8.11: Teleportation as a Quantum Computation

This tutorial works through the following teleportation circuit provided by Gilles Brassard in "Teleportation as a Quantum Computation" (arXiv:quant-ph/9605035v1). The computational methodology employed here is similar to that used in the other teleportation examples given in this series of tutorials.

$$\begin{array}{ccccccccccccc}
\text{Initial} & & 1 & 2 & 3 & 4 & & 5 & & 6 & 7 & & \text{Final} \\
|\Psi\rangle & \triangleright & \cdots & \cdots & \cdot & \boxed{\text{R}} & \cdots & \boxed{\text{S}} & \oplus & \boxed{\text{S}} & \oplus & \triangleright & \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
 & & & & | & & & & | & & | & & \\
|0\rangle & \triangleright & \boxed{\text{L}} & \cdot & \oplus & \cdots & \cdots & \cdots & | & \cdots & | & \triangleright & \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \\
 & & & | & & & & | & | & & | & & \\
|0\rangle & \triangleright & \cdots & \oplus & \cdots & \cdots & \cdots & \oplus & \cdot & \boxed{\text{T}} & \cdot & \triangleright & |\Psi\rangle
\end{array}$$

The necessary quantum bits or qubit states are:

Base states:

$$0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad 1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Superposition of base states:

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} = \alpha \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \beta \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

where

$$(|\alpha|)^2 + (|\beta|)^2 = 1$$

Using $\begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} = \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix}$ as the teleportee ($|\Psi>$), Brassard's circuit yields the final states shown in the circuit above. In other words $|\Psi>$ is teleported from the first wire on the left to the third wire on the right.

Initial state:

$$\begin{pmatrix} \sqrt{\frac{2}{3}} \\ \sqrt{\frac{1}{3}} \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sqrt{\frac{2}{3}} \\ 0 \\ 0 \\ 0 \\ \sqrt{\frac{1}{3}} \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

Final state:

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\otimes\begin{pmatrix}\sqrt{\frac{2}{3}}\\\sqrt{\frac{1}{3}}\end{pmatrix}=\begin{pmatrix}\sqrt{\frac{1}{6}}\\\sqrt{\frac{1}{12}}\\\sqrt{\frac{1}{6}}\\\sqrt{\frac{1}{12}}\\\sqrt{\frac{1}{6}}\\\sqrt{\frac{1}{12}}\\\sqrt{\frac{1}{6}}\\\sqrt{\frac{1}{12}}\end{pmatrix}$$

The identity operator and the following quantum gates are required to calculate the result of the teleportation circuit. L and R are single qubit rotations, S and T are single qubit phase shifts. The other gates (CNOT, CnNOT, and ICnNOT) are well-known in quantum circuitry.

$$I=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad L=\begin{pmatrix}1&-1\\1&1\end{pmatrix}\quad R=\frac{1}{\sqrt{2}}\begin{pmatrix}1&1\\-1&1\end{pmatrix}\quad S=\begin{pmatrix}i&0\\0&1\end{pmatrix}\quad T=\begin{pmatrix}-1&0\\0&-i\end{pmatrix}$$

$$\mathrm{CNOT}=\begin{pmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{pmatrix}\quad \mathrm{CnNOT}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&1&0&0&0&0&0&0\\0&0&1&0&0&0&0&0\\0&0&0&1&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&0&0&1&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&0&0&1&0\end{pmatrix}\quad \mathrm{ICnNOT}=\begin{pmatrix}1&0&0&0&0&0&0&0\\0&0&0&0&0&1&0&0\\0&0&1&0&0&0&0&0\\0&0&0&0&0&0&0&1\\0&0&0&0&1&0&0&0\\0&1&0&0&0&0&0&0\\0&0&0&0&0&0&1&0\\0&0&0&1&0&0&0&0\end{pmatrix}$$

$$\mathrm{Step1}=\mathrm{kronecker}(\mathrm{I},\mathrm{kronecker}(\mathrm{L},\mathrm{I}))\quad \mathrm{Step2}=\mathrm{kronecker}(\mathrm{I},\mathrm{CNOT})\quad \mathrm{Step3}=\mathrm{kronecker}(\mathrm{CNOT},\mathrm{I})$$
$$\mathrm{Step4}=\mathrm{kronecker}(\mathrm{R},\mathrm{kronecker}(\mathrm{I},\mathrm{I}))\quad \mathrm{Step5}=\mathrm{kronecker}(\mathrm{S},\mathrm{CNOT})\quad \mathrm{Step6}=\mathrm{ICnNOT}$$
$$\mathrm{Step7}=\mathrm{kronecker}(\mathrm{S},\mathrm{kronecker}(\mathrm{I},\mathrm{T}))\quad \mathrm{Step8}=\mathrm{ICnNOT}$$

According to this result |Ψ > has indeed been teleported to the bottom wire on the right, so the goal has been achieved. However, Brassard suggests that measurements on the top wires in the |0 >-|1 > basis are also instructive. There are four possible measurement outcomes: |00>, |01>, |10 > and |11 >. The projection operators for |0 > and |1 > are as follows.

Projection operator for |0>:

$$\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1&0\end{pmatrix}=\begin{pmatrix}1&0\\0&0\end{pmatrix}$$

Projection operator for |1>:

$$\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0&1\end{pmatrix}=\begin{pmatrix}0&0\\0&1\end{pmatrix}$$

These calculations also show successful teleportation.

$$\begin{array}{cc} \text{Measurement result for qubits x and y} & \text{Final 3-qubit state} \end{array}$$

$$2\text{kronecker}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{kronecker}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, I\right]\right]\Psi_{final} = \begin{pmatrix} 0.816 \\ 0.577 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \left[\begin{matrix} x & y & z \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix} \end{matrix}\right]$$

$$2\text{kronecker}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, I\right]\right]\Psi_{final} = \begin{pmatrix} 0 \\ 0 \\ 0.816 \\ 0.577 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \left[\begin{matrix} x & y & z \\ \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix} \end{matrix}\right]$$

$$2\text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{kronecker}\left[\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, I\right]\right]\Psi_{final} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.816 \\ 0.577 \\ 0 \\ 0 \end{pmatrix} \quad \left[\begin{matrix} x & y & z \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 1 \\ 0 \end{pmatrix} & \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix} \end{matrix}\right]$$

$$2\text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, \text{kronecker}\left[\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}, I\right]\right]\Psi_{final} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.816 \\ 0.577 \end{pmatrix} \quad \left[\begin{matrix} x & y & z \\ \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0 \\ 1 \end{pmatrix} & \begin{pmatrix} 0.816 \\ 0.577 \end{pmatrix} \end{matrix}\right]$$

Appendix

In the inverse controlled-NOT gate of steps 6 and 8 of Brassard's teleportation circuit, c is the control, a is the target and b is unchanged.

$$\begin{pmatrix} a & b & c & ' & a' & b' & c' \\ 0 & 0 & 0 & ' & 0 & 0 & 0 \\ 0 & 0 & 1 & ' & 1 & 0 & 1 \\ 0 & 1 & 0 & ' & 0 & 1 & 0 \\ 0 & 1 & 1 & ' & 1 & 1 & 1 \\ 1 & 0 & 0 & ' & 1 & 0 & 0 \\ 1 & 0 & 1 & ' & 0 & 0 & 1 \\ 1 & 1 & 0 & ' & 1 & 1 & 0 \\ 1 & 1 & 1 & ' & 0 & 1 & 1 \end{pmatrix}$$