8.55: Quantum Circuit for the Generation of GHZ States

The following circuit generates the eight GHZ maximally entangled states. A similar NMR circuit can be found on page 287 of *The Quest for the Quantum Computer* by Julian Brown.

$$
\begin{array}{ccccccc}
\text{Index} & & & & & & \\
|a\rangle & \triangleright & \cdots & \oplus & \cdots & \triangleright & \text{G} \\
 & & & | & & & \\
|b\rangle & \triangleright & \text{H} & \cdot & \cdot & \triangleright & \text{H} \\
 & & & & | & & \\
|c\rangle & \triangleright & \cdots & \cdots & \oplus & \triangleright & \text{Z}
\end{array}
$$

Given the quantum gates in matrix form the quantum circuit is formed using Kronecker multiplication.

$$
\text{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{H} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \quad \text{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \text{ICNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}
$$

$$
\text{GHZ} = \text{kronecker}(\text{I}, \text{CNOT})\ \text{kronecker}(\text{ICNOT}, \text{I})\ \text{kronecker}(\text{I}, \text{kronecker}(\text{H}, \text{I}))
$$

Using the index as input the quantum circuit generates the corresponding GHZ state.

$$
G_0 = \text{GHZ}\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]
$$

$$
G_1 = \text{GHZ}\begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.707 \\ 0 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right]
$$

$$
G_2 = \text{GHZ}\begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ -0.707 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]
$$

$$
\begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ \vdots \end{pmatrix} \quad \begin{pmatrix} 0 \\ 0.707 \\ 0 \\ 0 \\ \vdots \end{pmatrix}
$$

$$G_3 = \mathrm{GHZ}\begin{pmatrix}1\\0\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\-0.707\\0\end{pmatrix} \qquad \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$G_4 = \mathrm{GHZ}\begin{pmatrix}0\\0\\0\\0\\1\\0\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\0.707\\0.707\\0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

$$G_5 = \mathrm{GHZ}\begin{pmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0.707\\0\\0\\0.707\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$G_6 = \mathrm{GHZ}\begin{pmatrix}0\\0\\0\\0\\0\\0\\1\\0\end{pmatrix} = \begin{pmatrix}0\\0\\0\\-0.707\\0.707\\0\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right]$$

$$G_7 = \mathrm{GHZ}\begin{pmatrix}0\\0\\0\\0\\0\\0\\0\\1\end{pmatrix} = \begin{pmatrix}0\\0\\-0.707\\0\\0\\0.707\\0\\0\end{pmatrix} \qquad \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{\sqrt{2}}\left[\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix} - \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right]$$

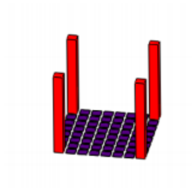
Given the following truth tables the operation of the circuit is followed algebraically for G_2 and G_4.

$$\text{Identity} \quad \begin{pmatrix}0 \text{ to } 0\\1 \text{ to } 1\end{pmatrix} \qquad \text{Hadamard} \quad \mathrm{H} = \begin{bmatrix}0 \text{ to } \frac{(0+1)}{\sqrt{2}} \text{ to } 0\\1 \text{ to } \frac{(0-1)}{\sqrt{2}} \text{ to } 1\end{bmatrix}$$

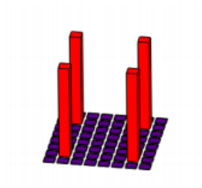
$$\text{ICNOT} \begin{pmatrix}\text{Decimal} & \text{Binary} & \text{to} & \text{Binary} & \text{Decimal}\\0 & 00 & \text{to} & 00 & 0\\1 & 01 & \text{to} & 11 & 3\\2 & 10 & \text{to} & 10 & 2\\3 & 11 & \text{to} & 01 & 1\end{pmatrix} \qquad \text{CNOT} \begin{pmatrix}\text{Decimal} & \text{Binary} & \text{to} & \text{Binary} & \text{Decimal}\\0 & 00 & \text{to} & 00 & 0\\1 & 01 & \text{to} & 01 & 1\\2 & 10 & \text{to} & 11 & 3\\3 & 11 & \text{to} & 10 & 2\end{pmatrix}$$

$$\begin{array}{cc} |010\rangle & |100\rangle \\ \mathrm{I} \otimes \mathrm{H} \otimes \mathrm{I} & \mathrm{I} \otimes \mathrm{H} \otimes \mathrm{I} \\ |0\rangle \left(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\right) |0\rangle = \frac{1}{\sqrt{2}}[|000\rangle - |010\rangle] & |1\rangle \left(\frac{|0\rangle + |1\rangle}{\sqrt{2}}\right) |0\rangle = \frac{1}{\sqrt{2}}[|100\rangle + |110\rangle] \\ \mathrm{ICNOT} \otimes \mathrm{I} & \mathrm{ICNOT} \otimes \mathrm{I} \\ \frac{1}{\sqrt{2}}[|000\rangle - |110\rangle] & \frac{1}{\sqrt{2}}[|100\rangle + |010\rangle] \\ \mathrm{I} \otimes \mathrm{CNOT} & \mathrm{I} \otimes \mathrm{CNOT} \\ \frac{1}{\sqrt{2}}[|000\rangle - |111\rangle] & \frac{1}{\sqrt{2}}[|100\rangle + |011\rangle] \end{array}$$

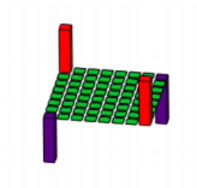
Next the GHZ density matrices are presented graphically.



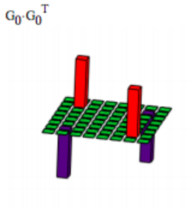
$G_0 \cdot G_0^T$



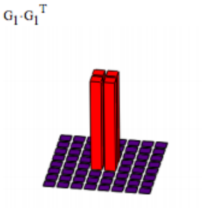
$G_1 \cdot G_1^T$



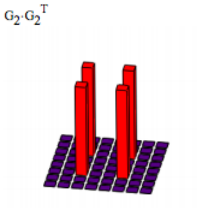
$G_2 \cdot G_2^T$



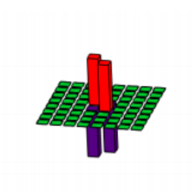
$G_3 \cdot G_3^T$



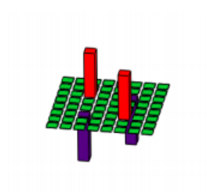
$G_4 \cdot G_4^T$



$G_5 \cdot G_5^T$



$G_6 \cdot G_6^T$



$G_7 \cdot G_7^T$

Using the GHZ states as input and running the circuit in reverse yields the indices.

$$\mathrm{IGHZ} = \mathrm{GHZ}^{-}1$$

$$\mathrm{IGHZ\ G_0} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{IGHZ\ G_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{IGHZ\ G_2} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{IGHZ\ G_3} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\mathrm{IGHZ\ G_4} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{IGHZ\ G_5} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad \mathrm{IGHZ\ G_6} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \mathrm{IGHZ\ G_7} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$