9.7: Time Evolution of the State Vector

The time evolution of the state vector is prescribed by the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t}|\Psi(t)\rangle = H|\Psi(t)\rangle$$

where H is the Hamiltonian operator. This equation can be solved, in principle, yielding

$$|\Psi(t)\rangle = e^{-iHt/\hbar}|\Psi(0)\rangle$$

where $|\Psi(0)\rangle$ is the initial state vector. The operator

$$U(t) = e^{-iHt\hbar}$$

is the **time evolution operator** or **quantum propagator**. Let us introduce the eigenvalues and eigenvectors of the Hamiltonian H that satisfy

$$H|E_i\rangle = E_i|E_i\rangle$$

The eigenvectors for an orthonormal basis on the Hilbert space and therefore, the state vector can be expanded in them according to

$$|\Psi(t)\rangle = \sum_i c_i(t)|E_i\rangle$$

where, of course, $c_i(t) = \langle E_i|\Psi(t)\rangle$, which is the amplitude for obtaining the value E_i at time t if a measurement of H is performed. Using this expansion, it is straightforward to show that the time evolution of the state vector can be written as an expansion:

$$\begin{aligned} |\Psi(t)\rangle &= e^{-iHt\hbar}|\Psi(0)\rangle \\ &= e^{-iHt/\hbar}\sum_i |E_i\rangle\langle E_i|\Psi(0)\rangle \\ &= \sum_i e^{-iE_it/\hbar}|E_i\rangle\langle E_i|\Psi(0)\rangle \end{aligned}$$

Thus, we need to compute all the initial amplitudes for obtaining the different eigenvalues E_i of H, apply to each the factor $\exp(-iE_it/\hbar)|E_i\rangle$ and then sum over all the eigenstates to obtain the state vector at time t.

If the Hamiltonian is obtained from a classical Hamiltonian $H(x, p)$, then, using the formula from the previous section for the action of an arbitrary operator $A(X, P)$ on the state vector in the coordinate basis, we can recast the Schrödiner equation as a partial differential equation. By multiplying both sides of the Schrödinger equation by $\langle x|$, we obtain

$$\begin{aligned} \langle x|H(X,P)|\Psi(t)\rangle &= i\hbar\frac{\partial}{\partial t}\langle x|\Psi(t)\rangle \\ H\left(x, \frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi(x,t) &= i\hbar\frac{\partial}{\partial t}\Psi(x,t) \end{aligned}$$

If the classical Hamiltonian takes the form

$$H(x,p) = \frac{p^2}{2m} + U(x)$$

then the Schrödinger equation becomes

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + U(x)\right]\Psi(x,t) = i\hbar\frac{\partial}{\partial t}\Psi(x,t)$$

which is known as the Schrödinger *wave equation* or the *time-dependent* Schrödinger equation. In a similar manner, the eigenvalue equation for H can be expressed as a differential equation by projecting it into the X basis:

$$\begin{aligned}\langle x|H|E_i\rangle &= E_i\langle x|E_i\rangle \\ H\left(x,\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\psi_i(x) &= E_i\psi_i(x) \\ \left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2}+U(x)\right]\psi_i(x) &= E_i\psi_i(x)\end{aligned}$$

where $\psi_i(x) = \langle x|E_i\rangle$ is an eigenfunction of the Hamiltonian.