1.33: Basic Matrix Mechanics

A quon (an entity that exhibits both wave and particle aspects in the peculiar quantum manner - Nick Herbert, *Quantum Reality*, page 64) has a variety of properties each of which can take on two values. For example, it has the property of **hardness** and can be either *hard* or *soft*. It also has the property of **color** and can be either *black* or *white*, and the property of **taste** and be *sweet* or *sour*. The treatment that follows draws on material from Chapter 3 of David Z Albert's book, *Quantum Mechanics and Experience*.

The basic principles of matrix and vector math are provided in Appendix A. An examination of this material will demonstrate that most of the calculations presented in this tutorial can easily be performed without the aid of Mathcad or any other computer algebra program. In other words, they can be done by hand.

In the matrix formulation of quantum mechanics the hardness and color states are represented by the following vectors.

$$\mathrm{Hard} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \mathrm{Soft} := \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \mathrm{Black} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} \quad \mathrm{White} := \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$$

Hard and *Soft* represent an orthonormal basis in the two-dimensional **Hardness** vector space.

Likewise *Black* and *White* are an orthonormal basis in the two-dimensional **Color** vector space.

$$\mathrm{Black}^T \cdot \mathrm{Black} = 1 \qquad \mathrm{White}^T \cdot \mathrm{White} = 1 \qquad \mathrm{Black}^T \cdot \mathrm{White} = 0$$

$$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 1 \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 1 \quad \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0$$

The relationship between the two bases is reflected in the following projection calculations.

Note

$$\frac{1}{\sqrt{2}} = 0.707$$

$$\mathrm{Hard}^T \cdot \mathrm{Black} = 0.707 \qquad \mathrm{Hard}^T \cdot \mathrm{White} = 0.707 \qquad \mathrm{Soft}^T \cdot \mathrm{Black} = 0.707 \qquad \mathrm{Soft}^T \cdot \mathrm{White} = -0.707$$

$$\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0.707 \quad \begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = 0.707 \quad \begin{pmatrix} 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = 0.707 \quad \begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = -0.707$$

The values calculated above are probability amplitudes. The absolute square of those values is the probability. In other words, the probability that a black quon will be found to be hard is 0.5. The probability that a white quon will be found to be soft is also 0.5.

$$\left(\left|\mathrm{Hard}^T \cdot \mathrm{Black}\right|\right)^2 = 0.5 \qquad \left(\left|\mathrm{Hard}^T \cdot \mathrm{White}\right|\right)^2 = 0.5 \qquad \left(\left|\mathrm{Soft}^T \cdot \mathrm{Black}\right|\right)^2 = 0.5 \qquad \left(\left|\mathrm{Soft}^T \cdot \mathrm{White}\right|\right)^2 = 0.5$$

$$\left[\left|\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|\right]^2 = 0.5 \quad \left[\left|\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}\right|\right]^2 = 0.5 \quad \left[\left|\begin{pmatrix} 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}\right|\right]^2 = 0.5 \quad \left[\left|\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}\right|\right]^2 = 0.5$$

Clearly *Black* and *White* can be written as superpositions of *Hard* and *Soft*, and vice versa. This means hard and soft quons do not have a well-defined color, and black and white quons do not have a well-defined hardness.

$\frac{1}{\sqrt{2}} \cdot (\mathrm{Hard} + \mathrm{Soft}) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \cdot \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$
$\frac{1}{\sqrt{2}} \cdot (\mathrm{Hard} - \mathrm{Soft}) = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right] = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$

$\frac{1}{\sqrt{2}} \cdot (\text{Black} + \text{White}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \left[\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} + \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \right] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\frac{1}{\sqrt{2}} \cdot (\text{Black} - \text{White}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\frac{1}{\sqrt{2}} \left[\begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} - \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} \right] = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Hard, *Soft*, *Black* and *White* are measurable properties and the vectors representing them are eigenstates of the *Hardness* and *Color* operators with **eigenvalues** $\pm$ 1. The **Identity** operator is also given and will be discussed later. Of course, the *Hardness* and *Color* operators are just the Pauli spin operators in the z- and x-directions. Later the *Taste* operator will be introduced; it is the y-direction Pauli spin operator.

Operators

$\text{Hardness} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$	$\text{Color} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\text{I} := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Eigenvalue +1	Eigenvalue -1

$\text{Hardness} \cdot \text{Hard} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\text{Hardness} \cdot \text{Soft} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$
$\text{Color} \cdot \text{Black} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$	$\text{Color} \cdot \text{White} = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} -0.707 \\ 0.707 \end{pmatrix}$

Another way of showing this is by calculating the **expectation (or average) value**. Every time the hardness of a hard quon is measured the result is +1. Every time the hardness of a soft quon is measured the result is -1.

$\text{Hard}^T \cdot \text{Hardness} \cdot \text{Hard} = 1$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1$	$\text{Soft}^T \cdot \text{Hardness} \cdot \text{Soft} = -1$	$\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = -1$
$\text{Black}^T \cdot \text{Color} \cdot \text{Black} = 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} =$	$\text{White}^T \cdot \text{Color} \cdot \text{White} = 1$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} =$

If a quon is in a state which is an eigenfunction of an operator, it means it has a well-defined value for the observable represented by the operator. If the quon is in a state which is not an eigenfunction of the operator, it does not have a well-defined value for the observable.

Hard and *Soft* are not **eigenfunctions** of the *Color* operator, and *Black* and *White* are not eigenfunctions of the **Hardness** operator. Hard and soft quons do not have a well-defined color, and black and white quons do not have a well-defined hardness.

$\text{Hardness} \cdot \text{Black} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$	$\text{Hardness} \cdot \text{White} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$
$\text{Color} \cdot \text{Hard} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\text{Color} \cdot \text{Soft} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

Therefore their **expectation values** are zero. In other words if the hardness of a black quon is measured, half the time it will register hard and half the time soft. If the color of a soft quon is measured, half the time it will register white and half the time black.

$\text{Black}^T \cdot \text{Hardness} \cdot \text{Black} = 0$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{pmatrix}$	$\text{White}^T \cdot \text{Hardness} \cdot \text{White} = 0$	$\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} \end{pmatrix}$
$\text{Hard}^T \cdot \text{Color} \cdot \text{Hard} = 0$	$\begin{pmatrix} 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 0$	$\text{Soft}^T \cdot \text{Color} \cdot \text{Soft} = 0$	$\begin{pmatrix} 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = 0$

As the **Hardness-Color** commutator shows, the **Hardness** and **Color** operators do not commute. They represent incompatible observables; observables that cannot simultaneously have well-defined values.

$\text{Hardness} \cdot \text{Color} - \text{Color} \cdot \text{Hardness} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ -2 & 0 \end{pmatrix}$

This means that the measurement of the color and then the hardness of a hard quon gives a different result than the measurement of the hardness and then the color.

$\text{Hardness} \cdot \text{Color} \cdot \text{Hard} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \end{pmatrix}$	$\text{Color} \cdot \text{Hardness} \cdot \text{Hard} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

We can also look at this from the perspective of the uncertainty principle. The uncertainty in a measurement is the square root of the difference between *the mean of the square* and *the square of the mean*.

Suppose we measure the color of a Black or White quon. Because Black and White are eigenfunctions of the Color operator the uncertainty in the measurement results are zero.

$$\sqrt{\text{Black}^T \cdot \text{Color}^2 \cdot \text{Black} - (\text{Black}^T \cdot \text{Color} \cdot \text{Black})^2} = 0$$

$$\sqrt{\text{White}^T \cdot \text{Color}^2 \cdot \text{White} - (\text{White}^T \cdot \text{Color} \cdot \text{White})^2} = 0$$

However, the measurement of the color of a Soft or Hard quon is by the same criterion uncertain.

$$\sqrt{\text{Soft}^T \cdot \text{Color}^2 \cdot \text{Soft} - (\text{Soft}^T \cdot \text{Color} \cdot \text{Soft})^2} = 1 \qquad \sqrt{\text{Hard}^T \cdot \text{Color}^2 \cdot \text{Hard} - (\text{Hard}^T \cdot \text{Color} \cdot \text{Hard})^2} = 1$$

The calculations of **Hardness** and **Color** reveal the strange behavior of quons. In the macro world we frequently find objects that simultaneously have well-defined values for these physical attributes. But we see this is not possible in the quantum world.

Mathcad has high-level commands which find the eigenvalues and eigenvectors of matrices which in quantum mechanics are operators. Below it is shown that they give the same results as were demonstrated above. See the Appendix for additional computational methods.

$\text{eigenvals(Hardness)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\text{eigenvals(Hardness}, -1) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$	$\text{eigenvals(Hardness}, 1) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\text{eigenvals(Color)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\text{eigenvals(Color}, -1) = \begin{pmatrix} -0.707 \\ 1 \end{pmatrix}$	$\text{eigenvals(Color}, 1) = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$

Besides the properties of hardness and color, suppose the quon also has the property of taste, tasting either *Sweet* or *Sour*. The **Taste** operator is defined below and its eigenvalues and eigenvectors calculated.

Operator	Eigenvalues	Sweet/Sour Eigenvectors
$\text{Taste} := \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$	$\text{eigenvals(Taste)} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\text{eigenvals(Taste)} = \begin{pmatrix} -0.707i & 0.707 \\ 0.707 & -0.707i \end{pmatrix}$

Squaring the **Hardness**, **Color** and **Taste** operators gives the **Identity** operator, that is they are unitary matrices. The **Identity** operator leaves the vector it operates on unchanged.

$$\text{Hardness}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Color}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Taste}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & \\ \mathrm{i} & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Another important property of these operators is that they are equal to their Hermitian conjugate as shown below. The physical significance of this is that they have real eigenvalues, something we know from earlier calculations.

$$\overline{\text{Hardness}}^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \left[\overline{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}\right]^T = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\overline{\text{Color}}^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \left[\overline{\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}}\right]^T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\overline{\text{Taste}}^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \left[\overline{\begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}}\right]^T = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

The Hadamard matrix is another operator which is important in quantum optics and quantum computing.

$$\text{Hadamard} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

The Hadamard matrix performs a Fourier transform between the **Hardness** and **Color** basis vectors.

$\text{Hadamard} \cdot \text{Hard} = \text{Black}$	$\text{Hadamard} \cdot \text{Hard} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix}$	$\text{Hadamard} \cdot \text{Black} = \text{Hard}$	$\text{Hadamard} \cdot \text{Black} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
$\text{Hadamard} \cdot \text{Soft} = \text{White}$	$\text{Hadamard} \cdot \text{Soft} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix}$	$\text{Hadamard} \cdot \text{White} = \text{Soft}$	$\text{Hadamard} \cdot \text{White} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

The eigenvalues and eigenvectors of the Hadamard matrix:

$\text{eigenvals}(\text{Hadamard}) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$	$\text{eigenvals}(\text{Hadamard}, 1) = \begin{pmatrix} 0.924 \\ 0.383 \end{pmatrix}$	$\text{eigenvals}(\text{Hadamard}, -1) = \begin{pmatrix} -0.383 \\ 0.924 \end{pmatrix}$

The Hadamard matrix is also unitary and its own Hermitian conjugate like the other matrices.

$$\text{Hadamard}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \overline{\text{Hadamard}}^T = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$

In addition to performing a Fourier transform between the **Hardness** and **Color** basis vectors, it has been reported in the *Journal of Olfactory Science* that the Hadamard matrix is the operator representing the property of **Odor**. It's eigenstates, shown above, are **Pleasant** and **Foul**, with eigenvalues +1 and -1, respectively. It is left to the interested reader to return to the beginning of this tutorial to explore the quantum relationship of **Odor** to **Hardness**, **Color** and **Taste**.

Concluding Remarks

The reason for using the properties of hardness, color and taste in these exercises is to emphasize how different the quantum world is from the macro world that we occupy. It is not an uncommon experience (it has happened to me) to eat a piece of candy that is hard, white and sweet. But this is not possible for *quantum candy* because the matrix operators representing these observables do not commute. Therefore, the observables cannot simultaneously be well defined.

In quantum mechanics these operators,

$$\text{Hardness} := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{Color} := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{Taste} := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

are actually the Pauli spin matrices and represent the observables for spin in the z-, x- and y-directions as mentioned earlier.

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \sigma_{\mathrm{x}} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_{\mathrm{y}} := \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}$$

They are also the operators for the rectilinear, diagonal and circular polarization properties of photons. In this case the eigenvectors are vertical, horizontal, diagonal, anti-diagonal, and right and left circular polarization.

$$\mathrm{V} := \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \mathrm{H} := \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\mathrm{D} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \mathrm{A} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\mathrm{R} := \frac{1}{\sqrt{2}} \cdot \begin{pmatrix} 1 \\ \mathrm{i} \end{pmatrix} \qquad \mathrm{L} := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -\mathrm{i} \end{pmatrix}$$

Appendix: Vector and Matrix Math

Vector inner product:

$$(ab) \cdot \begin{pmatrix} c \\ d \end{pmatrix} \rightarrow a \cdot c + b \cdot d$$

Vector outer product:

$$\begin{pmatrix} c \\ d \end{pmatrix} - (ab) \rightarrow \begin{pmatrix} a \cdot c & b \cdot c \\ a \cdot d & b \cdot d \end{pmatrix}$$

$$\operatorname{tr}\left[\begin{pmatrix} c \\ d \end{pmatrix} \cdot (ab)\right] \rightarrow a \cdot c + b \cdot d$$

Matrix-vector product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \rightarrow \begin{pmatrix} a \cdot x + b \cdot y \\ c \cdot x + d \cdot y \end{pmatrix}$$

$$(x, y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \rightarrow (a \cdot x + b \cdot y \quad c \cdot x + d \cdot y)$$

Expectation value:

$$(x \quad y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ simplify } \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$$

$$(x, y) \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ simplify } \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$$

$$\operatorname{tr}\left[\begin{pmatrix} x \\ y \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right] \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$$

$$\operatorname{tr}\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} \cdot (x \quad y)\right] \text{ simplify } \rightarrow a \cdot x^2 + d \cdot y^2 + b \cdot x \cdot y + c \cdot x \cdot y$$

Matrix product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} w & x \\ y & z \end{pmatrix} \rightarrow \begin{pmatrix} a \cdot w + b \cdot y & a \cdot x + b \cdot z \\ c \cdot w + d \cdot y & c \cdot x + d \cdot z \end{pmatrix}$$

Vector tensor product:

$$\begin{pmatrix} a \\ b \end{pmatrix} \otimes \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a \begin{pmatrix} c \\ d \end{pmatrix} \\ b \begin{pmatrix} c \\ d \end{pmatrix} \end{pmatrix} = \begin{pmatrix} ac \\ ad \\ bc \\ bd \end{pmatrix}$$

Matrix tensor product:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} w & x \\ y & z \end{pmatrix} = \begin{pmatrix} a\begin{pmatrix} w & x \\ y & z \end{pmatrix} & b\begin{pmatrix} w & x \\ y & z \end{pmatrix} \\ c\begin{pmatrix} w & x \\ y & z \end{pmatrix} & d\begin{pmatrix} w & x \\ y & z \end{pmatrix} \end{pmatrix} = \begin{pmatrix} aw & ax & bw & bx \\ ay & az & by & bz \\ cw & cx & dw & dx \\ cy & cz & dy & dz \end{pmatrix}$$

Matrix eigenvalues and eigenvectors (unnormalized):

$$\text{eigenvals}\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) \rightarrow \begin{pmatrix} a-b \\ a+b \end{pmatrix}$$

or

$$\left|\begin{pmatrix} a-\lambda & b \\ b & a-\lambda \end{pmatrix}\right| = 0 \text{ solve }, \lambda \rightarrow \begin{pmatrix} a+b \\ a-b \end{pmatrix}$$

or

$$\begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}^{-1} \begin{pmatrix} a & b \\ b & a \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} a-b & 0 \\ 0 & a+b \end{pmatrix}$$

using

$$\text{eigenvecs}\left(\begin{pmatrix} a & b \\ b & a \end{pmatrix}\right) \rightarrow \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} a & b \\ b & a \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = (a-b) \cdot \begin{pmatrix} x \\ y \end{pmatrix} \text{ solve,} y \rightarrow -x \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\begin{pmatrix} \mathrm{a} & \mathrm{b} \\ \mathrm{b} & \mathrm{a} \end{pmatrix} \cdot \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = (\mathrm{a}+\mathrm{b}) \cdot \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} \text{ solve, y} \rightarrow \mathrm{x} \qquad \begin{pmatrix} \mathrm{x} \\ \mathrm{y} \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Completeness relations:

$$\text{Black} \cdot \text{Black}^T + \text{White} \cdot \text{White}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\text{Hard} \cdot \text{Hard}^T + \text{Soft} \cdot \text{Soft}^T = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$