8.22: Elements of Reality- Another GHZ Gedanken Experiment Analyzed

Twenty years ago N. David Mermin published two articles (***Physics Today***, June 1990; ***American Journal of Physics***, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (***American Journal of Physics***, December 1990; ***Nature***, 3 February 2000) gedanken experiment involving spins that sharply revealed the clash between local realism and the quantum view of reality. In what follows I present Mermin's gedanken experiment using tensor algebra.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of h/4Ψ with spin operators in the x- and y-directions.

The z-basis eigenfunctions are:

$$Sz_{up} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Sz_{down} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The x- and y-direction spin operators and eigenvalues:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{eigenvals}(\sigma_x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \quad \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \text{eigenvals}(\sigma_y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

The initial spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The Appendix shows how to carry out vector tensor products in Mathcad.

The following operators represent the actual measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \quad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The three operators in tensor format are formed as follows.

$$\sigma_{xyy} = \text{kronecker}(\sigma_x, \text{kronecker}(\sigma_y,\ \sigma_y)) \quad \sigma_{xyy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{yxy} = \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_x,\ \sigma_y)) \quad \sigma_{yxy} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\sigma_{yyx} = \text{kronecker}(\sigma_y, \text{kronecker}(\sigma_y,\ \sigma_x)) \quad \sigma_{yyx} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

That the initial state is an eigenfunction of these operators with eigenvalue +1 is now demonstrated.

$$\Psi^T \sigma_{xyy} \Psi = 1 \quad \Psi^T \sigma_{yxy} \Psi = 1 \quad \Psi^T \sigma_{yyx} \Psi = 1$$

The fact that the operators commute means that they can have simultaneous eigenvalues.

$$\sigma_{xyy}\sigma_{yxy} - \sigma_{yxy}\sigma_{xyy} \rightarrow 0 \quad \sigma_{xyy}\sigma_{yyx} - \sigma_{yyx}\sigma_{xyy} \rightarrow 0 \quad \sigma_{yxy}\sigma_{yyx} - \sigma_{yyx}\sigma_{yxy} \rightarrow 0$$

The significance of this is evident when we consider the eigenvalue of the product of the three operators which obviously must be +1.

$$\left(\sigma_x^1 \sigma_y^2 \sigma_y^3\right)\left(\sigma_y^1 \sigma_x^2 \sigma_y^3\right)\left(\sigma_y^1 \sigma_y^2 \sigma_x^3\right) = 1 \quad \Psi^T \sigma_{xyy}\sigma_{yxy}\sigma_{yyx} \Psi = 1$$

If it is assumed that this result occurs because the particles are in well-defined spin states (s_x, s_y) prior to measurement the following must be accepted,

$$\left(s_x^1 {\color{blue}s_y^2} {\color{green}s_y^3}\right)\left({\color{red}s_y^1} s_x^2 {\color{green}s_y^3}\right)\left({\color{red}s_y^1} {\color{blue}s_y^2} s_x^3\right) = 1$$

Given that the spin measurement values can be +/- 1 and that the individual operators can have simultaneous eigenvalues, the following must be true,

$$s_y^1 s_y^1 = s_y^2 s_y^2 = s_y^3 s_y^3 = 1$$

This reduces the previous equation involving the three operators to the following local realistic prediction.

$$\left(s_x^1 s_x^2 s_x^3\right) = \left(\sigma_x^1 \sigma_x^2 \sigma_x^3\right) = 1$$

The disagreement between quantum mechanics and the local realistic view becomes starkly apparent when the $\sigma_x^1\sigma_x^2\sigma_x^3$ measurement outcome is calculated.

$$\sigma_{xxx} = \text{kronecker}(\sigma_x,\ \text{kronecker}(\sigma_x,\ \sigma_x)) \quad \Psi^T \sigma_{xxx} \Psi = -1$$

Thus we see absolute disagreement between local realism and quantum mechanics. Local realism predicts an eigenvalue of +1 and quantum mechanics an eigenvalue of -1.

The validity of the reasoning above requires that $\sigma_x^{\ 1}\sigma_x^{\ 2}\sigma_x^{\ 3}$ commutes with the other operators, which we now demonstrate.

$$\sigma_{xyy}\sigma_{xxx} - \sigma_{xxx}\sigma_{xyy} \rightarrow 0 \quad \sigma_{yxy}\sigma_{xxx} - \sigma_{xxx}\sigma_{yxy} \rightarrow 0 \quad \sigma_{yyx}\sigma_{xxx} - \sigma_{xxx}\sigma_{yyx} \rightarrow 0$$

Appendix

The tensor product of three vectors is shown below.

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} \otimes \begin{pmatrix} e \\ f \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} ce \\ cf \\ de \\ df \end{pmatrix} = \begin{pmatrix} ace \\ acf \\ ade \\ adf \\ bce \\ bcf \\ bde \\ bdf \end{pmatrix}$$

Mathcad does not have a command for the vector tensor product, so it is necessary to develop a way of implementing it using kronecker, which requires square matrices. For this reason the spin vector is stored in the left column of a 2x2 matrix by augmenting the spin vector with the null vector. After all the matrix tensor products have been carried out using kronecker the final spin vector resides in the left column of the final square matrix. Next the submatrix cammand is used to save this column, discarding the rest of the matrix.

The Mathcad syntax for the tensor multiplication of three vectors is as follows.

$$\Psi(a,\ b,\ c) = \text{submatrix}\left[\text{kronecker}\left[\text{augment}\left[a,\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right],\ \text{kronecker}\left[\text{augment}\left[b,\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right],\ \text{augment}\left[c,\ \begin{pmatrix} 0 \\ 0 \end{pmatrix}\right]\right]\right],1,\ 8,\ 1,\ 1\right]$$

The initial spin state:

$$\frac{1}{\sqrt{2}}\left(\Psi\left(Sz_{up},\ Sz_{up},\ Sz_{up}\right) - \Psi\left(Sz_{down},\ Sz_{down},\ Sz_{down}\right)\right) = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \end{pmatrix}$$