2.E: The Classical Wave Equation (Exercises)

Solutions to select questions can be found online.

2.1A

Find the general solutions to the following differential equations:

a. $\frac{d^2y}{dx^2} - 4y = 0$
b. $\frac{d^2y}{dx^2} - 3\frac{dy}{dx} - 54y = 0$
c. $\frac{d^2y}{dx^2} + 9y = 0$

2.1B

Find the general solutions to the following differential equations:

a. $\frac{d^2y}{dx^2} - 16y = 0$
b. $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 27y = 0$
c. $\frac{d^2y}{dx^2} + 100y = 0$

2.1C

Find the general solutions to the following differential equations:

a. $\frac{dy}{dx} - 4\sin(x)y = 0$
b. $\frac{d^2y}{dx^2} - 5\frac{dy}{dx} + 6y = 0$
c. $\frac{d^2y}{dx^2} = 0$

2.2A

Practice solving these first and second order homogeneous differential equations with given boundary conditions:

a. $\frac{dy}{dx} = ay$ with $y(0) = 11$
b. $\frac{d^2y}{dt^2} = ay$ with $y(0) = 6$ and $y'(0) = 4$
c. $\frac{d^2y}{dt^2} + \frac{dy}{dt} - 42y = 0$ with $y(0) = 2$ and $y'(0) = 0$

2.3A

Prove that $x(t)$ = cos(θ) oscillates with a frequency

$$\nu = \frac{1}{2\pi}\sqrt{\frac{k}{m}}$$

Prove that $x(t)$ = cos(θ) also has a period

$$T = 2\pi\sqrt{\frac{m}{k}}$$

where k is the force constant and m is mass of the body.

2.3B

Try to show that

$$x(t) = \sin(\omega t)$$

oscillates with a frequency

$$\nu = \omega/2\pi$$

Explain your reasoning. Can you give another function of x(t) that have the same frequency.

2.3C

Which two functions oscillate with the same frequency?

a. $x(t) = \cos(\omega t)$
b. $x(t) = \sin(2\omega t)$
c. $x(t) = A\cos(\omega t) + B\sin(\omega t)$

2.3D

Prove that $x(t) = \cos(\omega(t))$ oscillates with a frequency

$$\nu = \frac{\omega}{2\pi}.$$

Prove that $x(t) = A\cos(\omega(t) + B\sin(\omega(t))$ oscillates with the same frequency:

$$\nu = \frac{\omega}{2\pi}.$$

2.4

Show that the differential equation:

$$\frac{d^2y}{dx^2} + y(x) = 0$$

has a solution

$$y(x) = 2\sin x + \cos x$$

2.7

For a classical harmonic oscillator, the displacement is given by

$$\xi(t) = v_0\sqrt{\frac{m}{k}}\sin\sqrt{\frac{k}{m}}t$$

where $\xi = x - x_0$. Derive an expression for the velocity as a function of time, and determine the times at which the velocity of the oscillator is zero.

2.11

Verify that

$$Y(x,t) = A\sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

has a frequency ν = v/λ and wavelength λ traveling right with a velocity v.

2.13A

Explain (in words) how to expand the Hamiltonian into two dimensions and use it solve for the energy

2.13B

Given that the Schrödinger equation for a two-dimensional box, with sides a and b, is

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}+\frac{(8\pi^2 mE)}{h^2}\Psi(x,y)=0$$

and it has the boundary conditions of

$$\Psi(0,y)=\Psi(a,y)=0 \text{ and } \Psi(o,x)=\Psi(x,b)=0$$

for all x and y values, show that

$$E_{2,2}=\left(\frac{h^2}{2ma^2}\right)+\left(\frac{h^2}{2mb^2}\right).$$

2.14

Explain, in words, how to expand the Schrödinger Equations into a three-dimensional box

2.18

Solving for the differential equation for a pendulum gives us the following equation,

$$\phi(x)=c_1\cos\sqrt{\frac{g}{L}}+c_2\sin\sqrt{\frac{g}{L}}$$

Assuming $c_1=2$, $c_3=5$, $g=7$ and $L=3$, what is the position of the pendulum initially? Does this make sense in the real world. Why or why not? (We can ignore units for this problem).

2.23

Consider a Particle of mass m in a one-dimensional box of length a. Its average energy is given by

$$\langle E\rangle=\frac{1}{2m}\langle p^2\rangle$$

Because

$$\begin{aligned}\langle p\rangle &=0\\ \langle p^2\rangle &=\sigma_p^2\end{aligned}$$

where σ_p can be called the uncertainty in p. Using the Uncertainty Principle, show that the energy must be at least as large as $\hbar/8ma^2$ because σ_x, the uncertainty in x, cannot be larger than a.

2.33

Prove $y(x,t)=A\cos[2\pi/\lambda(x-vt)]$ is a wave traveling to the right with velocity v, wavelength λ, and period λ/v.
