8.59: An Entanglement Swapping Protocol

In the field of quantum information, interference, superpositions and entangled states are essential resources. Entanglement, a non-factorable superposition, is routinely achieved when two photons are emitted from the same source, perhaps a parametric down converter (PDC). Entanglement swapping involves the transfer (teleportation) of entanglement to two photons that were produced independently and never previously interacted. The Bell states are the four maximally entangled two-qubit entangled basis for a four-dimensional Hilbert space and play an essential role in quantum information theory and technology, including teleportation and entanglement swapping. This analysis attempts to provide the essential matrix math needed to understand parts of "Entangled delayed-choice entanglement swapping" (arXiv1203.4834) and "Delayed-choice gedanken experiments and their realizations" (arXiv1407.2930). The following analysis deals exclusively with entanglement swapping and does not consider the delayed-choice aspect of the research presented in these papers.

Bell states and the identity operator:

$$\Phi_p = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix} \quad \Phi_m = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \quad \Psi_p = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix} \quad \Psi_m = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \quad I = \begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}$$

A four-qubit state is prepared in which photons 1 and 2 are entangled in Bell state Ψ_m, and photons 3 and 4 are also entangled in Bell state Ψ_m. The state multiplication below is understood to be tensor vector multiplication.

$$\Psi_{1234} = \Psi_{m12}\Psi_{m34} = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix} \quad \Psi = \frac{1}{2}\begin{pmatrix}0 & 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0\end{pmatrix}^T$$

The authors write this state as a superposition of Bell state products to suggest a way to transfer entanglement from 1&2-3&4 to 1&4-2&3: perform a Bell state measurement on photons 2 and 3.

$$|\Psi\rangle_{1234} = \frac{1}{2}[|\Psi_p\rangle_{14} \otimes |\Psi_P\rangle_{23} - |\Psi_m\rangle_{14} \otimes \Psi_m\rangle_{23} - |\Psi_m\rangle_{23} - |\Phi_p\rangle_{14} \otimes \Phi_p\rangle_{23} + |\Phi_m\rangle_{14} \otimes \Phi_m\rangle_{23}]$$

The following calculations agree with this product of entangled photon pairs 1&4 and 2&3. Projection of photons 2 and 3 onto Ψ_p projects photons 1 and 4 onto Ψ_p.

Projection of photons 2 and 3 onto Φ_p projects photons 1 and 4 onto Ψ_m.

$$\begin{gathered}\left(\text{kronecker}\left(\mathrm{I}, \text{kronecker}\left(\Psi_p,\ \Psi_p^T,\ \mathrm{I}\right)\right)\Psi\right)^T \\ = \begin{pmatrix}0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0.25 & 0 & 0.25 & 0 & 0 & 0\end{pmatrix} \\ \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^T \\ = \frac{1}{4}\begin{pmatrix}0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0\end{pmatrix}\end{gathered}$$

Projection of photons 2 and 3 onto Ψ_m projects photons 1 and 4 onto -Ψ_m.

$$\begin{gathered}\left(\text{kronecker}\left(\mathrm{I}, \text{kronecker}\left(\Psi_m,\ \Psi_m^T,\ \mathrm{I}\right)\right)\Psi\right)^T \\ = \begin{pmatrix}0 & 0 & 0 & -0.25 & 0 & 0.25 & 0 & 0 & 0 & 0 & 0.25 & 0 & -0.25 & 0 & 0 & 0\end{pmatrix} \\ \frac{-1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\-1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]^T \\ = \frac{1}{4}\begin{pmatrix}0 & 0 & 0 & -1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0\end{pmatrix}\end{gathered}$$

Projection of photons 2 and 3 onto Φ_p projects photons 1 and 4 onto -Φ_p.

$$\left(\text{kronecker}\left(\mathrm{I},\, \text{kronecker}\left(\Phi_p,\, \Phi_p^T,\, \mathrm{I}\right)\right)\Psi\right)^T = \begin{pmatrix} -0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 \end{pmatrix}$$

$$\frac{-1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\frac{-1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}+\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right]^T$$

$$=\frac{1}{4}\begin{pmatrix}-1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1\end{pmatrix}$$

Projection of photons 2 and 3 onto Φ_m projects photons 1 and 4 onto Φ_m.

$$\left(\text{kronecker}\left(\mathrm{I},\, \text{kronecker}\left(\Phi_m,\, \Phi_m^T,\, \mathrm{I}\right)\right)\Psi\right)^T$$

$$=\begin{pmatrix}0.25 & 0 & 0 & 0 & 0 & 0 & -0.25 & 0 & 0 & -0.25 & 0 & 0 & 0 & 0 & 0 & 0.25\end{pmatrix}$$

$$\frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}-\begin{pmatrix}0\\1\end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right]^T$$

$$=\frac{1}{4}\begin{pmatrix}1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix}$$

The initial four-particle state can be written in the H/V, A/D and R/L polarization bases. See the Appendix for details.

$$\Psi_{1234}=\frac{1}{2}(H_1V_2-V_1H_2)=(H_3V_4-V_3H_4)=\frac{1}{2}(H_1V_2V_3V_4-H_1V_2V_3H_4-V_1H_2H_3V_4+V_1H_2V_3H_4$$
$$+V_1H_2V_3H_4)$$

$$\Psi_{1234}=\frac{1}{2}(A_1D_2-D_1A_2)=(A_3D_4-D_3A_4)=\frac{1}{2}(A_1D_2A_3D_4-A_1D_2A_3D_4-A_1D_2D_3A_4+D_1A_2A_3D_4$$
$$+D_1A_2D_3A_4)$$

$$\Psi_{1234}=\frac{1}{2}(L_1R_2-R_1L_2)=(L_3R_4-R_3L_4)=\frac{1}{2}(L_1R_2R_3R_4-L_1R_2R_3L_4-R_1L_2L_3R_4+R_1L_2R_3L_4)$$

Where,

$$H=\begin{pmatrix}1\\0\end{pmatrix}\quad V=\begin{pmatrix}0\\1\end{pmatrix}\quad R=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\i\end{pmatrix}\quad L=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\quad A=\frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}$$

$$H=\frac{1}{\sqrt{2}}(R+L)\quad V=\frac{i}{\sqrt{2}}(L-R)\quad H=\frac{1}{\sqrt{2}}(D+A)\quad V=\frac{1}{\sqrt{2}}(D-A)$$

$$R=\frac{1}{\sqrt{2}}(H+iV)\quad L=\frac{1}{\sqrt{2}}(H-iV)\quad R=\frac{1}{\sqrt{2}}(D+A)$$

After production, photon 1 is sent to Alice and photon 4 is sent to Bob. Photons 2 and 3 are sent to Victor. Alice and Bob can measure their photons in either the H/V, R/L or A/D bases. Victor can measure his photons separately in the H/V basis or he can carry out one of the four Bell state measurements on his photon pairs. This later choice as shown earlier projects photons 1 and 4 into the corresponding entangled Bell state.

The following table shows the possible measurement results that Alice and Bob will obtain, depending on the type of measurement Victor makes on his photons. In experiments 1-4 Victor measures his photons separately in the H/V basis and Alice and Bob do the same. In the remaining experiments Victor does a Bell state measurement on his photons and Alice and Bob measure in any of the three bases.

$$\begin{pmatrix}
\text{Experiment} & 1 & 2 & 3 & 4 & ' & 5 & 6 & 7 & 8 & ' & 9 & 10 & 11 & 12 & ' & 13 & 14 & 15 & 16 \\
\text{Alice1} & H & H & V & V' & H & V & H & V & ' & L & R & L & R & ' & D & A & A & D \\
\text{Bob4} & V & H & V & H & ' & V & H & H & V & ' & L & L & R & R & ' & D & D & A & A \\
\text{Victor23} & VH & VV & HH & HV & ' & \Psi_p & \Psi_m & \Phi_p & \Phi_m & ' & \Psi_p & \Psi_m & \Phi_p & \Phi'_m & \Psi_p & \Psi_m & \Phi_p & \Phi_m
\end{pmatrix}$$

Results 1-4 are consistent with the original state expressed in the H/V basis.

$$\Psi_{1234} = \frac{1}{2}(H_1V_2H_3V_4 - H_1V_2V_3H_4 - V_1H_2H_3V_4 + V_1H_2V_3H_4)$$

In the remaining experiments Victor makes a Bell state measurement on photons 2 and 3, and projects photons 1 and 4 into the following Bell states. The table shows measurement results that Alice and Bob could make on their photons given these states. See the Appendix for more detail.

$$\Psi_{p14} = \frac{1}{\sqrt{2}}(H_1V_4 + V_1H_4) = \frac{i}{\sqrt{2}}(L_1L_4 - R_1R_4) = \frac{1}{\sqrt{2}}(D_1D_4 - A_1A_4)$$
$$\Psi_{m14} = \frac{1}{\sqrt{2}}(H_1V_4 - V_1H_4) = \frac{i}{\sqrt{2}}(L_1L_4 - R_1R_4) = \frac{1}{\sqrt{2}}(A_1D_4 - D_1A_4)$$
$$\Phi_{p14} = \frac{1}{\sqrt{2}}(H_1H_4 + V_1V_4) = \frac{i}{\sqrt{2}}(L_1R_4 + R_1L_4) = \frac{1}{\sqrt{2}}(A_1A_4 - D_1D_4)$$
$$\Phi_{m14} = \frac{1}{\sqrt{2}}(H_1H_4 - V_1V_4) = \frac{i}{\sqrt{2}}(L_1L_4 + R_1R_4) = \frac{1}{\sqrt{2}}(A_1D_4 + D_1A_4)$$

Appendix

The Bell states are written in the H/V, R/L, and A/D bases.

$$\Psi_p = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}(H_iV_j + V_iH_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(R_i + L_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(R_j - L_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(L_i - R_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(R_j + L_j) \end{array} \right. \rightarrow \sqrt{2}\left(-\frac{R_iR_ji}{2} + \frac{L_iL_ji}{2}\right)$$

$$\frac{1}{\sqrt{2}}(H_iV_j + V_iH_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(D_i + A_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(D_j - A_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(D_i - A_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(D_j + A_j) \end{array} \right. \rightarrow \sqrt{2}\left(\frac{A_iA_ji}{2} - \frac{D_iD_ji}{2}\right)$$

$$\Psi_m = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}(H_iV_j - V_iH_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(R_i + L_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(L_j - R_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(L_i - R_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(R_j + L_j) \end{array} \right. \rightarrow \sqrt{2}\left(-\frac{L_iR_ji}{2} + \frac{L_iR_ji}{2}\right)$$

$$\frac{1}{\sqrt{2}}(H_iV_j - V_iH_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(D_i + A_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(D_j - A_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(D_i - A_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(D_j + A_j) \end{array} \right. \rightarrow \sqrt{2}\left(\frac{A_iD_ji}{2} - \frac{A_iD_ji}{2}\right)$$

$$\Phi_p = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}(H_iH_j + V_iV_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(R_i + L_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(L_j - R_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(L_i - R_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(R_j + L_j) \end{array} \right. \rightarrow \sqrt{2}\left(\frac{L_iR_ji}{2} + \frac{L_iR_ji}{2}\right)$$

$$\frac{1}{\sqrt{2}}(H_iV_j + V_iH_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(D_i + A_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(D_j - A_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(D_i - A_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(D_j + A_j) \end{array} \right. \rightarrow \sqrt{2}\left(\frac{A_iA_ji}{2} - \frac{D_iD_ji}{2}\right)$$

$$\Phi_m = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}}(H_iH_j - V_iV_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(R_i + L_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(L_j - R_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(L_i - R_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(R_j + L_j) \end{array} \right. \rightarrow \sqrt{2}\left(\frac{L_iL_ji}{2} + \frac{R_iR_ji}{2}\right)$$

$$\frac{1}{\sqrt{2}}(H_iV_j - V_iH_j) \left| \begin{array}{l} \text{substitute, } H_i = \frac{1}{\sqrt{2}}(D_i + A_i) \\ \text{substitute, } V_j = \frac{i}{\sqrt{2}}(D_j - A_j) \\ \text{substitute, } V_i = \frac{i}{\sqrt{2}}(D_i - A_i) \\ \text{substitute, } H_j = \frac{1}{\sqrt{2}}(D_j + A_j) \end{array} \right. \rightarrow \sqrt{2}\left(\frac{A_iD_ji}{2} - \frac{A_jD_ii}{2}\right)$$

The initial state is written in the H/V, R/L, and A/D bases.

$$\frac{1}{2}(H_1V_2 - V_1H_2)(H_3V_4 - V_3H_4) \left| \begin{array}{l} \text{substitute, } \mathrm{H}_1 = \frac{1}{\sqrt{2}}(D_1 + A_1) \\ \text{substitute, } \mathrm{V}_2 = \frac{1}{\sqrt{2}}(D_2 - A_2) \\ \text{substitute, } \mathrm{V}_1 = \frac{1}{\sqrt{2}}(D_1 - A_1) \\ \text{substitute, } \mathrm{H}_2 = \frac{1}{\sqrt{2}}(D_2 + A_2) \\ \text{substitute, } \mathrm{H}_3 = \frac{1}{\sqrt{2}}(D_3 + A_3) \\ \text{substitute, } \mathrm{V}_3 = \frac{1}{\sqrt{2}}(D_3 - A_3) \\ \text{substitute, } \mathrm{H}_4 = \frac{1}{\sqrt{2}}(D_4 + A_4) \\ \text{substitute, } \mathrm{V}_4 = \frac{1}{\sqrt{2}}(D_4 - A_4) \end{array} \right. \rightarrow \frac{A_1D_2 - A_2D_1)(A_3D_4 - A_4D_3)}{2}$$

$$\frac{1}{2}(H_1V_2 - V_1H_2)(H_3V_4 - V_3H_4) \left| \begin{array}{l} \text{substitute, } \mathrm{H}_1 = \frac{1}{\sqrt{2}}(R_1 + L_1) \\ \text{substitute, } \mathrm{V}_2 = \frac{i}{\sqrt{2}}(L_2 - R_2) \\ \text{substitute, } \mathrm{V}_1 = \frac{i}{\sqrt{2}}(L_1 - R_1) \\ \text{substitute, } \mathrm{H}_2 = \frac{1}{\sqrt{2}}(R_2 + L_2) \\ \text{substitute, } \mathrm{H}_3 = \frac{1}{\sqrt{2}}(R_3 + L_3) \\ \text{substitute, } \mathrm{V}_3 = \frac{i}{\sqrt{2}}(L_3 - R_3) \\ \text{substitute, } \mathrm{H}_4 = \frac{1}{\sqrt{2}}(R_4 + L_4) \\ \text{substitute, } \mathrm{V}_4 = \frac{1}{\sqrt{2}}(L_4 - R_4) \end{array} \right. \rightarrow \frac{L_1R_2 - L_2R_1)(L_3R_4 - L_4R_3)}{2}$$