1.35: Matrix Mechanics

The basic principles of quantum theory can be demonstrated very simply by exploring the properties of electron spin using Heisenberg's formulation of quantum mechanics which is usually referred to as matrix mechanics. The matrix formulation provides clear illustrations of the following essential quantum mechanical concepts: eigenvector, operator, eigenvalue, expectation value, the linear superposition, and the commutation relations.

Four quantum numbers are required to describe the electron in quantum mechanics. The last of these is the spin quantum number, s. The electron has a spin component in the x-, y-, and z-directions and for each of these directions the electron can have a value of spin-up or spin-down, or +1 and -1 in units of $\frac{h}{4\pi}$. These spin states are represented by vectors as is shown below.

$S_{xu} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$	$S_{xd} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\left(\overline{S_{xu}}\right)^T = \begin{pmatrix} 0.707 & 0.707 \end{pmatrix}$	$\left(\overline{S_{xd}}\right)^T = \begin{pmatrix} 0.707 & -0.707 \end{pmatrix}$
$S_{yu} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ i \end{pmatrix}$	$S_{yd} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ -i \end{pmatrix}$	$\left(\overline{S_{yu}}\right)^T = \begin{pmatrix} 0.707 & -0.707i \end{pmatrix}$	$\left(\overline{S_{yd}}\right)^T = \begin{pmatrix} 0.707 & 0.707i \end{pmatrix}$
$S_{zu} := \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$S_{zd} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\left(\overline{S_{zu}}\right)^T = \begin{pmatrix} 1 & 0 \end{pmatrix}$	$\left(\overline{S_{yd}}\right)^T = \begin{pmatrix} 0 & 1 \end{pmatrix}$

Let's look at the the y-direction spin states because they are complex, and therefore are slightly more difficult to deal with. In Dirac notation these four vectors are written as $|S_{yu}>$, $|S_{yd}>$, $<S_{yu}|$, and $<S_{yd}|$. Note that the bra-vectors are the transpose of the complex conjugate of the ket-vectors. It is also easy to show that these spin vectors in the x-, y-, and z-directions form orthonormal basis sets. That means they are normalized and orthogonal to each other.

$\left(\overline{S_{xu}}\right)^T \cdot S_{xu} = 1$	$\left(\overline{S_{xd}}\right)^T \cdot S_{xd} = 1$	$\left(\overline{S_{xu}}\right)^T \cdot S_{xd} = 0$
$\left(\overline{S_{yu}}\right)^T \cdot S_{yu} = 1$	$\left(\overline{S_{yd}}\right)^T \cdot S_{yd} = 1$	$\left(\overline{S_{yu}}\right)^T \cdot S_{yd} = 0$
$\left(\overline{S_{zu}}\right)^T \cdot S_{zu} = 1$	$\left(\overline{S_{zd}}\right)^T \cdot S_{zd} = 1$	$\left(\overline{S_{zu}}\right)^T \cdot S_{zd} = 0$

In Dirac notation we would write the first row as: $<S_{xu}|S_{xu}> = <S_{xd}|S_{xd}> = 1$, $<S_{xu}|S_{xd}> = 0$. In other words the projection of the spin states onto themselves is 1 (normalized) and the projection onto the other state is zero (orthogonal).

The calculations above for the y-direction spin vectors are shown explicitly below. You should do hand calculations on all of the above for practice and to have some appreciation for what the computer is doing.

$$\begin{pmatrix} 0.707 & -0.707i \end{pmatrix} \cdot \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix} = 1$$

$$\begin{pmatrix} 0.707 & 0.707i \end{pmatrix} \cdot \begin{pmatrix} 0.707 \\ -0.707i \end{pmatrix} = 1$$

$$\begin{pmatrix} 0.707 & -0.707i \end{pmatrix} \cdot \begin{pmatrix} 0.707 \\ -0.707i \end{pmatrix} = 0$$

It is easy to show that x- and z-spin states are not orthogonal to one another. This is true of any two different spin directions. $<S_{xu}|S_{zu}> = 0.707$, for example.

$$\left(\overline{S_{xu}}\right)^T \cdot S_{zu} = 0.707 \quad \left(\overline{S_{xu}}\right)^T \cdot S_{zd} = 0.707 \quad \left(\overline{S_{xd}}\right)^T \cdot S_{zu} = 0.707 \quad \left(\overline{S_{xd}}\right)^T \cdot S_{zd} = -0.707$$

This of course means that $|S_{xu}>$ and $|S_{xd}>$ can be written as linear superpositions of $|S_{zu}>$ and $|S_{zd}>$, and $|S_{zu}>$ and $|S_{zd}>$ can be written as linear superpositions of $|S_{xu}>$ and $|S_{xd}>$.

$$S_{xu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot S_{zu} + \frac{1}{\sqrt{2}} \cdot S_{zd} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

$$S_{xd} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot S_{zu} - \frac{1}{\sqrt{2}} \cdot S_{zd} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$$

$$S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot S_{xu} + \frac{1}{\sqrt{2}} \cdot S_{xd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_{zd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot S_{xu} - \frac{1}{\sqrt{2}} \cdot S_{xd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The concept of the linear superposition is central in quantum theory and has no classical analog. For example, if by measurement an electron is found to have spin-up in the z-direction, this means that the electron does not have a definite spin in either the x- or the y-direction because $|S_{zu}\rangle$ is a linear superposition of the x- and y-direction spin states.

$$S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot S_{xu} + \frac{1}{\sqrt{2}} \cdot S_{xd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \frac{1}{\sqrt{2}} \cdot S_{yu} + \frac{1}{\sqrt{2}} \cdot S_{yd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

In spite of its appearance, a linear superposition is not a mixture. In other words $|S_{zu}\rangle$ is not 50% $|S_{xu}\rangle$ and 50% $|S_{xd}\rangle$, or 50% $|S_{yu}\rangle$ and 50% $|S_{yd}\rangle$.

Another central dogma of quantum theory is that the wavefunction or state vector contains all the physical information available for the system. Quantum mechanics therefore consists, in large part, of extracting physical information from the wavefunction or state vector. Quantum mechanics consists of a small set of rules for carrying this procedure out mathematically.

For every observable of the system there is an operator. Since electrons can spin in the x-, y-, or z-directions, there are spin operators in those directions, or for that matter in any other arbitrary direction you might think of. (See Appendix B for the construction of a general spin operator.) In quantum mechanics states are vectors and operators are matrices. The spin operators in units of $\frac{h}{4\pi}$ are shown below. Note that squaring these operators gives the identity operator.

$S_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$S_y := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$S_z := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$
$S_x^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$S_y^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	$S_2^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

The square of the total spin operator in units of $\frac{h}{4\pi}$ is

$$S2 := S_x^2 + S_y^2 + S_z^2 \quad S2 = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$$

A measurement operator extracts information about the system by operating on the wavefunction or state vector. One possible outcome is that the operation returns the state vector multiplied by a numerical constant. For example,

$$S_x \cdot S_{xu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$$

$$S_x \cdot S_{xd} = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$$

$$S_y \cdot S_{yu} = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$$

$$S_y \cdot S_{yd} = \begin{pmatrix} -0.707 \\ 0.707i \end{pmatrix}$$

$$S_z \cdot S_{zu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S_z \cdot S_{zd} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

or, for example:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$$

In Dirac notation we would summarize these calculations as follows: $S_x|S_{xu}> = +1|S_{xu}>$, $S_x|S_{xd}> = -1|S_{xd}>$, $Sy|S_{yu}> = +1|S_{yu}>$, $S_y|S_{yd}> = -1|S_{yd}>$, $S_z|S_{zu}> = +1|S_{zu}>$, $S_z|S_{zd}> = -1|S_{zd}>$. In each of these cases, the state vector is an eigenfunction of the measurement operator with eigenvalue of either +1 or -1 (in units of $\frac{h}{4\pi}$). We say, for example, that $|S_{xu}>$ is an eigenfunction of $\mathbf{S_x}$ with eigenvalue +1. The electron has a well-defined value for spin in the x-direction (spin-up) and subsequent measurements of the x-direction spin will yield the value of +1 as long as no intervening measurements in another spin direction are made.

The other possible outcome of the measurement operation is that it yields another state vector.

$S_x \cdot S_{yu} = \begin{pmatrix} 0.707i \\ 0.707 \end{pmatrix}$	$S_x \cdot S_{yd} = \begin{pmatrix} -0.707i \\ 0.707 \end{pmatrix}$	$S_x \cdot S_{zu} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$S_x \cdot S_{zd} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$S_y \cdot S_{xu} = \begin{pmatrix} -0.707i \\ 0.707i \end{pmatrix}$	$S_y \cdot S_{xd} = \begin{pmatrix} 0.707i \\ 0.707i \end{pmatrix}$	$S_y \cdot S_{zu} = \begin{pmatrix} 0 \\ i \end{pmatrix}$	$S_y \cdot S_{zd} = \begin{pmatrix} -i \\ 0 \end{pmatrix}$
$S_z \cdot S_{xu} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$	$S_Z \cdot S_{xd} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$	$S_z \cdot s_{yu} = \begin{pmatrix} 0.707 \\ -0.707i \end{pmatrix}$	$S_z \cdot S_{yd} = \begin{pmatrix} 0.707 \\ 0.707i \end{pmatrix}$

In Dirac notation these operations appear as: $S_x|S_{yu}> = i|S_{yd}>$, $S_x|S_{yd}> = -i|S_{yu}>$, $S_x|S_{zu}> = |S_{zd}>$, $S_x|S_{zd}> = |S_{zu}>$, etc. In each case the resulting vector is different than the vector operated on. We say, for example, $|S_{yu}>$ is not an eigenfunction of $\mathbf{S_x}$, and therefore an electron in this state does not have a definite value for spin in the x-direction. X-direction spin measurements on a system known to be in state $|S_{yu}>$ will yield completely random results.

To put it another way, quantum mechanical principles state that a system can be in a well-defined state, $|S_{yu}>$, and yet the outcome of all experiments are not unikely determined. While a measurement of spin in the y-direction will yield a predictable result, +1, measurement of spin in the x- or z-direction is completely unpredictable and all we can calculate is the average value, or expectation value for a large number of measurements. This is completely different than classical physics where if you know the state of the system, you know the values of all physical observables.

As another example, consider the ground state of the hydrogen atom for which the electron's wave function is $\Psi = \pi^{-1/2} \exp(-r)$. When the electron is in this state it has a precise energy, but not a well-defined position or momentum. This, of course, makes the concept of an electron trajectory impossible and it is, therefore, meaningless to think of the electron as moving in any traditional sense.

The quantum mechanical algorithm for calculating the expectation value is to execute the following matrix multiplication: <State Vector | **Operator** | State Vector>. This formalism is quite general and can be used whether the state vector is an eigenfunction of the operator or not. This is demonstrated below for the spin states that we have been studying.

$\left(\overline{S_{xu}}\right)^T \cdot S_x \cdot S_{xu} = 1$	$\left(\overline{S_{xd}}\right)^T \cdot S_x \cdot S_{xd} = -1$	$\left(\overline{S_{zu}}\right)^T \cdot S_x \cdot S_{zu} = 0$
$\left(\overline{S_{zd}}\right)^T \cdot S_x \cdot S_{zd} = 0$	$\left(\overline{S_{yu}}\right)^T \cdot S_x \cdot S_{yu} = 0$	$\left(\overline{S_{yd}}\right)^T \cdot S_x \cdot S_{yd} = 0$
$\left(\overline{S_{xu}}\right)^T \cdot S_y \cdot S_{xu} = 0$	$\left(\overline{S_{xd}}\right)^T \cdot S_y \cdot S_{xd} = 0$	$\left(\overline{S_{zu}}\right)^T \cdot S_y \cdot S_{zu} = 0$
$\left(\overline{S_{zd}}\right)^T \cdot S_y \cdot S_{zd} = 0$	$\left(\overline{S_{yu}}\right)^T \cdot S_y \cdot S_{yu} = 1$	$\left(\overline{S_{yd}}\right)^T \cdot S_y \cdot S_{yd} = -1$
$\left(\overline{S_{xu}}\right)^T \cdot S_z \cdot S_{xu} = 0$	$\left(\overline{S_{xd}}\right)^T \cdot S_z \cdot S_{xd} = 0$	$\left(\overline{S_{zu}}\right)^T \cdot S_z \cdot S_{zu} = 1$
$\left(\overline{S_{zd}}\right)^T \cdot S_z \cdot S_{zd} = -1$	$\left(\overline{S_{yu}}\right)^T \cdot S_z \cdot S_{yu} = 0$	$\left(\overline{S_{yd}}\right)^T \cdot S_z \cdot S_{yd} = 0$

Let's look at the first six enteries because they are representative of the remaining results. If the electron is in the state $|S_{xu}>$ measurement of $\mathbf{S_x}$ will always yield the value of +1 (in units of $\frac{h}{4\pi}$)). If the electron is in the state $|S_{xd}>$ measurement of $\mathbf{S_x}$ will

always yield the value of -1 (in units of $\frac{h}{4\pi}$)). If instead $\mathbf{S_y}$ or $\mathbf{S_z}$ are measured, the measurement results will be a statistically random collection of +1 and -1, and the average value will, of course, be zero. Only when the system is in an eigenstate of the measurement operator is the outcome of the experiment certain.

This brings us to the concept of probability and how it is calculated in quantum mechanics. The projection of one state on to another, $<S_{zu} \mid S_{xd}> = 0.707$, is a **probability amplitude**. Its absolute square, $<S_{xd} \mid S_{zu}><S_{zu} \mid S_{xd}> = |<S_{zu} \mid S_{xd}>|^2 = 0.5$ (remember $<S_{xd} \mid S_{zu}> = <S_{zu} \mid S_{xd}>^*$), is the **probability** that an electron in state $|S_{xd}>$ will be found by measurement in the state $|S_{zu}>$. Representative calculations are shown below. (See the Appendix A for another computational method.)

$$\left[\left|\left(\overline{S_{zu}}\right)^{T} \cdot S_{xu}\right|\right]^2 = 0.5 \quad \left[\left|\left(\overline{S_{zd}}\right)^{T} \cdot S_{xu}\right|\right]^2 = 0.5 \quad \left[\left|\left(\overline{S_{xu}}\right)^{T} \cdot S_{yu}\right|\right]^2 = 0.5 \quad \left[\left|\left(\overline{S_{xu}}\right)^{T} \cdot S_{xd}\right|\right]^2 = 0.5$$

Let's review these concepts by taking a specific example. The electron is in the state $|S_{xu}>$ and we wish to measure $\mathbf{S_z}$. According to quantum mechanical procedures the average value for a statistically meaningful number of measurements is zero -$<S_{xu} \mid \mathbf{S_z} \mid S_{xu}> = 0$. The eigenstates (eigenfunctions) for $\mathbf{S_z}$ are $|S_{zu}>$ and $|S_{zd}>$ with eigenvalues +1 and -1, respectively. As the first two entries above show, the probability that an electron in state $|S_{xu}>$ will be found in $|S_{zu}>$ with eigenvalue +1 is 0.5, and the probaility that it will be found in state $|S_{zd}>$ with eigenvalue -1 is 0.5. Thus, the **average value is expected** to be zero, and the two ways of determining the average or expectation value of a measurement are consistent and equivalent.

There is yet another way to look at this issue. In quantum mechanics for most pairs of observables the order of measurement is important. Quantum mechanical operators don't generally commute. For example, as shown below, $\mathbf{S_xS_y}|S_{zu}>$ does not equal $\mathbf{S_yS_x}|S_{zu}>$. This means that if the electron is in the state $|S_{zu}>$ the combined operators $\mathbf{S_xS_y}$ and S_yS_x yield different measurement results.

$$S_x \cdot S_y \cdot S_{zu} = \begin{pmatrix} i \\ 0 \end{pmatrix} \quad S_y \cdot S_x \cdot S_{zu} = \begin{pmatrix} -i \\ 0 \end{pmatrix} \quad (S_x \cdot S_y - S_y \cdot S_x) \cdot S_{zu} = \begin{pmatrix} 2i \\ 0 \end{pmatrix}$$

Operators that do not commute have incompatible eigenstates. If a state vector is an eigenstate of one of the operators, it is not an eigenstate of the other. The fact that $\mathbf{S_x}$ and **Sy** do not commute means that an electron cannot simultaneously have well-defined values for $\mathbf{S_x}$ and $\mathbf{S_y}$. It is not surprising that there is a deep connection between these properties of operators and the Uncertainty Principle. The commutators for the spin operators are shown below.

$$S_x \cdot S_y - S_y \cdot S_x = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix} \qquad 2 \cdot i \cdot S_z = \begin{pmatrix} 2i & 0 \\ 0 & -2i \end{pmatrix}$$

$$S_z \cdot S_x - S_x \cdot S_z = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix} \qquad 2 \cdot i \cdot S_y = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$$

$$S_y \cdot S_z - S_z \cdot S_y = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix} \qquad 2 \cdot i \cdot S_x = \begin{pmatrix} 0 & 2i \\ 2i & 0 \end{pmatrix}$$

The Uncertainty Principle can also be illustrated by calculating ΔS_x and ΔS_y for an electron known to be in the S_{zu} state. Since we are working in units of $\frac{h}{4\pi}$, the uncertainty relation is: $\Delta S_x \cdot \Delta S_y \geq 1$.

$$\sqrt{S_{zu}^{T} \cdot S_x \cdot S_x \cdot S_{zu} - \left(S_{zu}^{T} \cdot S_x \cdot S_{zu}\right)^2} \cdot \sqrt{S_{zu}^{T} \cdot S_y \cdot S_y \cdot S_{zu} - \left(S_{zu}^{T} \cdot S_y \cdot S_{zu}\right)^2} = 1$$

We have been dealing with matrix operators and their associated eigenvectors and eigenvalues. The eigenvectors and eigenvalues can be obtained from the matrix operators with Mathcad's **eigenvecs** and **eigenvals** commands as is shown below.

$$\text{eignevals}\left(S_x\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{eigenvec}\left(S_x, 1\right) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \qquad \text{eigenvec}\left(S_x, -1\right) = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$$

$$\text{eignevals}\left(S_y\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \text{eigenvec}\left(S_y\right) = \begin{pmatrix} -0.707i & 0.707 \\ 0.707 & -0.707i \end{pmatrix}$$

$$\text{eignevals}\left(S_z\right) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad \text{eigenvec}\left(S_z\right) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

One final thing we will do is to demonstrate the completeness relationship. For example, $|S_{zu}><S_{zu}| + |S_{zd}><S_{zd}| = \mathbf{I}$, the identity operator. This demonstrates that the spin eigenfunction for the various Cartesian directions span the two-dimensional space.

$$S_{zu} \cdot S_{zu}^{T} + S_{zd} \cdot S_{zd}^{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_{xu} \cdot S_{xu}^{T} + S_{xd} \cdot S_{xd}^{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$S_{yu} \cdot \left(\overline{S_{yu}}\right)^{T} + S_{yd} \cdot \left(\overline{S_{yd}}\right)^{T} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Appendix A

By the method outlined above we can show that the probability that a S_{zu} electron will be found in the S_{xu} spin state is 0.5.

$$\left[\left|\left(\overline{S_{xu}}\right)^{T} \cdot S_{zu}\right|\right]^{2} = 0.5$$

This calculation can be rewritten in terms of the trace of the product of the $|S_{zu}\rangle \langle S_{zu}|$ and $|S_{xu}\rangle \langle S_{xu}|$ projection operators.

$$|\langle S_{xu}|S_{zu}\rangle|^{2} = \langle S_{zu}|S_{xu}\rangle \langle S_{xu}|S_{zu}\rangle = \sum_{i} \langle S_{zu}|i\rangle \langle i|S_{xu}\rangle \langle S_{xu}|S_{zu}\rangle = \sum_{i} \langle i|S_{xu}\rangle \langle S_{xu}|S_{zu}\rangle \langle S_{zu}|i\rangle = \text{Trace}$$

$$\left(|S_{xu}\rangle \langle S_{xu}|S_{zu}\rangle \langle S_{zu}|\right)$$

where the completeness relation $\sum_{i} |i\rangle\langle i| = 1$ has been employed.

$$\text{tr}\left[\left(S_{xu} \cdot S_{xu}^{T}\right) \cdot \left(S_{zu} \cdot S_{zu}^{T}\right)\right] = 0.5$$

Appendix B

A general spin operator can be constructed using the spherical coordinate system, where θ is the angle relative to z-axis and f is the angle relative to the x-axis.

$$S(\theta, \phi) := \cos(\phi) \cdot \sin(\theta) \cdot S_{x} + \sin(\phi) \cdot \sin(\theta) \cdot S_{y} + \cos(\theta) \cdot S_{z}$$

To confirm the validity of this general operator, we generate the traditional x-, y- and z-direction spin operators.

$$S\left(\frac{\pi}{2}, 0\right) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad S\left(\frac{\pi}{2}, \frac{\pi}{2}\right) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad S(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

For the Hadamard gate (operator), which is important in quantum computing, $\theta = \frac{\pi}{4}$ and $\phi = 0$.

$$S\left(\frac{\pi}{4}, 0\right) = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

As shown below it represents a Fourier transform between the x and z spin eigenstates.

$$S\left(\frac{\pi}{4}, 0\right) S_{zu} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad S\left(\frac{\pi}{4}, 0\right) S_{xu} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$S\left(\frac{\pi}{4}, 0\right) S_{zd} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \quad S\left(\frac{\pi}{4}, 0\right) S_{xd} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$