2.3: Oscillatory Solutions to Differential Equations

Learning Objectives

- Explore the basis of the oscillatory solutions to the wave equation
- Understand the consequences of boundary conditions on the possible solutions
- Rationalize how satisfying boundary conditions forces quantization (i.e., only solutions with specific wavelengths exist)

The boundary conditions for the string held to zero at both ends argue that $u(x,t)$ collapses to zero at the extremes of the string (Figure 2.3.1).

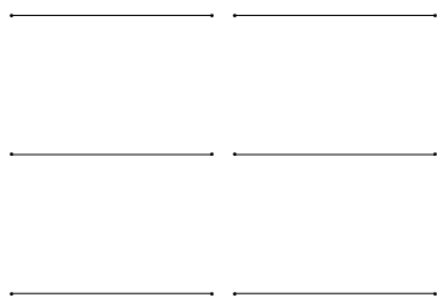

Figure 2.3.1 : Standing waves in a string (both spatially and temporally). from Wikipedia.

Unfortunately, when $K > 0$, the general solution (Equation 2.2.7) results in a sum of exponential decays and growths that cannot achieve the boundary conditions (except for the trivial solution); hence $K < 0$. This means we must introduce complex numbers due to the $\sqrt{K}$ terms in Equation 2.2.5. So we can rewrite K:

$$K = -p^2 \tag{2.3.1}$$

and Equation 2.2.4b can be

$$\frac{d^2X(x)}{dx^2} + p^2X(x) = 0 \tag{2.3.2}$$

The general solution to differential equations of the form of Equation 2.3.2 is

$$X(x) = Ae^{ipx} + Be^{-ipx} \tag{2.3.3}$$

Example 2.3.1

Verify that Equation 2.3.3 is the general form for differential equations of the form of Equation 2.3.2, which when substituted with Equation 2.3.1 give

$$X(x) = Ae^{ipx} + Be^{-ipx}$$

Solution

Expand the complex exponentials into trigonometric functions via Euler formula ($e^{i\theta} = \cos\theta + i\sin\theta$)

$$X(x) = A\left[\cos(px) + i\sin(px)\right] + B\left[\cos(px) - i\sin(px)\right]$$

collecting like terms

$$X(x) = (A+B)\cos(px) + i(A-B)\sin(px) \tag{2.3.4}$$

Introduce new *complex* constants $c_1 = A+B$ and $c_2 = i(A-B)$ so that the general solution in Equation 2.3.4 can be expressed as oscillatory functions

$$X(x) = c_1\cos(px) + c_2\sin(px) \tag{2.3.5}$$

Now let's apply the boundary conditions from Equation 2.2.7 to determine the constants c_1 and c_2. Substituting the first boundary condition ($X(x=0)=0$) into the general solutions of Equation 2.3.5 results in

$$\begin{aligned} X(x=0)=c_1\cos(0)+c_2\sin(0) &=0 \\ c_1+0 &=0 \\ c_1 &=0 \end{aligned} \tag{2.3.6}$$

and substituting the second boundary condition ($X(x=L)=0$) into the general solutions of Equation 2.3.5 results in

$$X(x=L)=c_1\cos(pL)+c_2\sin(pL)=0 \tag{2.3.7}$$

we already know that $c_1=0$ from the first boundary condition so Equation 2.3.7 simplifies to

$$c_2\sin(pL)=0 \tag{2.3.8}$$

Given the properties of sines, Equation 2.3.7 simplifies to

$$pL=n\pi \tag{2.3.9}$$

with $n=0$ is the *trivial solution* that we ignore so $n=1,2,3\ldots$.

$$p=\frac{n\pi}{L} \tag{2.3.10}$$

Substituting Equations 2.3.10 and 2.3.6 into Equation 2.3.5 results in

$$X(x)=c_2\sin\left(\frac{n\pi x}{L}\right)$$

which can simplify to

$$X(x)=c_2\sin(\omega x)$$

with

$$\omega=\frac{n\pi}{L}$$

A similar argument applies to the other half of the *ansatz* ($T(t)$).

? Exercise 2.3.1

Given two traveling waves:

$$\psi_1=\sin(c_1x+c_2t) \text{ and } \psi_2=\sin(c_1x-c_2t)$$

a. Find the wavelength and the wave velocity of ψ_1 and ψ_2
b. Find the following and identify nodes:

$$\psi_+=\psi_1+\psi_2 \text{ and } \psi_-=\psi_1-\psi_2$$

Solution a

ψ_1 is a sin function. At every integer $n\pi$ where $n=0,\pm1,\pm2,\ldots$, a sin function will be zero. Thus, $\psi_1=0$ when $c_1x+c_2t=\pi n$. Solving for the x, while ignoring trivial solutions:

$$x=\frac{n\pi-c_2t}{c_1}$$

The velocity of this wave is:

$$\frac{dx}{dt}=-\frac{c_2}{c_1}$$

Similarly for ψ_2. At every integer $n\pi$ where $n=0,\pm1,\pm2,\ldots$, a sin function will be zero. Thus, $\psi_2=0$ when $c_1x-c_2t=\pi n$. Solving for x, for ψ_2:

$$x = \frac{n\pi + c_2 t}{c_1}$$

The velocity of this wave is:

$$\frac{dx}{dt} = \frac{c_2}{c_1}$$

The wavelength for each wave is twice the distance between two successive nodes. In other words,

$$\lambda = 2(x_n - x_{n-1}) = \frac{2\pi}{c_1}$$

Solution b

Find $\psi_+ = \psi_1 + \psi_2$ and $\psi_- = \psi_1 - \psi_2$.

$$\begin{aligned} \psi_+ &= \sin(c_1 x + c_2 t) + \sin(c_1 x - c_2 t) \\ &= \sin(c_1 x)\cos(c_2 t) + \cancel{\cos(c_1 x)\sin(c_1 t)} + \sin(c_1 x)\cos(c_2 t) - \cancel{\cos(c_1 x)\sin(c_1 t)} \\ &= 2\sin(c_1 x)\cos(c_2 t) \end{aligned}$$

This should have a node at every $x = n\pi/c_1$ and

$$\begin{aligned} \psi_- &= \sin(c_1 x + c_2 t) - \sin(c_1 x - c_2 t) \\ &= \cancel{\sin(c_1 x)\cos(c_2 t)} + \cos(c_1 x)\sin(c_1 t) - \cancel{\sin(c_1 x)\cos(c_2 t)} + \cos(c_1 x)\sin(c_1 t) \\ &= 2\cos(c_1 x)\sin(c_2 t) \end{aligned}$$