

1.15: Exact and Inexact Differentials

So far, we discussed how to calculate the total differential of a function. If you are given a function of more than one variable, you can calculate its total differential using the definition of a total differential of a function u : ($du = \left(\frac{\partial u}{\partial x_1}\right)_{x_2 \dots x_n} dx_1 + \left(\frac{\partial u}{\partial x_2}\right)_{x_1, x_3 \dots x_n} dx_2 + \dots + \left(\frac{\partial u}{\partial x_n}\right)_{x_1 \dots x_{n-1}} dx_n$). You will have one term for each independent variable. What if we are given a differential (e.g.

$$dz = (9x^2 + 6xy + y^2)dx + (3x^2 + 2xy)dy$$

see Example 9.1) and we are asked to calculate the function whose total differential is dz ? This is basically working Example 9.1 backwards: we know the differential, and we are looking for the function. Things are a little bit more complicated than this, because not all differentials are the total differentials of a function. For example, from the example above we know that

$$dz = (9x^2 + 6xy + y^2)dx + (3x^2 + 2xy)dy$$

is the total differential of

$$z(x, y) = 3x^3 + 3xy^2 + xy^2.$$

However, the differential $dz = xydx + x^2dy$ is **not** the total differential of any function $z(x, y)$. You can write down every single function $z(x, y)$ in this planet, calculate their total differentials, and you will never see $dz = xydx + x^2dy$ in your list.

Therefore, the question we are addressing is the following: given a differential, 1) is it the total differential of any function? 2) if it is, which function?

To illustrate the question, let's say we are given the differential below (notice that I switched to P , V , and T , which are variables you will encounter often in thermodynamics):

$$dP = \frac{RT}{V-b} dT + \left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2} \right] dV \quad (1.15.1)$$

The question is whether this is the total differential of a function $P = P(T, V)$ (we are told that a and b are constants, and we already know that R is a constant). By definition of total differential, if the function exists, its total differential will be:

$$dP = \left(\frac{\partial P}{\partial T} \right)_V dT + \left(\frac{\partial P}{\partial V} \right)_T dV \quad (1.15.2)$$

Comparing Equation 1.15.1 and 1.15.2, if the function exists, its derivatives will have to be:

$$\left(\frac{\partial P}{\partial T} \right)_V = \frac{RT}{V-b} \quad (1.15.3)$$

$$\left(\frac{\partial P}{\partial V} \right)_T = \left[\frac{RT}{(V-b)^2} - \frac{a}{TV^2} \right] \quad (1.15.4)$$

If we find a function $P = P(T, V)$ that satisfies these equations at the same time, we know that Equation 1.15.1 will be its total differential.

From Equation 1.15.3, we can calculate P by integrating with respect to T at constant V :

$$\int dP = \int \frac{RT}{V-b} dT \rightarrow P = \frac{R}{V-b} \frac{T^2}{2} + f(V) \quad (1.15.5)$$

where we included an integration constant ($f(V)$) that can be any function of V (we are integrating at constant V).

In order to get an expression for $P(T, V)$, we need to find out $f(V)$ so we can complete the right side of Equation 1.15.5. To do that, we are going to take the derivative of P (Equation 1.15.5 with respect to V , and compare with Equation 1.15.4

$$\left(\frac{\partial P}{\partial V} \right)_T = -\frac{RT^2}{2(V-b)^2} + \frac{df(V)}{dV} \quad (1.15.6)$$

Looking at Equation 1.15.4 and 1.15.6, we see that the two expressions do not match, regardless of which function we chose for $f(V)$. This means that Equation 1.15.1 does not represent the total differential of any function $P(V, T)$. We call these differentials **inexact differentials**. If a differential is the total differential of a function, we will call the differential **exact**.

What we did so far is correct, but it is not the easiest way to test whether a differential is exact or inexact. There is, in fact, a very easy way to test for exactness. We'll derive the procedure below, but in the future we can use it without deriving it each time.

Given the differential $dz = f_1(x, y)dx + f_2(x, y)dy$, the differential is exact if

$$\left(\frac{\partial f_1(x, y)}{\partial y}\right)_x = \left(\frac{\partial f_2(x, y)}{\partial x}\right)_y \quad (1.15.7)$$

If Equation 1.15.7 does not hold, the differential is inexact. For instance, if $dz = (9x^2 + 6xy + y^2)dx + (3x^2 + 2xy)dy$, the functions f_1 and f_2 are $f_1 = 9x^2 + 6xy + y^2$ and $f_2 = 3x^2 + 2xy$. To test this differential, we perform the partial derivatives

$$\left(\frac{\partial f_1(x, y)}{\partial y}\right)_x = 6x + 2y$$

and

$$\left(\frac{\partial f_2(x, y)}{\partial x}\right)_y = 6x + 2y$$

The two derivatives are the same, and therefore the differential is said to be exact.

Let's prove why the test of Equation 1.15.7 works. Let's consider a differential of the form $dz = f_1(x, y)dx + f_2(x, y)dy$. If the differential is exact, it is the total differential of a function $z(x, y)$, and therefore:

$$dz = f_1(x, y)dx + f_2(x, y)dy = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy \quad (1.15.8)$$

We know that the mixed partial derivatives of a function are independent of the order they are computed:

$$\left(\frac{\partial^2 z}{\partial y \partial x}\right) = \left(\frac{\partial^2 z}{\partial x \partial y}\right)$$

From Equation 1.15.8

$$\begin{aligned} f_1(x, y) &= \left(\frac{\partial z}{\partial x}\right)_y \rightarrow \left(\frac{\partial f_1(x, y)}{\partial y}\right)_x = \left(\frac{\partial^2 z}{\partial x \partial y}\right) \\ f_2(x, y) &= \left(\frac{\partial z}{\partial y}\right)_x \rightarrow \left(\frac{\partial f_2(x, y)}{\partial x}\right)_y = \left(\frac{\partial^2 z}{\partial y \partial x}\right) \end{aligned}$$

Because the mixed partial derivatives are the same, for an exact differential:

$$\left(\frac{\partial f_1(x, y)}{\partial y}\right)_x = \left(\frac{\partial f_2(x, y)}{\partial x}\right)_y$$

This equation is true only for an exact differential because we derived it by assuming that the function $z = z(x, y)$ exists, so its mixed partial derivatives are the same. We can use this relationship to test whether a differential is exact or inexact. If the equality of Equation 1.15.7 holds, the differential is exact. If it does not hold, it is inexact.

✓ Example 1.15.1

Test whether the following differential is exact or inexact:

$$dz = \frac{1}{x^2} dx - \frac{y}{x^3} dy$$

Solution

To test whether dz is exact or inexact, we compare the following derivatives

$$\left(\frac{\partial(1/x^2)}{\partial y}\right)_x \stackrel{?}{=} \left(\frac{\partial(y/x^3)}{\partial x}\right)_y$$

$$\left(\frac{\partial(1/x^2)}{\partial y}\right)_x = 0$$

$$\left(\frac{\partial(y/x^3)}{\partial x}\right)_y = -3yx^{-4}$$

We conclude that dz is inexact, and therefore there is no function $z(x, y)$ whose total differential is dz .

✓ Example 1.15.2

Determine whether the following differential is exact or inexact. If it is exact, determine $z = z(x, y)$.

$$dz = (2x + y)dx + (x + y)dy$$

Solution

To test whether dz is exact or inexact, we compare the following derivatives

$$\left(\frac{\partial(2x + y)}{\partial y}\right)_x \stackrel{?}{=} \left(\frac{\partial(x + y)}{\partial x}\right)_y$$

If this equality holds, the differential is exact.

$$\left(\frac{\partial(2x + y)}{\partial y}\right)_x = 1$$

$$\left(\frac{\partial(x + y)}{\partial x}\right)_y = 1$$

Therefore, the differential is exact. Because it is exact, it is the total differential of a function $z(x, y)$. The total differential of $z(x, y)$ is, by definition,

$$dz = \left(\frac{\partial z}{\partial x}\right)_y dx + \left(\frac{\partial z}{\partial y}\right)_x dy$$

Comparing this expression to the differential $dz = (2x + y)dx + (x + y)dy$:

$$\left(\frac{\partial z}{\partial x}\right)_y = (2x + y)$$

$$\left(\frac{\partial z}{\partial y}\right)_x = (x + y) \tag{1.15.9}$$

To find $z(x, y)$, we can integrate the first expression partially with respect to x keeping y constant:

$$\int dz = z = \int (2x + y)dx = x^2 + xy + f(y)$$

So far we have

$$z = x^2 + xy + f(y) \tag{1.15.10}$$

so we need to find the function $f(y)$ to complete the expression above and finish the problem. To do that, we'll take the derivative of z with respect to y , and compare with Equation 1.15.9. The derivative of Equation 1.15.10 is:

$$\left(\frac{\partial z}{\partial y}\right)_x = x + \frac{df(y)}{dy}$$

comparing with Equation 1.15.9 we notice that $\frac{df(y)}{dy} = y$, and integrating, we obtain $f(y) = y^2/2 + c$

Therefore, $dz = (2x + y)dx + (x + y)dy$ is the total differential of $z = x^2 + xy + y^2/2 + c$.

We can check our result by working the problem in the opposite direction. If we are given $z = x^2 + xy + y^2/2 + c$ and we are asked to calculate its total differential, we would apply the definition:

$$dz = \left(\frac{\partial z}{\partial x} \right)_y dx + \left(\frac{\partial z}{\partial y} \right)_x dy$$

and because

$$\left(\frac{\partial z}{\partial x} \right)_y = y + 2x$$

and

$$+ \left(\frac{\partial z}{\partial y} \right)_x = y + x$$

we would write $dz = (2x + y)dx + (x + y)dy$, which is the differential we were given in the problem.

Check two extra solved examples in this video: <http://tinyurl.com/kq4qecu>

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