1.110: The Gram-Schmidt Procedure

In this exercise the Gram-Schmidt method will be used to create an orthonormal basis set from the following vectors which are neither normalized nor orthogonal.

$$u1 = \begin{pmatrix} 1+i \\ 1 \\ i \end{pmatrix} \quad u2 = \begin{pmatrix} i \\ 3 \\ 1 \end{pmatrix} \quad u3 = \begin{pmatrix} 0 \\ 28 \\ 0 \end{pmatrix}$$

Demonstrate that the vectors are not normalized and are not orthogonal.

$$\left(\overline{u1}\right)^T u1 = 4 \quad \left(\overline{u2}\right)^T u2 = 11 \quad \left(\overline{u3}\right)^T u3 = 784$$
$$\left(\overline{u1}\right)^T u2 = 4 \quad \left(\overline{u1}\right)^T u3 = 28 \quad \left(\overline{u2}\right)^T u3 = 84$$

Using the first vector make u2 orthogonal to it by subtracting its projection on u1.

$$u2 = u2 - \frac{\left(\overline{u1}\right)^T u2}{\left(\overline{u1}\right)^T u1} u1$$

Make u3 orthogonal to u1 and u2 by subtracting its projection on u1 and u2.

$$u3 = u3 - \frac{\left(\overline{u1}\right)^T u3}{\left(\overline{u1}\right)^T u1} u1 - \frac{\left(\overline{u2}\right)^T u3}{\left(\overline{u2}\right)^T u2} u2$$

Finally, normalize the new orthogonal vectors.

$$u1 = \frac{u1}{\sqrt{\left(\overline{u1}\right)^T u1}} \quad u2 = \frac{u2}{\sqrt{\left(\overline{u2}\right)^T u2}} \quad u3 = \frac{u3}{\sqrt{\left(\overline{u3}\right)^T u3}}$$

Demonstrate that an orthonormal basis set has been created.

$$\left(\overline{u1}\right)^T u1 = 1 \quad \left(\overline{u2}\right)^T u2 = 1 \quad \left(\overline{u3}\right)^T u3 = 1$$
$$\left(\overline{u1}\right)^T u2 = 0 \quad \left(\overline{u1}\right)^T u3 = 0 \quad \left(\overline{u2}\right)^T u3 = 0$$

Display the orthonormal basis set.

$$u1 = \begin{pmatrix} 0.5+0.5i \\ 0.5 \\ 0.5i \end{pmatrix} \quad u2 = \begin{pmatrix} -0.378 \\ 0.756 \\ 0.378-0.378i \end{pmatrix} \quad u3 = \begin{pmatrix} 0.085-0.592i \\ 0.423 \\ -0.676+0.085i \end{pmatrix}$$