10.9: Momentum-Space Variation Method for the Quartic Oscillator

For unit mass the quartic oscillator has the following energy operator in atomic units in coordinate space.

$$H = -\frac{1}{2}\frac{d^2}{dx^2}$$

Suggested trial wavefunction:

$$\psi(x,\beta) = \left(\frac{2\beta}{\pi}\right)^{\frac{1}{4}} exp(-\beta x^2)$$

Demonstrate that the wavefunction is normalized.

$$\int_{-\infty}^{\infty} \psi(x,\beta)^2 dx \; assume, \beta > 0 \rightarrow 1$$

Fourier transform the coordinate wavefunction into the momentum representation.

$$\Phi(p,\beta) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} exp(-ipx)\psi(x,\beta)dx \Big|_{simplify}^{assume,\, \beta>1} \rightarrow \frac{1}{2}\frac{2^{\frac{3}{4}}}{\pi^{\frac{1}{4}}}\frac{e^{\frac{-1}{4}\frac{p^2}{\beta}}}{\beta^{\frac{1}{4}}}$$

Demonstrate that the momentum wavefunction is normalized.

$$\int_{-\infty}^{\infty} \overline{\Phi(p,\beta)}\Phi(p,\beta)dp assume,\ \beta > 0 \rightarrow 1$$

The quartic oscillator energy operator in momentum space:

$$H = \frac{p^2}{2}\blacksquare + \frac{d^4}{dp^4}\blacksquare$$

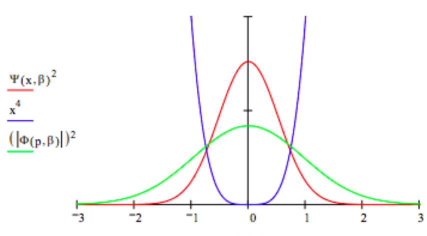
Evaluate the variational energy integral.

$$E(\beta) = \int_{-\infty}^{\infty} \overline{\Phi(p,\beta)}\frac{p^2}{2}\Phi(p,\beta)dp + \int_{-\infty}^{\infty} \overline{\Phi(p,\beta)}\frac{d^4}{dp^4}\Phi(p,\beta)dp \Big|_{simplify}^{assume,\, \beta>0} \rightarrow \frac{1}{16}\frac{8\beta^3+3}{\beta^2}$$

Minimize the energy with respect to the variational parameter β and report its optimum value and the ground-state energy.

$$\beta = 1 \;\; \beta = Minimize(E,\beta) \;\; \beta = 0.90856 \;\; E(\beta) = 0.688142$$

Plot the coordinate and momentum wavefunctions and the potential energy on the same graph.



These results demonstrate the uncertainty principle. For the harmonic potential, $x^2/2$, the coordinate and momentum wavefunctions are identical. Compared to the harmonic potential the quartic potential, x^4, constrains the spatial wavefunction leading to less uncertainty in position. The uncertainty principle, therefore, requires an increase in the momentum uncertainty. This is clearly revealed in graph above.