32.5: Determinants

The determinant is a useful value that can be computed from the elements of a square matrix

Consider row reducing the standard 2x2 matrix. Suppose that a is nonzero.

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\frac{1}{a} R_1 \rightarrow R_1, \quad R_2 - cR_1 \rightarrow R_2$$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ c & d \end{pmatrix}$$

$$\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & d - \frac{cb}{a} \end{pmatrix}$$

Now notice that we cannot make the lower right corner a 1 if

$$d - \frac{cb}{a} = 0$$

or

$$ad - bc = 0.$$

Definition: The Determinant

We call $ad - bc$ the determinant of the 2 by 2 matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

it tells us when it is possible to row reduce the matrix and find a solution to the linear system.

Example 32.5.1 :

The determinant of the matrix

$$\begin{pmatrix} 3 & 1 \\ 5 & 2 \end{pmatrix}$$

is

$$3(2) - 1(5) = 6 - 5 = 1.$$

Determinants of 3 x 3 Matrices

We define the determinant of a triangular matrix

$$\begin{pmatrix} a & d & e \\ 0 & b & f \\ 0 & 0 & c \end{pmatrix}$$

by

$$\det = abc.$$

Notice that if we multiply a row by a constant k then the new determinant is k times the old one. We list the effect of all three row operations below.

> **Theorem**
>
> The effect of the the three basic row operations on the determinant are as follows
>
> 1. Multiplication of a row by a constant multiplies the determinant by that constant.
> 2. Switching two rows changes the sign of the determinant.
> 3. Replacing one row by that row + a multiply of another row has no effect on the determinant.

To find the determinant of a matrix we use the operations to make the matrix triangular and then work backwards.

> **Example 32.5.2 :**
>
> Find the determinant of
>
> $$\begin{pmatrix} 2 & 6 & 10 \\ 2 & 4 & -3 \\ 0 & 4 & 2 \end{pmatrix}$$
>
> We use row operations until the matrix is triangular.
>
> $$\frac{1}{2}R_1 \rightarrow R_1 (\text{Multiplies the determinant by } \frac{1}{2})$$
>
> $$\begin{pmatrix} 1 & 3 & 5 \\ 2 & 4 & -3 \\ 0 & 4 & 2 \end{pmatrix}$$
>
> $$R_2 - 2R_1 \rightarrow R_2 \text{ (No effect on the determinant)}$$
>
> $$\begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -13 \\ 0 & 4 & 2 \end{pmatrix}$$
>
> Note that we do not need to zero out the upper middle number. We only need to zero out the bottom left numbers.
>
> $$R_3 + 2R_2 \rightarrow R_3 \text{ (No effect on the determinant)}.$$
>
> $$\begin{pmatrix} 1 & 3 & 5 \\ 0 & -2 & -13 \\ 0 & 0 & -24 \end{pmatrix}$$
>
> Note that we do not need to make the middle number a 1.
>
> The determinant of this matrix is 48. Since this matrix has $\frac{1}{2}$ the determinant of the original matrix, the determinant of the original matrix has
>
> $$\text{determinant} = 48(2) = 96.$$

Inverses

We call the square matrix I with all 1's down the diagonal and zeros everywhere else the *identity* matrix. It has the unique property that if A is a square matrix with the same dimensions then

$$AI = IA = A.$$

Definition

If A is a square matrix then the inverse A^{-1} of A is the unique matrix such that

$$AA^{-1} = A^{-1}A = I.$$

Example 32.5.3 :

Let

$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$$

then

$$A^{-1} = \begin{pmatrix} 3 & -5 \\ -1 & 2 \end{pmatrix}$$

Verify this!

Theorem: ExistEnce

The inverse of a matrix exists if and only if the determinant is nonzero.

To find the inverse of a matrix, we write a new extended matrix with the identity on the right. Then we completely row reduce, the resulting matrix on the right will be the inverse matrix.

Example 32.5.4 :

$$\begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}$$

First note that the determinant of this matrix is

$$-2+1=-1$$

hence the inverse exists. Now we set the augmented matrix as

$$\left(\begin{array}{cc|cc} 2 & -1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{array}\right)$$

$$R_1 \leftrightarrow R_2$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 2 & -1 & 1 & 0 \end{array}\right)$$

$$R_2 - 2R_1 \rightarrow R_2$$

$$\left(\begin{array}{cc|cc} 1 & -1 & 0 & 1 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

$$R_1 + R_2 \rightarrow R_1$$

$$\left(\begin{array}{cc|cc} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & -2 \end{array}\right)$$

Notice that the left hand part is now the identity. The right hand side is the inverse. Hence

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

Solving Equations Using Matrices

Example 32.5.5 :

Suppose we have the system

$$2x - y = 3$$

$$x - y = 4$$

Then we can write this in matrix form

$$Ax = b$$

where

$$A = \begin{pmatrix} 2 & -1 \\ 1 & -1 \end{pmatrix}, \quad x = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \text{and } b = \begin{pmatrix} 3 \\ 4 \end{pmatrix}$$

We can multiply both sides by A^{-1}:

$$A^{-1}Ax = A^{-1}b$$

or

$$x = A^{-1}b$$

From before,

$$A^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & -2 \end{pmatrix}$$

Hence our solution is

$$\begin{pmatrix} -1 & -5 \end{pmatrix}$$

or

$$x = -1 \text{ and } y = 5$$

Contributors and Attributions

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