8.18: Another Simulation of a GHZ Gedanken Experiment

Many years ago N. David Mermin published two articles (Physics Today, June 1990; American Journal of Physics, August 1990) in the general physics literature on a Greenberger-Horne-Zeilinger (American Journal of Physics, December 1990; Nature, 3 February 2000) thought experiment involving spins that sharply revealed the clash between local realism and the quantum view of reality.

Three spin-1/2 particles are created in a single event and move apart in the horizontal y-z plane. Subsequent spin measurements will be carried out in units of h/4π with spin operators in the x- and y-directions. The z-basis eigenfunctions are:

$$Sz_{up} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad Sz_{down} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

The x- and y-direction spin operators:

\[\begin{matrix} \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} & \text{eigenvals} (\sigma_x) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} & \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} & \text{eigenvals} (\sigma_y) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \end{matrix}\

The initial entangeld spin state for the three spin-1/2 particles in tensor notation is:

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix} \qquad \Psi = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -1 \end{pmatrix}$$

The following operators represent the measurements to be carried out on spins 1, 2 and 3, in that order.

$$\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3 \qquad \sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3 \qquad \sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3$$

The matrix tensor product is also known as the Kronecker product, which is available in Mathcad. The four operators in tensor format are formed as follows.

$$\begin{matrix} \sigma_{xyy} = \text{kronecker}(\sigma_x,\ \text{kronecker}(\sigma_y,\ \sigma_y)) & \sigma_{yxy} = \text{kronecker}(\sigma_y,\ \text{kronecker}(\sigma_x,\ \sigma_y)) \\ \sigma_{yyx} = \text{kronecker}(\sigma_y,\ \text{kronecker}(\sigma_y,\ \sigma_x)) & \sigma_{xxx} = \text{kronecker}(\sigma_x,\ \text{kronecker}(\sigma_x,\ \sigma_x)) \end{matrix}$$

These composite operators are Hermitian and mutually commute which means they can have simultaneous eigenvalues.

$$\begin{matrix} \sigma_{xyy}\sigma_{yxy} - \sigma_{yxy}\sigma_{xyy} \rightarrow 0 & \sigma_{xyy}\sigma_{yyx} - \sigma_{yyx}\sigma_{xyy} \rightarrow 0 & \sigma_{xyy}\sigma_{xxx} - \sigma_{xxx}\sigma_{xyy} \rightarrow 0 \\ \sigma_{yxy}\sigma_{yyx} - \sigma_{yyx}\sigma_{yxy} \rightarrow 0 & \sigma_{yxy}\sigma_{xxx} - \sigma_{xxx}\sigma_{yxy} \rightarrow 0 & \sigma_{yyx}\sigma_{xxx} - \sigma_{xxx}\sigma_{yyx} \rightarrow 0 \end{matrix}$$

The expectation values of the operators are now calculated.

$$\Psi^T\sigma_{xyy}\Psi = 1 \quad \Psi^T\sigma_{yxy}\Psi = 1 \quad \Psi^T\sigma_{yyx}\Psi = 1 \quad \Psi^T\sigma_{xxx}\Psi = 1$$

Consequently the product of the four operators has the expectation value of -1.

$$\Psi^T\sigma_{xyy}\sigma_{yxy}\sigma_{yyx}\sigma_{xxx}\Psi = -1$$

Local realism assumes that objects have definite properties independent of measurement. In this example it assumes that the x- and y-components of the spin have definite values prior to measurement. This position leads to a contradiction with the above result as demonstrated by Mermin (Physics Today, June 1990). Looking again at the measurement operators, notice that there is a σx measurement on the first spin in the first and fourth experiment. If the spin state is well-defined before measurement those results have to be the same, either both +1 or both -1, so that the product of the two measurements is +1.

$$\left(\sigma_x^1 \otimes \sigma_y^2 \otimes \sigma_y^3\right)\left(\sigma_y^1 \otimes \sigma_x^2 \otimes \sigma_y^3\right)\left(\sigma_y^1 \otimes \sigma_y^2 \otimes \sigma_x^3\right)\left(\sigma_x^1 \otimes \sigma_x^2 \otimes \sigma_x^3\right)$$

Likewise there is a σ_y measurement on the second spin in experiments one and three. By similar arguments those results will lead to a product of +1 also. Continuing with all pairs in the total operator using local realistic reasoning unambiguously shows that its expectation value should be +1, in sharp disagreement with the quantum mechanical result of -1. This result should cause all mathematically literate local realists to renounce and recant their heresy. However, they may resist saying this is just a thought experiment. It hasn't actually been performed. However, if you believe in quantum simulation it has been performed.

Quantum Simulation

"Quantum simulation is a process in which a quantum computer simulates another quantum system. Because of the various types of quantum weirdness, classical computers can simulate quantum systems only in a clunky, inefficient way. But because a quantum computer is itself a quantum system, capable of exhibiting the full repertoire of quantum weirdness, it can efficiently simulate other quantum systems. **The resulting simulation can be so accurate that the behavior the computer will be indistinguishable from the behavior of the simulated system itself.**" (Seth Lloyd, Programming the Universe, page 149.) The thought experiment can be simulated using the quantum circuit shown below which is an adaptation of one that can be found at: arXiv:1712.06542v2.

$$\begin{matrix} |1\rangle & \triangleright & \mathrm{H} & \cdot & \cdots & \cdots & \mathrm{H} & \triangleright & \text{Measure, 0 or 1} \\ & & & | & & & & & \\ |0\rangle & \triangleright & \cdots & \oplus & \cdot & \mathrm{S} & \mathrm{H} & \triangleright & \text{Measure, 0 or 1} \\ & & & & | & & & & \\ |0\rangle & \triangleright & \cdots & \cdots & \oplus & \mathrm{S} & \mathrm{H} & \triangleright & \text{Measure, 0 or 1} \end{matrix}$$

The matrix operators required for the implementation of the quantum circuit:

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad S = \begin{pmatrix} 1 & 0 \\ 0 & -i \end{pmatrix} \quad \mathrm{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\begin{matrix} \mathrm{HII} = \mathrm{kronecker}(\mathrm{H}, \mathrm{kronecker}(\mathrm{I}, \mathrm{I})) & \mathrm{CNOTI} = \mathrm{kronecker}(\mathrm{CNOT}, \mathrm{I}) & \mathrm{ICNOT} = \mathrm{kronecker}(\mathrm{I}, \mathrm{CNOT}) \\ \mathrm{ISS} = \mathrm{kronecker}(\mathrm{I}, \mathrm{kronecker}(\mathrm{S}, \mathrm{S})) & \mathrm{SIS} = \mathrm{kronecker}(\mathrm{S}, \mathrm{kronecker}(\mathrm{I}, \mathrm{S})) & \mathrm{SSI} = \mathrm{kronecker}(\mathrm{S}, \mathrm{kronecker}(\mathrm{S}, \mathrm{I})) \\ \mathrm{HHH} = \mathrm{kronecker}(\mathrm{H}, \mathrm{kronecker}(\mathrm{H}, \mathrm{H})) & & \end{matrix}$$

First it is demonstrated that the first three steps of the circuit create the initial state

$$\left[\mathrm{ICNOT\ CNOTI\ HII} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T\right]^T = \begin{pmatrix} 0.707 & 0 & 0 & 0 & 0 & 0 & 0 & -0.707 \end{pmatrix}$$

The complete circuit shown above simulates the expectation value of the $\sigma_x\sigma_y\sigma_y$ operator. The presence of S on a line before the final H gates indicates the measurement of the σy, its absence a measurement of σ_x. The subsequent simulations show the absence of S on the middle and last line, and finally on all three lines for the simulation of the expectation value for $\sigma_x\sigma_x\sigma_x$.

Eigenvalue |0 > = +1; eigenvalue |1 > = -1

$$\left[\mathrm{HHH\ ISS\ ICNOT\ CNOT\ HII} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T\right]^T = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \Rightarrow \langle\sigma_x\sigma_y\sigma_y\rangle = 1$$

Given the eigenvalue assignments above the expectation value associated with this measurement outcome is 1/4[(1)(1)(1)+(1)(-1)(-1)+(-1)(1)(-1)+(-1)(-1)(1)] = 1. Note that 1/2 is the probability amplitude for the product state. Therefore the probability of each member of the superposition being observed is 1/4. The same reasoning is used for the remaining simulations.

$$\left[\mathrm{HHH\ ISS\ ICNOT\ CNOT\ HII} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T\right]^T = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \Rightarrow \langle\sigma_y\sigma_x\sigma_y\rangle = 1$$

$$\left[\mathrm{HHH\ ISS\ ICNOT\ CNOT\ HII} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}^T\right]^T = \begin{pmatrix} 0.5 & 0 & 0 & 0.5 & 0 & 0.5 & 0.5 & 0 \end{pmatrix}$$

$$\frac{1}{2}(|000\rangle + |011\rangle + |101\rangle + |110\rangle) \Rightarrow \langle\sigma_y\sigma_y\sigma_x\rangle = 1$$

$$\left[\text{ HHH ISS ICNOT CNOT HII}\begin{pmatrix}0 & 0 & 0 & 0 & 1 & 0 & 0 & 0\end{pmatrix}^T\right]^T = \begin{pmatrix}0 & 0.5 & 0.5 & 0 & 0.5 & 0 & 0 & 0.5\end{pmatrix}$$

$$\frac{1}{2}(|001\rangle + |010\rangle + |100\rangle + |111\rangle) \Rightarrow \langle\sigma_x\sigma_x\sigma_x\rangle = -1$$

Individually and in product form the simulated results are in agreement with the previous quantum mechanical calculations.

$$\langle\sigma_x\sigma_x\sigma_x\rangle\langle\sigma_x\sigma_y\sigma_y\rangle\langle\sigma_y\sigma_x\sigma_y\rangle\langle\sigma_y\sigma_y\sigma_x\rangle = -1$$

The appendix provides algebraic calculations of $\langle\sigma_x\sigma_y\sigma_y\rangle$ and $\langle\sigma_x\sigma_x\sigma_x\rangle$.

Appendix

Truth tables for the operation of the circuit elements:

$$I = \begin{pmatrix}\text{0 to 0} \\ \text{1 to 1}\end{pmatrix} \quad H = \begin{bmatrix}\text{0 to } \frac{(0+1)}{\sqrt{2}} \\ \text{1 to } \frac{(0-1)}{\sqrt{2}}\end{bmatrix} \quad \text{CNOT} = \begin{pmatrix}\text{00 to 00} \\ \text{01 to 01} \\ \text{10 to 11} \\ \text{11 to 10}\end{pmatrix} \quad S = \begin{pmatrix}\text{0 to 0} \\ \text{1 to -i}\end{pmatrix}$$

$$\begin{array}{cc} |100\rangle & |100\rangle \\ H \otimes I \otimes I & H \otimes I \otimes I \\ \frac{1}{\sqrt{2}}[|000\rangle - |100\rangle] & \frac{1}{\sqrt{2}}[|000\rangle - |100\rangle] \\ CNOT \otimes I & CNOT \otimes I \\ \frac{1}{\sqrt{2}}[|000\rangle - |110\rangle] & \frac{1}{\sqrt{2}}[|000\rangle - |110\rangle] \\ I \otimes CNOT & I \otimes CNOT \\ \frac{1}{\sqrt{2}}[|000\rangle - |111\rangle] & \frac{1}{\sqrt{2}}[|000\rangle - |111\rangle] \\ I \otimes S \otimes S & H \otimes H \otimes H \\ \frac{1}{\sqrt{2}}[|000\rangle - |1 - i - i\rangle] & \frac{1}{\sqrt{2}}[|001\rangle + |010\rangle + |100\rangle + |111\rangle] \\ H \otimes H \otimes H & \langle\sigma_x\sigma_x\sigma_x\rangle = -1 \\ \frac{1}{2}[|000\rangle + |011\rangle + |101\rangle + |110\rangle] & \\ \langle\sigma_x\sigma_y\sigma_y\rangle = 1 & \end{array}$$