8.31: An Extension of Bohm's EPR Experiment

Quantum theory is both stupendously successful as an account of the small-scale structure of the world and it is also the subject of an unresolved debate and dispute about its interpretation. J. C. Polkinghorne, *The Quantum World*, p. 1.

In 1951 David Bohm proposed a gedanken experiment that further illuminated the conflict between local realism and quantum mechanics first articulated by Einstein, Podolsky and Rosen (EPR) in 1935. In this thought experiment a spin-1/2 pair is prepared in a singlet state and the individual particles travel in opposite directions on the y-axis to a pair of observers set up to measure spin in either the x- or z-direction.

In this summary tensor algebra will be used to analyze Bohm′s thought experiment. The vector states and matrix operators required are provided below.

Spin eigenvectors:

$$S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad S_{zd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad S_{xu} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad S_{xd} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

Spin operators in units of h/4π:

$$S_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

Identity:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

According to established quantum principles the singlet state for fermions is an entangled superposition written as follows in tensor format in the z-direction spin axis.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|S_{zu}\rangle_1 |S_{zd}\rangle_2 - |S_{zd}\rangle_1 |S_{zu}\rangle_2] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

First we calculate the expectation values for spin measurements in the z- and x-directions using the eigenvectors of the respective spin operators. We see that spin-up yields +1 and spin-down -1.

$$S_{zu}^T S_z S_{zu} = 1 \quad S_{zd}^T S_z S_{zd} = -1 \quad S_{xu}^T S_x S_{xu} = 1 \quad S_{xd}^T S_x S_{xd} = -1$$

Consequently the expectation value observed when both observers jointly measure the same spin direction is -1, because in the singlet state the spins have opposite orientations (kronecker performs the tensor multiplication of two matrices, as illustrated in the Appendix).

$$\Psi^T \text{kronecker}(S_z,\ S_z)\Psi = -1 \quad \Psi^T \text{kronecker}(S_x,\ S_x)\Psi = -1$$

Note that while the entangled singlet spin state shown above is written in the z-basis, it is the same to an over-all phase using the x-direction eigenvectors.

$$\Psi = \frac{1}{\sqrt{2}}[|S_{xd}\rangle_1 |S_{xu}\rangle_2 - |S_{xu}\rangle_1 |S_{xd}\rangle_2] = \frac{1}{\sqrt{2}}\left[\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} - \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \otimes \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

Next we show that in spite of the strong correlation shown above, individual spin measurements are totally random yielding expectation values of zero in both the z- and x-directions.

$$\Psi^T \text{kronecker}(S_z,\ I)\Psi = 0 \quad \Psi^T \text{kronecker}(I,\ S_z)\Psi = 0$$
$$\Psi^T \text{kronecker}(S_x,\ I)\Psi = 0 \quad \Psi^T \text{kronecker}(I,\ S_x)\Psi = 0$$

Quantum mechanics predicts the following results when the observers make different spin measurements on the particles.

$$\Psi^T \text{kronecker}(S_z,\ S_x)\Psi = 0 \quad \Psi^T \text{kronecker}(S_x,\ S_z)\Psi = 0$$

From a classical realist position the results for all the previous quantum calculations can be explained by assigning specific x- and z-spin states to the particles, as shown in the table below on the left (see Townsend's *A Modern Approach to Quantum Mechanics*, page 135). Each particle can be in any one of four equally probable spin states, and taken together the particles form four equally probable joint spin states. In other words, the particles are in well-defined, although unknown, spin states prior to measurement. Measurement, according to a realist, simply reveals a pre-existing state. The right-hand part of the table gives the joint measurement results expected given the spin states specified on the left. The bottom row gives the expectation (average) values under the assumptions stated above.

$$\begin{pmatrix} \text{Particle 1} & \text{Particle 2} & ' & S_z(1)S_z(2) & S_x(1)S_x(2) & S_z(1)S_x(2) & S_x(1)S_z(2) \\ S_{zu}S_{xu} & S_{zd}S_{xd} & ' & -1 & -1 & -1 & -1 \\ S_{zu}S_{xd} & S_{zd}S_{xu} & ' & -1 & -1 & 1 & 1 \\ S_{zd}S_{xu} & S_{zu}S_{xd} & ' & -1 & -1 & 1 & 1 \\ S_{zd}S_{xd} & S_{zu}S_{xu} & ' & -1 & -1 & -1 & -1 \\ \text{Expectation} & \text{Value} & ' & -1 & -1 & 0 & 0 \end{pmatrix}$$

At this point one may ask, "Where's the problem? The quantum and classical pictures are in agreement on the prediction of experimental results." The difficulty is that quantum mechanics does not accept the legitimacy of the states shown in the table on the left. One way to state the problem is to note that Sx and Sz are noncommuting operators. This means that according to quantum mechanics spin in the x- and z-directions cannot simultaneously have well-defined values (like position and momentum, they are conjugate observables).

The S_x-S_z commutator:

$$S_z S_x - S_x S_z = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

For example, if measurement reveals that particle 1 has S_{zu} then particle 2 is definitely S_{zd}. But that means it cannot have a well-defined value for the x-direction spin. In fact S_{zd} is a superposition of S_{xu} and S_{xd}, as shown below, with $\langle S_x \rangle = 0$.

$$S_{zd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \frac{1}{\sqrt{2}}(S_{xu} - S_{xd}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad S_{sd}^T S_x S_{zd} = 0$$

Or, suppose particle 1 is found to have S_{xd}, then particle 2 is S_{xu} which is a superposition of spin up and down in the z-direction and therefore $\langle S_x \rangle = 0$.

$$S_{xu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad \frac{1}{\sqrt{2}}(S_{zu} + S_{zd}) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad S_{xu}^T S_z S_{xu} = 0$$

To summarize, the states in the table are not valid, in spite of their agreement with the predictions of quantum mechanics, because they give well-defined values to incompatible (according to quantum theory) observables. The quantum objection to the classical spin states can also be expressed using the Uncertainty Principle. For a particle known to have spin-up in the z-direction, quantum theory requires that its x-direction spin be uncertain, as shown by the following calculation.

While Bohm's 1951 gedanken experiment clarified the conflict between quantum theory and classical realism, it did not provide for a direct experimental adjudication of the disagreement. That changed in 1964 with the theoretical analysis of John Bell who recognized the potential of Bohm's thought experiment to decide the issue one way or the other empirically. As is well known, the subsequent experimental work based on Bell's theorem decided the conflict between the two views in favor of quantum theory. See the other entries in this section for examples of the impact of Bell's work and the experimentalist who verified the implications of his theorem.

A modified version of the thought experiment shows that there are experiments involving entangled spin systems for which a local hidden-variable theory gives predictions which are incompatible with those of quantum theory. Instead of measuring the spins in the z- and x-directions, measure one in the z-direction and the other at some non-orthogonal angle to the z-axis, say 45 degrees (π/4). The appropriate spin operator for this diagonal direction (an even superposition of Sx and Sz) and its eigenvalues and eigenvectors are given below, and the singlet spin state is written in the diagonal spin basis.

$$S_d = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{eigenvals}(S_d) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{eigenvecs}(S_d) = \begin{pmatrix} 0.924 & -0.383 \\ 0.383 & 0.924 \end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}[|S_{du}\rangle_A |\, S_{dd}\rangle_B - |S_{dd}\rangle_A |\, S_{du}\rangle_B] = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \otimes \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} - \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \otimes \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

As expected, perfect anti-correlation is observed if both spins are measured in the diagonal direction while the individual spin measurements are totally random with an expectation value of zero.

$$\Psi^T \text{kronecker}(S_d,\ S_d)\Psi = -1 \quad \Psi^T \text{kronecker}(S_d,\ I)\Psi = 0 \quad \Psi^T \text{kronecker}(I,\ S_d)\Psi = 0$$

Up to this point the calculations are consistent with those for the x- and z-directions, and would appear to justify the local realist in providing the following hidden-variable interpretation of the results.

$$\begin{pmatrix} \text{Particle 1} & \text{Particle 2} & ' & S_z(1)S_z(2) & S_d(1)S_d(2) & S_z(1)S_d(2) & S_d(1)S_z(2) \\ S_{zu}S_{du} & S_{zd}S_{dd} & ' & -1 & -1 & -1 & -1 \\ S_{zu}S_{dd} & S_{zd}S_{du} & ' & -1 & -1 & 1 & 1 \\ S_{zd}S_{du} & S_{zu}S_{dd} & ' & -1 & -1 & 1 & 1 \\ S_{zd}S_{dd} & S_{zu}S_{du} & ' & -1 & -1 & -1 & -1 \\ \text{Expectation} & \text{Value} & ' & -1 & -1 & 0 & 0 \end{pmatrix}$$

However when the spins are measured in different directions (the z- and d-directions) quantum mechanics predicts that the expectation values are not zero as predicted by the hidden-variable model. According to quantum theory there is significant anti-correlation in the joint spin measurements.

$$\Psi^T \text{kronecker}(S_z,\ S_d)\Psi = -0.707 \quad \Psi^T \text{kronecker}(S_d,\ S_z)\Psi = -0.707$$

This disagreement has been examined experimentally and the experimental evidence supports quantum theory.

Appendix: Vector and Matrix Math

This addendum reviews basic vector and matrix operations. At the end it illustrates how vector and matrix tensor multiplication are implemented in Mathcad.

Tensor multiplication of several of the spin operators using Mathcad′s kronecker command:

$$\text{kronecker}(S_z,\ S_z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad \text{kronecker}(S_x,\ S_x) = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

$$\text{kronecker}(S_d,\ S_d) = \begin{pmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.5 & -0.5 & 0.5 & -0.5 \\ 0.5 & 0.5 & -0.5 & -0.5 \\ 0.5 & -0.5 & -0.5 & 0.5 \end{pmatrix} \qquad \text{kronecker}(S_z,\ S_x) = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$\text{kronecker}(S_z,\ S_d) = \begin{pmatrix} 0.707 & 0.707 & 0 & 0 \\ 0.707 & -0.707 & 0 & 0 \\ 0 & 0 & -0.707 & -0.707 \\ 0 & 0 & -0.707 & 0.707 \end{pmatrix} \qquad \text{kronecker}(S_z,\ I) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$\text{kronecker}(S_x,\ I) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \text{kronecker}(S_d,\ I) = \begin{pmatrix} 0.707 & 0 & 0.707 & 0 \\ 0 & 0.707 & 0 & 0.707 \\ 0.707 & 0 & -0.707 & 0 \\ 0 & 0.707 & 0 & -0.707 \end{pmatrix}$$

Mathcad′s kronecker command is only useful for matrix tensor multiplication, but can be adapted to carry out vector tensor multiplication in the manner shown below. Two matrices are created using the null vector, tensor multiplied and everything but the first column of the product matrix is discarded.

Null vector:

$$N = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Psi(a,\ b) = \text{submatrix}(\text{kronecker}(\text{augment}(\text{a, N}), \text{augment}(\text{b, N})), 1, 4, 1, 1)$$

$$\frac{\Psi(S_{zu}, S_{zd}) - \Psi(S_{zd}, S_{zu})}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix} \qquad \frac{\Psi(S_{xd}, S_{xu}) - \Psi(S_{xu}, S_{xd})}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$

$$S_{du} = \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix} \quad S_{dd} = \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix} \quad \frac{\Psi(S_{du}, S_{dd}) - \Psi(S_{dd}, S_{du})}{\sqrt{2}} = \begin{pmatrix} 0 \\ 0.707 \\ -0.707 \\ 0 \end{pmatrix}$$