8.10: A Simple Teleportation Exercise

This circuit teleports |Ψ> from the top wire to the bottom wire. Measurement on the top wire yields |0> or |1>, the power to which the Z matrix on the bottom wire is raised. As shown below Z^0 is the identity matrix, in other words do nothing because |Ψ> is already on the bottom wire.

$$\begin{array}{ccccccc} |\Psi\rangle & \triangleright & \cdot & H & \triangleright & \text{Measure m} = 0 \text{ or } 1 \\ |0\rangle & \triangleright & \oplus & & \triangleright & Z^m \rightarrow |\Psi\rangle \end{array}$$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad |\Psi\rangle = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

The required quantum gates in matrix format:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad Z^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Form the quantum circuit for teleportation:

$$\text{QC} = \text{kronecker}(\text{H}, \text{I})\ \text{CNOT}) \quad \text{QC} = \begin{pmatrix} 0.707 & 0 & 0 & 0.707 \\ 0 & 0.707 & 0.707 & 0 \\ 0.707 & 0 & 0 & -0.707 \\ 0 & 0.707 & -0.707 & 0 \end{pmatrix} \quad \text{QC} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix}$$

Write the initial state as a 4-vector using tensor multiplication.

$$\begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix}$$

Calculate the output state of the circuit and write it as a superposition of |0> and |1> on the top wire.

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ 0 \\ \sqrt{\frac{2}{3}} \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix}$$

$$\frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \\ \sqrt{\frac{2}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} = \frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} \right]$$

It is clear that if |0> is measured on the top wire, |Ψ> is on the bottom wire without further action. If |1> is measured on the top wire the qubit on the bottom wire is converted to |Ψ> by multiplication by Z.

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}=\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}$$

Matrix summary:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1&0&0&1\\0&1&1&0\\1&0&0&-1\\0&1&-1&0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\0\\ \sqrt{\frac{2}{3}}\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}=\frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}\right]$$

Algebraic summary:

$$Z=\begin{pmatrix}0&\text{to}&0\\1&\text{to}&-1\end{pmatrix}\quad H=\begin{bmatrix}0&\text{to}&\frac{(0+1)}{\sqrt{2}}\\1&\text{to}&\frac{(0-1)}{\sqrt{2}}\end{bmatrix}\quad \text{CNOT}=\begin{pmatrix}00&\text{to}&00\\01&\text{to}&01\\10&\text{to}&11\\11&\text{to}&10\end{pmatrix}$$

$$\left(\sqrt{\frac{1}{3}}\middle|0\right\rangle+\sqrt{\frac{2}{3}}\left|1\right\rangle\Big)|0\rangle=\sqrt{\frac{1}{3}}|00\rangle+\sqrt{\frac{2}{3}}|10\rangle$$

$$\text{CNOT}$$

$$\sqrt{\frac{1}{3}}|00\rangle+\sqrt{\frac{2}{3}}|11\rangle$$

$$\text{H}\otimes\text{I}$$

$$\frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{3}}(|0\rangle+|1\rangle)|0\rangle+\sqrt{\frac{2}{3}}(|0\rangle-|1\rangle)|1\rangle\right]=\frac{1}{\sqrt{2}}\left[|0\rangle\left(\sqrt{\frac{1}{3}}|0\rangle+\sqrt{\frac{2}{3}}|1\rangle\right)|1\rangle+\left(\sqrt{\frac{1}{3}}|0\rangle-\sqrt{\frac{2}{3}}|1\rangle\right)\right]$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\left[|0\rangle\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}+|1\rangle\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}\right]\xrightarrow{Action}\frac{1}{\sqrt{2}}\left[I\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}+Z\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}\right]$$

Total matrix operator approach:

m = 0

$$\text{kronecker}(I,\ Z^0)\text{kronecker}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\ I\right]\text{kronecker}(H,\ I)CNOT\begin{pmatrix}\sqrt{\frac{1}{3}}\\0\\ \sqrt{\frac{2}{3}}\\0\end{pmatrix}=\begin{pmatrix}0.408\\0.577\\0\\0\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}$$

m = 1

$$\text{kronecker}(I,\ Z^1)\text{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\ I\right]\text{kronecker}(H,\ I)CNOT\begin{pmatrix}\sqrt{\frac{1}{3}}\\0\\ \sqrt{\frac{2}{3}}\\0\end{pmatrix}=\begin{pmatrix}0\\0\\0.408\\0.577\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}$$

The measurement projection operators used above in the next to the last step are:

m = 0

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

m = 1

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$