3.5: Momentum Operators

One of the tasks we must be able to do as we develop the quantum mechanical representation of a physical system is to replace the classical variables in mathematical expressions with the corresponding quantum mechanical operators. In the preceding section, operators were identified for the total energy and the kinetic energy. Potential energy operators will be introduced case by case in the following chapters. In the remaining paragraphs, we will focus on the momentum operator.

Momentum operators now can be obtained from the kinetic energy operator. Since the classical expression for the kinetic energy of a particle moving in one dimension, along the x-axis, is

$$T_x = \frac{P_x^2}{2m} \tag{3.5.1}$$

we expect that

$$\hat{T}_x = \frac{P_x^2}{2m} = -\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} \tag{3.5.2}$$

so we can identify the operator for the square of the x-momentum as

$$\hat{P}_x^2 = -\hbar^2 \frac{\partial^2}{\partial x^2} \tag{3.5.3}$$

Since $\hat{P}_x^2$, can be interpreted to mean $\hat{P}_x . \hat{P}_x$, there are two possibilities for $\hat{P}_x$, namely

$$\hat{P}_x = i\hbar\frac{\partial}{\partial x} \tag{3.5.4}$$

or

$$\hat{P}_x = -i\hbar\frac{\partial}{\partial x} \tag{3.5.5}$$

where $i = \sqrt{-1}$. The second possibility is the best choice, as explained below.

In making this choice, consider the e^{ikx} function. This function is an eigenfunction of both possible forms for the momentum operator. This fact can be used to choose which form of the momentum operator to use.

Problems

Exercise 3.5.11 Demonstrate that the function e^{ikx} is an eigenfunction of either momentum operator.

Plan: Start with $\hat{P}_x\psi(x) = P_x\psi(x)$ where $\psi(x) = e^{ikx}$.

Operate on $\psi(x) = e^{ikx}$ with $\pm i\hbar\frac{\partial}{\partial x}$ to show that $P_x = \mp\hbar k$.

Which do you prefer, $p_x = +\hbar k$ or $p_x = -\hbar k$?

If we use the momentum operator that has the - sign, we get the momentum and the wave vector pointing in the same direction, $p_x = +\hbar k$, which is the preferred result corresponding to the de Broglie relation.

The review of vectors and scalar products may help you with the following exercises.

Exercise 3.5.12 Show graphically, using a unit vector diagram, that $\vec{x}\cdot\vec{x} = 1 \text{and} \vec{x}\cdot\vec{y} = 0$.

Exercise 3.5.13 Consider a particle moving in three dimensions. The total momentum, which is a vector, is $p = \vec{x}P_x + \vec{y}P_y + \vec{z}P_z$

where $\vec{x}, \vec{y}, and \vec{z}$ are unit vectors pointing in the x, y, and z directions, respectively. Write the operators for the momentum of this particle in the x, y, and z directions, and show that the total momentum operator is $-i\hbar\nabla = -i\hbar\left(\vec{x}\frac{\partial}{\partial x} + \vec{y}\frac{\partial}{\partial y} + \vec{z}\frac{\partial}{\partial z}\right)$ is the vector operator called del(nabla). Show that the scalar product $\nabla\cdot\nabla$ produces the Laplacian operator.

Exercise 3.5.14 Following Exercise 3.5.11, show that the de Broglie relation $p = \frac{h}{\lambda}$ follows from the definition of the momentum operator and the momentum eigenfunction for a one-dimensional space.

Exercise 3.5.15 Write the wavefunction for an electron moving in the z-direction with an energy of 100 eV. The form of the wavefunction is e^{ikz}. You need to find the value for k. Obtain the electron's momentum by operating on the wavefunction with the momentum operator.
