9.1: Measurement

The result of a measurement of the observable A must yield one of the eigenvalues of $\hat{A}$. Thus, we see why A is required to be a hermitian operator: Hermitian operators have *real* eigenvalues. If we denote the set of eigenvalues of $\hat{A}$ by $\{a_i\}$, then each of the eigenvalues a_i satisfies an eigenvalue equation

$$\hat{A}|a_i\rangle = a_i|a_i\rangle$$

where $|a_i\rangle$ is the corresponding eigenvector. Since the operator $\hat{A}$ is hermitian and a_i is therefore real, we have also the left eigenvalue equation

$$\langle a_i|\hat{A} = \langle a_i|a_i$$

The probability amplitude that a measurement of A will yield the eigenvalue a_i is obtained by taking the inner product of the corresponding eigenvector $|a_i\rangle$ with the state vector $|\Psi(t)\rangle$, $\langle a_i|\Psi(t)\rangle$. Thus, the probability that the value a_i is obtained is given by

$$P_{a_i} = |\langle a_i|\Psi(t)\rangle|^2$$

Another useful and important property of hermitian operators is that their eigenvectors form a complete orthonormal basis of the Hilbert space, when the eigenvalue spectrum is non-degenerate. That is, they are linearly independent, span the space, satisfy the orthonormality condition

$$\langle a_i|a_j\rangle = \delta_{ij}$$

and thus any arbitrary vector $|\phi\rangle$ can be expanded as a linear combination of these vectors:

$$|\phi\rangle = \sum_i c_i|a_i\rangle$$

By multiplying both sides of this equation by $\langle a_j|$ and using the orthonormality condition, it can be seen that the expansion coefficients are

$$c_i = \langle a_i|\phi\rangle$$

The eigenvectors also satisfy a closure relation:

$$I = \sum_i |a_i\rangle\langle a_i|$$

where I is the identity operator.

Averaging over many individual measurements of A gives rise to an average value or expectation value for the observable A, which we denote $\langle A\rangle$ and is given by

$$\langle A\rangle = \langle\Psi(t)|A|\Psi(t)\rangle$$

That this is true can be seen by expanding the state vector $|\Psi(t)\rangle$ in the eigenvectors of A:

$$|\Psi(t)\rangle = \sum_i \alpha_i(t)|a_i\rangle$$

where a_i are the amplitudes for obtaining the eigenvalue a_i upon measuring A, i.e., $\alpha_i = \langle a_i|\Psi(t)\rangle$. Introducing this expansion into the expectation value expression gives

$$\begin{aligned}\langle A\rangle(t) &= \sum_{i,j} \alpha_i^*(t)\alpha_j(t)\langle a_i|A|a_i\rangle \\ &= \sum_{i,j} \alpha_i^*(t)\alpha_j a_i(t)\delta_{ij} \\ &= \sum_i a_i|\alpha_i(t)|^2\end{aligned}$$

The interpretation of the above result is that the expectation value of A is the sum over possible outcomes of a measurement of A weighted by the probability that each result is obtained. Since $|\alpha_i|^2 = |\langle a_i|\Psi(t)\rangle|^2$ is this probability, the equivalence of the expressions can be seen.

Two observables are said to be compatible if $AB = BA$. If this is true, then the observables can be diagonalized simultaneously to yield the same set of eigenvectors. To see this, consider the action of BA on an eigenvector $|a_i\rangle$ of A. $BA|a_i\rangle = a_i B|a_i\rangle$. But if this must equal $AB|a_i\rangle$, then the only way this can be true is if $B|a_i\rangle$ yields a vector proportional to $|a_i\rangle$ which means it must also be an eigenvector of B. The condition $AB = BA$ can be expressed as

$$AB - BA = 0$$

that is

$$[A, B] = 0$$

where, in the second line, the quantity $[A, B] \equiv AB - BA$ is know as the commutator between A and B. If $[A, B] = 0$, then A and B are said to commute with each other. That they can be simultaneously diagonalized implies that one can simultaneously predict the observables A and B with the same measurement.

As we have seen, classical observables are functions of position x and momentum P (for a one-particle system). Quantum analogs of classical observables are, therefore, functions of the operators X and P corresponding to position and momentum. Like other observables X and P are linear hermitian operators. The corresponding eigenvalues x and P and eigenvectors $|x\rangle$ and $|P\rangle$ satisfy the equations

$$X|x\rangle = x|x\rangle$$

$$P|p\rangle = p|p\rangle$$

which, in general, could constitute a *continuous* spectrum of eigenvalues and eigenvectors. The operators X and P are not compatible. In accordance with the Heisenberg uncertainty principle (to be discussed below), the commutator between X and P is given by

$$[X, P] = i\hbar I$$

and that the inner product between eigenvectors of X and P is

$$\langle x|p\rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{ipx/\hbar}$$

Since, in general, the eigenvalues and eigenvectors of X and P form a continuous spectrum, we write the **orthonormality** and **closure relations** for the eigenvectors as:

$$\langle x|x'\rangle = \delta(x - x')$$

$$\langle p|p'\rangle = \delta(p - p')$$

$$|\phi\rangle = \int dx|x\rangle\langle x|\phi\rangle$$

$$|\phi\rangle = \int dp|p\rangle\langle p|\phi\rangle$$

$$I = \int dx|x\rangle\langle x|$$

$$I = \int dp|p\rangle\langle p|$$

The probability that a measurement of the operator X will yield an eigenvalue x in a region dx about some point is

$$P(x, t)dx = |\langle x|\Psi(t)\rangle|^2 dx$$

The object $\langle x|\Psi(t)\rangle$ is best represented by a continuous function $\Psi(x,t)$ often referred to as the *wave function*. It is a representation of the inner product between eigenvectors of X with the state vector. To determine the action of the operator X on the state vector in the basis set of the operator X, we compute

$$\langle x|X|\Psi(t)\rangle = x\Psi(x,t)$$

The action of P on the state vector in the basis of the X operator is consequential of the incompatibility of x and P and is given by

$$\langle x|P|\Psi(t)\rangle = \frac{\hbar}{i}\frac{\partial}{\partial x}\Psi(x,t)$$

Thus, in general, for any observable $A(X,P)$, its action on the state vector represented in the basis of X is

$$\langle x|A(X,P)|\Psi(t)\rangle = A\left(x,\frac{\hbar}{i}\frac{\partial}{\partial x}\right)\Psi(x,t)$$

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