8.13: Teleportation of Two Qubits

The following quantum circuit teleports |a>|b> from the top two wires on the left to the bottom two wires on the right.

Input	1	2	3	4	5	6		Output
$\lvert a\rangle$	$\cdots$	$\cdots$	$\cdots$	$\cdot$	$\boxed{H}$	$\cdots$	$\triangleright$	Measure, 0 or 1
$\lvert b\rangle$	$\cdots$	$\cdots$	$\cdots$	$\vert$	$\cdot$	$\boxed{H}$	$\triangleright$	Measure, 0 or 1
$\lvert 0\rangle$	$\boxed{H}$	$\cdot$	$\cdots$	$\oplus$	$\vert$	$\cdots$	$\triangleright$	Measure, 0 or 1
$\lvert 0\rangle$	$\boxed{H}$	$\vert$	$\cdot$	$\cdots$	$\oplus$	$\cdots$	$\triangleright$	Measure, 0 or 1
$\lvert 0\rangle$	$\cdots$	$\oplus$	$\vert$	$\cdots$	$\cdots$	$\cdots$		$\lvert a\rangle$
$\lvert 0\rangle$	$\cdots$	$\cdots$	$\oplus$	$\cdots$	$\cdots$	$\cdots$		$\lvert b\rangle$

There are 16 possible measurement results on the top four wires on the right ranging from |0> to |15> (|000000> to |111111> in binary notation), and these results determine the action required to transform the bottom two wires to |a>|b>.

As an initial example, if the teleportee in decimal notation is |3> the total input state is |48>.

Teleportee:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} = \lvert 3\rangle$$

Total input state:

$$\Psi = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \lvert 48\rangle$$

Ψ_{48} is the only non-zero element in the input state vector.

$$i = 0..63 \quad \Psi_i = 0 \quad \Psi_4 8 = 1$$

	44	45	46	47	48	49	50	51	52	53
0	0	0	0	0	0	1	0	0	0	0 $\cdots$

With the assistance of truth tables provided in the Appendix and a bit of tedious algebra the operation of the teleportation circuit on this input state can be summarized as follows.

Given $\lvert \Psi\rangle_{input} = \lvert 1\rangle\lvert 1\rangle\lvert 0\rangle\lvert 0\rangle\lvert 0\rangle\lvert 0\rangle = \lvert 48\rangle$ the teleportation circuit yields

$$\lvert \Psi\rangle_{output} = \frac{1}{4}\left[\begin{array}{l} (\lvert 3\rangle - \lvert 7\rangle - \lvert 11\rangle + \lvert 15\rangle)\lvert 00\rangle + (\lvert 2\rangle - \lvert 6\rangle - \lvert 10\rangle + \lvert 14\rangle)\lvert 01\rangle \\ + (\lvert 1\rangle - \lvert 5\rangle - \lvert 9\rangle + \lvert 13\rangle)\lvert 10\rangle + (\lvert 0\rangle - \lvert 4\rangle - \lvert 8\rangle + \lvert 12\rangle)\lvert 11\rangle \end{array} \right]$$

The output state shows that if the measurement result is 3, 7, 11 or 15, the NOT operator (|0> to |1> and |1> to |0>) should be applied to wires **5** and **6**. If the result is 2, 6, 10 or 14, the NOT operator is applied to wire **5**; if the result 1, 5, 9 or 13, the NOT operator is applied to wire **6**; if the result is 0, 4, 8 or 12, no action is required.

Now an example teleportation calculation will be carried out on the teleportee state given above. The following matrix operators are required for the teleportation circuit. They are the identity, the Hadamard gate, the measurement operators for |0> and |1>, and the CnNOT gate.

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad M_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad M_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \quad CnNOT = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

The matrix operators for the six steps in the circuit are formed using tensor (Kronecker) multiplication.

Step1= kronecker(I, kronecker(I, kronecker(H, kronecker(H, kronecker(I, I)))))

Step2= kronecker(I, kronecker(I, kronecker(CnNOT, I)))   Step3= kronecker(I, kronecker(I, kronecker(I, CnNOT)))

Step4= kronecker(CnNOT, kronecker(I, kronecker(I, I)))   Step5= kronecker(H, kronecker(CnNOT, kronecker(I, I)))

Step6= kronecker(I, kronecker(H, kronecker(I, kronecker(I, kronecker(I, I)))))

Step7 = kronecker(M_0, kronecker(M_0, kronecker(M_0, kronecker(M_0, kronecker(I, I)))))

The output vector is calculated for the simple case of measuring |0>|0>|0>|0> on the top four wires. The factor of 4 takes into account that measuring |0>|0>|0>|0> has a probability amplitude of 1/4. It therefore normalizes the calculation.

$$\Psi_{out} = 4\,\text{Step7 Step6 Step 5 Step4 Step3 Step2 Step1}\,\Psi$$

$$\Psi_{out}^{T} = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

$$\Psi_{out} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = |3\rangle$$

In a second example a superposition of |0 > + |1 > + |2 > + |3> is teleported.

Teleportee:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle + |2\rangle + 3\rangle}{2}$$

Total input vector:

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Non-zero vector elements:

$$\Psi_0 = \tfrac{1}{2} \quad \Psi_{16} = \tfrac{1}{2} \quad \Psi_{32} = \tfrac{1}{2} \quad \Psi_{48} = \tfrac{1}{2}$$

The output vector is calculated for the case of measuring |0>|0>|0>|0> on the top four wires.

$$\Psi_{out} = 4\,\text{Step7 Step6 Step 5 Step4 Step3 Step2 Step1}\,\Psi$$

$$\Psi_{out}^{T} = \begin{array}{c|c|c|c|c|c|c|c|c|c|c|c} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \\ \hline 0 & 0.5 & 0.5 & 0.5 & 1 & 0.5 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

$$\Psi_{out} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{2}\begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \frac{|0\rangle + |1\rangle + |2\rangle + 3\rangle}{2}$$

A final example:

Teleportee:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} \sqrt{\tfrac{1}{3}} \\ \sqrt{\tfrac{2}{3}} \end{pmatrix}$$

Total input vector:

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} \sqrt{\tfrac{1}{3}} \\ \sqrt{\tfrac{2}{3}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Non-zero vector elements:

$$\Psi_0 = \sqrt{\tfrac{1}{6}} \quad \Psi_{16} = \sqrt{\tfrac{1}{3}} \quad \Psi_{32} = \sqrt{\tfrac{1}{6}} \quad \Psi_{48} = \sqrt{\tfrac{1}{3}}$$

$$\Psi_{out} = 4\,\text{Step7Step6Step5Step4Step3Step2Step1}\,\Psi$$

$$\Psi_{out}^{T} = \begin{array}{c|c|c|c|c|c|c|c|c|c} & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \hline 0 & 0.408 & 0.577 & 0.408 & 0.577 & 0 & 0 & 0 & 0 & \cdots \end{array}$$

$$\Psi_{out} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} \sqrt{\tfrac{1}{3}} \\ \sqrt{\tfrac{2}{3}} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0.408 \\ 0.577 \\ 0.408 \\ 0.577 \end{pmatrix}$$

Appendix

$$\text{NOT} = \begin{pmatrix} 0 \text{ to } 1 \\ 1 \text{ to } 0 \end{pmatrix} \qquad H = \begin{bmatrix} 0 \text{ to } \frac{1}{\sqrt{2}}(0+1) \\ 1 \text{ to } \frac{1}{\sqrt{2}}(0-1) \end{bmatrix} \qquad \text{CnNOT} =$$

Decimal	Binary to Binary	Decimal
0	000 to 000	0
1	001 to 001	1
2	010 to 010	2
3	011 to 011	3
4	011 to 011	5
5	101 to 100	4
6	110 to 111	7
7	111 to 110	6