10.8: Variation Method for the Quartic Oscillator

Approximate Methods: The Quartic Oscillator

For unit mass the quartic oscillator has the following energy operator in atomic units.

$$H = -\frac{1}{2}\frac{d^2}{dx^2}\blacksquare + kx^4\blacksquare \quad \int_{-\infty}^{\infty}\blacksquare dx$$

Suggested trial wavefunction: $\psi(x;\beta) := (\frac{2\beta}{\pi})^{\frac{1}{4}} exp(-\beta x^2)$

Demonstrate that the wavefunction is normalized.

$$\int_{-\infty}^{\infty}\psi(x;\beta)^2 dx \; assume, \; \beta > 0 \rightarrow 1$$

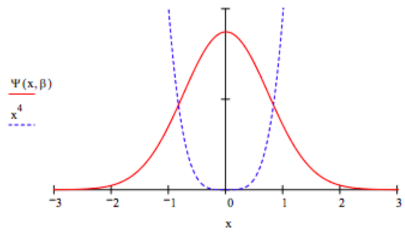
Evaluate the variational energy integral.

$$E(\beta) := \int_{-\infty}^{\infty}\psi(x,\beta) - \frac{1}{2}\frac{d^2}{dx^2}\psi(x,\beta)dx + \int_{-\infty}^{\infty}\psi(x,\beta)x^4\psi(x,\beta)dx|^{assume,\, \beta>0}_{simplify} \rightarrow \frac{1}{16}\frac{8\beta^3}{\beta^2}$$

Minimize the energy with respect to the variational parameter β and report its optimum value and the ground-state energy.

β := 1 β := Minimize(E, β) β = 0.90856 E(β) = 0.68142

Plot the optimum wavefunction and the potential energy on the same graph.



Calculate the classical turning point and the probability that tunneling is occurring.

$$x_{ctp} = 0.68142^{\frac{1}{4}} \tag{10.8.1}$$

$$= 0.90856 \tag{10.8.2}$$

$$2\int_{x_{ctp}}^{\infty}\psi(x,\beta)^2 dx \approx 0.083265$$

Compare the variational result to energy obtained by numerically integrating Schrödinger's equation for the quartic oscillator using the numerical integration algorithm provided below.

Numerical Solutions for Schrödinger's Equation

Integration limit: x_{max} := 3 Effective mass: μ := 1 Force constant: k := 1

Potential energy: $V(x) := kx^4$

Numerical integration of Schrödinger's equation:

Given

$$\frac{-1}{2\mu}\frac{d^2}{dx^2}\Phi(x) + V(x)\Phi(x) = energy\Phi(x)$$

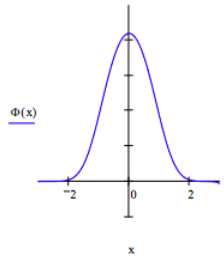
$$\Phi(-x_{max} = 0$$

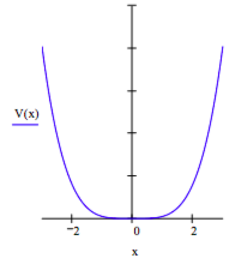
$$\Phi'(-x_{max} = 0.1$$

$$\Phi := Odesolve(x, x_{max}$$

Normalize wavefunction: $\Phi(x) := \frac{\Phi(x)}{\sqrt{\int_{-x_{max}}^{x_{max}} \Phi(x)^2 dx}}$

Enter energy guess: Energy = 0.6679864





Compare the variational and numerical solutions for the quartic oscillator by putting them on the same graph.