

8.23: Brief Elements of Reality

In the '90s N. David Mermin published two articles in the general physics literature (Physics Today, June 1990; American Journal of Physics, August 1990) on the Greenberger-Horne-Zeilinger (GHZ) gedanken experiment (American Journal of Physics, December 1990; Nature, 3 February 2000) involving three spin-1/2 particles that illustrated the clash between local realism and the quantum view of reality for the quantum nonspecialist.

The three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. It will be shown that a consideration of spin measurements (in units of $\hbar/4\pi$ in the x- and y-directions reveals the impossibility of assigning values to the spin observables independent of measurement.

The x- and y-direction spin operators are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The eigenvalues of the Pauli matrices are +/- 1:

$$\text{eigenvals}(\sigma_x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{eigenvals}(\sigma_y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The following operators represent the measurement protocols for spins 1, 2 and 3.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \quad \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

The tensor matrix product, also known as the Kronecker product, is available in Mathcad. The four operators in tensor format are formed as follows.

$$\begin{aligned} \sigma_{xyy} &= \text{kroncker}(\sigma_x, \text{kroncker}(\sigma_y, \sigma_y)) & \sigma_{yxy} &= \text{kroncker}(\sigma_y, \text{kroncker}(\sigma_x, \sigma_y)) \\ \sigma_{yyx} &= \text{kroncker}(\sigma_y, \text{kroncker}(\sigma_y, \sigma_x)) & \sigma_{xxx} &= \text{kroncker}(\sigma_x, \text{kroncker}(\sigma_x, \sigma_x)) \end{aligned}$$

These operators mutually commute, meaning that they can be assigned simultaneous eigenstates with simultaneous eigenvalues.

$$\begin{aligned} \sigma_{xyy}\sigma_{yxy} - \sigma_{yxy}\sigma_{xyy} &\rightarrow 0 & \sigma_{xyy}\sigma_{yyx} - \sigma_{yyx}\sigma_{xyy} &\rightarrow 0 & \sigma_{xyy}\sigma_{xxx} - \sigma_{xxx}\sigma_{xyy} &\rightarrow 0 \\ \sigma_{yxy}\sigma_{yyx} - \sigma_{yyx}\sigma_{yxy} &\rightarrow 0 & \sigma_{yxy}\sigma_{xxx} - \sigma_{xxx}\sigma_{yxy} &\rightarrow 0 & \sigma_{yyx}\sigma_{xxx} - \sigma_{xxx}\sigma_{yyx} &\rightarrow 0 \end{aligned}$$

The next step is to compare the matrix for the product of the first three operators ($\sigma_{xyy}\sigma_{yxy}\sigma_{yyx}$) with that of the fourth (σ_{xxx}).

$$\sigma_{xyy}\sigma_{yxy}\sigma_{yyx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad \sigma_{xxx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

This indicates the following relationship between the four operators and leads quickly to a refutation of the concept of noncontextual, hidden values for quantum mechanical observables.

$$(\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3) (\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3) (\sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3) = -(\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3)$$

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes that the x- and y-components of the spin have definite values prior to measurement. This position leads to a contradiction with the above result. There is no way to assign eigenvalues (+/-1) to the operators that is consistent with the above result.

Concentrating on the operator on the left side, we notice that there is a σ_y measurement on the first spin in the second and third term (blue). If the spin state is well-defined before measurement those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1. There is a σ_y measurement on the second spin in terms one and three (red). By similar arguments those results will lead to a product of +1 also. Finally there is a σ_y measurement on the third spin in terms one and two

(green). By similar arguments those results will lead to a product of +1 also. Incorporating these observations into the expression above leads to the following contradiction.

$$\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3 = -\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

This result should cause all mathematically literate local realists to renounce and recant their heresy.

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