

8.82: The Discrete or Quantum Fourier Transform

The continuous-variable Fourier transforms involving position and momentum are well known. In Dirac notation (see chapter 6 in *A Modern Approach to Quantum Mechanics* by John S. Townsend) they are,

$$\langle \Psi | = \int \langle x | \langle x | \Psi \rangle dx \quad \text{and} \quad \langle x | \Psi \rangle = \int \langle x | p \rangle \langle p | \Psi \rangle dp$$

where

$$\langle x | p \rangle = \langle p | x \rangle^* = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(i \frac{2\pi p x}{h}\right) = \frac{1}{\sqrt{2\pi\hbar}} \exp\left(i \frac{p x}{\hbar}\right)$$

Using the coordinate and momentum completeness relations

$$\int |x\rangle \langle x| dx = 1 \quad \text{and} \quad \int |p\rangle \langle p| dp = 1$$

we can write the following generic Fourier transforms.

$$\langle p | = \int \langle p | x \rangle \langle x | dx \quad \text{and} \quad \langle x | = \int \langle x | p \rangle \langle p | dp$$

By analogy a discrete Fourier transform between the k and j indices can be created.

$$\langle k | = \sum_{j=0}^{N-1} \langle k | j \rangle \langle j |$$

were, again, by analogy

$$\langle k | j \rangle = \frac{1}{\sqrt{N}} \exp\left(i \frac{2\pi}{N} k j\right)$$

so that

$$\langle k | = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} \exp\left(i \frac{2\pi}{N} k j\right) \langle j |$$

Summing over the k index and projecting on to $|\Psi\rangle$ yields a system of linear equations.

$$\sum_{k=0}^{N-1} \langle k | \Psi \rangle = \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} \sum_{j=0}^{N-1} \exp\left(i \frac{2\pi}{N} k j\right) \langle j | \Psi \rangle$$

Like all systems it is expressible in matrix form. For example, with $N=2$ and $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ as the operand we have,

$$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

Here the matrix operator is the well-known Hadamard transform. In this case it transforms spin-up in the z-direction to spin-up in the x-direction, or horizontal polarization to diagonal polarization, etc. Naturally it transforms spin-up in the x-direction to spin-up in the z-direction.

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

This, of course, also occurs with the continuous-variable Fourier transform.

$$\langle x | \Psi \rangle \xrightarrow{FT} \langle p | \Psi \rangle \rightarrow x | \Psi \rangle$$

The Mathcad implementation of the discrete or quantum Fourier transform (QFT) is now demonstrated.

$$N = 2 \quad m = 0..N-1 \quad n = 0..N-1 \quad QFT_{m,n} = \frac{1}{\sqrt{N}} \exp\left(i \frac{2\pi m n}{N}\right)$$

$$QFT = \begin{pmatrix} 0.707 & 0.707 \\ 0.707 & -0.707 \end{pmatrix}$$
$$QFT \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0.707 \\ 0.707 \end{pmatrix} \quad QFT \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$
$$QFT \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0.707 \\ -0.707 \end{pmatrix} \quad QFT \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

These calculations demonstrate that the QFT is a unitary operator:

$$QFT \, QFT = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

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