3.2: A Classical Wave Equation

The easiest way to find a differential equation that will provide wavefunctions as solutions is to start with a wavefunction and work backwards. We will consider a sine wave, take its first and second derivatives, and then examine the results. The amplitude of a sine wave can depend upon position, x, in space,

$$A(x) = A_0 \sin\left(\frac{2\pi x}{\lambda}\right) \tag{3.2.1}$$

or upon time, t,

$$A(t) = A_0 \sin(2\pi\nu t) \tag{3.2.2}$$

or upon both space and time,

$$A(x,t) = A_0 \sin\left(\frac{2\pi x}{\lambda} - 2\pi\nu t\right) \tag{3.2.3}$$

We can simplify the notation by using the definitions of a wave vector, $k = \frac{2\pi}{\lambda}$, and the angular frequency, $\omega = 2\pi\nu$ to get

$$A(x,t) = A_0 \sin(kx - \omega t) \tag{3.2.4}$$

When we take partial derivatives of A(x,t) with respect to both x and t, we find that the second derivatives are remarkably simple and similar.

$$\frac{\partial^2 A(x,t)}{\partial x^2} = -k^2 A_0 \sin(kx - \omega t) = -k^2 A(x,t) \tag{3.2.5}$$

$$\frac{\partial^2 A(x,t)}{\partial t^2} = -\omega^2 A_0 \sin(kx - \omega t) = -\omega^2 A(x,t) \tag{3.2.6}$$

By looking for relationships between the second derivatives, we find that both involve $A(x,t)$; consequently an equality is revealed.

$$k^{-2} \frac{\partial^2 A(x,t)}{\partial x^2} = -A(x,t) = \omega^{-2} \frac{\partial^2 A(x,t)}{\partial t^2} \tag{3.2.7}$$

Recall that ν and λ are related; their product gives the velocity of the wave, $\nu\lambda = v$. Be careful to distinguish between the similar but different symbols for frequency ν and the velocity v. If in ω = 2πv we replace v with v/λ, then

$$\omega = \frac{2\pi\nu}{\lambda} = \nu k \tag{3.2.8}$$

and Equation 3.2.7 can be rewritten to give what is known as the classical wave equation in one dimension. This equation is very important. It is a differential equation whose solution describes all waves in one dimension that move with a constant velocity (e.g. the vibrations of strings in musical instruments) and it can be generalized to three dimensions. The classical wave equation in one-dimension is

$$\frac{\partial^2 A(x,t)}{\partial x^2} = \nu^{-2} \frac{\partial^2 A(x,t)}{\partial t^2} \tag{3.2.9}$$

Example 3.2.1

Complete the steps leading from Equation 3.2.3 to Equations 3.2.5 and 3.2.6 and then to Equation 3.2.9.