12.3: Generalized Equations of Motion

The most general way a system can be driven away from equilibrium by a forcing function $F_e(t)$ is according to the equations of motion:

$$\dot{q}_i = \frac{\partial H}{\partial p_i} + C_i(\mathrm{x})F_e(t)$$

$$\dot{P}_i = -\frac{\partial H}{\partial p_i} + D_i(\mathrm{x})F_e(t)$$

where the $3N$ functions C_i and D_i are required to satisfy the incompressibility condition

$$\sum_{i=1}^{3N}\left[\frac{\partial C_i}{\partial q_i} + \frac{\partial D_i}{\partial p_i}\right] = 0$$

in order to insure that the Liouville equation for $f(x,t)$ is still valid. These equations of motion will give rise to a distribution function $f(x,t)$ satisfying

$$\frac{\partial f}{\partial t} + iLf = 0$$

with $\partial f/\partial t \neq 0$. (We assume that f is normalized so that $\int d\mathrm{x} f(\mathrm{x},t) = 1$.)

What does the Liouville equation say about the nature of $f(x,t)$ in the limit that C_i and D_i are small, so that the displacement away from equilibrium is, itself, small? To examine this question, we propose to solve the Liouville equation perturbatively. Thus, let us assume a solution of the form

$$f(\mathrm{x},t) = f_0(H(\mathrm{x})) + \Delta f(\mathrm{x},t)$$

Note, also, that the equations of motion $\dot{x}$ take a perturbative form

$$\dot{\mathrm{x}}(t) = \dot{\mathrm{x}}_0 + \Delta\dot{\mathrm{x}}(t)$$

and as a result, the Liouville operator contains two pieces:

$$iL = \dot{\mathrm{x}}\cdot\nabla_{\mathrm{x}} = \dot{\mathrm{x}}_0\cdot\nabla_x + \Delta\dot{\mathrm{x}}\cdot\nabla_{\mathrm{x}} = iL_0 + i\Delta L$$

where $iL_0 = \{\ldots, H\}$ and $f_0(H)$ is assumed to satisfy

$$iL_0 f_0(H(\mathrm{x})) = 0$$

$\dot{x}_0$ means the Hamiltonian part of the equations of motion

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$

$$\dot{P}_i = -\frac{\partial H}{\partial q_i}$$

For an observable $A(x)$, the ensemble average of A is a time-dependent quantity:

$$\langle A(t)\rangle = \int d\mathrm{x} A(\mathrm{x}) f(\mathrm{x},t)$$

which, when the assumed form for $f(x,t)$ is substituted in, gives

$$\langle A(t)\rangle = \int d\mathrm{x} A(\mathrm{x}) f_0(\mathrm{x}) + \int dx A(x)\Delta f(x,t) = \langle A\rangle_0 + \int d\mathrm{x} A(\mathrm{x})\Delta f(\mathrm{x},t)$$

where $\langle\cdot\rangle_0$ means average with respect to $f_0(x)$.

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