# 8.48: Another Bell Theorem Analysis - Shorter Version

The purpose of this tutorial is to review Jim Baggott's analysis of Bell's theorem as presented in Chapter 4 of *The Meaning of Quantum Theory* using matrix and tensor algebra.

A two-stage atomic cascade emits entangled photons (A and B) in opposite directions with the same circular polarization according to observers in their path.

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|L\rangle_A|L\rangle_B + |R\rangle_A|R\rangle_B]$$

The experiment involves the measurement of photon polarization states in the vertical/horizontal measurement basis, and allows for the rotation of the right-hand detector through an angle of θ, in order to explore the consequences of quantum mechanical entanglement. PA stands for polarization analyzer and could simply be a calcite crystal.
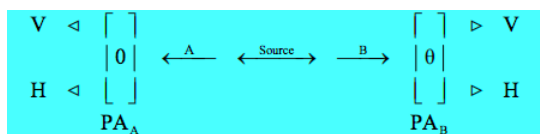
In vector notation the left- and right- circular polarization states are expressed as follows:

Left circular polarization:

$$L = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

Right circular polarization:

$$R = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

In tensor notation the initial state is the following entangled superposition,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}[|L\rangle_A|L\rangle_B + |R\rangle_A|R\rangle_B] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ i \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ i \end{pmatrix}_B + \begin{pmatrix} 1 \\ -i \end{pmatrix}_A \otimes \begin{pmatrix} 1 \\ -i \end{pmatrix}_B\right]$$

$$= \frac{1}{2\sqrt{2}}\left[\begin{pmatrix} 1 \\ i \\ i \\ -1 \end{pmatrix} + \begin{pmatrix} 1 \\ -i \\ -i \\ -1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

However, as mentioned above, the photon polarization measurements will actually be made in the vertical/horizontal basis. These polarization measurement states for photons A and B in vector representation are given below. Θ is the angle through which the PA2 has been rotated.

Vertical polarization:

$$V_A = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad V_B = \begin{pmatrix} \cos\theta \\ \sin\theta \end{pmatrix}$$

Horizontal polarization:

$$V_A = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad V_B = \begin{pmatrix} \sin\theta \\ \cos\theta \end{pmatrix}$$

It is easy to show that |Ψ> in the vertical/horizontal basis is,

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[|V\rangle_A|V\rangle_B + |H\rangle_A|H\rangle_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}_A \otimes \begin{pmatrix}1\\0\end{pmatrix}_B + \begin{pmatrix}0\\1\end{pmatrix}_A \otimes \begin{pmatrix}0\\1\end{pmatrix}_B\right] = \frac{1}{2\sqrt{2}}\left[\begin{pmatrix}1\\0\\0\\0\end{pmatrix} + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}\right]$$

$$= \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix}$$

A joint measurement operator is created using the A and B photon measurement eigenstates, with vertical polarization assigned an eigenvalue of +1 and horizontal polarization an eigenvalue of -1. This operator is then used to calculate the expectation value or correlation function.

$$V_A V_A^T - H_A H_A^T \rightarrow \begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \quad V_B V_B^T - H_B H_B^T \text{simplify} \rightarrow \begin{pmatrix}\cos(2\theta) & -\sin(2\theta)\\-\sin(2\theta) & 2\sin(\theta)^2 - 1\end{pmatrix}$$

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}\cos(2\theta) & -\sin(2\theta)\\-\sin(2\theta) & 2\sin^2(\theta) - 1\end{pmatrix} = \begin{pmatrix}\cos(2\theta) & -\sin(2\theta) & 0 & 0\\-\sin(2\theta) & 2\sin^2(\theta) - 1 & 0 & 0\\0 & 0 & -\cos(2\theta) & \sin(2\theta)\\0 & 0 & \sin(2\theta) & 2\sin^2(\theta) - 1\end{pmatrix}$$

$$\Psi = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\\0\\-1\end{pmatrix} \quad E(\theta) = \Psi\begin{pmatrix}\cos(2\theta) & -\sin(2\theta) & 0 & 0\\-\sin(2\theta) & 2\sin^2(\theta) - 1 & 0 & 0\\0 & 0 & -\cos(2\theta) & \sin(2\theta)\\0 & 0 & \sin(2\theta) & 2\sin^2(\theta) - 1\end{pmatrix}\Psi\text{simplify} \rightarrow \cos(2\theta)$$

As shown above the evaluation of $E(\theta)$ yields $\cos(2\theta)$. For $\theta = 0^0$ there is perfect correlation; for $\theta = 90^0$ perfect anti-correlation; for $\theta = 45^0$ no correlation.

$$E(0 \text{ deg}) = 1 \quad E(90 \text{ deg}) = -1 \quad E(45 \text{ deg}) = 0$$

Baggott presented a correlation function for this experiment based on a local hidden variable model of reality (pp. 110-113, 127-131). It (linear blue line) and the quantum mechanical correlation function, $E(\theta)$, are compared on the graph below. Quantum theory and local realism disagree at all angles except 0, 45 and 90 degrees.
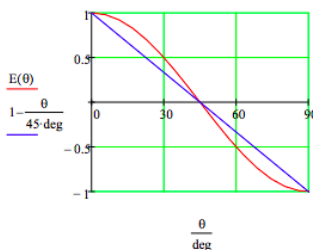
This example illustrates Bell's theorem: no local hidden-variable theory can reproduce all the predictions of quantum mechanics for entangled composite systems. As the quantum predictions are confirmed experimentally, the local hidden-variable approach to reality must be abandoned.

In spite of the correlation shown in the joint polarization measurements, the individual measurements on photons 1 and 2 are totally random for all values of θ, i.e. $E(\theta) = 0$. For example, the following operator is used to calculate the expectation value for measurements on photon 2 as a function of its analyzer's angle θ. The identity operator represents no measurement on spin 1.

$$\begin{pmatrix}1 & 0\\0 & -1\end{pmatrix} \otimes \begin{pmatrix}\cos(2\theta) & -\sin(2\theta)\\-\sin(2\theta) & 2\sin^2(\theta) - 1\end{pmatrix} = \begin{pmatrix}\cos(2\theta) & -\sin(2\theta) & 0 & 0\\-\sin(2\theta) & 2\sin^2(\theta) - 1 & 0 & 0\\0 & 0 & -\cos(2\theta) & \sin(2\theta)\\0 & 0 & \sin(2\theta) & 2\sin^2(\theta) - 1\end{pmatrix}$$

$$E(\theta) = \Psi^T \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & 2\sin^2(\theta)-1 \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) & 0 & 0 \\ -\sin(2\theta) & 2\sin^2(\theta)-1 & 0 & 0 \\ 0 & 0 & -\cos(2\theta) & \sin(2\theta) \\ 0 & 0 & \sin(2\theta) & 2\sin^2(\theta)-1 \end{pmatrix}$$

Naturally the same is true for photon 1.

$$\begin{pmatrix} \cos(2\theta) & -\sin(2\theta) \\ -\sin(2\theta) & 2\sin^2(\theta)-1 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \cos(2\theta) & 0 & -\sin(2\theta) & 0 \\ 0 & \cos(2\theta) & 0 & -\sin(2\theta) \\ -\sin(2\theta) & 0 & 2\sin^2(\theta)-1 & 0 \\ 0 & -\sin(2\theta) & 0 & 2\sin^2(\theta)-1 \end{pmatrix} \Psi \text{ simplify}$$

$$\to 0$$

The expectation value calculations can also be performed using the trace function as shown below.

$$\langle\Psi|\hat{O}|\Psi\rangle = \sum_i \langle\Psi|\hat{O}|i\rangle\langle i|\Psi\rangle = \sum_i \langle i|\Psi\rangle\langle\Psi|\hat{O}|i\rangle = Trace\left(|\Psi\rangle\langle\Psi|\hat{O}\right) \text{ where } \sum_i |i\rangle\langle i| = \text{Identity}$$

$$tr\left[\Psi\Psi^T \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) & 0 & 0 \\ -\sin(2\theta) & 2\sin(\theta)^2-1 & 0 & 0 \\ 0 & 0 & -\cos(2\theta) & \sin(2\theta) \\ 0 & 0 & \sin(2\theta) & 1-2\sin(\theta)^2 \end{pmatrix}\right] \text{simplify} \to \cos(2\theta)$$

$$tr\left[\Psi\ \Psi^T \begin{pmatrix} \cos(2\theta) & -\sin(2\theta) & 0 & 0 \\ -\sin(2\theta) & 2\sin(\theta)^2-1 & 0 & 0 \\ 0 & 0 & -\cos(2\theta) & \sin(2\theta) \\ 0 & 0 & \sin(2\theta) & 1-2\sin(\theta)^2 \end{pmatrix}\right] \text{simplify} \to \cos(2\theta)$$

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