reality and that no reality for observable properties on quantum mechanical system can exist independent of their observation. (Of course, Sheldon Cooper would also point out that one can be beaten up simply for referring to oneself as "one.) [19]

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