# 8.8: Quantum Teleportation - Another Look

A graphic representation of the teleportation procedure is accompanied by a quantum circuit for its implementation.

$$\begin{array}{cccccccc} & \text{Initial} & & & & & & \text{Final} \\ & |\Psi\rangle & \cdots & \cdot & \cdots & \boxed{\text{H}} & \triangleright & \text{Measure } |\text{a}\rangle \text{ 0 or 1} \\ \text{Alice} & & & | & & & & \text{Bell state measurement} \\ & \cdot & \cdots & \oplus & \cdots & \cdots & \triangleright & \text{Measure } |\text{b}\rangle \text{ 0 or 1} \\ & \beta_{00} & & & & & & \\ \text{Bob} & \cdot & \cdots & \cdots & \cdots & \cdots & \triangleright & X^b Z^a \rightarrow |\Psi\rangle \end{array}$$

Alice has the photon to be teleported (teleportee) and another photon belonging to an entangled pair that she shares with Bob.

Teleportee photon state:

$$\Psi = \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix}$$

Entangled Bell state photon pair:

$$\beta_{00} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

The shared Bell state and the remaining Bell states shown below are indexed in binary notation. The Appendix contains an explanation of the indexing scheme.

$$\beta_{01} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\beta_{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

$$\beta_{11} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

In the teleportation circuit, the teleportee is initially on the top wire and the Bell state photons are on the two lower wires. Alice controls the top two wires, while Bob has the bottom wire. The initial product state is formed by vector tensor multiplication.

$$\Psi_{in} = \Psi \beta_{00}$$

$$\Psi_{in} = \frac{1}{\sqrt{2}} \begin{pmatrix} \sqrt{\frac{1}{3}} & 0 & 0 & \sqrt{\frac{1}{3}} & \sqrt{\frac{2}{3}} & 0 & 0 & \sqrt{\frac{2}{3}} \end{pmatrix}^T$$

The matrix operators required to run the circuit and carry out the subsequent measurements and transformations are now identified.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Measurement operator for |0 >:

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

Measurement operator for |1 >:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

The foward slashes shown on the top two wires of the circuit represent measurements done by Alice. The matrix operator representing the quantum circuit prior to Alice's measurements is formed using tensor multiplication.

$$QC = \text{kronecker}(\text{H}, \text{kronecker } (\text{I}, \text{I}))\ \text{kronecker}(\text{CNOT}, \text{I})$$

After the controlled-not and Hadamard gates, but before measurement by Alice, the system is in a superposition state involving the Bell state indices on the top two registers. The third register contains a state that can be easily transformed into the teleportee once Alice tells Bob which Bell state she observed.

$$QC\Psi_{in} = \begin{pmatrix} 0.289 \\ 0.408 \\ 0.408 \\ 0.289 \\ 0.289 \\ -0.408 \\ -0.408 \\ 0.289 \end{pmatrix}$$

which can be written

$$\frac{1}{2}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ \sqrt{\frac{2}{3}} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \begin{pmatrix} \sqrt{\frac{1}{3}} \\ -\sqrt{\frac{2}{3}} \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \begin{pmatrix} -\sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix}$$

If Alice observes $\beta_{00}$ Bob does nothing (the identity operation) because he has $\Psi$ on his register. If Alice observes $\beta_{01}$ Bob applies the X operator, if she finds $\beta_{10}$ he uses the Z operator, and finally if Alice observes $\beta_{11}$ Bob applies the X operator followed by the Z operator. Further mathematical detail is provided by showing explicitly the four equally probable measurement outcomes that Alice observes, and Bob's subsequent action on his register.

$$2\text{kronecker}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\text{kronecker}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},I\right]\right]QC\Psi_{in}=\begin{pmatrix}0.577\\0.816\\0\\0\\0\\0\\0\\0\end{pmatrix}\qquad\begin{pmatrix}1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\\sqrt{\frac{2}{3}}\end{pmatrix}\qquad I\begin{pmatrix}\sqrt{\frac{1}{3}}\\\sqrt{\frac{2}{3}}\end{pmatrix}=\begin{pmatrix}0.577\\0.816\end{pmatrix}$$

$$2\text{kronecker}\left[\begin{pmatrix}1&0\\0&0\end{pmatrix},\text{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},I\right]\right]QC\Psi_{in}=\begin{pmatrix}0\\0\\0.816\\0.577\\0\\0\\0\\0\end{pmatrix}\qquad\begin{pmatrix}1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{3}}\end{pmatrix}\qquad X\begin{pmatrix}\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{3}}\end{pmatrix}=\begin{pmatrix}0.577\\0.816\end{pmatrix}$$

$$2\text{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\text{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},I\right]\right]QC\Psi_{in}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\-0.816\\0.577\end{pmatrix}\qquad\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}\sqrt{\frac{1}{3}}\\-\sqrt{\frac{2}{3}}\end{pmatrix}\qquad Z\begin{pmatrix}\sqrt{\frac{1}{3}}\\-\sqrt{\frac{2}{3}}\end{pmatrix}=\begin{pmatrix}0.577\\0.816\end{pmatrix}$$

$$2\text{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},\text{kronecker}\left[\begin{pmatrix}0&0\\0&1\end{pmatrix},I\right]\right]QC\Psi_{in}=\begin{pmatrix}0\\0\\0\\0\\0\\0\\-0.816\\0.577\end{pmatrix}\qquad\begin{pmatrix}0&1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}-\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{3}}\end{pmatrix}\qquad ZX\begin{pmatrix}-\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{3}}\end{pmatrix}=\begin{pmatrix}0.577\\0.816\end{pmatrix}$$

The teleportation circuit can also be analyzed algebraically.

$$\left(\sqrt{\frac{1}{3}}\left|0\right\rangle+\sqrt{\frac{2}{3}}\left|1\right\rangle\right)\frac{1}{\sqrt{2}}(|00\rangle+|11\rangle)=\frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{3}}(|000\rangle+|011\rangle)+\sqrt{\frac{2}{3}}(|100\rangle+|111\rangle)\right]$$

$$\text{CNOT}\otimes\text{I}$$

$$\frac{1}{\sqrt{2}}\left[\sqrt{\frac{1}{3}}|0\rangle(|00\rangle+|11\rangle)+\sqrt{\frac{2}{3}}|1\rangle(|10\rangle+|01\rangle)\right]$$

$$\text{H}\otimes\text{I}\otimes\text{I}$$

$$\frac{1}{\sqrt{2}}\left[(\sqrt{\frac{1}{3}}|0\rangle(|1\rangle)+(|00\rangle+|11\rangle)+\sqrt{\frac{2}{3}}(0\rangle-|1\rangle)(|10\rangle+|01\rangle)\right]$$

$$\downarrow$$

$$\frac{1}{\sqrt{2}}\left[|00\rangle\left(\sqrt{\frac{1}{3}}|0\rangle+\sqrt{\frac{2}{3}}|1\rangle\right)+|01\rangle\left(\sqrt{\frac{2}{3}}|0\rangle+\sqrt{\frac{1}{3}}|1\rangle\right)+|10\rangle\left(\sqrt{\frac{1}{3}}|0\rangle-\sqrt{\frac{2}{3}}|1\rangle\right)\right.$$
$$\left.+|11\rangle\left(-\sqrt{\frac{2}{3}}|0\rangle+\sqrt{\frac{1}{3}}|1\rangle\right)\right]$$

$$\downarrow$$

$$\frac{1}{2}\left[|00\rangle\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}+|01\rangle\begin{pmatrix}\sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\end{pmatrix}+|10\rangle\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}+|11\rangle\begin{pmatrix}-\sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\end{pmatrix}\right]$$

$$\overset{Action}{\rightarrow}\frac{1}{2}\left[I\begin{pmatrix}\sqrt{\frac{1}{3}}\\ \sqrt{\frac{2}{3}}\end{pmatrix}+X\begin{pmatrix}\sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\end{pmatrix}+Z\begin{pmatrix}\sqrt{\frac{1}{3}}\\ -\sqrt{\frac{2}{3}}\end{pmatrix}+ZX\begin{pmatrix}-\sqrt{\frac{2}{3}}\\ \sqrt{\frac{1}{3}}\end{pmatrix}\right]$$