# 1.62: Examining the Wigner Distribution Using Dirac Notation

abstract

Expressing the Wigner distribution function in Dirac notation reveals its resemblance to a classical trajectory in phase space.

References to the Wigner distribution function [1-3] and the phase-space formulation of quantum mechanics are becoming more frequent in the pedagogical and review literature [4-26]. There have also been several important applications reported in the recent research literature [27, 28]. Other applications of the Wigner distribution are cited in Ref. 25.

The purpose of this note is to demonstrate that when expressed in Dirac notation the Wigner distribution resembles a classical phase-space trajectory. The Wigner distribution can be generated from either the coordinate- or momentum-space wave function. The coordinate-space wave function will be employed here and the Wigner transform using it is given in equation (1) for a one-dimensional example in atomic units.

$$W(p, x)=\frac{1}{2 \pi} \int_{-\infty}^{\infty} \Psi^{*}\left(x+\frac{s}{2}\right) \Psi\left(x-\frac{s}{2}\right) e^{i p s} d s$$

In Dirac notation the first two terms of the integrand are written as follows,

$$\Psi^{*}\left(x+\frac{s}{2}\right)=\left\langle\Psi | x+\frac{s}{2}\right\rangle \qquad \Psi\left(x-\frac{s}{2}\right)=\left\langle x-\frac{s}{2} | \Psi\right\rangle$$

Assigning $1/2\pi$ to the third term and utilizing the momentum eigenfunction in coordinate space and its complex conjugate we have,

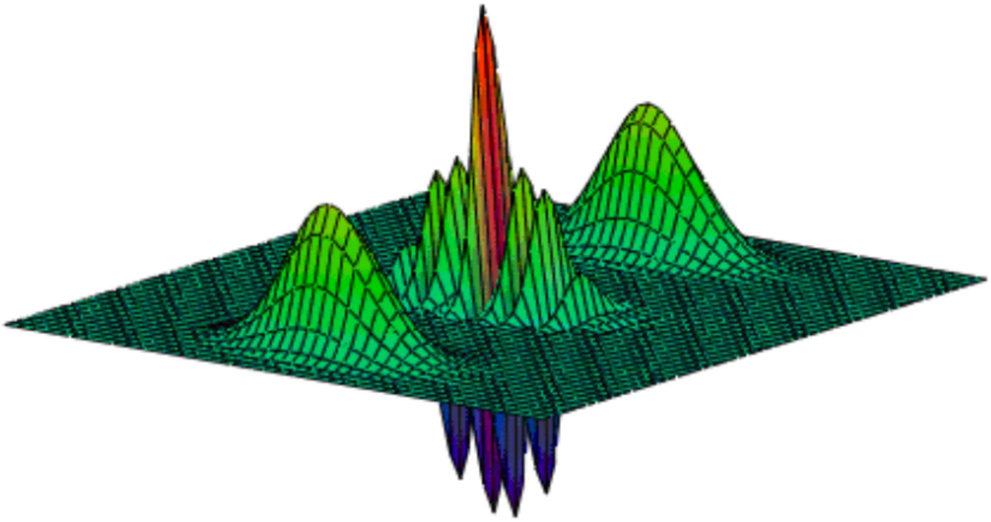
$$\frac{1}{2 \pi} \mathrm{e}^{i p s}=\frac{1}{\sqrt{2 \pi}} \mathrm{e}^{i p\left(x+\frac{s}{2}\right)} \frac{1}{\sqrt{2 \pi}} \mathrm{e}^{-i p\left(x-\frac{s}{2}\right)}=\left\langle x+\frac{s}{2} | p\right\rangle\left\langle p | x-\frac{s}{2}\right\rangle$$

Substituting equations (2) and (3) into equation (1) yields after arrangement

$$W(x, p)=\int_{-\infty}^{+\infty}\left\langle\Psi | x+\frac{s}{2}\right\rangle\left\langle x+\frac{s}{2} | p\right\rangle\left\langle p | x-\frac{s}{2}\right\rangle\left\langle x-\frac{s}{2} | \Psi\right\rangle d s$$

The four Dirac brackets are read from right to left as follows: (1) is the amplitude that a particle in the state $\Psi$ has position (x - s/2); (2) is the amplitude that a particle with position (x - s/2) has momentum p; (3) is the amplitude that a particle with momentum p has position (x + s/2); (4) is the amplitude that a particle with position (x + s/2) is (still) in the state $\Psi$. Thus, in Dirac notation the integrand is the quantum equivalent of a classical phase-space trajectory for a quantum system in the state $\Psi$.

Integration over s creates a superposition of all possible quantum trajectories of the state $\Psi$, which interfere constructively and destructively, providing a quasi-probability distribution in phase space. As an example, the Wigner probability distribution for a double-slit experiment is shown in the figure below [14, 27]. The oscillating positive and negative values in the middle of the Wigner distribution signify the interference associated with a quantum superposition, distinguishing it from a classical phase-space probability distribution. In the words of Leibfried et al. [14], the Wigner function is a “mathematical construct for visualizing quantum trajectories in phase space.”



Wigner distribution function for the double-slit experiment.

The Wigner double- and triple-slit distribution functions are calculated in the following tutorials.

Wigner Distribution for the Double Slit Experiment

Wigner Distribution for the Triple Slit Experiment

Examples of the generation and use of the Wigner distribution are available in the following tutorials.

Wigner Distribution for the Particle in a Box

Quantum Calculations on the Hydrogen Atom in Coordinate, Momentum and Phase Space

Variation Method Using the Wigner Function: The Feshbach Potential

## The Wigner Distribution Function for the Harmonic Oscillator

Given the quantum number this Mathcad file calculates the Wigner distribution function for the specified harmonic oscillator eigen state.

Quantum number: $\mathrm{n} := 2$

Harmonic oscillator eigenstate:

$$\Psi(x) := \frac{1}{\sqrt{2^n \cdot n! \sqrt{\pi}}} \cdot \mathrm{Her}(n, x) \cdot \exp\left(-\frac{x^2}{2}\right)$$

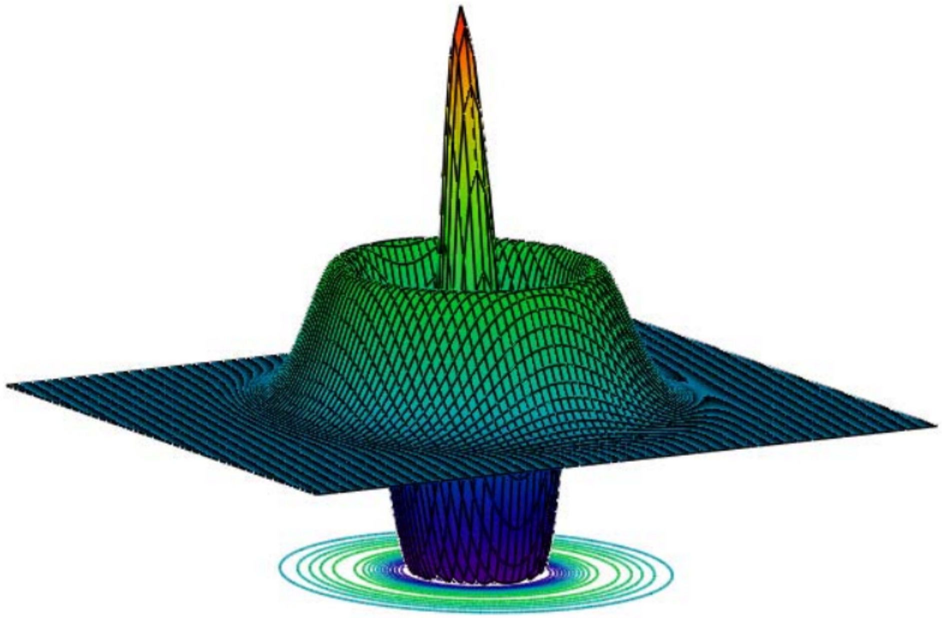
Calculate the Wigner distribution:

$$\mathrm{W}(\mathrm{x}, \mathrm{p}) := \frac{1}{\pi^{\frac{3}{2}}} \cdot \int_{-\infty}^{\infty} \Psi\left(\mathrm{x} + \frac{\mathrm{s}}{2}\right) \cdot \exp(\mathrm{i} \cdot \mathrm{s} \cdot \mathrm{p}) \cdot \Psi\left(\mathrm{x} - \frac{\mathrm{s}}{2}\right) \mathrm{ds}$$

Display the Wigner distribution:

$$\mathrm{N} := 80 \qquad \mathrm{i} := 0 \ldots \mathrm{N} \qquad x_i := -4 + \frac{8 \cdot i}{N}$$

$$\mathrm{j} := 0 \ldots \mathrm{N} \qquad \mathrm{p_j} := -5 + \frac{10 \cdot \mathrm{j}}{\mathrm{N}} \qquad \mathrm{Wigner}_{i,j} := W\left(x_i, p_j\right)$$



Wigner, Wigner

Phase-space quantum mechanical calculations using the Wigner distribution are compared with coordinate- and momentum-space calculations in the following tutorial.

The Cliff Notes version of the above can be accessed in the following tutorial.

## Literature cited: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