10.31: Momentum-Space Variation Method for Particle in a Gravitational Field

The following problem deals with a particle of unit mass in a gravitational field with acceleration due to gravity equal to 1.

Energy operator for particles near Earth's surface:

$$\frac{-1}{2\mu}\frac{d^2}{dz^2}\blacksquare + z\blacksquare$$

Trial wave function:

$$\Psi(\alpha, z) := 2\alpha^{\frac{3}{2}} z exp(-\alpha z)$$

Fourier the position wave function into momentum space:

$$\Phi(\alpha, p) := \frac{1}{\sqrt{2\pi}}\int_0^\infty exp(-1pz)\Psi(\alpha, z)dz|_{simplify}^{assume,\ \alpha>0} \rightarrow \frac{2^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\frac{\alpha^{\frac{3}{2}}}{(ip+a)^2}$$

Demonstrate that the momentum wave function is normalized.

$$\int_{-\infty}^{\infty}\overline{\Phi(\alpha, p)}\Phi(\alpha, p)dp\ \ assume,\ \alpha > 0 \rightarrow 1$$

Energy operator in momentum space:

$$\frac{p^2}{2}\blacksquare + i\frac{d}{dp}\blacksquare$$

Evaluate the variational expression for the energy:

$$E(\alpha) := \int_{-\infty}^{\infty}\overline{\Phi(\alpha, p)}\frac{p^2}{2}\Phi(\alpha, p)dp ... + \int_{-\infty}^{\infty}\overline{\Phi(\alpha, p)}i(\frac{d}{dp}\Phi(\alpha, p))dp$$

Minimize energy with respect to variational parameter α:

α := 1 α := Minimize (E, α α = 1.145 E(α) = 1.966

This momentum space result is in exact agreement with the coordinate-space result. The exact value for the energy is 1.856.

$$\frac{E(\alpha) - 1.856}{1.856} = 5.9$$