

1.70: Wigner Distribution for the Particle in a Box

The Wigner function is a quantum mechanical phase-space quasi-probability function. It is called a quasi-probability function because it can take on negative values, which have no classical meaning in terms of probability.

The PIB eigenstates for a box of unit dimension are given by:

$$\Psi(x, n) := \sqrt{2} \cdot \sin(n \cdot \pi \cdot x)$$

For these eigenstates the Wigner distribution function is:

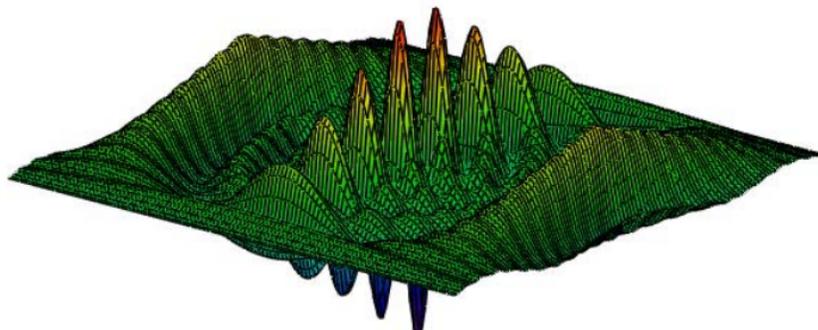
$$W(x, p, n) := \frac{1}{\pi} \cdot \int_{-x}^x \sqrt{2} \cdot \sin[n \cdot \pi \cdot (x+s)] \cdot \exp(2 \cdot i \cdot s \cdot p) \cdot \sqrt{2} \cdot \sin[n \cdot \pi \cdot (x-s)] ds$$

Integration with respect to s yields the following function:

$$W(x, p, n) := \frac{2}{\pi} \cdot \left[\frac{\sin[2 \cdot (p - n \cdot \pi) \cdot x]}{4 \cdot (p - n \cdot \pi)} + \frac{\sin[2 \cdot (p + n \cdot \pi) \cdot x]}{4 \cdot (p + n \cdot \pi)} - \cos(2 \cdot n \cdot \pi \cdot x) \cdot \frac{\sin(2 \cdot p \cdot x)}{2 \cdot p} \right]$$

The Wigner distribution for the n th eigenstate is calculated below:

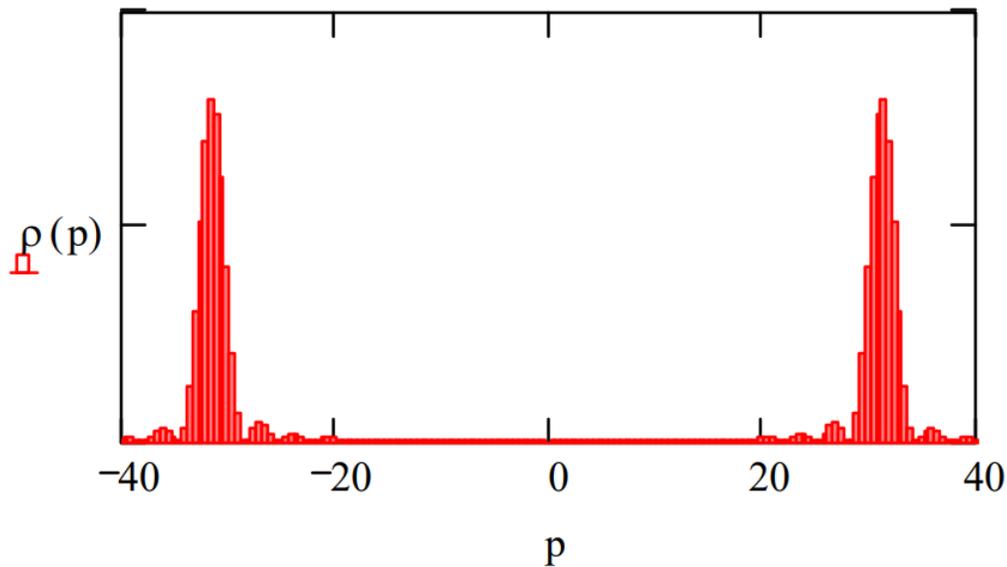
$$\begin{aligned} n &:= 10 & N &:= 115 & i &:= 0..N \\ x_i &:= \frac{i}{N} & j &:= 0..N & p_j &:= -40 + \frac{80 \cdot j}{N} \\ \text{Wigner}_{i,j} &:= \text{if}[x_i \leq 0.5, W(x_i, p_j, n), W[(1 - x_i), p_j, n]] \end{aligned}$$



Wigner

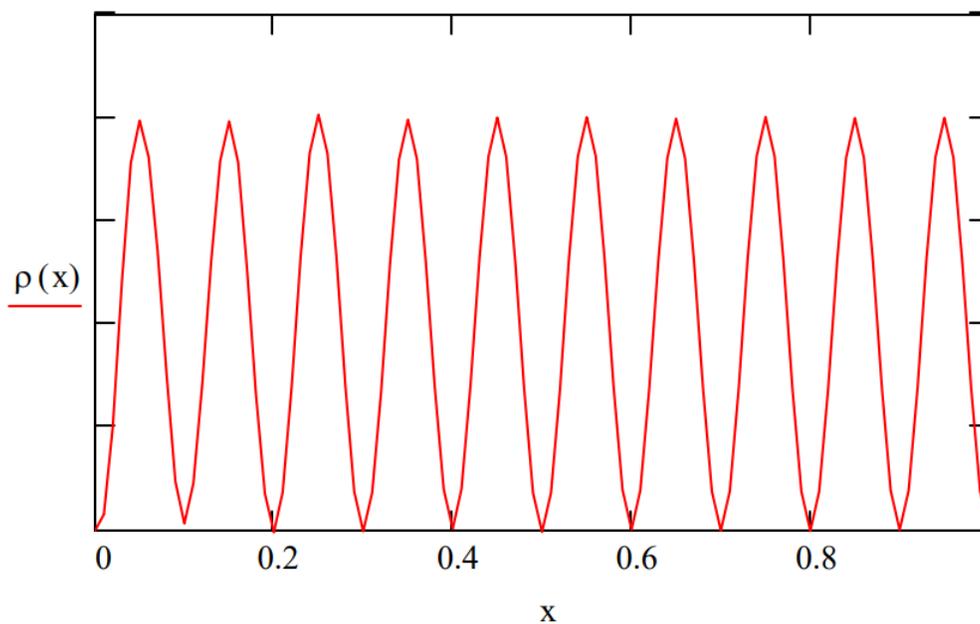
Integration of the Wigner function over the spatial coordinate yields the momentum distribution function as is shown below.

$$\rho(p) := \int_0^1 W(x, p, n) dx \quad p := -40, -39.5 \dots 40$$



Integration of the Wigner function over the momentum coordinate yields the spatial distribution function as is shown below.

$$\rho(x) := \int_{-51}^{50} W(x, p, n) dp \quad x := 0, 01 \dots 1$$



The Wigner distribution can be used to calculate the expectation values for position, momentum and kinetic energy.

$$x_{\text{bar}} = \int_{-\infty}^{\infty} \int_0^1 W(x, p, 1) \cdot x \, dx \, dp \text{ simplify } \rightarrow x_{\text{bar}} = \frac{1}{2}$$

$$p_{\text{bar}} = \int_{-\infty}^{\infty} \int_0^1 W(x, p, 1) \cdot p \, dx \, dp \text{ simplify } \rightarrow p_{\text{bar}} = 0$$

$$T_{\text{bar}} = \int_{-\infty}^{\infty} \int_0^1 W(x, p, 1) \cdot \frac{p^2}{2} \, dx \, dp \text{ simplify } \rightarrow T_{\text{bar}} = \frac{1}{2} \cdot \pi^2$$

References:

"Wigner quasi-probability distribution for the infinite square well: Energy eigenstates and time-dependent wave packets," by Belloni, Docheski and Robinett; *American Journal of Physics* **72(9)**, 1183-1192 (2004).

"Wigner functions and Weyl transforms for pedestrians," by William Case, *American Journal of Physics* **76(10)**, 937-946 (2008).

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