8.96: Grover's Search Algorithm- Four-Card Monte

Grover's search algorithm is great at playing four-card monte. As the following quantum circuit shows it can determine which card is the queen in one pass.

$$\begin{array}{ccccccccccc} |0\rangle & H & \triangleright & \lceil & & \rceil & H & \lceil & & \rceil & H & \triangleright & \text{Measure} \\ & & & | & \text{Oracle} & | & & | & J & | & & & \\ |0\rangle & H & \triangleright & \lfloor & & \rfloor & H & \lfloor & & \rfloor & H & \triangleright & \text{Measure} \end{array} \quad \text{where} \quad \begin{array}{ccc} \lceil & & \rceil \\ | & J & | \\ \lfloor & & \rfloor \end{array} = \begin{array}{ccc} X & \cdot & X \\ & | & \\ X & \boxed{\text{Z}} & X \end{array}$$

The following matrix operators are required to construct the circuit. Giving |10> a negative phase in the Oracle designates it as the queen. The Appendix shows the calculation of J as shown on the right side above.

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{HH} = \text{kronecker(H, H)} \quad \text{Oracle} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Operating on the input state, which creates a superposition of all queries, enables the algorithm to identify which card is the queen in one operation of the circuit.

$$\text{GroverSearch} = \text{HH J HH Oracle HH} \qquad \text{GroverSearch}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -|10\rangle$$

Now the operation of the algorithm is carried out in stages to show the importance of constructive and destructive interference in quantum computers.

$$\text{Step 1} \qquad \text{HH}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \frac{1}{2}[|00\rangle + |01\rangle + |10\rangle + |11\rangle]$$

$$\text{Step 2} \qquad \text{Oracle}\begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{pmatrix} = \frac{1}{2}[|00\rangle + |01\rangle - |10\rangle + |11\rangle]$$

$$\text{Step 3} \qquad \text{HH}\begin{pmatrix} 0.5 \\ 0.5 \\ -0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \frac{1}{2}[|00\rangle - |01\rangle + |10\rangle + |11\rangle]$$

$$\text{Step 4} \qquad \text{J}\begin{pmatrix} 0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \frac{1}{2}[-|00\rangle - |01\rangle + |10\rangle + |11\rangle]$$

$$\text{Step 5} \qquad \text{HH}\begin{pmatrix} -0.5 \\ -0.5 \\ 0.5 \\ 0.5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} = -|10\rangle$$

Appendix

$$\begin{array}{ccc} \lceil & & \rceil \\ | & J & | \\ \lfloor & & \rfloor \end{array} = \begin{array}{ccc} X & \cdot & X \\ & | & \\ X & \boxed{\text{Z}} & X \end{array}$$

$$J = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \mathrm{CZ} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad \mathrm{kronecker(X, X)\,CZ\,kronecker(X, X)} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$