

9.4: The Heisenberg Picture

In all of the above, notice that we have formulated the postulates of quantum mechanics such that the state vector $|\Psi(t)\rangle$ evolves in time, but the operators corresponding to observables are taken to be stationary. This formulation of quantum mechanics is known as the *Schrödinger picture*. However, there is another, completely equivalent, picture in which the state vector remains stationary and the operators evolve in time. This picture is known as the *Heisenberg picture*. This particular picture will prove particularly useful to us when we consider quantum time correlation functions.

The Heisenberg picture specifies an evolution equation for any operator A , known as the Heisenberg equation. It states that the time evolution of A is given by

$$\frac{dA}{dt} = \frac{1}{i\hbar} [A, H]$$

While this evolution equation must be regarded as a postulate, it has a very immediate connection to classical mechanics. Recall that any function of the phase space variables $A(x, p)$ evolves according to

$$\frac{dA}{dt} = \{A, H\}$$

where $\{\dots, \dots\}$ is the [Poisson bracket](#). The suggestion is that in the classical limit (\hbar small), the commutator goes over to the Poisson bracket. The Heisenberg equation can be solved in principle giving

$$\begin{aligned} A(t) &= e^{iHt/\hbar} A e^{-iHt/\hbar} \\ &= U^\dagger(t) A U(t) \end{aligned}$$

where A is the corresponding operator in the Schrödinger picture. Thus, the expectation value of A at any time t is computed from

$$\langle A(t) \rangle = \langle \Psi | A(t) | \Psi \rangle$$

where $|\Psi\rangle$ is the stationary state vector.

Let's look at the Heisenberg equations for the operators X and P . If H is given by

$$H = \frac{P^2}{2m} + U(X)$$

then Heisenberg's equations for X and P are

$$\begin{aligned} \frac{dX}{dt} &= \frac{1}{i\hbar} [X, H] \\ &= \frac{P}{m} \\ \frac{dP}{dt} &= \frac{1}{i\hbar} [P, H] \\ &= -\frac{\partial U}{\partial X} \end{aligned}$$

Thus, Heisenberg's equations for the operators X and P are just Hamilton's equations cast in operator form. Despite their innocent appearance, the solution of such equations, even for a one-particle system, is *highly* nontrivial and has been the subject of a considerable amount of research in physics and mathematics.

Note that any operator that satisfies $[A(t), H] = 0$ will not evolve in time. Such operators are known as constants of the motion. The Heisenberg picture shows explicitly that such operators do not evolve in time. However, there is an analog with the Schrödinger picture: Operators that commute with the Hamiltonian will have associated probabilities for obtaining different eigenvalues that do not evolve in time. For example, consider the Hamiltonian, itself, which is trivially a constant of the motion. According to the evolution equation of the state vector in the Schrödinger picture,

$$|\Psi(t)\rangle = \sum_i e^{-iE_i t/\hbar} |E_i\rangle \langle E_i | \Psi(0)\rangle$$

the amplitude for obtaining an energy eigenvalue E_j at time t upon measuring H will be

$$\begin{aligned}\langle E_j | \Psi(t) \rangle &= \sum_i e^{-iE_i t / \hbar} \langle E_j | E_i \rangle \langle E_i | \Psi(0) \rangle \\ &= \sum_i e^{-iE_i t / \hbar} \delta_{ij} \langle E_i | \Psi(0) \rangle \\ &= e^{-iE_j t / \hbar} \langle E_j | \Psi(0) \rangle\end{aligned}$$

Thus, the squared modulus of both sides yields the probability for obtaining E_j , which is

$$|\langle E_j | \Psi(t) \rangle|^2 = |\langle E_j | \Psi(0) \rangle|^2$$

Thus, the probabilities do not evolve in time. Since any operator that commutes with H can be diagonalized simultaneously with H and will have the same set of eigenvectors, the above arguments will hold for any such operator.

This page titled [9.4: The Heisenberg Picture](#) is shared under a [CC BY-NC-SA 4.0](#) license and was authored, remixed, and/or curated by [Mark Tuckerman](#).