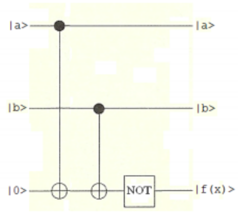
8.73: Another Illustration of the Deutsch-Jozsa Algorithm

The following circuit, U_f, produces the table of results to its right. The top wires carry the value of x and the circuit places f(x) on the bottom wire. As is shown in the previous tutorial this circuit can also operate in parallel accepting as input all x-values and returning on the bottom wire a superposition of all values of f(x).

U_f =



$$\begin{pmatrix} \mathrm{x} & 0 & 1 & 2 & 3 \\ \mathrm{f(x)} & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{where} \quad \begin{array}{l} |0\rangle = |0\rangle|0\rangle \\ |1\rangle = |0\rangle|1\rangle \\ |2\rangle = |1\rangle|0\rangle \\ |3\rangle = |1\rangle|1\rangle \end{array}$$

The function belongs to the balanced category because it produces 0 and 1 with equal frequency. The modification of this circuit (Deutsch-Jozsa algoritm) highlighted below answers the question of whether the function is constant or balanced (see Julian Brown, *The Quest for the Quantum Computer*, page 298). Naturally we already know the answer, so this is a simple demonstration that the circuit works.

$$\begin{array}{cccccccccccccc} |0\rangle & \cdots & \boxed{\mathrm{H}} & \cdots & \cdot & \cdots & \cdots & \cdots & & \cdots & \cdots & \boxed{\mathrm{H}} & \triangleright & \text{Measure, 0 or 1} \\ & & & & | & & & & & & & & & \\ |0\rangle & \cdots & \boxed{\mathrm{H}} & \cdots & | & \cdots & \cdot & \cdots & & \cdots & \cdots & \boxed{\mathrm{H}} & \triangleright & \text{Measure, 0 or 1} \\ & & & & | & & | & & & & & & & \\ |1\rangle & \cdots & \boxed{\mathrm{H}} & \cdots & \oplus & \cdots & \oplus & \cdots & \boxed{\mathrm{NOT}} & & \cdots & \cdots & & \end{array}$$

The input is $|0\rangle|0\rangle|1\rangle$: $\Psi_{in} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}^T$

The following matrices are required to execute the circuit.

$$\mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \mathrm{NOT} = \begin{pmatrix} 0 & 1 1 & 0 \end{pmatrix} \quad \mathrm{H} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

$$\mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \mathrm{CnNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The quantum circuit is assembled out of these matrices using tensor (kronecker) multiplication.

$$\mathrm{U_f} = \mathrm{kronecker(I, kronecker(I, NOT))\ kronecker(I, CNOT)\ CnNOT}$$
$$\mathrm{QuantumCircuit} = \mathrm{kronecker(H, kronecker(H, I))}\ \mathrm{U_f kronecker(H, kronecker(H, H))}$$

Operation of the quantum circuit on the input vector yields the following result which is written as a product of three qubits on the right.

$$\mathrm{QuantumCircuit}\Psi_{\mathrm{in}} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \\ 0.707 \end{pmatrix} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

According to the Deutsch-Jozsa scheme, if both wires are |0> the function is constant, but if at least one wire is |1> the function is balanced. We see by inspection that both wires are |1> indicating that the function is balanced.

The measurements on the top wires can be simulated with projection operators |0><1|, and confirm that the function is not constant but belongs to the balanced category.

$$\text{The first qubit is not } |0\rangle \qquad \left[\text{kronecker}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}^T, \text{kronecker}(\mathrm{I},\mathrm{I})\right]\text{QuantumCircuit}\Psi_{\text{in}}\right]^T = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$\text{The second qubit is not } |0\rangle \qquad \left[\text{kronecker}\left[\mathrm{I}, \text{kronecker}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}^T, \mathrm{I}\right]\right]\text{QuantumCircuit}\Psi_{\text{in}}\right]^T = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\end{pmatrix}$$

$$\text{The first qubit is } |1\rangle \qquad \left[\text{kronecker}\left[\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}^T, \text{kronecker}(\mathrm{I},\mathrm{I})\right]\text{QuantumCircuit}\Psi_{\text{in}}\right]^T = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 0.707\end{pmatrix}$$

$$\text{The second qubit is } |1\rangle \qquad \left[\text{kronecker}\left[\mathrm{I}, \text{kronecker}\left[\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}^T, \mathrm{I}\right]\right]\text{QuantumCircuit}\Psi_{\text{in}}\right]^T = \begin{pmatrix}0 & 0 & 0 & 0 & 0 & 0 & -0.707 & 0.707\end{pmatrix}$$