33.1: Deriving Planck's Distribution Law

Albert Einstein developed a simple but effective analysis of induced emission and absorption of radiation along with spontaneous emission that can be used to derive the Planck formula for thermal radiation.

Consider two energy levels for the molecules in a material. The lower of the two is denoted as E_1 and the higher as E_2. The probability of a transition from level 1 up to level 2 through induced absorption is assumed to be proportional to the energy density per unit frequency interval, $(du/d\nu)$. Likewise the probability of an induced transition from level 2 down to level 1 is assumed also to be proportional to $(du/d\nu)$. These two probabilities are taken to be $B_{12}(du/d\nu)$ and $B_{21}(du/d\nu)$, respectively, where B_{12} and B_{21} are constants. The probability of a spontaneous emission is assumed to be a constant A_{21}.

Let N_1 and N_2 be the number of molecules in energy states 1 and 2, respectively. For **equilibrium** the number of transitions from 1 to 2 has to be equal to the number from 2 to 1; i.e.,

$$\underbrace{N_1\left[B_{12}\left(\frac{du}{d\nu}\right)\right]}_{\text{flow up}} = \underbrace{N_2\left[B_{21}\left(\frac{du}{d\nu}\right)+A_{21}\right]}_{\text{flow down}}$$

This means that the ratio of the occupancies of the energy levels must be

$$\frac{N_2}{N_1} = \frac{B_{12}\left(\frac{du}{d\nu}\right)}{B_{21}\left(\frac{du}{d\nu}\right)+A_{21}} \tag{33.1.1}$$

But the occupancies are given by the Boltzmann distribution as

$$N_1 = N_0 \exp\left(-\frac{E_1}{kT}\right)$$

and

$$N_2 = N_0 \exp\left(-\frac{E_2}{kT}\right)$$

where k is Boltzmann's constant and T is absolute temperature. N_0 is just a constant that is irrelevant for the rest of the analysis.

Thus according to the Boltzmann distribution

$$\frac{N_2}{N_1} = \exp\left(-\frac{E_2-E_1}{kT}\right) \tag{33.1.2}$$

Therefore for radiative equilibrium, Equations 33.1.2 and 33.1.1 can be set to each other and

$$\exp\left(-\frac{E_2-E_1}{kT}\right) = \frac{B_{12}\left(\frac{du}{d\nu}\right)}{B_{21}\left(\frac{du}{d\nu}\right)+A_{21}}$$

This condition can be solved for $(du/d\nu)$; i.e.,

$$\frac{du}{d\nu} = \frac{A_{21}}{B_{12}\exp\left(\frac{E_2-E_1}{kT}\right)-B_{21}}$$

Consider what happens to the above expression for as $T\rightarrow\infty$. It goes to

$$\lim_{T\rightarrow\infty}\frac{du}{d\nu} = \frac{A_{21}}{B_{12}-B_{21}}$$

Einstein maintained that $(du/d\nu)$ must go to infinity as T goes to infinity. This requires that B_{12} be equal to B_{21}.

Thus

$$\frac{du}{d\nu} = \frac{A_{21}/B_{21}}{\exp\left(\frac{E_2 - E_1}{kT}\right) - 1} \tag{33.1.3}$$

Now Planck's assumption is introduced:

$$E_2 - E_1 = h\nu$$

Thus Equation 33.1.3 becomes

$$\frac{du}{d\nu} = \frac{A_{21}/B_{21}}{\exp\left(\frac{hv}{kT}\right) - 1} \tag{33.1.4}$$

The Rayleigh-Jeans Radiation Law says

$$\frac{du}{d\nu} = \frac{8\pi kT\nu^2}{c^3} \tag{33.1.5}$$

The Planck formula must coincide with the Rayleigh-Jeans Law for sufficiently small ν. Note that the exponent in the denominator of Equation 33.1.4 can be expanded (via a Taylor expansion):

$$\exp\left(\frac{h\nu}{kT}\right) \approx 1 + \frac{h\nu}{kT}$$

for sufficiently small ν.

This means that Equation 33.1.4 simplifies to

$$\frac{du}{d\nu} = \frac{A_{21}/B_{21}}{1 + (h\nu/kT) - 1} = \frac{A_{21}/B_{21}}{h\nu/kT}$$

and hence

$$\frac{du}{d\nu} = \left(\frac{A_{21}}{B_{21}}\right)\left(\frac{kT}{h\nu}\right) \tag{33.1.6}$$

Equating Equations 33.1.5 and 33.1.6 for $(du/d\nu)$ gives

$$\left(\frac{A_{21}}{B_{21}}\right)\left(\frac{kT}{h\nu}\right) = \frac{8\pi kT\nu^2}{c^3}$$

which reduces to

$$\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}$$

Thus

$$\frac{du}{d\nu} = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp(h\nu/kT) - 1}$$

This is Planck's formula in terms of frequency.

Reference

1. K.D. Möller, *Optics*, University Science Books, Mill Valley, California, 1988.