# 8.75: Qubit Quantum Mechanics

In this document I reproduce most of the results presented in Professor Galvez's paper using the Mathcad programming environment.

**State Vector**

$$\text{Photon moving horizontally:} \quad x = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Photon moving vertically:} \quad y = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{Null vector:} \quad n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\text{Horizontal polarization:} \quad h = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{Vertical polarization:} \quad v = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{Diagonal polarization:} \quad d = \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$$

**Single mode operators:**

Projection operators for motion in the x- and y-directions:

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Operator for polarizing film oriented at an angle of θ to the horizontal.

$$\Theta_{op}(\theta) = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix} \begin{pmatrix} \cos\theta & \sin\theta \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta^2 & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin\theta^2 \end{pmatrix}$$

Beam splitter:

$$BS = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

Mirror:

$$M = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Phase shift:

$$A(\delta) = \begin{pmatrix} e^{i\delta} & 0 \\ 0 & 1 \end{pmatrix}$$

Half and quarter wave plate:

$$W_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad W_4 = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix}$$

Identity:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Rotated half wave plate:

$$W_2(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) \\ \sin(2\theta) & -\cos(2\theta) \end{pmatrix} \quad W(\theta) = \begin{pmatrix} \cos(2\theta) & \sin(2\theta) & 0 & 0 \\ \sin(2\theta) & -\cos(2\theta) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Mach-Zehnder interferometer:

$$MZ(\delta) = BS\, A(\delta) M\, BS$$

**Two mode states and operators:**

Single-photon direction of propagation and polarization states:

$$xh = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad xv = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad yh = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad yv = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Two-photon direction of propagation states.

$$xx = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad xy = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad yx = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad yy = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Two-photon polarization states:

$$hh = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad hv = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad vh = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad vv = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

Polarizing beam splitter which transmits horizontally polarized photons and reflects vertically polarized photons.

$$PBS = xh\, xh^T + yv\, xv^T + yh\, yh^T + xv\, yv^T \qquad PBS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

Polarization M-Z interferometer:

$$MZp(\delta) = PBS\, \text{kronecker}(A(\delta),\ I) \text{kronecker}(M, I)\, PBS$$

Kronecker is Mathcad's command for tensor multiplication of square matrices.

Mach-Zehnder interferometer for direction of propagation and polarization, which places a rotatable half-wave plate in the upper path.

$$\mathrm{MZ_{dp}}(\theta,\delta) = \mathrm{kronecker\,(BS, I)\; kronecker(A\,(\delta)\,,I)\; W\,(\theta)\; kronecker\,(M, I)\; kronecker\,(BS, I)}$$

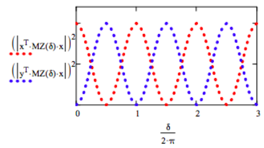
Mach-Zehnder two-photon direction-of-propagation interferometer.

$$\mathrm{BSBS} = \mathrm{kronecker(BS, BS)} \qquad \mathrm{MM} = \mathrm{kronecker(M, M)} \qquad \mathrm{AA}(\delta) = \mathrm{kronecker(A(\delta), A(\delta))}$$

$$\mathrm{MZ_{dd}}(\delta) = \mathrm{BSBS\; AA}(\delta)\mathrm{MM\; BSBS}$$

Confirm the results in Figure 2 for the Mach-Zehnder interferometer:

$$\delta = 0, .125\pi \ldots 6\pi$$

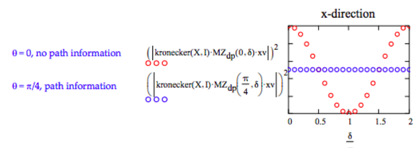


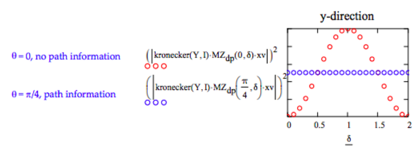
Demonstrate that a superposition is formed after the first beam splitter.

$$\mathrm{BS\,x} = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} \qquad \frac{1}{\sqrt{2}}(\mathrm{x} + \mathrm{iy}) = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$$

Confirmation that path information destroys interference.

$$\delta = 0, .1\pi\,..\,2\pi$$



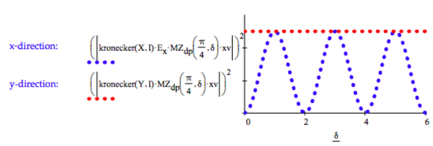




Erasure of path information restores interference. Erasers for the x- and y-directions place diagonal polarizers in those directions after the interferometer.

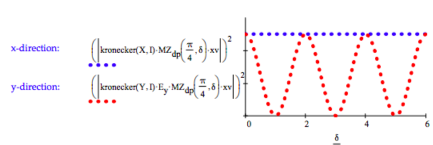
$$\mathrm{E_x} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \mathrm{E_y} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

The x-direction has an erase and the y-direction does not.

$$\delta = 0, .1\pi\,..\,6\pi$$




The y-direction has an eraser and the x-direction does not.




For the MZ polarization interferometer diagonally polarized light enters in the x-direction, |xd>. Tensor vector multiplication is awkward in Mathcad as shown below.

$$\Psi_{\mathrm{in}} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \mathrm{submatrix(kronecker(augment(x, n), augment(d, n)), 1, 4, 1, 1)} = \begin{pmatrix} 0.707 \\ 0.707 \\ 0 \\ 0 \end{pmatrix}$$

No light, however, exits in the x-direction. It exits in the y-direction showing no interference effects.

$$\delta = 0, .2\pi \ldots \pi$$

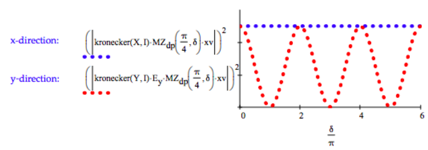
x-direction: y-direction:

$$\left(\left|\text{kronecker}(X, I)\ MZ_p(\delta)\Psi_{in}\right|\right)^2 = \qquad \left(\left|\text{kronecker}(Y, I)\ MZ)_p(\delta)\Psi_{in}\right|\right)^2 =$$

$$\begin{array}{|c|}\hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \qquad \begin{array}{|c|}\hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline 1 \\ \hline \end{array}$$

Placement of a D polarizer in the y-direction output erases distinguishing information and interference appears.

$$\delta = 0, .1\pi .. 6\pi$$



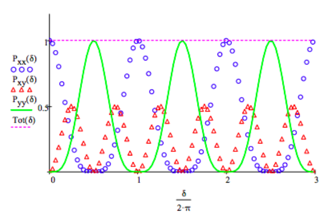

Calculation of exit probabilities for two photons in direction-of-propagation modes:

$$P_{xx}(\delta) = \left(\left|xx^T MZ_{dd}(\delta)xx\right|\right)^2 \quad P_{xy}(\delta) = \left[\left|\frac{1}{\sqrt{2}}(xy + yx)^T MZ_{dd}(\delta)xx\right|\right]$$

$$P_{yy}(\delta) = \left(\left|yy^T MZ_{dd}(\delta)xx\right|\right)^2 \quad Tot(\delta) = P_{xx}(\delta) + P_{xy}(\delta) + P_{yy}(\delta)$$

Reproduction of Figure 5b with the addition of $P_{yy}$.

$$\delta = 0, .07\pi .. 6\pi$$




"The striking result is that the ($P_{xy}$) interference pattern has twice the frequency of the single-photon interference pattern. Nonclassical interference shows new quantum aspects: two photons acting as a single quantum object (a biphoton)."

Hong-Ou-Mandel interference (right column, page 516):

$$BSBS\frac{1}{\sqrt{2}}(xy + yx) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix} \qquad \frac{i}{\sqrt{2}}(xx + yy) = \begin{pmatrix} 0.707i \\ 0 \\ 0 \\ 0.707i \end{pmatrix}$$

Section III.D deals with distinguishing between pure and mixed states experimentally. The pure state and it density matrix are given below.

$$\Psi_{pure} = \frac{1}{\sqrt{2}}(hh + vv) \qquad \Psi_{pure} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \qquad \Psi_{pure}\Psi_{pure}^T = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

The density matrix for the mixed state is calculated as follows.

$$\frac{1}{2}hh\ hh^T + \frac{1}{2}vv\ vv^T = \begin{pmatrix} 0.5 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0.5 \end{pmatrix}$$

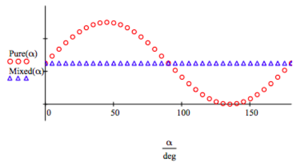
The following calculations and their graphical representation are in complete agreement with section III.D

$$Pure(\alpha) = tr\left[\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\frac{1}{2}\left[\begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \sin\alpha \\ \sin\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \sin\alpha \\ sin\alpha \end{pmatrix}^T\right]\right] \text{simplify} \rightarrow \frac{\sin(2\alpha)}{4} + \frac{1}{4}$$

$$Mixed(\alpha) = tr\left[\frac{1}{2}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}\frac{1}{2}\left[\begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \sin\alpha \\ \sin\alpha \end{pmatrix}\begin{pmatrix} \cos\alpha \\ \cos\alpha \\ \sin\alpha \\ sin\alpha \end{pmatrix}^T\right]\right] \text{simplify} \rightarrow \frac{1}{4}$$

Reproduce Figure 6 results.

$$\alpha = 0\text{deg},\ 5\text{deg}\,..\,180\text{deg}$$




The following calculation are in agreement with the math in the final paragraph of the section IV.D.

$$\text{kronecker}\left(\text{W}_2(0),\ \text{I}\right)\Psi_{\text{pure}} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \quad \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -0.5 & 0 & 0 & 0.5 \end{pmatrix}$$

```
\begin{matrix} \text{kronecker} \left( \text{W}_2 (0),~ \text{I}} \right) \Psi_{ \text{pure}} \Psi_{ \text{pure}}^T \text{kronecker} \left( \text{W}_2 (0),~ \text{I} \right)^T = \begin{pmatrix} 0.5 & 0 & 0 & -0.5 \\ 0 & 0 &
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The paper shows this as $\frac{[1-\sin\alpha}{4}$ which I am confident is a typographical error.

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