1.68: Another Look at the Wigner Function

The Wigner function, W(x,p), is a phase space distribution function which behaves similarly to the coordinate $\left(|\Psi(x)|^2\right)$ and momentum $\left(|\tilde{\Psi}(p)|^2\right)$ distribution functions. For example, its integral over phase space is normalized.

$$\iint W(x,p)dxdp = 1$$

In phase space, position and momentum are represented by multiplicative operators, so the calculation of their expectation values has a classical appearance. This, naturally, is part of the appeal of phase space quantum mechanical calculations.

$$\langle x\rangle = \iint xW(x,p)dxdp$$

$$\langle p\rangle = \iint pW(x,p)dxdp$$

While the Wigner function is real, unlike $|\Psi(x)|^2$ and $|\tilde{\Psi}(p)|^2$, it can take on negative values making it impossible to interpret it as a genuine probability distribution function. For this reason it is frequently referred to as a quasi-probability function, and loses some of its classical appeal. In any case, the Wigner function is redundant in the sense that it is generated from a Schrödinger coordinate or momentum wave equation.

In what follows, the quantum mechanical Wigner distribution function will be rationalized by reference to familiar classical concepts, such as position, momentum and trajectory.

In classical physics, a trajectory is a temporal sequence of position and momentum states. Let us try to represent a classical trajectory in a quantum mechanical formalism. Suppose a quantum mechanical object, a quon (thank you Nick Herbert), in state |Ψ> moves from position x –s/2 to position x + s/2. We might represent this transition quantum mechanically as the product of two coordinate space probability amplitudes (reading from left to right).

$$\langle x - \frac{s}{2}|\Psi\rangle\langle\Psi|x + \frac{s}{2}\rangle$$

Thus far we have a coordinate representation of a transition from one spatial location to another. However, a phase space description also requires a dynamic (or motional) parameter such as momentum. We can introduce momentum by first rearranging the above product of amplitudes as follows.

$$\langle\Psi|x + \frac{s}{2}\rangle\langle x - \frac{s}{2}|\Psi\rangle$$

This convolution of positional states takes on the coherent character of a trajectory with the insertion of the following momentum projector (see Feynman Lectures Volume 3) coupling the two spatial states.

$$\langle x + \frac{s}{2}|p\rangle\langle p|x - \frac{s}{2}\rangle$$

This gives us a quantum trajectory expressed in the following product of Dirac brackets,

$$\langle\Psi|x + \frac{s}{2}\rangle\langle x + \frac{s}{2}|p\rangle\langle p|x - \frac{s}{2}\rangle\langle x - \frac{s}{2}|\Psi\rangle$$

The four Dirac brackets are read now from right to left as follows: (1) is the amplitude that a particle in the state Ψ has position (x - $\frac{s}{2}$)); $(2) istheamplitudethataparticlewithposition (x - \backslash(\frac{s}{2})$ has momentum p; (3) is the amplitude that a particle with momentum p has position (x + $\frac{s}{2}$); (4) is the amplitude that a particle with position (x + $\frac{s}{2}$) is (still) in the state Ψ.

Integration over s yields the Wigner distribution function, which is a superposition of all possible quantum trajectories of the state Ψ, which interfere constructively and destructively, providing a quasi-probability distribution in phase space.

$$\int\langle\Psi|x + \frac{s}{2}\rangle\langle x + \frac{s}{2}|p\rangle\langle p|x - \frac{s}{2}\rangle\langle x - \frac{s}{2}|\Psi\rangle ds = \frac{1}{h}\int\Psi(x + \frac{s}{2})^* \exp\left(i\frac{ps}{\hbar}\right)\Psi(x - \frac{s}{2})ds$$

given that

$$\langle x+\frac{s}{2}|p\rangle\langle p|x-\frac{s}{2}\rangle=\frac{1}{\sqrt{h}}\exp\left(i\frac{p(x+\frac{s}{2})}{\hbar}\right)\frac{1}{\sqrt{h}}\exp\left(-i\frac{p(x-\frac{s}{2})}{\hbar}\right)=\frac{1}{h}\exp\left(i\frac{ps}{\hbar}\right)$$

While the Wigner distribution is more than a quantum mechanical curiosity and plays an important role in current research (see references below), it is also true, as mentioned above, that it is redundant because it is generated from either a coordinate or momentum wave function. In Dan Styer's words it is useful in exploring the quantum/classical transition, but it does not eliminate quantum weirdness – it simply repackages it (see reference 12).

Having said this it should be acknowledged that the Wigner phase-space distribution has been measured for the double slit experiment using tomographic techniques (see references 17-19).

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