# 13.3: The Interaction Picture

Consider a quantum system described by a time-dependent Hamiltonian of the form

$$H(t) = H_0 + H_1(t)$$

In the language of perturbation theory, $H_0$ is known as the unperturbed Hamiltonian and describes a system of interest such as a molecule or a condensed-phase sample such as a pure liquid or solid or a solution. $H_1(t)$ is known as the perturbation, and it often describes an external system, such as a laser field, that will be used to probe the energy levels and other properties of $H_0$.

We now seek a solution to the time-dependent Schrödinger equation

$$\mathrm{H}(t)|\Psi(t)\rangle = (\mathrm{H}_0 + \mathrm{H}_1(t))\,|\Psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Psi(t)\rangle \tag{13.3.1}$$

subject to an initial state vector $|\Psi(t_0)\rangle$. In order to solve the equation, we introduce a new state vector $|\Phi(t)\rangle$ related to $|\Psi(t)\rangle$ by

$$|\Psi(t)\rangle = e^{-i\mathrm{H}_0(t-t_0)}\,|\Phi(t)\rangle \tag{13.3.2}$$

The new state vector $|\Phi(t)\rangle$ is an equally valid representation of the state of the system. In Chapter 10, we introduced the concept of *pictures* in quantum mechanics and discussed the difference between the Schrödinger and Heisenberg pictures. Equation 13.3.2 represents yet another picture of quantum mechanics, namely the **interaction picture**. Like the Schrödinger and Heisenberg pictures, the interaction picture is a perfectly valid way of representing a quantum mechanical system. The interaction picture can be considered as "intermediate" between the Schrödinger picture, where the state *evolves in time* and the operators are *static*, and the Heisenberg picture, where the state vector is *static* and the operators *evolve*.

However, as we will see shortly, in the interaction picture, both the state vector and the operators evolve in time, however, the time-evolution is determined by the perturbation $H_1(t)$. Equation 13.3.2 specifies how to transform between the Schrödinger and interaction picture state vectors. The transformation of operators proceeds in an analogous fashion. If $A$ denotes an operator in the Schrödinger picture, its representation in the interaction picture is given by

$$\mathrm{A}_I(t) = e^{i\mathrm{H}_0(t-t_0)/\hbar}\mathrm{A}e^{-i\mathrm{H}_0(t-t_0)/\hbar} \tag{13.3.3}$$

which is equivalent to an equation of motion of the form

$$\frac{d\mathrm{A}_I(t)}{dt} = \frac{1}{i\hbar}[\mathrm{A}_I(t), \mathrm{H}_0] \tag{13.3.4}$$

Substitution of Equation 13.3.2into the time-dependent Schrödinger equation yields

$$(\mathrm{H}_0 + \mathrm{H}_1(t))\,e^{-i\mathrm{H}_0(t-t_0)/\hbar}|\Phi(t)\rangle \;= \mathrm{H}_0 e^{-i\mathrm{H}_0(t-t_0)/\hbar}|\Phi(t)\rangle + e^{-i\mathrm{H}_0(t-t_0)/\hbar}i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle \tag{13.3.5}$$

$$\mathrm{H}_1(t)e^{-iH_0(t-t_0)/\hbar}|\Phi(t)\rangle = e^{-iH_0(t-t_0)/\hbar}i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle \tag{13.3.6}$$

$$e^{iH_0(t-t_0)/\hbar}\mathrm{H}_1(t)e^{-iH_0(t-t_0)/\hbar}|\Phi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle \tag{13.3.7}$$

According to Equation 13.3.3, the $\exp[i\mathrm{H}_0(t-t_0)/\hbar]\mathrm{H}_1(t)\exp[-i\mathrm{H}_0(t-t_0)/\hbar]$ is the interaction-picture representation of the perturbation Hamiltonian, and we will denote this operator as $H_I(t)$. Thus, the time-evolution of the state vector in the interaction picture is given a Schrödinger equation of the form

$$H_I(t)|\Phi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\Phi(t)\rangle \tag{13.3.8}$$

The initial condition to Equation 13.3.8, $|\Phi(t_0)\rangle$ is, according to Equation 13.3.2, also $|\Phi(t_0)\rangle$. In the next section, we will develop an iterative solution to Equation 13.3.8, which will reveal a rich structure of the propagator for time-dependent systems.