

# MECHANICAL ENGINEERING VIBRATIONS



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# Mechanical Engineering Vibrations

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## Licensing

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A detailed breakdown of this resource's licensing can be found in [Back Matter/Detailed Licensing](#).

## Topic 01. Introduction to Vibrations

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Topic 01 covers: 1) Explaining the fundamental concepts in vibration; 2) Understanding vibration and vibration systems; 3) Defining degrees of freedom in mechanical systems; 4) Classifying different types of vibrations; 5) Identifying the standard procedure for vibration analysis.

Understanding vibrations is fundamental in Mechanical Engineering, as they play a crucial role in the design, analysis, and maintenance of mechanical systems. Vibrations occur in a wide range of systems, from simple pendulums to complex machinery, and can significantly affect performance, safety, and longevity. By understanding and analyzing vibrations, engineers can predict and mitigate potential issues, ensuring systems operate effectively under various conditions.

## Introduction to Vibrations

### Learning Objectives

- Explain the fundamental concepts in vibration.
- Explain vibration and vibration systems.
- Explain degrees of freedom.
- Classify vibrations.
- Identify vibration analysis procedure.

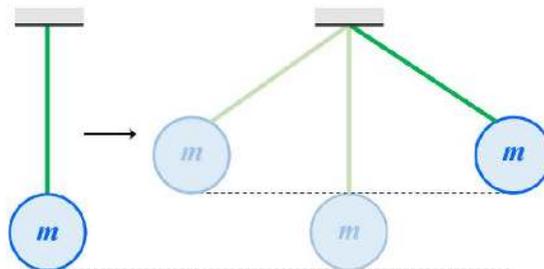
## Understanding of Vibrations

### I. Importance of Study of Vibrations

Positive Effects	Adverse Effects
sound production music instrument electric toothbrushes or massaging units, clocks, etc.	noise pollution earthquake, structure damage in earthquakes _____, _____, and/or _____ in mechanical systems

### II. Vibrations

- Study of \_\_\_\_\_ of \_\_\_\_\_ and the \_\_\_\_\_ associated with them
- Any \_\_\_\_\_ that \_\_\_\_\_ itself after an interval of \_\_\_\_\_  
 e.g. the swinging of a pendulum

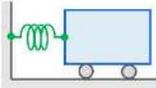
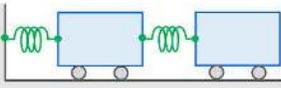
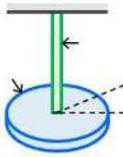
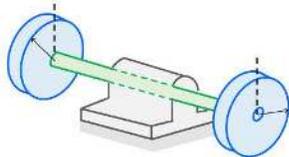


#### • Vibration System

- Potential Energy Storing System: \_\_\_\_\_ or elasticity
- Kinetic Energy Storing System: \_\_\_\_\_ or inertia
- Energy Dissipation : \_\_\_\_\_

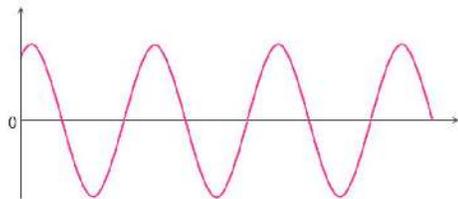
• **Number of Degrees of Freedom**

: Minimum \_\_\_\_ of independent \_\_\_\_\_ required for the \_\_\_\_\_ of all parts of a system at any instant of \_\_\_\_\_

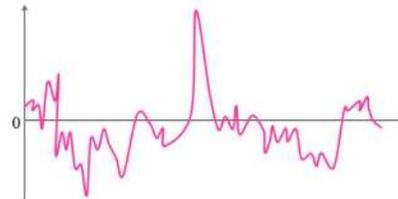
Model	Coord.	DOF	Model	Coord.	DOF
					
					

• **Classification of Vibrations**

- [ \_\_\_\_\_ ] **Free** Vibration: If a system is left to vibrate on its own after an initial disturbance
- [ \_\_\_\_\_ ] **Forced** Vibration: If a system is subjected to an external force
- [ \_\_\_\_\_ ] **Undamped** Vibration: If no energy is lost or dissipated during oscillation
- [ \_\_\_\_\_ ] **Damped** Vibration: If any energy is lost
- [ \_\_\_\_\_ ] **Linear** Vibration: If the basic components behave linearly
- [ \_\_\_\_\_ ] **Nonlinear** Vibration: If the basic components behave nonlinearly
- [ \_\_\_\_\_ ] **Harmonic** Vibration: If the magnitude of the motion acting on a system is known at a given time
- [ \_\_\_\_\_ ] **Stochastic** Vibration: If the magnitude of the motion acting on a system cannot be predicted



Periodic Vibration



Random Vibration

• **Vibration Problem Analysis Procedure**

- Step 1. Draw the \_\_\_\_\_ and the \_\_\_\_\_ for the system.
- Step 2. Derive the \_\_\_\_\_.
- Step 3. \_\_\_\_\_ the EOM.
- Step 4. Interpret the results.

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## Topic 02. Free Vibration of Single DOF: an Undamped Translational System

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Topic 02 covers: 1) Formulating the equation of motion (EOM) for an undamped translational single-degree-of-freedom (DOF) system for both horizontal and vertical systems using fundamental principles; 2) Solving the EOM analytically to determine the system's time-domain response; 3) Mathematically describing harmonic motion using sinusoidal functions; 4) Determining key harmonic motion parameters, including natural frequency and amplitude; 5) Interpreting the physical implications of results obtained from spring-mass models.

Understanding free vibration of single-DOF systems is fundamental in Mechanical Engineering. This topic focuses on the dynamics of undamped translational systems, equipping students with the analytical tools necessary to describe and predict system behavior. By mastering these principles, students build a solid foundation for analyzing more complex vibration systems and applying these concepts to practical engineering challenges.

At the end of this chapter, an animation illustrating simple harmonic motion for a single-degree-of-freedom (1 DOF) spring-mass system under free vibration is provided. Additionally, supplementary materials, titled "Extra Sheets," are included to review the fundamental knowledge required for this topic, covering: 1) Important Terminologies in Vibrations; 2) Differential Equations (ODE vs. PDE); and 3) Mathematical Identities. Additionally, to provide the required background for the practice problem, an animation is included to illustrate the determination of the force resolution angle for a body on an inclined surface. An extra sheet is provided to review the required angle relationships for parallel lines.

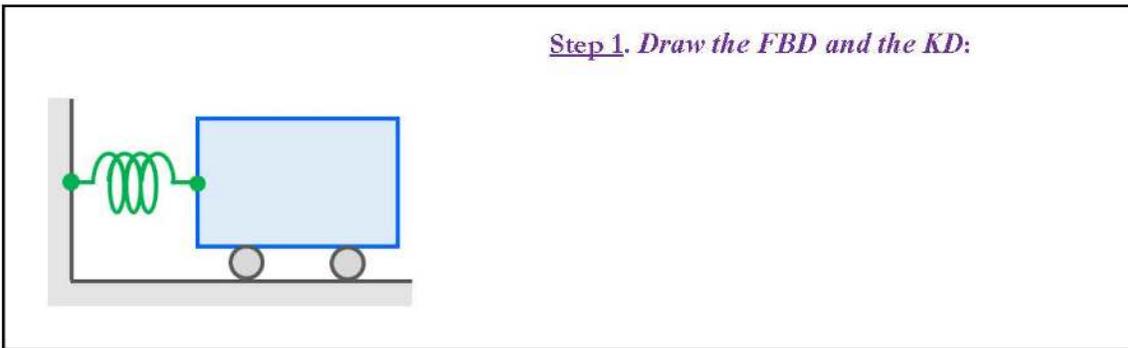
## Free Vibration of Single DOF: an Undamped Translational System

### Learning Objectives

- Formulate the equation of motion (EOM) for an undamped translational single DOF system for both horizontal and vertical systems using fundamental principles.
- Solve the EOM analytically to determine the system's time-domain response.
- Mathematically define and characterize harmonic motion using sinusoidal functions.
- Quantify key parameters of harmonic motion.
- Analyze and interpret the physical implications of results derived from spring-mass models.

## Undamped Translational Single DOF System

### I. Spring-Mass Model (Horizontal)



Step 2. Derive the equation of motion (EoM):

Before motion: =

With motion: By Newton's 2nd law of motion, = \_\_\_\_\_ = :

Hence, the EOM is:

: a \_\_\_\_\_ - order \_\_\_\_\_ differential equation

**Step 3. Solve the EoM:**

Let  $x =$  \_\_\_\_\_ : then  $\dot{x} =$  \_\_\_\_\_  $\ddot{x} =$  \_\_\_\_\_

Substituting them into the EOM,  $($  \_\_\_\_\_  $+$  \_\_\_\_\_  $)Ce^{st} = 0$

Hence, the auxiliary equation:

Roots of the auxiliary equation:  $s = \pm \sqrt{\text{_____}} =$  \_\_\_\_\_, where  $\omega_n =$

Therefore, **the general solution** of the system is:

$x(t) =$  \_\_\_\_\_

By using the identities  $e^{\pm i\omega t} = \cos \omega t \pm i \sin \omega t$ ,

**the general solution** can be written as:  $x(t) = A_1 \sin \omega_n t + A_2 \cos \omega_n t$ .

Hence, **the system response** can be expressed as

$x(t) =$  \_\_\_\_\_

, where  $A_1 = A \cos \phi$  and  $A_2 = A \sin \phi$ .

- $A$  is the \_\_\_\_\_ (\_\_\_\_\_, Displacement)
- $\omega_n$  is the \_\_\_\_\_ [ \_\_\_\_\_ ],  $\omega_n =$  \_\_\_\_\_
- $\phi$  is the \_\_\_\_\_ [ \_\_\_\_\_ ]

To obtain the constants  $A$  and  $\phi$ , apply \_\_\_\_\_ conditions:  $x( ) = x_0$ ,  $\dot{x}( ) = \dot{x}_0$

$\rightarrow x(0) = x_0 =$  \_\_\_\_\_  $\dot{x}(0) = \dot{x}_0 =$  \_\_\_\_\_  $\rightarrow \frac{\dot{x}_0}{\omega_n} =$  \_\_\_\_\_

i) For  $A$ :

$$\rightarrow x_0^2 + \left(\frac{\dot{x}_0}{\omega_n}\right)^2 = A^2 \sin^2 \phi + A^2 \cos^2 \phi$$

$$= A^2(\sin^2 \phi + \cos^2 \phi)$$

ii) For  $\phi$ :

$$\rightarrow \frac{x_0}{\frac{\dot{x}_0}{\omega_n}} = \frac{A \sin \phi}{A \cos \phi} =$$

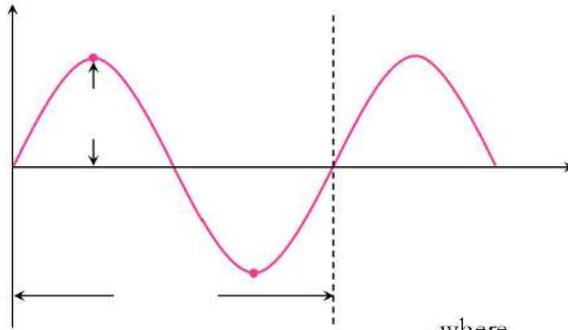
$\therefore A = \sqrt{\text{_____} + \left(\frac{\text{_____}}{\text{_____}}\right)^2}$

$\therefore \phi = \left(\frac{\text{_____}}{\text{_____}}\right)$



### III. Harmonic Motion

- \_\_\_\_\_ motion: motion which is repeated after \_\_\_\_\_ intervals of \_\_\_\_\_
- \_\_\_\_\_ motion: simplest periodic motion



, where

- The **displacement** can be expressed as:

- \_\_\_\_\_ : time
- \_\_\_\_\_ :
- \_\_\_\_\_ :

- Important Terminologies:

$\tau$ : \_\_\_\_\_ to complete \_\_\_\_\_ of motion

$f = \frac{1}{\tau}$

frequency: \_\_\_\_\_ of oscillations per \_\_\_\_\_

- One Cycle of Harmonic Motion

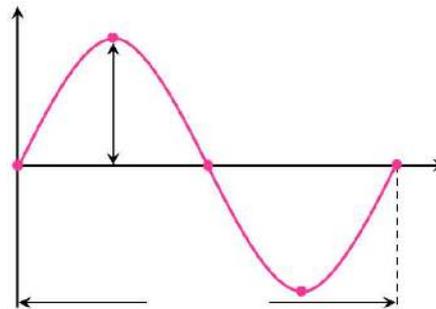
One cycle: \_\_\_\_\_ =

Hence, the circular frequency

$$\omega = \frac{2\pi}{\tau} =$$

Recall:

$$\tau = \frac{2\pi}{\omega}$$



### IV. Interpret the Results (Step 4): Spring-Mass Model

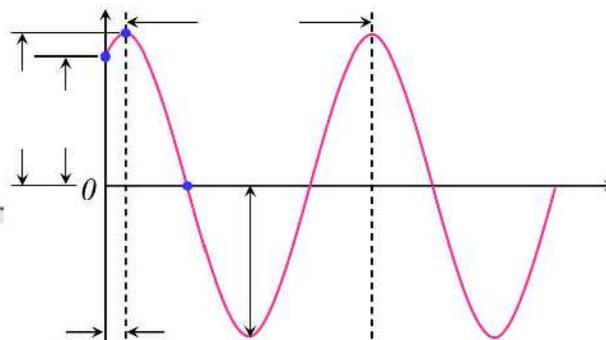
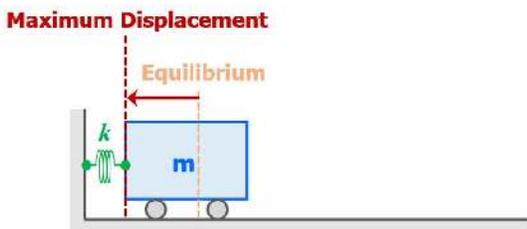
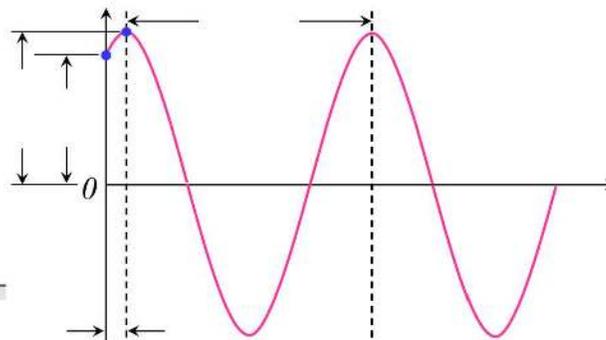
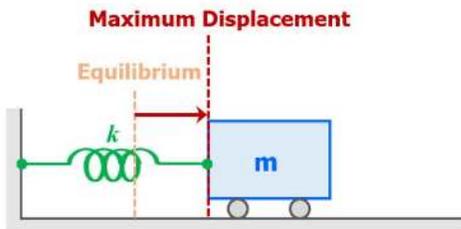
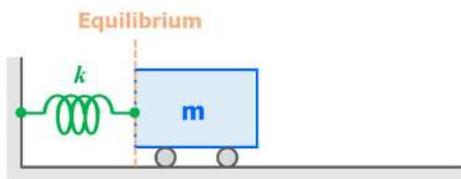
- Recall the system response:

$$x(t) =$$

, where  $\theta$  :

: angle between \_\_\_\_\_ and the \_\_\_\_\_

The motion is \_\_\_\_\_ since it is \_\_\_\_\_ about the solution which is the equilibrium position of the mass. The velocity is a maximum and the acceleration is \_\_\_\_\_ each time the mass passes through this position.



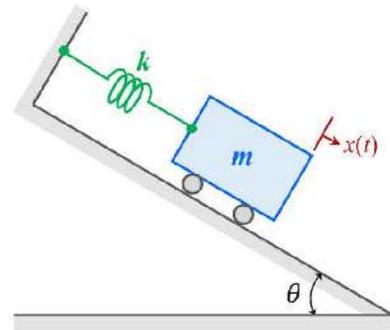
**Practice Problem**

The system undergoes free vibration on the frictionless surface. Derive the equation of motion for the system.

**Given:**

**Find:**

**Solution**



**Answer:**

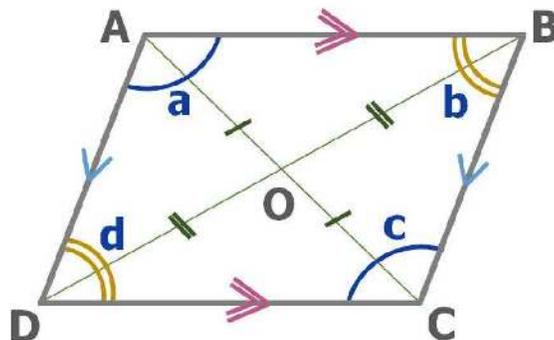
**Normal Force  $N$  on a Titled Surface** : How To determine the  $x$  and  $y$  components using the Given Angle



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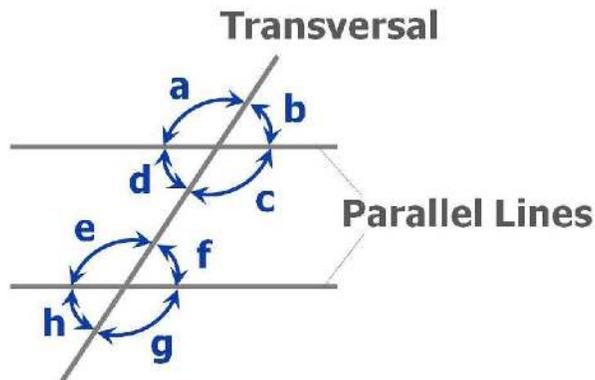


### Parallelogram Properties



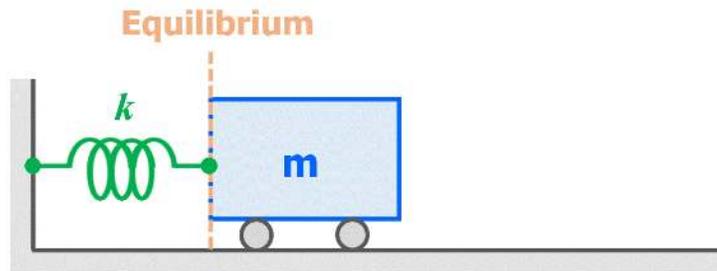
- Two pairs of Parallel Lines:  $\overline{AB} // \overline{CD}$   $\overline{AD} // \overline{BC}$
- Congruent Opposite Sides:  $\overline{AB} = \overline{CD}$   $\overline{AD} = \overline{BC}$
- Congruent Opposite Angles:  $a = c$   $b = d$
- Bisected Diagonals:  $\overline{AO} = \overline{OC}$   $\overline{BO} = \overline{OD}$

### Angles in Parallel Lines



- Corresponding Angles:  $a = e$   $b = f$   $c = g$   $d = h$
- Vertical Angles:  $a = c$   $b = d$   $e = g$   $f = h$
- Alternate Interior Angles:  $d = f$   $c = e$
- Alternate Exterior Angles:  $a = g$   $b = h$

Animations for Simple Harmonic Motion for a 1-DOF Spring-Mass System under Free Vibration



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## Extra Sheet for Important Terminology in Vibrations

## Important Terminology in Vibrations

### Amplitude $A$

The maximum displacement of a vibrating body from its equilibrium position

### (Time) Period $\tau$

The time taken to complete one cycle of motion. When a harmonic motion has an angular velocity  $\omega$ , the time period can be represented as the time required to rotate through an angle of  $2\pi$ :  $\tau = \frac{2\pi}{\omega}$

### Frequency $f$ or $\omega$

The number of cycles or occurrence of a repeating event per unit time.

The period ( $\tau$ ) is the duration of time of one cycle in a repeating event, so the period is the reciprocal of the frequency.

$f$ : the linear frequency:  $f = \frac{1}{\tau} = \frac{\omega}{2\pi}$  [cycles/s] = [Hz]

$\omega$ : the circular frequency: the angular velocity of the cyclic motion [rad/s]

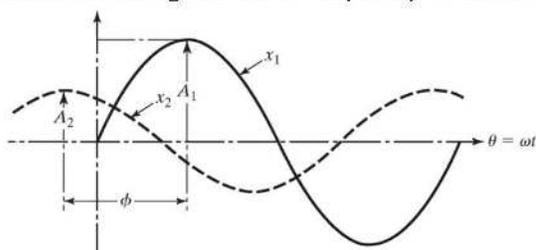
### Natural Frequency $f_n$ or $\omega_n$

After an initial disturbance, if a system is left to vibrate on its own, the frequency with which it oscillates without external forces is called as its natural frequency. In general,  $n$  distinct natural frequencies of vibration.

### Phase (Angle) $\phi$

The angle difference between a measured point and a reference point. It describes the relative timing between two signals.

Phase can be expressed of relative displacement between two corresponding features of two waveforms having the same frequency or same angular velocity  $\omega$ .



For two harmonic motions  $x_1 = A_1 \sin \omega t$

$x_2 = A_1 \sin(\omega t + \phi)$

the two motions are said to have a phase difference  $\phi$ .

## Extra Sheet for Differential Equation

## Differential Equation

### ODE vs PDE

- Ordinary Differential Equation (ODE)

A differential equation that involves a function of a "single variable" and some of its derivatives.

e.g.  $\frac{d^2q}{dx^2} + k(x) = 0$       where  $q = q(x)$

- Partial Differential Equation (PDE)

A differential equation that involves a function of "several variable" and some of its derivatives.

e.g.  $\frac{\partial q}{\partial t} + \frac{\partial^2 q}{\partial x^2} + \frac{\partial^2 q}{\partial y^2} = 0$       where  $q = q(x, y, t)$

### Linear vs Non-linear

- Linear Differential Equation

A differential equation that has the variable appears only with a power of ONE. Linear equations are simple enough to be solvable, whereas non-linear equations can usually not be solved exactly.

e.g.  $x'' + x = 0$  : Linear  
 $x'' + 1/x = 0$  : Non-linear  
 $x'' + \cos x = 0$  : Non-linear

### Homogeneous vs Non-homogeneous

In the linear differential equation  $a \frac{d^2y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$

- Homogeneous

If  $f(x) = 0$ , the linear differential equation is homogenous.

- Non-homogeneous

If  $f(x) \neq 0$ , the linear differential equation is non-homogenous.

## Second Order Linear Homogeneous Equation

$$a \frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = 0 \quad \text{To solve a secondary order problem, two initial conditions are needed.}$$

or

$$a y'' + b y' + cy = 0$$

**H**ow to solve:

**Step 1.** Find the auxiliary equation and its roots.

$$\left. \begin{array}{l} \text{Let } y = e^{mx} \\ \text{If then, } y' = m e^{mx} \\ y'' = m^2 e^{mx} \end{array} \right\} \begin{array}{l} \text{: substitute them into the equation, } a y'' + b y' + cy = 0 \\ (a m^2 + b m + c) e^{mx} = 0 \end{array}$$

The Auxiliary equation is  $a m^2 + b m + c = 0$

	$b^2 - 4ac > 0$	$b^2 - 4ac = 0$	$b^2 - 4ac < 0$
Auxiliary Eqn Roots	2 real and distinct roots: $m, n$	2 real and equal roots: $m$	complex roots: $p + qi, p - qi$
General Solution	$y(x) = C_1 e^{mx} + C_2 e^{nx}$	$y(x) = e^{mx}(C_1 + C_2 x)$	$y(x) = C_1 e^{px} \cos qx + C_2 e^{px} \sin qx$

**Step 2.** Write down the general solution.

Since  $e^{mx} \neq 0$ ,

$$y = e^{mx} \text{ is a solution if } a m^2 + b m + c = 0$$

By the principle of superposition, the general solution is  $y(x) = C_1 y_1(x) + C_2 y_2(x)$ .

**Step 3.** Substitute the initial conditions and work out the constants.

## Extra Sheet for Identities

## Euler's Identities

$$e^{ix} = \cos x + i \sin x \qquad \cos x = \frac{e^{ix} + e^{-ix}}{2}$$

$$e^{-ix} = \cos x - i \sin x \qquad \sin x = \frac{e^{ix} - e^{-ix}}{2i}$$

## Trigonometric Identities

### Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

### Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

### Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

### Opposite Angle Identities

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

### Sum & Difference Identities

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

### Double-Angle Identities

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\begin{aligned} \cos 2\theta &= \cos^2 \theta - \sin^2 \theta \\ &= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta \end{aligned}$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

### Cofunction Identities

$$\sin(90^\circ - \theta) = \cos \theta \qquad \sin(90^\circ + \theta) = \cos \theta$$

$$\cos(90^\circ - \theta) = \sin \theta \qquad \cos(90^\circ + \theta) = -\sin \theta$$

$$\tan(90^\circ - \theta) = \cot \theta \qquad \cot(90^\circ - \theta) = \tan \theta$$

$$\cot(90^\circ + \theta) = -\cot \theta$$

$$\sec(90^\circ - \theta) = \csc \theta \qquad \csc(90^\circ - \theta) = \sec \theta$$

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## Topic 03. Free Vibration of Single DOF: an Undamped Torsional System

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Topic 03 covers: 1) Deriving the equation of motion (EOM) for an undamped torsional single-degree-of-freedom (DOF) system using fundamental principles; 2) Applying the Energy Method to derive the equation of motion based on energy conservation principles; 3) Using Lagrange's Method to systematically formulate the equation of motion.

Understanding free vibration of single-DOF torsional systems is essential in Mechanical Engineering, as many mechanical systems, such as rotating shafts, automotive drivetrain components, turbine rotors, and precision instrumentation mechanisms, can be modeled as torsional systems. This topic explores the dynamics of undamped torsional systems, equipping students with analytical tools to model rotational motion and predict system behavior. Mastering these principles provides a strong foundation for analyzing more complex vibration systems and applying these concepts to real-world engineering applications.

At the end of this chapter, supplementary materials, titled "Extra Sheets," are included to review essential background knowledge, covering 1) Moment of Inertia, and 2) Linearization.

## Free Vibration of Single DOF: Undamped Torsional System

### Learning Objectives

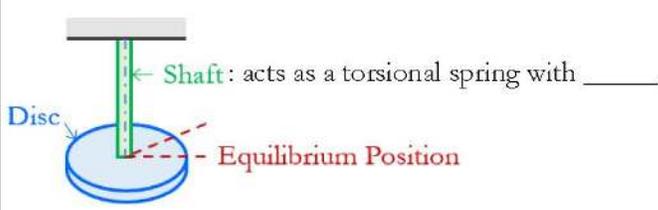
- Derive an equation of motion for an undamped torsional system.
- Derive equations of motion using the Energy Method.
- Derive equations of motion using the Lagrange's Method.

## Torsional Vibration

: Resulting motion of a rigid body which oscillates about a specified reference \_\_\_\_\_

: The body displacement is measured by an \_\_\_\_\_ coordinate, \_\_\_\_\_.

### I. Torsional System Model



Step 1. Draw the FBD and the KD:

### Step 2. Derive the equation of motion (EoM):

Before twisting,  $\sum \quad =$

After twisting, by Newton's 2<sup>nd</sup> law of motion,  $\sum \quad = \quad =$

$\rightarrow \quad =$

Hence, **the EOM** is:

where

- : (polar) \_\_\_\_\_ of the \_\_\_\_\_
- : \_\_\_\_\_ stiffness of the \_\_\_\_\_
- : torsional moment ( \_\_\_\_\_ ) about the \_\_\_\_\_
- : angular \_\_\_\_\_: represents the shaft's \_\_\_\_\_ of \_\_\_\_\_ [ \_\_\_\_\_ ]
- : angular \_\_\_\_\_

Step 3. Solve the EoM:

By approaching similarly with the mass-spring transitional system, *the general solution* is

Step 4. Interpret the results:

From the EOM

- the natural frequency in rad/s:  $= \sqrt{\frac{\quad}{\quad}} \text{ [ } \quad \text{ ]}$

- the natural frequency in Hz:  $= \frac{1}{2\pi} \sqrt{\frac{\quad}{\quad}} \text{ [ } \quad \text{ ] or [ } \quad \text{ ]}$

- the period:  $= \frac{1}{\sqrt{\frac{\quad}{\quad}}} \text{ [ } \quad \text{ ]}$

Recall:

$\omega = \frac{1}{2\pi} \sqrt{\frac{\quad}{\quad}} \text{ [ } \quad \text{ ]}$

$f = \frac{1}{2\pi} \sqrt{\frac{\quad}{\quad}} \text{ [ } \quad \text{ ]}$

$\therefore \omega = \frac{1}{2\pi} \sqrt{\frac{\quad}{\quad}}$

$\therefore f = \frac{1}{2\pi} \sqrt{\frac{\quad}{\quad}}$

## II. Other Methods to Derive EOM

- Principle of Conservation of Energy ( \_\_\_\_\_ Method)

: If a system has \_\_\_\_\_ E dissipation and \_\_\_\_\_ external force, total energy remains \_\_\_\_\_.

$$\text{Kinetic Energy} + \text{Potential Energy} = \quad + \quad =$$

Hence,  $\frac{d}{dt}(\quad) =$

	Spring-Mass System	Torsional System
Displacement		
Kinetic Energy, $T$		
Potential Energy, $U$		
Spring Force or Torque	Spring Force, $F$	Torque, $M_t$

- Lagrange's Equation

- Extended Newtonian mechanics
- Use \_\_\_\_\_ and \_\_\_\_\_ energy to solve for the motion.
- Use the generalized coordinate, \_\_\_\_\_, for any coordinates.
- Motion is determined by \_\_\_\_\_ equations.
- \_\_\_\_\_ need to solve for accelerations.
- \_\_\_\_\_ need to solve for inertial velocities.
- Good for complicated or real systems.
- By the definition of Lagrangian \_\_\_\_\_:

$$\quad$$

- Lagrange's Equation:

$$\frac{d}{dt}(\quad) - \quad = 0 \rightarrow$$

$$\frac{d}{dt}(\quad) - \quad + \quad = 0$$

$q_i$  :

• Gravitational Potential Energy ( $U$ )



, where

$m$ : the mass

$g$ : the gravity

$h$ : the height of an object relative to a chosen reference level, where gravitational  $U = 0$

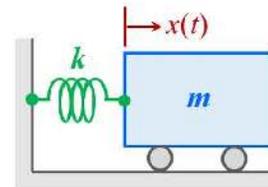
How to Determine “ $h$ ”

Select the \_\_\_\_\_ position, where  $U$  is \_\_\_\_\_, often at the \_\_\_\_\_ of the lowest part of the system. \_\_\_\_\_ is then measured vertically upward from this reference point.

i) Pendulum Rod	ii) Pendulum with a Mass
iii) Inverted Pendulum	

**Practice Problem 1**

Derive the equation of motion (EOM) for the given system using 1) the energy method, and 2) Lagrange's Equation.



**Given:**

**Find:** EOM using 1) the energy method; 2) Lagrange's equation.

**Solution:**

1) using the energy method

Recall:  
 By the Chain Rule,  
 If  $F(x) = f(g(x))$ ,  
 $F'(x) =$

By the energy method,

2) using Lagrange's equation

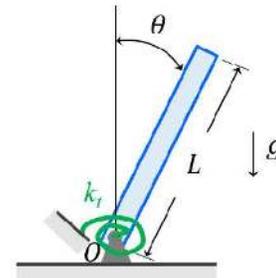
For the spring-mass system,

Putting them into Lagrange's equation for the EOM of the system,

**Answer:**

**Practice Problem 2**

Inverted pendulum consisting of a uniform rigid bar of mass  $m$  and length  $L$  with  $J_0$  the mass moment of inertia about the axis passing through pivot point  $O$  where there is a torsion spring (spring constant  $k_t$ ) attached. Determine the equation of motion using Lagrange's equation and the natural (circular) frequency and period.



**Given:**  $m, L, J_0, k_t$

**Find:** (1) EOM *using Lagrange's equation*; (2) ; (3)

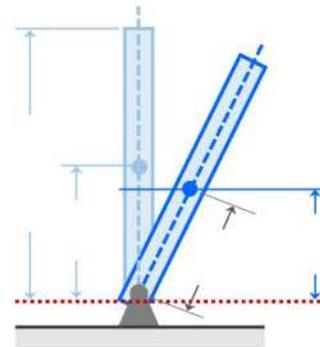
**Solution:**

i) By Lagrange's equation,

$$\frac{d}{dt} \left( \frac{\partial T}{\partial \dot{q}_i} \right) - \frac{\partial T}{\partial q_i} + \frac{\partial U}{\partial q_i} = 0$$

Kinetic Energy of the Bar:

Potential Energy of the Bar:



Set the generalized coordinate:

$$\frac{\partial T}{\partial \dot{\theta}} = \quad \rightarrow \quad \frac{d}{dt} \left( \frac{\partial T}{\partial \dot{\theta}} \right) = \quad ; \quad \frac{\partial T}{\partial \theta} =$$

$$\frac{\partial U}{\partial \theta} =$$

ii) Substitute them into the Lagrange's equation **for the EOM:**

iii) Linearize the EOM by the **small angle approximation**,

→

$$\rightarrow J_0 \ddot{\theta} + \left( k_t - mg \frac{L}{2} \right) \theta = 0$$

$$\omega_n = \sqrt{\quad} \quad ; \quad \tau = \quad$$

Recall:

If  $\theta$  is very small,

$\sin \theta \approx$

$\cos \theta \approx$

**Answer:** (1) EOM: (2)  $\omega_n = \sqrt{\frac{k_t - mg L/2}{J_0}}$  rad/s; (3)  $\tau = \frac{2\pi}{\sqrt{\frac{k_t - mg L/2}{J_0}}}$  [s]



## Moment of Inertia

### Second Moment of Inertia (MOI) or Area MOI, $I$

- Refers **to resistance to bending**.
- A measure for the resistance of a shape to bending about a certain axis.

e.g. the second MOI about the neutral axis  $x$ ,  $I_x$

$$\sigma_x = \frac{M_z y}{I_x} \quad \text{where } \sigma: \text{stress} \quad | \quad M_z: \text{moment about the neutral axis}$$

$y$ : the perpendicular distance to the neutral axis

#### Area MOI about $x$ -axis

$$I_x = \int_A r^2 dA \quad [\text{m}^4]$$

where  $r$ : distance to the axis

$A$ : area of the object

### (Polar) Mass Moment of Inertia, $J$

- Refers **to resistance to angular acceleration**.
- A measure for prediction of an object's ability to resist torsion

e.g. the angular acceleration  $\ddot{\theta}$

$$\ddot{\theta} = \frac{T}{J_z} \quad \text{where } T: \text{torque about } z\text{-axis}$$

#### Mass MOI about $z$ -axis (torque axis)

$$J_z = \int_m r^2 dm \quad [\text{kg} \cdot \text{m}^2]$$

where  $r$ : distance to the torque axis

$m$ : mass of the object

: depends on how mass is distributed around the rotation axis.

## Extra Sheet for Linearization

## Linearization

To approximate a complicated function.

If a differentiable function  $f$  is defined near  $a$ , the linearization of  $f$  at  $a$  is expressed as:

$$L(x) = f(a) + f'(a)(x - a)$$

e.g. Linearization of  $f(x) = \sin x$  at  $x = 0$

$$L(x) = \sin(0) + \cos(0)x = x$$

This also can be explained **by the small-angle approximation**.

**If angle  $\theta$  is very small**, the Taylor series for the trigonometric functions are approximated:

$$\sin \theta \approx \theta$$

$$\cos \theta \approx 1$$

$$\tan \theta \approx \theta$$

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## Topic 04. Free Vibration: Stiffness

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Topic 04 covers: Identifying the stiffness of structural elements in transverse, longitudinal, and rotational vibrations; determining the equivalent stiffness for given spring combinations; understanding how stiffness influences the natural frequency of vibratory systems.

Many mechanical systems are composed of structural elements such as rods, beams, shafts, and springs, each exhibiting stiffness that governs their response to external forces and vibrations. Stiffness plays a crucial role in free vibration analysis, as it directly influences a system's natural frequency, stability, and deformation behavior. Understanding stiffness in various configurations provides essential analytical tools for modeling vibratory systems and predicting their dynamic response. This knowledge enables engineers to design and optimize mechanical structures for improved performance and stability.

## Free Vibration: Stiffness

### Learning Objectives

- Identify stiffness of structural elements in transverse, longitudinal, and rotational vibrations.
- Determine the equivalent stiffness for the given spring combination.

### Stiffness

: In a vibrating system, any \_\_\_\_\_ or \_\_\_\_\_ element—such as \_\_\_\_\_, \_\_\_\_\_, or \_\_\_\_\_—can be modeled as a \_\_\_\_\_ with an equivalent stiffness \_\_\_\_\_, representing its resistance to deformation.

: Most practical vibratory system can be simplified as a \_\_\_\_\_ system.

### I. Applications



Equivalent System



II. Stiffness ( ) in Vibration

: related more directly to \_\_\_\_\_ and \_\_\_\_\_ properties of the spring

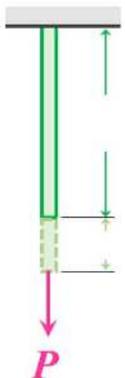
$$k = \frac{\text{Applied}}{\text{Resulting}} = \text{_____}$$

(1) Longitudinal Vibration

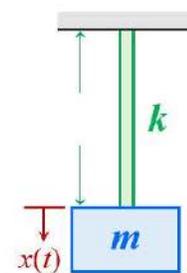
- Prismatic Structural Elements

Equivalent Spring Stiffness

Elongation  $\delta$  of the Rod:



$\delta = \text{_____} = \text{_____} = \text{_____}$



$\therefore \delta = \text{_____}$

$\therefore P = \text{_____}$

Recall:

Strain:  $\epsilon = \text{_____} = \text{_____}$

Stress:  $\sigma = \text{_____}$

The spring force is given by  $F = \text{_____} = \text{_____}$ . Thus, *the stiffness of the rod* in longitudinal vibration is expressed as:

$$k = \text{_____} = \text{_____} = \text{_____}$$

$E$ : elastic modulus of rod  
 $A$ : cross-sectional area of rod  
 $l$ : length of rod

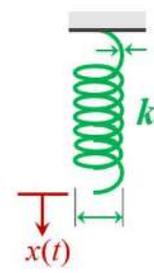
- Helical Spring

: stiffness is dependent on the “\_\_\_\_\_” of the metal rod

:  $k = f(\text{_____, _____, _____, _____})$

$$k = \text{_____}$$

$d$ : dia of spring material  
 $2r$ : dia of turns  
 $n$ : # of turns  
 $G$ : shear modulus



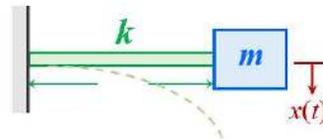
(2) Transverse Vibration (or \_\_\_\_\_ Vibration)

- Beam

: In general, the \_\_\_\_\_ of the beam is negligible.

$$k = \frac{3EI}{l^3}$$

$E$ : elastic modulus of beam  
 $I$ : MoI of cross-sectional area about the neutral axis  
 $l$ : length of beam

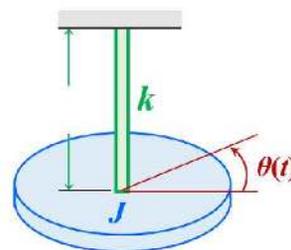


(3) Torsional Vibration

- Prismatic Structural Elements

$$k = \frac{JG}{l}$$

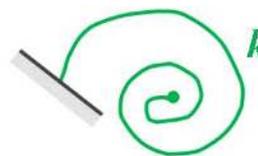
$J$ : MMI of the disk  
 $J_F$ : MMI of the rod  
 $G$ : shear modulus of rigidity of the rod  
 $l$ : length of rod



- Coil Spring

$$k = \frac{E}{l}$$

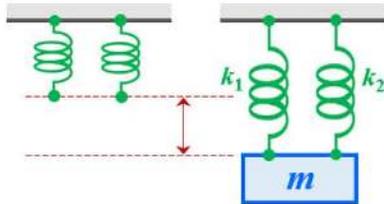
$E$ : elastic modulus of spring  
 $I$ : MoI of cross-sectional area  
 $l$ : total length of spring



### III. Spring Combination

#### • Parallel

- When a load ( ) is applied, the springs undergo a \_\_\_\_\_ ( ).



$$W = \quad + \quad =$$

$$k_{eq} =$$

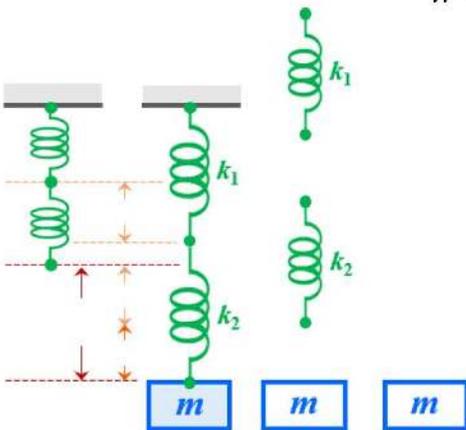
$\delta_{st}$ : static deflection

**Equivalent Spring Stiffness:**

$$\left( \quad \right)_{parallel} =$$

#### • Series

- When a load ( ) is applied, the springs are subjected to the \_\_\_\_\_ ( ).



$$W = \quad \rightarrow \delta_1 = \quad ; \quad W = \quad \rightarrow \delta_2 = \quad$$

Using the equivalent spring constant ( ),

$$W = \quad =$$

$$\rightarrow W = \quad \left( \quad + \quad \right)$$

$$\therefore \quad = \quad + \quad$$

$$\rightarrow k_{eq} = \quad \left( \quad + \quad \right)$$

$\delta_{st}$ : static deflection

**Equivalent Spring Stiffness:**

$$\left( \quad \right)_{series} =$$

**Practice Problem 1**

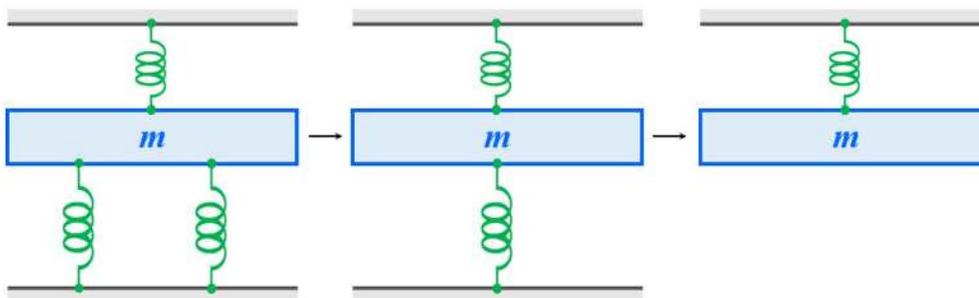
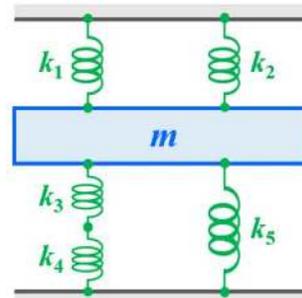
Determine the natural (circular) frequency of the system.

**Given:**

**Find:**

**Solution:**

To find the equivalent stiffness:



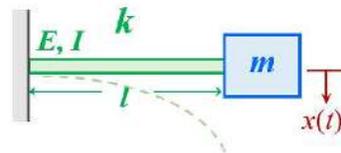
$$\therefore k_{eq} = \frac{(k_1 + k_2 + k_5)(k_3 + k_4) + k_3k_4}{(k_3 + k_4)}$$

**Answer:**

$$\omega_n = \sqrt{\frac{k_1k_3 + k_2k_3 + k_3k_5 + k_1k_4 + k_2k_4 + k_4k_5 + k_3k_4}{m(k_3 + k_4)}} \text{ rad/s}$$

**Practice Problem 2**

The oscillation frequency of a diver on a diving board can be modeled as the transverse vibration of a cantilever beam. Given that the diver's mass is  $m = 100$  kg and the effective length of the board is  $l = 1.5$  m, determine the required flexural rigidity  $EI$  of the diving board material to achieve a desired oscillation frequency of 3 Hz.



**Given:**

**Find:**

**Solution:**

Recall:

$$\omega = \text{---} =$$

Recall:

$k$  for the transverse vibration of a beam

$$k = \text{---}$$

**Answer:**

$$EI = 4.00 \times 10^4 \text{ Nm}^2$$

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## Topic 05. Free Vibration of Single DOF: Viscous Damping

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Topic 05 covers: Explaining the characteristics and behavior of viscous damping elements; deriving the equation of motion (EOM) for a damped single-degree-of-freedom (DOF) system; analyzing the response of damped single-DOF systems.

Damping is a fundamental aspect of mechanical vibrations, governing the dissipation of energy in oscillatory systems. Many engineering structures and mechanical components incorporate damping mechanisms to control vibrations, enhance stability, and prevent excessive oscillations. Viscous damping, in particular, plays a crucial role in reducing vibrational amplitude by converting mechanical energy into heat through fluid resistance or internal material deformation. Understanding damping behavior enables engineers to model and predict system response accurately, ensuring optimal performance and longevity in various applications, including machinery, vehicles, and structural systems.

At the end of this topic, an extra sheet is provided for "Exponential Function".

## Free Vibration of Single DOF: Viscous Damping

### Lecture Objectives

- Explain the characteristics and function of viscous damping elements in mechanical systems.
- Derive the equation of motion (EOM) for a damped single-degree-of-freedom (DOF) system.
- Analyze the dynamic response of damped single-DOF systems.

## Viscous Damping System

### I. Damping

- the mechanism by which vibration energy is \_\_\_\_\_, typically converted into heat, sound, or other forms of energy loss.
- used to \_\_\_\_\_ the response of mechanical systems.
- reduces vibrational \_\_\_\_\_ by converting mechanical energy into heat or sound.
- helps in attenuating excessive oscillations and preventing sustained \_\_\_\_\_.
- essential for accurately predicting and analyzing the vibration \_\_\_\_\_ of mechanical systems.
- used in structures, machines, and vehicles to \_\_\_\_\_ performance, \_\_\_\_\_, and durability

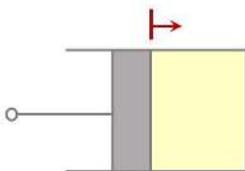
#### • How

- \_\_\_\_\_ | Symbol:

: A damper is assumed to have neither \_\_\_\_\_ nor \_\_\_\_\_.

#### • Types

- **Viscous Damping:** force is \_\_\_\_\_ to the \_\_\_\_\_ of the vibrating body.

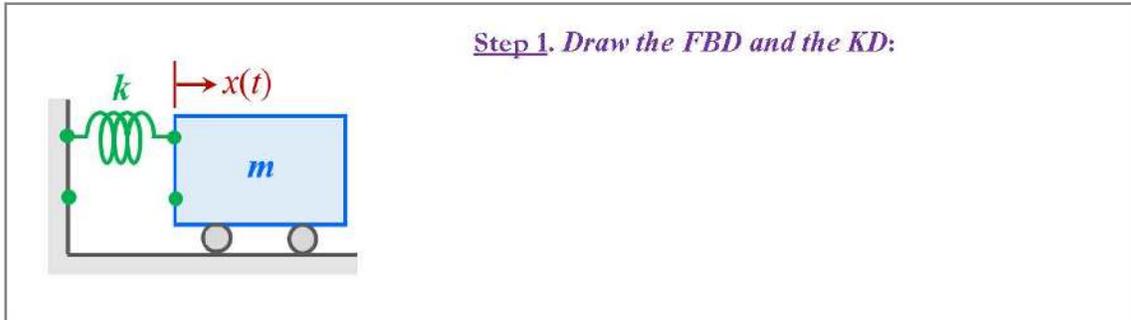


Viscous Damping Force:

$$F = \text{---} = \text{---} [ \quad ]$$

$c$ : [ ] or [ ]

## II. Spring-Mass-Damper Model



Step 2. Derive the equation of motion (EoM):

By Newton's 2<sup>nd</sup> law of motion,  $\sum F = ma$  :

Hence, *the EOM* is:

: a \_\_\_\_\_ order \_\_\_\_\_ differential equation

Step 3. Solve the EoM:

- Assuming the solution  $x = e^{\lambda t}$  :  $\dot{x} = \lambda e^{\lambda t}$  |  $\ddot{x} = \lambda^2 e^{\lambda t}$

- Substitute them into the EoM:

→

→

: *The differential equation* was turned into an \_\_\_\_\_ one.

- Solve for  $\lambda$  with quadratic formula:

$$\lambda_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Recall: Quadratic Formula

$$ax^2 + bx + c = 0$$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

*By the superposition principle, the general solution is*

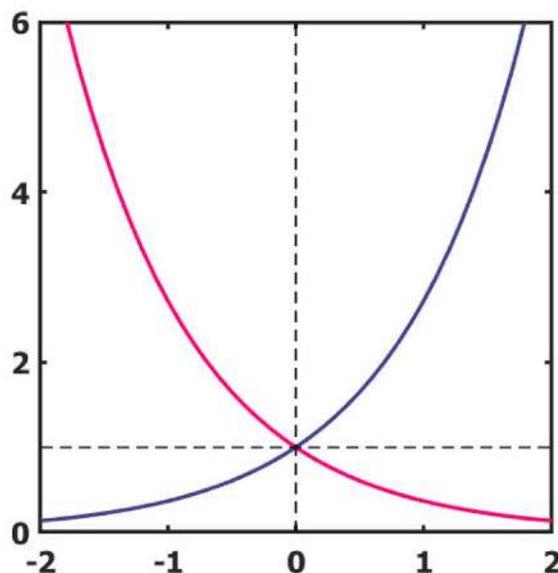
$x =$

, where the constants  $C_1$  and  $C_2$  are obtained from \_\_\_\_\_.

**Step 4. Interpret the results.**

- Enter solutions for  $\lambda_1$  and  $\lambda_2$  into the general solution and re-arrange:

$$x = \left[ C_1 e^{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} t} + C_2 e^{-\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}} t} \right]$$



: the term  $\left(\frac{c}{2m}\right)t$  is a \_\_\_\_\_ in exponential.

: the effect of the terms in the bracket depends on whether the radical becomes \_\_\_\_\_, \_\_\_\_\_, or \_\_\_\_\_.

- if \_\_\_\_\_,  $\left(\frac{c}{2m}\right)^2 > \frac{k}{m} \rightarrow$  the exponents are \_\_\_\_\_ : \_\_\_\_\_ **damped**

- if \_\_\_\_\_,  $\left(\frac{c}{2m}\right)^2 = \frac{k}{m} \rightarrow$  the exponents are \_\_\_\_\_ : \_\_\_\_\_ **damped**

Recall  $e^{i\theta} = \cos \theta + i \sin \theta \rightarrow$  \_\_\_\_\_

- if \_\_\_\_\_,  $\left(\frac{c}{2m}\right)^2 < \frac{k}{m} \rightarrow$  \_\_\_\_\_ **damped**

### III. Critical Damping and Damping Ratio

- **Critical Damping Constant** \_\_\_\_\_

: the value of  $c$  for which the radical becomes \_\_\_\_\_.

$$\left(\frac{c}{2m}\right)^2 - \frac{k}{m} = 0 \rightarrow \left(\frac{c}{2m}\right)^2 = \frac{k}{m} \rightarrow \therefore$$

$$\boxed{c = 2\sqrt{km} = 2m\omega_n \quad [ \quad ]}$$

Recall:

$$\omega_n = \sqrt{\frac{k}{m}} \quad [ \quad ]$$

- **Damping Ratio** \_\_\_\_\_

: the ratio of the damping constant \_\_\_\_\_ to the critical damping constant \_\_\_\_\_

$$\boxed{\zeta = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n} \quad ;}$$

: Recall the general solution  $x = e^{-\left(\frac{c}{2m}\right)t} \left[ C_1 e^{\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}t} + C_2 e^{-\sqrt{\left(\frac{c}{2m}\right)^2 - \frac{k}{m}}t} \right]$

: Since  $\frac{c}{2m} = \frac{c}{2m} = \frac{c}{2m} =$

: Now **the general solution** can be written as

$$\boxed{x = e^{-\zeta\omega_n t} \left[ C_1 e^{(\sqrt{\zeta^2 - 1})\omega_n t} + C_2 e^{(-\sqrt{\zeta^2 - 1})\omega_n t} \right]}$$

Note that **the behavior of the general solution** depends on the \_\_\_\_\_ of damping.

- **Important Ranges of the Damping Ratio**

-  $\zeta = 0$  :

-  $\zeta < 1$  :

-  $\zeta = 1$  :

-  $\zeta > 1$  :

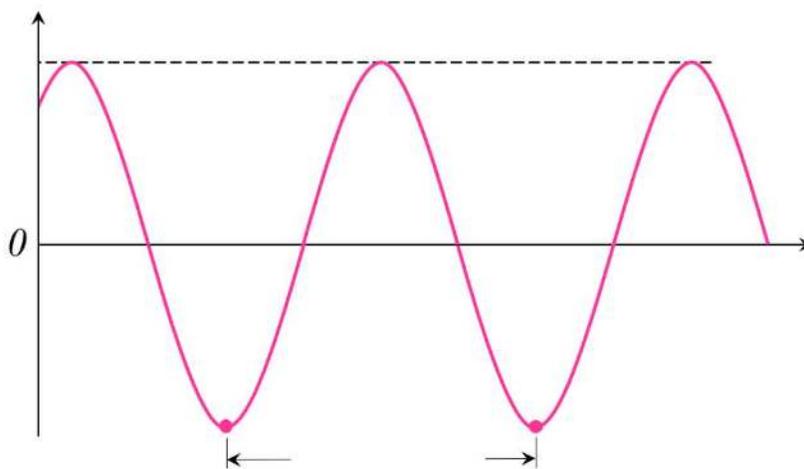
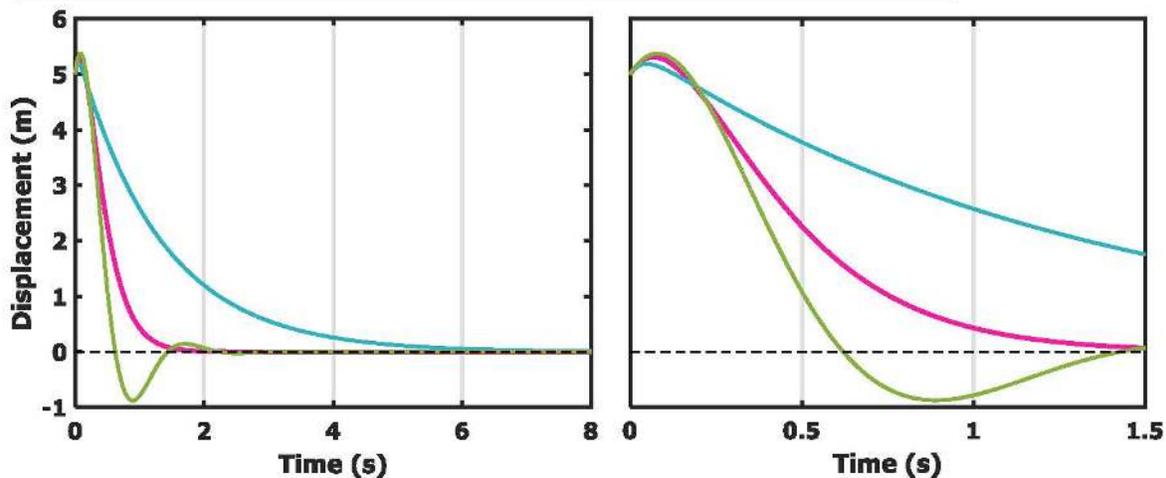
Recall:

$$\zeta = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$$

### IV. Damping Response

Displacement of the damped system:

$$x = e^{-\zeta\omega_n t} \left[ C_1 e^{(\sqrt{\zeta^2-1})\omega_n t} + C_2 e^{(-\sqrt{\zeta^2-1})\omega_n t} \right]$$



• Messages:

- *Critically damped system* will have *the* \_\_\_\_\_ *damping* required for aperiodic motion.
- *If critically damped*, the mass *returns to the position of rest in the* \_\_\_\_\_ *time*.



Extra Sheet for Exponential Function

## Exponential Function

### Exponential Function

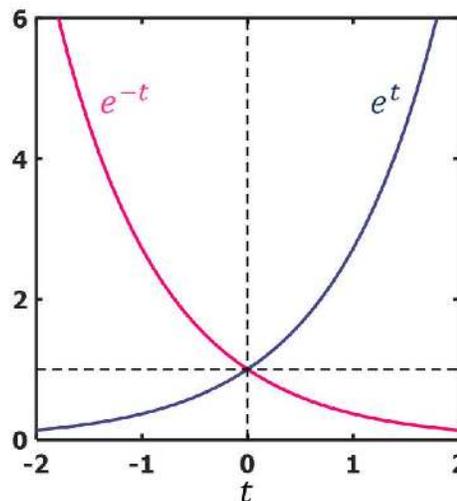
$$f(t) = \exp(t) = e^t$$

where  $e^0 = 1$

### Euler's Number, $e$

: a mathematical constant,  $e = 2.71828 \dots$

: the base of Natural Logarithm,  $\ln x = \log_e x$



	Exponent	Logarithm
	$x = b^t$	$t = \log_b x$
Common Log (based 10)	$x = 10^t$	$t = \log_{10} x$
Natural Log (base e)	$x = e^t$	$t = \log_e x = \ln x$

### Logarithm

$$t = \log_b x$$

: the inverse function to exponentiation

: Exponentiation is when the base  $b$  is raised to the power  $t$ , the exponent to give a value  $x$ :  $x = b^t$

: The base  $b$  is the inverse operation, that provides the output  $t$  from the input  $x$ .

### Imaginary Number $i$

$$i = \sqrt{-1}$$

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## Topic 06. Free Vibration of Single DOF: Damping Estimation from Underdamped Response

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Topic 06 covers: explaining the importance of determining the damping ratio and damping constant from an underdamped system response; analyzing the dynamic response of underdamped single-degree-of-freedom (DOF) systems; explaining the concept of logarithmic decrement; demonstrating how logarithmic decrement is applied in vibration system analysis to quantify damping; determining the logarithmic decrement of damped systems based on measured oscillation amplitudes.

Damping estimation plays a critical role in vibration analysis, as it directly influences the accuracy of system modeling and performance evaluation. In underdamped systems, the displacement response provides valuable insights into the damping characteristics, enabling engineers to quantify energy dissipation through the damping ratio and damping constant. Logarithmic decrement offers a practical method for determining these parameters by analyzing the decay of successive oscillation amplitudes. Understanding and applying these concepts equips engineers with essential tools for evaluating system behavior, optimizing

designs, and ensuring structural integrity across a wide

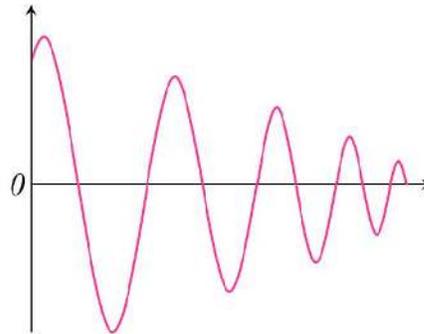
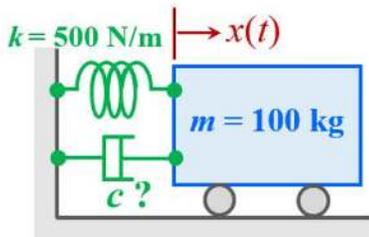
**Free Vibration of Single DOF: Damping Estimation from Underdamped Response**

**Lecture Objectives**

- Explain the importance of determining the damping ratio and damping constant from an underdamped system response.
- Analyze the dynamic response of underdamped single-degree-of-freedom (DOF) systems.
- Explain the concept of logarithmic decrement.
- Demonstrate how logarithmic decrement is applied in vibration system analysis to quantify damping.
- Determine the logarithmic decrement of damped systems based on measured oscillation amplitudes.

# Damping Estimation from Underdamped Response

## I. Motivation and Applications



**• Importance of Vibration Measurements**

: to identify key system parameters, such as \_\_\_\_\_, \_\_\_\_\_, and \_\_\_\_\_, for vibration analysis.

: to \_\_\_\_\_ and refine analytical models of vibratory motion.

**• Challenges in Vibration Measurement**

: Many vibrations exhibit \_\_\_\_\_ amplitudes and \_\_\_\_\_ responses, making accurate measurement challenging

: While \_\_\_\_\_ and \_\_\_\_\_ can be determined through \_\_\_\_\_ testing, \_\_\_\_\_ requires \_\_\_\_\_ testing for reliable evaluation.

**• Damping Identification:**

: The \_\_\_\_\_ response ( \_\_\_\_\_ ) of an \_\_\_\_\_ system provides a practical method for determining \_\_\_\_\_ and the corresponding \_\_\_\_\_.

## II. Underdamped Response ( $0 < \zeta < 1$ )

For the underdamped condition, *the solution* can be expressed as:

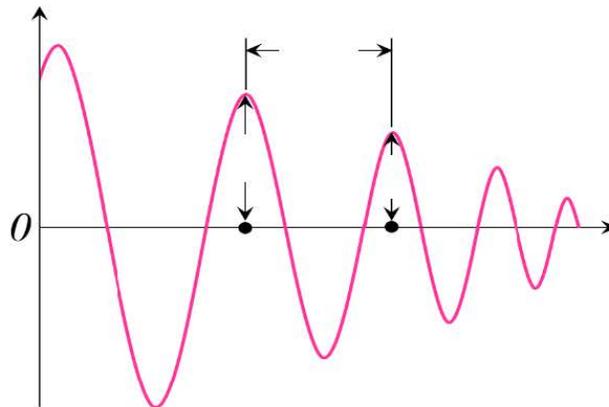
$$x(t) = \left( \sqrt{1 - \zeta^2} \omega_n t \right) \quad (\text{see the Appendix.})$$

- Amplitude: 
$$= \frac{\sqrt{x_0^2 \omega_n^2 + \dot{x}_0^2 + 2x_0 \dot{x}_0 \zeta \omega_n}}{\omega_n \sqrt{1 - \zeta^2}}$$

- Phase Angle: 
$$= \tan^{-1} \left( \frac{\dot{x}_0 + \zeta \omega_n x_0}{x_0 \omega_n \sqrt{1 - \zeta^2}} \right)$$

- \_\_\_\_\_:  $x(t = 0) = x_0$ ,  $\dot{x}(t = 0) = \dot{x}_0$

Note that due to the factor \_\_\_\_\_, the *amplitude* \_\_\_\_\_ exponentially with \_\_\_\_\_.



• Circular \_\_\_\_\_ of \_\_\_\_\_ vibration  $\omega_d =$  \_\_\_\_\_  $\therefore$  always  $\omega_d < \omega_n$

• \_\_\_\_\_ of the \_\_\_\_\_ damped  $\tau_d =$  \_\_\_\_\_ [ \_\_\_\_\_ ]

• Note that only \_\_\_\_\_ damped case can lead to *an oscillatory motion*.

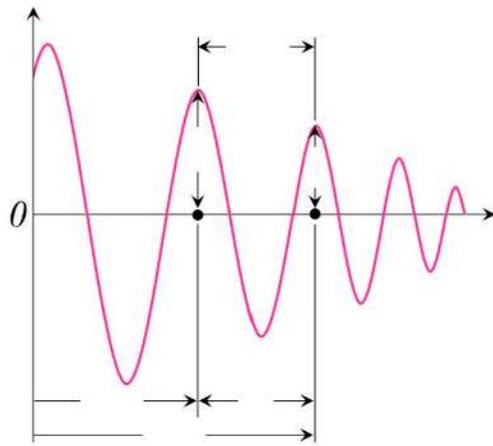
### III. Logarithmic Decrement ( )

• **What**

: the natural logarithm of the ratio of any \_\_\_\_\_ amplitudes

: dimension \_\_\_\_\_

: the \_\_\_\_\_ at which the \_\_\_\_\_ of a \_\_\_\_\_-damped vibration \_\_\_\_\_.



: period of \_\_\_\_\_ vibration:  $\tau_d = \text{_____}$

$$\text{_____} = \frac{Xe^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}{Xe^{-\zeta\omega_n t} \sin(\omega_d t + \phi)}$$

where  $t_2 =$

$$= \frac{Xe^{-\zeta\omega_n t_1} \sin(\omega_d t_1 - \phi)}{Xe^{-\zeta\omega_n (t_1 + \tau_d)} \sin[\omega_d (t_1 + \tau_d) - \phi]}$$

$$= \frac{1}{\text{_____}}$$

$$\begin{aligned} \text{where } \sin[\omega_d(t_1 + \tau_d) + \phi] &= \sin[\omega_d(t_1 + \text{_____}) + \phi] \\ &= \sin(\omega_d t_1 + \text{_____} + \phi) = \sin(\text{_____}) \end{aligned}$$

Hence, *the ratio of two consecutive amplitudes* is:

$$\text{_____} =$$

By the definition, *logarithmic decrement*:

$$= \left[ \text{_____} \right] =$$

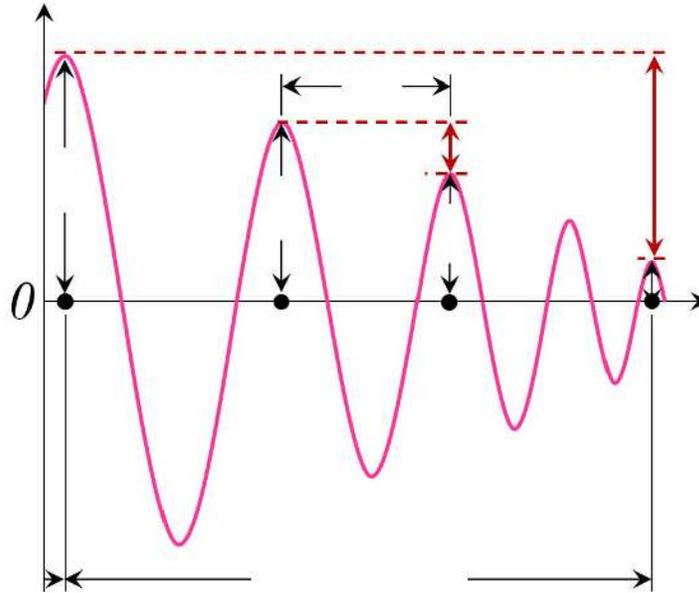
• **How to use**

- to determine the damping *experimentally* by measuring any \_\_\_\_\_ consecutive \_\_\_\_\_

if the \_\_\_\_\_ in the given system is \_\_\_\_\_ known.

• **H**ow to improve \_\_\_\_\_

- For more accuracy, measure \_\_\_\_\_ cycles instead of measuring \_\_\_\_\_ successive cycles.



- Ratio for \_\_\_\_\_ - cycles:

$$\frac{\text{Amplitude of } n\text{-th cycle}}{\text{Amplitude of } 1\text{-st cycle}} = \dots$$

$$= \dots =$$

$$\rightarrow \delta = \ln \left[ \frac{\text{Amplitude of } n\text{-th cycle}}{\text{Amplitude of } 1\text{-st cycle}} \right]$$

Hence, *logarithmic decrement for n-cycles* is:

$$\delta = \ln \left[ \frac{\text{Amplitude of } n\text{-th cycle}}{\text{Amplitude of } 1\text{-st cycle}} \right]$$

Recall:

$$= \left[ \frac{\text{Amplitude of } n\text{-th cycle}}{\text{Amplitude of } 1\text{-st cycle}} \right]$$

$$\tau_d = \frac{\delta}{\omega_d} \quad [ \quad ]$$

$$\omega_d = \frac{\delta}{\tau_d} \quad [ \quad ]$$

$$\omega_n = \frac{\delta}{\tau_d} \quad [ \quad ]$$

$$\zeta = \frac{\delta}{\omega_n \tau_d} = \frac{\delta}{\omega_d \tau_d}$$

**Practice Problem 1**

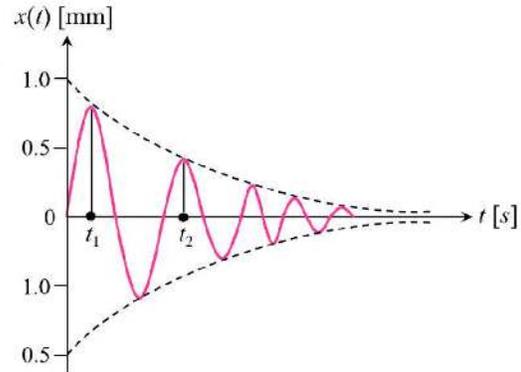
A system responds as shown. The system displacements are 9 mm and 1 mm at  $t_1$  and  $t_2$ , respectively.  $m = 2$  kg and  $k = 1500$  N/m. Find the damping constant.

**Given:**  $m = 2$  kg,  $k = 1500$  N/m  
 ; 9 mm at  $t_1$ , 1 mm at  $t_2$

**Find:**

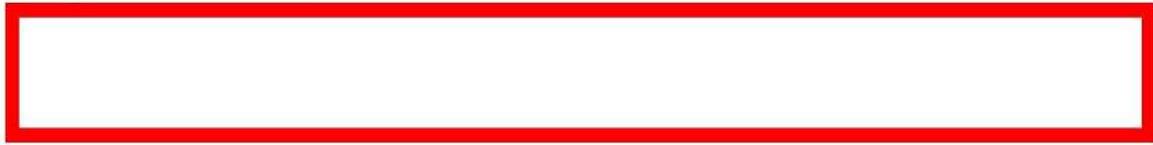
**Solution:**

i) Damping Ratio



$\zeta = \text{---} = \text{---}$

ii) Logarithmic Decrement



$\delta = \text{---} = \text{---}$

$\rightarrow \delta^2 = \text{---} \rightarrow$

$\therefore \delta^2 =$

$\rightarrow \zeta = \frac{2.197}{\sqrt{4\pi^2 + (2.197)^2}} = 0.33$

$\therefore \zeta = 0.33 :$

$\therefore c = 2(2 \text{ kg}) \sqrt{\frac{1500 \text{ N/m}}{2 \text{ kg}}} 0.33 = 36.2$

Recall:

$\omega_d = \text{---} = \text{---} \quad [ \quad ]$

$\tau_d = \text{---} \quad [ \quad ]$

$\omega_n = \text{---} \quad [ \quad ]$

**Answer:**

$c = 36.2 \text{ kg/s}$

**Practice Problem 2**

Express the logarithmic decrement in terms of the damping constant, damped frequency, and mass.

**Given:**

**Find:**

**Solution:**

Recall: What is Logarithmic Decrement?

- \_\_\_\_\_ at which the \_\_\_\_\_ of a \_\_\_\_\_ - \_\_\_\_\_ vibration \_\_\_\_\_
- \_\_\_\_\_ of the \_\_\_\_\_ of

Recall:

$$\delta = \left( \frac{\pi}{\omega_d} \right) =$$

$$x(t) = X e^{-\zeta \omega_n t} \sin \left( \sqrt{1 - \zeta^2} \omega_n t + \phi \right)$$

$$\omega_d =$$

$$\frac{x_1}{x_2} =$$

$$\zeta = \frac{c}{2m} = \frac{\pi}{\omega_d}$$

Answer: 
$$\delta = \left( \frac{c}{m} \right) \left( \frac{\pi}{\omega_d} \right)$$

## Appendix

For the underdamped condition ( $0 < \zeta < 1$ ), the equation of motion is

$$m\ddot{x} + c\dot{x} + kx = 0$$

Assume a solution of the form  $x(t) = Ce^{\lambda t}$  and substitution gives the characteristic equation:

$$m\lambda^2 + c\lambda + k = 0.$$

By the superposition principle, the general solution can be expressed as

$$x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$$

, where  $\lambda_1$  and  $\lambda_2$  are the complex numbers for the underdamped conditions

Define the natural frequency  $\omega_n$  and damping ratio  $\zeta$ :

$$\omega_n = \sqrt{\frac{k}{m}} \quad , \quad \zeta = \frac{c}{2m\omega_n}$$

Using the damping ratio  $\zeta$  and Euler's relations,  $e^{i\theta} = \cos\theta + i\sin\theta$ ;  $e^{-i\theta} = \cos\theta - i\sin\theta$ , the general solution can be expressed as

$$x(t) = e^{-\zeta\omega_n t} [(C_1 + C_2) \cos(\omega_d t) + i(C_1 - C_2) \sin(\omega_d t)], \text{ where } \omega_d = \sqrt{1 - \zeta^2} \omega_n$$

Because physical displacement  $x(t)$  must be real, choose real constants  $A$  and  $B$ , where  $A = C_1 + C_2$  and  $B = C_1 - C_2$ ,

$$x(t) = e^{-\zeta\omega_n t} (A \cos \omega_d t + B \sin \omega_d t)$$

Therefore, *the underdamped system response* can be expressed as

$$x(t) = X e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$$

, where  $X = \sqrt{A^2 + B^2}$

$$\phi = \tan^{-1}\left(\frac{A}{B}\right)$$

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## Topic 07. Forced Vibration of Single DOF: Undamped System Under Harmonic Excitation

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Topic 07 covers: deriving the equation of motion (EOM) for an undamped single-degree-of-freedom (DOF) system subjected to harmonic excitation; solving the EOM to determine the system's steady-state response; interpreting the dynamic behavior of the vibrating system under harmonic excitation.

Harmonic excitation is a common form of external forcing encountered in mechanical systems, arising from periodic inputs such as rotating machinery, engines, or unbalanced loads. Understanding how an undamped system responds to harmonic excitation is fundamental for predicting system behavior, identifying resonance conditions, and ensuring structural integrity. This chapter provides a comprehensive analysis of the system's steady-state response, emphasizing the relationship between excitation frequency, amplitude, and phase. Mastering these concepts enables engineers to evaluate system performance, design effective vibration isolation strategies, and optimize mechanical systems to operate safely and efficiently under periodic loading conditions.

At the end of the chapter, an Extra Sheet is provided on "the Method of Undetermined Coefficients", an approach for determining particular solutions to nonhomogeneous ordinary differential equations.

## Forced Vibration of Single DOF: Undamped System

### Learning Objectives

- Derive the equation of motion (EOM) for an undamped single-degree-of-freedom (DOF) system subjected to harmonic excitation.
- Solve the EOM to determine the system's total response.
- Interpret the dynamic behavior of the vibrating system, with a focus on resonance, phase relationships, and steady-state response characteristics.

## Undamped Single DOF System under Harmonic Excitation

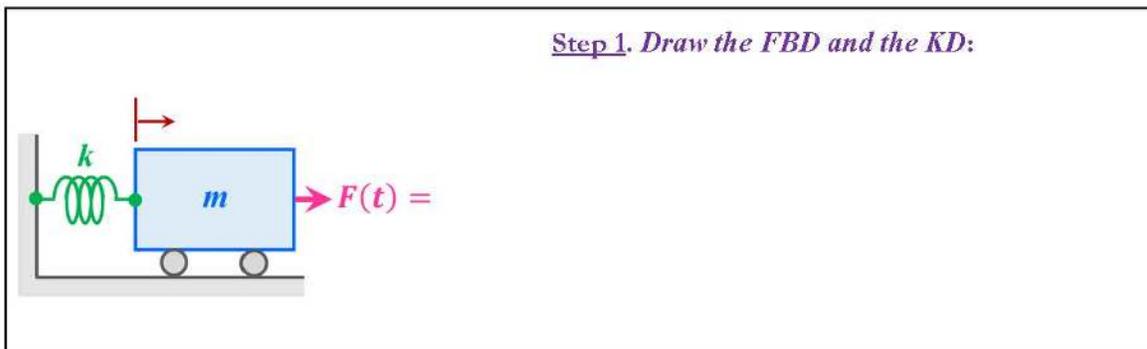
### I. Forced Vibration

: Harmonic Excitation induces \_\_\_\_\_ response of a system.

: Form of Harmonic Excitations:  $F(t) =$

$F_0$ : \_\_\_\_\_ of the harmonic excitation |  $\omega$ : \_\_\_\_\_ of the harmonic excitation

### II. Undamped Spring-Mass System under Harmonic Excitation



Step 2. Derive the equation of motion (EoM):

By Newton's 2nd law of motion, \_\_\_\_\_ = \_\_\_\_\_ :

Hence, *the EOM* is:

: a \_\_\_\_\_ - order \_\_\_\_\_ homogenous differential equation

Step 3. Solve the EoM:

The general solution of the \_\_\_\_\_-homogenous differential equation is

$$x(t) =$$

The homogeneous solution is:

where

$$x_h(t) =$$

$$\omega_n =$$

Assume that the particular solution is:

$$x_p(t) =$$

where  $X$  : \_\_\_\_\_ of the forced response

$\omega$  : \_\_\_\_\_ of the applied force

: then  $\dot{x} =$

$\ddot{x} =$

Substituting them into the EOM:  $m\ddot{x} + kx = F_0 \cos \omega t$ ,

→

$$\text{amplitude: } X = \frac{F_0}{k - m\omega^2} = \frac{F_0}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2} = \frac{F_0}{k} \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$\text{deflection: } \delta_{st} = \frac{F_0}{k}$$

$$\text{ratio: } \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

$$\text{ratio: } \frac{X}{\delta_{st}} = \frac{1}{1 - \left(\frac{\omega}{\omega_n}\right)^2}$$

: ratio of \_\_\_\_\_ to \_\_\_\_\_ amplitude of motion

Thus, the total solution of the system is:

$$x(t) = \left[ \frac{F_0}{k - m\omega^2} \cos \omega t \right] + \left[ \frac{\dot{x}_0}{\omega_n} \sin \omega_n t + \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t \right]$$

Using the \_\_\_\_\_ :  $x(0) = x_0$ ,  $\dot{x}(0) = \dot{x}_0$ , the system response is:

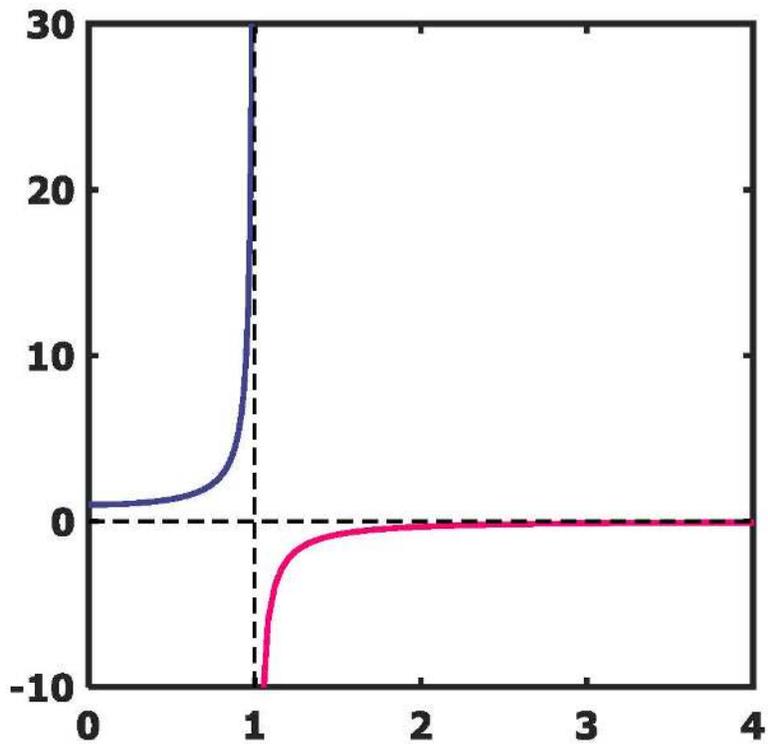
$$x(t) = \left[ \left( \frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t \right] + \left[ \frac{F_0}{k - m\omega^2} \cos \omega t \right]$$

**Step 4. Interpret the results:**

For the undamped spring-mass system under harmonic excitation, *the system response* is:

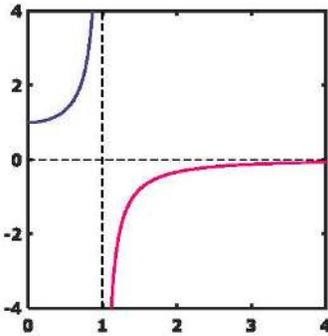
$$x(t) = \left[ \left( \frac{\dot{x}_0}{\omega_n} \right) \sin \omega_n t + \left( x_0 - \frac{F_0}{k - m\omega^2} \right) \cos \omega_n t \right] + \left[ \frac{F_0}{k - m\omega^2} \cos \omega t \right]$$

*The variation of the amplitude ratio with the frequency ratio* is shown:

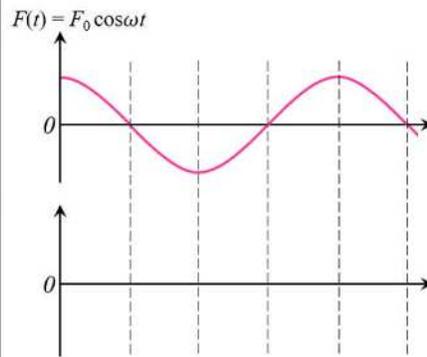


- Response w.r.t different ranges of the frequency ratio

Recall: the amplitude ratio  $\frac{X}{\delta_{st}} = \frac{1}{1-r^2}$  where  $r = \frac{\omega}{\omega_n}$



i) If  $0 < r < 1$ :  $\frac{1}{1-r^2} \rightarrow \frac{\omega}{\omega_n}$   
 Hence, the harmonic response of the system  $x_p(t) =$



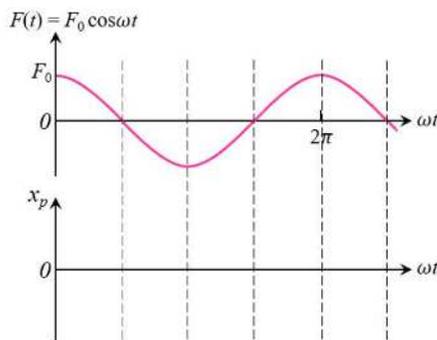
ii) If  $r > 1$ :  $\frac{1}{1-r^2} \rightarrow \frac{\omega}{\omega_n}$

: amplitude of the system  $X$  becomes \_\_\_\_\_, hence the harmonic response is

$x_p(t) = \frac{F_0}{1-r^2} \cos \omega t =$

Need to redefine  $X$  to be a positive quantity:

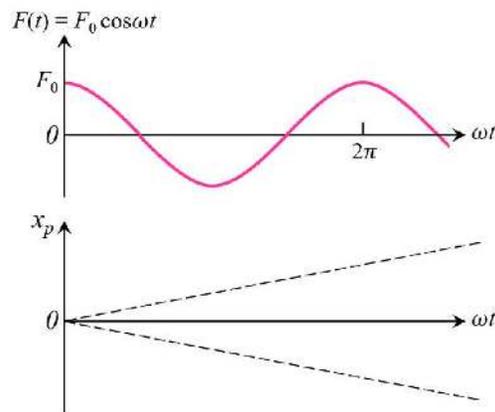
$X = \frac{\delta_{st}}{1-r^2}$



Note: If  $r \rightarrow \infty$ ,  $X \rightarrow 0$ : cannot keep up with force

iii) If  $r = 1$ :  $\omega = \omega_n$  :

: amplitude of the system  $X$  becomes \_\_\_\_\_.



• **Total Response of the System**

- Recall that  $x_p(t) =$  \_\_\_\_\_ is called a \_\_\_\_\_ solution ( ).

- Is this the general solution of the EOM? Y / N

- What else? \_\_\_\_\_ solution ( ):  $x_h(t) =$

- *the total response of the system*  $x(t) =$

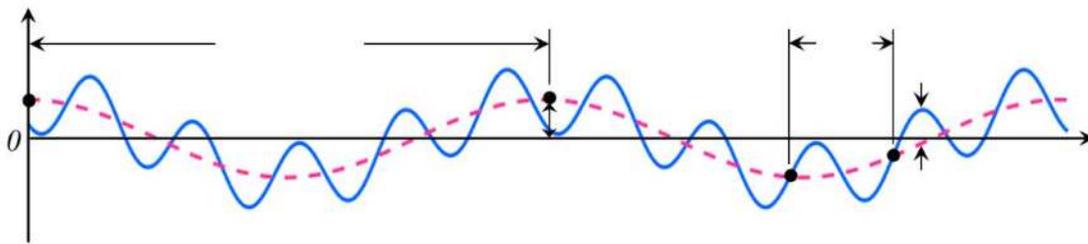
- Hence, *the total response of the system*  $x(t)$  is:

$$x(t) = A \cos(\omega_n t - \emptyset) + \frac{\delta_{st}}{1 - \left(\frac{\omega}{\omega_n}\right)^2} \cos \omega t \quad \text{for } \frac{\omega}{\omega_n} < 1$$

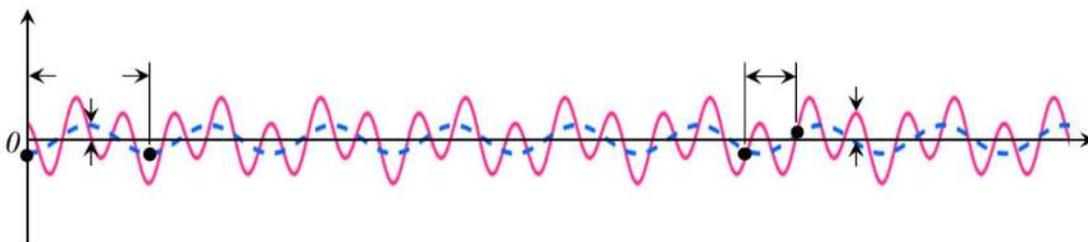
$$x(t) = A \cos(\omega_n t - \emptyset) - \frac{\delta_{st}}{-1 + \left(\frac{\omega}{\omega_n}\right)^2} \cos \omega t \quad \text{for } \frac{\omega}{\omega_n} > 1$$

*The complete motion* can be expressed as the \_\_\_\_\_ of the \_\_\_\_\_ *cosines of different frequencies.*

i)  $\omega < \omega_n$



ii)  $\omega > \omega_n$





## Extra Sheet for Method of Undetermined Coefficients

## Method of Undetermined Coefficients

An approach for determining particular solutions to nonhomogeneous ordinary differential equations and recurrence relations.

Nonhomogeneous second order linear equation is:

$$Ax'' + Bx' + Cx = F(t)$$

The general solution can be expressed in the form:  $x(t) = x_h(t) + x_p(t)$

The homogenous (or complimentary) solution  $x_h(t)$  is obtained from the corresponding homogenous equation:  $Ax'' + Bx' + Cx = 0$

The particular solution  $x_p(t)$  will have the form depending the form of  $F(t)$  as follows:

Function	$x_p(t)$ form
$k e^{ax}$	$C e^{ax}$
$k \cos(ax)$	$C_1 \cos(ax) + C_2 \sin(ax)$
$k \sin(ax)$	$C_1 \cos(ax) + C_2 \sin(ax)$

Then, undetermined coefficient(s)  $C, C_1$  or  $C_2$  is to be found.

This method is called the Method of Undetermined Coefficients.

### Example

EOM for a single DOF under harmonic excitation:

$$m\ddot{x} + kx = F_0 \cos \omega t$$

The general solution is  $x(t) = x_h(t) + x_p(t)$ .

*By the method of undetermined coefficients*, the particular solution  $x_p(t)$  can be expressed as:

$$x_p(t) = X_1 \cos \omega t + X_2 \sin \omega t$$

$$\dot{x}_p(t) = -X_1 \omega \sin \omega t + X_2 \omega \cos \omega t$$

$$\ddot{x}_p(t) = -X_1 \omega^2 \cos \omega t - X_2 \omega^2 \sin \omega t$$

Substituting them into the EOM:

$$m(-X_1 \omega^2 \cos \omega t - X_2 \omega^2 \sin \omega t) + k(X_1 \cos \omega t + X_2 \sin \omega t) = F_0 \cos \omega t$$

$$\rightarrow (-mX_1 \omega^2 + kX_1) \cos \omega t + (-X_2 \omega^2 + kX_2) = F_0 \cos \omega t$$

Hence,  $-mX_1 \omega^2 + kX_1 = F_0$  and  $-X_2 \omega^2 + kX_2 = 0 \rightarrow X_2(-\omega^2 + k) = 0 \rightarrow \mathbf{X_2 = 0}$

Therefore, for this case the particular solution can be directly expressed as:

$$x_p(t) = X \cos \omega t$$

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## Topic 08. Forced Vibration of Single DOF: Damped System Under Harmonic Excitation

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Topic 08 covers: deriving the equation of motion (EOM) for a damped single-degree-of-freedom (DOF) system subjected to harmonic excitation; solving the EOM to determine the system's total response, which includes both the transient and steady-state components; interpreting the dynamic behavior of the vibrating forced damped system under harmonic excitation.

Damping is an inherent characteristic of most mechanical systems, arising from internal material properties, structural connections, and energy dissipation mechanisms such as friction and fluid resistance. When a damped system is subjected to harmonic excitation, its response is influenced by both the system's natural properties and the nature of the external forcing. Unlike undamped systems, where resonance leads to unbounded oscillations, damping introduces energy dissipation, affecting the amplitude, phase, and stability of the steady-state response. Understanding the behavior of a damped system under harmonic excitation is essential for designing practical engineering systems, mitigating excessive vibrations, and optimizing dynamic performance.

At the end of the chapter, an Extra Sheet is provided on "Trigonometric Identities."

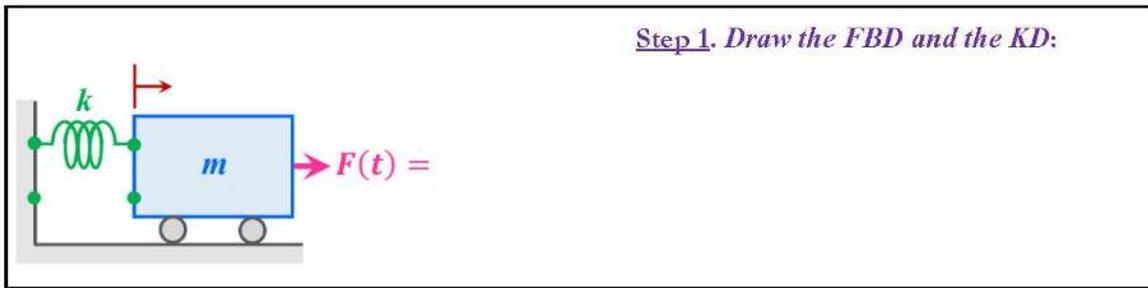
## Forced Vibration of Single DOF: Damped System

### Learning Objectives

- Derive the equation of motion (EOM) for a damped single-degree-of-freedom (DOF) system subjected to harmonic excitation.
- Solve the EOM to determine the total response, distinguishing between the transient and steady-state components.
- Analyze the dynamic behavior of a forced damped system, including the effects of damping on amplitude, phase, and resonance.

## Damped Single DOF System under Harmonic Excitation

### I. Damped Spring-Mass System under Harmonic Excitation



#### Step 2. Derive the equation of motion (EoM):

By Newton's 2nd law of motion,  $\sum F = ma$  ;  $\sum F = m\ddot{x}$

Hence, the EOM is:

: a \_\_\_\_\_ - order \_\_\_\_\_ homogenous differential equation

#### Step 3. Solve the EoM:

*The general solution* of the non-homogenous equation:  $x(t) =$

i) *The homogeneous solution* is:

$m\ddot{x} + c\dot{x} + kx =$

Recall:

 $\zeta = \frac{c}{2\sqrt{km}} = \frac{c}{2m\omega_n}$

$x_h(t) = C_1 e^{\left(\frac{-c \pm \sqrt{c^2 - 4km}}{2m}\right)t} + C_2 e^{\left(\frac{-c \mp \sqrt{c^2 - 4km}}{2m}\right)t}$

ii) According to *the method of the undetermined coefficient, the particular solution* is:

$x_p(t) =$  \_\_\_\_\_  $=$  \_\_\_\_\_

Then,  $\dot{x}_p =$  \_\_\_\_\_ ,  $\ddot{x}_p =$  \_\_\_\_\_

Substituting them into the EOM,  $m\ddot{x} + c\dot{x} + kx = F_0 \cos \omega t$  :

$$X[(k - m\omega^2) \cos(\omega t - \phi') - c\omega \sin(\omega t - \phi')] = F_0 \cos \omega t$$

$$\rightarrow X[(k - m\omega^2)(\cos \omega t \cos \phi' + \sin \omega t \sin \phi') - c\omega(\sin \omega t \cos \phi' - \cos \omega t \sin \phi')] = F_0 \cos \omega t$$

$$\rightarrow X\{[(k - m\omega^2) \cos \phi' + (c\omega \sin \phi')]\cos \omega t + [(k - m\omega^2) \sin \phi' - (c\omega \cos \phi')]\sin \omega t\} = F_0 \cos \omega t$$

Hence,

$$X[(k - m\omega^2) \cos \phi' + c\omega \sin \phi'] =$$

$$X[(k - m\omega^2) \sin \phi' - c\omega \cos \phi'] =$$

Recall:  $\sin^2 \phi' + \cos^2 \phi' = 1$   
 $\tan \phi' = \frac{\sin \phi'}{\cos \phi'}$

Squaring the both sides,

$$X^2 [(k - m\omega^2) \cos \phi' + c\omega \sin \phi']^2 =$$

$$X^2 [(k - m\omega^2) \sin \phi' - c\omega \cos \phi']^2 =$$

i) Adding them for  $X$ , and ii) dividing  $X[(k - m\omega^2) \sin \phi' - c\omega \cos \phi'] = 0$  by  $\cos \phi'$  for  $\phi'$ :

**Amplitude**  $X = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}}$

**Phase Angle**  $\phi' = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$

$$X[(k - m\omega^2) \sin \phi' - c\omega \cos \phi'] = 0$$

$$\rightarrow (k - m\omega^2) \sin \phi' = c\omega \cos \phi'$$

$$\therefore \frac{\sin \phi'}{\cos \phi'} = \frac{c\omega}{(k - m\omega^2)}$$

Recall: **Undamped Natural Frequency:**  $\omega_n = \sqrt{\frac{k}{m}}$

**Damping Ratio:**  $\zeta = \frac{c}{2m\omega_n} = \frac{c}{2\sqrt{km}}$   $\rightarrow \frac{c}{m} = 2\zeta\omega_n$

**Static Deflection** under  $F_0$ :  $\delta_{st} = \frac{F_0}{k}$  | **Frequency Ratio:**  $r = \frac{\omega}{\omega_n}$

Using them (see APPENDIX on page 3 for their derivations),

$$X = \frac{F_0}{\sqrt{\left\{ \left[ 1 - \left(\frac{\omega}{\omega_n}\right)^2 \right]^2 + \left[ 2\zeta \left(\frac{\omega}{\omega_n}\right) \right]^2 \right\}}} = \frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} =$$

$$\frac{F_0}{\sqrt{(k - m\omega^2)^2 + (c\omega)^2}} : \text{dimension } \frac{N}{N} = 1$$

$$\phi' = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) = \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)$$

## APPENDIX

**H**ow to express the amplitude and phase angle using the dimensionless parameters

$$\begin{aligned}
 X &= \frac{F_0}{\sqrt{(k - m\omega^2)^2 + c^2\omega^2}} \\
 &= \frac{\frac{F_0}{k}}{\sqrt{\left(1 - \frac{m}{k}\omega^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}} = \frac{\delta_{st}}{\sqrt{\left(1 - \left(\frac{\omega}{\omega_n}\right)^2\right)^2 + \left(\frac{c\omega}{k}\right)^2}} = \frac{\delta_{st}}{\sqrt{(1 - r^2)^2 + (2\zeta r)^2}}
 \end{aligned}$$

where

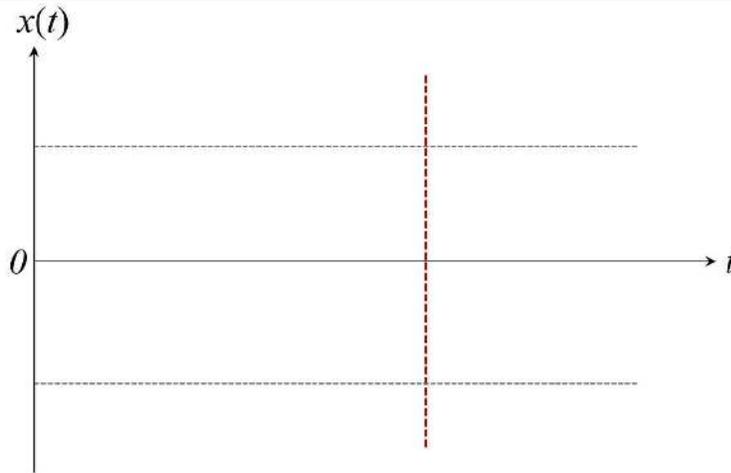
$$\begin{aligned}
 \frac{c\omega}{k} &= \frac{2\sqrt{mk}\zeta\omega}{k} = 2\zeta\omega\sqrt{\frac{mk}{k^2}} = 2\zeta\omega\sqrt{\frac{m}{k}} = 2\zeta\frac{\omega}{\omega_n} = 2\zeta r \\
 \zeta &= \frac{c}{2m\omega_n} = \frac{c}{2m\sqrt{mk}}
 \end{aligned}$$

$$\begin{aligned}
 \phi' &= \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right) \\
 &= \tan^{-1}\left(\frac{\frac{c}{m}\omega}{\frac{k}{m} - \omega^2}\right) = \tan^{-1}\left(\frac{\frac{c}{m}\omega}{\omega_n^2 - \omega^2}\right) = \tan^{-1}\left(\frac{\frac{c}{m}\frac{\omega}{\omega_n^2}}{1 - \left(\frac{\omega}{\omega_n}\right)^2}\right) = \tan^{-1}\left(\frac{2\zeta\omega_n\frac{\omega}{\omega_n^2}}{1 - r^2}\right) \\
 &= \tan^{-1}\left(\frac{2\zeta r}{1 - r^2}\right)
 \end{aligned}$$

Step 4. Interpret the results:

• System response for the Underdamped Case

$x(t) =$   
 $=$   
 \_\_\_\_\_ Response                      \_\_\_\_\_ - state Response  
 For large time, \_\_\_\_\_  
 \_\_\_\_\_ on \_\_\_\_\_                      \_\_\_\_\_ dependent on \_\_\_\_\_



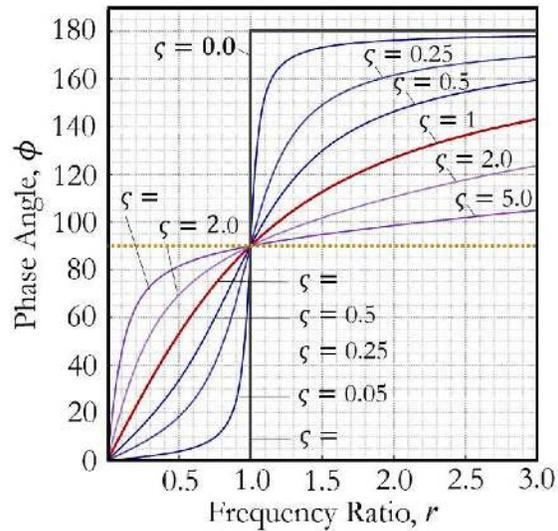
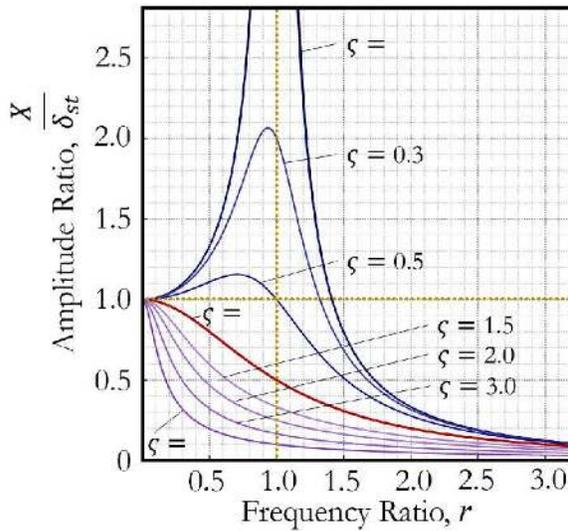
**M**essages from the result:

- Damping \_\_\_\_\_ the transient solution \_\_\_\_\_ and  $x_h(t)$  eventually \_\_\_\_\_.
- If damped, it will have \_\_\_\_\_ vibrations.
- For large time, the frequency of the system becomes same as \_\_\_\_\_ frequency:  
 $x_p(t) = \cos( \quad t - \quad )$
- With increasing time, the energy put into system by initial displacement and velocity is dissipated through damping force. The motion then becomes the \_\_\_\_\_ response of the system to the \_\_\_\_\_.
- The amplitude \_\_\_\_\_ depends on \_\_\_\_\_.
- For low frequency,  $\lim X = \text{_____}$
- For high frequency,  $\lim X = \text{_____}$

Recall:

$$X = \frac{F_0/k}{\left\{ \left[ 1 - \left( \frac{\omega}{\omega_n} \right)^2 \right]^2 + \left[ 2 \zeta \frac{\omega}{\omega_n} \right]^2 \right\}^{1/2}}$$

- Graphical Analysis for  $\frac{X}{\delta_{st}}$  and  $\phi$  with frequency ratio  $r$



**M**essages

- Peaks of the amplitude tend to get \_\_\_\_\_ as damping \_\_\_\_\_.
- With higher damping, the peaks tend to \_\_\_\_\_ below  $r =$  \_\_\_\_\_.
- When  $r \rightarrow \infty$ , the amplitude goes to \_\_\_\_\_.
- When  $r \rightarrow 1$ , the amplitude goes to \_\_\_\_\_; \_\_\_\_\_ when \_\_\_\_\_.
- At resonance, the phase angle is \_\_\_\_\_.  
: The phase difference between the excitation and the response is \_\_\_\_\_.
- For larger  $r$ ,  $\phi$  approaches \_\_\_\_\_.  
: The phase difference between the excitation and the response is \_\_\_\_\_ - \_\_\_\_\_ - \_\_\_\_\_.



## Extra Sheet for Trigonometric Identities

## Trigonometric Identities

### Reciprocal Identities

$$\sin \theta = \frac{1}{\csc \theta}$$

$$\cos \theta = \frac{1}{\sec \theta}$$

$$\tan \theta = \frac{1}{\cot \theta}$$

### Quotient Identities

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

### Pythagorean Identities

$$\sin^2 \theta + \cos^2 \theta = 1$$

$$\tan^2 \theta + 1 = \sec^2 \theta$$

$$1 + \cot^2 \theta = \csc^2 \theta$$

### Opposite Angle Identities

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

### Cofunction Identities

$$\sin(90^\circ - \theta) = \cos \theta$$

$$\cos(90^\circ - \theta) = \sin \theta$$

$$\tan(90^\circ - \theta) = \cot \theta$$

$$\sin(90^\circ + \theta) = \cos \theta$$

$$\cos(90^\circ + \theta) = -\sin \theta$$

$$\tan(90^\circ + \theta) = -\cot \theta$$

### Sum & Difference Identities

$$\sin(\alpha \pm \beta) = \sin \alpha \cos \beta \pm \cos \alpha \sin \beta$$

$$\cos(\alpha \pm \beta) = \cos \alpha \cos \beta \mp \sin \alpha \sin \beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan \alpha \pm \tan \beta}{1 \mp \tan \alpha \tan \beta}$$

### Double-Angle Identities

$$\sin 2\theta = 2 \sin \theta \cos \theta$$

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta$$

$$= 2 \cos^2 \theta - 1 = 1 - 2 \sin^2 \theta$$

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}$$

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## Topic 09. Forced Vibration – Base Excitation

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Topic 09 covers: deriving the equation of motion (EOM) and solving for the response of systems subjected to harmonic base excitation; defining and applying the concept of displacement transmissibility; interpreting the behavior of vibratory systems under base excitation using magnitude and phase plots; and applying the response equation to practical engineering problems.

In many real-world applications, mechanical systems experience motion induced not by direct external forces but by the movement of their supporting base—such as vehicles traveling over uneven terrain or equipment mounted on vibrating platforms. This type of excitation is known as base excitation. Understanding the dynamic response of a system under base excitation is essential for evaluating performance, minimizing transmitted vibrations, and preventing potential damage to sensitive components.

This topic introduces the formulation of the equation of motion under harmonic base excitation, focusing on both the relative and absolute motions of the system. The concept of displacement transmissibility is also presented to quantify how motion is transmitted from the base to the mass. Engineers use this information to design effective vibration isolation systems and optimize system stability under dynamic loading environments.

## Forced Vibration: Base Excitation

### Learning Objectives

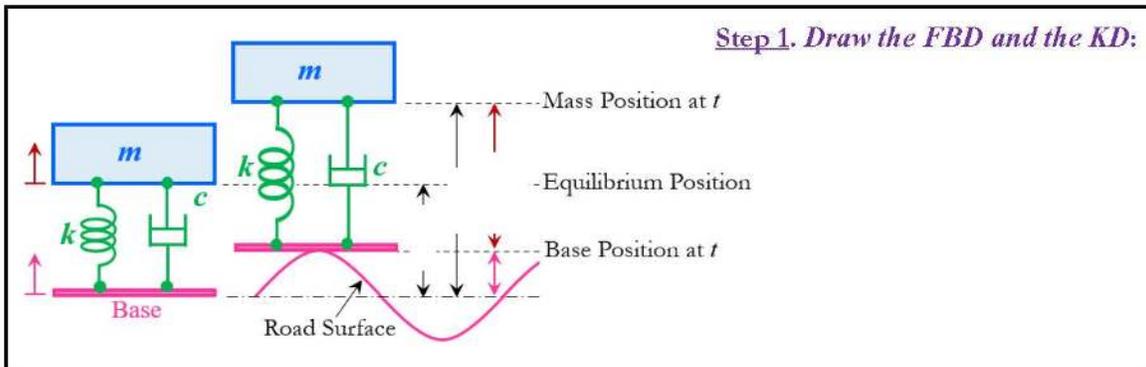
- Derive the equation of motion (EOM) for single-degree-of-freedom systems under harmonic base excitation.
- Solve the EOM to determine the total response, distinguishing between the transient and steady-state components.
- Define and apply the concept of displacement transmissibility.
- Interpret the key features of magnitude and phase plots.

## Base Excitation

### I. Applications



### II. Single DOF Model under Base Excitation



*Step 2. Derive the equation of motion (EoM):*

By Newton's 2nd law of motion,  $\Sigma F = ma$  :

Hence, the EOM is:



where

$x(t)$ : the displacement of the \_\_\_\_\_ from the static equilibrium position at time  $t$

*the displacement of the base*  $y(t) =$

$Y$ : amplitude of the \_\_\_\_\_ motion |  $\omega$ : \_\_\_\_\_ of the \_\_\_\_\_ oscillation

Step 3. Solve the EoM:

$$y(t) =$$

$$\dot{y}(t) =$$

Substituting them into the EoM,

$$m\ddot{x} + c\dot{x} + kx = kY \sin \omega t + c\omega Y \cos \omega t =$$

where  $A = Y\sqrt{k^2 + (c\omega)^2}$  and  $\alpha = \tan^{-1}\left(-\frac{c\omega}{k}\right)$

Hence, with similar approach for the damping system under forced vibration, **the steady-state response of the mass  $x_p(t)$** :

$$x_p(t) = \frac{Y\sqrt{k^2 + (c\omega)^2}}{[(k - m\omega^2)^2 + (c\omega)^2]^{1/2}} \sin(\omega t - \phi_1 - \alpha), \text{ where } \phi_1 = \tan^{-1}\left(\frac{c\omega}{k - m\omega^2}\right)$$

Using trigonometric identities:

$$x_p(t) =$$

Recall:

$$\frac{c}{m} =$$

$$\frac{k}{m} =$$

$$\frac{\omega}{\omega_n} =$$

where

$$\text{---} = \left[ \frac{k^2 + (c\omega)^2}{(k - m\omega^2)^2 + (c\omega)^2} \right]^{1/2} = \left[ \text{---} \right]^{1/2} =$$

:

: the ratio of the \_\_\_\_\_ response magnitude to the \_\_\_\_\_ displacement magnitude

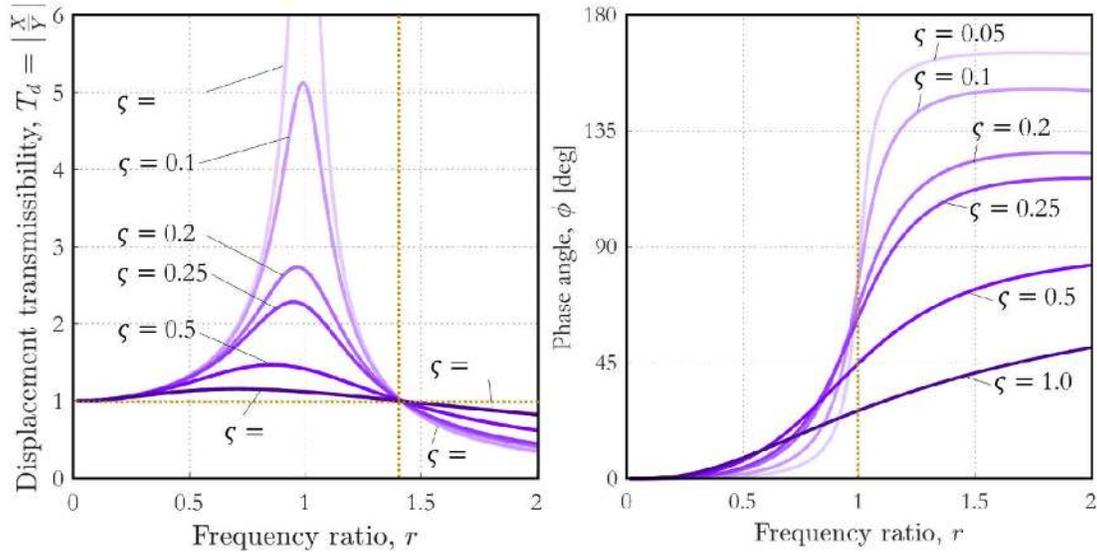
: the ratio of the response amplitude \_\_\_\_\_ to the base motion amplitude \_\_\_\_\_

: explains how motion is \_\_\_\_\_ from the \_\_\_\_\_ to the \_\_\_\_\_ as a function of \_\_\_\_\_ at various driving frequencies.

$$\phi = \tan^{-1} \left[ \frac{mc\omega^3}{k(k - m\omega^2) + (c\omega)^2} \right] = \tan^{-1} \left[ \frac{2}{1 + (4 \text{ ---} - 1)} \right]$$

**Step 4. Interpret the results:**

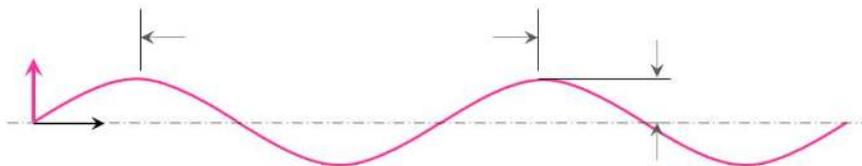
**Graphical Analysis for  $T_d$  and  $\phi$  with frequency ratio  $r$**



**M**essages from the result:

- For a fixed  $\zeta$  (i.e.,  $\zeta = 0.05$ ),
  - i) At  $r = 0$ ,  $T_d =$  \_\_\_\_\_ and for \_\_\_\_\_  $r$ ,  $T_d$  \_\_\_\_\_.
  - ii) For  $r = \sqrt{2}$ ,  $T_d =$  \_\_\_\_\_  
 : the \_\_\_\_\_ experiences \_\_\_\_\_ amplitude oscillations than the \_\_\_\_\_ excitation.
  - iii) At  $r = \sqrt{2}$ ,  $T_d =$  \_\_\_\_\_
  - iv) For  $r = \sqrt{2}$ , always  $T_d =$  \_\_\_\_\_  
 : the motion of the \_\_\_\_\_ will be of \_\_\_\_\_ amplitude than the amplitude of the exciting \_\_\_\_\_ motion.  
 : In this higher frequency range, the effect of increasing damping is just the \_\_\_\_\_ of that in the low frequency case.  
 : the amplitude is always \_\_\_\_\_ 1
  - v)  $r =$  \_\_\_\_\_, the \_\_\_\_\_ amount of \_\_\_\_\_ motion is transferred to the \_\_\_\_\_.
- How the non-dimensional response  $T_d$  can be below 1?  
 : Operate at \_\_\_\_\_ for any damping

• Road Surface Model



- *Input response*  $y =$

Determine the input response  $y$  as a sine function of  $z$  and  $\lambda$ .

- Time to complete one cycle:  $= \text{---} [ \quad ]$

- Number of cycles per second:  $= \text{---} = \text{---} [ \quad ]$

- **Frequency**  $\omega$  of the base excitation can be expressed as:

$$\omega = \quad = \quad \text{---} [ \quad ]$$

Recall:

Distance =

$\tau = \text{---} = \text{---}$

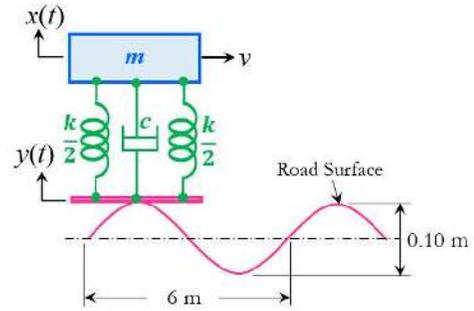
$\omega = \text{---} =$

- Convert  $z$  as a function of time  $t$ , and express  $y$  as a function of  $\omega$  and  $t$ .

Hence, the input response  $y$  as a sine function of  $z$  and  $\lambda$  can be expressed as:

**Practice Problem**

A vehicle with mass  $m = 1200$  kg travels at a constant speed of  $v = 20$  km/h over a road surface modeled as a sinusoidal wave, as shown in the figure. The suspension system is represented as a single-degree-of-freedom system with an effective spring stiffness of  $k = 400$  kN/m and a damping ratio  $\zeta = 0.5$ . The effect of the unsprung mass is neglected in this model.



- (1) Express the road surface profile  $y(t)$  as a sinusoidal function in terms of the vehicle's velocity  $v$  and the road wavelength  $\lambda$ .
- (2) Derive an equation of motion using Newton's second law, expressing your answer symbolically.
- (3) Determine the frequency ratio,  $r$ .
- (4) Determine the amplitude  $X$  of the vehicle. Use the amplitude ratio expression provided below.

$$\left| \frac{X}{Y} \right| = \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2}$$

**Given:**  $m = 1200$  kg,  $k = 400$  kN/m,  $\zeta = 0.5$ ;  $v = 20$  km/h,  $\lambda =$  \_\_\_\_\_,  $Y =$  \_\_\_\_\_

**Find:** (1)  $y(t)$ ; (2) EOM; (3)  $r$ ; (4)  $X$

**Solution:** (1) To express the road surface profile  $y(t)$ :

$y(t) =$  \_\_\_\_\_  $=$  \_\_\_\_\_  
 where  $(0.05) \sin\left(\frac{2\pi(5.5556)}{6}t\right) = 0.05 \sin(5.8178 t)$

$v =$  \_\_\_\_\_  $= 5.5556$

Recall:

$\omega =$  \_\_\_\_\_

Distance = \_\_\_\_\_

$\tau =$  \_\_\_\_\_

(2) To derive the EOM:

i) Draw the FBD and KD:

ii) Apply Newton's 2<sup>nd</sup> law of motion:

$\sum F = m \ddot{x} :$

**EOM:**  $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$

(3)  $r =$  \_\_\_\_\_  $= \frac{5.8178}{18.2574} = 0.3187$ , where  $\omega_n =$  \_\_\_\_\_  $= \sqrt{\text{_____}} = 18.2574$

(4)  $X = Y \left[ \frac{1 + (2\zeta r)^2}{(1 - r^2)^2 + (2\zeta r)^2} \right]^{1/2} = (0.05) \sqrt{\frac{1 + [2(\text{_____})(0.3187)]^2}{(1 - 0.3187^2)^2 + [2(\text{_____})(0.3187)]^2}} = 0.05505$

**Answer:**

(1)  $y(t) = 0.050 \sin(5.82 t)$  m; (2)  $m\ddot{x} + c\dot{x} + kx = c\dot{y} + ky$ ; (3)  $r = 0.319$ ; (4)  $X = 0.055$  m

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## Topic 12. Multiple-Degree-of-Freedom System

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Topic 12 covers: an overview of vibrations in rigid body dynamics; the formulation of equations of motion (EOMs) for systems with multiple degrees of freedom (DOF); presentation of the EOMs in matrix form; and the analysis of free vibration behavior through characteristic equations. This topic introduces the concept of natural frequencies and mode shapes, including how they are determined from amplitude ratios and how they govern system behavior.

Mechanical systems in the real world often exhibit multiple degrees of freedom due to their interconnected components and distributed mass properties. Unlike single-DOF systems, multi-DOF systems involve coupled motions that require a matrix-based formulation for systematic analysis. Understanding these systems is essential for predicting resonant behavior, analyzing mode interactions, and designing for dynamic stability.

This chapter focuses on deriving the governing equations of motion for linear undamped systems, interpreting the mathematical structure of these equations, and solving for natural frequencies and mode shapes. These foundational tools are critical for modal analysis, structural dynamics, and the design of complex mechanical systems subjected to vibratory loading.

At the end of the chapter, Extra Sheets are provided at the end of the chapter to review the fundamental concepts of (1) Matrix and Inverse Matrix and (2) the Eigenvalue Problem.

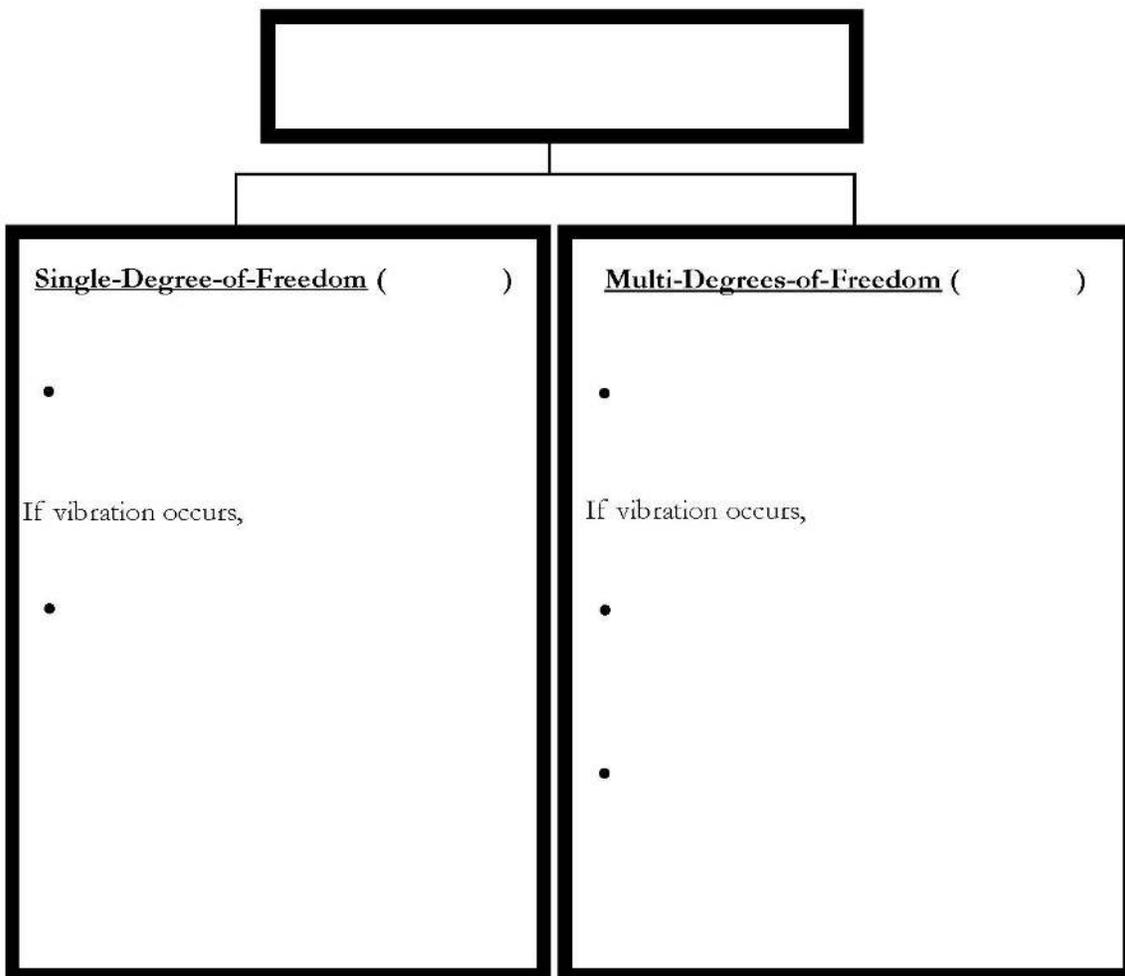
## Multi-Degree-of-Freedom System

### Learning Objectives

- Gain an overview of vibration in rigid body dynamics and its role in multi-DOF systems.
- Derive the equations of motion (EOMs) for multi-degree-of-freedom systems and express them in matrix form.
- Understand the characteristics of free vibration responses for linear systems.
- Analyze the properties of the characteristic equation associated with multi-DOF systems.
- Determine natural frequencies and mode shapes using amplitude ratios.

## Multi-DOF System

### I. Overview of Vibration w/ Rigid Body Dynamics



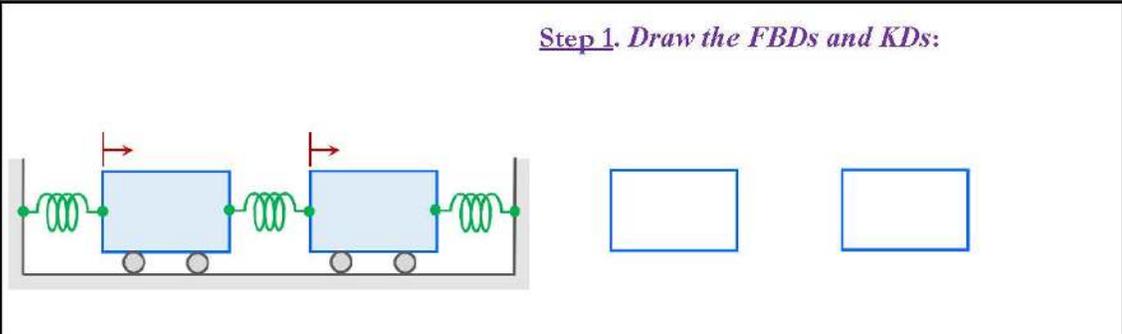
## II. Multi-DOF System Model under Free Vibration

- Characteristic of Free Vibration Solutions of Linear EOM

: \_\_\_\_\_

: The system response will be made up of the \_\_\_\_\_ of \_\_\_\_\_ normal modes.

Step 1. Draw the FBDs and KDs:



Step 2. Derive the equations of motion (EoM):

By Newton's 2nd law of motion,

i) For  $m_1$  :

$$\sum \quad = \quad :$$

EOM for  $m_1$  is

ii) For  $m_2$  :

$$\sum \quad = \quad :$$

EOM for  $m_2$  is

Note:

Each EOM involves all the coordinates \_\_\_\_\_ and \_\_\_\_\_: *Two \_\_\_\_\_ Differential Equations*

• Matrix Notation of the EOMs

• Compact matrix notation of the EOMs

where

$\underline{M}$ : \_\_\_\_\_

$\underline{\ddot{X}}$ : \_\_\_\_\_

$\underline{K}$ : \_\_\_\_\_

$\underline{X}$ : \_\_\_\_\_

{0}: \_\_\_\_\_

If the system is damped, the EOM matrix notation will also have:

: \_\_\_\_\_

: \_\_\_\_\_

• Characteristics of the matrices in the EOM matrix notation

i) The mass matrix  $\underline{M}$  is \_\_\_\_\_; the \_\_\_\_\_-\_\_\_\_\_ terms are \_\_\_\_\_.

ii) The mass and stiffness matrices,  $\underline{M}$  and  $\underline{K}$  are \_\_\_\_\_.

Because

iii) The off-diagonal stiffness terms are \_\_\_\_\_-\_\_\_\_\_: this \_\_\_\_\_ the EOMs.

: If  $\quad = \quad$ , the EOMs become \_\_\_\_\_(or \_\_\_\_\_coupled), which implies that the two masses are not physically connected. Then,  $\underline{K}$  also becomes \_\_\_\_\_.

Recall: *If undamped and free vibration, the compact matrix notation of the EOMs*



**Step 3. Solve the EoMs:**

Assume that the masses are oscillating harmonically with the \_\_\_\_\_ frequency \_\_\_\_\_ and phase angle \_\_\_\_\_, but with \_\_\_\_\_.

:

Hence, *i*) the responses are:

$$\{ \quad \} = \left\{ \begin{array}{c} \quad \\ \quad \end{array} \right\} = \left\{ \begin{array}{c} \quad \\ \quad \end{array} \right\}$$

→  $\underline{x}(t) =$

Example: For 2 DOF system, the responses are:

$$\{ \quad \} = \left\{ \begin{array}{c} \quad \\ \quad \end{array} \right\}$$

*ii*) Differentiating,

$\underline{\dot{x}}(t) =$

Example: For 2 DOF system, differentiating:

$$\left\{ \begin{array}{c} \quad \\ \quad \end{array} \right\} = \left\{ \begin{array}{c} \quad \\ \quad \end{array} \right\}$$

iii) Substituting into the EOM matrix,

EOM:

$$\rightarrow \begin{bmatrix} & \\ & \end{bmatrix} =$$

Note: For any linearized N-DOF system, this form will be available to find the vibration solutions.

- 1) If  $\underline{a} = 0$ : a \_\_\_\_\_ solution  $\rightarrow$  no motion, not useful.
- 2) If  $\underline{a} \neq 0$ : a nontrivial solution

$$\det \begin{bmatrix} & \\ & \end{bmatrix} = \quad : \text{Characteristic Equation ( )}$$

Note: 2 DOF system  $\rightarrow$  \_\_\_\_\_<sup>th</sup> order CE  $\rightarrow$  \_\_\_\_\_ roots for the natural frequencies  
 3 DOF system  $\rightarrow$  \_\_\_\_\_<sup>th</sup> order CE  $\rightarrow$  \_\_\_\_\_ roots for the natural frequencies  
 :  
 N DOF system  $\rightarrow$  \_\_\_\_\_<sup>th</sup> order CE  $\rightarrow$  \_\_\_\_\_ roots for the natural frequencies

Example: For 2 DOF system, substituting into the EOM matrix,

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} \\ \end{Bmatrix} + \begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} \\ \end{Bmatrix} = \begin{Bmatrix} \\ \end{Bmatrix}$$

Dividing by  $\cos(\omega t - \phi)$ ,

$$\begin{bmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\rightarrow$  A set of \_\_\_\_\_ equation e.g.,

For a nontrivial solution of  $a_1$  and  $a_2$ ,

$$\det \begin{bmatrix} & \\ & \end{bmatrix} =$$

$$\rightarrow ( \quad ) \omega^4 - \{ ( \quad ) m_2 + ( \quad ) m_1 \} \omega^2 + \{ (k_1 + k_2)(k_2 + k_3) - k_2^2 \} = 0$$

: “\_\_\_\_\_ Equation” because its solution yields the frequencies of the system

- How To Determine the Natural Frequencies and Natural Modes (or Normal Modes)

### 1. Method 1: Using Direct Calculations

#### - Natural Frequencies

Recall: The characteristic equation of the system for \_\_\_\_ DOF is:

$$(m_1 m_2) \omega^4 - \{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\} \omega^2 + \{(k_1 + k_2)(k_2 + k_3) - k_2^2\} = 0$$

Using the quadratic formula, *the roots* of the \_\_\_\_<sup>th</sup> - order characteristic equation for \_\_\_\_ DOF:

$$\omega = \frac{\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\} \mp \sqrt{\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}^2 - 4(m_1 m_2)\{(k_1 + k_2)(k_2 + k_3) - k_2^2\}}}{2m_1 m_2}$$

Recall:

$$ax^2 + bx + c = 0$$

Quadratic Formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

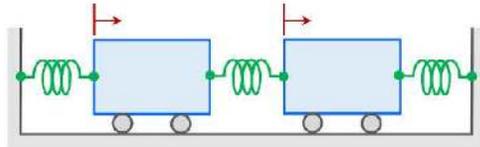
Hence, *the natural frequencies are:*

$$\begin{aligned} &= \sqrt{\frac{\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\} \mp \sqrt{\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}^2 - 4(m_1 m_2)\{(k_1 + k_2)(k_2 + k_3) - k_2^2\}}}{2m_1 m_2}} \\ &= \sqrt{\frac{\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\} \mp \sqrt{\{(k_1 + k_2)m_2 + (k_2 + k_3)m_1\}^2 - 4(m_1 m_2)\{(k_1 + k_2)(k_2 + k_3) - k_2^2\}}}{2m_1 m_2}} \end{aligned}$$

Note: The natural frequencies are characterized by the combinations of \_\_\_\_\_ and \_\_\_\_\_.

- **Natural Modes** ( \_\_\_\_\_ Modes)

: a \_\_\_\_\_ of \_\_\_\_\_



Recall: the System Responses for 2 DOF are

$$\begin{Bmatrix} x_1(t) \\ x_2(t) \end{Bmatrix} = \begin{Bmatrix} \quad \\ \quad \end{Bmatrix}$$

Recall: the EOM matrix for 2 DOF

$$\begin{bmatrix} -m_1\omega^2 + (k_1 + k_2) & -k_2 \\ -k_2 & -m_2\omega^2 + (k_2 + k_3) \end{bmatrix} \begin{Bmatrix} \quad \\ \quad \end{Bmatrix} = \begin{Bmatrix} \quad \\ \quad \end{Bmatrix}$$

From the 1<sup>st</sup> row:

$$\{-m_1\omega^2 + (k_1 + k_2)\} \quad =$$

From the 2<sup>nd</sup> row:

$$-k_2 \quad = \{m_2\omega^2 - (k_2 + k_3)\}$$

: The equations are \_\_\_\_\_; only \_\_\_\_\_ can be found, NOT absolute magnitudes.

Hence, *the Amplitude* \_\_\_\_\_ is obtained as:

$$\begin{aligned} i) \text{ For } \omega & : \quad \quad \quad = \frac{-m_1\omega^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega^2 + (k_2 + k_3)} = \\ ii) \text{ For } \omega & : \quad \quad \quad = \frac{-m_1\omega^2 + (k_1 + k_2)}{k_2} = \frac{k_2}{-m_2\omega^2 + (k_2 + k_3)} = \end{aligned}$$

: They are called \_\_\_\_\_ or \_\_\_\_\_.

### Why Normal?

- The modes are \_\_\_\_\_ in the sense that they can move independently.
- An excitation of on mode will \_\_\_\_\_ cause motion of a different mode.
- Hence, they are \_\_\_\_\_ to each other.

Hence, *the Normal Modes* corresponding to \_\_\_\_\_ and \_\_\_\_\_ can be expressed as:

$$\vec{A} = \begin{Bmatrix} \quad \\ \quad \end{Bmatrix} = \begin{Bmatrix} \quad \\ \quad \end{Bmatrix}, \quad \vec{A} = \begin{Bmatrix} \quad \\ \quad \end{Bmatrix} = \begin{Bmatrix} \quad \\ \quad \end{Bmatrix}$$

2. Method 2: Using Eigenvalue Problem Approaches

Note that the characteristic equation leads the characteristic value, \_\_\_\_\_. It is known as the \_\_\_\_\_. The \_\_\_\_\_ square roots give the \_\_\_\_\_ natural frequencies of the system. Hence, it is \_\_\_\_\_ problem. To find the solution \_\_\_\_\_, recall Eq. (1)

$$\begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} = \phantom{0} \quad x(t) =$$

$$\begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} = \phantom{0} \quad x(t) =$$

→  $\begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} =$  where  $\lambda =$

Premultiplying by \_\_\_\_\_:

$$\begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} =$$

→

where

$$\underline{I} = \begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} :$$

Recall

$$\omega_n =$$

$\underline{D} =$  \_\_\_\_\_ : \_\_\_\_\_ which is analogous to \_\_\_\_\_ = \_\_\_\_\_

Eq (2) can be expressed as the \_\_\_\_\_ eigenvalue problem:

$$=$$

For a nontrivial solution of  $\underline{u}$ , and to obtain \_\_\_\_\_:

$$\det \begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} =$$

: called the \_\_\_\_\_ equation or \_\_\_\_\_ equation.

: \_\_\_\_\_ is called the \_\_\_\_\_.

• Total Response Solutions

$$= \left\{ \begin{matrix} \\ \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} \\ \end{matrix} \right\}$$

Normalizing the mode shapes,

$$= \left\{ \begin{matrix} \\ \end{matrix} \right\}$$

$$= \left\{ \begin{matrix} \\ \end{matrix} \right\}$$

Recall:

$$\left(\frac{a_2}{a_1}\right)^{(1)} =$$

$$\left(\frac{a_2}{a_1}\right)^{(2)} =$$

By the mode superposition, *the total solution*, for example, for 2 DOF, is

$$= \left\{ \begin{matrix} \\ \end{matrix} \right\} = \left\{ \begin{matrix} \\ \end{matrix} \right\} \cos(\omega t - \phi) + \left\{ \begin{matrix} \\ \end{matrix} \right\} \cos(\omega t - \phi)$$

The constants  $a_1^{(1)}, a_1^{(2)}, \phi_1, \phi_2$  are determined by the \_\_\_\_\_.

\_\_\_\_\_ unknowns: \_\_\_\_\_ equations from the initial displacements from the two modes  
 \_\_\_\_\_ equations from the initial velocities

At time  $t = 0$ ,

i)  $x_1$  at  $t = 0$ :  $x_{10} =$

ii)  $x_2$  at  $t = 0$ :  $x_{20} = a_1 \cos \phi_1 + a_2 \cos \phi_2$

Note that  $\frac{d}{dt}[\cos(\omega t - \phi)] =$

iii)  $v_1$  at  $t = 0$ :  $\dot{x}_{10} =$

iv)  $v_2$  at  $t = 0$ :  $\dot{x}_{20} = a_1 \omega_1 R_1 \sin \phi_1 + a_2 \omega_2 R_2 \sin \phi_2$

Recall:

$$\cos(-x) =$$

$$\sin(-x) =$$

See the equations (5.18) for the values of  $a_1^{(1)}, a_1^{(2)}, \phi_1$ , and  $\phi_2$ .

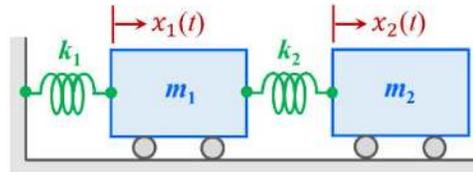
Step 4. Interpret the results:

• **C**haracteristics of an Multi-Degree-of-Freedom System Model

- \_\_\_\_\_ ODE
- The EOM give \_\_\_\_\_ natural frequency: \_\_\_\_\_ for the system.
- With a suitable initial excitation, the system oscillates at \_\_\_\_\_ of the \_\_\_\_\_ natural frequencies.
- The \_\_\_\_\_ are related in a specific shape:
- Normal (or natural) modes are \_\_\_\_\_, not absolute magnitudes.
- An excitation of one mode will \_\_\_\_\_ cause motion of a different mode.  
: \_\_\_\_\_ each other. Hence, the modes are \_\_\_\_\_  
because they move \_\_\_\_\_.
- N-DOF system has \_\_\_\_\_ normal modes of vibration corresponding the \_\_\_\_\_ natural frequencies.
- The \_\_\_\_\_ system response will contain \_\_\_\_\_ natural frequencies, and therefore the total solutions can be found from the \_\_\_\_\_ of each mode if the system is the \_\_\_\_\_ EOM.

**Practice Problem**

Consider a 2-DOF mass-spring system consisting of two masses,  $m_1 = m_2 = m$  kg, connected by linear springs with stiffness values  $k_1 = k_2 = k$  N/m. The system undergoes free vibration.



- (1) Derive the equations of motion (EOM) for the system and express them in matrix form.
- (2) Using the direct calculation method, determine the natural circular frequencies of the system.
- (3) Determine the mode shapes of the system.

**Given:**  $m_1 = m_2 = m$  kg;  $k_1 = k_2 = k$  N/m; free vibration

**Find:** (1) EOM in matrix form; (2) ; (3)

**Solution: (1) To derive the EOM:**

Draw the FBDs and KDs:



Apply Newton's 2<sup>nd</sup> law of motion:

i) For  $m_1$ ,  $\sum F = m \ddot{x}$  :

$$\text{EOM: } m_1 \ddot{x}_1 + (k_1 + k_2)x_1 - k_2x_2 = 0$$

ii) For  $m_2$ ,  $\sum F = m \ddot{x}$  :

$$\text{EOM: } m_2 \ddot{x}_2 - k_2x_1 + k_2x_2 = 0$$

Express the EOM in matrix form:

(2) To determine : solve the EOM matrix to **find a characteristic equation** for

Assuming the system undergoes harmonic motion with a same frequency, the response can be expressed as:

$$\begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} =$$

$$\text{Differentiating } \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} =$$

Substituting into the EOM matrix,

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos(\omega t - \phi) + \begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} \cos(\omega t - \phi) =$$

Dividing by  $\cos(\omega t - \phi)$ ,

$$\begin{bmatrix} & \\ & \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = 0 \quad \text{Eq (1)}$$

For nontrivial solutions, solving for the amplitudes  $a_1$  and  $a_2$ ,

$$\begin{bmatrix} \phantom{0} \\ \phantom{0} \end{bmatrix} \rightarrow$$

Hence, the characteristic equation (CE) can be expressed as:

$$\boxed{\phantom{0}}$$

Find the roots of the CE using the quadratic formula to determine the natural frequencies of the system:

$$\omega^2 = \frac{-3 \pm \sqrt{5}}{2}$$

Taking the positive value, the natural frequencies of the system are given by:

$$= \left( \frac{3 + \sqrt{5}}{2} \right) \rightarrow \therefore \omega_1 = 0.618$$

$$= \left( \frac{3 - \sqrt{5}}{2} \right) \rightarrow \therefore \omega_2 = 1.618$$

(3) To determine the mode shapes corresponding to :

Recall Eq (1): 
$$\begin{bmatrix} -m\omega^2 + 2k & -k \\ -k & -m\omega^2 + k \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

From the 1<sup>st</sup> row of Eq (1),

The amplitude ratio is given by:

For  $\omega_1 = 0.618 \sqrt{\frac{k}{m}}$  \_\_\_\_\_ = \_\_\_\_\_ = \_\_\_\_\_ = \_\_\_\_\_

For  $\omega_2 = 1.618 \sqrt{\frac{k}{m}}$  \_\_\_\_\_ = \_\_\_\_\_ = \_\_\_\_\_ = \_\_\_\_\_

**Answer:**

$$\boxed{\omega_1 = 0.618 \sqrt{\frac{k}{m}} \quad ; \quad \omega_2 = 1.618 \sqrt{\frac{k}{m}} \quad ; \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \phantom{0} \\ \phantom{0} \end{Bmatrix} \quad ; \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \phantom{0} \\ \phantom{0} \end{Bmatrix}}$$



Extra Sheet for Matrix and Inverse Matrix

## Matrix

### Terminology

- matrix: rectangular array of elements

- $m \times n$  matrix: 
$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

- square matrix:  $n \times n$  matrix

### Diagonal Matrix

A matrix in which the entries outside the main diagonal are all zero.

e.g.,  $2 \times 2$  diagonal matrix:  $\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix}$      $3 \times 3$  diagonal matrix:  $\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix}$

### Transpose of a Matrix

The transpose of a  $m \times n$  matrix,  $A$  is  $n \times m$  matrix,  $A^T$ . If  $A = [a_{ij}]$ , then  $A^T = [a_{ji}]$ .

e.g.  $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$      $A^T = \begin{bmatrix} 1 & 3 & 5 \\ 2 & 4 & 6 \end{bmatrix}$

If  $A = A^T$ , then the matrix  $A$  is symmetric.

### Determinant of a Matrix

Value that can be computed from the elements of a square matrix.

e.g.,  $2 \times 2$  matrix  $A$ ,  $\det[A] = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

$3 \times 3$  matrix  $A$ ,  $\det[A] = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = a \begin{vmatrix} e & f \\ h & i \end{vmatrix} - b \begin{vmatrix} d & f \\ g & i \end{vmatrix} + c \begin{vmatrix} d & e \\ g & h \end{vmatrix}$   
 $= aei + bfg + cdh - ceg - bdi - afh$

## Inverse of a Matrix

### For a square matrix $A$ ,

- $2 \times 2$  matrix  $A$ :  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$
- The inverse of the matrix  $A$ :  $A^{-1}$

$$A^{-1} = \frac{1}{\det[A]} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

where  $\det[A] = ad - bc$ .

- If then,  $A A^{-1} = A^{-1} A = I$

$$\begin{aligned} A A^{-1} &= \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \\ &= \frac{1}{ad - bc} \begin{bmatrix} ad - bc & bd - bd \\ ac - ac & ad - bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

where  $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$  is the identity matrix. It is a diagonal matrix.

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Extra Sheet for the Eigenvalue Problem

## Eigenvalue Problem

### Eigenvalue ( $\lambda$ )

An eigenvalue is a special scalar associated with a linear system of equations or a matrix transformation. It is also referred to as a characteristic value or characteristic root. Mathematically, if a square matrix  $\underline{A}$  and a nonzero vector  $\underline{u}$  satisfy the equation

$$\underline{A} \underline{u} = \lambda \underline{u} \quad \text{when } \underline{u} \neq 0$$

then  $\lambda$  is an eigenvalue of  $\underline{A}$  and  $\underline{u}$  is the corresponding eigenvector.

### Eigenvector ( $\underline{u}$ )

An eigenvector is a nonzero vector that only changes by a scalar factor when a linear transformation is applied. It is also called a characteristic vector.

Geometrically, for real and nonzero eigenvalues, the eigenvector points in a direction that is stretched (or compressed) by the transformation. The eigenvalue represents the scaling factor. If the eigenvalue is negative, the direction of the vector is reversed.

### Why Important?

Eigenvalues and eigenvectors are fundamental in physics and engineering, particularly in the analysis of dynamic systems. Their computation is equivalent to matrix diagonalization and is essential in:

- Stability analysis
- Vibration and modal analysis
- Rotational dynamics
- Structural dynamics and control systems
- Principal component analysis (PCA) in data science

In the context of vibrating systems, each eigenvalue corresponds to a natural frequency, and the associated eigenvector defines the mode shape of that vibration.

### Eigenvalue Problem: Linear Transformation

$$\underline{D} \underline{u} = \lambda \underline{u} \quad \text{when } \underline{u} \neq 0$$

where

$\underline{u}$  : eigenvector

: system response  $\underline{a}$  in vibration

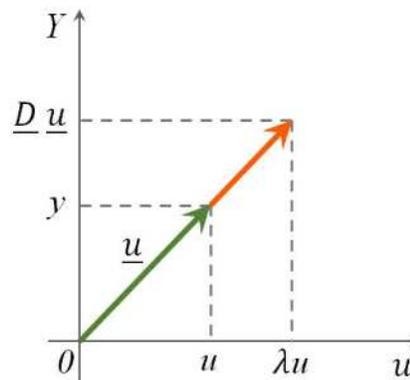
$\lambda$  : eigenvalue corresponding the eigenvector

: scaling factor: scalar satisfying the equation for nonzero  $\underline{u}$

:  $\lambda = \frac{1}{\omega^2}$  in vibration

$\underline{D}$  : dynamical matrix in vibration

:  $\underline{D} = \underline{K}^{-1} \underline{M}$  analogous to  $\frac{m}{k} = \frac{1}{\omega^2}$



Example:

$$x = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}, \quad y = \begin{bmatrix} -20 \\ -60 \\ -80 \end{bmatrix} \quad \rightarrow \quad x = \lambda y$$

Find the eigenvalue  $\lambda$ .

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## Topic 13. Modal Analysis of Multi-DOF System: Orthonormality of Modal Vectors

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Topic 13 covers: the core principles of modal analysis for linear undamped multi-degree-of-freedom (MDOF) systems; the theoretical foundation and properties of modal vectors; and the orthonormalization of mode shapes relative to the mass and stiffness matrices.

Modal analysis plays a central role in vibration engineering by transforming coupled differential equations into a decoupled form using modal coordinates. This allows each vibrational mode to be analyzed independently, revealing the system's natural frequencies and mode shapes—key characteristics that govern its dynamic response.

This chapter introduces the eigenvalue problem in the context of MDOF systems and explores the orthogonality and normalization properties of modal vectors. These mathematical tools form the foundation for simplifying the vibration analysis of complex mechanical systems and are essential for applications in structural dynamics, design optimization, and vibration control.

## Modal Analysis of Multi-DOF System: Orthonormality of Modal Vectors

### Learning Objectives

- Explain the concept and purpose of modal analysis in multi-degree-of-freedom (MDOF) systems.
- Describe the fundamental properties of mode shapes (modal vectors), including orthogonality and normalization.
- Perform orthonormalization of modal vectors with respect to the mass and stiffness matrices.

## Modal Analysis

A method to solve a \_\_\_\_DOF problem, which has a set of \_\_\_\_ \_\_\_\_\_<sup>nd</sup> order ODEs

### • Why?

- As the # of DOF increases, the solution of the frequency equation becomes more \_\_\_\_\_.

: The N DOF system has \_\_\_\_ natural frequencies, each associated with its own \_\_\_\_\_.

: To obtain the \_\_\_\_ natural frequencies, the characteristic equation \_\_\_\_\_ should be solved.



### • How to resolve?

- \_\_\_\_couple the system of \_\_\_\_\_ equations of motions (EOM).

: Use the property of the \_\_\_\_\_ known as \_\_\_\_\_.

### • Methodology?

- Apply the \_\_\_\_\_ **Theorem** to decouple the system of coupled EOM.

: Any modal vectors in the \_\_\_\_- dimensional space can be expressed by a \_\_\_\_\_ combination of the \_\_\_\_ linearly \_\_\_\_dependent vectors since the modal vectors are linearly independent due to their property of \_\_\_\_\_.

- *According to the expansion theorem*, the \_\_\_\_\_ of the masses are expressed as a \_\_\_\_\_ combination of the \_\_\_\_\_ of the system.

∴ **This linear transformation decouples the EOM as a set of \_\_\_\_coupled 2<sup>nd</sup> order ODEs.**

## Modal Vector ( )

: a vector that represents the \_\_\_\_\_ of a vibrating system corresponding to a specific natural frequency.

: describes the relative motion (amplitude and direction) of all degrees of freedom (DOFs) in the system when it vibrates in one of its \_\_\_\_\_.

: represents how the system moves in the \_\_\_\_<sup>th</sup> natural modes.

### Properties of the Modal Vectors:

- Each modal vector corresponds to \_\_\_\_\_ natural frequency.
- Modal vectors are linearly \_\_\_\_\_ dependent and can be used to form the modal matrix:
- Modal vectors are \_\_\_\_\_ with respect to the mass and stiffness matrices:
- Modal vectors can be \_\_\_\_\_ (unity-normalized) to simplify analysis.

### • Orthogonality of the Modal Vectors

- If the two vectors are *orthogonal*,



<p><u>Recall</u></p> <p><math>\hat{i} \cdot \hat{i} =</math></p> <p><math>\hat{i} \cdot \hat{j} =</math></p>
--

### • Orthonormality of the Modal Vectors

- Each modal vector is a \_\_\_\_\_ vector, and therefore its magnitude is \_\_\_\_\_.



- If the modal vectors satisfy the orthogonality and normality relations, the modal vectors are said to be \_\_\_\_\_.

Using the important property - \_\_\_\_\_ - of the modal vectors and recall the equation:

$$\boxed{\phantom{M \underline{u}^{(i)} = K \underline{u}^{(i)} \equiv K}} \rightarrow \boxed{\phantom{M \underline{u}^{(i)} = K \underline{u}^{(i)} \equiv K}}$$

i) For the  $i^{\text{th}}$  mode of the natural frequency \_\_\_\_\_ and the corresponding modal vector \_\_\_\_\_ :

$$\rightarrow \underline{M} \underline{u}^{(i)} = \underline{K} \underline{u}^{(i)} \equiv \underline{K}$$

ii) For the  $j^{\text{th}}$  mode of the natural frequency \_\_\_\_\_ and the corresponding modal vector \_\_\_\_\_ :

$$\rightarrow \underline{M} \underline{u}^{(j)} \equiv \omega_j^2 \underline{M} = \underline{K} \underline{u}^{(j)}$$

Premultiplying i) and ii) by  $\underline{u}^{(j)T}$  and  $\underline{u}^{(i)T}$ , respectively, considering the \_\_\_\_\_ of the \_\_\_\_\_ and \_\_\_\_\_, and subtracting them,

Recall

- $\hat{i} \cdot \hat{j} = \delta_{ij} \therefore \underline{u}^{(i)T} \underline{u}^{(j)} = \delta_{ij}$
- $\underline{M}$  and  $\underline{K}$  are symmetric,
- $(\underline{u}^T)^T = \underline{u}$ , where  $\underline{u}$  is a vector
- $(ABC)^T = C^T B^T A^T$ , where  $A, B,$  and  $C$  are matrices.

In general  $\omega_i^2 \neq \omega_j^2$ , hence

i) when  $i \neq j$ :

$$\boxed{\phantom{\underline{u}^{(j)T} (\underline{M} \underline{u}^{(i)} - \underline{K} \underline{u}^{(i)}) = \underline{u}^{(j)T} (\omega_j^2 \underline{M} \underline{u}^{(j)} - \underline{K} \underline{u}^{(j)})}}$$

It indicates that the modal vectors  $\underline{u}^{(i)}$  and  $\underline{u}^{(j)}$  are \_\_\_\_\_ with respect to both \_\_\_\_\_ .

ii) when  $i = j$ , they yield the generalized mass and spring constants of the  $i^{\text{th}}$  mode for  $n$ -DOF system:

$$\boxed{\phantom{\underline{u}^{(i)T} (\underline{M} \underline{u}^{(i)} - \underline{K} \underline{u}^{(i)}) = \underline{u}^{(i)T} (\omega_i^2 \underline{M} \underline{u}^{(i)} - \underline{K} \underline{u}^{(i)})}}$$

where  $i =$



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## Topic 14. Modal Analysis of Multi-DOF System: Response via Modal Superposition

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Topic 14 covers: the use of modal analysis to determine the time-domain response of linear undamped multi-degree-of-freedom (Multi-DOF) systems. This chapter focuses on the method of modal superposition, a fundamental tool in vibration analysis that enables the transformation of coupled equations into a set of decoupled single-degree-of-freedom systems.

In practical engineering applications, understanding the dynamic response of a Multi-DOF system is essential for evaluating performance and predicting behavior under initial disturbances or external forces. By decomposing the motion into modal contributions, engineers can analyze each mode independently, reconstruct the full system response, and gain insight into resonance, mode shapes, and energy distribution.

This chapter introduces the modal superposition method, shows how to compute the modal coordinates, and demonstrates how each mode contributes to the total displacement. The approach is essential for advanced dynamic analysis, structural design, and control of complex vibratory systems.

## Modal Analysis of Multi-DOF System: Response via Modal Superposition

### Learning Objectives

- Determine the system response of a multi-degree-of-freedom (Multi-DOF) system using modal analysis through modal superposition.
- Apply the modal decomposition method to solve the decoupled equations of motion.
- Interpret the contributions of each mode to the overall system response under free and forced vibration conditions.

## Modal Analysis

### How to?

: Use a new coordinate system, called \_\_\_\_\_ (natural or principle) coordinate, \_\_\_\_\_

For a N-DOF spring-mass system under forced vibration, *the equations of the motion* can be expressed as:

According to the modal expansion theorem, *the solution vector* can be expressed by

Using a compact expression,

where

$\underline{x}(t)$  : (actual) \_\_\_\_\_ coordinate     |      $\underline{q}(t)$  : \_\_\_\_\_ coordinate

$\underline{u}$  :

Recall that  $\underline{x}(t) = \underline{u} q(t)$

Then, take time derivative:  $\dot{\underline{x}}(t) = \underline{u} \dot{q}(t)$  |  $\ddot{\underline{x}}(t) = \underline{u} \ddot{q}(t)$

Substituting them into the EOM  $\underline{M} \ddot{\underline{x}}(t) + \underline{K} \underline{x}(t) = \underline{F}(t)$

$$\underline{M} \underline{u} \ddot{q}(t) + \underline{K} \underline{u} q(t) = \underline{F}(t)$$

Premultiplying by  $\underline{u}^T$ :

$$\underline{u}^T \underline{M} \underline{u} \ddot{q}(t) + \underline{u}^T \underline{K} \underline{u} q(t) = \underline{u}^T \underline{F}(t)$$

Recall the modal vectors' property:

i)  $\underline{u}^T \underline{M} \underline{u} = M_i$  for  $i = j$

$$\rightarrow \underline{u}^T \underline{M} \underline{u} = \begin{bmatrix} M_1 \\ \vdots \\ M_n \end{bmatrix};$$

ii)  $\underline{u}^T \underline{K} \underline{u} = K_i$  for  $i = j$

$$\rightarrow \underline{u}^T \underline{K} \underline{u} = \begin{bmatrix} K_1 \\ \vdots \\ K_n \end{bmatrix};$$

Hence, we obtain:

$$M_i \ddot{q}_i(t) + K_i q_i(t) = \underline{u}_i^T \underline{F}(t)$$

where

$\underline{u}_i(t)$  : \_\_\_\_\_ vector associated with the modal coordinates

$$Q_i(t) =$$

OR

$$M_i \ddot{q}_i(t) + K_i q_i(t) = F_i(t)$$

where

$M_i$  : \_\_\_\_\_ Mass

$K_i$  : \_\_\_\_\_ Stiffness

Therefore, we can have a set of \_\_\_\_\_ second-order differential equations.

Normalizing by

$$\underline{u}^T \underline{M} \underline{u} = \quad \text{and} \quad \underline{u}^T \underline{K} \underline{u} = \left[ \quad \right]$$

Then, we obtain:

Note: Again, we have now \_\_\_\_\_ coupled \_\_\_\_\_ DOF as:

$i =$

If the system is *under free vibration*,

$$\ddot{q}_i(t) + \omega_i^2 q_i = \quad \quad \quad i = 1, 2, \dots, N$$

The solution *free-vibration system response in modal coordinates* can be expressed as:

$i = 1, 2, \dots, N$

where

$q_i(0)$  :

$\dot{q}_i(0)$  : initial modal \_\_\_\_\_

*The values of the initial modal displacements and velocities* can be obtained from the initial values of the \_\_\_\_\_ physical displacement \_\_\_\_\_ and physical velocities \_\_\_\_\_.

i) To do so, recall the \_\_\_\_\_ displacement:

ii) To obtain the initial modal values, premultiply

→

Then, the initial modal values are

\_\_\_\_\_ and \_\_\_\_\_

Hence, according to the mode superposition the system response in actual physical coordinates can be expressed as:

• **S**ummary of Steps for Modal Analysis

Step 1. Determine *the natural frequencies* from

where  $\lambda =$   
 $\underline{D} =$

Step 2. Determine *the modal vectors*:

Step 3. Find *the orthonormalized modal vectors* by:

This orthonormalizing makes  $\underline{M} =$  \_\_\_\_\_ and  $\underline{K} =$  \_\_\_\_\_

Step 4. Make *the modal matrix* by:

Step 5. Set *the physical response* using the modal coordinates as:

Step 6. Calculate *the initial modal values*, premultiplying

$$\underline{u}^T \underline{M} \underline{x}(t) = \underline{u}^T \underline{M} \underline{u} \underline{q}(t)$$

$= \underline{u}^T \underline{M}$  \_\_\_\_\_ ;  $= \underline{u}^T \underline{M}$  \_\_\_\_\_

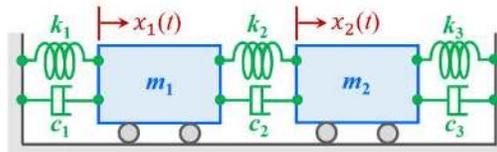
Step 7. Plug them into *the modal response equation*:

$$q_i(t) = \text{_____} \cos \omega_i t + \frac{\text{_____}}{\omega_i} \sin \omega_i t \quad \text{where } i = 1, 2, \dots, N$$

Step 8. Transform the modal coordinates into *the physical coordinates* by:

**Practice Problem**

Consider a 2-DOF mass-spring damper system consisting of two masses,  $m_1 = 9$  kg and  $m_2 = 1$  kg, connected by linear springs with stiffness values  $k_1 = 24$  N/m,  $k_2 = 3$  N/m, and  $k_3 = 3$  N/m. Assume that all damping effects are negligible ( $c_1 = c_2 = c_3 = 0$ ). The system undergoes free vibration. The matrix form of the equations of motion is:



$$\begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{Bmatrix} + \begin{bmatrix} 27 & -3 \\ -3 & 6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The modal vectors and natural frequencies are given as:

$$\underline{u}^{(1)} = u_1^{(1)} \begin{Bmatrix} 1 \\ 0.909 \end{Bmatrix}, \quad \omega_1 = 1.64 \text{ rad/s}$$

$$\underline{u}^{(2)} = u_1^{(2)} \begin{Bmatrix} 1 \\ -9.91 \end{Bmatrix}, \quad \omega_2 = 2.51 \text{ rad/s}$$

Initial conditions are  $\underline{x}(0) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$  and  $\dot{\underline{x}}(0) = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$ .

- (1) Orthonormalize the modal vectors, and find the modal matrix.
- (2) Find the orthonormalized modal mass and modal stiffness matrices.
- (3) Find the initial modal conditions.
- (4) Find the modal responses.
- (5) Find the physical displacements.

**Given:**  $M = \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix}$ ;  $K = \begin{bmatrix} 27 & -3 \\ -3 & 6 \end{bmatrix}$ ;  $\underline{u}^{(1)} = u_1^{(1)} \begin{Bmatrix} 1 \\ 0.909 \end{Bmatrix}$ ;  $\omega_1 = 1.64$  rad/s;  $\underline{u}^{(2)} = u_1^{(2)} \begin{Bmatrix} 1 \\ -9.91 \end{Bmatrix}$ ;  $\omega_2 = 2.51$  rad/s

- Find:** (1) orthonormalized (2) orthonormalized  
 (4) (5) (6)

**Solution:**

(1) Orthonormalize the modal vectors:



i) Orthonormalize  $\underline{u}^{(1)} = u_1^{(1)} \begin{Bmatrix} 1 \\ 0.909 \end{Bmatrix}$

$$\rightarrow = u_1^{(1)} \left[ \begin{array}{c} 1 \\ 0.909 \end{array} \right] \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0.909 \end{bmatrix} =$$

$$\rightarrow u_1^{(1)} =$$

Hence, the orthonormalized modal vector  $\underline{u}^{(1)}$ :





**(3) Find the initial modal conditions:**

Recall that  $\underline{x}(t) =$  and premultiplying by

→

$$\rightarrow \underline{q}(t) = \left\{ \begin{matrix} \\ \end{matrix} \right\} = \underline{u}^T \underline{M} \underline{x}(t) = \begin{bmatrix} \\ \\ \end{bmatrix} \begin{bmatrix} \\ \\ \end{bmatrix} \left\{ \begin{matrix} \\ \\ \end{matrix} \right\}$$

$$= \begin{Bmatrix} 0.2899 \\ -0.9571 \end{Bmatrix}$$

$$\rightarrow \underline{\dot{q}}(t) = \left\{ \begin{matrix} \\ \end{matrix} \right\} = \begin{bmatrix} 0.3190 & 0.2899 \\ 0.0966 & -0.9571 \end{bmatrix} \begin{bmatrix} 9 & 0 \\ 0 & 1 \end{bmatrix} \left\{ \begin{matrix} \\ \\ \end{matrix} \right\} = \left\{ \begin{matrix} \\ \\ \end{matrix} \right\}$$

**(4) Find the modal solutions:**

Recall that if no forcing, the solution can be expressed as:

where  $i =$

Hence, using  $\omega_1 = 1.64$  rad/s,  $\omega_2 = 2.51$  rad/s

$$i)$$

$$ii)$$

**(5) Find the physical displacements:**

Transforming back to the physical generalized coordinates :

$$\underline{x}(t) = \left\{ \begin{matrix} \\ \end{matrix} \right\} = \begin{bmatrix} 0.3190 & 0.0966 \\ 0.2899 & -0.9571 \end{bmatrix} \left\{ \begin{matrix} \\ \\ \end{matrix} \right\}$$

Hence, *the physical displacements* can be expressed as:

$$x_1(t) = 0.093 \cos(1.64t) - 0.093 \cos(2.51t)$$

$$x_2(t) = 0.084 \cos(1.64t) + 0.916 \cos(2.51t)$$

Note that *the total solution* can be found by *the mode superposition* as:

$$\underline{x}(t) =$$

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Prof. J Lee, Copyright.

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