8.79: Solving Equations Using a Quantum Circuit

This tutorial demonstrates the solution of two linear simultaneous equation using a quantum circuit. The circuit is taken from arXiv:1302.1210. See this reference for details on the experimental implementation of the circuit and also for a discussion of the potential of quantum solutions for systems of equations. Two other sources (arXiv:1302.1946 and 1302.4310) provide alternative quantum circuits and methods of implementation.

First we consider the conventional method of solving systems of linear equation for a particular matrix A and three different |b> vectors.

$$A|x\rangle = b\rangle \qquad |x\rangle = A^{-1}|b\rangle$$

$$A = \begin{pmatrix} 1.5 & 0.5 \\ 0.5 & 1.5 \end{pmatrix} \qquad b_1 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad b_2 = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} \qquad b_3 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$A^{-1}b_1 = \begin{pmatrix} 0.354 \\ 0.354 \end{pmatrix} \quad A^{-1}b_2 = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \quad A^{-1}b_3 = \begin{pmatrix} 0.75 \\ -0.25 \end{pmatrix}$$

Next we show the quantum circuit (arXiv:1302.1210) that generates the same solutions. The Appendix considers two other equivalent circuits from this reference.

$$\begin{array}{ccccccc} |b\rangle & \triangleright & \boxed{\mathrm{R}} & \cdot & \boxed{\mathrm{R^T}} & \triangleright & |x\rangle \\ & & & | & & & \\ |1\rangle & \triangleright & \cdots & \boxed{\mathrm{Ry}(\theta)} & \boxed{\mathrm{M_1}} & \triangleright & |1\rangle \end{array}$$

In this circuit, R is the matrix of eigenvectors of matrix A and R^T its transpose. The last step on the bottom wire is the measurement of |1>, which is represented by the projection operator M_1. The identity operator is required for cases in which a quantum gate operation is occurring on one wire and no operation is occurring on the other wire.

$$\mathrm{R} = \backslash\mathrm{eigenvecs(A)} \quad \mathrm{R} = \begin{pmatrix} 0.707 & -0.707 \\ 0.707 & 0.707 \end{pmatrix} \quad \mathrm{R}^T = \begin{pmatrix} 0.707 & 0.707 \\ -0.707 & 0.707 \end{pmatrix} \quad \mathrm{M}_1 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \leftarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \end{pmatrix} \quad \mathrm{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The controlled rotation, CR(θ), is the only two-qubit gate in the circuit. The rotation angle required is determined by the ratio of the eigenvalues of A as shown below.

$$\mathrm{CR}(\theta) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\left(\frac{\theta}{2}\right) & -\sin\left(\frac{\theta}{2}\right) \\ 0 & 0 & \sin\left(\frac{\theta}{2}\right) & \cos\left(\frac{\theta}{2}\right) \end{pmatrix} \quad \mathrm{eigenvals(A)} = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \theta = -2\mathrm{acos}\left(\frac{1}{2}\right)$$

The input (|b>|1>) and output (|x>|1>) states are expressed in tensor format. Kronecker is Mathcad's command for the tensor product of matrices.

Input $|b\rangle|1\rangle$	Quantum Circuit	Output $|x\rangle|1\rangle$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix}$	$\mathrm{kronecker}(R^T,\ M_1)\mathrm{CR}(\theta)\mathrm{kronecker(R, I)}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.354 \\ 0 \\ 0.354 \end{pmatrix}$	$\begin{pmatrix} 0.354 \\ 0.354 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}$	$\mathrm{kronecker}(R^T,\ M_1)\mathrm{CR}(\theta)\mathrm{kronecker(R, I)}\frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.707 \\ 0 \\ -0.707 \end{pmatrix}$	$\begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$	$\mathrm{kronecker}(R^T,\ M_1)\mathrm{CR}(\theta)\mathrm{kronecker(R, I)}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.75 \\ 0 \\ -0.25 \end{pmatrix}$	$\begin{pmatrix} 0.75 \\ -0.25 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Appendix

The alternative two-wire circuit shown in Fig. 1C requires CNOT and Ry rotation matrices.

$$\begin{array}{cccccccccc} |b\rangle & \triangleright & \boxed{\mathrm{R}} & \cdot & \cdots & \cdot & \boxed{\mathrm{R^T}} & \cdots & \triangleright & |x\rangle \\ & & & | & & | & & & & \\ |1\rangle & \triangleright & \cdots & oplus & \boxed{\mathrm{Ry}(\theta/2)} & \oplus & \boxed{\mathrm{Ry}\ (\theta/2)} & \boxed{\mathrm{M_1}} & \triangleright & |1\rangle \end{array}$$

The quantum circuit is set up as follows. The input and output states are the same as the previous circuit.

$$\begin{gathered}\text{QuantumCircuit}\\ =\text{kronecker}(\mathrm{I}, \mathrm{M}_1)\text{kronecker}\left(\mathrm{R}^T,\ \mathrm{Ry}\left(\frac{\theta}{2}\right)\right)\text{CNOT kronecker}\left(\mathrm{I},\ \mathrm{Ry}\left(\frac{-\theta}{2}\right)\right)\text{CNOT kronecker}(\mathrm{R}, \mathrm{I})\end{gathered}$$

The three-wire circuit Fig. 1B produces the following transformation: $|b\rangle|0\rangle|1\rangle \rightarrow |x\rangle|0\rangle|1\rangle$.

$$\begin{array}{cccccccccc} |b\rangle & \triangleright & \boxed{\mathrm{R}} & \cdot & \cdots & \cdot & \boxed{\mathrm{R^T}} & \cdots & \triangleright & |x\rangle \\ & & & | & & | & & & & \\ |0\rangle & \triangleright & \cdots & \oplus & \cdot & \oplus & \cdots & & \triangleright & |0\rangle \\ & & & & | & & & & & \\ |1\rangle & \triangleright & \cdots & oplus & \boxed{\mathrm{Ry}(\theta/2)} & \oplus & \boxed{\mathrm{Ry}\ (\theta/2)} & \boxed{\mathrm{M_1}} & \triangleright & |1\rangle \end{array}$$

$$\begin{array}{ccc} \text{Input } |b\rangle|0\rangle|1\rangle & \text{Quantum Circuit} & \text{Output } |x\rangle|0\rangle|1\rangle \\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix} & \mathrm{QC}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\0\\1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0.354\\0\\0\\0\\0.354\\0\\0\end{pmatrix} & \begin{pmatrix}0.354\\0.354\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\0\\-1\\0\\0\end{pmatrix} & \mathrm{QC}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\0\\-1\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0.707\\0\\0\\0\\0.707\\0\\0\end{pmatrix} & \begin{pmatrix}0.707\\-0.707\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \\ \frac{1}{\sqrt{2}}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}=\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix} & \mathrm{QC}\frac{1}{\sqrt{2}}\begin{pmatrix}0\\1\\0\\0\\0\\0\\0\\0\end{pmatrix}=\begin{pmatrix}0\\0.75\\0\\0\\0\\-0.25\\0\\0\end{pmatrix} & \begin{pmatrix}0.75\\-0.25\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} \end{array}$$