9.5: The Heisenberg Uncertainty Principle

Because the operators x and p are not compatible, $[\hat{X}, \hat{P}] \neq 0$, there is **no** measurement that can precisely determine both x and p simultaneously. Hence, there must be an uncertainty relation between them that specifies how uncertain we are about one quantity given a definite precision in the measurement of the other. Presumably, if one can be determined with infinite precision, then there will be an infinite uncertainty in the other. Recall that we had defined the uncertainty in a quantity by

$$\Delta A = \sqrt{\langle A^2 \rangle - \langle A \rangle^2} \tag{1}$$

Thus, for x and p, we have

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{2a}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2} \tag{2b}$$

These quantities can be expressed explicitly in terms of the wave function $\Psi(x,t)$ using the fact that

$$\langle x \rangle = \langle \Psi(t)|x|\Psi(t)\rangle = \int dx \langle \Psi(t)|x\rangle \langle |x|X|\Psi(t)\rangle = \int dx \Psi^*(x,t) x \Psi(x,t) \tag{3}$$

and

$$\langle x^2 \rangle = \langle \Psi(t)|x^2|\Psi(t)\rangle = \int \Psi^*(x,t) x^2 \Psi(x,t) \tag{4}$$

Similarly,

$$\langle p \rangle = \langle \Psi(t)|p|\Psi(t)\rangle = \int dx \langle \Psi(t)|x\rangle \langle |p|\Psi(t)\rangle = \int dx \Psi^*(x,t) \frac{\hbar}{i} \frac{\partial}{\partial x} \Psi(x,t) \tag{5}$$

and

$$\langle p^2 \rangle = \langle \Psi(t)|p^2|\Psi(t)\rangle = \int dx \Psi^*(x,t) \left(-\hbar^2 \frac{\partial^2}{\partial x^2}\right) \Psi(x,t) \tag{6}$$

Then, the Heisenberg uncertainty principle states that

$$\Delta x \Delta p \gtrsim \hbar \tag{7}$$

which essentially states that the greater certainty with which a measurement of x or p can be made, the greater will be the *uncertainty* in the other.