11.2: Mass-Energy Equivalence

This derivation of the mass-energy equivalence equation is based on the analyses of an elementary photon emission event by two sets of inertial observers as shown in the figures below.

A block which is stationary with respect to observers A^o and B^o emits photons of equal frequency in opposite directions.

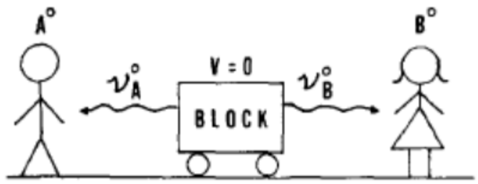

Fig. 1. Photon emission as viewed by two observers at rest relative to the block.

According to these observers the energy and momentum changes of the block are as follows:

$$\Delta E^o = -2 \cdot h\nu \quad \Delta p^o = -\Delta p\gamma^o = -(\nu_B^o - \nu_A^o) \cdot \frac{h}{c} = 0 \quad \nu_A^o = \nu_B^o = \nu$$

The block is moving with velocity v with respect to observers A and B.

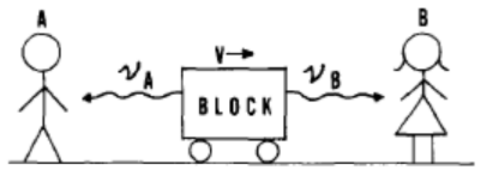

Fig. 2. Photon emission as viewed by viewed two observers relative to which the block has velocity *v*.

A and B observe photon frequencies shifted by 1/k and k respectively, where k is the optical Doppler shift factor. Their determinations of ΔE and Δp are given below. As there was no block recoil for A^o and B^o, the relativity principle requires the same for A and B. This means that the momentum change for these observers is due to a change in mass of the block, Δp = vΔm.

$$\Delta E = -h\nu_A - h\nu_B = -(k + \frac{1}{k}) \cdot h\nu$$

$$\Delta p = v\Delta m = -(\nu_B - \nu_A) \cdot \frac{h}{c} = -(k - \frac{1}{k}) \cdot \frac{h\nu}{c}$$

These equations and the optical Doppler shift factor k yield the mass-energy equivalence relation.

$$k = \sqrt{\frac{1+\frac{\nu}{c}}{1-\frac{\nu}{c}}}$$

$$\Delta E = -(k + \frac{1}{k}) \cdot h\nu$$

$$\text{substitute, } hv = \frac{-c \cdot \nu \cdot \Delta m}{k - \frac{1}{k}} \mid \text{substitute, } k = \sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}$$

$$\text{simplify: } \Delta E = \Delta m \cdot c^2$$

The differences in the energy and momentum changes determined by the two sets of observers are restated below (recall that k > 0).

$$\Delta E^o = -2 \cdot h\nu \quad \Delta p^o = 0 \quad \Delta E = -(\frac{k^2+1}{k}) \cdot h\nu \quad \Delta p = -(\frac{k^2-1}{k})) \cdot \frac{h\nu}{c}$$

These results bring the principles of energy and momentum conservation into question. However, they are the foundation of modern science, so they are preserved by recognizing that energy and mass are equivalent to two currencies with an exchange rate of c^2, $E = mc^2$. In releasing energy (photons) the block is also releasing mass ($\Delta m = \Delta E/c^2$). Therefore the kinetic energy of the moving block decreases. This is the cause of the extra energy loss in the reference frame in which the block is moving. It is also the explanation for the negative change in momentum in that reference frame.

**This presentation is based on the following paper: Daniel J. Steck and Frank Rioux, "An elementary development of mass-energy equivalence," Am. J. Phys. 51(5), 461 (1983).*