10.32: Momentum-Space Variation Method for the Abs(x) Potential

The energy operator in atomic units in coordinate space for a unit mass particle with potential energy V = |x| is given below.

$$H = \frac{-1}{2}\frac{d^2}{dx^2}\blacksquare + |x|\blacksquare$$

Suggested trial wave function:

$$\Psi(x,\beta) := \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} exp(-\beta x^2)$$

Demonstrate that the wave function is normalized.

$$\int_{-\infty}^{\infty} \Psi(x,\beta)^2 dx \;\; assume,\; \beta > 0 \rightarrow 1$$

Carry out Fourier transform to get momentum wave function:

$$\Phi(p,\beta) := \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} exp(-ipx)\Psi(x,\beta)dx\Big|_{simplify}^{assume,\;\beta>1} \rightarrow \frac{1}{2}\frac{2^{\frac{3}{4}}}{\pi^{\frac{1}{4}}}\frac{e^{\frac{-1}{4}\frac{p^2}{\beta}}}{\beta^{\frac{1}{4}}}$$

Demonstrate that the momentum wave function is normalized.

$$\int_{-\infty}^{\infty} \overline{\Phi(p,\beta)}\Phi(p,\beta)dp \;\; assume,\; \beta > 0 \rightarrow 1$$

The energy operator in momentum space is:

$$H = \frac{p^2}{2}\blacksquare + |i + \frac{d}{dp}\blacksquare|$$

Evaluate the variational energy integral:

$$E(\beta) := \int_{-\infty}^{\infty} \overline{\Phi(p,\beta)}\frac{p^2}{2}\Phi(p,\beta)dp + \int_{-\infty}^{\infty} \overline{\Phi(p,\beta)}|i\frac{d}{dp}\Phi(p,\beta)|dp\Big|_{simplify}^{assume,\;\beta>0} \frac{1}{2}\frac{\pi^{\frac{1}{2}}\beta^{\frac{3}{2}} + 2^{\frac{1}{2}}}{\beta^{\frac{1}{2}}\pi^{\frac{1}{2}}}$$

Minimize the energy with respect to the variational parameter β and report its optimum value and the ground-state energy.

$\beta := 1$ $\beta :=$ Minimize (E, β) $\beta = 0.542$ E(β) = 0.813

Plot the coordinate and momentum wave functions and the potential energy on the same graph.