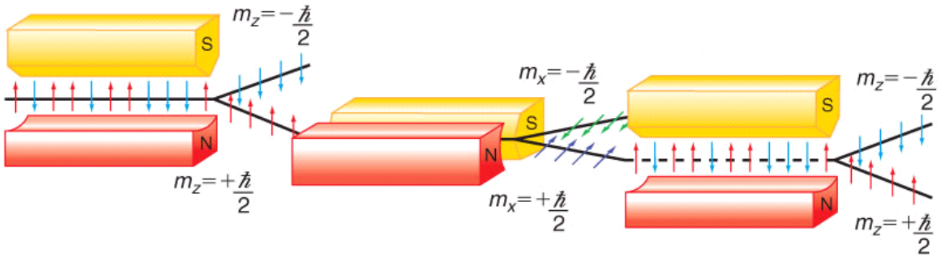
1.101: Related Analysis of the Stern-Gerlach Experiment

Silver atoms are deflected by an inhomogeneous magnetic field because of the two-valued magnetic moment associated with their unpaired 5s electron ([Kr]$5s^1 4d^{10}$). The beam of silver atoms entering the Stern-Gerlach magnet oriented in the z-direction (SGZ) on the left is unpolarized. This means it is a mixture of randomly polarized Ag atoms. A mixture cannot be represented by a wave function, it requires a density matrix, as will be shown later.



This situation is exactly analogous to the three-polarizer demonstration. Light emerging from an incadescent light bulb is unpolarized, a mixture of all possible polarization angles, so we can't write a wave function for it. The first Stern-Gerlach magnet plays the same role as the first polarizer, it forces the Ag atoms into one of measurement eigenstates - spin-up or spin-down in the z-direction. The only difference is that in the three-polarizer demonstration only one state was created - vertical polarization. Both demonstrations illustrate that the only values that are observed in an experiment are the eigenvalues of the measurement operator.

To continue with the analysis of the Stern-Gerlach demonstration we need vectors to represent the various spin states of the Ag atoms.

Spin Eigenfunctions

Spin-up in the z-direction:	$\alpha_Z := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
Spin-down in the z-direction:	$\beta_z := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$
Spin-up in the x-direction:	$\alpha_x := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$
Spin-down in the x-direction:	$\beta_x := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

In the next step, the spin-up beam (deflected toward by the magnet's north pole) enters a magnet oriented in the x-direction, SGX. The α_z beam splits into α_x and β_x beams of equal intensity. This is because it is a superposition of the x-direction spin eigenstates as shown below.

$$\frac{1}{\sqrt{2}} \cdot \left[\frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right] \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (\alpha_x + \beta_x) \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Next the α_x beam is directed toward a second SGZ magnet and splits into two equal α_z and β_z beams. This happens because α_x is a superposition of the α_z and β_z spin states.

$$\frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right] = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot (\alpha_z + \beta_z) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

Operators

We can also use the Pauli operators (in units of h/4π) to analyze this experiment.

SGZ operator:

$$\mathrm{SGZ} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

SGX operator:

$$\mathrm{SGX} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The probability that an α_z Ag atom will emerge spin-up after passing through a SGX magnet:

Probability amplitude:

$$\alpha_{\mathrm{x}}^{\mathrm{T}} \cdot \mathrm{SGX} \cdot \alpha_{\mathrm{Z}} = 0.707$$

Probability:

$$\left(\alpha_x^T \cdot \mathrm{SGX} \cdot \alpha_z\right)^2 = 0.5$$

The probability that an α_z Ag atom will emerge spin-down after passing through a SGX magnet:

Probability amplitude:

$$\beta_{\mathrm{x}}^{\mathrm{T}} \cdot \mathrm{SGX} \cdot \alpha_{\mathrm{z}} = -0.707$$

Probability:

$$\left(\beta_{\mathrm{x}}^{\mathrm{T}} \cdot \mathrm{SGX} \cdot \alpha_{\mathrm{z}}\right)^2 = 0.5$$

The probability that an α_x Ag atom will emerge spin-up after passing through a SGZ magnet:

Probability amplitude:

$$\alpha_z^T \cdot \mathrm{SGX} \cdot \alpha_x = 0.707$$

Probability:

$$\left(\alpha_z^T \cdot \mathrm{SGX} \cdot \alpha_x\right)^2 = 0.5$$

The probability that an α_x Ag atom will emerge spin-down after passing through a SGZ magnet:

Probability amplitude:

$$\beta_{\mathrm{z}}^{\mathrm{T}} \cdot \mathrm{SGX} \cdot \alpha_{\mathrm{x}} = 0.707$$

Probability:

$$\left(\beta_{\mathrm{z}}^{\mathrm{T}} \cdot \mathrm{SGX} \cdot \alpha_{\mathrm{x}}\right)^2 = 0.5$$

In examining the figure above we note that the SGX magnet distroys the entering α_z state, creating a superposition of spin-up and spin-down in the x-direction. Again measurement forces the system into one of the eigenstates of the measurement operator.

Density Operator (Matrix) Approach

A more general analysis is based on the concept of the density operator (matrix), in general given by the following outer product $|\Psi><\Psi|$. It is especially important because it can be used to represent mixtures, which cannot be represented by wave functions as noted above.

For example, the probability that an α_z spin system will emerge in the α_x channel of a SGX magnet is equal to the trace of the product of the density matrices representing the α_z and α_x states as shown below.

$$\begin{aligned} |\langle\alpha_x|\alpha_z\rangle|^2 &= \langle\alpha_z|\alpha_x\rangle\langle\alpha_x|\alpha_z\rangle \\ &= \sum_i \langle\alpha_z|i\rangle\langle i|\alpha_x\rangle\langle\alpha_x|\alpha_z\rangle \\ &= \sum_i \langle i|\alpha_x\rangle\langle\alpha_x|\alpha_z\rangle\langle\alpha_z|i\rangle \\ &= \mathrm{Tr}(|\alpha_x\rangle\langle\alpha_x|\alpha_z\rangle\langle\alpha_z|) = \mathrm{Tr}(\widehat{\rho_{\alpha_x}\rho_{\alpha_z}}) \end{aligned}$$

where the completeness relation $\sum_i |i\rangle\langle i| = 1$ has been employed.

Density matrices for spin-up and spin-down in the z-direction:

$$\rho_{\alpha z} := \alpha_z \cdot \alpha_z^T \qquad \rho_{\beta z} := \beta_z \cdot \beta_z^T$$

Density matrices for spin-up and spin-down in the x-direction:

$$\rho_{\mathrm{Qx}} := \alpha_{\mathrm{x}} \cdot \alpha_{\mathrm{x}}^{\mathrm{T}} \qquad \rho_{\beta \mathrm{x}} := \beta_{\mathrm{x}} \cdot \beta_{\mathrm{x}}^{\mathrm{T}}$$

An unpolarized spin system can be represented by a 50-50 mixture of any two orthogonal spin density matrices. Below it is shown that using the z-direction and the x-direction give the same answer.

$$\rho_{\mathrm{mix}} := \frac{1}{2} \cdot \rho_{\alpha z} + \frac{1}{2} \cdot \rho_{\beta z} = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}$$

Now we re-analyze the Stern-Gerlach experiment using the density operator (matrix) approach.

The probability that an unpolarized spin system will emerge in the α_z channel of a SGZ magnet is 0.5:

$$\operatorname{tr}(\rho_{\alpha z} \cdot \rho_{\mathrm{mix}}) = 0.5$$

The probability that the α_z beam will emerge in the α_x channel of a SGX magnet is 0.5:

$$\operatorname{tr}(\rho_{\alpha \mathrm{x}} \cdot \rho_{\alpha \mathrm{z}}) = 0.5$$

The probability that the α_x beam will emerge in the α_z channel of the final SGZ magnet is 0.5:

$$\operatorname{tr}(\rho_{\alpha z} \cdot \rho_{\alpha x}) = 0.5$$

The probability that the α_x beam will emerge in the β_z channel of the final SGZ magnet is 0.5:

$$\operatorname{tr}(\rho_{\beta z} \cdot \rho_{\alpha \mathrm{x}}) = 0.5$$

After the final SGZ magnet, 1/8 of the original Ag atoms emerge in the α_z channel and 1/8 in the β_z channel.

$$\operatorname{tr}(\rho_{\alpha z} \cdot \rho_{\alpha x}) \cdot \operatorname{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) \cdot \operatorname{tr}(\rho_{\alpha z} \cdot \rho_{\mathrm{mix}}) = 0.125$$
$$\operatorname{tr}(\rho_{\beta z} \cdot \rho_{\alpha x}) \cdot \operatorname{tr}(\rho_{\alpha x} \cdot \rho_{\alpha z}) \cdot \operatorname{tr}(\rho_{\alpha z} \cdot \rho_{\mathrm{mix}}) = 0.125$$
