2.35: Addition of Spin Angular Momentum- A Tensor Algebra Approach

In this entry I work through section 4.4.3 of David Griffiths' *Introduction to Quantum Mechanics* (2nd ed.) in which he treats the addition of angular momentum for two identical spin-1/2 particles. The tensor algebra approach is illustrated.

The four spin states of two spin-1/2 particles are written below in the spin-z basis in tensor format.

$$|\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix} \quad |\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}$$

$$|\uparrow\uparrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad |\uparrow\downarrow\rangle = \begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad |\downarrow\uparrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad |\downarrow\downarrow\rangle = \begin{pmatrix}0\\1\end{pmatrix}\otimes\begin{pmatrix}0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The two middle states are not permissible because they distinguish between identical particles. The solution is to form symmetric and anti-symmetric superpositions of them, but let's follow Griffith's approach for the time being. The four spin states are labeled as shown below.

$$\mathrm{a} = \begin{pmatrix}1\\0\\0\\0\end{pmatrix} \quad \mathrm{b} = \begin{pmatrix}0\\1\\0\\0\end{pmatrix} \quad \mathrm{c} = \begin{pmatrix}0\\0\\1\\0\end{pmatrix} \quad \mathrm{d} = \begin{pmatrix}0\\0\\0\\1\end{pmatrix}$$

The identity and spin operators in units of h/2π are now defined. The final two are the up and down spin ladder operators.

$$I = \begin{pmatrix}1&0\\0&1\end{pmatrix} \quad S_x = \tfrac{1}{2}\begin{pmatrix}0&1\\1&0\end{pmatrix} \quad S_y = \tfrac{1}{2}\begin{pmatrix}0&-i\\i&0\end{pmatrix} \quad S_z = \tfrac{1}{2}\begin{pmatrix}1&0\\0&-1\end{pmatrix} \quad S = S_x + S_y + S_z$$

$$S_u = \begin{pmatrix}0&1\\0&0\end{pmatrix} \quad S_d = \begin{pmatrix}0&0\\1&0\end{pmatrix}$$

Next the total spin operator and total spin operator in the z-direction are defined using kronecker, Mathcad's command for tensor matrix multiplication.

$$S_{tot} = \mathrm{kronecker}(\mathrm{S},\mathrm{I}) + \mathrm{kronecker}(\mathrm{I},\mathrm{S}) \qquad Sz_{tot} = \mathrm{kronecker}\left(S_z,\ I\right) + \mathrm{kronecker}\left(I,\ S_z\right)$$

Calculation of the expectation values for total spin in the z-direction for spin states a, b, c and d reveals the problem mentioned above. For S = 1 we expect S_z values of -1, 0 and 1. The extra value for Sz = 0 indicates an interpretive problem.

$$\mathrm{a}^T Sz_{tot}\mathrm{a} = 1 \quad \mathrm{b}^T Sz_{tot}\mathrm{b} = 0 \quad \mathrm{c}^T Sz_{tot}\mathrm{c} = 0 \quad \mathrm{d}^T Sz_{tot}\mathrm{d} = -1$$

Griffiths solves the problem by operating with the lowering operator on spin state a, which yields an in-phase superposition (unnormalized) of spin states b and c.

$$\left(\mathrm{kronecker}\left(S_d,\ I\right) + \mathrm{kronecker}\left(I,\ S_d\right)\right)\begin{pmatrix}1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\\0\end{pmatrix}$$

Repeating with the result from above yields an unnormalized d spin state.

$$\left(\mathrm{kronecker}\left(S_d,\ I\right) + \mathrm{kronecker}\left(I,\ S_d\right)\right)\begin{pmatrix}0\\1\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\2\end{pmatrix}$$

Operating on the unnormalized d spin state yield a null vector suggesting that the original spin states might be reformulated as triplet and singlet states.

$$\left(\mathrm{kronecker}\left(S_d,\ I\right) + \mathrm{kronecker}\left(I,\ S_d\right)\right)\begin{pmatrix}0\\0\\0\\2\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0\end{pmatrix}$$

Given this hint and the fact that the initial spin states are orthonormal, we preserve this property in the new spin states by constructing an out-of-phase superposition of b and c. This give us a revised set of orthonormal spin vectors. In conventional notation these states are |11 >, |10 >, |1-1 > and |00 >, where the first number is total spin value and the second the spin in the z-direction. Thus, we have a triplet and a singlet state as will be confirmed below.

$$\Psi_{1p1} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \Psi_{10} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{pmatrix} \quad \Psi_{1m1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \quad \Psi_{00} = \begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \\ 0 \end{pmatrix}$$

We now establish by calculation of expectation values for S^2 (constructed below) and S_z that the first three spin states are members of a triplet state (S = 1) and the final spin state is a singlet (S = 0). The eigenvalues for the S^2 operator are S(S + 1) which is 2 for the triplet state and 0 for the singlet state.

$$\mathrm{SS} = \mathrm{kronecker}\left(S^2,\ I\right) + \mathrm{kronecker}\left(I,\ S^2\right) + 2\mathrm{kronecker}\left(S_x,\ S_x\right) + \mathrm{kronecker}\left(S_y,\ S_y\right) + \mathrm{kronecker}\left(S_z,\ S_z\right)$$

$$\left(\Psi_{1p1}\right)^T \mathrm{SS}\Psi_{1p1} = 2 \quad \left(\Psi_{10}\right)^T \mathrm{SS}\Psi_{10} = 2 \quad \left(\Psi_{1m1}\right)^T \mathrm{SS}\Psi_{1m1} = 2 \quad \left(\Psi_{00}\right)^T \mathrm{SS}\Psi_{00} = 0$$

$$\left(\Psi_{1p1}\right)^T Sz_{tot}\Psi_{1p1} = 1 \quad \left(\Psi_{10}\right)^T Sz_{tot}\Psi_{10} = 0 \quad \left(\Psi_{1m1}\right)^T Sz_{tot}\Psi_{1m1} = -1 \quad \left(\Psi_{00}\right)^T Sz_{tot}\Psi_{00} = 0$$

Next, we look at this problem from the energy perspective and use the spin-spin interaction Hamiltonian to calculate the energy eigenvalues and eigenvectors for the two spin-1/2 particle system

$$H_{SpinSpin} = \mathrm{kronecker}\left(S_x,\ S_x\right) + \mathrm{kronecker}\left(S_y,\ S_y\right) + \mathrm{kronecker}\left(S_z,\ S_z\right)$$

$$H_{SpinSpin} = \begin{pmatrix} 0.25 & 0 & 0 & 0 \\ 0 & -0.25 & 0.5 & 0 \\ 0 & 0.5 & -0.25 & 0 \\ 0 & 0 & 0 & 0.25 \end{pmatrix}$$

We now ask Mathcad to calculate the eigenvalues and eigenvectors of the spin-spin operator. These results are displayed by constructing a matrix which contains the eigenvalues in the top row, and their eigenvalues in the columns below the eigenvalues.

$$i = 1..4 \quad \mathrm{E} = \mathrm{eigenvals}\left(H_{SpinSpin}\right) \quad \mathrm{EigenvalsEigenvec} = \mathrm{rsort}\left(\mathrm{stack}\left(E^T,\ \mathrm{eigenvecs}\left(H_{SpinSpin}\right)\right),\ 1\right)$$

$$\mathrm{EigenvalEigenvec} = \begin{pmatrix} -0.75 & 0.25 & 0.25 & 0.25 \\ 0 & 1 & 0 & 0 \\ 0.707 & 0 & 0 & 0.707 \\ -0.707 & 0 & 0 & 0.707 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

These calculations are consistent with those that preceded and with the results presented on page 284 of Griffiths' text.