4.4: Preservation of Phase Space Volume and Liouville's Theorem

Consider a phase space volume element dx_0 at t=0, containing a small collection of initial conditions on a set of trajectories. The trajectories evolve in time according to Hamilton's equations of motion, and at a time t later will be located in a new volume element dx_t as shown in the figure below:

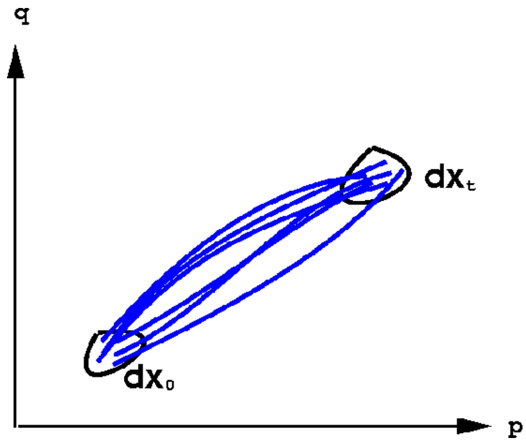

Figure 4.4.1

How is dx_0 related to dx_tdxdd ? To answer this, consider a trajectory starting from a phase space vector x_0 in dx_0 and having a phase space vector x_t at time t in dx_t. Since the solution of Hamilton's equations depends on the choice of initial conditions, x_t depends on x_0 :

$$\begin{aligned} x_0 &= (p_1(0), \cdots, p_N(0), r_1(0), \cdots, r_N(0)) \\ x_0 &= (p_1(t), \cdots, p_N(t), r_1(t), \cdots, r_N(t)) \\ x_t^i &= x_t^i\left(x_0^1, \cdots, x_0^{6N}\right) \end{aligned}$$

Thus, the phase space vector components can be viewed as a coordinate transformation on the phase space from $t = 0$ to time t. The phase space volume element then transforms according to

$$dx_t = J(x_t; x_0) dx_0$$

where $J(x_t; x_0)$ is the Jacobian of the transformation:

$$J(x_t; x_0) = \frac{\partial(x_t^1 \cdots x_t^n)}{\partial(x_0^1 \cdots x_0^n)}$$

where $n = 6N$. The precise form of the Jacobian can be determined as will be demonstrated below.

The Jacobian is the determinant of a matrix M,

$$J(x_t; x_0) = \det(M) = e^{TrlnM}$$

whose matrix elements are

$$M_{ij} = \frac{\partial x_t^i}{\partial x_0^j}$$

Taking the time derivative of the Jacobian, we therefore have

$$\begin{aligned} \frac{dJ}{dt} &= Tr\left(M^{-1}\frac{dM}{dt}\right)e^{TrlnM} \\ &= J\sum_{i=1}^{n}\sum_{j=1}^{n} M_{ij}^{-1}\frac{dM_{ij}}{dt} \end{aligned}$$

The matrices M_{-1} and $\frac{dM}{dt}$ can be seen to be given by

$$M_{ij}^{-1} = \frac{\partial x_0^i}{\partial x_t^j}$$

$$\frac{dM_{ji}}{dt} = \frac{\partial \dot{x}_t^i}{\partial x_0^i}$$

Substituting into the expression for dJ/dt gives

$$\begin{aligned}\frac{dJ}{dt} &= J\sum_{i,j=1}^{n} \frac{\partial x_0^i}{\partial x_t^j}\frac{\partial \dot{x}_t^i}{\partial x_0^i} \\ &= J\sum_{i,j,k=1}^{n} \frac{\partial x_0^i}{\partial x_t^j}\frac{\partial \dot{x}_t^i}{\partial x_t^k}\frac{\partial x_t^k}{\partial x_0^i}\end{aligned}$$

where the chain rule has been introduced for the derivative $\frac{\partial x_t^j}{\partial x_0^i}$. The sum over i can now be performed:

$$\sum_{i=1}^{n} \frac{\partial x_0^i}{\partial x_t^j}\frac{\partial x_t^k}{\partial x_0^i} = \sum_{i=1}^{n} M_{ij}^{-1} M_{ki} = \sum_{i=1}^{n} M_{ki} M_{ij}^{-1} = \delta_{kj}$$

Thus,

$$\frac{dJ}{dt} = J\sum_{j,k=1}^{n} \delta_{jk}\frac{\partial \dot{x}_t^j}{\partial x_0^k}$$

$$J\sum_{j=1}^{n} \frac{\partial \dot{x}_t^j}{\partial x_t^j} = J\nabla_x \cdot \dot{x}$$

or

$$\frac{dJ}{dt} = J\nabla_x \cdot \dot{x}$$

The initial condition on this differential equation is $J(0) \equiv J(x_0; x_0) = 1$. Moreover, for a Hamiltonian system $\nabla_x \cdot \dot{x} = 0$. This says that $dJ/dt = 0$ and $J(0) = 1$. Thus, $J(x_t; x_0) = 1$. If this is true, then the phase space volume element transforms according to

$$dx_o = dx_t$$

which is another conservation law. This conservation law states that the phase space volume occupied by a collection of systems evolving according to Hamilton's equations of motion will be preserved in time. This is one statement of Liouville's theorem.

Combining this with the fact that $df/dt = 0$, we have a conservation law for the phase space probability:

$$f(x_o, o)dx_o = f(x_t, t)dx_t$$

which is an equivalent statement of Liouville's theorem.