8.24: A Brief Analysis of Mermin's GHZ Thought Experiment

Twenty years ago N. David Mermin published two articles (Physics Today, June 1990; American Journal of Physics, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (American Journal of Physics, December 1990; Nature, 3 February 2000) gedanken experiment involving spins that sharply revealed the clash between local realism and the quantum view of reality.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of h/4π in the z-basis with spin operators in the x- and y-directions.

The z-basis eigenfunctions are:

$$Sz_{up} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Sz_{down} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The x- and y-direction spin operators in the z-basis are the Pauli matrices:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The initial entangled spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The following operators represent the measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \quad \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The four operators in tensor format are formed as follows.

$$\sigma_{xyy} = \text{kronecker}(\sigma_x, \text{ kronecker}(\sigma_y, \sigma_y)) \quad \sigma_{yxy} = \text{kronecker}(\sigma_y, \text{ kronecker}(\sigma_x, \sigma_y))$$
$$\sigma_{yyx} = \text{kronecker}(\sigma_y, \text{ kronecker}(\sigma_y, \sigma_x)) \quad \sigma_{xxx} = \text{kronecker}(\sigma_x, \text{ kronecker}(\sigma_x, \sigma_x))$$

The expectation values of the operators are now calculated.

$$\Psi^T \sigma_{xyy} \Psi = 1 \quad \Psi^T \sigma_{yxy} \Psi = 1 \quad \Psi^T \sigma_{yyx} \Psi = 1 \quad \Psi^T \sigma_{xxx} \Psi = -1$$

Consequently the product of the four operators has the expectation value of -1.

$$\Psi^T \sigma_{xyy} \sigma_{yxy} \sigma_{yyx} \sigma_{xxx} \Psi = -1$$

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes that the x- and y-components of the spin have definite values prior to measurement. This position leads to a contradiction with the above result. The following analysis is taken from "Quantum Information Science" by Seth Lloyd.

Looking again at the measurement operators, notice that there is a σ_x measurement on the first spin in the first and fourth experiment. If the spin state is well-defined before measurement those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \quad \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

Likewise there is a σ_y measurement on the second spin in experiments one and three. By similar arguments those results will lead to a product of +1 also. Continuing with all the pairs in the total operator using local realistic reasoning unambiguously shows that

its expectation value should be +1, in sharp disagreement with the quantum mechanical result of -1.