INTRODUCTORY DIFFERENTIAL EQUATIONS

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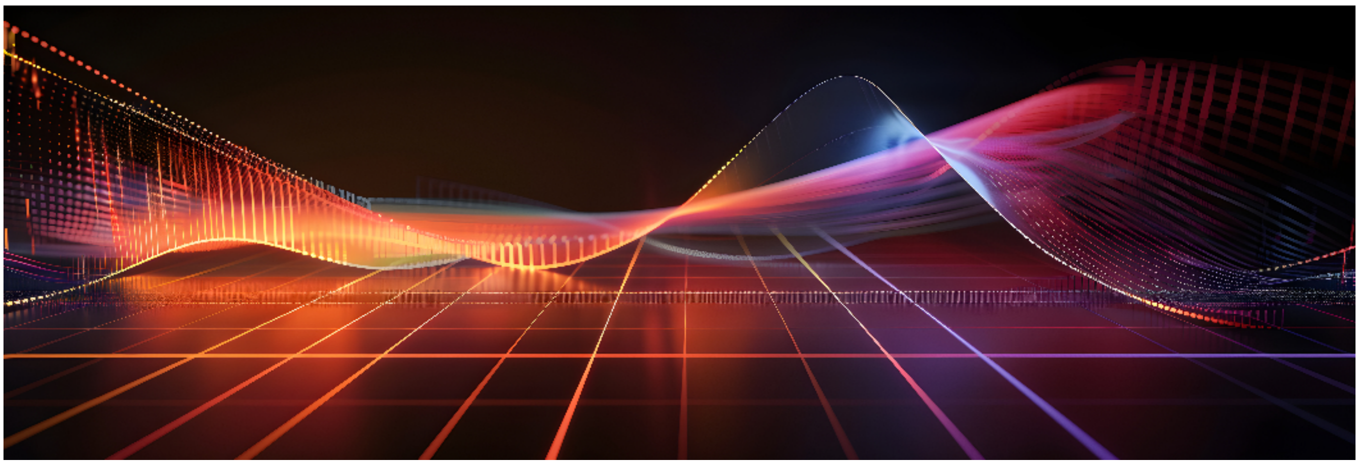

Math 2A: Introduction to Differential Equations

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Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing***.

About the Book

This book offers an accessible and thorough introduction to differential equations, focusing on both the theory and applications of first-order and higher-order differential equations, power series solutions, Laplace transforms, and systems of differential equations.

Through detailed explanations, real-world examples, and interactive elements, students learn how differential equations are used to model physical systems, population dynamics, electrical circuits, mechanical vibrations, and more. Key analytical techniques—such as separation of variables, integrating factors, exact equations, undetermined coefficients, variation of parameters, and the eigenvalue method—are presented alongside numerical methods and qualitative tools like slope fields, phase portraits, and autonomous systems.

Topics include the classification and solution of first-order differential equations using various methods, applications such as exponential growth and decay, Newton's law of cooling, mixing problems, and pursuit curves, as well as the study of second-order and higher-order linear differential equations with constant coefficients. Students will explore nonhomogeneous equations and their applications in physical systems like spring-mass-damper models and forced oscillations. The book also introduces power series methods for solving differential equations near ordinary and singular points, including the Frobenius method, and provides a comprehensive treatment of the Laplace transform and its use in solving initial value problems. Finally, the text covers systems of differential equations, matrix methods, and the eigenvalue method for solving linear systems.

This book builds upon the foundation established in *Introductory Differential Equations* by Jiří Lebl and *Elementary Differential Equations with Boundary Value Problems* by William F. Trench. We also acknowledge Lake Tahoe Community College's derivative version of Lebl's textbook, which served as an important starting point for our own work. In creating this resource, we have aimed to uphold the principles of open education while enhancing accessibility for a diverse range of learners.

About the Authors

Information about the authors

Author Images	Author Bios
	**Vinh Kha Nguyen** is a dedicated math instructor at De Anza College who is passionate about making mathematics accessible to everyone. With a deep commitment to education, he strives to help students develop a strong foundation and appreciation for the subject. Outside the classroom, Vinh enjoys hiking and traveling with his family, embracing new adventures together. He is especially grateful to his wife for her unwavering support in caring for their child, allowing him the time to focus on completing his Open Educational Resources (OER) textbook to benefit students and educators alike.
	**Neelam R. Shukla** an accomplished educator at De Anza College with an M.Phil. in Mathematics, has dedicated over two decades of her career to making mathematical concepts accessible and engaging for students, teachers, and enthusiasts alike. With years of experience in the field, she authored a comprehensive mathematics textbook to share her deep understanding of the subject and her passion for accessible education. Having taught internationally for two decades before relocating to the USA in 2014, Neelam educated diverse learners across various academic settings. Along the way, she expanded her logical skills in creating math exams for bachelor-level courses, coding, and thesis writing, while also earning her M.Phil and a Post Graduate Diploma in Computer Science and Applications (PGDCSA). Passionate about enhancing accessibility in education, Neelam stays updated with Educational and Instructional Technology (EDIT) courses and advocates for Open Educational Resources (OER), believing they offer affordable, high-quality learning materials for all students. Her book reflects her commitment to fostering a deeper appreciation for mathematics and inspiring a love for problem-solving.
	**Fatemeh Yarahmadi** is a dedicated math instructor at De Anza College, whose passion for mathematics extends far beyond the textbook. She believes that math is not just a subject, but a language, a tool for understanding the world, and she is committed to helping every student unlock its power. With a deep commitment to education, she strives to build a nurturing and stimulating learning environment, where students feel empowered to ask questions and explore complex concepts without fear. Her teaching philosophy is rooted in the belief that mathematics is accessible to everyone, regardless of their background or prior experiences. She employs innovative teaching methods, incorporating real-world examples and interactive exercises, to make abstract concepts tangible and relatable. She understands that each student learns differently, and she dedicates herself to providing personalized support, ensuring that no one is left behind.

CHAPTER OVERVIEW

1: First Order ODEs

1.1: Introduction to Differential Equations

Learning Objectives

- Solve first-order differential equations using analytical techniques and verify solutions against given initial conditions.
- Examine and interpret the role of differential equations in real-world scenarios, identifying the assumptions and implications of mathematical models.

Differential Equations

The laws of physics are generally written down as differential equations. Therefore, all of the science and engineering use differential equations to some degree. Understanding differential equations is essential to understanding almost anything you will study in your science and engineering classes. You can think of mathematics as the language of science, and differential equations are one of the most important parts of this language as far as science and engineering are concerned.

You saw many differential equations already without perhaps knowing about them. You solved simple differential equations when you took calculus such as modeling the velocity and distance traveled of a free falling object. Let us see an example you may not have seen:

$$\frac{dx}{dt} + x = 2\cos t. \tag{1.1.1}$$

Here x is the *dependent variable* and t is the *independent variable*. Equation $(1.1.1)$ is a basic example of a *differential equation*. It is an example of a *first-order differential equation* since it involves only the first derivative of the dependent variable. This equation arises from Newton's law of cooling where the ambient temperature oscillates with time.

Solutions of Differential Equations

Solving the differential equation means finding x in terms of t. That is, we want to find a function of t, which we call x, such that when we plug x, t, and $\frac{dx}{dt}$ into Equation $(1.1.1)$, the equation holds; that is, the left-hand side equals the right-hand side. It is the same idea as it would be for a normal (algebraic) equation of just x and t. We claim that

$$x = x(t) = \cos t + \sin t$$

is a *solution*. How do we check? We simply plug x into Equation $(1.1.1)$! First we need to compute $\frac{dx}{dt}$. We find that $\frac{dx}{dt} = -\sin t + \cos t$. Now let us compute the left-hand side of Equation $(1.1.1)$.

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t)}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t)}_{x} = 2\cos t.$$

Yay! We got precisely the right-hand side. But there is more! We claim $x = \cos t + \sin t + e^{-t}$ is also a solution. Let us try,

$$\frac{dx}{dt} = -\sin t + \cos t - e^{-t}.$$

We plug into the left-hand side of Equation $(1.1.1)$

$$\frac{dx}{dt} + x = \underbrace{(-\sin t + \cos t - e^{-t})}_{\frac{dx}{dt}} + \underbrace{(\cos t + \sin t + e^{-t})}_{x} = 2\cos t.$$

And it works yet again!

So there can be many different solutions. For this equation all solutions can be written in the form

$$x = \cos t + \sin t + Ce^{-t},$$

for some constant C. Different constants C will give different solutions, so there are infinitely many possible solutions. See Figure $1.1.1$ for the graph of a few of these solutions. We will see how we find these solutions a few lectures from now.

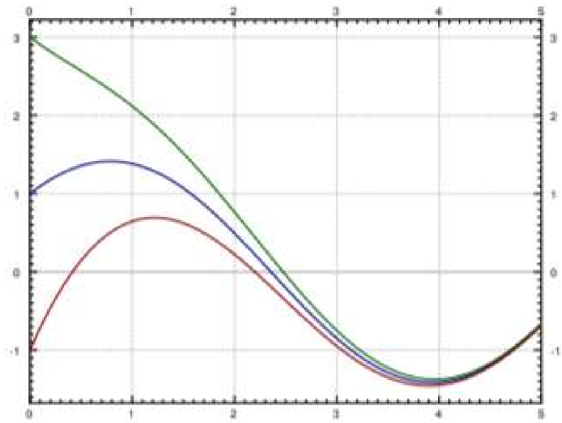


Figure 1.1.1: A graph displaying values of t starting from 0 and progressing every 1 to 5 in total on the x-axis and value of the solution of the function $\frac{dx}{dt}+x=2\cos t$ range of 1 starting from -1 and progressing every 1 to 3 in total on the y-axis. Solutions of $\frac{dx}{dt}+x=2\cos t$ is $x=\cos t+\sin t+Ce^{-t}$: Red graph line is the value of x when $c=-2$, Blue graph line is the value of x when $c=0$, green graph line is the value of x when $c=2$. (CC BY-SA 4.0; Larry Green via Introduction to Differential Equations)

Solving differential equations can be quite hard. There is no general method that solves every differential equation. We will generally focus on how to get exact formulas for solutions of certain differential equations, but we will also spend a little bit of time on getting approximate solutions. And we will spend some time to understand the equations without solving them.

Most of this book is dedicated to *ordinary differential equations* or ODEs, that is, equations with only one independent variable, where derivatives are only with respect to this one variable. If there are several independent variables, we get *partial differential equations* or PDEs.

Even for ODEs, which are very well understood, it is not a simple question of turning a crank to get answers. When you can find exact solutions, they are usually preferable to approximate solutions. It is important to understand how such solutions are found. Although in real applications you will leave much of the actual calculations to computers, you need to understand what they are doing. It is often necessary to simplify or transform your equations into something that a computer can understand and solve. You may even need to make certain assumptions and changes in your model to achieve this.

To be a successful engineer or scientist, you will be required to solve problems in your job that you never saw before. It is important to learn problem-solving techniques, so that you may apply those techniques to new problems. A common mistake is to expect to learn some prescription for solving all the problems you will encounter in your later career. This course is no exception.

Differential Equations in Practice

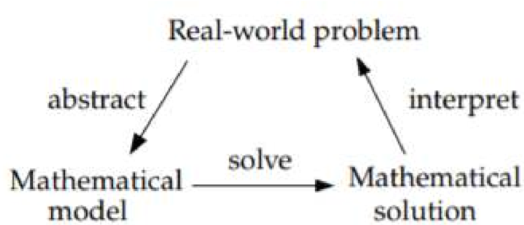


Figure 1.1.2: A flowchart illustrating the process of applying differential equations to real-word problems, involving abstraction, modeling, solving, and interpretation. (CC BY-SA 4.0; Larry Green via Introduction to Differential Equations)

So how do we use differential equations in science and engineering? First, we have some *real-world problem* we wish to understand. We make some simplifying assumptions and create a *mathematical model*. That is, we translate the real-world situation into a set of differential equations. Then we apply mathematics to get some sort of a *mathematical solution*. There is still something left to do. We have to interpret the results. We have to figure out what the mathematical solution says about the real-world problem we started with.

Learning how to formulate the mathematical model and how to interpret the results is what your physics and engineering classes do. In this course, we will focus mostly on the mathematical analysis. Sometimes we will work with simple real-world examples so that we have some intuition and motivation about what we are doing.

Let us look at an example of this process. One of the most basic differential equations is the standard *exponential growth model*. Let P denote the population of some bacteria on a Petri dish. We assume that there is enough food and enough space. Then the rate of growth of bacteria is proportional to the population—a large population grows quicker. Let t denote time (say in seconds) and P the population. Our model is

$$\frac{dP}{dt} = kP,$$

for some positive constant $k > 0$.

Example 1.1.1

Suppose there are 100 bacteria at time 0 and 200 bacteria 10 seconds later. How many bacteria will there be 1 minute from time 0 (in 60 seconds)?

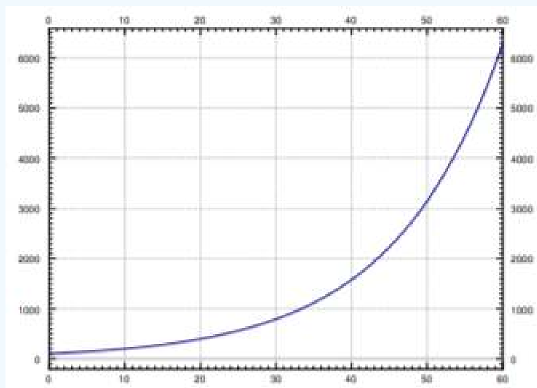


Figure 1.1.3: A graph displaying time in seconds with a range of 60 seconds starting from 0 and progressing every 10 to 60 in total on the x-axis and number of bacteria with a range of 1000 bacteria starting from 0 and progressing every 1000 to 6000 in total on the y-axis. The curve shows a steady growth of bacteria. (CC BY-SA 4.0; Larry Green via Introduction to Differential Equations)

Solution

First we need to solve the equation. We claim that a solution is given by

$$P(t) = Ce^{kt},$$

where C is a constant. Let us try:

$$\frac{dP}{dt} = Cke^{kt} = kP.$$

And it really is a solution.

OK, now what? We do not know C, and we do not know k. But we know something. We know $P(0) = 100$, and we know $P(10) = 200$. Let us plug these conditions in and see what happens.

$$\begin{aligned} 100 &= P(0) = Ce^{k0} = C, \\ 200 &= P(10) = 100\,e^{k10}. \end{aligned} \tag{1.1.2}$$

Therefore, $2 = e^{10k}$ or $\frac{\ln 2}{10} = k \approx 0.069$. So

$$P(t) = 100\,e^{(\ln 2)t/10} \approx 100\,e^{0.069t}.$$

At one minute, $t = 60$, the population is $P(60) = 6400$. See Figure 1.1.3.

Let us talk about the interpretation of the results. Does our solution mean that there must be exactly 6400 bacteria on the plate at 60s? No! We made assumptions that might not be true exactly, just approximately. If our assumptions are reasonable, then there will be approximately 6400 bacteria. Also, in real life P is a discrete quantity, not a real number. However, our model has no problem saying that for example at 61 seconds, $P(61) \approx 6859.35$.

Normally, the k in $P' = kP$ is known, and we want to solve the equation for different *initial conditions*. What does that mean? Take $k = 1$ for simplicity. Suppose we want to solve the equation $\frac{dP}{dt} = P$ subject to $P(0) = 1000$ (the initial condition). Then the solution turns out to be

$$P(t) = 1000\, e^t.$$

We call $P(t) = Ce^t$ *the general solution*, as every solution of the equation can be written in this form for some constant C. We need an initial condition to find out what C is, in order to find the *particular solution* we are looking for. Generally, when we say "particular solution," we just mean some solution.

a. Verify that the function $y = x^2 + \frac{c}{x^2}$ is a solution to the differential equation

$$xy' + 2y = 4x^2 \quad (x > 0)$$

b. Find the value of C for which the solution satisfies the initial condition $y(5) = 8$

Solution

- **Video Length:** 4 minutes 12 seconds
- **Context:** verify a solution to a particular differential equation, find a particular solution.

Fundamental Equations

A few equations appear often and it is useful to just memorize what their solutions are. Let us call them the four fundamental equations. Their solutions are reasonably easy to guess by recalling the properties of exponential, sines, and cosines functions. They are also simple to check, which is something that you should always do. No need to wonder if you remembered the solution correctly.

First such equation is

$$\frac{dy}{dx} = ky,$$

for some constant $k > 0$. Here y is the dependent and x the independent variable. The general solution for this equation is

$$y(x) = Ce^{kx}.$$

We saw above that this function is a solution, although we used different variable names.

Next,

$$\frac{dy}{dx} = -ky,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = Ce^{-kx}.$$

? Exercise 1.1.1

Check that the y given is really a solution to the equation.

Next, take the *second order differential equation*

$$\frac{d^2y}{dx^2} = -k^2 y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 \cos(kx) + C_2 \sin(kx).$$

Since the equation is a second-order differential equation, we have two constants in our general solution.

? Exercise 1.1.2

Check that the y given is really a solution to the equation.

Finally, consider the second order differential equation

$$\frac{d^2y}{dx^2} = k^2 y,$$

for some constant $k > 0$. The general solution for this equation is

$$y(x) = C_1 e^{kx} + C_2 e^{-kx},$$

or

$$y(x) = D_1 \cosh(kx) + D_2 \sinh(kx).$$

For those that do not know, cosh and sinh are defined by

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}.$$

They are called the *hyperbolic cosine* and *hyperbolic sine*. These functions are sometimes easier to work with than exponentials. They have some nice familiar properties such as $\cosh 0 = 1$, $\sinh 0 = 0$, and $\frac{d}{dx}\cosh x = \sinh x$ (no that is not a typo) and $\frac{d}{dx}\sinh x = \cosh x$.

? Exercise 1.1.3

Check that both forms of the y given are the real solutions to the equation.

In equations of higher order, you get more constants you must solve for to get a particular solution. The equation $\frac{d^2y}{dx^2} = 0$ has the general solution $y = C_1 x + C_2$; simply integrate twice and don't forget about the constant of integration. Consider the initial conditions $y(0) = 2$ and $y'(0) = 3$. We plug in our general solution and solve for the constants:

$$2 = y(0) = C_1 \cdot 0 + C_2 = C_2, \qquad 3 = y'(0) = C_1.$$

In other words, $y = 3x + 2$ is the particular solution we seek.

An interesting note about cosh: The graph of cosh is the exact shape of a hanging chain. This shape is called a *catenary*. Contrary to popular belief this is not a parabola. If you invert the graph of cosh, it is also the ideal arch for supporting its weight. For example, the gateway arch in Saint Louis is an inverted graph of cosh—if it were just a parabola it might fall. The formula used in the design is inscribed inside the arch:

$$y = -127.7 \cosh\left(\frac{x}{127.7}\right) + 757.7 \text{ ft}$$

1.1E: Exercises for Section 1.1

Exercises 1-17

Solve the exercises below by verifying solutions, solving differential equations, or finding constants to match initial conditions.

Exercise $1.1E.1$

Show that $x = e^{4t}$ is a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise $1.1E.2$

Show that $x = e^{t}$ is not a solution to $x''' - 12x'' + 48x' - 64x = 0$.

Exercise $1.1E.3$

Is $y = \sin t$ a solution to $\left(\frac{dy}{dt}\right)^2 = 1 - y^2$? Justify.

Exercise $1.1E.4$

Let $y'' + 2y' - 8y = 0$. Now try a solution of the form $y = e^{rx}$ for some constant r. Is this a solution for some r? If so, find all such r.

Exercise $1.1E.5$

Verify that $x = Ce^{-2t}$ is a solution to $x' = -2x$. Find C to solve for the initial condition $x(0) = 100$.

Exercise $1.1E.6$

Verify that $x = C_1e^{-t} + C_2e^{2t}$ is a solution to $x'' - x' - 2x = 0$. Find C_1 and C_2 to solve for the initial conditions $x(0) = 10$ and $x'(0) = 0$.

Exercise $1.1E.7$

Find a solution to $(x')^2 + x^2 = 4$ using your knowledge of derivatives of functions that you know from basic calculus.

Exercise $1.1E.8$

Solve the differential equations with given initial conditions:

a. $\frac{dA}{dt} = -10A, \quad A(0) = 5$
b. $\frac{dH}{dx} = 3H, \quad H(0) = 1$
c. $\frac{d^2y}{dx^2} = 4y, \quad y(0) = 0, \quad y'(0) = 1$
d. $\frac{d^2x}{dy^2} = -9x, \quad x(0) = 1, \quad x'(0) = 0$

Exercise $1.1E.9$

Is there a solution to $y' = y$, such that $y(0) = y(1)$?

Exercise $1.1E.10$

The population of city X was 100 thousand 20 years ago, and the population of city X was 120 thousand 10 years ago. Assuming constant growth, you can use the exponential population model. What do you estimate the population is now?

Exercise $1.1E.11$

Suppose that a football coach gets a salary of one million dollars now, and a raise of 10% every year. Let s be the salary in millions of dollars, and t is time in years.

a. What is $s(0)$ and $s(1)$?
b. Approximately how many years will it take for the salary to be 10 million.
c. Approximately how many years will it take for the salary to be 20 million.
d. Approximately how many years will it take for the salary to be 30 million.

Exercise $1.1E.12$

Show that $x = e^{-2t}$ is a solution to $x'' + 4x' + 4x = 0$.

Answer

Compute $x' = -2e^{-2t}$ and $x'' = 4e^{-2t}$. Then $(4e^{-2t}) + 4(-2e^{-2t}) + 4(e^{-2t}) = 0$.

Exercise $1.1E.13$

Is $y = x^2$ a solution to $x^2y'' - 2y = 0$? Justify.

Answer

Yes.

Exercise $1.1E.14$

Let $xy'' - y' = 0$. Try a solution of the form $y = x^r$. Is this a solution for some r? If so, find all such r.

Answer

$y = x^r$ is a solution for $r = 0$ and $r = 2$.

Exercise $1.1E.15$

Verify that $x = C_1e^t + C_2$ is a solution to $x'' - x' = 0$. Find C_1 and C_2 so that x satisfies $x(0) = 10$ and $x'(0) = 100$.

Answer

$C_1 = 100$, $C_2 = -90$

Exercise $1.1E.16$

Solve $\frac{d\varphi}{ds} = 8\varphi$ given initial condition $\varphi(0) = -9$.

Answer

$\varphi = -9e^{8s}$

Exercise $1.1E.17$

Solve the differential equations with given initial conditions:

a. $\dfrac{dx}{dt} = -4x, \quad x(0) = 9$
b. $\dfrac{d^2x}{dt^2} = -4x, \quad x(0) = 1, \quad x'(0) = 2$
c. $\dfrac{dp}{dq} = 3p, \quad p(0) = 4$
d. $\dfrac{d^2T}{dx^2} = 4T, \quad T(0) = 0, \quad T'(0) = 6$

Answer

a. $x = 9e^{-4t}$
b. $x = \cos(2t) + \sin(2t)$
c. $p = 4e^{3q}$
d. $T = 3\sinh(2x)$

1.2: Classification of Differential Equations

Learning Objectives

- Differentiate between types of differential equations.
- Analyze and categorize properties of differential equations.

There are many types of differential equations, and we classify them into different categories based on their properties. Let us quickly go over the most basic classification. We already saw the distinction between ordinary and partial differential equations:

- *Ordinary differential equations* or (ODE) are equations where the derivatives are taken with respect to only one variable. That is, there is only one independent variable.
- *Partial differential equations* or (PDE) are equations that depend on partial derivatives of several variables. That is, there are several independent variables.

Let us see some examples of ordinary differential equations:

$$\begin{aligned} &\frac{dy}{dt} = ky, && \text{(Exponential growth)} \\ &\frac{dy}{dt} = k(A-y), && \text{(Newton's law of cooling)} \\ &m\frac{d^2x}{dt^2} + c\frac{dx}{dt} + kx = f(t). && \text{(Mechanical vibrations)} \end{aligned} \tag{1.2.1}$$

And of partial differential equations:

$$\begin{aligned} &\frac{\partial y}{\partial t} + c\frac{\partial y}{\partial x} = 0, && \text{(Transport equation)} \\ &\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, && \text{(Heat equation)} \\ &\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}. && \text{(Wave equation in 2 dimensions)} \end{aligned} \tag{1.2.2}$$

If there are several equations working together, we have a so-called *system of differential equations*. For example,

$$y' = x, \qquad x' = y$$

is a simple system of ordinary differential equations. Maxwell's equations for electromagnetics,

$$\begin{aligned} &\nabla \cdot \mathbf{D} = \rho, && \nabla \cdot \mathbf{B} = 0, \\ &\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, && \nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}, \end{aligned} \tag{1.2.3}$$

are a system of partial differential equations. The divergence operator $\nabla\cdot$ and the curl operator $\nabla\times$ can be written out in partial derivatives of the functions involved in the x, y, and z variables.

The next bit of information is the *order* of the equation (or system). The order is simply the order of the largest derivative that appears. If the highest derivative that appears is the first derivative, the equation is of first order. If the highest derivative that appears is the second derivative, then the equation is of second order. For example, Newton's law of cooling is a first order equation, while the mechanical vibrations equation is a second order equation. The equation governing transversal vibrations in a beam,

$$a^4\frac{\partial^4 y}{\partial x^4} + \frac{\partial^2 y}{\partial t^2} = 0,$$

is a fourth order partial differential equation. It is fourth order as at least one derivative is the fourth derivative. It does not matter that the derivative in t is only of second order.

In the first chapter, we will start attacking first order ordinary differential equations, that is, equations of the form $\frac{dy}{dx} = f(x,y)$. In general, lower order equations are easier to work with and have simpler behavior, which is why we start with them.

We also distinguish how the dependent variables appear in the equation (or system). In particular, we say an equation is *linear* if the dependent variable (or variables) and their derivatives appear linearly, that is only as first powers, they are not multiplied together, and no other functions of the dependent variables appear. In other words, the equation is a sum of terms, where each term is some function of the independent variables or some function of the independent variables multiplied by a dependent variable or its derivative. Otherwise, the equation is called *nonlinear*. For example, an ordinary differential equation is linear if it can be put into the form

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=b(x). \tag{1.2.4}$$

The functions a_0, a_1, …, a_n are called the *coefficients*. The equation is allowed to depend arbitrarily on the independent variable. So

$$e^x\frac{d^2y}{dx^2}+\sin(x)\frac{dy}{dx}+x^2y=\frac{1}{x} \tag{1.2.5}$$

is still a linear equation as y and its derivatives only appear linearly.

All the equations and systems above as examples are linear. It may not be immediately obvious for Maxwell's equations unless you write out the divergence and curl in terms of partial derivatives. Let us see some nonlinear equations. For example ,

$$\frac{\partial y}{\partial t}+y\frac{\partial y}{\partial x}=\nu\frac{\partial^2 y}{\partial x^2},$$

is a nonlinear second order partial differential equation. It is nonlinear because y and $\frac{\partial y}{\partial x}$ are multiplied together. The equation

$$\frac{dx}{dt}=x^2 \tag{1.2.6}$$

is a nonlinear first order differential equation as there is a second power of the dependent variable x.

A linear equation may further be called *homogenous* if all terms depend on the dependent variable. That is, if no term is a function of the independent variables alone. Otherwise, the equation is called *nonhomogeneous*. For example, the exponential growth equation, the wave equation, or the transport equation above are homogeneous. The mechanical vibrations equation above is nonhomogeneous as long as $f(t)$ is not the zero function. Similarly, if the ambient temperature A is nonzero, Newton's law of cooling is nonhomogeneous. A homogeneous linear ODE can be put into the form

$$a_n(x)\frac{d^ny}{dx^n}+a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1(x)\frac{dy}{dx}+a_0(x)y=0.$$

Compare to Equation $(1.2.4)$ and notice there is no function $b(x)$.

If the coefficients of a linear equation are actually constant functions, then the equation is said to have *constant coefficients*. The coefficients are the functions multiplying the dependent variable(s) or one of its derivatives, not the function $b(x)$ standing alone. A constant coefficient nonhomogeneous ODE is an equation of the form

$$a_n\frac{d^ny}{dx^n}+a_{n-1}\frac{d^{n-1}y}{dx^{n-1}}+\cdots+a_1\frac{dy}{dx}+a_0y=b(x),$$

where $a_0, a_1, \ldots, a_n$ are all constants, but b may depend on the independent variable x. The mechanical vibrations equation above is a constant coefficient nonhomogeneous second order ODE. The same nomenclature applies to PDEs, so the transport equation, heat equation and wave equation are all examples of constant coefficient linear PDEs.

Finally, an equation (or system) is called *autonomous* if the equation does not depend on the independent variable. For autonomous ordinary differential equations, the independent variable is then thought of as time. Autonomous equation means an equation that does not change with time. For example, Newton's law of cooling is autonomous, so is Equation $(1.2.6)$. On the other hand, mechanical vibrations or Equation $(1.2.5)$ are not autonomous.

Example 1.2.1

Classify the following differential equations by their type, order, and linearity:

a. $\frac{dy}{dx} = 2x - 5$
b. $\cos(y)\frac{d^2y}{dx^2} - 5\frac{dy}{dx} = x - 2$
c. $\frac{d^4y}{dx^4} - 3\frac{d^3y}{dx^3} + e^y = 7x$
d. $(\frac{d^2y}{dx^2})^3 + 4(\frac{dy}{dx})^5 = \sin(y)$
e. $\frac{\partial y}{\partial x} - 4xy = 0$
f. $\frac{\partial^2 y}{\partial x^2} + 6\frac{\partial y}{\partial t} = 0$

Solution

- **Video Length:** 7 minutes 4 seconds
- **Context:** Classify the differential equations by their type, order, and linearity.

1.2E: Exercises for Section 1.2

Exercises 1-6

Classify and solve the differential equations as specified in each exercise.

Exercise $1.2E.1$

Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a. $\sin(t)\dfrac{d^2x}{dt^2}+\cos(t)x=t^2$
b. $\dfrac{\partial u}{\partial x}+3\dfrac{\partial u}{\partial y}=xy$
c. $y''+3y+5x=0, \quad x''+x-y=0$
d. $\dfrac{\partial^2 u}{\partial t^2}+u\dfrac{\partial^2 u}{\partial s^2}=0$
e. $x''+tx^2=t$
f. $\dfrac{d^4x}{dt^4}=0$

Exercise $1.2E.2$

If $\mathbf{u}=(u_1,u_2,u_3)$ is a vector, we have the divergence $\nabla\cdot\mathbf{u}=\frac{\partial u_1}{\partial x}+\frac{\partial u_2}{\partial y}+\frac{\partial u_3}{\partial z}$ and curl $\nabla\times\mathbf{u}=\left(\frac{\partial u_3}{\partial y}-\frac{\partial u_2}{\partial z},\frac{\partial u_1}{\partial z}-\frac{\partial u_3}{\partial x},\frac{\partial u_2}{\partial x}-\frac{\partial u_1}{\partial y}\right)$. Notice that curl of a vector is still a vector. Write out Maxwell's equations in terms of partial derivatives and classify the system.

Exercise $1.2E.3$

Suppose F is a linear function, that is, $F(x,y)=ax+by$ for constants a and b. What is the classification of equations of the form $F(y',y)=0$.

Exercise $1.2E.4$

Write down an explicit example of a third order, linear, nonconstant coefficient, nonautonomous, nonhomogeneous system of two ODE such that every derivative that could appear, does appear.

Exercise $1.2E.5$

Classify the following equations. Are they ODE or PDE? Is it an equation or a system? What is the order? Is it linear or nonlinear, and if it is linear, is it homogeneous, constant coefficient? If it is an ODE, is it autonomous?

a. $\dfrac{\partial^2 v}{\partial x^2}+3\dfrac{\partial^2 v}{\partial y^2}=\sin(x)$
b. $\dfrac{dx}{dt}+\cos(t)x=t^2+t+1$
c. $\dfrac{d^7F}{dx^7}=3F(x)$
d. $y''+8y'=1$
e. $x''+tyx'=0, \quad y''+txy=0$
f. $\dfrac{\partial u}{\partial t}=\dfrac{\partial^2 u}{\partial s^2}+u^2$

Answer

a. PDE, equation, second order, linear, nonhomogeneous, constant coefficient.
b. ODE, equation, first order, linear, nonhomogeneous, not constant coefficient, not autonomous.
c. ODE, equation, seventh order, linear, homogeneous, constant coefficient, autonomous.
d. ODE, equation, second order, linear, nonhomogeneous, constant coefficient, autonomous.
e. ODE, system, second order, nonlinear. PDE, equation, second order, nonlinear.

Exercise $1.2E.6$

Write down the general 0^{th} order linear ordinary differential equation. Write down the general solution.

Answer

Equation: $a(x)y = b(x)$.

Solution: $y = \frac{b(x)}{a(x)}$.

Exercise $1.2E.7$

For which k is $\frac{dx}{dt} + x^k = t^{k+2}$ linear?

Hint: there are two answers.

Answer

$k = 0$ or $k = 1$

1.3: Integrals as Solutions

Learning Objectives

- Understand and solve first-order ODEs using integration.
- Apply differential equations to real-world problems.

A first order ODE is an equation of the form

$$\frac{dy}{dx} = f(x, y)$$

or just

$$y' = f(x, y)$$

In general, there is no simple formula or procedure one can follow to find solutions. In the next few lectures we will look at special cases where solutions are not difficult to obtain. In this section, let us assume that f is a function of x alone, that is, the equation is

$$y' = f(x) \tag{1.3.1}$$

We could just integrate (antidifferentiate) both sides with respect to x.

$$\int y'(x)dx = \int f(x)dx + C$$

that is

$$y(x) = \int f(x)dx + C$$

This $y(x)$ is actually the general solution. So to solve Equation $(1.3.1)$, we find some antiderivative of $f(x)$ and then we add an arbitrary constant to get the general solution.

Now is a good time to discuss a point about calculus notation and terminology. Calculus textbooks muddy the waters by talking about the integral as primarily the so-called **indefinite integral**. The indefinite integral is really the **antiderivative** (in fact the whole one-parameter family of antiderivatives). There really exists only one integral and that is the definite integral. The only reason for the indefinite integral notation is that we can always write an antiderivative as a (definite) integral. That is, by the fundamental theorem of calculus we can always write $\int f(x)dx + C$ as

$$\int_{x_0}^{x} f(t)dt + C$$

Hence the terminology to integrate when we may really mean to antidifferentiate. Integration is just one way to compute the antiderivative (and it is a way that always works, see the following examples). Integration is defined as the area under the graph, it only happens to also compute antiderivatives. For sake of consistency, we will keep using the indefinite integral notation when we want an antiderivative, and you should always think of the definite integral.

Example 1.3.1

Find the general solution of $y' = 3x^2$.

Solution

Elementary calculus tells us that the general solution must be $y = x^3 + C$. Let us check: $y' = 3x^2$. We have gotten precisely our equation back.

Normally, we also have an initial condition such as $y(x_0) = y_0$ for some two numbersx_0 and y_0 x_0 is usually 0, but not always. We can then write the solution as a definite integral in a nice way. Suppose our problem is $y' = f(x)$, $y(x_0) = y_0$. Then the solution is

$$y(x)=\int_{x_0}^{x} f(s)ds+y_0 \tag{1.3.2}$$

Let us check! We compute $y'=f(x)$, via the fundamental theorem of calculus, and by Jupiter, y is a solution. Is it the one satisfying the initial condition? Well, $y(x_0)=\int_{x_0}^{x_0} f(x)dx+y_0=y_0$. It is!

Do note that the definite integral and the indefinite integral (antidifferentiation) are completely different beasts. The definite integral always evaluates to a number. Therefore, Equation $(1.3.2)$ is a formula we can plug into the calculator or a computer, and it will be happy to calculate specific values for us. We will easily be able to plot the solution and work with it just like with any other function. It is not so crucial to always find a closed form for the antiderivative.

Example $1.3.2$

Solve

$$y'=e^{-x^2}, \quad y(0)=1.$$

By the preceding discussion, the solution must be

$$y(x)=\int_{0}^{x} e^{-s^2} ds+1.$$

Solution

Here is a good way to make fun of your friends taking second semester calculus. Tell them to find the closed form solution. Ha ha ha (bad math joke). It is not possible (in closed form). There is absolutely nothing wrong with writing the solution as a definite integral. This particular integral is in fact very important in statistics.

Using this method, we can also solve equations of the form

$$y'=f(y)$$

Let us write the equation in Leibniz notation.

$$\frac{dy}{dx}=f(y)$$

Now we use the inverse function theorem from calculus to switch the roles of x and y to obtain

$$\frac{dy}{dx}=\frac{1}{f(y)}$$

What we are doing seems like algebra with dx and dy. It is tempting to just do algebra with dx and dy as if they were numbers. And in this case it does work. Be careful, however, as this sort of hand-waving calculation can lead to trouble, especially when more than one independent variable is involved. At this point we can simply integrate,

$$x(y)=\int \frac{1}{f(y)}dy+C$$

Finally, we try to solve for y.

Example $1.3.3$

Show $y'=ky$ (for some $k>0$) has the solution $y=Ce^{kx}$

Solution

Previously, we guessed $y'=ky$ (for some $k>0$) has the solution $y=Ce^{kx}$. We can now find the solution without guessing. First we note that $y=0$ is a solution. Henceforth, we assume $y\neq 0$. We write

$$\frac{dx}{dy}=\frac{1}{ky}$$

We integrate to obtain

$$x(y) = x = \frac{1}{k}\ln|y| + D$$

where D is an arbitrary constant. Now we solve for y (actually for $|y|$).

$$|y| = e^{kx-kD} = e^{-kD}e^{kx}$$

If we replace e^{-kD} with an arbitrary constant C we can get rid of the absolute value bars (which we can do as D was arbitrary). In this way, we also incorporate the solution $y = 0$. We get the same general solution as we guessed before, $y = Ce^{kx}$.

Example 1.3.4

Find the general solution of $y' = y^2$.

Solution

First we note that $y = 0$ is a solution. We can now assume that $y \neq 0$. Write

$$\frac{dx}{dy} = \frac{1}{y^2}$$

We integrate to get

$$x = \frac{-1}{y} + C$$

We solve for $y = \dfrac{1}{C-x}$. So the general solution is

$$y = \frac{1}{C-x} \; or \; y = 0$$

Note the singularities of the solution. If for example $C = 1$, then the solution as we approach $x = 1$ as in Figure 1.3.1. Generally, it is hard to tell from just looking at the equation itself how the solution is going to behave. The Equation $y' = y^2$ is very nice and defined everywhere, but the solution is only defined on some interval $(-\infty, C)$ or (C, ∞). Usually when this happens we only consider one of these the solution. For example if we impose a condition $y(0) = 1$, then the solution is $y = \frac{1}{1-x}$, and we would consider this solution only for x on the interval $(-\infty, 1)$. In the figure 1.3.1, it is the left side of the graph.

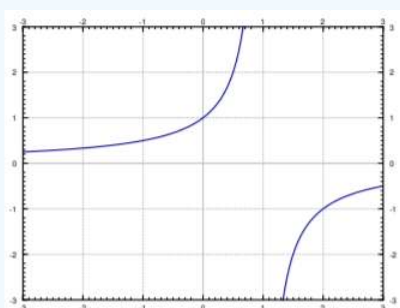


Figure 1.3.1: This is a general solution of $y' = y^2$ by considering the constant of integration C=1. It is the plot of function $y = \frac{1}{1-x}$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Classical problems leading to differential equations solvable by integration are problems dealing with velocity, acceleration, and distance. You have surely seen these problems before in your calculus class.

Example 1.3.5

Suppose a car drives at a speed $e^{t/2}$ m/s, where t is time in seconds. How far did the car get in 2 seconds (starting at $t = 0$)? How far in 10 seconds?

Solution

Let x denote the distance the car traveled. The equation is

$$x' = e^{t/2}$$

We can just integrate this equation to get

$$x(t) = 2e^{t/2} + C$$

We still need to figure out C. We know that when $t = 0$, then $x = 0$. That is, $x(0) = 0$. So

$$0 = x(0) = 2e^{0/2} + C = 2 + C$$

Thus $C = -2$ and

$$x(t) = 2e^{t/2} - 2$$

Now we just plug in to get where the car is at 2 and at 10 seconds. We obtain

$$x(2) = 2e^{2/2} - 2 \approx 3.44\text{~meters}, \quad x(10) = 2e^{10/2} - 2 \approx 294\text{~meters}$$

Example 1.3.6

Suppose that the car accelerates at a rate of $t^2\ m/s^2$. At time $t = 0$ the car is at the 1 meter mark and is traveling at 10 m/s. Where is the car at time $t = 10$?

Solution

Well this is actually a second order problem. If x is the distance traveled, then x' is the velocity, and x'' is the acceleration. The equation with initial conditions is

$$x'' = t^2, \quad x(0) = 1, \quad x'(0) = 10$$

What if we say $x' = v$. Then we have the problem

$$v' = t^2, \quad v(0) = 10$$

Once we solve for v, we can integrate and find x.

1.3E: Exercises for Section 1.3

Exercises 1-19

Solve each exercise using appropriate methods. Provide clear and concise solutions.

? Exercise $1.3E.1$

Consider Example 1.3.6 in Section 1.3, solve for v and then solve for x. Next, find $x(10)$.

? Exercise $1.3E.2$

Solve $\frac{dy}{dx} = x^2 + x$ for $y(1) = 3$.

? Exercise $1.3E.3$

Solve $\frac{dy}{dx} = \sin(5x)$ for $y(0) = 2$.

? Exercise $1.3E.4$

Solve $\frac{dy}{dx} = \frac{1}{x^2 - 1}$ for $y(0) = 0$.

? Exercise $1.3E.5$

Solve $y' = y^3$ for $y(0) = 1$.

? Exercise $1.3E.6$

Solve $y' = (y-1)(y+1)$ for $y(0) = 3$.

? Exercise $1.3E.7$

Solve $\frac{dy}{dx} = \frac{1}{y+1}$ for $y(0) = 0$.

? Exercise $1.3E.8$

Solve $y'' = \sin x$ for $y(0) = 0$, $y'(0) = 2$.

? Exercise $1.3E.9$

A spaceship is traveling at the speed $2t^2 + 1$ $^{\text{km}}/_{\text{s}}$ (t is time in seconds). It is pointing directly away from earth and at time $t = 0$ it is 1000 kilometers from earth. How far from earth is it at one minute from time $t = 0$?

? Exercise $1.3E.10$

Solve $\frac{dx}{dt} = \sin(t^2) + t$, $x(0) = 20$. It is OK to leave your answer as a definite integral.

❓ Exercise $1.3E.11$

A dropped ball accelerates downwards at a constant rate 9.8 meters per second squared. Set up the differential equation for the height above ground h in meters. Then supposing $h(0) = 100$ meters, how long does it take for the ball to hit the ground?

❓ Exercise $1.3E.12$

Find the general solution of $y' = e^x$, and then $y' = e^y$.

❓ Exercise $1.3E.13$

Solve $\dfrac{dy}{dx} = e^x + x$ and $y(0) = 10$.

Answer

$y = e^x + \frac{x^2}{2} + 9$

❓ Exercise $1.3E.14$

Solve $x' = \dfrac{1}{x^2}$, $x(1) = 1$.

Answer

$x = (3t - 2)^{1/3}$

❓ Exercise $1.3E.15$

Solve $x' = \frac{1}{\cos(x)}$, $x(0) = \frac{\pi}{2}$.

Answer

$\mathrm{A}x = \sin^{-1}(t + 1)$

❓ Exercise $1.3E.16$

Sid is in a car traveling at speed $10t + 70$ miles per hour away from Las Vegas, where t is in hours. At $t = 0$ the Sid is 10 miles away from Vegas. How far from Vegas is Sid 2 hours later?

Answer

170

❓ Exercise $1.3E.17$

Solve $y' = y''$, $y(0) = 1$, where n is a positive integer. Hint: You have to consider different cases.

Answer

If $n \neq 1$, then $y = ((1 - n)x + 1)^{1/(1-n)}$. If $n = 1$, then $y = e^x$.

Exercise $1.3E.18$

The rate of change of the volume of a snowball that is melting is proportional to the surface area of the snowball. Suppose the snowball is perfectly spherical. The volume (in centimeters cubed) of a ball of radius r centimeters is $\frac{4}{3}\pi r^3$. The surface area is $4\pi r^2$. Set up the differential equation for how the radius r is changing. Then, suppose that at time $t = 0$ minutes, the radius is 10 centimeters. After 5 minutes, the radius is 8 centimeters. At what time t will the snowball be completely melted?

Answer

The equation is $r' = -C$ for some constant C. The snowball will be completely melted in 25 minutes from time $t = 0$.

Exercise $1.3E.19$

Find the general solution to $y^{(4)} = 0$. How many distinct constants do you need?

Answer

$y = Ax^3 + Bx^2 + Cx + D$, so 4 constants.

1.4: Slope Fields

Learning Objectives

- Interpret and construct slope fields for first-order differential equations.
- Evaluate the existence and uniqueness of solutions using Picard's Theorem.

The general first order equation we are studying looks like

$$y' = f(x, y)$$

In general, we cannot simply solve these kinds of equations explicitly. It would be nice if we could at least figure out the shape and behavior of the solutions, or if we could find approximate solutions.

Slope Fields

The equation $y' = f(x, y)$ gives you a slope at each point in the (x, y)-plane. And this is the slope a solution $y(x)$ would have at x if its value was y. In other words, $f(x, y)$ is the slope of a solution whose graph runs through the point (x, y). At a point (x, y), we plot a short line with the slope $f(x, y)$. For example, if $f(x, y) = xy$, then at point $(2, 1.5)$ we draw a short line of slope $xy = 2 \times 1.5 = 3$. So, if $y(x)$ is a solution and $y(2) = 1.5$, then the equation mandates that $y'(2) = 3$ (see Figure $1.4.1$).

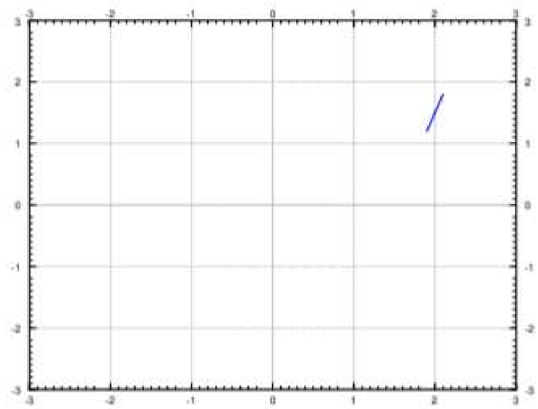


Figure $1.4.1$: The slope of a solution to the differential equation y' = xy at the point (2, 1.5). It is short line with slope 3, indicating the direction of the solution curve at that point. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

To get an idea of how solutions behave, we draw such lines at lots of points in the plane, not just the point $(2, 1.5)$. We would ideally want to see the slope at every point, but that is just not possible. Usually we pick a grid of points fine enough so that it shows the behavior, but not too fine so that we can still recognize the individual lines. We call this picture the of the equation. See Figure $1.4.2$ for the slope field of the equation $y' = xy$. Usually in practice, one does not do this by hand, but has a computer to do the drawing.

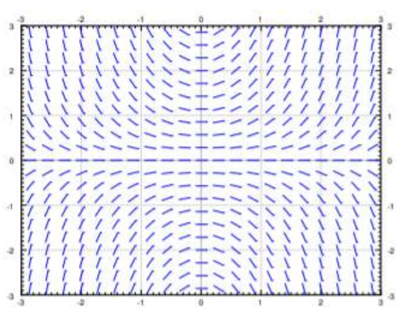


Figure $1.4.2$: A grid of short line segments curving upward above the line $y = 0$ and curving downward below the line $y = 0$, which are the slopes of solutions to the differential equation $y' = xy$ at various points. By analyzing the direction of these line segments, one can visualize the behavior of solutions to the equation. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

❓ Example 1.4.1

Sketch slope field/directional field of the differential equations.

a. $\frac{dy}{dx} = 2x$
b. $\frac{dy}{dx} = x - y$
c. $\frac{dy}{dx} = \frac{1}{x}$

Solution

- **Video Length:** 9 minutes 1 second
- **Context:** Describe the slope field, directional field, and what it represents.

Suppose we are given a specific initial condition $y(x_0) = y_0$. A solution, that is, the graph of the solution, would be a curve that follows the slopes we drew. For a few sample solutions, see Figure $1.4.3$. It is easy to roughly sketch (or at least imagine) possible solutions in the slope field, just from looking at the slope field itself. You simply sketch a line that roughly fits the little line segments and goes through your initial condition.

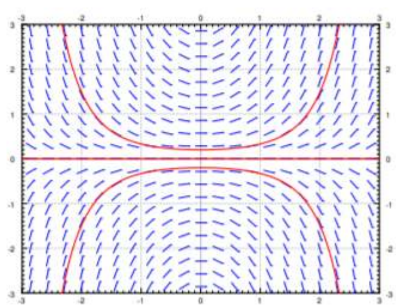


Figure $1.4.3$: Slope field of (y' = xy) with a graph of solutions satisfying (y(0) = 0.2), (y(0) = 0), and (y(0) = -0.2). It shows a slope field with three solution curves plotted on it. The slope field indicates the direction of the solution curves at various points, and the solution curves themselves show how the solutions to the differential equation behave.(CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

By looking at the slope field we get a lot of information about the behavior of solutions without having to solve the equation. For example, in Figure $1.4.3$ we see what the solutions do when the initial conditions are $y(0) > 0$, $y(0) = 0$ and $y(0) < 0$. A small change in the initial condition causes quite different behavior. We see this behavior just from the slope field and imagining what solutions ought to do.

We see a different behavior for the equation $y' = -y$. The slope field and a few solutions are in Figure $1.4.4$. If we think of moving from left to right (perhaps x is time and time is usually increasing), then we see that no matter what $y(0)$ is, all solutions tend to zero as x tends to infinity. Again that behavior is clear from simply looking at the slope field itself. A slope field with all curves going towards the positive x-axis in the limit is in Figure $1.4.4$.

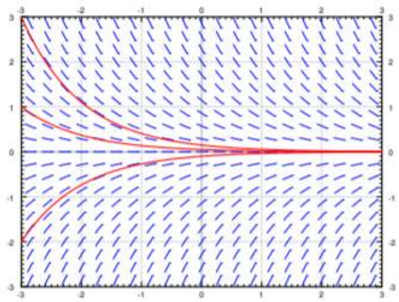


Figure 1.4.4: The slope field of (y' = -y) with a graph of a few solutions. It displays a grid of short line segments representing the slope of solutions to the differential equation at various points. The red curves represent specific solutions to the differential equation, illustrating how their behavior is influenced by the slope field. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Example 1.4.2

Determine which of the following differential equations would produce the given direction field:

a. $\frac{dy}{dx} = x^2 y$
b. $\frac{dy}{dx} = xy^2$
c. $\frac{dy}{dx} = x^2 y^2$
d. $\frac{dy}{dx} = xy$

Solution

- **Video Length:** 4 minutes 51 seconds
- **Context:** Identify the differential equation that corresponds to a given direction field by analyzing the behavior and slopes of the field. This involves comparing how each equation's structure influences the slope values at various points in the xy-plane.

Existence and uniqueness

We wish to ask two fundamental questions about the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

i. Does a solution exist?
ii. Is the solution unique (if it exists)?

What do you think is the answer? The answer seems to be yes to both does it not? Well, pretty much. But there are cases when the answer to either question can be no.

Since generally the equations we encounter in applications come from real life situations, it seems logical that a solution always exists. It also has to be unique if we believe our universe is deterministic. If the solution does not exist, or if it is not unique, we have probably not devised the correct model. Hence, it is good to know when things go wrong and why.

Example 1.4.3

Solve

$$y' = \frac{1}{x}, \quad y(0) = 0.$$

Solution

Integrate to find the general solution $y = \ln|x| + C$. Note that the solution does not exist at $x = 0$ (see Figure 1.4.5). The equation may have been written as the seemingly harmless $xy' = 1$. A slope field with all curves emanating from the negative y-axis. Five are shown that all eventually depart from the y-axis

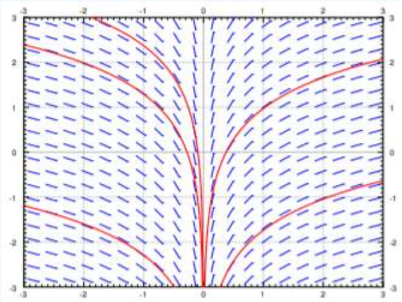


Figure 1.4.5: The slope field of $y' = \frac{1}{x}$. It displays a slope field with vertical lines, indicating that the slopes of solutions to the differential equation become increasingly steep as x approaches 0. This is a visual representation of the fact that the solution to the differential equation, $y = \ln|x| + C$, is not defined at x = 0. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Example 1.4.4

Solve

$$y' = 2\sqrt{|y|}, \quad y(0) = 0.$$

Solution

Note that $y = 0$ is a trivial solution. But another solution is the function

$$y(x) = \begin{cases} x^2 & \text{if } x \geq 0 \\ -x^2 & \text{if } x < 0 \end{cases}$$

See Figure 1.4.6 for both solutions.

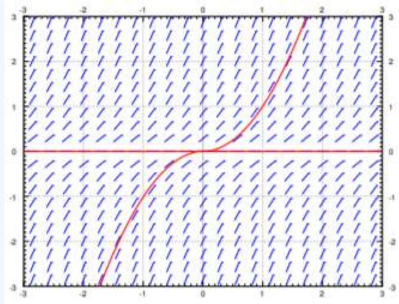


Figure 1.4.6: The slope field of $y' = 2\sqrt{|y|}$ with two solutions satisfying initial condition $y(0) = 0$. One solution appears to be a flat line, and another is an increasing curve. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

It is hard to tell by staring at the slope field that the solution is not unique. Is there any hope? Of course there is. We have the following theorem, known as **Picard's theorem**.

Theorem 1.4.1: Picard's Theorem on Existence and Uniqueness

If $f(x,y)$ is *continuous (as a function of two variables) and* $\frac{\partial f}{\partial y}$ *exists and is continuous near some* (x_0, y_0), then a solution to

$$y' = f(x,y), \quad y(x_0) = y_0$$

exists (at least for x *in some small interval) and is unique.*

Note that the problems $y' = \frac{1}{x}$, $y(0) = 0$ and $y' = 2\sqrt{|y|}$, $y(0) = 0$ do not satisfy the hypothesis of the theorem. Even if we can use the theorem, we ought to be careful about this existence business. It is quite possible that the solution only exists for a short while.

Example 1.4.5

For some constant A, solve:

$$y' = y^2 \quad y(0) = A$$

.

Solution

We know how to solve this equation. First assume that $A \neq 0$, so y is not equal to zero at least for some x near 0. So $x' = \frac{1}{y^2}$, so $x = -\frac{1}{y} + C$, so $y = \frac{1}{C - x}$. If $y(0) = A$, then $C = \frac{1}{A}$ so

$$y = \frac{1}{\frac{1}{A} - x}$$

For example, when $A = 1$ the solution "blows up" at $x = 1$. Hence, the solution does not exist for all x even if the equation is nice everywhere. The equation $y' = y^2$ certainly looks nice.

For most of this course we will be interested in equations where existence and uniqueness holds, and in fact holds "globally" unlike for the equation $y' = y^2$.

Footnotes

[1] Named after the French mathematician Charles Émile Picard (1856 – 1941)

- **1.2: Slope fields** by Jiří Lebl is licensed CC BY-SA 4.0. Original source: https://www.jirka.org/diffyqs.

1.4E: Exercises for Section 1.4

Exercises 1-18

Analyze the given equations and questions thoroughly. For slope fields, provide a visual sketch if needed.

Exercise $1.4E.1$

Sketch slope field for $y' = e^{x-y}$. How do the solutions behave as x grows? Can you guess a particular solution by looking at the slope field?

Exercise $1.4E.2$

Sketch slope field for $y' = x^2$.

Exercise $1.4E.3$

Sketch slope field for $y' = y^2$.

Exercise $1.4E.4$

Is it possible to solve the equation $y' = \frac{xy}{\cos x}$ for $y(0) = 1$? Justify.

Exercise $1.4E.5$

Is it possible to solve the equation $y' = y\sqrt{|x|}$ for $y(0) = 0$? Is the solution unique? Justify.

Exercise $1.4E.6$

Match equations $y' = 1 - x$, $y' = x - 2y$, $y' = x(1 - y)$ to slope fields. Justify.

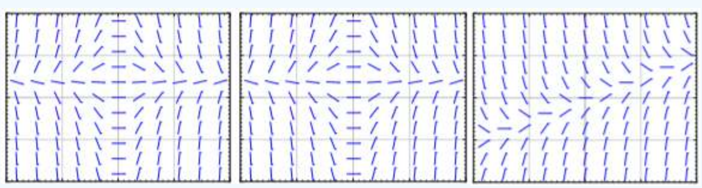


Figure $1.4E.1$: Slope fields (a) on the left, (b) in the middle , and (c) on the right.

Exercise $1.4E.7$

Take $y' = f(x, y)$, $y(0) = 0$, where $f(x, y) > 1$ for all x and y. If the solution exists for all x, can you say what happens to $y(x)$ as x goes to positive infinity? Explain.

Exercise $1.4E.8$

Given the differential equation $(y - x)y' = 0$ with initial condition $x(0) = 0$.

a. Find two distinct solutions.
b. Explain why this does not violate Picard's theorem.

Exercise $1.4E.9$

Suppose $y' = f(x, y)$. What will the slope field look like, explain and sketch an example, if you know the following about $f(x, y)$?

a. f does not depend on y.
b. f does not depend on x.
c. $f, (t,t) = 0$ for any number t.
d. $f(x,0) = 0$ and $f(x,1) = 1$ for all x.

? Exercise $1.4E.10$

Find a solution to $y' = |y|$, $y(0) = 0$. Does Picard's theorem apply?

? Exercise $1.4E.11$

Take an equation $y' = (y-2x)g(x,y)+2$ for some function $g(x,y)$. Can you solve the problem for the initial condition $y(0) = 0$, and if so what is the solution?

? Exercise $1.4E.12$

Suppose $y' = f(x,y)$ is such that $f(x,1) = 0$ for every x, f is continuous and $\frac{\partial f}{\partial y}$ exists and is continuous for every x and y.

a. Guess a solution given the initial condition $y(0) = 1$.
b. Can graphs of two solutions of the equation for different initial conditions ever intersect?
c. Given $y(0) = 0$, what can you say about the solution. In particular, can $y(x) > 1$ for any x? Can $y(x) = 1$ for any x? Why or why not?

? Exercise $1.4E.13$

Sketch the slope field of $y' = y^3$. Can you visually find the solution that satisfies $y(0) = 0$?

Answer

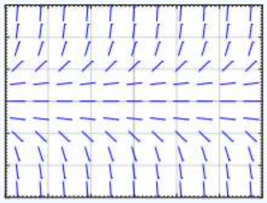


Figure $1.4E.2$: Slope field for $y' = y^3$. Line segments appear to be flat in the middle, increasing and curving upward above the middle line segments, decreasing and curving downward below the middle line segments.

? Exercise $1.4E.14$

Is it possible to solve $y' = xy$ for $y(0) = 0$? Is the solution unique?

Answer

Yes, a solution exists. The equation is $y' = f(x,y)$ where $f(x,y) = xy$. The function $f(x,y)$ is continuous and $\frac{\partial f}{\partial y} = x$, which is also continuous near $(0,0)$. So a solution exists and is unique. In fact, $y = 0$ is the solution.

? Exercise $1.4E.15$

Is it possible to solve $y' = \frac{x}{x^2-1}$ for $y(1) = 0$?

Answer

No, the equation is not defined at $(x,y) = (1,0)$.

Exercise 1.4E.16

Match equations $y' = \sin x$, $y' = \cos y$, $y' = y\cos(x)$ to slope fields. Justify.

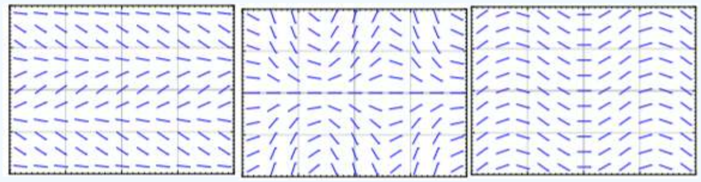


Figure 1.4E.3: Slope fields (a) on the left, (b) in the middle, and (c) on the right of trigonometric functions.

Answer

a. $y' = \cos y$
b. $y' = y\cos(x)$
c. $y' = \sin x$

Justification left to the reader.

Exercise 1.4E.17

Suppose

$$f(y) = \begin{cases} 0 & if\ y > 0, \\ 1 & if\ y \leq 0. \end{cases}$$

Does $y' = f(y)$, $y(0) = 0$ have a continuously differentiable solution? Does Picard's theorem apply? Why, or why not?

Answer

Picard's Theorem does not apply as f is not continuous at $y = 0$. The equation does not have a continuously differentiable solution. Suppose it did. Notice that $y'(0) = 1$. By the first derivative test, $y(x) > 0$ for small positive x. But then for those x we would have $y'(x) = 0$, so clearly the derivative cannot be continuous.

Exercise 1.4E.18

Consider an equation of the form $y' = f(x)$ for some continuous function f, and an initial condition $y(x_0) = y_0$. Does a solution exist for all x? Why or why not?

Answer

The solution is $y(x) = \int_{x_0}^{x} f(s)ds + y_0$, and this does indeed exist for every x.

1.5: Separable Equations

Learning Objectives

- Recognize and solve separable differential equations by correctly separating variables.
- Apply separable differential equations to practical problems, such as exponential growth/decay, Newton's Law of Cooling, and motion, and interpret the physical meaning of the solutions.

When a differential equation is of the form $y' = f(x)$, we can just integrate: $y = \int f(x)dx + C$. Unfortunately this method no longer works for the general form of the equation $y' = f(x, y)$. Integrating both sides yields

$$y = \int f(x, y)dx + C$$

Notice the dependence on y in the integral.

Separable equations

Let us suppose that the equation is *separable*. That is, let us consider

$$y' = f(x)g(y),$$

for some functions $f(x)$ and $g(y)$. Let us write the equation in the Leibniz notation

$$\frac{dy}{dx} = f(x)g(y)$$

Then we rewrite the equation as

$$\frac{dy}{g(y)} = f(x)dx$$

Now both sides look like something we can integrate. We obtain

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

If we can find closed form expressions for these two integrals, we can, perhaps, solve for y.

Example 1.5.1

Solve

$$y' = xy$$

Solution

First note that $y = 0$ is a solution, so assume $y \neq 0$ from now on, so that we can divide by y. Write the equation as $\frac{dy}{dx} = xy$, then

$$\int \frac{dy}{y} = \int x dx + C.$$

We compute the antiderivatives to get

$$\ln|y| = \frac{x^2}{2} + C$$

Or

$$|y| = e^{\frac{x^2}{2}+C} = e^{\frac{x^2}{2}} e^C = De^{\frac{x^2}{2}}$$

where $D > 0$ is some constant. Because $y = 0$ is a solution and because of the absolute value we actually can write:

$$y = De^{\frac{x^2}{2}}$$

for any number D (including zero or negative).

We check:

$$y' = Dxe^{\frac{x^2}{2}} = x\left(De^{\frac{x^2}{2}}\right) = xy$$

Yay!

We should be a little bit more careful with this method. You may be worried that we were integrating in two different variables. We seemingly did a different operation to each side. Let us work through this method more rigorously. Take

$$\frac{dy}{dx} = f(x)g(y)$$

We rewrite the equation as follows. Note that $y = y(x)$ is a function of x and so is $\frac{dy}{dx}$

$$\frac{1}{g(y)}\frac{dy}{dx} = f(x)$$

We integrate both sides with respect to x.

$$\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx + C$$

We use the change of variables formula (substitution) on the left hand side:

$$\int \frac{1}{g(y)}dy = \int f(x)dx + C$$

And we are done.

Implicit solutions

It is clear that we might sometimes get stuck even if we can do the integration. For example, take the separable equation

$$y' = \frac{xy}{y^2+1}$$

We separate variables,

$$\frac{y^2+1}{y}dy = \left(y + \frac{1}{y}\right)dy = xdx$$

We integrate to get

$$\frac{y^2}{2} + ln|y| = \frac{x^2}{2} + C$$

or perhaps the easier looking expression (where $D = 2C$)

$$y^2 + 2ln|y| = x^2 + D$$

It is not easy to find the solution explicitly as it is hard to solve for y. We, therefore, leave the solution in this form and call it an *implicit solution*. It is still easy to check that an implicit solution satisfies the differential equation. In this case, we differentiate with respect to x, and remember that y is a function of x, to get

$$y'\left(2y + \frac{2}{y}\right) = 2x$$

Multiply both sides by y and divide by $2(y^2+1)$ and you will get exactly the differential equation. We leave this computation to the reader.

If you have an implicit solution, and you want to compute values for y, you might have to be tricky. You might get multiple solutions y for each x, so you have to pick one. Sometimes you can graph x as a function of y, and then flip your paper. Sometimes you have to do more.

Computers are also good at some of these tricks. More advanced mathematical software usually has some way of plotting solutions to implicit equations. For example, for $C = 0$ if you plot all the points (x, y) that are solutions to $y^2 + 2\ln|y| = x^2$, you find the two curves in Figure $1.5.1$. This is not quite a graph of a function. For each x there are two choices of y. To find a function you would have to pick one of these two curves. You pick the one that satisfies your initial condition if you have one. For example, the top curve satisfies the condition $y(1) = 1$. So for each C we really got two solutions. As you can see, computing values from an implicit solution can be somewhat tricky. But sometimes, an implicit solution is the best we can do.

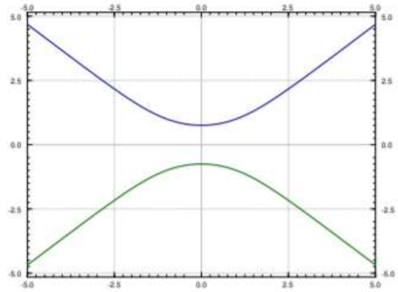


Figure $1.5.1$: Diagram illustrating the implicit solution $y^2 + 2\ln|y| = x^2$ to the differential equation $y' = \frac{xy}{y^2+1}$. The top blue curve is concave upwards, and the bottom green curve is concave downwards. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

The equation above also has the solution $y = 0$. So the general solution is

$$y^2 + 2\ln|y| = x^2 + C, \qquad \text{and} \qquad y = 0.$$

These outlying solutions such as $y = 0$ are sometimes called *singular solutions*.

? Example $1.5.2$

Solve the following differential equations by separation of variables:

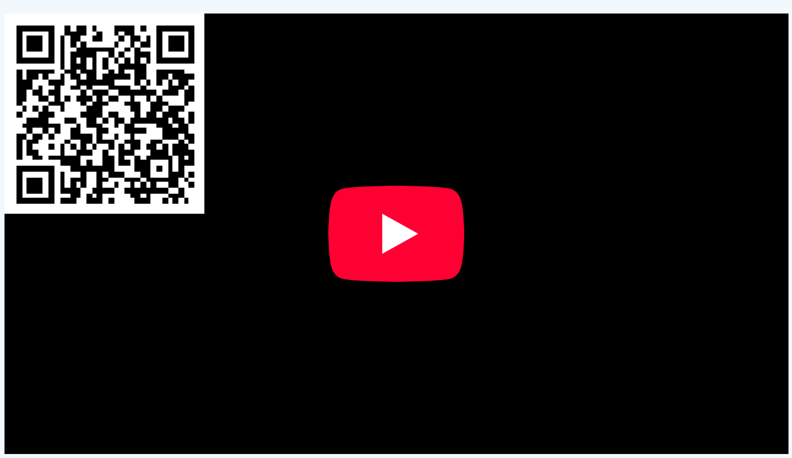
a. $dx - e^{3x} dy = 0$

b. $\frac{dy}{dx} = \frac{y+1}{x}$

c. $(4y + yx^2)dy - (2x + xy^2)dx = 0$

Solution

- **Video Length:** 10 minutes 28 seconds.
- **Context:** This video demonstrates step-by-step solutions to the equations listed above by separating variables and integrating each side appropriately.



Example 1.5.3

Solve $x^2y' = 1 - x^2 + y^2 - x^2y^2$, $y(1) = 0$.

Solution

First factor the right hand side to obtain

$$x^2y' = (1 - x^2)(1 + y^2)$$

Separate variables, integrate, and solve for y

$$\begin{aligned}\frac{y'}{1+y^2} &= \frac{1-x^2}{x^2},\\ \frac{y'}{1+y^2} &= \frac{1}{x^2} - 1,\\ \arctan(y) &= -\frac{1}{x} - x + C,\\ y &= \tan\left(-\frac{1}{x} - x + C\right)\end{aligned} \tag{1.5.1}$$

Solve for the initial condition, $0 = \tan(-2 + C)$ to get $C = 2$ (or $C = 2 + \pi$, or $C = 2 + 2\pi$, etc.). The particular solution we seek is, therefore,

$$y = \tan\left(\frac{-1}{x} - x + 2\right).$$

Example 1.5.4

Juan made a cup of coffee, and Juan likes to drink coffee only once reaches 60 degrees Celsius and will not burn him. Initially at time $t = 0$ minutes, Juan measured the temperature and the coffee was 89 degrees Celsius. One minute later, Juan measured the coffee again and it had 85 degrees. The temperature of the room (the ambient temperature) is 22 degrees. When should Juan start drinking?

Solution

Let T be the temperature of the coffee in degrees Celsius, and let A be the ambient (room) temperature, also in degrees Celsius. The rate at which the temperature of the coffee is changing is proportional to the difference between the ambient temperature and the temperature of the coffee. That is,

$$\frac{dT}{dt} = k(A - T),$$

for some constant k. For our setup $A = 22$, $T(0) = 89$, $T(1) = 85$. We separate variables and integrate (let C and D denote arbitrary constants)

$$\begin{aligned}\frac{1}{T-A}\frac{dT}{dt} &= -k,\\ \ln(T - A) &= -kt + C, \quad (\text{note that } T - A > 0)\\ T - A &= De^{-kt},\\ T &= A + De^{-kt}\end{aligned} \tag{1.5.2}$$

That is, $T = 22 + D\,e^{-kt}$. We plug in the first condition: $89 = T(0) = 22 + D$, and hence $D = 67$. So $T = 22 + 67\,e^{-kt}$. The second condition says $85 = T(1) = 22 + 67\,e^{-k}$. Solving for k we get $k = -\ln\frac{85-22}{67} \approx 0.0616$. Now we solve for the time t that gives us a temperature of 60 degrees. Namely, we solve

$$60 = 22 + 67e^{-0.0616t}$$

to get $t = -\frac{\ln\frac{60-22}{67}}{0.0616} \approx 9.21$ minutes. So Juan can begin to drink the coffee at just over 9 minutes from the time Juan made it. That is probably about the amount of time it took us to calculate how long it would take. See Figure 1.5.2.

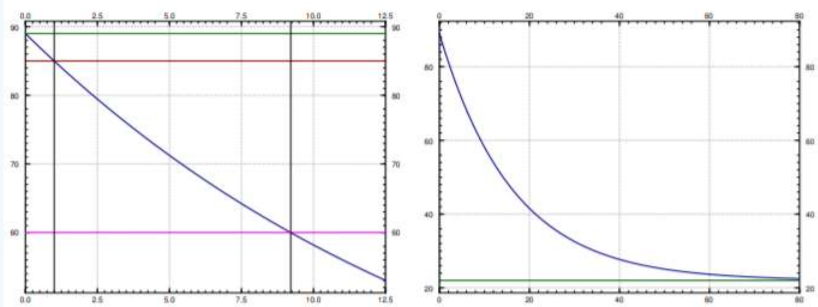


Figure 1.5.2: Graphs of the coffee temperature function $T(t)$. On the left, horizontal lines are drawn at temperatures 60, 85, and 89. Vertical lines are drawn at $t = 1$ and $t = 9.21$. Notice that the temperature of the coffee hits 85 at $t = 1$, and 60 at $t \approx 9.21$. On the right, the graph is over a longer period of time, with a horizontal line at the ambient temperature 22. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Example 1.5.5

Find the general solution to $y' = \frac{-xy^2}{3}$ (including singular solutions).

Solution

First note that $y = 0$ is a solution (a singular solution). Now assume that $y \neq 0$.

$$\begin{aligned} -\frac{3}{y^2}y' &= x, \\ \frac{3}{y} &= \frac{x^2}{2} + C, \\ y &= \frac{3}{\frac{x^2}{2} + C} = \frac{6}{x^2 + 2C}. \end{aligned} \tag{1.5.3}$$

So the general solution is,

$$y = \frac{6}{x^2 + 2C}, \quad \text{and} \quad y = 0.$$

1.5E: Exercises for Section 1.5

Exercises 1-20

Solve the differential equations of first order, and first degree, which can be written in the separable form.

Exercise $1.5E.1$

Solve $y' = \frac{x}{y}$.

Exercise $1.5E.2$

Solve $y' = x^2 y$.

Exercise $1.5E.3$

Solve $\frac{dx}{dt} = \left(x^2 - 1\right)$, for $x(0) = 0$.

Exercise $1.5E.4$

Solve $\frac{dx}{dt} = x\sin(t)$, for $x(0) = 1$.

Exercise $1.5E.5$

Solve $\frac{dy}{dx} = xy + x + y + 1$.

Hint: Factor the right-hand side.

Exercise $1.5E.6$

Solve $xy' = y + 2x^2 y$, where $y(1) = 1$.

Exercise $1.5E.7$

Solve $\frac{dy}{dx} = \frac{y^2+1}{x^2+1}$, for $y(0) = 1$.

Exercise $1.5E.8$

Find an implicit solution for $\frac{dy}{dx} = \frac{x^2+1}{y^2+1}$, for $y(0) = 1$.

Exercise $1.5E.9$

Find an explicit solution for $y' = xe^{-y}$, $y(0) = 1$.

Exercise $1.5E.10$

Find an explicit solution for $xy' = e^{-y}$, for $y(1) = 1$.

Exercise $1.5E.11$

Find an explicit solution for $y' = ye^{-x^2}$, $y(0) = 1$. It is alright to leave a definite integral in your answer?

Exercise $1.5E.12$

Suppose a cup of coffee is at 100 degrees Celsius at time $t = 0$, it is at 70 degrees at $t = 10$ minutes, and it is at 50 degrees at $t = 20$ minutes. Compute the ambient temperature.

Exercise $1.5E.13$

Solve $y' = 2xy$.

Answer

$y = Ce^{x^2}$

Exercise $1.5E.14$

Solve $x' = 3xt^2 - 3t^2$, $x(0) = 2$.

Answer

$x = e^{t^3} + 1$

Exercise $1.5E.15$

Find an implicit solution for $x' = \frac{1}{3x^2+1}$ $x(0) = 1$.

Answer

$x^3 + x = t + 2$

Exercise $1.5E.16$

Find an explicit solution to $xy' = y^2$, $y(1) = 1$.

Answer

$y = \frac{1}{1 - \ln x}$

Exercise $1.5E.17$

Find an implicit solution to $y' = \frac{\sin(x)}{\cos(y)}$.

Answer

$\sin(y) = -\cos(x) + C$

Exercise $1.5E.18$

Take Example 1.3.3 with the same numbers: 89 degrees at $t = 0$, 85 degrees at $t = 1$, and ambient temperature of 22 degrees. Suppose these temperatures were measured with precision of ± 0.5 degrees. Given this imprecision, the time it takes the coffee to cool to (exactly) 60 degrees is also only known in a certain range. Find this range.

Hint: Think about what kind of error makes the cooling time longer and what shorter.

Answer

The range is approximately $7.45$to 12.15minutes.

Exercise $1.5E.19$

A population x of rabbits on an island is modeled by $x' = x - (\frac{1}{1000})x^2$, where the independent variable is time in months. At time $t = 0$, there are 40 rabbits on the island.

a. Find the solution to the equation with the initial condition.
b. How many rabbits are on the island in 1 month, 5 months, 10 months, 15 months (round to the nearest integer)?

Answer

a. $x = \frac{1000e^t}{e^t+24}$.
b. 102 rabbits after one month, 861 after 5 months, 999 after 10 months, 1000 after 15 months.

Exercise $1.5E.20$

Newton's law of cooling states that $\frac{dx}{dt} = -k(x - A)$ where x is the temperature, t is time, A is the ambient temperature, and $k > 0$ is a constant. Suppose that $A = A_0 \cos(\omega t)$ for some constants A_0 and ω. That is, the ambient temperature oscillates (for example night and day temperatures).

a. Find the general solution.
b. In the long term, will the initial conditions make much of a difference? Why or why not?

1.6: Linear Equations and the Integrating Factor

Learning Objectives

- Recognize and solve linear differential equations.
- Apply linear equations to figure out the concentration of chemicals in bodies of water.

One of the most important types of equations we will learn how to solve are the so-called linear equations. In fact, the majority of the course is about linear equations. In this lecture we focus on the first order linear equation. A first order equation is linear if we can put it into the form:

$$y' + p(x)y = f(x). \tag{1.6.1}$$

Here the word "linear" means linear in y and y'; no higher powers nor functions of y or y' appear. The dependence on x can be more complicated.

Solutions of linear equations have nice properties. For example, the solution exists wherever $p(x)$ and $f(x)$ are defined, and has the same regularity (read: it is just as nice). But most importantly for us right now, there is a method for solving linear first order equations. The trick is to rewrite the left hand side of $(1.6.1)$ as a derivative of a product of y with another function. To this end we find a function $r(x)$ such that

$$r(x)y' + r(x)p(x)y = \frac{d}{dx}[r(x)y]$$

This is the left hand side of $(1.6.1)$ multiplied by $r(x)$. So if we multiply $(1.6.1)$ by $r(x)$, we obtain

$$\frac{d}{dx}[r(x)y] = r(x)f(x)$$

Now we integrate both sides. The right hand side does not depend on y and the left hand side is written as a derivative of a function. Afterwards, we solve for y. The function $r(x)$ is called the integrating factor and the method is called the integrating factor method.

We are looking for a function $r(x)$, such that if we differentiate it, we get the same function back multiplied by $p(x)$. That seems like a job for the exponential function! Let

$$r(x) = e^{\int p(x)dx}$$

We compute:

$$\begin{aligned} y' + p(x)y &= f(x), \\ e^{\int p(x)dx}y' + e^{\int p(x)dx}p(x)y &= e^{\int p(x)dx}f(x), \\ \frac{d}{dx}\left[e^{\int p(x)dx}y\right] &= e^{\int p(x)dx}f(x), \\ e^{\int p(x)dx}y &= \int e^{\int p(x)dx}f(x)dx + C, \\ y &= e^{-\int p(x)dx}\left(\int e^{\int p(x)dx}f(x)dx + C\right). \end{aligned} \tag{1.6.2}$$

Of course, to get a closed form formula for y, we need to be able to find a closed form formula for the integrals appearing above.

Example 1.6.1

Solve

$$y' + 2xy = e^{x-x^2}, \quad y(0) = -1$$

Solution

First note that $p(x) = 2x$ and $f(x) = e^{x-x^2}$. The integrating factor is $r(x) = e^{\int p(x)dx} = e^{x^2}$. We multiply both sides of the equation by $r(x)$ to get

$$\begin{aligned} e^{x^2} y' + 2xe^{x^2} y &= e^{x-x^2} e^{x^2}, \\ \frac{d}{dx}\left[e^{x^2} y\right] &= e^x. \end{aligned} \tag{1.6.3}$$

We integrate

$$\begin{aligned} e^{x^2} y &= e^x + C, \\ y &= e^{x-x^2} + Ce^{-x^2}. \end{aligned} \tag{1.6.4}$$

Next, we solve for the initial condition $-1 = y(0) = 1 + C$, so $C = -2$. The solution is

$$y = e^{x-x^2} - 2e^{-x^2}.$$

Note that we do not care which antiderivative we take when computing $e^{\int p(x)dx}$. You can always add a constant of integration, but those constants will not matter in the end.

Example 1.6.2

Solve the following first-order linear differential equations.

a. $\frac{dy}{dx} + \frac{2}{x} y = x - 1$
b. $x\frac{dy}{dx} - 4y = x^6 e^x$

Solution

- **Video Length:** 11 minutes 48 seconds.
- **Context:** Solve the first-order linear differential equations of the type $y' + p(x)y = f(x)$ by calculating the integrating factors.

Exercise 1.6.1

Try it! Add a constant of integration to the integral in the integrating factor and show that the solution you get in the end is the same as what we got above. An advice: Do not try to remember the formula itself, that is way too hard. It is easier to remember the process and repeat it.

Since we cannot always evaluate the integrals in closed form, it is useful to know how to write the solution in definite integral form. A definite integral is something that you can plug into a computer or a calculator. Suppose we are given

$$y' + p(x)y = f(x), \quad y(x_0) = y_0$$

. Look at the solution and write the integrals as definite integrals.

$$y(x) = e^{\int_{-x_0}^{x} p(s)ds} \left(\int_{x_0}^{x} e^{\int_{x_0}^{t} p(s)ds} f(t)dt + y_0 \right) \tag{1.6.5}$$

You should be careful to properly use dummy variables here. If you now plug such a formula into a computer or a calculator, it will be happy to give you numerical answers.

Exercise 1.6.2

Check that $y(x_0) = y_0$ in Formula $(1.6.5)$.

Exercise 1.6.3

Write the solution to the linear differential equation as a definite integral, but try to simplify as far as you can. You will not be able to find the solution in closed form.

$$y' + y = e^{x^2 - x}, \quad y(0) = 10$$

Note

Before we move on, we should note some interesting properties of linear equations. First, for the linear initial value problem $y' + p(x)y = f(x)$, $y(x_0) = y_0$, there is always an explicit formula $(1.6.5)$ for the solution. Second, it follows from the formula $(1.6.5)$ that if $p(x)$ and $f(x)$ are continuous on some interval (a, b), then the solution $y(x)$ exists and is differentiable on (a, b). Compare with the simple nonlinear example we have seen previously, $y' = y^2$, and compare to Theorem 1.4.1.

Linear equations are used in figuring out the concentration of chemicals in bodies of water such as rivers and lakes. Let's take a look at Example $1.6.3$.

Example 1.6.3

A 100 liter tank contains 10 kilograms of salt dissolved in 60 liters of water. Solution of water and salt (brine) with concentration of 0.1 kilograms per liter is flowing in at the rate of 5 liters a minute. The solution in the tank is well stirred and flows out at a rate of 3 liters a minute. See Figure $1.6.1$ for an illustration of the problem. How much salt is in the tank when the tank is full?

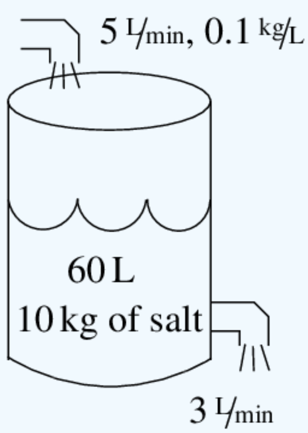


Figure $1.6.1$: The inflow and outflow of different concentration fluids inside a cylindirical tank as described in the example. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Solution

Let us come up with the equation. Let x denote the kilograms of salt in the tank, let t denote the time in minutes. For a small change Δt in time, the change in x (denoted Δx) is approximately

$$\Delta x \approx (\text{rate in x concentration in})\Delta t - (\text{rate out x concentration out})\Delta t.$$

Dividing through by Δt and taking the limit $\Delta t \to 0$ we see that

$$\frac{dx}{dt} = (\text{rate in x concentration in}) - (\text{rate out x concentration out})$$

In our example, we have

$$\begin{aligned} \text{rate in} &= 5, \\ \text{concentration in} &= 0.1, \\ \text{rate out} &= 3, \\ \text{concentration out} &= \frac{x}{\text{volume}} = \frac{x}{60+(5-3)t}. \end{aligned} \tag{1.6.6}$$

Our equation is, therefore,

$$\frac{dx}{dt} = (5 \text{ x } 0.1) - \left(3\frac{x}{60+2t}\right)$$

Or in the form $(1.6.1)$

$$\frac{dx}{dt} + \frac{3}{60+2t}x = 0.5$$

Let us solve. The integrating factor is

$$r(t) = \exp\left(\int \frac{3}{60+2t}dt\right) = \exp\left(\frac{3}{2}\ln(60+2t)\right) = (60+2t)^{3/2}$$

We multiply both sides of the equation to get

$$\begin{aligned} (60+2t)^{3/2}\frac{dx}{dt} + (60+2t)^{3/2}\frac{3}{60+2t}x &= 0.5(60+2t)^{3/2}, \\ \frac{d}{dt}\left[(60+2t)^{3/2}x\right] &= 0.5(60+2t)^{3/2}, \\ (60+2t)^{3/2}x &= \int 0.5(60+2t)^{3/2}dt + C, \\ x &= (60+2t)^{-3/2}\int \frac{(60+2t)^{3/2}}{2}dt + C(60+2t)^{-3/2}, \\ x &= (60+2t)^{-3/2}\frac{1}{10}(60+2t)^{5/2} + C(60+2t)^{-3/2}, \\ x &= \frac{(60+2t)}{10} + C(60+2t)^{-3/2}. \end{aligned} \tag{1.6.7}$$

We need to find C. We know that at $t=0$, $x=10$. So

$$10 = x(0) = \frac{60}{10} + C(60)^{-3/2} = 6 + C(60)^{-3/2}$$

or

$$C = 4(60^{3/2}) \approx 1859.03.$$

We are interested in x when the tank is full. So we note that the tank is full when $60+2t=100$, or when $t=20$. So

$$\begin{aligned} x(20) &= \frac{60+40}{10} + C(60+40)^{-3/2} \\ &\approx 10 + 1859.03(100)^{-3/2} \approx 11.86. \end{aligned} \tag{1.6.8}$$

See Figure $1.6.2$ for the graph of x over t.

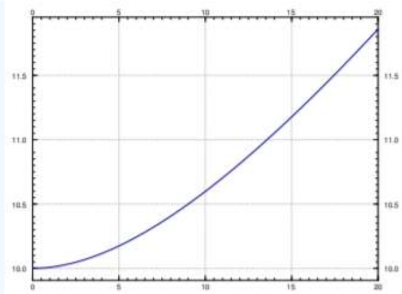


Figure 1.6.2: Graph of the solution x kilograms of salt in the tank at time t. Notice that at $t = 0$, $x = 10$ and tank is full when $t = 20$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

The concentration at the end is approximately $0.1186\,{}^{\text{kg}}/_{\text{liter}}$ and we started with $\frac{1}{6}$ or $0.167\,{}^{\text{kg}}/_{\text{liter}}$.

1.6E: Exercises for the section 1.6

Exercises 1-12

Leave the answer as a definite integral if a closed form of the solution can not be found. If you can find a closed form of the solution, then you should provide that closed-form solution instead.

Exercise $1.6E.1$

Solve $y' + xy = x$.

Exercise $1.6E.2$

Solve $y' + 6y = e^x$.

Exercise $1.6E.3$

Solve $y' + 3x^2 y = \sin(x)\, e^{-x^3}$ with $y(0) = 1$.

Exercise $1.6E.4$

Solve $y' + \cos(x)\, y = \cos(x)$.

Exercise $1.6E.5$

Solve $\frac{1}{x^2+1} y' + xy = 3$ with $y(0) = 0$

Exercise $1.6E.6$

Suppose there are two lakes located on a stream. Clean water flows into the first lake, then the water from the first lake flows into the second lake, and then water from the second lake flows further downstream. The in and out flow from each lake is 500 liters per hour. The first lake contains 100 thousand liters of water and the second lake contains 200 thousand liters of water. A truck with 500 kg of toxic substance crashes into the first lake. Assume that the water is being continually mixed perfectly by the stream.

a. Find the concentration of toxic substance as a function of time in both lakes.
b. When will the concentration in the first lake be below 0.001 kg per liter?
c. When will the concentration in the second lake be maximal?

Exercise $1.6E.7$

Initially 5 grams of salt are dissolved in 20 liters of water. Brine with concentration of salt 2 grams of salt per liter is added at a rate of 3 liters a minute. The tank is mixed well and is drained at 3 liters a minute. How long does the process have to continue until there are 20 grams of salt in the tank?

Exercise $1.6E.8$

Initially a tank contains 10 liters of pure water. Brine of unknown (but constant) concentration of salt is flowing in at 1 liter per minute. The water is mixed well and drained at 1 liter per minute. In 20 minutes there are 15 grams of salt in the tank. What is the concentration of salt in the incoming brine?

❓ Exercise $1.6E.9$

Solve $y' + 3x^2 y = x^2$.

Answer

$y = Ce^{-x^3} + \frac{1}{3}$

❓ Exercise $1.6E.10$

Solve $y' + 2\sin(2x)y = 2\sin(2x)$ with $y(\pi/2) = 3$

Answer

$y = 2e^{\cos(2x)+1} + 1$

❓ Exercise $1.6E.11$

Suppose a water tank is being pumped out at $3\ \frac{\mathrm{L}}{\mathrm{min}}$. The water tank starts at $10\ \mathrm{L}$ of clean water. Water with toxic substance is flowing into the tank at $2\ \frac{\mathrm{L}}{\mathrm{min}}$, with concentration $20t\ \frac{\mathrm{g}}{\mathrm{L}}$ at time t. When the tank is half empty, how many grams of toxic substance are in the tank (assuming perfect mixing)?

Answer

$250\,\text{grams}$

❓ Exercise $1.6E.12$

Suppose we have bacteria on a plate and suppose that we are slowly adding a toxic substance such that the rate of growth is slowing down. That is, suppose that $\frac{dP}{dt} = (2 - 0.1\,t)P$. If $P(0) = 1000$, find the population at $t = 5$.

Answer

$P(5) = 1000e^{2\times5 - 0.05\times5^2} = 1000e^{8.75} \approx 6.31 \times 10^6$

❓ Exercise $1.6E.13$

A cylindrical water tank has water flowing in at I cubic meters per second. Let A be the area of the cross section of the tank in meters. Suppose water is flowing from the bottom of the tank at a rate proportional to the height of the water level. Set up the differential equation for h, the height of the water, introducing and naming constants that you need. You should also give the units for your constants.

Answer

$Ah' = I - kh$, where k is a constant with units $\frac{\mathrm{m}^2}{\mathrm{s}}$.

1.7: Existence and Uniqueness of Solutions of Nonlinear Equations

Learning Objectives

- Understand and apply the existence and uniqueness theorems for nonlinear differential equations.
- Analyze scenarios where existence or uniqueness may fail.

Although there are methods for solving some nonlinear equations, it is impossible to find useful formulas for the solutions of most. Whether we are looking for exact solutions or numerical approximations, it is useful to know conditions that imply the existence and uniqueness of solutions of initial value problems for nonlinear equations. In this section, we state such a condition and illustrate it with examples.

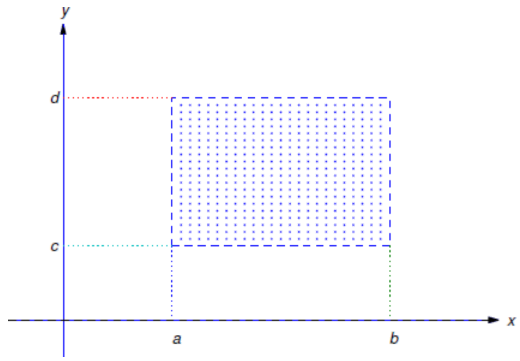


Figure 1.7.1 : An open rectangle, which is a set of points (x, y) where x is between two values 'a' and 'b', and y is between two values 'c' and 'd'. There are dots inside the open rectangle. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation).

Some terminology: an *open rectangle* R (see Figure 1.7.1) is a set of points (x, y) such that

$$a < x < b \quad \text{and} \quad c < y < d$$

We'll denote this set by $R : \{a < x < b, c < y < d\}$. "Open" means that the boundary rectangle (indicated by the dashed lines in Figure 1.7.1) is not included in R.

The next theorem gives sufficient conditions for existence and uniqueness of solutions of initial value problems for first order nonlinear differential equations. We omit the proof, which is beyond the scope of this book.

Theorem 1.7.1 : Existence and Uniqueness

a. If f is continuous on an open rectangle

$$R : \{a < x < b, c < y < d\}$$

that contains (x_0, y_0) then the initial value problem

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1.7.1}$$

has at least one solution on some open subinterval of (a, b) that contains x_0.

b. If both f and f_y are continuous on R then Equation (1.7.1) has a unique solution on some open subinterval of (a, b) that contains x_0

It's important to understand exactly what Theorem 1.7.1 says.

- (a) is an *existence theorem*. It guarantees that a solution exists on some open interval that contains x_0, but provides no information on how to find the solution, or to determine the open interval on which it exists. Moreover, (a) provides no information on the number of solutions that Equation (1.7.1) may have. It leaves open the possibility that Equation (1.7.1) may have two or more solutions that differ for values of x arbitrarily close to x_0. We will see in Example 1.7.6 that this can happen.
- (b) is a *uniqueness theorem*. It guarantees that Equation (1.7.1) has a unique solution on some open interval (a,b) that contains x_0. However, if $(a, b) \neq (-\infty, \infty)$, Equation (1.7.1) may have more than one solution on a larger interval that contains (a, b).

For example, it may happen that $b < \infty$ and all solutions have the same values on (a, b), but two solutions y_1 and y_2 are defined on some interval (a, b_1) with $b_1 > b$, and have different values for $b < x < b_1$; thus, the graphs of the y_1 and y_2 "branch off" in different directions at $x = b$. (See Example 1.7.7 and Figure 1.7.3). In this case, continuity implies that $y_1(b) = y_2(b)$ (call their common value y), and y_1 and y_2 are both solutions of the initial value problem

$$y = f(x, y), \quad y(b) = \overline{y} \tag{1.7.2}$$

that differ on every open interval that contains b. Therefore f or f_y must have a discontinuity at some point in each open rectangle that contains (b, y), since if this were not so, Equation (1.7.2) would have a unique solution on some open interval that contains b. We leave it to you to give a similar analysis of the case where $a > -\infty$.

Example 1.7.1

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{1 + x^2 + y^2}, \quad y(x_0) = y_0. \tag{1.7.3}$$

Show that it has a unique solution on some open interval that contains x_0.

Solution

Since

$$f(x, y) = \frac{x^2 - y^2}{1 + x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{2y(1 + 2x^2)}{(1 + x^2 + y^2)^2}$$

are continuous for all (x, y), Theorem 1.7.1 implies that if (x_0, y_0) is arbitrary, then Equation (1.7.3) has a unique solution on some open interval that contains x_0.

Example 1.7.2

Consider the initial value problem

$$y' = \frac{x^2 - y^2}{x^2 + y^2}, \quad y(x_0) = y_0. \tag{1.7.4}$$

Show that it has a unique solution on some open interval that contains x_0.

Solution

Here

$$f(x, y) = \frac{x^2 - y^2}{x^2 + y^2} \quad \text{and} \quad f_y(x, y) = -\frac{4x^2 y}{(x^2 + y^2)^2}$$

are continuous everywhere except at $(0, 0)$. If $(x_0, y_0) \neq (0, 0)$, there's an open rectangle R that contains (x_0, y_0) that does not contain $(0, 0)$. Since f and f_y are continuous on R, Theorem 1.7.1 implies that if $(x_0, y_0) \neq (0, 0)$ then Equation (1.7.4) has a unique solution on some open interval that contains x_0.

Example 1.7.3

Consider the initial value problem

$$y' = \frac{x + y}{x - y}, \quad y(x_0) = y_0. \tag{1.7.5}$$

Show that it has a unique solution on some open interval that contains x_0.

Solution

Here

$$f(x, y)=\frac{x+y}{x-y} \quad \text{and} \quad f_y(x, y)=\frac{2x}{(x-y)^2}$$

are continuous everywhere except on the line $y=x$. If $y_0 \neq x_0$, there's an open rectangle R that contains (x_0, y_0) that does not intersect the line $y=x$. Since f and f_y are continuous on R, Theorem 1.7.1 implies that if $y_0 \neq x_0$, Equation (1.7.5) has a unique solution on some open interval that contains x_0.

The following differential equation is of the separable form $y'=f(x)g(y)$

$$y'=2xy^2 \tag{1.7.6}$$

one such solution can be found by inspection $y=0$ now suppose y is a solution of Equation (1.7.6) that is not identically zero. since y is continuous there must be an interval on which y is never zero. Since division by y^2 is legitimate for x in this interval we can separate variables in this interval.

$$\frac{y'}{y^2}=2x$$

Integrate

$$-\frac{1}{y}=x^2+c$$

which is equivalent to

$$y=-\frac{1}{x^2+c}$$

Hence the two solutions are

$$y= \quad \text{and} \quad y=-\frac{1}{x^2+c},$$

where c is an arbitrary constant. In particular, this implies that no solution of Equation (1.7.6) other than $y=0$ can equal zero for any value of x. Show that Theorem $1.7.1b$ implies this.

We'll obtain a contradiction by assuming that Equation (1.7.6) has a solution y_1 that equals zero for some value of x, but is not identically zero. If y_1 has this property, there's a point x_0 such that $y_1(x_0)=0$, but $y_1(x) \neq 0$ for some value of x in every open interval that contains x_0. This means that the initial value problem

$$y'=2xy^2, \quad y(x_0)=0 \tag{1.7.7}$$

has two solutions $y=0$ and $y=y_1$ that differ for some value of x on every open interval that contains x_0. This contradicts Theorem 1.7.1 (b), since in Equation (1.7.6) the functions

$$f(x, y)=2xy^2 \quad \text{and} \quad f_y(x, y)=4xy.$$

are both continuous for all (x, y), which implies that Equation (1.7.7) has a unique solution on some open interval that contains x_0.

Example 1.7.4

Find the values of t where existence and uniqueness of the solution to the initial value problem is not guaranteed

$$y'=\frac{\sqrt[3]{5y-8}}{4t^2-7}, \quad y(t_0)=y_0$$

Solution

- **Video Length:** 5 minutes 36 seconds.
- **Context:** Determine the critical values of t where the conditions of the **Existence and Uniqueness Theorem** fail for the given initial value problem.

✔ Example 1.7.5

Consider the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(x_0) = y_0. \tag{1.7.8}$$

a. For what points (x_0, y_0) does Theorem $1.7.1a$ imply that Equation (1.7.8) has a solution?
b. For what points (x_0, y_0) does Theorem $1.7.1b$ imply that Equation (1.7.8) has a unique solution on some open interval that contains x_0?

Solution a

Since

$$f(x, y) = \frac{10}{3}xy^{2/5}$$

is continuous for all (x, y), Theorem 1.7.1 implies that Equation (1.7.8) has a solution for every (x_0, y_0).

Solution b

Here

$$f_y(x, y) = \frac{4}{3}xy^{-3/5}$$

is continuous for all (x, y) with $y \neq 0$. Therefore, if $y_0 \neq 0$ there's an open rectangle on which both f and f_y are continuous, and Theorem 1.7.1 implies that Equation (1.7.8) has a unique solution on some open interval that contains x_0.

If $y = 0$ then $f_y(x, y)$ is undefined, and therefore discontinuous; hence, Theorem 1.7.1 does not apply to Equation (1.7.8) if $y_0 = 0$.

✔ Example 1.7.6

Example 1.7.5 leaves open the possibility that the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = 0 \tag{1.7.9}$$

has more than one solution on every open interval that contains $x_0 = 0$. Show that this is true.

Solution

By inspection, $y = 0$ is a solution of the differential equation

$$y' = \frac{10}{3}xy^{2/5}. \tag{1.7.10}$$

Since $y = 0$ satisfies the initial condition $y(0) = 0$, it is a solution of Equation (1.7.9).

Now suppose y is a solution of Equation (1.7.10) that is not identically zero. Separating variables in Equation (1.7.10) yields

$$y^{-2/5}y' = \frac{10}{3}x$$

on any open interval where y has no zeros. Integrating this and rewriting the arbitrary constant as $5c/3$ yields

$$\frac{5}{3}y^{3/5} = \frac{5}{3}(x^2+c).$$

Therefore

$$y = (x^2+c)^{5/3}. \tag{1.7.11}$$

Since we divided by y to separate variables in Equation (1.7.10), our derivation of Equation (1.7.11) is legitimate only on open intervals where y has no zeros. However, Equation (1.7.11) actually defines y for all x, and differentiating Equation (1.7.11) shows that

$$y' = \frac{10}{3}x(x^2+c)^{2/3} = \frac{10}{3}xy^{2/5}, \; -\infty < x < \infty$$

Therefore Equation (1.7.11) satisfies Equation (1.7.10) on $(-\infty, \infty)$ even if $c \leq 0$, so that $y(\sqrt{|c|}) = y(-\sqrt{|c|}) = 0$. In particular, taking $c = 0$ in Equation (1.7.11) yields

$$y = x^{10/3}$$

as a second solution of Equation (1.7.9). Both solutions are defined on $(-\infty, \infty)$, and they differ on every open interval that contains $x_0 = 0$ (see Figure 1.7.2). In fact, there are *four* distinct solutions of Equation (1.7.9) defined on $(-\infty, \infty)$ that differ from each other on every open interval that contains $x_0 = 0$. Can you identify the other two?

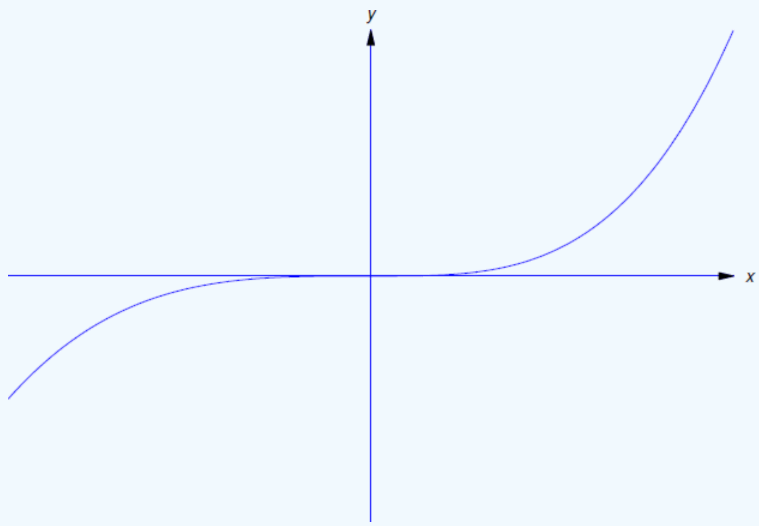


Figure 1.7.2 : Two solutions $y = 0$ and $y = x^{10/3}$ of $y' = \frac{10}{3}xy^{2/5}$ given $y(0) = 0$ that differ on every interval containing $x_0 = 0$ (CC BY-NC-SA 3.0, William F. Trench via Elementary Differential equation).

Example 1.7.7

From Example 1.7.5 , the initial value problem

$$y' = \frac{10}{3}xy^{2/5}, \quad y(0) = -1 \tag{1.7.12}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find a solution and determine the largest open interval (a, b) on which it is unique.

Solution

Let y be any solution of Equation (1.7.12). Because of the initial condition $y(0) = -1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation (1.7.11). Setting $x = 0$ and $y = -1$ in Equation (1.7.11) yields $c = -1$, so

$$y = (x^2 - 1)^{5/3} \tag{1.7.13}$$

for x in I. Therefore every solution of Equation (1.7.12) differs from zero and is given by Equation (1.7.13) on $(-1, 1)$; that is, Equation (1.7.13) is the unique solution of Equation (1.7.12) on $(-1, 1)$. This is the largest open interval on which Equation (1.7.12) has a unique solution. To see this, note that Equation (1.7.13) is a solution of Equation (1.7.12) on $(-\infty, \infty)$. There are infinitely many other solutions of Equation (1.7.12) that differ from Equation (1.7.13) on every open interval larger than $(-1, 1)$. One such solution is

$$y = \begin{cases} (x^2 - 1)^{5/3}, & -1 \le x \le 1, \\ 0, & |x| > 1. \end{cases}$$

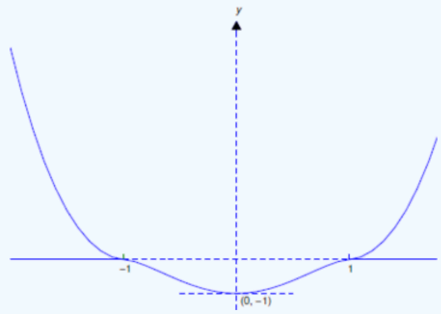


Figure 1.7.3 : Two solutions of $y' = \frac{10}{3} x y^{2/5}$, on $(-1, 1)$ that coincide on $(-1, 1)$, but on no larger open interval. One solution is the line $y = 0$, and another one is the curve $y = (x^2 - 1)^{5/3}$. (CC BY-NC-SA 3.0, William F. Trench via Elementary Differential equation).

Example 1.7.8

From Example 1.7.5 , the initial value problem

$$y' = \frac{10}{3} x y^{2/5}, \quad y(0) = 1 \tag{1.7.14}$$

has a unique solution on some open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

Solution

Let y be any solution of Equation (1.7.14). Because of the initial condition $y(0) = 1$ and the continuity of y, there's an open interval I that contains $x_0 = 0$ on which y has no zeros, and is consequently of the form Equation (1.7.11). Setting $x = 0$ and $y = 1$ in Equation (1.7.11) yields $c = 1$, so

$$y = (x^2 + 1)^{5/3} \tag{1.7.15}$$

for x in I. Therefore every solution of Equation (1.7.14) differs from zero and is given by Equation (1.7.15) on $(-\infty, \infty)$; that is, Equation (1.7.15) is the unique solution of Equation (1.7.14) on $(-\infty, \infty)$. Figure 1.7.4) shows the graph of this solution.

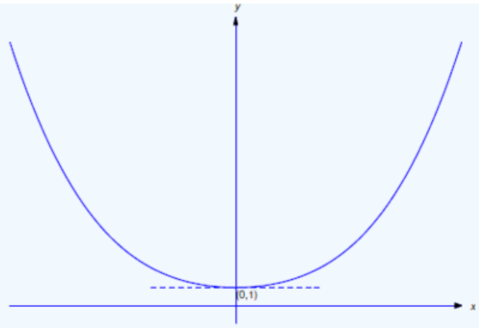


Figure 1.7.4 : The unique solution of $y' = \frac{10}{3}xy^{2/5}$ with initial condition $y(0) = 1$. The solution is an upward curve with vertex at the point $(0, 1)$. (CC BY-NC-SA 3.0, William F. Trench via Elementary Differential equation).

1.7E: Exercises for the section 1.7

Exercises 1-13

Find all (x_0, y_0) for which Theorem 1.7.1 implies that the initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ has (a) a solution and (b) a unique solution on some open interval that contains x_0.

Exercise $1.7E.1$

$y' = \frac{x^2+y^2}{\sin x}$

Exercise $1.7E.2$

$y' = \frac{e^x+y}{x^2+y^2}$

Answer

$f(x, y) = \frac{e^x+y}{x^2+y^2}$ and $f_y(x, y) = \frac{1}{x^2+y^2} - \frac{2y(e^x+y)}{(x^2+y^2)^2}$ are both continuous at all $(x, y) \neq (0, 0)$. Hence, Theorem 1.7.1 implies that if $(x_0, y_0) \neq (0, 0)$, then the initial value problem has a unique solution on some open interval containing x_0. Theorem 1.7.1 does not apply if $(x_0, y_0) = (0, 0)$.

Exercise $1.7E.3$

$y' = \tan xy$

Exercise $1.7E.4$

$y' = \frac{x^2+y^2}{\ln xy}$

Exercise $1.7E.5$

$y' = (x^2 + y^2)y^{1/3}$

Exercise $1.7E.6$

$y' = 2xy$

Answer

$f(x, y) = 2xy$ and $f_y(x, y) = 2x$ are both continuous at all (x, y). Hence, Theorem 1.7.1 implies that if (x_0, y_0) is arbitrary, then the initial value problem has a unique solution on some open interval containing x_0.

Exercise $1.7E.7$

$y' = \ln(1 + x^2 + y^2)$

Exercise $1.7E.8$

$y' = \frac{2x+3y}{x-4y}$

Answer

$f(x,y)=\frac{2x+3y}{x-4y}$ and $f_y(x,y)=\frac{3}{x-4y}+4\frac{2x+3y}{(x-4y)^2}$ are both continuous at all (x,y) such that $x\neq 4y$. Hence, Theorem 1.7.1 implies that if $x_0\neq 4y_0$, then the initial value problem has a unique solution on some open interval containing x_0. Theorem 1.7.1 does not apply if $x_0=4y_0$.

Exercise $1.7E.9$

$y'=(x^2+y^2)^{1/2}$

Exercise $1.7E.10$

$y'=x(y^2-1)^{2/3}$

Answer

$f(x,y)=x(y^2-1)^{2/3}$ is continuous at all (x,y), but $f_y(x,y)=\frac{4}{3}xy(y^2-1)^{1/3}$ is continuous at (x,y) if and only if $y\neq\pm 1$. Hence, Theorem 1.7.1 implies that if $y_0\neq\pm 1$, then the initial value problem has a unique solution on some open interval containing x_0, while if $y_0=\pm 1$, then the initial value problem has at least one solution (possibly not unique on any open interval containing x_0).

Exercise $1.7E.11$

$y'=(x^2+y^2)^2$

Exercise $1.7E.12$

$y'=(x+y)^{1/2}$

Answer

$f(x,y)=(x+y)^{1/2}$ and $f_y(x,y)=\frac{1}{2(x+y)^{1/2}}$ are both continuous at all (x,y) such that $x+y>0$. Hence, Theorem 1.7.1 implies that if $x_0+y_0>0$, then the initial value problem has a unique solution on some open interval containing x_0. Theorem 1.7.1 does not apply if $x_0+y_0\leq 0$.

Exercise $1.7E.13$

$y'=\frac{\tan y}{x-1}$

Exercises 14-15

Solve the following exercises with instructions in each question.

Exercise $1.7E.14$

Apply Theorem 1.7.1 to the initial value problem

$$y'+p(x)y=q(x),\quad y(x_0)=y_0$$

for a linear equation.

Answer

To apply Theorem 1.7.1, rewrite the given initial value problem as (A) $y'=f(x,y),\ y(x_0)=y_0$, where $f(x,y)=-p(x)y+q(x)$ and $f_y(x,y)=-p(x)$. If p and f are continuous on some open interval (a,b) containing x_0, then f and f_y are continuous on some open rectangle containing (x_0,y_0), so Theorem 1.7.1 implies that (A) has a unique solution on some open interval containing x_0.

Exercise $1.7E.15$

a. Verify that the function

$$y=\begin{cases}(x^2-1)^{5/3}, & -1<x<1,\\ 0, & |x|\ge 1,\end{cases}$$

is a solution of the initial value problem

$$y'=\frac{10}{3}xy^{2/5},\quad y(0)=-1$$

on $(-\infty,\infty)$. HINT: You'll need the definition

$$y'(\bar{x})=\lim_{x\to\bar{x}}\frac{y(x)-y(\bar{x})}{x-\bar{x}}$$

to verify that y satisfies the differential equation at $\bar{x}=\pm 1$.

b. Verify that if $\epsilon_i=0$ or 1 for $i=1,2$ and $a,b>1$, then the function

$$y=\begin{cases}\epsilon_1(x^2-a^2)^{5/3}, & -\infty<x<-a,\\ 0, & -a\le x\le -1,\\ (x^2-1)^{5/3}, & -1<x<1,\\ 0, & 1\le x\le b,\\ \epsilon_2(x^2-b^2)^{5/3}, & b<x<\infty,\end{cases}$$

is a solution of the initial value problem of a on $(-\infty,\infty)$.

Exercises 16-21

Solve the following exercises with instructions in each question.

Exercise $1.7E.16$

Use the ideas developed in **Exercise 1.7E.15** to find infinitely many solutions of the initial value problem

$$y'=y^{2/5},\quad y(0)=1$$

on $(-\infty,\infty)$.

Exercise $1.7E.17$

Consider the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(x_0)=y_0. \tag{A}$$

a. For what points (x_0,y_0) does Theorem 1.7.1 imply that (A) has a solution?
b. For what points (x_0,y_0) does Theorem 1.7.1 imply that (A) has a unique solution on some open interval that contains x_0?

Exercise $1.7E.18$

Find nine solutions of the initial value problem

$$y'=3x(y-1)^{1/3},\quad y(0)=1$$

that are all defined on $(-\infty,\infty)$ and differ from each other for values of x in every open interval that contains $x_0=0$.

Exercise $1.7E.19$

From Theorem 1.7.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(0) = 9$$

has a unique solution on an open interval that contains $x_0 = 0$. Find the solution and determine the largest open interval on which it is unique.

Exercise $1.7E.20$

a. From Theorem 1.7.1, the initial value problem

$$y' = 3x(y-1)^{1/3}, \quad y(3) = -7 \tag{A}$$

has a unique solution on some open interval that contains $x_0 = 3$. Determine the largest such open interval, and find the solution on this interval.

b. Find infinitely many solutions of (A), all defined on $(-\infty, \infty)$.

Exercise $1.7E.21$

Prove:

a. If

$$f(x, y_0) = 0, \quad a < x < b, \tag{A}$$

andx_0 is in (a, b), then $y = y_0$ is a solution of

$$y' = f(x, y), \quad y(x_0) = y_0$$

on (a, b).

b. If f and f_y are continuous on an open rectangle that contains (x_0, y_0) and (A) holds, no solution of $y' = f(x, y)$ other than $y \equiv y_0$ can equal y_0 at any point in (a, b).

1.8: Substitution

Learning Objectives

- Apply substitution to simplify and solve non-separable differential equations.
- Solve Bernoulli equations using the substitution

Just as when solving integrals, one method to try is to change variables to end up with a simpler equation to solve.

Substitution

The equation

$$y' = (x - y + 1)^2 \tag{1.8.1}$$

is neither separable nor linear. What can we do? How about trying to change variables, so that in the new variables the equation is simpler. We use another variable v, which we treat as a function of x. Let us try

$$v = x - y + 1. \tag{1.8.2}$$

We need to figure out y' in terms of v', v and x. We differentiate (in x) to obtain $v' = 1 - y'$. So $y' = 1 - v'$. We plug this into the equation to get

$$1 - v' = v^2 \tag{1.8.3}$$

In other words, $v' = 1 - v^2$. Such an equation we know how to solve by separating variables:

$$\frac{1}{1 - v^2} dv = dx \tag{1.8.4}$$

So

$$\frac{1}{2} \ln \left| \frac{v+1}{v-1} \right| = x + C, \quad \text{or} \quad \left| \frac{v+1}{v-1} \right| = e^{2x+2C}, \quad \text{or} \quad \frac{v+1}{v-1} = De^{2x},$$

for some constant D. Note that $v = 1$ and $v = -1$ are also solutions.

Now we need to “unsubstitute” to obtain

$$\frac{x - y + 2}{x - y} = De^{2x} \tag{1.8.5}$$

and also the two solutions $x - y + 1 = 1$ or $y = x$, and $x - y + 1 = -1$ or $y = x + 2$. We solve the first equation for y.

$$\begin{aligned} x - y + 2 &= (x - y)\, De^{2x}, \\ x - y + 2 &= Dxe^{2x} - yDe^{2x}, \\ -y + yDe^{2x} &= Dxe^{2x} - x - 2, \\ y(-1 + De^{2x}) &= Dxe^{2x} - x - 2, \\ y &= \frac{Dxe^{2x} - x - 2}{De^{2x} - 1}. \end{aligned} \tag{1.8.6}$$

Note that $D = 0$ gives $y = x + 2$, but no value of D gives the solution $y = x$.

Substitution in differential equations is applied in much the same way that it is applied in calculus. You guess. Several different substitutions might work. There are some general things to look for. We summarize a few of these in Table 1.8.1.

Table 1.8.1: Substitution in differential equations

When you see	Try substituting
yy'	$v = y^2$
$y^2 y'$	$v = y^3$

When you see	Try substituting
$(\cos y)y'$	$v=\sin y$
$(\sin y)y'$	$v=\cos y$
$y'e^y$	$v=e^y$

Usually you try to substitute in the "most complicated" part of the equation with the hopes of simplifying it. The above table is just a rule of thumb. You might have to modify your guesses. If a substitution does not work (it does not make the equation any simpler), try a different one.

Bernoulli Equations

There are some forms of equations where there is a general rule for substitution that always works. One such example is the so-called Bernoulli equation.[1]

$$y'+p(x)\,y=q(x)\,y^n \tag{1.8.7}$$

This equation looks a lot like a linear equation except for the y^n. If $n=0$ or $n=1$, then the equation is linear and we can solve it. Otherwise, the substitution $v=y^{1-n}$ transforms the Bernoulli equation into a linear equation. Note that n need not be an integer.

Example 1.8.1:

Solve

$$xy'+y(x+1)+xy^5=0,\;\; y(1)=1$$

Solution

First, the equation is Bernoulli $p(x)=\dfrac{x+1}{x}$ (and $q(x)=-1$). We substitute

$$v=y^{1-5}=y^{-4},\quad v'=-4y^{-5}y'$$

In other words, $\left(\dfrac{-1}{4}\right)y^5v'=y'$. So

$$\begin{aligned} xy'+y(x+1)+xy^5 &=0,\\ \frac{-xy^5}{4}v'+y(x+1)+xy^5 &=0,\\ \frac{-x}{4}v'+y^{-4}(x+1)+x &=0,\\ \frac{-x}{4}v'+v(x+1)+x &=0, \end{aligned} \tag{1.8.8}$$

and finally

$$v'-\frac{4(x+1)}{x}v=4$$

Now the equation is linear. We can use the integrating factor method. In particular, we use Formula (1.6.2). Let us assume that $x>0$ so $|x|=x$. This assumption is OK, as our initial condition is $x=1$. Let us compute the integrating factor. Here $p(s)$ from Formula (1.6.2) is $\dfrac{-4(s+1)}{s}$.

$$\begin{aligned} e^{\int_1^x p(s)ds} &=\exp\left(\int_1^x \frac{-4(s+1)}{s}ds\right)=e^{-4x-4ln(x)+4}=e^{-4x+4}x^{-4}=\frac{e^{-4x+4}}{x^4},\\ e^{-\int_1^x p(s)ds} &=e^{4x+4ln(x)-4}=e^{4x-4}x^4 \end{aligned} \tag{1.8.9}$$

We now plug in to Formula (1.6.2)

$$v(x) = e^{-\int_1^x p(s)ds}\left(\int_1^x e^{\int_1^t p(s)ds} 4dt + 1\right), \\ = e^{4x-4}x^4\left(\int_1^x 4\frac{e^{-4t+4}}{t^4}dt + 1\right) \tag{1.8.10}$$

Note that the integral in this expression is not possible to find in closed form. As we said before, it is perfectly fine to have a definite integral in our solution. Now “unsubstitute”

$$y^{-4} = e^{4x-4}x^4\left(4\int_1^x \frac{e^{-4t+4}}{t^4}dt + 1\right), \\ y = \frac{e^{-x+1}}{x\left(4\int_1^x \frac{e^{-4t+4}}{t^4}dt + 1\right)^{1/4}} \tag{1.8.11}$$

Note

Remember Formula (1.6.2) when we solve $y' + p(x)y = f(x)$

if we let

$$r(x) = e^{\int p(x)dx}$$

then

$$y = e^{-\int p(x)dx}\left(\int e^{\int p(x)dx} f(x)dx + C\right)$$

Homogeneous Equations

Another type of equations we can solve by substitution are the so-called homogeneous equations. Suppose that we can write the differential equation as

$$y' = F\left(\frac{y}{x}\right) \tag{1.8.12}$$

Here we try the substitutions

$$v = \frac{y}{x} \quad \text{and therefore} \quad y' = v + xv' \tag{1.8.13}$$

We note that the equation is transformed into

$$v + xv' = F(v) \quad \text{or} \quad xv' = F(v) - v \quad \text{or} \quad \frac{v'}{F(v) - v} = \frac{1}{x} \tag{1.8.14}$$

Hence an implicit solution is

$$\int \frac{1}{F(v) - v}dv = \ln|x| + C \tag{1.8.15}$$

Example 1.8.2

Solve

$$x^2y' = y^2 + xy, \quad y(1) = 1$$

Solution

We put the equation into the form $y' = \left(\frac{y}{x}\right)^2 + \frac{y}{x}$. We substitute $v = \frac{y}{x}$ to get the separable equation

$$xv' = v^2 + v - v = v^2$$

$$v^{-2} dv = \frac{1}{x} dx$$

which has a solution

$$\begin{aligned} \int \frac{1}{v^2} dv &= ln\,|x| + C, \\ \frac{-1}{v} &= ln\,|x| + C, \\ v &= \frac{-1}{ln\,|x| + C}. \end{aligned} \tag{1.8.16}$$

We unsubstitute

$$\begin{aligned} \frac{y}{x} &= \frac{-1}{ln\,|x| + C}, \\ y &= \frac{-x}{ln\,|x| + C} \end{aligned} \tag{1.8.17}$$

We want $y(1) = 1$, so

$$1 = y(1) = \frac{-1}{ln\,|1| + C} = \frac{-1}{C}$$

Thus $C = -1$ and the solution we are looking for is

$$y = \frac{-x}{ln\,|x| - 1}$$

Footnotes

[1] There are several things called Bernoulli equations, this is just one of them. The Bernoullis were a prominent Swiss family of mathematicians. These particular equations are named for Jacob Bernoulli (1654–1705).

Contributors and Attributions

- Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

1.8E: Exercises for Section 1.8

Exercises 1-10

Solve the differential equations step by step, and provide detailed explanations when necessary. If a closed-form solution is not possible, represent the solution in integral form.

Exercise $1.8E.1$

Solve $y'+y(x^2-1)+xy^6=0$, with $y(1)=1$.

Exercise $1.8E.2$

Solve $2yy'+1=y^2+x$, with $y(0)=1$.

Exercise $1.8E.3$

Solve $y'+xy=y^4$, with $y(0)=1$.

Exercise $1.8E.4$

Solve $yy'+x=\sqrt{x^2+y^2}$.

Exercise $1.8E.5$

Solve $y'=(x+y-1)^2$.

Exercise $1.8E.6$

Solve $y'=\dfrac{x^2-y^2}{xy}$, with $y(1)=2$.

Exercise $1.8E.7$

Solve $xy'+y+y^2=0$, $y(1)=2$.

Answer

$y=\frac{2}{3x-2}$

Exercise $1.8E.8$

Solve $xy'+y+x=0$, $y(1)=1$.

Answer

$y=\frac{3-x^2}{2x}$

Exercise $1.8E.9$

Solve $y^2y'=y^3-3x$, $y(0)=2$.

Answer

$y=\left(7e^{3x}+3x+1\right)^{1/3}$

Exercise $1.8E.10$

Solve $2yy' = e^{y^2 - x^2} + 2x$.

Answer

$$y = \sqrt{x^2 - \ln(C - x)}$$

1.9: Autonomous Equations

Learning Objectives

- Understand and analyze the qualitative behavior of autonomous equations.
- Solve and sketch phase diagram of autonomous equations and their applications.

Let us consider general differential equation problems of the form

$$\frac{dx}{dt} = f(x)$$

where the derivative of solutions depends only on x (the dependent variable). Such equations are called **autonomous** equations. If we think of t as time, the naming comes from the fact that the equation is independent of time.

Let us come back to the cooling coffee problem (Example 1.5.3). Newton's law of cooling says that

$$\frac{dx}{dt} = -k(x-A)$$

where x is the temperature, t is time, k is some constant, and A is the ambient temperature. See Figure $1.9.1$ for an example with $k = 0.3$ and $A = 5$.

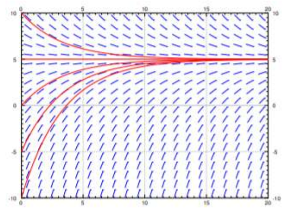


Figure $1.9.1$: Diagram illustrating the slope field and some solutions of $x' = 0.3(5-x)$. This image displays a slope field and several solution curves for the given differential equation. The slope field indicates the direction of the solution curves at various points, while the solution curves themselves illustrate how solutions to the differential equation evolve over time. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Note the solution $x = A$ (in the figure $x = 5$). We call these constant solutions the equilibrium solutions. The points on the x-axis where $f(x) = 0$ are called critical points. The point $x = A$ is a critical point. In fact, each critical point corresponds to an equilibrium solution. Note also, by looking at the graph, that the solution $x = A$ is "stable" in that small perturbations in x do not lead to substantially different solutions as t grows. If we change the initial condition a little bit, then as $t \to \infty$ we get $x \to A$. We call such critical points stable. In this simple example it turns out that all solutions in fact go to A as $t \to \infty$. If a critical point is not stable we would say it is unstable.

Let us consider the logistic equation

$$\frac{dx}{dt} = kx(M-x)$$

for some positive k and M. This equation is commonly used to model population if we know the limiting population M, that is the maximum sustainable population. The logistic equation leads to less catastrophic predictions on world population than $x' = kx$. In the real world there is no such thing as negative population, but we will still consider negative x for the purposes of the math (see Figure $1.9.2$ for an example).

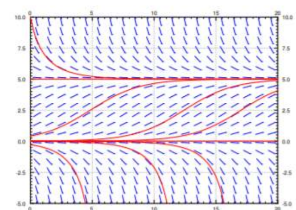


Figure $1.9.2$: Diagram illustrating the slope field and some solutions of $x' = 0.1x(5-x)$. The slope field indicates the direction of the solution curves at various points, while the solution curves themselves illustrate how solutions to the differential equation evolve over time. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Note two critical points, $x = 0$ and $x = 5$. The critical point at $x = 5$ is stable. On the other hand the critical point at $x = 0$ is unstable.

It is not really necessary to find the exact solutions to talk about the long term behavior of the solutions. For example, from the above slope field plot, we can easily see that

$$\lim_{t\to\infty} x(t) = \begin{cases} 5 & \text{if } x(0) > 0, \\ 0 & \text{if } x(0) = 0, \\ \text{DNE or } -\infty & \text{if } x(0) < 0. \end{cases}$$

Where DNE means "does not exist." From just looking at the slope field we cannot quite decide what happens if $x(0) < 0$. It could be that the solution does not exist for t all the way to ∞. Think of the equation $x' = x^2$, we have seen that it only exists for some finite period of time. Same can happen here. In our example equation above it will actually turn out that the solution does not exist for all time, but to see that we would have to solve the equation. In any case, the solution does go to $-\infty$, but it may get there rather quickly.

If we are interested only in the long term behavior of the solution, we would be doing unnecessary work if we solved the equation exactly. We could draw the slope field, but it is easier to just look at the or , which is a simple way to visualize the behavior of autonomous equations. In this case there is one dependent variable x. We draw the x-axis, we mark all the critical points, and then we draw arrows in between. Since x is the dependent variable we draw the axis vertically, as it appears in the slope field diagrams above. If $f(x) > 0$, we draw an up arrow. If $f(x) < 0$, we draw a down arrow. To figure this out, we could just plug in some x between the critical points, $f(x)$ will have the same sign at all x between two critical points as long $f(x)$ is continuous. For example, $f(6) = -0.6 < 0$, so $f(x) < 0$ for $x > 5$, and the arrow above $x = 5$ is a down arrow. Next, $f(1) = 0.4 > 0$, so $f(x) > 0$ whenever $0 < x < 5$, and the arrow points up. Finally, $f(-1) = -0.6 < 0$ so $f(x) < 0$ when $x < 0$, and the arrow points down.

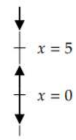


Figure 1.9.3: Diagram illustrating the phase line for the differential equation (x' = f(x)). This image shows a vertical line with labeled critical points and arrows indicating the direction of change for the dependent variable (x). The phase line is a useful tool for analyzing the long-term behavior of solutions to autonomous differential equations. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Armed with the phase diagram, it is easy to sketch the solutions approximately. As time t moves from left to right, the graph of a solution goes up if the arrow is up, and it goes down if the arrow is down.

Below is a video on solving an autonomous differential equation that describes logistic growth.

Example 1.9.1

A population P obeys the logistic model. It satisfies the equation: $\frac{dP}{dt} = \frac{3}{1100}P(11 - P)$, for $P > 0$ where

P_0 = initial population, K = carrying capacity, r = intrinsic growth rate

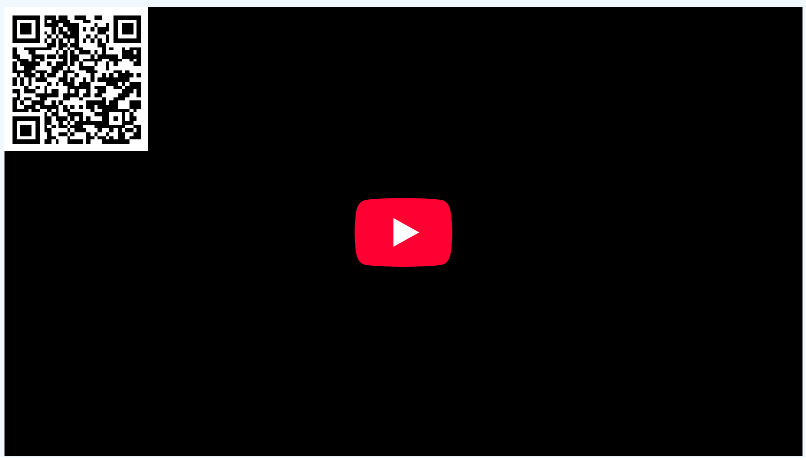
a. Find the time when the population is increasing.

b. Find the time when the population is decreasing.

c. Assume that $P(0) = 4$, find P(64).

Solution

- **Video Length:** 12 minutes 30 seconds
- **Context:** Analyze the behavior of a population following a logistic growth model and solve for specific conditions where the population increases, decreases, and its value at a given time.



Try sketching a few solutions simply from looking at the phase diagram. Check with the preceding graphs if you are getting the type of curves.

Once we draw the phase diagram, we can easily classify critical points as stable or unstable.

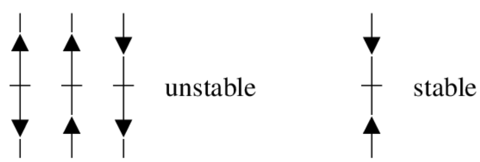


Figure 1.9.4: Diagram illustrating the phase line for the logistic equation with harvesting. This image shows a vertical line with labeled critical points and arrows indicating the direction of change for the population x. The phase line helps analyze the long-term behavior of the population under different harvesting scenarios. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Since any mathematical model we cook up will only be an approximation to the real world, unstable points are generally bad news.

Let us think about the logistic equation with harvesting. Suppose an alien race really likes to eat humans. They keep a planet with humans on it and harvest the humans at a rate of h million humans per year. Suppose x is the number of humans in millions on the planet and t is time in years. Let M be the limiting population when no harvesting is done and $k > 0$ is some constant depending on how fast humans multiply. Our equation becomes

$$\frac{dx}{dt} = kx(M - x) - h$$

We expand the right hand side and solve for critical points

$$\frac{dx}{dt} = -kx^2 + kMx - h$$

Solving for the critical points A and B from the quadratic equations:

$$A = \frac{kM + \sqrt{(kM)^2 - 4hk}}{2k}, \quad B = \frac{kM - \sqrt{(kM)^2 - 4hk}}{2k}$$

Example 1.9.2

Sketch a phase diagram for different possibilities. Note that these possibilities are $A > B$, or $A = B$, or A and B both complex (i.e. no real solutions).

Hint: Fix some simple k and M and then vary h.

Solution

For example, let $M = 8$ and $k = 0.1$. When $h = 1$, then A and B are distinct and positive. The slope field we get is in Figure 1.9.5. As long as the population starts above B, which is approximately 1.55 million, then the population will not die out. It will in fact tend towards $A \approx 6.45$ million. If ever some catastrophe happens and the population drops below B, humans will die out, and the fast food restaurant serving them will go out of business.

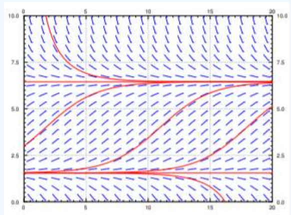


Figure 1.9.5: Diagram illustrating the slope field and some solutions of $x' = 0.1x(8 - x) - 1$. The slope field indicates the direction of the solution curves at various points, while the solution curves themselves illustrate how solutions to the differential equation evolve over time. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

When $h = 1.6$, then $A = B = 4$ and there is only **one** critical point and it is **unstable**. When the population starts above 4 million it will tend towards 4 million. If it ever drops below 4 million, humans will die out on the planet. This scenario is not one that we (as the human fast food proprietor) want to be in. A small perturbation of the equilibrium state and we are out of business; there is no room for error (see Figure 1.9.6).

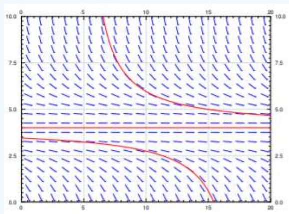


Figure 1.9.6: Diagram illustrating the slope field and some solutions of $x' = 0.1x(8 - x) - 1.6$. The slope field indicates the direction of the solution curves at various points, while the solution curves themselves illustrate how solutions to the differential equation evolve over time. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Finally if we are harvesting at 2 million humans per year, there are no critical points. The population will always plummet towards zero, no matter how well stocked the planet starts (see Figure 1.9.7).

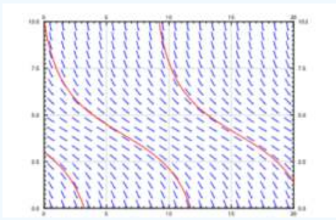


Figure 1.9.7: Diagram illustrating the slope field and some solutions of $x' = 0.1x(8 - x) - 2$. The slope field indicates the direction of the solution curves at various points, while the solution curves themselves illustrate how solutions to the differential equation evolve over time. (CC BY-SA 4.0, Jiří Lebl via Differential Equations for Engineers).

Footnotes

[1] Unstable points with one of the arrows pointing towards the critical point are sometimes called semistable.

References

1. Paul W. Berg and James L. McGregor, Elementary Partial Differential Equations, Holden-Day, San Francisco, CA, 1966.
2. William E. Boyce, Richard C. DiPrima, Elementary Differential Equations and Boundary Value Problems, 9th edition, John Wiley & Sons Inc., New York, NY, 2008.
3. C.H. Edwards and D.E. Penney, Differential Equations and Boundary Value Problems: Computing and Modeling, 4th edition, Prentice Hall, 2008.
4. Stanley J. Farlow, An Introduction to Differential Equations and Their Applications, McGraw-Hill, Inc., Princeton, NJ, 1994. (Published also by Dover Publications, 2006.)
5. E.L. Ince, Ordinary Differential Equations, Dover Publications, Inc., New York, NY, 1956.

1.9E: Exercises for Section 1.9

Exercises 1-9

Analyze the differential equations by constructing phase diagrams, finding critical points, classifying their stability, and sketching solutions.

Exercise $1.9E.1$

Consider $x' = x^2$.

a. Draw the phase diagram, find the critical points and mark them stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = -1$.

Exercise $1.9E.2$

Let $x' = \sin x$.

a. Draw the phase diagram for $-4\pi \le x \le 4\pi$. On this interval mark the critical points stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 1$.

Exercise $1.9E.3$

Suppose $f(x)$ is positive for $0 < x < 1$, it is zero when $x = 0$ and $x = 1$, and it is negative for all other x.

a. Draw the phase diagram for $x' = f(x)$, find the critical points and mark them stable or unstable.
b. Sketch typical solutions of the equation.
c. Find $\lim_{t\to\infty} x(t)$ for the solution with the initial condition $x(0) = 0.5$.

Exercise $1.9E.4$

Start with the logistic equation $\frac{dx}{dt} = kx(M-x)$. Suppose that we modify our harvesting. That is we will only harvest an amount proportional to current population. In other words we harvest hx per unit of time for some $h > 0$ (Similar to earlier example with h replaced with hx).

a. Construct the differential equation.
b. Show that if $kM > h$, then the equation is still logistic.
c. What happens when $kM < h$?

Exercise $1.9E.5$

A disease is spreading through the country. Let x be the number of people infected. Let the constant S be the number of people susceptible to infection. The infection rate $\frac{dx}{dt}$ is proportional to the product of already infected people, x, and the number of susceptible but uninfected people, $S - x$.

a. Write down the differential equation.
b. Supposing $x(0) > 0$, that is, some people are infected at time $t = 0$, what is $\lim_{t\to\infty} x(t)$.
c. Does the solution to part b) agree with your intuition? Why or why not?

Exercise $1.9E.6$

Let $x' = (x-1)(x-2)x^2$.

a. Sketch the phase diagram and find critical points.
b. Classify the critical points.
c. If $x(0) = 0.5$ then find $\lim_{t\to\infty} x(t)$.

Answer

a. $0,\ 1,\ 2$ are critical points.
b. $x = 0$ is unstable (semistable), $x = 1$ is stable, and $x = 2$ is unstable.
c. 1

Exercise $1.9E.7$

Let $x' = e^{-x}$.

a. Find and classify all critical points.
b. Find $\lim_{t\to\infty} x(t)$ given any initial condition.

Answer

a. There are no critical points.
b. ∞

Exercise $1.9E.8$

Assume that a population of fish in a lake satisfies $\frac{dx}{dt} = kx(M-x)$. Now suppose that fish are continually added at A fish per unit of time.

a. Find the differential equation for x.
b. What is the new limiting population?

Answer

a. $\frac{dx}{dt} = kx(M-x) + A$
b. $\frac{kM + \sqrt{(kM)^2 + 4Ak}}{2k}$

Exercise $1.9E.9$

Suppose $\frac{dx}{dt} = (x-\alpha)(x-\beta)$ for two numbers $\alpha < \beta$. For (b), (c), and (d), find $\lim_{t\to\infty} x(t)$ based on the phase diagram.

a. Find the critical points, and classify them.
b. $x(0) < \alpha$,
c. $\alpha < x(0) < \beta$,
d. $\beta < x(0)$.

Answer

a. α is a stable critical point, β is an unstable one.
b. α
c. α
d. ∞ or DNE.

1.10: Numerical Methods - Euler’s Method

Learning Objectives

- Understand and implement Euler’s method for numerical approximation.
- Analyze the accuracy and limitations of Euler’s method.

As we said before, unless $f(x, y)$ is of a special form, it is generally very hard if not impossible to get a nice formula for the solution of the problem

$$y' = f(x, y), \quad y(x_0) = y_0$$

If the equation can be solved in closed form, we should do that. But what if we have an equation that cannot be solved in closed form? What if we want to find the value of the solution at some particular x? Or perhaps we want to produce a graph of the solution to inspect the behavior. In this section we will learn about the basics of numerical approximation of solutions.

The simplest method for approximating a solution is Euler's Method.[1] It works as follows: Take x_0 and compute the slope $k = f(x_0, y_0)$. The slope is the change in y per unit change in x. Follow the line for an interval of length h on the x-axis. Hence if $y = y_0$ at x_0, then we say that y_1 (the approximate value of y at $x_1 = x_0 + h$) is $y_1 = y_0 + hk$. Rinse, repeat! Let $k = f(x_1, y_1)$, and then compute $x_2 = x_1 + h$, and $y_2 = y_1 + hk$. Now compute x_3 and y_3 using x_2 and y_2, etc. Consider the equation $y' = \frac{y^2}{3}$, $y(0) = 1$, and $h = 1$. Then $x_0 = 0$ and $y_0 = 1$. We compute

$$\begin{aligned} x_1 &= x_0 + h = 0 + 1 = 1, \qquad & y_1 &= y_0 + h\, f(x_0, y_0) = 1 + 1 \cdot \frac{1}{3} = \frac{4}{3} \approx 1.333, \\ x_2 &= x_1 + h = 1 + 1 = 2, \qquad & y_2 &= y_1 + h\, f(x_1, y_1) = \frac{4}{3} + 1 \cdot \frac{(\frac{4}{3})^2}{3} = \frac{52}{27} \approx 1.926. \end{aligned} \tag{1.10.1}$$

We then draw an approximate graph of the solution by connecting the points (x_0, y_0), (x_1, y_1), (x_2, y_2),…. The first two steps of the method (see Figure $1.10.1$) have two slope fields. The first has the line segment from $(0, 1)$ to $(1, 4/3)$ and the second has this and the next one from $(1, 4/3)$ to $(2, 1.926)$.

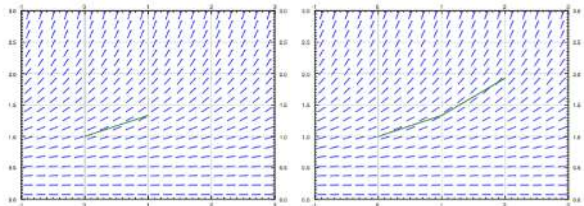


Figure 1.10.1: The first two steps of Euler's method with $h = 1$ for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$. The left diagram shows the first step, which is a line segment from $(0, 1)$ to $(1, 4/3)$. The right digram shows the second step which has the line segment in the first step and the next line segment from $(1, 4/3)$ to $(2, 1.926)$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

More abstractly, for any $i = 0, 1, 2, 3, \ldots$, we compute

$$x_{i+1} = x_i + h, \qquad y_{i+1} = y_i + h\, f(x_i, y_i).$$

The line segments we get are an approximate graph of the solution. Generally it is not exactly the solution. See Figure $1.10.2$ for the plot of the real solution and the approximation.

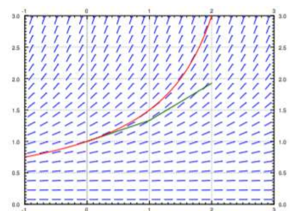


Figure 1.10.2: Diagram illustrating the two steps of Euler’s method (step size 1) in two connected green line segments and the exact solution for the equation $y' = \frac{y^2}{3}$ with initial conditions $y(0) = 1$. Th exact solution is the red curve (above th green line segments) increasing and concaving up(CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

We continue with the equation $y' = \frac{y^2}{3}$, $y(0) = 1$. Let us try to approximate $y(2)$ using Euler’s method. In Figures $1.10.1$ and $1.10.2$ we have graphically approximated $y(2)$ with step size 1. With step size 1, we have $y(2) \approx 1.926$. The real answer is 3. We are approximately 1.074 off. Let us halve the step size. Computing y_4 with $h = 0.5$, we find that $y(2) \approx 2.209$, so an error of about 0.791. Table $1.10.1$ gives the values computed for various parameters.

Example 1.10.1

Solve $y' = \frac{y^2}{3}$ given $y(0) = 1$ and show that $y(2) = 3$.

Solution

The difference between the actual solution and the approximate solution we will call the error. We will usually talk about just the size of the error and we do not care much about its sign. The main point is, that we usually do not know the real solution, so we only have a vague understanding of the error. If we knew the error exactly …what is the point of doing the approximation?

Table 1.10.1: The Euler’s method approximation of $y(2)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
1	1.92593	1.07407	
0.5	2.20861	0.79139	0.73681
0.25	2.47250	0.52751	0.66656
0.125	2.68034	0.31966	0.60599

h	Approximate $y(2)$	Error	$\frac{\text{Error}}{\text{Previous error}}$
0.0625	2.82040	0.17960	0.56184
0.03125	2.90412	0.09588	0.53385
0.015625	2.95035	0.04965	0.51779
0.0078125	2.97472	0.02528	0.50913

We notice that except for the first few times, every time we halved the interval the error approximately halved. This halving of the error is a general feature of Euler's method as it is a first order method. A second order method reduces the error to approximately one quarter every time we halve the interval (second order as $\frac{1}{4} = \frac{1}{2} \cdot \frac{1}{2}$).

To get the error to be within 0.1 of the answer we had to already do 64 steps. To get it to within 0.01 we would have to halve another three or four times, meaning doing 512 to 1024 steps. That is quite a bit to do by hand. The improved Euler method should quarter the error every time we halve the interval, so we would have to approximately do half as many "halvings" to get the same error. This reduction can be a big deal. With 10 halvings (starting at $h = 1$) we have 1024 steps, whereas with 5 halvings we only have to do 32 steps, assuming that the error was comparable to start with. A computer may not care about this difference for a problem this simple, but suppose each step would take a second to compute (the function may be substantially more difficult to compute than $\frac{y^2}{3}$). Then the difference is 32 seconds versus about 17 minutes.

Note: We are not being altogether fair, a second order method would probably double the time to do each step. Even so, it is 1 minute versus 17 minutes. Next, suppose that we have to repeat such a calculation for different parameters a thousand times. You get the idea.

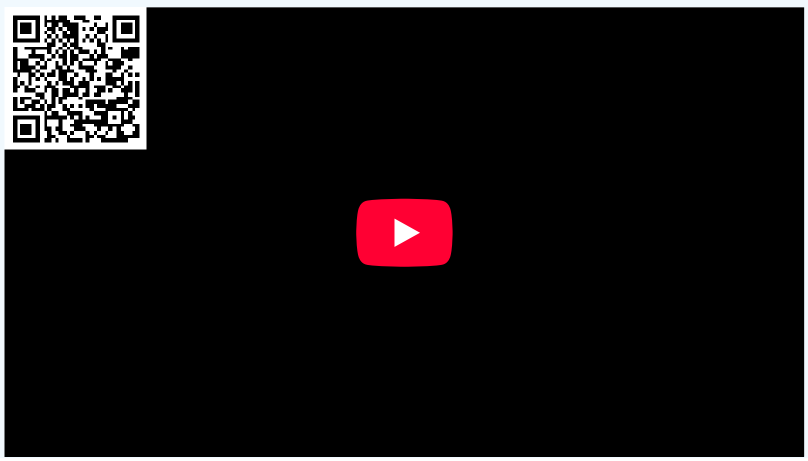
Example 1.10.2

Approximate the solution to a differential equation using Euler's method:

$$\frac{dy}{dx} = f(x,y) = 2 + 5x + 2y, \quad y(0) = 4, \quad (x_0, y_0) = (0, 4).$$

Solution

- **Video Length:** 9 minutes 19 seconds
- **Context:** Use Euler's method to approximate the solution to a first-order differential equation, starting from an initial condition, and iteratively compute values based on the slope function $f(x, y)$.



Note that in practice we do not know how large the error is! How do we know what is the right step size? Well, essentially we keep halving the interval, and if we are lucky, we can estimate the error from a few of these calculations and the assumption that the error goes down by a factor of one half each time (if we are using standard Euler).

Example 1.10.3

In Table 1.10.1, suppose you do not know the error. Take the approximate values of the function in the last two lines, assume that the error goes down by a factor of 2. Can you estimate the error in the last time from this? Does it (approximately) agree with the table? Now do it for the first two rows. Does this agree with the table?

Solution

Let us talk a little bit more about the example $y' = \frac{y^2}{3}$, $y(0) = 1$. Suppose that instead of the value $y(2)$ we wish to find $y(3)$. The results of this effort are listed in Table 1.10.2 for successive halvings of h. What is going on here? Well, you should solve the equation exactly and you will notice that the solution does not exist at $x = 3$. In fact, the solution goes to infinity when you approach $x = 3$.

Table 1.10.2: Euler's method to approximate $y(3)$ where of $y' = \frac{y^2}{3}$, $y(0) = 1$.

h	Approximate $y(3)$
1	3.16232
0.5	4.54329
0.25	6.86079
0.125	10.80321
0.0625	17.59893
0.03125	29.46004

h	Approximate $y(3)$
0.015625	50.40121
0.0078125	87.75769

Another case when things can go bad is if the solution oscillates wildly near some point. The solution may exist at all points, but even a much better numerical method than Euler would need an insanely small step size to approximate the solution with reasonable precision. And computers might not be able to easily handle such a small step size.

In real applications we would not use a simple method such as Euler's. The simplest method that would probably be used in a real application is the standard Runge-Kutta method (see Exercise 1.10E.4). That is a fourth order method, meaning that if we halve the interval, the error generally goes down by a factor of 16 (it is fourth order as $\frac{1}{16} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$).

Choosing the right method to use and the right step size can be very tricky. There are several competing factors to consider.

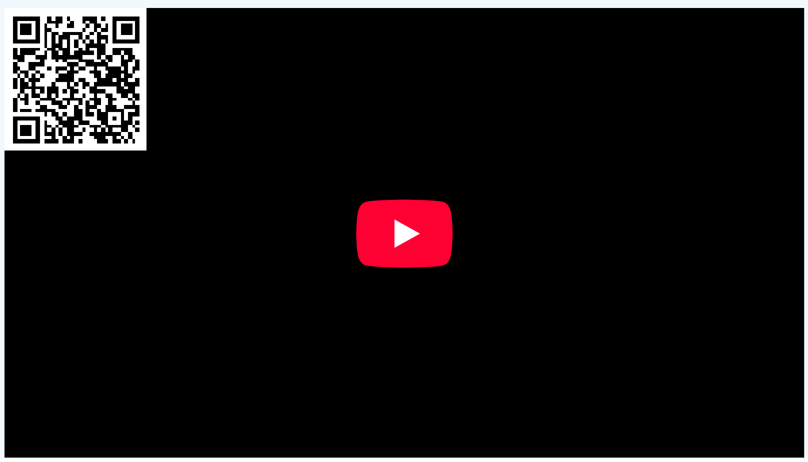
- **Computational time**: Each step takes computer time. Even if the function f is simple to compute, we do it many times over. Large step size means faster computation, but perhaps not the right precision.
- **Roundoff errors**: Computers only compute with a certain number of significant digits. Errors introduced by rounding numbers off during our computations become noticeable when the step size becomes too small relative to the quantities we are working with. So reducing step size may in fact make errors worse.
- **Stability**: Certain equations may be numerically unstable. What may happen is that the numbers never seem to stabilize no matter how many times we halve the interval. We may need a ridiculously small interval size, which may not be practical due to roundoff errors or computational time considerations. Such problems are sometimes called stiff. In the worst case, the numerical computations might be giving us bogus numbers that look like a correct answer. Just because the numbers have stabilized after successive halving, does not mean that we must have the right answer.

Example 1.10.4

Approximate the solution to the differential equation $y' = 4x + y^2$ with the initial condition $y(0) = 0$ using Euler's method with a step size of $h = 0.4$.

Solution

- **Video Length:** 7 minutes 41 seconds
- **Context:** Apply Euler's method with a step size of 0.4 to approximate the solution of a first-order differential equation with a given initial condition, using iterative calculations based on the slope function $f(x, y)$.



We have seen just the beginnings of the challenges that appear in real applications. Numerical approximation of solutions to differential equations is an active research area for engineers and mathematicians. For example, the general purpose method used for the ODE solver in Matlab and Octave (as of this writing) is a method that appeared in the literature only in the 1980s.

Footnotes

[1] Named after the Swiss mathematician Leonhard Paul Euler (1707–1783). The correct pronunciation of the name sounds more like "oiler."

1.10E: Exercises for Section 1.10

Exercises 1-9

Use numerical methods, including Euler's method, Improved Euler's method, or Runge-Kutta method, to approximate or solve the given differential equations.

Exercise 1.10.1

Consider $\frac{dx}{dt} = (2t - x)^2$, $x(0) = 2$. Use Euler's method with step size $h = 0.5$ to approximate $x(1)$.

Exercise 1.10.2

Consider $\frac{dx}{dt} = t - x$, $x(0) = 1$.

a. Use Euler's method with step sizes $h = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}$ to approximate $x(1)$.
b. Solve the equation exactly.
c. Describe what happens to the errors for each h you used. That is, find the factor by which the error changed each time you halved the interval.

Exercise 1.10.3

Approximate the value of e by looking at the initial value problem $y' = y$ with $y(0) = 1$ and approximating $y(1)$ using Euler's method with a step size of 0.2.

Exercise 1.10.4

Example of numerical instability: Take $y' = -5y$, $y(0) = 1$. We know that the solution should decay to zero as x grows. Using Euler's method, start with $h = 1$ and compute y_1, y_2, y_3, y_4 to try to approximate $y(4)$. What happened? Now halve the interval. Keep halving the interval and approximating $y(4)$ until the numbers you are getting start to stabilize (that is, until they start going towards zero). Note: You might want to use a calculator.

The simplest method used in practice is the **Runge-Kutta method**. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$ and a step size h. Everything is the same as in Euler's method, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), \\ k_2 &= f(x_i + \frac{h}{2}, y_i + k_1 \frac{h}{2}) \qquad\qquad x_{i+1} = x_i + h, \\ k_3 &= f(x_i + \frac{h}{2}, y_i + k_2 \frac{h}{2}) \qquad y_{i+1} = y_i + \frac{k_1 + 2k_2 + 2k_3 + k_4}{6} h, \\ k_4 &= f(x_i + h, y_i + k_3 h). \end{aligned} \tag{1.10.1}$$

Exercise 1.10.5

Consider $\frac{dy}{dx} = yx^2$, $y(0) = 1$.

a. Use Runge-Kutta (see Exercise 1.10E.4) with step sizes $h = 1$ and $h = \frac{1}{2}$ to approximate $y(1)$.
b. Use Euler's method with $h = 1$ and $h = \frac{1}{2}$.
c. Solve exactly, find the exact value of $y(1)$, and compare.

Exercise 1.10.6

Let $x' = \sin(xt)$, and $x(0) = 1$. Approximate $x(1)$ using Euler's method with step sizes 1, 0.5, 0.25. Use a calculator and compute up to 4 decimal digits.

Answer

Approximately: $1.0000,\ 1.2397,\ 1.382$

? Exercise 1.10.7

Let $x' = 2t$, and $x(0) = 0$.

a. Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
b. Solve exactly, and compute the errors.
c. Compute the factor by which the errors changed.

Answer

a. $0,\ 8,\ 12$
b. $x(4) = 16$, so errors are: $16,\ 8,\ 4$
c. Factors are $0.5,\ 0.5,\ 0.5$

? Exercise 1.10.8

Let $x' = xe^{xt+1}$, and $x(0) = 0$.

a. Approximate $x(4)$ using Euler's method with step sizes 4, 2, and 1.
b. Guess an exact solution based on part (a) and compute the errors.

Answer

a. $0,\ 0,\ 0$
b. $x = 0$ is a solution so errors are: $0,\ 0,\ 0$.

Note: The Improved Euler's Method

There is a simple way to improve Euler's method to make it a second order method by doing just one extra step. Consider $\frac{dy}{dx} = f(x, y)$, $y(x_0) = y_0$, and a step size h. What we do is to pretend we compute the next step as in Euler, that is, we start with (x_i, y_i), we compute a slope $k_1 = f(x_i, y_i)$, and then look at the point $(x_i + h, y_i + k_1 h)$. Instead of letting our new point be $(x_i + h, y_i + k_1 h)$, we compute the slope at that point, call it k_2, and then take the average of k_1 and k_2, hoping that the average is going to be closer to the actual slope on the interval from x_i to $x_i + h$. And we are correct, if we halve the step, the error should go down by a factor of $2^2 = 4$. To summarize, the setup is the same as for regular Euler, except the computation of y_{i+1} and x_{i+1}.

$$\begin{aligned} k_1 &= f(x_i, y_i), & x_{i+1} &= x_i + h, \\ k_2 &= f(x_i + h, y_i + k_1 h), & y_{i+1} &= y_i + \frac{k_1 + k_2}{2} h. \end{aligned} \tag{1.10.2}$$

? Exercise 1.10.9

Consider $\frac{dy}{dx} = x + y$, $y(0) = 1$.

a. Use the improved Euler's method with step sizes $h = \frac{1}{4}$ and $h = \frac{1}{8}$ to approximate $y(1)$.
b. Use Euler's method with $h = \frac{1}{4}$ and $h = \frac{1}{8}$.
c. Solve exactly, find the exact value of $y(1)$.
d. Compute the errors, and the factors by which the errors changed.

Answer

a. Improved Euler: $y(1) \approx 3.3897$for $h = 1/4$, $y(1) \approx 3.4237$for $h = 1/8$,
b. Standard Euler: $y(1) \approx 2.8828$for $h = 1/4$, $y(1) \approx 3.1316$for $h = 1/8$,

c. $y = 2e^x - x - 1$, so $y(2)$ is approximately 3.4366
d. Approximate errors for improved Euler: 0.046852 for $h = 1/4$, and 0.012881 for $h = 1/8$. For standard Euler: 0.55375 for $h = 1/4$, and 0.30499 for $h = 1/8$. Factor is approximately 0.27 for improved Euler, and 0.55 for standard Euler.

1.11: Exact Equations

Learning Objectives

- Identify and solve exact differential equations.
- Utilize integrating factors to solve non-exact equations.

Another type of equation that comes up quite often in physics and engineering is an *exact equation*. Suppose $F(x,y)$ is a function of two variables, which we call the *potential function*. The naming should suggest potential energy, or electric potential. Exact equations and potential functions appear when there is a conservation law at play, such as conservation of energy. Let us make up a simple example. Let

$$F(x,y) = x^2 + y^2.$$

We are interested in the lines of constant energy, that is lines where the energy is conserved; we want curves where $F(x,y) = C$, for some constant C. In our example, the curves $x^2 + y^2 = C$ are circles. See Figure 1.11.1.

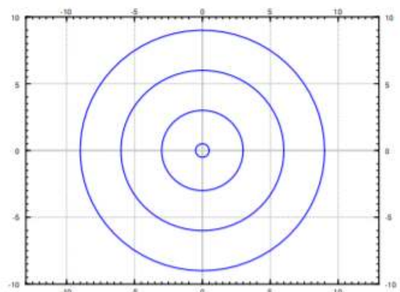


Figure 1.11.1: Diagram illustrating the solutions to $F(x,y) = x^2 + y^2 = C$. Four circles with center at (0,0) and radius approximately 0.5, 3.5, 6,and 8.(CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

We take the *total derivative* of F:

$$dF = \frac{\partial F}{\partial x} dx + \frac{\partial F}{\partial y} dy.$$

For convenience, we will make use of the notation of $F_x = \frac{\partial F}{\partial x}$ and $F_y = \frac{\partial F}{\partial y}$. In our example,

$$dF = 2x\,dx + 2y\,dy.$$

We apply the total derivative to $F(x,y) = C$, to find the differential equation $dF = 0$. The differential equation we obtain in such a way has the form

$$M\,dx + N\,dy = 0, \qquad \text{or} \qquad M + N\,\frac{dy}{dx} = 0.$$

An equation of this form is called *exact* if it was obtained as $dF = 0$ for some potential function F. In our simple example, we obtain the equation

$$2x\,dx + 2y\,dy = 0, \qquad \text{or} \qquad 2x + 2y\,\frac{dy}{dx} = 0.$$

Since we obtained this equation by differentiating $x^2 + y^2 = C$, the equation is exact. We often wish to solve for y in terms of x. In our example,

$$y = \pm\sqrt{C - x^2}.$$

An interpretation of the setup is that at each point $\mathbf{v} = (M,N)$ is a vector in the plane, that is, a direction and a magnitude. As M and N are functions of (x,y), we have a *vector field*. The particular vector field $\mathbf{v}$ that comes from an exact equation is a so-called *conservative vector field*, that is, a vector field that comes with a potential function $F(x,y)$, such that

$$\mathbf{v} = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y} \right).$$

Let γ be a path in the plane starting at (x_1, y_1) and ending at (x_2, y_2). If we think of $\mathbf{v}$ as force, then the work required to move along γ is

$$\int_\gamma \mathbf{v}(\mathbf{r}) \cdot d\mathbf{r} = \int_\gamma M\,dx + N\,dy = F(x_2, y_2) - F(x_1, y_1).$$

That is, the work done only depends on endpoints, that is where we start and where we end. For example, suppose F is gravitational potential. The derivative of F given by $\mathbf{v}$ is the gravitational force. What we are saying is that the work required to move a heavy box from the ground floor to the roof, only depends on the change in potential energy. That is, the work done is the same no matter what path we took; if we took the stairs or the elevator. Although if we took the elevator, the elevator is doing the work for us. The curves $F(x, y) = C$ are those where no work need be done, such as the heavy box sliding along without accelerating or breaking on a perfectly flat roof, on a cart with incredibly well oiled wheels.

An exact equation is a conservative vector field, and the implicit solution of this equation is the potential function.

Solving exact equations

Now you, the reader, should ask: Where did we solve a differential equation? Well, in applications we generally know M and N, but we do not know F. That is, we may have just started with $2x + 2y\frac{dy}{dx} = 0$, or perhaps even

$$x + y\frac{dy}{dx} = 0.$$

It is up to us to find some potential F that works. Many different F will work; adding a constant to F does not change the equation. Once we have a potential function F, the equation $F\big(x, y(x)\big) = C$ gives an implicit solution of the ODE.

✔ Example 1.11.1

Let us find the general solution to $2x + 2y\frac{dy}{dx} = 0$. Forget we knew what F was.

Solution

If we know that this is an exact equation, we start looking for a potential function F. We have $M = 2x$ and $N = 2y$. If F exists, it must be such that $F_x(x, y) = 2x$. Integrate with respect to the x variable to find

$$F(x, y) = x^2 + A(y), \tag{1.11.1}$$

for some function $A(y)$; though it is only constant as far as x is concerned, and it may still depend on y. Now differentiate Equation (1.11.1) in y and set it equal to N, which is what F_y is supposed to be:

$$2y = F_y(x, y) = A'(y).$$

Integrating, we find $A(y) = y^2$. We could add a constant of integration if we wanted to, but there is no need. We found $F(x, y) = x^2 + y^2$. Next for a constant C, we solve

$$F\big(x, y(x)\big) = C.$$

for y in terms of x. In this case, we obtain $y = \pm\sqrt{C - x^2}$ as we did before.

The procedure, once we know that the equation is exact, is:

i. Integrate $F_x = M$ in x resulting in $F(x, y) = \text{something} + A(y)$.
ii. Differentiate this F in y, and set that equal to N, so that we may find $A(y)$ by integration.

The procedure can also be done by first integrating in y and then differentiating in x. Pretty easy huh? Let's try this again.

Example 1.11.2

Solve $2x + y + xy\frac{dy}{dx} = 0$.

Solution

$M = 2x + y$ and $N = xy$. We try to proceed as before. Suppose F exists. Then $F_x(x, y) = 2x + y$. We integrate:

$$F(x,y) = x^2 + xy + A(y)$$

for some function $A(y)$. Differentiate in y and set equal to N:

$$N = xy = F_y(x,y) = x + A'(y).$$

But there is no way to satisfy this requirement! The function xy cannot be written as x plus a function of y. The equation is not exact; no potential function F exists.

Is there an easier way to check for the existence of F, other than failing in trying to find it? Turns out there is. Suppose $M = F_x$ and $N = F_y$. Then as long as the second derivatives are continuous,

$$\frac{\partial M}{\partial y} = \frac{\partial^2 F}{\partial y \partial x} = \frac{\partial^2 F}{\partial x \partial y} = \frac{\partial N}{\partial x}.$$

Let us state it as a theorem. Usually this is called the Poincaré Lemma.

Theorem 1.11.1: Poincare Lemma

If M and N are continuously differentiable functions of (x, y), and $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$, then near any point there is a function $F(x, y)$ such that $M = \frac{\partial F}{\partial x}$ and $N = \frac{\partial F}{\partial y}$.

The theorem doesn't give us a global F defined everywhere. In general, we can only find the potential locally, near some initial point. By this time, we have come to expect this from differential equations.

Let us return to Example 1.11.2 where $M = 2x + y$ and $N = xy$. Notice $M_y = 1$ and $N_x = y$, which are clearly not equal. The equation is not exact.

Example 1.11.3

Solve

$$\frac{dy}{dx} = \frac{-2x - y}{x - 1}, \qquad y(0) = 1.$$

Solution

We write the equation as

$$(2x + y) + (x - 1)\frac{dy}{dx} = 0,$$

so $M = 2x + y$ and $N = x - 1$. Then

$$M_y = 1 = N_x.$$

The equation is exact. Integrating M in x, we find

$$F(x,y) = x^2 + xy + A(y).$$

Differentiating in y and setting to N, we find

$$x - 1 = x + A'(y).$$

So $A'(y) = -1$, and $A(y) = -y$ will work. Take $F(x, y) = x^2 + xy - y$. We wish to solve $x^2 + xy - y = C$. First let us find C. As $y(0) = 1$ then $F(0, 1) = C$. Therefore $0^2 + 0 \times 1 - 1 = C$, so $C = -1$. Now we solve $x^2 + xy - y = -1$ for y

to get

$$y = \frac{-x^2 - 1}{x - 1}.$$

Example 1.11.4

Solve

$$-\frac{y}{x^2+y^2} dx + \frac{x}{x^2+y^2} dy = 0, \qquad y(1) = 2.$$

Solution

We leave to the reader to check that $M_y = N_x$.

This vector field (M, N) is not conservative if considered as a vector field of the entire plane minus the origin. The problem is that if the curve γ is a circle around the origin, say starting at $(1, 0)$ and ending at $(1, 0)$ going counterclockwise, then if F existed we would expect

$$0 = F(1,0) - F(1,0) = \int_\gamma F_x \, dx + F_y \, dy = \int_\gamma \frac{-y}{x^2+y^2} \, dx + \frac{x}{x^2+y^2} \, dy = 2\pi.$$

That is nonsense! We leave the computation of the path integral to the interested reader, or you can consult your multivariable calculus textbook. So there is no potential function F defined everywhere outside the origin $(0, 0)$.

If we think back to the theorem, it does not guarantee such a function anyway. It only guarantees a potential function locally, that is only in some region near the initial point. As $y(1) = 2$ we start at the point $(1, 2)$. Considering $x > 0$ and integrating M in x or N in y, we find

$$F(x, y) = \arctan\left(\frac{y}{x}\right).$$

The implicit solution is $\arctan\left(\frac{y}{x}\right) = C$. Solving, $y = \tan(C)x$. That is, the solution is a straight line. Solving $y(1) = 2$ gives us that $\tan(C) = 2$, and so $y = 2x$ is the desired solution. See Figure 1.11.1, and note that the solution only exists for $x > 0$.

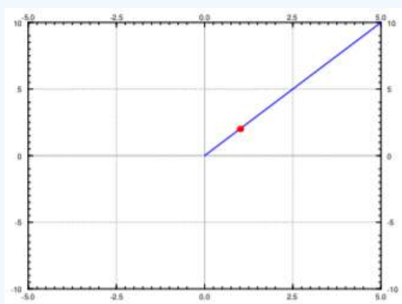


Figure 1.11.1: Diagram illustrating the solution to $-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$, $y(1) = 2$, with initial point marked. It is a line segment from the origin to the point $(5, 10)$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Example 1.11.5

Solve

$$x^2 + y^2 + 2y(x+1)\frac{dy}{dx} = 0.$$

Solution

The reader should check that this equation is exact. Let $M = x^2 + y^2$ and $N = 2y(x+1)$. We follow the procedure for exact equations

$$F(x, y) = \frac{1}{3}x^3 + xy^2 + A(y),$$

and

$$2y(x+1) = 2xy + A'(y).$$

Therefore $A'(y) = 2y$ or $A(y) = y^2$ and $F(x,y) = \frac{1}{3}x^3 + xy^2 + y^2$. We try to solve $F(x,y) = C$. We easily solve for y^2 and then just take the square root:

$$y^2 = \frac{C - (\frac{1}{3})x^3}{x+1}, \qquad \text{so} \qquad y = \pm\sqrt{\frac{C - (\frac{1}{3})x^3}{x+1}}.$$

When $x = -1$, the term in front of $\frac{dy}{dx}$ vanishes. You can also see that our solution is not valid in that case. However, one could in that case try to solve for x in terms of y starting from the implicit solution $\frac{1}{3}x^3 + xy^2 + y^2 = C$. The solution is somewhat messy and we leave it as implicit.

Integrating factors

Sometimes an equation $M\,dx + N\,dy = 0$ is not exact, but it can be made exact by multiplying with a function $u(x,y)$. That is, perhaps for some nonzero function $u(x,y)$,

$$u(x,y)M(x,y)\,dx + u(x,y)N(x,y)\,dy = 0$$

is exact. Any solution to this new equation is also a solution to $M\,dx + N\,dy = 0$.

In fact, a linear equation

$$\frac{dy}{dx} + p(x)y = f(x), \qquad \text{or} \qquad \big(p(x)y - f(x)\big)\,dx + dy = 0$$

is always such an equation. Let $r(x) = e^{\int p(x)\,dx}$ be the integrating factor for a linear equation. Multiply the equation by $r(x)$ and write it in the form of $M + N\frac{dy}{dx} = 0$.

$$r(x)p(x)y - r(x)f(x) + r(x)\frac{dy}{dx} = 0.$$

Then $M = r(x)p(x)y - r(x)f(x)$, so $M_y = r(x)p(x)$, while $N = r(x)$, so $N_x = r'(x) = r(x)p(x)$. In other words, we have an exact equation. Integrating factors for linear functions are just a special case of integrating factors for exact equations.

But how do we find the integrating factor u? Well, given an equation

$$M\,dx + N\,dy = 0,$$

u should be a function such that

$$\frac{\partial}{\partial y}\big[uM\big] = u_y M + u M_y = \frac{\partial}{\partial x}\big[uN\big] = u_x N + u N_x.$$

Therefore,

$$(M_y - N_x)u = u_x N - u_y M.$$

At first it may seem we replaced one differential equation by another. True, but all hope is not lost.

A strategy that often works is to look for a u that is a function of x alone, or a function of y alone. If u is a function of x alone, that is $u(x)$, then we write $u'(x)$ instead of u_x, and u_y is just zero. Then

$$\frac{M_y - N_x}{N}u = u'.$$

In particular, $\frac{M_y - N_x}{N}$ ought to be a function of x alone (not depend on y). If so, then we have a linear equation

$$u' - \frac{M_y - N_x}{N}u = 0.$$

Letting $P(x) = \frac{M_y - N_x}{N}$, we solve using the standard integrating factor method, to find $u(x) = Ce^{\int P(x)\,dx}$. The constant in the solution is not relevant, we need any nonzero solution, so we take $C = 1$. Then $u(x) = e^{\int P(x)\,dx}$ is the integrating factor.

Similarly we could try a function of the form $u(y)$. Then

$$\frac{M_y - N_x}{M} u = -u'.$$

In particular, $\frac{M_y - N_x}{M}$ ought to be a function of y alone. If so, then we have a linear equation

$$u' + \frac{M_y - N_x}{M} u = 0.$$

Letting $Q(y) = \frac{M_y - N_x}{M}$, we find $u(y) = Ce^{-\int Q(y)\,dy}$. We take $C = 1$. So $u(y) = e^{-\int Q(y)\,dy}$ is the integrating factor.

Example 1.11.6

Solve

$$\frac{x^2 + y^2}{x + 1} + 2y\frac{dy}{dx} = 0.$$

Solution

Let $M = \frac{x^2+y^2}{x+1}$ and $N = 2y$. Compute

$$M_y - N_x = \frac{2y}{x+1} - 0 = \frac{2y}{x+1}.$$

As this is not zero, the equation is not exact. We notice

$$P(x) = \frac{M_y - N_x}{N} = \frac{2y}{x+1}\frac{1}{2y} = \frac{1}{x+1}$$

is a function of x alone. We compute the integrating factor

$$e^{\int P(x)\,dx} = e^{\ln(x+1)} = x + 1.$$

We multiply our given equation by $(x + 1)$ to obtain

$$x^2 + y^2 + 2y(x+1)\frac{dy}{dx} = 0,$$

which is an exact equation that we solved in Example 1.11.5. The solution was

$$y = \pm\sqrt{\frac{C - (\frac{1}{3})x^3}{x+1}}.$$

Example 1.11.7

Solve

$$y^2 + (xy + 1)\frac{dy}{dx} = 0.$$

Solution

First compute

$$M_y - N_x = 2y - y = y.$$

As this is not zero, the equation is not exact. We observe

$$Q(y) = \frac{M_y - N_x}{M} = \frac{y}{y^2} = \frac{1}{y}$$

is a function of y alone. We compute the integrating factor

$$e^{-\int Q(y)\,dy} = e^{-\ln y} = \frac{1}{y}.$$

Therefore we look at the exact equation

$$y + \frac{xy+1}{y}\frac{dy}{dx} = 0.$$

The reader should double check that this equation is exact. We follow the procedure for exact equations

$$F(x,y) = xy + A(y),$$

and

$$\frac{xy+1}{y} = x + \frac{1}{y} = x + A'(y).$$

Consequently $A'(y) = \frac{1}{y}$ or $A(y) = \ln y$. Thus $F(x,y) = xy + \ln y$. It is not possible to solve $F(x,y) = C$ for y in terms of elementary functions, so let us be content with the implicit solution:

$$xy + \ln y = C.$$

We are looking for the general solution and we divided by y above. We should check what happens when $y = 0$, as the equation itself makes perfect sense in that case. We plug in $y = 0$ to find the equation is satisfied. So $y = 0$ is also a solution.

Footnotes

[1] Named for the French polymath Jules Henri Poincaré (1854–1912).

1.11E: Exercises for Section 1.11

Exercises 1-9

Solve or analyze the given exact differential equations and related problems.

Exercise $1.11E.1$

Solve the following exact equations, implicit general solutions will suffice:

a. $(2xy + x^2)\,dx + (x^2 + y^2 + 1)\,dy = 0$
b. $x^5 + y^5 \frac{dy}{dx} = 0$
c. $e^x + y^3 + 3xy^2 \frac{dy}{dx} = 0$
d. $(x + y)\cos(x) + \sin(x) + \sin(x)y' = 0$

Exercise $1.11E.2$

Find the integrating factor for the following equations making them into exact equations:

a. $e^{xy}\,dx + \frac{y}{x}e^{xy}\,dy = 0$
b. $\frac{e^x + y^3}{y^2}\,dx + 3x\,dy = 0$
c. $4(y^2 + x)\,dx + \frac{2x + 2y^2}{y}\,dy = 0$
d. $2\sin(y)\,dx + x\cos(y)\,dy = 0$

Exercise $1.11E.3$

Suppose you have an equation of the form: $f(x) + g(y)\frac{dy}{dx} = 0$.

a. Show it is exact.
b. Find the form of the potential function in terms of f and g.

Exercise $1.11E.4$

Suppose that we have the equation $f(x)\,dx - dy = 0$.

a. Is this equation exact?
b. Find the general solution using a definite integral.

Exercise $1.11E.5$

Find the potential function $F(x, y)$ of the exact equation $\frac{1 + xy}{x}\,dx + \left(\frac{1}{y} + x\right)dy = 0$ in two different ways.

a. Integrate M in terms of x and then differentiate in y and set to N.
b. Integrate N in terms of y and then differentiate in x and set to M.

Exercise $1.11E.6$

A function $u(x, y)$ is said to be **harmonic** if $u_{xx} + u_{yy} = 0$. Show if u is harmonic, then $-u_y\,dx + u_x\,dy = 0$ is an exact equation. So there exists (at least locally) the so-called **harmonic conjugate** function $v(x, y)$ such that $v_x = -u_y$ and $v_y = u_x$.

Verify that the following u are harmonic and find the corresponding harmonic conjugates v:

a. $u = 2xy$
b. $u = e^x \cos y$
c. $u = x^3 - 3xy^2$

Exercise $1.11E.7$

Solve the following exact equations, implicit general solutions will suffice:

a. $\cos(x) + ye^{xy} + xe^{xy}y' = 0$
b. $(2x+y)\,dx + (x-4y)\,dy = 0$
c. $e^x + e^y \frac{dy}{dx} = 0$
d. $(3x^2+3y)\,dx + (3y^2+3x)\,dy = 0$

Answer

a. $e^{xy} + \sin(x) = C$
b. $x^2 + xy - 2y^2 = C$
c. $e^x + e^y = C$
d. $x^3 + 3xy + y^3 = C$

Exercise $1.11E.8$

Find the integrating factor for the following equations making them into exact equations:

a. $\frac{1}{y}\,dx + 3y\,dy = 0$
b. $dx - e^{-x-y}\,dy = 0$
c. $\left(\frac{\cos(x)}{y^2} + \frac{1}{y}\right)dx + \frac{x}{y^2}\,dy = 0$
d. $\left(2y + \frac{y^2}{x}\right)dx + (2y+x)\,dy = 0$

Answer

a. Integrating factor is y, equation becomes $dx + 3y^2dy = 0$.
b. Integrating factor is e^x, equation becomes $e^x dx - e^{-y}dy = 0$.
c. Integrating factor is y^2, equation becomes $(\cos(x)+y)dx + xdy = 0$.
d. Integrating factor is x, equation becomes $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$.

Exercise $1.11E.9$

a. Show that every separable equation $y' = f(x)g(y)$ can be written as an exact equation, and verify that it is indeed exact.
b. Using this rewrite $y' = xy$ as an exact equation, solve it.

Answer

a. The equation is $-f(x)dx + \frac{1}{g(y)}dy$, and this is exact because $M = -f(x)$, $N = \frac{1}{g(y)}$, so $M_y = 0 = N_x$.
b. $-xdx + \frac{1}{y}dy = 0$, leads to potential function $F(x,y) = -\frac{x^2}{2} + \ln|y|$.

1.12: Transformation of Nonlinear Equations into Separable Equations

Learning Objectives

- Understand the method of solving Bernoulli equations by transforming them into separable equations through substitution.
- Apply substitution techniques to solve homogeneous nonlinear differential equations by reducing them to separable equations.
- Identify and solve specific forms of nonlinear differential equations that can be transformed using known solutions or variable substitutions.

The solutions of a linear nonhomogeneous equation

$$y' + p(x)y = f(x)$$

are of the form $y = uy_1$, where y_1 is a nontrivial solution of the complementary equation

$$y' + p(x)y = 0 \tag{1.12.1}$$

and u is a solution of

$$u'y_1(x) = f(x).$$

Note that this last equation is separable, since it can be rewritten as

$$u' = \frac{f(x)}{y_1(x)}.$$

In this section we'll consider nonlinear differential equations that are not separable to begin with, but can be solved in a similar fashion by writing their solutions in the form $y = uy_1$, where y_1 is a suitably chosen known function and u satisfies a separable equation. We'll say in this case that we **transformed** the given equation into a separable equation.

Bernoulli Equations

A **Bernoulli equation** is an equation of the form

$$y' + p(x)y = f(x)y^r, \tag{1.12.2}$$

where r can be any real number other than 0 or 1. (Note that Equation (1.12.2) is linear if and only if $r = 0$ or $r = 1$.) We can transform Equation (1.12.2) into a separable equation by variation of parameters: if y_1 is a nontrivial solution of Equation (1.12.1), substituting $y = uy_1$ into Equation (1.12.2) yields

$$u'y_1 + u(y_1' + p(x)y_1) = f(x)(uy_1)^r,$$

which is equivalent to the separable equation

$$u'y_1(x) = f(x)(y_1(x))^r u^r \quad \text{or} \quad \frac{u'}{u^r} = f(x)(y_1(x))^{r-1},$$

since $y_1' + p(x)y_1 = 0$.

Example 1.12.1

Solve the Bernoulli equation

$$y' - y = xy^2. \tag{1.12.3}$$

Solution

Since $y_1 = e^x$ is a solution of $y' - y = 0$, we look for solutions of Equation (1.12.3) in the form $y = ue^x$, where

$$u'e^x = xu^2e^{2x} \quad \text{or equivalently} \quad u' = xu^2e^x.$$

Separating variables yields

$$\frac{u'}{u^2} = xe^x,$$

and integrating yields

$$-\frac{1}{u} = (x-1)e^x + c.$$

Hence,

$$u = -\frac{1}{(x-1)e^x + c}$$

and

$$y = -\frac{1}{x-1+ce^{-x}}.$$

Figure 1.12.1 shows direction field and some integral curves of Equation (1.12.3).

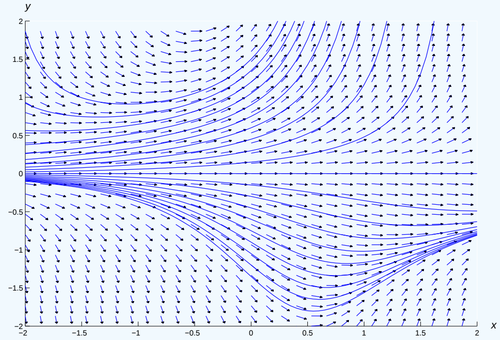


Figure 1.12.1: A direction field and integral curves for $y' - y = xy^2$. The set of directional vector arrows moves down toward the line $y = 0$ then curves upwards. Below the line $y = 0$, the set of arrows moves down then curves upwards. Matching integral curves are drawn in the direction of the arrows. (CC BY-NC-SA 3.0; William F. Trench via ELEMENTARY DIFFERENTIAL EQUATIONS).

Other Nonlinear Equations That Can be Transformed Into Separable Equations

We've seen that the nonlinear Bernoulli equation can be transformed into a separable equation by the substitution $y = uy_1$ if y_1 is suitably chosen. Now let's discover a sufficient condition for a nonlinear first order differential equation

$$y' = f(x, y) \tag{1.12.4}$$

to be transformable into a separable equation in the same way. Substituting $y = uy_1$ into Equation (1.12.4) yields

$$u'y_1(x) + uy_1'(x) = f(x, uy_1(x)),$$

which is equivalent to

$$u'y_1(x) = f(x, uy_1(x)) - uy_1'(x). \tag{1.12.5}$$

If

$$f(x, uy_1(x)) = q(u)y_1'(x)$$

for some function q, then Equation (1.12.5) becomes

$$u'y_1(x) = (q(u) - u)y_1'(x), \tag{1.12.6}$$

which is separable. After checking for constant solutions $u = u_0$ such that $q(u_0) = u_0$, we can separate variables to obtain

$$\frac{u'}{q(u) - u} = \frac{y_1'(x)}{y_1(x)}.$$

Homogeneous Nonlinear Equations

In the text we will consider only the most widely studied class of equations for which the method of the preceding paragraph works.

The differential equation Equation (1.12.4) is said to be **homogeneous** if x and y occur in f in such a way that $f(x, y)$ depends only on the ratio y/x; that is, Equation (1.12.4) can be written as

$$y' = q(y/x), \tag{1.12.7}$$

where $q = q(u)$ is a function of a single variable. For example,

$$y' = \frac{y + xe^{-y/x}}{x} = \frac{y}{x} + e^{-y/x}$$

and

$$y' = \frac{y^2 + xy - x^2}{x^2} = \left(\frac{y}{x}\right)^2 + \frac{y}{x} - 1$$

are of the form Equation (1.12.7), with

$$q(u) = u + e^{-u} \quad \text{and} \quad q(u) = u^2 + u - 1,$$

respectively. The general method discussed above can be applied to Equation (1.12.7) with $y_1 = x$ (and therefore $y_1' = 1$). Thus, substituting $y = ux$ in Equation (1.12.7) yields

$$u'x + u = q(u),$$

and separation of variables (after checking for constant solutions $u = u_0$ such that $q(u_0) = u_0$) yields

$$\frac{u'}{q(u) - u} = \frac{1}{x}.$$

Before turning to examples, we point out something that you may've already noticed: the definition of **homogeneous equation** given here is not the same as the definition given in Section 2.1, where we said that a linear equation of the form

$$y' + p(x)y = 0$$

is homogeneous. We make no apology for this inconsistency, since we didn't create it historically, **homogeneous** has been used in these two inconsistent ways. The one having to do with linear equations is the most important. This is the only section of the book where the meaning defined here will apply.

Since y/x is in general undefined if $x = 0$, we'll consider solutions of nonhomogeneous equations only on open intervals that do not contain the point $x = 0$.

✔ Example 1.12.2

Solve

$$y' = \frac{y + xe^{-y/x}}{x}. \tag{1.12.8}$$

Solution

Substituting $y = ux$ into Equation (1.12.8) yields

$$u'x + u = \frac{ux + xe^{-ux/x}}{x} = u + e^{-u}.$$

Simplifying and separating variables yields

$$e^u u' = \frac{1}{x}.$$

Integrating yields $e^u = \ln|x| + c$. Therefore $u = \ln(\ln|x| + c)$ and $y = ux = x\ln(\ln|x| + c)$.

Figure 1.12.2 shows a direction field and integral curves for Equation (1.12.8).

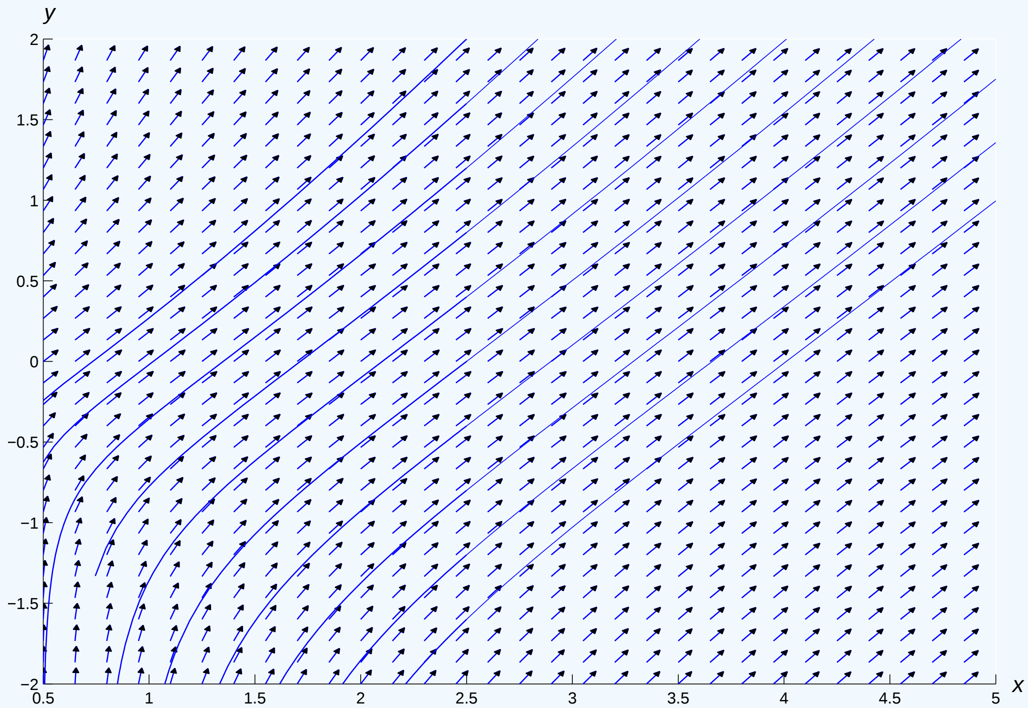


Figure 1.12.2: A direction field and some integral curves for $y' = \frac{y + xe^{-y/x}}{x}$. The set of directional vector arrows points in the North East direction. (CC BY-NC-SA 3.0; William F. Trench via ELEMENTARY DIFFERENTIAL EQUATIONS).

Example 1.12.3

a. Solve

$$x^2y' = y^2 + xy - x^2. \tag{1.12.9}$$

b. Solve the initial value problem

$$x^2y' = y^2 + xy - x^2, \quad y(1) = 2. \tag{1.12.10}$$

Solution a

We first find solutions of Equation (1.12.9) on open intervals that don't contain $x = 0$. We can rewrite Equation (1.12.9) as

$$y' = \frac{y^2 + xy - x^2}{x^2}$$

for x in any such interval. Substituting $y = ux$ yields

$$u'x + u = \frac{(ux)^2 + x(ux) - x^2}{x^2} = u^2 + u - 1,$$

so

$$u'x = u^2 - 1. \tag{1.12.11}$$

By inspection this equation has the constant solutions $u = 1$ and $u = -1$. Therefore $y = x$ and $y = -x$ are solutions of Equation (1.12.9). If u is a solution of Equation (1.12.11) that does not assume the values ± 1 on some interval, separating variables yields

$$\frac{u'}{u^2 - 1} = \frac{1}{x},$$

or, after a partial fraction expansion,

$$\frac{1}{2}\left[\frac{1}{u-1}-\frac{1}{u+1}\right]u'=\frac{1}{x}.$$

Multiplying by 2 and integrating yields

$$\ln\left|\frac{u-1}{u+1}\right|=2\ln|x|+k,$$

or

$$\left|\frac{u-1}{u+1}\right|=e^k x^2,$$

which holds if

$$\frac{u-1}{u+1}=cx^2 \tag{1.12.12}$$

where c is an arbitrary constant. Solving for u yields

$$u=\frac{1+cx^2}{1-cx^2}.$$

Therefore

$$y=ux=\frac{x(1+cx^2)}{1-cx^2} \tag{1.12.13}$$

is a solution of Equation (1.12.10) for any choice of the constant c. Setting $c=0$ in Equation (1.12.13) yields the solution $y=x$. However, the solution $y=-x$ can't be obtained from Equation (1.12.13). Thus, the solutions of Equation (1.12.9) on intervals that don't contain $x=0$ are $y=-x$ and functions of the form Equation (1.12.13).

The situation is more complicated if $x=0$ is the open interval. First, note that $y=-x$ satisfies Equation (1.12.9) on $(-\infty,\infty)$. If c_1 and c_2 are arbitrary constants, the function

$$y=\begin{cases}\frac{x(1+c_1x^2)}{1-c_1x^2}, & a<x<0\\ \frac{x(1+c_2x^2}{1-c_2x^2}, & 0\le x<b\end{cases} \tag{1.12.14}$$

is a solution of Equation (1.12.9) on (a,b), where

$$a=\begin{cases}-\frac{1}{\sqrt{c_1}} & \text{if } c_1>0,\\ -\infty & \text{if } c_1\le 0,\end{cases}\quad \text{and}\quad b=\begin{cases}\frac{1}{\sqrt{c_2}} & \text{if } c_2>0,\\ \infty & \text{if } c_2\le 0.\end{cases}$$

We leave it to you to verify this. To do so, note that if y is any function of the form Equation (1.12.13) then $y(0)=0$ and $y'(0)=1$.

Figure $1.12.3$ shows a direction field and some integral curves for Equation (1.12.9).

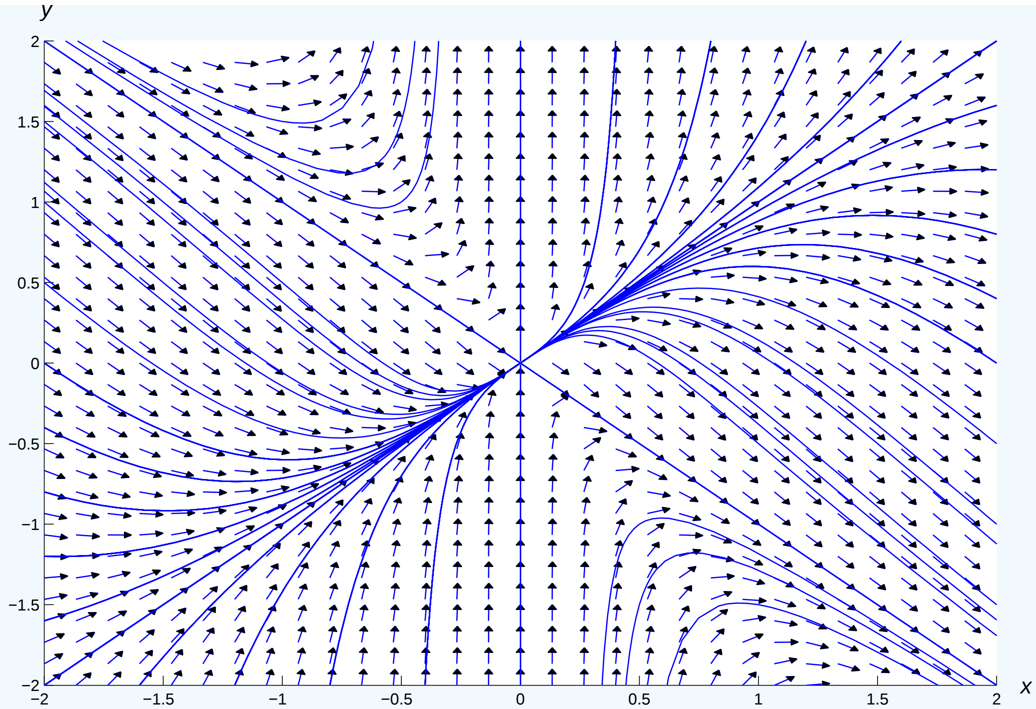


Figure 1.12.3: A direction field and integral curves for $x^2y' = y^2 + xy - x^2$. A set of arrows points in the diagonal direction from the point (-2,2) to (2,0) forming a diagonal line. Another set of arrows points straight up forming a vertical line. Other arrows curve around these two lines. (CC BY-NC-SA 3.0; William F. Trench via ELEMENTARY DIFFERENTIAL EQUATIONS).

Solution b

We could obtain c by imposing the initial condition $y(1) = 2$ in Equation (1.12.13), and then solving for c. However, it is easier to use Equation (1.12.12). Since $u = y/x$, the initial condition $y(1) = 2$ implies that $u(1) = 2$. Substituting this into Equation (1.12.12) yields $c = 1/3$. Hence, the solution of Equation (1.12.10) is

$$y = \frac{x(1 + x^2/3)}{1 - x^2/3}.$$

The interval of validity of this solution is $(-\sqrt{3}, \sqrt{3})$. However, the largest interval on which Equation (1.12.10) has a unique solution is $(0, \sqrt{3})$. To see this, note from Equation (1.12.14) that any function of the form

$$y = \begin{cases} \dfrac{x(1 + cx^2)}{1 - cx^2}, & a < x \le 0 \\ \dfrac{x(1 + x^2/3)}{1 - x^2/3} & 0 \le x < \sqrt{3} \end{cases} \tag{1.12.15}$$

is a solution of Equation (1.12.10) on $(a, \sqrt{3})$, where $a = -1/\sqrt{c}$ if $c > 0$ or $a = -\infty$ if $c \le 0$. Why does this not contradict Theorem 2.3.1?

Figure $1.12.4$ shows several solutions of the initial value problem Equation (1.12.10). Note that these solutions coincide on $(0, \sqrt{3})$.

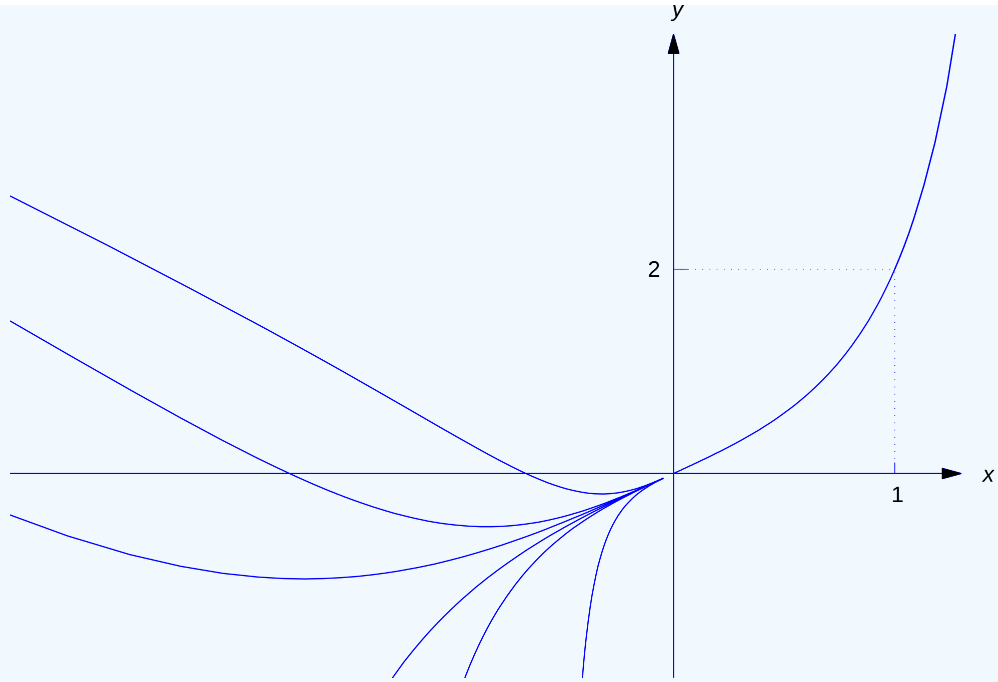


Figure 1.12.4: Solutions of $x^2y' = y^2 + xy - x^2$, $y(1) = 2$. One curve on the right intersects the point (1,2) and the origin. Three curves on the left are decreasing and concaves up. Another three curves on the left below them are increasing and concave down.(CC BY-NC-SA 3.0; William F. Trench via ELEMENTARY DIFFERENTIAL EQUATIONS).

In the last two examples we were able to solve the given equations explicitly. However, this is not always possible, as you'll see in the exercises.

1.12E: Exercises for Section 1.12

Exercises 1-4

Solve the given Bernoulli equation.

Exercise $1.12E.1$

$y' + y = y^2$

Exercise $1.12E.2$

$7xy' - 2y = -\frac{x^2}{y^6}$

Answer

$$y = x^{2/7}(C - \ln|x|)^{1/7}$$

Exercise $1.12E.3$

$x^2y' + 2y = 2e^{1/x}y^{1/2}$

Exercise $1.12E.4$

$(1+x^2)y' + 2xy = \frac{1}{(1+x^2)y}$

Answer

$$y = \frac{\pm\sqrt{2x+C}}{1+x^2}$$

Exercises 5-6

Find all solutions. Also, plot a direction field and some integral curves on the indicated rectangular region.

Exercise $1.12E.5$

$y' - xy = x^3y^3; \quad \{-3 \le x \le 3, -2 \le y \le 2\}$

Exercise $1.12E.6$

$y' - \frac{1+x}{3x}y = y^4; \quad \{-2 \le x \le 2, -2 \le y \le 2\}$

Answer

$$y = \left[\frac{x}{3(1-x)+Ce^{-x}}\right]^{1/3}$$

Exercises 7-11

Solve the initial value problem.

Exercise $1.12E.7$

$y' - 2y = xy^3, \quad y(0) = 2\sqrt{2}$

Exercise 1.12E.8

$y' - xy = xy^{3/2}, \quad y(1) = 4$

Answer

$y = \left(1 - \frac{3}{2} e^{-\frac{x^2-1}{4}}\right)^{-2}$

Exercise 1.12E.9

$xy' + y = x^4 y^4, \quad y(1) = 1/2$

Exercise 1.12E.10

$y' - 2y = 2y^{1/2}, \quad y(0) = 1$

Answer

$y = (2e^x - 1)^2$

Exercise 1.12E.11

$y' - 4y = \frac{48x}{y^2}, \quad y(0) = 1$

Exercises 12-**13**

Solve the initial value problem and graph the solution.

Exercise 1.12E.12

$x^2 y' + 2xy = y^3, \quad y(1) = 1/\sqrt{2}$

Answer

$y = \sqrt[2]{\frac{5x}{2(1+4x^2)}}$

Exercise 1.12E.13

$y' - y = xy^{1/2}, \quad y(0) = 4$

Exercise 14

Exercise 1.12E.14

You may have noticed that the logistic equation

$$P' = aP(1 - \alpha P)$$

from Verhulst's model for population growth can be written in Bernoulli form as

$$P' - aP = -a\alpha P^2.$$

This isn't particularly interesting, since the logistic equation is separable, and therefore solvable by the method studied in Section 2.2. So let's consider a more complicated model, where a is a positive constant and α is a positive continuous function of t on $[0, \infty)$. The equation for this model is

$$P' - aP = -a\alpha(t)P^2,$$

a non-separable Bernoulli equation.

a. Assuming that $P(0) = P_0 > 0$, find P for $t > 0$.
b. Verify that your result reduces to the known results for the Malthusian model where $\alpha = 0$, and the Verhulst model where α is a nonzero constant.
c. Assuming that

$$\lim_{t \to \infty} e^{-at} \int_0^t \alpha(\tau) e^{a\tau}\, d\tau = L$$

exists (finite or infinite), find $\lim_{t \to \infty} P(t)$.

Answer

$$\lim_{t \to \infty} P(t) = \begin{cases} \infty, & \text{if } L = 0, \\ 0, & \text{if } L = \infty, \\ \frac{1}{aL}, & \text{if } 0 < L < \infty. \end{cases}$$

Exercises 15-18

Solve the equation explicitly.

Exercise 1.12E.15

$y' = \frac{y+x}{x}$

Exercise 1.12E.16

$y' = \frac{y^2+2xy}{x^2}$

Answer

$y = \frac{cx^2}{1-cx}$

Exercise 1.12E.17

$xy^3y' = y^4 + x^4$

Exercise 1.12E.18

$y' = \frac{y}{x} + \sec\frac{y}{x}$

Answer

$y = x \arcsin(\ln|x| + c)$

Exercises 19-21

Solve the equation explicitly. Also, plot a direction field and some integral curves on the indicated rectangular region.

Exercise 1.12E.19

$x^2y' = xy + x^2 + y^2; \quad \{-8 \le x \le 8, -8 \le y \le 8\}$

Exercise $1.12E.20$

$xyy' = x^2 + 2y^2; \quad \{-4 \leq x \leq 4, -4 \leq y \leq 4\}$

Answer

$y = \pm x\sqrt{(}cx^2 - 1)$

Exercise $1.12E.21$

$y' = \frac{2y^2 + x^2 e^{-(y/x)^2}}{2xy}; \quad \{-8 \leq x \leq 8, -8 \leq y \leq 8\}$

Exercises 22-27

Solve the initial value problem.

Exercise $1.12E.22$

$y' = \frac{xy + y^2}{x^2}, \quad y(-1) = 2$

Answer

$y = -\frac{2x}{2\ln|x| + 1}$

Exercise $1.12E.23$

$y' = \frac{x^3 + y^3}{xy^2}, \quad y(1) = 3$

Exercise $1.12E.24$

$xyy' + x^2 + y^2 = 0, \quad y(1) = 2$

Answer

$y = \frac{1}{x}\sqrt{\frac{9 - x^4}{2}}$

Exercise $1.12E.25$

$y' = \frac{y^2 - 3xy - 5x^2}{x^2}, \quad y(1) = -1$

Exercise $1.12E.26$

$x^2 y' = 2x^2 + y^2 + 4xy, \quad y(1) = 1$

Answer

$y = \frac{-x(4x - 3)}{2x - 3}$

Exercise $1.12E.27$

$xyy' = 3x^2 + 4y^2, \quad y(1) = \sqrt{3}$

Exercises 28-34

Solve the given homogeneous equation implicitly.

Exercise 1.12E.28

$y' = \frac{x+y}{x-y}$

Answer

$y = \arctan \frac{y}{x} - \frac{1}{2} ln(x^2 + y^2) = c$

Exercise 1.12E.29

$(y'x - y)(\ln|y| - \ln|x|) = x$

Exercise 1.12E.30

$y' = \frac{y^3 + 2xy^2 + x^2y + x^3}{x(y+x)^2}$

Answer

$(y+3)^3 = 3x^3(\ln|x| + c)$

Exercise 1.12E.31

$y' = \frac{x+2y}{2x+y}$

Exercise 1.12E.32

$y' = \frac{y}{y-2x}$

Answer

$y^2(y-3x) = c$

Exercise 1.12E.33

$y' = \frac{xy^2 + 2y^3}{x^3 + x^2y + xy^2}$

Exercise 1.12E.34

$y' = \frac{x^3 + x^2y + 3y^3}{x^3 + 3xy^2}$

Answer

$\frac{y}{x} + \frac{y^3}{x^3} = \ln|x| + c$

Exercise 1.12E.35

a. Find a solution of the initial value problem

$$x^2y' = y^2 + xy - 4x^2, \quad y(-1) = 0 \tag{A}$$

on the interval $(-\infty, 0)$. Verify that this solution is actually valid on $(-\infty, \infty)$.

b. Use Theorem 2.3.1 to show that (A) has a unique solution on $(-\infty, 0)$.
c. Plot a direction field for the differential equation in (A) on a square

$$\{-r \leq x \leq r, -r \leq y \leq r\},$$

where r is any positive number. Graph the solution you obtained in (a) on this field.
d. Graph other solutions of (A) that are defined on $(-\infty, \infty)$.
e. Graph other solutions of (A) that are defined only on intervals of the form $(-\infty, a)$, where is a finite positive number.

Exercise $1.12E.36$

a. Solve the equation

$$xyy' = x^2 - xy + y^2 \tag{A}$$

implicitly.
b. Plot a direction field for (A) on a square

$$\{0 \leq x \leq r, 0 \leq y \leq r\}$$

where r is any positive number.
c. Let K be a positive integer. (You may have to try several choices for K.) Graph solutions of the initial value problems

$$xyy' = x^2 - xy + y^2, \quad y(r/2) = \frac{kr}{K},$$

for $k = 1$, 2, …, K. Based on your observations, find conditions on the positive numbers x_0 and y_0 such that the initial value problem

$$xyy' = x^2 - xy + y^2, \quad y(x_0) = y_0, \tag{B}$$

has a unique solution (i) on $(0, \infty)$ or (ii) only on an interval (a, ∞), where $a > 0$?
d. What can you say about the graph of the solution of (B) as $x \to \infty$? (Again, assume that $x_0 > 0$ and $y_0 > 0$.)

Answer

$e^{\frac{y}{x}}(y - x) = c$

Exercise $1.12E.37$

a. Solve the equation

$$y' = \frac{2y^2 - xy + 2x^2}{xy + 2x^2} \tag{A}$$

implicitly.
b. Plot a direction field for (A) on a square

$$\{-r \leq x \leq r, -r \leq y \leq r\}$$

where r is any positive number. By graphing solutions of (A), determine necessary and sufficient conditions on (x_0, y_0) such that (A) has a solution on (i) $(-\infty, 0)$ or (ii) $(0, \infty)$ such that $y(x_0) = y_0$.

Exercise $1.12E.38$

Follow the instructions of **Exercise 1.12E.37** for the equation

$$y' = \frac{xy + x^2 + y^2}{xy}.$$

Answer

$$y - x \ln|y - x| = cy$$

Exercise 1.12E.39

Pick any nonlinear homogeneous equation $y' = q(y/x)$ you like, and plot direction fields on the square $\{-r \le x \le r, \; -r \le y \le r\}$, where $r > 0$. What happens to the direction field as you vary r? Why?

Exercise 1.12E.40

Prove the statement:

If $ad - bc \ne 0$, then the equation

$$y' = \frac{ax + by + \alpha}{cx + dy + \beta}$$

can be transformed into the homogeneous nonlinear equation

$$\frac{dY}{dX} = \frac{aX + bY}{cX + dY}$$

by the substitution $x = X - X_0, \; y = Y - Y_0$, where X_0 and Y_0 are suitably chosen constants.

Exercises 41-43

Use a method suggested by Exercise 1.12E.40 to solve the given equation implicitly.

Exercise 1.12E.41

$y' = \frac{-6x + y - 3}{2x - y - 1}$

Exercise 1.12E.42

$y' = \frac{2x + y + 1}{x + 2y - 4}$

Answer

$$(y + x - 1)(y - x - 5)^3 = c$$

Exercise 1.12E.43

$y' = \frac{-x + 3y - 14}{x + y - 2}$

Exercises 44-51

Find a function y_1 such that the substitution $y = uy_1$ transforms the given equation into a separable equation of the form Equation 1.12.6. Then solve the given equation explicitly.

Exercise 1.12E.44

$3xy^2y' = y^3 + x$

Answer

$$y = x^{\frac{1}{3}} \sqrt[3]{(\ln|x| + c)}$$

Exercise $1.12E.45$

$xyy' = 3x^6 + 6y^2$

Exercise $1.12E.46$

$x^3y' = 2(y^2 + x^2y - x^4)$

Answer

$y = \frac{x^2(1+cx^4)}{1-cx^4}$

Exercise $1.12E.47$

$y' = y^2e^{-x} + 4y + 2e^x$

Exercise $1.12E.48$

$y' = \frac{y^2 + y\tan x + \tan^2 x}{\sin^2 x}$

Answer

$y = tan(x)tan(\ln|x| + c)$

Exercise $1.12E.49$

$x(\ln x)^2y' = -4(\ln x)^2 + y\ln x + y^2$

Exercise $1.12E.50$

$2x(y + 2\sqrt{x})y' = (y + \sqrt{x})^2$

Answer

$y = x^{\frac{1}{2}}(-2 \pm \sqrt{(}\ln|x| + c)$

Exercise $1.12E.51$

$(y + e^{x^2})y' = 2x(y^2 + ye^{x^2} + e^{2x^2}$

Exercises 52-55

Solve each problem as directed.

Exercise $1.12E.52$

Solve the initial value problem

$$y' + \frac{2}{x}y = \frac{3x^2y^2 + 6xy + 2}{x^2(2xy + 3)}, \quad y(2) = 2.$$

Answer

$y = \frac{-3 + \sqrt{(}1 + 60x)}{2x}$

Exercise $1.12E.53$

Solve the initial value problem

$$y' + \frac{3}{x}y = \frac{3x^4y^2 + 10x^2y + 6}{x^3(2x^2y + 5)}, \quad y(1) = 1.$$

Exercise $1.12E.54$

A **generalized Riccati equation** is of the form

$$y' = P(x) + Q(x)y + R(x)y^2. \tag{A}$$

If $R \equiv -1$ (R is equivalent to 1), (A) is a **Riccati equation**. Let y_1 be a known solution and y an arbitrary solution of (A). Let $z = y - y_1$. Show that z is a solution of a Bernoulli equation with $n = 2$.

Exercises 55-58

Given that y_1 is a solution of the given equation, use the method suggested by Exercise 1.12E.55 to find other solutions.

Exercise $1.12E.55$

$y' = 1 + x - (1 + 2x)y + xy^2$; $y_1 = 1$

Answer

$y = 1 + \frac{1}{x+1+ce^x}$

Exercise $1.12E.56$

$y' = e^{2x} + (1 - 2e^x)y + y^2$; $y_1 = e^x$

Exercise $1.12E.57$

$xy' = 2 - x + (2x - 2)y - xy^2$; $y_1 = 1$

Answer

$y = 1 - \frac{1}{x(1-cx)}$

Exercise $1.12E.58$

$xy' = x^3 + (1 - 2x^2)y + xy^2$; $y_1 = x$

CHAPTER OVERVIEW

2: Applications of First Order Equations

In this chapter, we consider applications of first order differential equations.

2.1: Growth and Decay

Learning Objectives

- Model and solve problems involving exponential growth and decay.
- Apply differential equations to mixed growth and decay scenarios.

This section begins with a discussion of exponential growth and decay, which you have probably already seen in calculus. We consider applications to radioactive decay, carbon dating, and compound interest. We also consider more complicated problems where the rate of change of a quantity is in part proportional to the magnitude of the quantity, but is also influenced by other other factors for example, a radioactive substance is manufactured at a certain rate, but decays at a rate proportional to its mass, or a saver makes regular deposits in a savings account that draws compound interest.

Since the applications in this section deal with functions of time, we'll denote the independent variable by t. If Q is a function of t, Q' will denote the derivative of Q with respect to t; thus,

$$Q' = \frac{dQ}{dt}.$$

Exponential Growth and Decay

One of the most common mathematical models for a physical process is the *exponential model*, where it is assumed that the rate of change of a quantity Q is proportional to Q; thus

$$Q' = aQ, \tag{2.1.1}$$

where a is the constant of proportionality.

The general solution of Equation (2.1.1) is

$$Q = ce^{at}$$

and the solution of the initial value problem

$$Q' = aQ, \quad Q(t_0) = Q_0$$

is

$$Q = Q_0 e^{a(t-t_0)}. \tag{2.1.2}$$

Since the solutions of $Q' = aQ$ are exponential functions, we say that a quantity Q that satisfies this equation *grows exponentially* if $a > 0$, or *decays exponentially* if $a < 0$ (Figure 2.1.1).

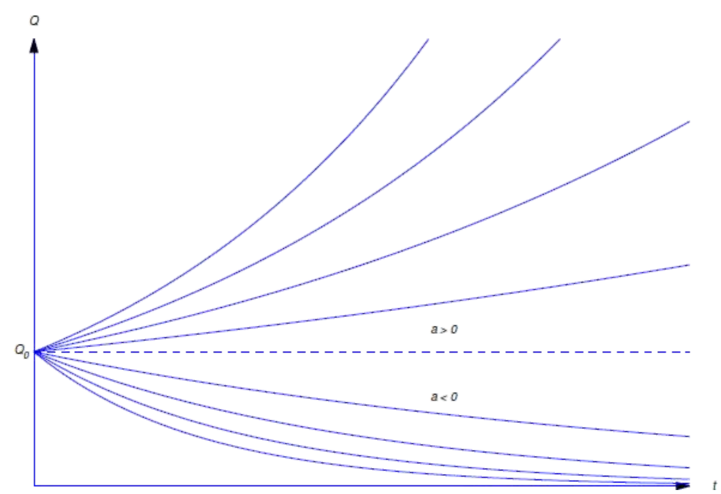


Figure 2.1.1 : Graphs of 8 exponential curves. Four graphs with a > 0 are increasing curve above the dotted line at Q_0. The other four graphs with a < 0 are increasing curve below the dotted line at Q_0. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation).

Example 2.1.1

Consider a bacteria population of weight 20 g. If the population doubles every 20 minutes, then what is the population after 30 minutes?

Note: It is easier to weigh this population than to count it.

Solution

One looks at the given information before trying to answer the question. First, we have the initial condition $P_0 = 20$ g. Since the population doubles every 20 minutes, then $P(20) = 2P_0 = 40$. Here we have take the time units as minutes. We are then asked to find $P(30)$.

We do not need to solve the differential equation. We will assume a simple growth model. Using the general solution, $P(t) = 20e^{kt}$, we have

$$P(20) = 20e^{20k} = 40$$

or

$$e^{20k} = 2$$

We can solve this for k,

$$20k = \ln 2, \quad \Rightarrow k = \frac{\ln 2}{20} \approx 0.035$$

This gives an approximate solution, $P(t) \approx 20e^{.035t}$. Now we can answer the original question. Namely, $P(30) \approx 57$.

Of course, we could get an exact solution. With some simple manipulations, we have

$$\begin{aligned} P(t) &= 20e^{kt} \\ &= 20e^{\left(\frac{\ln 2}{20}\right)t} \\ &= 20\left(e^{\ln 2}\right)^{\frac{t}{20}} \\ &= 20\left(2^{\frac{t}{20}}\right) \end{aligned}$$

This answer takes the general form for population doubling, $P(t) = P_0 2^{\frac{t}{\tau}}$, where τ is the doubling rate.

Radioactive Decay

Experimental evidence shows that radioactive material decays at a rate proportional to the mass of the material present. According to this model the mass $Q(t)$ of a radioactive material present at time t satisfies Equation (2.1.1), where a is a negative constant whose value for any given material must be determined by experimental observation. For simplicity, we'll replace the negative constant a by $-k$, where k is a positive number that we'll call the *decay constant* of the material. Thus, Equation (2.1.1) becomes

$$Q' = -kQ.$$

If the mass of the material present at $t = t_0$ is Q_0, the mass present at time t is the solution of

$$Q' = -kQ, \quad Q(t_0) = Q_0.$$

From Equation (2.1.2) with $a = -k$, the solution of this initial value problem is

$$Q = Q_0 e^{-k(t-t_0)}. \tag{2.1.3}$$

The *half–life* τ of a radioactive material is defined to be the time required for half of its mass to decay; that is, if $Q(t_0) = Q_0$, then

$$Q(\tau + t_0) = \frac{Q_0}{2}. \tag{2.1.4}$$

From Equation (2.1.3) with $t = \tau + t_0$, Equation (2.1.4) is equivalent to

$$Q_0e^{-k\tau} = \frac{Q_0}{2},$$

so

$$e^{-k\tau} = \frac{1}{2}.$$

Taking logarithms yields

$$-k\tau = \ln\frac{1}{2} = -\ln 2,$$

so the half-life is

$$\tau = \frac{1}{k}\ln 2. \tag{2.1.5}$$

(see Figure 2.1.2). The half-life is independent of t_0 and Q_0, since it is determined by the properties of material, not by the amount of the material present at any particular time.

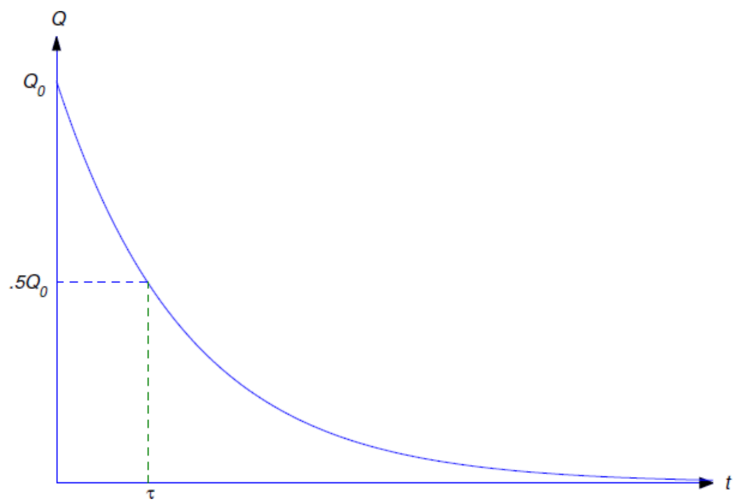


Figure 2.1.2: Graph illustrating the half-life decay of a radioactive substance over time, showing the exponential decrease in quantity (Q) relative to its initial quantity (Q_0). The half-life τ is marked at the point where the substance has decayed to half of its initial amount. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation).

Example 2.1.2

A radioactive substance has a half-life of 1620 years.

a. If its mass is now 4 g (grams), how much will be left 810 years from now?
b. Find the time t_1 when 1.5 g of the substance remain.

Solution a

From Equation (2.1.3) with $t_0 = 0$ and $Q_0 = 4$,

$$Q = 4e^{-kt}, \tag{2.1.6}$$

where we determine k from Equation (2.1.5), with τ= 1620 years:

$$k = \frac{\ln 2}{\tau} = \frac{\ln 2}{1620}.$$

Substituting this in Equation (2.1.6) yields

$$Q = 4e^{-(t\ln 2)/1620}. \tag{2.1.7}$$

Therefore the mass left after 810 years will be

$$Q(810) = 4e^{-(810\ln 2)/1620} = 4e^{-(\ln 2)/2} = 2\sqrt{2}\text{ g}.$$

Solution b

Setting $t = t_1$ in Equation (2.1.7) and requiring that $Q(t_1) = 1.5$ yields

$$\frac{3}{2} = 4e^{(-t_1 \ln 2)/1620}.$$

Dividing by 4 and taking logarithms yields

$$\ln\frac{3}{8} = -\frac{t_1 \ln 2}{1620}.$$

Since $\ln(3/8) = -\ln(8/3)$,

$$t_1 = 1620\frac{\ln(8/3)}{\ln 2} \approx 2292.4 \text{ years}.$$

Interest Compounded Continuously

Suppose we deposit an amount of money Q_0 in an interest-bearing account and make no further deposits or withdrawals for t years, during which the account bears interest at a constant annual rate r. To calculate the value of the account at the end of t years, we need one more piece of information: how the interest is added to the account, or—as the bankers say—how it is *compounded*. If the interest is compounded annually, the value of the account is multiplied by $1+r$ at the end of each year. This means that after t years the value of the account is

$$Q(t) = Q_0(1+r)^t.$$

If interest is compounded semiannually, the value of the account is multiplied by $(1+r/2)$ every 6 months. Since this occurs twice annually, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{2}\right)^{2t}.$$

In general, if interest is compounded n times per year, the value of the account is multiplied n times per year by $(1+r/n)$; therefore, the value of the account after t years is

$$Q(t) = Q_0\left(1+\frac{r}{n}\right)^{nt}. \tag{2.1.8}$$

Thus, increasing the frequency of compounding increases the value of the account after a fixed period of time. Table 2.1.1 shows the effect of increasing the number of compoundings over $t = 5$ years on an initial deposit of $Q_0 = 100$ (dollars), at an annual interest rate of 6%.

Table 2.1.1 : The effect of compound interest

n (number of compoundings per year)	$\$100\left(1+\frac{.06}{n}\right)^{5n}$ (value in dollars after 5 years)
1	\$133.82
2	\$134.39
4	\$134.68
8	\$134.83
364	\$134.98

You can see from Table 2.1.1 that the value of the account after 5 years is an increasing function of n. Now suppose the maximum allowable rate of interest on savings accounts is restricted by law, but the time intervals between successive compoundings isn't; then competing banks can attract savers by compounding often. The ultimate step in this direction is to *compound continuously*, by which we mean that $n \to \infty$ in Equation (2.1.8). Since we know from calculus that

$$\lim_{n\to\infty}\left(1+\frac{r}{n}\right)^n = e^r,$$

this yields

$$\begin{aligned} Q(t) &= \lim_{n\to\infty} Q_0\left(1+\tfrac{r}{n}\right)^{nt} = Q_0\left[\lim_{n\to\infty}\left(1+\tfrac{r}{n}\right)^n\right]^t \\ &= Q_0e^{rt}. \end{aligned}$$

Observe that $Q = Q_0e^{rt}$ is the solution of the initial value problem

$$Q' = rQ, \quad Q(0) = Q_0;$$

that is, with continuous compounding the value of the account grows exponentially.

Example 2.1.3

We wish to accumulate \$10,000 in 10 years by making a single deposit in a savings account bearing $5\frac{1}{2}\%$ annual interest compounded continuously. How much must we deposit in the account?

Solution

The value of the account at time t is

$$Q(t) = Q_0e^{.055t}. \tag{2.1.9}$$

Since we want $Q(10)$ to be \$10,000, the initial deposit Q_0 must satisfy the equation

$$10000 = Q_0e^{.55}, \tag{2.1.10}$$

obtained by setting $t = 10$ and $Q(10) = 10000$ in Equation (2.1.9). Solving Equation (2.1.10) for Q_0 yields

$$Q_0 = 10000e^{-.55} \approx \$5769.50.$$

Note

If \$150 is deposited in a bank that pays $5\frac{1}{2}\%$ annual interest compounded continuously, the value of the account after t years is

$$Q(t) = 150e^{.055t}$$

dollars. (Note that it is necessary to write the interest rate as a decimal; thus, $r = .055$.) Therefore, after $t = 10$ years the value of the account is

$$Q(10) = 150e^{.55} \approx \$259.99.$$

Mixed Growth and Decay

Example 2.1.4

A radioactive substance with decay constant k is produced at a constant rate of a units of mass per unit time.

a. Assuming that $Q(0) = Q_0$, find the mass $Q(t)$ of the substance present at time t.
b. Find $\lim_{t\to\infty} Q(t)$.

Solution a

Here

$$Q' = \text{ rate of increase of } Q - \text{ rate of decrease of } Q.$$

The rate of increase is the constant a. Since Q is radioactive with decay constant k, the rate of decrease is kQ. Therefore

$$Q' = a - kQ.$$

This is a linear first order differential equation. Rewriting it and imposing the initial condition shows that Q is the solution of the initial value problem

$$Q' + kQ = a, \quad Q(0) = Q_0. \tag{2.1.11}$$

Since e^{-kt} is a solution of the complementary equation, the solutions of Equation (2.1.11) are of the form $Q = ue^{-kt}$, where $u'e^{-kt} = a$, so $u' = ae^{kt}$. Hence,

$$u = \frac{a}{k}e^{kt} + c$$

and

$$Q = ue^{-kt} = \frac{a}{k} + ce^{-kt}.$$

Since $Q(0) = Q_0$, setting $t = 0$ here yields

$$Q_0 = \frac{a}{k} + c \quad \text{or} \quad c = Q_0 - \frac{a}{k}.$$

Therefore

$$Q = \frac{a}{k} + \left(Q_0 - \frac{a}{k}\right)e^{-kt}. \tag{2.1.12}$$

b. Since $k > 0$, $\lim_{t\to\infty} e^{-kt} = 0$, so from Equation (2.1.12)

$$\lim_{t\to\infty} Q(t) = \frac{a}{k}.$$

This limit depends only on a and k, and not on Q_0. We say that a/k is the *steady state* value of Q. From Equation 2.1.12 we also see that Q approaches its steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. If $Q_0 = a/k$, then Q remains constant (see Figure 2.1.3).

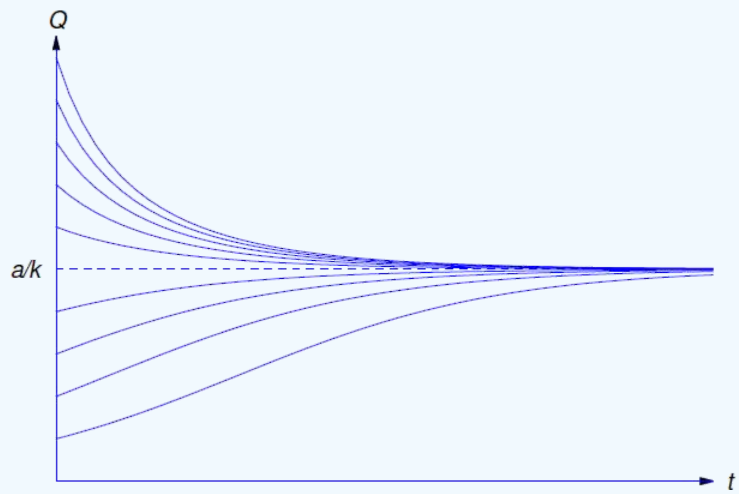


Figure 2.1.3 : Several graphs of $Q(t)$ approaches the steady state value from above if $Q_0 > a/k$, or from below if $Q_0 < a/k$. The steady state value is $\frac{a}{k}$ as $t \to \infty$ (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation).

Carbon Dating

The fact that Q approaches a steady state value in the situation discussed in Example 4 underlies the method of *carbon dating*, devised by the American chemist and Nobel Prize Winner *W.S. Libby*.

Carbon 12 is stable, but carbon-14, which is produced by cosmic bombardment of nitrogen in the upper atmosphere, is radioactive with a half-life of about 5570 years. Libby assumed that the quantity of carbon-12 in the atmosphere has been constant throughout time, and that the quantity of radioactive carbon-14 achieved its steady state value long ago as a result of its creation and decomposition over millions of years. These assumptions led Libby to conclude that the ratio of carbon-14 to carbon-12 has been nearly constant for a long time. This constant, which we denote by R, has been determined experimentally.

Living cells absorb both carbon-12 and carbon-14 in the proportion in which they are present in the environment. Therefore the ratio of carbon-14 to carbon-12 in a living cell is always R. However, when the cell dies it ceases to absorb carbon, and the ratio of carbon-14 to carbon-12 decreases exponentially as the radioactive carbon-14 decays. This is the basis for the method of carbon dating, as illustrated in the next example.

Example 2.1.5

An archaeologist investigating the site of an ancient village finds a burial ground where the amount of carbon-14 present in individual remains is between 42 and 44% of the amount present in live individuals. Estimate the age of the village and the length of time for which it survived.

Solution

Let $Q = Q(t)$ be the quantity of carbon-14 in an individual set of remains t years after death, and let Q_0 be the quantity that would be present in live individuals. Since carbon-14 decays exponentially with half-life 5570 years, its decay constant is

$$k = \frac{\ln 2}{5570}.$$

Therefore

$$Q = Q_0 e^{-t(\ln 2)/5570}$$

if we choose our time scale so that $t_0 = 0$ is the time of death. If we know the present value of Q we can solve this equation for t, the number of years since death occurred. This yields

$$t = -5570 \frac{\ln(Q/Q_0)}{\ln 2}.$$

It is given that $Q = .42Q_0$ in the remains of individuals who died first. Therefore these deaths occurred about

$$t_1 = -5570 \frac{\ln .42}{\ln 2} \approx 6971$$

years ago. For the most recent deaths, $Q = .44Q_0$; hence, these deaths occurred about

$$t_2 = -5570 \frac{\ln .44}{\ln 2} \approx 6597$$

years ago. Therefore it is reasonable to conclude that the village was founded about 7000 years ago, and lasted for about 400 years.

A Savings Program

Example 2.1.6

A person opens a savings account with an initial deposit of $1000 and subsequently deposits $50 per week. Find the value $Q(t)$ of the account at time $t > 0$, assuming that the bank pays 6% interest compounded continuously.

Solution

Observe that Q isn't continuous, since there are 52 discrete deposits per year of $50 each. To construct a mathematical model for this problem in the form of a differential equation, we make the simplifying assumption that the deposits are made continuously at a rate of $2600 per year. This is essential, since solutions of differential equations are continuous functions. With this assumption, Q increases continuously at the rate

$$Q' = 2600 + 0.06Q$$

and therefore Q satisfies the differential equation

$$Q' - .06Q = 2600. \tag{2.1.13}$$

(Of course, we must recognize that the solution of this equation is an approximation to the true value of Q at any given time. We'll discuss this further below.) Since $e^{.06t}$ is a solution of the complementary equation, the solutions of Equation (2.1.13) are of the form $Q = ue^{.06t}$, where $u'e^{.06t} = 2600$. Hence, $u' = 2600e^{-.06t}$,

$$u = -\frac{2600}{.06}e^{-0.06t} + c$$

and

$$Q = ue^{.06t} = -\frac{2600}{.06} + ce^{.06t}. \tag{2.1.14}$$

Setting $t = 0$ and $Q = 1000$ here yields

$$c = 1000 + \frac{2600}{0.06},$$

and substituting this into Equation (2.1.14) yields

$$Q = 1000e^{.06t} + \frac{2600}{.06}(e^{.06t} - 1) \tag{2.1.15}$$

where the first term is the value due to the initial deposit and the second is due to the subsequent weekly deposits.

Mathematical models must be tested for validity by comparing predictions based on them with the actual outcome of experiments. Example 2.1.6 is unusual in that we can compute the exact value of the account at any specified time and compare it with the approximate value predicted by Equation (2.1.15). Table 2.1.2 gives a comparison for a ten year period. Each exact answer corresponds to the time of the year-end deposit, and each year is assumed to have exactly 52 weeks.

Table 2.1.2 : Comparision for a ten year period

Year	Approximate Value of Q (Example 2.1.6)	Exact Value of P	Error $Q - P$	Percentage Error $(Q - P)/P$
1	\$3741.42	\$3739.87	\$1.55	.0413
2	6652.36	6649.17	3.19	.0479
3	9743.30	9738.37	4.93	.0506
4	13,025.38	13,018.60	6.78	.0521
5	16,510.41	16,501.66	8.75	.0530
6	20,210.94	20,200.11	10.83	.0536
7	24,140.30	24,127.25	13.05	.0541
8	28,312.63	28,297.23	15.40	.0544
9	32,742.97	32,725.07	17.90	.0547
10	37,447.27	37,426.72	20.55	.0549

2.1E: Exercises for Section 2.1

Exercises 1-27

Solve the following exercises involving growth, decay, financial, and production models using differential equations. Include the required solutions, whether exact, approximate, or qualitative analyses, as stated in the exercise.

Exercise $2.1E.1$

The half-life of a radioactive substance is 3200 years. Find the quantity $Q(t)$ of the substance left at time $t>0$ if $Q(0)=20$ g.

Exercise $2.1E.2$

The half-life of a radioactive substance is 2 days. Find the time required for a given amount of the material to decay to 1/10 of its original mass.

Answer

$k\tau=\ln 2$ and $\tau=2 \Rightarrow k=\frac{\ln 2}{2}$; $Q(t)=Q_0e^{-t\ln 2/2}$; if $Q(T)=\frac{Q_0}{10}$, then $\frac{Q_0}{10}=Q_0e^{-T\ln 2/2}$; $\ln 10=\frac{T\ln 2}{2}$; $T=\frac{2\ln 10}{\ln 2}$ days.

Exercise $2.1E.3$

A radioactive material loses 25% of its mass in 10 minutes. What is its half-life?

Exercise $2.1E.4$

A tree contains a known percentage p_0 of a radioactive substance with half-life τ. When the tree dies the substance decays and isn't replaced. If the percentage of the substance in the fossilized remains of such a tree is found to be p_1, how long has the tree been dead?

Answer

Let t_1 be the elapsed time since the tree died. Since $p(t)=e^{-(t\ln 2)/\tau}$, it follows that $p_1=p_0e^{-(t_1\ln 2)/\tau}$, so $\ln\left(\frac{p_1}{p_0}\right)=-\frac{t_1}{\tau}\ln 2$ and $t_1=\tau\frac{\ln(p_0/p_1)}{\ln 2}$.

Exercise $2.1E.5$

If t_p and t_q are the times required for a radioactive material to decay to $1/p$ and $1/q$ times its original mass (respectively), how are t_p and t_q related?

Exercise $2.1E.6$

Find the decay constant k for a radioactive substance, given that the mass of the substance is Q_1 at time t_1 and Q_2 at time t_2.

Exercise $2.1E.7$

A process creates a radioactive substance at the rate of 2 g/hr and the substance decays at a rate proportional to its mass, with constant of proportionality $k=0.1$. If $Q(t)$ is the mass of the substance at time t, find $\lim_{t\to\infty} Q(t)$.

Exercise $2.1E.8$

A bank pays interest continuously at the rate of 6%. How long does it take for a deposit of Q_0 to grow in value to $2Q_0$?

Answer

$Q' = 0.06Q, \quad Q(0) = Q_0; \quad Q = Q_0 e^{0.06t}$. We must find τ such that $Q(\tau) = 2Q_0$; that is, $Q_0 e^{0.06\tau} = 2Q_0$, so $0.06\tau = \ln 2$ and $\tau = \frac{\ln 2}{0.06} = \frac{50 \ln 2}{3}$ yr.

Exercise $2.1E.9$

At what rate of interest, compounded continuously, will a bank deposit double in value in 8 years?

Exercise $2.1E.10$

A savings account pays 5% per annum interest compounded continuously. The initial deposit is Q_0 dollars. Assume that there are no subsequent withdrawals or deposits.

a. How long will it take for the value of the account to triple?
b. What is Q_0 if the value of the account after 10 years is $100,000 dollars?

Answer

a. If T is the time to triple the value, then $Q(T) = Q_0 e^{0.05T} = 3Q_0$, so $e^{0.05T} = 3$. Therefore, $0.05T = \ln 3 \quad \text{and} \quad T = 20 \ln 3$.
b. If $Q(10) = 100000$, then $Q_0 e^{0.5} = 100000$, so $Q_0 = 100000e^{-0.5}$.

Exercise $2.1E.11$

A candymaker makes 500 pounds of candy per week, while his large family eats the candy at a rate equal to $Q(t)/10$ pounds per week, where $Q(t)$ is the amount of candy present at time t.

a. Find $Q(t)$ for $t > 0$ if the candymaker has 250 pounds of candy at $t = 0$.
b. Find $\lim_{t\to\infty} Q(t)$.

Exercise $2.1E.12$

Suppose a substance decays at a yearly rate equal to half the square of the mass of the substance present. If we start with 50 g of the substance, how long will it be until only 25 g remains?

Answer

$Q' = -\frac{Q^2}{2}, \quad Q(0) = 50; \quad \frac{Q'}{Q^2} = -\frac{1}{2}; \quad -\frac{1}{Q} = -\frac{t}{2} + c; \quad Q(0) = 50 \Rightarrow c = -\frac{1}{50}; \quad \frac{1}{Q} = \frac{t}{2} + \frac{1}{50}; \quad Q = \frac{50}{1+25t}$. Now, $Q(T) = 25 \Rightarrow 1 + 25T = 2 \Rightarrow 25T = 1 \Rightarrow T = \frac{1}{25}$ years.

Exercise $2.1E.13$

A super bread dough increases in volume at a rate proportional to the volume V present. If V increases by a factor of 10 in 2 hours and $V(0) = V_0$, find V at any time t. How long will it take for V to increase to $100V_0$?

Exercise $2.1E.14$

A radioactive substance decays at a rate proportional to the amount present, and half the original quantity Q_0 is left after 1500 years. In how many years would the original amount be reduced to $3Q_0/4$? How much will be left after 2000 years?

Answer

Since $\tau = 1500$, $k = \frac{\ln 2}{1500}$; hence $Q = Q_0 e^{-(t\ln 2)/1500}$. If $Q(t_1) = \frac{3Q_0}{4}$, then $e^{-(t_1 \ln 2)/1500} = \frac{3}{4}$; $-\frac{t_1 \ln 2}{1500} = \ln\left(\frac{3}{4}\right) = -\ln\left(\frac{4}{3}\right)$; $t_1 = 1500\frac{\ln(4/3)}{\ln 2}$. Finally, $Q(2000) = Q_0 e^{-\frac{4}{3}\ln 2} = 2^{-4/3} Q_0$.

Exercise $2.1E.15$

A wizard creates gold continuously at the rate of 1 ounce per hour, but an assistant steals it continuously at the rate of 5% of however much is there per hour. Let $W(t)$ be the number of ounces that the wizard has at time t. Find $W(t)$ and $\lim_{t\to\infty} W(t)$ if $W(0) = 1$.

Exercise $2.1E.16$

A process creates a radioactive substance at the rate of 1 g/hr, and the substance decays at an hourly rate equal to 1/10 of the mass present (expressed in grams). Assuming that there are initially 20 g, find the mass $S(t)$ of the substance present at time t, and find $\lim_{t\to\infty} S(t)$.

Exercise $2.1E.17$

A tank is empty at $t = 0$. Water is added to the tank at the rate of 10 gal/min, but it leaks out at a rate (in gallons per minute) equal to the number of gallons in the tank. What is the smallest capacity the tank can have if this process is to continue forever?

Exercise $2.1E.18$

A person deposits \$25,000 in a bank that pays 5% per year interest, compounded continuously. The person continuously withdraws from the account at the rate of \$750 per year. Find $V(t)$, the value of the account at time t after the initial deposit.

Answer

(A) $V' = -750 + \frac{V}{20}$, $V(0) = 25000$. Rewrite the differential equation in (A) as (B) $V' - \frac{V}{20} = -750$. Since $V_1 = e^{t/20}$ is a solution of the complementary equation, the solutions of (B) are given by $V = ue^{t/20}$, where $u'e^{t/20} = -750$. Therefore, $u' = -750e^{-t/20}$; $u = 15000e^{-t/20} + c$; $V = 15000 + ce^{t/20}$; $V(0) = 25000 \Rightarrow c = 10000$. Therefore, $V = 15000 + 10000e^{t/20}$.

Exercise $2.1E.19$

A person has a fortune that grows at the rate directly proportional to the square root of its worth. Find the worth W of the fortune as a function of t if it was \$1 million 6 months ago and is \$4 million today.

Exercise $2.1E.20$

Let $p = p(t)$ be the quantity of a product present at time t. The product is manufactured continuously at a rate proportional to p with the proportionality constant 1/2, and it is consumed continuously at a rate proportional to p^2, with proportionality constant 1/8. Find $p(t)$ if $p(0) = 100$.

Exercise $2.1E.21$

a. In the situation of **Example 2.1.6** find the exact value P(t) of the person's account after t years, where t is an integer. Assume that each year has exactly 52 weeks, and include the year-end deposit in the computation.

HINT: At time t the initial \$1000 has been on deposit for t years. There have been $52t$ deposits of \$50 each. The first \$50 has been on deposit for $t - 1/52$ years, the second for $t - 2/52$ years ... in general, the j^{th} \$50 has been on deposit for $t - j/52$ years ($1 \le j \le 52t$). Find the present value of each \$50 deposit assuming 6% interest compounded continuously, and use the formula

$$1+x+x^2+\ldots+x^n=\frac{1-x^{n+1}}{1-x}(x\neq 1)$$

to find their total value.

b. Let

$$p(t)=\frac{Q(t)-P(t)}{P(t)}$$

be the relative error after t years. Find

$$p(\infty)=\lim_{t\to\infty} p(t).$$

? Exercise $2.1E.22$

A homebuyer borrows P_0 dollars at an annual interest rate r, agreeing to repay the loan with equal monthly payments of M dollars per month over N years.

a. Derive a differential equation for the loan principal (amount that the homebuyer owes) $P(t)$ at time $t>0$, making the simplifying assumption that the homebuyer repays the loan continuously rather than in discrete steps. (See **Example 2.1.6**.)

b. Solve the equation derived in (a).

c. Use the result of (b) to determine an approximate value for M assuming that each year has exactly 12 months of equal length.

d. It can be shown that the exact value of M is given by

$$M=\frac{rP_0}{12}\left(1-(1+r/12)^{-12N}\right)^{-1}.$$

Compare the value of M obtained from the answer in (c) to the exact value if (i) $P_0=\$50,000$, $r=7.5\%$, $N=20$ (ii) $P_0=\$150,000$, $r=9.0\%$, $N=30$.

? Exercise $2.1E.23$

Assume that the homebuyer of **Exercise 2.1E.22** elects to repay the loan continuously at the rate of αM dollars per month, where α is a constant greater than 1. (This is called **accelerated payment**.)

a. Determine the time $T(\alpha)$ when the loan will be paid off and the amount $S(\alpha)$ that the homebuyer will save.
b. Suppose $P_0=\$50,000$, $r=8\%$, and $N=15$. Compute the savings realized by accelerated payments with $\alpha=1.05, 1.10$, and 1.15.

? Exercise $2.1E.24$

A benefactor wishes to establish a trust fund to pay a researcher's salary for T years. The salary is to start at S_0 dollars per year and increase at a fractional rate of a per year. Find the amount of money P_0 that the benefactor must deposit in a trust fund paying interest at a rate r per year. Assume that the researcher's salary is paid continuously, the interest is compounded continuously, and the salary increases are granted continuously.

? Exercise $2.1E.25$

A radioactive substance with decay constant k is produced at the rate of

$$\frac{at}{1+btQ(t)}$$

units of mass per unit time, where a and b are positive constants and $Q(t)$ is the mass of the substance present at time t; thus, the rate of production is small at the start and tends to slow when Q is large.

a. Set up a differential equation for Q.
b. Choose your own positive values for a, b, k, and $Q_0 = Q(0)$. Use a numerical method to discover what happens to $Q(t)$ as $t \to \infty$. (Be precise, expressing your conclusions in terms of a, b, k. However, no proof is required.)

? Exercise $2.1E.26$

Follow the instructions of **Exercise 2.1E.25**, assuming that the substance is produced at the rate of $at/(1+bt(Q(t))^2)$ units of mass per unit of time.

Answer

$$Q' = \frac{at}{1+btQ^2} - kQ; \quad \lim_{t\to\infty} Q(t) = \left(\frac{a}{bk}\right)^{1/3}.$$

? Exercise $2.1E.27$

Follow the instructions of **Exercise 2.1E.25**, assuming that the substance is produced at the rate of $at/(1+bt)$ units of mass per unit of time.

2.2: Cooling Problems

Learning Objectives

- Apply Newton's Law of Cooling to model temperature changes in various contexts.
- Solve differential equations to find temperature as a function of time and predict key values, such as time or temperature at specific points.

In this section we apply Newton's Law of Cooling to model temperature changes in various contexts.

Theorem 2.2.1: Newton's Law of Cooling

Newton's law of cooling states that if an object with temperature $T(t)$ at time t is in a medium with temperature $T_m(t)$, the rate of change of T at time t is proportional to $T(t)-T_m(t)$; thus, T satisfies a differential equation of the form

$$T' = -k(T - T_m). \tag{2.2.1}$$

Here $k > 0$, since the temperature of the object must decrease if $T > T_m$, or increase if $T < T_m$. We'll call k the *temperature decay constant of the medium*.

For simplicity, in this section we'll assume that the medium is maintained at a constant temperature T_m. This is another example of building a simple mathematical model for a physical phenomenon. Like most mathematical models it has its limitations. For example, it is reasonable to assume that the temperature of a room remains approximately constant if the cooling object is a cup of coffee, but perhaps not if it is a huge cauldron of molten metal.

To solve Equation (2.2.1), we rewrite it as

$$T' + kT = kT_m.$$

Since e^{-kt} is a solution of the complementary equation, the solutions of this equation are of the form $T = ue^{-kt}$, where $u'e^{-kt} = kT_m$, so $u' = kT_m e^{kt}$. Hence,

$$u = T_m e^{kt} + c,$$

so

$$T = ue^{-kt} = T_m + ce^{-kt}.$$

If $T(0) = T_0$, setting $t = 0$ here yields $c = T_0 - T_m$, so

$$T = T_m + (T_0 - T_m)e^{-kt}. \tag{2.2.2}$$

Note that $T - T_m$ decays exponentially, with decay constant k.

Example 2.2.1

A cup of coffee at 190°F is left in a room of 70°F. At time t=0, the coffee is cooling at 15°F per minute.

a) Find the function that models the cooling of the coffee.

b) How long will it take for the temperature to reach 143°F?

Solution

- **Video length**: 8 minutes 24 seconds.
- **Context**: This video demonstrates how to use Newton's Law of Cooling to model how the coffee's temperature decreases over time, relative to the room's temperature.

Example 2.2.2

A ceramic insulator is baked at $400^{\circ}\mathrm{C}$ and cooled in a room in which the temperature is $25^{\circ}\mathrm{C}$. After 4 minutes the temperature of the insulator is $200^{\circ}\mathrm{C}$. What is its temperature after 8 minutes?

Solution

Here $T_0 = 400$ and $T_m = 25$, so Equation (2.2.2) becomes

$$T = 25 + 375e^{-kt}. \tag{2.2.3}$$

We determine k from the stated condition that $T(4) = 200$; that is,

$$200 = 25 + 375e^{-4k};$$

hence,

$$e^{-4k} = \frac{175}{375} = \frac{7}{15}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{4}\ln\frac{7}{15} = \frac{1}{4}\ln\frac{15}{7}.$$

Substituting this into Equation (2.2.3) yields

$$T = 25 + 375e^{-\frac{t}{4}\ln\frac{15}{7}}$$

(see Figure $2.2.1$). Therefore the temperature of the insulator after 8 minutes is

$$\begin{aligned} T(8) &= 25 + 375e^{-2\ln\frac{15}{7}} \\ &= 25 + 375\left(\tfrac{7}{15}\right)^2 \approx 107^{\circ}\mathrm{C}. \end{aligned}$$

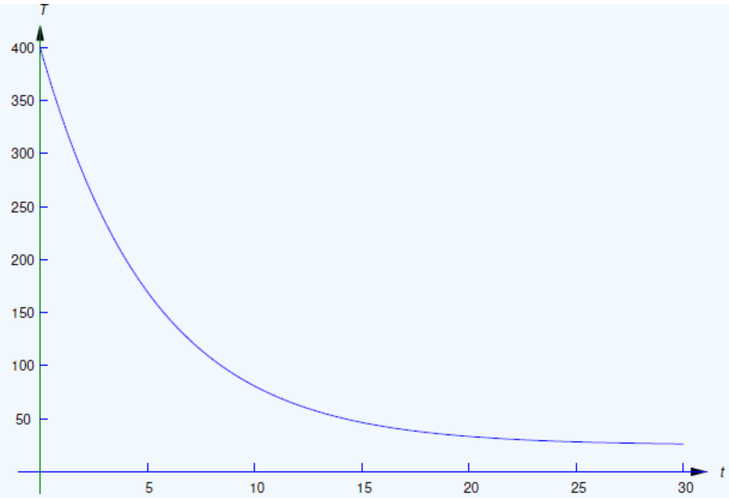


Figure 2.2.1: This graph illustrates the temperature of an object cooling over time, modeled by the equation $T = 25 + 375e^{-(t/4)\ln 15/7}$ (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation).

Example 2.2.3

An object with temperature 72°F is placed outside, where the temperature is -20°F. At 11:05 the temperature of the object is 60°F and at 11:07 its temperature is 50°F. At what time was the object placed outside?

Solution

Let $T(t)$ be the temperature of the object at time t. For convenience, we choose the origin $t_0 = 0$ of the time scale to be 11:05 so that $T_0 = 60$. We must determine the time τ when $T(\tau) = 72$. Substituting $T_0 = 60$ and $T_m = -20$ into Equation (2.2.2) yields

$$T = -20 + \big(60 - (-20)\big)e^{-kt}$$

or

$$T = -20 + 80e^{-kt}. \tag{2.2.4}$$

We obtain k from the stated condition that the temperature of the object is 50°F at 11:07. Since 11:07 is $t = 2$ on our time scale, we can determine k by substituting $T = 50$ and $t = 2$ into Equation (2.2.2) to obtain

$$50 = -20 + 80e^{-2k}$$

Hence,

$$e^{-2k} = \frac{70}{80} = \frac{7}{8}.$$

Taking logarithms and solving for k yields

$$k = -\frac{1}{2}\ln\frac{7}{8} = \frac{1}{2}\ln\frac{8}{7}.$$

Substituting this into Equation (2.2.4) yields

$$T = -20 + 80e^{-\frac{t}{2}\ln\frac{8}{7}},$$

(see Figure 2.2.2) and the condition $T(\tau) = 72$ implies that

$$72 = -20 + 80e^{-\frac{\tau}{2}\ln\frac{8}{7}};$$

hence,

$$e^{-\frac{\tau}{2}\ln\frac{8}{7}} = \frac{92}{80} = \frac{23}{20}.$$

Taking logarithms and solving for τ yields

$$\tau=-\frac{2\ln\frac{23}{20}}{\ln\frac{8}{7}}\approx -2.09 \text{ min.}$$

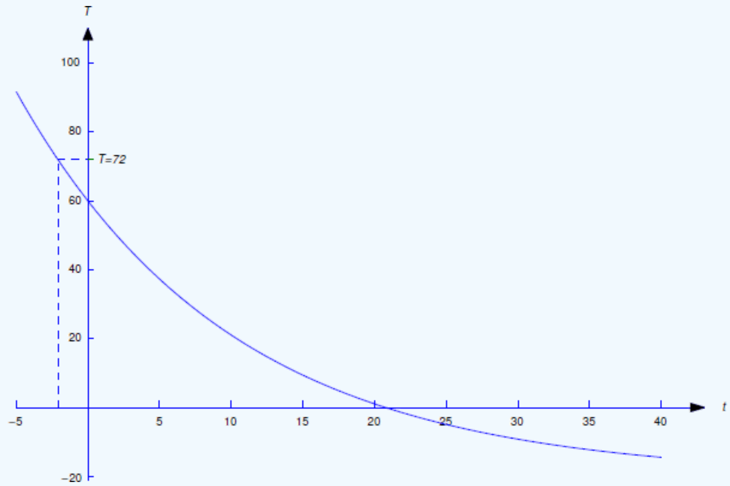


Figure 2.2.2 : This graph illustrates the temperature of an object cooling over time, modeled by the equation $-20+80e^{-\frac{t}{2}\ln\frac{8}{7}}$ (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation)

Therefore the object was placed outside about 2 minutes and 5 seconds before 11:05; that is, at 11:02:55.

2.2E: Exercises for Section 2.2

Exercises 1-9

Solve the following problems related to Newton's law of cooling.

? Exercise $2.2E.1$

A thermometer is moved from a room where the temperature is 70°F to a freezer where the temperature is $12^{\circ}F$. After 30 seconds the thermometer reads 40°F. What does it read after 2 minutes?

? Exercise $2.2E.2$

A fluid initially at 100°C is placed outside on a day when the temperature is -10°C, and the temperature of the fluid drops 20°C in one minute. Find the temperature $T(t)$ of the fluid for $t > 0$.

Answer

$$T = -10 + 110e^{-t\ln\frac{11}{9}}. \tag{2.2E.1}$$

? Exercise $2.2E.3$

At 12:00 pm a thermometer reading 10°F is placed in a room where the temperature is 70°F. It reads 56° when it is placed outside, where the temperature is 5°F, at 12:03. What does it read at 12:05 pm

? Exercise $2.2E.4$

A thermometer initially reading 212°F is placed in a room where the temperature is 70°F. After 2 minutes the thermometer reads 125°F.

a. What does the thermometer read after 4 minutes?
b. When will the thermometer read 72°F?
c. When will the thermometer read 69°F?

Answer

$T = 70 + 142e^{-\frac{t}{2}\ln\frac{142}{55}}$.

a. $T(4) = 70 + 142e^{-2\ln\frac{142}{55}} = 70 + 142\left(\frac{55}{142}\right)^2 \approx 91.30^{\circ}\text{F}$.
b. Let τ be the time when $T(\tau) = 72$. $\tau = 2\frac{\ln 71}{\ln\frac{142}{55}} \approx 8.99$ min.
c. The thermometer will never read 69°F.

? Exercise $2.2E.5$

An object with initial temperature 150°C is placed outside, where the temperature is 35°C. Its temperatures at 12:15 and 12:20 are 120°C and 90°C, respectively.

a. At what time was the object placed outside?
b. When will its temperature be 40°C?

? Exercise $2.2E.6$

An object is placed in a room where the temperature is 20°C. The temperature of the object drops by 5°C in 4 minutes and by 7°C in 8 minutes. What was the temperature of the object when it was initially placed in the room?

Answer

$T_0 = \left(\frac{85}{3}\right)^{\circ} C.$

Exercise $2.2E.7$

A cup of boiling water is placed outside at 1:00 pm. One minute later the temperature of the water is 152°F. After another minute its temperature is 112°F. What is the outside temperature?

Exercise $2.2E.8$

Suppose an object with initial temperature T_0 is placed in a sealed container, which is in turn placed in a medium with temperature T_m. Let the initial temperature of the container be S_0. Assume that the temperature of the object does not affect the temperature of the container, which in turn does not affect the temperature of the medium. (These assumptions are reasonable, for example, if the object is a cup of coffee, the container is a house, and the medium is the atmosphere.)

a. Assuming that the container and the medium have distinct temperature decay constants k and k_m respectively, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
b. Assuming that the container and the medium have the same temperature decay constant k, use Newton's law of cooling to find the temperatures $S(t)$ and $T(t)$ of the container and object at time t.
c. Find $\lim._{t\to\infty} S(t)$ and $\lim_{t\to\infty} T(t)$.

Exercise $2.2E.9$

In our previous examples and exercises concerning Newton's law of cooling we assumed that the temperature of the medium remains constant. This model is adequate if the heat lost or gained by the object is insignificant compared to the heat required to cause an appreciable change in the temperature of the medium. If this isn't so, we must use a model that accounts for the heat exchanged between the object and the medium. Let $T = T(t)$ and $T_m = T_m(t)$ be the temperatures of the object and the medium, respectively, and let T_0 and T_{m0} be their initial values. Again, we assume that T and T_m are related by Newton's law of cooling,

$$T' = -k(T - T_m). \tag{A}$$

We also assume that the change in heat of the object as its temperature changes from T_0 to T is $a(T - T_0)$ and that the change in heat of the medium as its temperature changes from T_{m0} to T_m is $a_m(T_m - T_{m0})$, where a and a_m are positive constants depending upon the masses and thermal properties of the object and medium, respectively. If we assume that the total heat of the system consisting of the object and the medium remains constant (that is, energy is conserved), then

$$a(T - T_0) + a_m(T_m - T_{m0}) = 0. \tag{B}$$

a. Equation (A) involves two unknown functions T and T_m. Use (A) and (B) to derive a differential equation involving only T.
b. Find $T(t)$ and $T_m(t)$ for $t > 0$.
c. Find $\lim_{t\to\infty} T(t)$ and $\lim_{t\to\infty} T_m(t)$.

2.3: Elementary Mechanics

Learning Objectives

- Understand Newton's Second Law of Motion and its application to modeling forces and motion.
- Analyze motion through a resisting medium, including terminal velocity and escape velocity scenarios.

Newton's Second Law of Motion

In this section we consider an object with constant mass m moving along a line under a force F. Let $y = y(t)$ be the displacement of the object from a reference point on the line at time t, and let $v = v(t)$ and $a = a(t)$ be the velocity and acceleration of the object at time t. Thus, $v = y'$ and $a = v' = y''$, where the prime denotes differentiation with respect to t. Newton's second law of motion asserts that the force F and the acceleration a are related by the equation

$$F = ma. \tag{2.3.1}$$

Note: Units

In applications there are three main sets of units in use for length, mass, force, and time: the cgs, mks, and British systems. All three use the second as the unit of time. Table 2.3.1 shows the other units. Consistent with Equation (2.3.1), the unit of force in each system is defined to be the force required to impart an acceleration of (one unit of length)$/s^2$ to one unit of mass.

Table 2.3.1 Unit of lenth, force, and mass. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation).

Set	Length	Force	Mass
cgs	centimeter (cm)	dyne (d)	gram (g)
mks	meter (m)	newton (N)	kilogram (kg)
British	foot (ft)	pound (lb)	slug (sl)

If we assume that Earth is a perfect sphere with constant mass density, Newton's law of gravitation (discussed later in this section) asserts that the force exerted on an object by Earth's gravitational field is proportional to the mass of the object and inversely proportional to the square of its distance from the center of Earth. However, if the object remains sufficiently close to Earth's surface, we may assume that the gravitational force is constant and equal to its value at the surface. The magnitude of this force is mg, where g is called the *acceleration due to gravity*. (To be completely accurate, g should be called the *magnitude of the acceleration due to gravity at Earth's surface*.) This quantity has been determined experimentally. Approximate values of g are

$$\begin{aligned} g &= 980 \text{ cm/s}^2 && \text{(cgs)} \\ g &= 9.8 \text{ m/s}^2 && \text{(mks)} \\ g &= 32 \text{ ft/s}^2 && \text{(British)}. \end{aligned}$$

In general, the force F in Equation (2.3.1) may depend upon t, y, and y'. Since $a = y''$, Equation (2.3.1) can be written in the form

$$my'' = F(t, y, y'), \tag{2.3.2}$$

which is a second order equation. We'll consider this equation with restrictions on F later; however, since Chapter 2 dealt only with first order equations, we consider here only problems in which Equation (2.3.2) can be recast as a first order equation. This is possible if F does not depend on y, so Equation (2.3.2) is of the form

$$my'' = F(t, y').$$

Letting $v = y'$ and $v' = y''$ yields a first order equation for v:

$$mv' = F(t, v). \tag{2.3.3}$$

Solving this equation yields v as a function of t. If we know $y(t_0)$ for some time t_0, we can integrate v to obtain y as a function of t.

Equations of the form Equation (2.3.3) occur in problems involving motion through a resisting medium.

Motion Through a Resisting Medium and Terminal Velocity

Now we consider an object moving vertically in some medium. We assume that the only forces acting on the object are gravity and resistance from the medium. We also assume that the motion takes place close to Earth's surface and take the upward direction to be positive, so the gravitational force can be assumed to have the constant value $-mg$. We'll see that, under reasonable assumptions on the resisting force, the velocity approaches a limit as $t\to\infty$. We call this limit the *terminal velocity*.

Example 2.3.1

An object with mass m moves under constant gravitational force through a medium that exerts a resistance with magnitude proportional to the speed of the object. (Recall that the speed of an object is $|v|$, the absolute value of its velocity v.) Find the velocity of the object as a function of t, and find the terminal velocity. Assume that the initial velocity is v_0.

Solution

The total force acting on the object is

$$F = -mg + F_1, \tag{2.3.4}$$

where $-mg$ is the force due to gravity and F_1 is the resisting force of the medium, which has magnitude $k|v|$, where k is a positive constant. If the object is moving downward ($v \leq 0$), the resisting force is upward (see Figure 2.3.1a), so

$$F_1 = k|v| = k(-v) = -kv.$$

On the other hand, if the object is moving upward ($v \geq 0$), the resisting force is downward (see Figure 2.3.1b), so

$$F_1 = -k|v| = -kv.$$

Thus, Equation (2.3.4) can be written as

$$F = -mg - kv, \tag{2.3.5}$$

regardless of the sign of the velocity.

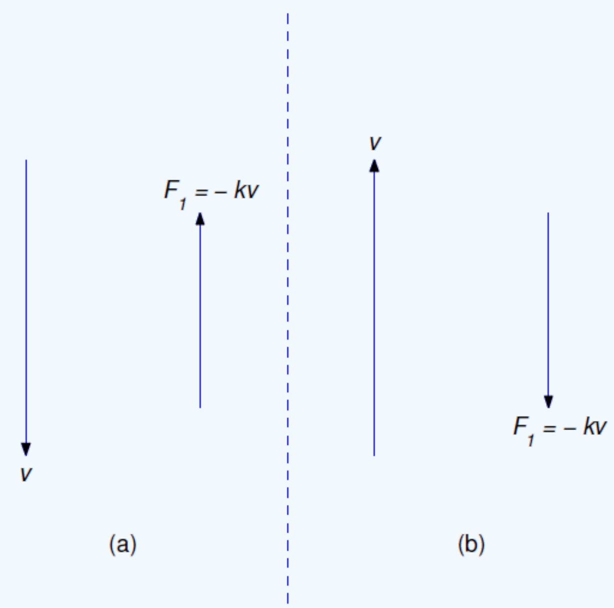


Figure 2.3.1 : This diagram illustrates the forces acting on an object falling through a fluid with resistance. Part (a) on left, down arrow for V and up arrow for $F_1 = -kv$. Part (b) on right, up arrow for V and down arrow for $F_1 = -kv$. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation)n).

From Newton's second law of motion,

$$F = ma = mv',$$

so Equation (2.3.5) yields

$$mv' = -mg - kv,$$

or

$$v' + \frac{k}{m}v = -g. \tag{2.3.6}$$

Since $e^{-kt/m}$ is a solution of the complementary equation, the solutions of Equation (2.3.6) are of the form $v = ue^{-kt/m}$, where $u'e^{-kt/m} = -g$, so $u' = -ge^{kt/m}$. Hence,

$$u = -\frac{mg}{k}e^{kt/m} + c,$$

so

$$v = ue^{-kt/m} = -\frac{mg}{k} + ce^{-kt/m}. \tag{2.3.7}$$

Since $v(0) = v_0$,

$$v_0 = -\frac{mg}{k} + c,$$

so

$$c = v_0 + \frac{mg}{k}$$

and Equation (2.3.7) becomes

$$v = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-kt/m}.$$

Letting $t \to \infty$ here shows that the terminal velocity is

$$\lim_{t\to\infty} v(t) = -\frac{mg}{k},$$

which is independent of the initial velocity v_0 (see Figure 2.3.2).

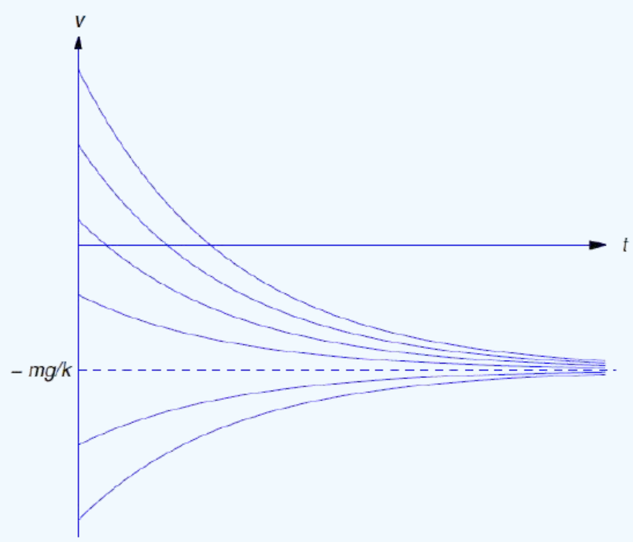


Figure 2.3.2 : Graph showing solutions to the differential equation mv' = -mg - kv. The y-axis represents velocity, and the t-axis represents time. Several curves illustrate different solution paths converging towards the terminal velocity -mg/k. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation)

Example 2.3.2

A 960-lb object is given an initial upward velocity of 60 ft/s near the surface of Earth. The atmosphere resists the motion with a force of 3 lb for each ft/s of speed. Assuming that the only other force acting on the object is constant gravity, find its velocity v as a function of t, and find its terminal velocity.

Solution

Since $mg = 960$ and $g = 32$, $m = 960/32 = 30$. The atmospheric resistance is $-3v$ lb if v is expressed in feet per second. Therefore

$$30v' = -960 - 3v,$$

which we rewrite as

$$v' + \frac{1}{10}v = -32.$$

Since $e^{-t/10}$ is a solution of the complementary equation, the solutions of this equation are of the form $v = ue^{-t/10}$, where $u'e^{-t/10} = -32$, so $u' = -32e^{t/10}$. Hence,

$$u = -320e^{t/10} + c,$$

so

$$v = ue^{-t/10} = -320 + ce^{-t/10}. \tag{2.3.8}$$

The initial velocity is 60 ft/s in the upward (positive) direction; hence, $v_0 = 60$. Substituting $t = 0$ and $v = 60$ in Equation (2.3.8) yields

$$60 = -320 + c,$$

so $c = 380$, and Equation (2.3.8) becomes

$$v = -320 + 380e^{-t/10} \text{ ft/s}$$

The terminal velocity is

$$\lim_{t\to\infty} v(t) = -320 \text{ ft/s}.$$

Example 2.3.3

A 10 kg mass is given an initial velocity $v_0 \le 0$ near Earth's surface. The only forces acting on it are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the resistance is 8 N if the speed is 2 m/s, find the velocity of the object as a function of t, and find the terminal velocity.

Solution

Since the object is falling, the resistance is in the upward (positive) direction. Hence,

$$mv' = -mg + kv^2, \tag{2.3.9}$$

where k is a constant. Since the magnitude of the resistance is 8 N when $v = 2$ m/s,

$$k(2^2) = 8,$$

so $k = 2$ N-s^2/m^2. Since $m = 10$ and $g = 9.8$, Equation (2.3.9) becomes

$$10v' = -98 + 2v^2 = 2(v^2 - 49). \tag{2.3.10}$$

If $v_0 = -7$, then $v \equiv -7$ for all $t \ge 0$. If $v_0 \ne -7$, we separate variables to obtain

$$\frac{1}{v^2 - 49}v' = \frac{1}{5}, \tag{2.3.11}$$

which is convenient for the required partial fraction expansion

$$\frac{1}{v^2-49}=\frac{1}{(v-7)(v+7)}=\frac{1}{14}\left[\frac{1}{v-7}-\frac{1}{v+7}\right]. \tag{2.3.12}$$

Substituting Equation (2.3.12) into Equation (2.3.11) yields

$$\frac{1}{14}\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{1}{5},$$

so

$$\left[\frac{1}{v-7}-\frac{1}{v+7}\right]v'=\frac{14}{5}.$$

Integrating this yields

$$\ln|v-7|-\ln|v+7|=14t/5+k.$$

Therefore

$$\left|\frac{v-7}{v+7}\right|=e^k e^{14t/5}.$$

If solution of a non-linear differential equation exists, then it is unique. This fact (from the Existence and Uniqueness Theorem) implies $(v-7)/(v+7)$ cannot change sign (why?), we can rewrite the last equation as

$$\frac{v-7}{v+7}=ce^{14t/5}, \tag{2.3.13}$$

which is an implicit solution of Equation (2.3.10). Solving this for v yields

$$v=-7\frac{c+e^{-14t/5}}{c-e^{-14t/5}}. \tag{2.3.14}$$

Since $v(0)=v_0$, Equation (2.3.13) implies that

$$c=\frac{v_0-7}{v_0+7}.$$

Substituting this into Equation (2.3.14) and simplifying yields

$$v=-7\frac{v_0(1+e^{-14t/5})-7(1-e^{-14t/5})}{v_0(1-e^{-14t/5})-7(1+e^{-14t/5})}.$$

Since $v_0\le 0$, v is defined and negative for all $t>0$. The terminal velocity is

$$\lim_{t\to\infty} v(t)=-7 \text{ m/s},$$

independent of v_0. More generally, it can be shown that if v is any solution of Equation (2.3.9) such that $v_0\le 0$ then

$$\lim_{t\to\infty} v(t)=-\sqrt{\frac{mg}{k}}.$$

This is demonstrated in Figure 2.3.3 .

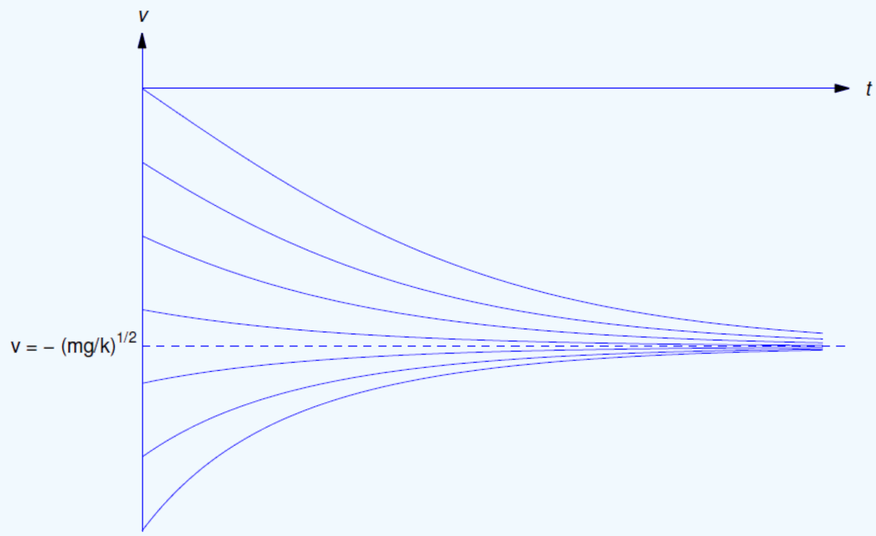


Figure 2.3.3 : This graph illustrates the solutions of the differential equation $mv' = -mg + kv^2$ for various $v_0 \leq 0$. Several curves are plotted, showing different trajectories that converge towards a horizontal dashed line which indicates the terminal velocity. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation)

Example 2.3.4

A 10-kg mass is launched vertically upward from Earth's surface with an initial velocity of v_0 m/s. The only forces acting on the mass are gravity and atmospheric resistance proportional to the square of the speed. Assuming that the atmospheric resistance is 8 N if the speed is 2 m/s, find the time T required for the mass to reach maximum altitude.

Solution

The mass will climb while $v > 0$ and reach its maximum altitude when $v = 0$. Therefore $v > 0$ for $0 \leq t < T$ and $v(T) = 0$; therefore, we replace Equation (2.3.10) by

$$10v' = -98 - 2v^2. \tag{2.3.15}$$

Separating variables yields

$$\frac{5}{v^2 + 49} v' = -1,$$

and integrating this yields

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + c.$$

(Recall that $\tan^{-1} u$ is the number θ such that $-\pi/2 < \theta < \pi/2$ and $\tan\theta = u$.) Since $v(0) = v_0$,

$$c = \frac{5}{7} \tan^{-1} \frac{v_0}{7},$$

so v is defined implicitly by

$$\frac{5}{7} \tan^{-1} \frac{v}{7} = -t + \frac{5}{7} \tan^{-1} \frac{v_0}{7}, \quad 0 \leq t \leq T. \tag{2.3.16}$$

Solving this for v yields

$$v = 7 \tan\left(-\frac{7t}{5} + \tan^{-1} \frac{v_0}{7}\right). \tag{2.3.17}$$

Using the identity

$$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

with $A = \tan^{-1}(v_0/7)$ and $B = 7t/5$, and noting that $\tan(\tan^{-1}\theta) = \theta$, we can simplify Equation (2.3.17) to

$$v = 7\frac{v_0 - 7\tan(7t/5)}{7 + v_0\tan(7t/5)}.$$

Since $v(T) = 0$ and $\tan^{-1}(0) = 0$, Equation (2.3.16) implies that

$$-T + \frac{5}{7}\tan^{-1}\frac{v_0}{7} = 0.$$

Therefore

$$T = \frac{5}{7}\tan^{-1}\frac{v_0}{7}.$$

Since $\tan^{-1}(v_0/7) < \pi/2$ for all v_0, the time required for the mass to reach its maximum altitude is less than

$$\frac{5\pi}{14} \approx 1.122 \text{ s}$$

regardless of the initial velocity. Figure 2.3.4 shows graphs of v over $[0, T]$ for various values of v_0.

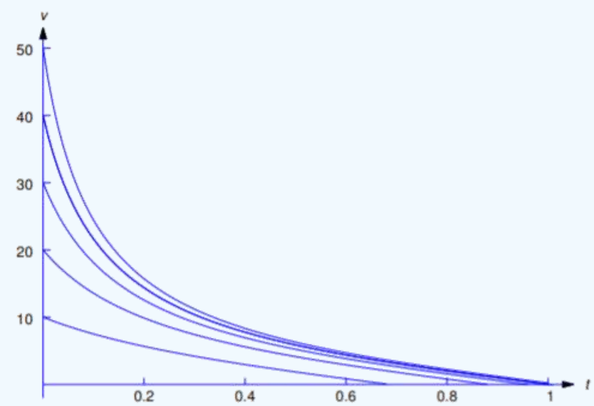


Figure 2.3.4 : This graph illustrates the solutions of $10v' = -98 - 2v^2$ for various $v_0 > 0$. The bottom solution curve is a straight line, and other solution curves follows an exponential-like decay pattern. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation)

Motion Through a Resisting Medium and Escape Velocity

Suppose a space vehicle is launched vertically and its fuel is exhausted when the vehicle reaches an altitude h above Earth, where h is sufficiently large so that resistance due to Earth's atmosphere can be neglected. Let $t = 0$ be the time when burnout occurs. Assuming that the gravitational forces of all other celestial bodies can be neglected, the motion of the vehicle for $t > 0$ is that of an object with constant mass m under the influence of Earth's gravitational force, which we now assume to vary inversely with the square of the distance from Earth's center; thus, if we take the upward direction to be positive then gravitational force on the vehicle at an altitude y above Earth is

$$F = -\frac{K}{(y+R)^2}, \tag{2.3.18}$$

where R is Earth's radius (see Figure 2.3.5).

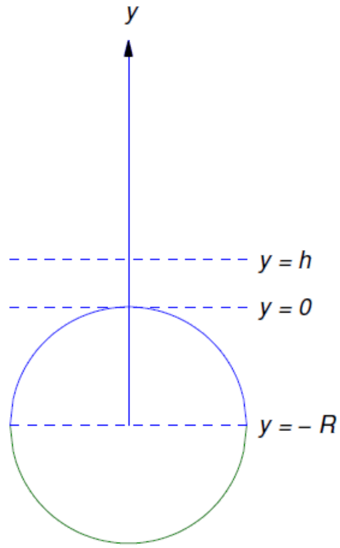


Figure 2.3.5 : This diagram illustrates the concept of escape velocity. Escape velocity showing circle centered at y = -R and top at y = 0. Horizontal line at y = h also shown. (CC BY-NC-SA 3.0; William F. Trench via Elementary Differential equation)

Since $F=-mg$ when $y=0$, setting $y=0$ in Equation (2.3.18) yields

$$-mg=-\frac{K}{R^2};$$

therefore $K=mgR^2$ and Equation (2.3.18) can be written more specifically as

$$F=-\frac{mgR^2}{(y+R)^2}. \tag{2.3.19}$$

From Newton's second law of motion,

$$F=m\frac{d^2y}{dt^2},$$

so Equation (2.3.19) implies that

$$\frac{d^2y}{dt^2}=-\frac{gR^2}{(y+R)^2}. \tag{2.3.20}$$

We'll show that there's a number v_e, called the *escape velocity*, with these properties:

1. If $v_0 \geq v_e$ then $v(t)>0$ for all $t>0$, and the vehicle continues to climb for all $t>0$; that is, it "escapes" Earth. (Is it really so obvious that $\lim_{t\to\infty} y(t)=\infty$ in this case?)
2. If $v_0<v_e$ then $v(t)$ decreases to zero and becomes negative. Therefore the vehicle attains a maximum altitude y_m and falls back to Earth.

Since Equation (2.3.20) is second order, we cannot solve it by methods discussed so far. However, we are concerned with v rather than y, and v is easier to find. Since $v=y'$ the chain rule implies that

$$\frac{d^2y}{dt^2}=\frac{dv}{dt}=\frac{dv}{dy}\frac{dy}{dt}=v\frac{dv}{dy}.$$

Substituting this into Equation (2.3.20) yields the first order separable equation

$$v\frac{dv}{dy}=-\frac{gR^2}{(y+R)^2}. \tag{2.3.21}$$

When $t=0$, the velocity is v_0 and the altitude is h. Therefore we can obtain v as a function of y by solving the initial value problem

$$v\frac{dv}{dy} = -\frac{gR^2}{(y+R)^2}, \quad v(h) = v_0.$$

Integrating Equation (2.3.21) with respect to y yields

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + c. \tag{2.3.22}$$

Since $v(h) = v_0$,

$$c = \frac{v_0^2}{2} - \frac{gR^2}{h+R},$$

so Equation (2.3.22) becomes

$$\frac{v^2}{2} = \frac{gR^2}{y+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right). \tag{2.3.23}$$

If

$$v_0 \geq \left(\frac{2gR^2}{h+R}\right)^{1/2},$$

the parenthetical expression in Equation (2.3.23) is nonnegative, so $v(y) > 0$ for $y > h$. This proves that there's an escape velocity v_e. We'll now prove that

$$v_e = \left(\frac{2gR^2}{h+R}\right)^{1/2}$$

by showing that the vehicle falls back to Earth if

$$v_0 < \left(\frac{2gR^2}{h+R}\right)^{1/2}. \tag{2.3.24}$$

If Equation (2.3.24) holds then the parenthetical expression in Equation (2.3.23) is negative and the vehicle will attain a maximum altitude $y_m > h$ that satisfies the equation

$$0 = \frac{gR^2}{y_m+R} + \left(\frac{v_0^2}{2} - \frac{gR^2}{h+R}\right).$$

The velocity will be zero at the maximum altitude, and the object will then fall to Earth under the influence of gravity.

Motion Through Air Resistance and Limiting Velocity

Example 2.3.5

A steel ball weighing 1 lb is dropped from 2500 ft with no velocity. As it falls, the air resistance is equal to v/8 in pounds where v is the velocity of the ball in feet per second. Find the limiting velocity and the time it takes for the ball to hit the ground.

Solution

- **Video length**: 11 minutes 51 seconds.
- **Context**: This video explores the motion of a falling object under the influence of gravity and air resistance.

2.3E: Exercises for Section 2.3

Exercises 1-16

For the following exercises, except where directed otherwise, assume that the magnitude of the gravitational force on an object with mass m is constant and equal to mg. In exercises involving vertical motion take the upward direction to be positive.

Exercise $2.3E.1$

A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to his speed, with $k = 2.5$ lb-s/ft. Assuming that he starts from rest, find his velocity as a function of time and find his terminal velocity.

Exercise $2.3E.2$

A firefighter who weighs 192 lb slides down an infinitely long fire pole that exerts a frictional resistive force with magnitude proportional to her speed, with constant of proportionality k. Find k, given that her terminal velocity is -16 ft/s, and then find her velocity v as a function of t. Assume that she starts from rest.

Answer

$$v = -16(1 - e^{-2t}).$$

Exercise $2.3E.3$

A boat weighs $64,000$ lb. Its propellor produces a constant thrust of $50,000$ lb and the water exerts a resistive force with magnitude proportional to the speed, with $k = 2000$ lb-s/ft. Assuming that the boat starts from rest, find its velocity as a function of time, and find its terminal velocity.

Exercise $2.3E.4$

A constant horizontal force of 10 N pushes a 20 kg-mass through a medium that resists its motion with $.5$ N for every m/s of speed. The initial velocity of the mass is 7 m/s in the direction opposite to the direction of the applied force. Find the velocity of the mass for $t > 0$.

Answer

$$v = 20 - 27e^{-t/40}.$$

Exercise $2.3E.5$

A stone weighing $1/2$ lb is thrown upward from an initial height of 5 ft with an initial speed of 32 ft/s. Air resistance is proportional to speed, with $k = 1/128$ lb-s/ft. Find the maximum height attained by the stone.

Exercise $2.3E.6$

A 3200-lb car is moving at 64 ft/s down a 30-degree grade when it runs out of fuel. Find its velocity after that if friction exerts a resistive force with magnitude proportional to the square of the speed, with $k = 1$ lb-s^2/ft^2. Also find its terminal velocity.

Exercise $2.3E.7$

A 96 lb weight is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the speed. Find its velocity as a function of time if its terminal velocity is -128 ft/s.

Exercise $2.3E.8$

An object with mass m moves vertically through a medium that exerts a resistive force with magnitude proportional to the speed. Let $y = y(t)$ be the altitude of the object at time t, with $y(0) = y_0$. Use the results of Example 4.3.1 to show that

$$y(t) = y_0 + \frac{m}{k}(v_0 - v - gt).$$

Exercise $2.3E.9$

An object with mass m is launched vertically upward with initial velocity v_0 from Earth's surface ($y_0 = 0$) in a medium that exerts a resistive force with magnitude proportional to the speed. Find the time T when the object attains its maximum altitude y_m. Then use the result of Exercise 2.3.8 to find y_m.

Exercise $2.3E.10$

An object weighing 256 lb is dropped from rest in a medium that exerts a resistive force with magnitude proportional to the square of the speed. The magnitude of the resisting force is 1 lb when $|v| = 4$ ft/s. Find v for $t > 0$, and find its terminal velocity.

Answer

$v = \frac{64(1-e^t)}{1+e^t}$, or $v = -\frac{64(1-e^{-t})}{1+e^{-t}}$. Therefore, $\lim_{t\to\infty} v(t) = -64$.

Exercise $2.3E.11$

An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force with magnitude proportional to the square of the speed. Find the velocity of the object for $t > 0$, and find its terminal velocity.

Exercise $2.3E.12$

An object with mass m is launched vertically upward with initial velocity v_0 in a medium that exerts a resistive force with magnitude proportional to the square of the speed.

a. Find the time T when the object reaches its maximum altitude.
b. Use the result of Exercise 2.3.11 to find the velocity of the object for $t > T$.

Exercise $2.3E.13$

An object with mass m is given an initial velocity $v_0 \le 0$ in a medium that exerts a resistive force of the form $a|v|/(1 + |v|)$, where a is positive constant.

a. Set up a differential equation for the speed of the object.
b. Use your favorite numerical method to solve the equation you found in (a), to convince yourself that there's a unique number a_0 such that $\lim_{t\to\infty} s(t) = \infty$ if $a \le a_0$ and $\lim_{t\to\infty} s(t)$ exists (finite) if $a > a_0$. (We say that a_0 is the **bifurcation value** of a. Try to find a_0 and $\lim_{t\to\infty} s(t)$ in the case where $a > a_0$.

Exercise $2.3E.14$

An object of mass m falls in a medium that exerts a resistive force $f = f(s)$, where $s = |v|$ is the speed of the object. Assume that $f(0) = 0$ and f is strictly increasing and differentiable on $(0, \infty)$.

a. Write a differential equation for the speed $s = s(t)$ of the object. Take it as given that all solutions of this equation with $s(0) \ge 0$ are defined for all $t > 0$ (which makes good sense on physical grounds).
b. Show that if $\lim_{s\to\infty} f(s) \le mg$ then $\lim_{t\to\infty} s(t) = \infty$.
c. Show that if $\lim_{s\to\infty} f(s) > mg$ then $\lim_{t\to\infty} s(t) = s_T$ (terminal speed), where $f(s_T) = mg$.

Answer

a. $mv' = -mg + f(|v|)$; (A) $ms' = mg - f(s)$.
b. Since f is increasing and $\lim_{s\to\infty} f(s) \leq mg$, we have $mg - f(s) > 0$ for all s. This and (A) imply that s is an increasing function of t, so either (B) $\lim_{t\to\infty} s(t) = \infty$, or (C) $\lim_{t\to\infty} s(t) = \bar{s} < \infty$. However, (A) and (C) imply that $s'(t) > K = \frac{g - f(\bar{s})}{m}$ for all $t > 0$. Consequently, $s(t) > s_0 + Kt$ for all $t > 0$, which contradicts (C) because $K > 0$.
c. There is a unique positive number s_T such that $f(s_T) = mg$, and $s = s_T$ is a constant solution of (A). Now suppose that $s(0) < s_T$. Then Theorem 1.7.1 implies that (D) $s(t) < s_T$ for all $t > 0$, so (A) implies that s is strictly increasing. This and (D) imply that $\lim_{t\to\infty} s(t) = \bar{s} \leq s_T$. If $\bar{s} < s_T$, then (A) implies that $s'(t) > K = \frac{g - f(\bar{s})}{m}$. Consequently, $s(t) > s(0) + Kt$, which contradicts (D) because $K > 0$. Therefore, $s(0) < s_T \Rightarrow \lim_{t\to\infty} s(t) = s_T$. A similar proof with inequalities reversed shows that $s(0) > s_T \Rightarrow \lim_{t\to\infty} s(t) = s_T$.

? Exercise $2.3E.15$

A 100-g mass with initial velocity $v_0 \leq 0$ falls in a medium that exerts a resistive force proportional to the fourth power of the speed. The resistance is .1 N if the speed is 3 m/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use **Exercise 2.3E.14** part (c) to determine the terminal velocity of the object.

? Exercise $2.3E.16$

A 64-lb object with initial velocity $v_0 \leq 0$ falls through a dense fluid that exerts a resistive force proportional to the square root of the speed. The resistance is 64 lb if the speed is 16 ft/s.

a. Set up the initial value problem for the velocity v of the mass for $t > 0$.
b. Use **Exercise 2.3E.14** part (c) to determine the terminal velocity of the object.

Answer

a. $v' = -32 + 8\sqrt{|v|}$.
b. From Exercise 2.3E.14 part (c), v_T is the negative number such that $-32 + 8\sqrt{|v_T|} = 0$; thus, $v_T = -16$ ft/s.

Exercises 17-20

Assume that the force due to gravity is given by Newton's law of gravitation. Take the upward direction to be positive.

? Exercise $2.3E.17$

A space probe is to be launched from a space station 200 miles above Earth. Determine its escape velocity in miles/s. Take Earth's radius to be 3960 miles.

? Exercise $2.3E.18$

A space vehicle is to be launched from the moon, which has a radius of about 1080 miles. The acceleration due to gravity at the surface of the moon is about 5.31 ft/s^2. Find the escape velocity in miles/s.

Answer

$$v_e = \sqrt{\frac{2 \cdot 5.31 \cdot 1080}{5280}} \approx 1.47 \text{ miles/s}.$$

Exercise $2.3E.19$

a. Show that Equation (2.3.27) can be rewritten as

$$v^2 = \frac{h-y}{y+R} v_e^2 + v_0^2.$$

b. Show that if $v_0 = \rho v_e$ with $0 \le \rho < 1$, then the maximum altitude y_m attained by the space vehicle is

$$y_m = \frac{h+R\rho^2}{1-\rho^2}.$$

c. By requiring that $v(y_m) = 0$, use Equation (2.3.26) to deduce that if $v_0 < v_e$ then

$$|v| = v_e \left[\frac{(1-\rho^2)(y_m - y)}{y+R}\right]^{1/2},$$

where y_m and ρ are as defined in (b) and $y \ge h$.

d. Deduce from (c) that if $v < v_e$, the vehicle takes equal times to climb from $y = h$ to $y = y_m$ and to fall back from $y = y_m$ to $y = h$.

Exercise $2.3E.20$

In the situation considered in the discussion of escape velocity, show that $\lim_{t\to\infty} y(t) = \infty$ if $v(t) > 0$ for all $t > 0$.

2.4: Mixing Problems

Learning Objectives

- Model and solve first-order differential equations in real-world applications involving mixture.
- Analyze and solve mixture problems to determine the concentration and quantity of substances over time.

In this section we will look at some simple applications which are modeled with first order differential equations. We will begin with simple rational model of mixture problem in a single tank.

Mixture problems often occur in a first course on differential equations as examples of first order differential equations. In such problems we consider a tank of brine, water containing a specific amount of salt with pure water entering and the mixture leaving, or the flow of a pollutant into, or out of, a lake. The goal is to predict the amount of salt, or pollutant, at some later time.

In general one has a rate of flow of some concentration of mixture entering a region and a mixture leaving the region. The goal is to determine how much stuff is in the region at a given time. This is governed by the equation

Theorem $2.4.1$

Rate of change of substance $=$ Rate In $-$ Rate Out.

The rates are not often given. One is generally given information about the concentration and flow rates in and out of the system. If one pays attention to the dimension and sketches the situation, then one can write out this rate equation as a first order differential equation. We consider a simple example.

Example $2.4.1$

A 50 gallon tank of pure water has a brine mixture with concentration of 2 pounds per gallon entering at the rate of 5 gallons per minute. At the same time the well-mixed contents drain out at the rate of 5 gallons per minute. Find the amount of salt in the tank at time t. In all such problems one assumes that the solution is well mixed at each instant of time. See Figure $2.4.1$ for an illustration of the problem.

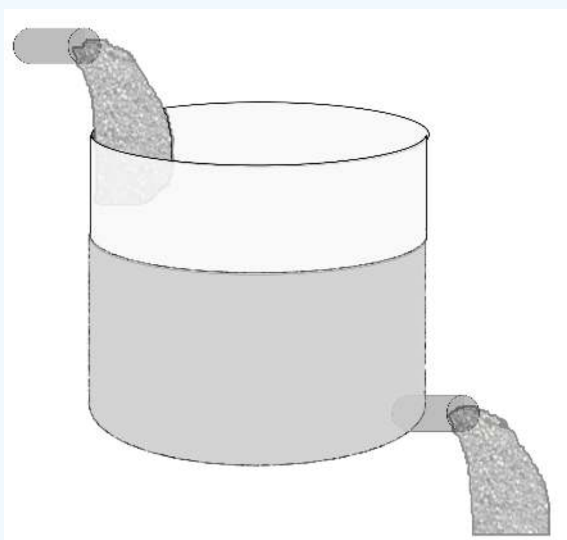


Figure $2.4.1$: A diagram illustrates a single tank mixing problem. Water is poured into the tank through one pipe at the top of the tank. Simultaneously, another pipe drains the well-mixed solution from the tank. (CC BY-NC-SA 3.0; Russell Herman via A First Course in Differential equations).

Solution

Let $x(t)$ be the amount of salt at time t. Then the rate at which the salt in the tank increases is due to the amount of salt entering the tank less that leaving the tank. To figure out these rates, one notes that dx/dt has units of pounds per minute. The amount of salt entering per minute is given by the product of the entering concentration times the rate at which the brine enters. This gives the correct units:

$$\left(2\frac{\text{pounds}}{\text{gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right)=10\frac{\text{pounds}}{\text{min}}.$$

Similarly, one can determine the rate out as

$$\left(\frac{x\text{ pounds}}{50\text{ gal}}\right)\left(5\frac{\text{gal}}{\text{min}}\right)=\frac{x}{10}\frac{\text{pounds}}{\text{min}}.$$

Thus, we have

$$\frac{dx}{dt}=10-\frac{x}{10}$$

This equation is solved using the methods for linear first order equations. The integrating factor is $\mu=e^{x/10}$, leading to the general solution

$$x(t)=100+Ae^{-t/10}$$

Using the initial condition, one finds the particular solution

$$x(t)=100\left(1-e^{-t/10}\right)$$

Often one is interested in the long time behavior of a system. In this case we have that $\lim_{t\to\infty} x(t)=100\text{ lb}$. This makes sense because 2 pounds per galloon enter during this time to eventually leave the entire 50 gallons with this concentration. Thus,

$$50\text{ gal}\times 2\frac{\text{lb}}{\text{gal}}=100\text{ lb}$$

Example 2.4.2

A tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0=0$, water that contains 1/2 pound of salt per gallon is poured into the tank at the rate of 4 gal/min and the mixture is drained from the tank at the same rate (see Figure 2.4.2).

a. Find a differential equation for the quantity $Q(t)$ of salt in the tank at time $t>0$, and solve the equation to determine $Q(t)$.
b. Find $\lim_{t\to\infty} Q(t)$.

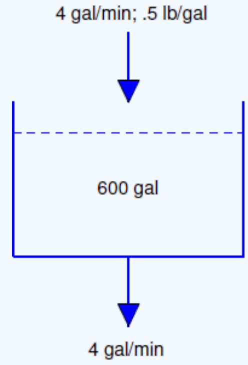


Figure 2.4.2 : A water tank with 600 gallon of water. Water is poured into the tank at 4 gal/min and it has 0.5lb of salt per gallon of water representing by the down arrow above the tank. Water is drained at a rate of 4 gal/min representing by the down arrow below the tank. (CC BY-NC-SA 3.0; Russell Herman via A First Course in Differential equations).

Solution a

To find a differential equation for Q, we must use the given information to derive an expression for Q'. But Q' is the rate of change of the quantity of salt in the tank changes with respect to time; thus, if *rate in* denotes the rate at which salt enters the tank and *rate out* denotes the rate by which it leaves, then

$$Q' = \text{rate in} - \text{rate out}. \tag{2.4.1}$$

The rate in is

$$\left(\frac{1}{2} \text{ lb/gal}\right) \times (4 \text{ gal/min}) = 2 \text{ lb/min}.$$

Determining the rate out requires a little more thought. We're removing 4 gallons of the mixture per minute, and there are always 600 gallons in the tank; that is, we are removing $1/150$ of the mixture per minute. Since the salt is evenly distributed in the mixture, we are also removing $1/150$ of the salt per minute. Therefore, if there are $Q(t)$ pounds of salt in the tank at time t, the rate out at any time t is $Q(t)/150$. Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out } &= (\text{concentration}) \times (\text{rate of flow out}) \\ &= (\text{lb/gal}) \times (\text{gal/min}) \\ &= \frac{Q(t)}{600} \times 4 \\ &= \frac{Q(t)}{150}. \end{aligned}$$

We can now write Equation (2.4.1) as

$$Q' = 2 - \frac{Q}{150}.$$

This first order equation can be rewritten as

$$Q' + \frac{Q}{150} = 2.$$

Since $e^{-t/150}$ is a solution of the complementary equation, the solutions of this equation are of the form $Q = ue^{-t/150}$, where $u'e^{-t/150} = 2$, so $u' = 2e^{t/150}$. Hence,

$$u = 300e^{t/150} + c,$$

so

$$Q = ue^{-t/150} = 300 + ce^{-t/150} \tag{2.4.2}$$

Since $Q(0) = 40$, $c = -260$; therefore,

$$Q = 300 - 260e^{-t/150}.$$

See Figure 2.4.3 for a graph of the solution.

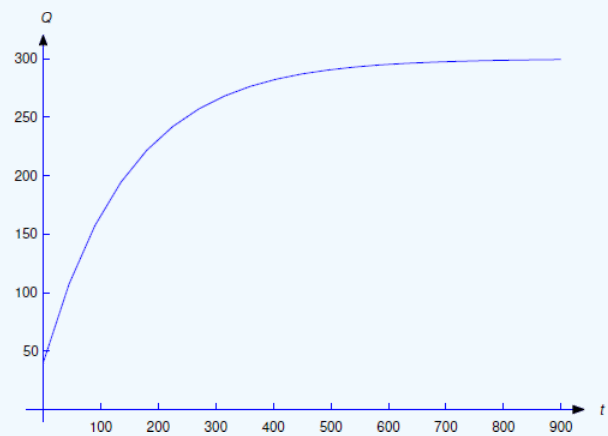


Figure 2.4.3 : The graph of $Q = 300 - 260e^{-t/150}$ illustrates how the amount of salt in the tank gradually increases towards a steady-state value of 300 pounds (CC BY-SA 4.0, Russell Herman via A First Course in Differential equations).

Solution b

From Equation (2.4.2), we see that that $\lim_{t\to\infty} Q(t) = 300$ for any value of $Q(0)$. This is intuitively reasonable, since the incoming solution contains 1/2 pound of salt per gallon and there are always 600 gallons of water in the tank.

Example 2.4.3

A 500-liter tank initially contains 10 g of salt dissolved in 200 liters of water. Starting at $t_0 = 0$, water that contains 1/4 g of salt per liter is poured into the tank at the rate of 4 liters/min and the mixture is drained from the tank at the rate of 2 liters/min (see Figure 2.4.4). Find a differential equation for the quantity $Q(t)$ of salt in the tank at time t prior to the time when the tank overflows and find the concentration $K(t)$ (g/liter) of salt in the tank at any such time.

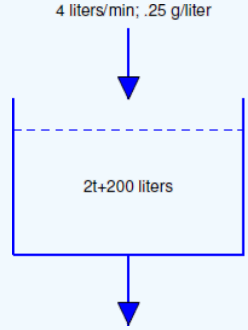


Figure 2.4.4 : A water tank with $2t+200$ litters of water. Water is poured into the tank at 4 litters/min and it has 0.25 gram of salt per litter of water representing by the down arrow above the tank. Another down arrow below the tank representing water is drained out of the tank. (CC BY-NC-SA 3.0; Russell Herman via A First Course in Differential equations).

Solution

We first determine the amount $W(t)$ of solution in the tank at any time t prior to overflow. Since $W(0) = 200$ and we are adding 4 liters/min while removing only 2 liters/min, there's a net gain of 2 liters/min in the tank; therefore,

$$W(t) = 2t + 200.$$

Since $W(150) = 500$ liters (capacity of the tank), this formula is valid for $0 \le t \le 150$.

Now let $Q(t)$ be the number of grams of salt in the tank at time t, where $0 \le t \le 150$. As in Example 2.4.2

$$Q' = \text{rate in} - \text{rate out}. \tag{2.4.3}$$

The rate in is

$$\left(\frac{1}{4}\text{ g/liter}\right) \times (4\text{ liters/min}) = 1\text{ g/min}. \tag{2.4.4}$$

To determine the rate out, we observe that since the mixture is being removed from the tank at the constant rate of 2 liters/min and there are $2t+200$ liters in the tank at time t, the fraction of the mixture being removed per minute at time t is

$$\frac{2}{2t+200} = \frac{1}{t+100}.$$

We're removing this same fraction of the salt per minute. Therefore, since there are $Q(t)$ grams of salt in the tank at time t,

$$\text{rate out} = \frac{Q(t)}{t+100}. \tag{2.4.5}$$

Alternatively, we can arrive at this conclusion by arguing that

$$\begin{aligned} \text{rate out} &= (\text{concentration}) \times (\text{rate of flow out}) = (\text{g/liter}) \times (\text{liters/min}) \\ &= \tfrac{Q(t)}{2t+200} \times 2 = \tfrac{Q(t)}{t+100}. \end{aligned}$$

Substituting Equation (2.4.4) and Equation (2.4.5) into Equation (2.4.3) yields

$$Q' = 1 - \frac{Q}{t+100}, \quad \text{so} \quad Q' + \frac{1}{t+100}Q = 1. \tag{2.4.6}$$

By separation of variables, $1/(t+100)$ is a solution of the complementary equation, so the solutions of Equation (2.4.6) are of the form

$$Q = \frac{u}{t+100}, \quad \text{where} \quad \frac{u'}{t+100} = 1, \quad \text{so} \quad u' = t+100.$$

Hence,

$$u = \frac{(t+100)^2}{2} + c. \tag{2.4.7}$$

Since $Q(0) = 10$ and $u = (t+100)Q$, Equation (2.4.7) implies that

$$(100)(10) = \frac{(100)^2}{2} + c,$$

so

$$c = 100(10) - \frac{(100)^2}{2} = -4000$$

and therefore

$$u = \frac{(t+100)^2}{2} - 4000.$$

Hence,

$$Q = \frac{u}{t+100} = \frac{t+100}{2} - \frac{4000}{t+100}.$$

Now let $K(t)$ be the concentration of salt at time t. Then

$$K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$$

This is shown in Figure 2.4.5 .

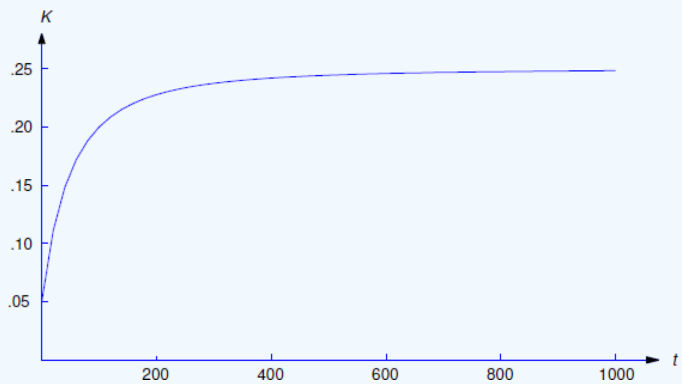


Figure 2.4.5: The graph of $K(t) = \frac{1}{4} - \frac{2000}{(t+100)^2}$ illustrates the concentration of salt in a tank over time. It is reaching toward a stead-value of 0.25 as t approaches infinity. (CC BY-NC-SA 3.0; Russell Herman via A First Course in Differential equations).

2.4E: Exercises for Section 2.4

Exercises 1-17

Solve the following mixture Problems.

Exercise $2.4E.1$

A tank initially contains a solution of 10 pounds of salt in 60 gallons of water. Water with $1/2$ pound of salt per gallon is added to the tank at 6 gal/min, and the resulting solution leaves at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$

Exercise $2.4E.2$

A tank initially contains 40 gallons of pure water. A solution with 1 gram of salt per gallon of water is added to the tank at 3 gal/min, and the resulting solution drains out at the same rate. Find the quantity $Q(t)$ of salt in the tank at time $t > 0$.

Answer

The quantity of salt in the tank at time $t > 0$ is:

$$Q(t) = 40(1 - e^{-3t/40}).$$

Exercise $2.4E.3$

A 150-gallon tank is initially filled with pure water. At time $t = 0$, a salt solution with a concentration of 2 pounds of salt per gallon is added to the tank at a rate of 5 gallons per minute. The well-stirred mixture is drained from the tank at the same rate of 5 gallons per minute.

a. Find the number of pounds of salt in the container as a function of time, $S(t)$.
b. How many minutes does it take for the salt concentration to reach 1.5 pounds per gallon?
c. What does the concentration in the container approach as $t \to \infty$? Does this agree with your intuition?
d. Suppose the tank is much larger than 150 gallons, and everything remains the same except that the mixture is drained at 4 gallons per minute instead of 5. Find the new function $S(t)$. Determine how long it takes for the salt concentration to reach 1.5 pounds per gallon.

Exercise $2.4E.4$

A tank initially contains 100 liters of a salt solution with a concentration of .1 g/liter. A solution with a salt concentration of .3 g/liter is added to the tank at 5 liters/min, and the resulting mixture is drained out at the same rate. Find the concentration $K(t)$ of salt in the tank as a function of t

Answer

The concentration of salt in the tank at time $t > 0$ is: $K(t) = 0.3 - 0.2e^{-t/20}$.

Exercise $2.4E.5$

A 200 gallon tank initially contains 100 gallons of water with 20 pounds of salt. A salt solution with $1/4$ pound of salt per gallon is added to the tank at 4 gal/min, and the resulting mixture is drained out at 2 gal/min. Find the quantity of salt in the tank as it is about to overflow.

Exercise $2.4E.6$

Suppose water is added to a tank at 10 gal/min, but leaks out at the rate of $1/5$ gal/min for each gallon in the tank. What is the smallest capacity the tank can have if the process is to continue indefinitely?

Answer

The smallest tank capacity that allows the process to continue indefinitely is: $V_{\text{min}} = 50 \text{ gallons}$.

Exercise $2.4E.7$

A chemical reaction in a laboratory with volume V (in ft^3) produces q_1 ft^3/min of a noxious gas as a byproduct. The gas is dangerous at concentrations greater than $\overline{c}$, but harmless at concentrations $\leq \overline{c}$. Intake fans at one end of the laboratory pull in fresh air at the rate of q_2 ft^3/min and exhaust fans at the other end exhaust the mixture of gas and air from the laboratory at the same rate. Assuming that the gas is always uniformly distributed in the room and its initial concentration c_0 is at a safe level, find the smallest value of q_2 required to maintain safe conditions in the laboratory for all time

Exercise $2.4E.8$

A 1200-gallon tank initially contains 40 pounds of salt dissolved in 600 gallons of water. Starting at $t_0 = 0$, water that contains $1/2$ pound of salt per gallon is added to the tank at the rate of 6 gal/min and the resulting mixture is drained from the tank at 4 gal/min. Find the quantity $Q(t)$ of salt in the tank at any time $t > 0$ prior to overflow.

Answer

The quantity of salt in the tank at any time $t > 0$ prior to overflow is: $Q(t) = \frac{600+2t}{2} - \frac{93600000}{(600+2t)^2}$.

Exercise $2.4E.9$

Tank T_1 initially contain 50 gallons of pure water. Starting at $t_0 = 0$, water that contains 1 pound of salt per gallon is poured into T_1 at the rate of 2 gal/min. The mixture is drained from T_1 at the same rate into a second tank T_2, which initially contains 50 gallons of pure water. Also starting at $t_0 = 0$, a mixture from another source that contains 2 pounds of salt per gallon is poured into T_2 at the rate of 2 gal/min. The mixture is drained from T_2 at the rate of 4 gal/min.

a. Find a differential equation for the quantity $Q(t)$ of salt in tank T_2 at time $t > 0$.
b. Solve the equation derived in (a) to determine $Q(t)$.
c. Find $\lim_{t\to\infty} Q(t)$.

Exercise $2.4E.10$

Identical tanks T_1 and T_2 initially contain W gallons each of pure water. Starting at $t_0 = 0$, a salt solution with constant concentration c is pumped into T_1 at r gal/min and drained from T_1 into T_2 at the same rate. The resulting mixture in T_2 is also drained at the same rate. Find the concentrations $c_1(t)$ and $c_2(t)$ in tanks T_1 and T_2 for $t > 0$.

Answer

Concentration in Tank T_1: $c_1(t) = c(1 - e^{-(r/W)t})$. Concentration in Tank T_2: $c_2(t) = c\left((1 - e^{-(r/W)t}) - (r/W)te^{-(r/W)t}\right)$.

Exercise $2.4E.11$

Tanks T_1 and T_2 have capacities W_1 and W_2 liters, respectively. Initially they are both full of dye solutions with concentrations c_1 and c_2 grams per liter. Starting at $t_0 = 0$, the solution from T_1 is pumped into T_2 at a rate of r liters per minute, and the solution from T_2 is pumped into T_1 at the same rate.

a. Find the concentrations $c_1(t)$ and $c_2(t)$ of the dye in T_1 and T_2 for $t > 0$.
b. Find $\lim_{t\to\infty} c_1(t)$ and $\lim_{t\to\infty} c_2(t)$.

Exercise $2.4E.12$

An infinite sequence of identical tanks T_1, T_2, ..., T_n, ..., initially contain W gallons each of pure water. They are hooked together so that fluid drains from T_n into T_{n+1} $(n = 1, 2, \cdots)$. A salt solution is circulated through the tanks so that it enters and leaves each tank at the constant rate of r gal/min. The solution has a concentration of c pounds of salt per gallon when it enters T_1.

a. Find the concentration $c_n(t)$ in tank T_n for $t > 0$.
b. Find $\lim_{t\to\infty} c_n(t)$ for each n.

Answer

Concentration in tank T_n: $c_n(t) = c\left(1 - e^{-(r/W)t} \sum_{k=0}^{n-1} \frac{(r/W)^k}{k!} t^k\right)$. Long-term behavior: $\lim_{t\to\infty} c_n(t) = c$.

Exercise $2.4E.13$

Consider the mixing problem of Example 2.4.2, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{150} Q = 2$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} Q(t)$?.
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 - \sin(e^{-t}).$$

Exercise $2.4E.14$

Consider the mixing problem of Example 2.4.3 in a tank with infinite capacity, but without the assumption that the mixture is stirred instantly so that the salt is always uniformly distributed throughout the mixture. Assume instead that the distribution approaches uniformity as $t \to \infty$. In this case the differential equation for Q is of the form

$$Q' + \frac{a(t)}{t+100} Q = 1$$

where $\lim_{t\to\infty} a(t) = 1$.

a. Let $K(t)$ be the concentration of salt at time t. Assuming that $Q(0) = Q_0$, can you guess the value of $\lim_{t\to\infty} K(t)$?
b. Use numerical methods to confirm your guess in the these cases:

$$\text{(i) } a(t) = t/(1+t) \quad \text{(ii) } a(t) = 1 - e^{-t^2} \quad \text{(iii) } a(t) = 1 + \sin(e^{-t}).$$

Answer

a. The given differential equation is: $Q' + \frac{a(t)}{t+100} Q = 1,$ where $\lim_{t\to\infty} a(t) = 1$. The concentration of salt is defined as:

$$K(t) = \frac{Q(t)}{t+100}. \tag{2.4E.1}$$

By analyzing the long-term behavior of the equation, we predict: $\lim_{t\to\infty} K(t) = 1.$ We solved the differential equation numerically for the following cases:

1. $a(t) = \frac{t}{1+t}$
2. $a(t) = 1 - e^{-t^2}$
3. $a(t) = 1 + \sin(e^{-t})$.

b. The numerical solutions confirm that: $\lim_{t\to\infty} K(t) = 1.$ Thus, regardless of the specific function $a(t)$, as long as $\lim_{t\to\infty} a(t) = 1$, the final steady-state concentration is: $\lim_{t\to\infty} K(t) = 1.$

? Exercise $2.4E.15$

A 250-gallon tank is initially filled with pure water. At time $t = 0$, a salt solution with a concentration of 4 pounds of salt per gallon is added to the tank at a rate of 3 gallons per minute. The well-stirred mixture is drained from the tank at the same rate of 3 gallons per minute.

a. Find the number of pounds of salt in the container as a function of time, $S(t)$.
b. How many minutes does it take for the salt concentration to reach 2.5 pounds per gallon?
c. What does the concentration in the container approach as $t \to \infty$? \item Does this agree with your intuition?
d. Suppose the tank is much larger than 250 gallons, and everything remains the same except that the mixture is drained at 2 gallons per minute instead of 3. Find the new function $S(t)$. Determine how long it takes for the salt concentration to reach 2.5 pounds per gallon. \end{enumerate}

? Exercise $2.4E.16$

A 200-gallon tank is initially filled with pure water. At time $t = 0$, a salt solution with a concentration of 3 pounds of salt per gallon is added to the tank at a rate of 4 gallons per minute. The well-stirred mixture is drained from the tank at the same rate of 4 gallons per minute.

a. Find the number of pounds of salt in the container as a function of time, $S(t)$.
b. How many minutes does it take for the salt concentration to reach 2 pounds per gallon?
c. What does the concentration in the container approach as $t \to \infty$? Does this agree with your intuition?
d. Suppose the tank is much larger than 200 gallons, and everything remains the same except that the mixture is drained at 3 gallons per minute instead of 4. Find the new function $S(t)$. Determine how long it takes for the salt concentration to reach 2 pounds per gallon.

Answer

a. Function for the amount of salt in the tank: $S(t) = 600(1 - e^{-t/50})$.
b. Time for the concentration to reach 2 pounds per gallon: $t \approx 55 \text{ minutes}$.
c. Long-term concentration as $t \to \infty$: $3 \text{ pounds per gallon}$.
d. If the outflow rate is 3 gallons per minute: New equation: $S(t) = 400(1 - e^{-3t/200})$. Time to reach 2 pounds per gallon: The concentration never reaches exactly 2 pounds per gallon in finite time.

? Exercise $2.4E.17$

You make two gallons of chili for a party. The recipe calls for two teaspoons of hot sauce per gallon, but you accidentally put in two tablespoons per gallon. You decide to serve the chili anyway. Assume that: Guests consume 1 cup per minute. You replace what was taken with beans and tomatoes without any hot sauce. Unit conversions: 1 gallon = 16 cups, and 1 tablespoon = 3 teaspoons. Answer the following:

a. Write down the differential equation and initial condition for the amount of hot sauce as a function of time in this mixture-type problem.
b. Solve the initial value problem.
c. Determine how long it will take to reduce the concentration to the recipe's suggested level.

Answer

a. Differential equation: $\frac{dS}{dt} = -\frac{S}{32}, \quad S(0) = 12.$
b. Solution to the initial value problem:} $S(t) = 12e^{-t/32}$.
c. Time to reach the recipe's concentration: $t \approx 35.2$ minutes.

2.5: Orthogonal Trajectories of Curves

Learning Objectives

- Understand and solve differential equations to find orthogonal trajectories of a given family of curves.

There are many problems from geometry which have lead to the study of differential equations. One such problem is the construction of orthogonal trajectories. Give a a family of curves, $y_1(x;a)$, we seek another family of curves $y_2(x;c)$ such that the second family of curves are perpendicular the to given family. This means that the tangents of two intersecting curves at the point of intersection are perpendicular to each other. The slopes of the tangent lines are given by the derivatives $y_1'(x)$ and $y_2'(x)$. We recall from elementary geometry that the slopes of two perpendicular lines are related by

$$y_2'(x) = -\frac{1}{y_1'(x)}$$

Example 2.5.1

Find a family of orthogonal trajectories to the family of parabolas $y_1(x;a) = ax^2$.

Solution

We note that the new collection of curves has to satisfy the equation

$$y_2'(x) = -\frac{1}{y_1'(x)} = -\frac{1}{2ax}$$

Before solving for $y_2(x)$, we need to eliminate the parameter a. From the given function, we have that $a = \frac{y}{x^2}$. Inserting this into the equation, we have

$$y'(x) = -\frac{1}{2ax} = -\frac{1}{2\left(\frac{y}{x^2}\right)x} = -\frac{x}{2y}$$

Thus, to find $y_2(x)$, we have to solve the differential equation

$$2yy' + x = 0$$

Noting that $\left(y^2\right)' = 2yy'$ and $\left(\frac{1}{2}x^2\right)' = x$, this (exact) equation can be written as

$$\frac{d}{dx}\left(y^2 + \frac{1}{2}x^2\right) = 0$$

Integrating, we find the family of solutions,

$$y^2 + \frac{1}{2}x^2 = k$$

In Figure 2.5.1 we plot both families of orthogonal curves.

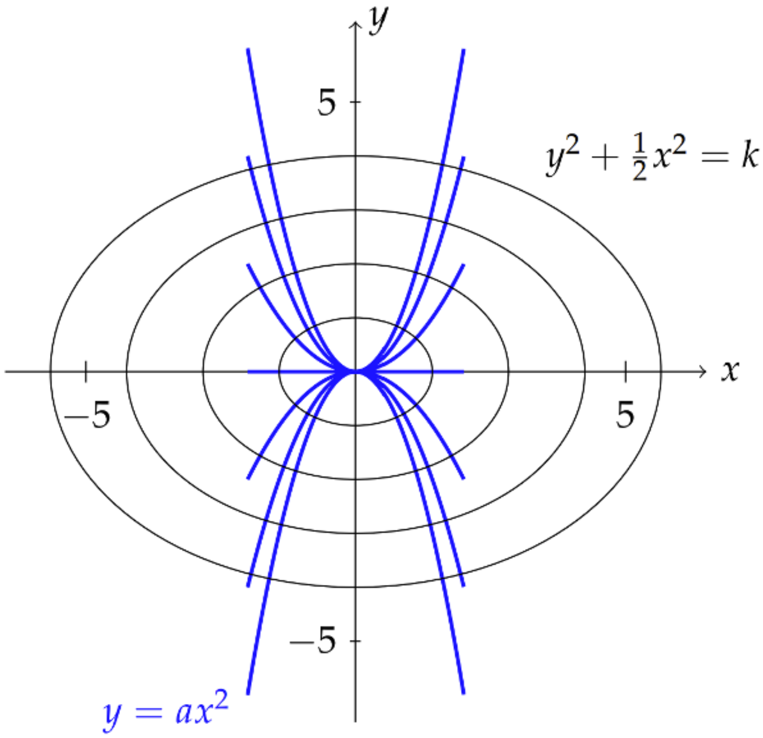


Figure 2.5.1: The family of blue parabolas $y = ax^2$ are orthogonal trajectories of the family of black ellipses $y^2 + \dfrac{1}{2}x^2 = k$. Multiple parabolas and ellipses are drawn on the xy-plane where $-5 \leq x \leq 5$ and $-5 \leq y \leq 5$ (CC BY-NC-SA 3.0, Russell Herman via A First Course in Differential equations).

2.5E: Exercises for Section 2.5

Exercises 1-10

Solve the following orthogonal trajectory curves.

Exercise $2.5E.1$

Find the family of orthogonal curves to the given family of curves: $y^2 = ax^3$.

Exercise $2.5E.2$

Find the family of orthogonal curves to the given family of curves: $y = ae^x$.

Answer

$y^2 + 2x = C$

Exercise $2.5E.3$

Find the family of orthogonal curves to the given family of curves: $y^2 = ax^4$.

Exercise $2.5E.4$

Find the family of orthogonal curves to the given family of curves: $y = ae^{2x}$.

Answer

$y^2 + x = C$

Exercise $2.5E.5$

Find the family of orthogonal curves to the given family of curves: $y = ax^2$.

Exercise $2.5E.6$

Find the family of orthogonal curves to the given family of curves: $x^2 + y^2 = ay$.

Exercise $2.5E.7$

Find the family of orthogonal curves to the given family of curves: $x^2 + 4xy + y^2 = c$

Answer

$(y - x)^3 (y + x) = k$

Exercise $2.5E.8$

Find the family of orthogonal curves to the given family of curves: $y = ce^2 x$

Exercise $2.5E.9$

Find the family of orthogonal curves to the given family of curves: $xye^{x^2} = c$

Answer

$y^2 = -\frac{1}{2}\ln(1+2x^2)+k$

? Exercise $2.5E.10$

Find the family of orthogonal curves to the given family of curves: $y = \frac{ce^x}{x}$

2.6: Pursuit Curves

Learning Objectives

- Understand and derive the pursuit curve equations for a moving body following a target, and determine conditions for interception.

Another application that is interesting is to find the path that a body traces out as it moves towards a fixed point or another moving body. Such curves are know as pursuit curves. These could model aircraft or submarines following targets, or predators following prey. We demonstrate this with an example.

Example $2.6.1$

A hawk at point (x, y) sees a sparrow traveling at speed v along a straight line. The hawk flies towards the sparrow at constant speed w but always in a direction along line of sight between their positions. If the hawk starts out at the point $(a, 0)$ at $t=0$ when the sparrow is at $(0,0)$, then what is the path the hawk needs to follow? Will the hawk catch the sparrow? The situation is shown in Figure $2.6.1$. We pick the path of the sparrow to be along the y-axis. Therefore, the sparrow is at position $(0, vt)$.

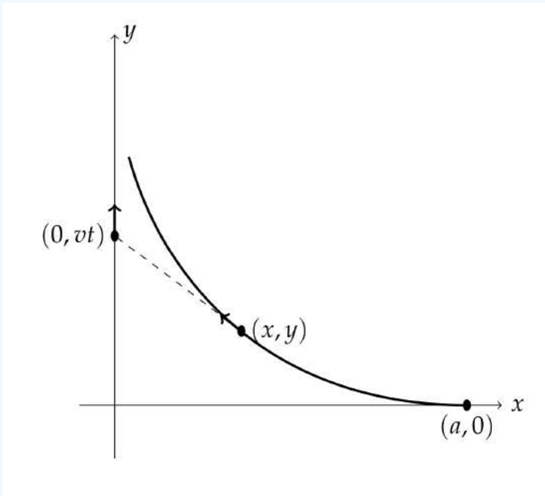


Figure $2.6.1$: A hawk at point (x, y) sees a sparrow at point $(0, vt)$ and always follows the straight line between these points. (CC BY-NC-SA 3.0; Russell Herman via A First Course in Differential equations).

Solution

First we need the equation of the line of sight between the points (x, y) and $(0, vt)$. Considering that the slope of the line is the same as the slope of the tangent to the path, $y=y(x)$, we have

$$y^{\prime}=\frac{y-vt}{x}$$

The hawk is moving at a constant speed, w. Since the speed is related to the time through the distance the hawk travels. We need to find the arclength of the path between $(a, 0)$ and (x, y). This is given by

$$L=\int ds=\int_x^a \sqrt{1+\left[y^{\prime}(x)\right]^2}\, dx .$$

The distance is related to the speed, w, and the time, t, by $L=wt$. Eliminating the time using $y^{\prime}=\frac{y-vt}{x}$, we have

$$\int_x^a \sqrt{1+\left[y^{\prime}(x)\right]^2}\, dx=\frac{w}{v}\left(y-x y^{\prime}\right)$$

Furthermore, we can differentiate this result with respect to x to get rid of the integral,

$$\sqrt{1+\left[y^{\prime}(x)\right]^2}=\frac{w}{v} x y^{\prime\prime}$$

Even though this is a second order differential equation for $y(x)$, it is a first order separable equation in the speed function $z(x)=y'(x)$. Namely,

$$\frac{w}{v} x z'=\sqrt{1+z^{2}}$$

Separating variables, we find

$$\frac{w}{v} \int \frac{dz}{\sqrt{1+z^{2}}}=\int \frac{dx}{x}$$

The integrals can be computed using standard methods from calculus. We can easily integrate the right hand side,

$$\int \frac{dx}{x}=\ln |x|+c_{1}.$$

The left hand side takes a little extra work using trigonometric substitution method. Let $z=\tan \theta$. Then $dz=\sec^{2} \theta d\theta$. The methods proceeds as follows:

$$\begin{aligned}
\int \frac{dz}{\sqrt{1+z^{2}}} &=\int \frac{\sec^{2} \theta}{\sqrt{1+\tan^{2} \theta}} d\theta \\
&=\int \sec \theta d\theta \\
&=\ln (\tan \theta+\sec \theta)+c_{2} \\
&=\ln \left(z+\sqrt{1+z^{2}}\right)+c_{2}
\end{aligned}$$

Putting these together, we have for $x>0$,

$$\ln \left(z+\sqrt{1+z^{2}}\right)=\frac{v}{w} \ln x+C$$

Using the initial condition $z=y'=0$ and $x=a$ at $t=0$,

$$0=\frac{v}{w} \ln a+C$$

or $C=-\frac{v}{w} \ln a$.

Using this value for c, we find

$$\begin{aligned}
\ln \left(z+\sqrt{1+z^{2}}\right) &=\frac{v}{w} \ln x-\frac{v}{w} \ln a \\
\ln \left(z+\sqrt{1+z^{2}}\right) &=\frac{v}{w} \ln \frac{x}{a} \\
\ln \left(z+\sqrt{1+z^{2}}\right) &=\ln \left(\frac{x}{a}\right)^{\frac{v}{w}} \\
z+\sqrt{1+z^{2}} &=\left(\frac{x}{a}\right)^{\frac{v}{w}}
\end{aligned}$$

We can solve for $z=y'$, to find

$$y'=\frac{1}{2}\left[\left(\frac{x}{a}\right)^{\frac{v}{w}}-\left(\frac{x}{a}\right)^{-\frac{v}{w}}\right]$$

Integrating,

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + k$$

The integration constant, k, can be found knowing $y(a)=0$. This gives

$$\begin{aligned} 0 &= \frac{a}{2}\left[\frac{1}{1+\frac{v}{w}} - \frac{1}{1-\frac{v}{w}}\right] + k \\ k &= \frac{a}{2}\left[\frac{1}{1-\frac{v}{w}} - \frac{1}{1+\frac{v}{w}}\right] \\ &= \frac{avw}{w^2 - v^2} \end{aligned}$$

The full solution for the path is given by

$$y(x) = \frac{a}{2}\left[\frac{\left(\frac{x}{a}\right)^{1+\frac{v}{w}}}{1+\frac{v}{w}} - \frac{\left(\frac{x}{a}\right)^{1-\frac{v}{w}}}{1-\frac{v}{w}}\right] + \frac{avw}{w^2 - v^2}$$

Can the hawk catch the sparrow? This would happen if there is a time when $y(0)=vt$. Inserting $x=0$ into the solution, we have $y(0) = \frac{avw}{w^2 - v^2} = vt$. This is possible if $w > v$.

2.6E: Exercises for Section 2.6

Exercise $2.6E.1$

A cat sitting in a field suddenly sees a standing dog. To save its life, the cat runs away in a straight line with speed u. Without any delay, the dog starts running with constant speed $v > u$ to catch the cat. Initially, v is perpendicular to u and L is the initial separation between the two. If the dog always changes its direction so that it is always heading directly at the cat, find the time the dog takes to catch the cat in terms of v, u, and L. Assume that:

- the dog starts at the point $S = (L, 0)$ and the cat at the origin $O = (0, 0)$.
- the cat moves in the positive direction along the y-axis, and the dog's path describes a curve of pursuit.

Answer

$$T = L\frac{v}{v^2 - u^2}.$$

CHAPTER OVERVIEW

3: Higher order linear ODEs

We have already studied the basics of differential equations, including separable first-order equations. In this chapter, we go a little further and look at second-order equations, which are equations containing second derivatives of the dependent variable. The solution methods we examine are different from those discussed earlier, and the solutions tend to involve trigonometric functions as well as exponential functions. Here we concentrate primarily on second-order equations with constant coefficients.

3.1: Second order linear ODEs

Learning Objectives

- Understand and solve second-order linear differential equations, both homogeneous and nonhomogeneous.
- Learn techniques such as superposition and reduction of order to construct general solutions.

Let us consider the general second order linear differential equation

$$A(x)y'' + B(x)y' + C(x)y = F(x).$$

We usually divide through by $A(x)$ to get

$$y'' + p(x)y' + q(x)y = f(x),$$

where $p(x) = \frac{B(x)}{A(x)}$, $q(x) = \frac{C(x)}{A(x)}$, and $f(x) = \frac{F(x)}{A(x)}$. The word *linear* means that the equation contains no powers nor functions of y, y', and y''.

In the special case when $f(x) = 0$ we have a so-called *homogeneous equation*

$$y'' + p(x)y' + q(x)y = 0, \tag{3.1.1}$$

We have already seen some second order linear homogeneous equations:

$$\begin{array}{lll} y'' + k^2 y = 0 & \text{Two solutions are:} & y_1 = \cos(kx), \quad y_2 = \sin(kx). \\ y'' - k^2 y = 0 & \text{Two solutions are:} & y_1 = e^{kx}, \quad y_2 = e^{-kx}. \end{array}$$

If we know two solutions of a linear homogeneous equation, we know a lot more of them.

Theorem 3.1.1 Superposition

Suppose y_1 and y_2 are two solutions of the homogeneous Equation $(3.1.1)$. Then

$$y(x) = C_1 y_1(x) + C_2 y_2(x),$$

for arbitrary constants C_1 and C_2.

That is, we can add solutions together and multiply them by constants to obtain new and different solutions. We call the expression $C_1 y_1 + C_2 y_2$ a *linear combination* of y_1 and y_2. Let us prove this theorem; the proof is very enlightening and illustrates how linear equations work.

Proof

Let $y = C_1 y_1 + C_2 y_2$. Then

$$\begin{aligned} y'' + py' + qy &= (C_1 y_1 + C_2 y_2)'' + p(C_1 y_1 + C_2 y_2)' + q(C_1 y_1 + C_2 y_2) \\ &= C_1 y_1'' + C_2 y_2'' + C_1 p y_1' + C_2 p y_2' + C_1 q y_1 + C_2 q y_2 \\ &= C_1 (y_1'' + p y_1' + q y_1) + C_2 (y_2'' + p y_2' + q y_2) \\ &= C_1 .0 + C_2 .0 = 0 \end{aligned}$$

The proof becomes even simpler to state if we use the operator notation. An *operator* is an object that eats functions and spits out functions (kind of like what a function is, which eats numbers and spits out numbers). Define the operator L by

$$Ly = y'' + py' + qy.$$

The differential equation now becomes $Ly = 0$. The operator (and the equation) L being linear means that $L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2$. The proof in Theorem 3.1.1 becomes

$$Ly = L(C_1 y_1 + C_2 y_2) = C_1 L y_1 + C_2 L y_2 = C_1 .0 + C_2 .0 = 0$$

Two different solutions to the second equation $y'' - k^2 y = 0$ are $y_1 = \cosh(kx)$ and $y_2 = \sinh(kx)$. Let us remind ourselves of the definition, $\cosh x = \frac{e^x + e^{-x}}{2}$ and $\sinh x = \frac{e^x - e^{-x}}{2}$. Therefore, these are solutions by superposition as they are linear combinations of the two exponential solutions.

The functions sinh and cosh are sometimes more convenient to use than the exponential. Let us review some of their properties.

$$\begin{aligned} &\cosh 0 = 1 && \sinh 0 = 0, \\ &\frac{d}{dx}(\cosh x) = \sinh x, && \frac{d}{dx}(\sinh x) = \cosh x, \\ &\cosh^2 x - \sinh^2 x = 1. \end{aligned}$$

Example 3.1.1

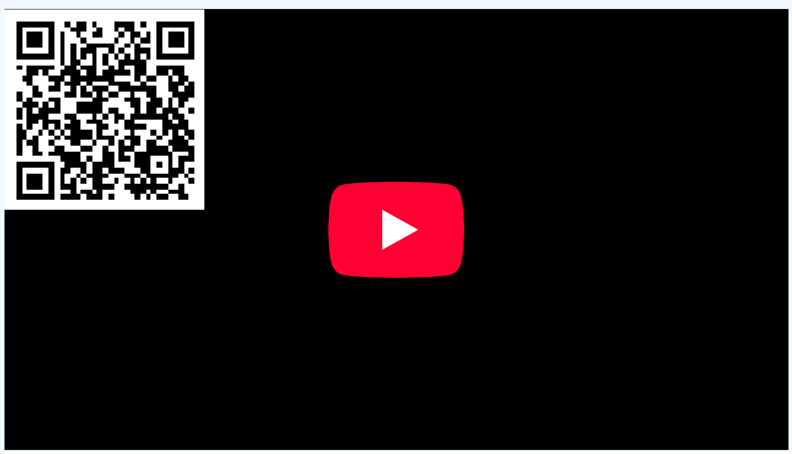
Find a particular solution to a differential equation. The solution of a certain differential equation is of the form

$$y = ae^{4t} + 6b^{6t} \text{ where a and b are constants}$$

The solution has initial conditions $y(0) = 2$ and $y'(0) = 3$. Find the solution by using the initial conditions to get linear equations for a and b.

Solution

- **Video Length**: 4 minutes 42 seconds.
- **Context**: The video demonstrates how to solve a linear differential equation with exponential terms by applying given initial conditions.



Theorem 3.1.2

Suppose $p(x)$, $q(x)$, and $f(x)$ are continuous functions on some interval I containing a with a, b_0 and b_1 constants. The equation

$$y'' + p(x)y' + q(x)y = f(x).$$

has exactly *one* solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$

For example, the equation $y'' + k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cos(kx) + \frac{b_1}{k} \sin(kx)$$

The equation $y'' - k^2 y = 0$ with $y(0) = b_0$ and $y'(0) = b_1$ has the solution

$$y(x) = b_0 \cosh(kx) + \frac{b_1}{k} \sinh(kx)$$

Using $\cosh$ and $\sinh$ in this solution allows us to solve for the initial conditions in a cleaner way than if we have used the exponentials.

The initial conditions for a second order ODE consist of two equations. Common sense tells us that if we have two arbitrary constants and two equations, then we should be able to solve for the constants and find a solution to the differential equation satisfying the initial conditions.

Theorem 3.1.3

Let $p(x)$ and $q(x)$ be continuous functions and let y_1 and y_2 be two linearly independent solutions to the homogeneous Equation (3.1.1). Then every other solution is of the form

$$y = C_1 y_1 + C_2 y_2.$$

That is, $y = C_1 y_1 + C_2 y_2$ is the general solution.

For example, we found the solutions $y_1 = \sin x$ and $y_2 = \cos x$ for the equation $y'' + y = 0$. It is not hard to see that sine and cosine are not constant multiples of each other. If $\sin x = A \cos x$ for some constant A, we let $x = 0$ and this would imply $A = 0$. But then $\sin x = 0$ for all x, which is preposterous. So y_1 and y_2 are linearly independent. Hence,

$$y = C_1 \cos x + C_2 \sin x$$

is the general solution to $y'' + y = 0$.

For two functions, checking linear independence is rather simple. Let us see another example. Consider $y'' - 2x^{-2} y = 0$. Then $y_1 = x^2$ and $y_2 = \frac{1}{x}$ are solutions. To see that they are linearly independent, suppose one is a multiple of the other: $y_1 = A y_2$, we just have to find out that A cannot be a constant. In this case we have $A = \frac{y_1}{y_2} = x^3$, this most decidedly not a constant. So $y = C_1 x^2 + C_2 \frac{1}{x}$ is the general solution.

If you have one solution to a second order linear homogeneous equation, then you can find another one. This is the *reduction of order method*. The idea is that if we somehow found y_1 as a solution of $y'' + p(x) y' + q(x) y = 0$ we try a second solution of the form $y_2(x) = y_1(x) v(x)$. We just need to find v. We plug y_2 into the equation:

$$\begin{aligned} 0 = y_2'' + p(x) y_2' + q(x) y_2 &= y_1'' v + 2 y_1' v' + y_1 v'' + p(x)(y_1' v + y_1 v') + q(z) y_1 v \\ &= y_1 v'' + (2 y_1' + p(x) y_1) v' + \overbrace{\left(y_1'' + p(x) y_1' + q(x) y_1\right)}^{0} v. \end{aligned} \tag{3.1.2}$$

In other words, $y_1 v'' + (2 y_1' + p(x) y_1) v' = 0$. Using $w = v'$ we have the first order linear equation $y_1 w' + (2 y_1' + p(x) y_1) w = 0$. After solving this equation for w (integrating factor), we find v by antidifferentiating w. We then form y_2 by computing $y_1 v$. For example, suppose we somehow know $y_1 = x$ is a solution to $y'' + x^{-1} y' - x^{-2} y = 0$. The equation for w is then $x w' + 3w = 0$. We find a solution, $w = C x^{-3}$, and we find an antiderivative $v = \frac{-C}{2x^2}$. Hence $y_2 = y_1 v = \frac{-C}{2x}$. Any C works and so $C = -2$ makes $y_2 = \frac{1}{x}$. Thus, the general solution is $y = C_1 x + C_2 \frac{1}{x}$.

Since we have a formula for the solution to the first order linear equation, we can write a formula for y_2:

$$y_2(x) = y_1(x) \int \frac{e^{-\int p(x)\, dx}}{\left(y_1(x)\right)^2}\, dx$$

However, it is much easier to remember that we just need to try $y_2(x) = y_1(x) v(x)$ and find $v(x)$ as we did above. Also, the technique works for higher order equations too: you get to reduce the order for each solution you find. So it is better to remember how to do it rather than a specific formula.

We will study the solution of nonhomogeneous equations in Section 3.9. We will first focus on finding general solutions to homogeneous equations.

3.1E: Exercises for Section 3.1

Exercises 1-4

Solve the following exercises by demonstrating key principles of differential equations, such as verifying solutions, proving linear independence, and applying reduction of order.

Exercise $3.1E.1$

Show that $y=e^x$ and $y=e^{2x}$ are linearly independent

Exercise $3.1E.2$

Take $y''+5y=10x+5$. Find (guess!) a solution.

Exercise $3.1E.3$

Prove the superposition principle for nonhomogeneous equations. Suppose that y_1 is a solution to $Ly_1=f(x)$ and y_2 is a solution to $Ly_2=g(x)$ (same linear operator L). Show that $y=y_1+y_2$ solves $Ly=f(x)+g(x)$

Exercise $3.1E.4$

For the equation $x^2y''-xy'=0$, find two solutions, show that they are linearly independent and find the general solution. Hint: Try $y=x'$.

Exercises 5-14

Equations of the form $ax^2y''+bxy'+cy=0$ are called *Euler's equations* or *Cauchy-Euler equations*. They are solved by trying $y=x^r$ and solving for r (we can assume that $x\geqslant 0$ for simplicity).

Exercise $3.1E.5$

Suppose that $(b-a)^2-4ac>0$.

a. Find a formula for the general solution of $ax^2y''+bxy'+cy=0$. Hint: Try $y=x^r$ and find a formula for r.
b. What happens when $(b-a)^2-4ac=0$ or $(b-a)^2-4ac<0$?

We will revisit the case when $(b-a)^2-4ac<0$ later.

Exercise $3.1E.6$

Suppose that $(b-a)^2-4ac>0$ and suppose $(b-a)^2-4ac=0$. Find a formula for the general solution of $ax^2y''+bxy'+cy=0$. Hint: Try $y=x^r\ln x$ for the second solution.

Exercise $3.1E.7$

Suppose y_1 is a solution to $y''+p(x)y'+q(x)y=0$. Show that

$$y_2(x)=y_1(x)\int\frac{e^{-\int p(x)dx}}{(y_1(x))^2}dx \tag{3.1E.1}$$

is also a solution.

Note: If you wish to come up with the formula for reduction of order yourself, start by trying $y_2(x)=y_1(x)v(x)$. Then plug y_2 into the equation, use the fact that y_1 is a solution, substitute $w=v'$, and you have a first order linear equation in w. Solve for w and then for v. When solving for w, make sure to include a constant of integration. Let us solve some famous equations using the method.

Exercise $3.1E.8$

Take $(1-x^2)y''-xy'+y=0$.

a. Show that $y=x$ is a solution.
b. Use reduction of order to find a second linearly independent solution.
c. Write down the general solution.

Exercise$3.1E.9$

Take $y''-2xy'+4y=0$.

a. Show that $y=1-2x^2$ is a solution.
b. Use reduction of order to find a second linearly independent solution.
c. Write down the general solution.

Exercise $3.1E.10$

Are $\sin(x)$ and e^x linearly independent? Justify.

Answer

Yes. To justify try to find a constant A such that $\sin(x)=Ae^x$ for all x.

Exercise $3.1E.11$

Are e^x and e^{x+2} linearly independent? Justify.

Answer

No. $e^{x+2}=e^2e^x$.

Exercise $3.1E.12$

Guess a solution to $y''+y'+y=5$.

Answer

$y=5$

Exercise $3.1E.13$

Find the general solution to $xy''+y'=0$. Hint: Notice that it is a first order ODE in y'.

Answer

$y=C_1\ln(x)+C_2$

Exercise $3.1E.14$

Write down an equation (guess) for which we have the solutions e^x and e^{2x}. Hint: Try an equation of the form $y''+Ay'+By=0$ for constants A and B, plug in both e^x and e^{2x} and solve for A and B.

Answer

$y''-3y'+2y=0$

3.2: The Method of Undetermined Coefficients I

Learning Objectives

- Understand the method of undetermined coefficients to find particular solutions of nonhomogeneous second-order linear differential equations with constant coefficients.
- Apply the principle of superposition to combine particular solutions for equations with multiple nonhomogeneous terms.

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\alpha x}G(x), \tag{3.2.1}$$

where α is a constant and G is a polynomial.

The general solution of Equation (3.2.1) is $y = y_p + c_1y_1 + c_2y_2$, where y_p is a particular solution of Equation (3.2.1) and $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation

$$ay'' + by' + cy = 0.$$

We showed how to find $\{y_1, y_2\}$. In this section we'll show how to find y_p. The procedure that we'll use is called *the method of undetermined coefficients*.

Example 3.2.1

Find a particular solution of

$$y'' - 7y' + 12y = 4e^{2x}. \tag{3.2.2}$$

Then find the general solution.

Solution

Substituting $y_p = Ae^{2x}$ for y in Equation (3.2.2) will produce a constant multiple of Ae^{2x} on the left side of Equation (3.2.2), so it may be possible to choose A so that y_p is a solution of Equation (3.2.2). Let's try it; if $y_p = Ae^{2x}$ then

$$y_p'' - 7y_p' + 12y_p = 4Ae^{2x} - 14Ae^{2x} + 12Ae^{2x} = 2Ae^{2x} = 4e^{2x}$$

if $A = 2$. Therefore $y_p = 2e^{2x}$ is a particular solution of Equation (3.2.2). To find the general solution, we note that the characteristic polynomial of the complementary equation

$$y'' - 7y' + 12y = 0 \tag{3.2.3}$$

is $p(r) = r^2 - 7r + 12 = (r-3)(r-4)$, so $\{e^{3x}, e^{4x}\}$ is a fundamental set of solutions of Equation (3.2.3). Therefore the general solution of Equation (3.2.2) is

$$y = 2e^{2x} + c_1e^{3x} + c_2e^{4x}.$$

Example 3.2.2

Find a particular solution of

$$y'' - 7y' + 12y = 5e^{4x}. \tag{3.2.4}$$

Then find the general solution.

Solution

Fresh from our success in finding a particular solution of Equation (3.2.2), where we chose $y_p = Ae^{2x}$ because the right side of Equation (3.2.2) is a constant multiple of e^{2x}, it may seem reasonable to try $y_p = Ae^{4x}$ as a particular solution of Equation (3.2.4). However, this will not work, since we saw in Example 3.2.1 that e^{4x} is a solution of the complementary Equation (3.2.3), so substituting $y_p = Ae^{4x}$ into the left side of Equation (3.2.4) produces zero on the left, no matter how we choose A. To discover a suitable form for y_p, we use the same approach that we used in Section 3.1 to find a second solution of

$$ay'' + by' + cy = 0$$

in the case where the characteristic equation has a repeated real root: we look for solutions of Equation (3.2.4) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting

$$y = ue^{4x}, \quad y' = u'e^{4x} + 4ue^{4x}, \quad \text{and} \quad y'' = u''e^{4x} + 8u'e^{4x} + 16ue^{4x} \tag{3.2.5}$$

into Equation (3.2.4) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 7(u' + 4u) + 12u = 5,$$

or

$$u'' + u' = 5.$$

By inspection we see that $u_p = 5x$ is a particular solution of this equation, so $y_p = 5xe^{4x}$ is a particular solution of Equation (3.2.4). Therefore

$$y = 5xe^{4x} + c_1e^{3x} + c_2e^{4x}$$

is the general solution.

Example 3.2.3

Find a particular solution of

$$y'' - 8y' + 16y = 2e^{4x}. \tag{3.2.6}$$

Solution

Since the characteristic polynomial of the complementary equation

$$y'' - 8y' + 16y = 0 \tag{3.2.7}$$

is $p(r) = r^2 - 8r + 16 = (r-4)^2$, both $y_1 = e^{4x}$ and $y_2 = xe^{4x}$ are solutions of Equation (3.2.7). Therefore Equation (3.2.6) does not have a solution of the form $y_p = Ae^{4x}$ or $y_p = Axe^{4x}$. As in Example 3.2.2 , we look for solutions of Equation (3.2.6) in the form $y = ue^{4x}$, where u is a function to be determined. Substituting from Equation (3.2.5) into Equation (3.2.6) and canceling the common factor e^{4x} yields

$$(u'' + 8u' + 16u) - 8(u' + 4u) + 16u = 2,$$

or

$$u'' = 2.$$

Integrating twice and taking the constants of integration to be zero shows that $u_p = x^2$ is a particular solution of this equation, so $y_p = x^2e^{4x}$ is a particular solution of Equation (3.2.4). Therefore

$$y = e^{4x}(x^2 + c_1x + c_2)$$

is the general solution.

The preceding examples illustrate the following facts concerning the form of a particular solution y_p of a constant coefficient equation

$$ay'' + by' + cy = ke^{\alpha x},$$

where k is a nonzero constant:

a. If $e^{\alpha x}$ isn't a solution of the complementary equation

$$ay'' + by' + cy = 0, \tag{3.2.8}$$

then $y_p = Ae^{\alpha x}$, where A is a constant. (See Example 3.2.1).

b. If $e^{\alpha x}$ is a solution of Equation (3.2.8) but $xe^{\alpha x}$ is not, then $y_p = Axe^{\alpha x}$, where A is a constant. (See Example 3.2.2).

c. If both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of Equation (3.2.8), then $y_p = Ax^2e^{\alpha x}$, where A is a constant. (See Example 3.2.3).

In all three cases you can just substitute the appropriate form for y_p and its derivatives directly into

$$ay_p'' + by_p' + cy_p = ke^{\alpha x},$$

and solve for the constant A, as we did in Example 3.2.1 . However, if the equation is

$$ay'' + by' + cy = ke^{\alpha x}G(x),$$

where G is a polynomial of degree greater than zero, we recommend that you use the substitution $y = ue^{\alpha x}$ as we did in Examples 3.2.2 and 3.2.3 . The equation for u will turn out to be

$$au'' + p'(\alpha)u' + p(\alpha)u = G(x), \tag{3.2.9}$$

where $p(r) = ar^2 + br + c$ is the characteristic polynomial of the complementary equation and $p'(r) = 2ar + b$. However, you shouldn't memorize this since it is easy to derive the equation for u in any particular case. Note, however, that if $e^{\alpha x}$ is a solution of the complementary equation then $p(\alpha) = 0$, so Equation (3.2.9) reduces to

$$au'' + p'(\alpha)u' = G(x),$$

while if both $e^{\alpha x}$ and $xe^{\alpha x}$ are solutions of the complementary equation then $p(r) = a(r-\alpha)^2$ and $p'(r) = 2a(r-\alpha)$, so $p(\alpha) = p'(\alpha) = 0$ and Equation (3.2.9) reduces to

$$au'' = G(x).$$

Example 3.2.4

Find a particular solution of

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2). \tag{3.2.10}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and} y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation (3.2.10) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 3(u' + 3u) + 2u = -1 + 2x + x^2,$$

or

$$u'' + 3u' + 2u = -1 + 2x + x^2. \tag{3.2.11}$$

As in Example 5.3.2, in order to guess a form for a particular solution of Equation (3.2.11), we note that substituting a second degree polynomial $u_p = A + Bx + Cx^2$ for u in the left side of Equation (3.2.11) produces another second degree polynomial with coefficients that depend upon A, B, and C; thus,

$$\text{if} \quad u_p = A + Bx + Cx^2 \quad \text{then} \quad u_p' = B + 2Cx \quad \text{and} \quad u_p'' = 2C.$$

If u_p is to satisfy Equation (3.2.11), we must have

$$\begin{aligned} u_p'' + 3u_p' + 2u_p &= 2C + 3(B + 2Cx) + 2(A + Bx + Cx^2) \\ &= (2C + 3B + 2A) + (6C + 2B)x + 2Cx^2 = -1 + 2x + x^2. \end{aligned}$$

Equating coefficients of like powers of x on the two sides of the last equality yields

$$\begin{aligned} 2C &= 1 \\ 2B + 6C &= 2 \\ 2A + 3B + 2C &= -1. \end{aligned}$$

Solving these equations for C, B, and A (in that order) yields $C = 1/2, B = -1/2, A = -1/4$. Therefore

$$u_p = -\frac{1}{4}(1+2x-2x^2)$$

is a particular solution of Equation (3.2.11), and

$$y_p = u_p e^{3x} = -\frac{e^{3x}}{4}(1+2x-2x^2)$$

is a particular solution of Equation (3.2.10)

Example 3.2.5

Find a particular solution of

$$y'' - 4y' + 3y = e^{3x}(6+8x+12x^2). \tag{3.2.12}$$

Solution

Substituting

$$y = ue^{3x}, \quad y' = u'e^{3x} + 3ue^{3x}, \quad \text{and } y'' = u''e^{3x} + 6u'e^{3x} + 9ue^{3x}$$

into Equation (3.2.12) and canceling e^{3x} yields

$$(u'' + 6u' + 9u) - 4(u' + 3u) + 3u = 6 + 8x + 12x^2,$$

or

$$u'' + 2u' = 6 + 8x + 12x^2. \tag{3.2.13}$$

There's no u term in this equation, since e^{3x} is a solution of the complementary equation for Equation (3.2.12). Therefore Equation (3.2.13) does not have a particular solution of the form $u_p = A + Bx + Cx^2$ that we used successfully in Example 3.2.4 , since with this choice of u_p,

$$u_p'' + 2u_p' = 2C + (B + 2Cx)$$

can't contain the last term ($12x^2$) on the right side of Equation (3.2.13). Instead, let's try $u_p = Ax + Bx^2 + Cx^3$ on the grounds that

$$u_p' = A + 2Bx + 3Cx^2 \quad \text{and} \quad u_p'' = 2B + 6Cx$$

together contain all the powers of x that appear on the right side of Equation (3.2.13).

Substituting these expressions in place of u' and u'' in Equation (3.2.13) yields

$$(2B + 6Cx) + 2(A + 2Bx + 3Cx^2) = (2B + 2A) + (6C + 4B)x + 6Cx^2 = 6 + 8x + 12x^2.$$

Comparing coefficients of like powers of x on the two sides of the last equality shows that u_p satisfies Equation (3.2.13) if

$$\begin{aligned} 6C &= 12 \\ 4B + 6C &= 8 \\ 2A + 2B &= 6. \end{aligned}$$

Solving these equations successively yields $C = 2$, $B = -1$, and $A = 4$. Therefore

$$u_p = x(4 - x + 2x^2)$$

is a particular solution of Equation (3.2.13), and

$$y_p = u_p e^{3x} = xe^{3x}(4 - x + 2x^2)$$

is a particular solution of Equation (3.2.12).

Example 3.2.6

Find a particular solution of

$$4y''+4y'+y=e^{-x/2}(-8+48x+144x^2). \tag{3.2.14}$$

Solution

Substituting

$$y=ue^{-x/2},\quad y'=u'e^{-x/2}-\frac{1}{2}ue^{-x/2},\quad \text{and}\quad y''=u''e^{-x/2}-u'e^{-x/2}+\frac{1}{4}ue^{-x/2}$$

into Equation (3.2.14) and canceling $e^{-x/2}$ yields

$$4\left(u''-u'+\frac{u}{4}\right)+4\left(u'-\frac{u}{2}\right)+u=4u''=-8+48x+144x^2,$$

or

$$u''=-2+12x+36x^2, \tag{3.2.15}$$

which does not contain u or u' because $e^{-x/2}$ and $xe^{-x/2}$ are both solutions of the complementary equation. To obtain a particular solution of Equation (3.2.15) we integrate twice, taking the constants of integration to be zero; thus,

$$u_p'=-2x+6x^2+12x^3\quad \text{and}\quad u_p=-x^2+2x^3+3x^4=x^2(-1+2x+3x^2).$$

Therefore

$$y_p=u_pe^{-x/2}=x^2e^{-x/2}(-1+2x+3x^2)$$

is a particular solution of Equation (3.2.14).

3.2E: Exercises for Section 3.2

Exercises 1-14

Find a particular solution.

Exercise $3.2E.1$

$y'' - 3y' + 2y = e^{3x}(1 + x)$

Exercise $3.2E.2$

$y'' - 6y' + 5y = e^{-3x}(35 - 8x)$

Answer

$$y_p = e^{-3x}\left(1 - \frac{x}{4}\right).$$

Exercise $3.2E.3$

$y'' - 2y' - 3y = e^{x}(-8 + 3x)$

Exercise $3.2E.4$

$y'' + 2y' + y = e^{2x}(-7 - 15x + 9x^2)$

Answer

$$y_p = e^{2x}\left(1 - 3x + x^2\right).$$

Exercise $3.2E.5$

$y'' + 4y = e^{-x}(7 - 4x + 5x^2)$

Exercise $3.2E.6$

$y'' - y' - 2y = e^{x}(9 + 2x - 4x^2)$

Answer

$$y_p = e^{x}(-2 + x + 2x^2).$$

Exercise $3.2E.7$

$y'' - 4y' - 5y = -6xe^{-x}$

Exercise $3.2E.8$

$y'' - 3y' + 2y = e^{x}(3 - 4x)$

Answer

$$y_p = xe^{x}\left(1 + 2x\right).$$

Exercise $3.2E.9$

$y'' + y' - 12y = e^{3x}(-6 + 7x)$

Exercise $3.2E.10$

$2y'' - 3y' - 2y = e^{2x}(-6 + 10x)$

Exercise $3.2E.11$

$y'' + 2y' + y = e^{-x}(2 + 3x)$

Exercise $3.2E.12$

$y'' - 2y' + y = e^{x}(1 - 6x)$

Answer

$$y_p = x^2 e^x \left(\frac{1}{2} - x\right).$$

Exercise $3.2E.13$

$y'' - 4y' + 4y = e^{2x}(1 - 3x + 6x^2)$

Exercise $3.2E.14$

$9y'' + 6y' + y = e^{-x/3}(2 - 4x + 4x^2)$

Exercises 15-19

Find the general solution.

Exercise $3.2E.15$

$y'' - 3y' + 2y = e^{3x}(1 + x)$

Exercise $3.2E.16$

$y'' - 6y' + 8y = e^{x}(11 - 6x)$

Answer

$$y = e^x(1 - 2x) + c_1 e^{2x} + c_2 e^{4x}$$

Exercise $3.2E.17$

$y'' + 6y' + 9y = e^{2x}(3 - 5x)$

Exercise $3.2E.18$

$y'' + 2y' - 3y = -16xe^{x}$

Answer

$$y = xe^x(1 - 2x) + c_1 e^x + c_2 e^{-3x}.$$

Exercise $3.2E.19$

$y'' - 2y' + y = e^x(2 - 12x)$

Exercises 20-23

Solve the initial value problem and plot the solution.

Exercise $3.2E.20$

$y'' - 4y' - 5y = 9e^{2x}(1 + x), \quad y(0) = 0, \quad y'(0) = -10$

Exercise $3.2E.21$

$y'' + 3y' - 4y = e^{2x}(7 + 6x), \quad y(0) = 2, \quad y'(0) = 8$

Exercise $3.2E.22$

$y'' + 4y' + 3y = -e^{-x}(2 + 8x), \quad y(0) = 1, \quad y'(0) = 2$

Answer

$y = e^{-x}(2 + x - 2x^2) - e^{-3x}.$

Exercise $3.2E.23$

$y'' - 3y' - 10y = 7e^{-2x}, \quad y(0) = 1, \quad y'(0) = -17$

Exercises 24-30

Use the principle of superposition to find a particular solution.

Exercise $3.2E.24$

$y'' + y' + y = xe^x + e^{-x}(1 + 2x)$

Exercise $3.2E.25$

$y'' - 7y' + 12y = -e^x(17 - 42x) - e^{3x}$

Exercise $3.2E.26$

$y'' - 8y' + 16y = 6xe^{4x} + 2 + 16x + 16x^2$

Exercise $3.2E.27$

$y'' - 3y' + 2y = -e^{2x}(3 + 4x) - e^x$

Exercise $3.2E.28$

$y'' - 2y' + 2y = e^x(1 + x) + e^{-x}(2 - 8x + 5x^2)$

Answer

$y_p = y_{p_1} + y_{p_2} = e^x(1 + x) + x^2e^{-x}.$

Exercise $3.2E.29$

$y''+y=e^{-x}(2-4x+2x^2)+e^{3x}(8-12x-10x^2)$

Exercise $3.2E.30$

a. Prove that y is a solution of the constant coefficient equation

$$ay''+by'+cy=e^{\alpha x}G(x) \tag{A}$$

if and only if $y=ue^{\alpha x}$, where u satisfies

$$au''+p'(\alpha)u'+p(\alpha)u=G(x) \tag{B}$$

and $p(r)=ar^2+br+c$ is the characteristic polynomial of the complementary equation

$$ay''+by'+cy=0.$$

For the rest of this exercise, let G be a polynomial. Give the requested proofs for the case where

$$G(x)=g_0+g_1x+g_2x^2+g_3x^3.$$

b. Prove that if $e^{\alpha x}$ isn't a solution of the complementary equation then (B) has a particular solution of the form $u_p=A(x)$, where A is a polynomial of the same degree as G, as in Example 3.2.4. Conclude that (A) has a particular solution of the form $y_p=e^{\alpha x}A(x)$.
c. Show that if $e^{\alpha x}$ is a solution of the complementary equation and $xe^{\alpha x}$ isn't , then (B) has a particular solution of the form $u_p=xA(x)$, where A is a polynomial of the same degree as G, as in Example 3.2.5. Conclude that (A) has a particular solution of the form $y_p=xe^{\alpha x}A(x)$.
d. Show that if $e^{\alpha x}$ and $xe^{\alpha x}$ are both solutions of the complementary equation then (B) has a particular solution of the form $u_p=x^2A(x)$, where A is a polynomial of the same degree as G, and $x^2A(x)$ can be obtained by integrating G/a twice, taking the constants of integration to be zero, as in Example 3.2.6. Conclude that (A) has a particular solution of the form $y_p=x^2e^{\alpha x}A(x)$.

Exercises 31-40

Treat the equations considered in Section 3.2 Examples 3.2.1-3.2.6. Substitute the suggested form of y_p into the equation and equate the resulting coefficients of like functions on the two sides of the resulting equation to derive a set of simultaneous equations for the coefficients in y_p. Then solve for the coefficients to obtain y_p. Compare the work you've done with the work required to obtain the same results in Section 3.2 Examples 3.2.1-3.2.6.

Exercise $3.2E.31$

Compare with Example 3.2.1:

$$y''-7y'+12y=4e^{2x};\quad y_p=Ae^{2x}$$

Exercise $3.2E.32$

Compare with Example 3.2.2:

$$y''-7y'+12y=5e^{4x};\quad y_p=Axe^{4x}$$

Exercise $3.2E.33$

Compare with Example 3.2.3:

$$y''-8y'+16y=2e^{4x};\quad y_p=Ax^2e^{4x}$$

Exercise $3.2E.34$

Compare with Example 3.2.4:

$$y'' - 3y' + 2y = e^{3x}(-1 + 2x + x^2), \quad y_p = e^{3x}(A + Bx + Cx^2)$$

Answer

$y_p = -\frac{e^{3x}}{4}\left(1 + 2x - 2x^2\right)$.

Exercise $3.2E.35$

Compare with Example 3.2.5:

$$y'' - 4y' + 3y = e^{3x}(6 + 8x + 12x^2), \quad y_p = e^{3x}(Ax + Bx^2 + Cx^3)$$

Exercise $3.2E.36$

Compare with Example 3.2.6:

$$4y'' + 4y' + y = e^{-x/2}(-8 + 48x + 144x^2), \quad y_p = e^{-x/2}(Ax^2 + Bx^3 + Cx^4)$$

Exercise $3.2E.37$

Write $y = ue^{\alpha x}$ to find the general solution.

a. $y'' + 2y' + y = \frac{e^{-x}}{\sqrt{x}}$
b. $y'' + 6y' + 9y = e^{-3x} \ln x$
c. $y'' - 4y' + 4y = \frac{e^{2x}}{1+x}$
d. $4y'' + 4y' + y = 4e^{-x/2}\left(\frac{1}{x} + x\right)$

Exercise $3.2E.38$

Suppose $\alpha \neq 0$ and k is a positive integer. In most calculus books integrals like $\int x^k e^{\alpha x}\,dx$ are evaluated by integrating by parts k times. This exercise presents another method. Let

$$y = \int e^{\alpha x} P(x)\,dx$$

with

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k$$

(where $p_k \neq 0$).

a. Show that $y = e^{\alpha x} u$, where

$$u' + \alpha u = P(x). \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A_0 + A_1 x + \cdots + A_k x^k,$$

where A_k, A_{k-1}, …, A_0 can be computed successively by equating coefficients of $x^k, x^{k-1}, \ldots, 1$ on both sides of the equation

$$u_p' + \alpha u_p = P(x).$$

c. Conclude that

$$\int e^{\alpha x} P(x)\, dx = \left(A_0 + A_1 x + \cdots + A_k x^k\right) e^{\alpha x} + c,$$

where c is a constant of integration.

Exercise $3.2E.39$

Use the method suggested in **Exercise 3.2E.38** to evaluate the integrals.

a. $\int e^x(4+x)dx$
b. $\int e^{-x}(-1+x^2)dx$
c. $\int x^3 e^{-2x}dx$
d. $\int e^x(1+x)^2dx$
e. $\int e^{3x}(-14+30x+27x^2)dx$
f. $\int e^{-x}(1+6x^2-14x^3+3x^4)dx$

Exercise $3.2E.40$

Use the method suggested in **Exercise 3.2E.38** to evaluate $\int x^k e^{\alpha x}\, dx$, where k is an arbitrary positive integer and $\alpha \neq 0$.

3.3: The Method of Undetermined Coefficients II

Learning Objectives

- Understand how to use the method of undetermined coefficients to find particular solutions for differential equations with sinusoidal forcing functions.
- Apply strategies to identify the correct form of a particular solution based on the complementary equation and the degree of polynomials in the forcing function.

In this section we consider the constant coefficient equation

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{3.3.1}$$

where λ and ω are real numbers, $\omega \neq 0$, and P and Q are polynomials. We want to find a particular solution of Equation (3.3.1). The procedure that we will use is called *the method of undetermined coefficients*.

Forcing Functions Without Exponential Factors

We begin with the case where $\lambda = 0$ in Equation (3.3.1) ; thus, we we want to find a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x, \tag{3.3.2}$$

where P and Q are polynomials.

Differentiating $x^r\cos\omega x$ and $x^r\sin\omega x$ yields

$$\frac{d}{dx}x^r\cos\omega x = -\omega x^r\sin\omega x + rx^{r-1}\cos\omega x$$

and

$$\frac{d}{dx}x^r\sin\omega x = \quad \omega x^r\cos\omega x + rx^{r-1}\sin\omega x.$$

This implies that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are polynomials, then

$$ay_p'' + by_p' + cy_p = F(x)\cos\omega x + G(x)\sin\omega x,$$

where F and G are polynomials with coefficients that can be expressed in terms of the coefficients of A and B. This suggests that we try to choose A and B so that $F = P$ and $G = Q$, respectively. Then y_p will be a particular solution of Equation (3.3.2). The next theorem tells us how to choose the proper form for y_p. (See section 3.3E *Exercise 3.3E.37 for an outline of the proof)*.

Theorem 3.3.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x$$

has a particular solution

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x, \tag{3.3.3}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation. The solutions of

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x$$

for which $\cos\omega x$ and $\sin\omega x$ are solutions of the complementary equation are of the form of Equation (3.3.2), where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

Example 3.3.1

Find a particular solution of

$$y''-2y'+y=5\cos 2x+10\sin 2x. \tag{3.3.4}$$

Solution

In Equation (3.3.4) the coefficients of $\cos 2x$ and $\sin 2x$ are both zero degree polynomials (constants). Therefore Theorem 3.3.1 implies that Equation (3.3.4) has a particular solution

$$y_p=A\cos 2x+B\sin 2x.$$

Since

$$y_p'=-2A\sin 2x+2B\cos 2x \quad \text{and} \quad y_p''=-4(A\cos 2x+B\sin 2x),$$

replacing y by y_p in Equation (3.3.4) yields

$$\begin{aligned} y_p''-2y_p'+y_p &= -4(A\cos 2x+B\sin 2x)-4(-A\sin 2x+B\cos 2x)+(A\cos 2x+B\sin 2x)\\ &=(-3A-4B)\cos 2x+(4A-3B)\sin 2x. \end{aligned}$$

Equating the coefficients of $\cos 2x$ and $\sin 2x$ here with the corresponding coefficients on the right side of Equation (3.3.4) shows that y_p is a solution of Equation (3.3.4) if

$$\begin{aligned} -3A-4B &= \ \ 5\\ 4A-3B &= 10. \end{aligned}$$

Solving these equations yields $A=1$, $B=-2$. Therefore

$$y_p=\cos 2x-2\sin 2x$$

is a particular solution of Equation (3.3.4).

Example 3.3.2

Find a particular solution of

$$y''+4y=8\cos 2x+12\sin 2x. \tag{3.3.5}$$

Solution

The procedure used in Example 3.3.1 doesn't work here; substituting $y_p=A\cos 2x+B\sin 2x$ for y in Equation (3.3.5) yields

$$y_p''+4y_p=-4(A\cos 2x+B\sin 2x)+4(A\cos 2x+B\sin 2x)=0$$

for any choice of A and B, since $\cos 2x$ and $\sin 2x$ are both solutions of the complementary equation for Equation (3.3.5). We're dealing with the second case mentioned in Theorem 3.3.1 , and should therefore try a particular solution of the form

$$y_p=x(A\cos 2x+B\sin 2x). \tag{3.3.6}$$

Then

$$\begin{aligned} y_p' &= A\cos 2x+B\sin 2x+2x(-A\sin 2x+B\cos 2x)\\ y_p'' &= -4A\sin 2x+4B\cos 2x-4x(A\cos 2x+B\sin 2x)\\ &= -4A\sin 2x+4B\cos 2x-4y_p, \end{aligned}$$

so

$$y_p''+4y_p=-4A\sin 2x+4B\cos 2x.$$

Therefore y_p is a solution of Equation (3.3.5) if

$$-4A\sin 2x+4B\cos 2x=8\cos 2x+12\sin 2x,$$

which holds if $A=-3$ and $B=2$. Therefore

$$y_p=-x(3\cos 2x-2\sin 2x)$$

is a particular solution of Equation (3.3.5).

Example 3.3.3

Find a particular solution of

$$y''+3y'+2y=(16+20x)\cos x+10\sin x. \tag{3.3.7}$$

Solution

The coefficients of $\cos x$ and $\sin x$ in Equation (3.3.7) are polynomials of degree one and zero, respectively. Therefore Theorem 3.3.1 tells us to look for a particular solution of Equation (3.3.7) of the form

$$y_p=(A_0+A_1x)\cos x+(B_0+B_1x)\sin x. \tag{3.3.8}$$

Then

$$y_p'=(A_1+B_0+B_1x)\cos x+(B_1-A_0-A_1x)\sin x \tag{3.3.9}$$

and

$$y_p''=(2B_1-A_0-A_1x)\cos x-(2A_1+B_0+B_1x)\sin x, \tag{3.3.10}$$

so

$$y_p''+3y_p'+2y_p \quad =[A_0+3A_1+3B_0+2B_1+(A_1+3B_1)x]\cos x+[B_0+3B_1-3A_0-2A_1+(B_1-3A_1)x]\sin x. \tag{3.3.11}$$

Comparing the coefficients of $x\cos x$, $x\sin x$, $\cos x$, and $\sin x$ here with the corresponding coefficients in Equation (3.3.7) shows that y_p is a solution of Equation (3.3.7) if

$$\begin{aligned} A_1+3B_1 &=20 \\ -3A_1+\ \ B_1 &=0 \\ A_0+3B_0+3A_1+2B_1 &=16 \\ -3A_0+\ \ B_0-2A_1+3B_1 &=10. \end{aligned}$$

Solving the first two equations yields $A_1=2$, $B_1=6$. Substituting these into the last two equations yields

$$\begin{aligned} A_0+3B_0 &=16-3A_1-2B_1=-2 \\ -3A_0+\ \ B_0 &=10+2A_1-3B_1=-4. \end{aligned}$$

Solving these equations yields $A_0=1$, $B_0=-1$. Substituting $A_0=1$, $A_1=2$, $B_0=-1$, $B_1=6$ into Equation (3.3.8) shows that

$$y_p=(1+2x)\cos x-(1-6x)\sin x$$

is a particular solution of Equation (3.3.7).

A Useful Observation

In Equations (3.3.9), (3.3.10), and (3.3.11) the polynomials multiplying $\sin x$ can be obtained by replacing A_0, A_1, B_0, and B_1 by B_0, B_1, $-A_0$, and $-A_1$, respectively, in the polynomials mutiplying $\cos x$. An analogous result applies in general, as follows.

Theorem 3.3.2

If

$$y_p=A(x)\cos\omega x+B(x)\sin\omega x,$$

where $A(x)$ and $B(x)$ are polynomials with coefficients $A_0 \dots, A_k$ and $B_0, \dots, B_k$, then the polynomials multiplying $\sin\omega x$ in

$$y_p', \quad y_p'', \quad ay_p''+by_p'+cy_p \quad \text{and} \quad y_p''+\omega^2y_p$$

can be obtained by replacing $A_0, \dots, A_k$ by $B_0, \dots, B_k$ and $B_0, \dots, B_k$ by $-A_0, \dots, -A_k$ in the corresponding polynomials multiplying $\cos\omega x$.

We will not use this theorem in our examples, but we recommend that you use it to check your manipulations when you work the exercises.

Example 3.3.4

Find a particular solution of

$$y''+y=(8-4x)\cos x-(8+8x)\sin x. \tag{3.3.12}$$

Solution

According to Theorem 3.3.1 , we should look for a particular solution of the form

$$y_p = (A_0x + A_1x^2)\cos x + (B_0x + B_1x^2)\sin x, \tag{3.3.13}$$

since $\cos x$ and $\sin x$ are solutions of the complementary equation. However, let's try

$$y_p = (A_0 + A_1x)\cos x + (B_0 + B_1x)\sin x \tag{3.3.14}$$

first, so you can see why it doesn't work. From Equation (3.3.10),

$$y_p'' = (2B_1 - A_0 - A_1x)\cos x - (2A_1 + B_0 + B_1x)\sin x,$$

which together with Equation (3.3.14) implies that

$$y_p'' + y_p = 2B_1\cos x - 2A_1\sin x.$$

Since the right side of this equation does not contain $x\cos x$ or $x\sin x$, Equation (3.3.14) can't satisfy Equation (3.3.12) no matter how we choose A_0, A_1, B_0, and B_1.

Now let y_p be as in Equation (3.3.13). Then

$$\begin{aligned} y_p' &= \left[A_0 + (2A_1 + B_0)x + B_1x^2\right]\cos x \\ &+ \left[B_0 + (2B_1 - A_0)x - A_1x^2\right]\sin x \end{aligned}$$

and

$$\begin{aligned} y_p'' &= \left[2A_1 + 2B_0 - (A_0 - 4B_1)x - A_1x^2\right]\cos x \\ &+ \left[2B_1 - 2A_0 - (B_0 + 4A_1)x - B_1x^2\right]\sin x, \end{aligned}$$

so

$$y_p'' + y_p = (2A_1 + 2B_0 + 4B_1x)\cos x + (2B_1 - 2A_0 - 4A_1x)\sin x.$$

Comparing the coefficients of $\cos x$ and $\sin x$ here with the corresponding coefficients in Equation (3.3.12) shows that y_p is a solution of Equation (3.3.12) if

$$\begin{aligned} 4B_1 &= -4 \\ -4A_1 &= -8 \\ 2B_0 + 2A_1 &= 8 \\ -2A_0 + 2B_1 &= -8. \end{aligned}$$

The solution of this system is $A_1 = 2$, $B_1 = -1$, $A_0 = 3$, $B_0 = 2$. Therefore

$$y_p = x\left[(3 + 2x)\cos x + (2 - x)\sin x\right]$$

is a particular solution of Equation (3.3.12).

Forcing Functions with Exponential Factors

To find a particular solution of

$$ay'' + by' + cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right) \tag{3.3.15}$$

when $\lambda \neq 0$, we recall from Section 3.2 that substituting $y = ue^{\lambda x}$ into Equation (3.3.15) will produce a constant coefficient equation for u with the forcing function $P(x)\cos\omega x + Q(x)\sin\omega x$. We can find a particular solution u_p of this equation by the procedure that we used in Examples 3.3.1 - 3.3.4 . Then $y_p = u_pe^{\lambda x}$ is a particular solution of Equation (3.3.15).

Example 3.3.5

Find a particular solution of

$$y'' - 3y' + 2y = e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right]. \tag{3.3.16}$$

Solution

Let $y = ue^{-2x}$. Then

$$\begin{aligned} y'' - 3y' + 2y &= e^{-2x}\left[(u'' - 4u' + 4u) - 3(u' - 2u) + 2u\right] \\ &= e^{-2x}(u'' - 7u' + 12u) \\ &= e^{-2x}\left[2\cos 3x - (34 - 150x)\sin 3x\right] \end{aligned}$$

if

$$u'' - 7u' + 12u = 2\cos 3x - (34 - 150x)\sin 3x. \tag{3.3.17}$$

Since $\cos 3x$ and $\sin 3x$ aren't solutions of the complementary equation

$$u'' - 7u' + 12u = 0,$$

Theorem 3.3.1 tells us to look for a particular solution of Equation (3.3.17) of the form

$$u_p = (A_0 + A_1x)\cos 3x + (B_0 + B_1x)\sin 3x. \tag{3.3.18}$$

Then

$$\begin{aligned} u_p' &= (A_1 + 3B_0 + 3B_1x)\cos 3x + (B_1 - 3A_0 - 3A_1x)\sin 3x \\ \text{and} \qquad u_p'' &= (-9A_0 + 6B_1 - 9A_1x)\cos 3x - (9B_0 + 6A_1 + 9B_1x)\sin 3x, \end{aligned}$$

so

$$\begin{aligned} u_p'' - 7u_p' + 12u_p &= [3A_0 - 21B_0 - 7A_1 + 6B_1 + (3A_1 - 21B_1)x]\cos 3x \\ &+ [21A_0 + 3B_0 - 6A_1 - 7B_1 + (21A_1 + 3B_1)x]\sin 3x. \end{aligned}$$

Comparing the coefficients of $x\cos 3x$, $x\sin 3x$, $\cos 3x$, and $\sin 3x$ here with the corresponding coefficients on the right side of Equation (3.3.17) shows that u_p is a solution of Equation (3.3.17) if

$$\begin{aligned} 3A_1 - 21B_1 \quad &= 0 \\ 21A_1 + \;\; 3B_1 \quad &= 150 \\ 3A_0 - 21B_0 - 7A_1 + \;\; 6B_1 \quad &= \quad 2 \\ 21A_0 + \;\; 3B_0 - 6A_1 - \;\; 7B_1 \quad &= -34. \end{aligned} \tag{3.3.19}$$

Solving the first two equations yields $A_1 = 7$, $B_1 = 1$. Substituting these values into the last two equations of Equation (3.3.19) yields

$$\begin{aligned} 3A_0 - 21B_0 \;\; &= \quad\; 2 + 7A_1 - 6B_1 = 45 \\ 21A_0 + \;\; 3B_0 \;\; &= -34 + 6A_1 + 7B_1 = 15. \end{aligned}$$

Solving this system yields $A_0 = 1$, $B_0 = -2$. Substituting $A_0 = 1$, $A_1 = 7$, $B_0 = -2$, and $B_1 = 1$ into Equation (3.3.18) shows that

$$u_p = (1 + 7x)\cos 3x - (2 - x)\sin 3x$$

is a particular solution of Equation (3.3.17). Therefore

$$y_p = e^{-2x}\left[(1 + 7x)\cos 3x - (2 - x)\sin 3x\right]$$

is a particular solution of Equation (3.3.16).

✔ Example 3.3.6

Find a particular solution of

$$y'' + 2y' + 5y = e^{-x}\left[(6 - 16x)\cos 2x - (8 + 8x)\sin 2x\right]. \tag{3.3.20}$$

Solution

Let $y = ue^{-x}$. Then

$$\begin{aligned} y'' + 2y' + 5y &= e^{-x}\left[(u'' - 2u' + u) + 2(u' - u) + 5u\right] \\ &= e^{-x}(u'' + 4u) \\ &= e^{-x}\left[(6 - 16x)\cos 2x - (8 + 8x)\sin 2x\right] \end{aligned}$$

if

$$u'' + 4u = (6 - 16x)\cos 2x - (8 + 8x)\sin 2x. \tag{3.3.21}$$

Since $\cos 2x$ and $\sin 2x$ are solutions of the complementary equation

$$u'' + 4u = 0,$$

Theorem 3.3.1 tells us to look for a particular solution of Equation (3.3.21) of the form

$$u_p = (A_0x + A_1x^2)\cos 2x + (B_0x + B_1x^2)\sin 2x.$$

Then

$$\begin{aligned} u_p' &= \left[A_0 + (2A_1 + 2B_0)x + 2B_1x^2\right]\cos 2x \\ &+ \left[B_0 + (2B_1 - 2A_0)x - 2A_1x^2\right]\sin 2x \end{aligned}$$

and

$$\begin{aligned} u_p'' &= \left[2A_1 + 4B_0 - (4A_0 - 8B_1)x - 4A_1x^2\right]\cos 2x \\ &+ \left[2B_1 - 4A_0 - (4B_0 + 8A_1)x - 4B_1x^2\right]\sin 2x, \end{aligned}$$

so

$$u_p''+4u_p=(2A_1+4B_0+8B_1x)\cos 2x+(2B_1-4A_0-8A_1x)\sin 2x.$$

Equating the coefficients of $x\cos 2x$, $x\sin 2x$, $\cos 2x$, and $\sin 2x$ here with the corresponding coefficients on the right side of Equation (3.3.21) shows that u_p is a solution of Equation (3.3.21) if

$$\begin{aligned} 8B_1 &= -16 \\ -8A_1 &= -\ 8 \\ 4B_0+2A_1 &= 6 \\ -4A_0+2B_1 &= -8. \end{aligned} \tag{3.3.22}$$

The solution of this system is $A_1=1$, $B_1=-2$, $B_0=1$, $A_0=1$. Therefore

$$u_p=x[(1+x)\cos 2x+(1-2x)\sin 2x]$$

is a particular solution of Equation (3.3.21), and

$$y_p=xe^{-x}\left[(1+x)\cos 2x+(1-2x)\sin 2x\right]$$

is a particular solution of Equation (3.3.20).

You can also find a particular solution of Equation (3.3.20) by substituting

$$y_p=xe^{-x}\left[(A_0+A_1x)\cos 2x+(B_0+B_1x)\sin 2x\right]$$

for y in Equation (3.3.20) and equating the coefficients of $xe^{-x}\cos 2x$, $xe^{-x}\sin 2x$, $e^{-x}\cos 2x$, and $e^{-x}\sin 2x$ in the resulting expression for

$$y_p''+2y_p'+5y_p$$

with the corresponding coefficients on the right side of Equation (3.3.20). This leads to the same system Equation (3.3.22) of equations for A_0, A_1, B_0, and B_1 that we obtained in Example 3.3.6 . However, if you try this approach you'll see that deriving Equation (3.3.22) this way is much more tedious than the way we did it in Example 3.3.6 .

3.3E: Exercises for Section 3.3

Exercises 1-17

Find a particular solution.

Exercise $3.3E.1$

$y''+3y'+2y=7\cos x-\sin x$

Exercise $3.3E.2$

$y''+3y'+y=(2-6x)\cos x-9\sin x$

Answer

$y_p=\cos x+(2-2x)\sin x.$

Exercise $3.3E.3$

$y''+2y'+y=e^x(6\cos x+17\sin x)$

Exercise $3.3E.4$

$y''+3y'-2y=-e^{2x}(5\cos 2x+9\sin 2x)$

Answer

$y_p=\frac{e^{2x}}{2}(\cos 2x-\sin 2x).$

Exercise $3.3E.5$

$y''-y'+y=e^x(2+x)\sin x$

Exercise $3.3E.6$

$y''+3y'-2y=e^{-2x}\left[(4+20x)\cos 3x+(26-32x)\sin 3x\right]$

Answer

$y_p=e^{-2x}(1-2x)(\cos 3x-\sin 3x).$

Exercise $3.3E.7$

$y''+4y=-12\cos 2x-4\sin 2x$

Exercise $3.3E.8$

$y''+y=(-4+8x)\cos x+(8-4x)\sin x$

Answer

$y_p=-x\left[(2-x)\cos x+(3-2x)\sin x\right].$

Exercise $3.3E.9$

$4y'' + y = -4\cos x/2 - 8x\sin x/2.$

Exercise $3.3E.10$

$y'' + 2y' + 2y = e^{-x}(8\cos x - 6\sin x)$

Exercise $3.3E.11$

$y'' - 2y' + 5y = e^{x}\left[(6+8x)\cos 2x + (6-8x)\sin 2x\right]$

Exercise $3.3E.12$

$y'' + 2y' + y = 8x^2\cos x - 4x\sin x$

Answer

$y_p = -(14-10x)\cos x - (2+8x-4x^2)\sin x.$

Exercise $3.3E.13$

$y'' + 3y' + 2y = (12+20x+10x^2)\cos x + 8x\sin x$

Exercise $3.3E.14$

$y'' + 3y' + 2y = (1-x-4x^2)\cos 2x - (1+7x+2x^2)\sin 2x$

Answer

$y_p = \frac{x^2}{2}(\cos 2x - \sin 2x).$

Exercise $3.3E.15$

$y'' - 5y' + 6y = -e^{x}\left[(4+6x-x^2)\cos x - (2-4x+3x^2)\sin x\right]$

Exercise $3.3E.16$

$y'' - 2y' + y = -e^{x}\left[(3+4x-x^2)\cos x + (3-4x-x^2)\sin x\right]$

Answer

$y_p = e^{x}(1-x^2)(\cos x + \sin x).$

Exercise $3.3E.17$

$y'' - 2y' + 2y = e^{x}\left[(2-2x-6x^2)\cos x + (2-10x+6x^2)\sin x\right]$

Exercises 18-21

Find a particular solution and graph it.

Exercise 3.3E.18

$$y'' + 2y' + y = e^{-x}\left[(5-2x)\cos x - (3+3x)\sin x\right]$$

Exercise 3.3E.19

$$y'' + 9y = -6\cos 3x - 12\sin 3x$$

Exercise 3.3E.20

$$y'' + 3y' + 2y = (1 - x - 4x^2)\cos 2x - (1 + 7x + 2x^2)\sin 2x$$

Answer

$$y_p = -x^3\cos x + (x + 2x^2)\sin x.$$

Exercise 3.3E.21

$$y'' + 4y' + 3y = e^{-x}\left[(2 + x + x^2)\cos x + (5 + 4x + 2x^2)\sin x\right]$$

Exercises 22-26

Solve the initial value problem.

Exercise 3.3E.22

$$y'' - 7y' + 6y = -e^x(17\cos x - 7\sin x), \quad y(0) = 4, \ y'(0) = 2$$

Answer

$$y = e^x(2\cos x + 3\sin x) + 3e^x - e^{6x}.$$

Exercise 3.3E.23

$$y'' - 2y' + 2y = -e^x(6\cos x + 4\sin x), \quad y(0) = 1, \ y'(0) = 4$$

Exercise 3.3E.24

$$y'' + 6y' + 10y = -40e^x\sin x, \quad y(0) = 2, \quad y'(0) = -3$$

Answer

$$y = e^x(\cos x - 2\sin x) + e^{-3x}(\cos x + \sin x).$$

Exercise 3.3E.25

$$y'' - 6y' + 10y = -e^{3x}(6\cos x + 4\sin x), \quad y(0) = 2, \quad y'(0) = 7$$

Exercise 3.3E.26

$$y'' - 3y' + 2y = e^{3x}\left[21\cos x - (11 + 10x)\sin x\right], \ y(0) = 0, \quad y'(0) = 6$$

Exercises 27-32

Use the principle of superposition to find a particular solution. Where indicated, solve the initial value problem.

Exercise $3.3E.27$

$y'' - 2y' - 3y = 4e^{3x} + e^x(\cos x - 2\sin x)$

Exercise $3.3E.28$

$y'' + y = 4\cos x - 2\sin x + xe^x + e^{-x}$

Answer

$y_p = y_{p_1} + y_{p_2} + y_{p_3} = x(\cos x + 2\sin x) - \frac{e^x}{2}(1-x) + \frac{e^{-x}}{2}.$

Exercise $3.3E.29$

$y'' - 3y' + 2y = xe^x + 2e^{2x} + \sin x$

Exercise $3.3E.30$

$y'' - 2y' + 2y = 4xe^x\cos x + xe^{-x} + 1 + x^2$

Exercise $3.3E.31$

$y'' - 4y' + 4y = e^{2x}(1+x) + e^{2x}(\cos x - \sin x) + 3e^{3x} + 1 + x$

Exercise $3.3E.32$

$y'' - 4y' + 4y = 6e^{2x} + 25\sin x, \quad y(0) = 5,\ y'(0) = 3$

Answer

$y = (1 - 2x + 3x^2)e^{2x} + 4\cos x + 3\sin x.$

Exercises 33-35

Solve the initial value problem and graph the solution.

Exercise $3.3E.33$

$y'' + 4y = -e^{-2x}\left[(4-7x)\cos x + (2-4x)\sin x\right],\ y(0) = 3, \quad y'(0) = 1$

Exercise $3.3E.34$

$y'' + 4y' + 4y = 2\cos 2x + 3\sin 2x + e^{-x}, \quad y(0) = -1,\ y'(0) = 2$

Exercise $3.3E.35$

$y'' + 4y = e^x(11 + 15x) + 8\cos 2x - 12\sin 2x, \quad y(0) = 3,\ y'(0) = 5$

Exercise $3.3E.36$

a. Verify that if

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x$$

where A and B are twice differentiable, then

$$\begin{aligned} y_p' &= (A' + \omega B)\cos\omega x + (B' - \omega A)\sin\omega x \quad \text{ and} \\ y_p'' &= (A'' + 2\omega B' - \omega^2 A)\cos\omega x + (B'' - 2\omega A' - \omega^2 B)\sin\omega x. \end{aligned}$$

b. Use the results of (a) to verify that

$$\begin{aligned} ay_p'' + by_p' + cy_p = \ & \left[(c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA''\right]\cos\omega x + \\ & \left[-b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB''\right]\sin\omega x. \end{aligned}$$

c. Use the results of (a) to verify that

$$y_p'' + \omega^2 y_p = (A'' + 2\omega B')\cos\omega x + (B'' - 2\omega A')\sin\omega x.$$

d. Prove Theorem 3.3.2.

Exercise $3.3E.37$

Let a, b, c, and ω be constants, with $a \neq 0$ and $\omega > 0$, and let

$$P(x) = p_0 + p_1 x + \cdots + p_k x^k \quad \text{ and } \quad Q(x) = q_0 + q_1 x + \cdots + q_k x^k,$$

where at least one of the coefficients p_k, q_k is nonzero, so k is the larger of the degrees of P and Q.

a. Show that if $\cos\omega x$ and $\sin\omega x$ are not solutions of the complementary equation

$$ay'' + by' + cy = 0,$$

then there are polynomials

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k \quad \text{ and } \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k \tag{A}$$

such that

$$\begin{aligned} (c - a\omega^2)A + b\omega B + 2a\omega B' + bA' + aA'' &= P \\ -b\omega A + (c - a\omega^2)B - 2a\omega A' + bB' + aB'' &= Q, \end{aligned}$$

where (A_k, B_k), (A_{k-1}, B_{k-1}), …,(A_0, B_0) can be computed successively by solving the systems

$$\begin{aligned} (c - a\omega^2)A_k + b\omega B_k &= p_k \\ -b\omega A_k + (c - a\omega^2)B_k &= q_k, \end{aligned}$$

and, if $1 \leq r \leq k$,

$$\begin{aligned} (c - a\omega^2)A_{k-r} + b\omega B_{k-r} &= p_{k-r} + \cdots \\ -b\omega A_{k-r} + (c - a\omega^2)B_{k-r} &= q_{k-r} + \cdots, \end{aligned}$$

where the terms indicated by "$\cdots$" depend upon the previously computed coefficients with subscripts greater than $k - r$. Conclude from this and **Exercise 3.3E.36b** that

$$y_p = A(x)\cos\omega x + B(x)\sin\omega x \tag{B}$$

is a particular solution of

$$ay'' + by' + cy = P(x)\cos\omega x + Q(x)\sin\omega x.$$

b. Conclude that the equation

$$a(y'' + \omega^2 y) = P(x)\cos\omega x + Q(x)\sin\omega x \tag{C}$$

does not have a solution of the form (B) with A and B as in (A). Then show that there are polynomials

$$A(x) = A_0 x + A_1 x^2 + \cdots + A_k x^{k+1} \quad \text{ and } \quad B(x) = B_0 x + B_1 x^2 + \cdots + B_k x^{k+1}$$

such that

$$\begin{aligned} a(A''+2\omega B') &= P \\ a(B''-2\omega A') &= Q, \end{aligned}$$

where the pairs (A_k, B_k), (A_{k-1}, B_{k-1}), …, (A_0, B_0) can be computed successively as follows:

$$\begin{aligned} A_k &= -\frac{q_k}{2a\omega(k+1)} \\ B_k &= \frac{p_k}{2a\omega(k+1)}, \end{aligned}$$

and, if $k \geq 1$,

$$\begin{aligned} A_{k-j} &= -\frac{1}{2\omega}\left[\frac{q_{k-j}}{a(k-j+1)} - (k-j+2)B_{k-j+1}\right] \\ B_{k-j} &= \frac{1}{2\omega}\left[\frac{p_{k-j}}{a(k-j+1)} - (k-j+2)A_{k-j+1}\right] \end{aligned}$$

for $1 \leq j \leq k$. Conclude that (B) with this choice of the polynomials A and B is a particular solution of (C).

Exercise $3.3E.38$

Show that Theorem 3.3.1 implies Theorem 3.3E.1.

Theorem 3.3E.1

Suppose ω is a positive number and P and Q are polynomials. Let k be the larger of the degrees of P and Q. Then the equation

$$ay''+by'+cy = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

has a particular solution

$$y_p = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right), \tag{A}$$

where

$$A(x) = A_0 + A_1x + \cdots + A_kx^k \quad \text{and} \quad B(x) = B_0 + B_1x + \cdots + B_kx^k,$$

provided that $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are not solutions of the complementary equation. The equation

$$a\left[y''-2\lambda y'+(\lambda^2+\omega^2)y\right] = e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)$$

(for which $e^{\lambda x}\cos\omega x$ and $e^{\lambda x}\sin\omega x$ are solutions of the complementary equation) has a particular solution of the form (A), where

$$A(x) = A_0x + A_1x^2 + \cdots + A_kx^{k+1} \quad \text{and} \quad B(x) = B_0x + B_1x^2 + \cdots + B_kx^{k+1}.$$

Exercise $3.3E.39$

This exercise presents a method for evaluating the integral

$$y = \int e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)\,dx$$

where $\omega \neq 0$ and

$$P(x) = p_0 + p_1x + \cdots + p_kx^k, \quad Q(x) = q_0 + q_1x + \cdots + q_kx^k.$$

a. Show that $y = e^{\lambda x}u$, where

$$u' + \lambda u = P(x)\cos\omega x + Q(x)\sin\omega x. \tag{A}$$

b. Show that (A) has a particular solution of the form

$$u_p = A(x)\cos\omega x + B(x)\sin\omega x,$$

where

$$A(x) = A_0 + A_1 x + \cdots + A_k x^k, \quad B(x) = B_0 + B_1 x + \cdots + B_k x^k,$$

and the pairs of coefficients (A_k, B_k), (A_{k-1}, B_{k-1}), …,(A_0, B_0) can be computed successively as the solutions of pairs of equations obtained by equating the coefficients of $x^r\cos\omega x$ and $x^r\sin\omega x$ for $r = k, k-1, \ldots, 0$.

c. Conclude that

$$\int e^{\lambda x}\left(P(x)\cos\omega x + Q(x)\sin\omega x\right)\,dx = e^{\lambda x}\left(A(x)\cos\omega x + B(x)\sin\omega x\right) + c,$$

where c is a constant of integration.

Exercise $3.3E.40$

Evaluate the integrals using information in Exercise 3.3E.39.

a. $\int x^2\cos x\,dx$
b. $\int x^2 e^x\cos x\,dx$
c. $\int x e^{-x}\sin 2x\,dx$
d. $\int x^2 e^{-x}\sin x\,dx$
e. $\int x^3 e^x\sin x\,dx$
f. $\int e^x[x\cos x - (1+3x)\sin x]dx$
g. $\int e^{-x}[(1+x^2)\cos x + (1^2_x)\sin x]dx$

3.4: Constant coefficient second order linear ODEs

Learning Objectives

- Understand and apply the method of solving second-order linear homogeneous differential equations with constant coefficients using the characteristic equation.
- Analyze and solve differential equations with distinct, repeated, or complex roots, and interpret the resulting solutions in terms of exponential and trigonometric functions.

Solving Constant Coefficient Equations

Suppose we have the problem

$$y'' - 6y' + 8y = 0, y(0) = -2, y'(0) = 6$$

This is a second order linear homogeneous equation with constant coefficients. Constant coefficients means that the functions in front of y'', y', and y are constants and do not depend on x.

To guess a solution, think of a function that you know stays essentially the same when we differentiate it, so that we can take the function and its derivatives, add some multiples of these together, and end up with zero.

Let us try a solution of the form $y = e^{rx}$. Then $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Plug in to get

$$\begin{aligned} y'' - 6y' + 8y &= 0, \\ \underbrace{r^2 e^{rx}}_{y''} - 6\underbrace{re^{rx}}_{y'} + 8\underbrace{e^{rx}}_{y} &= 0, \\ r^2 - 6r + 8 &= 0 \qquad \text{(divide through by } e^{rx}), \\ (r-2)(r-4) &= 0. \end{aligned} \tag{3.4.1}$$

Hence, if $r = 2$ or $r = 4$, then e^{rx} is a solution. So let $y_1 = e^{2x}$ and $y_2 = e^{4x}$.

Example 3.4.1

Check that y_1 and y_2 are solutions.

Solution

The functions e^{2x} and e^{4x} are linearly independent. If they were not linearly independent we could write $e^{4x} = Ce^{2x}$ for some constant C, implying that $e^{2x} = C$ for all x, which is clearly not possible. Hence, we can write the general solution as

$$y = C_1 e^{2x} + C_2 e^{4x}$$

We need to solve for C_1 and C_2. To apply the initial conditions we first find $y' = 2C_1 e^{2x} + 4C_2 e^{4x}$. We plug in $x = 0$ and solve.

$$\begin{aligned} -2 &= y(0) = C_1 + C_2 \\ 6 &= y'(0) = 2C_1 + 4C_2 \end{aligned} \tag{3.4.2}$$

Either apply some matrix algebra, or just solve these by high school math. For example, divide the second equation by 2 to obtain $3 = C_1 + 2C_2$, and subtract the two equations to get $5 = C_2$. Then $C_1 = -7$ as $-2 = C_1 + 5$. Hence, the solution we are looking for is

$$y = -7e^{2x} + 5e^{4x}$$

Let us generalize this example into a method. Suppose that we have an equation

$$ay'' + by' + cy = 0, \tag{3.4.3}$$

where a, b, c are constants. Try the solution $y = e^{rx}$ to obtain

$$ar^2 e^{rx} + bre^{rx} + ce^{rx} = 0$$

Divide by e^{rx} to obtain the so-called *characteristic equation* of the ODE:

$$ar^2 + br + c = 0$$

Solve for the r by using the quadratic formula.

$$r_1, r_2 = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Therefore, we have $e^{r_1 x}$ and $e^{r_2 x}$ as solutions. There is still a difficulty if $r_1 = r_2$, but it is not hard to overcome.

Theorem 3.4.1

Suppose that r_1 and r_2 are the roots of the characteristic equation.

If r_1 and r_2 are distinct and real (when $b^2 - 4ac > 0$), then Equation $(3.4.3)$ has the general solution

$$y = C_1 e^{r_1 x} + C_2 e^{r_2 x}$$

If $r_1 = r_2$ (happens when $b^2 - 4ac = 0$), then Equation $(3.4.3)$ has the general solution

$$y = (C_1 + C_2 x) e^{r_1 x}$$

For another example of the first case, take the equation $y'' - k^2 y = 0$. Here the characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions.

Example 3.4.2

Solve

$$y'' - k^2 y = 0.$$

Solution

The characteristic equation is $r^2 - k^2 = 0$ or $(r-k)(r+k) = 0$. Consequently, e^{-kx} and e^{kx} are the two linearly independent solutions, and the general solution is

$$y = C_1 e^{kx} + C_2 e^{-kx}.$$

Since $\cosh s = \frac{e^s + e^{-s}}{2}$ and $\sinh s = \frac{e^s - e^{-s}}{2}$, we can also write the general solution as

$$y = D_1 \cosh(kx) + D_2 \sinh(kx).$$

Example 3.4.3

Find the general solution of

$$y'' - 8y' + 16y = 0$$

Solution

The characteristic equation is $r^2 - 8r + 16 = (r-4)^2 = 0$. The equation has a double root $r_1 = r_2 = 4$. The general solution is, therefore,

$$y = (C_1 + C_2 x) e^{4x} = C_1 e^{4x} + C_2 x e^{4x}$$

Example 3.4.4

Check that e^{4x} and xe^{4x} *are linearly independent.*

Solution

That e^{4x} solves the equation is clear. If xe^{4x} solves the equation, then we know we are done. Let us compute $y' = e^{4x} + 4xe^{4x}$ and $y'' = 8e^{4x} + 16xe^{4x}$. Plug in

$$y'' - 8y' + 16y = 8e^{4x} + 16xe^{4x} - 8(e^{4x} + 4xe^{4x}) + 16xe^{4x} = 0$$

We should note that in practice, doubled root rarely happens. If coefficients are picked truly randomly we are very unlikely to get a doubled root.

Let us give a short proof for why the solution xe^{rx} works when the root is doubled. This case is really a limiting case of when the two roots are distinct and very close. Note that $\frac{e^{(r_2)x} - e^{(r_1)x}}{r_2 - r_1}$ is a solution when the roots are distinct. When we take the limit as r_1 goes to r_2, we are really taking the derivative of e^{rx} using r as the variable. Therefore, the limit is xe^{rx}, and hence this is a solution in the doubled root case.

Complex numbers and Euler's formula

It may happen that a polynomial has some complex roots. For example, the equation $r^2 + 1 = 0$ has no real roots, but it does have two complex roots. Here we review some properties of complex numbers.

Complex numbers may seem a strange concept, especially because of the terminology. There is nothing imaginary or really complicated about complex numbers. A complex number is simply a pair of real numbers, (a, b). We can think of a complex number as a point in the plane. We add complex numbers in the straightforward way, $(a, b) + (c, d) = (a + c, b + d)$. We define multiplication by

$$(a, b) \times (c, d) \overset{\text{def}}{=} (ac - bd, ad + bc).$$

It turns out that with this multiplication rule, all the standard properties of arithmetic hold. Further, and most importantly $(0, 1) \times (0, 1) = (-1, 0)$.

Generally we just write (a, b) as $(a + ib)$, and we treat i as if it were an unknown. We do arithmetic with complex numbers just as we would with polynomials. The property we just mentioned becomes $i^2 = -1$. So whenever we see i^2, we replace it by -1. The numbers i and $-i$ are the two roots of $r^2 + 1 = 0$.

Note that engineers often use the letter j instead of i for the square root of -1. We will use the mathematicians' convention and use i.

We can also define the exponential e^{a+ib} of a complex number. We do this by writing down the Taylor series and plugging in the complex number. Because most properties of the exponential can be proved by looking at the Taylor series, these properties still hold for the complex exponential. For example the property $e^{x+y} = e^x e^y$ implies that $e^{a+ib} = e^a e^{ib}$. Hence if we can compute e^{ib}, we can compute e^{a+ib}. For e^{ib} we use the so-called Euler's formula.

Theorem 3.4.2: Euler's Formula

$$e^{i\theta} = \cos\theta + i\sin\theta \quad \textit{and} \quad e^{-i\theta} = \cos\theta - i\sin\theta$$

In other words, $e^{a+ib} = e^a(\cos(b) + i\sin(b)) = e^a\cos(b) + ie^a\sin(b)$.

Example 3.4.5

Start with $e^{i(2\theta)} = \left(e^{i\theta}\right)^2$. Use Euler's Formula to deduce:

$$\cos(2\theta) = \cos^2\theta - \sin^2\theta \quad \text{and} \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

Answer

Video Length: 3 minutes and 56 seconds

Context: Use Euler's Formula to deduce the double angle formulas.

Complex roots

Suppose that the equation $ay'' + by' + cy = 0$ has the characteristic equation $ar^2 + br + c = 0$ that has complex roots. By the quadratic formula, the roots are $\dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$. These roots are complex if $b^2 - 4ac < 0$. In this case the roots are

$$r_1, r_2 = \frac{-b}{2a} \pm i\frac{\sqrt{4ac - b^2}}{2a}$$

As you can see, we always get a pair of roots of the form $\alpha \pm i\beta$. In this case we can still write the solution as

$$y = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

However, the exponential is now complex valued. We would need to allow C_1 and C_2 to be complex numbers to obtain a real-valued solution (which is what we are after). While there is nothing particularly wrong with this approach, it can make calculations harder and it is generally preferred to find two real-valued solutions.

Here we can use Euler's Formula. Let

$$y_1 = e^{(\alpha + i\beta)x} \quad \text{and} \quad y_2 = e^{(\alpha - i\beta)x}$$

Then note that

$$\begin{aligned} y_1 &= e^{ax}\cos(\beta x) + ie^{ax}\sin(\beta x) \\ y_2 &= e^{ax}\cos(\beta x) - ie^{ax}\sin(\beta x) \end{aligned} \tag{3.4.4}$$

Linear combinations of solutions are also solutions. Hence,

$$\begin{aligned} y_3 &= \frac{y_1 + y_2}{2} = e^{ax}\cos(\beta x) \\ y_4 &= \frac{y_1 - y_2}{2i} = e^{ax}\sin(\beta x) \end{aligned} \tag{3.4.5}$$

are also solutions. Furthermore, they are real-valued. It is not hard to see that they are linearly independent (not multiples of each other).

Theorem 3.4.3

For the homegneous second order ODE

$$ay'' + by' + cy = 0$$

If the characteristic equation has the roots $\alpha \pm i\beta$ (when $b^2 - 4ac < 0$), then the general solution is

$$y = C_1 e^{ax}\cos(\beta x) + C_2 e^{ax}\sin(\beta x)$$

Example 3.4.6

Find the general solution of $y'' + k^2 y = 0$, for a constant $k > 0$.

Solution

The characteristic equation is $r^2 + k^2 = 0$. Therefore, the roots are $r = \pm ik$ and by Theorem 3.4.3 we have the general solution

$$y = C_1 \cos(kx) + C_2 \sin(kx)$$

Example 3.4.7

Find the solution of $y'' - 6y' + 13y = 0, y(0) = 0, y'(0) = 10$.

Solution

The characteristic equation is $r^2 - 6r + 13 = 0$. By completing the square we get $(r-3)^2 + 2^2 = 0$ and hence the roots are $r = 3 \pm 2i$. By Theorem 3.4.3 we have the general solution

$$y = C_1 e^{3x} \cos(2x) + C_2 e^{3x} \sin(2x)$$

To find the solution satisfying the initial conditions, we first plug in zero to get

$$0 = y(0) = C_1 e^0 \cos 0 + C_2 e^0 \sin 0 = C_1$$

Hence $C_1 = 0$ and $y = C_2 e^{3x} \sin(2x)$. We differentiate

$$y' = 3C_2 e^{3x} \sin(2x) + 2C_2 e^{3x} \cos(2x)$$

We again plug in the initial condition and obtain $10 = y'(0) = 2C_2$, or $C_2 = 5$. Hence the solution we are seeking is

$$y = 5e^{3x} \sin(2x)$$

Footnotes

[1] Making an educated guess with some parameters to solve for is such a central technique in differential equations, that people sometimes use a fancy name for such a guess: *ansatz*, German for "initial placement of a tool at a work piece." Yes, the Germans have a word for that.

3.4E: Exercises for Section 3.4

Exercises 1-18

Solve the following second-order differential equations for their general or particular solutions.

Exercise $3.4E.1$

Find the general solution of $2y'' + 2y' - 4y = 0$.

Exercise $3.4E.2$

Find the general solution of $y'' + 9y' - 10y = 0$.

Exercise $3.4E.3$

Solve $y'' - 8y' + 16y = 0$ for $y(0) = 2, y'(0) = 0$.

Exercise $3.4E.4$

Solve $y'' + 9y' = 0$ for $y(0) = 1, y'(0) = 1$.

Exercise $3.4E.5$

Find the general solution of $2y'' + 50y = 0$.

Exercise $3.4E.6$

Find the general solution of $y'' + 6y' + 13y = 0$.

Exercise $3.4E.7$

Find the general solution of $y'' = 0$ using the methods of this section.

Exercise $3.4E.8$

The method of this section applies to equations of other orders than two. We will see higher orders later. Try to solve the first order equation $2y' + 3y = 0$ using the methods of this section.

Exercise $3.4E.9$

Suppose that $(b - a)^2 - 4ac < 0$. Find a formula for the general solution of $ax^2y'' + bxy' + cy = 0$.

Hint: Note that $x^r = e^{r \ln x}$.

Exercise $3.4E.10$

Find the solution to $y'' - (2\alpha)y' + \alpha^2 y = 0$, $y(0) = a$, $y'(0) = b$, where α, a, and b are real numbers.

Exercise $3.4E.11$

Construct an equation such that $y = C_1 e^{-2x} \cos(3x) + C_2 e^{-2x} \sin(3x)$ is the general solution.

Exercise $3.4E.12$

Find the general solution to $y'' + 4y' + 2y = 0$.

Answer

$y = C_1 e^{(-2+\sqrt{2})x} + C_2 e^{(-2-\sqrt{2})x}$

Exercise $3.4E.13$

Find the general solution to $y'' - 6y' + 9y = 0$.

Answer

$y = C_1 e^{3x} + C_2 x e^{3x}$

Exercise $3.4E.14$

Find the solution to $2y'' + y' + y = 0, y(0) = 1, y'(0) = -2$.

Answer

$y = e^{-x/4} \cos\left((\frac{\sqrt{7}}{4})x\right) - \sqrt{7} e^{-x/4} \sin\left((\frac{\sqrt{7}}{4})x\right)$

Exercise $3.4E.15$

Find the solution to $2y'' + y' - 3y = 0, y(0) = a, y'(0) = b$.

Answer

$y = \frac{2(a-b)}{5} e^{-3x/2} + \frac{3a+2b}{5} e^{x}$

Exercise $3.4E.16$

Find the solution to $z''(t) = -2z'(t) - 2z(t), z(0) = 2, z'(0) = -2$.

Answer

$z(t) = 2e^{-t} \cos(t)$

Exercise $3.4E.17$

Find the solution to $y'' - (\alpha + \beta)y' + \alpha\beta y = 0$, $y(0) = a$, $y'(0) = b$, where α, β, a, and b are real numbers, and $\alpha \neq \beta$.

Answer

$y = \frac{\alpha\beta - b}{\beta - \alpha} e^{\alpha x} + \frac{b - a\alpha}{\beta - \alpha} e^{\beta x}$

Exercise $3.4E.18$

Construct an equation such that $y = C_1 e^{3x} + C_2 e^{-2x}$ is the general solution.

Answer

$y'' - y' - 6y = 0$

3.5: Higher order linear ODEs

Learning Objectives

- Solve higher-order linear differential equations, including those with constant coefficients, using characteristic equations to find general and particular solutions.
- Understand and apply the concepts of linear independence and superposition to construct solutions for systems of differential equations.

Equations that appear in applications tend to be second order, although higher order equations do appear from time to time. Hence, it is a generally assumed that the world is “second order” from a modern physics perspective. The basic results about linear ODEs of higher order are essentially the same as for second order equations, with 2 replaced by n. The important concept of linear independence is somewhat more complicated when more than two functions are involved.

For higher order constant coefficient ODEs, the methods are also somewhat harder to apply, but we will not dwell on these complications. We can always use the methods for systems of linear equations to solve higher order constant coefficient equations. So let us start with a general homogeneous linear equation:

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_o(x)y=f(x) \tag{3.5.1}$$

Theorem 3.5.1

Superposition

Suppose $y_1, y_2, \ldots, y_n$ are solutions of the homogeneous Equation (2.3.1). Then

$$y(x)=C_1y_1(x)+C_2y_2(x)+\ldots+C_ny_n(x)$$

also solves Equation (2.3.1) for arbitrary constants $C_1, \ldots. C_n$.

In other words, a linear combination of solutions to Equation (2.3.1) is also a solution to Equation (2.3.1). We also have the existence and uniqueness theorem for nonhomogeneous linear equations.

Theorem 3.5.2

Existence and Uniqueness

Suppose p_o through p_{n-1}, and f are continuous functions on some interval I, a is a number in I, and $b_0, b_1, \ldots, b_{n-1}$ are constants. The equation

$$y^{(n)}+p_{n-1}(x)y^{(n-1)}+\ldots+p_1(x)y'+p_o(x)y=f(x)$$

has exactly one solution $y(x)$ defined on the same interval I satisfying the initial conditions

$$y(a)=b_0, \quad y'(a)=b_1, \quad \ldots, \quad y^{(n-1)}(a)=b_{n-1}$$

Linear Independence

When we had two functions y_1 and y_2 we said they were linearly independent if one was not the multiple of the other. Same idea holds for n functions. In this case it is easier to state as follows. The functions $y_1, y_2, \ldots, y_n$ are linearly independent if

$$c_1y_1+c_2y_2+\cdots+c_ny_n=0$$

has only the trivial solution $c_1=c_2=\cdots=c_n=0$, where the equation must hold for all x. If we can solve equation with some constants where for example $c_1 \neq 0$, then we can solve for y_1 as a linear combination of the others. If the functions are not linearly independent, they are linearly dependent.

Example 3.5.1

Show that e^x, e^{2x}, and e^{3x} are linearly independent functions.

Solution

Let us give several ways to show this fact. Many textbooks introduce Wronskians, but that is really not necessary to solve this example.

Method 1:

Let us write down

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

We use rules of exponentials and write $z = e^x$. Then we have

$$c_1 z + c_2 z^2 + c_3 z^3 = 0$$

The left hand side is a third degree polynomial in z. It can either be identically zero, or it can have at most 3 zeros. Therefore, it is identically zero, $c_1 = c_2 = c_3 = 0$, and the functions are linearly independent.

Method 2:

As before we write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

This equation has to hold for all x. What we could do is divide through by e^{3x} to get

$$c_1 e^{-2x} + c_2 e^{-x} + c_3 = 0$$

As the equation is true for all x, let $x \to \infty$. After taking the limit we see that $c_3 = 0$. Hence our equation becomes

$$c_1 e^x + c_2 e^{2x} = 0$$

Rinse, repeat!

Method 3:

We again write

$$c_1 e^x + c_2 e^{2x} + c_3 e^{3x} = 0$$

We can evaluate the equation and its derivatives at different values of x to obtain equations for c_1, c_2, and c_3. Let us first divide by e^x for simplicity.

$$c_1 + c_2 e^x + c_3 e^{2x} = 0$$

We set $x = 0$ to get the equation $c_1 + c_2 + c_3 = 0$. Now differentiate both sides

$$c_2 e^x + 2c_3 e^{2x} = 0$$

We set $x = 0$ to get $c_2 + 2c_3 = 0$. We divide by e^x again and differentiate to get $2c_3 e^x = 0$. It is clear that c_3 is zero. Then c_2 must be zero as $c_2 = -2c_3$, and c_1 must be zero because $c_1 + c_2 + c_3 = 0$.

There is no one best way to do it. All of these methods are perfectly valid. The important thing is to understand why the functions are linearly independent.

Example 3.5.2

Show that e^x, e^{-x} and $\cosh x$ are linearly independent functions.

Solution

Simply apply definition of the hyperbolic cosine:

$$\cosh x = \frac{e^x + e^{-x}}{2} \quad \text{or} \quad 2\cosh x - e^x - e^{-x} = 0$$

The sum of the three functions yields:

$$c_1 e^x + c_2 e^{-x} + \frac{c_3}{2}(e^x + e^{-x}) = 0$$

If we set $c_1 = -1$, $c_2 = -1$, and $c_3 = 2$, we have a non-trivial solution, which tells us the three functions are linearly independent.

Constant Coefficient Higher Order ODEs

When we have a higher order constant coefficient homogeneous linear equation, the song and dance is exactly the same as it was for second order. We just need to find more solutions. If the equation is n^{th} order we need to find n linearly independent solutions. It is best seen by example.

Example 3.5.3: Third order ODE with Constant Coefficients

Find the general solution to

$$y''' - 3'' - y' + 3y = 0 \tag{3.5.2}$$

Solution

Try: $y = e^{rx}$. We plug in and get

$$\underbrace{r^3 e^{rx}}_{y'''} - 3\underbrace{r^2 e^{rx}}_{y''} - \underbrace{re^{rx}}_{y'} + 3\underbrace{e^{rx}}_{y} = 0.$$

We divide through by e^{rx}. Then

$$r^3 - 3r^2 - r + 3 = 0$$

The trick now is to find the roots. There is a formula for the roots of degree 3 and 4 polynomials, but it is very complicated. There is no formula for higher degree polynomials. That does not mean that the roots do not exist. There are always n roots for an n^{th} degree polynomial. They may be repeated and they may be complex. Computers are pretty good at finding roots approximately for reasonable size polynomials.

A good place to start is to plot the polynomial and check where it is zero. We can also simply try plugging in. We just start plugging in numbers $r = -2, -1, 0, 1, 2, \ldots$ and see if we get a hit (we can also try complex numbers). Even if we do not get a hit, we may get an indication of where the root is. For example, we plug $r = -2$ into our polynomial and get -15; we plug in $r = 0$ and get 3. That means there is a root between $r = -2$ and $r = 0$, because the sign changed. If we find one root, say r_1, then we know $(r - r_1)$ is a factor of our polynomial. Polynomial long division can then be used.

A good strategy is to begin with $r = -1$, 1, or 0. These are easy to compute. Our polynomial happens to have two such roots, $r_1 = -1$ and $r_2 = 1$ and. There should be three roots and the last root is reasonably easy to find. The constant term in a monic polynomial such as this is the multiple of the negations of all the roots because $r^3 - 3r^2 - r + 3 = (r - r_1)(r - r_2)(r - r_3)$. So

$$3 = (-r_1)(-r_2)(-r_3) = (1)(-1)(-r_3) = r_3$$

You should check that $r_3 = 3$ really is a root. Hence we know that e^{-x}, e^x, and e^{3x} are solutions to Equation 3.5.2. They are linearly independent as can easily be checked, and there are three of them, which happens to be exactly the number we need. Hence the general solution is

$$y = C_1 e^{-x} + C_2 e^x + C_3 e^{3x}$$

Suppose we were given some initial conditions $y(0) = 1$, $y'(0) = 2$, and $y''(0) = 3$. Then

$$\begin{aligned} 1 &= y(0) = C_1 + C_2 + C_3 \\ 2 &= y'(0) = -C_1 + C_2 + 3C_3 \\ 3 &= y''(0) = C_1 + C_2 + 9C_3 \end{aligned} \tag{3.5.3}$$

It is possible to find the solution by high school algebra, but it would be a pain. The sensible way to solve a system of equations such as this is to use matrix algebra. For now we note that the solution is $C_1 = -\frac{1}{4}$, $C_2 = 1$, and $C_3 = \frac{1}{4}$. The specific solution to the ODE is

$$y = -\frac{1}{4}e^{-x} + e^x + \frac{1}{4}e^{3x}$$

Next, suppose that we have real roots, but they are repeated. Let us say we have a root r repeated k times. In the spirit of the second order solution, and for the same reasons, we have the solutions

$$e^{rx}, xe^{rx}, x^2e^{rx}, \ldots, x^{k-1}e^{rx}$$

We take a linear combination of these solutions to find the general solution.

✓ Example 3.5.4

Solve

$$y^{(4)} - 3y''' + 3y'' - y' = 0$$

Solution

We note that the characteristic equation is

$$r^4 - 3r^3 + 3r^2 - r = 0$$

By inspection we note that $r^4 - 3r^3 + 3r^2 - r = r(r-1)^3$. Hence the roots given with multiplicity are $r = 0, 1, 1, 1$. Thus the general solution is

$$y = \underbrace{(C_1 + C_2 + C_3x^2)e^x}_{\text{terms coming from r = 1}} + \underbrace{C_4}_{\text{from r = 0}}$$

The case of complex roots is similar to second order equations. Complex roots always come in pairs $r = \alpha \pm i\beta$. Suppose we have two such complex roots, each repeated k times. The corresponding solution is

$$(C_0 + C_1x + \cdots + C_{k-1}x^{k-1})e^{ax}\cos(\beta x) + (D_0 + D_1x + \cdots + D_{k-1}x^{k-1})e^{ax}\sin(\beta x)$$

where $C_0, \ldots, C_{k-1}, D_0, \ldots, D_{k-1}$ are arbitrary constants.

✓ Example 3.5.5

Find the solution of the given initial value problem:

$$4y'' - 8y' + 3y = 0, \quad y(0) = 2, \quad y'(0) = 1/2$$

Solution

- **Video Length**: 8 minutes 3 seconds.
- **Context**: This video demonstrates how to solve a second-order linear differential equation using characteristic roots and initial conditions.

Example 3.5.6

Solve

$$y^{(4)} - 4y''' + 8y'' - 8y' + 4y = 0$$

Solution

The characteristic equation is

$$\begin{aligned} r^4 - 4r^3 + 8r^2 - 8r + 4 &= 0 \\ (r^2 - 2r + 2)^2 &= 0 \\ ((r-1)^2 + 1)^2 &= 0 \end{aligned} \tag{3.5.4}$$

Hence the roots are $1 \pm i$, both with multiplicity 2. Hence the general solution to the ODE is

$$y = (C_1 + C_2 x)e^x \cos x + (C_3 + C_4 x)e^x \sin x$$

The way we solved the characteristic equation above is really by guessing or by inspection. It is not so easy in general. We could also have asked a computer or an advanced calculator for the roots.

Footnotes

[1] The word monic means that the coefficient of the top degree r^d, in our case r^3, is 1.

3.5E: Exercises for Section 3.5

Exercises 1-17

Solve the given differential equations or answer the questions regarding linear independence or constructing differential equations based on the provided solutions.

Exercise $3.5E.1$

Find the general solution for $y''' - y'' + y' - y = 0$.

Exercise $3.5E.2$

Find the general solution for $y^{(4)} - 5y''' + 6y'' = 0$.

Exercise $3.5E.3$

Find the general solution for $y''' + 2y'' + 2y' = 0$.

Exercise $3.5E.4$

Suppose the characteristic equation for a differential equation is $(r-1)^2(r-2)^2 = 0$.

a. Find such a differential equation.
b. Find its general solution.

Exercise $3.5E.5$

Suppose that a fourth order equation has a solution $y = 2e^{4x} x \cos x$.

a. Find such an equation.
b. Find the initial conditions that the given solution satisfies.

Exercise $3.5E.6$

Find the general solution for the equation of **Exercise 3.5E.5**.

Exercise $3.5E.7$

Let $f(x) = e^x - \cos x, g(x) = e^x + \cos x$ and $h(x) = \cos x$. Are $f(x), g(x)$, and $h(x)$ and linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $3.5E.8$

Let $f(x) = 0, g(x) = \cos x$, and $h(x) = \sin x$. Are $f(x), g(x)$, and $h(x)$ and linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $3.5E.9$

Are $x, x^2,\ and\ x^4$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $3.5E.10$

Are $e^x, xe^x,\ and\ x^2e^x$ linearly independent? If so, show it, if not, find a linear combination that works.

Exercise $3.5E.11$

Find an equation such that $y = xe^{-2x}\sin(3x)$ is a solution.

Exercise $3.5E.12$

Find the general solution of $y^{(5)} - y^{(4)} = 0$.

Answer

$y = C_1e^x + C_2x^3 + C_3x^2 + C_4x + C_5$

Exercise $3.5E.13$

Suppose that the characteristic equation of a third order differential equation has roots $3 \pm 2i$.

a. What is the characteristic equation?
b. Find the corresponding differential equation.
c. Find the general solution.

Answer

a. $r^3 - 3r^2 + 4r - 12 = 0$
b. $y''' - 3y'' + 4y' - 12y = 0$
c. $y = C_1e^{3x} + C_2\sin(2x) + C_3\cos(2x)$

Exercise $3.5E.14$

Solve $1001y''' + 3.2y'' + \pi y' - \sqrt{4}y = 0, y(0) = 0, y'(0) = 0, y''(0) = 0$.

Answer

$y = 0$

Exercise $3.5E.15$

Are $e^x, e^{x+1}, e^{2x}, \sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Answer

No. $e^1e^x - e^{x+1} = 0$.

Exercise $3.5E.16$

Are $\sin(x), x, x\sin(x)$ linearly independent? If so, show it, if not find a linear combination that works.

Answer

Yes. (Hint: First note that $\sin(x)$ is bounded. Then note that x and $x\sin(x)$ cannot be multiples of each other.)

Exercise $3.5E.17$

Find an equation such that $y = \cos(x)$, $y = \sin(x)$, $y = e^x$ are solutions.

Answer

$$y''' - y'' + y' - y = 0$$

3.6: Reduction of Order

Learning Objectives

- Understand and apply the reduction of order method to solve second-order linear differential equations when one solution is already known.
- Derive and solve the resulting first-order equation to find the general solution of the given differential equation.

In this section we give a method for finding the general solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{3.6.1}$$

if we know a nontrivial solution y_1 of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{3.6.2}$$

The method is called *reduction of order* because it reduces the task of solving Equation (3.6.1) to solving a first order equation. Unlike the method of undetermined coefficients, it does not require P_0, P_1, and P_2 to be constants, or F to be of any special form.

By now you shouldn't be surprised that we look for solutions of Equation (3.6.1) in the form

$$y = uy_1 \tag{3.6.3}$$

where u is to be determined so that y satisfies Equation (3.6.1). Substituting Equation (3.6.3) and

$$\begin{aligned} y' &= u'y_1 + uy_1' \\ y'' &= u''y_1 + 2u'y_1' + uy_1'' \end{aligned}$$

into Equation (3.6.1) yields

$$P_0(x)(u''y_1 + 2u'y_1' + uy_1'') + P_1(x)(u'y_1 + uy_1') + P_2(x)uy_1 = F(x).$$

Collecting the coefficients of u, u', and u'' yields

$$(P_0y_1)u'' + (2P_0y_1' + P_1y_1)u' + (P_0y_1'' + P_1y_1' + P_2y_1)u = F. \tag{3.6.4}$$

However, the coefficient of u is zero, since y_1 satisfies Equation (3.6.2). Therefore Equation (3.6.4) reduces to

$$Q_0(x)u'' + Q_1(x)u' = F, \tag{3.6.5}$$

with

$$Q_0 = P_0y_1 \quad \text{and} \quad Q_1 = 2P_0y_1' + P_1y_1.$$

(It isn't worthwhile to memorize the formulas for Q_0 and Q_1!) Since Equation (3.6.5) is a linear first order equation in $z = u'$, we can solve it for u' by using integrating factor as in Section 1.6, integrate the solution to obtain u, and then obtain y from Equation (3.6.3).

Example 3.6.1

a. Find the general solution of

$$xy'' - (2x+1)y' + (x+1)y = x^2, \tag{3.6.6}$$

given that $y_1 = e^x$ is a solution of the complementary equation

$$xy'' - (2x+1)y' + (x+1)y = 0. \tag{3.6.7}$$

b. As a byproduct of (a), find a fundamental set of solutions of Equation (3.6.7).

Solution

a. If $y = ue^x$, then $y' = u'e^x + ue^x$ and $y'' = u''e^x + 2u'e^x + ue^x$, so

$$\begin{aligned} xy''-(2x+1)y'+(x+1)y &= x(u''e^x+2u'e^x+ue^x)-(2x+1)(u'e^x+ue^x)+(x+1)ue^x \\ &= (xu''-u')e^x. \end{aligned}$$

Therefore $y=ue^x$ is a solution of Equation (3.6.6) if and only if

$$(xu''-u')e^x=x^2,$$

which is a first order equation in u'. We rewrite it as

$$u''-\frac{u'}{x}=xe^{-x}. \tag{3.6.8}$$

Let $z=u'$, so that Equation (3.6.8) becomes

$$z'-\frac{z}{x}=xe^{-x}. \tag{3.6.9}$$

We leave it to you to show (by separation of variables) that $z_1=x$ is a solution of the complementary equation

$$z'-\frac{z}{x}=0$$

for Equation (3.6.9). By Equation (3.6.3), every solution of Equation (3.6.9) is of the form

$$z=vx \quad \text{where} \quad v'x=xe^{-x}, \quad \text{so} \quad v'=e^{-x} \quad \text{and} \quad v=-e^{-x}+C_1.$$

Since $u'=z=vx$, u is a solution of Equation (3.6.8) if and only if

$$u'=vx=-xe^{-x}+C_1x.$$

Integrating this yields

$$u=(x+1)e^{-x}+\frac{C_1}{2}x^2+C_2.$$

Therefore the general solution of Equation (3.6.6) is

$$y=ue^x=x+1+\frac{C_1}{2}x^2e^x+C_2e^x. \tag{3.6.10}$$

b. By letting $C_1=C_2=0$ in Equation (3.6.10), we see that $y_{p_1}=x+1$ is a solution of Equation (3.6.6). By letting $C_1=2$ and $C_2=0$, we see that $y_{p_2}=x+1+x^2e^x$ is also a solution of Equation (3.6.6). Since the difference of two solutions of Equation (3.6.6) is a solution of Equation (3.6.7), $y_2=y_{p_1}-y_{p_2}=x^2e^x$ is a solution of Equation (3.6.7). Since y_2/y_1 is nonconstant and we already know that $y_1=e^x$ is a solution of Equation (3.6.6), Theorem 3.1.1 implies that $\{e^x, x^2e^x\}$ is a fundamental set of solutions of Equation (3.6.7).

Although Equation (3.6.10) is a correct form for the general solution of Equation (3.6.6), it is silly to leave the arbitrary coefficient of x^2e^x as $C_1/2$ where C_1 is an arbitrary constant. Moreover, it is sensible to make the subscripts of the coefficients of $y_1=e^x$ and $y_2=x^2e^x$ consistent with the subscripts of the functions themselves. Therefore we rewrite Equation (3.6.10) as

$$y=x+1+c_1e^x+c_2x^2e^x$$

by simply renaming the arbitrary constants. We'll also do this in the next two examples, and in the answers to the exercises.

Example 3.6.2

a. Find the general solution of

$$x^2y''+xy'-y=x^2+1,$$

given that $y_1=x$ is a solution of the complementary equation

$$x^2y''+xy'-y=0. \tag{3.6.11}$$

As a byproduct of this result, find a fundamental set of solutions of Equation (3.6.11).

b. Solve the initial value problem

$$x^2y'' + xy' - y = x^2 + 1, \quad y(1) = 2, \; y'(1) = -3. \tag{3.6.12}$$

Solution

a. If $y = ux$, then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' + xy' - y &= x^2(u''x + 2u') + x(u'x + u) - ux \\ &= x^3u'' + 3x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of Equation (3.6.12) if and only if

$$x^3u'' + 3x^2u' = x^2 + 1,$$

which is a first order equation in u'. We rewrite it as

$$u'' + \frac{3}{x}u' = \frac{1}{x} + \frac{1}{x^3}. \tag{3.6.13}$$

Let write $z = u'$, so that Equation (3.6.13) becomes

$$z' + \frac{3}{x}z = \frac{1}{x} + \frac{1}{x^3}. \tag{3.6.14}$$

We leave it to you to show (by separation of variables) that $z_1 = 1/x^3$ is a solution of the complementary equation

$$z' + \frac{3}{x}z = 0$$

for Equation (3.6.14). By Equation (3.6.3), every solution of Equation (3.6.14) is of the form

$$z = \frac{v}{x^3} \quad \text{where} \quad \frac{v'}{x^3} = \frac{1}{x} + \frac{1}{x^3}, \quad \text{so} \quad v' = x^2 + 1 \quad \text{and} \quad v = \frac{x^3}{3} + x + C_1.$$

Since $u' = z = v/x^3$, u is a solution of Equation (3.6.14) if and only if

$$u' = \frac{v}{x^3} = \frac{1}{3} + \frac{1}{x^2} + \frac{C_1}{x^3}.$$

Integrating this yields

$$u = \frac{x}{3} - \frac{1}{x} - \frac{C_1}{2x^2} + C_2.$$

Therefore the general solution of Equation (3.6.12) is

$$y = ux = \frac{x^2}{3} - 1 - \frac{C_1}{2x} + C_2x. \tag{3.6.15}$$

Reasoning as in the solution of Example $3.6.1a$, we conclude that $y_1 = x$ and $y_2 = 1/x$ form a fundamental set of solutions for Equation (3.6.11).

As we explained above, we rename the constants in Equation (3.6.15) and rewrite it as

$$y = \frac{x^2}{3} - 1 + c_1x + \frac{c_2}{x}. \tag{3.6.16}$$

b. Differentiating Equation (3.6.16) yields

$$y' = \frac{2x}{3} + c_1 - \frac{c_2}{x^2}. \tag{3.6.17}$$

Setting $x = 1$ in Equation (3.6.16) and Equation (3.6.17) and imposing the initial conditions $y(1) = 2$ and $y'(1) = -3$ yields

$$\begin{aligned} c_1 + c_2 &= \frac{8}{3} \\ c_1 - c_2 &= -\frac{11}{3}. \end{aligned}$$

Solving these equations yields $c_1 = -1/2$, $c_2 = 19/6$. Therefore the solution of Equation (3.6.12) is

$$y = \frac{x^2}{3} - 1 - \frac{x}{2} + \frac{19}{6x}.$$

Using reduction of order to find the general solution of a homogeneous linear second order equation leads to a homogeneous linear first order equation in u' that can be solved by separation of variables. The next example illustrates this.

✓ Example 3.6.3

Find the general solution and a fundamental set of solutions of

$$x^2y'' - 3xy' + 3y = 0, \tag{3.6.18}$$

given that $y_1 = x$ is a solution.

Solution

If $y = ux$ then $y' = u'x + u$ and $y'' = u''x + 2u'$, so

$$\begin{aligned} x^2y'' - 3xy' + 3y &= x^2(u''x + 2u') - 3x(u'x + u) + 3ux \\ &= x^3u'' - x^2u'. \end{aligned}$$

Therefore $y = ux$ is a solution of Equation (3.6.18) if and only if

$$x^3u'' - x^2u' = 0.$$

Separating the variables u' and x yields

$$\frac{u''}{u'} = \frac{1}{x},$$

so

$$\ln|u'| = \ln|x| + k, \quad \text{or equivalently} \quad u' = C_1x.$$

Therefore

$$u = \frac{C_1}{2}x^2 + C_2,$$

so the general solution of Equation (3.6.18) is

$$y = ux = \frac{C_1}{2}x^3 + C_2x,$$

which we rewrite as

$$y = c_1x + c_2x^3.$$

Therefore $\{x, x^3\}$ is a fundamental set of solutions of Equation (3.6.18).

3.6E: Exercises for Section 3.6

Exercises 1-17

Find the general solution, given that y_1 satisfies the complementary equation. As a byproduct, find a fundamental set of solutions of the complementary equation.

Exercise $3.6E.1$

$(2x+1)y''-2y'-(2x+3)y=(2x+1)^2; \quad y_1=e^{-x}$

Exercise $3.6E.2$

$x^2y''+xy'-y=\frac{4}{x^2}; \quad y_1=x$

Answer

$y=ux=\frac{4}{3x^2}-\frac{C_1}{2x}+C_2x, \quad \text{or} \quad y=\frac{4}{3x^2}+c_1x+\frac{c_2}{x}.$

As a byproduct, $\{x,1/x\}$ is a fundamental set of solutions of the complementary equation.

Exercise $3.6E.3$

$x^2y''-xy'+y=x; \quad y_1=x$

Exercise $3.6E.4$

$y''-3y'+2y=\frac{1}{1+e^{-x}}; \quad y_1=e^{2x}$

Answer

$y=(e^{2x}+e^x)\ln(1+e^{-x})+c_1e^{2x}+c_2e^x.$

As a byproduct, $\{e^{2x},e^x\}$ is a fundamental set of solutions of the complementary equation.

Exercise $3.6E.5$

$y''-2y'+y=7x^{3/2}e^x; \quad y_1=e^x$

Exercise $3.6E.6$

$4x^2y''+(4x-8x^2)y'+(4x^2-4x-1)y=4x^{1/2}e^x(1+4x); \quad y_1=x^{1/2}e^x$

Answer

$y=e^x\left(2x^{3/2}+x^{1/2}\ln x+c_1x^{1/2}+c_2x^{-1/2}\right).$

As a byproduct, $\{x^{1/2}e^x,x^{-1/2}e^{-x}\}$ is a fundamental set of solutions of the complementary equation.

Exercise $3.6E.7$

$y''-2y'+2y=e^x\sec x; \quad y_1=e^x\cos x$

Exercise $3.6E.8$

$y''+4xy'+(4x^2+2)y=8e^{-x(x+2)}; \quad y_1=e^{-x^2}$

Answer

$y=e^{-x^2}(2e^{-2x}+c_1+c_2x).$

As a byproduct, $\{e^{-x^2}, xe^{-x^2}\}$ is a fundamental set of solutions of the complementary equation.

Exercise $3.6E.9$

$x^2y''+xy'-4y=-6x-4; \quad y_1=x^2$

Exercise $3.6E.10$

$x^2y''+2x(x-1)y'+(x^2-2x+2)y=x^3e^{2x}; \quad y_1=xe^{-x}$

Answer

$y=\frac{xe^{2x}}{9}+xe^{-x}(c_1+c_2x).$

As a byproduct, $\{xe^{-x}, x^2e^{-x}\}$ is a fundamental set of solutions of the complementary equation.

Exercise $3.6E.11$

$x^2y''-x(2x-1)y'+(x^2-x-1)y=x^2e^x; \quad y_1=xe^x$.

Exercise $3.6E.12$

$(1-2x)y''+2y'+(2x-3)y=(1-4x+4x^2)e^x; \quad y_1=e^x$

Exercise $3.6E.13$

$x^2y''-3xy'+4y=4x^4; \quad y_1=x^2$

Exercise $3.6E.14$

$2xy''+(4x+1)y'+(2x+1)y=3x^{1/2}e^{-x}; \quad y_1=e^{-x}$

Answer

$y=e^{-x}\left(x^{3/2}+c_1+c_2x^{1/2}\right).$

As a byproduct, $\{e^{-x}, x^{1/2}e^{-x}\}$ is a fundamental set of solutions of the complementary equation.

Exercise $3.6E.15$

$xy''-(2x+1)y'+(x+1)y=-e^x; \quad y_1=e^x$

Exercise $3.6E.16$

$4x^2y''-4x(x+1)y'+(2x+3)y=4x^{5/2}e^{2x}; \quad y_1=x^{1/2}$

Exercise $3.6E.17$

$x^2y''-5xy'+8y=4x^2; \quad y_1=x^2$

Exercises 18-30

Find a fundamental set of solutions, given that y_1 is a solution.

Exercise $3.6E.18$

$xy''+(2-2x)y'+(x-2)y=0; \quad y_1=e^x$

Answer

$y=e^x\left(-\frac{C_1}{x}+C_2\right)$ is the general solution, and $\{e^x, e^x/x\}$ is a fundamental set of solutions.

Exercise $3.6E.19$

$x^2y''-4xy'+6y=0; \quad y_1=x^2$

Exercise $3.6E.20$

$x^2(\ln|x|)^2y''-(2x\ln|x|)y'+(2+\ln|x|)y=0; \quad y_1=\ln|x|$

Exercise $3.6E.21$

$4xy''+2y'+y=0; \quad y_1=\sin\sqrt{x}$

Exercise $3.6E.22$

$xy''-(2x+2)y'+(x+2)y=0; \quad y_1=e^x$

Answer

$y=\left(\frac{C_1x^3}{3}+C_2\right)e^x$ is the general solution, and $\{e^x, x^3e^x\}$ is a fundamental set of solutions.

Exercise $3.6E.23$

$x^2y''-(2a-1)xy'+a^2y=0; \quad y_1=x^a$

Exercise $3.6E.24$

$x^2y''-2xy'+(x^2+2)y=0; \quad y_1=x\sin x$

Exercise $3.6E.25$

$xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$

Exercise $3.6E.26$

$4x^2(\sin x)y''-4x(x\cos x+\sin x)y'+(2x\cos x+3\sin x)y=0; \quad y_1=x^{1/2}$

? Exercise $3.6E.27$

$4x^2y''-4xy'+(3-16x^2)y=0; \quad y_1=x^{1/2}e^{2x}$

? Exercise $3.6E.28$

$(2x+1)xy''-2(2x^2-1)y'-4(x+1)y=0; \quad y_1=1/x$

? Exercise $3.6E.29$

$(x^2-2x)y''+(2-x^2)y'+(2x-2)y=0; \quad y_1=e^x$

? Exercise $3.6E.30$

$xy''-(4x+1)y'+(4x+2)y=0; \quad y_1=e^{2x}$

Answer

$y=e^{2x}\left(\frac{C_1x^2}{2}+C_2\right)$ is the general solution, and $\{e^{2x}, x^2e^{2x}\}$ is a fundamental set of solutions.

Exercises 31-33

Solve the initial value problem, given that y_1 satisfies the complementary equation.

? Exercise $3.6E.31$

$x^2y''-3xy'+4y=4x^4, \quad y(-1)=7, \quad y'(-1)=-8; \quad y_1=x^2$

? Exercise $3.6E.32$

$(3x-1)y''-(3x+2)y'-(6x-8)y=0, \quad y(0)=2,\ y'(0)=3; \quad y_1=e^{2x}$

Answer

$y=2e^{2x}-xe^{-x}.$

? Exercise $3.6E.33$

$(x+1)^2y''-2(x+1)y'-(x^2+2x-1)y=(x+1)^3e^x, \quad y(0)=1, \quad y'(0)=\ -1; \quad y_1=(x+1)e^x$

Exercises 34-40

Solve the initial value problem and graph the solution, given that y_1 satisfies the complementary equation.

? Exercise $3.6E.34$

$x^2y''+2xy'-2y=x^2, \quad y(1)=\frac{5}{4},\ y'(1)=\frac{3}{2}; \quad y_1=x$

Answer

$y=\frac{x^2}{4}+x.$

Exercise $3.6E.35$

$(x^2-4)y''+4xy'+2y=x+2, \quad y(0)=-\frac{1}{3}, \quad y'(0)=-1; \quad y_1=\frac{1}{x-2}$

Exercise $3.6E.36$

Suppose p_1 and p_2 are continuous on (a,b). Let y_1 be a solution of

$$y''+p_1(x)y'+p_2(x)y=0 \tag{A}$$

that has no zeros on (a,b), and let x_0 be in (a,b). Use reduction of order to show that y_1 and

$$y_2(x)=y_1(x)\int_{x_0}^{x}\frac{1}{y_1^2(t)}\exp\left(-\int_{x_0}^{t}p_1(s)\,ds\right)dt$$

form a fundamental set of solutions of (A) on (a,b).

Exercise $3.6E.37$

The nonlinear first order equation

$$y'+y^2+p(x)y+q(x)=0 \tag{A}$$

is a **Riccati equation**. Assume that p and q are continuous.

a. Show that y is a solution of (A) if and only if $y=z'/z$, where

$$z''+p(x)z'+q(x)z=0. \tag{B}$$

b. Show that the general solution of (A) is

$$y=\frac{c_1z_1'+c_2z_2'}{c_1z_1+c_2z_2}, \tag{C}$$

where $\{z_1,z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

c. Does the formula (C) imply that the first order equation (A) has a two–parameter family of solutions? Explain your answer.

Exercise $3.6E.38$

Use a method suggested by **Exercise 3.6E.37** to find all solutions of the equation.

a. $y'+y^2+k^2=0$
b. $y'+y^2-3y+2=0$
c. $y'+y^2+5y-6=0$
d. $y'+y^2+8y+7=0$
e. $y'+y^2+14y+50=0$
f. $6y'+6y^2-y-1=0$
g. $36y'+36y^2-12y+1=0$

Exercise $3.6E.39$

Use the method of reduction of order to find all solutions of the equation, given that y_1 is a solution.

a. $x^2(y'+y^2)-x(x+2)y+x+2=0; \quad y_1=1/x$
b. $y'+y^2+4xy+4x^2+2=0; \quad y_1=-2x$
c. $(2x+1)(y'+y^2)-2y-(2x+3)=0; \quad y_1=-1$
d. $(3x-1)(y'+y^2)-(3x+2)y-6x+8=0; \quad y_1=2$
e. $x^2(y'+y^2)+xy+x^2-\frac{1}{4}=0; \quad y_1=-\tan x-\frac{1}{2x}$
f. $x^2(y'+y^2)-7xy+7=0; \quad y_1=1/x$

Exercise $3.6E.40$

The nonlinear first order equation

$$y' + r(x)y^2 + p(x)y + q(x) = 0 \tag{A}$$

is the **generalized Riccati equation**. Assume that p and q are continuous and r is differentiable.

a. Show that y is a solution of (A) if and only if $y = z'/rz$, where

$$z'' + \left[p(x) - \frac{r'(x)}{r(x)}\right]z' + r(x)q(x)z = 0. \tag{B}$$

b. Show that the general solution of (A) is

$$y = \frac{c_1 z_1' + c_2 z_2'}{r(c_1 z_1 + c_2 z_2)},$$

where $\{z_1, z_2\}$ is a fundamental set of solutions of (B) and c_1 and c_2 are arbitrary constants.

3.7: Variation of Parameters

Learning Objectives

- Understand the method of variation of parameters to find a particular solution of second-order linear nonhomogeneous differential equations.
- Apply variation of parameters to equations with known solutions of the complementary equation to derive the general solution.

In this section we give a method called *variation of parameters* for finding a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x) \tag{3.7.1}$$

if we know a fundamental set $\{y_1, y_2\}$ of solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0. \tag{3.7.2}$$

Having found a particular solution y_p by this method, we can write the general solution of Equation (3.7.1) as

$$y = y_p + c_1y_1 + c_2y_2.$$

Since we need only one nontrivial solution of Equation (3.7.2) to find the general solution of Equation (3.7.1) by reduction of order, it is natural to ask why we are interested in variation of parameters, which requires two linearly independent solutions of Equation (3.7.2) to achieve the same goal. Here's the answer:

- If we already know two linearly independent solutions of Equation (3.7.2) then variation of parameters will probably be simpler than reduction of order.
- Variation of parameters generalizes naturally to a method for finding particular solutions of higher order linear equations (Section 3.5) and linear systems of equations (Section 6.2), while reduction of order doesn't.
- Variation of parameters is a powerful theoretical tool used by researchers in differential equations.

Theorem 3.7.1

Suppose p and q are continuous on an open interval (a,b) and let y_1 and y_2 be solutions of

$$y'' + p(x)y' + q(x)y = 0$$

on (a,b). Then the following statements are equivalent; that is, they are either all true or all false.

1. The general solution on (a,b) is $y = c_1y_1 + c_2y_2$.
2. $\{y_1, y_2\}$ is a fundamental set of solutions on (a,b).
3. $\{y_1, y_2\}$ is linearly independent on (a,b).
4. The Wronskian of $\{y_1, y_2\}$, which is defined to be $y_1y_2' - y_1'y_2$, is nonzero at some point in (a,b).
5. The Wronskian of $\{y_1, y_2\}$, which is defined to be $y_1y_2' - y_1'y_2$, is nonzero at all points in (a,b).

We'll now derive the method. As usual, we consider solutions of Equation (3.7.1) and Equation (3.7.2) on an interval (a,b) where P_0, P_1, P_2, and F are continuous and P_0 has no zeros. Suppose that $\{y_1, y_2\}$ is a fundamental set of solutions of the complementary equation Equation (3.7.2). We look for a particular solution of Equation (3.7.1) in the form

$$y_p = u_1y_1 + u_2y_2 \tag{3.7.3}$$

where u_1 and u_2 are functions to be determined so that y_p satisfies Equation (3.7.1). You may not think this is a good idea, since there are now two unknown functions to be determined, rather than one. However, since u_1 and u_2 have to satisfy only one condition that y_p is a solution of Equation (3.7.1), we can impose a second condition that produces a convenient simplification, as follows.

Differentiating Equation (3.7.3) yields

$$y_p' = u_1y_1' + u_2y_2' + u_1'y_1 + u_2'y_2. \tag{3.7.4}$$

As our second condition on u_1 and u_2 we require that

$$u_1'y_1+u_2'y_2=0. \tag{3.7.5}$$

Then Equation (3.7.4) becomes

$$y_p'=u_1y_1'+u_2y_2'; \tag{3.7.6}$$

that is, Equation (3.7.5) permits us to differentiate y_p (once!) as if u_1 and u_2 are constants. Differentiating Equation (3.7.4) yields

$$y_p''=u_1y_1''+u_2y_2''+u_1'y_1'+u_2'y_2'. \tag{3.7.7}$$

There are no terms involving u_1'' and u_2'' here, as there would be if we hadn't required Equation (3.7.5). Substituting Equation (3.7.3), Equation (3.7.6), and Equation (3.7.7) into Equation (3.7.1) and collecting the coefficients of u_1 and u_2 yields

$$u_1(P_0y_1''+P_1y_1'+P_2y_1)+u_2(P_0y_2''+P_1y_2'+P_2y_2)+P_0(u_1'y_1'+u_2'y_2')=F.$$

As in the derivation of the method of reduction of order, the coefficients of u_1 and u_2 here are both zero because y_1 and y_2 satisfy the complementary equation. Hence, we can rewrite the last equation as

$$P_0(u_1'y_1'+u_2'y_2')=F. \tag{3.7.8}$$

Therefore y_p in Equation (3.7.3) satisfies Equation (3.7.1) if

$$\begin{aligned} u_1'y_1+u_2'y_2 &= 0 \\ u_1'y_1'+u_2'y_2' &= \tfrac{F}{P_0}, \end{aligned} \tag{3.7.9}$$

where the first equation is the same as Equation (3.7.5) and the second is from Equation (3.7.8).

We'll now show that you can always solve Equation (3.7.9) for u_1' and u_2'. (The method that we use here will always work, but simpler methods usually work when you're dealing with specific equations.) To obtain u_1', multiply the first equation in Equation (3.7.9) by y_2' and the second equation by y_2. This yields

$$\begin{aligned} u_1'y_1y_2'+u_2'y_2y_2' &= 0 \\ u_1'y_1'y_2+u_2'y_2'y_2 &= \frac{Fy_2}{P_0}. \end{aligned}$$

Subtracting the second equation from the first yields

$$u_1'(y_1y_2'-y_1'y_2)=-\frac{Fy_2}{P_0}. \tag{3.7.10}$$

Since $\{y_1,y_2\}$ is a fundamental set of solutions of Equation (3.7.2) on (a,b), Theorem 3.7.1 implies that the Wronskian $y_1y_2'-y_1'y_2$ has no zeros on (a,b). Therefore we can solve Equation (3.7.10) for u_1', to obtain

$$u_1'=-\frac{Fy_2}{P_0(y_1y_2'-y_1'y_2)}. \tag{3.7.11}$$

We leave it to you to start from Equation (3.7.9) and show by a similar argument that

$$u_2'=\frac{Fy_1}{P_0(y_1y_2'-y_1'y_2)}. \tag{3.7.12}$$

We can now obtain u_1 and u_2 by integrating u_1' and u_2'. The constants of integration can be taken to be zero, since any choice of u_1 and u_2 in Equation (3.7.3) will suffice.

You should not memorize Equation (3.7.11) and Equation (3.7.12). On the other hand, you don't want to rederive the whole procedure for every specific problem. We recommend:

a. Write

$$y_p=u_1y_1+u_2y_2 \tag{3.7.13}$$

to remind yourself of what you're doing.

b. Write the system

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0 \\ u_1'y_1' + u_2'y_2' &= \frac{F}{P_0} \end{aligned} \tag{3.7.14}$$

for the specific problem you're trying to solve.

c. Solve Equation (3.7.14) for u_1' and u_2' by any convenient method.
d. Obtain u_1 and u_2 by integrating u_1' and u_2', taking the constants of integration to be zero.
e. Substitute u_1 and u_2 into Equation (3.7.13) to obtain y_p.

✓ Example 3.7.1

Find a particular solution y_p of

$$x^2y'' - 2xy' + 2y = x^{9/2}, \tag{3.7.15}$$

given that $y_1 = x$ and $y_2 = x^2$ are solutions of the complementary equation

$$x^2y'' - 2xy' + 2y = 0.$$

Then find the general solution of Equation (3.7.15).

Solution

We set

$$y_p = u_1x + u_2x^2,$$

where

$$\begin{aligned} u_1'x + \ u_2'x^2 &= 0 \\ u_1' \ \ + 2u_2'x \ &= \frac{x^{9/2}}{x^2} = x^{5/2}. \end{aligned}$$

From the first equation, $u_1' = -u_2'x$. Substituting this into the second equation yields $u_2'x = x^{5/2}$, so $u_2' = x^{3/2}$ and therefore $u_1' = -u_2'x = -x^{5/2}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -\frac{2}{7}x^{7/2} \quad \text{and} \quad u_2 = \frac{2}{5}x^{5/2}.$$

Therefore

$$y_p = u_1x + u_2x^2 = -\frac{2}{7}x^{7/2}x + \frac{2}{5}x^{5/2}x^2 = \frac{4}{35}x^{9/2},$$

and the general solution of Equation (3.7.15) is

$$y = \frac{4}{35}x^{9/2} + c_1x + c_2x^2.$$

✓ Example 3.7.2

Find a particular solution y_p of

$$(x-1)y'' - xy' + y = (x-1)^2, \tag{3.7.16}$$

given that $y_1 = x$ and $y_2 = e^x$ are solutions of the complementary equation

$$(x-1)y'' - xy' + y = 0.$$

Then find the general solution of Equation (3.7.16).

Solution

We set

$$y_p = u_1x + u_2e^x,$$

where

$$\begin{aligned} u_1'x + u_2'e^x &= 0 \\ u_1' \quad + u_2'e^x &= \frac{(x-1)^2}{x-1} = x-1. \end{aligned}$$

Subtracting the first equation from the second yields $-u_1'(x-1) = x-1$, so $u_1' = -1$. From this and the first equation, $u_2' = -xe^{-x}u_1' = xe^{-x}$. Integrating and taking the constants of integration to be zero yields

$$u_1 = -x \quad \text{and} \quad u_2 = -(x+1)e^{-x}.$$

Therefore

$$y_p = u_1x + u_2e^x = (-x)x + (-(x+1)e^{-x})e^x = -x^2 - x - 1,$$

so the general solution of Equation (3.7.16) is

$$y = y_p + c_1x + c_2e^x = -x^2 - x - 1 + c_1x + c_2e^x = -x^2 - 1 + (c_1 - 1)x + c_2e^x. \tag{3.7.17}$$

However, since c_1 is an arbitrary constant, so is $c_1 - 1$; therefore, we improve the appearance of this result by renaming the constant and writing the general solution as

$$y = -x^2 - 1 + c_1x + c_2e^x. \tag{3.7.18}$$

There's nothing *wrong* with leaving the general solution of Equation (3.7.16) in the form Equation (3.7.17); however, we think you'll agree that Equation (3.7.18) is preferable. We can also view the transition from Equation (3.7.17) to Equation (3.7.18) differently. In this example the particular solution $y_p = -x^2 - x - 1$ contained the term $-x$, which satisfies the complementary equation. We can drop this term and redefine $y_p = -x^2 - 1$, since $-x^2 - x - 1$ is a solution of Equation (3.7.16) and x is a solution of the complementary equation; hence, $-x^2 - 1 = (-x^2 - x - 1) + x$ is also a solution of Equation (3.7.16). In general, it is always legitimate to drop linear combinations of $\{y_1, y_2\}$ from particular solutions obtained by variation of parameters. We'll do this in the following examples and in the answers to exercises that ask for a particular solution. Therefore, don't be concerned if your answer to such an exercise differs from ours only by a solution of the complementary equation.

✓ Example 3.7.3

Find a particular solution of

$$y'' + 3y' + 2y = \frac{1}{1+e^x}. \tag{3.7.19}$$

Then find the general solution.

Solution

The characteristic polynomial of the complementary equation

$$y'' + 3y' + 2y = 0 \tag{3.7.20}$$

is $p(r) = r^2 + 3r + 2 = (r+1)(r+2)$, so $y_1 = e^{-x}$ and $y_2 = e^{-2x}$ form a fundamental set of solutions of Equation (3.7.20). We look for a particular solution of Equation (3.7.19) in the form

$$y_p = u_1e^{-x} + u_2e^{-2x},$$

where

$$\begin{aligned} u_1'e^{-x} + \ \ u_2'e^{-2x} &= 0 \\ -u_1'e^{-x} - 2u_2'e^{-2x} &= \frac{1}{1+e^x}. \end{aligned}$$

Adding these two equations yields

$$-u_2'e^{-2x} = \frac{1}{1+e^x}, \quad \text{so} \quad u_2' = -\frac{e^{2x}}{1+e^x}.$$

From the first equation,

$$u_1' = -u_2'e^{-x} = \frac{e^x}{1+e^x}.$$

Integrating by means of the substitution $v = e^x$ and taking the constants of integration to be zero yields

$$u_1 = \int \frac{e^x}{1+e^x}\,dx = \int \frac{dv}{1+v} = \ln(1+v) = \ln(1+e^x)$$

and

$$\begin{aligned} u_2 &= -\int \frac{e^{2x}}{1+e^x}\,dx = -\int \frac{v}{1+v}\,dv = \int \left[\frac{1}{1+v} - 1\right] dv \\ &= \ln(1+v) - v = \ln(1+e^x) - e^x. \end{aligned}$$

Therefore

$$\begin{aligned} y_p &= u_1 e^{-x} + u_2 e^{-2x} \\ &= [\ln(1+e^x)]e^{-x} + [\ln(1+e^x) - e^x]\,e^{-2x}, \end{aligned}$$

so

$$y_p = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) - e^{-x}.$$

Since the last term on the right satisfies the complementary equation, we drop it and redefine

$$y_p = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x).$$

The general solution of Equation (3.7.19) is

$$y = y_p + c_1e^{-x} + c_2e^{-2x} = \left(e^{-x} + e^{-2x}\right)\ln(1+e^x) + c_1e^{-x} + c_2e^{-2x}.$$

Example 3.7.4

Solve the initial value problem

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}, \quad y(0) = -1, \quad y'(0) = -5, \tag{3.7.21}$$

given that

$$y_1 = \frac{1}{x-1} \quad \text{and} \quad y_2 = \frac{1}{x+1}$$

are solutions of the complementary equation

$$(x^2-1)y'' + 4xy' + 2y = 0.$$

Solution

We first use variation of parameters to find a particular solution of

$$(x^2-1)y'' + 4xy' + 2y = \frac{2}{x+1}$$

on $(-1, 1)$ in the form

$$y_p = \frac{u_1}{x-1} + \frac{u_2}{x+1},$$

where

$$\frac{u_1'}{x-1}+\frac{u_2'}{x+1}=0 \tag{3.7.22}$$

$$-\frac{u_1'}{(x-1)^2}-\frac{u_2'}{(x+1)^2}=\frac{2}{(x+1)(x^2-1)}$$

Multiplying the first equation by $1/(x-1)$ and adding the result to the second equation yields

$$\left[\frac{1}{x^2-1}-\frac{1}{(x+1)^2}\right]u_2'=\frac{2}{(x+1)(x^2-1)}. \tag{3.7.23}$$

Since

$$\left[\frac{1}{x^2-1}-\frac{1}{(x+1)^2}\right]=\frac{(x+1)-(x-1)}{(x+1)(x^2-1)}=\frac{2}{(x+1)(x^2-1)},$$

Equation (3.7.23) implies that $u_2'=1$. From Equation (3.7.22),

$$u_1'=-\frac{x-1}{x+1}u_2'=-\frac{x-1}{x+1}.$$

Integrating and taking the constants of integration to be zero yields

$$\begin{aligned} u_1 &= -\int\frac{x-1}{x+1}\,dx=-\int\frac{x+1-2}{x+1}\,dx \\ &= \int\left[\frac{2}{x+1}-1\right]dx=2\ln(x+1)-x \end{aligned}$$

and

$$u_2=\int dx=x.$$

Therefore

$$\begin{aligned} y_p &= \frac{u_1}{x-1}+\frac{u_2}{x+1}=[2\ln(x+1)-x]\frac{1}{x-1}+x\frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1}+x\left[\frac{1}{x+1}-\frac{1}{x-1}\right]=\frac{2\ln(x+1)}{x-1}-\frac{2x}{(x+1)(x-1)}. \end{aligned}$$

However, since

$$\frac{2x}{(x+1)(x-1)}=\left[\frac{1}{x+1}+\frac{1}{x-1}\right]$$

is a solution of the complementary equation, we redefine

$$y_p=\frac{2\ln(x+1)}{x-1}.$$

Therefore the general solution of Equation (3.7.24) is

$$y=\frac{2\ln(x+1)}{x-1}+\frac{c_1}{x-1}+\frac{c_2}{x+1}. \tag{3.7.24}$$

Differentiating this yields

$$y'=\frac{2}{x^2-1}-\frac{2\ln(x+1)}{(x-1)^2}-\frac{c_1}{(x-1)^2}-\frac{c_2}{(x+1)^2}.$$

Setting $x=0$ in the last two equations and imposing the initial conditions $y(0)=-1$ and $y'(0)=-5$ yields the system

$$\begin{aligned} -c_1+c_2 &= -1 \\ -2-c_1-c_2 &= -5. \end{aligned}$$

The solution of this system is $c_1 = 2,\ c_2 = 1$. Substituting these into Equation (5.7.24) yields

$$\begin{aligned} y &= \frac{2\ln(x+1)}{x-1} + \frac{2}{x-1} + \frac{1}{x+1} \\ &= \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1} \end{aligned}$$

as the solution of Equation (3.7.21). See Figure 3.7.1 for a graph of the solution.

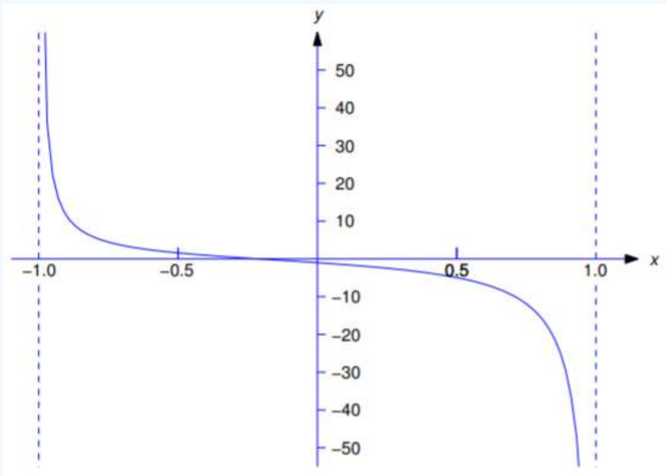


Figure 3.7.1 : The solution curve $y = \frac{2\ln(x+1)}{x-1} + \frac{3x+1}{x^2-1}$ to a nonhomogeneous linear differential equation. The curve has two vertical asymptotes, and it is decreasing. (CC BY-NC-SA 3.0, William F. Trench via Elementary Differential equation)

We've now considered three methods for solving nonhomogeneous linear equations: undetermined coefficients, reduction of order, and variation of parameters. It's natural to ask which method is best for a given problem. The method of undetermined coefficients should be used for constant coefficient equations with forcing functions that are linear combinations of polynomials multiplied by functions of the form $e^{\alpha x}$, $e^{\lambda x}\cos\omega x$, or $e^{\lambda x}\sin\omega x$. Although the other two methods can be used to solve such problems, they will be more difficult except in the most trivial cases, because of the integrations involved.

If the equation isn't a constant coefficient equation or the forcing function isn't of the form just specified, the method of undetermined coefficients does not apply and the choice is necessarily between the other two methods. The case could be made that reduction of order is better because it requires only one solution of the complementary equation while variation of parameters requires two. However, variation of parameters will probably be easier if you already know a fundamental set of solutions of the complementary equation.

3.7E: Exercises for Section 3.7

Exercises 1-6

Use variation of parameters to find a particular solution.

Exercise $3.7E.1$

$y'' + 9y = \tan 3x$

Exercise $3.7E.2$

$y'' + 4y = \sin 2x \sec^2 2x$

Answer

$y_p = -\frac{\sin 2x \ln|\cos 2x|}{4} + \frac{x\cos 2x}{2}.$

Exercise $3.7E.3$

$y'' - 3y' + 2y = \frac{4}{1+e^{-x}}$

Exercise $3.7E.4$

$y'' - 2y' + 2y = 3e^x \sec x$

Answer

$y_p = 3e^x(\cos x \ln|\cos x| + x\sin x).$

Exercise $3.7E.5$

$y'' - 2y' + y = 14x^{3/2}e^x$

Exercise $3.7E.6$

$y'' - y = \frac{4e^{-x}}{1-e^{-2x}}$

Answer

$y_p = e^x \ln(1-e^{-2x}) - e^{-x}\ln(e^{2x}-1).$

Exercises 7-29

Use variation of parameters to find a particular solution, given the solutions y_1, y_2 of the complementary equation.

Exercise $3.7E.7$

$x^2y'' + xy' - y = 2x^2 + 2; \quad y_1 = x, \quad y_2 = \frac{1}{x}$

Exercise $3.7E.8$

$xy'' + (2-2x)y' + (x-2)y = e^{2x}; \quad y_1 = e^x, \quad y_2 = \frac{e^x}{x}$

Answer

$y_p = \frac{e^{2x}}{x}.$

Exercise 3.7E.9

$4x^2y'' + (4x - 8x^2)y' + (4x^2 - 4x - 1)y = 4x^{1/2}e^x, \quad x > 0 \; ; y_1 = x^{1/2}e^x, \; y_2 = x^{-1/2}e^x$

Exercise 3.7E.10

$y'' + 4xy' + (4x^2 + 2)y = 4e^{-x(x+2)}; \quad y_1 = e^{-x^2}, \quad y_2 = xe^{-x^2}$

Answer

$y_p = e^{-x(x+2)}.$

Exercise 3.7E.11

$x^2y'' - 4xy' + 6y = x^{5/2}, \; x > 0; \quad y_1 = x^2, \; y_2 = x^3$

Exercise 3.7E.12

$x^2y'' - 3xy' + 3y = 2x^4 \sin x; \quad y_1 = x, \; y_2 = x^3$

Answer

$y_p = -2x^2 \sin x - 2x \cos x.$

Exercise 3.7E.13

$(2x + 1)y'' - 2y' - (2x + 3)y = (2x + 1)^2 e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = xe^x$

Exercise 3.7E.14

$4xy'' + 2y' + y = \sin\sqrt{x}; \quad y_1 = \cos\sqrt{x}, \quad y_2 = \sin\sqrt{x}$

Answer

$y_p = -\frac{\sqrt{x}\cos\sqrt{x}}{2}.$

Exercise 3.7E.15

$xy'' - (2x + 2)y' + (x + 2)y = 6x^3e^x; \quad y_1 = e^x, \quad y_2 = x^3e^x$

Exercise 3.7E.16

$x^2y'' - (2a - 1)xy' + a^2y = x^{a+1}; \quad y_1 = x^a, \quad y_2 = x^a \ln x$

Answer

$y_p = x^{a+1}.$

Exercise 3.7E.17

$x^2y'' - 2xy' + (x^2 + 2)y = x^3 \cos x; \quad y_1 = x \cos x, \quad y_2 = x \sin x$

Exercise $3.7E.18$

$xy'' - y' - 4x^3y = 8x^5; \quad y_1 = e^{x^2},\ y_2 = e^{-x^2}$

Answer

$y_p = -2x^2.$

Exercise $3.7E.19$

$(\sin x)y'' + (2\sin x - \cos x)y' + (\sin x - \cos x)y = e^{-x}; \quad y_1 = e^{-x}, \quad y_2 = e^{-x}\cos x$

Exercise $3.7E.20$

$4x^2y'' - 4xy' + (3 - 16x^2)y = 8x^{5/2}; \quad y_1 = \sqrt{x}e^{2x},\ y_2 = \sqrt{x}e^{-2x}$

Answer

$y_p = -\frac{\sqrt{x}}{2}.$

Exercise $3.7E.21$

$4x^2y'' - 4xy' + (4x^2 + 3)y = x^{7/2}; \quad y_1 = \sqrt{x}\sin x,\ y_2 = \sqrt{x}\cos x$

Exercise $3.7E.22$

$x^2y'' - 2xy' - (x^2 - 2)y = 3x^4; \quad y_1 = xe^x,\ y_2 = xe^{-x}$

Exercise $3.7E.23$

$x^2y'' - 2x(x+1)y' + (x^2 + 2x + 2)y = x^3e^x; \quad y_1 = xe^x, \quad y_2 = x^2e^x$

Exercise $3.7E.24$

$x^2y'' - xy' - 3y = x^{3/2}; \quad y_1 = 1/x, \quad y_2 = x^3$

Exercise $3.7E.25$

$x^2y'' - x(x+4)y' + 2(x+3)y = x^4e^x; \quad y_1 = x^2, \quad y_2 = x^2e^x$

Exercise $3.7E.26$

$x^2y'' - 2x(x+2)y' + (x^2 + 4x + 6)y = 2xe^x; \quad y_1 = x^2e^x, \quad y_2 = x^3e^x$

Exercise $3.7E.27$

$x^2y'' - 4xy' + (x^2 + 6)y = x^4; \quad y_1 = x^2\cos x, \quad y_2 = x^2\sin x$

Exercise $3.7E.28$

$(x-1)y'' - xy' + y = 2(x-1)^2e^x; \quad y_1 = x, \quad y_2 = e^x$

Exercise $3.7E.29$

$4x^2y'' - 4x(x+1)y' + (2x+3)y = x^{5/2}e^x; \quad y_1 = \sqrt{x}, \quad y_2 = \sqrt{x}e^x$

Exercises 30-32

Use variation of parameters to solve the initial value problem, given y_1, y_2 are solutions of the complementary equation.

Exercise $3.7E.30$

$(3x-1)y'' - (3x+2)y' - (6x-8)y = (3x-1)^2e^{2x}, \quad y(0) = 1, \ y'(0) = 2 \ ;$

$y_1 = e^{2x}, \ y_2 = xe^{-x}$

Answer

$y = \frac{e^{2x}(3x^2-2x+6)}{6} + \frac{xe^{-x}}{3}.$

Exercise $3.7E.31$

$(x-1)^2y'' - 2(x-1)y' + 2y = (x-1)^2, \quad y(0) = 3, \quad y'(0) = -6 \ ;$

$y_1 = x-1$, $y_2 = x^2-1$

Exercise $3.7E.32$

$(x-1)^2y'' - (x^2-1)y' + (x+1)y = (x-1)^3e^x, \quad y(0) = 4, \quad y'(0) = -6 \ ;$

$y_1 = (x-1)e^x, \quad y_2 = x-1$

Exercises 33-39

Use variation of parameters to solve the initial value problem and graph the solution, given that y_1, y_2 are solutions of the complementary equation.

Exercise $3.7E.33$

$(x^2-1)y'' + 4xy' + 2y = 2x, \quad y(0) = 0, \ y'(0) = -2; \quad y_1 = \frac{1}{x-1}, \ y_2 = \frac{1}{x+1}$

Exercise $3.7E.34$

$x^2y'' + 2xy' - 2y = -2x^2, \quad y(1) = 1, \ y'(1) = -1; \quad y_1 = x, \ y_2 = \frac{1}{x^2}$

Answer

$y = -\frac{x^2}{2} + x + \frac{1}{2x^2}.$

Exercise $3.7E.35$

$(x+1)(2x+3)y'' + 2(x+2)y' - 2y = (2x+3)^2, \quad y(0) = 0, \quad y'(0) = 0 \ ;$ $y_1 = x+2, \quad y_2 = \frac{1}{x+1}$

Exercise $3.7E.36$

Suppose

$$y_p = \overline{y} + a_1y_1 + a_2y_2$$

is a particular solution of

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = F(x), \tag{A}$$

where y_1 and y_2 are solutions of the complementary equation

$$P_0(x)y'' + P_1(x)y' + P_2(x)y = 0.$$

Show that $\bar{y}$ is also a solution of (A).

? Exercise 3.7E.37

Suppose p, q, and f are continuous on (a, b) and let x_0 be in (a, b). Let y_1 and y_2 be the solutions of

$$y'' + p(x)y' + q(x)y = 0$$

such that

$$y_1(x_0) = 1, \quad y_1'(x_0) = 0, \quad y_2(x_0) = 0, \quad y_2'(x_0) = 1.$$

Use variation of parameters to show that the solution of the initial value problem

$$y'' + p(x)y' + q(x)y = f(x), \quad y(x_0) = k_0, \; y'(x_0) = k_1,$$

is

$$\begin{aligned} y(x) &= k_0 y_1(x) + k_1 y_2(x) \\ &+ \int_{x_0}^{x} \left(y_1(t)y_2(x) - y_1(x)y_2(t)\right) f(t) \exp\left(\int_{x_0}^{t} p(s)\, ds\right) dt. \end{aligned}$$

Hint: **Use Abel's formula for the Wronskian of $\{y_1, y_2\}$, and integrate u_1' and u_2' from x_0 to x.**

Show also that

$$\begin{aligned} y'(x) &= k_0 y_1'(x) + k_1 y_2'(x) \\ &+ \int_{x_0}^{x} \left(y_1(t)y_2'(x) - y_1'(x)y_2(t)\right) f(t) \exp\left(\int_{x_0}^{t} p(s)\, ds\right) dt. \end{aligned}$$

? Exercise 3.7E.38

Suppose f is continuous on an open interval that contains $x_0 = 0$. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' - y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

? Exercise 3.7E.39

Suppose f is continuous on (a, ∞), where $a < 0$, so $x_0 = 0$ is in (a, ∞).

a. Use variation of parameters to find a formula for the solution of the initial value problem

$$y'' + y = f(x), \quad y(0) = k_0, \quad y'(0) = k_1.$$

Hint: **You will need the addition formulas for the sine and cosine.**

$$\begin{aligned} \sin(A + B) &= \sin A \cos B + \cos A \sin B \\ \cos(A + B) &= \cos A \cos B - \sin A \sin B \end{aligned}$$

For the rest of this exercise assume that the improper integral $\int_0^\infty f(t)\, dt$ is absolutely convergent.

b. Show that if y is a solution of

$$y'' + y = f(x) \tag{A}$$

on (a, ∞), then

$$\lim_{x\to\infty}\left(y(x) - A_0\cos x - A_1\sin x\right) = 0 \tag{B}$$

and

$$\lim_{x\to\infty}\left(y'(x) + A_0\sin x - A_1\cos x\right) = 0, \tag{C}$$

where

$$A_0 = k_0 - \int_0^\infty f(t)\sin t\,dt \quad \text{and} \quad A_1 = k_1 + \int_0^\infty f(t)\cos t\,dt.$$

Hint: **Recall from calculus that if $\int_0^\infty f(t)dt$ converges absolutely, then $\lim_{x\to\infty}\int_x^\infty |f(t)|dt = 0$.**

c. Show that if A_0 and A_1 are arbitrary constants, then there's a unique solution of $y'' + y = f(x)$ on (a, ∞) that satisfies (B) and (C).

3.8: Mechanical Vibrations

Learning Objectives

- Understand the mathematical modeling of physical systems such as mass-spring systems, RLC circuits, and pendulums using linear second-order differential equations.
- Analyze the behavior of solutions for damped and undamped systems, including the concepts of natural frequency, amplitude, and different damping scenarios.

Let us look at some applications of linear second-order constant coefficient equations.

Examples in Mass-Spring System, RLC Circuit, and Pendulum

Our first example is a mass on a spring Figure $3.8.1$. Suppose we have a mass $m>0$ (in kilograms) connected by a spring with spring constant $k>0$ (in newtons per meter) to a fixed wall. There may be some external force $F(t)$ (in newtons) acting on the mass. Finally, there is some friction measured by $c\geq 0$ (in newton-seconds per meter) as the mass slides along the floor (or perhaps there is a damper connected).

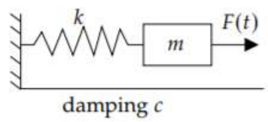


Figure $3.8.1$: A mass-spring system (a common example used to study mechanical vibrations) with a mass m attached to a spring with spring constant k. A force vector F(t) pulls the mass to the right. Below the A mass-spring system is a damping constant c. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Let x be the displacement of the mass ($x=0$ is the rest position), with x growing to the right (away from the wall). The force exerted by the spring is proportional to the compression of the spring by Hooke's law. Therefore, it is kx in the negative direction. Similarly the amount of force exerted by friction is proportional to the velocity of the mass. By Newton's second law we know that force equals mass times acceleration and hence $mx''=F(t)-cx'-kx$ or

$$mx''+cx'+kx=F(t)$$

This is a linear second order constant coefficient ODE. We set up some terminology about this equation. We say the motion is

i. Forced, if $F\not\equiv 0$ (if F is not identically zero),
ii. Unforced or free, if $F\equiv 0$ (if F is identically zero),
iii. Damped, if $c>0$, and
iv. Undamped, if $c=0$.

This system appears in lots of applications even if it does not at first seem like it. Many real-world scenarios can be simplified to a mass on a spring. For example, a bungee jump setup is essentially a mass and spring system (you are the mass). It would be good if someone did the math before you jump off the bridge, right? Let us give two other examples.

Here is an example for electrical engineers. Consider the RLC circuit as in Figure $3.8.2$. There is a resistor with a resistance of R ohms, an inductor with an inductance of L henries, and a capacitor with a capacitance of C farads. There is also an electric source (such as a battery) giving a voltage of $E(t)$ volts at time t (measured in seconds). Let $Q(t)$ be the charge in coulombs on the capacitor and $I(t)$ be the current in the circuit. The relation between the two is $Q'=I$. By elementary principles we find $LI'+RI+\frac{Q}{C}=E$. We differentiate to get

$$LI''(t)+RI'(t)+\frac{1}{C}I(t)=E'(t).$$

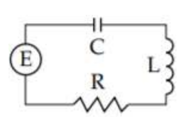


Figure $3.8.2$: A circuit labeled with resister R (bottom) , capacitor C (top), battery E (left), and inductor L (right). (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

This is a nonhomogeneous second order constant coefficient linear equation. As L, R, and C are all positive, this system behaves just like the mass and spring system. Position of the mass is replaced by current. Mass is replaced by inductance, damping is replaced by resistance, and the spring constant is replaced by one over the capacitance. The change in voltage becomes the forcing function—for constant voltage this is an unforced motion.

Our next example is a motion of mass swinging on a pendulum (see Figure $3.8.3$). Suppose a mass m hangs on a pendulum of length L. We seek an equation for the angle $\theta(t)$ (in radians). Let g be the force of gravity. Elementary physics mandates that the equation is

$$\theta'' + \frac{g}{L}\sin\theta = 0.$$

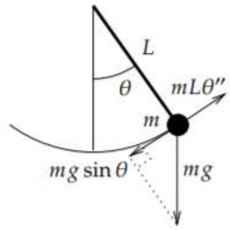


Figure $3.8.3$: A simple pendulum consisting of a mass mmm attached to a string of length L. The forces acting on the pendulum include: The gravitational force mg acting downward, the tangential component of gravity $mgsin\theta$ which acts as the restoring force, and the acceleration term $mL\theta''$ representing angular acceleration. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Let us derive this equation using Newton's second law: force equals mass times acceleration. The acceleration is $L\theta''$ and mass is m. So $mL\theta''$ has to be equal to the tangential component of the force given by the gravity, which is $mg\sin\theta$ in the opposite direction. So $mL\theta'' = -mg\sin\theta$. The m curiously cancels from the equation.

Now we make our approximation. For small θ we have that approximately $\sin\theta \approx \theta$. This can be seen by looking at the graph. In Figure $3.8.4$ we can see that for approximately $-0.5 < \theta < 0.5$ (in radians) the graphs of $\sin\theta$ and θ are almost the same.

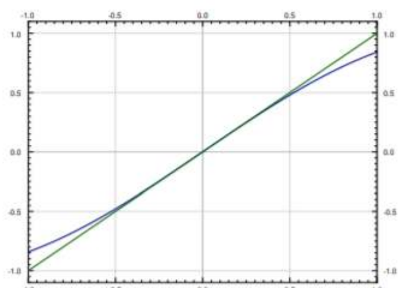


Figure $3.8.4$: The graphs of $\sin\theta$ and θ (in radians) on the interval in which $-1 \leq x \leq 1$ and $-1 \leq y \leq 1$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Therefore, when the swings are small, θ is small and we can model the behavior by the simpler linear equation

$$\theta'' + \frac{g}{L}\theta = 0.$$

The errors from this approximation build up. So after a long time, the state of the real-world system might be substantially different from our solution. Also we will see that in a mass-spring system, the amplitude is independent of the period. This is not true for a pendulum. Nevertheless, for reasonably short periods of time and small swings (that is, only small angles θ), the approximation is reasonably good.

In real-world problems it is often necessary to make these types of simplifications. We must understand both the mathematics and the physics of the situation to see if the simplification is valid in the context of the questions we are trying to answer.

Free Undamped Motion

In this section we will only consider free or unforced motion, as we cannot yet solve nonhomogeneous equations. Let us start with undamped motion where $c = 0$. We have the equation

$$mx'' + kx = 0$$

If we divide by m and let $w_0 = \sqrt{\frac{k}{m}}$, then we can write the equation as

$$x'' + w_0^2 x = 0$$

The general solution to this equation is

$$x(t) = A\cos(w_0 t) + B\sin(w_0 t)$$

By a trigonometric identity $\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$, we have that for two different constants C and γ, we have

$$A\cos(w_0 t) + B\sin(w_0 t) = C\cos(w_0 t - \gamma)$$

It is not hard to compute that $C = \sqrt{A^2 + B^2}$ and $\tan\gamma = \frac{B}{A}$. Therefore, we let C and γ be our arbitrary constants and write $x(t) = C\cos(w_0 t - \gamma)$.

While it is generally easier to use the first form with A and B to solve for the initial conditions, the second form is much more natural. The constants C and γ have very nice interpretation. We look at the form of the solution

$$x(t) = C\cos(w_0 t - \gamma)$$

We can see that the amplitude is C, w_0 is the (angular) frequency, and γ is the so-called phase shift. The phase shift just shifts the graph left or right. We call w_0 the natural (angular) frequency. This entire setup is usually called **simple harmonic motion**.

Let us pause to explain the word angular before the word frequency. The units of w_0 are radians per unit time, not cycles per unit time as is the usual measure of frequency. Because we know one cycle is 2π radians, the usual frequency is given by $\frac{w_0}{2\pi}$. It is simply a matter of where we put the constant 2π, and that is a matter of taste.

The period of the motion is one over the frequency (in cycles per unit time) and hence $\frac{2\pi}{w_0}$. That is the amount of time it takes to complete one full oscillation.

Example 3.8.1

Suppose that $m = 2kg$ and $k = 8\frac{N}{m}$. The whole mass and spring setup is sitting on a truck that was traveling at $1\frac{m}{s}$. The truck crashes and hence stops. The mass was held in place 0.5 meters forward from the rest position. During the crash the mass gets loose. That is, the mass is now moving forward at $1\frac{m}{s}$, while the other end of the spring is held in place. The mass therefore starts oscillating in mks units (meters-kilograms-seconds). What is the frequency of the resulting oscillation and what is the amplitude?

Solution

The setup means that the mass was at half a meter in the positive direction during the crash and relative to the wall the spring is mounted to, the mass was moving forward (in the positive direction) at $1\frac{m}{s}$. This gives us the initial conditions.

So the equation with initial conditions is

$$2x'' + 8x = 0, \quad x(0) = 0.5, \quad x'(0) = 1$$

We can directly compute $w_0 = \sqrt{\frac{k}{m}} = \sqrt{4} = 2$. Hence the angular frequency is 2. The usual frequency in Hertz (cycles per second) is $\frac{2}{2\pi} = \frac{1}{\pi} \approx 0.318$.

The general solution is

$$x(t) = A\cos(2t) + B\sin(2t)$$

Letting $x(0)=0.5$ means $A=0.5$. Then $x'(t)=-2(0.5)\sin(2t)+2B\cos(2t)$. Letting $x'(0)=1$ we get $B=0.5$. Therefore, the amplitude is $C=\sqrt{A^2+B^2} = \sqrt{0.25+0.25}=\sqrt{0.5}\approx 0.707$. The solution is

$$x(t)=0.5\cos(2t)+0.5\sin(2t)$$

A plot of $x(t)$ is shown in Figure $3.8.5$.

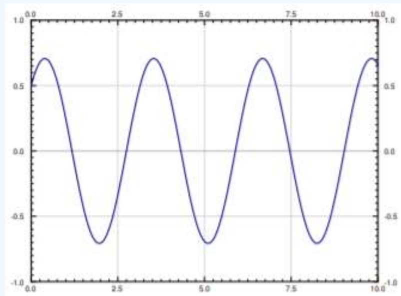


Figure $3.8.5$: A simple undamped oscillation showing a graph of a sine function which is shifted both vertically and horizontally. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

In general, for free undamped motion, a solution of the form

$$x(t)=A\cos(w_0t)+B\sin(w_0t)$$

corresponds to the initial conditions $x(0)=A$ and $x'(0)=w_0B$. Therefore, it is easy to figure out A and B from the initial conditions. The amplitude and the phase shift can then be computed from A and B. In the example, we have already found the amplitude C. Let us compute the phase shift. We know that $\tan\gamma=\dfrac{B}{A}=1$. We take the arctangent of 1 and get approximately 0.785. We still need to check if this γ is in the correct quadrant (and add π to γ if it is not). Since both A and B are positive, then γ should be in the first quadrant, and 0.785 radians really is in the first quadrant.

Note

Many calculators and computer software do not only have the atan function for arctangent, but also what is sometimes called `atan2` . This function takes two arguments, B and A, and returns a γ in the correct quadrant for you.

Free Damped Motion

Let us now focus on damped motion. Let us rewrite the equation

$$mx''+cx'+kx=0$$

as

$$x''+2px'+w_0^2x=0$$

where

$$w_0=\sqrt{\frac{k}{m}},\quad p=\frac{c}{2m}$$

The characteristic equation is

$$r^2+2pr+w_0^2=0$$

Using the quadratic formula we get that the roots are

$$r=-p\pm\sqrt{p^2-w_0^2}$$

The form of the solution depends on whether we get complex or real roots. We get real roots if and only if the following number is nonnegative:

$$p^2 - w_0^2 = (\frac{c}{2m})^2 - \frac{k}{m} = \frac{c^2 - 4km}{4m^2}$$

The sign of $p^2 - w_0^2$ is the same as the sign of $c^2 - 4km$. Thus we get real roots if and only if $c^2 - 4km$ is nonnegative, or in other words if $c^2 \geq 4km$.

Overdamping

When $c^2 - 4km > 0$, we say the system is overdamped. In this case, there are two distinct real roots r_1 and r_2. Notice that both roots are negative. As $\sqrt{p^2 - w_0^2}$ is always less than P, then $-P \pm \sqrt{P^2 - w_0^2}$ is negative.

The solution is

$$x(t) = C_1 e^{r_1 t} + C_2 e^{r_2 t}$$

Since r_1, r_2 are negative, $x(t) \to 0$ as $t \to \infty$. Thus the mass will tend towards the rest position as time goes to infinity. See Figure $3.8.6$ for a few sample plots given different initial conditions.

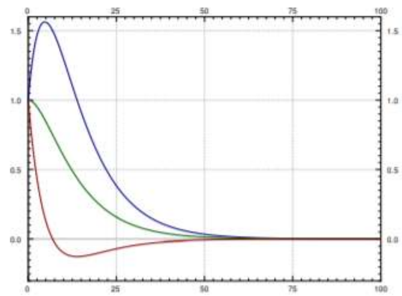


Figure $3.8.6$: An overdamped motion for several different initial conditions. The top blue curve goes up and down, then flats out to the x-axis. The middle green curve goes down and flats out to the x-axis. The bottom red curve goes down and up, then flats out to the x-axis. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

No oscillation happens! In fact, the graph will cross the x axis at most once. To see why, we try to solve $0 = C_1 e^{r_1 t} + C_2 e^{r_2 t}$. Therefore, $C_1 e^{r_1 t} = -C_2 e^{r_2 t}$ and using laws of exponents we obtain

$$\frac{-C_1}{C_2} = e^{(r_2 - r_1)t}$$

This equation has at most one solution $t \geq 0$. For some initial conditions the graph will never cross the x axis, as is evident from the sample graphs in Figure $3.8.6$.

Suppose the mass is released from rest. That is $x(0) = x_0$ and $x'(0) = 0$. Then

$$x(t) = \frac{x_0}{r_1 - r_2}(r_1 e^{r_2 t} - r_2 e^{r_1 t})$$

satisfies the initial conditions.

Critical damping

When $c^2 - 4km = 0$, we say the system is critically damped. In this case, there is one root of multiplicity 2 and this root is $-P$. Therefore, our solution is

$$x(t) = C_1 e^{-pt} + C_2 t e^{-pt}$$

The behavior of a critically damped system is very similar to an overdamped system. After all a critically damped system is in some sense a limit of overdamped systems. Since these equations are really only an approximation to the real world, in reality we are never critically damped, it is a place we can only reach in theory. We are always a little bit underdamped or a little bit overdamped. It is better not to dwell on critical damping.

Underdamping

When $c^2 - 4km < 0$, we say the system is underdamped. In this case, the roots are complex.

$$\begin{aligned} r &= -p \pm \sqrt{p^2 - w_0^2} \\ &= -p \pm \sqrt{-1}\sqrt{w_0^2 - p^2} \\ &= -p \pm i w_1 \end{aligned} \tag{3.8.1}$$

where $w_1 = \sqrt{w_0^2 - p^2}$. Our solution is

$$x(t) = e^{-pt}(A\cos(w_1 t) + B\sin(w_1 t)$$

or

$$x(t) = Ce^{-pt}\cos(w_1 t - \gamma)$$

An example plot is given in Figure 3.8.7. Note that we still have that $x(t) \to 0$ as $t \to \infty$.

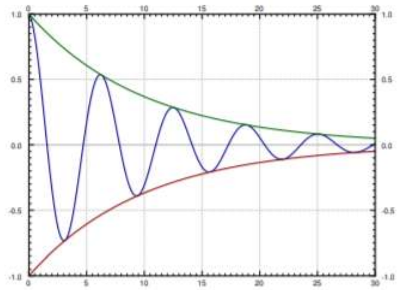


Figure 3.8.7: An underdamped motion with the envelope curves. A blue curve oscillates up and down between the top decreasing green curve and the bottom increasing red curve. The blue curve flats out towards the x-axis as x-values get bigger. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Figure 3.8.7 also shows the *envelope curves* Ce^{-pt} and $-Ce^{pt}$. The solution is the oscillating line between the two envelope curves. The envelope curves give the maximum amplitude of the oscillation at any given point in time. For example if you are bungee jumping, you are really interested in computing the envelope curve so that you do not hit the concrete with your head.

The phase shift γ just shifts the graph left or right but within the envelope curves (the envelope curves do not change if γ changes).

Finally note that the angular pseudo-frequency (we do not call it a frequency since the solution is not really a periodic function) w_1 becomes smaller when the damping c (and hence P) becomes larger. This makes sense. When we change the damping just a little bit, we do not expect the behavior of the solution to change dramatically. If we keep making c larger, then at some point the solution should start looking like the solution for critical damping or overdamping, where no oscillation happens. So if c^2 approaches $4km$, we want w_1 to approach 0.

On the other hand when c becomes smaller, w_1 approaches w_0 (w_1 is always smaller than w_0), and the solution looks more and more like the steady periodic motion of the undamped case. The envelope curves become flatter and flatter as c (and hence P) goes to 0.

Footnotes

[1] We do not call ω_1 a frequency since the solution is not really a periodic function.

3.8E: Exercises for Section 3.8

Exercises 1-9

Solve each exercise by applying the principles of mechanics, circuits, or oscillatory systems as specified.

Exercise $3.8E.1$

Consider a mass and spring system with a mass $m = 2$, spring constant $k = 3$, and damping constant $c = 1$.

a. Set up and find the general solution of the system.
b. Is the system underdamped, overdamped or critically damped?
c. If the system is not critically damped, find a c that makes the system critically damped.

Exercise $3.8E.2$

Consider a mass and spring system with a mass $m = 3$, spring constant $k = 12$, and damping constant $c = 12$.

a. Set up and find the general solution of the system.
b. Is the system underdamped, overdamped or critically damped?
c. If the system is not critically damped, find a c that makes the system critically damped.

Exercise $3.8E.3$

Using the mks units (meters-kilograms-seconds), suppose you have a spring with spring constant $4\frac{N}{m}$. You want to use it to weight items. Assume no friction. You place the mass on the spring and put it in motion.

a. You count and find that the frequency is 0.8 Hz (cycles per second). What is the mass?
b. Find a formula for the mass m given the frequency w in Hz.

Exercise $3.8E.4$

Suppose we add possible friction to Exercise 3.8E.3. Further, suppose you do not know the spring constant, but you have two reference weights 1 kg and 2 kg to calibrate your setup. You put each in motion on your spring and measure the frequency. For the 1 kg weight you measured 1.1 Hz, for the 2 kg weight you measured 0.8 Hz.

a. Find k (spring constant) and c (damping constant).
b. Find a formula for the mass in terms of the frequency in Hz. Note that there may be more than one possible mass for a given frequency.
c. For an unknown object you measured 0.2 Hz, what is the mass of the object? Suppose that you know that the mass of the unknown object is more than a kilogram.

Exercise $3.8E.5$

Suppose you wish to measure the friction a mass of 0.1 kg experiences as it slides along a floor (you wish to find c). You have a spring with spring constant $k = 5\frac{N}{m}$. You take the spring, you attach it to the mass and fix it to a wall. Then you pull on the spring and let the mass go. You find that the mass oscillates with frequency 1 Hz. What is the friction?

Exercise $3.8E.6$

A mass of 2 kilograms is on a spring with spring constant k newtons per meter with no damping. Suppose the system is at rest and at time $t = 0$ the mass is kicked and starts traveling at 2 meters per second. How large does k have to be to so that the mass does not go further than 3 meters from the rest position?

Answer

$k = \frac{8}{9}$ (and larger)

Exercise $3.8E.7$

Suppose we have an RLC circuit with a resistor of 100 miliohms (0.1 ohms), inductor of inductance of 50 millihenries (0.05 henries), and a capacitor of 5 farads, with constant voltage.

a. Set up the ODE equation for the current I.
b. Find the general solution.
c. Solve for $I(0) = 10$ and $I'(0) = 0$.

Answer

a. $0.05I'' + 0.1I' + \left(\frac{1}{5}\right) I = 0$
b. $I = Ce^{-t}\cos(\sqrt{3}t - \gamma)$
c. $I = 10e^{-t}\cos(\sqrt{3}t) + \frac{10}{\sqrt{3}}e^{-t}\sin(\sqrt{3}t)$

Exercise $3.8E.8$

A 5000 kg railcar hits a bumper (a spring) at $1\frac{m}{s}$, and the spring compresses by 0.1 m. Assume no damping.

a. Find k.
b. Find out how far does the spring compress when a 10000 kg railcar hits the spring at the same speed.
c. If the spring would break if it compresses further than 0.3 m, what is the maximum mass of a railcar that can hit it at $1\frac{m}{s}$?
d. What is the maximum mass of a railcar that can hit the spring without breaking at $2\frac{m}{s}$?

Answer

a. $k = 500000$
b. $\frac{1}{5\sqrt{2}} \approx 0.141$
c. 45000 kg
d. 11250 kg

Exercise $3.8E.9$

A mass of m kg is on a spring with $k = 3\frac{\text{N}}{\text{m}}$ and $c = 2\frac{\text{Ns}}{\text{m}}$. Find the mass m_0 for which there is critical damping. If $m < m_0$, does the system oscillate or not, that is, is it underdamped or overdamped?

Answer

$m_0 = \frac{1}{2}$. If $m < m_0$, then the system is overdamped and will not oscillate.

3.9: Nonhomogeneous Equations

Learning Objectives

- Learn to solve nonhomogeneous second-order linear differential equations using methods such as undetermined coefficients and variation of parameters.
- Understand the decomposition of the general solution into complementary and particular solutions and apply initial conditions to determine specific solutions.

Solving Nonhomogeneous Equations

We have solved linear constant coefficient homogeneous equations. What about nonhomogeneous linear ODEs? That is, suppose we have an equation such as

$$y'' + 5y' + 6y = 2x + 1 \tag{3.9.1}$$

We will write $Ly = 2x + 1$ when the exact form of the operator is not important. We solve Equation (3.9.1) in the following manner. First, we find the general solution y_c to the associated homogeneous equation

$$y'' + 5y' + 6y = 0 \tag{3.9.2}$$

We call y_c the *complementary solution*. Next, we find a single particular solution y_p to Equation $(3.9.1)$ in some way. Then

$$y = y_c + y_p$$

is the general solution to Equation $(3.9.1)$. We have $Ly_c = 0$ and $Ly_p = 2x + 1$. As L is a linear operator we verify that y is a solution, $Ly = L(y_c + y_p) = Ly_c + Ly_p = 0 + (2x + 1)$. Let us see why we obtain the general solution.

Let y_p and $\tilde{y}_p$ be two different particular solutions to Equation $(3.9.1)$. Write the difference as $w = y_p - \tilde{y}_p$. Then plug w into the left hand side of the equation to get

$$w'' + 5w' + 6w = (y_p'' + 5y_p' + 6y_p) - (\tilde{y}_p'' + 5\tilde{y}_p' + 6\tilde{y}_p) = (2x + 1) - (2x + 1) = 0$$

Using the operator notation the calculation becomes simpler. As L is a linear operator we write

$$Lw = L(y_p - \tilde{y}_p) = Ly_p - L\tilde{y}_p = (2x + 1) - (2x + 1) = 0$$

So $w = y_p - \tilde{y}_p$ is a solution to Equation $(3.9.2)$, that is $Lw = 0$. Any two solutions of Equation $(3.9.1)$ differ by a solution to the homogeneous Equation $(3.9.2)$. The solution $y = y_c + y_p$ includes all solutions to Equation $(3.9.1)$, since y_c is the general solution to the associated homogeneous equation.

Theorem 3.9.1

Let $Ly = f(x)$ be a linear ODE (not necessarily constant coefficient). Let y_c be the complementary solution (the general solution to the associated homogeneous equation $Ly = 0$) and let y_p be any particular solution to $Ly = f(x)$. Then the general solution to $Ly = f(x)$ is

$$y = y_c + y_p.$$

The moral of the story is that we can find the particular solution in any old way. If we find a different particular solution (by a different method, or simply by guessing), then we still get the same general solution. The formula may look different, and the constants we will have to choose to satisfy the initial conditions may be different, but it is the same solution.

Undetermined Coefficients

The trick is to somehow, in a smart way, guess one particular solution to Equation $(3.9.1)$. Note that $2x + 1$ is a polynomial, and the left hand side of the equation will be a polynomial if we let y be a polynomial of the same degree. Let us try

$$y_p = Ax + B$$

We plug in to obtain

$$\begin{aligned} y_p'' + 5y_p' + 6y_p &= (Ax+B)'' + 5(Ax+B)' + 6(Ax+B) \\ &= 0 + 5A + 6Ax + 6B = 6Ax + (5A+6B) \end{aligned} \tag{3.9.3}$$

So $6Ax + (5A+6B) = 2x+1$. Therefore, $A = \frac{1}{3}$ and $B = -\frac{1}{9}$. That means $y_p = \frac{1}{3}x - \frac{1}{9} = \frac{3x-1}{9}$. Solving the complementary problem we get

$$y_c = C_1 e^{-2x} + C_2 e^{-3x}$$

Hence the general solution to Equation $(3.9.1)$ is

$$y = C_1 e^{-2x} + C_2 e^{-3x} + \frac{3x-1}{9}$$

Now suppose we are further given some initial conditions. For example, $y(0) = 0$ and $y'(0) = \frac{1}{3}$. First find $y' = -2C_1 e^{-2x} - 3C_2 e^{-3x} + \frac{1}{3}$. Then

$$0 = y(0) = C_1 + C_2 - \frac{1}{9}, \frac{1}{3} = y'(0) = -2C_1 - 3C_2 + \frac{1}{3}$$

We solve to get $C_1 = \frac{1}{3}$ and $C_2 = -\frac{2}{9}$. The particular solution we want is

$$y(x) = \frac{1}{3}e^{-2x} - \frac{2}{9}e^{-3x} + \frac{3x-1}{9} = \frac{3e^{-2x} - 2e^{-3x} + 3x - 1}{9}$$

Note

A common mistake is to solve for constants using the initial conditions with y_c and only add the particular solution y_p after that. That will not work. You need to first compute $y = y_c + y_p$ and only then solve for the constants using the initial conditions.

A nonhomogeneous equation in which the right hand side consisting of exponentials, sines, and cosines can be handled similarly.

If the right hand side contains sine or cosine, we try sine and/or cosine. For example,

$$y'' + 2y' + 2y = \cos(2x)$$

Let us find some y_p. We start by guessing the solution includes some multiple of $\cos(2x)$. We may have to also add a multiple of $\sin(2x)$ to our guess since derivatives of cosines are sines. We try

$$y_p = A\cos(2x) + B\sin(2x)$$

We plug y_p into the equation and we get

$$\underbrace{-4A\cos(2x) - 4B\sin(2x)}_{y_p''} + 2\underbrace{\left(-2A\sin(2x) + 2B\cos(2x)\right)}_{y_p'} \\ +2\underbrace{\left(A\cos(2x) + 2B\sin(2x)\right)}_{y_p} = \cos(2x),$$

The left hand side must equal to right hand side. We group terms and we get that $-4A + 4B + 2A = 1$ and $-4B - 4A + 2B = 0$. So $-2A + 4B = 1$ and $2A + B = 0$ and hence $A = \frac{-1}{10}$ and $B = \frac{1}{5}$. So

$$y_p = A\cos(2x) + B\sin(2x) = \frac{-\cos(2x) + 2\sin(2x)}{10}$$

Similarly, **if the right hand side contains exponentials, we try exponentials**. For example, for

$$Ly = e^{3x}$$

we will try $y = Ae^{3x}$ as our guess and try to solve for A.

When the right hand side is a multiple of sines, cosines, exponentials, and polynomials, we can use the product rule for differentiation to come up with a guess. We need to guess a form for y_p such that Ly_p is of the same form, and has all the terms needed to for the right hand side. For example,

$$Ly = (1+3x^2)e^{-x}\cos(\pi x)$$

For this equation, we will guess

$$y_p = (A+Bx+Cx^2)e^{-x}\cos(\pi x) + (D+Ex+Fx^2)e^{-x}\sin(\pi x)$$

We will plug in and then hopefully get equations that we can solve for A, B, C, D, E and F. As you can see this can make for a very long and tedious calculation very quickly.

There is one hiccup in all this. It could be that our guess actually solves the associated homogeneous equation. That is, suppose we have

$$y'' - 9y = e^{3x}$$

We would love to guess $y = Ae^{3x}$, but if we plug this into the left hand side of the equation we get

$$y'' - 9y = 9Ae^{3x} - 9Ae^{3x} = 0 \neq e^{3x}$$

There is no way we can choose A to make the left hand side be e^{3x}. The trick in this case is to multiply our guess by x to get rid of duplication with the complementary solution. That is first we compute y_c (solution to $Ly = 0$)

$$y_c = C_1 e^{-3x} + C_2 e^{3x}$$

and we note that the e^{3x} term is a duplicate with our desired guess. We modify our guess to $y = Axe^{3x}$ and notice there is no duplication anymore. Let us try. Note that $y' = Ae^{3x} + 3Axe^{3x}$ and $y'' = 6Ae^{3x} + 9Axe^{3x}$. So

$$y'' - 9y = 6Ae^{3x} + 9Axe^{3x} - 9Axe^{3x} = 6Ae^{3x}$$

Thus $6Ae^{3x}$ is supposed to equal e^{3x}. Hence, $6A = 1$ and so $A = \frac{1}{6}$. We can now write the general solution as

$$y = y_c + y_p = C_1 e^{-3x} + C_2 e^{3x} + \frac{1}{6} x e^{3x}$$

Note

It is possible that multiplying by x does not get rid of all duplication. For example,

$$y'' - 6y' + 9y = e^{3x}$$

The complementary solution is $y_c = C_1 e^{3x} + C_2 x e^{3x}$. Guessing $y = Axe^{3x}$ would not get us anywhere. In this case we want to guess $y_p = Ax^2 e^{3x}$. Basically, we want to multiply our guess by x until all duplication is gone. But no more! Multiplying too many times will not work.

Finally, what if the right hand side has several terms, such as

$$Ly = e^{2x} + \cos x$$

In this case we find u that solves $Lu = e^{2x}$ and v that solves $Lv = \cos x$ (that is, do each term separately). Then note that if $y = u + v$, then $Ly = e^{2x} + \cos x$. This is because L is linear; we have $Ly = L(u+v) = Lu + Lv = e^{2x} + \cos x$.

Variation of Parameters

The method of undetermined coefficients will work for many basic problems that crop up. But it does not work all the time. It only works when the right hand side of the equation $Ly = f(x)$ has only finitely many linearly independent derivatives, so that we can write a guess that consists of them all. Some equations are a bit tougher. Consider

$$y'' + y = \tan x$$

Note that each new derivative of $\tan x$ looks completely different and cannot be written as a linear combination of the previous derivatives. If we start differentiating $\tan x$, we get

$$\sec^2 x, \quad 2\sec^2 x\, \tan x, \quad 4\sec^2 x\, \tan^2 x + 2\sec^4 x,$$
$$8\sec^2 x\, \tan^3 x + 16\sec^4 x\, \tan x, \quad 16\sec^2 x\, \tan^4 x + 88\sec^4 x \tan^2 x + 16\sec^6 x, \quad \ldots$$

This equation calls for a different method. We present the method of variation of parameters, which will handle any equation of the form $Ly = f(x)$, provided we can solve certain integrals. For simplicity, we restrict ourselves to second order constant coefficient equations, but the method works for higher order equations just as well (the computations become more tedious). The method also works for equations with nonconstant coefficients, provided we can solve the associated homogeneous equation.

Perhaps it is best to explain this method by example. Let us try to solve the equation

$$Ly = y'' + y = \tan x$$

First we find the complementary solution (solution to $Ly_c = 0$). We get $y_c = C_1 y_1 + C_2 y_2$, where $y_1 = \cos x$ and $y_2 = \sin x$. To find a particular solution to the nonhomogeneous equation we try

$$y_p = y = u_1 y_1 + u_2 y_2$$

where u_1 and u_2 are functions and not constants. We are trying to satisfy $Ly = \tan x$. That gives us one condition on the functions u_1 and u_2. Compute y' using the Product Rule, we get

$$y' = (u_1' y_1 + u_2' y_2) + (u_1 y_1' + u_2 y_2')$$

We can still impose one more condition at our discretion to simplify computations (we have two unknown functions, so we should be allowed two conditions). We require that $(u_1' y_1 + u_2' y_2) = 0$. This makes computing the second derivative easier.

$$\begin{aligned} y' &= u_1 y_1' + u_2 y_2' \\ y'' &= (u_1' y_1' + u_2' y_2') + (u_1 y_1'' + u_2 y_2'') \end{aligned} \tag{3.9.4}$$

Since y_1 and y_2 are solutions to $y'' + y = 0$, we know that $y_1'' = -y_1$ and $y_2'' = -y_2$. (Note: If the equation was instead $y'' + p(x)y' + q(x)y = 0$ we would have $y_i'' = -p(x)y_i' - q(x)y_i$.) So

$$y'' = (u_1' y_1' + u_2' y_2') - (u_1 y_1 + u_2 y_2)$$

We have $(u_1 y_1 + u_2 y_2) = y$ and so

$$y'' = (u_1' y_1' + u_2' y_2') - y$$

and hence

$$y'' + y = Ly = u_1' y_1' + u_2' y_2'$$

For y to satisfy $Ly = f(x)$ we must have $f(x) = u_1' y_1' + u_2' y_2'$.

So what we need to solve are the two equations (conditions) we imposed on u_1 and u_2

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= f(x) \end{aligned} \tag{3.9.5}$$

We can now solve for u_1' and u_2' in terms of $f(x), y_1$ and y_2. We will always get these formulas for any $Ly = f(x)$, where $Ly = y'' + p(x)y' + q(x)y$. There is a general formula for the solution we can just plug into, but it is better to just repeat what we do below. In our case the two equations become

$$\begin{aligned} u_1' \cos(x) + u_2' \sin(x) &= 0 \\ -u_1' \sin(x) + u_2' \cos(x) &= \tan(x) \end{aligned} \tag{3.9.6}$$

Hence

$$\begin{aligned} u_1' \cos(x)\sin(x) + u_2' \sin^2(x) &= 0 \\ -u_1' \sin(x)\cos(x) + u_2' \cos^2(x) &= \tan(x)\cos(x) = \sin(x) \end{aligned} \tag{3.9.7}$$

And thus

$$\begin{aligned} u_2'(\sin^2(x)+\cos^2(x)) &= \sin(x) \\ u_2' &= \sin(x) \\ u_1' &= \frac{-\sin^2(x)}{\cos(x)} = -\tan(x)\sin(x) \end{aligned} \tag{3.9.8}$$

Now we need to integrate u_1' and u_2' to get u_1 and u_2.

$$\begin{aligned} u_1 &= \int u_1' dx = \int -\tan(x)\sin(x) dx = \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| + \sin(x) \\ u_2 &= \int u_2' dx = \int \sin(x) dx = -\cos(x) \end{aligned} \tag{3.9.9}$$

So our particular solution is

$$\begin{aligned} y_p &= u_1y_1 + u_2y_2 = \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| + \cos(x)\sin(x) - \cos(x)\sin(x) \\ &= \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right| \end{aligned} \tag{3.9.10}$$

The general solution to $y'' + y = \tan x$ is, therefore,

$$y = C_1\cos(x) + C_2\sin(x) + \frac{1}{2}\cos(x)\ln\left|\frac{\sin(x)-1}{\sin(x)+1}\right|$$

Contributors and Attributions

- Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

3.9E: Exercises for Section 3.9

Exercises 1-16

Solve each differential equation or system using the specified method, such as undetermined coefficients, variation of parameters, or other techniques.

Exercise $3.9E.1$

Find a particular solution of $y'' - y' - 6y = e^{2x}$.

Exercise $3.9E.2$

Find a particular solution of $y'' - 4y' + 4y = e^{2x}$.

Exercise $3.9E.3$

Solve the initial value problem $y'' + 9y = \cos(3x) + \sin(3x)$ for $y(0) = 2, y'(0) = 1$.

Exercise $3.9E.4$

Set up the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x$.

Exercise $3.9E.5$

Set up the form of the particular solution but do not solve for the coefficients for $y^{(4)} - 2y''' + y'' = e^x + x + \sin x$.

Exercise $3.9E.6$

a. Using variation of parameters find a particular solution of $y'' - 2y' + y = e^x$.
b. Find a particular solution using undetermined coefficients.
c. Are the two solutions you found the same? What is going on?

Exercise $3.9E.7$

Find a particular solution of $y'' - 2y' + y = \sin(x^2)$. It is OK to leave the answer as a definite integral.

Exercise $3.9E.8$

For an arbitrary constant c find a particular solution to $y'' - y = e^{cx}$. Hint: Make sure to handle every possible real c.

Exercise $3.9E.9$

a. Using variation of parameters find a particular solution of $y'' - y = e^x$
b. Find a particular solution using undetermined coefficients.
c. Are the two solutions you found the same? What is going on?

Exercise $3.9E.10$

Find a polynomial $P(x)$, so that $y = 2x^2 + 3x + 4$ solves $y'' + 5y' + y = P(x)$.

Exercise $3.9E.11$

Find a particular solution to $y'' - y' + y = 2\sin(3x)$

Answer

$y = \frac{-16\sin(3x)+6\cos(3x)}{73}$

Exercise $3.9E.12$

a. Find a particular solution to $y'' + 2y = e^x + x^3$.
b. Find the general solution.

Answer

a. $y = \frac{2e^x+3x^3-9x}{6}$
b. $y = C_1\cos(\sqrt{2}x) + C_2\sin(\sqrt{2}x) + \frac{2e^x+3x^3-9x}{6}$

Exercise $3.9E.13$

Solve $y'' + 2y' + y = x^2, y(0) = 1, y'(0) = 2$.

Answer

$y(x) = x^2 - 4x + 6 + e^{-x}(x - 5)$

Exercise $3.9E.14$

Use variation of parameters to find a particular solution of $y'' - y = \frac{1}{e^x + e^{-x}}$.

Answer

$y = \frac{2xe^x - (e^x+e^{-x})\log(e^{2x}+1)}{4}$

Exercise $3.9E.15$

For an arbitrary constant c find the general solution to $y'' - 2y = \sin(x + c)$.

Answer

$y = \frac{-\sin(x+c)}{3} + C_1e^{\sqrt{2}x} + C_2e^{-\sqrt{2}x}$

Exercise $3.9E.16$

Undetermined coefficients can sometimes be used to guess a particular solution to other equations than constant coefficients. Find a polynomial $y(x)$ that solves $y' + xy = x^3 + 2x^2 + 5x + 2$.

Note: Not every right hand side will allow a polynomial solution, for example, $y' + xy = 1$ does not, but a technique based on undetermined coefficients does work.

Answer

$y = x^2 + 2x + 3$

3.10: Forced Oscillations and Resonance

Learning Objectives

- Analyze and solve forced oscillation problems using second-order differential equations, with a focus on undamped, damped, and resonance scenarios.
- Explore the concepts of pure and practical resonance and their implications in real-world systems, including the effects of damping on oscillatory behavior.

Let us consider to the example of a mass on a spring (see Figure $3.10.1$). We now examine the case of forced oscillations, which we did not yet handle. That is, we consider the equation

$$mx'' + cx' + kx = F(t)$$

for some nonzero $F(t)$. The setup is again: m is mass, c is friction, k is the spring constant, and $F(t)$ is an external force acting on the mass.

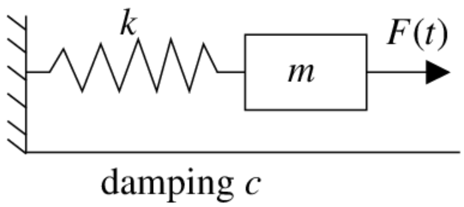


Figure $3.10.1$: A spring with spring constant connected to a wall and a box with mass m that has damping C and outward force F(t). (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

What we are interested in is periodic forcing, such as noncentered rotating parts, or perhaps loud sounds, or other sources of periodic force. Once we learn about Fourier series, we will see that we cover all periodic functions by simply considering $F(t) = F_0 \cos(\omega t)$ (or sine instead of cosine, the calculations are essentially the same).

Undamped Forced Motion and Resonance

First let us consider undamped $c = 0$ motion for simplicity. We have the equation

$$mx'' + kx = F_0 \cos(\omega t)$$

This equation has the complementary solution (solution to the associated homogeneous equation)

$$x_c = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

where $\omega_0 = \sqrt{\frac{k}{m}}$ is the *natural frequency* (angular), which is the frequency at which the system "wants to oscillate" without external interference.

Let us suppose that $\omega_0 \neq \omega$. We try the solution $x_p = A\cos(\omega t)$ and solve for A. Note that we need not have sine in our trial solution as on the left hand side we will only get cosines anyway. If you include a sine, it is fine; you will find that its coefficient will be zero.

We solve using the method of undetermined coefficients. We find that

$$x_p = \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

We leave it as an exercise to do the algebra required.

The general solution is

$$x = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) + \frac{F_0}{m(\omega_0^2 - \omega^2)} \cos(\omega t)$$

or written another way

$$x = C\cos(\omega_0 t - y) + \frac{F_0}{m(\omega_0^2 - \omega^2)}\cos(\omega t)$$

Hence it is a superposition of two cosine waves at different frequencies.

Example 3.10.1

Find a solution to

$$0.5x'' + 8x = 10\cos(\pi t), \quad x(0) = 0, \quad x'(0) = 0$$

Solution

First we read off the parameters: $\omega = \pi, \omega_0 = \sqrt{\frac{8}{0.5}} = 4, F_0 = 10, m = 0.5$. The general solution is

$$x = C_1\cos(4t) + C_2\sin(4t) + \frac{20}{16 - \pi^2}\cos(\pi t)$$

Solve for C_1 and C_2 using the initial conditions. It is easy to see that $C_1 = \frac{-20}{16-\pi^2}$ and $C_2 = 0$. Hence

$$x = \frac{20}{16 - \pi^2}(\cos(\pi t) - \cos(4t))$$

The graph of the solution is in Figure 3.10.2

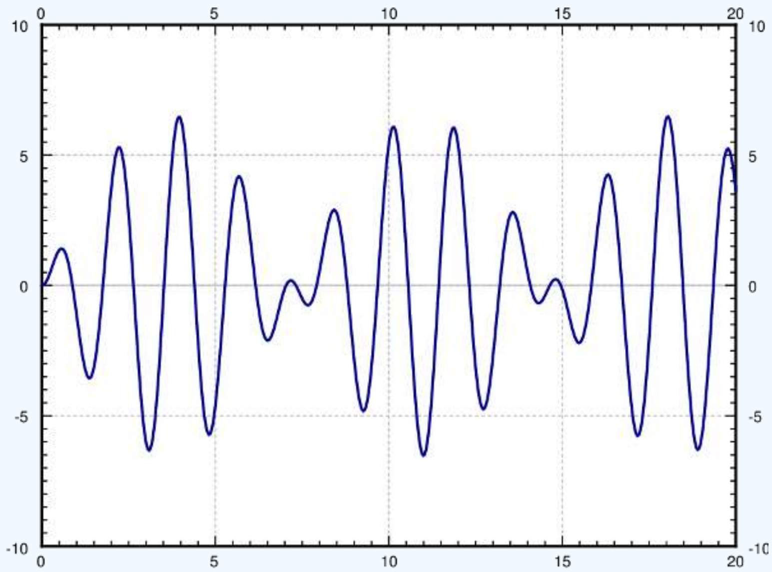


Figure 3.10.2: Graph of $x(t) = \frac{20}{16-\pi^2}(\cos(\pi t) - \cos(4t))$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Notice the “beating” behavior in Figure 3.10.2 First use the trigonometric identity

$$2\sin(\frac{A - B}{2})\sin(\frac{A + B}{2}) = \cos B - \cos A$$

to get that

$$x = \frac{20}{16 - \pi^2}(2\sin(\frac{4 - \pi}{2}t)\sin(\frac{4 + \pi}{2}t))$$

Notice that x is a high frequency wave modulated by a low frequency wave.

Now suppose that $\omega_0 = \omega$. Obviously, we cannot try the solution $A\cos(\omega t)$ and then use the method of undetermined coefficients. We notice that $\cos(\omega t)$ solves the associated homogeneous equation. Therefore, we need to try $x_p = At\cos(\omega t) + Bt\sin(\omega t)$. This time we do need the sine term since the second derivative of $t\cos(\omega t)$ does contain sines. We write the equation

$$x'' + \omega^2 x = \frac{F_0}{m} \cos(\omega t)$$

Plugging x_p into the left hand side we get

$$2B\omega \cos(\omega t) - 2A\omega \sin(\omega t) = \frac{F_0}{m} \cos(\omega t)$$

Hence $A = 0$ and $B = \frac{F_0}{2m\omega}$. Our particular solution is $\frac{F_0}{2m\omega} t \sin(\omega t)$ and our general solution is

$$x = C_1 \cos(\omega t) + C_2 \sin(\omega t) + \frac{F_0}{2m\omega} t \sin(\omega t)$$

The important term is the last one (the particular solution we found). We can see that this term grows without bound as $t \to \infty$. In fact it oscillates between $\frac{F_0 t}{2m\omega}$ and $\frac{-F_0 t}{2m\omega}$. The first two terms only oscillate between $\pm\sqrt{C_1^2 + C_2^2}$, which becomes smaller and smaller in proportion to the oscillations of the last term as t gets larger. In Figure $3.10.3$ we see the graph with $C_1 = C_2 = 0, F_0 = 2, m = 1, \omega = \pi$.

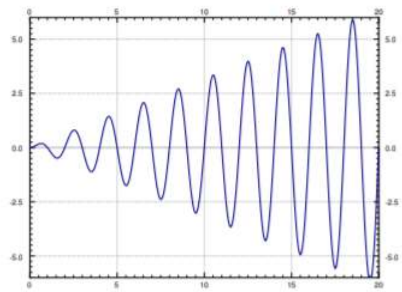


Figure 3.10.3: Graph of $x(t) = \frac{1}{\pi} t \sin(\pi t)$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

By forcing the system in just the right frequency we produce very wild oscillations. This kind of behavior is called resonance or perhaps pure resonance. Sometimes resonance is desired. For example, remember when as a kid you could start swinging by just moving back and forth on the swing seat in the "correct frequency"? You were trying to achieve resonance. The force of each one of your moves was small, but after a while it produced large swings.

On the other hand resonance can be destructive. In an earthquake some buildings collapse while others may be relatively undamaged. This is due to different buildings having different resonance frequencies. So figuring out the resonance frequency can be very important.

A common (but wrong) example of destructive force of resonance is the Tacoma Narrows bridge failure. It turns out there was a different phenomenon at play.[1]

Damped Forced Motion and Practical Resonance

In real life things are not as simple as they were above. There is, of course, some damping. Our equation becomes

$$mx'' + cx' + kx = F_0 \cos(\omega t), \tag{3.10.1}$$

for some $c > 0$. We have solved the homogeneous problem before. We let

$$p = \frac{c}{2m} \quad \omega_0 = \sqrt{\frac{k}{m}}$$

We replace Equation (3.10.1) with

$$x'' + 2px' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega t)$$

The roots of the characteristic equation of the associated homogeneous problem are $r_1, r_2 = -p \pm \sqrt{p^2 - \omega_0^2}$. The form of the general solution of the associated homogeneous equation depends on the sign of $p^2 - \omega_0^2$, or equivalently on the sign of $c^2 - 4km$, as we have seen before. That is,

$$x_c = \begin{cases} C_1 e^{r_1 t} + C_2 e^{r_2 t}, & \text{if } c^2 > 4km, \\ C_1 e^{pt} + C_2 t e^{-pt}, & \text{if } c^2 = 4km, \\ e^{-pt}(C_1 \cos(\omega_1 t) + C_2 \sin(\omega_1 t)), & \text{if } c^2 < 4km, \end{cases}$$

where $\omega_1 = \sqrt{\omega_0^2 - p^2}$. In any case, we can see that $x_c(t) \to 0$ as $t \to \infty$. Furthermore, there can be no conflicts when trying to solve for the undetermined coefficients by trying $x_p = A\cos(\omega t) + B\sin(\omega t)$. Let us plug in and solve for A and B. We get (the tedious details are left to reader)

$$((\omega_0^2 - \omega^2)B - 2\omega pA)\sin(\omega t) + ((\omega_0^2 - \omega^2)A + 2\omega pB)\cos(\omega t) = \frac{F_0}{m}\cos(\omega t)$$

Solving for A and B yield

$$A = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m\left(\omega_0^2 - \omega^2\right)^2}$$

$$B = \frac{2\omega pF_0}{m(2\omega p)^2 + m\left(\omega_0^2 - \omega^2\right)^2}$$

We also compute $C = \sqrt{A^2 + B^2}$ to be

$$C = \frac{F_0}{m\sqrt{(2\omega p)^2 + \left(\omega_0^2 - \omega^2\right)^2}}$$

Thus our particular solution is

$$x_p = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m\left(\omega_0^2 - \omega^2\right)^2}\cos(\omega t) + \frac{2\omega pF_0}{m(2\omega p)^2 + m\left(\omega_0^2 - \omega^2\right)^2}\sin(\omega t)$$

Or in the alternative notation we have amplitude C and phase shift γ where (if $\omega \neq \omega_0$)

$$\tan\gamma = \frac{B}{A} = \frac{2\omega p}{\omega_0^2 - \omega^2}$$

Hence we have

$$x_p = \frac{F_0}{m\sqrt{(2\omega p)^2 + \left(\omega_0^2 - \omega^2\right)^2}}\cos(\omega t - \gamma)$$

If $\omega = \omega_0$ we see that $A = 0, B = C = \frac{F_0}{2m\omega p},$ and $\gamma = \frac{\pi}{2}$.

The exact formula is not as important as the idea. Do not memorize the above formula, you should instead remember the ideas involved. For different forcing function F, you will get a different formula for x_p. So there is no point in memorizing this specific formula. You can always recompute it later or look it up if you really need it.

For reasons we will explain in a moment, we call $x_c$the transient solution and denote it by x_{tr}. We call the x_p we found above the steady periodic solution and denote it by x_{sp}. The general solution to our problem is

$$x = x_c + x_p = x_{tr} + x_{sp}$$

We note that $x_c = x_{tr}$ goes to zero as $t \to \infty$, as all the terms involve an exponential with a negative exponent. Hence for large t, the effect of x_{tr} is negligible and we will essentially only see x_{sp}. Hence the name transient. Notice that x_{sp} involves no arbitrary constants, and the initial conditions will only affect x_{tr}. This means that the effect of the initial conditions will be negligible after some period of time. Because of this behavior, we might as well focus on the steady periodic solution and ignore the transient solution. A graph of different initial conditions is given in Figure 3.10.4

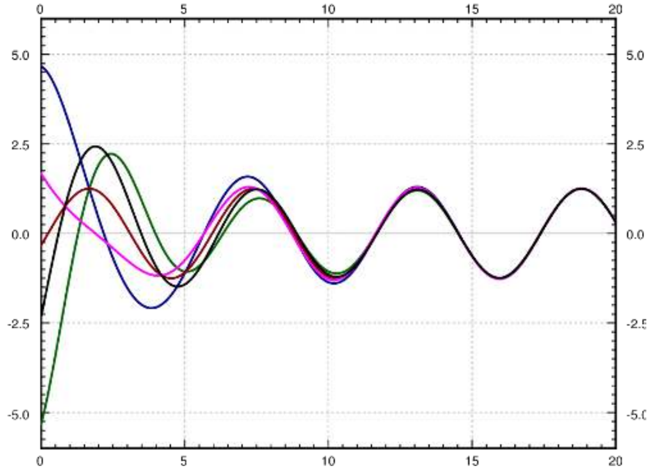


Figure 3.10.4: Several graphs that start all over the place on the y-axis. Solutions with different initial conditions for parameters $k=1, m=1, F_0=1, c=0.7,$ and $\omega=1.1$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

Notice that the speed at which x_{tr} goes to zero depends on P (and hence c). The bigger P is (the bigger c is), the "faster" x_{tr} becomes negligible. So the smaller the damping, the longer the "transient region." This agrees with the observation that when $c=0$, the initial conditions affect the behavior for all time (i.e. an infinite "transient region").

Let us describe what we mean by resonance when damping is present. Since there were no conflicts when solving with undetermined coefficient, there is no term that goes to infinity. What we will look at however is the maximum value of the amplitude of the steady periodic solution. Let C be the amplitude of x_{sp}. If we plot C as a function of ω (with all other parameters fixed) we can find its maximum. We call the ω that achieves this maximum the **practical resonance frequency**. We call the maximal amplitude $C(\omega)$ the **practical resonance amplitude**. Thus when damping is present we talk of practical resonance rather than pure resonance. A sample plot for three different values of c is given in Figure 3.10.5. As you can see the practical resonance amplitude grows as damping gets smaller, and any practical resonance can disappear when damping is large.

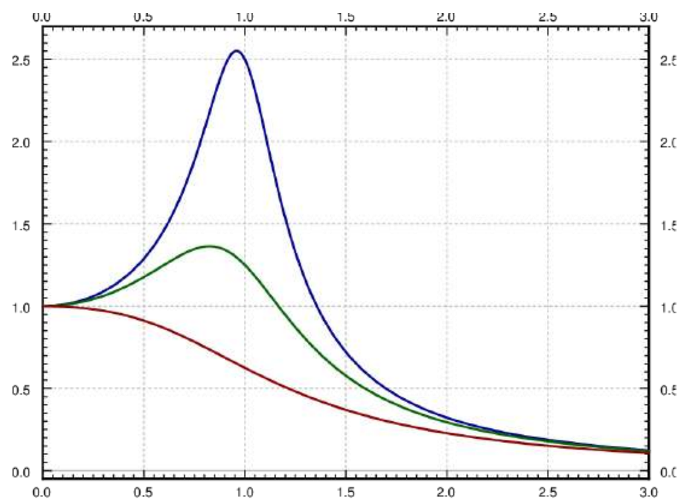


Figure 3.10.5: Graph of $C(\omega)$ showing practical resonance with parameters $k=1, m=1, F_0=1$. The top line is with $c=0.4$, the middle line with $c=0.8$, and the bottom line with $c=1.6$. (CC BY-SA 4.0; Jiří Lebl via Differential Equations for Engineers).

To find the maximum we need to find the derivative $C'(\omega)$. Computation shows

$$C'(\omega) = \frac{-4\omega(2p^2+\omega^2-\omega_0^2)F_0}{m\left((2\omega p)^2+(\omega_0^2-\omega^2)\right)^{3/2}}$$

This is zero either when $\omega=0$ or when $2p^2+\omega^2-\omega_0^2=0$. In other words, $C'(\omega)=0$ when

$$\omega=\sqrt{\omega_0^2-2p^2} \text{ or } \omega=0$$

It can be shown that if $\omega_0^2 - 2p^2$ is positive, then $\sqrt{\omega_0^2 - 2p^2}$ is the practical resonance frequency (that is the point where $C(\omega)$ is maximal, note that in this case $C'(\omega) > 0$ for small ω). If $\omega = 0$ is the maximum, then essentially there is no practical resonance since we assume that $\omega > 0$ in our system. In this case the amplitude gets larger as the forcing frequency gets smaller.

If practical resonance occurs, the frequency is smaller than ω_0. As the damping c (and hence p) becomes smaller, the practical resonance frequency goes to ω_0. So when damping is very small, ω_0 is a good estimate of the resonance frequency. This behavior agrees with the observation that when $c = 0$, then ω_0 is the resonance frequency.

Another interesting observation to make is that if $\omega \to \infty$, then $C \to 0$. This means that if the forcing frequency gets too high it does not manage to get the mass moving in the mass-spring system. This is quite reasonable intuitively. If we wiggle back and forth really fast while sitting on a swing, we will not get it moving at all, no matter how forceful. Fast vibrations just cancel each other out before the mass has any chance of responding by moving one way or the other.

The behavior is more complicated if the forcing function is not an exact cosine wave, but for example a square wave. A general periodic function will be the sum (superposition) of many cosine waves of different frequencies. The reader is encouraged to come back to this section once we have learned about the Fourier series.

Footnotes

[1]K. Billah and R. Scanlan, Resonance, Tacoma Narrows Bridge Failure, and Undergraduate Physics Textbooks, American Journal of Physics, 59(2), 1991, 118–124, textbook.

3.10E: Exercises for Section 3.10

Exercises 1-8

Solve the given problems using concepts of damped and forced oscillations in mass-spring systems, resonance phenomena, and differential equations.

Exercise 3.10.1

Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \sin(\omega t)$. Assume $c > 0$.

Exercise 3.10.2

Derive a formula for x_{sp} if the equation is $mx'' + cx' + kx = F_0 \cos(\omega t) + F_1 \cos(3\omega t)$. Assume $c > 0$.

Exercise 3.10.3

Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $m > 0$ and $k > 0$. Now think of the function $C(\omega)$. For what values of c (solve in terms of m, k, and F_0) will there be no practical resonance (that is, for what values of c is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise 3.10.4

Take $mx'' + cx' + kx = F_0 \cos(\omega t)$. Fix $c > 0$ and $k > 0$. Now think of the function $C(\omega)$. For what values of m (solve in terms of c, k, and F_0) will there be no practical resonance (that is, for what values of m is there no maximum of $C(\omega)$ for $\omega > 0$)?

Exercise 3.10.5

Suppose a water tower in an earthquake acts as a mass-spring system. Assume that the container on top is full and the water does not move around. The container then acts as a mass and the support acts as the spring, where the induced vibrations are horizontal. Suppose that the container with water has a mass of $m = 10,000 kg$. It takes a force of 1000 newtons to displace the container 1 meter. For simplicity assume no friction. When the earthquake hits the water tower is at rest (it is not moving).

Suppose that an earthquake induces an external force $F(t) = mA\omega^2 \cos(\omega t)$.

a. What is the natural frequency of the water tower?
b. If ω is not the natural frequency, find a formula for the maximal amplitude of the resulting oscillations of the water container (the maximal deviation from the rest position). The motion will be a high frequency wave modulated by a low frequency wave, so simply find the constant in front of the sines.
c. Suppose $A = 1$ and an earthquake with frequency 0.5 cycles per second comes. What is the amplitude of the oscillations? Suppose that if the water tower moves more than 1.5 meter, the tower collapses. Will the tower collapse?

Exercise 3.10.6

A mass of 4 kg on a spring with $k = 4$ and a damping constant $c = 1$. Suppose that $F_0 = 2$. Using forcing of $F_0 \cos(\omega t)$. Find the ω that causes practical resonance and find the amplitude.

Answer

$$\omega = \frac{\sqrt{31}}{4\sqrt{2}} \approx 0.984 \quad C(\omega) = \frac{16}{3\sqrt{7}} \approx 2.016$$

> **Exercise 3.10.7**
>
> Derive a formula for x_{sp} for $mx'' + cx' + kx = F_0 \cos(\omega t) + A$ where A is some constant. Assume $c > 0$.
>
> **Answer**
>
> $$x_{sp} = \frac{(\omega_0^2 - \omega^2)F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\cos(\omega t) + \frac{2\omega p F_0}{m(2\omega p)^2 + m(\omega_0^2 - \omega^2)^2}\sin(\omega t) + \frac{A}{k}$$, where $p = \frac{c}{2m}$ and $\omega_0 = \sqrt{\frac{k}{m}}$

> **Exercise 3.10.8**
>
> Suppose there is no damping in a mass and spring system with $m = 5, k = 20,$ and $F_0 = 5$. Suppose that ω is chosen to be precisely the resonance frequency.
>
> a. Find ω.
> b. Find the amplitude of the oscillations at time $t = 100$.
>
> **Answer**
>
> a. $\omega = 2$
> b. 25

CHAPTER OVERVIEW

4: Power series methods

Contributors and Attributions

- Jiří Lebl (Oklahoma State University).These pages were supported by NSF grants DMS-0900885 and DMS-1362337.

4.1: Power Series

Learning Objectives

- Define power series, understand its general form and to differentiate between finite and infinite series.
- Identify the components of a power series (e.g., coefficient, center, variable).
- Learn the concept of convergence of power series, to find the radius, and interval of convergence using appropriate tests (e.g. ratio test, root test).
- Express elementary functions (e.g., exponential, trigonometric, logarithmic functions) as power series. Taylor approximation and its connection with analytic functions.
- Perform algebraic operations with power series and to differentiate/ integrate power series term by term within the radius of convergence.

Many functions can be written in terms of a power series

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

If we assume that a solution of a differential equation is written as a power series, then perhaps we can use a method reminiscent of undetermined coefficients. That is, we will try to solve for the numbers a_k. Before we can carry out this process, let us review some results and concepts about power series.

Power Series

As we said, a power series is an expression such as

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \cdots, \tag{4.1.1}$$

where $a_0, a_1, a_2, \ldots, a_k, \ldots$ and x_0 are constants. Let

$$S_n(x) = \sum_{k=0}^{n} a_k (x-x_0)^k = a_0 + a_1 (x-x_0) + a_2 (x-x_0)^2 + a_3 (x-x_0)^3 + \cdots + a_n (x-x_0)^n,$$

denote the so-called *partial sum*. If for some x, the limit

$$\lim_{n\to\infty} S_n(x) = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x-x_0)^k$$

exists, then we say that the series

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

converges at x. Note that for $x = x_0$, the series always converges to a_0. When

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

converges at any other point $x \neq x_0$, we say that

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

is a convergent power series. In this case we write

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k = \lim_{n\to\infty} \sum_{k=0}^{n} a_k (x-x_0)^k$$

If the series does not converge for any point $x \neq x_0$, we say that the series is divergent.

Example 4.1.1

The series

$$\sum_{k=0}^{\infty} \frac{1}{k!} x^k = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

is convergent for any x. Explain.

Solution

Recall that $k! = 1 \cdot 2 \cdot 3 \cdots k$ is the factorial. By convention we define $0! = 1$. In fact, you may recall that this series converges to e^x.

We say that

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

converges absolutely at x whenever the limit

$$\lim_{n \to \infty} \sum_{k=0}^{n} |a_k| \, |x - x_0|^k$$

exists. That is, the series $\sum_{k=0}^{\infty} |a_k| \, |x - x_0|^k$ is convergent. If

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

converges absolutely at x, then it converges at x. However, the opposite implication is not true.

Example 4.1.2

Check the convergence of the power series

$$\sum_{k=1}^{\infty} \frac{1}{k} x^k$$

Solution

It converges absolutely for all x in the interval $(-1, 1)$.

It converges at $x = -1$, as

$\sum_{k=1}^{\infty} \frac{(-1)^k}{k}$ converges (conditionally) by the alternating series test.

But the power series does not converge absolutely at $x = -1$, because $\sum_{k=1}^{\infty} \frac{1}{k}$ does not converge. The series diverges at $x = 1$.

Radius of Convergence

If a power series converges absolutely at some x_1, then for all x such that $|x - x_0| \leq |x_1 - x_0|$ (that is, x is closer than x_1 to x_0) we have $\left|a_k (x - x_0)^k\right| \leq \left|a_k (x_1 - x_0)^k\right|$ for all k. As the numbers $\left|a_k (x_1 - x_0)^k\right|$ sum to some finite limit, summing smaller positive numbers $\left|a_k (x - x_0)^k\right|$ must also have a finite limit. Hence, the series must converge absolutely at x.

Theorem 4.1.1

For a power series

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

there exists a number ρ (we allow $\rho=\infty$) called the radius of convergence such that the series converges absolutely on the interval $(x_0-\rho,\ x_0+\rho)$ and diverges for $x<x_0-\rho$ and $x>x_0+\rho$. We write $\rho=\infty$ if the series converges for all x.

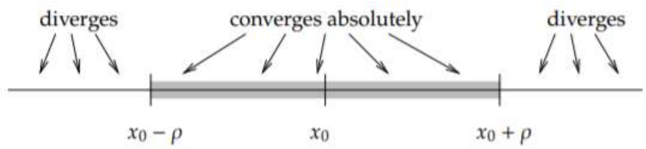


Figure 4.1.1: The diagram illustrates the radius of convergence of a power series centered at x_0. It consists of a horizontal number line representing real values of x. The central point x_0 is the center of the power series. Two boundary points, $x_0-\rho$ (to the left) and $x_0+\rho$ (to the right). This interval is visually represented by a shaded region between these points. Outside this interval, for $x<x_0-\rho$ and $x>x_0+\rho$, the series diverges, as indicated by arrows labeled "diverges." (CC BY-SA 4.0; Jiří Lebl via Differential Equation for Engineers)

In Example 4.1.1 the radius of convergence is $\rho=\infty$ as the series converges everywhere. In Example 4.1.2 the radius of convergence is $\rho=1$. We note that $\rho=0$ is another way of saying that the series is divergent. A useful test for convergence of a series is the *ratio test*. Suppose that

$$\sum_{k=0}^{\infty} c_k$$

is a series such that the limit

$$L=\lim_{n\to\infty}\left|\frac{c_{k+1}}{c_k}\right|$$

exists. Then the series converges absolutely if $L<1$ and diverges if $L>1$.

Let us apply this test to the series

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

. That is we let $c_k=a_k(x-x_0)^k$ in the test. Compute

$$L=\lim_{n\to\infty}\left|\frac{c_{k+1}}{c_k}\right|=\lim_{n\to\infty}\left|\frac{a_{k+1}(x-x_0)^{k+1}}{a_k(x-x_0)^k}\right|=\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right||x-x_0|$$

Define A by

$$A=\lim_{n\to\infty}\left|\frac{a_{k+1}}{a_k}\right|$$

Then if $1>L=A|x-x_0|$ the series

$$\sum_{k=0}^{\infty} a_k (x-x_0)^k$$

converges absolutely. If $A=0$, then the series always converges. If $A>0$, then the series converges absolutely if $|x-x_0|<\frac{1}{A}$, and diverges if $|x-x_0|>\frac{1}{A}$. That is, the radius of convergence is $\frac{1}{A}$.

A similar test is the *root test*. Suppose

$$L=\lim_{k\to\infty}\sqrt[k]{|c_k|}$$

exists. Then $\sum_{k=0}^{\infty} c_k$ converges absolutely if $L < 1$ and diverges if $L > 1$. We can use the same calculation as above to find A. Let us summarize.

Theorem 4.1.2: Ratio and Root Tests for Power Series

Let

$$\sum_{k=0}^{\infty} a_k (x - x_0)^k$$

be a power series such that

$$A = \lim_{n\to\infty} \left|\frac{a_{k+1}}{a_k}\right|$$

Or

$$A = \lim_{k\to\infty} \sqrt[k]{|a_k|}$$

exists. If $A = 0$, then the radius of convergence of the series is ∞. Otherwise the radius of convergence is $\frac{1}{A}$.

Example 4.1.3

Find the radius of convergence of the series centered at x=1

$$\sum_{k=0}^{\infty} 2^{-k} (x-1)^k$$

Solution

First we compute,

$$A = \lim_{k\to\infty} \left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\frac{2^{-k-1}}{2^{-k}}\right| = 2^{-1} = \frac{1}{2}.$$

Therefore the radius of convergence is 2, and the series converges absolutely on the interval $(-1, 3)$. And we could just as well have used the root test:

$$A = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{|2^{-k}|} = \lim_{k\to\infty} 2^{-1} = \frac{1}{2}.$$

Example 4.1.4

Find the radius of convergence of the series

$$\sum_{k=0}^{\infty} \frac{1}{k^k} x^k$$

Solution

Compute the limit for the root test,

$$A = \lim_{k\to\infty} \sqrt[k]{|a_k|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{1}{k^k}\right|} = \lim_{k\to\infty} \sqrt[k]{\left|\frac{1}{k}\right|^k} = \lim_{k\to\infty} \frac{1}{k} = 0$$

So the radius of convergence is ∞ it means that the series converges everywhere. The ratio test would also work here.

The root or the ratio test does not always apply. That is the limit of $\left|\frac{a_{k+1}}{a_k}\right|$ or $\sqrt[k]{|a_k|}$ might not exist. There exist more sophisticated ways of finding the radius of convergence, but those would be beyond the scope of this chapter. The two methods above cover many of the series that arise in practice. Often if the root test applies, so does the ratio test, and vice versa, though the limit might be easier to compute in one way than the other.

Analytic Functions

Functions represented by power series are called analytic functions. Not every function is analytic, although the majority of the functions you have seen in calculus are. An analytic function $f(x)$ is equal to its **Taylor series** near a point x_0. That is, for x near x_0 we have

$$f(x)=\sum_{k=0}^{\infty}\frac{f^{(k)}(x_0)}{k!}(x-x_0)^k, \tag{4.1.2}$$

where $f^{(k)}(x_0)$ denotes the k^{th} derivative of $f(x)$ at the point x_0.

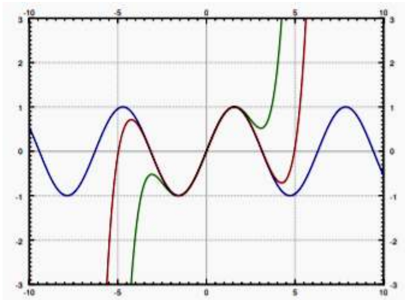


Figure $4.1.2$: The image presents the Taylor series representation of analytic function, a graph comparing $\sin(x)$ with its 5th and 9th-degree approximations. (CC BY-SA 4.0; Jiří Lebl via Differential Equation for Engineers)

$$\sin(x)=\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n+1)!}x^{2n+1}.$$

In Figure $4.1.2$ we plot $\sin(x)$ and the truncations of the series up to degree 5 and 9. You can see that the approximation is very good for x near 0, but gets worse for x further away from 0. This is what happens in general. To get a good approximation far away from x_0 you need to take more and more terms of the Taylor series.

Manipulating Power Series

One of the main properties of power series that we will use is that we can differentiate them term by term. That is, suppose that $\sum a_k(x-x_0)^k$ is a convergent power series. Then for x in the radius of convergence we have

$$\frac{d}{dx}\left[\sum_{k=0}^{\infty}a_k(x-x_0)^k\right]=\sum_{k=1}^{\infty}ka_k(x-x_0)^{k-1}.$$

Notice that the term corresponding to $k=0$ disappeared as it was constant. The radius of convergence of the differentiated series is the same as that of the original.

Example $4.1.5$

Show that the exponential function $y=e^x$ solves the first order differential equation $y'=y$.

Solution

First write

$$y=e^x=\sum_{k=0}^{\infty}\frac{1}{k!}x^k.$$

Now differentiate

$$y'=\sum_{k=1}^{\infty}k\frac{1}{k!}x^{k-1}=\sum_{k=1}^{\infty}\frac{1}{(k-1)!}x^{k-1}.$$

We *reindex* the series by simply replacing k with $k+1$. The series does not change, what changes is simply how we write it. After reindexing the series starts at $k=0$ again.

$$\sum_{k=1}^{\infty} \frac{1}{(k-1)!} x^{k-1} = \sum_{k+1=1}^{\infty} \frac{1}{((k+1)-1)!} x^{(k+1)-1} = \sum_{k=0}^{\infty} \frac{1}{k!} x^k.$$

That was precisely the power series for e^x that we started with, so we showed that $\frac{d}{dx}[e^x] = e^x$.

Convergent power series can be added and multiplied together, and multiplied by constants using the following rules. First, we can add series by adding term by term,

$$\left(\sum_{k=0}^{\infty} a_k (x-x_0)^k\right) + \left(\sum_{k=0}^{\infty} b_k (x-x_0)^k\right) = \sum_{k=0}^{\infty} (a_k + b_k)(x-x_0)^k.$$

We can multiply by constants,

$$\alpha \left(\sum_{k=0}^{\infty} a_k (x-x_0)^k\right) = \sum_{k=0}^{\infty} \alpha a_k (x-x_0)^k.$$

We can also multiply series together,

$$\left(\sum_{k=0}^{\infty} a_k (x-x_0)^k\right) \left(\sum_{k=0}^{\infty} b_k (x-x_0)^k\right) = \sum_{k=0}^{\infty} c_k (x-x_0)^k,$$

where $c_k = a_0 b_k + a_1 b_{k-1} + \cdots + a_k b_0$. The radius of convergence of the sum or the product is at least the minimum of the radii of convergence of the two series involved.

Power Series for Rational Functions

Polynomials are simply finite power series. That is, a polynomial is a power series where the a_k are zero for all k large enough. We can always expand a polynomial as a power series about any point x_0 by writing the polynomial as a polynomial in $(x-x_0)$. For example, let us write $2x^2 - 3x + 4$ as a power series around $x_0 = 1$:

$$2x^2 - 3x + 4 = 3 + (x-1) + 2(x-1)^2.$$

In other words $a_0 = 3$, $a_1 = 1$, $a_2 = 2$, and all other $a_k = 0$. To do this, we know that $a_k = 0$ for all $k \geq 3$ as the polynomial is of degree 2.

We write $a_0 + a_1(x-1) + a_2(x-1)^2$, we expand, and we solve for a_0, a_1, and a_2.

Let us look at rational functions, that is, ratios of polynomials. An important fact is that a series for a function only defines the function on an interval even if the function is defined elsewhere. For example, for $-1 < x < 1$ we have

$$\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k = 1 + x + x^2 + \cdots$$

This series is called the *geometric series*. The ratio test tells us that the radius of convergence is 1. The series diverges for $x \leq -1$ and $x \geq 1$, even though $\frac{1}{1-x}$ is defined for all $x \neq 1$.

We can use the geometric series together with rules for addition and multiplication of power series to expand rational functions around a point, as long as the denominator is not zero at x_0.

Example 4.1.6

Expand $\frac{x}{1+2x+x^2}$ as a power series around the origin $(x_0 = 0)$ and find the radius of convergence. First, write $1 + 2x + x^2 = (1+x)^2 = \left(1-(-x)\right)^2$. Now we compute

$$\begin{aligned}\frac{x}{1+2x+x^2} &= x\left(\frac{1}{1-(-x)}\right)^2 \\ &= x\left(\sum_{k=0}^{\infty}(-1)^k x^k\right)^2 \\ &= x\left(\sum_{k=0}^{\infty} c_k x^k\right) \\ &= \sum_{k=0}^{\infty} c_k x^{k+1},\end{aligned} \tag{4.1.3}$$

where using the formula for the product of series we obtain, $c_0 = 1$, $c_1 = -1-1 = -2$, $c_2 = 1+1+1 = 3$, $\ldots$.

Therefore

$$\frac{x}{1+2x+x^2} = \sum_{k=1}^{\infty}(-1)^{k+1} k x^k = x - 2x^2 + 3x^3 - 4x^4 + \cdots$$

The radius of convergence is at least 1. We use the ratio test

$$\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{(-1)^{k+2}(k+1)}{(-1)^{k+1}k}\right| = \lim_{k\to\infty}\frac{k+1}{k} = 1.$$

So the radius of convergence is actually equal to 1.

When the rational function is more complicated, it is also possible to use method of partial fractions. For example, to find the Taylor series for $\frac{x^3+x}{x^2-1}$, we write

$$\frac{x^3+x}{x^2-1} = x + \frac{1}{1+x} - \frac{1}{1-x} = x + \sum_{k=0}^{\infty}(-1)^k x^k - \sum_{k=0}^{\infty} x^k = -x + \sum_{\substack{k=3 \\ k \text{ odd}}}^{\infty}(-2)x^k.$$

Footnotes

[1] Named after the English mathematician Sir Brook Taylor (1685–1731).

4.1E: Exercises for Section 4.1

Exercises 1 - 16

Solve the following exercises related to power series and their convergence, expansion, and properties.

Exercise $4.1E.1$

Is the power series $\sum_{k=0}^{\infty} e^k x^k$ convergent? If so, what is the radius of convergence?

Exercise $4.1E.2$

Is the power series $\sum_{k=0}^{\infty} kx^k$ convergent? If so, what is the radius of convergence?

Exercise $4.1E.3$

Is the power series $\sum_{k=0}^{\infty} k!x^k$ convergent? If so, what is the radius of convergence?

Exercise $4.1E.4$

Is the power series $\sum_{k=0}^{\infty} \frac{1}{(2k)!}(x-10)^k$ convergent? If so, what is the radius of convergence?

Exercise $4.1E.5$

Determine the Taylor series for $\sin x$ around the point $x_0 = \pi$.

Exercise $4.1E.6$

Determine the Taylor series for $\ln x$ around the point $x_0 = 1$, and find the radius of convergence.

Exercise $4.1E.7$

Determine the Taylor series and its radius of convergence of $\dfrac{1}{1+x}$ around $x_0 = 0$.

Exercise $4.1E.8$

Determine the Taylor series and its radius of convergence of $\dfrac{x}{4-x^2}$ around $x_0 = 0$. Hint: You will not be able to use the ratio test.

Exercise $4.1E.9$

Expand $x^5 + 5x + 1$ as a power series around $x_0 = 5$.

Exercise $4.1E.10$

Suppose that the ratio test applies to a series $\sum_{k=0}^{\infty} a_k x^k$. Show, using the ratio test, that the radius of convergence of the differentiated series is the same as that of the original series.

Exercise $4.1E.11$

Suppose that f is an analytic function such that $f^{(n)}(0) = n$. Find $f(1)$.

Exercise $4.1E.12$

Is the power series $\sum_{n=1}^{\infty} (0.1)^n x^n$ convergent? If so, what is the radius of convergence?

Answer

Yes. Radius of convergence is 10.

Exercise $4.1E.13$

Is the power series $\sum_{n=1}^{\infty} \frac{n!}{n^n} x^n$ convergent? If so, what is the radius of convergence?

Answer

Yes. Radius of convergence is e.

Exercise $4.1E.14$

Using the geometric series, expand $\frac{1}{1-x}$ around $x_0 = 2$. For what x does the series converge?

Answer

$\frac{1}{1-x} = -\frac{1}{1-(2-x)}$ so $\frac{1}{1-x} = \sum\limits_{n=0}^{\infty} (-1)^{n+1} (x-2)^n$, which converges for $1 < x < 3$.

Exercise $4.1E.15$

Find the Taylor series for $x^7 e^x$ around $x_0 = 0$.

Answer

$$\sum_{n=7}^{\infty} \frac{1}{(n-7)!} x^n$$

Exercise $4.1E.16$

Imagine f and g are analytic functions such that $f^{(k)}(0) = g^{(k)}(0)$ for all large enough k. What can you say about $f(x) - g(x)$?

Answer

$f(x) - g(x)$ is a polynomial. Hint: Use Taylor series.

4.2: Series Solutions of Linear Second Order ODEs

> **Learning Objectives**
>
> - Understand the general form of second-order differential equation.
> - Distinguish between linear and nonlinear differential equations.
> - Recognize and classify second-order differential equations as homogenous or non-homogenous.
> - Find solutions to Airy's equation and Hermite's equation.

Suppose we have a linear second order homogeneous ODE of the form

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{4.2.1}$$

Suppose that $p(x)$, $q(x)$, and $r(x)$ are polynomials. We will try a solution of the form

$$y = \sum_{k=0}^{\infty} a_k (x - x_0)^k$$

and solve for the a_k to try to obtain a solution defined in some interval around x_0.

> **Definition: Ordinary and Singular Points**
>
> The point x_0 is called an *ordinary point* if $p(x_0) \neq 0$ in linear second order homogeneous ODE of the form $p(x)y'' + q(x)y' + r(x)y = 0$.
>
> That is, the functions
>
> $$\frac{q(x)}{p(x)} \quad \text{and} \quad \frac{r(x)}{p(x)}$$
>
> are defined for x near x_0.
>
> If $p(x_0) = 0$, then we say x_0 is a *singular point*.

Handling singular points is harder than ordinary points and so we now focus only on ordinary points.

> **Example 4.2.1: Expansion around an Ordinary Point**
>
> Let us start with a very simple example
>
> $$y'' - y = 0$$
>
> Find the power series solution near $x_0 = 0$, which is an ordinary point.
>
> **Solution**
>
> Every point is an ordinary point in fact, as the equation has constant coefficients. We already know we should obtain exponentials or the hyperbolic sine and cosine, but let us pretend we do not know this.
>
> We try
>
> $$y = \sum_{k=0}^{\infty} a_k x^k$$
>
> If we differentiate, the $k = 0$ term is a constant and hence disappears. We therefore get
>
> $$y' = \sum_{k=1}^{\infty} k\, a_k\, x^{k-1}$$
>
> We differentiate yet again to obtain (now the $k = 1$ term disappears)

$$y'' = \sum_{k=2}^{\infty} k\,(k-1)\,a_k\,x^{k-2}$$

We reindex the series (replace k with $k+2$) to obtain

$$y'' = \sum_{k=0}^{\infty} (k+2)\,(k+1)\,a_{k+2}\,x^{k}$$

Now we plug y and y'' into the differential equation.

$$\begin{aligned} 0 = y'' - y &= \left(\sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2}x^k\right) - \left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2}x^k - a_k x^k\right) \\ &= \sum_{k=0}^{\infty} \left((k+2)(k+1)a_{k+2} - a_k\right)x^k. \end{aligned} \tag{4.2.2}$$

As $y''-y$ is supposed to be equal to 0, we know that the coefficients of the resulting series must be equal to 0. Therefore,

$$(k+2)(k+1)a_{k+2} - a_k = 0, \qquad \text{or} \qquad a_{k+2} = \frac{a_k}{(k+2)(k+1)}.$$

The above equation is called a *recurrence relation* for the coefficients of the power series. It did not matter what a_0 or a_1 was. They can be arbitrary. But once we pick a_0 and a_1, then all other coefficients are determined by the recurrence relation.

Let us see what the coefficients must be. First, a_0 and a_1 are arbitrary

$$a_2 = \frac{a_0}{2}, \quad a_3 = \frac{a_1}{(3)(2)}, \quad a_4 = \frac{a_2}{(4)(3)} = \frac{a_0}{(4)(3)(2)}, \quad a_5 = \frac{a_3}{(5)(4)} = \frac{a_1}{(5)(4)(3)(2)}, \quad \ldots$$

So we note that for even k, that is $k=2n$ we get

$$a_k = a_{2n} = \frac{a_o}{(2n)!}$$

and for odd k that is $k=2n+1$ we have

$$a_k = a_{2n+1} = \frac{a_1}{(2n+1)!}$$

Let us write down the series

$$y = \sum_{k=0}^{\infty} a_k x^k = \sum_{n=0}^{\infty}\left(\frac{a_0}{(2n)!}\,x^{2n} + \frac{a_1}{(2n+1)!}\,x^{2n+1}\right) = a_0 \sum_{n=0}^{\infty} \frac{1}{(2n)!}\,x^{2n} + a_1 \sum_{n=0}^{\infty} \frac{1}{(2n+1)!}\,x^{2n+1}.$$

We recognize the two series as the hyperbolic sine and cosine. Therefore,

$$y = a_o \cosh x + a_1 \sinh x$$

Of course, in general we will not be able to recognize the series that appears, since usually there will not be any elementary function that matches it. In that case we will be content with the series.

Example 4.2.2

Let us do a more complex example. Suppose we wish to solve *Airy's equation, that is*

$$y\text{''} - xy = 0$$

near the point $x_0 = 0$, which is an ordinary point.

Solution

$$y=\sum_{k=0}^{\infty} a_k x^k$$

We differentiate twice (as above) to obtain

$$y''=\sum_{k=2}^{\infty} k(k-1)a_k x^{k-2}$$

We plug y into the equation

$$\begin{aligned} 0=y''-xy &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right)-x\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right)-\left(\sum_{k=0}^{\infty} a_k x^{k+1}\right). \end{aligned} \tag{4.2.3}$$

We reindex to make things easier to sum

$$\begin{aligned} 0=y''-xy &= \left(2a_2+\sum_{k=1}^{\infty}(k+2)\,(k+1)\,a_{k+2}x^k\right)-\left(\sum_{k=1}^{\infty} a_{k-1}x^k\right). \\ &= 2a_2+\sum_{k=1}^{\infty}\Big((k+2)\,(k+1)\,a_{k+2}-a_{k-1}\Big)\,x^k. \end{aligned} \tag{4.2.4}$$

Again $y''-xy$ is supposed to be 0, so first we notice that $a_2=0$ and also

$$(k+2)\,(k+1)\,a_{k+2}-a_{k-1}=0, \qquad \text{or} \qquad a_{k+2}=\frac{a_{k-1}}{(k+2)(k+1)}.$$

Now we jump in steps of three. First we notice that since $a_2=0$ we must have that, $a_5=0$, $a_8=0$, $a_{11}=0$, etc .... In general $a_{3n+2}=0$. The constants a_0 and a_1 are arbitrary and we obtain

$$a_3=\frac{a_0}{(3)(2)}, \quad a_4=\frac{a_1}{(4)(3)}, \quad a_6=\frac{a_3}{(6)(5)}=\frac{a_0}{(6)(5)(3)(2)}, \quad a_7=\frac{a_4}{(7)(6)}=\frac{a_1}{(7)(6)(4)(3)}, \quad \ldots$$

For a_k where k is a multiple of 3, that is $k=3n$ we notice that

$$a_{3n}=\frac{a_0}{(2)(3)(5)(6)\cdots(3n-1)(3n)}.$$

For a_k where $k=3n+1$, we notice

$$a_{3n+1}=\frac{a_1}{(3)(4)(6)(7)\cdots(3n)(3n+1)}.$$

In other words, if we write down the series for y we notice that it has two parts

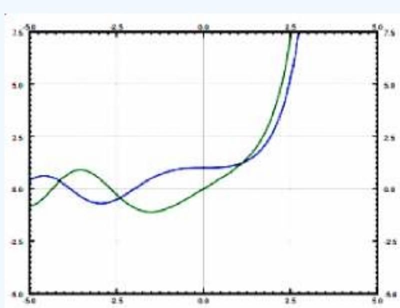


Figure 4.2.1: There are two solutions y_1 and y_2 to Airy's equation. y_1 is a concave up curve, starting below the x-axis, crossing the x-axis near $x=-1$ and increasing rapidly as x increases. y_2 is a concave down curve, starting above the x-axis and decreasing until around $x=0$, where it begins to increase slowly. *(CC BY-SA 4.0; Jiří Lebl via Differential Equation for Engineers)*

The functions y_1 and y_2 cannot be written in terms of the elementary functions that you know. See Figure $4.2.1$ for the plot of the solutions y_1 and y_2. These functions have many interesting properties. For example, they are oscillatory for negative x (like solutions to $y''+y=0$) and for positive x they grow without bound (like solutions to $y''-y=0$).

Note

The general form of the Hermite equation is: $y''-2xy'+2ny=0$ where "n" represents the order of the equation.

Sometimes a solution may turn out to be a polynomial.

Example $4.2.3$

Find a solution to the so-called Hermite equation of order n is the equation

$$y''-2xy'+2ny=0.$$

Find a solution around the point $x_0=0$.

Solution

We try

$$y=\sum_{k=0}^{\infty} a_k x^k.$$

We differentiate (as above) to obtain

$$\begin{aligned} y' &= \sum_{k=1}^{\infty} k a_k x^{k-1}, \\ y'' &= \sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}. \end{aligned} \tag{4.2.5}$$

Now we plug into the equation

$$\begin{aligned} 0 &= y''-2xy'+2ny \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - 2x\left(\sum_{k=1}^{\infty} k a_k x^{k-1}\right) + 2n\left(\sum_{k=0}^{\infty} a_k x^k\right) \\ &= \left(\sum_{k=2}^{\infty} k\,(k-1)\,a_k x^{k-2}\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(\sum_{k=0}^{\infty} 2n a_k x^k\right) \\ &= \left(2a_2+\sum_{k=1}^{\infty}(k+2)\,(k+1)\,a_{k+2}x^k\right) - \left(\sum_{k=1}^{\infty} 2k a_k x^k\right) + \left(2na_0+\sum_{k=1}^{\infty} 2n a_k x^k\right) \\ &= 2a_2+2na_0+\sum_{k=1}^{\infty}\left((k+2)\,(k+1)\,a_{k+2}-2ka_k+2na_k\right)x^k. \end{aligned} \tag{4.2.6}$$

As $y''-2xy'+2ny=0$ we have

$$(k+2)\,(k+1)\,a_{k+2}+(-2k+2n)a_k=0, \qquad \text{or} \qquad a_{k+2}=\frac{(2k-2n)}{(k+2)(k+1)}a_k.$$

This recurrence relation actually includes $a_2=-na_0$ (which comes about from $2a_2+2na_0=0$). Again a_0 and a_1 are arbitrary.

$$\begin{aligned} a_2 &= \frac{-2n}{(2)(1)}a_0, \qquad a_3 = \frac{2(1-n)}{(3)(2)}a_1, \\ a_4 &= \frac{2(2-n)}{(4)(3)}a_2 = \frac{2^2(2-n)(-n)}{(4)(3)(2)(1)}a_0, \\ a_5 &= \frac{2(3-n)}{(5)(4)}a_3 = \frac{2^2(3-n)(1-n)}{(5)(4)(3)(2)}a_1, \quad \dots \end{aligned} \tag{4.2.7}$$

Let us separate the even and odd coefficients. We find that

$$\begin{aligned} a_{2m} &= \frac{2^m(-n)(2-n)\cdots(2m-2-n)}{(2m)!}, \\ a_{2m+1} &= \frac{2^m(1-n)(3-n)\cdots(2m-1-n)}{(2m+1)!}. \end{aligned} \tag{4.2.8}$$

Let us write down the two series, one with the even powers and one with the odd.

$$\begin{aligned} y_1(x) &= 1 + \frac{2(-n)}{2!}x^2 + \frac{2^2(-n)(2-n)}{4!}x^4 + \frac{2^3(-n)(2-n)(4-n)}{6!}x^6 + \cdots, \\ y_2(x) &= x + \frac{2(1-n)}{3!}x^3 + \frac{2^2(1-n)(3-n)}{5!}x^5 + \frac{2^3(1-n)(3-n)(5-n)}{7!}x^7 + \cdots. \end{aligned} \tag{4.2.9}$$

We then write

$$y(x) = a_0 y_1(x) + a_1 y_2(x).$$

We also notice that if n is a positive even integer, then $y_1(x)$ is a polynomial as all the coefficients in the series beyond a certain degree are zero. If n is a positive odd integer, then $y_2(x)$ is a polynomial. For example, if $n = 4$, then

$$y_1(x) = 1 + \frac{2(-4)}{2!}x^2 + \frac{2^2(-4)(2-4)}{4!}x^4 = 1 - 4x^2 + \frac{4}{3}x^4.$$

Footnotes

[1] Named after the English mathematician Sir George Biddell Airy (1801 – 1892).

[2] Named after the French mathematician Charles Hermite (1822–1901).

4.2E: Exercises for Section 4.2

Exercises 1-11

In the following exercises, when asked to solve an equation using power series methods, you should find the first few terms of the series, and if possible find a general formula for the k^{th} coefficient.

Exercise $4.2E.1$

Use power series methods to solve $y''+y=0$ at the point $x_0=1$.

Exercise $4.2E.2$

Use power series methods to solve $y''+4xy=0$ at the point $x_0=0$.

Exercise $4.2E.3$

Use power series methods to solve $y''-xy=0$ at the point $x_0=1$.

Exercise $4.2E.4$

Use power series methods to solve $y''+x^2y=0$ at the point $x_0=0$.

Exercise $4.2E.5$

The methods work for other orders than second order. Try the methods of this section to solve the first order system $y'-xy=0$ at the point $x_0=0$.

Exercise $4.2E.6$

Chebyshev's equation of order p:

a. Solve $(1-x^2)y''-xy'+p^2y=0$ using power series methods at $x_0=0$.
b. For what p is there a polynomial solution?

Exercise $4.2E.7$

Find a polynomial solution to $(x^2+1)y''-2xy'+2y=0$ using power series methods.

Exercise $4.2E.8$

a. Use power series methods to solve $(1-x)y''+y=0$ at the point $x_0=0$.
b. Use the solution to part a) to find a solution for $xy''+y=0$ around the point $x_0=1$.

Exercise $4.2E.9$

Use power series methods to solve $y''+2x^3y=0$ at the point $x_0=0$.

Answer

$a_2=0$, $a_3=0$, $a_4=0$, recurrence relation (for $k\geq 5$): $a_k=\frac{-2a_{k-5}}{k(k-1)}$, so $y(x)=a_0+a_1x-\frac{a_0}{10}x^5-\frac{a_1}{15}x^6+\frac{a_0}{450}x^{10}+\frac{a_1}{825}x^{11}-\frac{a_0}{47250}x^{15}-\frac{a_1}{99000}x^{16}+\cdots$

Exercise $4.2E.10$

We can also use power series methods in nonhomogeneous equations.

a. Use power series methods to solve $y'' - xy = \frac{1}{1-x}$ at the point $x_0 = 0$. Hint: Recall the geometric series.
b. Now solve for the initial condition $y(0) = 0$, $y'(0) = 0$.

Answer

a. $a_2 = \frac{1}{2}$, and for $k \geq 1$ we have $a_k = \frac{a_{k-3}+1}{k(k-1)}$, so
$y(x) = a_0 + a_1 x + \frac{1}{2}x^2 + \frac{a_0+1}{6}x^3 + \frac{a_1+1}{12}x^4 + \frac{3}{40}x^5 + \frac{a_0+2}{30}x^6 + \frac{a_1+2}{42}x^7 + \frac{5}{112}x^8 + \frac{a_0+3}{72}x^9 + \frac{a_1+3}{90}x^{10} + \cdots$
b. $y(x) = \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{12}x^4 + \frac{3}{40}x^5 + \frac{1}{15}x^6 + \frac{1}{21}x^7 + \frac{5}{112}x^8 + \frac{1}{24}x^9 + \frac{1}{30}x^{10} + \cdots$

Exercise $4.2E.11$

Attempt to solve $x^2 y'' - y = 0$ at $x_0 = 0$ using the power series method of this section (x_0 is a singular point). Can you find at least one solution? Can you find more than one solution?

Answer

Applying the method of this section directly we obtain $a_k = 0$ for all k and so $y(x) = 0$ is the only solution we find.

4.3: Singular Points and the Method of Frobenius

Learning Objectives

- Behavior of ODEs at singular points.
- Use power series to solve differential equations. (e.g. method of Frobenius).
- Approximate solutions to problems in physics, engineering, and numerical methods. (e.g., Bessel functions).

While the behavior of ODEs at singular points is more complicated, certain singular points are not especially difficult to solve. Let us look at some examples before giving a general method. We may be lucky and obtain a power series solution using the method of the previous section, but in general, we may have to try other things.

Example 4.3.1

Solve the first order equation

$$2xy' - y = 0. \tag{4.3.1}$$

Solution

Note that $x = 0$ is a singular point. If we only try to plug in

$$y = \sum_{k=0}^{\infty} a_k x^k, \tag{4.3.2}$$

we obtain

$$\begin{aligned} 0 = 2xy' - y &= 2x \left(\sum_{k=1}^{\infty} k a_k x^{k-1} \right) - \left(\sum_{k=0}^{\infty} a_k x^k \right) \\ &= a_0 + \sum_{k=1}^{\infty} (2k a_k - a_k)\, x^k. \end{aligned} \tag{4.3.3}$$

First, $a_0 = 0$. Next, the only way to solve $0 = 2ka_k - a_k = (2k-1)\,a_k$ for $k = 1, 2, 3, \dots$ is for $a_k = 0$ for all k. Therefore we only get the trivial solution $y = 0$. We need a nonzero solution to get the general solution.

Let us try $y = x^r$ for some real number r. Consequently our solution may only make sense for positive x. Then $y' = rx^{r-1}$. So

$$0 = 2xy' - y = 2xrx^{r-1} - x^r = (2r-1)x^r.$$

Therefore $r = \frac{1}{2}$, or in other words $y = x^{1/2}$. Multiplying by a constant, the general solution for positive x is

$$y = Cx^{1/2}.$$

If $C \neq 0$ then the derivative of the solution "blows up" at $x = 0$ (the singular point). There is only one solution that is differentiable at $x = 0$ and that's the trivial solution $y = 0$.

Not every problem with a singular point has a solution of the form $y = x^r$, of course. But perhaps we can combine the methods. What we will do is to try a solution of the form

$$y = x^r f(x)$$

where $f(x)$ is an analytic function.

Example 4.3.2

Find the general solution of the differential equation

$$4x^2y'' - 4x^2y' + (1-2x)y = 0, \tag{4.3.4}$$

As we did in the above example note that $x = 0$ is a singular point. Let us try

$$y = x^r \sum_{k=0}^{\infty} a_k x^k = \sum_{k=0}^{\infty} a_k x^{k+r}, \tag{4.3.5}$$

where r is a real number, not necessarily an integer. Again if such a solution exists, it may only exist for positive x. First let us find the derivatives

$$\begin{aligned} y' &= \sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1}, \\ y'' &= \sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2}. \end{aligned} \tag{4.3.6}$$

Plugging Equations (4.3.5) - (4.3.6) into our original differential equation we obtain

$$\begin{aligned} 0 &= 4x^2 y'' - 4x^2 y' + (1-2x)y \\ &= 4x^2 \left(\sum_{k=0}^{\infty} (k+r)\,(k+r-1)\, a_k x^{k+r-2} \right) - 4x^2 \left(\sum_{k=0}^{\infty} (k+r)\, a_k x^{k+r-1} \right) + (1-2x) \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) \\ &= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) - \left(\sum_{k=0}^{\infty} 4(k+r)\, a_k x^{k+r+1} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=0}^{\infty} 2 a_k x^{k+r+1} \right) \\ &= \left(\sum_{k=0}^{\infty} 4(k+r)\,(k+r-1)\, a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 4(k+r-1)\, a_{k-1} x^{k+r} \right) + \left(\sum_{k=0}^{\infty} a_k x^{k+r} \right) - \left(\sum_{k=1}^{\infty} 2 a_{k-1} x^{k+r} \right) \\ &= 4r(r-1)\, a_0 x^r + a_0 x^r + \sum_{k=1}^{\infty} \left(4(k+r)\,(k+r-1)\, a_k - 4(k+r-1)\, a_{k-1} + a_k - 2a_{k-1} \right)\, x^{k+r} \\ &= \left(4r(r-1) + 1 \right)\, a_0 x^r + \sum_{k=1}^{\infty} \left(\left(4(k+r)\,(k+r-1) + 1 \right)\, a_k - \left(4(k+r-1) + 2 \right)\, a_{k-1} \right)\, x^{k+r}. \end{aligned} \tag{4.3.7}$$

To have a solution we must first have $\left(4r(r-1)+1\right)\, a_0 = 0$. Supposing that $a_0 \neq 0$ we obtain

$$4r(r-1)+1 = 0.$$

This equation is called the *indicial equation*. This particular indicial equation has a double root at $r = \frac{1}{2}$.

OK, so we know what r has to be. That knowledge we obtained simply by looking at the coefficient of x^r. All other coefficients of x^{k+r} also have to be zero so

$$\left(4(k+r)\,(k+r-1)+1\right)\, a_k - \left(4(k+r-1)+2\right)\, a_{k-1} = 0.$$

If we plug in $r = \frac{1}{2}$ and solve for a_k we get

$$a_k = \frac{4(k+\frac{1}{2}-1)+2}{4(k+\frac{1}{2})\,(k+\frac{1}{2}-1)+1}\, a_{k-1} = \frac{1}{k}\, a_{k-1}.$$

Let us set $a_0 = 1$. Then

$$a_1 = \frac{1}{1} a_0 = 1, \qquad a_2 = \frac{1}{2} a_1 = \frac{1}{2}, \qquad a_3 = \frac{1}{3} a_2 = \frac{1}{3 \cdot 2}, \qquad a_4 = \frac{1}{4} a_3 = \frac{1}{4 \cdot 3 \cdot 2}, \qquad \ldots$$

Extrapolating, we notice that

$$a_k = \frac{1}{k(k-1)(k-2)\cdots 3 \cdot 2} = \frac{1}{k!}.$$

In other words,

$$y = \sum_{k=0}^{\infty} a_k x^{k+r} = \sum_{k=0}^{\infty} \frac{1}{k!} x^{k+1/2} = x^{1/2} \sum_{k=0}^{\infty} \frac{1}{k!} x^k = x^{1/2} e^x.$$

That was lucky! In general, we will not be able to write the series in terms of elementary functions. We have one solution, let us call it $y_1 = x^{1/2} e^x$. But what about a second solution? If we want a general solution, we need two linearly independent solutions. Picking a_0 to be a different constant only gets us a constant multiple of y_1, and we do not have any other r to try; we only have one solution to the indicial equation. Well, there are powers of x floating around and we are taking derivatives, perhaps the logarithm (the antiderivative of x^{-1}) is around as well. It turns out we want to try for another solution of the form

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+r} + (\ln x) y_1,$$

which in our case is

$$y_2 = \sum_{k=0}^{\infty} b_k x^{k+1/2} + (\ln x) x^{1/2} e^x.$$

We now differentiate this equation, substitute into the differential equation and solve for b_k. A long computation ensues and we obtain some recursion relation for b_k. The reader can (and should) try this to obtain for example the first three terms

$$b_1 = b_0 - 1, \qquad b_2 = \frac{2b_1 - 1}{4}, \qquad b_3 = \frac{6b_2 - 1}{18}, \qquad \ldots$$

We then fix b_0 and obtain a solution y_2. Then we write the general solution as $y = Ay_1 + By_2$.

Method of Frobenius

Before giving the general method, let us clarify when the method applies. Let

$$p(x)y'' + q(x)y' + r(x)y = 0$$

be an ODE. As before, if $p(x_0) = 0$, then x_0 is a singular point. If, furthermore, the limits

$$\lim_{x \to x_0} (x - x_0) \frac{q(x)}{p(x)} \qquad \text{and} \qquad \lim_{x \to x_0} (x - x_0)^2 \frac{r(x)}{p(x)}$$

both exist and are finite, then we say that x_0 is a *regular singular point*.

✓ Example 4.3.3

Determine if $x = 0$ is a regular singular point of the second-degree ODE

$$x^2 y'' + x(1+x)y' + (\pi + x^2)y = 0.$$

Solution

Write

$$\begin{aligned} \lim_{x \to 0} x \frac{q(x)}{p(x)} &= \lim_{x \to 0} x \frac{x(1+x)}{x^2} = \lim_{x \to 0} (1+x) = 1, \\ \lim_{x \to 0} x^2 \frac{r(x)}{p(x)} &= \lim_{x \to 0} x^2 \frac{(\pi + x^2)}{x^2} = \lim_{x \to 0} (\pi + x^2) = \pi \end{aligned} \tag{4.3.8}$$

So $x = 0$ is a regular singular point.

✓ Example 4.3.4

Determine if $x = 0$ is a regular singular point of the second-degree ODE

$$x^2 y'' + (1+x)y' + (\pi + x^2)y = 0,$$

Solution

$$\lim_{x \to 0} x \frac{q(x)}{p(x)} = \lim_{x \to 0} x \frac{(1+x)}{x^2} = \lim_{x \to 0} \frac{1+x}{x} = \text{DNE}.$$

Here DNE stands for *does not exist*. The point 0 is a singular point, but not a regular singular point.

Let us now discuss the general *Method of Frobenius*. Let us only consider the method at the point $x = 0$ for simplicity. The main idea is the following theorem.

Theorem 4.3.1

Method of Frobenius

Suppose that

$$p(x)y'' + q(x)y' + r(x)y = 0 \tag{4.3.9}$$

has a regular singular point at $x = 0$, then there exists at least one solution of the form

$$y = x^r \sum_{k=0}^{\infty} a_k x^k.$$

A solution of this form is called a *Frobenius-type solution*.

The method usually breaks down like this.

i. We seek a Frobenius-type solution of the form

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}.$$

We plug this y into equation $p(x)y'' + q(x)y' + r(x)y = 0$. We collect terms and write everything as a single series.

ii. The obtained series must be zero. Setting the first coefficient (usually the coefficient of x^r) in the series to zero we obtain the *indicial equation*, which is a quadratic polynomial in r.

iii. If the indicial equation has two real roots r_1 and r_2 such that $r_1 - r_2$ is not an integer, then we have two linearly independent Frobenius-type solutions. Using the first root, we plug in

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and we solve for all a_k to obtain the first solution. Then using the second root, we plug in

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k,$$

and solve for all b_k to obtain the second solution.

iv. If the indicial equation has a doubled root r, then there we find one solution

$$y_1 = x^r \sum_{k=0}^{\infty} a_k x^k,$$

and then we obtain a new solution by plugging

$$y_2 = x^r \sum_{k=0}^{\infty} b_k x^k + (\ln x) y_1,$$

into equation $p(x)y'' + q(x)y' + r(x)y = 0$ and solving for the constants b_k.

v. If the indicial equation has two real roots such that $r_1 - r_2$ is an integer, then one solution is

$$y_1 = x^{r_1} \sum_{k=0}^{\infty} a_k x^k,$$

and the second linearly independent solution is of the form

$$y_2 = x^{r_2} \sum_{k=0}^{\infty} b_k x^k + C(\ln x) y_1,$$

where we plug y_2 into equation $p(x)y'' + q(x)y' + r(x)y = 0$ and solve for the constants b_k and C.

vi. Finally, if the indicial equation has complex roots, then solving for a_k in the solution

$$y = x^{r_1} \sum_{k=0}^{\infty} a_k x^k$$

results in a complex-valued function with all the a_k as complex numbers. We obtain our two linearly independent solutions by taking the real and imaginary parts of y.

The main idea is to find at least one Frobenius-type solution. If we are lucky and find two, we are done. If we only get one, we either use the ideas above or even a different method such as reduction of order to obtain a second solution.

Bessel Functions

An important class of functions that arises commonly in physics are the *Bessel functions*. For example, these functions appear when solving the wave equation in two and three dimensions. First we have *Bessel's equation* of order p:

$$x^2y'' + xy' + \left(x^2 - p^2\right) y = 0.$$

We allow p to be any number, not just an integer, although integers and multiples of $\frac{1}{2}$ are most important in applications. When we plug

$$y = \sum_{k=0}^{\infty} a_k x^{k+r}$$

into Bessel's equation of order p we obtain the indicial equation

$$r(r-1) + r - p^2 = (r-p)(r+p) = 0.$$

Therefore we obtain two roots $r_1 = p$ and $r_2 = -p$. If p is not an integer following the method of Frobenius and setting $a_0 = 1$, we obtain linearly independent solutions of the form

$$\begin{aligned} y_1 &= x^p \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k+p)(k-1+p)\cdots(2+p)(1+p)}, \\ y_2 &= x^{-p} \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{2^{2k} k!(k-p)(k-1-p)\cdots(2-p)(1-p)}. \end{aligned} \tag{4.3.10}$$

Exercise 4.3.1

a. Verify that the indicial equation of Bessel's equation of order p is $(r-p)(r+p) = 0$.
b. Suppose that p is not an integer. Carry out the computation to obtain the solutions y_1 and y_2 above.

Bessel functions will be convenient constant multiples of y_1 and y_2. First we must define the *gamma function*

$$\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt.$$

Notice that $\Gamma(1) = 1$. The gamma function also has a wonderful property

$$\Gamma(x+1) = x\Gamma(x).$$

From this property, one can show that $\Gamma(n) = (n-1)!$ when n is an integer, so the gamma function is a continuous version of the factorial. We compute:

$$\begin{aligned} \Gamma(k+p+1) &= (k+p)(k-1+p)\cdots(2+p)(1+p)\Gamma(1+p), \\ \Gamma(k-p+1) &= (k-p)(k-1-p)\cdots(2-p)(1-p)\Gamma(1-p). \end{aligned} \tag{4.3.11}$$

Exercise 4.3.2

Verify the above identities using $\Gamma(x+1) = x\Gamma(x)$.

We define the *Bessel functions of the first kind* of order p and $-p$ as

$$\begin{aligned} J_p(x) &= \frac{1}{2^p\Gamma(1+p)} y_1 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k+p+1)} \left(\frac{x}{2}\right)^{2k+p}, \\ J_{-p}(x) &= \frac{1}{2^{-p}\Gamma(1-p)} y_2 = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!\Gamma(k-p+1)} \left(\frac{x}{2}\right)^{2k-p}. \end{aligned} \tag{4.3.12}$$

As these are constant multiples of the solutions we found above, these are both solutions to Bessel's equation of order p. The constants are picked for convenience.

When p is not an integer, J_p and J_{-p} are linearly independent. When n is an integer we obtain

$$J_n(x)=\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+n)!}\left(\frac{x}{2}\right)^{2k+n}.$$

In this case it turns out that

$$J_n(x)=(-1)^nJ_{-n}(x),$$

and so we do not obtain a second linearly independent solution. The other solution is the so-called *Bessel function of second kind*. These make sense only for integer orders n and are defined as limits of linear combinations of $J_p(x)$ and $J_{-p}(x)$ as p approaches n in the following way:

$$Y_n(x)=\lim_{p\to n}\frac{\cos(p\pi)J_p(x)-J_{-p}(x)}{\sin(p\pi)}.$$

As each linear combination of $J_p(x)$ and $J_{-p}(x)$ is a solution to Bessel's equation of order p, then as we take the limit as p goes to n, $Y_n(x)$ is a solution to Bessel's equation of order n. It also turns out that $Y_n(x)$ and $J_n(x)$ are linearly independent. Therefore when n is an integer, we have the general solution to Bessel's equation of order n

$$y=AJ_n(x)+BY_n(x),$$

for arbitrary constants A and B. Note that $Y_n(x)$ goes to negative infinity at $x=0$. Many mathematical software packages have these functions $J_n(x)$ and $Y_n(x)$ defined, so they can be used just like say $\sin(x)$ and $\cos(x)$. In fact, they have some similar properties. For example, $-J_1(x)$ is a derivative of $J_0(x)$, and in general the derivative of $J_n(x)$ can be written as a linear combination of $J_{n-1}(x)$ and $J_{n+1}(x)$. Furthermore, these functions oscillate, although they are not periodic. See Figure $4.3.1$ for graphs of Bessel functions.

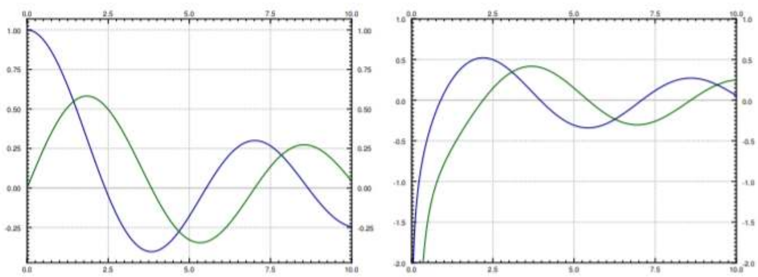


Figure 4.3.1: The image illustrates the plot of the Bessel functions $J_0(x)$ and $J_1(x)$ in the first graph and $Y_0(x)$ and $Y_1(x)$ in the second graph. (CC BY-SA 4.0; Jiří Lebl via Differential Equation for Engineers)

Example 4.3.5: Using Bessel Functions to Solve a ODE

Other equations can sometimes be solved in terms of the Bessel functions. For example, given a positive constant λ,

$$xy''+y'+\lambda^2xy=0,$$

can be changed to $x^2y''+xy'+\lambda^2x^2y=0$. Then changing variables $t=\lambda x$ we obtain via chain rule the equation in y and t:

$$t^2y''+ty'+t^2y=0,$$

which can be recognized as Bessel's equation of order 0. Therefore the general solution is $y(t)=AJ_0(t)+BY_0(t)$, or in terms of x:

$$y=AJ_0(\lambda x)+BY_0(\lambda x).$$

This equation comes up for example when finding fundamental modes of vibration of a circular drum, but we digress.

Footnotes

[1] Named after the German mathematician Ferdinand Georg Frobenius (1849 – 1917).

[2] See Joseph L. Neuringera, The Frobenius method for complex roots of the indicial equation, International Journal of Mathematical Education in Science and Technology, Volume 9, Issue 1, 1978, 71–77.

[3] Named after the German astronomer and mathematician Friedrich Wilhelm Bessel (1784 – 1846).

4.3E: Exercises for Section 4.3

Exercises 1-10

Solve the following exercises by finding the general solution or a specific Frobenius-type solution as required. Classify singular points where necessary, and use appropriate series expansions or solution methods.

Exercise $4.3E.1$

Find a particular (Frobenius-type) solution of $x^2y'' + xy' + (1+x)y = 0$.

Exercise $4.3E.2$

Find a particular (Frobenius-type) solution of $xy'' - y = 0$.

Exercise $4.3E.3$

Find a particular (Frobenius-type) solution of $y'' + \frac{1}{x}y' - xy = 0$.

Exercise $4.3E.4$

Find the general solution of $2xy'' + y' - x^2y = 0$.

Exercise $4.3E.5$

Find the general solution of $x^2y'' - xy' - y = 0$.

Exercise $4.3E.6$

In the following equations classify the point $x = 0$ as ordinary, regular singular, or singular but not regular singular.

a. $x^2(1+x^2)y'' + xy = 0$
b. $x^2y'' + y' + y = 0$
c. $xy'' + x^3y' + y = 0$
d. $xy'' + xy' - e^xy = 0$
e. $x^2y'' + x^2y' + x^2y = 0$

Exercise $4.3E.7$

In the following equations classify the point $x = 0$ as ordinary, regular singular, or singular but not regular singular.

a. $y'' + y = 0$
b. $x^3y'' + (1+x)y = 0$
c. $xy'' + x^5y' + y = 0$
d. $\sin(x)y'' - y = 0$
e. $\cos(x)y'' - \sin(x)y = 0$

Answer

a. ordinary,
b. singular but not regular singular,
c. regular singular,
d. regular singular,
e. ordinary.

Exercise $4.3E.8$

Find the general solution of $x^2y'' - y = 0$.

Answer

$$y = Ax^{\frac{1+\sqrt{5}}{2}} + Bx^{\frac{1-\sqrt{5}}{2}}$$

Exercise $4.3E.9$

Find a particular solution of $x^2y'' + (x - \frac{3}{4})y = 0$.

Answer

$y = x^{3/2}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!(k+2)!}x^k$ (Note that for convenience we did not pick $a_0 = 1$.)

Exercise $4.3E.10$

Find the general solution of $x^2y'' - xy' + y = 0$.

Answer

$$y = Ax + Bx\ln(x)$$

CHAPTER OVERVIEW

5: The Laplace Transform

The Laplace transform can also be used to solve differential equations and reduces a linear differential equation to an algebraic equation, which can then be solved by the formal rules of algebra.

Contributors and Attributions

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5.1: The Laplace Transform

Learning Objectives

- Understand the basics of Laplace transforms and their existence and uniqueness.
- Identify common functions and their Laplace transforms.
- Understand properties and theorems of Laplace transform.
- Find inverse Laplace transform.

Transform

In this chapter, we will discuss the Laplace transform. The Laplace transform turns out to be a very efficient method to solve certain ODE problems. In particular, the transform can take a differential equation and turn it into an algebraic equation. If the algebraic equation can be solved, applying the inverse transform gives us our desired solution. The Laplace transform also has applications in the analysis of electrical circuits, NMR spectroscopy, signal processing, and elsewhere. Finally, understanding the Laplace transform will also help with understanding the related Fourier transform, which, however, requires more understanding of complex numbers.

The Laplace transform also gives a lot of insight into the nature of the equations we are dealing with. It can be seen as converting between the time and the frequency domain. For example, take the standard equation

$$mx''(t) = cx'(t) + kx(t) = f(t).$$

We can think of t as time and $f(t)$ as incoming signal. The Laplace transform will convert the equation from a differential equation in time to an algebraic (no derivatives) equation, where the new independent variable s is the frequency.

We can think of the *Laplace transform* as a black box that eats functions and spits out functions in a new variable. We write $\mathcal{L}\{f(t)\} = F(s)$ for the Laplace transform of $f(t)$. It is common to write lower case letters for functions in the time domain and upper case letters for functions in the frequency domain. We use the same letter to denote that one function is the Laplace transform of the other. For example $F(s)$ is the Laplace transform of $f(t)$. Let us define the transform.

$$\mathcal{L}\{f(t)\} = F(s) \stackrel{\text{def}}{=} \int_0^\infty e^{-st} f(t)\, dt.$$

We note that we are only considering $t \geq 0$ in the transform. Of course, if we think of t as time there is no problem, we are generally interested in finding out what will happen in the future (Laplace transform is one place where it is safe to ignore the past)

Let us compute some simple transforms. Example $5.1.2$ and Example $5.1.3$ shows how to find the Laplace Transform of a constant function.

Example $5.1.1$

Evaluate the Laplace transform of $f(t) = e^{-at}$.

Solution

$$\mathcal{L}\{e^{-at}\} = \int_0^\infty e^{-st} e^{-at} dt = \int_0^\infty e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)}\right]_{t=0}^\infty = \frac{1}{s+a}.$$

The limit only exists if $s+a > 0$. So $\mathcal{L}\{e^{-at}\}$ is only defined for $s+a > 0$.

Example $5.1.2$

Evaluate the Laplace transformation of $f(t) = t$

Solution

Using integration by parts

$$\begin{aligned}\mathcal{L}\{t\} &= \int_0^\infty e^{-st}t\,dt \\ &= \left[\frac{-te^{-st}}{s}\right]_{t=0}^\infty + \frac{1}{s}\int_0^\infty e^{-st}\,dt \\ &= 0 + \frac{1}{s}\left[\frac{e^{-st}}{-s}\right]_{t=0}^\infty \\ &= \frac{1}{s^2}.\end{aligned}$$

Again, the limit only exists if $s > 0$.

Example 5.1.3

A common function is the unit step function, which is sometimes called the *Heaviside function*[2]. This function is generally given as

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

Evaluate the Laplace transform of $u(t-a)$, where $a \geq 0$ is some constant. That is, the function that is 0 for $t < a$ and 1 for $t \geq a$.

Solution

$$\mathcal{L}\{u(t-a)\} = \int_0^\infty e^{-st}u(t-a)dt = \int_a^\infty e^{-st}dt = \left[\frac{e^{-st}}{-s}\right]_{t=a}^\infty = \frac{e^{-as}}{s},$$

where of course $s > 0$ (and $a \geq 0$ as we said before).

By applying similar procedures we can compute the transforms of many elementary functions. Many basic transforms are listed in Table 5.1.1.

Table 5.1.1: Common Laplace Transforms of Basic Functions (C, ω, and a are constants).

$f(t)$	$\{f(t)\}$
C	$\frac{C}{s}$
t	$\frac{1}{s^2}$
t^2	$\frac{2}{s^3}$
t^3	$\frac{6}{s^4}$
t^n	$\frac{n!}{s^{n+1}}$
e^{-at}	$\frac{1}{s+a}$
$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
$\sinh(\omega t)$	$\frac{\omega}{s^2-\omega^2}$
$\cosh(\omega t)$	$\frac{s}{s^2-\omega^2}$

$f(t)$	$\{f(t)\}$
$u(t-a)$	$\frac{e^{-as}}{s}$

Since the transform is defined by an integral. We can use the linearity properties of the integral. For example, suppose C is a constant, then

$$\mathcal{L}\{f(t)\} = \int_o^\infty e^{-st} C f(t)\, dt = C \int_0^\infty e^{-st} f(t)\, dt = C\mathcal{L}\{f(t)\}.$$

So we can "pull out" a constant out of the transform. Similarly we have linearity. Since linearity is very important we state it as a theorem.

Theorem 5.1.1

Linearity of the Laplace Transform

Suppose that A, B, and C are constants, then

$$\mathcal{L}\{Af(t) + Bg(t)\} = A\mathcal{L}\{f(t)\} + B\mathcal{L}\{g(t)\}$$

and in particular

$$\mathcal{L}\{Cf(t)\} = C\mathcal{L}\{f(t)\}.$$

These rules together with Table 5.1.1 make it easy to find the Laplace transform of a whole lot of functions already. But be careful. It is a common mistake to think that the Laplace transform of a product is the product of the transforms. In general

$$\mathcal{L}\{f(t)g(t)\} \neq \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

It must also be noted that not all functions have a Laplace transform. For example, the function $1/t$ does not have a Laplace transform as the integral diverges for all s. Similarly, $\tan t$ or e^{t^2} do not have Laplace transforms.

Existence and Uniqueness

Let us consider when does the Laplace transform exist in more detail. First let us consider functions of exponential order. The function $f(t)$ is of *exponential order* as t goes to infinity if

$$|f(t)| \leq Me^{ct},$$

for some constants M and c, for sufficiently large t (say for all $t > t_o$ for some t_o). The simplest way to check this condition is to try and compute

$$\lim_{t\to\infty} \frac{f(t)}{e^{ct}}$$

If the limit exists and is finite (usually zero), then $f(t)$ is of exponential order.

For an exponential order function, we have the existence and uniqueness of the Laplace transform.

Theorem 5.1.2

Existence

Let $f(t)$ be continuous and of exponential order for a certain constant c. Then $F(s) = \mathcal{L}\{f(t)\}$ is defined for all $s > c$.

The existence is not difficult to see. Let $f(t)$ be of exponential order, that is $|f(t)| \leq Me^{ct}$ for all $t > 0$ (for simplicity $t_0 = 0$). Let $s > c$, or in other words $(c - s) < 0$. By the comparison theorem from calculus, the improper integral defining $\mathcal{L}\{f(t)\}$ exists if the following integral exists

$$\int_0^\infty e^{-st}(Me^{ct})dt = M\int_0^\infty e^{(c-s)t}dt = M\left[\frac{e^{(c-s)t}}{c-s}\right]_{t=0}^\infty = \frac{M}{c-s}.$$

The transform also exists for some other functions that are not of exponential order, but that will not be relevant to us. Before dealing with uniqueness, let us note that for exponential order functions we obtain that their Laplace transform decays at infinity:

$$\lim_{s\to\infty} F(s) = 0$$

Theorem 5.1.3

Uniqueness

Let $f(t)$ and $g(t)$ be continuous and of exponential order. Suppose that there exists a constant C, such that $F(s) = G(s)$ for all $s > C$. Then $f(t) = g(t)$ for all $t \geq 0$.

Both theorems hold for piecewise continuous functions as well. Recall that piecewise continuous means that the function is continuous except perhaps at a discrete set of points where it has jump discontinuities like the Heaviside function. Uniqueness however does not "see" values at the discontinuities. So we can only conclude that $F(s) = G(s)$ outside of discontinuities. For example, the unit step function is sometimes defined using $u(0) = 1/2$. This new step function, however, has the exact same Laplace transform as the one we defined earlier where $u(0) = 1$.

Inverse Transform

As we said, the Laplace transform will allow us to convert a differential equation into an algebraic equation. Once we solve the algebraic equation in the frequency domain we will want to get back to the time domain, as that is what we are interested in. If we have a function $F(s)$, to be able to find $f(t)$ such that $\mathcal{L}\{f(t)\} = F(s)$, we need to first know if such a function is unique. It turns out we are in luck by Theorem $5.1.3$. So we can without fear make the following definition.

If $F(s) = \mathcal{L}\{f(t)\}$ for some function $f(t)$. We define the *inverse Laplace transform* as

$$\mathcal{L}^{-1}\{F(s)\} \stackrel{\text{def}}{=} f(t)$$

There is an integral formula for the inverse, but it is not as simple as the transform itself—it requires complex numbers and path integrals. For us it will suffice to compute the inverse using Table $5.1.1$.

Example $5.1.4$

Find the inverse Laplace transform of $F(s) = \dfrac{1}{s+1}$

Solution

We look at the table to find

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t}$$

As the Laplace transform is linear, the inverse Laplace transform is also linear. That is,

$$\mathcal{L}^{-1}\{AF(s) + BG(s)\} = A\mathcal{L}^{-1}\{F(s)\} + B\mathcal{L}^{-1}\{G(s)\}$$

Of course, we also have $\mathcal{L}^{-1}\{AF(s) = A\mathcal{L}^{-1}\{F(s)\}$. Let us demonstrate how linearity can be used.

Example $5.1.5$

Find the inverse Laplace transform of

$$F(s) = \frac{s^2+s+1}{s^3+s}.$$

Solution

First we use the *method of partial fractions* to write F in a form where we can use Table $5.1.1$. We factor the denominator as $s(s^2+1)$ and write

$$\frac{s^2+s+1}{s^3+s} = \frac{A}{s} + \frac{Bs+C}{s^2+1}$$

Putting the right hand side over a common denominator and equating the numerators we get $A(s^2+1)+s(Bs+C)=s^2+s+1$. Expanding and equating coefficients we obtain $A+B=1$, $C=1$, $A=1$ and thus $B=0$. In other words,

$$F(s) = \frac{s^2+s+1}{s^3+s} = \frac{1}{s} + \frac{1}{s^2+1}$$

By linearity of the inverse Laplace transform we get

$$\mathcal{L}^{-1}\left\{\frac{s^2+s+1}{s^3+s}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\} = 1+\sin t.$$

Shifting Property of Laplace Transforms

Another useful property is the so-called *shifting property* or the *first shifting property*

$$\mathcal{L}\{e^{-at}f(t)\} = F(s+a)$$

where $F(s)$ is the Laplace transform of $f(t)$ and a is a constant.

The shifting property can be used, for example, when the denominator is a more complicated quadratic that may come up in the method of partial fractions. We complete the square and write such quadratics as $(s+a)^2+b$ and then use the shifting property.

✓ Example $5.1.6$

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+4s+8}\right\}.$$

Solution

First we complete the square to make the denominator $(s+2)^2+4$. Next we find

$$\mathcal{L}\left\{\frac{1}{s^2+4}\right\} = \frac{1}{2}\sin(2t).$$

Putting it all together with the shifting property, we find

$$\mathcal{L}\left\{\frac{1}{s^2+4s+8}\right\} = \mathcal{L}\left\{\frac{1}{(s+2)^2+4}\right\} = \frac{1}{2}e^{-2t}\sin(2t).$$

In general, we want to be able to apply the Laplace transform to *rational functions*, that is functions of the form

$$\frac{F(s)}{G(s)}$$

where $F(s)$ and $G(s)$ are polynomials. Since normally, for the functions that we are considering, the Laplace transform goes to zero as $s\to\infty$, it is not hard to see that the degree of $F(s)$ must be smaller than that of $G(s)$. Such rational functions are called *proper rational functions* and we can always apply the method of partial fractions. Of course this means we need to be able to factor the denominator into linear and quadratic terms, which involves finding the roots of the denominator,

Footnotes

[1] Just like the Laplace equation and the Laplacian, the Laplace transform is also named after Pierre-Simon, Marquis De Laplace (1749 – 1827).

[2] The function is named after the English mathematician, engineer, and physicist Oliver Heaviside (1850–1925). Only by coincidence is the function “heavy” on “one side.”

5.1E: Exercises for Section 5.1

Exercises 1-15

Find the Laplace Transform or inverse Laplace Transform of the functions as instructed

Exercise $5.1E.1$

Find the Laplace transform of $3+t^5+\sin(\pi t)$.

Exercise $5.1E.2$

Find the Laplace transform of $a+bt+ct^2$ for some constants a, b, and c.

Answer

$\frac{a}{s}+\frac{b}{s^2}+\frac{2c}{s^3}$

Exercise $5.1E.3$

Find the Laplace transform of $A\cos(\omega t)+B\sin(\omega t)$.

Exercise $5.1E.4$

Find the Laplace transform of $\cos^2(\omega t)$.

Answer

$\frac{1}{2s}+\frac{s}{2(s^2+4\omega^2)}$

Exercise $5.1E.5$

Find the inverse Laplace transform of $\dfrac{4}{s^2-9}$.

Exercise $5.1E.6$

Find the inverse Laplace transform of $\dfrac{2s}{s^2-1}$.

Answer

$2cosh(t)$

Exercise $5.1E.7$

Find the inverse Laplace transform of $\dfrac{1}{(s-1)^2(s+1)}$.

Exercise $5.1E.8$

Find the Laplace transform of $f(t)=\begin{cases} t & \text{if } t\geq 1, \\ 0 & \text{if } t<1. \end{cases}$

Answer

$e^{-s}\left(\frac{1}{s}+\frac{1}{s^2}\right)$

Exercise $5.1E.9$

Find the inverse Laplace transform of $\dfrac{s}{(s^2+s+2)(s+4)}$.

Exercise $5.1E.10$

Find the Laplace transform of $\sin(\omega(t-a))$.

Answer

$e^{-as}\frac{\omega}{s^2+\omega^2}$

Exercise $5.1E.11$

Find the Laplace transform of $t\sin(\omega t)$. Hint: Several integrations by parts.

Exercise $5.1E.12$

Find the Laplace transform of $4(t+1)^2$.

Answer

$\frac{8}{s^3}+\frac{8}{s^2}+\frac{4}{s}$

Exercise $5.1E.13$

Find the inverse Laplace transform of $\dfrac{8}{s^3(s+2)}$.

Answer

$2t^2-2t+1-e^{-2t}$

Exercise $5.1E.14$

Find the Laplace transform of te^{-t} (Hint: integrate by parts).

Answer

$\frac{1}{(s+1)^2}$

Exercise $5.1E.15$

Find the Laplace transform of $\sin(t)e^{-t}$ (Hint: integrate by parts).

Answer

$\frac{1}{s^2+2s+2}$

- **6.E: The Laplace Transform (Exercises)** has no license indicated.

5.2: Transforms of Derivatives and ODEs

Learning Objectives

- Derive and understand the standard formulas for transforms of first, second, and higher-order derivatives.
- Understand how linearity properties of transforms simplify the handling of derivatives and complex differential equations.

Transforms of Derivatives

Let us see how the Laplace transform is used for differential equations. First let us try to find the Laplace transform of a function that is a derivative. Suppose $g(t)$ is a differentiable function of exponential order, that is, $|g(t)| \leq Me^{ct}$ for some M and c. So $\mathcal{L}\{g(t)\}$ exists, and what is more, $\lim_{t\to\infty} e^{-st} g(t) = 0$ when $s > c$. Then

$$\mathcal{L}\{g'(t)\} = \int_0^\infty e^{-st} g'(t) dt = \left[e^{-st} g(t)\right]_{t=0}^{\infty} - \int_0^\infty (-s) e^{-st} g(t) dt = -g(0) + s\mathcal{L}\{g(t)\}.$$

We repeat this procedure for higher derivatives. The results are listed in Table $5.2.1$. The procedure also works for piecewise smooth functions, that is functions that are piecewise continuous with a piecewise continuous derivative. The fact that the function is of exponential order is used to show that the limits appearing above exist. We will not worry much about this fact.

Table $5.2.1$: Laplace Transforms of Derivatives

$f(t)$	$\mathcal{L}\{f(t)\} = F(s)$
$g'(t)$	$sG(s) - g(0)$
$g''(t)$	$s^2G(s) - sg(0) - g'(0)$
$g'''(t)$	$s^3G(s) - s^2g(0) - sg'(0) - g''(0)$

Solving ODEs with the Laplace Transform

Notice that the Laplace transform turns differentiation into multiplication by s. Let us see how to apply this fact to differential equations.

Example $5.2.1$

Solve the differential equation

$$x''(t) + x(t) = \cos(2t), \qquad x(0) = 0, \qquad x'(0) = 1.$$

Solution

We will take the Laplace transform of both sides. By $X(s)$ we will, as usual, denote the Laplace transform of $x(t)$.

$$\begin{aligned} \mathcal{L}\{x''(t) + x(t)\} &= \mathcal{L}\{\cos(2t)\}, \\ s^2X(s) - sx(0) + x'(0) + X(s) &= \frac{s}{s^2+4}. \end{aligned} \tag{5.2.1}$$

We plug in the initial conditions now—this makes the computations more streamlined—to obtain

$$s^2X(s) - 1 + X(s) = \frac{s}{s^2+4}.$$

We solve for $X(s)$,

$$X(s) = \frac{s}{(s^2+1)(s^2+4)} + \frac{1}{s^2+1}.$$

We use partial fractions (exercise) to write

$$X(s) = \frac{1}{3}\frac{s}{s^2+1} - \frac{1}{3}\frac{s}{s^2+4} + \frac{1}{s^2+1}.$$

Now take the inverse Laplace transform to obtain

$$x(t) = \frac{1}{3}\cos(t) - \frac{1}{3}\cos(2t) + \sin(t).$$

The procedure for linear constant coefficient equations is as follows. We take an ordinary differential equation in the time variable t. We apply the Laplace transform to transform the equation into an algebraic (non differential) equation in the frequency domain. All the $x(t)$, $x'(t)$, $x''(t)$, and so on, will be converted to $X(s)$, $sX(s) - x(0)$, $s^2X(s) - sx(0) - x'(0)$, and so on. We solve the equation for $X(s)$. Then taking the inverse transform, if possible, we find $x(t)$.

It should be noted that since not every function has a Laplace transform, not every equation can be solved in this manner. Also if the equation is not a linear constant coefficient ODE, then by applying the Laplace transform we may not obtain an algebraic equation.

Using the Heaviside Function

Before we move on to more general equations than those we could solve before, we want to consider the Heaviside function. See Figure $5.2.1$ for the graph.

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

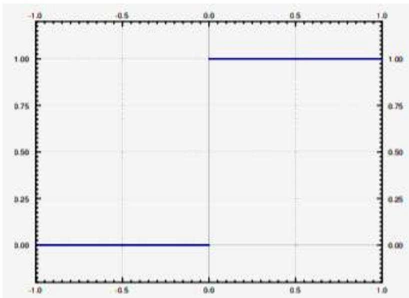


Figure $5.2.1$: Plot of the Heaviside (unit step) function $u(t)$. x-axis to the left of the origin and the horizontal line y = 1 to the right. (CC BY-SA 4.0; Jiří Lebl via Differential equations for engineers

This function is useful for putting together functions, or cutting functions off. Most commonly it is used as $u(t-a)$ for some constant a. This just shifts the graph to the right by a. That is, it is a function that is 0 when $t < a$ and 1 when $t \geq a$. Suppose for example that $f(t)$ is a "signal" and you started receiving the signal $\sin t$ at time $t = \pi$. The function $f(t)$ should then be defined as

$$f(t) = \begin{cases} 0 & \text{if } t < \pi, \\ \sin t & \text{if } t \geq \pi. \end{cases}$$

Using the Heaviside function, $f(t)$ can be written as

$$f(t) = u(t-\pi)\sin t$$

Similarly the step function that is 1 on the interval $[1, 2)$ and zero everywhere else can be written as

$$u(t-1) - u(t-2).$$

The Heaviside function is useful to define functions defined piecewise. If you want to define $f(t)$ such that $f(t) = t$ when t is in $[0, 1]$, $f(t) = -t + 2$ when t is in $[1, 2)$ and $f(t) = 0$ otherwise, you can use the expression

$$f(t) = t\left(u(t) - u(t-1)\right) + (-t+2)\left(u(t-1) - u(t-2)\right).$$

Hence it is useful to know how the Heaviside function interacts with the Laplace transform. We have already seen that

$$\mathcal{L}\{u(t-a)\} = \frac{e^{-as}}{s}.$$

Shifting Property

This can be generalized into a shifting property or second shifting property.

$$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}. \tag{5.2.2}$$

Example 5.2.2

Suppose that the forcing function is not periodic. For example, suppose that we had a mass-spring system

$$x''(t) + x(t) = f(t), \qquad x(0) = 0, \qquad x'(0) = 0,$$

where $f(t) = 1$ if $1 \le t < 5$ and zero otherwise. We could imagine a mass-spring system, where a rocket is fired for 4 seconds starting at $t = 1$. Or perhaps an RLC circuit, where the voltage is raised at a constant rate for 4 seconds starting at $t = 1$, and then held steady again starting at $t = 5$. Solve the differential equation.

Solution

We can write $f(t) = u(t-1) - u(t-5)$. We transform the equation and we plug in the initial conditions as before to obtain

$$s^2X(s) + X(s) = \frac{e^{-s}}{s} - \frac{e^{-5s}}{s}.$$

We solve for $X(s)$ to obtain

$$X(s) = \frac{e^{-s}}{s(s^2+1)} - \frac{e^{-5s}}{s(s^2+1)}.$$

We leave it as an exercise to the reader to show that

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\} = 1 - \cos t.$$

In other words $\mathcal{L}\{1-\cos t\} = \frac{1}{s(s^2+1)}$. So using Equation (5.2.2) we find

$$\mathcal{L}^{-1}\left\{\frac{e^{-s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-s}\mathcal{L}\{1-\cos t\}\} = (1-\cos(t-1))u(t-1).$$

Similarly

$$\mathcal{L}^{-1}\left\{\frac{e^{-5s}}{s(s^2+1)}\right\} = \mathcal{L}^{-1}\{e^{-5s}\mathcal{L}\{1-\cos t\}\} = (1-\cos(t-5))u(t-5).$$

Hence, the solution is

$$x(t) = (1-\cos(t-1))\,u(t-1) - (1-\cos(t-5))\,u(t-5).$$

The plot of this solution is given in Figure 5.2.2.

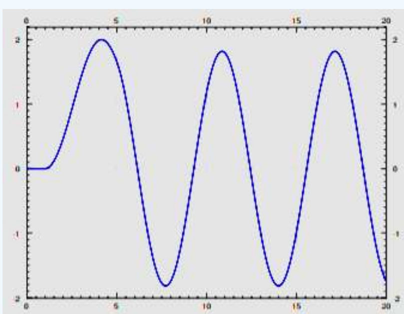


Figure 5.2.2: Plot of $x(t)$ as the solution of the mass-spring system. (CC BY-SA 4.0; Jiří Lebl via Differential equations for engineers)

Transfer Functions

Laplace transform leads to the following useful concept for studying the steady state behavior of a linear system. Suppose we have an equation of the form

$$Lx = f(t),$$

where L is a linear constant coefficient differential operator. Then $f(t)$ is usually thought of as input of the system and $x(t)$ is thought of as the output of the system. For example, for a mass-spring system the input is the forcing function and output is the behavior of the mass. We would like to have an convenient way to study the behavior of the system for different inputs.

Let us suppose that all the initial conditions are zero and take the Laplace transform of the equation, we obtain the equation

$$A(s)X(s) = F(s).$$

Solving for the ratio $\frac{X(s)}{F(s)}$ we obtain the so-called *transfer function* $H(s) = \frac{1}{A(s)}$.

$$H(s) = \frac{X(s)}{F(s)}$$

In other words, $X(s) = H(s)F(s)$. We obtain an algebraic dependence of the output of the system based on the input. We can now easily study the steady state behavior of the system given different inputs by simply multiplying by the transfer function.

Example 5.2.3

Given $x'' + \omega_0^2 x = f(t)$, find the transfer function (assuming the initial conditions are zero).

Solution

First, we take the Laplace transform of the equation.

$$s^2X(s) + \omega_0^2 X(s) = F(s).$$

Now we solve for the transfer function $\frac{X(s)}{F(s)}$.

$$H(s) = \frac{X(s)}{F(s)} = \frac{1}{s^2 + \omega_0^2}.$$

Let us see how to use the transfer function. Suppose we have the constant input $f(t) = 1$. Hence $F(s) = \frac{1}{s}$, and

$$X(s) = H(s)F(s) = \frac{1}{s^2 + \omega_0^2}\frac{1}{s}.$$

Taking the inverse Laplace transform of $X(s)$ we obtain

$$x(t) = \frac{1 - \cos(\omega_0 t)}{\omega_0^2}.$$

Transforms of Integrals

A feature of Laplace transforms is that it is also able to easily deal with integral equations. That is, equations in which integrals rather than derivatives of functions appear. The basic property, which can be proved by applying the definition and doing integration by parts, is

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}F(s).$$

It is sometimes useful (e.g. for computing the inverse transform) to write this as

$$\int_0^t f(\tau)\,d\tau = \mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\}.$$

Example 5.2.4

Evaluate $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+1)}\right\}$

Solution

$$\mathcal{L}^{-1}\left\{\frac{1}{s}\frac{1}{s^2+1}\right\}=\int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s^2+1}\right\}=\int_0^t \sin\tau\,d\tau=1-\cos t.$$

5.2E: Exercises for Section 5.2

Exercises 1-13

Solve the following exercises using Laplace transforms and related techniques for differential equations.

? Exercise $5.2E.1$

Verify the Laplace transforms given in Table 5.2.1 by deriving them using the definition of the Laplace transform.

? Exercise $5.2E.2$

Using the Heaviside function

$$u(t) = \begin{cases} 0 & \text{if } t < 0, \\ 1 & \text{if } t \geq 0. \end{cases}$$

write down the piecewise function that is 0 for $t < 0, t^2$ for t in $[0, 1]$ and t for $t > 1$.

Answer

$f(t) = t^2u(t) + t(1-t)u(t-1)$

? Exercise $5.2E.3$

Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b.$$

where $m > 0, c > 0, k > 0$, and $c^2 - 4km > 0$ (system is overdamped).

? Exercise $5.2E.4$

Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b.$$

where $m > 0, c > 0, k > 0$, and $c^2 - 4km < 0$ (system is underdamped).

Answer

$x(t) = e^{-\frac{c}{2m}t}\left(a\cos\left(\frac{\sqrt{(4km-c^2}}{2m}\right) + \frac{2mb+ca}{\sqrt{(4km-c^2)}}\sin\left(\frac{\sqrt{(4km-c^2}}{2m}\right)t\right).$

? Exercise $5.2E.5$

Using the Laplace transform solve

$$mx'' + cx' + kx = 0, \quad x(0) = a, \quad x'(0) = b.$$

where $m > 0, c > 0, k > 0$, and $c^2 = 4km$ (system is critically damped).

? Exercise $5.2E.6$

Solve $x'' + x = u(t-1)$ for initial conditions $x(0) = 0$ and $x'(0) = 0$.

Answer

$x(t) = u(t-1)(1 - \cos(t-1))$

? Exercise $5.2E.7$

Suppose $\mathcal{L}\{f(t)\} = F(s)$, then show

$$\mathcal{L}\{-tf(t)\} = F'(s).$$

Hint: Differentiate under the integral sign.

Exercise $5.2E.8$

Solve $x''' + x = t^3 u(t-1)$ for initial conditions $x(0) = 1$ and $x'(0) = 0$, $x''(0) = 0$.

Answer

$$x(t) = \frac{2}{3} e^{\frac{1}{2}t} \cos(\frac{\sqrt{3}}{2}t) + \frac{1}{3} e^{-t} + 4(t-1)[-5 + 3(t-1) + 3(t-1)^2 + (t-1)^3 + \frac{2}{3} e^{-(t-1)} + \frac{13}{2} e^{\frac{1}{2}(t-1)} \cos(\frac{\sqrt{3}(t-1)}{2} - 3\sqrt{3} e^{\frac{1}{2}(t-1)} \sin(\frac{\sqrt{3}(t-1)}{2}]$$

Exercise $5.2E.9$

Show the second shifting property: $\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as}\mathcal{L}\{f(t)\}$.

Exercise $5.2E.10$

Define

$$f(t) = \begin{cases} (t-1)^2 & if\ 1 \le t < 2, \\ 3-t & if\ 2 \le t < 3, \\ 0 & otherwise. \end{cases}$$

a. Sketch the graph of $f(t)$.
b. Write down $f(t)$ using the Heaviside function.
c. Solve $x'' + x = f(t), x(0) = 0, x'(0) = 0$ using Laplace transform.

Answer

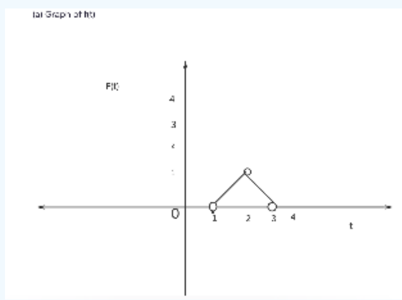
a. 

Figure $5.2E.1$: Graph of $f(t) = (t-1)^2 : 1 \le t < 2, = 3-t : 2 \le t < 3$ and $= 0$ otherwise.

b. $f(t) = (t-1)^2[u(t-1) - u(t-2)] + (3-t)[u(t-2) - u(t-3)]$

c. $x(t) = u_1(t)[-2 + (t-1)^2 + 2\cos(t-1)] + u_2(t)[2 - 3(t-2) - 9t - 2)^2 - 2\cos(t-2) + 3\sin(t-2)] + u_3(t)[t - 3 - \sin(t-3)]$

Exercise $5.2E.11$

Find the transfer function for $mx'' + cx' + kx = f(t)$ (assuming the initial conditions are zero).

Exercise $5.2E.12$

Solve $x'' - x = (t^2 - 1)u(t-1)$ for initial conditions $x(0) = 1, x'(0) = 2$ using the Laplace transform.

Answer

$$x(t) = (2e^{t-1} - t^2 - 1)u(t-1) - \frac{1}{2}e^{-t} + \frac{3}{2}e^t$$

Exercise $5.2E.13$

Find the transfer function for $x' + x = f(t)$ (assuming the initial conditions are zero).

Answer

$$H(s) = \frac{1}{s+1}$$

5.3: Convolution

Learning Objectives

- Understand the key idea of convolution as a process of combining two functions to produce a third function.
- Learn and verify key properties of convolution.
- Use the Convolution Theorem to find the Laplace Transform of the integral.
- Use the inverse form of the Convolution Theorem to find the inverse Laplace Transform.

Convolution

We said that the Laplace transformation of a product is not the product of the transforms. All hope is not lost however. We simply have to use a different type of a "product." Take two functions $f(t)$ and $g(t)$ defined for $t \geq 0$, and define the *convolution* of $f(t)$ and $g(t)$ as

$$(f * g)(t) \overset{\text{def}}{=} \int_0^t f(\tau) g(t-\tau) d\tau. \tag{5.3.1}$$

As you can see, the convolution of two functions of t is another function of t.

Example 5.3.1

Calculate $(f * g)(t)$ when $f(t) = e^t$ and $g(t) = t$ for $t \geq 0$.

Solution

$$(f * g)(t) = \int_0^t e^{\tau}(t-\tau) d\tau = e^t - t - 1.$$

To solve the integral we did one integration by parts.

Example 5.3.2

Calculate $(f * g)(t)$ when $f(t) = \sin(\omega t)$ and $g(t) = \cos(\omega t)$ for $t \geq 0$.

Solution

$$(f * g)(t) = \int_0^t \sin(\omega\tau) \cos(\omega(t-\tau)) d\tau.$$

We apply the identity

$$\cos(\theta)\sin(\psi) = \frac{1}{2}(\sin(\theta+\psi) - \sin(\theta-\psi))$$

Hence,

$$\begin{aligned} (f * g)(t) &= \int_0^t \frac{1}{2}(\sin(\omega t) - \sin(\omega t - 2\omega\tau)) d\tau \\ &= \left[\frac{1}{2}\tau\sin(\omega t) + \frac{1}{4\omega}\cos(2\omega\tau - \omega t)\right]_{\tau=0}^{t} \\ &= \frac{1}{2} t \sin(\omega t). \end{aligned} \tag{5.3.2}$$

The formula holds only for $t \geq 0$. We assumed that f and g are zero (or simply not defined) for negative t.

The convolution has many properties that make it behave like a product. Let c be a constant and f, g, and h be functions then

$$\begin{aligned} f*g &= g*f \\ (cf)*g &= f*(cg) = c(f*g) \\ (f*g)*h &= f*(g*h) \end{aligned} \tag{5.3.3}$$

The most interesting property for us, and the main result of this section is the following theorem.

Theorem 5.3.1

Let $f(t)$ and $g(t)$ be of exponential type, then

$$\mathcal{L}\{(f*g)(t)\} = \mathcal{L}\left\{\int_0^t f(\tau)g(t-\tau)d\tau\right\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}.$$

In other words, the Laplace transform of a convolution is the product of the Laplace transforms. The simplest way to use this result is in reverse.

Example 5.3.3

Find

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+s}\right\}$$

.

Solution

$$\frac{1}{(s+1)s^2} = \frac{1}{s+1}\frac{1}{s^2}.$$

Then

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} = t.$$

Therefore,

$$\mathcal{L}^{-1}\left\{\frac{1}{s+1}\frac{1}{s^2}\right\} = \int_0^t \tau e^{-(t-\tau)}\,d\tau = e^{-t} + t - 1.$$

The calculation of the integral involved an integration by parts.

Solving ODEs

The next example demonstrates the full power of the convolution and the Laplace transform. We can give the solution to the forced oscillation problem for any forcing function as a definite integral.

Example 5.3.4

Find the solution to

$$x'' + \omega_0^2 x = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$.

Solution

We first apply the Laplace transform to the equation. Denote the transform of $x(t)$ by $X(s)$ and the transform of $f(t)$ by $F(s)$ as usual.

$$s^2 X(s) + \omega_0^2 X(s) = F(s),$$

or in other words

$$X(s) = F(s)\frac{1}{s^2+\omega_0^2}.$$

We know

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2+\omega_0^2}\right\} = \frac{\sin(\omega_0 t)}{\omega_0}.$$

Therefore,

$$x(t) = \int_0^t f(\tau)\frac{\sin(\omega_0(t-\tau))}{\omega_0}d\tau,$$

or if we reverse the order

$$x(t) = \int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}f(t-\tau)d\tau.$$

Let us notice one more feature of this example. We can now see how Laplace transform handles resonance. Suppose that $f(t) = \cos(\omega_0 t)$. Then

$$x(t) = \int_0^t \frac{\sin(\omega_0\tau)}{\omega_0}\cos(\omega_0(t-\tau))d\tau = \frac{1}{\omega_0}\int_0^t \sin(\omega_0\tau)\cos(\omega_0(t-\tau))d\tau.$$

We have computed the convolution of sine and cosine in Example 6.3.2. Hence

$$x(t) = \left(\frac{1}{\omega_0}\right)\left(\frac{1}{2}t\sin(\omega_0 t)\right) = \frac{1}{2\omega_0}\sin(\omega_0 t).$$

Note the t in front of the sine. The solution, therefore, grows without bound as t gets large, meaning we get resonance.

Similarly, we can solve any constant coefficient equation with an arbitrary forcing function $f(t)$ as a definite integral using convolution. A definite integral, rather than a closed form solution, is usually enough for most practical purposes. It is not hard to numerically evaluate a definite integral.

Volterra Integral Equation

A common integral equation is the Volterra integral equation[2]

$$x(t) = f(t) + \int_0^t g(t-\tau)x(\tau)\,d\tau$$

where $f(t)$ and $g(t)$ are known functions and $x(t)$ is an unknown we wish to solve for. To find $x(t)$, we apply the Laplace transform to the equation to obtain

$$X(s) = F(s) + G(s)X(s),$$

where $X(s)$, $F(s)$, and $G(s)$ are the Laplace transforms of $x(t)$, $f(t)$, and $g(t)$, respectively. We find

$$X(s) = \frac{F(s)}{1-G(s)}.$$

To find $x(t)$ we now need to find the inverse Laplace transform of $X(s)$.

Example $5.3.5$

Apply Laplace transform to solve

$$x(t) = e^{-t} + \int_0^t \sinh(t-\tau)x(\tau)\,d\tau$$

Solution

We apply Laplace transform to obtain

$$X(s) = \frac{1}{s+1} + \frac{1}{s^2-1} X(s),$$

or

$$X(s) = \frac{\frac{1}{s+1}}{1 - \frac{1}{s^2-1}} = \frac{s-1}{s^2-2} = \frac{s}{s^2-2} - \frac{1}{s^2-2}.$$

It is not hard to find

$$x(t) = \cosh\left(\sqrt{2}\,t\right) - \frac{1}{\sqrt{2}} \sinh\left(\sqrt{2}\,t\right).$$

Footnotes

[1] For those that have seen convolution defined before, you may have seen it defined as $(f * g)(t) = \int_{-\infty}^{\infty} f(\tau) g(t-\tau) d\tau$. This definition agrees with (5.3.1) if you define $f(t)$ and $g(t)$ to be zero for $t < 0$. When discussing the Laplace transform the definition we gave is sufficient. Convolution does occur in many other applications, however, where you may have to use the more general definition with infinities.

[2] Named for the Italian mathematician Vito Volterra (1860–1940).

5.3E: Exercises for Section 5.3

Exercises 1-13

The following exercises are related to the convolution computation.

Exercise $5.3E.1$

Let $f(t) = t^2$ for $t \geq 0$, and $g(t) = u(t-1)$. Compute $f * g$.

Exercise $5.3E.2$

Let $f(t) = t$ for $t \geq 0$, and $g(t) = \sin t$ for $t \geq 0$. Compute $f * g$.

Answer

$(f * g)(t) = sint - t$ for $t \geq 0$

Exercise $5.3E.3$

Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0, c > 0, k > 0$, and $c^2 - 4km > 0$ (system is overdamped). Write the solution as a definite integral.

Exercise $5.3E.4$

Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0, c > 0, k > 0$, and $c^2 - 4km < 0$ (system is underdamped). Write the solution as a definite integral.

Answer

$$x(t) = \frac{2}{\sqrt{(4km - c^2)}} \int_0^t e^{-\frac{c}{2m}t} \sin(\frac{\sqrt{(4km - c^2)}}{2m} t) f(t - \tau) d\tau$$

Exercise $5.3E.5$

Find the solution to

$$mx'' + cx' + kx = f(t), \quad x(0) = 0, \quad x'(0) = 0,$$

for an arbitrary function $f(t)$, where $m > 0, c > 0, k > 0$, and $c^2 = 4km$ (system is critically damped). Write the solution as a definite integral.

Exercise $5.3E.6$

Solve

$$x(t) = e^{-t} + \int_0^t \cos(t - \tau) x(\tau)\, d\tau.$$

Answer

$x(t) = e^{-t} + \sin t + \cos t - 1$

Exercise $5.3E.7$

Solve

$$x(t) = \cos t + \int_0^t \cos(t-\tau)x(\tau)\, d\tau.$$

Answer

$x(t) = \sin t + 2\cos t - 1$

Exercise $5.3E.8$

Compute $\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\}$ using convolution.

Answer

$\mathcal{L}^{-1}\left\{\frac{s}{(s^2+4)^2}\right\} = \frac{1}{4}t\sin(2t).$

Exercise $5.3E.9$

Write down the solution to $x'' - 2x = e^{-t^2}, x(0) = 0, x'(0) = 0$ as a definite integral. Hint: Do not try to compute the Laplace transform of e^{-t^2}.

Exercise $5.3E.10$

Let $f(t) = \cos t$ for $t \geq 0$, and $g(t) = e^{-t}$. Compute $f * g$.

Answer

$\frac{1}{2}(\cos t + \sin t - e^{-t})$

Exercise $5.3E.11$

Compute $\mathcal{L}^{-1}\left\{\frac{5}{s^4+s^2}\right\}$ using convolution.

Answer

$5t - 5\sin t$

Exercise $5.3E.12$

Solve $x'' + x = \sin t, x(0) = 0, x'(0) = 0$ using convolution.

Answer

$\frac{1}{2}(\sin t - t\cos t)$

Exercise $5.3E.13$

Solve $x''' + x' = f(t), x(0) = 0, x'(0) = 0, x''(0) = 0$ using convolution. Write the result as a definite integral.

Answer

$\int_0^t f(\tau)(1 - \cos(t-\tau))d\tau$

CHAPTER OVERVIEW

6: Systems of ODEs

Contributors and Attributions

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6.1: Introduction to Systems of ODEs

Learning Objectives

- Solving the first-order system of differential equations for two variables.
- Changing the higher-order ordinary differential equations and system of ordinary differential equations to first-order systems.
- Solve the equilibrium solutions of Autonomous differential equations and check the stability of the equilibrium solutions.

Introduction:

Often we do not have just one dependent variable and just one differential equation, we may end up with systems of several equations and several dependent variables even if we start with a single equation.

If we have several dependent variables, suppose y_1, y_2, ..., y_n, then we can have a differential equation involving all of them and their derivatives. For example, $y_1'' = f(y_1', y_2', y_1, y_2, x)$. Usually, when we have two dependent variables we have two equations such as

$$\begin{aligned} y_1'' &= f_1(y_1', y_2', y_1, y_2, x) \\ y_2'' &= f_2(y_1', y_2', y_1, y_2, x) \end{aligned} \tag{6.1.1}$$

for some functions f_1 and f_2. We call the above a *system of differential equations*. More precisely, the above is a second order system of ODEs as second order derivatives appear. The system

$$\begin{aligned} x_1' &= g_1(x_1, x_2, x_3, t), \\ x_2' &= g_2(x_1, x_2, x_3, t), \\ x_3' &= g_3(x_1, x_2, x_3, t), \end{aligned} \tag{6.1.2}$$

is a *first order system*, where x_1, x_2, x_3 are the dependent variables, and t is the independent variable.

The terminology for systems is essentially the same as for single equations. For the system above, a *solution* is a set of three functions $x_1(t)$, $x_2(t)$, $x_3(t)$, such that

$$\begin{aligned} x_1'(t) &= g_1\big(x_1(t), x_2(t), x_3(t), t\big), \\ x_2'(t) &= g_2\big(x_1(t), x_2(t), x_3(t), t\big), \\ x_3'(t) &= g_3\big(x_1(t), x_2(t), x_3(t), t\big). \end{aligned} \tag{6.1.3}$$

We usually also have an *initial condition*. Just like for single equations we specify x_1, x_2, and x_3 for some fixed t. For example, $x_1(0) = a_1$, $x_2(0) = a_2$, $x_3(0) = a_3$. For some constants a_1, a_2, and a_3. For the second order system we would also specify the first derivatives at a point. And if we find a solution with constants in it, where by solving for the constants we find a solution for any initial condition, we call this solution the *general solution*. Best to look at a simple example.

Example 6.1.1

Sometimes a system is easy to solve by solving for one variable and then for the second variable. Take the first order system

$$\begin{aligned} y_1' &= y_1, \\ y_2' &= y_1 - y_2, \end{aligned} \tag{6.1.4}$$

with initial conditions of the form $y_1(0) = 1$ and $y_2(0) = 2$.

Solution

We note that $y_1 = C_1 e^x$ is the general solution of the first equation. We then plug this y_1 into the second equation and get the equation $y_2' = C_1 e^x - y_2$, which is a linear first order equation that is easily solved for y_2. By the method of integrating factor we obtain

$$e^x y_2 = \frac{C_1}{2} e^{2x} + C_2$$

or

$$y_2 = \frac{C_1}{2}e^x + C_2e^{-x}.$$

The general solution to the system is, therefore,

$$y_1 = C_1e^x, \quad \text{and} \quad y_2 = \frac{C_1}{2}e^x + C_2e^{-x}.$$

We now solve for C_1 and C_2 given the initial conditions. We substitute $x = 0$ and find that $C_1 = 1$ and $C_2 = \frac{3}{2}$. Thus the solution is: $y_1 = e^x$, and $y_2 = \frac{1}{2}e^x + \frac{3}{2}e^{-x}$.

Generally, we will not be so lucky to be able to solve for each variable separately as in the example above, and we will have to solve for all variables at once. While we won't generally be able to solve for one variable and then the next, we will try to salvage as much as possible from this technique. It will turn out that in a certain sense we will still (try to) solve a bunch of single equations and put their solutions together. Let's not worry right now about how to solve systems yet.

We will mostly consider the *linear systems*. The example above is a so-called *linear first order system*. It is linear as none of the dependent variables or their derivatives appear in nonlinear functions or with powers higher than one (x, y, x' and y', constants, and functions of t can appear, but not xy or $(y')^2$ or x^3). Another, more complicated, example of a linear system is

$$\begin{aligned} y_1'' &= e^t y_1' + t^2 y_1 + 5y_2 + \sin(t), \\ y_2'' &= t y_1' - y_2' + 2y_1 + \cos(t). \end{aligned} \tag{6.1.5}$$

Applications

Let us consider some simple applications of systems and how to set up the equations.

Example 6.1.2

Consider a system of two salt brine tanks connected by bidirectional water flow. The tanks are evenly mixed, meaning the salt concentration in each tank is uniform at any given time. Suppose we have two tanks, each containing volume V liters of salt brine. The amount of salt in the first tank is x_1 grams, and the amount of salt in the second tank is x_2 grams. The liquid is perfectly mixed and flows at the rate r liters per second out of each tank into the other. See Figure 6.1.1. Find a differential equation describing the rate of change of salt in each tank.

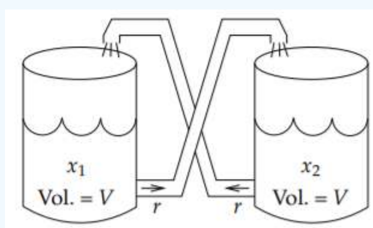

Figure 6.1.1: The image shows two cylindrical tanks each with a volume V. Water flows between them at a rate r, with arrows indicating bidirectional flow. (CC BY-SA 4.0; Jiří Lebl via Differential equations for engineers)

Solution

The rate of change of x_1, that is x_1', is the rate of salt coming in minus the rate going out. The rate coming in is the density of the salt in tank 2, that is $\frac{x_2}{V}$, times the rate r. The rate coming out is the density of the salt in tank 1, that is $\frac{x_1}{V}$, times the rate r. In other words it is

$$x_1' = \frac{x_2}{V}r - \frac{x_1}{V}r = \frac{r}{V}x_2 - \frac{r}{V}x_1 = \frac{r}{V}(x_2 - x_1).$$

Similarly we find the rate x_2', where the roles of x_1 and x_2 are reversed. All in all, the system of ODEs for this problem is

$$\begin{aligned} x_1' &= \frac{r}{V}(x_2 - x_1), \\ x_2' &= \frac{r}{V}(x_1 - x_2). \end{aligned} \tag{6.1.6}$$

In this system we cannot solve for x_1 or x_2 separately. We must solve for both x_1 and x_2 at once, which is intuitively clear since the amount of salt in one tank affects the amount in the other. We can't know x_1 before we know x_2, and vice versa.

We don't yet know how to find all the solutions, but intuitively we can at least find some solutions. Suppose we know that initially the tanks have the same amount of salt. That is, we have an initial condition such as $x_1(0) = x_2(0) = C$. Then clearly the amount of salt coming and out of each tank is the same, so the amounts are not changing. In other words, $x_1 = C$ and $x_2 = C$ (the constant functions) is a solution: $x_1' = x_2' = 0$, and $x_2 - x_1 = x_1 - x_2 = 0$, so the equations are satisfied.

Let us think about the setup a little bit more without solving it. Suppose the initial conditions are $x_1(0) = A$ and $x_2(0) = B$, for two different constants A and B. Since no salt is coming in or out of this closed system, the total amount of salt is constant. That is, $x_1 + x_2$ is constant, and so it equals $A + B$. Intuitively if A is bigger than B, then more salt will flow out of tank one than into it. Eventually, after a long time we would then expect the amount of salt in each tank to equalize. In other words, the solutions of both x_1 and x_2 should tend towards $\frac{A+B}{2}$. Once you know how to solve systems you will find out that this really is so.

Example 6.1.3

Consider a system consisting of two masses, m_1 and m_2, connected by a single spring with a spring constant k. If x_1 and x_2 represent the displacements of the masses from their equilibrium positions, respectively, what are the equations of motion for the system (i.e., what are the differential equations describing the motion of the masses)?

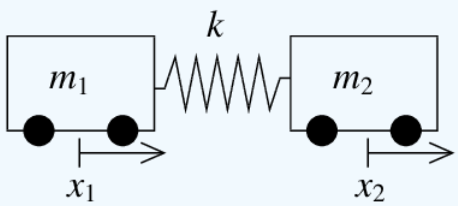

Figure 6.1.2 The image illustrates a mass-spring system, demonstrating the interaction of two rolling masses connected by a spring. The displacements of the masses are denoted as x_1 and x_2, with arrows indicating the direction of motion. (CC BY-SA 4.0; Jiří Lebl via Differential equations for engineers)

Solution

We can think of the masses as carts, and we will suppose that they ride along a straight track with no friction. Let x_1 be the displacement of the first cart and x_2 be the displacement of the second cart. That is, we put the two carts somewhere with no tension on the spring, and we mark the position of the first and second cart and call those the zero positions. Then x_1 measures how far the first cart is from its zero position, and x_2 measures how far the second cart is from its zero position. The force exerted by the spring on the first cart is $k(x_2 - x_1)$, since $x_2 - x_1$ is how far the string is stretched (or compressed) from the rest position. The force exerted on the second cart is the opposite, thus the same thing with a negative sign.

Newton's second law states that force equals mass times acceleration. So the system of equations governing the setup is

$$\begin{aligned} m_1 x_1'' &= k(x_2 - x_1) \\ m_2 x_2'' &= -k(x_2 - x_1) \end{aligned} \tag{6.1.7}$$

In this system we cannot solve for the x_1 or x_2 variable separately. That we must solve for both x_1 and x_2 at once is intuitively clear, since where the first cart goes depends exactly on where the second cart goes and vice-versa.

Changing to First Order

Before we talk about how to handle systems, let us note that in some sense we need only consider first order systems. Let us take an n^{th} order differential equation

$$y^{(n)} = F(y^{(n-1)}, \ldots, y', y, x).$$

We define new variables $u_1, \ldots, u_n$ and write the system

$$\begin{aligned} u_1' &= u_2 \\ u_2' &= u_3 \\ &\vdots \\ u_{n-1}' &= u_n \\ u_n' &= F(u_n, u_{n-1}, \ldots, u_2, u_1, x) \end{aligned} \tag{6.1.8}$$

We solve this system for $u_1, u_2, \ldots, u_n$. Once we have solved for the u's, we can discard u_2 through u_n and let $y = u_1$. We note that this y solves the original equation.

Example 6.1.4

Take $x''' = 2x'' + 8x' + x + t$. Letting $u_1 = x$, $u_2 = x'$, $u_3 = x''$, we find the system:

$$u_1' = u_2, \qquad u_2' = u_3, \qquad u_3' = 2u_3 + 8u_2 + u_1 + t.$$

A similar process can be followed for a system of higher order differential equations. For example, a system of k differential equations in k unknowns, all of order n, can be transformed into a first order system of $n \times k$ equations and $n \times k$ unknowns.

Example 6.1.5

Consider the system from the carts example,

$$m_1 x_1'' = k(x_2 - x_1), \qquad m_2 x_2'' = -k(x_2 - x_1).$$

Let $u_1 = x_1$, $u_2 = x_1'$, $u_3 = x_2$, $u_4 = x_2'$. The second order system becomes the first order system

$$u_1' = u_2, \qquad m_1 u_2' = k(u_3 - u_1), \qquad u_3' = u_4, \qquad m_2 u_4' = -k(u_3 - u_1).$$

Example 6.1.6

Sometimes we can use this idea in reverse as well. Let us take the system

$$x' = 2y - x, \quad y' = x,$$

where the independent variable is t. We wish to solve for the initial conditions $x(0) = 1$ and $y(0) = 0$.

If we differentiate the second equation we get $y'' = x'$. We know what x' is in terms of x and y, and we know that $x = y'$.

$$y'' = x' = 2y - x = 2y - y'.$$

We now have the equation $y'' + y' - 2y = 0$. We know how to solve this equation and we find that $y = C_1 e^{-2t} + C_2 e^t$. Once we have y we use the equation $y' = x$ to get x.

$$x = y' = -2C_1 e^{-2t} + C_2 e^t$$

We solve for the initial conditions $1 = x(0) = -2C_1 + C_2$ and $0 = y(0) = C_1 + C_2$. Hence, $C_1 = -C_2$ and $1 = 3C_2$. So $C_1 = -\frac{1}{3}$ and $C_2 = \frac{1}{3}$. Our solution is

$$x = \frac{2e^{-2t} + e^t}{3}, \quad y = \frac{-e^{-2t} + e^t}{3}$$

It is useful to go back and forth between systems and higher order equations for other reasons. For example, software for solving ODE numerically (approximation) is generally for first order systems. To use it, you take whatever ODE you want to solve and convert it to a first order system. It is not very hard to adapt computer code for the Euler or Runge–Kutta method for first order equations to handle first order systems. We simply treat the dependent variable not as a number but as a vector. In many mathematical computer languages there is almost no distinction in syntax.

Autonomous Systems and Vector Fields

A system where the equations do not depend on the independent variable is called an *autonomous system*. For example the system $x' = 2y - x$, $y' = x$ is autonomous as t is the independent variable but does not appear in the equations.

For autonomous systems we can draw the so-called *direction field* or *vector field*, a plot similar to a slope field, but instead of giving a slope at each point, we give a direction (and a magnitude). The previous example, $x' = 2y - x$, $y' = x$, says that at the point (x, y) the direction in which we should travel to satisfy the equations should be the direction of the vector $(2y - x, x)$ with the speed equal to the magnitude of this vector. So we draw the vector $(2y - x, x)$ at the point (x, y) and we do this for many points on the xy-plane. For example, at the point $(1, 2)$ we draw the vector $\big(2(2) - 1, 1\big) = (3, 1)$, a vector pointing to the right and a little bit up, while at the point $(2, 1)$ we draw the vector $\big(2(1) - 2, 2\big) = (0, 2)$ a vector that points straight up. When drawing the vectors, we will scale down their size to fit many of them on the same direction field. If we drew the arrows at the actual size, the diagram would be a jumbled mess once you would draw more than a couple of arrows. So we scale them all so that not even the longest one interferes with the others. We are mostly interested in their direction and relative size. See Figure $6.1.3$.

We can draw a path of the solution in the plane. Suppose the solution is given by $x = f(t)$, $y = g(t)$. We pick an interval of t (say $0 \leq t \leq 2$ for our example) and plot all the points $\big(f(t), g(t)\big)$ for t in the selected range. The resulting picture is called the *phase portrait* (or phase plane portrait). The particular curve obtained is called the *trajectory* or *solution curve*. See an example plot in Figure $6.1.4$. In the figure the solution starts at $(1, 0)$ and travels along the vector field for a distance of 2 units of t. We solved this system precisely, so we compute $x(2)$ and $y(2)$ to find $x(2) \approx 2.475$ and $y(2) \approx 2.457$. This point corresponds to the top right end of the plotted solution curve in the figure.

Notice the similarity to the diagrams we drew for autonomous systems in one dimension. But note how much more complicated things become when we allow just one extra dimension.

We can draw phase portraits and trajectories in the xy-plane even if the system is not autonomous. In this case, however, we cannot draw the direction field, since the field changes as t changes. For each t we would get a different direction field.

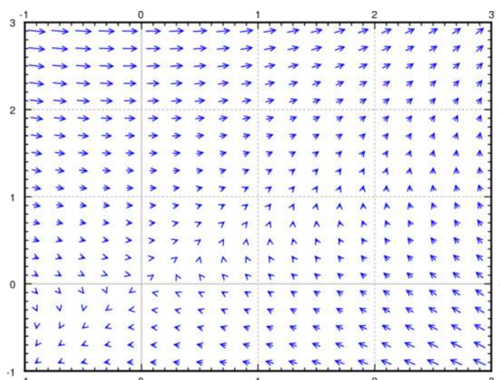

Figure $6.1.3$: This image is a direction field, also known as a slope field, visually representing the solutions to a system of differential equations $x' = 2y - x$ and $y' = x$. A grid is laid out, with x and y axes ranging from -1 to 3. At each point on this grid, there's an arrow representing the slope of the solution curve passing through that point. (CC BY-SA 4.0; Jiří Lebl via Differential equations for engineers)

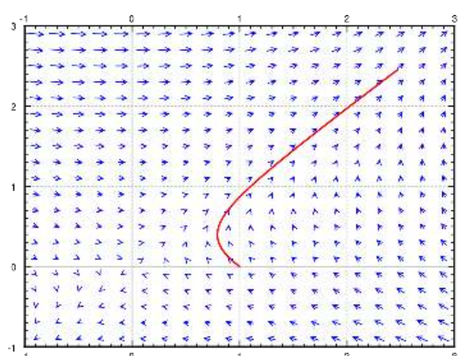

Figure $6.1.4$: This is a direction field, showing slopes of solutions to a system of differential equations on a grid from -1 to 3 on both the x and y axes. Small arrows at each grid point indicate the slope at that location. A red line shows a particular solution curve, starting at the point (1,0) and curving upwards and to the right. (CC BY-SA 4.0; Jiří Lebl via Differential equations for engineers)

6.1E: Exercises for Section 6.1

Exercises 1-10

The following exercises are related to the system of Differential Equations.

Exercise $6.1E.1$

Find the general solution of $x_1' = x_2 - x_1 + t, x_2' = x_2$.

Exercise $6.1E.2$

Find the general solution of $x_1' = 3x_1 - x_2 + e^t, x_2' = x_1$.

Exercise $6.1E.3$

Write $ay'' + by' + cy = f(x)$ as a first order system of ODEs.

Exercise $6.1E.4$

Write $x'' + y^2y' - x^3 = \sin(t), y'' + (x' + y')^2 - x = 0$ as a first order system of ODEs.

Exercise $6.1E.5$

Find the general solution to $y_1' = 3y_1, y_2' = y_1 + y_2, y_3' = y_1 + y_3$.

Answer

$y_1 = C_1e^{3x}$, $y_2 = y(x) = C_2e^x + \frac{C_1}{2}e^{3x}$, $y_3 = y(x) = C_3e^x + \frac{C_1}{2}e^{3x}$

Exercise $6.1E.6$

Solve $y' = 2x, x' = x + y, x(0) = 1, y(0) = 3$.

Answer

$x = \frac{5}{3}e^{2t} - \frac{2}{3}e^{-t}$, $y = \frac{5}{3}e^{2t} + \frac{4}{3}e^{-t}$

Exercise $6.1E.7$

Write $x''' = x + t$ as a first order system.

Answer

$x_1' = x_2$, $x_2' = x_3$, $x_3' = x_1 + t$

Exercise $6.1E.8$

Write $y_1'' + y_1 + y_2 = t, y_2'' + y_1 - y_2 = t^2$ as a first order system.

Answer

$y_3' + y_1 + y_2 = t$, $y_4' + y_1 - y_2 = t^2$, $y_1' = y_3$, $y_2' = y_4$

6.2: Linear systems of ODEs

Learning Objectives

- Understand the differentiation rules for matrix and vector-valued functions, and apply them to linear systems of ordinary differential equations (ODEs).
- Solve first-order linear systems of ODEs using the fundamental matrix and the principle of superposition.

First let us talk about matrix or vector valued functions. Such a function is just a matrix whose entries depend on some variable. If t is the independent variable, we write a *vector valued function* $\vec{x}(t)$ as

$$\vec{x}(t)=\begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$$

Similarly a matrix valued function $A(t)$ is

$$A(t)=\begin{bmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{bmatrix}$$

We can talk about the derivative $A'(t)$ or $\frac{dA}{dt}$. This is just the matrix valued function whose ij_{th} entry is $a'_{ij}(t)$.

Rules of differentiation of matrix valued functions are similar to rules for normal functions. Let $A(t)$ and $B(t)$ be matrix valued functions. Let c be a scalar and let C be a constant matrix. Then

$$\begin{aligned} (A(t)+B(t))' &= A'(t)+B'(t) \\ (A(t)B(t))' &= A'(t)B(t)+A(t)B'(t) \\ (cA(t))' &= cA'(t) \\ (CA(t))' &= CA'(t) \\ (A(t)C)' &= A'(t)C \end{aligned} \tag{6.2.1}$$

Note the order of the multiplication in the last two expressions.

A *first order linear system of ODEs* is a system that can be written as the vector equation

$$\overrightarrow{x'}(t)=P(t)\vec{x}(t)+\vec{f}(t)$$

where $P(t)$ is a matrix valued function, and $\vec{x}(t)$ and $\vec{f}(t)$ are vector valued functions. We will often suppress the dependence on t and only write $\overrightarrow{x'}=P\vec{x}+\vec{f}$. A solution of the system is a vector valued function $\vec{x}$ satisfying the vector equation.

For example, the equations

$$\begin{aligned} x'_1 &= 2tx_1+e^tx_2+t^2 \\ x'_2 &= \frac{x_1}{t}-x_2+e^t \end{aligned} \tag{6.2.2}$$

can be written as

$$\overrightarrow{x'}=\begin{bmatrix} 2t & e^t \\ \frac{1}{t} & -1 \end{bmatrix}\vec{x}+\begin{bmatrix} t^2 \\ e^t \end{bmatrix}$$

We will mostly concentrate on equations that are not just linear, but are in fact *constant coefficient* equations. That is, the matrix P will be constant; it will not depend on t.

When $\vec{f} = \vec{0}$ (the zero vector), then we say the system is *homogeneous*. For homogeneous linear systems we have the principle of superposition, just like for single homogeneous equations.

Theorem 6.2.1

Superposition

Let $\overrightarrow{x'} = P\vec{x}$ be a linear homogeneous system of ODEs. Suppose that $\vec{x}_1, \ldots, \vec{x}_n$ are n solutions of the equation, then

$$\vec{x} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$$

is also a solution.

Linear independence for vector valued functions is the same idea as for normal functions. The vector valued functions $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$ are linearly independent when

$$c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n = \vec{0} \tag{6.2.3}$$

has only the solution $c_1 = c_2 = \cdots = c_n = 0$, where the equation must hold for all t.

Example 6.2.1

Show that $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ 1+t \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$ are linearly dependent.

Solution

Since $\vec{x}_1 + \vec{x}_3 = \vec{x}_2$, and this holds for all t. So $c_1 = 1, c_2 = -1$ and $c_3 = 1$ above will work.

Example 6.2.2

Show that $\vec{x}_1 = \begin{bmatrix} t^2 \\ t \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 0 \\ t \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} -t^2 \\ 1 \end{bmatrix}$ are linearly independent.

Solution

First write $c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \vec{0}$ and note that it has to hold for all t. We get that

$c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 = \begin{bmatrix} c_1t^2 - c_3t^3 \\ c_1t + c_2t + c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

In other words $c_1t^2 - c_3t^3 = 0$ and $c_1t + c_2t + c_3 = 0$. If we set $t = 0$, then the second equation becomes $c_3 = 0$. However, the first equation becomes $c_1t^2 = 0$ for all t and so $c_1 = 0$. Thus the second equation is just $c_2t = 0$, which means $c_2 = 0$. So $c_1 = c_2 = c_3 = 0$ is the only solution and $\vec{x}_1, \vec{x}_2$ and $\vec{x}_3$ are linearly independent.

The linear combination $c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n$ could always be written as

$$X(t)\vec{c}$$

where $X(t)$ is the matrix with columns $\vec{x}_1, \ldots, \vec{x}_n$, and $\vec{c}$ is the column vector with entries $c_1, \ldots, c_n$. The matrix valued function $X(t)$ is called the *fundamental matrix*, or the *fundamental matrix solution*.

To solve nonhomogeneous first order linear systems, we use the same technique as we applied to solve single linear nonhomogeneous equations.

Theorem 6.2.2

Let $\vec{x}' = P\vec{x} + \vec{f}$ be a linear system of ODEs. Suppose $\vec{x}_p$ is one particular solution. Then every solution can be written as

$$\vec{x} = \vec{x}_c + \vec{x}_p$$

where $\vec{x}_c$ is a solution to the associated homogeneous equation $(\vec{x} = P\vec{x})$.

So the procedure will be the same as for single equations. We find a particular solution to the nonhomogeneous equation, then we find the general solution to the associated homogeneous equation, and finally we add the two together.

Alright, suppose you have found the general solution $\vec{x}' = P\vec{x} + \vec{f}$. Now you are given an initial condition of the form

$$\vec{x}t_0 = \vec{b}$$

for some constant vector $\vec{b}$. Suppose that $X(t)$ is the fundamental matrix solution of the associated homogeneous equation (i.e. columns of $X(t)$ are solutions). The general solution can be written as

$$\vec{x}(t) = X(t)\vec{c} + \vec{x}_p(t)$$

We are seeking a vector $\vec{c}$ such that

$$\vec{b} = \vec{x}(t_0) = X(t_0)\vec{c} + \vec{x}_p(t_0)$$

In other words, we are solving for $\vec{c}$ the nonhomogeneous system of linear equations

$$X(t_0)\vec{c} = \vec{b} - \vec{x}_p(t_0)$$

Example 6.2.3

Solve the system

$$\begin{aligned} x_1' &= x_1 \\ x_2' &= x_1 - x_2 \end{aligned} \tag{6.2.4}$$

with initial conditions $x_1(0) = 1, x_2(0) = 2$.

Solution

This is a homogeneous system, so $\vec{f}(t) = \vec{0}$. We write the system and the initial conditions as

$$\vec{x}' = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}\vec{x}, \quad \vec{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

We found the general solution was $x_1 = C_1 e^t$ and $x_2 = \frac{c_1}{2}e^t + c_2 e^{-t}$. Letting $C_1 = 1$ and $C_2 = 0$, we obtain the solution $\begin{bmatrix} e^t \\ \frac{1}{2}e^t \end{bmatrix}$. Letting $C_1 = 0$ and $C_2 = 1$, we obtain $\begin{bmatrix} 0 \\ e^{-t} \end{bmatrix}$. These two solutions are linearly independent, as can be seen by setting $t = 0$, and noting that the resulting constant vectors are linearly independent. In matrix notation, the fundamental matrix solution is, therefore,

$$X(t) = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix}$$

Hence to solve the initial problem we solve the equation

$$X(0)\vec{c} = \vec{b}$$

or in other words,

$$\begin{bmatrix} 1 & 0 \\ \frac{1}{2} & 1 \end{bmatrix}\vec{c} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\vec{x}(t) = X(t)\vec{c} = \begin{bmatrix} e^t & 0 \\ \frac{1}{2}e^t & e^{-t} \end{bmatrix}\begin{bmatrix} 1 \\ \frac{3}{2} \end{bmatrix} = \begin{bmatrix} e^t \\ \frac{1}{2}e^t + \frac{3}{2}e^{-t} \end{bmatrix}$$

6.2E: Exercises for Section 6.2

Exercises 1-9

Verify solutions, establish linear independence, rewrite systems in matrix form, and solve systems to express solutions in component and matrix notation.

Exercise $6.2E.1$

a. Verify that the system $\vec{x}' = \begin{bmatrix} 1 & 3 \\ 3 & 1 \end{bmatrix} \vec{x}$ has the two solutions $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{4t}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-2t}$.

b. Write down the general solution.

c. Write down the general solution in the form $x_1 = ?, x_2 = ?$ (i.e. write down a formula for each element of the solution).

Exercise $6.2E.2$

Verify that $\begin{bmatrix} 1 \\ 1 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix} e^t$ are linearly independent. Hint: Just plug in $t = 0$.

Exercise $6.2E.3$

Verify that $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^t$ and $\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} e^{2t}$ are linearly independent. Hint: You must be a bit more tricky than in the previous exercise.

Exercise $6.2E.4$

Verify that $\begin{bmatrix} t \\ t^2 \end{bmatrix}$ and $\begin{bmatrix} t^3 \\ t^4 \end{bmatrix}$ are linearly independent.

Exercise $6.2E.5$

Take the system $x_1' + x_2' = x_1$, $x_1' - x_2' = x_2$.

a. Write it in the form $A\vec{x}' = B\vec{x}$ for matrices A and B.

b. Compute A^{-1} and use that to write the system in the form $\vec{x}' = P\vec{x}$.

Exercise $6.2E.6$

Are $\begin{bmatrix} e^{2t} \\ e^t \end{bmatrix}$ and $\begin{bmatrix} e^t \\ e^{2t} \end{bmatrix}$ linearly independent? Justify.

Answer

Yes.

Exercise $6.2E.7$

Are $\begin{bmatrix} cosh(t) \\ 1 \end{bmatrix}$, $\begin{bmatrix} e^t \\ 1 \end{bmatrix}$ and $\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}$ linearly independent? Justify.

Answer

No. 2 $\left[\cosh(t) 1\right] - \begin{bmatrix} e^t \\ 1 \end{bmatrix} - \begin{bmatrix} e^{-t} \\ 1 \end{bmatrix} = \vec{0}$

Exercise $6.2E.8$

Write $x' = 3x - y + e^t$ and $y' = tx$ in matrix notation.

Answer

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} 3 & -1 \\ t & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \begin{bmatrix} e^t \\ 0 \end{bmatrix}$$

Exercise $6.2E.9$

a. Write $x_1' = 2t\, x_2$ and $x_2' = 2t\, x_2$ in matrix notation.

b. Solve and write the solution in matrix notation.

Answer

a. $\vec{x}' = \begin{bmatrix} 0 & 2t \\ 0 & 2t \end{bmatrix} \vec{x}$

b. $\vec{x} = \begin{bmatrix} C_2 e^{t^2} + C_1 \\ C_2 e^{t^2} \end{bmatrix}$

6.3: Eigenvalue Method

Learning Objectives

- Writing system of linear differential equations in the matrix form.
- Solve for eigenvalues and eigenvectors of a given matrix.
- Apply eigenvalue method to solve system of Linear Differential Equations to find general solutions.

In this section we will learn how to solve linear homogeneous constant coefficient systems of ODEs by the eigenvalue method. Suppose we have such a system

$$\vec{x}' = P\vec{x},$$

where P is a constant square matrix. We wish to adapt the method for the single constant coefficient equation by trying the function $e^{\lambda t}$. However, $\vec{x}$ is a vector. So we try $\vec{x} = \vec{v}e^{\lambda t}$, where $\vec{v}$ is an arbitrary constant vector. We plug this $\vec{x}$ into the equation to get

$$\underbrace{\lambda \vec{v} e^{\lambda t}}_{\vec{x}'} = \underbrace{P\vec{v}e^{\lambda t}}_{P\vec{x}}.$$

We divide by $e^{\lambda t}$ and notice that we are looking for a scalar λ and a vector $\vec{x}$ that satisfy the equation

$$\lambda \vec{v} = P\vec{v}.$$

To solve this equation we need a little bit more linear algebra, which we now review.

Eigenvalues and Eigenvectors of a Matrix

Let A be a constant square matrix. Suppose there is a scalar λ and a nonzero vector $\vec{v}$ such that

$$A\vec{v} = \lambda \vec{v}.$$

We then call λ an *eigenvalue* of A and $\vec{x}$ is said to be a corresponding *eigenvector*.

Example 6.3.1

Show the matrix $\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix}$ has an eigenvalue of $\lambda = 2$ with a corresponding eigenvector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Solution

$$\begin{bmatrix} 2 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Let us see how to compute the eigenvalues for any matrix. We rewrite the equation for an eigenvalue as

$$(A - \lambda I)\vec{v} = \vec{0}.$$

We notice that this equation has a nonzero solution $\vec{v}$ only if $A - \lambda I$ is not invertible. Were it invertible, we could write $(A-\lambda I)^{-1}(A-\lambda I)\vec{v} = (A-\lambda I)^{-1}\vec{0}$, which implies $\vec{v} = \vec{0}$. Therefore, A has the eigenvalue λ if and only if λ solves the equation

$$det(A - \lambda I) = 0.$$

Consequently, we will be able to find an eigenvalue of A without finding a corresponding eigenvector. An eigenvector will have to be found later, once λ is known.

Example 6.3.2

Find all eigenvalues of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$.

Solution

We write

$$\begin{aligned} det\left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right) &= det\left(\begin{bmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix}\right) \\ &= (2-\lambda)((2-\lambda)^2-1) = -(\lambda-1)(\lambda-2)(\lambda-3). \end{aligned} \tag{6.3.1}$$

So the eigenvalues are $\lambda = 1, \lambda = 2,$ and $\lambda = 3$.

Note that for an $n \times n$ matrix, the polynomial we get by computing $det(A - \lambda I)$ will be of degree n, and hence we will in general have n eigenvalues. Some may be repeated, some may be complex.

To find an eigenvector corresponding to an eigenvalue λ, we write

$$(A - \lambda I)\vec{v} = \vec{0},$$

and solve for a nontrivial (nonzero) vector $\vec{v}$. If λ is an eigenvalue, there will be at least one free variable, and so for each distinct eigenvalue λ, we can always find an eigenvector

Example 6.3.3

Find an eigenvector of $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ corresponding to the eigenvalue $\lambda = 3$.

Solution

We write

$$(A - \lambda I)\vec{v} = \left(\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} - 3\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \vec{0}.$$

It is easy to solve this system of linear equations. We write down the augmented matrix

$$\left[\begin{array}{ccc|c} -1 & 0 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{array}\right],$$

and perform row operations (exercise: which ones?) until we get:

$$\left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The entries of $\vec{v}$ have to satisfy the equations $v_1 - v_2 = 0, v_3 = 0$ and v_2 is a free variable. We can pick v_2 to be arbitrary (but nonzero), let $v_1 = v_2$, and of course $v_3 = 0$. For example, if we pick $v_2 = 1$, then $\vec{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. Let us verify that $\vec{v}$ really is an eigenvector corresponding to $\lambda = 3$:

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 0 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Eigenvalue Method with Distinct Real Eigenvalues

We have the system of equations

$$\vec{x}' = P\vec{x}.$$

We find the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ of the matrix P, and corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. Now we notice that the functions $\vec{v}_1 e^{\lambda_1 t}, \vec{v}_2 e^{\lambda_2 t}, \ldots, \vec{v}_n e^{\lambda_n t}$ are solutions of the system of equations and hence $\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}$ is a solution.

Theorem 6.3.1

Take $\vec{x}' = P\vec{x}$. If P is an $n \times n$ constant matrix that has n distinct real eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$, then there exist n linearly independent corresponding eigenvectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n$, and the general solution to $\vec{x}' = P\vec{x}$ can be written as

$$\vec{x} = c_1 \vec{v}_1 e^{\lambda_1 t} + c_2 \vec{v}_2 e^{\lambda_2 t} + \cdots + c_n \vec{v}_n e^{\lambda_n t}.$$

The corresponding fundamental matrix solution is

$$X(t) = [\vec{v}_1 e^{\lambda_1 t} \quad \vec{v}_2 e^{\lambda_2 t} \quad \cdots \quad \vec{v}_n e^{\lambda_n t}].$$

That is, $X(t)$ is the matrix whose j^{th} column is $\vec{v}_j e^{\lambda_j t}$.

Example 6.3.4

Consider the system

$$\vec{x}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \vec{x}.$$

Find the general solution.

Solution

Earlier, we found the eigenvalues are $1, 2, 3$. We found the eigenvector $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ for the eigenvalue 3. Similarly we find the eigenvector $\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ for the eigenvalue 1, and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ for the eigenvalue 2. Hence our general solution is

$$\vec{x} = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} e^{3t} = \begin{bmatrix} c_1 e^t + c_3 e^{3t} \\ -c_1 e^t + c_2 e^{2t} + c_3 e^{3t} \\ -c_2 e^{2t} \end{bmatrix}.$$

In terms of a fundamental matrix solution

$$\vec{x} = X(t)\vec{c} = \begin{bmatrix} e^t & 0 & e^{3t} \\ -e^t & e^{2t} & e^{3t} \\ 0 & -e^{2t} & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

- **3.4: Eigenvalue Method** by Jiří Lebl is licensed CC BY-SA 4.0. Original source: https://www.jirka.org/diffyqs.

6.3E: Exercises for Section 6.3

Exercises 1 - 11

Follow the instructions to solve each exercise. To find the general solution use eigenvalue method.

Exercise $6.3E.1$

Let A be a 3×3 matrix with an eigenvalue of 3 and a corresponding eigenvector $\vec{v} = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$. Find $A\vec{v}$.

Exercise $6.3E.2$

a. Find the general solution of $x_1' = 2x_1, x_2' = 3x_2$ using the eigenvalue method (first write the system in the form $\vec{x}' = A\vec{x}$).

b. Solve the system by solving each equation separately and verify you get the same general solution.

Exercise $6.3E.3$

Find the general solution of $x_1' = 3x_1 + x_2, x_2' = 2x_1 + 4x_2$ using the eigenvalue method.

Exercise $6.3E.4$

Find the general solution of $x_1' = x_1 - 2x_2, x_2' = 2x_1 + x_2$ using the eigenvalue method. Do not use complex exponentials in your solution.

Exercise $6.3E.5$

a. Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.

b. Find the general solution of $\vec{x}' = A\vec{x}$.

Exercise $6.3E.6$

Compute eigenvalues and eigenvectors of $\begin{bmatrix} -2 & -1 & -1 \\ 3 & 2 & 1 \\ -3 & -1 & 0 \end{bmatrix}$.

Exercise $6.3E.7$

Let a, b, c, d, e, f be numbers. Find the eigenvalues of $\begin{bmatrix} a & b & c \\ 0 & d & e \\ 0 & 0 & f \end{bmatrix}$.

Exercise $6.3E.8$

a. Compute eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 0 & 3 \\ -1 & 0 & 1 \\ 2 & 0 & 2 \end{bmatrix}$.

b. Solve the system $\vec{x}' = A\vec{x}$.

Answer

a. Eigenvalues: $4,\ 0,\ -1$ Eigenvectors: $\begin{bmatrix}1\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}3\\5\\-2\end{bmatrix}$

b. $\vec{x} = C_1\begin{bmatrix}1\\0\\1\end{bmatrix}e^{4t} + C_2\begin{bmatrix}0\\1\\0\end{bmatrix} + C_3\begin{bmatrix}3\\5\\-2\end{bmatrix}$

Exercise $6.3E.9$

a. Compute eigenvalues and eigenvectors of $A = \begin{bmatrix}1 & 1\\-1 & 0\end{bmatrix}$.

b. Solve the system $\vec{x}' = A\vec{x}$.

Answer

a. Eigenvalues: $\frac{1+\sqrt{3}i}{2}, \frac{1-\sqrt{3}i}{2}$, Eigenvectors: $\begin{bmatrix}-2\\1-\sqrt{3}i\end{bmatrix}, \begin{bmatrix}-2\\1+\sqrt{3}i\end{bmatrix}$

b. $\vec{x} = C_1e^{t/2}\begin{bmatrix}-2\cos\left(\frac{\sqrt{3}t}{2}\right)\\ \cos\left(\frac{\sqrt{3}t}{2}\right)+\sqrt{3}\sin\left(\frac{\sqrt{3}t}{2}\right)\end{bmatrix} + C_2e^{t/2}\begin{bmatrix}-2\sin\left(\frac{\sqrt{3}t}{2}\right)\\ \sin\left(\frac{\sqrt{3}t}{2}\right)-\sqrt{3}\cos\left(\frac{\sqrt{3}t}{2}\right)\end{bmatrix}$

Exercise $6.3E.10$

Solve $x_1' = x_2, x_2' = x_1$ using the eigenvalue method.

Answer

$\vec{x} = C_1\begin{bmatrix}1\\1\end{bmatrix}e^{t} + C_2\begin{bmatrix}1\\-1\end{bmatrix}e^{-t}$

Exercise $6.3E.11$

Solve $x_1' = x_2, x_2' = -x_1$ using the eigenvalue method.

Answer

$\vec{x} = C_1\begin{bmatrix}\cos(t)\\-\sin(t)\end{bmatrix} + C_2\begin{bmatrix}\sin(t)\\\cos(t)\end{bmatrix}$

6.4: Matrices and linear systems

> **Learning Objectives**
>
> - Understand the fundamental operations of matrix arithmetic, including addition, multiplication, and the computation of transposes and inverses.
> - Solve systems of linear equations using matrix methods such as row reduction, determinants, and matrix inverses.

Introduction:

A vector is an array of 1 row x n columns (row vector) or 1 column x n rows (column vector), while a matrix is an array of m rows x n columns. While ordinary variables hold a single value, arrays hold many values. Functions that expect a vector argument generally require a column vector.

Matrices and vectors

Before we can start talking about linear systems of ODEs, we will need to talk about matrices, so let us review these briefly. A matrix is an $m \times n$ array of numbers (m rows and n columns). For example, we denote a 3×5 matrix as follows

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \end{bmatrix}$$

The numbers a_{ij} are called *elements* or *entries*.

By a *vector* we will usually mean a *column vector*, that is an $m \times 1$ matrix. If we mean a *row vector* we will explicitly say so (a row vector is a $1 \times n$ matrix). We will usually denote matrices by upper case letters and vectors by lower case letters with an arrow such as $\vec{x}$ or $\vec{b}$. By $\vec{0}$ we will mean the vector of all zeros.

It is easy to define some operations on matrices. Note that we will want 1×1 matrices to really act like numbers, so our operations will have to be compatible with this viewpoint.

First, we can multiply by a scalar (a number). This means just multiplying each entry by the same number. For example,

$$2 \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 2 & 4 & 6 \\ 8 & 10 & 12 \end{bmatrix}$$

Matrix addition is also easy. We add matrices element by element. For example,

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 3 & 2 \\ 4 & 7 & 10 \end{bmatrix}$$

If the sizes do not match, then addition is not defined.

If we denote by 0 the matrix of with all zero entries, by c, d scalars, and by A, B, C matrices, we have the following familiar rules.

$$\begin{aligned} A+0 &= A = 0+A \\ A+B &= B+A \\ (A+B)+C &= A+(B+C) \\ c(A+B) &= cA+cB \\ (c+d)A &= cA+dA \end{aligned} \tag{6.4.1}$$

Another useful operation for matrices is the *transpose*. This operation just swaps rows and columns of a matrix. The transpose of A is denoted by A^T. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}$$

Matrix Multiplication

Let us now define matrix multiplication. First we define the *dot product* (or *inner product*) of two vectors. Usually this will be a row vector multiplied with a column vector of the same size. For the dot product we multiply each pair of entries from the first and the second vector and we sum these products. The result is a single number. For example,

$$[\, a_1 \quad a_2 \quad a_3 \,] \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = [\, a_1b_1 + a_2b_2 + a_3b_3 \,]$$

And similarly for larger (or smaller) vectors.

Armed with the dot product we can define the product of matrices. First let us denote by $\operatorname{row}_i(A)$ the i^{th} row of A and by $\operatorname{column}_j(A)$ the j^{th} column of A. For an $m \times n$ matrix A and an $n \times p$ matrix B we can define the product AB. We let AB be an $m \times p$ matrix whose ij^{th} entry is

$$\operatorname{row}_i(A) \cdot \operatorname{column}_j(B)$$

Do note how the sizes match up. Example:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix} =$$

$$= \begin{bmatrix} 1\cdot 1+2\cdot 1+3\cdot 1 & 1\cdot 0+2\cdot 1+3\cdot 0 & 1\cdot(-1)+2\cdot 1+3\cdot 0 \\ 4\cdot 1+5\cdot 1+6\cdot 1 & 4\cdot 0+5\cdot 1+6\cdot 0 & 4\cdot(-1)+5\cdot 1+6\cdot 0 \end{bmatrix} = \begin{bmatrix} 6 & 2 & 1 \\ 15 & 5 & 1 \end{bmatrix}$$

For multiplication we want an analog of a 1. This analog is the identity matrix. The identity matrix is a square matrix with 1s on the main diagonal and zeros everywhere else. It is usually denoted by I. For each size we have a different identity matrix and so sometimes we may denote the size as a subscript. For example, the I_3 would be the 3×3 identity matrix

$$I = I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

We have the following rules for matrix multiplication. Suppose that A, B, C are matrices of the correct sizes so that the following make sense. Let α denote a scalar (number).

$$\begin{aligned} A(BC) &= (AB)C \\ A(B+C) &= AB+AC \\ (B+C)A &= BA+CA \\ \alpha(AB) &= (\alpha A)B = A(\alpha B) \\ IA &= A = AI \end{aligned} \tag{6.4.2}$$

A few warnings are in order.

i. $AB \neq BA$ in general (it may be true by fluke sometimes). That is, matrices do not commute. For example take $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.
ii. $AB = AC$ does not necessarily imply $B = C$, even if A is not 0.
iii. $AB = 0$ does not necessarily mean that $A = 0$ or $B = 0$. For example take $A = B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

For the last two items to hold we would need to “divide” by a matrix. This is where the matrix inverse comes in. Suppose that A and B are $n \times n$ matrices such that

$$AB = I = BA$$

Then we call B the inverse of A and we denote B by A^{-1}. If the inverse of A exists, then we call A *invertible*. If A is not invertible we sometimes say A is *singular*.

If A is invertible, then $AB = AC$ does imply that $B = C$ (in particular the inverse of A is unique). We just multiply both sides by A^{-1} to get $A^{-1}AB = A^{-1}AC$ or $IB = IC$ or $B = C$. It is also not hard to see that $(A^{-1})^{-1} = A$.

Determinant

We can now talk about determinants of square matrices. We define the determinant of a 1×1 matrix as the value of its only entry. For a 2×2 matrix we define

$$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) \overset{\text{def}}{=} ad - bc$$

Before trying to compute the determinant for larger matrices, let us first note the meaning of the determinant. Consider an $n \times n$ matrix as a mapping of the n dimensional euclidean space $\mathbb{R}^n$ to $\mathbb{R}^n$. In particular, a 2×2 matrix A is a mapping of the plane to itself, where $\vec{x}$ gets sent to $A\vec{x}$. Then the determinant of A is the factor by which the area of objects gets changed. If we take the unit square (square of side 1) in the plane, then A takes the square to a parallelogram of area $|\det(A)|$. The sign of $\det(A)$ denotes changing of orientation (negative if the axes got flipped). For example, let

$$A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$$

Then $\det(A) = 1 + 1 = 2$. Let us see where the square with vertices $(0, 0)$, $(1, 0)$, $(0, 1)$ and $(1, 1)$ gets sent. Clearly $(0, 0)$ gets sent to $(0, 0)$.

$$\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

So the image of the square is another square. The image square has a side of length $\sqrt{2}$ and is therefore of area 2.

If you think back to high school geometry, you may have seen a formula for computing the area of a parallelogram with vertices $(0, 0)$, (a, c), (b, d) and $(a + b, c + d)$. And it is precisely

$$\left|\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)\right|$$

The vertical lines above mean absolute value. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ carries the unit square to the given parallelogram.

Now we can define the determinant for larger matrices. We define A_{ij} as the matrix A with the i^{th} row and the j^{th} column deleted. To compute the determinant of a matrix, pick one row, say the i^{th} row and compute.

$$\det(A) = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})$$

For the first row we get

$$\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13}) - \ldots \begin{cases} +a_{1n}\det(A_{1n} & \text{if n is odd} \\ -a_{1n}\det(A_{1n} & \text{if n even} \end{cases}$$

We alternately add and subtract the determinants of the submatrices A_{ij} for a fixed i and all j. For a 3×3 matrix, picking the first row, we would get $\det(A) = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + a_{13}\det(A_{13})$. For example,

$$\begin{aligned} \det\left(\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\right) &= 1 \cdot \det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) - 2 \cdot \det\left(\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix}\right) + 3 \cdot \det\left(\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix}\right) \\ &= 1(5 \cdot 9 - 6 \cdot 8) - 2(4 \cdot 9 - 6 \cdot 7) + 3(4 \cdot 8 - 5 \cdot 7) = 0 \end{aligned} \tag{6.4.3}$$

The numbers $(-1)^{i+j}\det(A_{ij})$ are called *cofactors* of the matrix and this way of computing the determinant is called the *cofactor expansion*. It is also possible to compute the determinant by expanding along columns (picking a column instead of a row above).

Note that a common notation for the determinant is a pair of vertical lines:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right)$$

One of the most important properties of determinants (in the context of this course) is the following theorem.

Think of the determinants telling you the scaling of a mapping. If B doubles the sizes of geometric objects and A triples them, then AB (which applies B to an object and then A) should make size go up by a factor of 6. This is true in general:

$$\det(AB) = \det(A)\det(B).$$

This property is one of the most useful, and it is employed often to actually compute determinants. A particularly interesting consequence is to note what it means for existence of inverses. Take A and B to be inverses of each other, that is $AB = I$. Then

$$\det(A)\det(B) = \det(AB) = \det(I) = 1.$$

Neither $\det(A)$ nor $\det(B)$ can be zero. Let us state this as a theorem as it will be very important in the context of this course.

Theorem 6.4.1

An $n \times n$ matrix A is invertible if and only if $\det(A) \neq 0$.

In fact, there is a formula for the inverse of a 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice the determinant of the matrix in the denominator of the fraction. The formula only works if the determinant is nonzero, otherwise we are dividing by zero.

Solving Linear Systems

One application of matrices we will need is to solve systems of linear equations. This is best shown by example. Suppose that we have the following system of linear equations

$$\begin{aligned} 2x_1 + 2x_2 + 2x_3 &= 2 \\ x_1 + x_2 + 3x_3 &= 5 \\ x_1 + 4x_2 + x_3 &= 10 \end{aligned} \tag{6.4.4}$$

Without changing the solution, we could swap equations in this system, we could multiply any of the equations by a nonzero number, and we could add a multiple of one equation to another equation. It turns out these operations always suffice to find a solution.

It is easier to write the system as a matrix equation. Note that the system can be written as

$$\begin{bmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$$

To solve the system we put the coefficient matrix (the matrix on the left hand side of the equation) together with the vector on the right and side and get the augmented matrix

$$\left[\begin{array}{ccc|c} 2 & 2 & 2 & 2 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right]$$

We apply the following three elementary operations.

i. Swap two rows.
ii. Add a multiple of one row to another row.
iii. Multiply a row by a nonzero number.

We will keep doing these operations until we get into a state where it is easy to read off the answer, or until we get into a contradiction indicating no solution, for example if we come up with an equation such as $0 = 1$.

Let us work through the example. First multiply the first row by $\frac{1}{2}$ to obtain

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 1 & 3 & 5 \\ 1 & 4 & 1 & 10 \end{array}\right]$$

Now subtract the first row from the second and third row.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 2 & 4 \\ 0 & 3 & 0 & 9 \end{array}\right]$$

Multiply the last row by $\frac{1}{3}$ and the second row by $\frac{1}{2}$.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

Swap rows 2 and 3.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

Subtract the last row from the first, then subtract the second row from the first.

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

If we think about what equations this augmented matrix represents, we see that $x_1 = -4$, $x_2 = 3$ and $x_3 = 2$. We try this solution in the original system and, voilà, it works!

We write this equation in matrix notation as

$$A\vec{x} = \vec{b},$$

where A is the matrix $\left[\begin{smallmatrix} 2 & 2 & 2 \\ 1 & 1 & 3 \\ 1 & 4 & 1 \end{smallmatrix}\right]$ and $\vec{b}$ is the vector $\left[\begin{smallmatrix} 2 \\ 5 \\ 10 \end{smallmatrix}\right]$. The solution can also be computed via the inverse,

$$\vec{x} = A^{-1}A\vec{x} = A^{-1}\vec{b}.$$

It is possible that the solution is not unique, or that no solution exists. It is easy to tell if a solution does not exist. If during the row reduction you come up with a row where all the entries except the last one are zero (the last entry in a row corresponds to the right-hand side of the equation), then the system is *inconsistent* and has no solution. For example, for a system of 3 equations and 3 unknowns, if you find a row such as $[\,0 \quad 0 \quad 0 \mid 1\,]$ in the augmented matrix, you know the system is inconsistent. That row corresponds to $0 = 1$.

You generally try to use row operations until the following conditions are satisfied. The first (from the left) nonzero entry in each row is called the *leading entry*.

i. The leading entry in any row is strictly to the right of the leading entry of the row above.
ii. Any zero rows are below all the nonzero rows.
iii. All leading entries are 1.
iv. All the entries above and below a leading entry are zero.

Such a matrix is said to be in *reduced row echelon form*. The variables corresponding to columns with no leading entries are said to be *free variables*. Free variables mean that we can pick those variables to be anything we want and then solve for the rest of the unknowns.

Example 6.4.1

Determine if he following augmented matrix is in reduced row echelon form, then solve it.

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Solution

The augmented matrix is in reduced row echelon form.

Suppose the variables are x_1, x_2 and x_3. Then x_2 is the free variable, $x_1 = 3 - 2x_2$, and $x_3 = 1$.

Note that if during the row reduction process you come up with the matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 13 & 3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 3 \end{array}\right]$$

there is no need to go further. The last row corresponds to the equation $0x_1 + 0x_2 + 0x_3 = 3$, which is preposterous. Hence, no solution exists.

Computing the Inverse

If the coefficient matrix is square and there exists a unique solution $\vec{x}$ to $A\vec{x} = \vec{b}$ for any $\vec{b}$, then A is invertible. In fact by multiplying both sides by A^{-1} you can see that $\vec{x} = A^{-1}\vec{b}$. So it is useful to compute the inverse if you want to solve the equation for many different right hand sides $\vec{b}$.

The 2×2 inverse can be given by a formula, but it is also not hard to compute inverses of larger matrices. While we will not have too much occasion to compute inverses for larger matrices than 2×2 by hand, let us touch on how to do it. Finding the inverse of A is actually just solving a bunch of linear equations. If we can solve $A\vec{x}_k = \vec{e}_k$ where $\vec{e}_k$ is the vector with all zeros except a 1 at the k^{th} position, then the inverse is the matrix with the columns $\vec{x}_k$ for $k = 1, \ldots, n$ (exercise: why?). Therefore, to find the inverse we can write a larger $n \times 2n$ augmented matrix $[A \mid I]$, where I is the identity. We then perform row reduction. The reduced row echelon form of $[A \mid I]$ will be of the form $[I \mid A^{-1}]$ if and only if A is invertible. We can then just read off the inverse A^{-1}.

6.4E: Exercises for Section 6.4

Exercises 1 - 14

Follow the instructions to solve each exercise using appropriate matrix operations.

Exercise $6.4E.1$

Solve $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \vec{x} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$ by using matrix inverse.

Exercise $6.4E.2$

Compute determinant of $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}$.

Answer

$\det(A) = 6.$

Exercise $6.4E.3$

Compute determinant of $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 4 & 0 & 5 & 0 \\ 6 & 0 & 7 & 0 \\ 8 & 0 & 10 & 1 \end{bmatrix}$. Hint: Expand along the proper row or column to make the calculations simpler.

Exercise $6.4E.4$

Compute inverse of $\begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Answer

$$B^{-1} = \begin{bmatrix} -\frac{1}{2} & 0 & \frac{1}{2} \\ \frac{3}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} \end{bmatrix}.$$

Exercise $6.4E.5$

For which h is $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & h \end{bmatrix}$ not invertible? Is there only one such h? Are there several? Infinitely many?

Exercise $6.4E.6$

For which h is $\begin{bmatrix} h & 1 & 1 \\ 0 & h & 0 \\ 1 & 1 & h \end{bmatrix}$ not invertible? Find all such h.

Answer

The matrix $C=\begin{bmatrix} h & 0 & 1 \\ 1 & h & 1 \\ 0 & h & h \end{bmatrix}$ is not invertible for: $h=0, \quad h=\frac{1}{2}-\frac{\sqrt{3}i}{2}, \quad h=\frac{1}{2}+\frac{\sqrt{3}i}{2}$.

Exercise 6.4E.7

Solve $\begin{bmatrix} 9 & -2 & -6 \\ -8 & 3 & 6 \\ 10 & -2 & -6 \end{bmatrix}\vec{x}=\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.

Exercise 6.4E.8

Solve $\begin{bmatrix} 5 & 3 & 7 \\ 8 & 4 & 4 \\ 6 & 3 & 3 \end{bmatrix}\vec{x}=\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$.

Answer

The given system of equations $A\mathbf{x}=\mathbf{b}$ where $A=\begin{bmatrix} 5 & 8 & 6 \\ 3 & 4 & 3 \\ 7 & 4 & 3 \end{bmatrix}$, $\mathbf{b}=\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}$ is inconsistent because: $\text{rank}(A)=2, \quad \text{rank}([A|b])=3, \quad \text{and} \quad \det(A)=0.$ Since $\text{rank}(A)\neq\text{rank}([A|b])$, the system has no solution.

Exercise 6.4E.9

Solve $\begin{bmatrix} 3 & 2 & 3 & 0 \\ 3 & 3 & 3 & 3 \\ 0 & 2 & 4 & 2 \\ 2 & 3 & 4 & 3 \end{bmatrix}\vec{x}=\begin{bmatrix} 2 \\ 0 \\ 4 \\ 1 \end{bmatrix}$.

Exercise 6.4E.10

Find 3 nonzero 2×2 matrices A, B, and C such that $AB=AC$ but $B\neq C$.

Answer

The matrices satisfying $AB=AC$ but $B\neq C$ are: $A=\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad B=\begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad C=\begin{bmatrix} 2 & 3 \\ 6 & 7 \end{bmatrix}$. Verifying: $AB=AC=\begin{bmatrix} 2 & 3 \\ 0 & 0 \end{bmatrix}, \quad \text{but} \quad B\neq C.$

Exercise 6.4E.11

Compute determinant of $\begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & -5 \\ 1 & -1 & 0 \end{bmatrix}$

Answer

-15

Exercise $6.4E.12$

Find t such that $\begin{bmatrix} 1 & t \\ -1 & 2 \end{bmatrix}$ is not invertible.

Answer

-2

Exercise $6.4E.13$

Solve $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \vec{x} = \begin{bmatrix} 10 \\ 20 \end{bmatrix}$.

Answer

$\vec{x} = \begin{bmatrix} 15 \\ -5 \end{bmatrix}$

Exercise $6.4E.14$

Suppose a, b, c are nonzero numbers. Let $M = \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}$, $N = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$.

a. Compute M^{-1}.

b. Compute N^{-1}.

Answer

a. $\begin{bmatrix} \frac{1}{a} & 0 \\ 0 & \frac{1}{b} \end{bmatrix}$

b. $\begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{1}{b} & 0 \\ 0 & 0 & \frac{1}{c} \end{bmatrix}$

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