8.54: Expressing Bell and GHZ States in Vector Format Using Mathcad

Mathcad provides the kronecker command for matrix tensor multiplication. It requires square matrices for its arguments and therefore cannot be used directly for vector tensor multiplication. However, if a vector is augmented with a null vector (or matrix) to produce a square matrix, vector tensor multiplication can be carried out using kronecker and a submatrix command that discards everything except the first column of the product matrix. This technique is illustrated by putting the Bell and GHZ states in vector format.

The z- and x-direction spin eigenfunctions and the appropriate null vector are required.

$$z_u = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad z_d = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad x_u = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad x_d = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad n = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

The Mathcad syntax for tensor multiplication of two 2-dimensional vectors.

$$\psi(\mathrm{a},\ \mathrm{b}) = \mathrm{submatrix}(\mathrm{kronecker}(\mathrm{augment}(\mathrm{a},\ \mathrm{n}),\ \mathrm{augment}(\mathrm{b},\ \mathrm{n})),\ 1,\ 4,\ 1,\ 1)$$

The four maximally entangled Bell states will be expressed in both the z- and x-basis.

$$|\Phi_p\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle + |\downarrow_1\rangle|\downarrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

$$\Phi_p = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u) + \psi(z_d,\ z_d)) \quad \Phi_p = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \quad \Phi_p = \frac{1}{\sqrt{2}}(\psi(x_u,\ x_u)) + (\psi(x_d,\ x_d)) \quad \Phi_p = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0.707 \end{pmatrix}$$

$$|\Phi_m\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle - |\downarrow_1\rangle|\downarrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\Phi_m = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u) - \psi(z_d,\ z_d)) \quad \Phi_m = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \quad \Phi_m = \frac{1}{\sqrt{2}}(\psi(x_u,\ x_u)) + (\psi(x_d,\ x_d)) \quad \Phi_m = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}$$

$$|\Psi_p\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle + |\downarrow_1\rangle|\downarrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\Psi_p = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u) + \psi(z_d,\ z_d)) \quad \Phi_p = \begin{pmatrix} 0 \\ 0.707 \\ 0.707 \\ 0 \end{pmatrix} \quad \Psi_p = \frac{1}{\sqrt{2}}(\psi(x_u,\ x_u)) - (\psi(x_d,\ x_d)) \quad \Psi_p = \begin{pmatrix} 0 \\ 0.707 \\ 0.707 \\ 0 \end{pmatrix}$$

$$|\Psi_m\rangle = \frac{1}{\sqrt{2}}[|\uparrow_1\rangle|\uparrow_2\rangle - |\downarrow_1\rangle|\downarrow_2\rangle] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Psi_m = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u) - \psi(z_d,\ z_d)) \quad \Psi_m = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix} \quad \Psi_m = \frac{1}{\sqrt{2}}(\psi(x_u,\ x_u)) - (\psi(x_d,\ x_d)) \quad \Psi_m = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

The Mathcad syntax for tensor multiplication of three 2-dimensional vectors.

$$\Psi(\mathrm{a, b, c}) = \mathrm{submatrix(kronecker(augment(a, n), kronecker(augment(b, n), augment(c, n))), 1, 8, 1, 1)}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & \pm 1 \end{pmatrix}^T$$

$$\Psi_1 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u,\ z_u) + \psi(z_d,\ z_d,\ z_d)) \qquad \Psi_1^T = \begin{pmatrix} 0.707 & 0 & 0 & 0 & 0 & 0 & 0 & 0.707 \end{pmatrix}$$

$$\Psi_2 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u,\ z_u) - \psi(z_d,\ z_d,\ z_d)) \qquad \Psi_2^T = \begin{pmatrix} 0.707 & 0 & 0 & 0 & 0 & 0 & 0 & -0.707 \end{pmatrix}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & \pm 1 & 0 \end{pmatrix}^T$$

$$\Psi_3 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u,\ z_d) + \psi(z_d,\ z_d,\ z_u)) \qquad \Psi_3^T = \begin{pmatrix} 0 & 0.707 & 0 & 0 & 0 & 0 & 0.707 & 0 \end{pmatrix}$$

$$\Psi_4 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_u,\ z_d) - \psi(z_d,\ z_d,\ z_u)) \qquad \Psi_4^T = \begin{pmatrix} 0 & 0.707 & 0 & 0 & 0 & 0 & -0.707 & 0 \end{pmatrix}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \pm \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \pm 1 & 0 & 0 \end{pmatrix}^T$$

$$\Psi_5 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_d,\ z_u) + \psi(z_d,\ z_u,\ z_d)) \qquad \Psi_5^T = \begin{pmatrix} 0 & 0 & 0.707 & 0 & 0 & 0.707 & 0 & 0 \end{pmatrix}$$

$$\Psi_6 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_d,\ z_u) - \psi(z_d,\ z_u,\ z_d)) \qquad \Psi_6^T = \begin{pmatrix} 0 & 0 & 0.707 & 0 & 0 & -0.707 & 0 & 0 \end{pmatrix}$$

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \pm \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 0 & 0 & 1 & \pm 1 & 0 & 0 & 0 \end{pmatrix}^T$$

$$\Psi_7 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_d,\ z_d) + \psi(z_d,\ z_u,\ z_u)) \qquad \Psi_7^T = \begin{pmatrix} 0 & 0 & 0 & 0.707 & 0.707 & 0 & 0 & 0 \end{pmatrix}$$

$$\Psi_8 = \frac{1}{\sqrt{2}}(\psi(z_u,\ z_d,\ z_d) - \psi(z_d,\ z_u,\ z_u)) \qquad \Psi_8^T = \begin{pmatrix} 0 & 0 & 0 & 0.707 & -0.707 & 0 & 0 & 0 \end{pmatrix}$$