2.3: The Liouville Operator and the Poisson Bracket

From the last lecture, we saw that Liouville's equation could be cast in the form

$$\frac{\partial f}{\partial t} + \nabla_x \cdot \dot{x} f = 0$$

The **Liouville equation** is the foundation on which statistical mechanics rests. It will now be cast in a form that will be suggestive of a more general structure that has a definite quantum analog (to be revisited when we treat the quantum Liouville equation).

Define an operator

$$iL = \dot{x} \cdot \nabla_x$$

known as the **Liouville operator** ($i = \sqrt{-1}$ - the *i* is there as a matter of convention and has the effect of making L a Hermitian operator). Then Liouville's equation can be written

$$\frac{\partial f}{\partial t} + iLf = 0$$

The Liouville operator also be expressed as

$$iL = \sum_{i=1}^{N} \left[\frac{\partial H}{\partial p_i} \cdot \frac{\partial}{\partial r_i} - \frac{\partial H}{\partial r_i} \cdot \frac{\partial}{\partial p_i} \right] \equiv \{\cdots, H\}$$

where $\{A, B\}$ is known as the **Poisson bracket** between $A(x)$ and $B(x)$:

$$\{A, B\} = \sum_{i=1}^{N} \left[\frac{\partial A}{\partial r_i} \cdot \frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i} \cdot \frac{\partial B}{\partial r_i} \right]$$

Thus, the Liouville equation can be written as

$$\frac{\partial f}{\partial t} + \{f, H\} = 0$$

The Liouville equation is a partial differential equation for the phase space probability distribution function. Thus, it specifies a general class of functions $f(x, t)$ that satisfy it. In order to obtain a specific solution requires more input information, such as an initial condition on *f*, a boundary condition on *f*, and other control variables that characterize the ensemble.