12.6: The Onsager Fluctuation Regression Theorem

Suppose that $F_e(t)$ is of the form

$$F_e(t) = F_0 e^{\epsilon t}\theta(-t)$$

which adiabatically induces a fluctuation in the system for $t<0$ and the lets the system evolve in time according to the unperturbed Hamiltonian for $t>0$. How will the induced fluctuation evolve in time? Combining the Kubo transform relation with the linear response result for $\langle B(t)\rangle$, we find that

$$\begin{aligned}\langle B(t)\rangle &= \int_{-\infty}^{0} ds\, e^{\epsilon s}\int_0^{\beta} d\lambda \langle \dot{B}(-i\hbar\lambda)B(t-s)\rangle_0 \\ &= -e^{\epsilon t}\int_0^{\beta} d\lambda \int_t^{\infty} du\, e^{-\epsilon u}\frac{d}{du}\langle B(-i\hbar\lambda)B(u)\rangle_0\end{aligned}$$

where the change of variables $u=t-s$ has been made. Taking the limit $\epsilon\rightarrow 0$, and performing the integral over u, we find

$$\langle B(t)\rangle = -\int_0^{\beta} d\lambda\ [\langle B(-i\hbar\lambda)B(\infty)\rangle_0 - \langle B(-i\hbar\lambda)B(t)\rangle_0]$$

Since we assumed that $\langle B\rangle_0 = 0$, we have $\langle B(-i\hbar\lambda)B(\infty)\rangle_0 = \langle B(-i\hbar\lambda)\rangle_0\langle B(\infty)\rangle_0 = 0$. Thus, dividing by $\langle B(0)\rangle$, we find

$$\frac{\langle B(t)\rangle}{\langle B(0)\rangle} = \frac{\int_0^{\beta} d\lambda B(-i\hbar\lambda)B(t)\rangle_0}{\int_0^{\beta} d\lambda B(-i\hbar\lambda)B(0)\rangle_0} \rightarrow_{\hbar\rightarrow 0} \frac{\langle B(0)B(t)\rangle_0}{\langle B(0)^2\rangle_0}$$

Thus at long times in the classical limit, the fluctuations decay to 0, indicting a complete *regression* or suppression of the induced fluctuation:

$$\frac{\langle B(t)\rangle}{\langle B(0)\rangle} \rightarrow 0$$