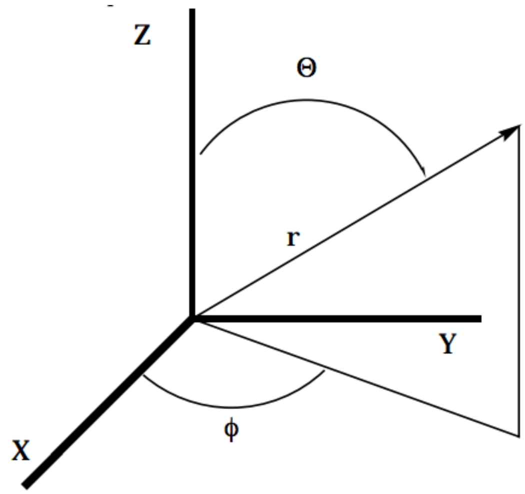
22.1.5: v. Review Exercise Solutions

Q1

The general relationships are as follows:



$$x = \text{rSin}\theta\,\text{Cos}\phi \qquad r^2 = x^2+y^2+z^2$$

$$y = \text{rSin}\theta\,\text{Sin}\phi \qquad \text{Sin}\theta\frac{\sqrt{x^2+y^2}}{\sqrt{x^2+y^2+z^2}}$$

$$z = \text{rCos}\theta \qquad \text{Cos}\theta = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

$$\text{Tan}\phi = \frac{y}{x}$$

$a.$

$$3x+y-4z=12$$
$$3(\text{rSin}\theta\text{Cos}\phi)+\text{Sin}\theta\text{Sin}\phi-4(\text{rCos}\theta)=12$$
$$r(3\text{Sin}\theta\text{Cos}\phi+\text{Sin}\theta\text{Sin}\phi-4\text{Cos}\theta)=12$$

$b.$

$$x = \text{rCos}\phi \qquad r^2 = x^2+y^2$$
$$y = \text{rSin} \qquad \text{Tan}\phi = \frac{y}{x}$$
$$z = z \qquad y^2+z^2=9$$
$$r^2\text{Sin}^2+z^2=9$$

$c.$

$$r = 2\text{Sin}\theta\text{Cos}\phi$$
$$r = 2\left(\frac{x}{r}\right)$$
$$r^2=2x$$
$$x^2+y^2+z^2=2x$$
$$x^2-2x+y^2+z^2=0$$
$$x^2-2x+1+y^2+z^2=1$$
$$(x-1)^2+y^2+z^2=1$$

Q2

a.

$$
\begin{gathered}
9x+16y\frac{\partial y}{\partial x}=0 \\
16ydy=-9xdx \\
\frac{16}{2}y^2=-\frac{9}{2}x^2+c \\
16y^2=-9x^2+c' \\
\frac{y^2}{9}+\frac{x^2}{16}=c''(\text{general equation for an ellipse})
\end{gathered}
$$

b.

$$
\begin{gathered}
2y+\frac{\partial y}{\partial x}+6=0 \\
2y+6=-\frac{dy}{dx} \\
-2dx=-\frac{dy}{2dx} \\
-2dx=\frac{dy}{y+3} \\
-2x=ln(y+3)+c \\
c'e^{-2x}=y+3 \\
y=c'e^{-2x}-3
\end{gathered}
$$

Q3

a. First determine the eigenvalues:

$$
\begin{gathered}
\det\begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix}=0 \\
(-1-\lambda)(2-\lambda)-2^2=0 \\
-2+\lambda-2\lambda+\lambda^2-4=0 \\
\lambda^2-\lambda-6=0 \\
(\lambda-3)(\lambda+2)=0 \\
\lambda=3 \text{ or } \lambda=-2.
\end{gathered}
$$

Next, determine the eigenvectors. First, the eigenvector associated with eigenvalue -2:

$$
\begin{gathered}
\begin{bmatrix} -1 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix}=-2\begin{bmatrix} C_{11} \\ C_{21} \end{bmatrix} \\
-C)11+2C_{21}=-2C_{11} \\
C_{11}=-2C_{21}
\end{gathered}
$$

(Note: The second row offers no new information, e.g. $2C_{11}+2C_{21}=-2C_{21}$)

$$C_{11}^2 + C_{21}^2 = 1 \text{ (from normalization)}$$
$$(-2C_{21})^2 + C_{21}^2 = 1$$
$$4C_{21}^2 + C_{21}^2 = 1$$
$$5C_{21}^2 = 1$$
$$C_{21}^2 = 0.2$$
$$C_{21} = \sqrt{0.2}$$
$$\text{(again the second row offers no new information)}$$
$$C_{12}^2 + C_{22}^2 = 1$$
$$0.25C_{22}^2 + C_{22}^2 = 1$$
$$1.25C_{22}^2 = 1$$
$$C_{22}^2 = 0.8$$
$$C_{22} = \sqrt{0.8} = 2\sqrt{0.2}, \text{ and therefore } C_{12} = \sqrt{0.2}.$$
$$\text{Therefore the eigenvector matrix becomes:}$$
$$\begin{bmatrix} -2\sqrt{0.2} & \sqrt{0.2} \\ \sqrt{0.2} & 2\sqrt{0.2} \end{bmatrix}$$

b. First determine the eigenvalues:

$$\det \begin{bmatrix} -2-\lambda & 0 & 0 \\ 0 & -1-\lambda & 2 \\ 0 & 2 & 2-\lambda \end{bmatrix} = 0$$
$$\det[-2-\lambda] \det \begin{bmatrix} -1-\lambda & 2 \\ 2 & 2-\lambda \end{bmatrix} = 0$$

From 3a, the solutions then become -2, -2, and 3. Next, determine the eigenvectors. First the eigenvector associated with eigenvalue 3 (the third root):

$$\begin{bmatrix} -2 & 0 & 0 \\ 0 & -1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix} = 3 \begin{bmatrix} C_{11} \\ C_{21} \\ C_{31} \end{bmatrix}$$
$$-2C_{13} = 3C_{13} \text{ (row one)}$$
$$C_{13} = 0$$
$$-C_{23} + 2C_{33} = 3C_{23} \text{ (row two)}$$
$$2C_{33} = 2C_{23}$$
$$C_{33} = 2C_{23} \text{ (again the third row offers no new information)}$$
$$C_{13}^2 + C_{23}^2 + C_{33}^2 = 1 \text{ (from normalization)}$$
$$0 + C_{23}^2 + (2C_{23})^2 = 1$$
$$5C_{23}^2 = 1$$
$$C_{23} = \sqrt{0.2}, \text{ and therefore } C_{33} = \sqrt{0.2}.$$

Next, find the pair of eigenvectors associated with the degenerate eigenvalue of -2. First, root one eigenvector one:

$$-2C_{11} = -2C_{11} \text{ (no new information from row one)}$$
$$-C_{21} + 2C_{31} = -2C_{21} \text{ (row two)}$$
$$C_{21} = -2C_{31} \text{ (again the third row offers no new information)}$$
$$C_{11}^2 + C_{21}^2 + C_{31}^2 = 1 \text{ (from normalization)}$$
$$C_{11}^2 + (-2C_{31})^2 + C_{31}^2 = 1$$
$$C_{11}^2 + 5C_{31}^2 = 1$$
$$C_{11} = \sqrt{1 - 5C_{31}^2}$$

Note: There are now two equations with three unkowns. Second, root two eigenvector two:

$$-2C_{12} = -2C_{12} \text{ (no new information from row one)}$$
$$-C_{21} + 2C_{31} = -2C_{21} \text{ (row two)}$$
$$C_{21} = -2C_{31} \text{ (again the third row offers no new information)}$$
$$C_{11}^2 + C_{21}^2 + C_{31}^2 = 1 \text{ (from normalization)}$$
$$C_{12}^2 + (-2C_{32})^2 + C_{32}^2 = 1$$
$$C_{12}^2 + 5C_{32}^2 = 1$$
$$C_{12} = \sqrt{1 - 5C_{32}^2}$$

Note: Again there are now two equations with three unknows.

$$C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32} = 0 \text{ (from orthogonalization)}$$

Now there are five equations with six unknowns.

$$\text{Arbitrarily choose } C_{11} = 0$$
$$C_{11} = 0 = \sqrt{1 - 5C_{31}^2}$$
$$5C_{31}^2 = 1$$
$$C_{31} = \sqrt{0.2}$$
$$C_{21} = -2\sqrt{0.2}$$
$$C_{11}C_{12} + C_{21}C_{22} + C_{31}C_{32} = 0 \text{ (from orthogonalization)}$$
$$0 + -2\sqrt{0.2}(-2C_{32}) + \sqrt{0.2}C_{32} = 0$$
$$5C_{32} = 0$$
$$C_{32} = 0, C_{22} = 0, \text{ and } C_{12} = 1$$

Therefore the eigenvector matrix becomes:

$$\begin{bmatrix} 0 & 1 & 0 \\ -2\sqrt{0.2} & 0 & \sqrt{0.2} \\ \sqrt{0.2} & 0 & 2\sqrt{0.2} \end{bmatrix}$$

Q4

Show: $\langle \phi_1 | \phi_1 \rangle = 1$, $\langle \phi_2 | \phi_2 \rangle = 1$, and $\langle \phi_1 | \phi_2 \rangle = 0$

$$\langle\phi_1|\phi_1\rangle \stackrel{?}{=} 1$$
$$(-2\sqrt{0.2})^2 + (\sqrt{0.2})^2 \stackrel{?}{=} 1$$
$$4(0.2) + 0.2 \stackrel{?}{=} 1$$
$$0.8 + 0.2 \stackrel{?}{=} 1$$
$$1 = 1$$
$$\langle\phi_2|\phi_2\rangle \stackrel{?}{=} 1$$
$$(\sqrt{0.2})^2 + (2\sqrt{0.2})^2 \stackrel{?}{=} 1$$
$$0.2 + 4(0.2) \stackrel{?}{=} 1$$
$$0.2 + 0.8 \stackrel{?}{=} 1$$
$$1 = 1$$
$$\langle\phi_1|_2\phi\rangle = \langle\phi_2|\phi_1\rangle \stackrel{?}{=} 0$$
$$-2\sqrt{0.2}\sqrt{0.2}_{\sqrt{0.2}}2\sqrt{0.2} \stackrel{?}{=} 0$$
$$-2(0.2) + 2(0.2) \stackrel{?}{=} 0$$
$$-0.4 + 0.4 \stackrel{?}{=} 0$$
$$0 = 0$$

Q5

Show (for the degenerate eigenvalue; λ = -2): $\langle\phi_2|\phi_2\rangle = 1$, $\langle\phi_2|\phi_2 = 1, \text{and}_1|e\phi\rangle = 0$

$$\langle\phi_1|\phi_1\rangle \stackrel{?}{=} 1$$
$$0 + (-2\sqrt{0.2})^2 + (\sqrt{0.2})^2 \stackrel{?}{=} 1$$
$$4(0.2) + 0.2 \stackrel{?}{=} 1$$
$$0.8 + 0.2 \stackrel{?}{=} 1$$
$$1 = 1$$
$$\langle\phi_1|\phi_2\rangle \stackrel{?}{=} 1$$
$$1^2 + 0 + 0 \stackrel{?}{=} 1$$
$$1 = 1$$
$$\langle\phi_1|\phi_2 = \langle\phi_2|\phi_1\rangle \stackrel{?}{=} 0$$
$$(0)(1) + (-2\sqrt{0.2})(0) + (\sqrt{0.2})(0) \stackrel{?}{=} 0$$

Q6

Suppose the solution is of the form x(t) = eat , with a unknown. Inserting this trial solution into the differential equation results in the following:

$$\frac{d^2}{dt^2}e^{\alpha t} + k^2 e^{\alpha t} = 0$$
$$\alpha^2 e^{\alpha t} + k^2 e^{\alpha t} = 0$$
$$\left(\alpha^2 + k^2\right) x(t) = 0$$
$$\left(\alpha^2 + k^2\right) = 0$$
$$\alpha^2 = -k^2$$
$$\alpha = \sqrt{-k^2}$$
$$\alpha = \pm ik$$

$\therefore$ Solutions are of the form e^{-ikt}, e^{ikt}, or a combination of both: $x(t) = C_1 e^{ikt} + C_2 e^{-ikt}$.
Euler's formula also states that: $e^{+i\theta} = Cos\Theta \pm iSin\Theta$, so the previous equation for x(t) can also be written as:

$$x(t) = C_1\left[Cos(kt) + iSin(kt)\right] + C_2\left[Cos(kt) - iSin(kt)\right]$$
$$x(t) = (C_1 + C_2)\,Cos(kt) + (C_1 + C_2)\,iSin(kt), \text{ or alternatively}$$
$$x(t) = C_3Cos(kt) + C_4Sin(kt).$$

We can determin these coefficients by making use of the "boundary conditions".

$$\text{at t } = 0, x(0) = L$$
$$x(0) = C_3Cos(0) + C_4Sin(0) = L$$
$$C_3 = L$$
$$\text{at t } = 0, \frac{dx(0)}{dt} = 0$$
$$\frac{d}{dt}x(t) = \frac{d}{dt}(C_3Cos(kt) + C_4Sin(kt))$$
$$\frac{d}{dt}x(t) = -C_3kSin(kt) + C_4kCos(kt)$$
$$\frac{d}{dt}x(0) = 0 = -C_3kSin(0) + C_4kCos(0)$$
$$C_4k = 0$$
$$C_4 = 0$$

∴ The solution is of the form: $x(t) = LCos(kt)$