## 22.1.6: vi. Exercise Solutions

### Q1

a.

$$K.E. = \frac{mv^2}{2} = \left(\frac{m}{m}\right)\frac{dmv^2}{2} = \frac{(mv)^2}{2m} = \frac{p^2}{2m} \tag{22.1.6.1}$$

$$K.E. = \frac{1}{2m}\left(p_x^2 + p_y^2 + p_z^2\right) \tag{22.1.6.2}$$

$$K.E. = \frac{1}{2m}\left[\left(\hbar\frac{\partial}{\partial x}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right)^2 + \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)^2\right] \tag{22.1.6.3}$$

$$K.E. = \frac{-\hbar^2}{2m}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\frac{\partial^2}{\partial z^2}\right] \tag{22.1.6.4}$$

b.

$$\mathbf{p} = m\mathbf{v} = \mathbf{i}p_x + \mathbf{j}p_y + \mathbf{k}p_z \tag{22.1.6.5}$$

$$p = \left[\mathbf{i}\left(\frac{\hbar}{i}\frac{\partial}{\partial x} + \mathbf{j}\left(\frac{\hbar}{i}\frac{\partial}{\partial y}\right) + \left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right)\right)\right] \tag{22.1.6.6}$$

where i, j, and k are unit vectors along the x, y, and z axes.

c.

$$L_y = zp_x - xp_z \tag{22.1.6.7}$$

$$L_y = z\left(\frac{\hbar}{i}\frac{\partial}{\partial x}\right) - x\left(\frac{\hbar}{i}\frac{\partial}{\partial z}\right) \tag{22.1.6.8}$$

### Q2

First derive the general formulas for $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}$ in terms of r, $\theta$, and $\phi$, and $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$, and $\frac{\partial}{\partial \phi}$ in terms of x,y, and z. The general relationships are as follows:

$$x = r\sin\theta\cos\phi \qquad r^2 = x^2y^2 + z^2 \tag{22.1.6.9}$$

$$y = r\sin\theta\sin\phi \qquad \sin\theta = \frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}} \tag{22.1.6.10}$$

$$z = r\cos\theta \qquad \cos\theta = \frac{z}{\sqrt{x^2+y^2+z^2}} \tag{22.1.6.11}$$

$$\tan\phi = \frac{y}{x} \tag{22.1.6.12}$$

First $\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$, and $\frac{\partial}{\partial z}$ from the chain rule:

$$\frac{\partial}{\partial x} = \left(\frac{\partial r}{\partial x}\right)_{y,z}\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial x}\right)_{y,z}\frac{\partial}{\partial \theta} + \left(\frac{\partial \phi}{\partial x}\right)_{y,z}\frac{\partial}{\partial \phi}, \tag{22.1.6.13}$$

$$\frac{\partial}{\partial y} = \left(\frac{\partial r}{\partial y}\right)_{y,z}\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial y}\right)_{y,z}\frac{\partial}{\partial \theta} + \left(\frac{\partial \phi}{\partial y}\right)_{y,z}\frac{\partial}{\partial \phi}, \tag{22.1.6.14}$$

$$\frac{\partial}{\partial z} = \left(\frac{\partial r}{\partial z}\right)_{y,z}\frac{\partial}{\partial r} + \left(\frac{\partial \theta}{\partial z}\right)_{y,z}\frac{\partial}{\partial \theta} + \left(\frac{\partial \phi}{\partial z}\right)_{y,z}\frac{\partial}{\partial \phi}, \tag{22.1.6.15}$$

Evaluation of the many "coefficients" gives the following:

$$\left(\frac{\partial r}{\partial x}\right)_{y,z} = \sin\theta\cos\phi, \qquad \left(\frac{\partial \theta}{\partial x}\right)_{y,z} = \frac{\cos\theta\cos\phi}{r}, \qquad \left(\frac{\partial \phi}{\partial x}\right)_{y,z} = -\frac{\sin\phi}{r\sin\theta} \tag{22.1.6.16}$$

$$\left(\frac{\partial r}{\partial y}\right)_{x,z} = \sin\theta\sin\phi, \qquad \left(\frac{\partial \theta}{\partial y}\right)_{x,z} = \frac{\cos\theta\sin\phi}{r}, \qquad \left(\frac{\partial \phi}{\partial y}\right)_{x,z} = \frac{\cos\phi}{r\sin\theta} \tag{22.1.6.17}$$

$$\left(\frac{\partial r}{\partial z}\right)_{x,z} = \cos\theta, \qquad \left(\frac{\partial \theta}{\partial z}\right)_{x,y} = -\frac{\sin\theta}{r}, \text{ and } \qquad \left(\frac{\partial \phi}{\partial z}\right)_{y,z} = 0. \tag{22.1.6.18}$$

Upon substitution of these "coefficients":

$$\frac{\partial}{\partial x} = \sin\theta\cos\phi\frac{\partial}{\partial r} + \frac{\cos\theta\cos\phi}{r}\frac{\partial}{\partial \theta} - \frac{\sin\phi}{r\sin\theta}\frac{\partial}{\partial \phi}, \tag{22.1.6.19}$$

$$\frac{\partial}{\partial y} = \sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{Cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}, \text{ and} \tag{22.1.6.20}$$

$$\frac{\partial}{\partial z} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta} + 0\frac{\partial}{\partial \phi}, \tag{22.1.6.21}$$

Next $\frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}$, and $\frac{\partial}{\partial \phi}$ from the chain rule:

$$\frac{\partial}{\partial r} = \left(\frac{\partial x}{\partial r}\right)_{\theta,\phi} \frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial r}\right)_{\theta,\phi} \frac{\partial}{\partial y} + \left(\frac{\partial z}{\partial r}\right)_{\theta,\phi} \frac{\partial}{\partial z}, \tag{22.1.6.22}$$

$$\frac{\partial}{\partial \theta} = \left(\frac{\partial x}{\partial \theta}\right)_{r,\phi} \frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial \theta}\right)_{r,\phi} \frac{\partial}{\partial y} + \left(\frac{\partial z}{\partial \theta}\right)_{r,\phi} \frac{\partial}{\partial z}, \text{ and} \tag{22.1.6.23}$$

$$\frac{\partial}{\partial \phi} = \left(\frac{\partial x}{\partial \phi}\right)_{r,\theta} \frac{\partial}{\partial x} + \left(\frac{\partial y}{\partial \phi}\right)_{r,\theta} \frac{\partial}{\partial y} + \left(\frac{\partial z}{\partial \phi}\right)_{r,\theta} \frac{\partial}{\partial z}. \tag{22.1.6.24}$$

Again evaluation of the the many "coefficients" results in:

$$\left(\frac{\partial x}{\partial r}\right)_{\theta,\phi} = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad \left(\frac{\partial y}{\partial r}\right)_{\theta,\phi} = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad \left(\frac{\partial z}{\partial r}\right)_{\theta,\phi} = \frac{z}{\sqrt{x^2+y^2+z^2}} \tag{22.1.6.25}$$

$$\left(\frac{\partial x}{\partial \theta}\right)_{r,\phi} = \frac{xz}{\sqrt{x^2+y^2}}, \quad \left(\frac{\partial y}{\partial \theta}\right)_{r,\phi} = \frac{yz}{\sqrt{x^2+y^2}}, \quad \left(\frac{\partial z}{\partial \theta}\right)_{r,\phi} = -\sqrt{x^2+y^2} \tag{22.1.6.26}$$

$$\left(\frac{\partial x}{\partial \phi}\right)_{r,\theta} = -y, \quad \left(\frac{\partial y}{\partial \phi}\right)_{r,\theta} = x, \quad \text{and } \left(\frac{\partial z}{\partial \phi}\right)_{r,\theta} = 0 \tag{22.1.6.27}$$

Upon substitution of these "coefficients":

$$\frac{\partial}{\partial r} = \frac{x}{\sqrt{x^2+y^2+z^2}} \frac{\partial}{\partial x} + \frac{y}{\sqrt{x^2+y^2+z^2}} \frac{\partial}{\partial y} + \frac{z}{\sqrt{x^2+y^2+z^2}} \frac{\partial}{\partial z} \tag{22.1.6.28}$$

$$\frac{\partial}{\partial \theta} = \frac{xz}{\sqrt{x^2+y^2}} \frac{\partial}{\partial x} + \frac{yz}{\sqrt{x^2+y^2}} \frac{\partial}{\partial y} - \sqrt{x^2+y^2} \frac{\partial}{\partial z} \tag{22.1.6.29}$$

$$\frac{\partial}{\partial \phi} = -y \frac{\partial}{\partial x} + x \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial z}. \tag{22.1.6.30}$$

Note, these many "coefficients" are the elements which make up the Jacobian matrix used whenever one wishes to transform a function from one coordinate representation to another. One very familiar result should be in transforming the volume element $dx\,dy\,dz$ to $r^2 sin\theta\, dr\, d\theta\, d\phi$. For example:

$$\int f(x,y,z)dxdydz = \int f(x(r,\theta,\phi), y(r,\theta,\phi), z(r,\theta,\phi)) \begin{vmatrix} \left(\frac{\partial x}{\partial r}\right)_{\theta\phi} & \left(\frac{\partial x}{\partial \theta}\right)_{r\phi} & \left(\frac{\partial x}{\partial \phi}\right)_{r\theta} \\ \left(\frac{\partial y}{\partial r}\right)_{\theta\phi} & \left(\frac{\partial y}{\partial \theta}\right)_{r\phi} & \left(\frac{\partial y}{\partial \phi}\right)_{r\theta} \\ \left(\frac{\partial z}{\partial r}\right)_{\theta\phi} & \left(\frac{\partial z}{\partial \theta}\right)_{r\phi} & \left(\frac{\partial z}{\partial \phi}\right)_{r\theta} \end{vmatrix} dr d\theta d\phi$$

a.

$$L_x = \frac{\hbar}{i}\left[y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right] \tag{22.1.6.31}$$

$$L_x = \frac{\hbar}{i}\left[r sin\theta sin\phi\left(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\right)\right] - \frac{\hbar}{i}\left[r\cos\theta\left(\sin\theta\sin\phi\frac{\partial}{\partial r} + \frac{\cos\theta\sin\phi}{r}\frac{\partial}{\partial \theta} + \frac{\cos\phi}{r\sin\theta}\frac{\partial}{\partial \phi}\right)\right] \tag{22.1.6.32}$$

$$L_x = -\frac{\hbar}{i}\left(\sin\phi\frac{\partial}{\partial \theta} + \cot\theta\cos\phi\frac{\partial}{\partial \phi}\right) \tag{22.1.6.33}$$

b.

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial \phi} = -i\hbar\frac{\partial}{\partial \phi} \tag{22.1.6.34}$$

$$L_z = \frac{\hbar}{i}\left(-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}\right) \tag{22.1.6.35}$$

## Q3

$$\begin{array}{rrrr} & B & B' & B'' \\ i. & 4x^4-12x^2+3 & 16x^3-24x & 48x^2-24 \\ ii. & 5x^4 & 20x^3 & 60x^2 \\ iii. & e^{3x}+e^{-3x} & 3\left(e^{3x}-e^{3x}\right) & 9\left(e^{3x}+e^{-3x}\right) \\ iv. & x^2-4x+2 & 2x-4 & 2 \\ v. & 4x^3-3 & 12x^2-3 & 24x \end{array} \tag{22.1.6.36–22.1.6.41}$$

B(v.) is an eigenfunction of A(i.):

$$\left(1-x^2\right)\frac{d^2}{dx^2}-x\frac{d}{dx}\text{B(v.)} = \tag{22.1.6.42}$$

$$=\left(1-x^2\right)(24x)-x\left(12x^2-3\right) \tag{22.1.6.43}$$

$$=24x-24x^3-12x^3+3x \tag{22.1.6.44}$$

$$=-36x^3+27x \tag{22.1.6.45}$$

$$=-9\left(4x^3-3x\right)\ \text{(eigenvalue is -9)} \tag{22.1.6.46}$$

B(iii.) is an eigenfunction of A(ii.):

$$\frac{d^2}{dx^2}\text{B(iii.)} = \tag{22.1.6.47}$$

$$=9\left(e^{3x}+e^{-3x}\right)\ \text{(eigenvalue is 9)} \tag{22.1.6.48}$$

B(ii.) is an eigenfunction of A(iii.):

$$x\frac{d}{dx}\text{B(ii.)} = \tag{22.1.6.49}$$

$$=x\left(20x^3\right) \tag{22.1.6.50}$$

$$=20x^4 \tag{22.1.6.51}$$

$$=4\left(5x^4\right)\ \text{(eigenvalue is 4)} \tag{22.1.6.52}$$

B(i.) is an eigenfunction of A(vi.):

$$\frac{d^2}{dx^2}-2x\frac{d}{dx}\text{B(i.)} = \tag{22.1.6.53}$$

$$=\left(48x^2-24\right)-2x\left(16x^3-24x\right) \tag{22.1.6.54}$$

$$=48x^2-24-32x^4+48x^2 \tag{22.1.6.55}$$

$$=-32x^4+6x^2-24 \tag{22.1.6.56}$$

$$=-8\left(4x^4-12x^2+3\right)\ \text{(eigenvalue is -8)} \tag{22.1.6.57}$$

B(iv.) is an eigenfunction of A(v.):

$$x\frac{d^2}{dx^2}-+(1-x)\frac{d}{dx}\text{B(iv.)} = \tag{22.1.6.58}$$

$$=x(2)+(1-x)(2x-4) \tag{22.1.6.59}$$

$$=2x+2x-4-2x^2+4x \tag{22.1.6.60}$$

$$=-2x^2+8x-4 \tag{22.1.6.61}$$

$$=-2\left(x^2-4x+2\right)\ \text{(eigenvalue is -2)} \tag{22.1.6.62}$$

## Q4

Show that:

$$\int f^*\mathbf{A}g\,d\tau=\int g(\mathbf{A}f)^*d\tau$$

a. Suppose f and g are functions of x and evaluate the integral on the left hand side by "integration by parts":

$$\int f(x)^*\left(-i\hbar\frac{\partial}{\partial x}\right)g(x)dx$$

let dv = $\frac{\partial}{\partial x}g(x)dx$ and $u=-i\hbar f(x)^*$

$v=g(x)du=-i\hbar\frac{\partial}{\partial x}f(x)^*dx$

Now, $\int udv=uv-\int vdu$,

so:

$$f(x)^*\left(-i\hbar\frac{\partial}{\partial x}\right)g(x)dx=-i\hbar f(x)^*g(x)+i\hbar\int g(x)\frac{\partial}{\partial x}f(x)^*dx.$$

Note that in, principle, it is impossible to prove hermiticity **unless** you are given knowledge of the type of function on which the operator is acting. Hermiticity requires (as can be seen in this example) that the term -iħf(x)*g(x) vanish when evaluated at the integral limits. This, in general, will occur for the "well behaved" functions (e.g., in **bound state** quantum chemistry, the wavefunctions will vanish as the distances among particles approaches infinity). So, in proving the hermiticity of an operator, one must be careful to specify the behavior of the functions on which the operator is considered to act. This means that an operator may be hermitian for one class of functions and non-hermitian for another class of functions. If we assume that f and g vanish at the boundaries, then we have

$$\int f(x)^* \left(-i\hbar \frac{\partial}{\partial x}\right) g(x)dx = \int g(x)\left(-\hbar \frac{\partial}{\partial x}(x)\right)^* dx$$

b. Suppose f and g are functions of y and z and evaluate the integral on the left hand side by "integration by parts" as in the previous exercise:

$$\int f(y,z)^* \left[-i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] g(y,z)dydz = \int f(y,z)^* \left[-i\hbar \left(y\frac{\partial}{\partial z}\right)\right] g(y,z)dydz - \int (y,z)^* \left[-i\hbar \left(z\frac{\partial}{\partial y}\right)\right] g(y,z)dydz$$

For the first integral, $\int f(z)^* \left(-i\hbar y \frac{\partial}{\partial z}\right) g(z)dz$,

$$\text{let dv } = \frac{\partial}{\partial z} g(z)dz \qquad u = -i\hbar y f(z)^* \tag{22.1.6.63}$$

$$v = g(z) \qquad du = -i\hbar y \frac{\partial}{\partial z}(z)^* dz \tag{22.1.6.64}$$

so:

$$\int f(z)^* \left(-i\hbar y \frac{\partial}{\partial z}\right) g(z)dz = -i\hbar y f(z)^* g(z) + i\hbar y \int g(z)\frac{\partial}{\partial z} f(z)^* dz = \int g(z)\left(-i\hbar y \frac{\partial}{\partial z} f(z)\right)^* dz.$$

For the second integral, $\int f(y)^* \left(-i\hbar y \frac{\partial}{\partial y}\right) g(y)dy$,

$$\text{let dv } = \frac{\partial}{\partial y} g(y)dy \qquad u = -i\hbar y f(y)^* \tag{22.1.6.65}$$

$$v = g(y) \qquad du = -i\hbar z \frac{\partial}{\partial y} f(y)^* dy \tag{22.1.6.66}$$

so:

$$\int f(y)^* \left(-i\hbar z \frac{\partial}{\partial y}\right) g(y)dy = -i\hbar z f(y)^* g(y) + i\hbar z \int g(y)\frac{\partial}{\partial y} f(y)^* dy = \int g(y)\left(-i\hbar z \frac{\partial}{\partial y} f(y)\right)^* dy.$$

$$\int f(y,z)^* \left[-i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right] g(y,z)dydz = \int g(z)\left(-i\hbar y \frac{\partial}{\partial z} f(z)\right)^* dz - \int g(y)\left(-i\hbar z \frac{\partial}{\partial y} f(y)\right)^* dy$$

$$= \int g(y,z)\left(-\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right) f(y,z)\right)^* dydz.$$

Again we have had to assume that the functions f and g vanish at the boundary.

## Q5

$$L_+ = \mathbf{L}_x + i\mathbf{L}_y$$

$$L_- = \mathbf{L}_x - i\mathbf{L}_y, \text{ so}$$

$$\mathbf{L}_+ + \mathbf{L}_- = 2\mathbf{L}_x, \text{ or } \mathbf{L}_x = \frac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-)$$

$$\mathbf{L}_+ Y_{l,m} = \sqrt{l(l+l) - m(m+l)}\, \hbar T_{l,m+1}$$

$$\mathbf{L}_- Y_{l,m} = \sqrt{l(l+l) - m(m-l)}\, \hbar T_{l,m-1}$$

Using these relationships:

$$\mathbf{L}_-\Psi_{2p_{-1}} = 0, \mathbf{L}_-\Psi_{2p_0} = \sqrt{2}\hbar\Psi_{2p_{-1}}, \mathbf{L}_-\Psi_{2p_{+1}} = \sqrt{2}\hbar\Psi_{2p_0}$$

$$\mathbf{L}_+\Psi_{2p_{-1}} = \sqrt{2}\hbar\Psi_{2p_0}, \mathbf{L}_+\Psi_{2p_0} = \sqrt{2}\hbar\Psi_{2p_{+1}}, \mathbf{L}_+\Psi_{2p_{+1}}$$

$= 0$ , and the following $\mathrm{L}_x$ matrix elements can be evaluated:

$$L_x(1,1) = \langle \Psi_{2p_{-1}} | \frac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_{-1}} \rangle = 0$$

$$L_x(1,2) = \langle \Psi_{2p_{-1}} | \frac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_0} \rangle = \frac{\sqrt{2}}{2}\hbar$$

$$L_x(1,3) = \langle \Psi_{2p_{-1}} | \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_{+1}} \rangle = 0$$

$$L_x(2,1) = \langle \Psi_{2p_0} | \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_{-1}} \rangle = \frac{\sqrt{2}}{2}\hbar$$

$$L_x(2,2) = \langle \Psi_{2p_0} | \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_0} \rangle = 0$$

$$L_x(2,3) = \langle \Psi_{2p_0} | \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-) big | \Psi_{2p_{+1}} \rangle = \frac{\sqrt{2}}{2}\hbar$$

$$L_x(3,1) = \langle \Psi_{2p_{+1}} | \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_{-1}} \rangle = 0$$

$$L_x(3,2) = \langle \Psi_{2p_{+1}} | \frac{1}{2} (\mathbf{L}_+ + \mathbf{L}_-) | \Psi_{2p_{-1}} \rangle = \frac{\sqrt{2}}{2}\hbar$$

$$L_x(3,3) = 0$$

This matrix:

$$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2}\hbar & 0 \\ \frac{\sqrt{2}}{2}\hbar & 0 & \frac{\sqrt{2}}{2}\hbar \\ 0 & \frac{\sqrt{2}}{2}\hbar & 0 \end{bmatrix} \tag{22.1.6.67}$$

, can now be diagonalized:

$$\begin{vmatrix} 0-\lambda & \frac{\sqrt{2}}{2}\hbar & 0 \\ \frac{\sqrt{2}}{2}\hbar & 0 & \frac{\sqrt{2}}{2}\hbar \\ 0 & \frac{\sqrt{2}}{2} & 0 \end{vmatrix} \tag{22.1.6.68} =$$

$$\begin{vmatrix} 0-\lambda & \frac{\sqrt{2}}{2}\hbar \\ \frac{\sqrt{2}}{2}\hbar & 0-\lambda \end{vmatrix} \tag{22.1.6.69}(-$$

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$$\begin{vmatrix} \frac{\sqrt{2}}{2}\hbar & \frac{\sqrt{2}}{2}\hbar \\ 0 & 0-\lambda \end{vmatrix} \tag{22.1.6.70}\Bigg($$

Expanding theses determinants yields:

$$\left(\lambda^2 - \frac{\hbar^2}{2}\right)(-\lambda) - \frac{\sqrt{2}\hbar}{2}(-\lambda)\left(\frac{\sqrt{2}\hbar}{2}\right) = 0$$

$$-\lambda\left(\lambda^2 - \hbar^2\right) = 0$$

$$-\lambda(\lambda - \hbar)(\lambda + \hbar) = 0$$

with roots: 0,$\hbar$, and $-\hbar$

Next, determine the corresponding eigenvectors:

For $\lambda = 0$

$$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2}\hbar & 0 \\ \frac{\sqrt{2}}{2}\hbar & 0 & \frac{\sqrt{2}}{2}\hbar \\ 0 & \frac{\sqrt{2}}{2}\hbar & 0 \end{bmatrix} \tag{22.1.6.71}\Bigg[$$

$$\frac{\sqrt{2}}{2}\hbar C_{21} = 0 \text{ (row one)}$$

$$C_21 = 0$$

$$\frac{\sqrt{2}}{2}\hbar C_{11}+\frac{\sqrt{2}}{2}\hbar C_{31}=0 \text{ (row two)}$$

$$C_{11}+C_{31}=0$$

$$C_{11}=-C_{31}$$

$$C_{11}^2+C_{21}^2+C_{31}^2=1 \text{ (normalization)}$$

$$C_{11}^2+(-C_{11})^2=1$$

$$2C_{11}^2=1$$

$$C_{11}=\frac{1}{\sqrt{2}}, C_{21}=0, \text{ and } C_{31}=-\frac{1}{\sqrt{2}}$$

For $\lambda=1\hbar$:

$$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2}\hbar & 0 \\ \frac{\sqrt{2}}{2}\hbar & 0 & \frac{\sqrt{2}}{2}\hbar \\ 0 & \frac{\sqrt{2}}{2}\hbar & 0 \end{bmatrix} \tag{22.1.6.72}$$

$$\frac{\sqrt{2}}{2}\hbar C_{22}=\hbar C_{12} \text{ (row one)}$$

$$C_{12}=\frac{\sqrt{2}}{2}C_{22}$$

$$\frac{\sqrt{2}}{2}\hbar C_{12}+\frac{\sqrt{2}}{2}\hbar C_{32}=\hbar C_{22} \text{ (row two)}$$

$$\frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2}C_{22}+\frac{\sqrt{2}}{2}C_{32}=C_{22}$$

$$\frac{1}{2}C_{22}+\frac{\sqrt{2}}{2}C_{32}=C_{22}$$

$$\frac{\sqrt{2}}{2}C_{32}=\frac{1}{2}C_{22}$$

$$C_{32}=\frac{\sqrt{2}}{2}C_{22}$$

$$C_{12}^2+C_{22}^2+C_{32}^2=1 \text{ (normalization)}$$

$$\left(\frac{\sqrt{2}}{2}C_{22}\right)^2+C_{22}^2+\left(\frac{\sqrt{2}}{2}C_{22}\right)^2=1$$

$$\frac{1}{2}C_{22}^2+C_{22}^2+\frac{1}{2}C_{22}^2=1$$

$$2C_{22}^2=1$$

$$C_{22}=\frac{\sqrt{2}}{2}$$

$$C_{12}=\frac{1}{2}, C_{22}=\frac{\sqrt{2}}{2}, \text{ and } C_{32}=\frac{1}{2}$$

For $\lambda=-1\hbar$

$$\begin{bmatrix} 0 & \frac{\sqrt{2}}{2}\hbar & 0 \\ \frac{\sqrt{2}}{2}\hbar & 0 & \frac{\sqrt{2}}{2}\hbar \\ 0 & \frac{\sqrt{2}}{2}\hbar & 0 \end{bmatrix} \tag{22.1.6.73}$$

$$\frac{\sqrt{2}}{2}\hbar C_{23}=-\hbar C_{13} \text{ (row one)}$$

$$C_{13}=-\frac{\sqrt{2}}{2}C_{23}$$

$$\frac{\sqrt{2}}{2}\hbar C_{13}+\frac{\sqrt{2}}{2}\hbar C_{33}=-\hbar C_{23}\text{ (row two)}$$

$$\frac{\sqrt{2}}{2}\left(-\frac{\sqrt{2}}{2}C_{23}\right)+\frac{\sqrt{2}}{2}C_{33}=-C_{23}$$

$$-\frac{1}{2}C_{23}+\frac{\sqrt{2}}{2}C_{33}=-C_{23}$$

$$\frac{\sqrt{2}}{2}C_{33}=-\frac{1}{2}C_{23}$$

$$C_{33}=-\frac{\sqrt{2}}{2}C_{23}$$

$$C_{13}^2+C_{23}^2+C_{33}^2=1\text{ (normalization)}$$

$$\left(-\frac{\sqrt{2}}{2}C_{23}\right)^2+C_{23}^2+\left(-\frac{\sqrt{2}}{2}C_{23}\right)^2=1$$

$$\frac{1}{2}C_{23}^2+C_{23}^2+\frac{1}{2}C_{23}^2=1$$

$$2C_{23}^2=1$$

$$C_{23}=\frac{\sqrt{2}}{2}$$

$$C_{13}=-\frac{1}{,}C_{23}=\frac{\sqrt{2}}{2}\text{, and }C_{33}=-\frac{1}{2}$$

Show: $\langle\phi_1|\phi_1\rangle=1$, $\langle\phi_2|\phi_2\rangle=1$, $\langle\phi_3|\phi_3\rangle=1$, $\langle\phi_1|\phi_2\rangle=0$, $\langle\phi_1|\phi_3\rangle=0$, and$\langle\phi_2|\phi_3\rangle=0$.

$$\langle\phi_1|\phi_1\rangle\stackrel{?}{=}1$$

$$\left(\frac{\sqrt{2}}{2}\right)^2+0+\left(-\frac{\sqrt{2}}{2}\right)^2\stackrel{?}{=}1$$

$$\frac{1}{2}+\frac{1}{2}\stackrel{?}{=}1$$

$$1=1$$

$$\langle\phi_2|\phi_2\rangle\stackrel{?}{=}1$$

$$\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{1}{2}\right)^2\stackrel{?}{=}1$$

$$\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\stackrel{?}{=}1$$

$$1=1$$

$$\langle\phi_3|\phi_3\rangle\stackrel{?}{=}1$$

$$\left(-\frac{1}{2}\right)^2+\left(\frac{\sqrt{2}}{2}\right)^2+\left(-\frac{1}{2}\right)^2\stackrel{?}{=}1$$

$$\frac{1}{4}+\frac{1}{2}+\frac{1}{4}\stackrel{?}{=}1$$

$$1=1$$

$$\langle\phi_1|\phi_2\rangle=\langle\phi_2|\phi_1\rangle\stackrel{?}{=}0$$

$$\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+(0)\left(\frac{\sqrt{2}}{2}\right)+\left(-\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)\stackrel{?}{=}0$$

$$\left(\frac{\sqrt{2}}{4}\right)-\left(\frac{\sqrt{2}}{4}\right)\stackrel{?}{=}0$$

$$0 = 0$$

$$\langle \phi_1 | \phi_3 \rangle = \langle \phi_3 | \phi_1 \rangle \stackrel{?}{=} 0$$

$$\left(\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) + (0)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{\sqrt{2}}{2}\right)\left(-\frac{1}{2}\right) \stackrel{?}{=} 0$$

$$\left(-\frac{\sqrt{2}}{4}\right) + \left(\frac{\sqrt{2}}{4}\right) \stackrel{?}{=} 0$$

$$0 = 0$$

$$\langle \phi_2 | \phi_3 \rangle = \langle \phi_3 | \phi_2 \rangle \stackrel{?}{=} 0$$

$$\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(-\frac{1}{2}\right) \stackrel{?}{=} 0$$

$$\left(-\frac{1}{4}\right) + \left(\frac{1}{2}\right) + \left(-\frac{1}{4}\right) \stackrel{?}{=} 0$$

$$0 = 0$$

## Q6

$$P_{2p_{+1}} = \left|\langle \phi_{2p_{+1}} \Phi^{0\hbar}_{L_x} \rangle\right|^2$$

$$\Psi^{0\hbar}_{L_x} = \frac{1}{\sqrt{2}}\phi_{2p_{-1}} - \frac{1}{\sqrt{2}}\phi_{2p_{+1}}$$

$$P_{2p_{+1}} = \left| -\frac{1}{\sqrt{2}}\langle \phi_{20_{+1}} \phi_{2p_{+1}} \rangle\right|^2 = \frac{1}{2} \text{ (or } 50\%)$$

## Q7

It is useful here to use some of the general commutator relations found in Appendix C.V.

a.

$$[L_x, L_y] = [yp_z - zp_y, zp_x - xp_z]$$
$$= [yp_z, zp_x] - [yp_z, xp_z] - [zp_y, zp_x] + [zp_y, xp_z]$$
$$= [y, z]p_xp_z + z[y, p_x]p_z + y[p_z, z]p_x + yz[p_z, p_x]$$
$$-[y, x]p_zp_z - x[y, p_z]p_z - y[p_z, x]p_z - yx[p_z, p_z]$$
$$-[z, z]p_xp_y - z[z, p_x]p_y - z[p_y, z]p_x - zz[p_y, p_x]$$
$$+[z, x]p_zp_y + x[z, p_z]p_y + z[p_y. x]p_z + zx[p_y, p_z]$$

As can be easily ascertained, the only non-zero terms are:

$$[L_x, L_y] = y\left[p_z, Z\right]p_x + x\left[z, p_z\right]p_y$$
$$= y(i\hbar)p_x + x(i\hbar)p_y$$
$$= i\hbar\left(-yp_x + xp_y\right)$$
$$= i\hbar L_z$$

b.

$$[L_y, L_z] = [zp_x - xp_z, xp_y - yp_z]$$
$$= [zp_x, xp_y] - [zp_x, yp_x] - [xp_z, xp_y] + [xp_z, yp_x]$$
$$= [z, x]p_yp_x + x\left[z, p_y\right]p_x + z\left[p_x, x\right]p_y + zx\left[p_x, p_z\right]$$
$$-[z, y]P_xp_x - y\left[z, p_x\right]p_x - z\left[p_x, y\right]p_x - zy\left[p_x, p_x\right]$$
$$-[x, x]p_yp_z - x\left[x, p_y\right]p_z - x\left[p_z, x\right]p_y - xx\left[p_z, p_y\right]$$
$$+[x, y]p_xp_z + y\left[x, p_x\right]p_z + x\left[p_z, y\right]p_x + xy\left[p_z, p_x\right]$$

Again, as can be easily ascertained, the only non-zero terms are:

$$
\begin{aligned}
[L_y, L_z] &= z\,[p_x, x]\,p_y + y\,[x, p_x]\,p_z \\
&= z(-i\hbar)p_y + y(i\hbar)p_z \\
&= i\hbar(-zp_y + yp_z) \\
&= i\hbar L_x
\end{aligned}
$$

c.

$$
\begin{aligned}
[L_z, L_x] &= [xp_y - yp_x, yp_z - zp_y] \\
&= [xp_y, yp_z] - [xp_y, zp_y] - [yp_x, yp_z] + [yp_x, zp_y] \\
&= [x, y]\,p_z p_y + y\,[x, p_z]\,p_y + x\,[p_y, y]\,p_z + xy\,[p_y, p_z] \\
&- [x, z]\,p_y p_y - z\,[x, p_y]\,p_y - x\,[p_y, z]\,p_y - xz\,[p_y, p_y] \\
&- [y, y]\,p_z p_x - z\,[y, p_y]\,p_x - y\,[p_x, y]\,p_z - yy\,[p_x, p_z] \\
&+ [y, z]\,p_y p_x + z\,[y, p_y]\,p_x + y\,[p_x, z]\,p_y + yz\,[p_x, p_z]
\end{aligned}
$$

Again, as can be easily ascertained, the only non-zero terms are:

$$
\begin{aligned}
[L_z, L_x] &= x\,[p_y, y]\,p_z + z\,[y, p_y]\,p_x \\
&= x(-i\hbar)p_z + z(i\hbar)p_x \\
&= i\hbar(-xp_z + zp_x) \\
&= i\hbar L_y
\end{aligned}
$$

d.

$$
\begin{aligned}
\left[L_x, L^2\right] &= \left[L_x. L_x^2 + L_y^2 + L_z^2\right] \\
&= \left[L_x, L_x^2\right] + \left[L_x, L_y^2\right] + \left[L_x, L_z^2\right] \\
&= \left[L_x, L_y^2\right] + \left[L_x, L_z^2\right] \\
&= [L_x, L_y]\,L_y + L_y\,[L_x. L_y] + [L_x, L_z]\,L_z + L_z\,[L_x, L_z] \\
&= (i\hbar L_z)\,L_y + L_y\,(i\hbar L_z) + (-i\hbar L_y)\,L_z + L_z\,(-i\hbar L_y) \\
&= (i\hbar)\,(-L_z L_x - L_x L_z + L_x L_z + L_z L_x) \\
&= (i\hbar)\,([L_z, L_y] + [L_y, L_z]) = 0
\end{aligned}
$$

e.

$$
\begin{aligned}
\left[L_y, L^2\right] &= \left[L_y, L_x^2 + L_y^2 + L_z^2\right] \\
&= \left[L_y, L_x^2\right] + \left[L_y, L_y^2\right] + \left[L_y, L_z^2\right] \\
&= \left[L_z, L_x^2\right] + \left[L_z, L_y^2\right] \\
&= [L_y, L_x]\,L_x + L_x\,[L_y, L_x] + [L_y, L_z]\,L_z + L_z\,[L_y, L_z] \\
&= (-i\hbar L_z)\,L_x + L_x\,(-i\hbar L_z) + (i\hbar L_x)\,L_z + L_z\,(i\hbar L_x) \\
&= (i\hbar)\,(-L_z L_x - L_x L_z + L_x L_z + L_z L_x) \\
&= (i\hbar)\,([L_x, L_z] + [L_z, L_x]) = 0
\end{aligned}
$$

f.

$$
\begin{aligned}
\left[L_z, L^2\right] &= \left[L_z, L_x^2 + L_y^2 + L_z^2\right] \\
&= \left[L_z, L_x^2\right] + \left[L_z, L_y^2\right] + \left[L_z, L_z^2\right] \\
&= \left[L_z, L_x^2\right] + \left[L_z, L_y^2\right] \\
&= [L_z, L_x]\,L_x + L_x\,[L_z, L_x] + [L_z, L_y]\,L_y + L_y\,[L_z, L_y] \\
&= (i\hbar L_y)\,L_x + L_x\,(i\hbar L_y) + (-i\hbar L_x)\,L_y + L_y\,(-i\hbar L_x) \\
&(i\hbar)\,(L_y L_x + L_x L_y - L_x L_y - L_y L_x)
\end{aligned}
$$

$$(i\hbar)\left([L_y, L_x] + [L_x, L_y]\right) = 0$$

## Q8

Use the general angular momentum relationships:

$$J^2 \big| j, m\rangle = \hbar^2 \left(j(j+1)\right) \big| j, m\rangle$$

$$J_z \big| j, m\rangle = \hbar m \big| j, m\rangle,$$

and the information used in exercise 5, namely that:

$$\mathbf{L}_x = \frac{1}{2}(\mathbf{L}_+ + \mathbf{L}_-)$$

$$\mathbf{L}_+ Y_{l,m} = \sqrt{l(l+l) - m(m+l)}\,\hbar Y_{l,m+1}$$

[ \textbf{L}_- Y_{l,m} = \sqrt{l(l + l) - m(m - l)}\hbar Y_{l,m-1} \nonumber \]

Given that:

$$Y_{0,0}(\theta, \phi) = \frac{1}{\sqrt{4\pi}} = \big|0, 0\rangle$$

$$Y_{1,0}(\theta, \phi) = \sqrt{\frac{3}{4\pi}} Cos\theta = \big|1, 0\rangle.$$

a.

$$\mathbf{L}_z \big|0, 0\rangle = 0$$

[ \textbf{L}^2 \big| 0,0\rangle = 0 \nonumber \]

Since $L^2$ and $L_z$ commute you would expect |0,0> to be simultaneous eigenfunctions of both.

b.

$$\mathbf{L}_x \big|0, 0\rangle = 0$$

$$\mathbf{L}_z \big|0, 0\rangle = 0$$

$L_x$ and $L_z$ **do not** commute. It is unexpected to find a simultaneous eigenfunction (|0,0>) of both ... for sure these operators do not have the same full set of eigenfunctions.

c.

$$\mathbf{L}_z \big|1, 0\rangle = 0$$

$$\mathbf{L}^2 \big|1, 0\rangle = 2\hbar^2 \big|1, 0\rangle$$

Again since $L^2$ and $L_z$ commute you would expect |1,0> to be simultaneous eigenfunctions of both.

d.

$$\mathbf{L}_x \big|1, 0\rangle = \frac{\sqrt{2}}{2}\hbar \big|1, -1\rangle + \frac{\sqrt{2}}{2}\hbar \big|1, 1\rangle$$

$$\mathbf{L}_z \big|1, 0\rangle = 0$$

Again, $L_x$ and $L_z$ **do not** commute. Therefore it is expected to find differing sets of eigenfunctions for both.

## Q9

For

$$\Psi(x, y) = \sqrt{\left(\frac{1}{2L_x}\right)\left(\frac{1}{2L_y}\right)} \left[ e^{\left(\frac{in_x \pi x}{L_x}\right)} - e^{\left(\frac{-in_x \pi x}{L_x}\right)} \right] \left[ e^{\left(\frac{in_y \pi y}{L_y}\right)} - e^{\left(\frac{-in_y \pi y}{L_y}\right)} \right]$$

$$\langle \Psi(x, y) \big| \Psi(x, y) \rangle \stackrel{?}{=} 1$$

Let: $a_x = \frac{n_x \pi}{L_x}$, and $a_y = \frac{n_y \pi}{L_y}$ and using Euler's formula, expand the exponentials into Sin and Cos terms.

$$\Psi(x,y)=\sqrt{\left(\frac{1}{2L_x}\right)\left(\frac{1}{2L_y}\right)}\left[\cos(a_x x)+i\sin(a_x x)-\cos(a_x x)+i\sin(a_x x)\right]\left[\cos(a_y y)+i\sin(a_y y)-\cos(a_y y)\right.$$
$$\left.+i\sin(a_y y)\right]$$
$$\Psi(x,y)=\sqrt{\left(\frac{1}{2L_x}\right)\left(\frac{1}{2L_y}\right)}2i\sin(a_x x)2i\sin(a_y y)$$
$$\Psi(x,y)=\sqrt{\left(-\frac{2}{L_x}\right)\left(\frac{2}{L_y}\right)}\sin(a_x x)\sin(a_y y)$$
$$\langle\Psi(x,y)|\Psi(x,y)\rangle=\int\left(\sqrt{\left(-\frac{2}{L_x}\right)\left(\frac{2}{L_y}\right)}\sin(a_x x)\sin(a_y y)\right)^2 dxdy$$
$$=\left(\frac{2}{L_x}\right)\left(\frac{2}{L_y}\right)\int\sin^2(a_x x)\sin^2(a_y y)dxdy$$

Using the integral:

$$\int_0^L \sin^2\frac{n\pi x}{L}dx=\frac{L}{2},$$
$$\angle\Psi(x,y)|\Psi(x,y)\rangle=\left(\frac{2}{L_x}\right)\left(\frac{2}{L_y}\right)\left(\frac{L_x}{2}\right)\left(\frac{L_y}{2}\right)=1$$

## Q10

$$\langle\Psi(x,y)|p_x|\Psi(x,y)\rangle=\left(\frac{2}{L_y}\right)\int_0^{L_y}\sin^2(a_y y)dy\left(\frac{2}{L_x}\right)\int_0^{L_x}\sin(a_x x)\left(-i\hbar\frac{\partial}{\partial x}\right)\sin(a_x x)dx=\left(\frac{-i\hbar 2a_x}{L_x}\right)\int_0^{L_x}\sin$$
$$(a_x x)\cos(a_x x)dx$$

But the integral:

$$\int_0^{L_x}\cos(a_x x)\sin(a_x x)dx=0,$$
$$\therefore\langle\Psi(x,y)|p_x|\Psi(x,y)\rangle=0$$

## Q11

$$\langle\Psi_0|x^2|\Psi_0\rangle=\sqrt{\frac{\alpha}{\pi}}\int_{-\infty}^{\infty}\left(e^{-\frac{1}{2}\alpha x^2}\right)\left(\mathbf{x}^2\right)\left(e^{-\frac{1}{2}\alpha x^2}\right)dx=2\sqrt{\frac{\alpha}{\pi}}\int_0^{\infty}x^2e^{-\frac{1}{2}\alpha x^2}dx$$

Using the integral:

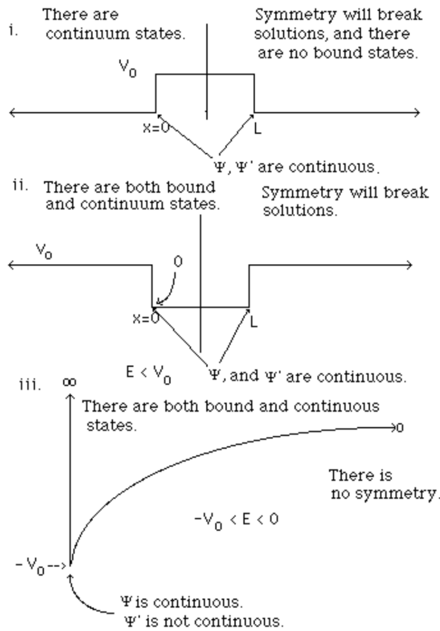
$$\int_0^{\infty}x^{2n}e^{-\beta x^2}dx=\frac{1\cdot 3\cdots(2n-1)}{2^{n+1}}\sqrt{\left(\frac{\pi}{\beta^{2n+1}}\right)}$$
$$\langle\Psi_0|\mathbf{x}^2|\Psi_0\rangle=2\sqrt{\frac{\alpha}{\pi}}\left(\frac{1}{2^2}\right)\sqrt{\frac{\pi}{\alpha^3}}$$
$$\langle\Psi_0|\mathbf{x}^2|\Psi_0\rangle=\left(\frac{1}{2\alpha}\right)$$
$$\langle\Psi_1|\mathbf{x}^2|\Psi_1\rangle=\sqrt{\frac{4\alpha^3}{\pi}}\int_{-\infty}^{\infty}\left(xe^{-\frac{1}{2}\alpha x^2}\right)\left(\mathbf{x}^2\right)\left(xe^{-\alpha\frac{1}{2}x^2}\right)dx$$
$$=2\sqrt{\frac{4\alpha^3}{\pi}}\int_0^{\infty}x^4e^{-\alpha\frac{1}{2}x^2}dx$$

Using the previously defined integral:

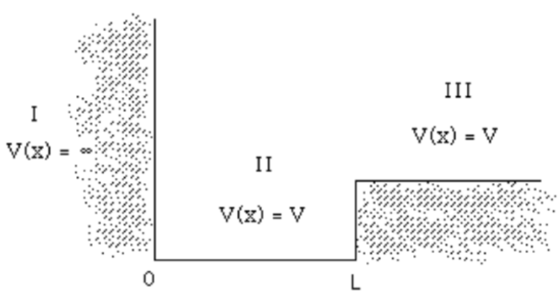
$$\langle \Psi_1 | \mathbf{x}^2 | \Psi_1 \rangle = 2\sqrt{\frac{4\alpha^3}{\pi}} \left(\frac{3}{2^3}\right) \sqrt{\frac{\pi}{\alpha^5}}$$

$$\langle \Psi_1 | \mathbf{x}^2 | \Psi_1 \rangle = \left(\frac{3}{2\alpha}\right)$$

## Q12




## Q13




a.

$$\Psi_I(x) = 0$$

$$\Psi_{II}(x) = Ae^{\left(i\sqrt{\frac{2mE}{\hbar^2}}x\right)} + Be^{\left(-i\sqrt{\frac{2mE}{\hbar^2}}x\right)}$$

$$\Psi_{III}(x) = A'e^{\left(i\sqrt{\frac{2m(V-E)}{\hbar^2}}x\right)} + B'e^{\left(-i\sqrt{\frac{2m(V-E)}{\hbar^2}}x\right)}$$

b.

$$I \leftrightarrow II$$

$$\Psi_I(0) = \Psi_{II}(0)$$

$$\Psi_I(0) = 0 = \Psi_{II}(0) = Ae^{\left(i\sqrt{\frac{2mE}{\hbar^2}}(0)\right)} + Be^{\left(-i\sqrt{\frac{2mE}{\hbar^2}}(0)\right)}$$

$$0 = A + B$$

$$B=-A$$

$$\Psi_I'(0)=\Psi_{II}'(0)$$

( this gives no useful information since $\Psi_I'(x)$ does not exist at x = 0 )

$$I \leftrightarrow II$$

$$\Psi_{II}(L)=\Psi_{III}(L)$$

$$Ae^{\left(i\sqrt{\frac{2mE}{\hbar^2}}L\right)}+Be^{\left(-i\sqrt{\frac{2mE}{\hbar^2}}L\right)}=A'e^{\left(i\sqrt{\frac{2m(V-E)}{\hbar^2}}L\right)}+B'e^{\left(-i\sqrt{\frac{2m(V-E)}{\hbar^2}}L\right)}$$

$$\Psi_{II}'(L)=\Psi_{III}'(L)A\left(e^{i\sqrt{\frac{2mE}{\hbar^2}}L}\right)+B\left(e^{-i\sqrt{\frac{2mE}{\hbar^2}}L}\right)=A'\left(e^{i\sqrt{\frac{2m(V-E)}{\hbar^2}}L}\right)+B'\left(e^{-i\sqrt{\frac{2m(V-E)}{\hbar^2}}L}\right)$$

$$\Psi_{II}'(L)=\Psi_{III}'(L)$$

$$A\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)e^{i\sqrt{\frac{2mE}{\hbar}}L}-B\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)e^{i\sqrt{\frac{2mE}{\hbar}}L}$$

$$=A'\left(i\sqrt{\frac{2m(V-E)}{\hbar^2}}\right)e^{i\sqrt{\frac{2m(V-E)}{\hbar}}L}-B'\left(i\sqrt{\frac{2m(V-E)}{\hbar^2}}\right)e^{i\sqrt{\frac{2m(V-E)}{\hbar}}L}$$

c.

$$asx\to-\infty,\Psi_I(x)=0$$

$$asx\to\infty,\Psi_{III}(x)=0\therefore A'=0$$

d. Rewrite the equations for $\Psi_I(0)$, $\Psi_{II}(0)$, $\Psi_{II}(L)=\Psi_{III}(L)$, and $\Psi_{II}'(L)=\Psi_{III}'(L)$ using the information in 13c:

$$B=-A(eqn.1)$$

$$Ae^{i\sqrt{\frac{2mE}{\hbar}}L}+Be^{-i\sqrt{\frac{2mE}{\hbar}}L}=B'e^{i\sqrt{\frac{2m(V-E)}{\hbar}}L}$$

(eqn. 2)

$$A\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)e^{i\sqrt{\frac{2mE}{\hbar}}L}-B\left(i\sqrt{\frac{2mE}{\hbar^2}}L\right)e^{-i\sqrt{\frac{2mE}{\hbar^2}}L}=-B'\left(i\sqrt{\frac{2m(V-E)}{\hbar^2}}\right)e^{-i\sqrt{\frac{2m(V-E)}{\hbar^2}}L}$$

substitution (eqn. 1) into (eqn. 2)

$$Ae^{i\sqrt{\frac{2mE}{\hbar^2}}L}-Ae^{-i\sqrt{\frac{2mE}{\hbar^2}}L}=B'e^{-i\sqrt{\frac{2m(V-E)}{\hbar^2}}L}$$

$$A\left[\cos\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)+i\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)\right]-A\left[\cos\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)-i\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)\right]=B'e^{-i\sqrt{\frac{(V-E)}{\hbar^2}}L}$$

$$2Ai\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)=B'e^{-i\sqrt{\frac{2m(V-E)}{\hbar^2}}L}$$

$$\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)=\frac{B'}{2Ai}e^{-i\sqrt{\frac{2m(V-E)}{\hbar^2}}L}\quad\text{(eqn. 4)}$$

substituting (eqn. 1) into (eqn. 3):

$$A\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)e^{\left(i\sqrt{\frac{2mE}{\hbar^2}}L\right)}+A\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)e^{\left(-i\sqrt{\frac{2mE}{\hbar^2}}L\right)}=-B'\left(i\sqrt{\frac{2m(V-E)}{\hbar^2}}\right)e^{\left(-1\sqrt{\frac{2m(V-E)}{\hbar^2}}L\right)}$$

$$A\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)\left(\cos\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)+i\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)\right)$$
$$+A\left(i\sqrt{\frac{2mE}{\hbar^2}}\right)\left(\cos\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)-i\sin\left(\sqrt{\frac{2mE}{\hbar^2}}L\right)\right)$$