MGF 1131: MATHEMATICS IN CONTEXT (FSW)

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Mathematics in Context

MGF 1131

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CHAPTER OVERVIEW

1: Number Representation in Different Bases and Cryptography

1.1: Hindu-Arabic Positional System

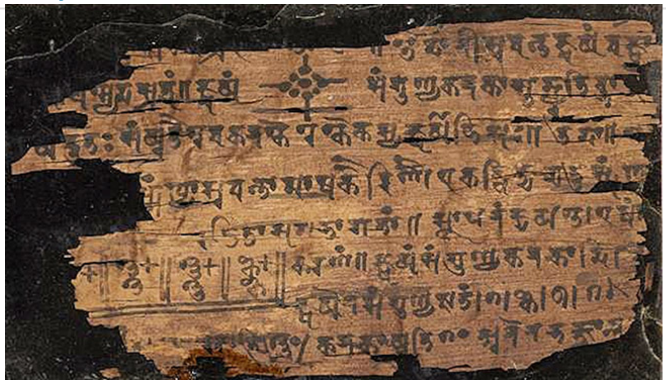

Figure 1.1.1 : This manuscript is an early example of Hindu numerals. (credit: modification of work “Bakshali manuscript”, Bodleian Libraries/ University of Oxford, public domain)

Learning Objectives

After completing this section, you should be able to:

- Evaluate an exponential expression.
- Convert a Hindu-Arabic numeral to expanded form.
- Convert a number in expanded form to a Hindu-Arabic numeral.

The modern system of counting and computing isn’t necessarily natural. The fact that different symbols are used to indicate different quantities or amounts is a relatively new invention. Simple marking by scratches or dots, one for each item being counted, was the norm long into human history. The modern system doesn’t use repeated symbols to indicate more than one thing. It uses the place of a digit in a numeral to determine what that digit represents. A numeral is a symbol used to represent a number. A number is an abstract idea that represents quantity or amount.

Being clear about the difference between a numeral and a number is important. Just like a person can be called by various names, such as brother, father, husband, uncle, they are all representing the same person, John Smith. The person John Smith is the number, and the names brother, father, husband, and uncle are the numerals.

Who Knew?: Hindu-Arabic Numerals

The numerals we currently use are referred to as Hindu-Arabic numerals, although they have changed as time has passed. Early forms of the numerals for $0, 1, 2, 3, 4, 5, 6, 7, 8$, and 9 began in India, and passed through Persia to the Middle East. Place value was also employed in India's early systems. Once this system was in North Africa and the Middle East, it spread to Europe, eventually replacing Roman numerals. Over time, the original symbols transformed into our modern ones.

The system we use for counting and computing uses place values based on powers of 10. In this section, we review exponents and our positional system.

Evaluating Exponential Expressions

Most modern numerical systems depend on place values, where the quantity represented depends not only on the digit, but also on where the digit is in the number. The place value is a power of some specific number, which means most numbering systems are actually exponential expressions. An exponential expression is any mathematical expression that includes exponents. So, evaluating such an expression means performing the calculation.

In this chapter, we will be using exponents that are positive integer values. Before we do so, let’s remind ourselves about exponents and what they represent. Suppose you want to multiply a number. Let’s label that number a, by itself some number of times. Let’s label the number of times b. We denote that as a^b. We say a, or the **base**, raised to the bth power, or the **exponent**. For example, if we are multiplying 13 by itself eight times, we write 13^8 and say 13 to the eighth power.

When computing exponential expressions, we should be careful to remember the order of operations. Using the order of operation rules, calculations inside the parentheses are done first, then exponents are calculated, then multiplication and division calculations are performed, and then addition and subtraction.

Example 1.1.1: Evaluating an Exponential Expression

Evaluate the following exponential expressions.

1. $4 \times 5^2 + 2 \times 6^3$
2. $6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0$
3. $3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0$

Answer 1

To evaluate or calculate this expression, we use the order of operations, which means the exponents are done first, then multiplications, and then addition.

$$4 \times 5^2 + 2 \times 6^3 = 4 \times 25 + 2 \times 216 = 100 + 432 = 532$$

Answer 2

To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, then the additions. Remember that any base raised to the exponent 0 is 1.

$$6 \times 8^2 + 3 \times 8^1 + 4 \times 8^0 = 6 \times 64 + 3 \times 8 + 4 \times 1 = 384 + 24 + 4 = 412$$

Answer 3

To evaluate the expression, we use the order of operations, which means the exponents are done first, then the multiplications, and then the additions. Remember that any base raised to the power 0 is 1.

$$3 \times 10^2 + 0 \times 10^1 + 6 \times 10^0 = 3 \times 100 + 0 \times 10 + 6 \times 1 = 300 + 0 + 6 = 306$$

Your Turn 1.1.1: Evaluate

Find the value of the expression: $6 \cdot 3^3 + 3 \cdot 3^2 + 2 \cdot 3$

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Converting Hindu-Arabic Numerals to Expanded Form

When you see the number 738, and you speak the number out loud, what do you say? You probably said "seven hundred thirty-eight" while wondering what point could possibly be made by asking this. What you didn't say was "seven, and three, and eight." A pre-K student might say that. Which should make you wonder, why?

The reason is that you've been taught place values, or the positions of digits in a number that determine the values of those digits. You know that in a three-digit number, the first digit is hundreds, the second digit is tens, and the last digit is ones. These place values rely on powers of 10, making this system a base 10 system.

Place Value in Hindu Arabic System

Place values in the Hindu-Arabic System rely on powers of 10. They are

$$\begin{array}{l} \ldots, 10^6, 10^5, 10^4, 10^3, 10^2, 10^1, 10^0 \\ \text{Or} \\ \ldots, 1000000, 100000, 10000, 1000, 100, 10, 1 \end{array} \tag{1.1.1}$$

$1,000,000$	$100,000$	$10,000$	$1,000$	100	10	1
millions	hundred thousands	ten thousands	thousands	hundreds	tens	ones

This sense of place value makes our system of numbers so useful. You've also been taught the Hindu-Arabic numeration system. This system, which uses the digits $0, 1, 2, 3, 4, 5, 6, 7, 8$, and 9, and also employs place value based on powers of 10, is in use today.

Example 1.1.2: Identify the Place Value

Write the place value of each digit below in the number $75,140,382$.

Answer

The place values of each digit are shown in the table.

Millions			Thousands			Ones		
Hundred millions	Ten millions	Millions	Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Ones
	7	5	1	4	0	3	8	2

Figure 1.1.2: Place value of $75,140,382$.

Your Turn 1.1.2: Find Place Values

Consider the number $389,454$

Write the digit for the given place value in the whole number above.

Ten thousands:

Hundreds:

Ones:

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People in Mathematics: Aryabhata of Kusumapura and Brahmagupta

The Hindu-Arabic numeral system developed in India, and Aryabhata of Kusumapura is credited with the place value notation in the 5th century. However, the system wasn't as complete as it could be until. Roughly a century later, when Brahmagupta introduced the symbol for 0. The 0 is necessary to indicate that a given place value has been skipped, as in $4,098$. In $4,098$, the 10^2 power is skipped. Without such a symbol, $4,098$ and 498 look similar. The value of both the place value notation and the introduction of the symbol 0 cannot be overstated, for math and the sciences.

Example 1.1.3: Writing a Number in Expanded Form

Write the following in expanded form.

1. 563
2. $4,821$
3. $903,786$

Answer 1

Multiply each digit by its respective place value and add them. From right to left, the place value of 3 is 10^0, the place value of 6 is 10^1, and the place value of 5 is 10^2. So expanded form of 563 is

$$5\times 10^2+6\times 10^1+3\times 10^0 \tag{1.1.2}$$

Answer 2

Multiply each digit by its respective place value and add them. From right to left, the place value of 1 is 10^0, the place value of 2 is 10^1, the place value of 8 is 10^2, and the place value of 4 is 10^3. So expanded form of $4,821$ is

$$4\times 10^3+8\times 10^2+2\times 10^1+1\times 10^0 \tag{1.1.3}$$

Answer 3

Similarly, we can expand $903,786$.

$$9 \times 10^5 + 0 \times 10^4 + 3 \times 10^3 + 7 \times 10^2 + 8 \times 10^1 + 6 \times 10^0 \tag{1.1.4}$$

Your Turn 1.1.3: Write in Expanded Form

Rewrite this base 10 numeral in expanded form. Use '*' for multiplication and '^' for exponents.

$$10,958$$

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Converting Numbers in Expanded Form to Hindu-Arabic Numerals

Converting from an expanded form back into a Hindu-Arabic numeral is the reverse process of expanding a number and is equivalent to evaluating the exponential expression.

Example 1.1.4: Converting a Number from Expanded Form to a Hindu-Arabic Numeral

Convert the following into Hindu-Arabic numerals.

1. $3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0$
2. $5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0$
3. $6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0$

Answer 1

Evaluating the expression results in:

$$3 \times 10^2 + 4 \times 10^1 + 8 \times 10^0 = 300 + 40 + 8 = 348$$

Answer 2

Evaluating the expression results in:

$$5 \times 10^3 + 0 \times 10^2 + 9 \times 10^1 + 9 \times 10^0 = 5000 + 0 + 90 + 9 = 5,099$$

Answer 3

Evaluating the expression results in:

$$6 \times 10^6 + 2 \times 10^5 + 0 \times 10^4 + 9 \times 10^3 + 1 \times 10^2 + 1 \times 10^1 + 7 \times 10^0 = 6,209,117$$

Your Turn 1.1.4: Write in Standard Form

Rewrite this expanded form numeral in decimal form

$$(4 \times 10^7) + (7 \times 10^6) + (9 \times 10^5) + (3 \times 10^4) + (9 \times 10^3) + (6 \times 10^2) + (5 \times 10^1) + (9 \times 10^0)$$

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Place Value to the Right of the Decimal Point

Place values after the decimal point represent **fractions of one** and follow a specific order, decreasing in size from left to right. Here's how they go: Tenths ($\frac{1}{10}$), Hundredths ($\frac{1}{100}$), Thousandths ($\frac{1}{1000}$), Ten-thousandths ($\frac{1}{10000}$), etc. and those place values are summarized in table.

$\frac{1}{10,000,000}$	$\frac{1}{1,000,000}$	$\frac{1}{100,000}$	$\frac{1}{10,000}$	$\frac{1}{1000}$	$\frac{1}{100}$	$\frac{1}{10}$
Ten Millionths	Millionths	Hundred Thousandths	Ten Thousandths	Thousandths	Hundredths	tenths

Example 1.1.5: Writing a Number in Expanded Form

Which number in 1642.895 represents the hundredths place value?

Answer

9 is in the hundredths place.

Your Turn 1.1.5: Find Place Value

Consider the following number:

802.64913

What is the place value of the digit 8?

Select an answer

What is the place value of the digit 4?

Select an answer

What is the place value of the digit 6?

Select an answer

What is the place value of the digit 2?

Select an answer

What is the place value of the digit 9?

Select an answer

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1.2: Early Numeration Systems

Figure 1.2.1 : Babylonians used clay tablets for writing and record keeping. (credit: modification of work by Osama Shukir Muhammed Amin FRCP(Glasg), CC BY 4.0 International)

Learning Objectives

1. Understand and convert Babylonian numerals to Hindu-Arabic numerals.
2. Understand and convert Mayan numerals to Hindu-Arabic numerals.
3. Understand and convert between Roman numerals and Hindu-Arabic numerals.

Each culture throughout history had to develop its own method of counting and recording quantity. The system used in Australia would necessarily differ from the system developed in Babylon, which would, in turn, differ from the system developed in sub-Saharan Africa. These differences arose due to cultural differences. In nearly all societies, knowing the difference between one and two would be useful. But it might not be useful to know the difference between 145 and 167, as those quantities never had a practical use. For example, a shepherd likely didn't manage more than 100 sheep, so quantities larger than 100 might never have been encountered. This can even be seen in our use of the term few, which is an inexact quantity that most would agree means more than two. However, as societies became more complex, as commerce arose, as military bodies developed, so did the need for a system to handle large numbers. No matter the system, the issues of representing multiple values and how many symbols to use had to be addressed. In this section, we explore how the Babylonians, Mayans, and Romans addressed these issues.

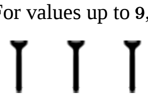
Understand and Convert Babylonian Numerals to Hindu-Arabic Numerals

Babylonian numerals are part of one of the earliest known number systems, developed by the ancient Babylonians around 2000 BCE. The time period for the Babylonian civilization is roughly from 1894 **BCE to** 539 **BCE.** Unlike our modern decimal (base - 10) system, Babylonian numerals were **sexagesimal**, meaning they were based on 60. This is why we still divide an hour into 60 minutes or a circle into 360 degrees.

The Babylonians used a mix of an additive system of numbers and a positional system of numbers. An additive system is a number system where the value of repeated instances of a symbol is added the number of times the symbol appears. A positional system is a system of numbers that multiplies a “digit” by a number raised to a power, based on the position of the “digit.”

The Babylonian place values didn’t use powers of 10, but instead powers of 60. They didn’t use 60 different symbols, though. For the value 1, they used the following symbol:

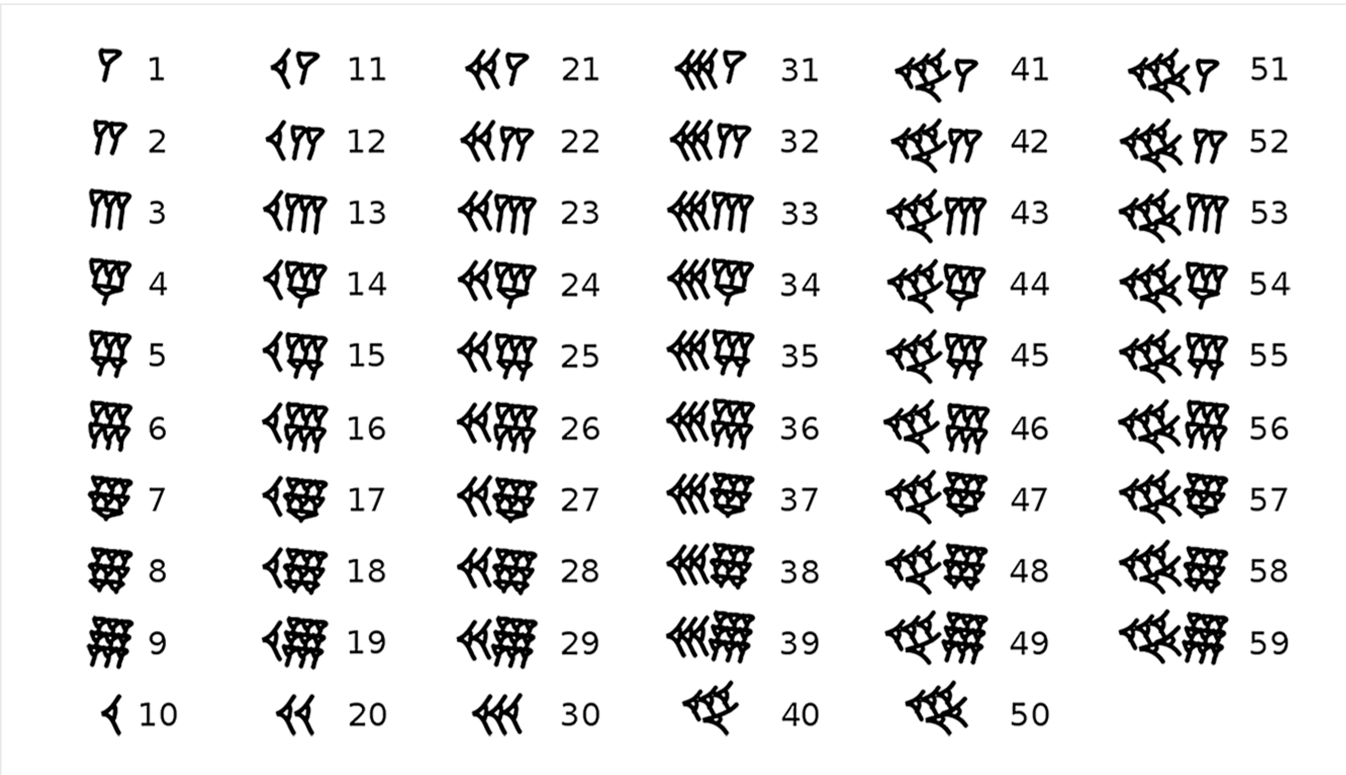
For values up to 9, that symbol would be repeated, so three would be written as



To represent the quantity 10, they used

For 20, 30, 40, and 50,they repeated the symbol for 10 however many times it was needed, so 40 would be written.

When they reached 60, they moved to the next place value. The complete list of the Babylonian numerals up to 59 is given in the following table.



You can see how Babylonians repeated the symbols to indicate multiples of a value. The number 6 is 6 of the symbol for 1 grouped together. The symbol for 30 is three of the symbols for 10 grouped together. However, their system doesn't go past 59. To go past 59, they used place values. As opposed to the Hindu-Arabic system, which was based on powers of 10, the Babylonian positional system was based on powers of 60.

You should also notice there is no symbol for 0, which has some impact on the number system. Since the Babylonian number system lacked a 0, they didn't have a placeholder when a power of 60 was absent. Without a 0, 101, 110, and 11 all look the same. However, there is some evidence that the Babylonians left a small space between "digits" where we would use a 0, allowing them to represent the absence of that place value.

To summarize, the **Babylonian system of numbers** used repeating a symbol to indicate more than one, used place values, and lacked a 0.

The place values in the Babylonian numeration system use a power of 60. The place values are

$$\ldots, 60^3, 60^2, 60^1, 60^0$$

We can expand the above exponential notation as follows.

$$\ldots, 216000, 3600, 60, 1$$

Who Knew?: Invention of 0

The idea of 0 is not a natural one. Most cultures failed to recognize the need for a 0. If someone asked a farmer in 300 B.C.E. how many cows they had, but they had none, they would not answer "zero." They'd say "I don't have any" and be done with it. It wasn't until roughly 3 B.C.E. that 0 appeared in Mesopotamia. It was independently discovered (or invented!) in the Mayan culture around 4 C.E. It made its appearance in India in the 400 s C.E., and began to spread at that point. It wasn't developed earlier, mostly because positional systems were not yet fully developed. Once positional systems arose, the need to represent a missing power had to be addressed.

The place values in the Babylonian system are based on powers of 60. If you have n digits in the Babylonian number, you multiply the first "digit" by 10 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 60 raised to a power that is one smaller. However, be careful of spaces, since they represent a zero in that place.

Step to Convert Babylonian Number to Hindu-Arabic Number

1. Identify how many digits a Babylonian number has.
2. Identify the Hindu-Arabic value and place value of each Babylonian digit.
3. Multiply each digit by its place value.
4. **Add them up** to get the Hindu-Arabic number.

Note: The rightmost Babylonian digit has place value 1, the next one has place value 60^1, and so on...

Example 1.2.1: Converting Two-Digit Babylonian Number to Hindu-Arabic Number

Convert the Babylonian number to the Hindu-Arabic Number.

Answer

This number has two digits (symbols). Note that the space between digits separates place values in a numeral from one another. Now convert each Babylonian numeral to an Arabic numeral, multiply them by their place value, and add those products.

Place value is Babylonian systems are $\ldots, 216000, 3600, 60, 1$

The left digit is 4, and it place value is 60 and the right digit is 27, and its place value is $1 = 60^1$.

$4 \times 60^1 + 27 \times 60^0 = 240 + 27 = 267$

So, the above Babylonian number equals 267 in the Hindu-Arabic number system.

Example 1.2.2: Converting Three-Digit Babylonian Number to Hindu-Arabic Number

Convert the Babylonian number into a Hindu-Arabic number.

Answer

This number has three digits (symbols). Note that the space between digits separates place values in a numeral from one another.

Now convert each Babylonian numeral to a Hindu-Arabic numeral, multiply them by their place value, and add those products.

Place value is Babylonian systems are $\ldots, 216000, 3600, 60, 1$

The left digit is 13, and its place value is 3600. The middle digit is 8, and its pace value is 60. The right digit is 54, and its place value is 1.

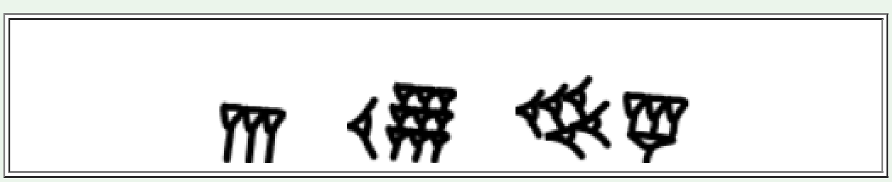
Multiplying and adding them yields

$13 \times 3{,}600 + 8 \times 60 + 54 \times 1 = 46{,}800 + 480 + 54 = 47{,}334$

So, the above Babylonian number equals $47,334$ in the Hindu-Arabic number system.

Your Turn 1.2.2: Babylonian to Hindu Arabic

Convert the Babylonian number shown below into Hindu-Arabic.



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Example 1.2.3: Converting Four-Digit Babylonian Number to Hindu-Arabic Number

Convert the Babylonian number into a Hindu-Arabic number.

Answer

This number has three digits (symbols). Note that the space between digits separates place values in a numeral from one another.

Now convert each Babylonian numeral to an Arabic numeral, multiply them by their place value, and add those products. And place value in Babylonian systems is $\ldots, 216,000, 3600, 60, 1$

$$\begin{aligned} & 11 \times 216000 + 40 \times 3600 + 3 \times 60 + 3 \times 1 \\ & = 2,376,000 + 144,000 + 180 + 3 \\ & = 2,520,183 \end{aligned} \tag{1.2.1}$$

So, the above Babylonian number equals $2,520,183$ in the Hindu-Arabic number system.

Your Turn 1.2.3: Babylonian to Hindu Arabic

Consider the following Babylonian cuneiform representation of a number.

Determine the equivalent decimal representation of the above Babylonian cuneiform number.

The decimal equivalent of the above value is ________.

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Who Knew?: The Legacy of Babylonian System

The Babylonian system can still be seen today. An hour is 60 minutes, and a minute is 60 seconds. Additionally, when measuring angles in degrees, each degree can be split into 60 minutes $(\frac{1}{60})^{\text{th}}$ of a degree) and 60 seconds $(\frac{1}{60})^{\text{th}}$ of minute.

Understand and Convert Mayan Numerals to Hindu-Arabic Numerals

The Mayans employed a positional system just as we do, and the Babylonians did, but they based their position values on powers of 20, and they had a dedicated symbol for zero. Like the Babylonians, the Mayans repeated symbols to indicate specific values. A single dot was a 1, two dots were a 2, up to four dots. Then, a five was a horizontal bar. The horizontal bars could be used three times since the fourth horizontal bar would make a 20, which was a new position in the number. The 0 was a unique picture, which appears like a turtle lying on its back. The shell would then be "empty," so maybe that's why the symbol was 0. The complete list is provided in Table $1.2.1$. Another feature of Mayan numbers was that they were written vertically. The powers of 20 increased from bottom to top.Adapt 1.2.1

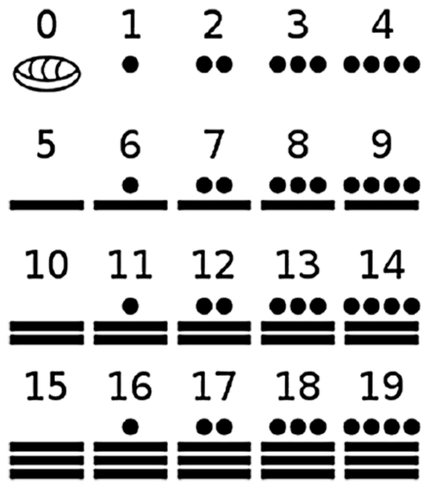


Table 3 1.2.1: Mayan Numerals. (Copyright; author via source)

To summarize, the **Mayan system of numbers** used a repeating symbol to indicate more than one, used place values, and employed a 0. So, how do we convert from Mayan numbers to Hindu-Arabic numbers? To do so, we need to use the symbols from Table 1.2.1 and then place values based on powers of 20. If you have n digits in the Mayan number, you multiply the first "digit" by 20 raised to one less than the number of "digits." You then continue through the "digits," multiplying each by 20 raised to a power that is one smaller than the previous power. Fortunately, there is an explicit 0, so there is no ambiguity about numbers like $110, 101$, and 11.

Mayan numerals are written vertically rather than horizontally, with the units positioned on the bottom.

The place values in the Mayan numeration systems are

$$\ldots, 18 \times 20^3, 18 \times 20^2, 18 \times 20^1, 20^1, 20^0$$

We can expand the above exponential notation as follows.

$$\ldots, 144000, 7200, 360, 20, 1$$

It is believed that the Mayans used 18×20 so that their numeration system would conform to their calendar of 18 months, 20 days each, plus 5 "ghost days" to complete the 365 day year.

Step to Convert Mayan Numeral to Hindu Arabic Numeral?

1. The number in the bottom row is to be multiplied by 1.
2. The number in the second row from the bottom is to be multiplied by 20.
3. The number in the third row will be multiplied by or 360.
4. The number in the fourth row will be multiplied by 7200 and so on, and then add all the products.

Example 1.2.4: Converting Two-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the Mayan number into a Hindu-Arabic number.

Answer

The first symbol (bottom) represents 9, so this is multiplied by $1 = 20^0$. The second symbol (second row from the bottom) represents 15 in the Mayan system. This is multiplied by 20. Therefore

$15 \times 20^1 + 9 \times 20^0 = 300 + 9 = 309$

So, the above Mayan number equals 309 in the Hindu-Arabic number system.

Your Turn 1.2.4

Convert the Mayan number into a Hindu-Arabic number.

Example 1.2.5: Converting Three-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the following Mayan number into a Hindu-Arabic number.

Answer

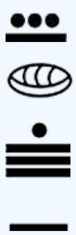
The first symbol (bottom) represents 4 in the Mayan system, and this is multiplied by 1. The second row from the bottom represents 8, so this is multiplied by 20, and the last digit (third from the bottom) represents 6, multiplied by 18×20.

$$6 \times 18 \times 20 + 8 \times 20 + 4 \times 1 = 2160 + 160 + 4 = 2324$$

So, the above Mayan number equals $2,234$ in the Hindu-Arabic number system.

Example 1.2.6: Converting Four-Digit Mayan Numbers to Hindu-Arabic Numbers

Convert the following Mayan number into a Hindu-Arabic number.



Answer

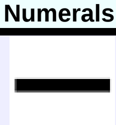
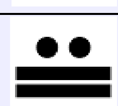
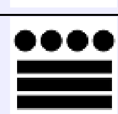
As we did in the previous examples,

$$8 \times 18 \times 20^2 + 0 \times 18 \times 20 + 16 \times 20 + 5 \times 1 = 57,600 + 0 + 320 + 5 = 57,925$$

So, the above Mayan number equals $57,925$ in the Hindu-Arabic number system.

Your Turn 1.2.6: Mayan to Hindu Arabic

Consider the Mayan numeral given in the left column of the table below. In the middle column, report the *expanded* form of each Mayan digit. In the right column, report the simplified Hindu-Arabic numeral equivalent to each Mayan numeral.

Mayan Numerals	Hindu Arabic Value	Product after Multiplying by place values
		
		
		

Determine the sum of the values in the right column.

The sum of the values in the right column is ______ .

Determine the Hindu-Arabic representation of the given Mayan numeral.

The Hindu-Arabic representation of the Mayan numeral is ______ .

Hint

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Who Knew?: The Mayan Calendar

The Mayans used this base 20 system for everyday situations. But their culturally important, and extremely accurate, calendar system used a slightly different system. For their calendars, they used a system where the place values were $1, 20$, then 20×18, then $20 \times 20 \times 18$. The reason for this is 20×18 is 360, which is closer to the number of days in a year. Had they used a purely base 20 system for their calendar, they'd be very far off with 400 days in a year.

Three hundred sixty days still left the Mayans a bit short, as there are 365 days in a year (ignoring leap years). The Mayan calendar also included 5 days, called Wayeb days, which brings their calendar to 365 days. As it happens, Wayeb is the Mayan god of misfortune, so these 5 days were considered the bad luck days.

Understand and Convert Between Roman Numerals to Hindu-Arabic Numerals

The Mayan and Babylonian systems shared two features, one of which we are familiar with (place value) and one that we don't use (repeated symbols). The **Roman system of numbers** used repeated symbols but did not employ a place value. It also lacks a 0. The Roman system is built on the following symbols in Table $1.2.2$.

Table $1.2.2$ Roman Numerals

Roman Numeral	Hindu-Arabic Value
I	1
V	5
X	10
L	50
C	100
D	500
M	$1,000$

As in the Mayan and Babylonian systems, a symbol may be repeated to indicate a larger value. However, at 4, they did not use $IIII$. They instead used IV. Since I came before the V, the number stands for "one before five." A similar process was used for 9, which was written IX, or "one before ten." The value 40 was written XL, or "ten before fifty," while 49 was written $XLIX$, or "forty plus nine."

Roman Numerals: Remember Those Rules.

1. Up to three symbols may be grouped together; for example, III for 3, or XXX for 30, or CC for 200. Only I, V, and C can be repeated.
2. A larger value followed by a smaller value indicated addition; for example, VII for 7, $XIII$ for 13, LV for 55, andMCC for 1200.
3. I can be placed before V and X. So $IV = 4$ and $IX = 9$. These are the only ways I is used as a subtraction.
4. X can be placed before L and C. So $XL = 40$ and $XC = 90$. These are the only ways X is used as a subtraction.
5. C can be placed before D and M. So $CD = 400$ and $CM = 900$. These are the only ways C is used as a subtraction.
6. If there is a smaller value in front of a larger value, those two values are grouped, and then we use the subtract property in the Roman numeral system. For example, $XXIX$ for 29 and $CCXC$ for 290.

7. If the values of the symbols decrease from left to right, add them to find the total value of the Roman numeral. If the values increase from left to right, subtract them instead."

Your Turn 1.2.7: Roman to Hindu Arabic

Fill in the blank to produce a true statement.

If the symbols in the Roman numeral representation of a number *decrease* in value from left to right, then one should [Select an answer] the values of the symbols to obtain the decimal representation of the Roman numeral.

For example, consider the Roman numeral

$$\text{MD}$$

The Roman numeral text(MD) may be converted to its Hindu-Arabic numeral representation by the following calculation.

$\text{MD} = \quad 1000$ [?] $500 =$ ______

Hint

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Who Knew?: Legacy of Roman Numerals

The Roman numbering system is still used today in some situations. Many cornerstones of buildings have the year written in Roman numerals. Movie titles often represent the year the movie was produced as Roman numerals. The most recognizable might be that the Super Bowl is numbered using Roman numerals.

Example 1.2.8: Converting Roman Numerals to Hindu-Arabic Numbers

Convert the following Roman numerals into Hindu-Arabic numerals.

1. $XXVII$
2. $XXXIV$
3. $MMCMXLVIII$

Answer

1. Is there a smaller value in front of the large value? No. So, add the value of each Roman numeral symbol to get the value of the whole. Therefore

$$\begin{aligned} XXVII &= 10+10+5+1+1 \\ &= 27 \end{aligned} \tag{1.2.2}$$

2. Is there a smaller value in front of the large value? Yes. The symbol I (smaller) is in front of V (bigger). So, we group IV, whose value is $5-1=4$. Therefore

$$\begin{aligned} XXXIV &= 10+10+10+(5-1) \\ &= 34 \end{aligned} \tag{1.2.3}$$

3. Is there a smaller value in front of the large value? Yes. The symbol X (smaller) is in front of L (bigger). So we group XL, whose value is $50-10=40$, and also the symbol C (smaller) is in front of M (bigger). So, we group CM whose value is $1000-100=900$. Therefore

$$\begin{aligned} MMCMXLVIII &= 1000+1000+900+40+5+1+1+1 \\ &= 2,948 \end{aligned} \tag{1.2.4}$$

Understand and Convert Hindu-Arabic Numerals to Roman Numerals

Example 1.2.8: Converting Hindu-Arabic Numbers to Roman Numerals

Convert the following Hindu-Arabic numerals into Roman numerals.

1. 38
2. 94
3. 846
4. $2,987$

Answer

1. Thirty is represented as three $X's$, and the 8 is represented with $VIII$, so 38 in Roman numerals is $XXXVIII$. Or You can write 38 as 30 plus 8, 30 = XXX and 8= $VIII$. So 38 = $XXXVIII$.
2. 94 is 90 plus 4. 90 is represented by XC, and 4 is represented by IV, so 94 in Roman numerals is $XCIV$.
3. The number is less than 900 and more than 500, so the first symbol to be used is D, which is 500. To get to 800, we need 300 more, which is represented by three C's. 40 is represented by XL, and 6 is represented by VI. The Roman numerals are $DCCCXLVI$. Or you can write 846 is 500 plus 300 plus 40 plus 6. So 846 = $DCCCXLVI$
4. The 2000 is represented by two $M's$. The 900 is represented by CM. The 80 is represented by $LXXX$ $(50 + 30)$. Finally, the 7 is represented by VII. We have that $2,987$ in Roman numerals is $MMCMLXXXVII$.

Your Turn 1.2.8: Roman to Hindu Arabic and Vice Versa

A. Convert the following Roman numerals to a Hindu-Arabic numeral.

1) CDLXXV = ______

2) DXIII = ______

3) DCCXIV = ______

B. Convert the following Hindu-Arabic numerals to Roman numerals.

4) 2453 = ______

5) 1732 = ______

6) 1016 = ______

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Vinculum

Roman numerals with a **bar over them** are a cool little feature! The bar—called a **vinculum**—is used to **multiply the numeral by 1,000**. For example $\overline{LV}III = 55 \times 1000(\overline{LV}) + 3$ (III) = $55,003$

Example 1.2.9

Convert the following Roman numeral to a Hindu-Arabic numeral.

1. $\overline{X}$
2. $\overline{CM}$

3. $\overline{X}IV$

Answer

1. $\overline{X} = 10 \times 1,000 = 10,000$
2. $\overline{CM} = (1000 - 100) \times 1,000 = 900,000$
3. $\overline{X}IV = 10 \times 1,000 + (5 - 1) = 10,004$

Your Turn 1.2.9: Roman number with Vinculum

Consider the following value expressed using Roman numerals.

$$\text{VIDCCCXIII}$$

Express the above value using Hindu-Arabic numerals.

$\text{VIDCCCXIII} =$ ______

Hint

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1.3: Converting to Different Base Systems

1.3: Converting to Different Base Systems

Learning Objectives

1. Convert base ten to different bases.
2. Convert different bases to base ten.

Introduction and Basic

During the previous discussions, we have been referring to positional base systems. In this chapter, we will explore exactly what a base system is and what it means if a system is "positional." We will do so by first looking at our own familiar base-ten system and then deepening our exploration by looking at other possible base systems.

A base system is a structure within which we count. The easiest way to describe a base system is to think about our own base-ten system. The base-ten system, which we call the "decimal" system, requires a total of ten different symbols/digits to write any number. They are, of course, $0, 1, 2, \ldots \ldots 9$.

The decimal system is also an example of a *positional* base system, which means that the position of a digit gives its place value. Not all civilizations had a positional system, even though they did have a base with which they worked.

In our base-ten system, a number like $5,783,216$ has meaning to us because we are familiar with the system and its places. As we know, there are six ones since there is a 6 in the ones' place. Likewise, there are seven hundred thousand since the 7 reside in that place. Each digit has a value explicitly determined by its position within the number. We distinguish between a digit, a symbol such as 5, and a number, which comprises one or more digits. We can take this number and assign each of its digits a value. One way to do this is with a table, which follows:

$5,000,000$	$= 5 \times 1,000,000$	$= 5 \times 10^6$	Five million
$+700,000$	$= 7 \times 100,000$	$= 7 \times 10^5$	Seven hundred thousand
$+80,000$	$= 8 \times 10,000$	$= 8 \times 10^4$	Eighty thousand
$+3,000$	$= 3 \times 1000$	$= 3 \times 10^3$	Three thousand
$+200$	$= 2 \times 100$	$= 2 \times 10^2$	Two hundred
$+10$	$= 1 \times 10$	$= 1 \times 10^1$	Ten
$+6$	$= 6 \times 1$	$= 6 \times 10^0$	Six
$5,783,216$	Five million, seven hundred eighty-three thousand, two hundred sixteen		

The third column in the table shows that each place is simply a multiple of ten. Of course, this makes sense, given that our base is ten. The digits that are multiplying each place tell us how many of that place we have. We are restricted to having at most 9 in any one place before we have to "carry" over to the next place. We cannot, for example, have 11 in the hundreds place. Instead, we would carry 1 to the thousands place and retain 1 in the hundreds place. This comes as no surprise to us since we readily see that 11 hundreds is the same as one thousand, one hundred. Carrying is a pretty typical occurrence in a base system.

However, base ten is not the only option we have. Practically any positive integer greater than or equal to 2 can be used as a base for a number system. Such systems can work just like the decimal system, except the number of symbols will differ, and each position will depend on the base itself.

Other Bases

For example, let's suppose we adopt a base-five system. The only modern digits we would need for this system are $0, 1, 2, 3$, and 4. What are the place values in such a system? To answer that, we start with the ones place, as most base systems do. However, if we were to count in this system, we could only get to four (4) before we had to jump up to the next place. Our base is 5, after all! What is the next place that we would jump to? It would not be tens since we are no longer in base ten. We're in a different numerical world. As the base-ten system progresses from 10^0 to 10^1, so the base-five system moves from 5^0 to $5^1 = 5$. Thus, we move from the ones to the fives.

After the fives, we would move to the 5^2 place, or the twenty-fives. Note that in base-ten, we would have gone from the tens to the hundreds, which is, of course, 10^2.

Let's take an example and build a table. Consider the number 30412 in base five. We will write this as 30412_5, where the subscript 5 is not part of the number but indicates the base we're using. First off, note that this is NOT the number "thirty thousand, four hundred twelve." We must be careful not to impose the base-ten system on this number.

You must read as 30412_5 as "three-zero-four-one-two" in base 5"

Here's what our table might look like. We will use it to convert this number to our more familiar base-ten system.

Place Value and Digits Used in Other Bases

The digits used in any base, b, can not reach b. For example, the digits used in the base b are $0, 1, 2, \ldots, b-1$. And the place values in base b are the power of b.

$$\ldots, b^4, b^3, b^2, b^1, 1$$

Base	Digit symbols	Place Value
2	$0, 1$	$\ldots, 2^3, 2^2, 2^1, 1$
3	$0, 1, 2$	$\ldots, 3^3, 3^2, 3^1, 1$
4	$0, 1, 2, 3$	$\ldots, 4^3, 4^2, 4^1, 1$
5	$0, 1, 2, 3, 4$	$\ldots, 5^3, 5^2, 5^1, 1$
6	$0, 1, 2, 3, 4, 5$	$\ldots, 6^3, 6^2, 6^1, 1$
7	$0, 1, 2, 3, 4, 5, 6$	$\ldots, 7^3, 7^2, 7^1, 1$
8	$0, 1, 2, 3, 4, 5, 6, 7$	$\ldots, 8^3, 8^2, 8^1, 1$
9	$0, 1, 2, 3, 4, 5, 6, 7, 8$	$\ldots, 9^3, 9^2, 9^1, 1$
10	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9$	$\ldots, 10^3, 10^2, 10^1, 1$

Be Careful!

It's important to use only the allowed digits for a given base. For example, in a base 6 system, we can only use the symbols $0, 1, 2, 3, 4$, and 5. Therefore, we cannot write 23570_6, because the digit 7 is not valid in base 6.

Also, you do not need to specify the base for the base 10 number. For example, $21048_{10} = 21048$.

Example 1.3.1

What is the place value of the digit 2 in each number below?

1. 17526_8
2. 203_5
3. 2110_9
4. 3012_4

Answer

1. The digit 2 is in the $8^1 = 8$'s place
2. The digit 2 is in the $5^2 = 25$'s place
3. The digit 2 is in the $9^3 = 729$'s place
4. The digit 2 is in the $4^0 = 1$'s place

Your Turn 1.3.1: Identify Bases

Find the place value of 2 in 624_{six}.

Place value: Select an answer

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Converting Other Bases to Base 10

To convert a number from any base to base 10, you need to understand how numbers are represented in different bases. In base b each digit has a place value as a power of b.. The step to convert to base 10 from other bases is summarized below.

Steps to Convert Other Bases to Base 10

1. Identify the place value of each digit in the given base.
2. Multiply each digit by its place value.
3. Sum the results from step 2 to get the number in base 10.

Example 1.3.2: Convert Other Bases to Base Ten

1. Convert 6234_7 to a base 10 number.
2. Convert the base-seven number 6234_9 to base 10.

Answer 1

We first note that we are given a base-7 number that we are to convert. Thus, our places will start at the ones (7^0, and then move up to the $7's, 49's\ (7^2)$, etc. Here's the breakdown:

	Base 7	Convert	Base 10
	$=6\times7^3$	$=6\times343$	$=2058$
$+$	$=2\times7^2$	$=2\times49$	$=98$
$+$	$=3\times7$	$=3\times7$	$=21$
$+$	$=4\times1$	$=4\times1$	$=4$
		Total	2181

Thus $6234_7=2181_{10}$.

Answer 2

The following is a slightly different way to do the conversion.

The place values in base 9 are the powers of 9.

$$9^0=1$$
$$9^1=9$$
$$9^2=81$$
$$9^3=729$$

Etc...

$$6234_9=(6\times729)+(2\times81)+(3\times9)+(4\times1)=4567_{10}$$

Thus $6234_9=4567_{10}$.

Example 1.3.3: Convert Other Bases to Base Ten

1. Convert Convert 110011_2 to a base 10 number.
2. Convert 30412_5 to base 10.

Answer 1

You can also make a table similar to what we did in example $1.3.2$. The following is a slightly different way to do the conversion. Here, we identify the place value of each digit in base 2, multiply those digits by their place value, and then ADD the product.

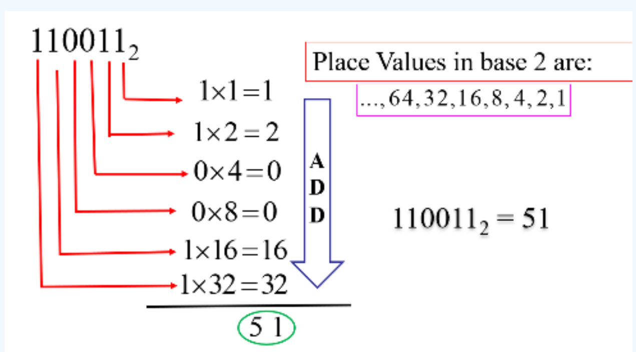

Figure $1.3.1$: Convert Base 2 to base 10.

Answer 2

Let's make a table to do the conversion.

	Base 5	This column coverts to base-ten	In Base-Ten
	3×5^4	$= 3 \times 625$	$= 1875$
+	0×5^3	$= 0 \times 125$	$= 0$
+	4×5^2	$= 4 \times 25$	$= 100$
+	1×5^1	$= 1 \times 5$	$= 5$
+	2×5^0	$= 2 \times 1$	$= 2$
		Total	1982

As you can see, the number $30412_5$is equivalent to $1,982$ in base-ten. We will say $30412_5 = 1982_{10}$.

Your Turn 1.3.3: Convert Other Bases to Base Ten

The following number is a **base-five numeral**. Convert this to a base ten numeral.

4615_{five}= ______ *ten*

The following number is a **base-eight numeral**. Convert this to a base ten numeral.

4615_{eight}= ______ *ten*

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Converting Base Ten to Other Bases

Converting from an unfamiliar base to a familiar decimal system is not that difficult once you get the hang of it. It's only a matter of identifying each place and multiplying each digit by the appropriate power. However, going the other direction can be a little trickier. Suppose you have a base-ten number and want to convert it to base-five. Let's start with some simple examples before we get to a more complicated one.

Steps to Convert Base 10 to Other Bases

1. Find the highest power of the base b that will divide into the given number at least once, and then divide.
2. Write down the whole number part, then use the remainder from the division in the next step.
3. Repeat step two, dividing by the next highest power of the base b, writing down the whole number part (including 0), and using the remainder in the next step.
4. Continue until the remainder is smaller than the base. This last remainder will be in the "ones" place.
5. Collect all your whole-number parts to get your number in base b notation.

Example 1.3.4: Convert Base 10 to Other Bases

1. Convert the base-ten number 16 to a base-five number.
2. Convert the base-ten number 2 to a base-two number.
3. Convert the base-ten number 25 to a base-nine number.

Answer 1

Place values in base five are $\ldots, 25, 5, 1$

The largest place value smaller than 16 that divides the given number 16 is 5.

$16 \div 5 = \quad 3 \quad \text{R} \quad 1$

We'll say that $16_{10} = 31_5$.

Answer 2

Place values in base two are $\ldots, 4, 2, 1$

The largest place value smaller than (or equal to) 2 that divides the given number 2 is 2.

$$2 \div 2 = \quad 1 \quad R \quad 0$$

We'll say that $2_{10} = 10_2$.

Answer 3

Place values in base nine are $\ldots, 81, 9, 1$

The largest place value smaller than 25 that divides the given number 25 is 9.

$$25 \div 9 = \quad 2 \quad R \quad 7$$

We'll say that $25_{10} = 27_9$.

Your Turn 1.3.4: Convert Base 10 to Other Bases

Consider the following base-10 numeral.

$$5$$

Convert the given base-10 numeral to base 2.

$5_{\text{ten}} =$ ______ $_{\text{two}}$

Consider the following base-10 numeral.

$$6$$

Convert the given base-10 numeral to base 4.

$6_{\text{ten}} =$ ______ $_{\text{four}}$

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Example 1.3.5: Convert Base 10 to Other Bases

1. Convert base 10 number 23 to a base four number.
2. Convert base 10 number 2763 to a base five number.

Answer 1

Place values in base four are $\ldots, 16, 4, 1$

The largest place value smaller than 23 that divides the given number 23 is 16.

$$\begin{array}{lccc} 23 \div 16 = & 1 & R & 7 \\ 7 \div 4 = & 1 & R & 3 \end{array}$$

We'll say that $23_{10} = 113_4$

We can check our work by converting 113_4 to base ten.

$$(1 \times 16) + (1 \times 4) + (3 \times 1) = 23$$

Here, we have 1 sixteen, 1 four, and 3 ones. Hence, we have 113 in base four Thus, $23_{10} = 113_4$.

Answer 2

Note: In general, when converting from base-ten to some other base, it is often helpful to determine the highest power of the base that will divide into the given number at least once. In the last example, $4^2 = 16$ is the largest power of five that is present in 23, so that was our starting point. If we had moved to $4^3 = 64$, then 64) would not divide into 23 at least once.

Place values in base five are $\ldots, 3125, 625, 125, 25, 5, 1$

The largest place value smaller than 2763 that divides the given number 2763 is 625.

$$\begin{array}{lll} 2763 \div 625 = & 4 & \text{R} \quad 263 \\ 263 \div 125 = & 2 & \text{R} \quad 13 \\ 13 \div 25 = & 0 & \text{R} \quad 13 \\ 13 \div 5 = & 2 & \text{R} \quad 3 \end{array}$$

Putting all of this together means that $2763_{10} = 42023_5$.

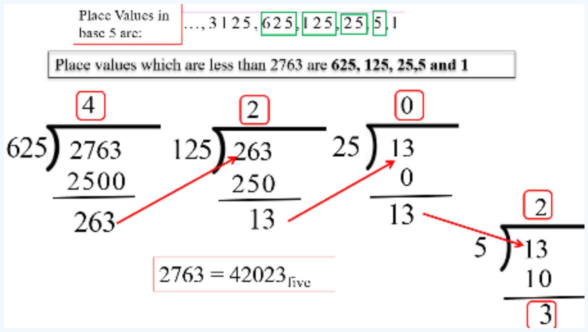

Figure 1.3.2: Convert 2763 to base 5.

We can use the above division in a long format as follows.

Your Turn 1.3.5: Convert Base 10 to Other Bases

Consider the following base-10 numeral.

$$617$$

Mentally convert the given base-10 numeral to base 4.

$617_{\text{ten}} =$ ______ $_{\text{four}}$

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Working on the Bases Higher Than Ten.

Using Base 12

As mentioned in the text, working in base 10 is mathematically awkward. Ten has only two natural number divisors: 2 and 5. This means dividing into groups is not easy. However, 12, or a dozen, has more divisors: $2, 3, 4$, and 6. The Dozenal Society recognizes this more mathematically pleasant detail. It advocates for a switch to using base 12 for numbers. Their argument is based on the divisibility of the number 12. But has there ever been a society that used such a system? The answer is yes. A dialect of the Gwandara language in Nigeria uses the base 12 system. It is unlikely, though, that the Dozenal Society will achieve their goal, as the base 10 system is so entrenched in our society.

A number base higher than ten and less than or equal to sixteen includes systems like base-9 through base-16, which use additional symbols beyond the standard digits 0 to 9. To represent values greater than 9, these systems introduce letters from the alphabet, starting with **A** for 10, **B** for 11, **C** for 12, **D** for 13, **E** for 14 and **F** for 15 in base-16 (hexadecimal). These bases are especially useful in fields like computer science and digital electronics. For example, hexadecimal (base-16) is commonly used to simplify binary code since each hex digit represents four binary digits (bits). These number systems allow for more compact and human-readable representations of data, making them practical for tasks such as memory addressing, color codes in web design, and machine-level programming.

Bases	Digits Used	Place Values
11	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A$	$\ldots, 11^3, 11^2, 11^1, 1$
12	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B$	$\ldots, 12^3, 12^2, 12^1, 1$
13	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C$	$\ldots, 13^3, 13^2, 13^1, 1$
14	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D$	$\ldots, 14^3, 14^2, 14^1, 1$
15	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$	$\ldots, 15^3, 15^2, 15^1, 1$
16	$0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F$	$\ldots, 16^3, 16^2, 16^1, 1$

Example 1.3.6: Convert Base Sixteen to Base Ten

Convert $FC52E_{16}$ into base 10.

Answer

The place values in ase 16 are powers of 16 :

$$\begin{aligned}16^0 &= 1\\16^1 &= 16\\16^2 &= 256\\16^3 &= 4096\\16^4 &= 65,536\end{aligned}$$

Etc...

$$FC52E_{16} = (F(15) \times 65,536) + (12 \times 4096) + (5 \times 256) + (2 \times 16) + (E(14) \times 1) = 1,033,518_{10}$$

Thus $FC52E_{16} = 1,033,518_{10}$.

Your Turn 1.3.6: Converst Base Sixteen to Base Ten

The following number is a **base-sixteen numeral**. Convert this to a base-ten numeral. (In base 16, the digits are 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E, F.)

401_{sixteen} = ______ *ten*

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1.4: Addition and Subtraction in Base Systems

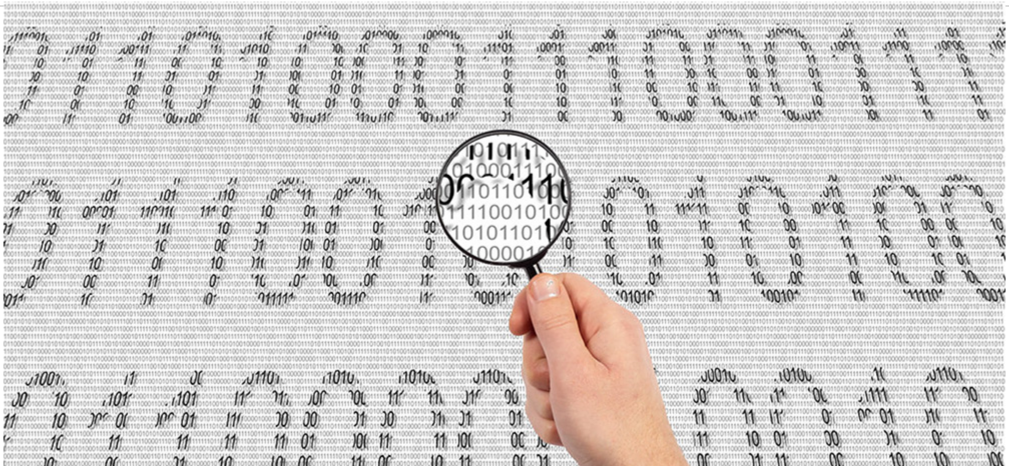

Figure 1.4.1: All information in computers is represented by 0's and 1's, including quantity, which means computers use Base 2 for arithmetic. (credit: modification of work "Magnifying glass and binary code" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

Add and subtract in bases 2 to base 16.

Once we decide on a system for counting, we need to establish rules for combining the numbers we're using. This begins with the rules for addition and subtraction. We are familiar with base 10 arithmetic, such as $2+5=7$ or $3\times 5=15$. How does that change if we use a different base instead? A larger base? A smaller one? In particular, computers use base 2 for all number representations. When your calculator adds or subtracts, multiplies or divides, it uses base 2. This is because the circuitry recognizes only two things, high current and low current, which means the system is uses only has two symbols. Which is what base 2 is.

In this section, we use addition and subtraction in bases other than 10 by referencing the processes of base 10, but applied to a new base system.

Addition in Bases Other Than Base 10

Now that we understand what it means for numbers to be expressed in a base other than 10, we can look at arithmetic using other bases, starting with addition. When you think back to when you first learned addition, it is very likely you learned the addition table. Once you knew the addition table, you moved on to the addition of numbers with more than one digit. The same process holds for addition in other bases. We begin with an addition table and then move on to adding numbers with two or more digits.

Let's create the base 6 addition table.

Look in the row that starts with "5" and select the number in the column that starts with "1". At the intersection of the selected row and column, you will find the number 10.
Is $5_6+1_6=6_6$? Since 6 is not one of the digits in base 6. So, we need to convert 6 to base 6.

$6\div 6= \quad 1 \quad R \quad 0$

We'll say that $6_{10}=10_6$

Is $5_6+2_6=7_6$? Since 7 is not one of the digits in base 6. So, we need to convert 7 to base 6.

$7\div 6= \quad 1 \quad R \quad 1$

We'll say that $7_{10}=11_6$

Similarly, $5_6+3_6=12_6$

And so it goes. Using that process, stepping one more along the list, we can create the addition table for base 6.

Be Careful!

When adding two single-digit numbers in the given base, if the sum is greater than or equal to the base, we need to convert the sum to the given base by dividing it by the base.

Base 6 : Addition Table

+	0	1	2	3	4	5
0	0	1	2	3	4	5
1	1	2	3	4	5	10
2	2	3	4	5	10	1
3	3	4	5	10	11	12
4	4	5	10	11	12	13
5	5	10	11	12	13	14

With this table and our understanding of "carrying the one," we can then use the addition table to do addition in base 6 for numbers with two or more digits, using the same processes you learned for addition when you did it by hand.

Your Turn 1.4.1: Adding Single Digit Number in Different Bases

$2_{\text{three}} + 2_{\text{three}} =$ ______ three	$3_{\text{five}} + 4_{\text{five}} =$ ______ five
$5_{\text{seven}} + 3_{\text{seven}} =$ ______ seven	$6_{\text{nine}} + 4_{\text{nine}} =$ ______ nine
$3_{\text{four}} + 2_{\text{four}} =$ ______ four	$5_{\text{six}} + 4_{\text{six}} =$ ______ six

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Example 1.4.2: Adding Base Six Numbers

Calculate $251_6 + 133_6$

Answer

$$\begin{array}{r} 251_{\text{six}} \\ +\ 133_{\text{six}} \\ \hline 424_{\text{six}} \end{array} \tag{1.4.1}$$

Let's do for the place of one's first. According to the base 6 addition table see above, $1_6 + 3_6 = 4_6$.

Now, we do the "tens" place (it's the sixes place). According to the base 6 addition table, we have $5_6 + 3_6 = 12_6$.

So, like in base 10, we use the 2 and carry the 1. Now the "hundreds" place (really, thirty-sixes place). There, we have $1(\text{carried}) + 2_6 + 1_6 = 4_6$.

Therefore

$$251_6 + 133_6 = 424_6$$

Note: When you add sixes places (middle column), do not write 8_6. Be Careful! We are adding in base 6.

Example 1.4.3: Adding Base Seven Numbers

Create the addition table for base 7. and add $536_7 + 433_7$.

Answer

In base 7, the number that follows 6 is 10 (since we've run out of symbols!). So, $6_7 + 1_7 = 10_7$.

In another word $6_7 + 1_7 = 7$, but 7 is not base 7 number so we need to convert 7 to base 7 dividing by 7 (See previous section).

Once that is established, 6_7 + 2_7 will be two numbers past 6, which is 11 in base 7 (That is 11_7). Continuing, we can fill in the rows as we would in base 10, but be aware that we are working in base 7.

+	0	1	2	3	4	5	6
0	0	1	2	3	4	5	6
1	1	2	3	4	5	6	10
2	2	3	4	5	6	10	11
3	3	4	5	6	10	11	12
4	4	5	6	10	11	12	13
5	5	6	10	11	12	13	14
6	6	10	11	12	13	14	15

$$\begin{array}{r} 536_{\text{seven}} \\ +\ 433_{\text{seven}} \\ \hline 1302_{\text{seven}} \end{array} \tag{1.4.2}$$

Let's do for the place of one's first. According to the base 7 addition table, $6_7 + 3_7 = 12_7$. We will carry the 1. Now, we do the "tens" place (it's the sevens place). According to the base 7 addition table, we have $1(\text{carried}) + 3_7 + 3_7 = 7 = 10_7$. We use the 0 and carry the 1. Now the "hundreds" place (really, forty-nines place). There, we have $1(\text{carried}) + 5_7 + 4_7 = 10 = 13_7$.

So,

$$536_7 + 433_7 = 1302_7$$

Your Turn 1.4.3: Add in Different Bases

The following numbers are written in base 8. Complete the sum and give your answer in base 8.

$160_8 + 247_8$ = ______ $_8$

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As seen previously, when performing addition in another base, set up the problem exactly as you would for addition in base 10. At each step, check the addition table for the base. As in base 10 addition, move right to the left, adding down the columns using the rules in the addition table. When necessary and just as in base 10, be sure to carry the 1.

Adding and Subtracting in a Base Higher Than Ten

As we know from the previous section, when the **base is higher than** 10, we usually use **letters** to represent digits beyond 9. For example:

Base 11 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A$, where $10 = A$

Base 12 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B$, where $10 = A, 11 = B$

Base 13 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C$, where $10 = A, 11 = B, 12 = C$

Base 14 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D$, where $10 = A, 11 = B, 12 = C, 13 = D$

Base 15 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, C, D, E$, where $10 = A, 11 = B, 12 = C, 13 = D, 14 = E$

Base 16 : Digits are $0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B, \ldots, F$, where $10 = A, 11 = B, 12 = C, 13 = D, 14 = E, 15 = F$

Let’s make an **addition table for base** 12, just as an example:

Example 1.4.4: Adding Base Twelve Numbers

Create the addition table for base 12 and calculate $3A7_{12} + 9BA_{12}$.

Answer

When the sum is up to 9, we are good. If sum is 10, we use A for 10 as a digit in base 12. When sum is 11 we use B for 11 as digit in base 12. What if sum is 12, for example, $6+6=12$, we do not have digits that represent 12 in base 10. (We only have up to 11, which is B)

In other words, since 12 is not one of the digits in base 12. So we need to convert 12 to base 12 by using division.

$12 \div 12 = \quad 1 \quad R \quad 0$

We'll say that $12 = 10_{12}$

Again, what is $9+7$ in base 12? It is 16, not one of the digits in base 12. So we need to convert 16 to base 12 by division.

$16 \div 12 = \quad 1 \quad R \quad 4$

We'll say that $16 = 14_{12}$.

Finally, what is $B+B$? $B(11) + B(11) = 22$, and 22 is more than base 12. So we need to convert 22 to base 12 by division.

$22 \div 12 = \quad 1 \quad R \quad 10(A)$

We'll say that $22 = 1A_{12}$

+	0	1	2	3	4	5	6	7	8	9	A (10)	B (11)
0	0	1	2	3	4	5	6	7	8	9	A	B
1	1	2	3	4	5	6	7	8	9	A	B	10
2	2	3	4	5	6	7	8	9	A	B	10	11
3	3	4	5	6	7	8	9	A	B	10	11	12
4	4	5	6	7	8	9	A	B	10	11	12	13
5	5	6	7	8	9	A	B	10	11	12	13	14
6	6	7	8	9	A	B	10	11	12	13	14	15
7	7	8	9	A	B	10	11	12	13	14	15	16
8	8	9	A	B	10	11	12	13	14	15	16	17
9	9	A	B	10	11	12	13	14	15	16	17	18
A (10)	A	B	10	11	12	13	14	15	16	17	18	19
B (11)	B	10	11	12	13	14	15	16	17	18	19	1A

$$\begin{array}{r} 3A7_{12} \\ +\ 9BA_{12} \\ \hline 11A5_{12} \end{array} \tag{1.4.3}$$

Let’s do the for the place of ones According to the base 12 addition table, $7 + A = 15_{12}$. We will carry the 1. Now, we do the “tens” place (it’s really a twelves place). According to the base 12 addition table, we have $1 + A + B = 22 = 1A_{12}$. So, like in base 10, we use the A and carry the 1. Now the “hundreds” place (really, 144 s place). There, we have $1 + 3 + 9 = 13 = 11_{12}$.

So,

$$3A7_{12} + 9BA_{12} = 11A5_{12}$$

Subtraction in Bases Other Than Base 10

Subtraction in bases other than base 10 follows the same processes as base 10 subtraction, but, as with addition, using the addition table for the base.

Example 1.4.5: Subtracting in Base Lower than Ten

1. Calculate $52_6 - 34_6$.
2. Calculate $563_7 - 164_7$.

Answer 1

For the place of ones (right column), when we borrow a 1 from the 5, we borrow one group of 6. After borrowing, 2 becomes $2+6=8$, so $8-4=4$. Now top 5 on the left column becomes new 4 and $4-3=2$.

$$\begin{array}{r} 52_6 \\ -\ 34_6 \\ \hline 14_6 \end{array} \tag{1.4.4}$$

Answer 2

For the place of ones (right column), when we borrow a 1 from the 6, we borrow one group of 7. After borrowing, 3 becomes $3+7=10$, so $10-4=6$. In the middle column, top 6 becomes new 5. Again, we borrow 1, from 5 (the left column's top number), we borrow one group of 7, new 5 becomes $5+7=12$, so $12-6=6$. In the left column (third column from the right), 5 becomes new 4, and $4-1=3$.

$$\begin{array}{r} 563_7 \\ -\ 164_7 \\ \hline 366_7 \end{array} \tag{1.4.5}$$

Your Turn 1.4.5: Subtraction in Different Bases

Consider the following difference.

$$511_{\text{nine}} - 217_{\text{nine}}$$

Determine the difference using base 9.

$511_{\text{nine}} - 217_{\text{nine}} =$ ______ $_{\text{nine}}$

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Example 1.4.6: Subtracting in Base 12

Calculate $A17_{12} - 4B3_{12}$

Answer

$$\begin{array}{r} A17_{12} \\ -\ 4B3_{12} \\ \hline 524_{12} \end{array} \tag{1.4.6}$$

For one place, $7-3=2$, all good. For middle column, we borrow 1, from $A=10$, we borrow one group of 12, after borrowing 1 becomes $1+12=13$, so $13-B(11)=2$. Now in third column from the right, $A=10$, becomes new 9, and $9-4=5$.

Your Turn 1.4.6: Subtraction in Base Twelve

Without converting to base ten, subtract the numbers in base TWELVE.

$3B\ 5_{\text{twelve}} - 8\ 9_{\text{twelve}} =$ ______ twelve

Hint: A denotes ten, B denotes eleven.

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Errors When Adding and Subtracting in Bases Other Than Base 10

When computing in bases other than 10, errors often involve applying base 10 rules or symbols to an arithmetic problem in a base other than base 10. The first type of error is using a symbol that is not in the symbol set for the base. For instance, if a 9 shows up when working in base 7, you know an error has happened because 9 is not a legal symbol in base 7.

✔ Example 1.4.7: Identifying an Illegal Symbol in Arithmetic in a Base Other Than Base 10

Explain the error in the following calculation:

$$15_6 + 34_6 = 49_6$$

Answer

Since the problem is in base 6, the symbol set available is $0, 1, 2, 3, 4$, and 5. The 9 in the answer is clearly not a legal symbol for base 6. Returning to the base 6 addition table, we see that $5_6 + 4_6 = 13_6$. Correcting the error, we know the sum is $15_6 + 34_6 = 53_6$.

Remember: The second type of error is using a base 10 rule when the numbers are not in base 10. For instance, if you are working in base 13, then $9_{13} + 9_{13}$ is not 18_{13}, even though 18 is the correct answer in base 10.

✔ Your Trun 1.4.8: Identifying an Arithmetic Error in a Base Other Than Base 10

Explain the error in the following calculation and correct the error:

$$\begin{array}{r} 89_{12} \\ +\ 76_{12} \\ \hline 165_{12} \end{array} \tag{1.4.7}$$

Answer

This would be the correct answer if this problem were a base 10 problem. However, in base 12, $9 + 6$ is not 15, but is instead 13. To correct this error, use the addition table for base 12. If properly used, the correct answer would be 143_{12}, as seen below:

$$\begin{array}{r} 89_{12} \\ +\ 76_{12} \\ \hline 143_{12} \end{array} \tag{1.4.8}$$

1.5: Cryptography

Learning Objectives

1. Encrypt and decrypt a message using a substitution cipher method.
2. Encrypt and decrypt a message using a transposition cipher method.
3. Encrypt and decrypt a message using a shared symmetric key cipher method.
4. Encrypt and decrypt a message using a Bifid cipher method.

When people need to secretly store or communicate messages, they turn to cryptography. Cryptography involves using techniques to obscure a message so outsiders cannot read the message. It is typically split into two steps: encryption, in which the message is obscured, and decryption, in which the original message is recovered from the obscured form.

Definitions

- **Plaintext** – the original, readable data (e.g., "Hello").
- **Key** – a secret value (like a password) used by the algorithm to encrypt and decrypt.
- **Ciphertext** – the scrambled, unreadable version of the data (e.g., "X9&@1z").
- **Encryption** is a process used to protect information by converting it into a format that can only be read or accessed by someone with the correct decryption key or password.
- **Decryption** is the process of converting encrypted data (**ciphertext**) back into its original, readable form (**plaintext**), using a **key** and an algorithm.

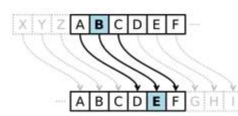
  In other words, **decryption reverses the process of encryption**.

Substitution Cipher

How does Substitution Cipher work?

A substitution cipher replaces each letter in the message with a different letter, following some established mapping.

A simple example of a substitution cipher is called the **Caesar cipher**, sometimes called a shift cipher. In this approach, each letter is replaced with a letter some fixed number of positions later in the alphabet. For example, if we use a shift of **3**, then the letter A would be replaced with D, the letter **3** positions later in the alphabet. The entire mapping would look as shown in the figure. [1]



Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Maps to: DEFGHIJKLMNOPQRSTUVWXYZABC

Example 1.5.1: Encryption

Use the Caesar cipher with a shift of **3** to encrypt the message: “We ride at noon”

Answer

We use the mapping above to replace each letter. W gets replaced with Z, and so forth, giving the encrypted message:

ZH ULGH DW QRRQ.

Notice that the length of the words could give an important clue to the cipher shift used. If we saw a single letter in the encrypted message, we would assume it must be an encrypted A or I, since they are the only single letters that form valid English words.

To obscure the message, the letters are often rearranged into equally sized blocks. The message ZH ULGH DW QRRQ could be written in blocks of three characters as

ZHU LGH DWQ RRQ.

Your Turn 1.5.1: Encrypt Using Caesar Shift

Encrypt the word CHILD using an alphabetic Caesar shift cipher with shift 6 (mapping A to G).

Alphabetic means we're only using the characters

ABCDEFGHIJKLMNOPQRSTUVWXYZ

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Example 1.5.2: Decryption

Decrypt the message GZD KNK YDX MFW JXA if it was encrypted using a shift cipher with a shift of 5.

Answer

We start by writing out the character mapping by shifting the alphabet, with A mapping to F, five characters later in the alphabet.

Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ

Maps to: FGHIJKLMNOPQRSTUVWXYZABCDE

We now work backwards to decrypt the message. The first letter G is mapped to by B, so B is the first character of the original message. Continuing, our decrypted message is

BUY FIF TYS HAR ESA.

Removing spaces, we get BUYFIFTYSHARESA. In this case, an extra character was added to the end to make the groups of three come out even, and the original message was “Buy fifty shares.”

Your Turn 1.5.2: Decrypt Using Caesar Shift

Decrypt the word RTZYM if it was encrypted using an alphabetic Caesar shift cipher with shift 5 (mapping A to F).

Alphabetic means we're only using the characters

ABCDEFGHIJKLMNOPQRSTUVWXYZ

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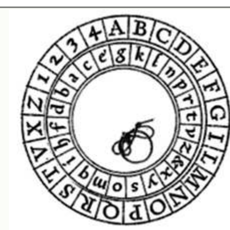

Notice that in both the ciphers above, the extra part of the alphabet wraps around to the beginning. Because of this, a handy version of the shift cipher is a cipher disc, such as the Alberti cipher disk shown here [2] from the 1400 s. In a cipher disc, the inner wheel could be turned to change the cipher shift. This same approach is used for “secret decoder rings.”

The security of a cryptographic method is very important to the person relying on their message being kept secret. The security depends on two factors:

1. The security of the method being used
2. The security of the encryption key used

In the case of a shift cipher, the method is “a shift cipher is used.” The encryption key is the specific amount of shift used.

Suppose an army is using a shift cipher to send their messages and one of their officers is captured by their enemy. It is likely the method and encryption key could become compromised. It is relatively hard to change encryption methods, but relatively easy to change encryption keys.

During World War II, the Germans’ Enigma encryption machines were captured, but having details on the encryption method only slightly helped the Allies, since the encryption keys were still unknown and hard to discover. Ultimately, the security of a message cannot rely on the method being kept secret; it needs to rely on the key being kept secret.

Encryption Security

The security of any encryption method should depend only on the encryption key being difficult to discover. It is not safe to rely on the encryption method (algorithm) being kept secret.

With that in mind, let's analyze the security of the Caesar cipher.

Example 1.5.3

Suppose you intercept a message and know the sender is using a Caesar cipher, but do not know the shift being used. The message begins EQZP. How hard would it be to decrypt this message?

Answer

Since there are only 25 possible shifts, we would only have to try 25 different possibilities to see which one produces results that make sense. While that would be tedious, one person could easily do this by hand in a few minutes. A modern computer could try all possibilities in under a second.

Shift	Message	Shift	Message	Shift	Message	Shift	Message
1	DPYO	7	XJSI	13	RDMC	19	LXGW
2	COXN	8	WIRH	14	QCLB	20	KWFV
3	BNWM	9	VHQG	15	PBKA	21	JVEU
4	AMVL	10	UGPF	16	OAJZ	22	IUDT
5	ZLUK	11	TFOE	17	NZIY	23	HTCS
6	YKTJ	**12**	**SEND**	18	MYHX	24	GSBR
						25	FRAQ

In this case, a shift of 12 (A mapping to M) decrypts EQZP to SEND. Because of the ease of trying all possible encryption keys, the Caesar cipher is not a very secure encryption method.

Brute Force Attack

A brute force attack is a method for breaking encryption by trying all possible encryption keys.

To make a brute force attack harder, we could make a more complex substitution cipher by using something other than a shift of the alphabet. By choosing a random mapping, we could get a more secure cipher, with the tradeoff that the encryption key is harder to describe; the key would now be the entire mapping, rather than just the shift amount.

Example 1.5.4: Encryption

Use the substitution mapping below to encrypt the message "March 12 0300"

Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789

Maps to: 2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P

Answer

Using the mapping, the message would be encrypted to 62SQT ZN Y1YY

Your Turn 1.5.4: Decrypt Using Mapping

Decrypt the word QBBKD if it was encrypted using the mapping below.

Original:	A B C D E F G H I J K L M N O P Q R S T U V W X Y Z
Maps to:	Q W T J C Y A U F H O B M G K R L I E Z P S D X V N

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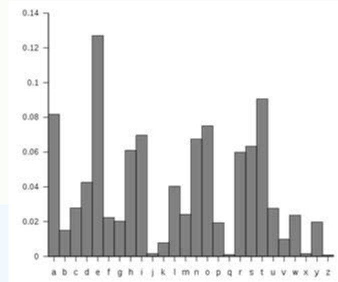
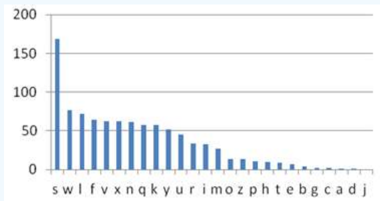
While there were only 25 possible shift cipher keys (35 if we had included numbers), there are about 10^{40} possible substitution ciphers[3]. That's much more than a trillion trillions. It would be essentially impossible, even with supercomputers, to try every possible combination. Having a huge number of possible encryption keys is one important part of key security.



Unfortunately, this cipher is still not secure because of a technique called frequency analysis, discovered by Arab mathematician Al-Kindi in the 9th century. English and other languages have certain letters that show up more often in writing than others.[4] For example, the letter E shows up the most frequently in English. The chart to the right shows the typical distribution of characters.

Example 1.5.5

The chart to the right shows the frequency of different characters in some encrypted text. What can you deduce about the mapping?



Answer

Because of the high frequency of the letter S in the encrypted text, it is very likely that the substitution maps E to S. Since W is the second most frequent character, it is likely that T or A maps to W. Because C, A, D, and J show up rarely in the encrypted text, it is likely they are mapped to from J, Q, X, and Z.

In addition to looking at individual letters, certain pairs of letters show up more frequently, such as the pair "th." The substitution mapping can be deduced by analyzing how often different letters and letter pairs show up in an encrypted message [5].

Transposition Cipher

A transposition cipher is one in which the order of characters is scrambled to obscure the message. The characters themselves are not changed.

An early version of a transposition cipher was a Scytale[6], in which paper was wrapped around a stick and the message was written. Once unwrapped, the message would be unreadable until it was wrapped around a stick of the same size again.

One modern transposition cipher is written in rows, then forms the encrypted message from the text in the columns.

Example 1.5.6: Encryption

Encrypt the message "Meet at First and Pine at midnight" using rows 8 characters long.

Answer

We write the message in rows of 8 characters each. Nonsense characters are added to the end to complete the last row.

MEETATFI

RSTANDPI

NEATMIDN

IGHTPXNR

We could then encode the message by recording down the columns. The first column, reading down, would be MRNI. All together, the encoded message would be MRNI ESEG ETAH TATT ANMP TDIX FPDN IINR. The spaces would be removed or repositioned to hide the size of the table used, since that is the encryption key in this message.

Your Turn 1.5.6: Encrypt Using Transposition

Encrypt the message AT ELEVEN HEAD WEST TO BASE CAMP using a tabular transposition cipher with rows of length 4 characters. **If necessary, pad the message with A's.**

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Example 1.5.7: Decryption

Decrypt the message CEE IAI MNL NOG LTR VMH NW using the method above with a table with rows of 5 characters.

Answer

Since there are total of 20 characters and each row should have 5 characters, then there will be $20/5 = 4$ rows.

We start writing, putting the first 4 letters, CEEI, down the first column.

CALLM

EINTH

EMORN

INGVW

We can now read the message: CALL ME IN THE MORNING VW. The VW is likely nonsense characters used to fill out the message.

Your Turn $1.5.7$: Decrypt Using Transposition

Decrypt the message AATH TDOE TNTS WORG OREC HTNZ EHCM if it was encrypted using a tabular transposition cipher with rows of length 7 characters.

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More complex versions of this rows-and-column-based transposition cipher can be created by specifying an order in which the columns should be recorded. For example, the method could specify that after writing the message out in rows that you should record the third column, then the fourth, then the first, then the fifth, then the second. This adds additional complexity that would make it harder to make a brute-force attack.

To make the encryption key easier to remember, a word could be used. For example, if the keyword was “MONEY”, it would specify that rows should have 5 characters each. The order of the letters in the alphabet would dictate which order to read the columns in. Since E, the 4^{th} letter in the word, is the earliest letter in the alphabet from the word MONEY, the 4^{th} column would be used first, followed by the 1^{st} column (M), the 3^{rd} column (N), the 2^{nd} column (O), and the 5^{th} column (Y).

Example 1.5.8: Encryption

Encrypt the message BUY SOME MILK AND EGGS using a transposition cipher with the keyword MONEY.

Answer

Writing out the message in rows of 5 characters:

BUYSO
MEMIL
KANDE
GGSPK

We now record the columns in order $4\ 1\ 3\ 2\ 5$:

SIDP BMKG YMNS UEAG OLEK

As before, we’d then remove or reposition the spaces to conceal evidence of the encryption key.

Your Turn 1.5.8: Encrypt Using Keyword

Encrypt the message AT FOUR AIR STRIKE ON TARGET using a tabular transposition cipher with encryption keyword MAINE. **If necessary, pad the message with A's.**

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To decrypt a keyword-based transposition cipher, we'd reverse the process. In the example above, the keyword MONEY tells us to begin with the 4^{th} column, so we'd start by writing SIDP down the 4^{th} column, then continue to the 1^{st} column, 3^{rd} column, etc.

Example 1.5.9: Decryption

Decrypt the message RHA VTN USR EDE AIE RIK ATS OQR using a row-and-column transposition cipher with keyword PRIZED.

Answer

The keyword PRIZED tells us to use rows with 6 characters. Since D comes first in the alphabet, we start with 6^{th} column. Since E is next in the alphabet, we'd follow with the 5^{th} column. Continuing, the word PRIZED tells us the message was recorded with the columns in order 4 5 3 6 2 1.

For the decryption, we set up a table with 6 characters in each row. Since the beginning of the encrypted message came from the last column, we start writing the encrypted message down the last column.

					R
					H
					A
					V

The 5^{th} column was the second one the encrypted message was read from, so is the next one we write to.

				T	R
				N	H
				U	A
				S	V

Continuing, we can fill out the rest of the message.

A	I	R	S	T	R
I	K	E	O	N	H
E	A	D	Q	U	A
R	T	E	R	S	V

Reading across the rows gives our decrypted message: AIRSTRIKEONHEADQUARTERSV

Your Turn 1.5.9: Decrypt Using Keyword

Decrypt the message EII BAW SAR NCV AES ESP TNT OEB VRK AMJ if it was encrypted using a tabular transposition cipher with encryption keyword GREAT.

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Unfortunately, since the transposition cipher does not change the frequency of individual letters, it is still susceptible to frequency analysis, though the transposition does eliminate information from letter pairs.

Shared **Symmetric-Key** Methods

Both the substitution and transposition methods discussed so far are shared **symmetric-key** methods, meaning that both sender and receiver would have to have agreed upon the same secret encryption key before any messages could be sent.

All of the methods so far have been susceptible to frequency analysis since each letter is always mapped to the same encrypted character. More advanced methods get around this weakness. For example, the Enigma machines used in World War II had wheels that rotated. Each wheel was a substitution cipher, but the rotation would cause the substitution used to shift after each character.

For a simplified example, in the initial setup, the wheel might provide the mapping

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789
- Maps to: 2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7P

After the first character is encrypted, the wheel rotates, shifting the mapping one space, resulting in a new shifted mapping:

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ0123456789
- Maps to: P2BQF5WRTD8IJ6HLCOSUVK3A0X9YZN1G4ME7

Using this approach, no letter gets encrypted as the same character over and over.

✔ Example 1.5.10: Encryption

Encrypt the message “See me”. Use a basic Caesar cipher with shift 3 as the initial substitution, but shift the substitution one place after each character.

Answer

The initial mapping is

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Maps to: DEFGHIJKLMNOPQRSTUVWXYZABC

This would map the first letter, S to V. We would then shift the mapping by one.

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Now maps to: EFGHIJKLMNOPQRSTUVWXYZABCD

Now the next letter, E, will map to I. Again we shift the cipher

- Original: ABCDEFGHIJKLMNOPQRSTUVWXYZ
- Now maps to: FGHIJKLMNOPQRSTUVWXYZABCDE

The next letter, E, now maps to J. Continuing this process, the final message would be VIJSL.

Notice that frequency analysis is much less useful now, since the character E has been mapped to three different characters due to the shifting of the substitution mapping.

Your Turn 1.5.11: Encrypt With Keyword and Caesar Shift

Encrypt the word WHERE using an alphabetic Caesar shift cipher that starts with shift 13 (mapping A to N), and shifts one additional space after each character is encrypted.

Alphabetic means we're only using the characters

ABCDEFGHIJKLMNOPQRSTUVWXYZ

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The actual Enigma machines used in WWII were more complex. Each wheel consisted of a complex substitution cipher, and multiple wheels were used in a chain[7]. The specific wheels used, the order of the wheels, and the starting position of the wheels formed the encryption key. While captured Enigma devices provided the Allied forces details on the encryption method, the keys still had to be broken to decrypt messages.

These code-breaking efforts led to the development of some of the first electronic computers by Alan Turing at Bletchley Park in the United Kingdom. This is generally considered the beginning of modern computing[8].

In the 1970s, the U.S. government had a competition and ultimately approved an algorithm, deemed DES (Data Encryption Standard), to be used for encrypting government data. It became the standard encryption algorithm used. This method used a combination of multiple substitution and transposition steps, along with other steps in which the encryption key is mixed with the message. This method uses an encryption key with a length of 56 bits, meaning there are 256 possible keys.

This number of keys make a brute force attack extremely difficult and costly, but not impossible. In 1998 a team was able to find the decryption key for a message in 2 days, using about $250,000 worth of hardware. However, the price and time will go down as computer power increases.

From 1997 to 2001, the government held another competition, ultimately adopting a new method, deemed AES (Advanced Encryption Standard). This method uses encryption keys with 128, 192, or 256 bits, providing up to 2^{256} possible keys, making brute force attacks essentially impossible.

Bifid Cipher

The **Bifid cipher** is a classical encryption technique that combines the **Polybius square** with **transposition**, using fractionation to break up plaintext letters and spread their information across the ciphertext. This method was invented by Félix Delastelle (early 20 th century). This method combines substitution (via Polybius square) and transposition (reordering digits)

The **Polybius square** is a simple substitution cipher used to convert letters into pairs of numbers based on a 5×5 grid. Since the alphabet has 26 letters and a 5×5 grid only allows for 25, **"I" and "J" are usually combined.**

	1	2	3	4	5
1	A	B	C	D	E
2	F	G	H	I / J	K
3	L	M	N	O	P
4	Q	R	S	T	U
5	V	W	X	Y	Z

Where each pair or number represents a letter. For example

A → 11 **B** → 12 **H** → 23 **J** or **I** →24 **Z** →55 and so on.

The word **"HELLO"** becomes: 23 15 31 31 34

You can customize the Polybius square using a **keyword** (e.g., "SECRET") to fill the square before the rest of the alphabet.Query 1.5.1

The following is Polybius square with the keyword "SECRET":

Start with the keyword: Remove duplicate letters from **SECRETE → SECRT. Complete the square** with the remaining unused letters of the alphabet (I and J are usually combined in a 5×5 square)

	1	2	3	4	5
1	S	E	C	R	T
2	A	B	D	F	G
3	H	I/J	K	L	M
4	N	O	P	Q	U
5	V	W	X	Y	Z

Decryption: Bifid Cipher

- Convert ciphertext letters back to coordinates.
- Split the sequence into two equal parts (first half = rows, second half = columns).
- Reconstruct the original letter coordinates.
- Use the Polybius square to map back to letters.

Example 1.5.11: Decryption

Decode the following word using the bifid cipher with the given key.

Word to decode: **KONOUPYVVY**
Key: **UNDER**

Answer

Polybius square using a **keyword** ("UNDER") to fill the square before the rest of the alphabet.

	1	2	3	4	5
1	U	N	D	E	R
2	A	B	C	F	G
3	H	I/J	K	L	M
4	O	P	Q	S	T
5	V	W	X	Y	Z

Ciphertext: **KONOUPYVVY**

We locate each letter in the Polybius square and note its pairs:

K	O	N	O	U	P	Y	V	V	Y
33	41	12	41	11	42	54	51	51	54

Break the above pairs in half as follows. Space those numbers out so that instead of writing 10 pairs horizontally, we really write 10 pairs vertically.

33 41 12 41 11 : 3 3 4 1 1 2 4 1 1 1

42 54 51 51 54 : 4 2 5 4 5 1 5 1 5 4

Convert back to Plaintext Letters, looking at the Polybius square.

34→ L 32 → I 45→ T 14→ E 15 → R 21 → A 45→ T 11 → U 15→ R 14→ E

LITERATURE is the final decoded message.

Your Turn 1.5.11: Decrypt Using Bifid Cipher

Decode the following word using the bifid cipher with given key.

Word to decode: **PRPINKUKKT**
Key: **LEVEL**

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Example 1.5.12: Encryption

Encode the following word using the Bifid cipher with the given key.

Word to encode: **TELEVISION**
Key: **TOTAL**

Answer

Polybius square using a **keyword** ("TOTAL") to fill the square before the rest of the alphabet.

	1	2	3	4	5
1	T	O	A	L	B
2	C	D	E	F	G
3	H	I/J	K	M	N
4	P	Q	R	S	U
5	V	W	X	Y	Z

Word to encode: **TELEVISION**

We locate each letter in the Polybius square and note its pairs and place them vertically.

T	E	L	E	V	I	S	I	O	N
1	2	1	2	5	3	4	3	1	3
1	3	4	3	1	2	4	2	2	5

Separate the above into two rows of numbers as follows. We want to move the second row next to the first row, the fourth row next to the third row, and so on.

12 12 53 43 13

13 43 12 42 25

Now we put above two rows above together

12 12 53 43 13 13 43 12 42 25

Convert back to Plaintext Letters, looking at the Polybius square.

12 → O 12 → O 53 → X 43 → R 13 →A 13 → A 43 → R 12 → O 42 → Q 25 → G

OOXRAAROQG is the final decoded message.

Your Turn 1.5.12: Encrypt Using Bifid Cipher

Encode the following word using the bifid cipher with given key.

Word to encode: **HELICOPTER**
Key: **CLOSE**

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Your Turn 1.5.13: Understanding Definition

Cryptography Terms - Matching
Match each topic with the correct description.
- Key - Encryption - Ciphertext - Plaintext

- Transposition Cipher
- Decryption
- Bifid Cipher

a. is the process of converting encrypted data (ciphertext) back into its original, readable form.
b. the scrambled, unreadable version of the data
c. a secret value (like a password) used by the algorithm to encrypt and decrypt.
d. is a classical encryption technique that combines the Polybius square with transposition
e. the original, readable data
f. is a process used to protect information by converting it into a format that can only be read or accessed by someone
g. is one in which the order of characters is scrambled to obscure the message. The characters themselves are not changed.

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[1] en.Wikipedia.org/w/index.php?title=File:Caesar3.svg&page=1. PD

[2] en.Wikipedia.org/wiki/File:Alberti_cipher_disk.JPG

[3] There are 35 choices for what *A* maps to, then 34 choices for what *B* maps to, and so on, so the total number of possibilities is 35*34*33*…*2*1 = 35! = about 10^{40}

[4] en.Wikipedia.org/w/index.php?title=File:English_letter_frequency_(alphabetic).svg&page=1 PD

[5] For an example of how this is done, see en.Wikipedia.org/wiki/Frequency_analysis

[6] en.Wikipedia.org/wiki/File:Skytala%26EmptyStrip-Shaded.png

[7] http://en.Wikipedia.org/wiki/File:En...abet_rings.jpg

[8] For a good overview, see http://www.youtube.com/watch?v=5nK_ft0Lf1s

1.6: Modular Arithmetic

Figure 1.6.1 : If a credit card number is entered incorrectly, error checking algorithms will often catch the mistake. (credit: modification of work "Senior couple at home checking finance on credit card from above" by Nenad Stojkovic/Flickr, CC BY 2.0)

Learning Objectives

After completing this section, you should be able to:

1. Add, subtract using modulo arithmetic.
2. Apply modular arithmetic to calculate real-world applications.

Working with Mod

Online shopping requires you to enter your credit card number, which is then sent electronically to the vendor. Using an ATM involves sliding your bank card into a reader, which then reads, sends, and verifies your card. Swiping or tapping for a purchase in a brick–and-mortar store is how your card sends its information to the machine, which is then communicated to the store's computer and your credit card company. This information is read, recorded, and transferred many times. Each instance provides one more opportunity for error to creep into the process, a misrecorded digit, transposed digits, or missing digits. Fortunately, these card numbers have a built-in error-checking system that relies on modular arithmetic, which is often referred to as clock arithmetic. In this section, we explore clock, or modular, arithmetic.

We want to create a new system of arithmetic based on remainders, always keeping in mind the number we are dividing by, known as the modulus. Modular arithmetic is a system of arithmetic for integers where numbers "wrap around" upon reaching a certain value—the modulus. This concept is often referred to as "clock arithmetic" because it resembles how we tell time on a 12-hour clock.

For example, in a "days of the week" scenario, the modulus is 7. The possible remainders we can get are $0, 1, 2, 3, 4, 5$, and 6, with each remainder representing a different day of the week. Similarly, in a "months of the year" problem, the modulus is 12. We typically assign numbers to days like the following.

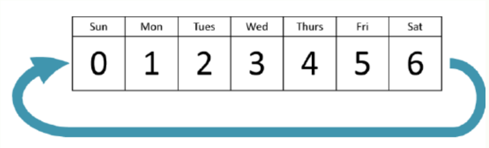

Figure 1.6.2 : Assigning numbers to days

Mod Example

What is $17 \bmod 5$?

To find $17 \bmod , 5$ we divide 17 by 5 and find the remainder.

$17 \div 5 = 3 \text{ remainder of } 2$

Because $5 \times 3 = 15$ and $17 - 15 = 2$. So $17 \bmod 5 = 2$

Mod of Negative number

$-a$ mod b is same as $[b - (a \bmod b)]$
For example, $-12 \bmod 5 = 5 - (12 \bmod 5) = 5 - 2 = 3$

When we do arithmetic, numbers can become larger and larger. But when we work with time, specifically with clocks, the numbers cycle back on themselves. It will never be 49 o'clock. Once 12 o'clock is reached, we go back to 1 and repeat the numbers. If it's 11 AM and someone says, "See you in four hours," you know that 11AM plus 4 hours is 3 PM, not 15 AM (ignoring military time for now). Math worked on the clock, where numbers restart after passing 12, is called clock arithmetic.

Clock arithmetic hinges on the number 12. Each cycle of 12 hours returns to the original time Figure $1.6.3$. Imagine going around the clock one full time. Twelve hours pass, but the time is the same. So, if it is $3:00$, 14 hours later and two hours later, both read the same on the clock, $5:00$. Adding 14 hours and adding 2 hours are identical. As is adding 26 hours. And adding 38 hours.

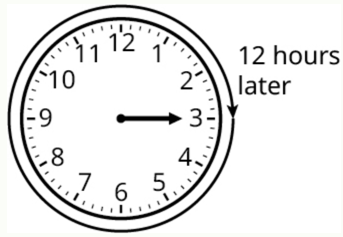

Figure $1.6.3$: lock showing $3:00$ o'clock with arrow going around the clock one full time, or 12 hours

What do $2, 14, 26$, and 38 have in common in relation to 12? When they are divided by 12, they each have a remainder of 2. That's the key. When you add a number of hours to a specific time on the clock, first divide the number of hours being added by 12 and determine the remainder. Add that remainder to the time on the clock to know what time it will be.

A good visualization is to wrap a number line around the clock, with the 0 at the starting time. Then, each time 12 on the number line passes, the number line passes the starting spot on the clock. This is referred to as modulo 12 arithmetic. Even though the process says to divide the number being added by 12 first, perform the addition; the result will be the same if you add the numbers first, and then divide by 12 and determine the remainder.

a mod b

Let a be a positive integer. Then a modulo b, written a mod b, is the remainder when a is divided by b.

Checkpoint 1.6.1

Caution: 12 mod 12 is 0. So, if a mod 12 problem ends at 0, that would be 12 on the clock.

Example 1.6.1

Find the value of the following numbers modulo.

1. $88 \bmod 7$
2. $539 \bmod 12$
3. $-40 \bmod 7$

Answer a

To find $88 \bmod 7$, we divide 88 by 7 and find the remainder.

$$88 \div 7 = 12 \text{ remainder of } 4\backslash). So\backslash[88 \bmod 7 = \tag{1.6.1}$$

Answer b

To find $539 \bmod 12$ we divide 539 by 12 and find the remainder.

$$539 \div 12 = 44 \text{ remainder of } 11 \tag{1.6.2}$$

So

$$539 \bmod , 12 = 11 \tag{1.6.3}$$

Answer c

Note that $-a$ mod b is same as $b-(a \bmod b)$ So $-40 \bmod 7 = (7-(40 \bmod 7)) = 7-5 = 2$

Your Turn 1.6.1: Find Mod

For each part, find a mod b.

$a = 163$ and $b = 11$

$a = 447$ and $b = 17$

$a = 90$ and $b = 10$

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Working with **Clock Using Modular Arithmetic**

Clock arithmetic is modulo 12 arithmetic, but applied to time. As time is divided into 12 hours that repeat a cycle, we use modulo 12 for clock arithmetic.

Example $1.6.2$: Find Future Time or Past Time

1. If it is $3:00$ o'clock now, what time will it be in 89 hours?
2. If it is $4:00$ o'clock now, what time was it 67 hours ago?

Answer

1. To find that future time, we may determine the value of $89 \text{ mod } 12$

$89 \div 12 = 4 \text{ remainder of } 5$. So $89 \text{ mod } 12 = 5$. Counting 5 hours from to $3:00$ o'clcok results in $8:00$.

2. To find that past time, we may determine the value of $67 \text{ mod } 12$.

$67 \div 12 = 5 \text{ remainder of } 7$. So $67 \text{ mod } 12 = 7$. Counting 7 hours backward from to $4:00$ o'clcok results in $9:00$.We see this in the Figure $1.6.4$.

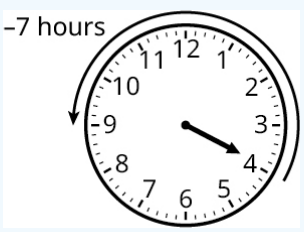

Figure $1.6.4$: Clock showing 7 hours subtracted from $4:00$

Your Turn $1.6.2$: Find the Time in Past and Future

If a 12-hour clock currently reads 1 o'clock, what time will it read in 172 hours?

______ o'clock.

If a 12-hour clock currently reads 8 o'clock, what time will it read in 36 hours ago?

______ o'clock.

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24 Hours Clock

Clock arithmetic in modulo 24 (also called mod 24 arithmetic) is used to represent time on a 24-hour clock. It is similar to clock 12 arithmetic.

Modulo 24 means we only care about the remainder after dividing by 24. For example:

1. $25 \bmod 24 = 1$
2. $48 \bmod 24 = 0$

Example 1.6.3 Applying Clock Arithmetic

1. Suppose it is $3:00$, and you decide to check your email every 5 hours. What time will it be when you check your email the ninth time?
2. Your family has a cat, and no one wants to empty the litter box. However, it has to be done daily. The six of you agree to take turns, so everyone has to empty the litter box every 6 days. You empty the box on a Thursday. What day will you empty the box for the 10th time?

Answer

1. If you check your email every 5 hours nine times, that ninth check will occur 45 hours after $3:00$, which is an addition of 45 hours to $3:00$. So, we find 45 modulo 12, which is 9. Nine hours after $3:00$ is $12:00$. It will be $12:00$ when you check your email the ninth time.

2. The first time you emptied the litter box was on a Thursday. So, the 10 th time you empty the litter box will be 9 times later (you've already had your first turn, so 9 turns left!). This will happen $54 (9 \times 6)$ days later. Finding the value of 54 modulo 7, that is $54 \bmod 7$ gives the answer 5. Five days after a Thursday is Tuesday.

Your Turn 1.6.3

1. You have agreed to text your friend every 3 hours while driving across the country. You began your trip at 8 AM. What time will it be when you text your friend the 15th time?

2. Your family shares the cooking duties in the home. You've agreed to prepare the meal every 5 days. The last time you prepared dinner was on a Tuesday. What day of the week will it be after you've prepared the meals 20 more times?

Working with Days Using Modular Arithmetic

Clock arithmetic processes can be applied to days of the week. Every 7 days the day of the week repeats, much like every 12 hours the time on the clock repeats. The only difference will be that we work with remainders after dividing by 7. In technical terms, this is referred to as modulo 7. More generally, let n be a positive integer. Then n modulo 7, written n mod 7, is the remainder when n is divided by 7.

If today is Tuesday, what day will it be in 10 days?

In Figure 1.6.2, we know Tuesday = 2

$(2+10) \bmod 7 = 12 \bmod 7 = 5$. From the table in the Figure 1.6.1, 5 corresponds to Friday.

Example 1.6.4: Find Future Day or Past Days

1. If today is Thursday, what day is it 1000 days from today?
2. If today is Saturday, what day was it 100 days ago?

Answer

1. Since there are 7 days in a week, every 7 days we will return to Thursday, if today is Thursday. If we divide 1000 by 7, we get a remainder of 6. This is the same as finding 1000 mod 7, which is 6. So we count 6 days forward starting from Thursday.

1 day after Thursday is → Friday

2 days after Thursday is → Saturday

..

..

6 days after Thursday is → **Wednesday**

There is another way to solve this problem. Thursday is 4, see the above table. Now $4+1000=1004$. $1004 \bmod 7$ is 3 (This is the same as saying when we divide 1004 by 7, we get a remainder of 3). From table in the Figure 1.6.2, 3 corresponds to Wednesday.

2. We divide 100 by 7, and we get a remainder of 2. This is the same as finding $100 \bmod 7 = 2$. So we count 2 days backward starting from Saturday.

1 day before Saturday → Friday

2 days before Saturday → **Thursday**

There is another way to solve this problem. Thursday is 6, see the above table. Now $6-100=-94$.

Since $-a$ mod b is same as $b-(a \bmod b)$
$-94 \bmod 7 = 7-(94 \bmod 7) = 7-3 = 4$

From table in the Figure 1.6.2, the number 4 corresponds to Thursday.

Your Turn 1.6.4: Find the Future and Past Day

Sun	Mon	Tues	Wed	Thurs	Fri	Sat
0	1	2	3	4	5	6

If this day is Thursday, what day will it be 90 days from now?

If today is Thursday, what day of the week was it 123 days ago?

Enter the full name of the day (e.g., Monday)

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Working with months using Modular Arithmetic

When working with months of the year, we use modulo 12 arithmetic, since there are 12 months in a year.

Note: Assigning Number to Months

We can number the months with our possible remainders, 0 through 11. We could start with 0 for January, 1 for February, etc., but this wouldn't match up with our standard numbering. Instead, we'll start with 1 for January, 2 for February, etc., up to 11 for November. We can't use 12 for December because 12 isn't a valid remainder. We'll use 0 for December.

JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEP	OCT	NOV	DEC
1	2	3	4	5	6	7	8	9	10	11	0

Figure $1.6.5$: Assigning numbers to months

Example $1.6.5$: Find the Future and Past Month

1. If this month is April, what month will it be 100 months from this month?
2. If the month is April, what month was it 27 months ago?

Answer

1. To find what month it will be 100 months from April, we can use modular arithmetic. There are 12 months in a year, so:

$100 \bmod 12 = 4$. This means 100 months from now is 4 **months ahead** of April.

Starting from April and counting 4 months forward:

1 month after April → May

2 months after April is → June

3 months after April is → July

4 months after April is → **August**

There is another way to solve this problem. April is 4, see the above table in Figure $1.6.5$. Now $4 + 100 = 104$. $104 \bmod 12$ is 8 (This is the same as saying when we divide 104 by 12, we get a remainder of 8). From the same table, the number 8 corresponds to August.

2. To find the month that is 27 months ago from **April**, we use modular arithmetic again. There are 12 months in a year:

$27 \bmod 12 = 3$. This means 27 months ago is 3 **months before April.**

So, 27 months ago is 3 **months before April**.

Starting from April and counting 3 months backward:

1 month before April → March

2 months before April is → February

3 months before April is → **January**

There is another way to solve this problem. April is 4. Now $4 - 27 = -23$.

Since $-a \bmod b$ is same as $b - (a \bmod b)$
$-23 \bmod 12 = 12 - (23 \bmod 12) = 12 - 11 = 1$

From table in the Figure $1.6.5$, the number 1 corresponds to January.

✎ Your Turn 1.6.5: Find the Future and Past Month

JAN	FEB	MAR	APR	MAY	JUNE	JULY	AUG	SEP	OCT	NOV	DEC
1	2	3	4	5	6	7	8	9	10	11	0

If this month is August, what month was it 274 months ago?

If this month is October, what month will it be 90 months from now?

Enter the full name of the day (e.g., July):

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CHAPTER OVERVIEW

2: Modeling in Mathematics

2.1: Linear Growth

Learning Objectives

1. Apply a recursive formula to model the linear relationship between two variables.
2. Apply an explicit formula to model the linear relationship between two variables.

Introduction

Marco is a collector of antique soda bottles. His collection currently contains 437 bottles. Every year, he budgets enough money to buy 32 new bottles. Can we determine how many bottles he will have in 5 years and how long it will take for his collection to reach 1000 bottles?

While both of these questions you could probably solve without an equation or formal mathematics, we are going to formalize our approach to this problem to provide a means to answer more complicated questions.

Suppose that P_n represents the number, or population, of bottles Marco has after n years. So P_0 would represent the number of bottles now, P_1 would represent the number of bottles after 1 year, P_2 would represent the number of bottles after 2 years, and so on. We could describe how Marco's bottle collection is changing using:

$$\begin{aligned} P_0 &= 437 \\ P_n &= P_{n-1} + 32 \end{aligned}$$

This is called a **recursive relationship**. A recursive relationship is a formula that relates the next value in a sequence to the previous values. Here, the number of bottles in year n can be found by adding 32 to the number of bottles in the previous year, P_{n-1}. Using this relationship, we could calculate:

$$\begin{aligned} P_1 &= P_0 + 32 = 437 + 32 = 469 \\ P_2 &= P_1 + 32 = 469 + 32 = 501 \\ P_3 &= P_2 + 32 = 501 + 32 = 533 \\ P_4 &= P_3 + 32 = 533 + 32 = 565 \\ P_5 &= P_4 + 32 = 565 + 32 = 597 \end{aligned}$$

We have answered the question of how many bottles Marco will have in 5 years. However, solving how long it will take for his collection to reach 1000 bottles would require a lot more calculations.

While recursive relationships are excellent for describing simply and cleanly *how* a quantity is changing, they are not convenient for making predictions or solving problems that stretch far into the future. For that, a closed or explicit form of the relationship is preferred.

An **explicit equation** allows us to calculate P_n directly without needing to know P_{n-1}. While you may already be able to guess the explicit equation, let us derive it from the recursive formula. We can do so by selectively not simplifying as we go:

$$\begin{aligned} P_1 &= 437 + 32 && = 437 + 1(32) \\ P_2 &= P_1 + 32 = 437 + 32 + 32 && = 437 + 2(32) \\ P_3 &= P_2 + 32 = (437 + 2(32)) + 32 && = 437 + 3(32) \\ P_4 &= P_3 + 32 = (437 + 3(32)) + 32 && = 437 + 4(32) \end{aligned} \tag{2.1.1}$$

You can probably see the pattern now and generalize that

$$P_n = 437 + n(32) = 437 + 32n$$

Using this equation, we can calculate how many bottles he'll have after 5 years:

$$P_5 = 437 + 32(5) = 437 + 160 = 597$$

We can now also solve for when the collection reaches 1000 bottles by substituting in 1000 for P_n and solving for n.

$$\begin{aligned} 1000 &= 437 + 32n \\ 563 &= 32n \\ n &= \frac{563}{32} \\ n &= 17.59 \end{aligned}$$

So Marco will reach 1000 bottles in 18 years.

In the previous example, Marco's collection grew by the *same number* of bottles every year. This constant change is the defining characteristic of linear growth. Plotting the values we calculated for Marco's collection, we can see the values form a straight line, the shape of linear growth.

Linear Growth or Depreciation

Linear Growth:

If a quantity starts at size P_0 and grows by d every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$$P_n = P_{n-1} + d$$

Explicit form:

$$P_n = P_0 + dn$$

This equation represents the **common difference** – the amount that the population changes each time period n increases by 1.

Linear Depreciation:

If a quantity starts at size P_0 and decreases (or depreciates) by d every time period, then the quantity after n time periods can be determined using the following relations:

Recursive form:

$$P_n = P_{n-1} - d$$

Explicit form:

$$P_n = P_0 - dn$$

Note: $P(t)$ is the same as $P(t)$.

Check Point:

The cost, in dollars, of a gym membership for n months can be described by the explicit equation $P_n = 70 + 30n$. What does this equation tell us?

The value for P_0 in this equation is 70, so the initial starting cost is $\$70$. This tells us that there must be an initiation or start-up fee of $\$70$ to join the gym.

The value for d in the equation is 30, so the cost increases by $\$30$ each month. This tells us that the monthly membership fee for the gym is $\$30$ a month.

The value for P_n in this equation tells us what would be the total cost you pay in n months after you join the gym.

Connection to Prior Learning – Slope and Intercept

You may recognize the common difference, d, in our linear equation as the slope **(rate of change)**. The entire explicit equation should look familiar – it is the same linear equation you learned in algebra, probably stated as $y = mx + b$ for $f(x) = mx + b$

In the algebraic equation $y = mx + b$, b was the y-intercept, or the y value when x was zero. In the form of the equation we're using, we are using $P\text{-}_0$ to represent that initial amount.

In the $y = mx + b$ equation, recall that m was the slope. You might remember this as "rise over run", or the change in y divided by the change in x. Either way, it represents the same thing as the common difference, d, we are using – the amount the output P_n changes when the input n increases by 1.

The equations $y = mx + b$ and $P_n = P_0 + dn$ mean the same thing and can be used in the same ways. We're just writing it somewhat differently.

Note: The slope, m, in $y = mx + b$ is negative when the rate of change is negative. In $P_n = P_0 + dn$, when d is negative, the value of P_n is decreasing.

Your Turn 2.1.1: Find Linear Function

A person has 310 dollars in a cookie jar. Every month, the person puts 54 additional dollars into the jar.

Write a linear equation, in slope-intercept form, that models this situation.

(Note: " y =" is already given -- just enter the right side of the equation.)

y = ______

In the equation, the variable x represents which of the following? (Choose one)

- ○ the total amount of money in the jar (dollars)
- ○ the time spent saving money (months)
- ○ the rate of saving money (dollars per month)
- ○ the amount of money the person started with (dollars)

In the equation, the variable y represents which of the following? (Choose one)

- the rate of saving money (dollars per month)
- the amount of money the person started with (dollars)
- the total amount of money in the jar (dollars)
- the time spent saving money (months)

How much money will be in the jar after 9 months?

[ ] dollars

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Example 2.1.2

A city currently has 130 streetlights. As part of an urban renewal program, the city council has decided to install 2 additional streetlights at the end of each week for the next year. Write an equation that models the number of streetlights, $P(t)$, after t weeks. How many streetlights will the city have at the end of 35 weeks? When will there be 230 streetlights?

Answer

The initial streetlights are $P_0 = 130$, and the common difference is $d = 2$, considered a slope. The linear growth model for this problem is:

$$P(t) = P_t = P_0 + dt = 130 + 2t \tag{2.1.2}$$

The above equation models the number of streetlights after t weeks.

The next question asks how many streetlights are in the city at the end of 35 weeks, $t = 35$.

$$P(35) = 130 + 2(35) = 130 + 70 = 200 \tag{2.1.3}$$

Finally, we are asked to find the time when there is 230 straight light. So, $P(t) = 230$, we will solve for t.

$$\begin{aligned} 230 &= 130 + 2t \\ 230 - 130 &= 2t \\ 50 &= t \end{aligned} \tag{2.1.4}$$

There will be 230 streetlights after 50 weeks.

Your Turn 2.1.2: Find the Linear Function

A city currently has 127 streetlights. As part of an urban renewal program, the city council has decided to install 2 additional streetlights at the end of each week for the next year.

Write an equation that models the number of streetlights, y , after x weeks.

y = [ ] x + [ ]

How many streetlights will the city have at the end of 45 weeks?

[ ] streetlights

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Slope of the line containing to Point

If the line passes through two points (x_1, y_1) and (x_2, y_2), the slope of the line is given by the following formula

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1}$$

Example 2.1.3: Linear Function Using Two Points

The population of elk in a national forest was measured to be $12,000$ in 2003 and was measured again to be $15,000$ in 2007. If the population continues to grow linearly at this rate. Find the function that models the elk population in time t. What will the elk population be in 2014?

Answer

To begin, we need to define how we're going to measure t. Remember that P_0 is the population when $t = 0$, so we probably don't want to literally use the year 0. Since we already know the population in 2003, let us define $t = 0$ to be the year 2003. Then

$P_0 = P(0) = 12,000$, this is the y-intercept of the linear equation.

Next, we need to find d. Remember, d is the growth per time period, in this case, growth per year (slope, m.) Between the two measurements, the population grew by $15,000 - 12,000 = 3,000$, but it took $2007 - 2003 = 4$ years to grow that much. To find the annual growth, we can divide 3000 elk by 4 years, giving 750 elk in 1 year.

Alternatively, you can use the slope formula from algebra to determine the common difference or slope. Here $(x_1, y_1) = (2003, 12,000)$ and $(x_2, y_2) = (2007, 15,000)$

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{15,000 - 12,000}{2007 - 2003} \\ &= \frac{3000}{4} \\ &= 750 \end{aligned}$$

The model equation in explicit form is

$$P_t = P(t) = 12,000 + 750t$$

To answer the question, we need to first note that the year 2014 will be $t = 11$, since 2014 is 11 years after 2003. The explicit form will be easier to use for this calculation:

$$\begin{aligned} P_{11} &= 12,000 + 750(11) \\ &= 20,250 \end{aligned}$$

The elk population be in 2014 will be $20,250$.

Your Turn 2.1.3: Linear Function

The cost of attending college continues to increase every year. In 2012 the cost to take a course was \$260. In 2016 the cost to take the same course was \$293.

(A) Assuming this is a linear trend which of the equations below models this situation? The variable t is the number of years since 2012.

- ○ $C = 260 + 293t$
- ○ $C = 260 + 8.25t$
- ○ $C = 8.25 + 293t$
- ○ $C = 8.25 + 260t$
- ○ $C = 293 + 8.25t$

(B) Using the model from above, how much will it cost to take a course at this college in the year 2027?

\$

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Example 2.1.4

In 1990, approximately $1,820,000$ violent crimes were reported in the U.S. By 2019, this number had fallen to approximately $1,200,000$.

1. Write a linear model to describe the number of violent crimes in the U.S. from 1990 onward.
2. Using this linear model, predict the number of violent crimes in 2027. Round your answer to the nearest whole number.
3. When do you expect the number of violent crimes to reach $1,000,000$? Give your answer in a calendar year (e.g., 2020).

Answer 1

Letting $n = 0$ correspond with 1990 would give $P_0 = 1,820,000$ violent crime.

Next, we need to find m. Remember, m is the rate of change per year (slope, m). Between the two years from 1990 to 2019, violent crime decreased by $1,820,000 - 1,200,000 = 620,000$, but it took $2019 - 1990 = 29$ years to decrease that much. So, to find the rate by which violent crime decreased, we can divide $620,000$ by 29 years, which gives 21379.31034.

We say, slope, $m = -21379.31034$

Equivalently, we can find the slope in the following way

Here $[x_1, y_1) = (1990, 1,820,000)$ and (\(x_2,y_2)=(2019,1,200,000)\nonumber\]

$$\begin{aligned} \text{slope} &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{1,200,000 - 1,820,000}{2019 - 1990} \\ &= \frac{620,000}{29} \\ &= -21379.31034 \end{aligned}$$

The explicit form for the model is:

$$P(t) = P_t = P_0 + dt = 1,820,000 - 21379.31034t$$

Answer 2

We can now use our model to make predictions about the future, assuming that the previous trend continues unchanged. To predict violent crime in $2027, t = 2027–1990 = 37$ year later.

$$\begin{aligned} P_{37} &= 1,820,000 - 21379.31034(37) \\ &= 1,028,965.5172 \end{aligned}$$

Our model predicts that there will be $1,028,965.5172$ violent crimes in 2016 if the current trend continues.

Answer 3

To find when we expect the number of violent crimes to reach $1,000,000$, we would set $P_n = 1,000,000$ and solve for t:

$$\begin{aligned} 1,000,000 &= 1,820,000 - 21,379.31034t \\ 1,000,000 - 1,820,000 &= -21,379.31034t \\ -820,000 &= -21,379.31034t \\ 38.354 &= t \end{aligned}$$

This tells us that violent crimes will reach $1,000,000$ about 39 years after 1990, (Round UP to the nearest year) which would be in the year $1,990 + 39 = 2,028$. So in 2029, violent crime will reach $1,000,000$.

Your Turn 2.1.4: Violent Crime

In 1990, approximately 1,820,000 violent crimes were reported in the U.S. By 2019, this number had fallen to approximately 1,200,000.

(a) Write a linear model to describe the number of violent crimes in the U.S. from 1990 onward. Round your slope to 3 decimal places slope to 3 decimal places.

P_t = [ ]

(b) Using this linear model, predict the number of violent crimes in 2035. Round your answer to the nearest whole number.

[ ] violent crimes

(c) When do you expect the number of violent crimes to reach 800,000? Give your answer as a calendar year (ex: 2020).

During the year [ ]

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Example 2.1.5: Recursive Formula

The number of cars sold weekly by a new automobile dealership grows according to a linear growth model. In the first week, the dealership sold seven cars, $P_0 = 7$. In the second week, the dealership sold thirteen cars, $P_1 = 13$.

1. Write the recursive formula for the number of cars sold, P_n, in the $(n+1)\text{th}$ week.
2. Write the explicit formula for the number of cars sold, P_n, in the $(n+1)\text{th}$ week.
3. How many cars will be sold in the sixth week if this trend continues?

Answer

Given $P_0 = 7$, car sold in first week, here $n = 0$ week.

$P_1 = 13$, car sold in second week, here $n = 1$ week.

So $d = 13 - 7 = 6$, car sales increased by 6 car per week.

Recursive formula

$$P_n = P_{n-1} + d = 1 * P_{n-1} + 6$$

Explicit formula:

$$P_n = P_0 + dn = 7 + 6n \tag{2.1.5}$$

Predicting for car sales in 6th week, we use $n = 5$

$$P_6 = 7 + 6(5) = 7 + 30 = 37$$

37 cars will be sold in 6th week.

Your Turn 2.1.5: Recursive Formula

The number of cars sold weekly by a new automobile dealership grows according to a linear growth model. The first week the dealership sold four cars ($P_0 = 4$). The second week the dealership sold six cars ($P_1 = 6$).

Write the recursive formula for the number of cars sold, P_n, in the $(n+1)^{\text{th}}$ week.

$P_n = P_{n-1} +$ [ ]

Write the explicit formula for the number of cars sold, P_n, in the $(n+1)^{\text{th}}$ week.

$P_n =$ [ ]

P_n =

If this trend continues, how many cars will be sold in the 4th week?

cars

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✓ Example 2.1.6

A city currently has 1.89 thousand residents. Each year, the city's population grows by around 220 persons.

1. Write a function to model this situation where t is the number of years and $P(t)$) the population after t years.
2. After 3 years, what will the approximate population of the city be?

Answer

(a) The initial population is $P_0 = 1.89 \times 1,000 = 1890$ and slope, m, is 220 person grow per year. Note that the slope is positive here. Using an explicit form of the

$$P(t) = P_t = P_0 + mt = 1890 + 220t$$

The above equation models the number of people after t years.

(b) To find the approximate population of the city after 3 years, put $t = 3$.

$$P(3) = 1,890 + 3(220) = 1,890 + 660 = 2,550$$

So the city population after 3 years is $2,250$ or 2.25 thousand.

Your Turn 2.1.6: Gift Card

Suppose you have a $230 gift card for gas. You spend on average $15 per week for gas.

a) Write a function to model this situation where w is the number of weeks and G(w) is the value in dollars remaining on the card.

$G(w)$ =

b) How much value (in dollar) remains in the card after 5 week? Or Evaluate $g(5)$ = $

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2.2: Exponential, Natural, and Logistic Models

Learning Objectives

1. Apply a recursive formula to model the exponential relationship between two variables.
2. Apply an explicit formula to model the exponential relationship between two variables.
3. Apply the natural growth formula to solve the problem.
4. Apply the logistic growth formula to solve the problem.

Exponential Growth

Suppose that every year, only 10% of the fish in a lake have surviving offspring. Assuming that none of the adult fish dies, if there were 100 fish in the lake last year, there would now be 110 fish. If there were 1000 fish in the lake last year, there would now be 1100 fish. Absent any inhibiting factors, populations of people and animals tend to grow by a percentage of the existing population each year. Whenever a quantity grows by a fixed percentage rather than a fixed amount, the growth is exponential, like compound interest.

Suppose our lake began with 1000 fish, and 10% of the fish have surviving offspring each year. Since we start with 1000 fish, $P_0 = 1000$. How do we calculate P_1? The new population will be the old population plus an additional 10%. Symbolically:

$$P_1 = P_0 + 0.10P_0$$

Notice this could be condensed to a shorter form by factoring:

$$P_1 = P_0 + 0.10P_0 = (1 + 0.10)P_0 = 1.10P_0$$

While 10% is the **growth rate**, 1.10 is the **growth multiplier**. Notice that 1.10 can be thought of as "the original 100% plus an additional 10%", which is 110% (of the original population).

For our fish population,

$$P_0 = 1.10(1000) = 1100$$

We could then calculate the population in later years:

$$P_2 = 1.10P_1 = 1.10(1100) = 1210$$

$$P_3 = 1.10P_2 = 1.10(1210) = 1331$$

Notice that in the first year, the population grew by 100 fish; in the second year, the population grew by 110 fish; and in the third year, the population grew by 121 fish. While there is a constant *percentage* growth, the actual increase in number of fish is increasing each year.

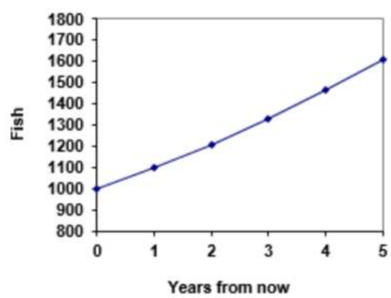

Figure $2.2.1$: Fish growth graph

Graphing these values shows that this growth doesn't appear linear.

To get a better picture of how this percentage-based growth affects things, we need an explicit form to quickly calculate values further out in the future.

Like we did for the linear model, we will start building from the recursive equation:

$$\begin{aligned}
P_1 &= 1.10P_0 = 1.10(1000) \\
P_2 &= 1.10P_1 = 1.10(1.10(1000)) = 1.10^2(1000) \\
P_3 &= 1.10P_2 = 1.10\left(1.10^2(1000)\right) = 1.00^3(1000) \\
P_4 &= 1.10P_3 = 1.10\left(1.10^3(1000)\right) = 1.10^4(1000)
\end{aligned}$$

Observing a pattern, we can generalize the explicit form to be:

$P_n = 1.10^n(1000)$ or equivalently, $P_n = 1000\left(1.10^n\right)$. This is exactly like the compound interest formula with $r = 10\%$ annually.

From this, we can quickly calculate the number of fish in $10, 20$, or 30 years:

$$P_{10} = 1.10^{10}(1000) = 2594$$
$$P_{20} = 1.10^{20}(1000) = 6727$$
$$P_{30} = 1.10^{30}(1000) = 17,449$$

Adding these values to our graph reveals a shape that is definitely not linear. If our fish population had been growing linearly, by 100 fish each year, the population would have only reached 4000 in 30 years, compared to almost $18,000$ with this percent-based growth, called **exponential growth**.

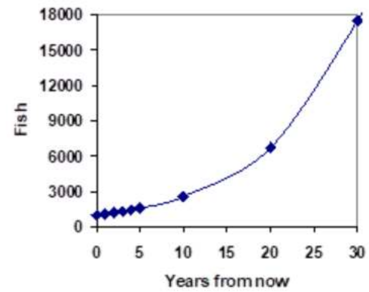

Figure $2.2.2$: Fish population exponential growth graph

In exponential growth, the population grows proportionally to its own size, so as the population increases, the same percentage growth will yield a larger numeric growth.

Exponential Growth Formula: Explicit From

If a quantity starts at size P_0 and grows by r (written as a decimal) every time period, then the quantity after n time periods can be determined using either of these relations:

Recursive form:

$$P_n = P(n) = P_0(1+r)^n$$

$P(n)$: sometimes can be written as P_n, which is the population (or amount) after n time.

$P(0)$: sometimes can be written as P_0, which is population (or amount) in the beginning or initial amount.

Growth Rate: We call r the growth rate. Usually, it is expressed as a percentage, but you need to convert it to a decimal when you put it into the equation.
Growth Factor: The term $(1+r)$ is called the growth multiplier, or growth factor, or common ratio.

Note: If there is $(1-r)$ in the equation, rather than $1+r$, the exponential function is called exponential decay.

Note that the notation $P(n)$ sometimes can be written as P_n, which is the population (or amount) after n time.

Note that the notation $P(0)$ sometimes can be written as P_0, which is population (or amount) in the beginning or initial amount.

Additionally, we can express the explicit form slightly differently, as shown below, with n time periods denoted as t.

$$f(t) = ab^t$$

Where $f(t)$ is similar to $P(t)$, $P(0)$ as a and $1+r$ as a b.

Interpretation of Exponential Growth Equation

A friend is using the equation $P_n = 4600(1.072)^n$ to predict the annual tuition at a local college. She says the formula is based on years after 2010. What does this equation tell us?

In the equation, $P_0 = 4600$ is the tuition's starting value when $n = 0$. This tells us that the tuition in 2010 was $\$4,600$.

The growth multiplier or growth factor is 1.072, so the growth rate is 0.072, or 7.2%. This tells us that the tuition is expected to grow by 7.2% yearly.

Putting this together, we could say that the tuition in 2010 was $\$4,600$ and is expected to grow by 7.2% each year.

Your Turn 2.2.1: Exponential Model

Given the exponential growth function $y = 78(1.8)^x$

What is the initial value of the function?

What is the growth factor of the function?

What is the growth factor of the function?

What is the growth rate of the function?

%

What is the y value when x = 10?

(Round your answer to two decimal places)

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Example 2.2.2: Use Explicit Formula

Between 2007 and 2008, Olympia, WA, grew by almost 3% to a population of 245 thousand people. If this growth rate were to continue, what would the population of Olympia be in 2014?

Answer

As we did before, we first need to define what year will correspond to $n = 0$. Since we know the population in 2008, it would make sense to have 2008 correspond to $n = 0$, so $P_0 = 245,000$..The year 2014 would then be $2014 - 2008 = 6$.

We know the growth rate is 3%, giving $r = 0.03$.

Using the explicit form:

$$\begin{aligned} P(6) = P_6 &= 245,000(1+0.03)^6 \\ &= 245,000 \times 1.19405 \\ &= 292,542.25 \end{aligned}$$

The model predicts that in 2014, Olympia would have a population of about 293 thousand people.

Your Turn 2.2.2: Use Explicit Formula

Tacoma's population in 2005 was about 200 thousand, and has been growing by about 8.2% each year. If this continues, what will Tacoma's population be in 2038?

1. What is the growth factor of the function?

2. Tacoma's population be people in 2038.

   (Round your answer to the nearest whole number.)

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Here's a simple table showing **exponential growth** over time.

Time, t	0	1	2	3	4	5	6	7	8	9	10
Value, $P(t)$	100	150	225	337.5	506.25	759.38	1139.06	1708.59	2562.89	3844.34	5766.51

What is an exponential model equation representing the above table?

Here, $P_0 = 100$, $P_1 = 150$, $P_2 = 225$ and so on.

Since the growth factor is a common ratio. To find it, we have to divide the one value by the value that follows it.

Growth factor = $\frac{P_1}{P_0} = \frac{150}{100} = 1.5$, which is the same as $1 + r = 1.5$. In the above exponential model, $r = 0.50$, the growth rate is 50%.

So, the exponential model in explicit form is

$$P_t = P_0(1+r)^t = 100(1.5)^t$$

The graph of the above model is shown below.

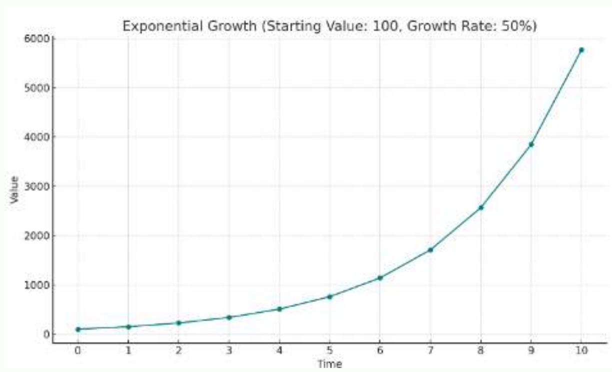

Figure 2.2.3: Exponential Growth, Growth rate 50%

What is the predicted value when $t = 25$. To find it, we put in the model equation

$$P_{25} = P_0(1+r)^t = 100(1.5)^{25} = 2,525,116.829 \tag{2.2.1}$$

Your Turn 2.2.3: Exponential Model

In 2018, Dazzline bought a rare first edition book for $\$260$. Every year since then, it has increased in value by 13%.

Write an equation for an exponential function $f(t) = ab^t$ that models the value of this rare first edition book t years after her purchase.

Find $a$and b

a = [ ] and b = [ ]

$f(t)$ = [ ]

What is the price of vintage watch after 18 years? Round your answer to two decimal places.

$ [ ]

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Rounding

As a note on rounding, if we had rounded the growth rate to 2.1 our calculation for the emissions in 2050 would have been 3347. Rounding to 2% would have changed our result to 3156. A very small difference in the growth rates gets magnified greatly in exponential growth. For this reason, it is recommended to round the growth rate as little as possible.

Example 2.2.3 : Use Explicit Formula

Year	1998	1999	2000	2001	2002	2003
Balance ($)	8,000.00	8,440.00	8,904.20	9,393.93	9,910.60	10,455.68

Figure 2.2.4: Balance over Time

The table above shows a person's bank account balance for 5 years.

1. Find an exponential model for this data, with y representing the balance t years after 1998 and $t = 0$ corresponding to 1998.
2. Find this person's balance in 2010 if the bank account continues to grow at the same rate.

Answer

Here, $P_0 = 8000$, $P_1 = 8440$, $P_2 = 8904.20$ and so on.

Growth factor $= \frac{P_1}{P_0} = \frac{8440}{8000} = 1.055$, which is the same as $1 + r = 1.055$. In the above exponential model, $r = 0.055$, the growth rate is 5.5%.

So, the exponential model in explicit form is

$$y = P_t = P(t) = P_0(1+r)^t = 8000(1.055)^t$$

To find the predicted balance in 2010, which is $t = 2010 - 1998 = 12$ years after 1998, we put $t = 12$ into the equation

$$\begin{aligned} P(t) &= P_0(1+r)^t \\ P(12) &= 8000(1.055)^{12} \\ &= 15209.65989 \end{aligned} \tag{2.2.2}$$

So this person's balance in 2010 will be $15209.66. Round to two decimal places.

Your Turn 2.2.4: Exponential Model

40 individuals of an invasive species is introduced into a new environment. Thereafter, the population doubles every 9 years. In this story, the variables are:

t: The time in years since the species was introduced
$P(t)$:- The population of the invasive species

Complete the table below using the information in the story. Use 0 as starting time.

t	$P(t)$

What type of model fits this story?

- ◯ Linear
- ◯ Exponential

○ Logistic

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✔ Example 2.2.5 : Use Explicit Formula

The population of the world in 1987 was 5 billion and the annual growth rate was estimated at 1.4% per year. Assuming that the world population follows an exponential growth model, find the projected world population in 2018. Round your answer to one decimal place.

Answer

We will correspond $n = 0$ with 1987, as that is the year for the first piece of data we have. That will make $P_0 = 5$ billion. In this problem, we are not given the growth rate but instead are given that Extra close brace or missing open brace as a decimal.

Grwoth factor is $1 + r = 1 + 0.013 = 1.014$, the explicit equation looks like:

$$P_t = P_0(1+r)^t = 5(1.014)^t$$

From 1987 to 2018), *thereis*\(31 years

$$t = 1987 - 2018 = 31$$

So to find the projected world population, we use the above formula for $P(t)$

$$\begin{aligned} P(t) &= 5(1.014)^t \\ &= 5(1.014)^{31} \\ &= 7.7 \end{aligned}$$

So in 2018)*theworldpopulationwillbe*\(7.7 billions.

✎ Your Turn 2.2.5: Use Exponential Formula

Exponential Model
Peyton, a professional wrestler, went on a very strict liquid diet for 26 weeks to lose weight. When he began the diet, he weighed in at a healthy 265 pounds and during the diet, he consistently lost 0.7% of his body weight each week. His weight loss can be modeled by the function $W(t) = 265(0.993)^t$ where W is his weight in pounds and t is the time in weeks that he has been on the diet. Use the function to answer the following questions.
What is decay factor? ____   Determine how much Peyton weighed after 9 weeks. Round your answers to the nearest tenth of a pound.   After 9 weeks, Peyton weighed ____ pounds.

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Exponential Decay: Recursive Form

Recursive form:

$$P_n = (1+r)P_{n-1}$$

We call r the growth rate. Usually in percentage, but you have to change it to decimal when you put it into the equation.
The term $(1+r)$ is called the growth multiplier, or growth factor, or common ratio. If there is $(1-r)$ in the equation, rather than \((1+r)\), the exponential

function is called exponential decay.

In the formula, $P_n = (1+r)P_{n-1}$, growth factor $1+r$ can be found by using the the following

$$\text{Growth Factor} = \frac{P_1}{P_0} = \frac{P_2}{P_1} = \frac{P_3}{P_2} = \ldots$$

Sometimes, if a growth factor is given, to find the growth rate, you use the following formula

$$\text{Growth Rate} = \text{Growth Factor} - 1$$

In the function, $f(t) = ab^t$, b is a growth factor and a is initial value.

Example 2.2.6 : Use Recursive Formula

Starting in the year 2006, the number of speeding tickets issued each year in Middletown is predicted to grow according to an exponential growth model. During the year 2006, Middletown issued 240 speeding tickets. Every year thereafter, the number of speeding tickets issued is predicted to grow by 10. If P_n denotes the predicted number of speeding tickets n year after 2006.

1. Write the recursive formula for P_n.
2. Write the explicit formula for P_n.
3. If this trend continues, how many speeding tickets will be issued in 2022?

Answer

1. Year 2006 corresponds to $n=0$, so $P_0 = 240$.

   We know the growth rate is 10, giving $r = 0.10$.

   The recursive formula will be

$$P_n = (1+r)P_{n-1} = (1+0.10)P_{n-1} = 1.10P_{n-1} + 0$$

2. Using the explicit form:

$$P_n = P_0(1+r)^n = 240(1+0.10)^n = 240(1.10)^n$$

3. To find the predicted number of speeding tickets in 2022, which is $t = 2022 - 2006 = 16$ years after 2006, we put $t = 16$ into equation

$$\begin{aligned} P_{16} &= 240 \times (1+0.10)^{16} \\ &= 240(1.10)^{16} \\ &= 1102.7935 \end{aligned} \tag{2.2.3}$$

   The predicted number of speeding tickets in 2022 is $1,103$.

Your Turn 2.2.6: Use Recursive Formula

Starting in the year 2006, the number of speeding tickets issued each year in Middletown is predicted to grow according to an exponential growth model. During the year 2006, Middletown issued 150 speeding tickets ($P_0 = 150$). Every year thereafter, the number of speeding tickets issued is predicted to grow by 10%.

If P_N denotes the predicted number of speeding tickets during the year $2006 + N$, then

Write the recursive formula for P_N

P_N = [ ] $P_{N-1}+$ [ ]

Write the explicit formula for P_N

P_N = [ ] $\times$ [ ] ^{N}

If this trend continues, how many speeding tickets are predicted to be issued in 2025?

[ ] tickets. (Round to 2 decimal places.)

In each box, enter a single number. Do not leave any boxes blank (enter 1 or 0 as appropriate if necessary).

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Natural Growth

In this section, we will examine natural and logistic growth. Population growth is usually modeled using natural growth models, which are also exponential. However, they use the mathematical constant ***e* as the base**. The growth rate per given period, *k*, is a part of the exponent, which is multiplied by the number of periods.

Natural growth is **a form of exponential growth** involving the mathematical constant **e, approximately equal to ≈ 2.718.**

Explicit Formula: The Natural Growth Model

The Natural Growth Model is

$$P(t) = P_0 e^{kt}$$

where P_0 is the initial population, k is the growth rate per unit of time, and t is the number of time periods.
Note: $P(t)$ is same as P_t.

Note: Given $P_0 > 0$, if *k > 0*, this is an exponential growth model; if *k < 0*, this is an exponential decay model.

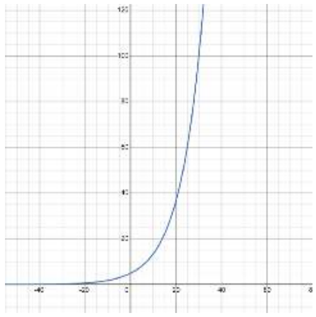

Figure 2.2.1: Natural Growth, $P(t) = P_0 e^{kt}, k > 0$

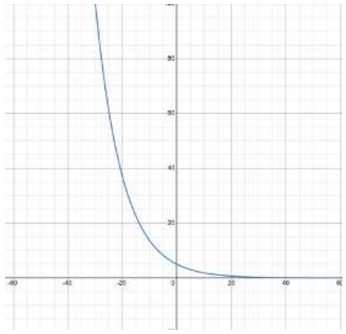

Figure 2.2.2: Natural Decay, $P(t) = P_0 e^{-kt}, k > 0$

Example 2.2.7

When a certain drug is administered to a patient, the number of milligrams remaining in the bloodstream after t hours is given by the model

$$P(t) = 40e^{-0.25t}$$

How many milligrams are in the blood after two hours?

Answer

To solve this problem, we use the given equation with $t = 2$. Here growth rate is 0.25, which is same as 25%.

$$\begin{aligned} P(2) &= 40e^{-0.25(2)} \\ &= 24.26 \end{aligned} \tag{2.2.4}$$

There are approximately 24.6 milligrams of the drug in the patient's bloodstream after two hours.

Your Turn \(\PageIndex{7 \): Natural Growth

The number of bacteria in a culture is given by the function $n(t) = 990e^{0.1t}$, where t is measured in hours.

(a) What is the relative rate of growth of this bacterium population?

Your answer is [ ] percent

(b) What is the initial population of the culture (at t = 0)?

Your answer is [ ]

(c) How many bacteria will the culture contain at time t = 5?

Your answer is [ ]

(Round to two decimal places.)

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Example 2.2.8

Bob has an ant problem. On the first day of May, Bob discovers he has a small red ant hill in his backyard, with a population of about 100 ants. If conditions are just right, red ant colonies will have a growth rate of 240% per year during the first four years. If Bob does nothing, how many ants will he have next May? How many in five years?

Answer

We solve this problem using the natural growth model. Here $k = 2.40$.

$$P(t) = 100e^{2.4t}$$

In one year, $t = 1$, we have

$$P(1) = 100e^{2.4(1)} = 1102 \text{ ants}$$

In one year, $t = 5$, we have

$$P(5) = 100e^{2.4(5)} = 16,275,479 \text{ ants}$$

That is a lot of ants! Bob will not let this happen in his backyard!

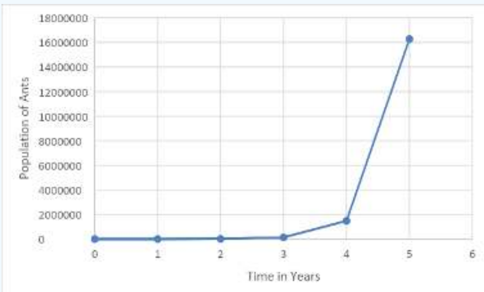

Figure 2.2.3 : Graph of Ant Population Growth in Bob's Yard.

Note: The population of ants in Bob's backyard follows an exponential (or natural) growth model.

Your Turn 2.2.8: Natural Decay

Evaluating and Solving Exonential Functions
Since 1997, the number of fish in Lake Beckett has been decreasing at a rate of 1.7% per year. In 1997, the population of fish was estimated to be 146 million. Use this information to answer the following:
a) Write the exponential function $P(t)$ for this scenario where $P(t)$ is the fish population in millions t years after 1997. Exponential Function: $P(t) =$ ______
b) Determine the number of fish in Lake Beckett in 2002. Round your answer to two decimal places. The population of fish in Lake Beckett in 2002 will be ______ million fish.

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The problem with exponential growth is that the population grows without bounds, and, at some point, the model will no longer predict what is actually happening since the amount of resources available is limited. Populations cannot continue to grow on a purely physical level; eventually, death occurs, and a limiting population is reached.

Logistic Growth Model

Another growth model for living organisms is the logistic growth model. The logistic growth model has a maximum population called the carrying capacity. As the population grows, the number of individuals grows to the carrying capacity and stays there. This is the maximum population the environment can sustain. **Logistic growth** is used to model situations where growth starts out exponentially but then **slows down** as it approaches a **maximum limit** (called the **carrying capacity**).

For our fish, the carrying capacity is the largest population that the resources in the lake can sustain. If the population in the lake is far below the carrying capacity, then we would expect the population to grow essentially exponentially. However, as the population approaches the carrying capacity, there will be a scarcity of food and space available, and the growth rate will decrease. If the population exceeds the carrying capacity, there won't be enough resources to sustain all the fish and there will be a negative growth rate, causing the population to decrease back to the carrying capacity.

Carrying Capacity

The carrying capacity, or maximum sustainable population, is the largest population that an environment can support.

Explicit Formula: Logistic Growth

$$P_t = P(t) = \frac{M}{1+Ke^{-rt}}$$

where $P(t)$ = Population over the time t

M = Carrying capacity

r = growth rate

t = time

$K = \frac{M}{P_0} - 1$

$P(0) = P_0$ = Population over the time t

Note: $P(t)$ is same as P_t.

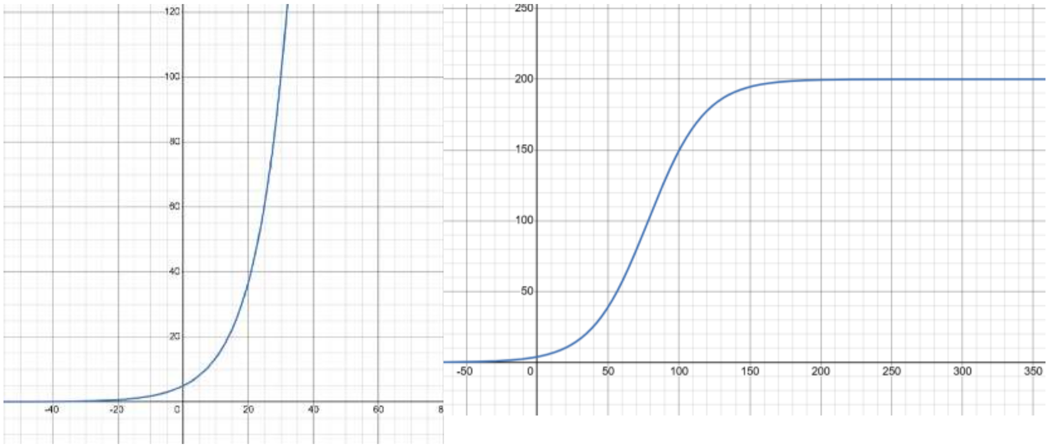

Figure 2.2.4: Logistic (Right) and Exponential (Left) Model Graph

The horizontal line M on this graph illustrates the carrying capacity.

(Logistic Growth Image 1, n.d.)

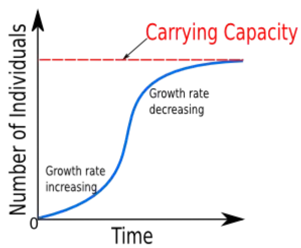

Figure 2.2.5: Logistic Growth Model

(Logistic Growth Image 2, n.d.)

The graph for logistic growth starts with a small population. When the population is small, the growth is fast because there is more elbow room in the environment. As the population approaches the carrying capacity, the growth slows.

Example 2.2.9

Suppose 16 blackberry plants started growing in a yard. Absent constraint, the number of blackberry plants will increase continuously at a monthly rate of 80%. If the yard can only sustain 150 plants, use a logistic growth model to estimate the number of plants after 4 months.

Answer

The following formula gives the logistic growth model.

$$P(t)=\frac{M}{1+Ke^{-rt}}$$

Where

$M=150$

$P_0=16$

$r=80\%=0.8$

$K=\frac{150}{16}-1=8.375$

$$\begin{gathered}P(t)=\frac{150}{1+8.375e^{-0.80t}}\\P(4)=\frac{150}{1+8.375e^{-0.80\times 4}}\\P(4)=111.82484\end{gathered}\tag{2.2.5}$$

After 4 month, there will be about 112 blackberry plants.

Your Turn 2.2.9: Logistic Growth

Assume there is a certain population of fish in a pond whose growth is described by the logistic equation. It is estimated that the carrying capacity for the pond is 1800 fish. Absent constraints, the population would grow by 190% per year.

If the starting population is given by $P_0 = 600$, then after one breeding season the population of the pond is given by

P_1 = [ ]

After two breeding seasons the population of the pond is given by

P_2 = [ ]

(Round your answer to the nearest whole number.)

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Example 2.2.10: Bird Population

The population of an endangered bird species on an island grows according to the logistic growth model.

$$P(t) = \frac{3640}{1 + 25e^{-0.04t}}$$

Identify the initial population. What will the bird population be in five years? What will the population be like in 150 years? What will the population be like in 500 years?

Answer

We know the initial population, P_0, occurs when $t = 0$.

$$P_0 = P(0) = \frac{3640}{1 + 25e^{-0.04(0)}} = \frac{3640}{1 + 25 \times 1} = 140$$

Note: $e^{-0.04(0)} = e^0 = 1$

When $t = 5$, calculate the population in five years.

$$P(5) = \frac{3640}{1 + 25e^{-0.04(5)}} = 169.6$$

The island will be home to approximately 170 birds in five years

Calculate the population in 150 years, when $t = 150$.

$$P(150) = \frac{3640}{1 + 25e^{-0.04(150)}} = 3427.6$$

The island will be home to approximately 3428 birds in 150 years.

Calculate the population in 500 years when $t = 500$.

$$P(500) = \frac{3640}{1 + 25e^{-0.04(500)}} = 3640.0$$

The island will be home to approximately 3640 birds in 500 years.

This example shows that the population grows quickly between five years and 150 years, with an overall increase of over 3000 birds, but slows dramatically between 150 years and 500 years (a longer span of time) with an increase of just over 200 birds.

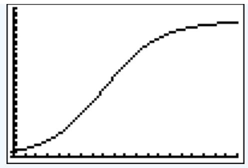

Figure 2.2.6 : Bird Population over a 200-Year Span

Your Turn 2.2.10: Logistic Growth

A rumor is spreading around a middle school. The number of people who have heard the rumor t hours after 9 am is modeled by:

$$A = \frac{750}{1+749e^{-0.25t}}$$

where A is the number of people who knew the rumor after t minutes.

(a) How many people had heard the rumor at time t = 0? Round to whole number.

______ people.

(b) How many people had heard the rumor at time t = 55? Round to whole number.

______ people.

(c) According to the model, what is the maximum number of people who will hear the rumor?

______ people. Round to whole number.

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CHAPTER OVERVIEW

3: Counting and Probability

3.1: The Multiplication Rule for Counting

Figure $3.1.1$: The Multiplication Rule for Counting allows us to compute more complicated probabilities, like drawing two aces from a deck. (credit: "Pair of Aces – Poker" by Poker Photos/Flickr, CC BY 2.0)

Learning Objectives

Apply the Multiplication Rule for Counting to solve problems.

One of the first bits of mathematical knowledge children learn is how to count objects by pointing to them in turn and saying: "one, two, three, …" That's a useful skill. Still, when the number of things that we need to count grows large, that method becomes onerous (or, for *very* large numbers, impossible for humans to accomplish in a typical human lifespan). So, mathematicians have developed shortcuts to counting big numbers. These techniques fall under the mathematical discipline of combinatorics, which is devoted to counting.

Multiplication as a Combinatorial Shortcut

One of the first combinatorial shortcuts to counting students learn in school has to do with areas of rectangles. If we have a set of objects to be counted that can be physically arranged into a rectangular shape, then we can use multiplication to do the counting for us. Consider this set of objects (Figure $3.1.2$):

Figure $3.1.2$

Indeed, we can count them by pointing and running through the numbers, but it's more efficient to group them (Figure $3.1.3$.)

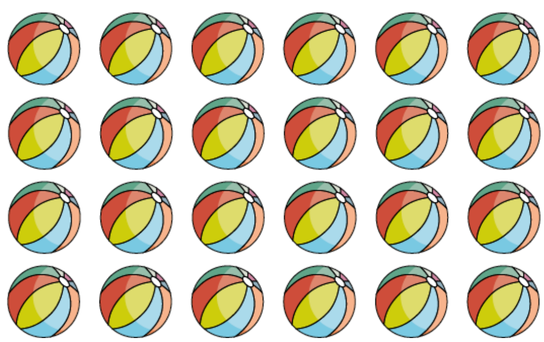

Figure $3.1.3$

If we group the balls by 4s, we see that we have 6 groups (or, we can see this arrangement as 4 groups of 6 balls). Since multiplication is repeated addition (i.e., $6 \times 4 = 4+4+4+4+4+4$, we can use this grouping to quickly see that there are 24 balls.

Let's generalize this idea a little bit. Let's say that we're visiting a bakery that offers customized cupcakes. For the cake, we have three choices: vanilla, chocolate, and strawberry. Each cupcake can be topped with one of four types of frosting: vanilla, chocolate, lemon, and strawberry. How many different cupcake combinations are possible? We can think of laying out all the possibilities in a grid, with cake choices defining the rows and frosting choices defining the columns (Figure $3.1.4$.)

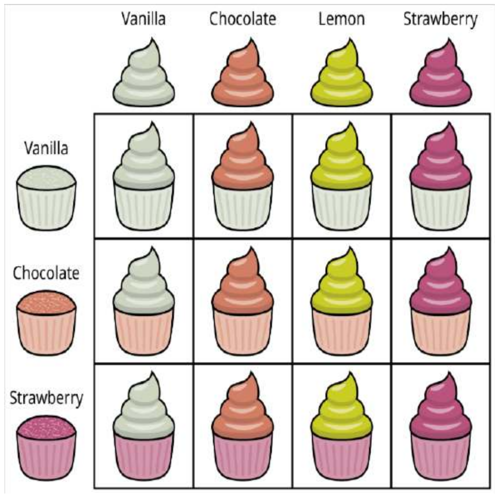

Figure 3.1.4: 12 Possible Combination

Since there are 3 rows (cakes) and 4 columns (frostings), we have $3 \times 4 = 12$ possible combinations. This is the reasoning behind the Multiplication Rule for Counting, which is also known as the Fundamental Counting Principle. This rule says that if there are n ways to accomplish one task and m ways to accomplish a second task, then there are $n \times m$ ways to accomplish both tasks. We can tackle additional tasks by multiplying the number of ways to accomplish those tasks using our previous product.

Multiplication Rule for Counting

Let's ask to choose one item from each of two separate categories where there are m items in the first category and n items in the second category. The total number of available choices is $m \times n$.

Multiplication Rule: Key Point

To use the Multiplication rule for counting, you need to look for two things

1. How many groups are you choosing items from?
2. How many ways can you choose each item from that group?

For example, in the above figure 3.1.4, there are two groups: "cake group" and "frosting group". In the cake group, we can choose any one item at a time out of 3 and in the frosting group, we can choose any one item at a time out of 4.

Example 3.1.1 Using the Multiplication Rule for Counting for 4 Groups

Every card in a standard deck of cards has two identifying characteristics: a suit (clubs, diamonds, hearts, or spades; these are indicated by these symbols, respectively: ♣, ♢, ♡, ♠) and a rank (ace, 2, 3, 4, 5, 6, 7, 8, 9, 10, jack, queen, and king; the letters A, J, Q, and K are used to represent the words). Each possible pair of suits and ranks appears exactly once on the deck. How many cards are in the standard deck?

Answer

Since there are two group "suits" ((4 items) and "rank" (13) items, the number of cards must be $4 \times 13 = 52$

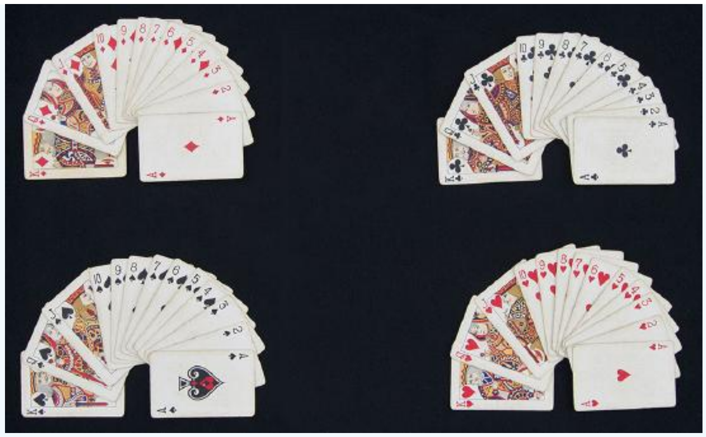

Figure 3.1.5 : Standard Deck of Cards, Sorted by Rank and Suit (credit: "Playing Cards, USS Arkansas" by Naval History & Heritage Command/Flickr, CC BY 2.0)

Your Turn 3.1.1: Multiplication Rule

A pizza shop is running a special on pizzas. You can choose from 3 crusts, 2 sauces, 5 cheeses, 9 vegetables, and 5 meats. How many variations of **thin crust** pizzas are possible if you must choose 1 from each category?

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Example 3.1.2 Using the Multiplication Rule for Counting for 4 Groups

The University Combinatorics Club has 31 members: 8 seniors, 7 juniors, 5 sophomores, and 11 first-years. How many possible 4-person committees can be formed by selecting one member from each class?

Answer

Since we have four groups: seniors, juniors, sophomores, and first-year students. And number of choices: 8 choices for the senior, 7 choices for the junior, 8 for the sophomore, and 11 for the first year, there are

$$8 \times 7 \times 5 \times 11 = 3,080$$

There are $3,080$ different ways to fill out the committee members.

Your Turn 3.1.2: Multiplication Rule

You are opening a T-shirt store. You can have long sleeves or short sleeves, six different colors, five different designs, and four different sizes. How many different shirts can you make?

The number of shirts =

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Example 3.1.3: Using the Multiplication Rule for Counting for More Groups

1. The standard license plates for vehicles in a certain state are six characters: 3 letters followed by 3 digits. There are 26 letters in the alphabet and 10 digits (0 through 9) to choose from. How many license plates can be made using this format?
2. 10 people are to be seated in a row of 10 chairs. How many different seating arrangements are there?

Answer

1. Since we are choosing 6 characters (items): There are 6 group: First three group are all 26 alphabets and last 3 group are 10 number. Since there are 26 different letters and 10 different digits. Each of the first three characters can be selected in 26 different ways, and the last three can be chosen in 10 different ways. (Note that there is **no restriction**. Digits and letters can be repeated. The total number of possible license plates

$$26 \times 26 \times 26 \times 10 \times 10 \times 10 = 17,576,000$$

2. Here we have to choose 10 items (person) for each chair. So we consider there are ten groups. In the first chair, there are 10 choices. Since one person (the first person that is selected) can sit in that chair, there are now 9 persons left, and there are 9 choices for the second chair (in other words, we have 9 intem in this group), and so on. The total number of setting arrangements will be

$$10 \times 9 \times 8 \times 7 \times 5 \times 6 \times 4 \times 3 \times 2 \times 1 = 3,628,800$$

Your Turn 3.1.3: Multiplication Rule

For some particular website, the password must start with one of the letters P, Q, R, S, or T. Then there must be 6 more characters, and each of those characters may be a numeric digit or a letter of the alphabet. How many different passwords are possible?

There are ______ different passwords.

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Example 3.1.4: Using the Multiplication Rule for Counting for More Groups

A quiz consists of 3 true-or-false questions. In how many ways can a student answer the quiz?

Answer

There are 3 questions. Each question has 2 possible answers (true or false). There are two ways to answer each of the three questions. So, the quiz can be answered in $2 \times 2 \times 2 = 8$ different ways.

Your Turn 3.1.4: Multiplication Rule

A quiz has 19 true/false questions. In how many different ways can the 19-question quiz be answered?

______

(Write your answer in the exponential form.)

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Example 3.1.5

How many license plates can one create using the DDLLLL format, in which D stands for any odd digit and L stands for any capital letter A-N?

1. If the repetition of digits and letters is allowed.
2. If the repetition of digits is allowed, but the repetition of letters is not allowed.
3. If the repetition of digits is not allowed, but the repetition of letters is allowed.
4. If the repetition of digits and letters is not allowed.

Answer

There are a total of 6 symbols on the license plate. The first two are odd digits, and the last 4 are alphabet from A to N.

1. Note that repetition is allowed. Since there are 5 odd digits, the number of choices for the first two D is 5, and since there are 14 capital letters A-N, the number of choices for the last four L is 14, thus:

$$5 \times 5 \times 14 \times 14 \times 14 \times 14 = 960,400 \tag{3.1.1}$$

2. If the repetition of digits is allowed, but the repetition of letters is not allowed.

$$5 \times 5 \times 14 \times 13 \times 12 \times 11 = 600,600 \tag{3.1.2}$$

3. If the repetition of digits is not allowed, but the repetition of letters is allowed.

$$5 \times 4 \times 14 \times 14 \times 14 \times 14 = 768,320 \tag{3.1.3}$$

4. If the repetition of digits and letters is not allowed.

$$5 \times 4 \times 14 \times 13 \times 12 \times 11 = 480,480 \tag{3.1.4}$$

Your Turn 3.1.5: Multiplication Rule

How many license plates one can create using the $DDDLL$ -format in which D stands for any even digit and L stands for any capital vowel letter?

i. If the repetition of digits and letters is allowed.

ii. If the repetition of digits is allowed but the repetition of letters is not allowed.

iii. If the repetition of digits is not allowed but the repetition of letters is allowed.

iv. If the repetition of digits and letters is not allowed.

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3.2: Permutations and Permutation

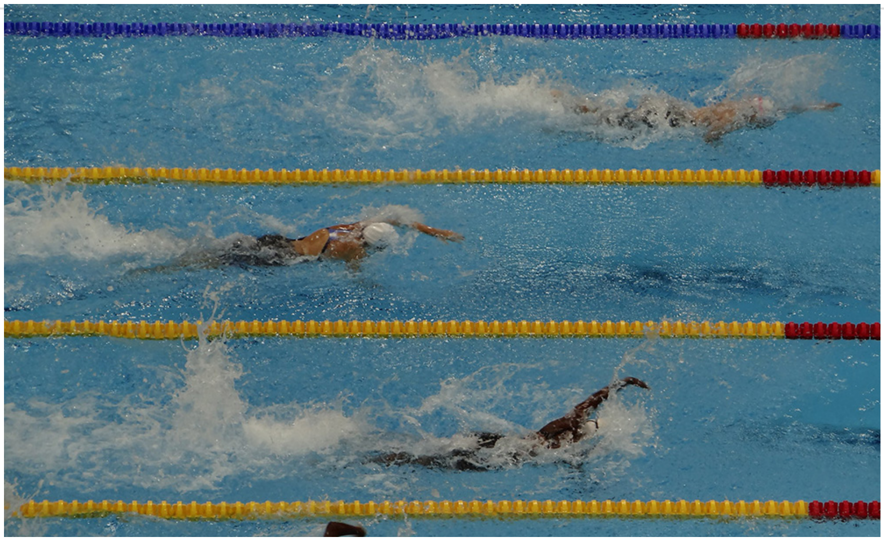

Figure $3.2.1$: We can use permutations to calculate the number of different orders of finish in an Olympic swimming heat. (credit: "London 2012 Olympics Park Stratford London" by Gary Bembridge/Flickr, CC BY 2.0)

Learning Objectives

1. Use the Multiplication Rule for Counting to determine the number of permutations.
2. Compute expressions containing factorials.
3. Compute permutations.
4. Apply permutations to solve problems.
5. Distinguish between permutation and combination uses.
6. Compute combinations.
7. Apply combinations to solve applications.

Swimming events are some of the most popular events at the Summer Olympic Games. In the finals of each event, 8 swimmers compete at the same time, making for some exciting finishes. How many different orders of finish are possible in these events? In this section, we'll extend the Multiplication Rule for Counting to help answer questions like this one related to permutations. A permutation is an ordered list of objects taken from a given population. The length of the list is given, and the list cannot contain any repeated items.

Applying the Multiplication Rule for Counting to Permutations

In the case of the swimming finals, one possible permutation of length 3 would be the list of medal winners (first, second, and third place finishers). A permutation of length 8 would be the full order of finish (first place through eighth place). Let's use the Multiplication Rule for Counting to figure out how many of each of these permutations there are.

Example 3.2.1: Using the Multiplication Rule for Counting to Find the Number of Permutations

The final heat of Olympic swimming events features eight swimmers (or teams of swimmers).

1. How many podium placements (first, second, and third place) are possible?
2. How many complete orders of finish (first place through eighth place) are possible?

Answer 1

Let's start with the first-place finisher. How many options are there? Since eight swimmers compete, there are 8 possibilities. Once that first swimmer completes the race, there are 7 swimmers left competing for second place. After the second finisher is decided, there are 6 swimmers remaining who could possibly finish in third place. Thus, there are 8 possibilities for first place, 7 for second place, and 6 for third place.

The Multiplication Rule for Counting then tells us there are

$$8 \times 6 \times 7 = 336$$

There are 336 different ways the winners' podium can be filled out.

Answer 2

To look at the complete order of finish, we can continue the pattern we can see in part 1 of this example: There are 5 possibilities for fourth place, 4 for fifth place, 3 for sixth place, 2 for seventh place, and then just 1 swimmer is left to finish in eighth place.

Using the Multiplication Rule for Counting, we see that there are

$$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 40,320$$

There are $40,320$ possible orders of finish

Your Turn $3.2.1$: Find Permutation

Eight bands are to perform at a weekend festival.
How many different ways are there to schedule their appearances?

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Factorials

A **factorial** is a mathematical operation denoted by an exclamation point (!), and it's used to multiply a number by all the whole numbers below it.

Definition: Factorial

For any positive whole number n, we define the factorial of n (denoted $n!$ read " n factorial") to be the product of every whole number less than or equal to n.

$$n! = 1 \times 2 \times \ldots \times n$$

And $0! = 1$

We will use factorials in a couple of different contexts, so let's get some practice doing computations with them.

How to Split Factorial?

Note that we can split (!) differently according to our needs. For example

$$\begin{aligned} 100! &= 99! \times 100 \\ 100! &= 98! \times 99 \times 100 \\ 100! &= 97! \times 98 \times 99 \times 100 \\ 100! &= 90! \times 91 \times 92 \times \ldots \times 100 \end{aligned} \tag{3.2.1}$$

Example $3.2.2$: Computing Factorials

Compute the following:

1. $4!$
2. $\frac{8!}{6!}$
3. $\frac{9!}{3!4!}$

Answer 1

$4! = 4 \times 3 \times 2 \times 1 = 24$

Answer 2

There are two ways to approach this calculation. The first way is to compute the factorials first, then divide.

One way

$$\begin{aligned} \frac{8!}{6!} &= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \times 5 \times 6} \\ &= 54 \end{aligned} \tag{3.2.2}$$

Let's approach this one using our canceling technique by splitting factorial. When we see two factorials in either the numerator or denominator, we should focus on the larger one first. So:

$$\begin{aligned}\frac{9!}{3!4!} &= \frac{4! \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4!} \\ &= \frac{15,120}{6} \\ &= 2,520\end{aligned} \tag{3.2.3}$$

With that in mind, we can proceed this way by canceling out the $6!$.That's much easier!

Answer 3

$$\begin{aligned}\frac{9!}{3!4!} &= \frac{4! \times 5 \times 6 \times 7 \times 8 \times 9}{1 \times 2 \times 3 \times 4!} \\ &= \frac{15,120}{6} \\ &= 2,520\end{aligned} \tag{3.2.4}$$

Your Turn $3.2.2$: Evaluate Factorial

What is the value of each expression

$4!$ =

$\frac{14!}{5! \cdot 9!}$=

$(7!) \cdot (6!)$=

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Permutations

As we've seen, factorials can pop up when we're computing permutations. There is a formula that we can use to make that connection explicit. Let's define some notation first. If we have a collection of n objects and we wish to create an ordered list of r of the objects (where $1 \leq r \leq n$), we'll call the number of these **arrangements** a permutation. The order of an arrangement matters in a permutation. **The order of arrangement matters in permutation.**

The symbol ${}_nP_r$ can be read as "the number of permutations of n objects taken r at a time". We formalize the formula we'll use to compute permutations below.

FORMULA: Permutation

The number of permutations of n objects taken r at a time is

$${}_nP_r = \frac{n!}{(n-r)!}$$

Sometimes ${}_nP_r$ is also denoted by $P(n, r)$.

Example $3.2.3$: Computing Permutations

Find the following numbers.

1. The number of permutations of 12 objects taken 3 at a time.
2. The number of permutations of 8 objects taken 5 at a time.
3. The number of permutations of 32 objects taken 2 at a time.

Answer 1

Here $n = 12$ and $r = 3$

$$
\begin{aligned}
{}_{12}P_3 = \frac{12!}{(12-3)!} &= \frac{12!}{9!} \\
&= \frac{9!(10)(11)(12)}{9!} \\
&= 1,320
\end{aligned} \tag{3.2.5}
$$

Answer 2

Here $n = 8$ and $r = 5$

$$
\begin{aligned}
{}_8P_5 = \frac{8!}{(8-5)!} &= \frac{8!}{3!} \\
&= \frac{3!(4)(5)(6)(7)(8)}{3!} \\
&= 6,720
\end{aligned} \tag{3.2.6}
$$

Answer 3

Here $n = 32$ and $r = 2$

$$
\begin{aligned}
{}_{32}P_2 = \frac{32!}{(32-2)!} &= \frac{32!}{30!} \\
&= \frac{30!(31)(32)}{30!} \\
&= 992
\end{aligned} \tag{3.2.7}
$$

Your Turn 3.2.3: Evaluate Permutation

Evaluate each expression.

1. Choose 11 out of 9: ${}_{11}P_9$ =
2. Choose 5 out of 2: ${}_5P_2$ =
3. Choose 8 out of 8: ${}_8P_8$ =
4. Choose 10 out of 1: ${}_{10}P_1$ =

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Example 3.2.4: Applying Permutations

1. A high school graduating class has 312 students. The top student is declared valedictorian, and the second-best is named salutatorian. How many possible outcomes are there for the valedictorian and salutatorian?
2. In the card game blackjack, the dealer’s hand of 2 cards is dealt with one face up and one facedown. If the game is being played with a single deck of (52) cards, how many possible hands could the dealer get?
3. The University Combinatorics Club has three officers: president, vice president, and treasurer. If there are 18 club members, how many ways are there to fill the officer positions?

Answer 1

This is the number of permutations of 312 students taken 2 at a time, and the chosen students' order matters. So

$$
\begin{aligned}
P(312,2) = {}_{312}P_2 = \frac{312!}{(312-2)!} &= \frac{312!}{310!} \\
&= \frac{310!(311)(312)}{310!} = 97,032
\end{aligned} \tag{3.2.8}
$$

Answer 2

We want the number of permutations of cards taken two at a time, and the order of the cards matters. So

$$P(52,2) = {}_{52}P_2 = \frac{52!}{(52-2)!} = \frac{52!}{50!} \\ = \frac{50! \times 51 \times 52}{50!} = 2,652 \tag{3.2.9}$$

Answer 3

Here, we're looking for the number of permutations of 18 members taken three at a time, and the officers' order matters. So

$$P(18,3) = {}_{18}P_3 = \frac{18!}{(18-3)!} = \frac{18!}{15!} \\ = \frac{15! \times 16 \times 17 \times 18}{15!} = 4,896 \tag{3.2.10}$$

Your Turn \(\PageIndex{4 \): Problem Involving Permutation

Evaluating Permutations
Evaluate the following
There are 8 children that are taking turns using the swing on a playground. How many different ways are there in which they each can use the swing where everyone gets exactly one turn?
Kaden has 14 books on his summer reading list. He wants to choose 5 books to read in order from first to last. In how many ways can he do this?
The student council has 16 students. They need to choose a President, Vice President, Secretary and Treasurer for student council. In how many ways can the 4 different positions be appointed from the 16 students?
Lloyd is creating a 6 digit passcode using the digits 0 through 9. How many 6 digit passcodes are possible if the numbers cannot be repeated?

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Who Knew?: Very Big Permutations

Permutations involving relatively small sets of objects can get very big very quickly. A standard deck contains 52 cards. So, the number of different ways to shuffle the cards—in other words, the number of permutations of 52 objects taken 52 at a time—is $52! \approx 8 \times 10^{67}$ (written out, that's an 8 followed by 67 zeroes). The estimated age of the universe is *only* about 4×10^{17} seconds. So, if a very bored, all-powerful being started shuffling cards at the instant the universe began, it would have to have averaged at least $\frac{8 \times 10^{67}}{4 \times 10^{17}} \approx 2 \times 10^{50}$ shuffles *per second since the beginning of time* to have covered every possible arrangement of a deck of cards. That means the next time you pick up a deck of cards and give it a good shuffle, it's almost certain that the particular arrangement you created has never been created before and likely never will be created again.

Permutation with Duplicate Items

In some cases, some of the choices available are identical. Suppose someone was given the chance to choose five gift cards out of 10 available gift cards, and they noticed that 3 were from their favorite coffee shop. They might choose those 3 first and then decide on the other two. The number of choices changes when duplicate items are available.

FORMULA

The number of distinct permutations of n items when p times are identical, q items are identical, r items are identical, and so on... is given by,

$$\text{Number of choices} = \frac{n!}{p!q!r!\ldots} \tag{3.2.11}$$

Note: Here, the number of choices is also the number of arrangements.

Example $3.2.5$: Computing Permutations for Duplicate Items

1. How many distinct ways could you arrange the letters in the word MISSISSIPPI?
2. Sarah bought 12 plants to arrange along the border of her garden. How many distinguishable arrangements can she make if the plants comprise three tulips, six roses, and three daisies?

Answer 1

Consider first the total number of items. There are 11 letters in the word MISSISSIPPI. This means $n = 11$. Next, consider the duplicate items.

There are four of the letter I, four of the letter S, and two of the letter P.

$$\text{Choices} = \frac{n!}{p!q!r!\ldots} = \frac{11!}{4!4!2!} = 34,650$$

In comparison, if we were asked to arrange the letters in an 11 -letter word with no duplicate letters, the answer would be

$${}_{11}P_{11} = \frac{11!}{(11-11)!} = 11! = 39,916,800$$

Answer 2

There are 12 plants. This means $n = 12$

Next, consider the duplicate items. There are three tulips, six roses, and three daisies. The number of distinguishable arrangements will be

$$\text{Choices} = \frac{n!}{p!q!r!\ldots} = \frac{12!}{3!6!3!} = 18,480$$

Your Turn $3.2.5$: Permuation of Duplicate Items

Determine the number of distinguishable arrangements you can make using the letters in the word "MISSISSIPPI".

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10

Figure 3.2.1: Combinations help us count things like the number of possible card hands, when the order in which the cards were drawn doesn't matter. (credit: "IMG_3177" by Zanaca/Flickr, CC BY 2.0)

In Permutations, we studied permutations, which we use to count the number of ways to generate an ordered list of a given length from a group of objects. An important property of permutations is that the order of the list matters: The results of a race and the selection of club officers are examples of lists where the order is important. In other situations, the order is not important. For example, in most card games where a player receives a hand of cards, the order in which the cards are received is irrelevant; in fact, players often rearrange the cards in a way that helps them keep the cards organized.

When the order of a list of objects doesn't matter, the lists are no longer permutations. Instead, we call them combinations. A **combination** refers to the selection of objects without regard to the order. Unlike permutations, the order of selection does not matter in combinations.

Example 3.2.6: Distinguishing Between Permutations and Combinations

For each of the following situations, decide whether the chosen subset is a permutation or a combination.

1. A social club selects 3 members to form a committee. Each of the members has an equal share of responsibility.
2. You are prompted to reset your email password; select a password consisting of 10 characters without repeats.
3. At a dog show, the judge must choose first, second, and third-place finishers from 16 dogs.
4. At a restaurant, the day's special comes with the customer's choice of 3 sides taken from a list of 6 possibilities.
5. A club has 5 members. How many ways can you choose a president and a vice president?

Answer

1. Since there is no distinction among the responsibilities of the 3 committee members, the order isn't necessary. So, this is a combination.
2. The order of the characters in a password matters, so this is a permutation.
3. The order of finish matters in a dog show, so this is a permutation.
4. A plate with mashed potatoes, peas, and broccoli is functionally the same as a plate with peas, broccoli, and mashed potatoes, so this is a combination.
5. **Order matters** because the roles of **president** and **vice president** are different, so this is a permutation.

Your Turn 3.2.6: Permutation of Combination

Determine if the following scenarios represent combinations or permutations.

a. How many ways are there to pick 3 astronauts to go to the moon out of 10 candidates?
Select an answer

b. How many different ways are there to pick 3 astronauts to go to the moon, when one will be captain, one navigator, and one science officer? Select an answer

c. If there are 12 runners in a race, how many different orders could the all 12 runners finish?
Select an answer

d. If there are 12 runners in a race, and prizes are given for first, second, and third places, how many different orders are there for the top 3 places? Select an answer

e. If there are 12 runners in a race, and 3 runners are randomly chosen out of the 12 for drug testing, how many different ways can these 3 be chosen? Select an answer

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Note: Permutation and Combination: How they are Different?

If you have the letters A, B, and C, the possible ***permutations*** of 2 letters are:
AB, BA, AC, CA, BC, CB.
Here, the order of the letters makes a difference. There are six permutations.

If you have the letters A, B, and C, the possible ***combinations*** of 2 letters are:
AB, AC, BC.
In this case, AB and BA are considered the same combination because the order doesn't matter. Similarly, AC and CA are the same, BC and CB are the same.

These examples show the difference clearly: **permutations** are about **arrangements** (order matters), while **combinations** are about **selections** (order doesn't matter)

FORMULA: Combination

The number of combinations when r objects are chosen from n objects is

$$_{n}C_{r}=\frac{n!}{r!(n-r)!} \tag{3.2.12}$$

Sometime $_{n}C_{r}$ is also denoted by $C(n,r)$.

Example 3.2.7: Using the Combination Formula

Compute the following:

1. $_{8}C_{3}$
2. $_{12}C_{5}$
3. $_{15}C_{9}$

Answer

1. Here $n=8$ and $r=3$

$$C(8,3)={}_{8}C_{3}=\frac{8!}{3!(8-3)!}=\frac{8\times 7\times \mathbf{6}\times \mathbf{5!}}{\mathbf{3}\times \mathbf{2}\times \mathbf{1}\times \mathbf{5!}}=8\times 7=56 \tag{3.2.13}$$

2. Here $n=12$ and $r=5$

$$\begin{aligned}C(12,5)={}_{12}C_{5}&=\frac{12!}{5!(12-5)!}\\&=\frac{12\times 11\times 10\times 9\times 8\times 7!}{5\times 4\times 3\times 2\times 1\times 7!}\\&=\frac{12\times 11\times 10\times 9\times 8}{120}\\&=\frac{95,040}{120}\\&=792\end{aligned} \tag{3.2.14}$$

3. Here $n=15$ and $r=9$

$$\begin{aligned} C(15,9) = {}_{15}C_9 &= \frac{15!}{9!(15-9)!} \\ &= \frac{\mathbf{15} \times 14 \times 13 \times \mathbf{12} \times 11 \times 10 \times \mathbf{9!}}{\mathbf{9!} \times \mathbf{6} \times \mathbf{5} \times 4 \times \mathbf{3} \times \mathbf{2} \times 1} \\ &= \frac{15 \times 14 \times 13 \times 11 \times 10}{720} \\ &= 5,005 \end{aligned} \tag{3.2.15}$$

Your Turn 3.2.7: Evaluate Combination

Evaluate the following.

$_{22}C_4$ = ____________

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Example 3.2.8: Applying the Combination Formula

1. In the card game Texas Hold'em (a variation of poker), players are dealt 2 cards from a standard deck to form their hands. How many different hands are possible?
2. The board game *Clue* uses a deck of 21 cards. If 3 people are playing, each person gets 6 cards for their hand. How many different 6 -card *Clue* hands are possible?
3. Palmetto Cash 5 is a game offered by the South Carolina Education Lottery. Players choose 5 numbers from the whole numbers between 1 and 38 (inclusive); the player wins the jackpot of $\$100,000$ if the randomizer selects those numbers in any order. How many different sets of winning numbers are possible?

Answer

1. A standard deck has 52 cards, and a hand has 2 cards. Since the order doesn't matter, we use the formula for counting combinations:

$$\begin{aligned} {}_{52}C_2 &= \frac{52!}{2!(52-2)!} \\ &= \frac{52!}{2! \times 50!} \\ &= \frac{50! \times 51 \times 52}{1 \times 2 \times 50!} \\ &= 1,326 \end{aligned} \tag{3.2.16}$$

2. Again, the order doesn't matter, so the number of combinations is:

$$\begin{aligned} {}_{21}C_6 &= \frac{21!}{6!(21-6)!} \\ &= \frac{21!}{6! \times 15!} \\ &= \frac{15! \times 16 \times \ldots \times 21}{1 \times \ldots \times 6 \times 15!} \\ &= 54,264 \end{aligned} \tag{3.2.17}$$

3. There are 38 numbers to choose from, and we must pick 5. Since order doesn't matter, the number of combinations is:

$$\begin{aligned} {}_{38}C_5 &= \frac{38!}{5!(38-5)!} \\ &= \frac{38!}{5! \times 33!} \\ &= \frac{33! \times 34 \times \ldots \times 38}{1 \times \ldots \times 5 \times 33!} \\ &= 501,942 \end{aligned} \tag{3.2.18}$$

Your Turn 3.2.8: Use Combination

Does the following question involve permutation or combination? Find n and r and solve the problem.

There are 27 members on a city council. On a recent agenda item, 6 the council members voted in favor of a budget increase. How many possible groups of council members could have voted in favor?

○ Combination

Permutation

$n =$

$r =$

There can be ______ groups.

Hint

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Checkpoint

The notation and nomenclature used for the number of combinations is not standard across all sources. You'll sometimes see $\binom{n}{r}$ *instead of* msub $_nC_r$ *Sometimes you'll hear that expression read as "n choose r" as shorthand for "the number of combinations of n objects taken r at a time."*

People in Mathematics: Early Eastern Mathematicians

Although combinations weren't studied in Europe until around the 13th century, mathematicians of the Middle and Far East had already been working on them for hundreds of years. The Indian mathematician known as Pingala had described them by the second century BCE; Varāhamihira (fl. sixth century) and Halayudha (fl. 10th century) extended Pingala's work. In the ninth century, an Indian mathematician named Mahāvīra gave the formula for combinations that we use today.

In 10th-century Baghdad, a mathematician named Al-Karaji also knew formulas for combinations;. However, his work is now lost, it was known to (and repeated by) Persian mathematician Omar Khayyam, whose work survives. Khayyam is probably best remembered as a poet, with his *Rubaiyat* being his most famous work.

Meanwhile, in 11th-century China, Jia Xian was also working with combinations, as was his 13th-century successor, Yang Hui.

It is not known whether the discoveries of any of these men were known in the other regions or if the Indians, Persians, and Chinese all came to their discoveries independently. We know that mathematical knowledge and sometimes texts got passed along trade routes, so it can't be ruled out.

Example 3.2.9: Combining Combinations with the Multiplication Rule for Counting

The student government at a university consists of 10 seniors, 8 juniors, 6 sophomores, and 4 freshmen.

1. How many ways are there to choose a committee of 8 people from this group?
2. How many ways are there to choose a committee of 8 people if the committee must consist of 2 people from each class?

Answer

1. There are 28 people to choose from, and we need 8. So, the number of possible committees is

$$_{28}C_8 = \frac{28!}{8!(28-8)!} = \frac{28!}{8! \times 20!} = 3,108,105 \tag{3.2.19}$$

msubmsub

2. Break the selection of the committee members down into a 4-step process: Choose the seniors, then choose the juniors, then the sophomores, and then the freshmen. The number of ways to choose 2 people from 10 seniorsmsub

$$_{10}C_2 = \frac{10!}{2!(10-2)!} = 45 \tag{3.2.20}$$

The number of ways to choose 2 people from 8 juniors

$$_8C_2 = \frac{8!}{2!(8-2)!} = 28 \tag{3.2.21}$$

The number of ways to choose 2 people from 6 sophomores

$$ {}_6C_2 = \frac{6!}{2!(6-2)!} = 15 \tag{3.2.22} $$

The number of ways to choose 2 people from 4 freshmen

$$ {}_4C_2 = \frac{4!}{2!(4-2)!} = 6 \tag{3.2.23} $$

The Multiplication Rule for Counting tells us that we can get the total number of ways to complete this task by multiplying the number of ways to do each of the four subtasks. So, there are

$$ 45 \times 28 \times 15 \times 6 = 113{,}400 $$

There are $113,400$ possible committees with these restrictions.

Your Turn $3.2.9$: Combining Combination

A company is forming a team of 5 employees from different departments. The company has 10 marketing employee and 10 engineering employees. How many ways can they select a team of 5 people if the team must consist of 3 marketing employees and 2 engineering employees?

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3.3: Fundamental Concepts of Probability

Learning Objectives

1. Determine outcomes, sample spaces, and probabilities of events in probability experiments.
2. Calculate basic theoretical probabilities.
3. Calculate basic empirical probabilities.

Introducing Probability

It all comes down to this. The game *Monopoly*, which started hours ago, is in the home stretch. Your sister has the dice, and if she rolls a 4, 5, or 7, she'll land on one of your best spaces, and the game will be over. How likely is it that the game will end on the next turn? Is it more likely than not? How can we measure that likelihood? This section addresses this question by introducing a way to measure uncertainty.

If you roll a die, pick a card from the deck of playing cards, or randomly select a person and observe their hair color, we are executing an experiment or procedure. In probability, we look at the likelihood of different outcomes. We begin with some terminology

Uncertainty is, almost by definition, a nebulous concept. In order to put enough constraints on it that we can mathematically study it, we will focus on uncertainty strictly in the context of experiments. Recall that experiments are processes whose outcomes are unknown; the sample space for the experiment is the collection of all those possible outcomes. When we want to talk about the likelihood of particular outcomes, we sometimes group outcomes together; for example, in the *Monopoly* example at the beginning of this section, we were interested in the roll of 4 dice that might fall as a 4, 5, or 7. A grouping of outcomes that we're interested in is called an event. In other words, an event is a subset of the sample space of an experiment; it often consists of the outcomes of interest to the experimenter.

When we are talking about combinatorics or probability, the word "experiment" has a slightly different meaning than it does in the sciences. Experiments can range from very simple ("flip a coin") to very complex ("count the number of uranium atoms that undergo nuclear fission in a sample of a given size over the course of an hour"). Experiments have unknown outcomes that generally rely on something random, so that if the experiment is repeated (or replicated), the outcome might be different. No matter what the experiment, though, analysis of the experiment typically begins with identifying its sample space.

Outcome

The possible result of an experiment is called an **outcome**.

Sample Space

The **sample space** of an experiment is the set of all possible outcomes of the experiment.

It's often expressed as a set (i.e., as a list bound by braces; if the experiment is "randomly select a number between 1 and 4," the sample space would be written $\{1, 2, 3, 4\}$.

Event

An **event** is any particular outcome or group of outcomes. The event is a subset of the sample space.

Example $3.3.1$: Finding the Sample Space (Dice and Coin)

For each of the following experiments, identify the sample space.

1. Flip a coin (which has 2 faces, typically called "heads" and "tails").
2. Flip a coin 2 times.
3. Roll a 6-sided die.
4. Roll two 6-sided dice.

Answer

1. If we use "H" to denote "heads is facing up" and "T" to denote "tails is facing up", then the sample space is $\{H, T\}$.
2. {HH, HT, TH, TT}. Note that each sequence shows the result of the **first flip** followed by the **second flip**.
3. There are 6 numbers on the die: $1, 2, 3, 4, 5$, and 6. So, the sample space for a single roll of the die is $\{1, 2, 3, 4, 5, 6\}$.
4. Note that each ordered pair below shows the result of the first die followed by the second die.

SECOND DIE

FIRST DIE

	⚀	⚁	⚂	⚃	⚄	⚅
⚀	(1, 1)	(1, 2)	(1, 3)	(1, 4)	(1, 5)	(1, 6)
⚁	(2, 1)	(2, 2)	(2, 3)	(2, 4)	(2, 5)	(2, 6)
⚂	(3, 1)	(3, 2)	(3, 3)	(3, 4)	(3, 5)	(3, 6)
⚃	(4, 1)	(4, 2)	(4, 3)	(4, 4)	(4, 5)	(4, 6)
⚄	(5, 1)	(5, 2)	(5, 3)	(5, 4)	(5, 5)	(5, 6)
⚅	(6, 1)	(6, 2)	(6, 3)	(6, 4)	(6, 5)	(6, 6)

Figure $3.3.1$: Possible outcomes of rolling two dice

Your Turn $3.3.1$: Find Sample Space

An engineering firm is hired to determine if certain waterways in New York are safe for fishing. Samples are taken from 3 watersources. Let the letters

F = safe for fishing,   N = not safe for fishing

a) List all the outcomes in the sample space, the results from the 3 water samples.

b) List all the outcomes in the event E: at least 2 waterways are safe for fishing.

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Example $3.3.2$: Finding the Event (Die and Deck of Cards)

Identify the event in the following experiment.

1. Roll one 6-sided die. Find the event rolling even number.
2. Roll two 6-sided dice. Find the event rolling two same numbers.
3. Draw a card from a deck of 52 cards. Find the event drawing king.

Answer

1. The sample space is the set of all possible outcomes: $\{1,2,3,4,5,6\}$. There are 6 outcomes in the sample space. The **event** here is $\{2,4,6\}$.
2. All 36 possible outcomes while rolling two dice are given in the table in the example $3.3.1$.

   So, for example, $(3,2)$ represents the outcome where one 6-sided roll results in a 3, and another 6-sided roll gives us a 2. The sample space of the experiment in the above table can also be written as

   Event: Rolling two same numbers is $\{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

3. Recall that a standard deck of cards consists of 52 unique cards which are labeled with a rank (the whole numbers from 2 to 10, plus J, Q, K, and A) and a suit (♣ , ♢, ♡, or ♠). A standard deck is thoroughly shuffled, and you draw one card at random (so every card has an equal chance of being drawn). Find the theoretical probability of each of these events: There are 52 possible outcomes while drawing a card from a deck of cards. (See the image.)

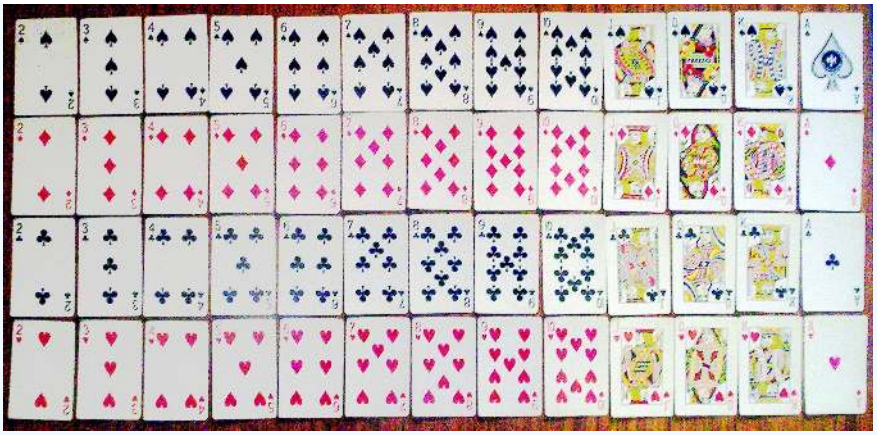

Figure 3.3.2: Set of Playing Cards (credit: via Wikimedia Commons, CC BY_SA 3.0)

The **event** for drawing a king will be {♣K , ♢K, ♡K, ♠K}.

Using Tree Diagrams to Identify Sample Spaces

In experiments where there are two or more stages, or where the stages are dependent, a tree diagram is a helpful tool for systematically identifying the sample space. Tree diagrams are built by first drawing a single point (or *node*), then from that node we draw one branch (a short line segment) for each outcome of the first stage. Each branch gets its own node at the other end (which we typically label with the corresponding outcome for that branch); from each of these, we draw another branch for each outcome of the second stage, assuming that the outcome of the first stage matches the branch we were on. If there are other stages, we can continue from there by continuing to add branches and nodes. This sounds really complicated, but it's easier to understand through an example.

✔ Example 3.3.3: Finding the Sample Space Using Tree Diagram

Use a tree diagram to find the sample spaces of each of the following experiments:

1. You flip a coin 3 times.
2. You flip a coin. If the result is a head, you roll a 4 -sided die. If it's a tail, you roll a 6-sided die.

Answer

Let H represent the head outcome, and T represent the tail outcome. See figure 3.3.3.

The sample space is: S = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.

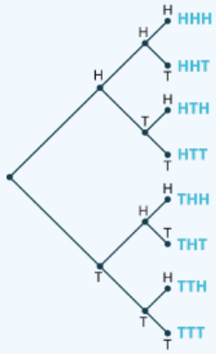

Figure 3.3.3: Sample Space of Flipping Three Coins

Answer

If the result is H, you roll either $1, 2, 3, \text{and } 4$. If result is T, you roll either $1, 2, 3, 4, 5, \text{and} 6$. See figure 3.3.4.

The sample space is {H1, H2, H3, H4, T1, T2, T3, T4, T5, T6}

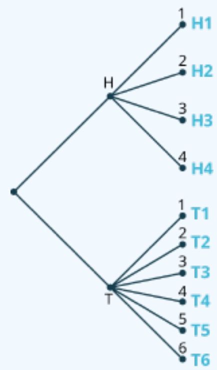

Figure 3.3.4: Sample Sapce

Your Turn 3.3.3: Indentify Sample Space

Determine the sample space of these events.

a) Kasa has a box of buttons containing 2 black buttons (B), 2 white buttons (W), and 2 pink buttons (P). She draws 1 button at a time from the box. Write the sample space if Kasa draws out 1 button.

b) Write the sample space for a coin flipped, and a six-sided die rolled. Example: H1 means the coin lands heads and the die lands 1.

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Theoretical Probability

The theoretical method gives the most reliable results, but cannot always be used. Suppose the sample space of an experiment consists of equally likely outcomes. In that case, the theoretical probability of an event is defined as the ratio of the number of outcomes in the event to the number of outcomes in the sample space.

Probability Notation: Possible Values of Probability

For any event A, the notation $P(A)$ is for the probability of event A occurring. The smallest possible probability is 0, if no outcomes correspond with the event. The largest possible probability is 1, if all possible outcomes correspond with the event.

The probability of any event must be between zero and one, inclusive. That is

$$0 \leq P(A) \leq 1$$

.

Note: You can express **probability as a percentage** by simply converting the decimal value to a percentage. For example, the probability of tossing heads when we flip one coin is 0.5. We can also say that when we flip a coin, there is a 50% chance that the coin lands as a head.

In the course of this chapter, if you compute a probability and get an answer that is negative or greater than 1, you have made a mistake and should check your work.

An **impossible event** has a probability of 0. In another word, if $P(A)=0$, the event A is impossible.

A **certain event** has a probability of 1 or 100%. In another word, if $P(A)=1$, the event A is certain.

Probabilities are essentially fractions and can be reduced to lower terms, like fractions, or we can express probability as a decimal as well.

Your Turn 3.3.4: Classify the Event

Describe the likelyhood of each event given its probability.

a) $P=0.5$ Select an answer

b) $P=0.39$ Select an answer

c) $P=0$ Select an answer

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FORMULA: THEORETICAL PROBABILITY

For an experiment whose sample space S consists of equally likely outcomes, the **theoretical probability** of the event E is the ratio

$$P(\text{Event})=\frac{\text{number of outcomes in the event}}{\text{number of outcomes in sample space}}$$

Example 3.3.5: Compute Theoretical Probability (Die)

If we roll a 6-sided die, calculate

1. P(rolling a 1).
2. P(rolling a number bigger than 4).

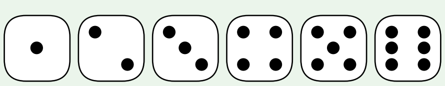
Answer

Recall that the sample space is $\{1,2,3,4,5,6\}$.

1. There is one outcome corresponding to "rolling a 1", so P(rolling a 1) = $\frac{1}{6}$
2. There are two outcomes bigger than a 4, so P(rolling a number bigger than 4) = $\frac{2}{6}=\frac{1}{3}$

Your Turn 3.3.5: Roll a Die

A single, six-sided fair die is rolled. Find the following probabilities.



1. The number showing is a 3:
   $P(3)=$ ____
2. The number showing is an even number:
   $P(\text{even})=$ ____
3. The number showing is greater than 3:
   $P(\textit{greater than } 3)=$ ____

(Round your asnwer to 4 decimal places)

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Example 3.3.6: Determining Certain and Impossible Events

A. Consider an experiment that consists of rolling a single standard 6-sided die (with faces numbered $1-6$). Decide if these probabilities are equal to zero, equal to one, or somewhere in between.

1. P(roll 4).
2. P(roll 7).
3. P(roll a positive number).
4. P(roll $\frac{1}{3}$).
5. P(roll an even number).

B. Consider an experiment of drawing a card from a deck of cards. Decide if these probabilities are equal to zero, equal to one, or somewhere in between.

1. A card is a king.
2. A card is a king or queen.
3. A card is red and black.
4. A card is red or black.

Answer

A. Let's start by identifying the sample space. For one roll of this die, the possible outcomes are $\{1,2,3,4,5,6\}$. We can use that to assess these probabilities:

1. We see that 4 is in the sample space, so it's possible that it will be the outcome. It's not certain to be the outcome, though. So, $0 < P(\text{roll a }4) < 1$.
2. Notice that 7 is not in the sample space. So, $P(\text{roll a }7) = 0$.
3. Every outcome in the sample space is a positive number, so this event is certain. Thus, $P(\text{roll a positive number}) = 1$.
4. Since $\frac{1}{3}$ is not in the sample space, P(roll $\frac{1}{3}$) = 0, so this is impossible event.
5. Some outcomes in the sample space are even numbers $2, 4$, and 6, but the others aren't. So, $0 < P(\text{roll an even number}) < 1$

B. There are 52 cards in the deck, so the sample space for each of these experiments has 52 elements. That will be the denominator for each of our probabilities.

1. There are 4 king cards in total in the deck, so $P(\text{king}) = \frac{4}{52} = \frac{1}{13}$, so this event probability is in between zero and one.
2. There are 8 king or queen cards in the deck, so $P(\text{king or queen}) = \frac{8}{52} = \frac{2}{13}$, so this event probability is in between zero and one.
3. There are 0 red and black cards in the deck, so $P(\text{red and black}) = \frac{0}{52} = 0$, so this event is impossible.
4. There are 52 red or black cards in the deck, so $P(\text{red or black}) = \frac{52}{52} = 1$, so this event is certain.

Your Turn 3.3.6: 12-sided Die

You are about to roll a standard 8-sided die (with faces labeled 1-8).

Find the probability of each event and classify it as certain, impossible, or something in between. Enter probabilities as integers or reduced fractions.

Event	Probability	Classification
Roll a positive number		Select an answer
Roll a number less than 1		Select an answer

Roll a 8		Select an answer
Roll a number less than 3		Select an answer
Not roll a 8		Select an answer
Roll a multiple of 2		Select an answer

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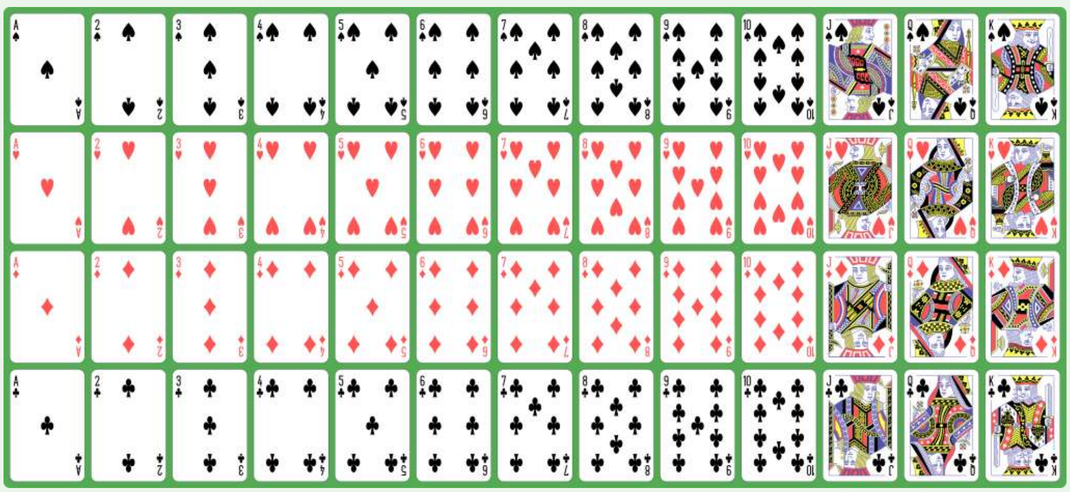
Your Turn 3.3.7: Deck of Cards

Here is a full deck of cards.



You will randomly draw one card from a well-shuffled deck.

(a) Find the chances of these events written as probabilities. *Write answers as simplified fractions.*

i. P(draw a card **without a number**)

ii. P(draw a card that is a **black queen**)

iii. P(draw a card that contains an **odd number**)

(Note that ace and picture card has no number value)

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Example 3.3.8: Spinner

Suppose the spinner shown below is used for a game. Find the following probability.

Figure 3.3.6: Spinning Wheel with Six number 1 to 6

1. P(spin 5).
2. P(spin multiple of 2).
3. P(spin a number greater than 4).
4. P(spin 2 or 3).

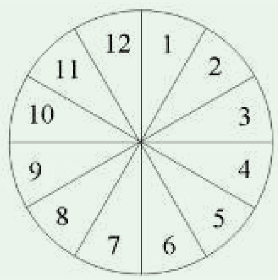
Answer

There are 6 possible outcomes $\{1, 2, 3, 4, 5, 6\}$.

1. P(spin 5) = $\frac{1}{6}$
2. P(spin multiple of 2) = $\frac{3}{6}$ = $\frac{1}{2}$
3. P(spin a number greater than 4) = $\frac{2}{6}$ = $\frac{1}{3}$
4. P(spin 2 or 3) = $\frac{2}{6}$ = $\frac{1}{3}$

Your Turn 3.3.8: Babylonian to Hindu Arabic

Suppose the spinner shown below is used for a game.



Find the probability of each event below. Write all probabilities as fractions in simplest form.

(a) *P(spin 2)* =

(b) *P(spin 5 or 8)* =

(c) *P(spin a multiple of 3)* =

(d) *P(spin a number greater than 5)* =

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The theoretical method gives the most reliable results, but it cannot always be used. If the sample space of an experiment consists of equally likely outcomes, then the theoretical probability of an event is defined to be the ratio of the number of outcomes in the event to the number of outcomes in the sample space.

Checkpoint

It is critical that you make sure that every outcome in a sample space is equally likely before you compute theoretical probabilities!

Example 3.3.9: Find Theoretical Probabilities (Roll of Two Die)

In the Basic Concepts of Probability, we were considering a *Monopoly* game where, if your sister rolled a sum of 4, 5, or 7 with 2 standard dice, you would win the game. What is the probability of this event? Use tables to determine your answer.

Answer

We should think of this experiment as occurring in two stages: (1) one die roll, then (2) another die roll. Even though these two stages will usually occur simultaneously in practice, since they're independent, it's okay to treat them separately. Now, each of the 36 ordered pairs in the table represents an equally likely outcome.

Since we have two independent stages, let's create a table (Figure 3.3.8), which is probably the most efficient method for determining the sample space.

Second Die \ First Die	1	2	3	4	5	6
1	(1, 1)	(2, 1)	(3, 1)	(4, 1)	(5, 1)	(6, 1)
2	(1, 2)	(2, 2)	(3, 2)	(4, 2)	(5, 2)	(6, 2)
3	(1, 3)	(2, 3)	(3, 3)	(4, 3)	(5, 3)	(6, 3)
4	(1, 4)	(2, 4)	(3, 4)	(4, 4)	(5, 4)	(6, 4)
5	(1, 5)	(2, 5)	(3, 5)	(4, 5)	(5, 5)	(6, 5)
6	(1, 6)	(2, 6)	(3, 6)	(4, 6)	(5, 6)	(6, 6)

Figure 3.3.8: Sample Space of Rolling Two Dice

To make our analysis easier, let's replace each ordered pair with the sum, and since the event we're interested in is the one consisting of rolls of 4, 5, or 7. Let's shade those in (Figure ✔ 3.3.9.)

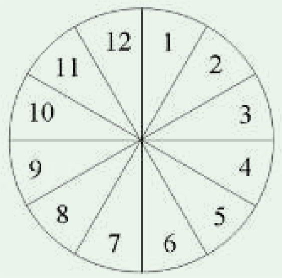
Second Die \ First Die	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Figure 3.3.9: Possible Sum When Rolling Two Dice

Our event contains 13 outcomes, so the probability that your sister rolls a losing number is $\frac{13}{36}$.

Your Turn 3.3.9: Spinner

Suppose the spinner shown below is used for a game.



Find the probability of each event below. Write all probabilities as fractions in simplest form.

(a) *P(spin 1)* =

(b) *P(spin 4 or 11)* =

(c) *P(spin a multiple of 4)* =

(d) *P(spin a number greater than 5)* =

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Example 3.3.10: Tree Diagrams to Compute Theoretical Probability (Flip Coins 3 times)

If you flip a fair coin 3 times, what is the probability of each event? Use a tree diagram to determine your answer

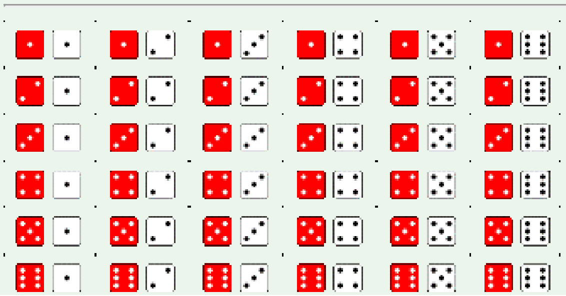
1. You flip exactly 2 heads.
2. You flip 2 consecutive heads at some point in the three flips.
3. All 3 flips show the same result.

Answer

As we see in figure 3.3.3, the sample space is {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}, which has 8 outcomes.

1. Flipping exactly 2 heads occurs three times (HHT, HTH, THH), so the probability is $\frac{3}{8}$.
2. Flipping 2 consecutive heads at some point in the experiment happens 3 times: HHH, HHT, THH. So, the probability is $\frac{3}{8}$.
3. There are 2 outcomes that all show the same result: HHH and TTT. So, the probability is $\frac{2}{8} = \frac{1}{4}$.

Your Turn 3.3.10: Roll Die Two Times



Two dice are rolled. What is the probability of getting a sum of 4?

Use 4 decimal places.

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People in Mathematics: Gerolamo Cardano

The first known text that provided a systematic approach to probabilities was written in 1564 by Gerolamo Cardano ($1501–1576$). Cardano was a physician whose illegitimate birth closed many doors that would have otherwise been open to someone with a medical degree in 16th-century Italy. As a result, Cardano often turned to gambling to help ends meet. He was a remarkable mathematician, and he used his knowledge to gain an edge when playing cards or dice. His 1564work, titled *Liber de ludo aleae* (which translates as *Book on Games of Chance*), summarized everything he knew about probability. Of course, if that book fell into the hands of those he played against, his advantage would disappear. That's why he never allowed it to be published in his lifetime (it was eventually published in 1663). Cardano made other contributions to mathematics; he was the first person to publish the third-degree analogue of the Quadratic Formula (though he didn't discover it himself), and he popularized the use of negative numbers.

Empirical Probability

Theoretical probabilities are precise but can't be found in every situation. If the outcomes in the sample space are not equally likely, then we're out of luck. Suppose you're watching a baseball game, and your favorite player is about to step up to the plate. What is the probability that he will get a hit?

In this case, the sample space is {hit, not a hit}. That doesn't mean that the probability of a hit is \frac{1}{2}, since those outcomes aren't equally likely. The theoretical method simply can't be used in this situation. Instead, we might look at the player's statistics up to this point in the season, and see that he has 122 hits in 531 opportunities. So, we might think that the probability of a hit in the next plate appearance would be about $\frac{122}{531} \simeq 0.23$. When we use the outcomes of previous replications of an experiment to assign a probability to the next replication, we're defining an empirical probability. Empirical probability is assigned using the outcomes of previous replications of an experiment by finding the ratio of the number of times the event occurred in the previous replications to the total number of previous replications.

Empirical probabilities aren't exact, but when the number of previous replications is large, we expect them to be close. Also, if the previous runs of the experiment were not conducted under the exact set of circumstances as the one we're interested in, the empirical probability is less reliable. For instance, in the case of our favorite baseball player, we might try to get a better estimate of the probability of a hit by looking only at his history against left- or right-handed pitchers (depending on the handedness of the pitcher he is about to face).

FORMULA: EMPIRICAL PROBABILITY

Empirical probability is the probability of an event occurring based on actual experiments or historical data rather than theoretical calculations. It is calculated using the formula:

$$P(\text{Event}) = \frac{\text{number of times event occurs}}{\text{total number of trials}}$$

If you flip a coin 100 times and it lands on heads 48 times, the empirical probability of getting heads is:

$$P(\text{Heads}) = \frac{48}{100} = 0.48$$

Empirical probability differs from theoretical probability because it is based on observed outcomes rather than assumed equal likelihoods. It is useful in real-world scenarios where probabilities cannot be easily determined theoretically.

Example 3.3.11: Finding Empirical Probabilities

Find the empirical probability of the following events:

1. Jose is on the basketball court practicing his shots from the free-throw line. He made 47 out of his last 80 attempts. What is the probability he will make his next shot?
2. Amy is about to begin her morning commute. Over her last 60 commutes, she arrived at work 12 times in under half an hour. What is the probability that she arrives at work in 30 minutes or less?
3. Felix is playing *Yahtzee* with his sister. Felix won 14 of the last 20 games he played against her. How likely is he to win this game?

Answer

1. Since Jose made 47 out of his last 80 attempts, assign this event an empirical probability of $\frac{47}{80} \approx 59\%$.
2. Amy completed the commute in under 30 minutes in 12 of the last 60 commutes, so we can estimate her probability of making it in under 30 minutes this time at $\frac{12}{60} = 20\%$.
3. Since Felix has won 14 of the last 20 games, assign a probability for a win this time of $\frac{14}{20} = 70\%$

Your Turn 3.3.11: Empirical Probability

A group of people were asked if they had run a red light in the last year:

- 138 responded "Yes."
- 332 responded "No."

What is the empirical probability that a person has run a red light in the last year?

Give your answer as a fraction or decimal rounded to at least 3 decimal places.

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Example 3.3.12: Empirical Probabillity

A group of people was asked if they had run a red light the previous year. 479 responded 'yes', and 341 responded 'no'. Find the Probability that if a person chooses at random,

1. The person ran a red light.
2. The person did not run a red light.

Answer

There are $479 + 341 = 820$ people surveyed

1. P(ran in red light) = $\frac{479}{820} = 0.584 = 58.4\%$
2. P(did not run in red light) = $\frac{341}{820} = 41.6\%$

Your Turn 3.3.12: Probability from Two Way Table

Giving a test to a group of students, the grades and gender are summarized below

	A	B	C	Total
Male	16	8	17	41
Female	2	14	20	36
Total	18	22	37	77

If one student was chosen at random, determine the following probabilities. Write your answers as reduced fractions.

P(Student was male) = ______

P(Student was female) = ______

P(Student was male and got an "A") = ______

P(Student was female and got a "B") = ______

P(Student got a "C") = ______

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3.4: The Addition and Complement Rule for Probability

Learning Objectives

1. Find the probability of the complement of an event.
2. Determine if two events are mutually exclusive.
3. Use the Addition Rule to find the probability of "or" events.

Complementary Events

Now, let us examine the probability that an event does **not** happen. As in the previous section, consider the situation of rolling a 6-sided die and first compute the probability of rolling a 6: the answer is $P(6) = \frac{1}{6}$. Now consider the probability that we do *not* roll a 6: there are 5 outcomes that are not a 6, so the answer is $P(\texttt{not a } 6) = \frac{5}{6}$. Notice that

$$P(6) + P(\texttt{not a } 6) = \frac{1}{6} + \frac{5}{6} = \frac{6}{6} = 1$$

So we have a rule

$$P(\text{not } A) = 1 - P(A)$$

Let's summarize the above discussion.

Complement of an Event

The **complement** of event A is the event "A does not happen (not A)." It is the set of outcomes in the sample space S that are not in event A.

- The notation A' is used for the complement of event A.
- The probability of the complement of event A is $P(A') = 1 - P(A)$.
- Notice also that $P(A) = 1 - P(A')$ since event A and its complement make up the entire sample space, $P(A) + P(A') = 1$.
- Note sometime A' is also written as A^c

Note that the complement of an event in probability is essentially the same as that of a set since an event is a subset of the sample space.

Example 3.4.1: Complement of an Event

1. If you pick a random card from a deck of playing cards, what is the probability it is not a heart?
2. ABC Airlines reports that 90.6% of its flights arrive on time. What percent of its flights do not arrive on time?

Answer

1. There are 13 hearts in the deck, so

$$P(\text{heart}) = \frac{13}{52} = \frac{1}{4}$$

The probability of *not* drawing a heart is the complement of drawing a heart:

$$P(\text{not heart}) = 1 - P(\text{heart}) = 1 - \frac{1}{4} = \frac{3}{4}$$

2. Let A = arrive on time. The probability of A is $P(A) = 0.906$. The probability of not arriving on time is is

$$P(A') = 1 - P(A) = 1 - 0.906 = 0.094 = 9.4\%$$

Your Turn 3.4.1: Find the Probabilty of Complement

The chance that Farrell hits the basket with a ball to win a prize at the county fair is $\frac{1}{2}$.

What is the probability that Farrell DOES NOT hit the basket?

A poll showed that $53.2\ \%$ of Americans say they believe that some psychics can help solve murder cases. What is the probability of randomly selecting someone who does not believe that some psychics can help solve murder cases.

Express your answer as a percent. %

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Addition Rule for “Or” Probabilities

Many probabilities in real life involve more than one event. If we draw a single card from a deck we might want to know the probability that it is either red or a jack. If we look at a group of students, we might want to know the probability that a single student has brown hair and blue eyes. When we combine two events, we make a single event called a **compound event**. To create a compound event, we can use the word “and” or the word “or” to combine events. It is very important in probability to pay attention to the words “and” and “or” if they appear in a problem. The word “and” restricts the field of possible outcomes to only those outcomes that simultaneously describe all events. The word “or” broadens the field of possible outcomes to those that describe one or more events. This is essentially the same as unions and intersections in sets.

Note that the non-exclusive use of the word "or" will double-count certain events. Say we look at the event of drawing a single card from a deck and the outcome of drawing a "red card or a jack". If you count the number of options for red cards and the number of options for a jack, you are counting the red jacks twice. This double counting needs to be accounted for.

Look at an example.

Suppose a teacher wants to know the probability that a student in her class of 30 is taking either Art or English. She asks the class to raise their hands if they are taking Art and counts 13 hands. Then she asks the class to raise their hands if they are taking English and counts 21 hands. The teacher then calculates (the incorrect answer)

$$P(\text{Art or English}) = \frac{13}{30} + \frac{21}{30} = \frac{13+21}{30} = \frac{34}{30} > 1$$

The teacher knows this is wrong because probabilities must be inclusive and between 0 and 1. After thinking about it, she remembered nine students taking both Art and English. These students raised their hands each time she counted, so the teacher counted them twice. When we calculate probabilities, we have to be careful to count each outcome only once.

The correct answer would be:

$$P(\text{Art or English}) = \frac{13+21-9}{30} = \frac{25}{30} = \frac{5}{6}$$

We will work out how to get the correct answer in general below.

Mutually Exclusive Events

An experiment involves drawing one card from a well-shuffled deck of 52 cards. Consider the events E = the card is red, F = the card is a 5, and G = the card is a spade. A card can be both red and a 5 simultaneously, but it is not possible for a card to be both red and a spade simultaneously. It would be easy to accidentally count a red 5 twice by mistake. It is not possible to count a red spade twice.

If two events have events in common or can happen simultaneously, the overlap is called the **intersection** of the events. The intersection of events is denoted as A "and" B, and is the same as the intersection of two sets A and B.

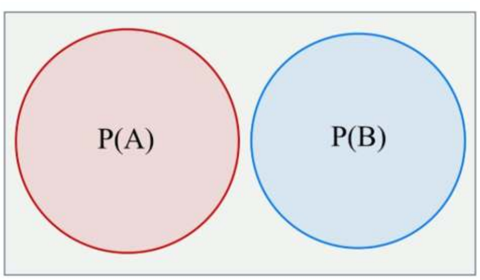

Figure $3.4.1$: Mutually exclusive events

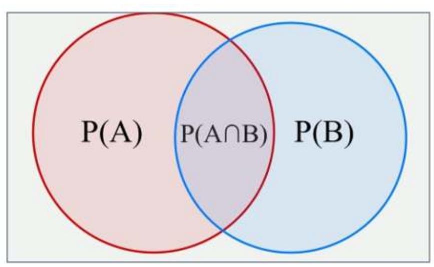

Figure $3.4.2$: Not mutually exclusive events

> Two events are **mutually exclusive** if they have no outcomes in common. In other words, **mutually exclusive events** are those that **cannot occur simultaneously**.

The probability of two mutually exclusive events A and B is $P(A \text{ and } B) = 0$.

Example of mutually exclusive events:

1. Drawing one card from a deck, A: Drawing a red card, and B: Drawing a black card
2. Rolling a die, A: Rolling a 2 and B: Rolling a 3
3. A student is chosen randomly: A: Student passes the exam, and B: Student fails the exam

Examples of not mutually exclusive events:

1. Rolling a die, A: Rolling an even number, and B: Rolling a number greater than 3
2. Drawing one card from a deck, A: Drawing a red card (hearts or diamonds), and B: Drawing a king
3. A student is chosen randomly. A: Student is in the drama club, and B: Student is in the soccer team

✔ Example $3.4.2$: Identify if the Events are Mutually Exclusive or not?

Two fair dice are tossed, and different events are recorded. Let the events A, B and C be as follows:

$A = \text{the sum is } 5 = \{(1,4),(2,3),(3,2),(4,1)\}$

$B = \text{both numbers are even } = \{(2,2),(2,4),(2,6),(4,2),(4,4),(4,6),(6,2),(6,4),(6,6)\}$

$C = \text{both numbers are less than } 5 = \{(1,1),(1,2),(1,3),(1,4),(2,1),(2,2),(2,3),(2,4),(3,1),(3,2),(3,3),(3,4),(4,1),(4,2),(4,3),(4,4)\}$

a. Are events A and B mutually exclusive?
b. Are events A and C mutually exclusive?
c. Are events B and C mutually exclusive?

Answer

a. Yes. A and B are mutually exclusive because they have no outcomes in common. It is not possible to add 2 even numbers to get a sum of 5.
b. No. A and C are not mutually exclusive because they have some outcomes in common. The pairs $(1,4)$, $(2,3)$, $(3,2)$, and $(4,1)$ all have sums of 5, and both numbers are less than 5.
c. No. B and C are not mutually exclusive because they have some outcomes in common. The pairs $(2,2)$, $(2,4)$, $(4,2)$, and $(4,4)$ all have two even numbers that are less than 5.

✎ Your Turn $3.4.2$: Disjoint or Not

For each scenario, determine whether the events are mutually exclusive or not mutually exclusive.

(A) *Scenario:* A spinner is divided into eight equal-sized regions and numbered from 1 to 8. You spin the spinner.

Event A: The spinner lands on the number 4.

Event B: The spinner lands on the number 6.

○ not mutually exclusive

○ mutually exclusive

(B) *Scenario:* A bag contains six yellow jerseys numbered 1 to 6 . The bag also contains 4 purple jerseys numbered 1 to 4. You randomly pick a jersey.

Event C: The jersey is purple.

Event D: The jersey has a number greater than 5.

○ not mutually exclusive

○ mutually exclusive

(C) *Scenario:* You select a single card from a standard deck of cards.

Event E: The card is a club.

Event F: The card is a red card.

○ not mutually exclusive
○ mutually exclusive

(D) *Scenario:* You select one single day out of the year.

Event G: The day selected is a Tuesday.

Event H: The day selected is Thanksgiving.

○ not mutually exclusive
○ mutually exclusive

(E) *Scenario*: Select a student taking classes at a community college.

Event I: The student is taking Calculus I.

Event J: The student is taking Biology.

○ mutually exclusive
○ not mutually exclusive

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Do Not Count the Same Outcome Twice.

The addition rule for probabilities is used when the word " or " connects the events .” Remember teacher in her class wanted to know the probability that her students took either Art or English. Her problem was that she counted some students twice. She needed to add the number of students taking Art to the number of students taking English, and then subtract the number of students she counted twice. After dividing the result by the total number of students, she will find the desired probability. The calculation is as follows:

$$\begin{aligned} P(\text{Art or English}) &= \frac{\#\text{ taking Art} + \#\text{ taking English - }\#\text{ taking both}}{\text{total number of students}} \\ &= \frac{13+21-9}{30} \\ &= \frac{25}{30} \approx 0.833 \end{aligned}$$

The probability that a student takes Art or English is 0.833 or $\$83.3\%$.

When calculating the probability for compound events connected by the word “or,” we must be careful not to count the same thing twice.

If we want the probability of drawing a red card or a 5 we cannot count the red 5s twice. If we want to know the probability of a person being blonde-haired or blue-eyed, we cannot count the blue-eyed blondes twice. The addition rule for probabilities adds the number of blonde-haired people to the number of blue-eyed people and then subtracts the number of people we counted twice.

Formula: Addition Rule for "Or" Probabilities

If A and B are any events, then the probability of either A or B occurring (or both) is

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are mutually exclusive events then $P(A \text{ and } B) = 0$, so then

$$P(A \text{ or } B) = P(A) + P(B)$$

Slightly different way, we can see the above formula

$$P(\text{A or B}) = \frac{\text{\# of outcomes in A} \ + \ \text{\#of outcomes in B} \ - \ \text{\# of outcomes in both A and B}}{\text{total number outcomes}}$$

Example 3.4.3: Deck of Card Mutually Exclusive

A single card is drawn from a well-shuffled deck of 52 cards. Find the probability that the card is a club or a face card.

Answer

There are 3 cards that are clubs, 12 face cards (J, Q, K in each suit), and 3 face cards that are clubs.

$$\begin{aligned} P(\text{club or face card}) &= P(\text{club}) + P(\text{face card}) - P(\text{club and face card}) \\ &= \frac{13}{52} + \frac{12}{52} - \frac{3}{52} \\ &= \frac{22}{52} = \frac{11}{26} \approx 0.423 \end{aligned}$$

The probability that the card is a club or a face card is approximately 0.423 or 42.3%.

OR you can do the following directly

$$\begin{aligned} P(\text{club card or face card}) &= \frac{\text{\# of club card} + \text{\#of face card} - \text{\# of both club and face card}}{\text{total number cards}} \\ &= \frac{13+12-3}{52} \\ &= \frac{22}{52} \approx 0.423 \end{aligned}$$

A simple way to check this answer is to take the 52 card deck and count the number of physical cards that are either clubs or face cards. If you were to set aside all of the clubs and face cards in the deck, you would end up with the following:

OR you can also do the following

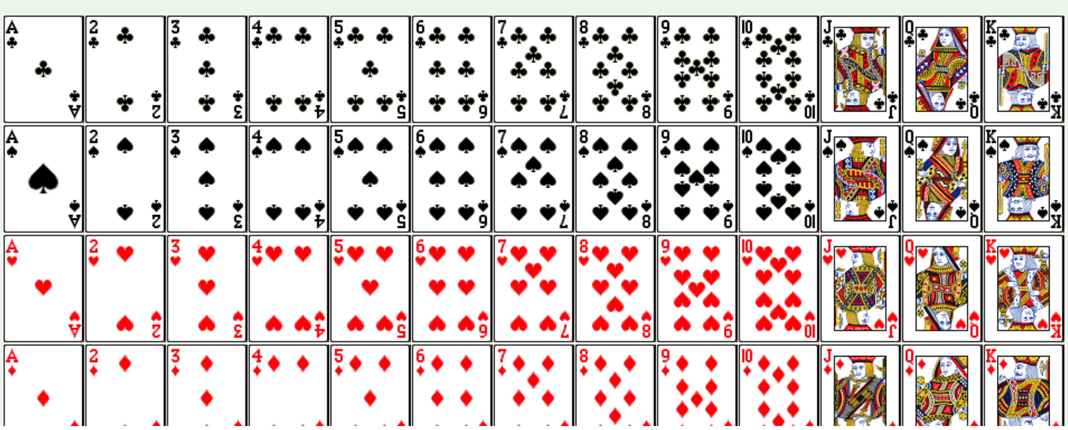
{2 Clubs, 3 Clubs, 4 Clubs, 5 Clubs, 6 Clubs, 7 Clubs, 8 Clubs, 9 Clubs, 10 Clubs, J Clubs, Q Clubs, K Clubs, A Clubs, J Hearts, Q Hearts, K Hearts, J Spades, Q Spades, K Spades, J Diamonds, Q Diamonds, K Diamonds}

That is 22 cards out of the 52 card deck, which gives us a probably of:

$$\frac{22}{52} = \frac{11}{26} \approx 0.423$$

This confirms our earlier answer using the formal Addition Rule.

Your Turn 3.4.3: Find Or Probability



As shown above, a classic deck of cards is made up of 52 cards, 26 are black, 26 are red. Each color is split into two suits of 13 cards each, clubs and spades are black, and hearts and diamonds are red. Each suit is split into 13 individual cards (Ace, 2-10, Jack, Queen, and King). Use fractions, or a decimal accurate to at least 4 decimals.

If you select one card at random, what is the probability of:

1. Drawing a black card?
   $P(\text{black}) =$ [ ]
2. Drawing a Queen?
   $P(\text{Queen}) =$ [ ]
3. Drawing a card that is black and a Queen?
   $P(\text{black and Queen}) =$ [ ]
4. Drawing a card that is black or a Queen?
   $P(\text{black or Queen}) =$ [ ]

Hint

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✔ Example 3.4.4: Deck of Cards (Mutually Exclusive and Not Mutually Exclusive)

1. Suppose we draw one card from a standard deck. What is the probability that we get a queen or a king?
2. Suppose we draw one card from a standard deck. What is the probability that we will get a red card or a king?

Answer 1

The event of drawing a king card and drawing a queen is mutually exclusive. There are 4 queens and 4 kings in the deck, hence 8 outcomes corresponding to a queen or king out of 52 possible outcomes. Thus, the probability of drawing a queen or a king is:

$$P(\text{ King or Queen}) = \frac{8}{52} = \frac{2}{13}$$

Note that in this case, there are no cards that are both a queen and a king, so $P(\text{King and Queen}) = 0$. Using the addition rule, we could have said:

$$P(\text{ King or Queen}) = P(\text{ King }) + P(\text{ Queen}) - P(\text{ King and Queen}) = \frac{4}{52} + \frac{4}{52} - 0 = \frac{8}{52} = \frac{2}{13}$$

Note that you should not simplify the individual fractions before adding/subtracting them, since they naturally have a common denominator. Simplify after the addition.

OR you can do the following directly

$$\begin{aligned} P(\text{King or Queen}) &= \frac{\#\text{ of Kings } + \#\text{of Queen } - \#\text{ of King and Queen}}{\text{total number of outcomes in sample space}} \\ &= \frac{4+4-0}{52} \\ &= \frac{8}{52} = \frac{2}{13} \end{aligned}$$

Answer 2

The two events, drawing a red card and drawing a king, are not mutually exclusive. Half the cards are red, so

$$P(\text{red}) = \frac{26}{52} \tag{3.4.1}$$

There are 4 kings, so

$$P(\text{King}) = \frac{4}{52} \tag{3.4.2}$$

There are 2 red kings, so

$$P(\text{red and King}) = \frac{2}{52} \tag{3.4.3}$$

We can then calculate

$$\begin{aligned} P(\text{red or King}) &= P(\text{Red}) + P(\text{King}) - P(\text{Red and King}) \\ &= \frac{26}{52} + \frac{4}{52} - \frac{2}{52} \\ &= \frac{28}{52} = \frac{7}{13} \end{aligned} \tag{3.4.4}$$

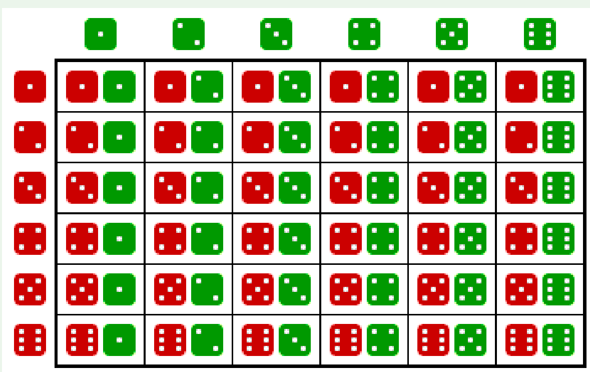
Note that you should not simplify the individual fractions before adding/subtracting them, since they naturally have a common denominator. Simplify after the addition.

OR you can do the following directly

$$\begin{aligned} P(\text{King or Red}) &= \frac{\#\text{ of King } + \#\text{of Red } - \#\text{ of King and Red}}{\text{total number of outcomes in sample space}} \\ &= \frac{4+26-2}{52} \\ &= \frac{28}{52} = \frac{7}{13} \end{aligned}$$

Your Turn 3.4.4: Find Or Probability

A pair of dice is rolled:



Let events A = the sum on the dice is 12, and B = the dice show the same number.

Use Additive Rule

$$P(A \quad \text{or} \quad B) = P(A) + P(B) - P(A \quad \text{and} \quad B)$$

to find the probability of A or B. Express your answers as *reduced fractions*.

(a) $P(A)$ =

(b) $P(B)$ =

(c) $P(A \quad \text{and} \quad B)$ =

(d) $P(A \quad \text{or} \quad B)$ =

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Find the Probability Using a Two-way Table

A **two-way table** (also called a **contingency table**) is a way to organize data that shows the relationship between **two categorical variables**. It helps you see how different categories intersect. It's like a grid with one variable across the **top (columns)** and another variable down the **side (rows)**, and each **cell** shows the **frequency/count** of observations for that pair of categories.

KEEP IN MIND

When you find a probability using a **two-way table**, you **must always use the total number of observations** (the **grand total**) in the **denominator**, unless the problem specifically tells you to find a *conditional probability*.

Example 3.4.5: Calculate Probability Using Two Way Table

A survey was conducted among 250 people who had recently purchased a car, and the results are summarized in the following table.

Satisfaction of Car Buyers

	Satisfied	Not Satisfied	Total
New Car	92	28	120
Used Car	83	47	130
Total	175	75	250

Find the probability that a person bought a new car or was not satisfied.

Answer

$$P(\text{new car or not satisfied}) = P(\text{new car}) + P(\text{not satisfied}) - P(\text{new car and not satisfied})$$
$$= \frac{120}{250} + \frac{75}{250} - \frac{28}{250} = \frac{167}{250} \approx 0.668$$

The probability that a person bought a new car or was not satisfied is approximately 0.668 or 66.8%.

OR you can find the probability directly as shown below

$$P(\text{new car or not satisfied}) = \frac{\#\text{ of new car } + \#\text{of not satisfied} - \#\text{ of new are and not satisfied}}{\text{total number people}}$$
$$= \frac{120 + 75 - 28}{250}$$
$$= \frac{167}{250} \approx 0.668$$

Your Turn 3.4.5: Probabilty Using Two Way Table

According to a recent Pew Research study, majority of Americans now oppose rules aimed at dramatically increasing electric vehicle sales in the United States. In a recent survey, participants were asked whether they support or oppose measures to expand electric vehicle sales. The table below shows the results of a recent survey by party affiliation.

	Oppose	Favor	Totals
Republicans	84	18	
Democrats	37	63	
Totals			

(a) Fill out the totals in the table above.

Suppose a respondent from this study is randomly selected.

(b) What is the probability that the respondent is a Democrat?

Round to 4 decimal places.

(c) What is the probability that the respondent opposes rules aimed at dramatically expanding electric vehicle sales?

Round to 4 decimal places.

(d) What is the probability that the respondent is a Democrat who opposes rules aimed at dramatically expanding electric vehicle sales?

Round to 4 decimal places.

(e) Use the addtion rule (OR rule) to find the probability that the respondent is a Democrat or someone who opposes rules aimed at dramatically expanding electric vehicle sales?

Addition Rule: $P(A \quad \text{or} \quad B) = P(A) + P(B) - P(A \quad \text{and} \quad B)$

Round to 4 decimal places.

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Example 3.4.6: Calculate Probability Using Two Way Table

The table below shows the number of survey subjects who have received and have not received a speeding ticket in the last year, and the color of their car. Find the probability that a randomly chosen person:

1. Has a red car *and* got a speeding ticket
2. Has a red car *or* got a speeding ticket.

	Speeding ticket	No speeding ticket	Total
Red car	15	135	150
Not red car	45	470	515
Total	60	605	665

Answer

1. We can see that 15 people of the 665 surveyed had both a red car and got a speeding ticket, so the probability is

$$P(\text{red car and ticket}) = \frac{15}{665} \approx 0.0226 \tag{3.4.5}$$

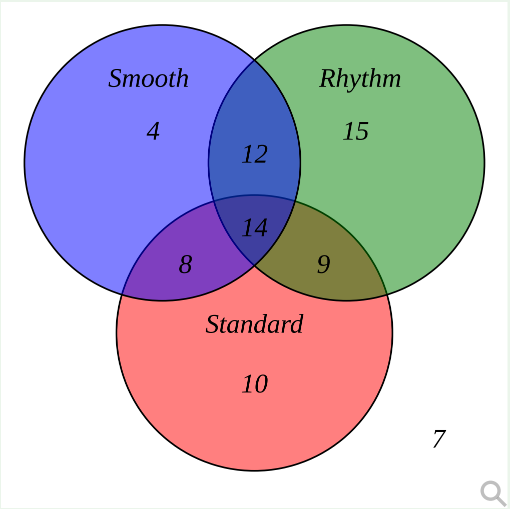
Notice that having a red car and getting a speeding ticket are not independent events, so the probability of both of them occurring is not simply the product of the probabilities of each one occurring.

2. Of the 665 surveyed, we can see that 150 people had red cars, 60 people had speeding tickets, and 15 people had red cars and speeding tickets. So

$$\begin{aligned} P(\text{red car or ticket}) &= \frac{\text{\# of red car + \#of ticket} - \text{\# of red car and got ticket}}{\text{total number people surveyed}} \\ &= \frac{150 + 60 - 15}{665} \\ &= \frac{195}{665} = \frac{39}{133} = 0.2932 \end{aligned}$$

Your Turn 3.4.7: Calculate Probability Using Venn Diagram

A survey was conducted at a local ballroom dance studio asking students if they had ever competed in the following dance categories: Smooth, Rhythm, or Standard. The results were then presented to the owner in the following Venn Diagram.



Determine the following probabilities.
Write your answers in percent form, rounded to the nearest tenth.

a) P(Smooth) = ____ %

b) P(Rhythm and Standard) = ____ %

c) P(Smooth or Standard) = ____ %

d) P(Rhythm or Smooth) = ____ %

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3.5: Odd and Expected Value

Learning Objectives

After completing this section, you should be able to:

1. Determine odds from probabilities.
2. Determine probabilities from odds.
3. Calculate the expected value of an experiment.
4. Interpret the expected value of an experiment.
5. Use expected value to analyze applications.

Figure $3.5.1$: Scratch-off lottery tickets, as well as many other games, represent the likelihood of winning using odds. (credit: “My Scratch-off Winnings” by Shoshanah/Flickr, CC BY 2.0)

A particular lottery instant-win game has 2 million tickets available. Of those, $500,000$ win a prize. If there are $500,000$ winners, then it follows that there are $1,500,000$ losing tickets. When we evaluate the risk associated with a game like this, it can be useful to compare the number of ways to win the game to the number of ways to lose. In the case of this game, we would compare the $500,000$ wins to the $1,500,000$ losses. In other words, there are three losing tickets for every winning ticket. Comparisons of this type are the focus of this section.

Computing Odds

The ratio of the number of equally likely outcomes in an event, E, to the number of equally likely outcomes *not* in the event, E', odds for (or odds in favor of) the event. The opposite ratio (the number of outcomes not in the event to the number in the event to the number in the event is called the odds against the event. The formula is given below

FORMULA: Odd in Favor and Odd Against

For an event E

$$\begin{aligned} \text{P(not E)} &= 1 - P(E) \\ \text{Odd in favor of E } &= \frac{P(E)}{P(\text{not} E)} \\ \text{Odd against E } &= \frac{P(\text{not} E)}{P(E)} \end{aligned}$$

Odd in Favor and Odd Against: Interpretation

Type	Formula	Interpretation
Odds in Favor	Favorable: Unfavorable	It compares how often an event happens vs. how often it does not.
Odds Against	Unfavorable: Favorable	It compares how often an event does *not* happen vs. how often it does.

Let's say, based on past data, in a certain city, it rains 60 days in a year, and it doesn't rain 305 days a year. Find the **odds in favor** of a rainy day and interpret your answer**.**

Now, suppose you're trying to find the **odds in favor** of **a rainy day**:

Favorable outcomes (rainy days) = 60

Unfavorable outcomes (non-rainy days) = 305

$$\text{Odds in favor of rain} = 60{:}305 = 12{:}61 \tag{3.5.1}$$

Interpretation: The odds in favor of rain on a randomly chosen day are $12{:}61$, meaning for every 12 rainy day, there are 61 days without rain..

Checkpoint

Both odds and probabilities are calculated as ratios. To avoid confusion, we will always use fractions, decimals, or percents for probabilities and colons to indicate odds. The rules for simplifying fractions apply to odds, too. Thus, the odds for winning a prize in the game described in the section opener are $500{,}000{:}1{,}500{,}000 = 1{:}3$ *and the odds against winning a prize are* $3{:}1$. *These would often be expressed in words as "the odds of winning are one to three in favor" or "the odds of winning are three to one against."*

Example 3.5.1: Computing Odds

1. If you roll a fair 6-sided die, what are the odds of rolling a five or higher?
2. If you draw a card at random from a standard deck, what are the odds of drawing a face card?

Answer 1

The sample space for this experiment is $\{1, 2, 3, 4, 5, 6\}$. Two of those outcomes are in the event, E, "roll a five or higher," so E = $\{5, 6\}$.

$$\begin{aligned} P(E) &= \frac{2}{6} \\ P(\text{not } E) &= 1 - \frac{2}{6} = \frac{4}{6} \\ \text{Odd in favor of } E &= \frac{\frac{2}{6}}{\frac{4}{6}} \\ &= \frac{2}{6} \times \frac{6}{4} = \frac{2}{4} = \frac{1}{2} = 1{:}2 \end{aligned} \tag{3.5.2}$$

Answer 2

There are 12 of those outcomes in the event, E, "drawing a face card," so E ={J♥, Q♥, K♥,J♥, Q♦, K♦, J♣, Q♣, K♣ J♠, Q♠, K♠}. Note that a deck has 52 cards.

$$\begin{aligned} P(E) &= \frac{12}{52} \\ &= \frac{3}{13} \\ P(\text{not } E) &= 1 - \frac{3}{13} = \frac{10}{13} \end{aligned} \tag{3.5.3}$$

$$\text{Odd in favor of } E = \frac{\frac{3}{13}}{\frac{10}{13}} = \frac{3}{13} \times \frac{13}{10} = \frac{3}{10} = 3{:}10 \tag{3.5.4}$$

Your Turn 3.5.1: Finding Odd

A card is drawn randomly from a standard 52-card deck. Find the following:

Write all answers as simplified fractions (not mixed numbers).

A. The probability the card drawn is a face card.

P(face card drawn) is equal to

P(face card drawn) is equal to [ ] .

B. The probability the card drawn is not a face card.

P(not a face card drawn) is equal to [ ] .

C. The odds in favor of drawing a face card.

The odds in favor of drawing a face card are [ ] .

D. Odds against drawing a face card.

The odds in favor of drawing a face card are [ ] .

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Odds as a Ratio of Probabilities

We can use these formulas to convert probabilities to odds and vice versa.

We can also think of odds as a ratio of probabilities. Consider again the instant-win game from the section opener, with $500,000$ winning tickets out of $2,000,000$ total tickets. If a player buys one ticket, the probability of winning is $\frac{5000,000}{2,000,000} = \frac{1}{4}$, and the probability of losing is $1 - \frac{1}{4} = \frac{3}{4}$. Notice that the ratio of the probability of winning to the probability of losing is $\frac{1}{4} \div \frac{3}{4} = \frac{1}{3} = 1:3$, which matches the odds in favor of winning

Example 3.5.2: Converting Probabilities to Odds

Given the following probabilities of an event, find the corresponding odds for and against that event.

1. $\mathrm{P(E\ )} = \frac{3}{5}$
2. $\mathrm{P(E\ )} = 17\%$

Answer 1

We need to find P(E) and P(not E) to find the odds for:

$\mathrm{P(E\ )} = \frac{3}{5}$

$\mathrm{P(not\ E\ )} = 1 - \frac{3}{5} = \frac{2}{5}$

$$\begin{aligned}\text{Odd in favor of E } &= \frac{\frac{3}{5}}{\frac{2}{5}} \\ &= \frac{3}{5} \times \frac{5}{2} \\ &= 3:2\end{aligned} \tag{3.5.5}$$

Since the odds for E are 3:2, the odds against E must be 2:3.

Answer 2

Again, we'll use the formula:

$\mathrm{P(E\ )} = 0.17 = \frac{17}{100}$

$\mathrm{P(not\ E\ )} = 1 - 0.17 = 0.83 = \frac{83}{100}$

$\text{Odd in favor of E } = \frac{\frac{17}{100}}{\frac{83}{100}} = \frac{17}{83} = 17{:}83$

So odds against E is $83{:}17$.

Your Turn $3.5.2$: Finding Odd

Answer the questions relating probability to odds.

(a) The probability that a person will get the flu in a given year is 18%. What are the *odds in favor* of a person getting the flu?

_____ to _____ *Give the odds as a ratio in simplest form.*

(b) The probability winning a prize in a game is 0.17. What are the *odds against* winning a prize?

_____ to _____ *Give the odds as a ratio in simplest form.*

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Converting Odds to Probability

Now, let's convert odds to probabilities. Let's say the odds for an event are $a : b$, where a is the number of favorable outcomes and b is the number of unfavorable outcomes. Now, we have the following formula.

FORMULA: Converting Odd to Probability

If the odds in favor of E are $a{:}b$, then the probability of E is

$$\mathrm{P(E\,)} = \frac{a}{a+b}$$

If the odds against of E are $a{:}b$, then the probability of E is

$$\mathrm{P(E\,)} = \frac{b}{a+b}$$

Example $3.5.3$: Converting Odds to Probability

Find $P(E)$ if

1. The odds in favor of E are $2{:}1$.
2. The odds against E are $6{:}1$.

Answer

1. $$\mathrm{P(E\,)} = \frac{a}{a+b} = \frac{2}{2+1} = \frac{2}{3}$$
2. $$\mathrm{P(E\,)} = \frac{b}{a+b} = \frac{1}{6+1} = \frac{1}{7}$$

Your Turn $3.5.3$: Converting Odd to Probability

Answer the questions relating probability to odds.

(a) The odds in favor of your favorite sports team winning a game are 29 to 12. What is the *probability* they will win?

_____ *Give the probability as a fraction in simplest form.*

(b) The odds in favor of a tornado watch being issued are 41 to 92. What is the *probability* of a tornado watch being issues?

% *Give the probability as a percentage rounded to two decimal places.*

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Checkpoint

*Some places, particularly state lottery websites, will use the words "odds" and "probability" interchangeably. Never assume that the word "odds" is being used correctly! Compute one of the odds/probabilities yourself to make sure you know how the wor*Expected Value*d is being used!*

Expected Value

Figure $3.5.1$: The concept of expected value allows us to analyze games that involve randomness, like Roulette. (credit: "Roulette Table and Roulette Wheel in a Casino with People betting on numbers" by Marco Verch/Flickr, CC BY 2.0)

The casino game roulette has dozens of different bets that can be made. These bets have different probabilities of winning but also have different payouts. In general, the lower the probability of winning a bet, the more money a player wins for that bet. With so many options, is there one bet that's "smarter" than the rest? What's the best play to make at a roulette table? This section will develop the tools we need to answer these questions.

Many experiments have numbers associated with their outcomes. Some are easy to define; if you roll 2 dice, the sum of the numbers showing is a good example. In some card games, cards have different point values associated with them; for example, in some forms of the game rummy, aces are worth 15 points; 10 s, jacks, queens, and kings are worth 10; and all other cards are worth 6. The outcomes of casino and lottery games are all associated with an amount of money won or lost. These outcome values are used to find the expected value of an experiment: the mean of the values associated with the outcomes that we would observe over a large number of repetitions of the experiment.

That definition is a little vague; How many is "a large number?" In practice, it depends on the experiment; the number must be large enough that every outcome is expected to appear at least a few times. For example, if we're rolling a standard 6-sided die and noting the number that shows, a few dozen replications should be sufficient to ensure the mean is representative. Since the probability of each outcome is $\frac{1}{6}$, we would expect to see each outcome about 8 times over the course of 48 replications. However, if we're talking about the Powerball lottery, where the probability of winning the jackpot is about $\frac{1}{292{,}000{,}000}$, we would need several *billion* replications to ensure that every outcome appears a few times. Luckily, we can find the theoretical expected value before we even run the experiment for the first time.

FORMULA: Expected Value

If O represents an outcome of an experiment and $n(O)$ represents the value of that outcome, $P(O)$ represents the probability of the outcome. The expected value of the experiment is:

$$\Sigma(n(O) \times P(O))$$

where Σ is the "sum," meaning we add up the results of the formula that follows over all possible outcomes.

Key Points to Remember for Finding Expected Value

1. **Identify all possible outcomes and their values.**
   These values can be positive or negative depending on the situation.

2. **Determine the probability of each outcome.**
3. **Set up a table:**
   - First column → outcome values
   - Second column → corresponding probabilities
4. **Multiply each outcome value by its probability**, then **add all the products together** to find the **expected value**.

Here are the most common areas where people use expected value:

Gambling and Games: Used to determine the average gain or loss in activities like lottery, cards, dice, and slot machines, and to decide if a game is fair or profitable.

Insurance: Insurance companies use expected value to set premiums by estimating the expected cost of accidents, health claims, or property damage. Finance and Investments: Used to evaluate risky investments and predict the average profit or loss over time.

Sports: Helps predict average points, wins, or scoring outcomes in a game.

Weather and Natural Events: Used to estimate expected rainfall, average temperature, or the likelihood of natural disasters.

✔ Example 3.5.4: Finding Expected Values

Find the expected values of the following experiments.

1. An **unfair** six-sided die has sides numbered $1, 2, 3, 4, 5, 6$ with probabilities given in the table below. Find the expected value of one roll of the die.

Value of Outcome	1	2	3	4	5	6
Probability	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{8}$

2. Draw a card from a well-shuffled standard deck of cards and note its rummy value (15 for aces; 10 for tens, jacks, queens, and kings; 5 for everything else).

Answer 1

We multiply the values and their probability and add them.

$$\begin{aligned}\text{Expected value} &= 1 \times \frac{1}{8} + 2 \times \frac{1}{8} + 3 \times \frac{1}{8} + 4 \times \frac{1}{4} + 5 \times \frac{1}{4} + 6 \times \frac{1}{8} \\ &= \frac{1}{8} + \frac{2}{8} + \frac{3}{8} + \frac{4}{4} + \frac{5}{4} + \frac{6}{8} \\ &= \frac{30}{8} = 3.75\end{aligned} \tag{3.5.6}$$

So, the expected value of one roll of the dice is 3.750.

Answer 2

Let’s find the value of the outcome and its corresponding probabilities.

Outcome	Probability	Value of Outcome
$\{A\}$	$\frac{4}{52}$	15
$\{10, J, K, Q\}$	$\frac{16}{52}$	10
$\{2, 3, 4, 5, 6, 7, 8, 9\}$	$\frac{32}{52}$	5

We multiply the values and their probability and add them.

$$\begin{aligned}\text{Expected value} &= 15 \times \frac{4}{52} + 10 \times \frac{16}{52} + 5 \times \frac{32}{52} \\ &= \frac{60}{52} + \frac{160}{52} + \frac{160}{52} \\ &= \frac{380}{52} = 7.3\end{aligned} \tag{3.5.7}$$

✎ Your Turn 3.5.4: Calculated Expected Value

Calculating Expected Value using a Table

Gwen created a game for the school fair. Each game costs $\$5$. The player is shown a jar filled with different colored marbles. Then the jar is hidden from view and well shaken so the player cannot see the colors of colored marbles. Then the jar is hidden from view and well shaken so the player cannot see the colors of the marbles and so the marbles are distributed randomly. Then the player draws a mabrle.

The table below shows the colors of the marbles in the jar, the probability of selecting the color, and how much is won by the player selecting a marble of that color.

Complete the table by computing the amounts indicated in the top row, and then answer the remaining questions to find the expected value of the game for the player.

Note: If the player wins less than the $5 they pay, then they will lose money. This means the Total Winnings will be written as a negative number.

Marble Color	Probability of Choosing the Marble	Amount Won	Total Winnings Hint	Probability $\times$ Total Winnings
silver	0.39	$\$3$	$	$
gold	0.05	$\$7$	$	$
pink	0.03	$\$12$	$	$
orange	0.42	$\$0$	$	$
red	0.11	$\$6$	$	$

To find the expected value of the game for the player, compute

the sum of all the values in the ***Probability*** $\times$ ***Total Winnings*** column.

Expected value of game= $

Round your answer to the nearest cent

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As we noted, the expected value of an experiment is the mean of the values we would observe if we repeated the experiment a large number of times. (This interpretation is due to an important theorem in the theory of probability called the Law of Large Numbers.)

Example 3.5.5: Finding Expected Value (Sum of Two Dice)

A bag contains 3 gold marbles, 9 silver marbles, and 26 black marbles. Someone offers to play this game: You randomly select one marble from the bag. If it is gold, you win $\$4.00$. If it is silver, you win $\$3.00$. If it is black, you lose $\$1.00$. What is your expected value if you play this game?

Answer

To find the expected value, you need to find the value of the outcomes and the probability of the outcomes. Those are listed in the following table.

Outcome	Value	Probability	Value $\times$ probability
Gold	$\$4$	$\frac{3}{38}$	$\frac{12}{38}$
Silver	$\$3$	$\frac{9}{38}$	$\frac{27}{38}$
Black	$-\$1$	$\frac{26}{38}$	$\frac{-26}{38}$

$$\begin{aligned} \text{Expected value} &= \$\frac{12}{38} + \$\frac{27}{38} + \$\frac{-26}{38} \\ &= \frac{-4}{38} \\ &= -\$0.3421 \end{aligned} \tag{3.5.8}$$

Your Turn 3.5.5: Find Expected Value for Card Game

Azarias and Winona are playing a game. Azarias selects one card from a standard 52-card deck. If Azarias selects a face card (Jack, Queen, or King), Winona pays him \$6. If Azarias selects any other type of card, he pays Winona \$3.

a) What is Azarias's expected value for this game? Round your answer to the nearest cent.

\$ ______

b) What is Winona's expected value for this game? Round your answer to the nearest cent.

\$ ______

c) Who has the advantage in this game? Select an answer

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Example 3.5.6: Risk of Dying

A 40-year-old man in the U.S. has a 0.24% risk of dying during the next year. An insurance company charges \$300 annually for a life insurance policy that pays a \$100,000 death benefit. What is the expected value for the person buying the insurance? Round your answer to the nearest dollar.

Answer 1

To find the expected value, we need to determine the value of each outcome and its corresponding probability. Those are listed in the following table.

Outcomes (Events)	Values $n(O)$	Probability $P(O)$
Survive	$-\$300$	$1 - 0.0024 = 0.9976$
Die	$\$100,000 - \$300 = \$99,700$	0.0024

$$\text{Expected value} = (-\$300)(0.9976) + (\$99,700)(0.0024) = -\$60$$

Note: The negative sign in front of \$300 is because we are looking for the insured person's expected value; if that person survives, she/he will lose \$300. You also see that we subtract \$300 to find the value of outcome die. It is because the insured person gets \$300 less than the policy value because the insured person has already paid \$300 for the policy.

Your Turn 3.5.6: Risk of Dying

A 40-year-old man in the U.S. has a 0.243% risk of dying during the next year . An insurance company charges \$250 per year for a life-insurance policy that pays a \$110689 death benefit. What is the expected value for the person buying the insurance? Round your answer to the nearest dollar.

Complete the table with the appropriate values. Be sure to include negatives where appropriate.

Outcomes	Person does not survive	Person survive
Probability of each		

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outcome		
Payoff of each outcome	$	$

Use the table to calculate the expected value.

Expected value of the person buying insurance = $

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Example 3.5.7: Every Game Has a Negative Expected Value to the player

Suppose someone offers to play a game with you. If you roll a die and get a 6, you get $\$10$. However, getting a five or below means you lose $\$1$. Is this a game you'd want to play?

Answer

Let's look at the expected value: The probability of winning is $\frac{1}{6}$, and the probability of losing is $\frac{5}{6}$, so the expected value is

$$\$10 \times \frac{1}{6} + (-\$1) \times \frac{5}{6} = \frac{5}{6} \approx \$0.83.$$

That means, on average, you'll come out ahead by about 83 cents every time you play this game.

It's a great deal! On the other hand, if the winnings for rolling a 6 drop to $\$3$, the expected value becomes

$$\$3 \times \frac{1}{6} + (-\$1) \times \frac{5}{6} = -\frac{1}{3} \approx -\$0.33.$$

This means you should expect to lose about 33 cents on average for every time you play.

Playing that game is not a good idea! This is how casinos and lottery corporations make money:

Your Turn 3.5.7: Expected Value of Drawing Marbles

A bag contains 2 gold marbles, 9 silver marbles, and 29 black marbles. Someone offers to play this game: You randomly select one marble from the bag. If it is gold, you win $3. If it is silver, you win $2. If it is black, you lose $1.

What is your expected value if you play this game?

$

(Round to two decimal places)

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Who Knew?: Expected Values in Football

In the 21st century, data analytics tools have revolutionized the way sports are coached and played. One tool in particular is used in football at crucial moments in the game. When a team faces a fourth down (the last possession in a series of four possessions, a fairly common occurrence), the coach faces a decision: Run one play to try to gain a certain number of yards, or kick the ball away to the other team. Here's the interesting part of the decision: If the team "goes for it" and runs the play and they are successful, then they keep possession of the ball and can continue in their quest to score more points. If they are unsuccessful, then they lose possession of the ball, giving the other team an opportunity to score points. If, instead, the team punts, or kicks the ball away, then the other team gets possession of the ball, but in a worse position for them than if the original team goes for it and fails. To analyze this situation, data analysts have generated empirical probabilities for every fourth-down situation and computed the expected value (in terms of points) for each decision. Coaches frequently use those calculations when they decide which option to take!

People in Mathematics: Pierre de Fermat and Blaise Pascal

In 1654, a French writer and amateur mathematician named Antoine Gombaud (who called himself the Chevalier du Mére) reached out to his gambling buddy Blaise Pascal to answer a question that he'd read about called the "problem of points." The question goes like this: Suppose you're playing a game that is scored using points, and the first person to earn 5 points is the winner. The game is interrupted with the score 4 points to 2. If the winner stood to win \$100, how should the prize money be divided between the players? Certainly, the person who is 1 point away from victory should get more, but *how much* more?

We have developed tools in this section to answer this question. At its heart, it's a question about conditional probabilities and expected value. Pascal first started thinking about it at the time, though those ideas hadn't yet been invented. Pascal reached out to a colleague named Pierre de Fermat, and over the course of a couple of months, their correspondence with each other would eventually solve the problem. In the process, they first described conditional probabilities and expected values!

Pascal is remembered for the "arithmetical triangle" that is named after him (though he wasn't the first person to discover it; see the section on the binomial distribution for more), as well as his work in geometry. In physics, Pascal worked on hydrodynamics and air pressure (the SI unit for pressure is named after him), and in philosophy, Pascal advocated a mathematical approach to philosophical problems.

Example 3.5.8: Using Expected Values

In the casino game keno, a machine randomly chooses 20 numbers between 1 and 80 (inclusive) without replacement. Players try to predict which numbers will be selected. Players don't try to guess all 20; generally, they'll try to predict between 1 and 10 of the chosen numbers. The amount won depends on the number of guesses they made and the number of guesses that were correct. At one casino, a player can try to guess just one number. If that number is among the 20 selected, the player wins \$2; otherwise, the player loses \$1. What is the expected value?

Answer

There are 20 winning numbers out of 80, so if we try to guess one of them, the probability of guessing correctly is $\frac{20}{80}=\frac{1}{4}$. If the player guesses correctly, the player wins \$2. The probability of losing is then $\frac{3}{4}$. If the player guesses incorrectly, the player loses \$1.

Outcomes (Events)	Values $n(O)$	Probability $P(O)$
Winner	\$2	$\frac{1}{4}$
Die	$-\$1$	$\frac{3}{4}$

$$\begin{aligned}\text{Expected value} &= \$2\times\frac{1}{4}+(-\$1)\times\frac{3}{4}\\ &= -\$0.25\end{aligned} \tag{3.5.9}$$

Your Turn 3.5.8: Expected Value Raffle Ticket

Need Calculator?

The PTO is selling raffle tickets to raise money for classroom supplies. A raffle ticket costs \$2. There is 1 winning ticket out of the 140 tickets sold. The winner gets a prize worth \$80. *Round your answers to the nearest cent.*

What is the expected value (to you) of one raffle ticket? \$

Calculate the expected value (to you) if you purchase 6 raffle tickets. \$

What is the expected value (to the PTO) of one raffle ticket? \$

If the PTO sells all 140 raffle tickets, how much money can they expect to raise for the classroom supplies? \$

supplies? $

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WORK IT OUT: Make Your Own Lottery

By yourself or with a partner, devise your own lottery scheme. Assume you would have access to one or more machines that choose numbers randomly. What will a lottery draw look like? How many numbers are players choosing from? How many will be drawn? Will they be drawn with replacement or without replacement? What conditions must be met for a player to win the first or second (or more!) prize? Once you've decided on that, determine the payoff structure for winners and how much the game will cost. Try to make the game enticing enough that people will want to play it but with enough negative expected value that the lottery will make money. Aim for the expected value to be about -0.25 times the cost of playing the game.

CHAPTER OVERVIEW

4: Financial Mathematics

4.1: Understanding Percent

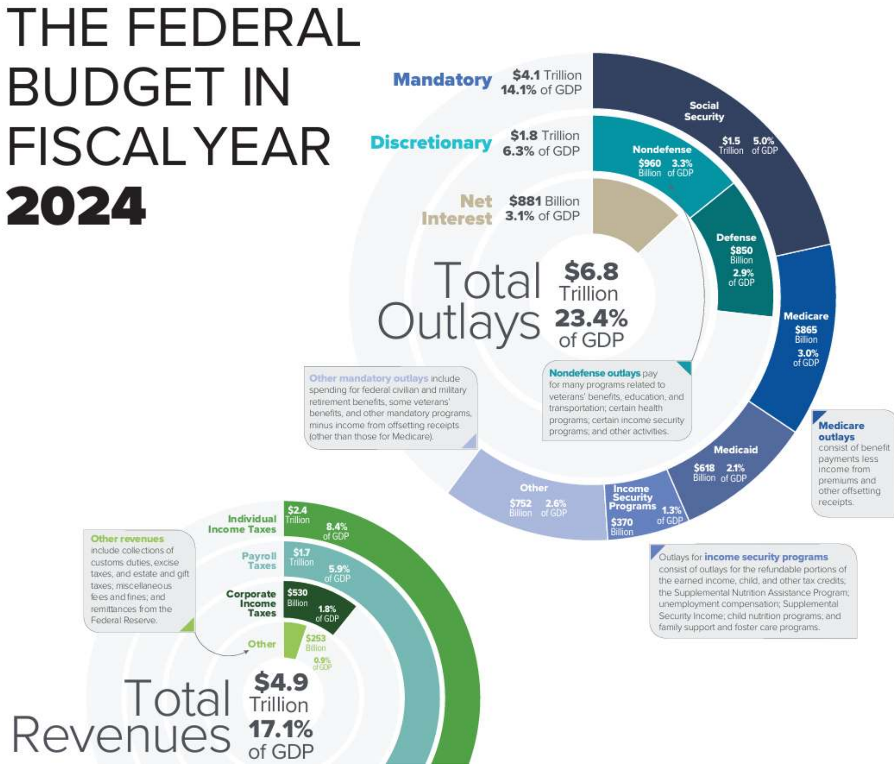

Figure 4.1.1 : “The federal budget describes how money is spent (Total Outlays) and how money is earned (Total Revenues)”.. (credit: [1])

Learning Objectives

1. Convert between percent, decimal, and fractional values.
2. Calculate the total, percent, or part.
3. Solve application problems involving percent.
4. Calculate the percent increase and percent decrease.

In 2024, the U.S. federal government budgeted $3.8 billion for the National Park Service, which appears to be a very large number (and is!) and a large portion of the total federal budget. However, the total outlays from the U.S. federal government in 2024 were $6.8 trillion. So, the amount budgeted for the National Park Service was less than one-tenth of 1 percent, or $\frac{1}{10}\%$, of the total outlays. This **percentage** describes a specific number. Understanding that ratio puts the $3.8 billion budgeted to the National Park Service in perspective.

This chapter 4 focuses on percent as a primary tool for understanding money management. The interest paid on debt, the interest earned through investments, and even taxes are entirely determined using percentages. This section introduces the basics of working with this invaluable tool.

Convert Between Percent, Decimal, and Fractional Values

The word percent comes from the Latin phrase per centum, which means "by the hundred." So any percent is a number divided by 100. Changing a percent to a fraction is to write the percent in its fractional form. To write $n\%$ in its fractional form is to write the percent as the fraction $\frac{n}{100}$.

Formula: Percent to Decimal

The fraction form of $n\%$ is found by dividing $\frac{n}{100}$ and calculating the decimal value of the fraction.

Do You Know?

A percent need not be an integer and does not have to be less than 100.

Example 4.1.1: Rewriting a Percent as a Fraction or Decimal Form

Rewrite the following as fractions and decimal form:

1. 8%
2. 84%

3. $\frac{1}{4}\%$
4. 213%

Answer

1. Using the definition and $n = 8$, 8% in fractional form is $\frac{8}{100} = 0.08$.
2. Using the definition and $n = \frac{1}{4}$, 0.25% in fractional form is $\frac{0.25}{100} = 0.0025$.
3. Using the definition and $n = 38.7$, 38.7% in fractional form is $\frac{38.7}{100} = 0.387$.
4. Using the definition and $n = 213$, 213% in fractional form is $\frac{213}{100} = 2.13$.

When any calculation with a percent is to be performed, the form of the percent must be changed to its fractional form or its **decimal form**. We can change a percent into decimal form by dividing the percent by 100 and representing the result as a decimal.

FORMULA

To convert the number x from decimal form (or fraction form) to percent, multiply x by 100 and place a percent sign, %, after the number, $(x \times 100)\%$.

Example 4.1.2: Converting the Decimal Form or Fraction to Percent

Convert each of the following to a percent:

1. 0.34
2. 4.15
3. 0.0391
4. $\frac{3}{8}$

Answer

1. Using the formula and $x = 0.34$, we calculate $(0.34 \times 100)\%$, which gives us 34%.
2. Using the formula and $x = 4.15$, we calculate $(4.15 \times 100)\%$, which gives us 415%.
3. Using the formula and $x = 0.0391$, we calculate $(0.0391 \times 100)\%$, which gives us 3.91%.
4. Using the formula and $x = \frac{3}{8}$, we calculate $(\frac{3}{8} \times 100\%) = 37.5\%$.

To Convert Decimal to Percent...

You should notice that to convert from decimal form to percent form, you can simply move the decimal two places to the right without performing the multiplication.

Your Turn 4.1.2: Percent as Decimals and Vice Versa

Percents as Decimals and Decimals as Percents
Complete the table below.
Percent = Decimal
$13\% =$ ____
$3\% =$ ____
$8.5\% =$ ____
____ $= 0.43$
____ $= 0.155$
____ $= 1.05$
Don't forget to enter your answer with a percent sign, %, when appropriate.

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Calculate the Total, Percent, or Part

The word “of” indicates multiplication using fractions, as in “one-fourth of 56.” To find “one-fourth of 56” we would multiply 56 by one-fourth. We can think of percents as fractions with a specific denominator - 100.

So, to calculate “25% of 52,” we multiply 52 by 25%. But first, we need to convert the percent to decimal form, 0.25. Using the decimal form of 25%, we have 0.25×52, which equals 13.

In this problem, 52 is the **total** or **base**, 25 is the **percentage**, and 13 is **the percentage of** 52 or the **part (amount)** of 52.

FORMULA: Percent, Total and Part

The mathematical formula relates the total (base), the percent in decimal form, and the part (amount).

Part = Percent × Total

Checkpoint

1. *In all calculations, the percent is expressed in decimal form.*
2. In Calculation, the word **" of"** implies multiplication.

Knowing any two of the values in our formula allows us to calculate the third value. In the following example, we know the total and the percent, and are asked to find the percentage of the total.

Example 4.1.3: Finding the Percent of a Total

Determine 156% of 720.

Answer

The total is 720, and the percent is 156%. The decimal form of 156% is 1.56. To find the part, or percent of the total, substitute those values into the formula and calculate.

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ &= 1.56 \times 720 \\ &= 1,123.2 \end{aligned}$$

From this, we say that 156% of 720 is $1,123.20$.

Your Turn 4.1.3: Find Part

Identify the percent, total, and part in this problem.

For the unknown value use x. And find the value of x.

What is 61 $\%$ of of 190 ?

Percent = ______ $\%$

Part = ______

Total = ______

x = ______

(Round your answer to two decimal places)

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In the previous example, we knew the total and the percent and found the part using our formula. We may instead know the percent and the part, but not the total. We can use our formula again to solve for the total.

Example 4.1.4: Finding the Total from the Percent and the Part

What is the total if 10% of the total is $4,000$?

Answer

The percent is 10, which in decimal form is 0.10. We were given that 10% of the total is $4,000$, so the part is $4,000$. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 4,000 &= 0.10 \times \text{ total} \end{aligned}$$

To find the total, we solve the equation for the total.

$$\begin{aligned} 4,000 &= 0.10 \times \text{ total} \\ \frac{4,000}{0.10} &= \frac{\cancel{0.10} \times \text{ total}}{\cancel{0.10}} \\ 40,000 &= \text{ total} \end{aligned}$$

From this we see that $40,000$ is the total, or that 10% of $40,000$ is $4,000$.

Your Turn 4.1.4: Find the Total

Identify the percent, part, and total in this problem.

For the unknown value use x. And find the value of x.

25% of what is 595 ?

Percent = ______ $\%$

Total = ______

Part = ______

x = ______

(Round your answer to two decimal places)

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Similarly, the percent can be found if the total and the percent of the total (the part) are known. This will result in the decimal form of the percent, so it must be converted to percent form.

Example 4.1.5: Finding the Percent from the Total and the Part

What percent of 500 is 175?

Answer

The total is 500, the percent of the total is 175. Substituting into the formula, we have

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 175 &= \text{percent} \times 500 \end{aligned}$$

To find the percent, we solve the equation for the percent.

$$\begin{aligned} 175 &= \text{percent} \times 500 \\ \frac{175}{500} &= \frac{\text{percent} \times \cancel{500}}{\cancel{500}} \\ 0.35 &= \text{percent} \end{aligned}$$

We see the percent in decimal form is 0.35. Converting from the decimal form yields 35%. We say that 175 is 35% of 500.

Your Turn 4.1.5: Find Percentage

Identify the percent, part, and total in this problem.

For the unknown value use x. And find the value of x.

310 is what percent of 510 ?

Percent = ____ %

Part = ____

Total = ____

x = ____ %

(Round your answer to two decimal places)

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Solve Application Problems Involving Percent

Percentages are frequently used in finance, research, science experiments, and casual conversations. Understanding these values helps when consuming media or discussing finances, for instance. Effectively working with and interpreting numbers and percentages will help you become an informed consumer of this information.

In most cases, working through what is presented requires you to identify that you are indeed working with a question of percent, which two of the three values that are related through percent are known, and which of the three values you need to find.

Example 4.1.6: Retention Rate at College

Justine applied to a medium-sized university outside her hometown and found out that the retention rate (percent of students who return for their sophomore year) for the 2021 academic year at the university was 84%. During a visit to the registrar's office, she found out that $1,350$ people had enrolled in the academic year 2021. How many students from the academic year 2021 are returning for the 2022 academic year?

Answer

The percentage of students who will return for the 2022 academic year (the retention rate) is 84%. The total number of students enrolled in the 2021 academic year was $1,350$. This means the percent is known, and the total is known. From this, we can determine the number of students who will return (percent of the total) for the 2022 academic year using the formula.

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ \text{part} &= 0.84 \times 1,350 \\ \text{part} &= 1,134 \end{aligned}$$

So, $1,134$ students will return for the 2022 academic year.

Your Turn 4.1.6: Find the Parts

The sales tax rate for the state of Washington was 7.7%.

What is the state sales tax on a $2,800 car in Washington?

$ ______ (Round your answer to two decimal places.)

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Example 4.1.7: Percent of Chemistry Majors

Cameron enrolls in a calculus class. In this class of 45 students, there are 18 chemistry majors. What percent of the class are chemistry majors?

Answer

In this situation, the percentage is to be determined. We know the total number of students is 45, and the part of the students who are chemistry majors is 18. Using that information and the formula.

$$\text{part} = \text{percent} \times \text{total}$$

The percentage can be found. Substituting and solving, we have

$$\begin{aligned} 18 &= \text{percent} \times 45 \\ \frac{18}{45} &= \frac{\text{percent} \times \cancel{45}}{\cancel{45}} \\ 0.4 &= \text{percent} \end{aligned}$$

Converting the 0.4 from decimal form, we find that 40% of the students in the calculus class are chemistry majors.

Your Turn 4.1.7: Find Percentage

On a multiple-choice test over English vocabulary, Cole got 36 out of 40 correct. What percent of the problems did she miss? Round your answer to the nearest tenth of a percent, if necessary.

______ %

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Example 4.1.8: Total Sales and Commission

Mariel makes a 20% commission on every sale she makes. One week, her commission check is for $\$153.00$. What were her total sales that week?

Answer

In this problem, Mariel's total sales are to be determined. We know the percent she earns is 20%. We also know that her sales commission was $\$153.00$, which is the percent of the total. Using this information and the formula

$$\text{part} = \text{percent} \times \text{total}$$

we can find Mariel's total sales. The decimal form of 20% is 0.2. The part, or percent of the total, is 153. Substituting and solving, we obtain

$$\begin{aligned} \text{part} &= \text{percent} \times \text{total} \\ 153 &= 0.2 \times \text{ total} \\ \frac{153}{0.2} &= \frac{\cancel{0.2} \times \text{ total}}{\cancel{0.2}} \end{aligned}$$

Mariel's total sales were $\$765.00$.

Your Turn 4.1.8: Find Total

At a restaurant, the bill comes to $32. If you decide to leave a 15% tip, how much total do you need to pay? Give your answer to the nearest cent.

$

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Who Knew?: LED Lightbulbs

According to the energy website from the U.S. government, LED lightbulbs use at least 75% less energy than incandescent bulbs. They also last up to 25 times as long as an incandescent bulb. If lighting accounts for a significant percentage of your electricity use, replacing all incandescent bulbs with LED bulbs can significantly reduce your electricity bill.

Definition: Percent Increase or Decrease

Percent increase and percent decrease are ways to express the change in a quantity as a percentage of the original quantity. To calculate the percent decrease, you use the following formula:

$$\text{Change in Value} = \text{New Value - Old Value}$$

The value of the Increase (Decrease) is also called the **absolute change**. If there is a decrease in value, the absolute change will be negative.

$$\text{Percent Increase (Decrease)} = \frac{\text{Change in Value}}{\text{Old Value}} \times 100\%$$

The percent Increase (or Decrease) is also called **a relative change.**

Example 4.1.9: Percent Increase or Decrease

1. If the price of a shirt increases from $\$20$ to $\$25$, what is the percent increase?
2. If the price of a shirt decreases from $\$25$ to $\$20$, what is the percent decrease?

Answer

1. Old value = $\$20$ AND New value = $\$25$

$$\begin{aligned} \text{Change in Value} &= \$25 - \$20 = \$5 \\ \text{Percent Increase} &= \frac{\text{Change in Value}}{\text{Old Value}} \times 100\% \\ &= \frac{\$5}{\$20} \times 100\% = 25\% \end{aligned} \tag{4.1.1}$$

So, the percent increase is 25%.

2. Old value = $\$25$ AND New value = $\$20$

$$\begin{aligned} \text{Change in Value} &= \$20 - \$25 = -\$5 \\ \text{Percent Decrease} &= \frac{\text{Change in Value}}{\text{Old Value}} \times 100\% \\ &= \frac{-\$5}{\$25} \times 100\% = -20\% \end{aligned} \tag{4.1.2}$$

So, the percent decrease is 20%. Negative signs indicate that there is a decrease.

Your Turn 4.1.9: Find Percent Increase or Percentage Decrease

The tuition at a private college X is increasing from $\$54,500$ to $\$60,500$. Find the absolute change and relative change in tuition.

Absolute change: $

Relative change: %

Round your answer to the nearest hundredth.

The tuition at a another private college Y is decreasing from $\$60,500$ to $\$54,500$. Find the absolute change and relative change in tuition.

Absolute change: $

Relative change: %

Round your answer to the nearest hundredth. Do not type negative.

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[1] https://www.cbo.gov/system/files/202...ral-Budget.pdf

4.2: Discounts, Markups, and Sales Tax

Figure 4.2.1: Sale prices are often described as percent discounts. (credit: "Close-up of a discount sign" by Ivan Radic/Flickr, CC BY 2.0)

Learning Objectives

1. Compute discounts, markups, and sales tax.
2. Solve application problems involving discounts, markups, and sales tax.

Many people first encounter percentages during a retail transaction such as a percent discount (SALE! 25% off!!), or through sales tax ("Wait, I thought this was $\$1.99$?"), a report that something has increased by some percentage of the previous value (NOW! 20 more!!). These are examples of percent decreases and percent increases. In this section, we discuss the decrease, increase, and then the case of sales tax.

Calculating Discounts

Retailers frequently hold sales to help move merchandise. The sale price is almost always expressed as some amount off the original price. These are discounts, a reduction in the price of something. The price after the discount is sometimes referred to as the reduced price or the sale price.

When a reduction is a percent discount, it is an application of percent, which was introduced in Understanding Percent. The formula used was "part=percentage×total." In the formula, it would be helpful to remind that the present is expressed as a decimal. For 25% discount, use 0.25 as percent discount.In a discount application, the discount plays the role of the part, the percent discount is the percentage, and the original price plays the role of the total.

FORMULA: Discount

The formula for a discount based on a percentage is

$$\begin{aligned} \text{discount} &= \text{percent discount} \times \text{original price} \\ \text{sale price} &= \text{original price} - \text{discount} \end{aligned} \tag{4.2.1}$$

NOTE: You can think of the discount as a part and the original price as a total, as we introduced those terms in the previous section.

Also, we can find the sale price directly using the following formula

$$\text{sale price} = \text{original price } \times (1 - \text{percent discount}) \tag{4.2.2}$$

Sometimes the sale price and the percent discount of an item are known. We use the following formula to find the original price

$$\text{original price} = \frac{\text{sale price}}{\text{1- percent discount}} \tag{4.2.3}$$

Example 4.2.1: Calculating Discount for a Percent Discount

Calculate the discount for the given price and discount percentage. Then calculate the sale price.

original price = $\$75.80$; percent discount is 25%.

Answer

Substituting the values into the formula

$$\begin{aligned} \text{discount} &= \text{percent discount} \times \text{original price} \\ &= 0.25 \times 75.80 \\ &= 18.95 \end{aligned}$$

The sale price of the item is then

$$\text{Sale price} = \$75.80 - \$18.95 = \$56.85 \tag{4.2.4}$$

The sale price is $\$56.85$.

Your Turn $4.2.1$: Find Sale Price

A color printer regularly priced at \$410 is on sale for 25% off. What is the amount of the discount? What is the sale price? The discount is \$ _____ The sale price is \$ _____	Need Calculator? Show Calculator

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Example $4.2.2$: Calculating the Percent Discount from the Original and Sale Prices

Determine the percent discount based on the given original and sale prices.

original price = $\$1,200.00$; sale price = $\$900.00$

Answer

Find the discount. Using the original price and the sale price, we can find the discount with the formula

discount = original price − sale price

Substituting and calculating using the above formula, we find the discount to be $\$300.00$.

Find the percent discount. Substituting the discount of $\$300.00$ and the original price of $\$1,200.00$ into the formula, we can find the percent discount.

$$\begin{aligned} \text{discount} &= \text{percent discount} \times \text{original price} \\ 300 &= \text{percent discount} \times 1200 \\ \frac{300}{1200} &= \text{percent discount} \\ 0.25 &= \text{percent discount} \end{aligned}$$

Converting to percent form, the percent discount is 25%.

Your Turn $4.2.2$: Find Percent Discount

A TV sells for $\$213$. However, this weekend it is on sale for 125.67 . What is the percent discount? The discount is _____ %	Need Calculator? Show Calculator

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✔ Example 4.2.3: Calculating the Original Price from the Percent Discount and Sale Price

Determine the original price based on the percent discount and sale price.

percent discount 10%; sale price = $\$450.00$

Answer

Using the percent discount and the sale price, we can find the original price with the formula

$$\begin{aligned} \text{original price} &= \frac{\text{sale price}}{\text{1- percent discount}} \\ &= \frac{450}{1-0.10} \\ &= \frac{450}{0.90} \\ &= 500 \end{aligned}$$

Your Turn 4.2.3: Find Original Price Before Discount

Joyce paid $\$60.00$ for an item at the store that was 50 percent off the original price. What was the original price? $ [ ] Give your answer to the nearest cent. ;	Need Calculator? Show Calculator

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WORK IT OUT

There are cases where retailers allow multiple discounts to be applied. However, it is rare that the discount percentages are added together. For example, if you have a 15% coupon and qualify for a 20% price reduction, the retailer typically does not add those two percentages together to determine the new price. The retailer instead applies one discount and then applies the second discount to the price obtained after the first discount was deducted.

Research the original prices of two different laptops offered by one retail outlet. Assume you will receive a student discount of 12% and your outlet of choice is having a 15% off sale on all laptops.

For each laptop:

1. List the original price and calculate the price after applying the student discount 12% only.
2. Then find the price after applying the sale discount (15% off) to the price found in Step 1.
3. Determine the total saved on the laptop and what percent discount the total savings represent.
4. Now, apply the discounts in reverse order (first the sale discount, then the student discount).
5. Note anything interesting about your findings.

Calculate Markups

When retailers purchase goods to sell, they pay a certain price, called the **cost**. The retailer then charges more than that amount for the goods. This increase is called the **markup**. This selling price, or **retail price**, is what the retailer charges the consumer in order to pay their own costs and make a profit. **Markup, then, is very similar to discount, except we add the markup, while we subtract the discount.**

FORMULA: Mark Up

The formula for a markup is

$$\begin{aligned} \text{markup} &= \text{percent markup} \times \text{cost} \\ \text{retail price (marked up price)} &= \text{cost+mark up} \end{aligned} \tag{4.2.5}$$

Also, we can find the retail price directly using the following formula

$$\text{retail price (marked up price)} = \text{cost} \times (1\text{+percentage mark up}) \tag{4.2.6}$$

Sometimes the retail price and the markup percentage of an item are known. We use the following formula to find the cost (before markup).

$$\text{cost} = \frac{\text{retail parice}}{1\text{+percent markup}} \tag{4.2.7}$$

It should be noted that the formulas used for a markup are very similar to those for a discount, with addition replacing the subtraction.

Example 4.2.4: Determining the Retail Price Based on the Cost and the Percent Markup

Calculate the markup for the given cost and markup percentage. Then calculate the retail price.

cost = $\$62.00$; percent markup is 15%.

Answer

Substituting the values into the formula

$$\begin{aligned} \text{retail price (mark up price)} &= \text{cost} \times (1\text{+percentage mark up}) \\ &= 62 \times (1 + 0.15) \\ &= 62 \times (1.15) \\ &= 71.30 \end{aligned}$$

So retail price of item is $\$71.30$.

Your Turn 4.2.4: Price after Mark Up

Cask'n'Cork wine shop has a markup rate of 15% . What price will the shop charge for a bottle of zinfandel wine that costs $\$15.00$ wholesale?

$ [ ]

(Round your answer to two decimal places.)

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Example 4.2.5: Calculating the Percent Markup from the Cost and Retail Price

Determine the percent markup based on the given cost and retail price. Round percentages to two decimal places.

cost = $\$5.20$; retail price = $\$9.90$

Answer

Using the cost and the retail price, we can find the markup $\$4.70$.
After substituting the markup $\$4.70$, and the cost $\$5.20$, into the formula,

$$\begin{aligned} \text{markup} &= \text{percent markup} \times \text{cost} \\ 4.70 &= \text{percent markup} \times 5.20 \\ \frac{4.70}{5.20} &= \text{percent markup} \\ 0.9038 &= \text{percent markup} \end{aligned}$$

Converting to percent form, the percent markup is 90.38%.

Your Turn 4.2.5: Mark Up Percentage

A television originally priced at $\$100.00$ is sold for $\$600.00$, calculate the rate of markup based on the cost. Round the answer to two decimal places. **Rate of Markup =** ______ **%**	Need Calculator? Show Calculator

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Example 4.2.6: Calculating the Cost from the Percent Markup and Retail Price

Determine the cost based on the percent markup and retail price.

percent markup 225%; retail price =$\$26.55$

Answer

Using the percent markup and the retail price, we can find the cost with the given formula

$$\begin{aligned} \text{retail price} &= \text{cost} \times (1{+}\text{percent markup}) \\ 26.55 &= \text{cost} \times (1 + 2.25) \\ \frac{26.55}{3.25} &= \text{cost} \\ 8.17 &= \text{cost} \end{aligned}$$

The cost of the item before markup is $\$8.17$.

Or you can use another way

$$\begin{aligned} \text{cost} &= \frac{\text{retail parice}}{1{+}\text{percent markup}} \\ &= \frac{26.55}{1{+}2.25} \\ &= \frac{26.55}{3.25} \\ &= 8.17 \end{aligned} \tag{4.2.8}$$

Your Turn 4.2.6: Price Before Mark Up

Bob decided to order a t-shirt for his gaming friend online for $\$ 27.46$. He knows the markup on such t-shirts is 38%. What was the t-shirt's cost before the markup? T-shirts cost before the markup is $ (Round your answer to the nearest cent.)	Need Calculator? Show Calculator ;

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Compute Sales Tax

Sales tax is applied to the sale or lease of some goods and services in the United States but is not determined by the federal government. It is most often set, collected, and spent by individual states, counties, parishes, and municipalities. None of these sales tax revenues go to the federal government.

For example, North Carolina has a state sales tax of 4.75% while New Mexico has a state sales tax of 5%. Additionally, many counties in North Carolina charge an additional 2% sales tax, bringing the total sales tax for most 72 of the 100 counties in North Carolinians to 6.75%. However, in Durham, the county sales tax is 2.25% plus an additional 0.5% tax used to fund public transportation, bringing Durham County's sales tax to 7%. To find the sales tax in a particular place, add other locality sales taxes to the base state sales tax rate.

How much we pay in sales tax depends on where we are and what we are buying.

To determine the amount of sales tax on a taxable purchase, we need to find the purchase price and add the sales tax for that locality

FORMULA: Sales Tax

To calculate the amount of sales tax paid on the purchase price and to calculate the total price (after sales tax). We use the following formula.

$$\begin{aligned} \text{sales tax } &= \text{purchase price} \times \text{tax rate} \\ \text{total price} &= \text{purchase price} + \text{sales tax} \end{aligned} \tag{4.2.9}$$

Also, we can find the total price (after tax) directly using the following formula

$$\text{total price} = \text{purchase price} \times (1{+}\text{tax rate}) \tag{4.2.10}$$

Sometimes the total price and tax rate of an item are known. We use the following formula to find the purchase price (before sale tax), using the following formula

$$\text{purchase price} = \frac{\text{total price}}{1{+}\text{tax rate}} \tag{4.2.11}$$

Checkpoint

Round to the nearest penny or round to the nearest cent means round to two decimal places.

You should notice that this is the same as markup, except using a different term. Sales tax plays the role of markup, the purchase price plays the role of cost, and the tax rate plays the role of percent markup. This means all the strategies developed for markups apply to this situation, with the changes indicated.

Example $4.2.7$: Calculating the Sales Tax from the Purchase Price

The sales tax in Kankakee, Illinois, is 8.25%. Find the sales tax and total price of items based on the purchase price listed. Purchase price = $\$428.99$

Answer

The sales tax is found using

sales tax = purchase price × tax rate.

The purchase price is $\$428.99$ and the tax rate is 8.25%. Substituting and calculating, the sales tax is

$$\text{sales tax} = 428.99 \times 0.0825 = \$35.39 \tag{4.2.12}$$

The total price = purchase price + sales tax = $\$428.99 + \$35.39 = \$464.38$

Or Directly

$$\begin{aligned} \text{total price} &= \text{purchase price} \times (1+\text{tax rate}) \\ &= 428.99 \times (1 + 0.0825) \\ &= 428.99 \times (1.0825) \\ &= 464.38 \end{aligned}$$

Your Turn 4.2.7: Price After Sales Tax

The sales tax rate for South Salt Lake is 7.05%. What is the state sales tax on a \$5,600 car in South Salt Lake? \$ ____ What is the final cost of the car, including tax? \$ ____	The sales tax rate for Morgan County is 5.95%. What is the state sales tax on that same car purchase in Morgan County? \$ ____ What is the final cost of the car in Morgan County, including tax? \$ ____
How much money would you save by driving to Morgan County to purchase the vehicle? \$ ____	Need Calculator? Show Calculator

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Be aware that almost all sales tax rates are structured as full percentages, half percentages, one-quarter percentages, or three-quarter percentages. This means the decimal value of the sales tax rate, written as a percent, will be either 0, as in 5.0%, 5 as in 7.5%, 25 as in 3.25%, or 75 as in 4.75%. When rounding for the sales tax percentage, be sure to use this guideline.

Example 4.2.8: Calculating the Sales Tax from the Purchase Price and the Total Price

Find the sales tax rate for the indicated purchase price and total price. Round using the guideline for sales tax percentages.

purchase price =$\$329.50$; total price = $\$354.21$

Answer

Find the sales tax paid. First, the amount of sales tax must be found. Subtracting the purchase price from the total price, the amount of sales tax is $\$24.71$. Find the sales tax rate. Using the purchase price, the sales tax, and the formula

$$\text{sales tax} = \$354.21 - \$329.50 = \$24.71 \tag{4.2.13}$$

Substituting and solving yields

$$\begin{aligned} \text{Sales Tax} &= \text{purchase price} \times \text{tax rate} \\ \$24.71 &= \$329.50 \times \text{tax rate} \\ \frac{\$24.71}{\$329.50} &= \text{tax rate} \\ 0.07499 &= \text{tax rate} \end{aligned}$$

Keeping in mind the guideline for rounding sales tax rate, the sales tax rate is 7.50%.

Your Turn 4.2.8: Find Tax Rate

The sale tax on a used car priced at $1300 is $68.90. What percent is the sales tax rate?

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Example 4.2.9: Calculating the Purchase Price from the Sales Tax and Total Price

Determine the purchase price for the specified sales tax rate and total cost.

sales tax rate = 5.75%; total price = $\$36.56$

Answer

When the sales tax rate and the total price are known, the formula is. Substituting the tax rate and total price into the formula and solving, we find

$$\begin{aligned} \text{total price} &= \text{purchase price} \times (1{+}\text{tax rate}) \\ 36.56 &= \text{purchase price} \times (1+0.0575) \\ 36.56 &= \text{purchase price} \times 1.0575 \\ \frac{36.56}{1.0575} &= \text{purchase price} \\ 34.57 &= \text{purchase price} \end{aligned} \tag{4.2.14}$$

The purchase price before tax was $\$34.57$.

Or you can use the following formula directly to find the purchase price

$$\begin{aligned} \text{purchase price} &= \frac{\text{total price}}{1{+}\text{tax rate}} \\ &= \frac{36.56}{1{+}0.0575} \\ &= \frac{36.56}{1.0575} \\ &= 34.57 \end{aligned} \tag{4.2.15}$$

Your Turn 4.2.9: Price Before Tax

You purchased a new car for the total cost of $32,400.00, which includes the 11% sales tax. What was the original price of the car? Original price of the car is $ ______ (Round your answer to two decimal places.)	Need Calculator? Show Calculator

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Who Knew?

West Virginia was the first state to impose a sales tax. This happened on May 3, 1921.

Look up your locality on this website that lists standard state-level sales tax rates and compare the sales tax structure in your state to two nearby states.

4.3: Simple and Compound Interest

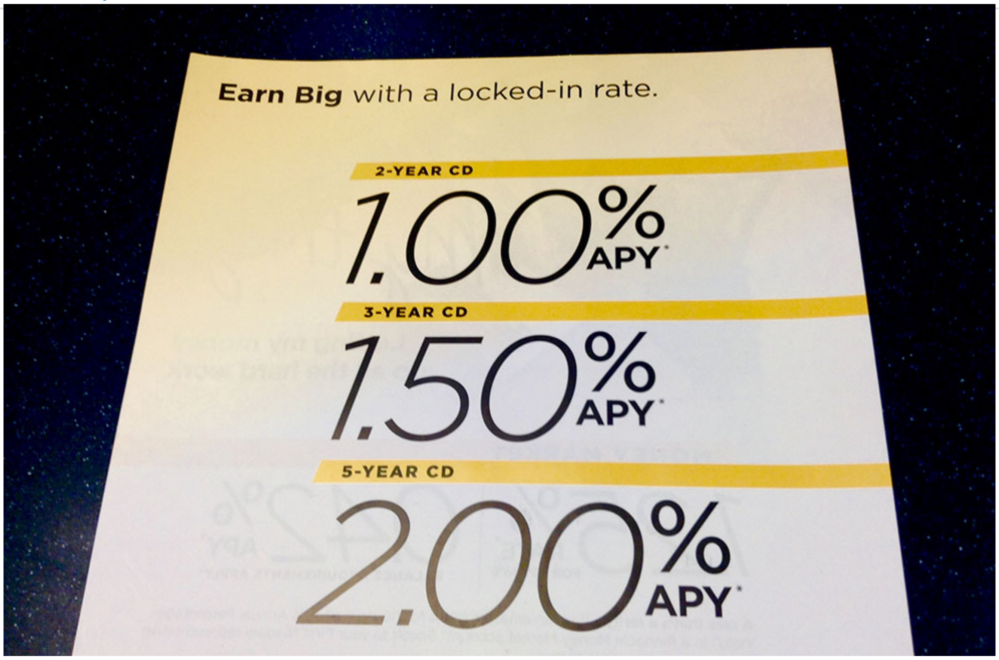

Figure 4.3.1: Interest is how savings earn money. (credit: "Interest Rates" by Mike Mozart/Flickr, CC BY 2.0)

Learning Objectives

1. Compute simple interest.
2. Understand and compute future value.
3. Compute simple interest loans with partial payments.
4. Understand and compute present value.
5. Understand and compute future value.
6. Compute present value.
7. Compute and interpret effective annual yield.

There is truth in the phrase "You need to have money to make money." In essence, if you have money to lend, you can lend it at a cost to a borrower and make money on that transaction.

When money is borrowed, the person borrowing the money (the borrower) typically has to pay the person or entity that lent the money (the lender) more than the amount borrowed. This extra money is the interest that is to be paid. Interest is sometimes referred to as the cost of borrowing, the cost of the loan, or the finance charge.

This idea also applies when someone deposits money into a bank account or invests in another form of investment. That person is essentially lending the money to the bank or company. The money earned by the depositor is also referred to as interest. The interest is typically based on the amount borrowed, or the principal.

The pairing of borrower and lender can take various forms. The borrower may be a consumer using a credit card or taking out a loan from a bank, the lender. Companies also borrow from lending banks. Someone who invests in a company's stock is the lender in this case; the company is essentially the borrower.

In this section, we examine the basic building block of interest paid on loans and borrowed credit, as well as the returns on investments such as bank accounts and simple interest.

Simple Interest and Future Value

Let's get some terminology understood. The interest to be paid by a borrower is often expressed as an annual percentage rate, which is the percentage of the principal that is paid as interest for each year the money is borrowed. This means that the more that is borrowed, the more that must be paid back. Sometimes, the interest to be paid back is simple interest, which means that the interest is calculated on the amount borrowed only.

The length of time until the loan must be paid off is the **term** of the loan. The date when the loan must be paid off is the due **date**. The day that the loan is issued is the origination date. We'll put this terminology to use in the following examples. Note that in this section, we will use letters, called variables, to represent the different parts of the formulas we'll be using. This will help keep our formulas and calculations manageable.

Calculating simple interest is similar to the percent calculations we made in Understanding Percent and Discounts, Markups, and Sales Tax, but it must be multiplied by the term of the loan (in years, if dealing with an annual percentage rate).

FORMULA: Simple Interest

The simple interest, I, to be paid on a loan with annual interest rate r, with principal P, is found using

$$I = P \times r \times t$$

where the decimal form of the interest rate, r, is used and time, t, is expressed in years.

The amount that is loaned is called the present value, PV, or just P.

FORMULA: Future Value

The future value, FV, of an investment that yields simple interest is

$$FV = P + I$$
$$Or$$
$$FV = P(1 + rt)$$

Also, the interest that is earned by investing P can be calculated as

P is the principal (amount invested at the start), r is the annual interest rate in decimal form, and t is the length of time the money is invested (in years).

$$I = FV - P$$

Note: The future value of the amount is also denoted by just A.

The Money can also be invested for a certain period, earning simple interest while it is invested. The terms and formulas we use are the same as before. The investor wants to know how much the investment will be worth after interest is added. This is called the **future value**.

As we see above, **future value** is the amount of money you end up with after the principal (the amount originally invested) and all the interest earned during that time are added together.

Example 4.3.1: Simple Interest on Loans with Whole Number Year Terms

1. Calculate the simple interest to be paid on a loan with the given principal, annual percentage rate, and number of years. Then, calculate the loan payoff amount. (Round your answers to the nearest cent.): Principal P = $\$4,000$, annual interest rate r = 5.5%, and number of years t = 4.
2. Riley runs an auto repair shop and needs to purchase a new brake lathe, which costs $\$11,995$. She takes out a two-year, simple interest loan at an annual interest rate of 14.9%. How much interest will she pay, and how much will she repay on the loan? (Round your answers to the nearest cent)

Answer

1. Substitute the principal P = $\$4,000$, the decimal form of the annual interest rate r = 0.055, and number of years $t = 4$ into the formula for simple interest, and calculate.

$$I = 4,000 \times 0.055 \times 4 = 880$$

The simple interest, or cost of the loan, to be paid on the loan is $\$880$, which is I. The loan payoff amount, or the total to be repaid, is

$$A = P + I = \$4,000 + 880 = \$4,880$$

So $\$4,880.00$ is the loan payoff amount.

2. Determine the variables or parts of the formula. The principal P = $\$11,995$. The interest rate Riley pays is 14.9%, or $r = 0.149$ in decimal form. The length of the loan is two years, so $t = 2$. We are first asked to find I, the interest Riley will pay. Substitute the known variables into the formula for simple interest $I = P \times r \times t$ and solve for I.

$$I = 11,995 \times 0.149 \times 2 = 3,574.51$$

This tells us that the simple interest, or cost to borrow, to be paid on the loan is $\$3,574.51$.

Use the formula $A = P + I$ to determine the total amount Riley will repay.

$$A = P + I = \$11,995 + \$3,574.51 = \$15,569.51$$

The total to be repaid is $\$15,569.51$.

Your Turn 4.3.1: Use Simple Interest

Daisy invests $14,000.00 at 6% simple interest for 4 years.

How much interest is earned over the 4 year period?

The interest earned over the 4 year period is $ ____ .

How much is in the account at the end of the 4 year period?

Daisy will have $ ____ in the account at the end of the 4 year period.

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Simple Interest Loans with Other Lengths of Terms

In the previous example and the Your Turn exercise, the loans were repaid in a single payment after an integer number of years. However, there are also loans lasting a length of time not equal to an integer number of years (like 1, 2, or 3 years or more), but in a number of months (like 4 months, 18 months, and so on). What model would apply to these situations?

When the loan is repaid after a term that is not a year but is instead a number of months, days, or weeks, t must be expressed in a year. Here are some examples.

1. For a 5-month loan, the time in years is $t = \frac{5}{12}$.
2. For a 18-month loan, the time, in years, is $t = \frac{18}{12}$.
3. For a 100-day term, the term in years is $t = \frac{100}{365}$.
4. For a 30-week loan, the time in years is $t = \frac{30}{52}$.

Example 4.3.2: Loan to Purchase Equipment

1. Abeje needs a loan to purchase equipment for the gym she is going to open. She visits the bank and secures a 4-month loan of $20,000. Her annual percentage rate is 6.75%. How much interest will Abeje pay, and what is her loan payoff amount?
2. David plans to move his family from Raleigh, North Carolina, to Tempe, Arizona. His company will reimburse (pay after the move) David for the move. David does research and determines that movers will cost $5,600 to move his family's belongings to Tempe. He takes out a simple interest, 45-day loan at 11.75% interest to pay this cost. How much interest will be paid on this 45-day loan, and what is David's loan payoff amount? (Round your answers to the nearest cent)

Answer

1. Abeje's loan is for $20,000, so her principal is $P = \$20,000$. The interest rate Abeje will pay is 6.75%, or $r = 0.0675$ in decimal form. The length of the loan is 4 months, so $t = \frac{4}{12}$. Substituting these in the formula for simple interest, we find her interest to be

$$I = 20,000 \times 0.0675 \times \frac{4}{12} = 450$$

The simple interest, or cost to borrow, to be paid on the loan is $450.00. The payoff is $\$20,000 + \$450 = \$20,450$.

2. The principal for the loan is the moving cost, or $P = \$5,600$. The annual interest rate that David will pay is 11.75%, which in decimal is $r = 0.1175$. The length of time for the loan is 45 days, so $t = \frac{45}{365}$ year. Substituting these values into the formula and calculating, we find that the interest to be paid is

$$I = 5600 \times 0.1175 \times \frac{45}{365} = 81.12$$

$81.12 is the interest for 45 days.

For Pay off

$$A = P + I = \$5,600 + \$81.12 = \$5,681.12$$

The payoff for the loan is $5,681.12.

Your Turn 4.3.2: Use Simple Interest

Booker invests $10,000.00 at $5\frac{3}{4}$ % simple interest for 1345 days.

How much interest is earned over the 1345 day period?

The interest earned over the 1345 day period is $ ______.

How much is in the account at the end of the 1345 day period?

Booker will have $ ______ in the account at the end of the 1345 day period.

Hint

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Who Knew?

It seems reasonable to use 365 as the number of days in the year since there are 365 days in most years. However, sometimes, banks have used (and continue to use) 360 as the number of days in a year. They may also treat all months as if they have 30 days. These differences lead to (sometimes small) differences in how much interest is paid. Since the number of days is in the denominator, a smaller denominator (360) will result in larger numbers (interest); therefore, 365 is used for the denominator. See this page from ACRE for a comparison.

Example 4.3.3: Simple Interest on a Investment

In the following, determine how much interest was earned on the investment and the future value of the investment, if the investment yields simple interest. Round your answer to the nearest cent.

1. Principal is $1,000, annual interest rate is 2.01%, and time is 5 years.
2. Principal is $10,000, annual interest rate is 1.25%, and time is 18 months.
3. Principal is $7,000, annual interest rate is 3.26%, and time is 100 days.

Answer

1. The principal is $P = \$1,000$, the annual interest rate, in decimal form, is $r = 0.0201$, and the term is 5 years, or $t = 5$.

$$FV = P(1 + rt) = 1000(1 + 0.0201 \times 5) = 1100.50$$

The interest earned on the investment is

$$I = FV - P = 1100.50 - 1000 = 100.50$$

The future value of the investment at the end of 5 years is $1,100.50, where $100.50 comes from interest and $1000 comes from principal.

2. The principal is $P = \$10,000$, the annual interest rate, in decimal form, is 0.0125, and the term is 18 months. Since the term is in months, we have to write the months in terms of years. For 18 months, we have used $\frac{18}{12}$ as t.

$$\begin{aligned} FV &= P(1 + rt) \\ &= 10,000\left(1 + 0.0125 \times \frac{18}{12}\right) \\ &= 10,187.50 \end{aligned}$$

The interest earned on the investment is

$$I = FV - P = 10,187.50 - 10,000 = 187.50$$

The future value of the investment at the end of 18 months is $10,187.50, where $187.50 comes from interest and $10,000 comes from principal.

3. The principal is $P = \$7,000$, the annual interest rate, in decimal form, is $r = 0.0326$, and the term is 100 days. Since the term is in days, we have to write the time in years, or $t = \frac{100}{365}$.

$$\begin{aligned} FV &= P(1+rt) \\ &= 7,000\left(1+0.0326 \times \frac{100}{365}\right) \\ &= 7,062.52 \end{aligned}$$

The interest earned on the investment is

$$I = FV - P = \$7,062.52 - \$7,000 = \$62.52$$

The future value of the investment at the end of 100 days is $\$7,062.52$, where $\$62.52$ comes from interest and $\$7,000$ comes from principal.

Your Turn 4.3.3: Use Simple Interest for Investment

Lloyd invests $19,000.00 at 3% simple interest for 40 years.Round your answer to the nearest cent.

How much will the investment be worth at the end of the 40 year period?

The future value of the investment at the end of the 40 year period will be $ ______ .

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A certificate of deposit (CD) is a savings account that holds a single deposit (the principal) for a fixed term at a fixed interest rate. Once the term of the CD is over, the CD may be redeemed (cashed in or withdrawn), and the owner of the CD receives the original principal plus the interest earned. The deposit often cannot be withdrawn until the term is up; if it can be withdrawn early, a penalty is typically imposed for doing so.

For example: Jonas deposits $\$2,500$ in a CD bearing 3.25% simple interest for a term of 3 years. When he redeems his CD at the end of the 3 years, how much will he receive?

This is a future value example. We know that P = $\$2,500$ is the amount deposited. The annual simple interest rate in decimal form is r = 0.0325. The term of the investment is t=3 years.

$$FV = P(1+rt) = 2,500(1+0.0325 \times 3) = 2,743.75$$

When the CD is redeemed, Jonas will receive $\$2,743.75$.

WORK IT OUT

The reason CD (certificate of deposit) rates look so small is that they are extremely safe investments. Though overall interest rates for CDs change over time and individual returns vary with the terms of the CD, investors are offered predictable interest income for their investments.

To investigate this yourself, search online to determine the strengths and weaknesses of CDs (investopedia.com offers good, basic information on investing). Then, online, identify five national banks and two local banks that offer CDs.

1. Track the interest rates for the CDs at various terms (1 year, 3 years, 5 years) for each of the banks you found that offer CDs.
2. Calculate the amount of interest earned for a $\$10,000$ deposit for each CD at each of the terms.
3. Compare the results from the various banks, CDs, and terms and decide which is the best investment. You may want to consider both the length of time that the money is locked up and the return.

Understand and Compute Present Value for Simple Interest Investments

When finding the future value of an investment, we know how much is deposited, but we have no idea how much that money will be worth in the future. If we set a goal for the future, it would be useful to know how much to deposit now so that the account reaches the goal. The amount that needs to be deposited now to reach a goal in the future is called the **present value, PV, or simply P,** which is a notation used for a loan.

FORMULA: Present Value

The present value, PV of money deposited at an annual, simple interest rate of r (in decimal form) for time t (in years) with a specified future value of FV is calculated with the formula

$$\mathrm{PV} = \frac{FV}{(1+rt)}$$

Note: Present value, in this calculation, is always **rounded UP**. Otherwise, future value may fall short of the target future value. Sometimes, present value may be denoted by just P.

Understanding what this tells you is important. When you find the present value, that is how much you need to invest now to reach the goal, under the conditions (time and rate) at which the money will be invested.

Example 4.3.4: Find Present Value

Compute the present value of the investment described. Interpret the result.

Beatriz will invest some money in a CD that yields 3.99% simple interest when invested for 30 years. How much must Beatriz invest so that after those 30 years, her CD is worth $\$300,000$?

Answer

Beatriz needs to know how much to deposit now so that her CD is worth $\$300,000$ after 30 years. This means she needs to know the present value of that $\$300,000$. The time is 30 years and the annual simple interest rate, in decimal form, is $r = 0.0399$. Using that information and the formula for present value, we calculate the present value of that $\$300,000$.

$$\begin{aligned} PV &= \frac{FV}{(1+rt)} \\ &= \frac{300,000}{(1+0.0399 \times 30)} \\ &= \frac{300,000}{2.197} \\ &= 136,459.8407 \end{aligned}$$

Rounding up, Beatriz needs to invest $\$136,549.85$ so that she has $\$300,000$ in 30 years.

Your Turn 4.3.4: Find Present Value

How much would you need to deposit in an account now in order to have $5,000.00 in the account in 30 months? Assume the account earns 7.84% simple interest. Round your answer to the nearest cent.

You would need to deposit $ ________ in your account now.

Hint

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Determining Simple Interest Rate and Time

Using the same formula, we can also calculate the simple interest rate, r, and the time, t.

Rate and Time Formula: Simple Interest

$$\text{Interest rate: } r = \frac{FV - P}{Pt}$$

$$\text{Time: } t = \frac{FV - P}{Pr}$$

Example 4.3.5: Find r and t

1. A simple interest loan for $\$20,000$ is taken at a 12.6% annual rate when the loan's future value is $\$25,000$.
2. A simple interest loan for $\$1500$ is taken, and you pay it back to double in 4 years? What simple interest rate do you pay?

Answer

Here we have, $PV = \$20,000, \;\; FV = \$25,000, \, r = 0.126$, and t = Unknown

Use the formula for t

$$\begin{aligned}\text{Time: } t &= \frac{FV-P}{Pr}\\ &= \frac{25,000-20,000}{20,000\times 0.126}\\ &= \frac{5,000}{2520}\\ &= 1.98\end{aligned}$$

So A simple interest loan for $\$20,000$ will be $\$25,000$ in almost 2 years with a 12.6% interest rate.

Answer

Here we have, $PV = \$1500, FV = \$3000, t = 4$, and r = Unknown

Use the formula for r

$$\begin{aligned}\text{Time: } r &= \frac{FV-P}{Pt}\\ &= \frac{3,000-1,500}{1,500\times 4}\\ &= \frac{1,500}{6000}\\ &= 0.25\end{aligned}$$

So $\$1500$ loan will be doubled in 4 year with 25% simple interest rate.

Your Turn 4.3.5: Find Simple Interest Rate

Leire wants to invest $\$5,200.00$ in a savings account. Determine the simple interest rate required for Leire's investment to grow to $\$14,600.00$ in 18 years.

Round your answer to the nearest tenth of a percent.

The interest rate required to grow the investment to $\$14,600.00$ is ______ %.

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CAREFUL: When solving problems in the Simple Interest section, make sure to read each question carefully. Look for the word **"simple"** so you know to use the **simple interest formula**. Always ensure you understand **which formula** is required to solve the problem correctly.

Compound Interest

For a very long time in certain parts of the world, interest was not charged due to religious dictates. Once this restriction was relaxed, loans that earned interest became possible. Initially, such loans had short terms, so only simple interest was applied to the loan. However, when loans began to stretch out for years, it was natural to add the interest at the end of each year, and add the interest to the principal of the loan. After another year, the interest was calculated on the initial principal plus the interest from year 1, or the interest earned. Each year, more interest was added to the money owed, and that interest continued to earn interest.

Since the amount in the account grows each year, more money earns interest, increasing the account faster. This growth follows a geometric series (Geometric Sequences). It is this feature that gives compound interest its power. This module covers the mathematics of compound interest.

Understand and Compute Future Value

As we saw in Simple Interest, an account that pays simple interest only pays based on the original principal and the term of the loan. Accounts offering compound interest pay interest at regular intervals. After each interval, the interest is added to the original principal. Later, interest is calculated on the original principal plus the interest that has been added previously.

After each period, the interest on the account is calculated and then added to the account balance. Then, after the next period, when interest is computed, it is computed based on the original principal AND the interest that was added in the previous periods. The period of time between two interest payments is called the compounding period

What is Compound Interest?

Compound interest is interest that is calculated **on both the principal (the original amount of money) *and* any interest that has already been earned**. In other words, the interest "compounds," meaning it grows on top of previous interest.

With compound interest, your money grows faster because each time interest is added, the total balance increases—and the next interest calculation uses that new, larger amount.

Compound Interest: To Find the Future Value Using the Compound Interest

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

A: The balance (future value) in the account after t years.

P: The starting balance of the account (also called initial deposit, or principal)

r: The annual interest rate in decimal form.

n: The number of compounding periods in one year.

Additionally, we can calculate the interest earned in compound interest by subtracting the future value from the principal.

$$I = A - P$$

Note: Sometimes, future value A can also be represented as FV, and P can also be represented as PV.

Compounding Period

In finance and mathematics, **a compounding period** refers to the time interval between two interest calculations.

When you invest money or take out a loan, interest is usually not just added once at the end—it's added many times along the way. Each time interest is added (or "compounded"), that's a **compounding period**. We use n to denote the compounding period.

Examples of Compound Periods:

Annually → Once per year, $n = 1$.

Semiannually → Twice per year (every 6 months), $n = 2$.

Quarterly → Four times per year (every 3 months), $n = 4$.

Weekly → 52 times per year , $n = 52$.

Bi-weekly → 26 times per year, $n = 26$.

Monthly → Twelve times per year (every month), $n = 12$.

Daily → 365 times per year (every day), $n = 365$.

The most important thing to remember about using this formula is that it assumes we deposit money into the account once and let it sit there earning interest.

Example $4.3.6$: Computing Future Value for Compound Interest

In the following, compute the future value of the investment with the given conditions.

1. Principal is $\$5,000$, annual interest rate is 3.8%, compounded monthly, for 5 years. How much interest is included in the future value? (Round your final answer to the nearest cent)

2. Cody invests $\$18,500$, in an account that earns 6.25%, compounded quarterly, for 17 years. Determine the value of Cody's investment after 17 years. How much interest is included in Cody's investment? (Round your final answer to the nearest cent.)

Answer

1. The principal is $P = \$5,000$, interest rate, in decimal form, 0.038, compounded monthly so $=12$, and for $=5$ years. Substituting these values into the formula, we find

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 5000\left(1+\frac{0.038}{12}\right)^{12\times 5} \\ &= 5000(1+0.00316666)^{60} \\ &= 5000(1.00316666)^{60} \\ &= 5000 \times 1.20888615 \\ &= 6044.4307 \end{aligned} \tag{4.3.1}$$

The future value of the investment is $\$6,044.43$.

Interest earned

$$I = A - P = \$6,044.43 - \$5,000 = \$1,044.43 \tag{4.3.2}$$

2. The principal is $P = \$18,500$, interest rate, in decimal form, $r = 0.0625$, compounded quarterly so $n = 4$, and for $t = 17$ years. Substituting these values into the formula, we

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} \\ &= 18,500\left(1+\frac{0.0625}{4}\right)^{4\times 17} \\ &= 18,500(1+0.015625)^{68} \\ &= 18,500(1.015625)^{68} \\ &= 18,500 \times 2.86992152 \\ &= 53,093.54812 \end{aligned} \tag{4.3.3}$$

The future value of the investment is $\$53093.55$.

Interest earned

$$I = A - P = \$53,093.55 - \$18,500 = \$34,593.55 \tag{4.3.4}$$

You can also find future value in 1 and 2 using the following TVM (Time Value Money) Calculator. **But you should learn how to do it manually using a formula.** For example, 2 input the following in the TVM solver.

- ✔ **N:** The number of payments = P/Y × # of years = $4 \times 17 = 68$
- ✔ **I%:** The interest rate, as a percentage. Put I $=6.25$
- ✔ **PV:** Present value. Put PV = $18,500$
- ✔ **PMT:** Periodic payment (We don't have this information, so we entered zero for it.)
- ✔ **FV:** Future value (We need to solve for it, so click it after you put all the values)
- ✔ **P/Y:** Payments per year. Put P/Y = 4 (quarterly)
- ✔ **C/Y:** Times per year that interest is compounded. Put C/Y = 4 (quarterly)

TVM Solver

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

Your Turn 4.3.6: Future Value and Interst from Compound Interest

You deposit $\$4,000.00$ in an account earning $9\frac{3}{4}\%$ interest compounded quarterly.

How much will you have in the account in 15 years?

There will be $ ______ in the account in 15 years.

How much interest will have been earned?

The amount of interest that will have been earned is $ ______ . Hint

(Round your answer to the nearest cent.)

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

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Let us compare the amount of money earned from compounding against the amount you would earn from simple interest

Years	Simple Interest ($15 per month)	6% compounded monthly $=0.5\%$ each month.
5	$3900	$4046.55
10	$4800	$5458.19
15	$5700	$7362.28
20	$6600	$9930.61
25	$7500	$13394.91
30	$8400	$18067.73
35	$9300	$24370.65

As you can see, over a long period of time, compounding makes a large difference in the account balance. You may recognize this as the difference between linear growth and exponential growth.

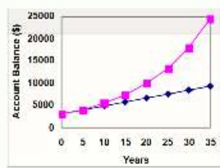

Figure 4.3.1: Copy and Paste Caption here. (Copyright; author via source)

Understand and Compute Present Value

When investing, there is often a goal to reach, such as “after 20 years, I’d like the account to be worth $100,000.” The question to be answered in this case is “How much money must be invested now to reach the goal?” As with simple interest, this is referred to as the present value.

FORMULA: Compute Present Value from Simple Interest

The money invested in an account bearing an annual interest rate of r (in decimal form), compounded n times per year for t years, is called the present value, PV (or just P), of the account (or of the money) and found using the formula

$$P=\frac{A}{(1+\frac{r}{n})^{nt}} \tag{4.3.5}$$

where A is the value of the account at the investment’s end. Always round this value **UP** to the nearest cent unless mentioned otherwise.

Example 4.3.7: Computing Present Value

Pilar plans early for retirement, believing she will need $1,500,000 to live comfortably after the age of 67. How much will she need to deposit at age 23 in an account bearing 6.35% annual interest compounded monthly. How much of $1,500,000 comes for interest.

Answer

She’s 23 now and will leave the money in the account until the age of 67. She leaves the money for $67-23=44$ years, making $t=44$. $r=6.35\%=0.0635$, and $n=12$ because compounded monthly. Using this information and substituting in the formula for present value given above.

$$\begin{aligned} P &= \frac{A}{(1+\frac{r}{n})^{nt}} \\ &= \frac{1,150,000}{(1+\frac{0.0635}{12})^{12\times 44}} \\ &= \frac{1,150,000}{(1.005291667)^{528}} \\ &= 70,872.5737 \end{aligned}$$

Round up to the nearest cent, we get $P = \$70,872.58$.

So, for this account to reach $\$1,150,000$ after 44 years, $\$70,872.58$ needs to be invested with the given interest rate, and compounding must be monthly. Interest earned during that period is:

$$I = A - P = \$1,150,000 - \$70,872.58 = \$1,079,127.42$$

Also, you can find the present value using the TVM solver

- N: The number of payments = $12 \times 44 = 528$
- I%: The interest rate, as a percentage. Put I = 6.35
- PV: Present value. We need to solve it, so click it after you put all the values in.
- PMT: Periodic payment (We don't have this information, so we have entered zero for it.)
- FV: Future value. Put FV =$1,500,000$
- P/Y: Payments per year. Put P/Y = 12 (monthly)
- C/Y: Times per year that interest is compounded. Put C/Y = 12 (monthly)

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

Your Turn 4.3.7: Effective Annual Yield

You deposit $\$6,000.00$ **in an account earning** $7\frac{7}{8}\%$ **interest compounded monthly.**

How much will you have in the account in 12 years?

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

Present Value

There will be $ [ ] in the account in 12 years.

How much interest will have been earned?

The amount of interest that will have been earned is $ [ ] . Hint

(Round your answer to the nearest cent.)

PMT: = 0 Solve
Payment

FV: = 0 Solve
Future Value

P/Y: 12
Payments per Year

C/Y: 12
Compounding Periods per Year

PMT: = END
Payments are made at the end of the period

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CAREFUL: When solving problems in the **Compound Interest** section, make sure to read each question carefully. Look for the word **"compounded,"** so you know to use the *compound interest formula*. Always be sure you understand **which formula** is required to solve the problem correctly.

If you are using the **TVM Solver**:

- Do **not** use commas and dollar signs in the input boxes.
- Do **not** convert the interest rate to a decimal.
- If your answer comes out **negative**, enter the final answer as a **positive** value.
- Remember that $N = n \times t$.

Compute and Interpret Effective Annual Yield

As we've seen, quarterly compounding pays interest 4 times a year or every 3 months; monthly compounding pays 12 times a year; daily compounding pays interest every day, and so on. Effective annual yield allows direct comparisons between simple interest and compound interest by converting compound interest to its equivalent simple interest rate. We can even directly compare different compound interest situations. This gives information that can be used to identify the best investment from a yield perspective.

Using a formula, we can interpret compound interest as simple interest. The effective annual yield formula stems from the compound interest formula and is based on an investment of $\$1$ for one year.

Formula: Effective Annual Yield

$$Y = \left(1 + \frac{r}{n}\right)^n - 1$$

where Y = effective annual yield, r = interest rate in decimal form, and n= number of times the interest is compounded in a year.

Y is interpreted as the equivalent annual simple interest rate.

Note: Effective annual yield is also called Annual Percentage Yield (APY).

Example 4.3.8: Determine and Interpret Effective Annual Yield for 6% Compounded Quarterly

Suppose you have an investment paying a rate of 6% compounded quarterly. Determine and interpret the effective annual yield of the investment.

Answer

Here, $n = 4$ (quarterly) and $r = 0.06$ (decimal form). Substituting into the formula, we find that the effective annual yield is

$$\begin{aligned} Y &= \left(1+\frac{r}{n}\right)^n - 1 \\ &= \left(1+\frac{0.06}{4}\right)^n - 1 \\ &= (1.015)^4 - 1 \\ &= 0.06136 \end{aligned}$$

Therefore, a rate of 6% compounded quarterly is equivalent to a simple interest rate of 6.14%.

Your Turn 4.3.8: Effective Annual Yield

A bank features a savings account that has an annual percentage rate of 5.5% with interest compounded quarterly. Mariana deposits $\$10,500$ into the account.

How much money will Mariana have in the account in 1 years?

Answer = $ ________. *Round answer to the nearest penny.*

What is the annual percentage yield (APY) for the savings account?

$APY =$ ________%. *Round to the nearest hundredth of a percent.*

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4.4: Methods of Savings (Annuities, Stock, and Bonds)

Figure 4.4.1 : Money wisely invested grows over time. (credit: "Stack of Cash" by Janak Raja/Flickr, Public Domain Mark 1.0)

Learning Objectives

1. Distinguish various basic forms of savings plans.
2. Compute return on investment for basic forms of savings plans.
3. Compute the payment to reach a financial goal.
4. Distinguish between basic forms of investments, including stocks, bonds, and mutual funds.
5. Read and derive information from a stock table
6. Compute return on investment for basic forms of investments.
7. Identify and distinguish between retirement savings accounts.

The stock market crash of 1929 led to the Great Depression, a decade-long global downturn in productivity and employment. A state of shock swept through the United States; the damage to people's lives was immeasurable. Americans no longer trusted established financial institutions. By October 1931, the banking industry's biggest challenge was restoring confidence to the American public. In the next 10 years, the federal government would impose strict regulations and guidelines on the financial industry. The Emergency Banking Act of 1933 created the Federal Deposit Insurance Corporation (FDIC), which insures bank deposits. The new federal guidelines helped ease suspicions among the general public about the banking industry. Gradually, things returned to normal, and today we have more investment instruments, many insured through the FDIC, than ever before.

In this section, we will first look at the different types of savings accounts and proceed to discuss the various types of investments. There is some overlap, but we will try to differentiate among these financial instruments. Saving money should be a goal of every adult, but it can also be a difficult goal to attain.

Various Basic Forms of Savings Plans

There are at least three types of savings accounts. Traditional savings accounts, certificates of deposit (CDs), and money market accounts are the primary options for saving.

Savings Account

A savings account is probably the most well-known type of investment, and for many people, it is their first experience with a bank. A savings account is a deposit account held at a bank or other financial institution that bears interest on the deposited money. Savings accounts are intended as a place to save money for emergencies or to achieve short-term goals. They typically pay a low interest rate, but there is virtually no risk involved, and they are insured by the FDIC for up to $\$250,000$

Savings accounts have some strengths. They are highly flexible. Generally, there are no limitations on the number of withdrawals allowed and no limit on how much you can deposit. It is not unusual, however, for a savings account to have a minimum balance in order for the bank to cover maintenance costs. If your account balance drops below the minimum, fees may be applied.

Who Knew?

Many banks are covered by FDIC insurance. The FDIC, or Federal Deposit Insurance Corporation, is an independent agency created by the U.S. Congress. One of its purposes is to provide insurance for deposits in banks, including savings accounts and other deposit types. Be aware, not all banks are FDIC insured. The FDIC insures up to $\$250,000$ for a savings account, so you do not want your balance to exceed that federally insured limit.

Having your savings account at the same bank as your checking account does offer a real advantage. For example, if your checking account is approaching its lower limit, you can transfer funds from your savings account and avoid any bank fees. Similarly, if you have an excess of funds in your checking account, you can transfer funds to your savings account and earn some interest. Checking accounts rarely pay interest.

50-30-20 Rule

The **50-30-20 rule** is a **simple personal budgeting method** designed to help individuals manage their finances by dividing **after-tax income** into three main spending categories:

50% — **Needs (Essential expenses)**

- Utilities (electricity, water, gas)
- Groceries
- Transportation (car payment, gas, public transit)
- Insurance (health, auto)
- Minimum loan payments

30% — **Wants (Non-essential expenses)**

- Entertainment (movies, streaming, events)
- Shopping (clothes, gadgets)
- Vacations
- Hobbies and subscriptions

20% — **Savings & Debt Repayment**

- Emergency fund
- Retirement accounts (IRA, 401(k))
- Investments
- Extra payments toward credit cards or loans
- Saving for a home, education, etc

Look at an example

David gathers his pay stubs and bills from the past 6 months. His income, after taxes, is $\$3,450$ per month. His rent, utilities included, is $\$925$. His car payments are $\$178.54$ per month, his car insurance is $\$129.49$ per month, his credit cards cost him $\$117.00$ per month, he spends $\$195$ per month on gas, and his food costs are $\$290$ per month. He also spends $\$21.99$ on Amazon Prime, $\$49.99$ on his internet bill, and $\$400$ per month going out. Create David's monthly budget, including totals, based on that information.

1. Using David's Budget, how much income does he have per month after accounting for his expenses?
2. Apply the $50-30-20$ budget philosophy to David's budget.

3. Evaluate David's budget with respect to the $50-30-20$ budget philosophy.

Amount ($)	Expense	Amount ($)
3450.00	Rent, with utilities	925.00
	Car payments	178.54
	Car Insurance	129.49
	Credit card debt	117.00
	Gas	195.00
	Food	290.00
	Amazon prime	21.99
	Internet	49.99
	Going out	400.00
3450.00	**Total**	2307.01

David saved in one month

$$\$3450.00–\$2307.01=\$1142.99$$

Necessities 50% :

$$0.50\times\$3450=\$1725$$

Wants 30% :

$$0.30\times\$3450=\$1035$$

Savings and Debt Repyment 20% :

$$0.20\times\$3450=\$690$$

Necessities: $\$925+\$178.54+\$129.49+\$117+\$195+\$290=\$1835.03,$ This is very close to the 50%.

Wants: $\$21.99+\$49.99+\$400=\$471.98,$ This is much lower than the 30%.

Saving and Debt Repayment: $\$1142.99,$ This is much greater than the 20%.

People in Mathematics: J.P. Morgan

J.P. Morgan was a wealthy banker around the turn of the 20 th century. His business interests included railroads and the steel industry. However, it was in 1907 that a financial crisis, caused by poor banking decisions and followed by such great distrust in the banking system that a frenzy of withdrawals from banks occurred, that J.P. Morgan and other wealthy bankers lent from their own funds to help stabilize and save the system.

There are some weaknesses to savings accounts. Primarily, it is because savings accounts earn very low interest rates. This means they are not the best way to grow your money. Experts, though, recommend keeping a savings account balance to cover 3 to 6 months of living expenses in case you should lose your job, have a sudden medical expense, or other emergency.

What is 1099 form?

Around tax time, you will receive a 1099-INT form stating the amount of interest earned on your savings, which is the amount that must be reported when you file your tax return. A 1099 form is a tax form that reports earnings that do not come from your employer, including interest earned on savings accounts. These 1099 forms have the suffix INT to indicate that the income is interest income.

Savings accounts earn interest, and those earnings can be found using the interest formulas from previous sections. The final value of these accounts is sometimes referred to as the future value of the account.

Who Knew?

Banks have not always offered interest on savings accounts. An 1836 publication from Indiana noted that banks in other states allow small interest on deposits. It specifically says that in these other states, these deposits are what business transactions are based upon. And that giving interest would encourage deposits, and thus increase the business that banks can do.

Journal of the House of Representatives of the State of Indiana

Certificates of Deposit, or CDs, and Money Market Accounts

We discussed **certificates of deposit** (CDs) in earlier sections. CDs differ from savings accounts in a few ways. First, the investment lasts for a fixed period of time, agreed to when the money is invested in the CD. These time periods often range from 6 months to 5 years. Money from the CD cannot be withdrawn (without penalty) until the end of the investment period. Also, money cannot be added to an existing CD.

Certificates of deposit share features similar to those of savings accounts. They are insured by the FDIC. They are entirely safe. They do, though, offer a better interest rate. The trade-off is that once the money is invested in a CD, that money is unavailable until the investment period ends.

A money market account is similar to a savings account, except the number of transactions (withdrawals and transfers) is generally limited to six each month. Money market accounts typically require a minimum balance to be maintained. If the balance in the account falls below the minimum, a penalty may be incurred. Money market accounts offer the flexibility of checks and ATM cards. Ultimately, the interest rate on a money market account is generally higher than that of a savings account.

FORMULA: Return on Investment

The return on investment, often denoted ROI, is the percent difference between the initial investment, P, and the final value of the investment, FV.

\begin{align*}
\text{ROI}=\frac{FV-P}{P}\times100\%
\end{align*}

Note that *the length of time of the investment is not considered in ROI.*

Example 4.4.1: 5-Year CD ROI

Silvio deposits $\$10,000$ in a CD that yields 2.17% compounded semiannually for 5 years. How much is the CD worth after 5 years? How much did she earn in interest?

Answer

This also uses the compound interest formula from the Compound Interest section.
Substituting the values $P=\$10,000$, $r=0.0217$, $n=2$ (semiannually means twice per year), and $t=5$. We find the account will be worth

$$\begin{aligned} A&=P\left(1+\frac{r}{n}\right)^{nt}\\ &=10,000\left(1+\frac{0.0217}{2}\right)^{2\times5}\\ &=10,000(1.01085)^{10}\\ &=10,000\times1.113953736\\ &=11,139.53736 \end{aligned} \tag{4.4.1}$$

The CD will be worth $\$11,139.54$ after 5 years.

Interest earned during that period is:

$$I=A-P=\$11,139.54-\$10,000=\$1,139.54$$

The initial deposit in the CD was $\$10,000$, so $P=P=\$10,000$. The value at the end of 5 years was $\$11,139.54$. and \(FV = $11,139.54.\) Substituting and computing, we find the return on investment.

$$\begin{aligned} \text{ROI}&=\frac{FV-P}{P}\times100\%\\ &=\frac{\$11,139.54-\$10,000}{\$10,000}\times100\%\\ &=\frac{\$1,139.54}{\$10,000}\times100\%\\ &=11.3954\% \end{aligned} \tag{4.4.2}$$

The ROI is 11.40%.

Note: As we did in the compound interest section, you can find the future value using the TVM solver. Put the following values in the given variables

$$\begin{aligned} \text{N}&=n\times t=2\times5=10\\ \text{I}&=2.17\ \text{(Do not put percent)}\\ \text{PV}&=10,000\ \text{(Deposited amount or present value)}\\ \text{FV}&=\text{We need to find it.}\\ \text{PMT}&=0\\ \text{P/Y}&=2\\ \text{C/Y}&=2 \end{aligned}$$

And click solve for FV.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

Your Turn 4.4.1: Find the Future Value for Annuity

Craig deposited $\$7,500$ in a savings account that compounded monthly with a yearly interest rate of 2.5%. What was Craig's balance at the end of the fourth year? *(Round to the nearest cent.)*

Balance after fourth year: $

How much interest did Craig earn on the account during that year? *(Round to the nearest cent.)*

Interest earned in 4 years: $

What is the return on investment (ROI) on the account in year? *(Round to two decimal places when necessary.)*

ROI: %

Question Help: Written Example

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Annuities as Savings

In Compound Interest, we talked about the future value of a single deposit. In reality, people often open accounts that allow them to add deposits or payments to the account at regular intervals. When a deposit is made at the end of each compounding period, such a savings account is called an ordinary annuity. The

$-30-20$ is a simple budgeting guideline that helps individuals manage their money effectively. It divides your **after-tax income** into three broad categories:

Formula: Future Value of Annuity and Interest Earned

$$FV = PMT\left[\frac{(1+\frac{r}{n})^{nt}-1}{(\frac{r}{n})}\right]$$

where

FV: The future value of the annuity.

PMT: The payment or periodic deposit.

r: The annual interest rate (in decimal form).

n: The number of compounding periods per year.

t: The number of years.

$$\text{Total deposit} = PMT \times n \times t$$
$$\text{Interest earned from annuity} = FV - \text{Total deposit}$$

Checkpoint: Annuity Due

Another form of annuity is the annuity due, which has deposits at the start of each compounding period. This other annuity type has different formulas and is not addressed in this text.

Example 4.4.2: Future Value of an Ordinary Annuity

Jill has an account that bears 5% interest compounded quarterly. She decides to deposit $\$500$ quarterly into this account.

1. What is the future value of this account, after 35 years?
2. How much money will Jill have put into the account in 35 years?
3. How much interest will Jill have earned?

Answer 1

Here

FV is the future value of the annuity (We do not know)

$PMT = \$500$ is the payment (Periodic deposit)

$r = 0.05$ is the annual interest rate (in decimal form)

$n = 4$ is the number of compounding periods per year

$t = 35$ is the number of years

$$\begin{aligned} FV &= PMT\left[\frac{(1+\frac{r}{n})^{nt}-1}{(\frac{r}{n})}\right] \\ &= 500\left[\frac{(1+\frac{0.05}{4})^{4\times 35}-1}{(\frac{0.05}{4})}\right] \\ &= 500\left[\frac{(1.0125)^{140}-1}{0.0125}\right] \\ &= 500\left[\frac{4.692518676}{0.0125}\right] \\ &= 500 \times 375.4014941 \\ &= 187,700.747 \end{aligned} \tag{4.4.3}$$

So the future value of this account, after 35 years, will be $\$187,700.75$.

Answer 2

$$\begin{aligned} \text{Total deposit} &= PMT \times n \times t \\ &= \$500 \times 4 \times 35 \\ &= \$70,000 \end{aligned}$$

Answer 3

The interest Jill will have earned will be

$$\begin{aligned}\text{Interest earned} &= FV - \text{Total deposit}\\ &= \$187,700.75 - \$70,000\\ &= \$117,700.75\end{aligned}$$

Note: As we did before, you can check your answer using the TVM solver. Put the following values in the given variables to find the future value of the annuity.

$$\begin{aligned}\text{N} &= n \times t = 4 \times 35 = 140\\ \text{I} &= 5 \ \ \text{(Do not put percent)}\\ \text{PV} &= 0 \ \ \text{(We don't have it, so put it as zero)}\\ \text{FV} &= \text{We need to find it.}\\ \text{PMT} &= 500\\ \text{P/Y} &= 4\\ \text{C/Y} &= 4\end{aligned}$$

And solve for FV.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

Your Turn 4.4.2: Find the Future Value from Annuity

For 15 years, you deposit $\$400.00$ every quarter into an account earning 4% interest compounded quarterly. Round your answer to the nearest cent.

How much will you have in the account in 15 years? $ ____

How much money will you have put into the account? $ ____ Hint

How much interest will you have earned? $ ____ Hint

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Who Knew?: Setting Savings Account Interest Rates

There are a number of factors that contribute to the interest rates a bank gives for savings accounts. The interest rate reflects how much the bank values deposits. It also reflects the money that the bank will earn when it lends out money. Finally, interest rates are impacted by the Federal Reserve Bank. When the Fed raises interest rates, so do banks.

People in Mathematics: The Federal Reserve Chairperson

The Federal Reserve Board monitors the risks in the financial system to help ensure a healthy economy for individuals, companies, and communities. The Board oversees the 12 regional reserve banks. The Chairperson of the Federal Reserve Board testifies to Congress twice per year, meets with the Secretary of the Treasury, chairs the Federal Open Market Committee, and is the face of federal monetary policy. Mr. Jerome Powell is the current Chairman of the Federal Reserve Board. Jerome Powell was sworn in as chairman on February $5, 2018$. He had been first nominated to the position by President Donald Trump on November $2, 2017$, and confirmed by the Senate. He was nominated to a second term by President Joe Biden, confirmed by the Senate, and sworn in on May $23, 2022$.

The formula used to calculate the future value of an ordinary annuity is useful for determining the final amount in the account. However, that isn't how planning works. To plan, we need to determine how much to invest in the ordinary annuity each compounding period in order to reach our goal. Fortunately, that formula exists.

Periodic Payment or Periodic Deposits (Or Annuity)

The formula for the amount that needs to be deposited per period.

$$PMT = FV\left[\frac{(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}\right]$$

where

FV: The future value of the annuity,

PMT: The periodic payment or deposits

r: The annual interest rate (in decimal form)

n: The number of compounding periods per year

t: The number of years

Example $4.4.3$: Saving for a Car

Yaroslava wants to save in order to buy a car in 3 years, without taking out a loan. She determines that she'll need $\$35,500$ for the purchase. If she deposits money into an ordinary annuity that yields 4.25% interest compounded monthly.

1. How much will she need to deposit each month?
2. How much money will she have put into the account?
3. How much interest will she have earned?

Answer 1

Yaroslava has a goal and needs to know the payments to make to reach the goal. Her goal is $FV = \$35,500$, with an interest rate $r = 0.0425$, compounded per month so $n = 12$, and for 3 years, making $t = 3$. Substituting into the formula, Yaroslava finds the necessary payment.

$$\begin{aligned} PMT &= FV\left[\frac{(\frac{r}{n})}{(1+\frac{r}{n})^{nt}-1}\right] \\ &= 35,500\left[\frac{(\frac{0.0425}{12})}{(1+\frac{0.0425}{12})^{12\times 3}-1}\right] \\ &= 35,500\left[\frac{0.003541667}{(1.003541667)^{36}-1}\right] \\ &= \frac{125.72916}{0.13572907} \\ &= 926.3248 \end{aligned} \tag{4.4.4}$$

That means if Yaroslava deposits $\$926.33$ every months, after 3 years, her account will grow to $\$35,500$ with 4.25% interest rate.

Answer 2

Yaroslava puts $$926.33 monthly. So here the total deposit will be

$$\begin{aligned}\text{Total deposit} &= PMT \times n \times t \\ &= \$926.33 \times 12 \times 3 \\ &= \$33,347.88\end{aligned}$$

Answer 3

The interest Yaroslava will have earned will be

$$\begin{aligned}\text{Interest earned} &= FV - \text{Total deposit} \\ &= \$35,500 - \$33,347.88 \\ &= \$2,152.12\end{aligned}$$

You can check your work by using the TVM solver. Put the following

$$\begin{aligned}\mathrm{N} &= n \times t = 12 \times 3 = 36 \\ \mathrm{I} &= 4.25 \ \text{(Do not put percent)} \\ \mathrm{PV} &= 0 \ \text{(We don't have it, so put it as a zero.)} \\ \mathrm{FV} &= 35500 \\ \mathrm{PMT} &= \text{We need to find it.} \\ \mathrm{P/Y} &= 12 \\ \mathrm{C/Y} &= 12\end{aligned}$$

And click solve for PMT.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

You will find PMT = $926.33

To reach her goal, Yaroslava would need to deposit $926.33 (ROUND UP) in her account each month.

Your Turn4.4.3: Find Periodic Deposit

Suppose you want to have $700,000.00 for retirement in 25 years. You plan to make regular monthly deposits into an account earning 7% interest compounded monthly. Round your answer to the nearest cent.

How much would you need to deposit in the account each month?

$

How much money will you have put into the account? Hint

$

How much interest will you have earned? Hint

$

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This has been rounded up so that the deposits don't fall short of the goal. However, some round off uses the standard rounding rules:

Distinguish Between Basic Forms of Investments

You can save your money in a safe or a vault (or worse, under the mattress!), but that money does not grow. It would be hard to save enough for retirement that way. What can be done to increase the value of the money you already have?

The answer is to invest it. Use the money that you have to earn more money back. For instance, as we saw in Methods of Savings, you can save it in a bank. Or, to reach loftier goals, invest in something more likely to grow, such as stocks.

A great example of this is Apple stock. Anyone who bought stock in Apple in 1997 and held onto the shares earned a lot of money. To be more specific, $\$100$ worth of Apple shares bought in 1980, when it was first sold to the public, was valued at $\$67,564$ in 2019, or 676 times more! Perhaps you have heard a story like that, of an investment opportunity taken that paid off, or the story of an investment opportunity missed. But such stories are the exceptions.

In this section, we'll investigate bonds, stocks, and mutual funds and their comparative strengths and weaknesses. We close the section with a discussion of retirement savings accounts.

Bonds, stocks, and mutual funds tend to offer higher returns, but to varying degrees, come with higher risks. Stocks and mutual funds also vary in the amount of earnings they generate. Their predicted rates of return on investment are not guaranteed, but educated guesses based on market trends and historical performance.

Bonds

Bonds are issued by big companies and by governments. Selling bonds is an alternative to an institution borrowing from a bank. The funds from the sale of bonds are often used for large projects, such as funding the construction of a new highway or hospital.

Bonds are considered a conservative investment. They are bought for what is known as the issue price. The interest is fixed (does not change) at the time of purchase and is based on the issue price of the bond. The interest rate is often referred to as the coupon rate; the interest paid is often called the coupon yield. The interest paid is often higher than that of savings accounts, and the risk is exceptionally low. The bond is for a fixed term. The end of this time is the maturity date of the bond.

A **bond** is like a loan you give to a company or government. The **face value** (or par value) is the amount of money the bond will pay you back when it matures, usually $1,000. The **coupon rate** is the interest rate the bond pays based on its face value. For example, if a bond has a face value of $1,000 and a 5% coupon rate, it will pay $50 in interest each year. The coupon rate tells you how much income the bond will generate, while the face value tells you how much you will receive at the end of the bond's term.

There are several types of bonds and their return level

- **Treasury bonds** are issued by the federal government: safer, lower returns
- **Municipal bonds** are issued by state and local governments: tax-free, moderate returns
- **Corporate bonds** are issued by major corporations: riskier, higher returns

The **maturity date of a bond** is the **specific date in the future when the bond issuer must repay the full face value (principal) to the investor**

Short-term: 1–3 years

Medium-term: 4–10 years

Long-term: 10–30 years or more

If you buy a 10-year bond on January $1, 2025$. The **maturity date** is **January** $1, 2035$. You will receive coupon payments each year until 2035. On Jan $1, 2035$, you get the **full** $\$1,000$ **face value back**

Who Knew?: Trading Bonds

Bonds are often part of larger investment portfolios. These bonds may be traded. However, the interest paid is based on the price when the bond was bought (the issue price). These bonds can be bought and sold for more or less money than the issue price. If the bond is bought for more than the issue price, the interest is still paid on the issue price, not on the purchase price at the time of the trade. This means the actual return on the bond decreases. If the bond is purchased for less than the issue price, the return on the bond increases.

Example 4.4.4: Bond Investment

Muriel purchases a $\$3,000$ bond with a maturity of 4 years at a fixed coupon rate of 5.5% paid annually.

1. How much is Muriel paid each year, and what is the total amount earned with the bond?
2. What is Mauriel's return on investment?
3. What was Muriel's annual return on investment? Interpret this as compound interest.

Answer 1

The coupon rate is 5.5%. per year.

Money paid in one year is

$$0.055 \times \$3,000 = \$165$$

Each year Muriel receives $\$165$. Total amount earned in the bond is $\$165 \times 4 = \$660$.

Answer 2

Each year, Muriel received $\$165$. She received this money four times, so she earned a total of $\$660$. Muriel's bond face value is

$\$3,000 + \$660 = \$3,660$ and investment on the bond is $3,000$, which is P.

$$\begin{aligned} \text{ROI} &= \frac{\text{FV-P}}{\text{P}} \times 100\% \\ \text{ROI} &= \frac{660}{3000} \times 100\% \\ \text{ROI} &= 22\% \end{aligned} \tag{4.4.5}$$

So 22% is Muriel's ROI.

Answer 3

Muriel purchases a $\$3,000$ bond so her investment $P = \$3,000$. Muriel earned a total of $\$660$.

$FV = \$3,000 + \$660 = \$3,660$. Using that, we find that the annual return is

$$\begin{aligned} \textbf{Annual return} &= \left[\left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1\right] \times 100\% \\ &= \left[\left(\frac{3,660}{3000}\right)^{\frac{1}{4}} - 1\right] \times 100\% \\ &= (1.22^{\frac{1}{4}} - 10 \times 100\% \\ &== 5.0969\% \end{aligned} \tag{4.4.6}$$

5.1% is Muriel's annual return.

The 5.5% bond earned the equivalent of 5.10% compounded annually.

Your Turn 4.4.4: Return on Bond

Alexa purchases a $2,500$ bond with a maturity of 5 years at a fixed coupon rate of 10% paid annually.

1. How much is Muriel paid each year, and what is the total amount earned with the bond?

2. What is Mauriel's return on investment? [ ] %
3. What was Muriel's annual return on investment? [ ] %

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FORMULA: Annual Return (AR)

As you see in the ROI formula above, it does not account for the duration of the investment. A good way to do that is to equate the ROI to an account bearing interest that is compounded annually. The annual return is the **average annual rate**, or the annual percentage yield (APY) that would result in the same amount **if** the interest **were** paid once a year.

The formula for the annual return of the investment

$$\text{AR} = \left[\left(\frac{FV}{P}\right)^{\frac{1}{t}} - 1\right] \times 100\%$$

where

t = the number of years

FV = new value

P = starting principal

AR = Annual Return

Also, if you know the annual return (AR), we can find the value of investment (FV) after t years by the following formula

$$\text{Future Value: FV} = P(1+r)^t \times 100\%$$

Stocks

Stocks are part ownership in a company. They come in units called shares. The performance and earnings of stocks is not guaranteed, which makes them riskier than any other investment discussed earlier. However, they can offer a higher return on investment than other investments. Their value grows in two ways. They offer dividends, which is a portion of the profit made by the company. And the price per share can increase based on how others see the value of the company changing. If the value of the company drops or the company folds, the money invested in the stock also drops.

Most stock transactions are executed through a broker. Brokers' commissions can be a percentage of the value of the trades made or a flat fee. There are full-service brokers who charge higher commission rates, but they also offer financial advice and perform the research that you may not have the time or the expertise to do on your own. A discount broker only executes the stock transactions, buying or selling, so they charge lower rates than full-service brokers. There are also brokers that offer commission-free trading.

An important thing to remember is that stocks may provide a substantial return on investment, but the trade-off is the risk associated with owning them.

Who Knew?: Chapter 11 Bankruptcy and Stocks

In the fall of 2022, the parent company of Regal Theaters, named Cineworld, filed for Chapter 11 bankruptcy. According to news articles, the bankruptcy was necessitated due to its heavy debt load. Generally, a company can file for Chapter 11 bankruptcy to allow them time to reorganize and restructure debts. When this happens, the company, after the Chapter 11 process is over, offers new stock. This makes the previous stock worthless. However, the company may allow an exchange of old stock for a discounted amount of the new stock. This, in effect, reduces (maybe vastly) the wealth held by those who owned the original stock.

Who Knew?: Risk and Volkswagen

The question of risk hovers over every investment. How risky can it get? Volkswagen seems to be a rather safe investment. But in 2015, Volkswagen's stock tumbled 30% over a few days when it was revealed that the company had installed software that altered the emission performance of some of its diesel engines. Volkswagen's hope was that lower emissions would bolster US sales of some of their diesel models. This was a drastic drop, and many investors lost a lot of money. However, the stock has since recovered. This was mild compared to the 65% drop in the Martha Stewart Living Omnimedia stocks.

People in Mathematics: Warren Buffett

Warren Buffett is an investment legend. He began his career as an investment salesman in the $1950s$. He formed Buffett Associates in 1956. In 1965, he was in control of Berkshire Hathaway, which began as a merger between two textile companies. In his role there, he began to invest in a variety of companies. It is now a conglomerate holding company, and fully owns GEICO, Duracell, Diary Queen, and other large companies.

His investment philosophy involves finding stocks and bonds from companies that have high intrinsic worth compared to their stock or bond prices. This means he focuses not on the supply and demand side of stock investing, but instead on the company's worth in total. Using this philosophy, he has become one of the world's most successful investors.

Trading Platform

A stock trading platform is a digital tool—typically a website or mobile app—that enables investors to buy and sell stocks, mutual funds, bonds, and other securities. These platforms connect users to financial markets and provide the necessary tools to manage their investments. Some popular stock trading platforms include Robinhood, E*TRADE, Charles Schwab, Fidelity, TD Ameritrade, Vanguard, and Webull.

On the other hand, a crypto trading platform is an online exchange—also available as a website or mobile app—where you can buy, sell, trade, and store cryptocurrencies such as Bitcoin, Ethereum, Solana, and many others. Some well-known crypto trading platforms include Coinbase, Binance, Kraken, and Crypto.com.

Reading Stock Tables

Information about particular stocks is contained in **stock tables**. This information includes how much the stock is selling for, and its high and low values from the past year 52 weeks).

52-Week High Low	The highest and the lowest share prices during the past 52 weeks.
SYM	The company name and the ticker symbol are used to identify the company.
DIV	The current annual dividend per share.
Yld %	$\frac{\text{annual divided}}{\text{share price}} \times 100\%$
Vol	The number of shares that have been traded today.
Open/High/Low	Opening, highest, and lowest prices so far today
Close	The price at which the stock traded at the close of the in prior trading day.
Net Chg	The difference between the prior trading period's closing price and the current trading period's closing price
Market Cap	Total stock value of the company.
Share Outstanding	The total number of shares that exist for the company to trade.
P/E	The share price divided by the earnings per share over the past year (dd indicates loss.)

The formulas for yield and price to earnings is a good way to measure how much the stock returns per share. Their values are calculated in the stock table, but deserve attention here.

FORMULA: P/E Ratio and EPS

$$\begin{aligned} \mathrm{P/E} &= \frac{\text{Share price}}{\text{Dividend}} \\ \mathrm{Yld} &= \frac{\text{Annual Dividend}}{\text{Share price}} \times 100\% \\ \text{Earning per share} &= \frac{\text{Share price in opening day}}{\text{P/E ratio}} \end{aligned} \tag{4.4.7}$$

It should be noted that the price of a stock increases and decreases every moment, and so these values change as the share price changes.

Example $4.4.5$: Computing Percent Yield

Find the Annual percent yield for a stock with a price of $\$30.69$ and quarterly dividends of $\$1.48$ per share.

Answer

Substituting the values for price, $\$30.69$, and annual dividend, $\$4 \times 1.48$,, becuase quarterly dividends of $\$1.48$ per share. We find the Annual percent yield for the stock to be

$$\begin{aligned} \text{Yld} &= \frac{4 \times \$1.48}{\$30.69} \times 100\% \\ &= 19.28\% \end{aligned} \tag{4.4.8}$$

So the percent yield for a stock is 19.28%

Your Turn \ (\PageIndex{5}\): Annual Percent Yield

In 2020, Caterpillar Inc. paid four quarterly dividends of $\$1.03$ per share. In March of 2020, shares of Caterpillar Inc. once traded for $\$100.38$.

a. What was the annual percent yield on the stock when it traded for $100.38? *(Round to the nearest thousandths of a percent.)*

Annual percent yield: ______ %

b. What was the annual percent yield when the stock traded at 151.26? *(Round to the nearest thousandths of a percent.)*

Annual percent yield: ______ %

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Example 4.4.6: Reading an Online Stock Table

Consider the stock table Figure $4.4.1$, and answer the questions based on the table.

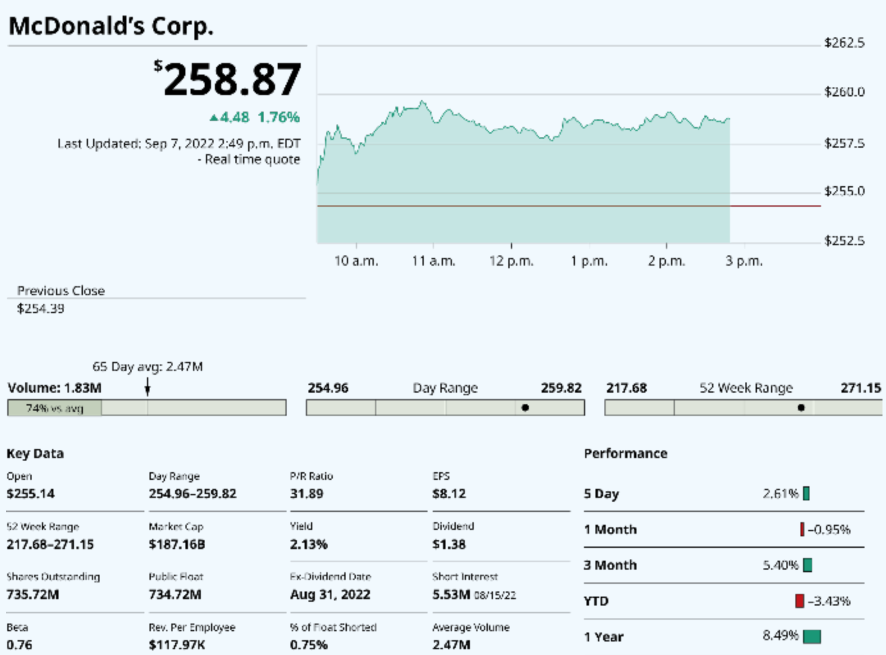

Figure $4.4.1$: Key data for McDonald's stock 9-7-2022 (data source: marketwatch.com)

1. What is the current price of McDonald's Corp on this date?
2. What is the price of the stock (share) at the start of the day?
3. What is the 52-week high? 52-week low?

4. When is the dividend expected?
5. What is its yield?
6. Sujan owns 500 shares of McDonald's Corp stock. How much will Sujan earn in dividends in the quarter?
7. What are the earnings per share (EPS)?
8. What is the total value of the company?

Answer

1. Looking at the table, the current price of a share is $\$258.87$.
2. The high was $\$271.15$, and the low was $\$217.68$.
3. $\$255.14$
4. August $15, 2022$
5. 2.13
6. $\$500 \times 1.38 = \$690$
7. The EPS value is $\$8.12$.
8. $\$187.16$ billion

✔ Example 4.4.7: Stock Price Increases

Vincent buys 100 stocks in the REM company for $\$21.87$ per share. One year later, he sells those 100 shares for $\$29.15$ per share.

1. How much money did Vincent make?
2. What was his return on investment for that one year?

Answer

1. Vincent spent $\$21.87$ per share to buy the stock. The total he spent on the stock was $\$21.87 \times 100 = \$2,187$. When he sold the stock, the price was $\$29.15$, so he receive $\$29.15 \times 100 = \$2,915$.
2. He made $\$2,915 - 2,187 = \$728$. His return on investment was

$$\begin{aligned} \text{ROI} &= \frac{\text{earning}}{\text{original price}} \\ &= \frac{\$728}{\$2,187} \\ &= 0.3329 \\ &= 33.29\% \end{aligned} \tag{4.4.9}$$

✎ Your Turn \(\PageIndex{7): Find ROI

You buy 200 shares of stock in a company.

a. During the first 6 months, the stocks paid **dividends** of $1.35 per share. How much did you earn **in dividends** during the first 6 months? Round your answer to two decimal places if necessary.

$ ____________

b. Two years later, you decide to sell all 200 shares of the stock. If you purchased the stock for $29.41 per share, and the selling price is now $39.8 per share, how much will you make off the sale? Round your answer to two decimal places if necessary.

$ ____________

c. The "return on investment" is a percentage, computed as:

$$\frac{\text{Current Value - Cost of Investment}}{\text{Cost of Investment}}.$$

Based on your work in part (b), what is the approximate return on investment for those 2 years? *Give your answer as a percent, rounded to the nearest whole number.*

____________ %

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Mutual Funds

A **mutual fund** is a collection of investments that are combined into a single portfolio. When you buy shares of a mutual fund, your money is pooled with the assets of other investors. This pooled money is invested in stocks, bonds, money market instruments, and other assets. Mutual funds are typically managed by professional money managers who allocate the fund's assets and strive to generate capital gains or income for the fund's investors. Some mutual funds are

Examples of some mutual funds

1. Vanguard 500 Index Fund (VFIAX)
2. Fidelity 500 Index Fund (FXAIX)
3. Schwab S&P 500 Index Fund (SWPPX)

An example of a mutual fund, VFIAX, is provided online below, with some parts labeled.

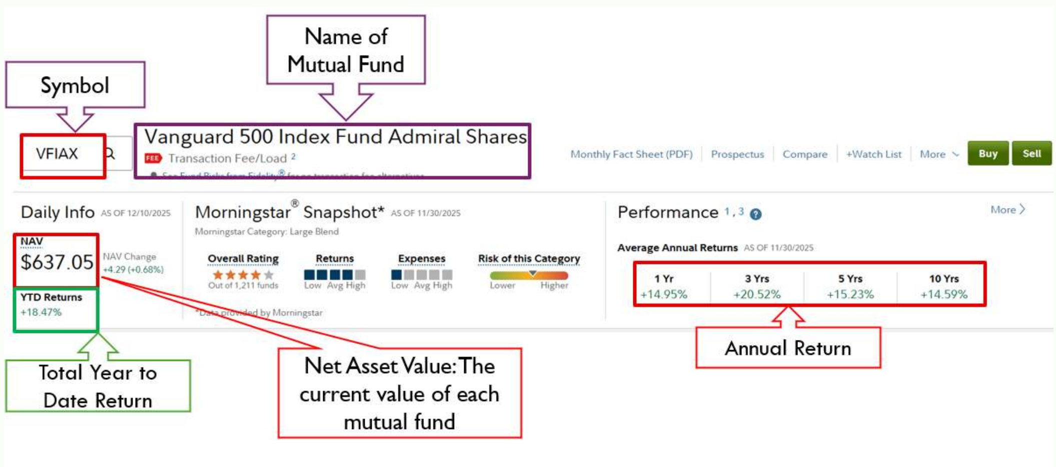
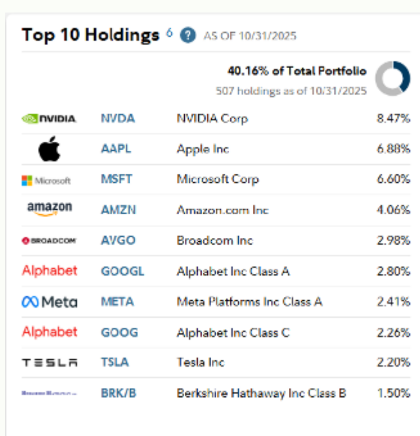


Figure 4.4.1: VFIAX Mutual Fund



Top 10 Holdings AS OF 10/31/2025

40.16% of Total Portfolio
507 holdings as of 10/31/2025

NVIDIA	NVDA	NVIDIA Corp	8.47%
	AAPL	Apple Inc	6.88%
Microsoft	MSFT	Microsoft Corp	6.60%
amazon	AMZN	Amazon.com Inc	4.06%
BROADCOM	AVGO	Broadcom Inc	2.98%
Alphabet	GOOGL	Alphabet Inc Class A	2.80%
Meta	META	Meta Platforms Inc Class A	2.41%
Alphabet	GOOG	Alphabet Inc Class C	2.26%
TESLA	TSLA	Tesla Inc	2.20%
	BRK/B	Berkshire Hathaway Inc Class B	1.50%

Figure 4.4.1:Top 10 holding

As we mentioned, a **mutual fund** is a collection of investments. Figure 4.4.2 shows the top 10 collection of stocks in the VFIAX mutual fund. Those collection is called holdings in a mutual fund.

A key benefit of mutual funds is that they allow small or individual investors to invest in professionally managed portfolios of equities, bonds, and other securities. This means that each shareholder participates proportionally in the fund's gains or losses. The performance of a mutual fund is usually stated as how much the mutual fund's total value has increased or decreased. Since there are many different investments within the mutual fund, the risk is significantly reduced compared to direct ownership of stocks. Even so, mutual funds historically perform well and can earn more than 10% annually.

The investments that comprise a mutual fund are structured and managed to align with the stated investment objectives, as specified in its **prospectus**. A prospectus is a document that provides information about a mutual fund. Before buying shares of a mutual fund, consult its prospectus to consider its goals and strategies and determine if they align with your objectives and values. Additionally, research any associated fees.

✔ Example 4.4.8: Reading Mutual Fund Table

Answer the following questions based on an actual **Fidelity Select Communication Service mutual** fund.

a. If you invest $\$5,000$ in this fund today, how many shares will you be able to buy?
b. If you had invested $\$5,000$ in this fund three years ago, how much would it be worth today?
c. If you had invested $\$5,000$ in this fund ten years ago, how much would it be worth today?

Answer

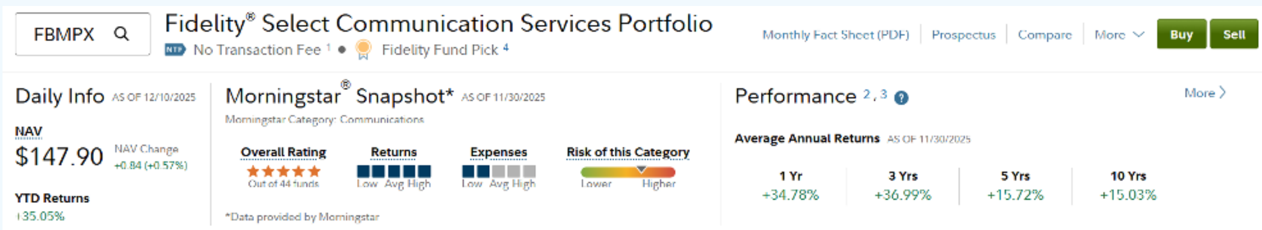


Figure $4.4.1$: Copy and Paste Caption here. (Copyright; author via source)

a. Since each shares cost $\$147.90$, the umber of share we can buy with $\$5000$ is: $\frac{5000}{147.90} = 33.8066$

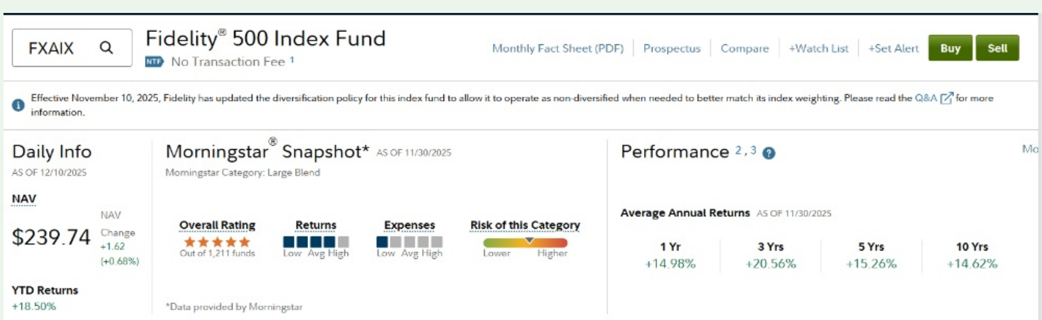
b. Here we need to find future value of $\$5,000$ investment after 3 years.

$$\begin{aligned}\text{Future Value: FV} &= P(1+r)^t \\ &= 5,000(1+0.3699)^3 \\ &= 12853.94986\end{aligned}$$

c. Here we need to find future value of $\$5,000$ investment after 10 years.

$$\begin{aligned}\text{Future Value: FV} &= P(1+r)^t \\ &= 5,000(1+0.1503)^{10} \\ &= 20280.6188\end{aligned}$$

✎ Your Turn 4.4.8: Find Future Value of Mutual Fund Investment



(You can enlarge the image if you did not see the number in the image. Sorry about that.)

Answer the following questions based on an actual Fidelity Select Communication Service (FXAIX) mutual fund. Round all answers to two decimal places.

1. If you invest $\$7,000$ in this fund today, how many shares will you be able to buy?

(It could be a fractional share)

2. If you had invested $\$7,000$ in this fund three years ago, how much would it be worth today?

$

3. If you had invested $\$7,000$ in this fund ten years ago, how much would it be worth today?

$

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Retirement Savings Plans

We close this section by investigating the three main forms of retirement savings accounts: traditional individual retirement accounts (IRAs), Roth IRAs, and $401(\mathrm{k})$ accounts. Each has distinct characteristics that are suited to the needs of different investors.

Individual Retirement Accounts

A **traditional IRA** lets you contribute up to an amount set by the government, which may change from year to year. For 2026, if you're 49 or younger, you can contribute up to $\$7,500$, and if you're 50 or older, you can contribute up to $\$8,600$.

Anyone is eligible to contribute to a traditional IRA, regardless of their income level. Your money grows tax-deferred(You pay taxes later when you withdraw the money in retirement), but withdrawals after age $59½$ are taxed at current rates. Traditional IRAs also allow you to use the contribution itself as a deduction on a current-year tax return.

Roth IRAs allow contributions at the same levels as traditional IRAs. For 2026, if you're 49 or younger, you can contribute up to $\$7,500$, and if you're 50 or older, you can contribute up to $\$8,600$. However, to be eligible to make contributions, your earned income must be below a certain level. A Roth IRA allows after-tax contributions. In other words, the contribution itself is not tax-deductible, unlike the traditional IRA. However, your money grows tax-free. If you make no withdrawals until you are age $59½$, there are no penalties. IRAs pay a modest interest rate.

In either case, IRA deposits have to be from earned income, which in effect means if your earned income is over $\$6,000$ ($\$7,000$) then you can deposit the maximum.

401(k) Accounts

Your employer may offer a retirement account to you. These are often in the form of a **401(k)** account. There are traditional and Roth $401(\mathrm{k})$ accounts, which differ in how they are taxed, much as with other IRAs. In the traditional $401(\mathrm{k})$ plans, the money is deposited before tax is assessed, which means you do not pay taxes on this money. However, that means when money is withdrawn, it is taxed. These accounts are similar to mutual funds in that the money is invested in a wide range of assets, spreading the risk.

One of the perks some employers offer is to match some amount of your contributions to the $401(\mathrm{k})$ plan. For instance, they may match your deposits up to 5% of your income. This is an instant 100% return on the money that was matched.

$401(\mathrm{k})$ plans with matching funds provide great value, as their rates of return are high compared to savings accounts, and are less risky than stocks since such funds invest across many investment vehicles. The next example demonstrates the power of constant deposits into a $401(\mathrm{k})$ plan that has some employer match.

✔ Example 4.4.9: Contribution in 401K

Alice signs up for her employer-based $401(\mathrm{k})$. The employer matches any $401(\mathrm{k})$ contribution up to 6% of the employee's salary. Alice's annual salary is $\$51,600$.

1. What is the maximum amount that Alice can deposit, which will be fully matched by the company?
2. How much total will be deposited into Alice's account if she deposits the full 6%?

Answer

1. The employer will match up to 6% of any employee's salary. The 6% of Alice's salary is

$$0.06 \times \$51,600 = \$3,096$$

So Alice can deposit up to $\$3,096$ and receive that amount in matching funds in her account.

2. Alice's contribution plus the company's contribution is

$$\$3,096+\$3,096=\$6,192$$

which is the total that is deposited into Alice's account.

3. She earns a 100% return on the day she deposits her $\$3,096$.

Your Turn $4.4.9$: Contribution to 401 K

James signs up for his employer-based 401 (k). The employer matches any 401 (k) contribution up to 9.25% of the employee's salary. Jameis' annual salary is $\$59,000$. Round your answer to two decimal places.

What is the most money that Jamie can deposit that will be fully matched by the company?

$ ______

How much total will be deposited into James' account if he deposits the full 9.25% ?

$ ______

How much total will be deposited into James' account if he deposits only 5.25% ?

$ ______

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Note: Which Investment is Riskier?

Which of the types of investments listed would be considered to have the **highest** level of risk?

Bonds are considered the lowest-risk investment, while stocks are considered the highest-risk investment.

4.5: Basics of Loans (Mortgage, Cars, and Credit Cards)

Figure 4.5.1: Loans are contracts that allow people to buy now but require them to pay more. (credit: "Closing" by Tim Pierce/Flickr, CC BY 2.0)

Learning Objectives

1. Understand how credit scoring works.
2. Calculate the payment necessary to pay off a loan.
3. Read an amortization table.
4. Calculate the monthly payment for a mortgage and related interest costs.
5. Solve application problems involving the affordability of a mortgage.
6. Read and understand the basic parts of a credit card statement.
7. Compute interest, balance due, and minimum payment due for a credit card.

New car envy is real. Some people look at a new car and feel that they, too, should have a new car. The search begins. They find the model they want, in the color they want, with the features they want, and then they look at the price. That's often the point where the new car fever breaks, and the reality of borrowing money to purchase the car enters the picture. This borrowing takes the form of a loan.

In this section, we look at the basics of loans, including terminology, credit scores, payments, and the cost of borrowing money.

Reasons for Loans

Even if you want a new car because you need one, or if you need a new computer since your current one no longer runs as fast or smoothly as you would like, or you need a new chimney because the one on your house is crumbling, it's likely you do not have that cost in cash. Those are very large purchases. How do you buy that if you don 't have the cash? You borrow the money. And to help you with your purchase, the company or bank charges you interest.

Loans are taken out to pay for goods or services when a person does not have the cash to pay for the goods or services. We are most familiar with loans for the big purchases in our lives, such as cars, homes, and a college education. Loans are also taken out to pay for repairs, smaller purchases, and home goods like furniture and computers.

Loans can come from a bank, or from the company selling the goods or providing the service. The borrower agrees to pay back more than the amount borrowed. So there is a cost to borrowing that should be considered when deciding on a purchase bought with credit or borrowed money. Even using a credit card is a form of a loan.

Essentially, a loan can be obtained for just about any purchase, large or small, that has a cost beyond a person's cash on hand.

The Terminology of Loans

There are many words and acronyms that are used in relation to loans. A few are below.

APR is the annual percentage rate. It is the annual interest paid on the money that was borrowed. The principal is the total amount of the loan that has been financed. A fixed-interest rate loan has an interest rate that remains constant throughout the life of the loan. A variable interest rate loan has an interest rate that may change during the life of the loan. The **term** of the loan refers to the duration for which the borrower is required to repay the loan. An installment loan is a loan with a fixed period, and the borrower pays a fixed amount per period until the loan is paid off. The periods are almost uniformly monthly. Loan amortization is the process used to calculate how much of each payment will be applied to the principal and how much is applied to interest. Revolving credit, also known as open-end credit, is how most credit cards work, but is also a kind of loan account. (We will learn about credit cards in Credit Cards) You can use up to some specified value, called the limit, any way you want, and as long as you pay the issuer of the credit according to their terms, you can keep borrowing from this account.

These and other terminologies can be researched further at Forbes.

Who Knew?: Credit Scores

Not everyone pays the same interest for the same loan. One person might get an APR of 2.9% while another pays 6.9%. These rates are based on your credit score.

Data about you and your credit is collected by three credit bureaus—Experian, Equifax, and TransUnion. They calculate your score using one of two main models: FICO and Vantage Score. The score they develop is based on the following categories:

- **Payment History:** Making your payments on time and not missing payments is by far the most important factor. All three credit types—revolving, installment, and open—contribute to this factor.
- **Credit Utilization or Amount Owed:** How much do you owe on your credit card accounts? This category is concerned with the ratio of how much you owe on revolving credit accounts relative to your available credit, also known as your credit utilization ratio.
  This is the only category that depends solely on your revolving credit accounts.
- **Length of Credit History:** This is the average age of your credit history, including the age of the oldest and newest accounts. All three types of credit accounts play a role in this category.
- **Credit Mix:** This number represents the different types of credit accounts you have, such as credit cards, car loans, mortgages, and whether you are successful managing both revolving and installment accounts.
- **New Credit:** Have you recently opened a new account or applied for new credit? Lenders want to know how much new credit you are taking on. If you plan to buy a car and make another large purchase with a credit card, consider spacing these purchases out.

If you have done well in these categories, your credit score will be high, and you will qualify for lower interest rates because you are not perceived as being a risky investment. However, if you do poorly in these categories, your score will be low, and you will pay higher interest rates since you present a greater risk.

Calculating Loan Payments

Loan payments are made up of two components. One component is the interest that accrued during the payment period. The other component is part of the principal. This should remind you of partial payments from Simple Interest.

Over the course of the loan, the amount of principal remaining to be paid decreases. The interest you pay in a month is based on the remaining principal, just as in the partial payments of Simple Interest.

The payment of the loan has to be such that the principal of the loan is paid off with the last payment. In any period, the amount of interest is defined by the formula above, but changes from period to period since the principal is decreasing with each payment. The trick is knowing how much principal should be paid each payment so that the loan is paid off at the stated time. Fortunately, that is found using the following formula.

FORMULA: Monthly Payment for Loans

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

The payment, PMT, per period to pay down a loan with beginning principal, P is where r is the annual interest rate in decimal form, t is the number of years of the loan, and n is the number of payments per year (typically, loans are paid monthly, making $n = 12$.) Note, payment to lenders is always **rounded UP** to the next penny unless otherwise mentioned.

Example 4.5.1: Calculating the Payment for a Loan

In the following, calculate the payment necessary to pay off the loan with the given details. The payments are monthly.

1. A car loan was taken out for $\$28,500$ at an annual interest rate of 3.99% for 5 years.

2. A home loan was taken out for $\$136,700$ with an annual interest rate of 5.75% for 15 years.

Answer

1. The loan is for $\$28,500$, which is the principal. The rate is 3.99%, so $r = 0.0399$. The term of the loan is 5 years, so $t = 5$. Monthly payments mean $n = 12$.

   Substituting these values into the formula

$$\begin{aligned} PMT &= \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \\ &= \frac{28,500\left(\frac{0.0399}{12}\right)}{\left[1-\left(1+\frac{0.0399}{12}\right)^{-12\times 5}\right]} \\ &= \frac{28,500\,(0.003325)}{\left[1-(1+0.003325)^{-60}\right]} \\ &= \frac{94.7625}{\left[1-(1.003325)^{-60}\right]} \\ &= \frac{94.7625}{1-0.8194113482} \\ &= \frac{94.7625}{0.1805886518} \\ &= 524.7422751 \end{aligned} \tag{4.5.1}$$

The monthly payment needed is $\$524.75$.

We can also find PMT using the TVM calculator: Put the following values for the variable

$$\begin{aligned} \mathrm{N} &= n\times t = 12\times 5 = 60 \\ \mathrm{I} &= 3.99 \ \text{(Do not put percent)} \\ \mathrm{PV} &= 28,500 \ \text{(LOAN)} \\ \mathrm{FV} &= 0 \\ \mathrm{PMT} &= \ \text{We need to find it.} \\ \mathrm{P/Y} &= 12 \\ \mathrm{C/Y} &= 12 \end{aligned}$$

And click Solve for PMT.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

2. As we did above, we use the same formula to find the monthly payment, and we can verify your answer using the TVM calculator.

Put the following values for the variable

$$\begin{aligned}
N &= n \times t = 12 \times 15 = 180 \\
I &= 5.75 \;\; \text{(Do not put percent)} \\
PV &= 136,700 \text{ (LOAN)} \\
FV &= 0 \\
PMT &= \text{ We need to find it.} \\
P/Y &= 12 \\
C/Y &= 12
\end{aligned}$$

And click "Solve for PMT". The monthly payment needed is $\$1,135.18$.

Your Turn $4.5.1$: Find the Monthly Payment

You want to buy a house. The loan amount will be $\$220,000.00$. A bank is offering a 4% interest rate for 180 months (15 years). What will your monthly payments be?

My monthly payment will be $ ______.

Round to the nearest cent as appropriate.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

Question Help: Video Video

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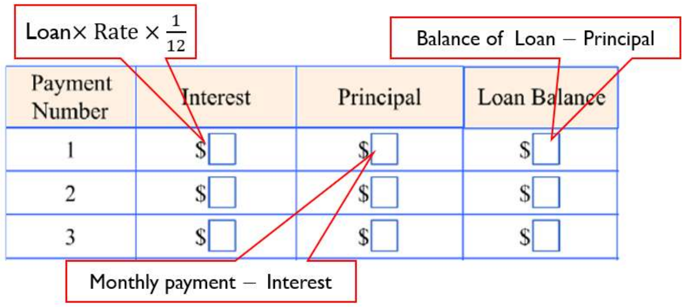
Reading Amortization Tables

An **amortization table** or amortization schedule is a table that provides the details of the periodic payments for a loan, where the payments are applied to both the principal and the interest. The principal of the loan is paid down over the life of the loan. Typically, the payments each period are equal. Importantly, the second column will show how much of each payment is used for interest, the third column will show how much is applied to the outstanding principal, and the fourth column will show the remaining principal or balance. As you see in the following table.



Formula to Find the Amortization Table

Interest Payment = $P \times r \times \frac{1}{12}$: For one month.

Principal Payment = Monthly PMT – Interest Payment

Balance of loan = Principal BAL (Loan) - Principal payment

Where P is the loan amount, r is the annual interest rate, and t is one month

We will find the first two rows of example $4.5.3$ using the above formula.

For the **FIRST** month

$$\begin{aligned}\text{Interest} &= P \times r \times \frac{1}{12} \\ &= \$10,000 \times 0.0475 \times \frac{1}{12} \\ &= \$39.58\end{aligned}$$

$$\begin{aligned}\text{Principal Payment} &= \text{Monthly PMT–Interest Payment} \\ &= \$64.62 - \$39.58 \\ &= \$25.04\end{aligned}$$

$$\begin{aligned}\text{Balance of loan} &= \text{Principal BAL (Loan)} - \text{Principal payment} \\ &= \$10,000 - \$25.04 \\ &= \$9,974.96\end{aligned}$$

For the **SECOND** month

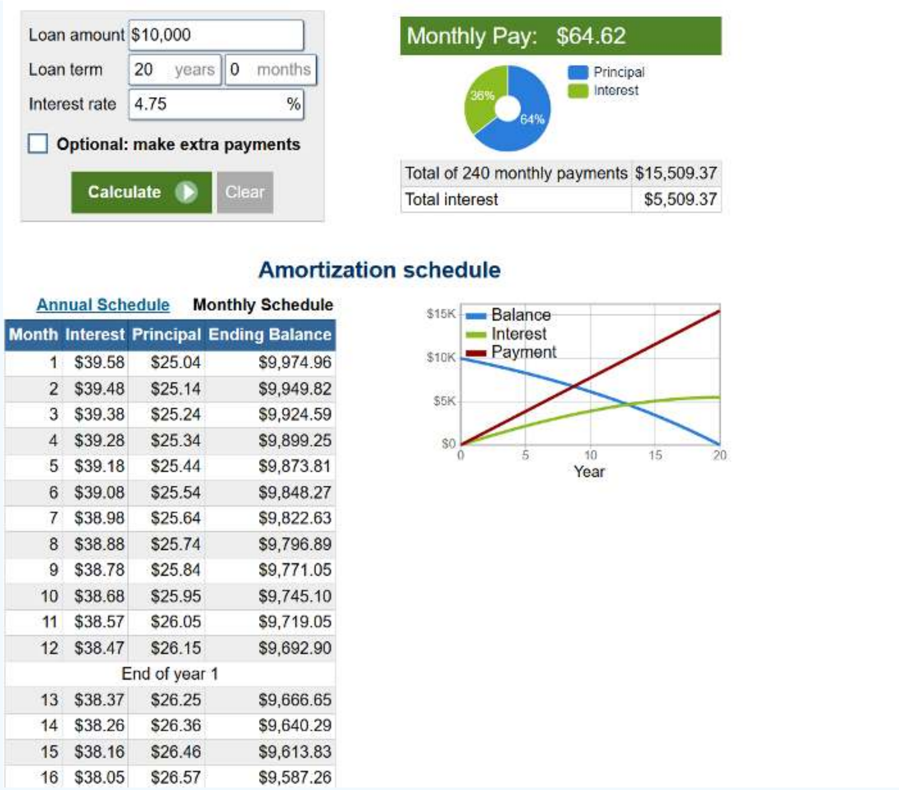
$$\begin{aligned}\text{Interest} &= P \times r \times \frac{1}{12} \\ &= \$9,974.96 \times 0.0475 \times \frac{1}{12} \\ &= \$39.48\end{aligned}$$

$$\begin{aligned}\text{Principal Payment} &= \text{Monthly PMT–Interest Payment} \\ &= \$64.62 - \$39.48 \\ &= \$25.14\end{aligned}$$

$$\begin{aligned}\text{Balance of loan} &= \text{Principal BAL (Loan)} - \text{Principal payment} \\ &= \$9,974.96 - \$25.14 \\ &= \$9,949.82\end{aligned}$$

Example $4.5.2$: Reading from an Amortization Table

Using the partial amortization table $4.5.2$, answer the following questions.



Month	Interest	Principal	Ending Balance
1	$39.58	$25.04	$9,974.96
2	$39.48	$25.14	$9,949.82
3	$39.38	$25.24	$9,924.59
4	$39.28	$25.34	$9,899.25
5	$39.18	$25.44	$9,873.81
6	$39.08	$25.54	$9,848.27
7	$38.98	$25.64	$9,822.63
8	$38.88	$25.74	$9,796.89
9	$38.78	$25.84	$9,771.05
10	$38.68	$25.95	$9,745.10
11	$38.57	$26.05	$9,719.05
12	$38.47	$26.15	$9,692.90
End of year 1			
13	$38.37	$26.25	$9,666.65
14	$38.26	$26.36	$9,640.29
15	$38.16	$26.46	$9,613.83
16	$38.05	$26.57	$9,587.26

Figure $4.5.2$: Amortization table

1. What is the loan amount (principal), the interest rate, and the term of the loan?
2. How much is the monthly payment?
3. How much remaining balance is there after the payment in month 15?
4. How much was the interest on payment 10?
5. Looking at the table or graph, what do you conclude about the interest payment and the principal payment in each month?
6. How much interest is paid when the loan is paid of after 20 years?

Answer

1. Reading the values at the top of the table, we see the principal is $\$10,000$, the interest rate is 4.75% and the term is 20 years.
2. The monthly payment is listed below the term of the loan, and is $\$64.62$.
3. $\$9,613.83$
4. $\$38.68$
5. Interest payment decreases each month, while the principal payment increases each month
6. $\$5509.37$

Your Turn $4.5.2$: Amortization Table

Alicia graduates from college with a loan of $25,000, which she will repay with equal monthly payments over the next 15 years. The interest rate is 4.2%.

How much is Alicia's monthly payment? **Round your answer to the nearest cent.**

$

Fill in the amortization table below for the first two months of the loan. **Round entries to the nearest cent.**

Payment Number	Interest Payment($)	Principal Payment($)	Balance
			$25,000
1			
2			

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Buying House

After renting an apartment for 10 years, you realize that it may be time to purchase a home. Your job is stable, and you could use more space. It is time to investigate becoming a homeowner. What are the things that you must consider, and what is the financial benefit of owning as opposed to renting? This section is about the advantages, disadvantages, and costs of homeownership as opposed to renting.

Advantages and Disadvantages of Renting

When renting, you will likely sign a lease, which is a contract between a renter and a landlord. A landlord is the person or company that owns property that is rented. The lease will detail your responsibilities, restrictions on activities, deposits, fees, maintenance, repairs, and rent during the term of the lease. It also defines what your landlord can and cannot do with the property while you occupy it.

There are advantages to renting, but also some disadvantages.

Advantages	Disadvantages
• Lower cost. • Short-term commitment. • Little to no maintenance cost. The landlord pays for or performs most maintenance. • You need not stay at the end of the lease. Once the lease term is over (the lease is up), you are not obligated to stay. • If renting in an apartment complex, there may be a pool, gym, or community room for renters to use.	• No tax incentives. • Housing cost is not fixed. When the lease is up, the rent can change. • No equity. When you are done living in a rental, you have built no value. • Restrictions on occupants. There may be a limit on how many can live in the apartment. • Restrictions on decorating. The property is not yours, so any decorating or improvements need the landlord's permission. • Limits on pets. Permission for pets and their number and type will be set forth in the lease. • May not be able to remain when the lease term is over. The landlord can, at the end of your lease, invite you to leave. • The building may be sold, and the new landlord may institute changes to the lease when the previous lease expires.

Renting has fees to be paid at the start of the lease. Typically, when you rent, you will pay the first month's rent, last month's rent, and a security deposit. A security deposit is a sum of money that the landlord holds until the renter leaves the rental property. The deposit will cover repairs for any damage to the apartment during the renter's stay, but it may be returned if the apartment is in good condition upon departure. If your landlord runs a credit check on you, the landlord may charge you for that.

Advantages and Disadvantages of Buying a Home

The advantages to buying a home mirror the disadvantages of renting, and the disadvantages of home ownership mirror the advantages of renting.

Advantages	Disadvantages
• There are tax incentives. The interest you pay for your mortgage (more on that later) is deductible on your federal income tax. • There are no restrictions on pets or occupants unless laws in your area specify limits for homes. • You can redecorate any way you wish, limited only by the laws in your area. • Once your mortgage is set with a fixed interest rate, your housing cost is fixed. • Your home grows equity, that is, the difference between what you owe and what the house is worth grows. You can use the equity to secure loans, and you recover the equity (and more if you're fortunate) when you sell the house. • As long as you pay your mortgage and maintain the home to the standards of your community, you can stay as long as you wish.	• The cost is higher than renting. Mortgages and associated costs are typically higher than rent for a similar living space. • The owner is responsible for upkeep, maintenance, and repairs. These can be extremely costly. • The owner cannot walk away from the property. It can be sold, but simply leaving the property, especially if not paid off yet, has serious consequences.

The big question of affordability looms large over the decision to rent or buy. Renting, strictly from an affordability viewpoint, comes with much less initial outlay and smaller commitment. If you do not have sufficient income to regularly save for possibly expensive repairs, or your credit isn't quite as good as it needs to be, then renting may be the best choice. Of course, even if you can afford to buy a home, you may choose to rent based on the comparative advantages.

Buying a home really involves two buyers. You and the mortgage company. The mortgage company has an interest in the home, as they are providing the funds for the home. They want to protect their investment, and many fees are about the bank as much as the buyer. They fund a mortgage based on the value they assign the property. Not you. This means they will want some certainty that the home is sound and that you are a good investment.

Who Knew?: Closing Costs

When a home is bought, there are many costs that need to be paid at the time of purchase, which are lumped under the term closing costs. At the start of 2022, the average closing costs for a single-family home exceeded $\$6,800$. These costs include:

- The appraisal fee is what is paid to someone to establish the home's worth. The value of the home to the bank may differ from what the home is listed for, or what an app tells you the home is worth. It may run approximately $\$350$.
- The home inspection fee. The inspection should reveal any problems with the house that will need to be fixed either before or after you obtain the home.
- The title search. There is a records search to ensure there are no issues with who actually owns the property. It can cost about 0.5% to 1% of the amount you are financing.
- Prepaid taxes. You will need to pay about 6 months of taxes at the time of purchase.
- The credit report fee. This is a fee for checking your credit. You might pay $\$25$ or more for this.
- The origination fee. This is the price the mortgage company charges you to cover the costs of creating the mortgage. This could be 0.5% to 1% (or more) of the amount you are borrowing.
- The application fee. This is just a processing fee and could come to several hundred dollars.
- The underwriting fee. This covers the cost of verifying your financial qualifications. It could be a flat fee or some small percentage of the amount financed. Such as 0.5% or 1%.
- Attorney fees. If you use an attorney, you will have to pay the attorney.
- State of local fees. This may include a filing fee charged by the county or municipality in which you reside.

That's a long list, and it is not even complete. When buying, be prepared to see these costs. It can be surprising. But in the end, you will have equity in the home, which means when you sell your home, you will get some of your money back.

In the end, you must weigh your options and carefully consider your priorities in choosing to rent or buy a home.

Mortgages

Some people will purchase a home or condo with cash, but the majority of people will apply for a mortgage. A mortgage is a long-term loan, and the property itself is the security. The bank decides the minimum down payment (with your input), the payment schedule, the duration of the loan, whether the loan can be assumed by another party, and the penalty for late payments. The title of the home belongs to the bank.

Since a mortgage is a loan, everything about loans from The Basics of Loans holds true, including the formula for the payments. The term related to mortage.

Down Payment	This is the **initial amount of money** you pay upfront when purchasing something such as a house or a car.
Mortgage	This is a **loan used to buy a home or property**. You borrow money from a bank or lender, and in return, you agree to pay it back over time with interest. $$\text{Morgage} = \text{Sale Price} - \text{down payment}$$
Monthly Payment	This is the amount you pay each month, which includes principal plus interest for the loan you took.
Closing points	Generally paid by the buyer at the time of closing the home. It is part of the loan. 1 point = 1% of the loan amount. 2 point = 2% of the loan amount.... so on.
Total paid	This is the total amount of your payments over the life of the loan
Interest paid (Cost of financing)	This is the difference between the principal of the mortgage and the total paid over the life of the mortgage.
Amortization table	A table that shows how a loan is paid off over time through regular payments. It breaks each payment into three parts: principal payment, Interest payment, and remaining balance.
Escrow account	An account is a separate account held by a neutral third party (usually your mortgage lender) to manage and pay certain property-related expenses on your behalf, mainly property taxes and homeowners' insurance.

Monthly Mortgage Payments

The formula to calculate your monthly payments of principal and interest uses APR as the annual interest rate.

FORMULA: Monthly Mortgage Payment

The payment, PMT, per month to pay down a mortgage with a beginning principal P (Loan or mortgage)is

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]}$$

where r is the annual interest rate in decimal form and t is the number of years of the payment. Here $n=12$. Note, payment to lenders is always rounded UP to the next penny, unless it says otherwise.

Example 4.5.3: 30-Year Mortgage

Evan buys a house. His 30-year mortgage comes to $\$132,650$ with 4.8% interest. Find Evan's monthly payments.

Answer

Using the information above, $P=\$132,650$, $r=0.048$ and $< t=30$. Substituting those values into the formula

$$\begin{aligned} PMT &= \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \\ &= \frac{132,650\left(\frac{0.048}{12}\right)}{\left[1-\left(1+\frac{0.048}{12}\right)^{-12\times30}\right]} \\ &= \frac{132,650\,(0.004)}{\left[1-(1+0.004)^{-360}\right]} \\ &= \frac{530.6}{\left[1-(1.004)^{-360}\right]} \\ &= \frac{530.6}{1-0.2376092748} \\ &= \frac{530.6}{0.7623907252} \\ &= 695.9685926 \end{aligned} \tag{4.5.2}$$

His mortgage payment is $\$695.97$.

We can check our answer using TVM calculator:

$$N = n \times t = 12 \times 30 = 360$$
$$\text{I} = 4.80 \ \ \text{(Do not put percent)}$$
$$PV = 132,650 \text{ (LOAN)}$$
$$FV = 0$$
$$\text{PMT} = \text{(We need to find it)}$$
$$\text{P/Y} = 12$$
$$\text{C/Y} = 12$$

And click Solve for PMT.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

With the principal of the mortgage and the total amount paid over the life of the mortgage, the cost of financing can be found by subtracting the principal from the total paid.

Formula: Interest Paid and Total Paid

The total paid on a t years mortgage with monthly payments PMT is

$$\text{Total Paid} = PMT \times 12 \times t$$

The cost of financing a mortgage (Total Interest Paid),

$$\text{Interest Paid} = \text{Total Paid} - P \tag{4.5.3}$$

where P is the mortgage's starting principal.

Example 4.5.4: Total Paid and Interest

In an example 4.5.5,

1. How much is Evan's total payment over the life of the mortgage?
2. What is the cost of financing a mortgage, or how much interest Evan pays over the life of the mortgage?

Answer

1. Using the mortgage payment of $\$695.97$ and $t = 30$ years in the formula

$$\begin{aligned} \text{Total Paid} &= PMT \times 12 \times t \\ &= 695.97 \times 12 \times 30 \\ &= 250,549.20 \end{aligned} \tag{4.5.4}$$

Evan will pay for the mortgage, which is $\$250,549.20$. This is his total payment.

2. We found that the total Evan will pay for the $\$132,650$ and mortgage (Principal) is $\$250,549.20$.

$$\begin{aligned} \text{Intrest Paid} &= \text{Total Paid} - P \\ &= \$250,549.20 - \$132,650 \\ &= \$117,899.20 \end{aligned} \tag{4.5.5}$$

So Evan's interest will be $\$117,899.20$ over the lifetime mortgage.

Your Turn 4.5.4: Mortgage

The cost of a home you want to purchase is $225,000.00. To qualify for a mortgage, your lender wants a 20% down payment. Your mortgage interest rate is 5% for 15 years and you have to pay 2 points.

Round to the nearest cent as appropriate.

A. How much money do you need for the down payment?

I need $ ______ for the down payment.

B. How much will your mortgage be?

My mortgage amount will be $ ______.

C. How much will you have to pay for the points?

I will have to pay $ ______ for the 2 points. Hint

D. What is the monthly payment?

My payment will be $ ______ a month.

E. By the end of the loan what will be the total of all your payments?

By the end of the loan, the total of all my payment will be $ ______.

F. What is the total interest you will have paid on your mortgage by the end of the loan?

I will have paid $ ______ in interest by the end of the loan. Hint

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With the principal of the mortgage and how much total is paid over the life of the mortgage, the cost of financing can be found by subtracting the principal of the mortgage from the total paid over the life of the mortgage.

Who Knew?: Private Mortgage Insurance (PMI)

When you purchase a home, you will have to pay a down payment. This means you have money tied to the property, which lenders believe makes you less likely to default on the loan. The amount of the down payment will be decided between you and the mortgage company. However, if your down payment is less than 20% of the property value, you will be required to pay private mortgage insurance (PMI). This is insurance you pay for so that the mortgage company is protected if *you* default on the loan. It often comes to between 0.5% and 2.25% of the original loan amount. It increases your monthly payment. Once you reach 20% of the loan value, you can request that the PMI be dropped. Even if you do not request cancelling the PMI, it will eventually and automatically be dropped.

For more information, see this article on ways to eliminate PMI.

Credit Cards

It can be difficult to get along these days without at least one credit card. Most hotels and rental car agencies require that a credit card is used. There are even a number of retailers and restaurants that no longer accept cash. They make online purchasing easier. And nothing contributes more to a good credit rating than a solid history of making credit card payments on time.

Being granted a credit card is a privilege. Used unwisely, that privilege can become a curse, and the privilege may be withdrawn. In this section, we will talk about the different types of credit cards and their advantages and disadvantages. The more knowledge a cardholder has about the credit card industry, the better able credit accounts can be managed, and that knowledge may cause major adjustments to a cardholder's lifestyle.

All credit cards are not equal, but they all represent consumers borrowing money, usually from a bank, to pay for needs and "wants." As such, they are a type of loan, and your repayment may include interest. (You might want to review Section 6.8, which discusses loans and repayment plans.)

There are many institutions and credit cards to choose from. Use caution as you shop around for a credit card that suits you. Your top concern is likely the interest rates on purchases and cash advances. But be careful to also read the small print regarding charges for late payments, and other fees such as an **annual fee,** where the credit card charges you (the cardholder) a fee each year for the privilege of using the cards. Many cards charge no such fee, but there are many that charge modest to heavy fees. Make sure to understand rules for **reward programs**, where the credit card issuer grants benefits based on one's spending. Finally, once one applies for and is granted a credit card, pay attention to the credit limit the bank offers. Once a company owes that much money, use of the card for purchases should be curtailed until some of the debt is paid off.

Checkpoint

The interest rate will not matter if the balance is paid every month. When the balance is paid every month, there is NO INTEREST charged.

Types of Credit Cards

There are basically three types of credit cards: bank-issued credit cards, store-issued credit cards, and travel/entertainment credit cards. We will examine all three and discuss the strengths and weaknesses of each.

Bank-Issued Credit Cards

Perhaps the most widely used credit card type is the **bank-issued credit card**, like Visa or MasterCard (and even American Express and Discover cards). These types of cards are an example of revolving credit, meaning that additional credit is extended before the previous balance is paid—but only up to the assigned credit limit. Bank-issued cards are considered the most convenient, as they can be used to purchase anything, including apparel, furniture, groceries, fuel for automobiles, meals, hotel bills, and so on, just as if paying with cash. The interest rates on bank-issued credit cards are usually lower than those for other credit cards we'll discuss, and the credit limits are generally higher. Currently, bank-issued cards have an average 20.09% APR.

Store-Issued Credit Cards

Store-issued credit cards are issued by retailers. One can hardly walk into a store these days without being offered a discount on purchases if one applies for the store credit card. These cards can only be used in that store or family of stores that issues the card. However, if a store credit card is associated with Visa, MasterCard, or American Express, then the card might be used the same way that the bank-issued cards are used. This is called cobranding. The logo of the bank-issued card will be present on the store card. Many stores offer both types. Like other credit cards, they may come with an annual fee.

Store credit cards usually charge higher interest rates than bank-issued cards. Currently, store credit cards have an APR (annual percentage rate) of 24.15%. Any rewards offered by store credit cards are usually limited to purchases made in their own store, and it typically takes longer to accumulate enough rewards or points to redeem them, whereas cobranded credit cards offer opportunities to earn rewards on all purchases, regardless of whether purchases are made in the issuing store or not.

Store credit cards usually offer lower credit limits, at least in the beginning. After being proved to be a responsible credit card owner, credit limits can be raised. Nevertheless, store credit cards are a good choice for those new to the credit card industry. If on-time payments are consistently made, it is an excellent way to get started building a credit history.

Travel/Entertainment Cards, or Charge Cards

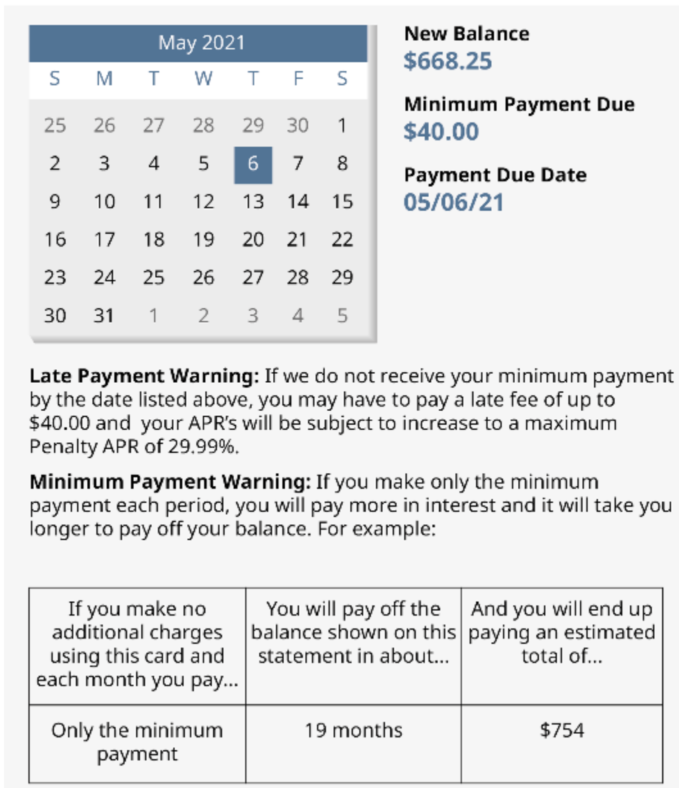
This is the third type of credit card. The **travel and entertainment cards,** also known as **charge cards,** first and foremost offer very high limits or unlimited credit, but they must be paid in full every month. They generally charge high annual fees and impose expensive penalties should a payment be late. On the other hand, they typically have longer grace periods and offer many and various kinds of rewards.

> **Checkpoint**
>
> *An interest rate will greatly depend on a credit score. Responsible use of credit cards will increase a credit score. See the WHO KNEW? from The Basics of Loans.*

Credit Card Statements

Cardholders usually receive monthly statements and have 21 days to pay the minimum amount due. The statements itemize and summarize activity on the credit card for that statement's billing period. The billing period for a credit card is generally a month long, but typically does not start and end on the first and last days of the month. The statement will include the current balance, interest rate, the minimum payment due, and the due date. Be aware, different companies produce statements that are laid out differently. The information will be clearly labeled, though.

The due date is a top concern. Missing a due date is one of the worst things a cardholder can do financially, and this is by far the biggest downfall of owning a credit card. Not only is the cardholder subject to late fees, but when a payment is late more than once, there is a high probability that the cardholder will be negatively reported to the credit bureaus, which can quickly erode a credit score. Figure 4.5.4 shows an excerpt from an actual statement from a Chase Bank Visa card, based on the current $668.25 balance.



May 2021

S	M	T	W	T	F	S
25	26	27	28	29	30	1
2	3	4	5	6	7	8
9	10	11	12	13	14	15
16	17	18	19	20	21	22
23	24	25	26	27	28	29
30	31	1	2	3	4	5

New Balance
$668.25

Minimum Payment Due
$40.00

Payment Due Date
05/06/21

Late Payment Warning: If we do not receive your minimum payment by the date listed above, you may have to pay a late fee of up to $40.00 and your APR's will be subject to increase to a maximum Penalty APR of 29.99%.

Minimum Payment Warning: If you make only the minimum payment each period, you will pay more in interest and it will take you longer to pay off your balance. For example:

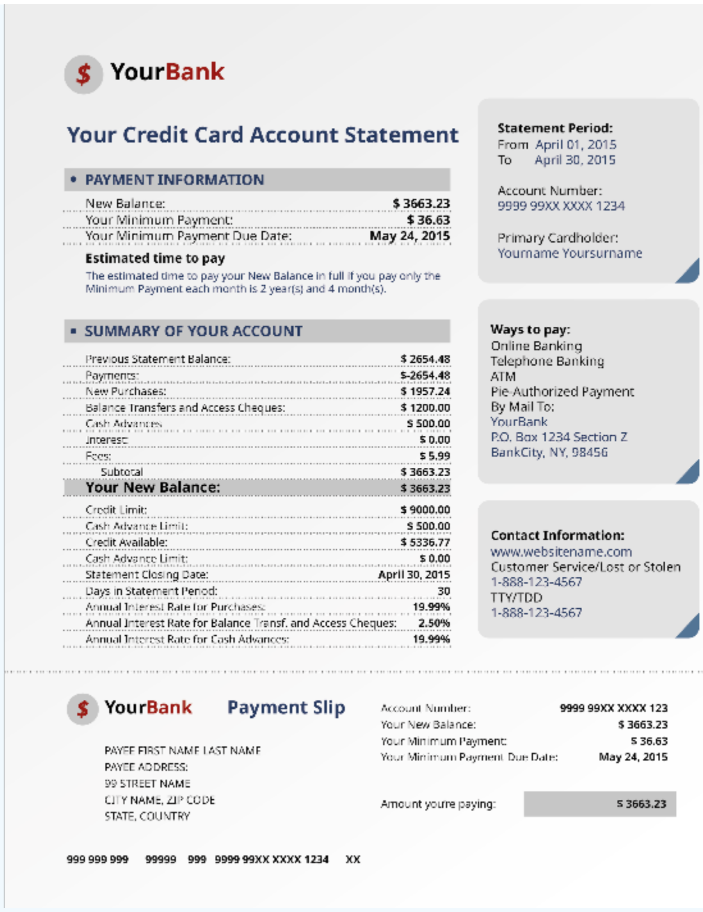
If you make no additional charges using this card and each month you pay...	You will pay off the balance shown on this statement in about...	And you will end up paying an estimated total of...
Only the minimum payment	19 months	$754

Figure 4.5.4 : Credit card statement

Specifically, pay attention to the late payment penalty and minimum payment warning statements. stating that if no other purchases are made and you continue making only the minimum payment, it will take 19 months to pay off the balance and you will pay $754.00. You can't say you were not warned.

It is critical that you examine your statement every month because it is always a possibility that your account may have been compromised. If you should notice fraudulent charges on your statement, notifying the credit card company is often enough to have those charges researched by the company and removed. The card with the fraudulent charges will be canceled, and a new card with a new account number will be sent to you.

> **Example 4.5.5: Reading a Credit Card Statement**
>
> On the credit card statement Figure 4.5.5,
>
> 1. What is the balance due?
> 2. What is the minimum required payment?
> 3. What is the length of time it takes to pay off the balance by paying the minimum payments and without charging more to the card?
> 4. What is the interest rate for purchases?
> 5. What is the maximum amount you can spend on the credit card?



YourBank

Your Credit Card Account Statement

PAYMENT INFORMATION

New Balance:	$ 3663.23
Your Minimum Payment:	$ 36.63
Your Minimum Payment Due Date:	May 24, 2015

Estimated time to pay

The estimated time to pay your New Balance in full if you pay only the Minimum Payment each month is 2 year(s) and 4 month(s).

SUMMARY OF YOUR ACCOUNT

Previous Statement Balance:	$ 2654.48
Payments:	$-2654.48
New Purchases:	$ 1957.24
Balance Transfers and Access Cheques:	$ 1200.00
Cash Advances:	$ 500.00
Interest:	$ 0.00
Fees:	$ 5.99
Subtotal	$ 3663.23
Your New Balance:	$ 3663.23
Credit Limit:	$ 9000.00
Cash Advance Limit:	$ 500.00
Credit Available:	$ 5336.77
Cash Advance Limit:	$ 0.00
Statement Closing Date:	April 30, 2015
Days in Statement Period:	30
Annual Interest Rate for Purchases:	19.99%
Annual Interest Rate for Balance Transf. and Access Cheques:	2.50%
Annual Interest Rate for Cash Advances:	19.99%

Statement Period:
From April 01, 2015
To April 30, 2015

Account Number:
9999 99XX XXXX 1234

Primary Cardholder:
Yourname Yoursurname

Ways to pay:
Online Banking
Telephone Banking
ATM
Pie-Authorized Payment
By Mail To:
YourBank
P.O. Box 1234 Section Z
BankCity, NY, 98456

Contact Information:
www.websitename.com
Customer Service/Lost or Stolen
1-888-123-4567
TTY/TDD
1-888-123-4567

YourBank Payment Slip

PAYEE FIRST NAME LAST NAME
PAYEE ADDRESS:
99 STREET NAME
CITY NAME, ZIP CODE
STATE, COUNTRY

Account Number:	9999 99XX XXXX 123
Your New Balance:	$ 3663.23
Your Minimum Payment:	$ 36.63
Your Minimum Payment Due Date:	May 24, 2015
Amount you're paying:	$ 3663.23

999 999 999 99999 999 9999 99XX XXXX 1234 XX

Figure 4.5.5 : Credit card account statement

Answer

1. The balance due is under the payment information heading and is $3,663.23.

2. The minimum payment due is also under the payment information heading, and is $36.63.

3. The time to pay off the balance using only minimum payments is below the payment information, and says it takes 2 years and 4 months to pay off the balance.

4. The interest rate for purchases is toward the bottom of the statement. It is 19.99%.

5. The maximum amount you can spend on the credit card is $9,000. This is the credit limit.

Your Turn 4.5.5

Referring to the statement above in Figure 4.5.5, answer the following:

1. What is the statement period?

2. Is the previous month's balance completely paid on the credit card?

3. How much in fees were charged?

4. Does the cardholder take cash from the credit card? If so, how much?

Compute Interest, Balance Due, and Minimum Payment Due for a Credit Card

Computing all of these values depends on understanding and computing the average daily balance on a credit card. Once that is known, the interest, balance due, and minimum payment can be found.

Above all else, if you pay off the entire balance each month, interest is not charged.

Averge Daily Balance

Most credit card companies compute interest using the **average daily balance** method.

To find the average daily balance on your credit card, determine the balance on the card each day of the billing period (often that month), and take the average. One process to find that average daily balance follows these steps:

1. Start with a list of transactions with their dates and amounts.
2. For each day that had transactions, find the total of the transactions for the day. Expenditures are treated as positive values, payments are treated as negative values.
3. Create a table containing each day with a different balance. The balance is the previous balance plus the day's total transactions.
4. Add a column for the number of days those balances until the balance changed.
5. Add a column that contains the balances multiplied by the number of days until the balance changed.
6. Find the sum of that last column.
7. Divide the sum by the number of days in the billing period (often the number of days in the month). This is the average daily balance.

Interest and Balance on Credit Card

The interest charge, I , for a credit card during a billing cycle is

$$\text{Interest Charge } = \text{average daily balance} \times \text{interest rate} \times \text{time} \tag{4.5.6}$$

As before, interest is rounded **up** to the next penny; otherwise, it is stated.

Balance on Credit Card

The **balance,** or sometimes balance due, on a credit card is the previous balance, plus all expenses, minus all payments and credits, plus the interest on the card. As stated earlier, if the card is paid off, no interest is due.

$$\text{Balance Due} = \text{balance in the end of the month} + \text{interest charge} \tag{4.5.7}$$

Minimum Payment Due

The **minimum payment** due is the smallest required amount to be paid on a credit card to avoid late fees and penalties, such as an increased interest rate. The calculations for this may differ from card to card. They also depend on the balance of the credit card. General guidelines for minimum payment due are:

- For larger balances (usually over $\$1,000$), the minimum payment will be some percentage of the balance due.
- For moderate balances (between $\$25$ and $\$1,000$), the minimum would be a specified dollar amount. $\$25$ seems to be a common value.
- If the balance is small (under $\$25$ for instance), then the minimum payment is the balance.

Those are just guidelines. Individual cards may vary in these values.

Example 4.5.6: Computing Average Daily Balance

Assume the annual interest rate is 13.7% of the average daily balance and the billing period is from July 1 - July 31. Complete the following, rounded to the nearest cent.

1. Calculate the average daily balance.
2. Calculate the interest to be paid at the beginning of the next month.
3. Calculate the balance due at the beginning of the next month.
4. This credit card requires a $\$20$ minimum payment or $\frac{1}{36}$ of the amount due, whichever is higher. What is the minimum monthly payment due for this month?

Transaction	Amount
Previous balance: $\$669.69$	
July 1: Billing date	
July 4: Carnival tickets	$\$71$ charge
July 15: Payment	$\$139$ credit
July 19: Bookstore purchase	$\$109.62$ charge
July 25: Gym membership	$\$59$ charge
July 31: End of billing cycle	

Answer

1. To find the average daily balance, we use the following steps.

Date	Balance	Days Until Balance Changes	Balance Times Days
July 1	$\$669.69$	$4-1=3$	$\$669.69 \times 3 = \$2,009.07$
July 4	$\$669.69 + \$71 = \$740.69$	$15-4=11$	$\$740.69 \times 11 = \$8,147.59$
July 15	$\$740.69 - \$139 = \$601.69$	$19-15=4$	$\$601.69 \times 4 = \$2,406.76$
July 19	$\$601.69 + \$109.62 = \$711.31$	$25-19=6$	$\$711.31 \times 6 = \$4,267.86$
July 25	$\$711.31 + \$59 = \$770.31$	$31-25=6$ and $6+1=7$	$\$770.31 \times 7 = \$5,392.17$

Find the sum of that last column. Adding that last column, we have a sum of $\$22,223.45$. There are 31 days in July, so divide the sum by 31, which gives an average of $\$716.89$, which is the average daily balance.

$$\text{average daily balance} = \frac{\$22,223.45}{31} = \$716.89 \tag{4.5.8}$$

2. We use $\$716.89$, as a principal and time will be 1 month with is same as $\frac{1}{12}$ year.

$$\begin{aligned} \text{Interst } &= \text{average daily balance} \times \text{interest rate} \times \text{time} \\ &= \$716.89 \times 0.137 \times \frac{1}{12} \\ &= \$8.18 \end{aligned} \tag{4.5.9}$$

Or we can find monthly interest rate $\frac{13.7\%}{12} = 1.1416666$

$$\begin{aligned} \text{Interst } &= \text{average daily balance} \times \text{interest rate} \times \mathit{time} \\ &= \$716.89 \times 0.01416666 \times 1 \\ &= \$8.18 \end{aligned} \tag{4.5.10}$$

Interest to pay is $\$8.18$.

3. The balance due at the beginning of the next month is the sum of the unpaid balance in the given month, which is $\$770.31$ plus interest, $\$8.18$.

$$\begin{aligned} \text{Balance due } &= \$770.31 + \$8.18 \\ &= \$778.49 \end{aligned} \tag{4.5.11}$$

4. $\frac{1}{36}$ of $\$778.49 = \$21.62$, which is greater than $\$20$. So minimum payment due is $\$21.62$.

Your Turn 4.5.6: Average Daily Balance

Consider the following credit card transactions. Fill in the blanks for the statement.

Date	Transaction	Amount	Balance	Days
Apr 1	Previous Balance	$7,000	$7,000	8
Apr 9	books	$191	$ ____	____
Apr 13	shoes	$127	$ ____	____
Apr 14	clothes	$135	$ ____	____
Apr 24	zoo membership	$190	$ ____	____
Apr 27	chair	$95	$ ____	____

1. Compute the average daily balance. Round to the nearest cent.

   Average daily balance = $ ____

2. Calculate the interest to be paid at the beginning of the next month if the monthly interest rate is 0.0032.

   Interest paid = $ ____

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Who Knew?: Credit Cards Charge Stores Fees

The interest you pay is not the only way a credit card company generates revenue. It also charges fees to the retailers, online stores, and service providers that allow you, the consumer, to use your credit card to pay them. These are called processing fees. Currently they typically range from 2.87% to 4.35% of each transaction. That means if you use your credit card at a store and spend $\$100$, the store will have to pay the credit card company somewhere between $\$2.87$ and $\$4.35$.

The Basics of Car Purchasing

There are people who don't need a car and won't purchase one. However, for many people, the question of whether to have a car is not a matter of choice. Having a car is a basic necessity for these people.

Obtaining a car can be daunting. The models, features, additional costs, and securing funding are all steps that need to be taken. One of the big decisions is whether to buy the car or to lease the car. This section will address some of the issues associated with each option.

The biggest questions you will answer before purchasing a car are, what do you want and what do you need?

Does it have to be new? Does it have to be a make and model you are familiar with? Does it require assisted driving? What other details are important to you? For a new vehicle, every feature beyond standard features comes with additional cost, which leads to the question that constrains all of your decisions about a car. How much can you afford to spend on a car?

What you can afford must include insurance costs (discussed later in this section), as well as maintenance and upkeep. Once you have this in mind, you can search for a car that matches, as closely as possible, what you want and can afford. Most, if not all, dealers have websites that you can search through to identify the car you want. If new cars are not affordable, used cars are often less expensive but come with the wear and tear of use.

The sticker price of the car, called the manufacturer's suggested retail price (MSRP), or the negotiated price you arrive at, isn't the end of the cost to buying a car.

Fees accociate with the Purchase of the Car

There are several fees associated with purchasing the car, including sales tax. These include, but aren't necessarily limited to, the following:

- The title and registration fee include registering your car with the state, getting the license plate, and assigning the title of the car to the lender. This cannot be avoided.
- a destination fee, which covers the cost of delivering the vehicle to the dealer
- a documentation fee, sometimes referred to a processing fee of handling fee, is the cost of all the paperwork the dealer did to get you the car
- a dealer preparation fee, which is for washing the car and other preparation of that sort. You should try to negotiate that out of the cost of the dealer tries to charge for those.
- extended warranties and maintenance plans, which help cover some of the costs of caring for the car.
- Sales tax.

You could pay for these immediately, but they are often added to the financing of the car, meaning they become part of the principal of your loan.

One way to bring down payments on a car is to provide a down payment or a trade-in. This is money applied to the purchase price before financing happens. Be warned, the sales tax applies to the full purchase price! If you reduce the amount financed, the payments will go down. This often becomes part of the negotiating process.

Example $4.5.7$: Total Cost to Purchase a Car with Down Payment

Sophia negotiates a $\$19,800$ price for her new car. The sales tax is 9.5% in her area, and the dealership charges her $\$300$ in documentation fees. Her title, plates, and registration come to $\$321.50$. The dealership adds to this a destination fee of $\$1,100$. If she places a down payment of $\$5,000$ on the car, what is the total she will finance for the car?

Answer

The price was $\$19,800$. The sales tax of 9.5% is based on this number. The sales tax comes to

$$\$19,800 \times 0.095 = \$1,881$$

Adding all the fees to the price and the sales tax brings the total cost of the car to

$$19,800 + \$1,88 + \$300 + \$321.50 + \$1,100 = \$23,402.50$$

Her down payment is applied to this number, so the $\$5,000$ is subtracted from $\$23,402.50$.
The subtraction yields the amount to be financed, which is $\$18,402.50$.

Your Turn 4.5.7: Cost on Purchasing Car

Sophia negotiates a $\$29,350.00$ price for her new car. The sales tax is 8.75% in her area, and the dealership charges her $\$450$ in documentation fees. Her title, plates, and registration come to $\$250$. The dealership adds to this a destination fee of $\$810$.

If she places a down payment of $\$9,000.00$ on the car, what is the total she will finance for the car?

a. Find the total cost of the car, including sales tax and other fees.

b. Find the amount she needs to finance to buy a car.

Round to the nearest cent when appropriate.

Submit | Try a similar question

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When purchasing a car, the total cost to obtain the car is not the only factor in your monthly price. You will also pay an interest rate for the loan you obtain. The interest rate you will get is dependent on your credit score (see The Basics of Loans). But you can choose from different lenders. The dealership will likely offer to finance your car loan. Frequently, dealerships offer special financing with very low rates. This is to help move inventory, and may indicate their desire to make sales. This might make negotiating easier. Even if the dealership offers financing, check with your bank or credit union to determine the interest rates they are offering. To reduce your payments, choose the lowest rate you can find.

Purchase Payments and Interest

Whether or not you buy a new car or a used car, if you finance the purchase, you are taking out a loan. The interest rates available for used cards are frequently higher than those for new cards. These loan payments work exactly the same way as other loans do as far as payments are concerned. The payment function comes from The Basics of Loans. The difference between financing a new car and a used car is that financing a new car typically comes with a lower interest rate and a longer term than financing a used car.

FORMULA: Monty Payment For Car Loan

The payment, PMT, per month to pay for a car loan with a beginning principal P is

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \tag{4.5.12}$$

where r the annual interest rate in decimal form and t is the number of years of the payment.
Please note that payments to lenders are always rounded up to the next penny, unless otherwise specified.

Example 4.5.8: New Car Payments

You decide to finance a $\$18,000$ car at a 6% interest rate for 5 years.

1. How much will your monthly payment?
2. By the end of the loan, what will be the total of all your payments?
3. How much interest will you pay over the life of the loan?

Answer

1. The loan is for $\$18,000$, which is the principal. The rate is 6%, so $r = 0.06$. The term of the loan is 5 years, Monthly payments mean $n = 12$. Substituting these values into the formula as we did in the previous example.

$$PMT = \frac{P\left(\frac{r}{n}\right)}{\left[1-\left(1+\frac{r}{n}\right)^{-nt}\right]} \tag{4.5.13}$$

Your monthly payment needed is $\$348$.

We can check our answer using the TVM solver.

$$\begin{aligned} \mathrm{N} &= n \times t = 12 \times 5 = 60 \\ \mathrm{I} &= 6 \ \text{ (Do not put percent)} \\ \mathrm{PV} &= 18,000 \text{ (LOAN)} \\ \mathrm{FV} &= 0 \\ \mathrm{PMT} &= \text{We need to find it.} \\ \mathrm{P/Y} &= 12 \\ \mathrm{C/Y} &= 12 \end{aligned}$$

And click Solve for PMT.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

2. Using the mortgage payment of $\$348$ and $t = 5$ years in the formula

$$\begin{aligned} \text{Total Paid} &= PMT \times 12 \times t \\ &= 348 \times 12 \times 5 \\ &= 20,880.00 \end{aligned} \tag{4.5.14}$$

You will pay for a total of $\$20,880$.

3. We found that the total you will pay for the $\$18,000$ loan (Principal) is $\$20,880$.

$$\begin{aligned} \text{Intrest Paid} &= \text{Total Paid} - P \\ &= 20,880 - 18,000 \\ &= 2,880 \end{aligned} \tag{4.5.15}$$

So your interest will be $\$2,880$ over the life of the loan.

Your Turn $4.5.8$: Car Payment

You decide to finance a $\$20,000.00$ car at a 5.25% interest rate for 5 years.

A. How much will your monthly payment?

My payment will be $ [ ] a month. Hint

B. By the end of the loan, what will be the total of all your payments?

The total of all my payments will be $ [ ] . Hint

C. How much interest will you pay over the life of the loan?

I will pay $ [ ] in interest over the life of the loan. Hint

Round to the nearest cent as appropriate.

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The Basics of Leasing a Car

Leasing a car is an alternative to purchasing a car. It is still a loan and acts like one in many respects. They typically last either 24 months or 36 months, though other terms are available. Leases also come with mileage limits, frequently $10,000$, $12,000$, or $15,000$ miles per year. When the lease is over, the car is returned to the dealer. At that time, there may be fees that have to be paid, such as for damage to the car or for extra miles driven over the limit.

There are two components to lease costs. One is the monthly payment for the lease. The other is the fees for leasing, These often are paid before the lease is complete. These include:

- a down payment, which is your initial payment that is applied to the price of the car. It reduces the amount you finance, much the same as when you purchase a car. It is recommended that this be negotiated away.
- The acquisition fee, sometimes called the bank fee. This is the money charged for the company to set up the lease. It is essentially a paperwork fee. It is not likely that this can be negotiated.
- a security deposit, which might be required. It is about the same as 1 month's payment for the lease. The deposit is returned to you if the car is in good shape at the end. This can be negotiated away.
- disposition fees, which cover the cost the company will incur when they take your car back and are typically between $\$200$ and $\$450$.
- the title, registration, and license fees, just as with the purchase of a car.
- sales tax, which will likely be applied. The sales tax only covers the depreciated portion of the car (more on depreciation later) in many states. Since this depends on the state in which the car is leased, you should determine the sales tax rules for where you lease the car.

As you can imagine, this can come to a fairly high dollar amount.

You have some obligations when you lease a car. You must keep the car in good condition, cleaned, maintained, and free of anything more than minor damage. If the car is in poor condition when the car is returned, you will be responsible for the cost to bring the car to an acceptable condition. You are also expected to keep the mileage under its limit. If you go over, you will pay 10 to 25 cents per mile over.

Lease Payments

Lease payments are similar to regular loan payments, but have some other details. Calculating a lease payment involves knowing the following values:

- **The price of the car.** This is the cost you would pay for the car after applying all discounts, incentives, and negotiations.
- **Residual Value.** This is the manufacturer's estimate of the car's value after a set period of time. The residual value is expressed as a percentage of the manufacturer's suggested retail price (MSRP).
- **Months.** This is the length of the lease. Most leases are either 24- or 36-month leases, but other terms are available.

- **Monthly Depreciation**. The monthly depreciation is the difference between the price of the car and the residual value, divided by the number of months of the lease, and represents the monthly loss of value of the car while it's being used

Comparing Purchasing and Leasing

When deciding to buy or lease a car, the differences between the two options should be carefully evaluated. The following is a list of points of comparison between the two.

- The payments for a lease are likely less than the payments for purchasing.
- When leasing, you get a new car after the lease term is over, typically 24 or 36 months. Buying the car means the same car is driven until it is resold, and a new one is bought. Essentially, leasing a car is equivalent to renting a car.
- The leased car is new, so all warranties are in force, and you drive the car during its best years. When the car is purchased, it may be kept past its warranty and may be driven until it is quite old.
- Each time you lease a new car, all the fees and taxes must be paid again. When buying a car, these fees are only paid once.
- Leasing contracts carry restrictions on the mileage you can drive per year, and going over incurs more cost at the end of the lease. Buying the car means no mileage limits.
- When leasing, you are obligated to keep the vehicle in good condition and maintain it according to the dealer's schedule. Some dealerships will even pay for oil changes over the life of the lease. When the car is purchased, the upkeep schedule is the choice of the owner.
- When a car is purchased and kept for long enough, the warranty expires, and the owner is responsible for all maintenance items and repairs. The warranty for a car won't expire during the lease term.
- When a new car is purchased and the loan is paid off, the car is still owned by the buyer and may be traded in when a new car is purchased. When leasing, the car is returned to the dealer when the lease term is over.

When deciding between the two, you are choosing between these features. If you aren't willing to drive an older car or deal with the upkeep that accompanies an older car, you may want to lease. This means you will need to pay those beginning costs each time the lease is up. If you want to own the car after the payments are over, then you may want to buy a car. This means you are paying for all the upkeep after the warranties expire, but you have no limits on mileage and own the car at the end. It really depends on your preferences.s.

Car Insurance

Car insurance is meant to cover costs associated with accidents involving cars. Most states (all except New Hampshire and Virginia) require some insurance. Without insurance, the state may not let you get a license for your car or register your car. Your state's requirements can be hard to follow. Fortunately, insurance companies and brokers will make sure your insurance is sufficient for your state and will warn you if you try to not meet the requirements. Of course, they may offer more than what is sufficient, so it is your responsibility to determine how much coverage you want, as long as the minimum insurance requirements are met. The cost of insurance should be accounted for when evaluating the affordability of buying or leasing a car.

Whether your car is leased or owned, you do need insurance. This contributes to the cost of having the vehicle. Leasing or owning makes no difference to the insurance company you choose, because they are insuring you based on what you are driving, your driving record, and other information about you, including where you live and your age. These insurance policies have many components that address different costs that can come from auto accidents. This may make details confusing, and you may not realize what you are paying for until you must use it. Here is a brief outline of the different components of auto insurance, many of which are required by the state that issues your driver's license.

- **Liability insurance** is mandatory coverage in most states. Liability insurance covers property damage and injuries to others should you be found legally responsible for an accident. You are required to have the minimum amount of coverage, as determined by your state, in both areas.
- **Collision insurance** is insurance that covers the damage caused to your car if it is involved in an accident with another vehicle.
- **Comprehensive insurance** is an extra level of coverage if involved in an accident with another vehicle and covers other things like theft, vandalism, fire, or weather events as outlined in your policy. There is a deductible assigned to each type of insurance, an amount that you pay out of pocket before your comprehensive coverage takes effect. Comprehensive insurance is often required if you lease or finance the purchase of a vehicle.
- **Uninsured or underinsured motorist insurance:** If you are hit by an uninsured or underinsured motorist, this insurance will help pay medical bills and damage to your car.
- **Medical payments insurance** is mandatory in some states and helps cover the costs of medical expenses associated with an accident, regardless of who is at fault.
- **Personal injury protection insurance** is coverage for certain medical bills and other expenses due to a car accident. Other covered expenses may include loss of income or childcare, depending on the specific terms of your policy.
- **Gap insurance** is designed to cover the gap between what is owed on the car and what the car is worth in the event your car is a total loss.
- **Rental reimbursement insurance** provides coverage for a rental car while your vehicle is under repair due to an accident.

You can also purchase other special insurance policies, such as classic car insurance, new car replacement insurance, and sound system replacement insurance, to name a few. It is essential to determine exactly what you need, as insurance policies can be expensive and vary based on your age, driving history, and location.

Equity

Equity is the difference between the market value of a property (car, house, etc) and the amount still owed on the loan.

Example 4.5.9: Equity in House

You bought a house 15 years ago, taking out a $\$179,000$ mortgage at a 4% interest rate for 30 years. Your monthly payments are $\$854.58$.

1. How much will still be owed after making payments for 15 years?
2. If the house's value is now $\$299,000$, then how much equity do you have after making payments for 15 years?

Answer

1. We use TVM calculator to find the amount that you still owe.

$$\begin{aligned} N &= n \times t = 12 \times 15 = 180 \\ \mathrm{I} &= 4 \;\; \text{(Do not put percent)} \\ PV &= 179,000 \text{ (LOAN)} \\ \mathrm{PMT} &= -854.58 \text{ (Do not forget to put negative symbol)} \\ \mathrm{P/Y} &= 12 \\ \mathrm{C/Y} &= 12 \end{aligned}$$

Click Solve for FV. The face value (FV) will be $\$115,529.98$, and this is the value you still owe after 15 years.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

2. The house equity is $\$299,000 - \$115,529.98 = \$183,470.02$

Your Turn 4.5.9: Find Equity

You bought a house 20 years ago, taking out a \$336,000.00 mortgage at a 4.8% interest rate for 25 years. Your monthly payments are \$1,925.27 .

A. How much will still be owed after making payments for 20 years?

I will still owe ________ after making payments for 20 years.

B. If the house's value is now $406,000.00 then how much equity do you have after making payments for 20 years?

I will have ______ equity after making payments for 20 years.

Time Value of Money Solver

Enter the given values.

N: = 0 Solve

Number of Compounding Periods

I:% = 0 Solve

Annual Interest Rate as a Percent

PV: = 0 Solve

Present Value

PMT: = 0 Solve

Payment

FV: = 0 Solve

Future Value

P/Y: 12

Payments per Year

C/Y: 12

Compounding Periods per Year

PMT: = END

Payments are made at the end of the period

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4.6: Income Tax and FICA Taxes

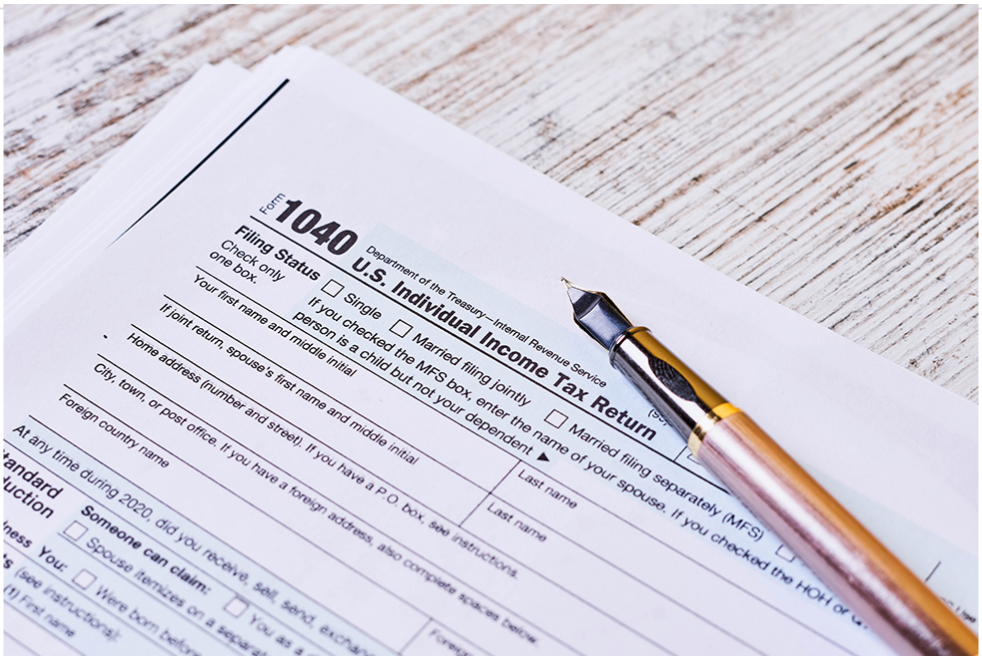

Figure 4.6.1: Federal income tax is a concern for most US citizens. (credit: "1040 US tax form" by Marco Verch Professional Photographer/Flickr, CC BY 2.0)

Learning Objectives

1. Determine gross, adjusted gross, and taxable income.
2. Apply deductions and credits to basic income tax calculations.
3. Compute FICA tax and income tax.

Before the start of the American Civil War in 1861, most of the country’s revenue came from tariffs on trade and excise taxes. However, this fell far short of the high cost of the war. Because of this, the federal government enacted the nation’s first income tax with the Revenue Act of 1861, which created the Internal Revenue Service as we know it today.

No one likes paying income tax, but it is a reality of life. In this section, we will learn about Form 1040, the U.S. Individual Income Tax Return, and ways to prepare for tax time.

The U.S. tax code may change from year to year. Because of this, this section includes examples of how taxes, deductions, and exemptions might be computed. The types of income, deductions, and exemptions that are used in the examples are used in the current tax code.

Gross (Total) Income, Adjusted Gross Income, and Taxable Income

Your income drives how much you pay in taxes. The more you earn, the more you are likely to pay. But your income alone is not the full story. When you add all the money you earn from your job, freelance work, interest from savings, and other sources, you have your gross income. If you are an employee, your income from your job will be reported on a W-2, which is sent to you by your employer. Income from freelance work will be reported on a 1099-MISC form, and is sent by the company that paid you. Income from interest is reported on a 1099 -INT form and comes from the entity that paid the interest.

Before you determine how much you owe in taxes, you will make certain adjustments to that gross income. You will deduct or subtract some of the income from the gross income. That’s your adjusted gross income or AGI. That is still not what you are taxed on. Next, you need to apply exemptions to your income. These are pieces of income that the government does not tax. After that is done, you reach your taxable income. We will look at each of these parts of the taxable income.

Tax Withholding: W-2 Form

"Tax withholding" refers to the portion of an employee’s wages that an employer deducts and sends directly to the government as partial payment of the employee’s income tax. It serves as a way to pay income taxes gradually over the course of the year rather than all at once when you file your return. You can find **federal income tax withholding** on your **Form W-2.**

Who Knew? Gifts and Winning

Money given as a gift may be taxed if the gift amount is high enough. If you win $50,000 in the lottery, that money is taxed as income. If you give a family member a large cash gift, that gift will be subject to tax, provided that the gift exceeds the federally set limits.

You will notice that your paycheck already has taxes taken out of it. Your employer will withhold some of your income, sending it directly to the federal, state, and local governments. It is an estimate of how much you will owe in income tax. Ultimately, it reduces the amount you will pay when your taxes are due. If they withhold too much income, you will receive the extra they withheld in the form of a refund.

States with No State Tax

Here are the U.S. states that **do not have a state income tax** as of 2025: **Alaska, Florida, Nevada, South Dakota, Texas, Washington, and Wyoming**

Gross Income (Total Income)

Gross income (GI) refers to the total earnings an individual or business receives before any deductions or taxes are taken out. Gross income includes wages, salaries, bonuses, rental income, tips, investment income, unemployment compensation, profit from business, and game-show winnings.

Adjusted Gross Income

Adjusted Gross Income (AGI) is your **gross income minus certain allowable deductions**, known as **adjustments**. Some of the adjustments include student loan interest, contributions to a traditional IRA, health savings account (HSA) contributions, alimony paid, educator expenses, tuition, and fees.

$$\text{Adjusted Gross Income (AGI)} = \text{Gross Income} - \text{adjustment}$$

What is Deduction? Which Deduction We Use?

A **deduction** is an amount you can subtract from your **adjusted gross income (AGI)** to reduce the amount of income that's subject to tax. Deductions help lower your **taxable income**, which can decrease the amount of tax you owe. There are two types of deductions:

Standard Deduction: A fixed amount set by the IRS based on your **filing status** (e.g., single, married, filing jointly).

Itemize Deduction: If your deductible expenses are higher than the standard deduction, you can itemize. Common itemized deductions include: Mortgage interest that you paid, charitable contributions if you made any, medical bills over a threshold, medical insurance under certain circumstances, and property taxes. If you add all these up, and they are all legal deductions, the sum is subtracted from your gross adjusted gross income.

During calculation, you have to use only one deduction, whichever is higher, **NOT BOTH.**

Your Turn 4.6.1: Which Deduction

Filing Status	Standard Deduction
Head of Household	$21,900
Single	$14,600
Married Filing Separately	$14,600
Married Filing Jointly	$29,200

Tax rate	Single	Head of household	Married filing jointly or qualified surviving spouse	Married fili separately
10%	$0 to $11,600	**$0–$16,550**	$0–$23,200	$0–$11,600
12%	$11,600 to $47,150	$16,550–$63,100	$23,200–$94,300	$11,600–$4
22%	$47,150 to $100,525	$63,100–$100,500	$94,300–$201,050	$47,150–$1
24%	$100,525 to $191,950	$100,500–$191,950	$201,050–$383,900	$100,525–$
32%	$191,500 to $243,725	$191,950–$243,700	$383,900–$487,450	$191,950–$
35%	$243,725 to $609,350	$243,700–$609,350	$487,450–$731,200	$243,725–$

37%	$609,350 or more	$609,350 or more	$731,200+ or more	$365,600 or

Tarana had $63,000 of income from wages and $2,950 of taxable interest. Tarana also made contributions of $2,700 to a tax-deferred retirement account. Tarana files as head of household.

What is Tarana's total income?

$

What is Tarana's adjusted gross income?

$

For Tarana's filing status, the standard deduction is $21,900. What is Tarana's taxable income?

$

Use the above tax table to find the income tax for Tarana filing as head of household. Round to the nearest dollar.

$

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Ho to Find Taxable Income?

Taxable income (TI) is the portion of your income that is subject to income tax after all **deductions** (standard or itemized) and **adjustments** (like those used to calculate AGI) have been subtracted from your gross income.

$$\text{Taxable Income} = \text{AGI} - \text{Deduction}$$

Remember that your AGI is not your **taxable income**.

Example 4.6.2: Find GI, AGI, and TI

Puspa and Sami were calculating their taxes for 2022. Their income for that year was $\$176,815$. They contributed $\$7,900$ to tax-deferred retirement plans. They also have a rental income of $\$19,800$ for a home they rent and $\$18,200$ for profits from a small business. They have the following deduction: mortgage interest payments of $\$15,600$. State and local taxes deduction of $\$7,400$. And charitable contribution amount of $\$10,200$. The standard deduction for a married couple filing jointly is $\$25,900$.

1. What is their gross income?
2. What is their adjusted gross income?
3. What is their taxable income?

Answer

1. $\text{Adjusted Gross Income (AGI)} = \$176,815 + \$19,800 + \$18,200 = \$214,815$

2. Here, $\$7,900$ to a tax-deferred retirement account is the adjustment. So

$$\begin{aligned}\text{Adjusted Gross Income (AGI)} &= \text{Gross Income} - \text{adjustment}\\ &= \$214,815 - \$7,900\\ &= \$206,915\end{aligned}$$

3. Since itemized deduction = $\$15,600 + \$7,400 + \$10,200 = \$33,200$ is more than the standard deduction, we must subtract the itemized deduction to find taxable income.

$$\begin{aligned}\text{Taxable Income} &= AGI - \text{deduction}\\ &= \$206,915 - \$33,200\\ &= \$173,715\end{aligned}$$

Note: For deductions, always find both options (itemize and standard) and use the higher one. If you subtract the higher deduction, your taxable income will be less. That's what all taxpayers want.

Your Turn 4.6.2: Taxable Income

	Single	Married filed jointly
Standard Deduction	**$6,100**	$**12,200**

Clarks is single and income of $38,500. And she has $7,500 for profits from a small business. Her $2,500 contribution to her retirement account is an adjustment. She can claim the following deductions: $2,500 in property tax, $3000 in mortgage interest, and $1,750 in charitable contributions.

Find Clarks's gross income. $

Find Clarks's adjusted gross income. $

Will Clarks use the standard deduction or itemized deductions?

- Itemized
- Standard

Find Clarks's taxable income: $

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Tax Credits

Another piece of the tax puzzle is tax credits. This is money subtracted from the tax you owe.

Tax credits are very different from deductions or exemptions. Deductions and exemptions are taken away from your gross income before the tax you owe is calculated. A tax credit is subtracted, dollar for dollar, from your tax bill. Once the tax you owe is calculated, subtract any tax credits from that calculated tax.

Some of the tax credits are refundable. This means that if subtracting them from your tax results in a negative number, you receive a tax refund. For more details, see this article about tax credits.

Refundable Tax Credit	Non-Refundable Tax Credit
1. Can reduce your tax bill **below zero**. 2. If the credit is larger than the tax you owe, the **extra amount is refunded to you**	1. Can reduce your tax bill **to zero**, but **not below zero**. 2. If the credit is more than the tax you owe, the extra amount is **lost**.

The federal government has imposed income limits and restrictions on those eligible to receive tax credits because the value of these credits is so high. Here is a partial list of tax credits that you might qualify for:

- **Earned income credit** is a refundable tax credit for low- to moderate-income workers and ranges from $\$632$ to $\$7,830$ depending on dependents and income. This is **refundable**.
- **American Opportunity Credit** is a credit taken by parents who have children enrolled in college at least half-time and pursuing a degree. This credit is worth $\$2,500$ per student for the first 4 years of undergraduate school, subject to income limits. This is a **refundable** tax credit.
- **The lifetime learning credit** is a credit equivalent to 20% of educational expenses, up to $\$2,000$ per year, subject to income limits. There is no cap on how many years you can apply for this credit. This is a **nonrefundable** tax credit.
- **The child tax credit** is worth $\$2,000$ per child under the age of 17 if that child lives at home for at least half the year, subject to income limits. This is a refundable tax credit.
- **The Child and Dependent Care Tax Credit** was designed to help pay for child care while the parent is working. The amount of the credit is dependent on your income. However, the maximum amount that can be received is, in 2026, $\$3,000$ for one eligible person, or $\$6,000$ for two or more qualifying people. A dependent qualifies if they are a child under 13 years old, a spouse who is unable to care for themselves, or some other qualifying person. This is a **refundable** tax credit.
- **The premium tax credit** was created by the Affordable Care Act and is received by many people throughout the year. In essence, it is a health insurance premium subsidy. The amount of the credit is based on your income and the price of health insurance in your area. This is a **refundable** tax credit.

Example $4.6.3$: Apply a Tax Credit

$1.$ Kaitlyn has calculated the tax she owes to be $\$5,200$. However, she receives an earned income tax credit of $\$1,715$. How much does Kaitlyn owe after applying the earned income tax credit?

$2.$ Chanajah calculated their tax owed, which came to $\$4,300$. They have an earned income tax credit of $\$2,190$, a child tax credit of $\$2,000$, and a child and dependent care tax credit of $\$4,000$. How much tax does Chanajah owe, or how much will they get in a refund?

Answer

$1.$ The amount of taxes Katilyn owes is her calculated tax of $\$5,200$ minus the credit she receives. The amount she owes in taxes is

$$\$5,200-\$1,715=\$3,485$$

$2.$ Adding Chanajah's tax credits together, we find their total to be $\$8,190$. That is more than the tax they owe, which was $\$4,300$. Each of those credits is refundable, which means they will receive a refund. Subtracting the credits from the tax owed yields

$$\$4,300-\$8,190=-\$3,890$$

This is negative, so it represents a refund of $\$3,890$.

Your Turn $4.6.3$: Applying Tax Credit

Bradley taxes owed based on his income are $11,080. However, he qualifies for the earned income tax credit of $2,570 and a child tax credit of $2000. How much does Bradley owe after applying the tax credits?

Bradley's owe $ ________ after applying the tax credits.

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> **Deduction vs Credits**

- **Deduction**: Lowers how much of your income is *taxable*.
  Example: If you make $\$60,000$ and have a $\$5,000$ deduction, you now only pay taxes on $\$55,000$.
- **Tax credit**: Lowers your *actual tax bill*, dollar for dollar.
  Example: If you owe $\$5,000$ in taxes and you have a $\$1,000$ credit, you only owe $\$4,000$.

Big difference:

- A **deduction** reduces the *amount of income* taxed.
- A **credit** directly reduces *how much tax* you owe.

> **Example 4.6.4: Tax Credit or Deduction**

Determine how much the following individuals or couples will save in taxes with the specified tax credits or deductions.

1. Max is in 12% tax bracket. How much will his tax bill be reduced if he makes a $\$2000$ contribution to charity?

2. Kris is in 12% tax bracket. How much will his tax bill be reduced if he qualifies for a $\$2,000$ tax credit?

Answer

1. If you itemize your deduction, the $\$2000$ contribution to the charity reduces your taxable income by $\$2000$. Since you are in 12% tax bracket, your tax bill will be reduced by $\$2000 \times 0.12 = \$240$. But if you use the standard deduction (when your itemized deduction is less than the standard deduction), your tax bill is reduced by nothing.

2. Whether you use itemize or standard deduction, your tax bill is reduced by $\$2000$.

> **Your Turn 4.6.4: Deduction or Tax Credit**

Determine how much the following individuals or couples will save in taxes with the specified tax credits or deductions.

Vanessa is in the 15% tax bracket and claims the standard deduction ($\$6,100$). How much will her tax bill be reduced if she qualifies for a $\$1500$ tax credit?

$ ______

Determine how much the following individuals or couples will save in taxes with the specified tax credits or deductions.

Shiro is in the 39.6% tax bracket and itemizes his deductions. How much will his tax bill be reduced by if he makes a $\$2000$ contribution to charity?

$ ______

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Computing FICA Taxes

FICA stands for the **Federal Insurance Contributions Act** of 1935. FICA taxes are used solely to fund Social Security and Medicare and are separate from federal income tax.

FICA taxes are the taxes taken out of your paycheck to fund **Social Security** and **Medicare**.

- **Social Security** helps people when they retire, become disabled, or die (their family can sometimes get benefits).

- **Medicare** helps mainly older people $(65+)$ with health insurance.

2022 FICA Tax Rates[1]

Employee's Rates	Self-Employed Rates
◦ SS: 6.20% on the first $\$147,000$ of wages ◦ Medicare: 1.45% on all wages ◦ Tota	◦ 12.4% on the first $\$147,000$ of wages ◦ Medicare: 2.90% on all wages ◦ Tota FICA tax: 15.3% on upto $\$147,000$ plus medicare tax above $\$147,000$

Employee's Rates	Self-Employed Rates
FICA tax: 7.65% on up to $\$147,000$ plus medicare tax above $\$147,000$	

FICA taxes include Social Security and Medicare. The Social Security tax applies to the first $\$147,000$ of income, while the Medicare tax applies to all income.

It amounts to 7.65% of your gross pay, which is withheld from your paycheck automatically. Your employer is required to match the 7.65% amount. Of the 7.65%, 6.2% goes to Social Security (SSI), and 1.45% goes to Medicare.

As of 2022, SSI tax only applies to the first $\$147,000$ of earnings. Any gross income above that is not taxed for Social Security. This limit changes every year.

Medicare tax, on the other hand, applies to the entirety of your gross income.

Example 4.6.5: Computed FICA Taxes

1. McKenzi earned $\$2,700$ in gross income, before taxes, in a given 2-week period. How much does she owe in FICA taxes, and how much of that is for SSI?

2. Suppose you earn $\$50,000$ from your own business in 2022. How much do you owe in FICA taxes, and how much of that is for SSI?

3. Renard earned $\$195,000$ in wages for the year as an employee. How much in SSI taxes does Renard owe for the year? What are the total FICA taxes?

Answer

1. The FICA tax is 7.65% of her gross earnings. 7.65% of her $\$2,700$ is $\$206.55$. Also, the SSI is 6.2% of her income, which is $0.062 \times \$2700 = \$167.40$
2. The FICA tax is $15.3\% \times \$50,000 = \$7,650$. SSI is $12.4\% \times \$50,000 = \$6,200$.
3. Since Renard earned more than the taxable limit of $\$147,000$, he only pays the 6.2% SSI tax on $\$147,000$. This comes to $\$9,114 = 0.062 \times \$147,000$.

$$\begin{aligned}\text{FICA Tax} &= 7.65\% \text{ of } \$147,000 + 1.45\% \text{ of } (\$195,000 - \$147,000) \\ &= 0.0765 \times \$147,000 + 0.0145 \times \$48,000 \\ &= \$11,941.50\end{aligned}$$

Your Turn 4.6.5: Calculate FICA Taxes

2024 FICA Tax Rates[1]

Employee's Rates	Matching Rates Paid by the Employer	Self-Employed Rates
• 7.65% on the first \$168,600 of income • 1.45% on income above \$168,600	• 7.65% on the first \$168,600 paid in wages • 1.45% on wages above \$168,600	• 15.3% on the first \$168,600 of net earnings • 2.9% on earnings above \$168,600

[1]FICA taxes include Social Security and Medicare. The Social Security tax applies to the first \$168,600 of income, while the Medicare tax applies to all income.

Suppose a graphic designer earns \$32,000 and is self-employed. How much will the designer have to pay in FICA taxes?

\$

Suppose Gracie has \$175,000 of income from work and is self-employed. How much will Gracie have to pay in FICA taxes?

\$

(Round your answer to the nearest cent.)

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Calculating Your Income Tax

Your income tax bill and your income tax rate are based on your taxable income. The tax system in the United States is progressive, meaning that the tax rates are marginal, so the higher your taxable income, the higher the tax rate you will pay.

Taxable income is broken into brackets, or ranges of income. Each bracket has a different tax rate. The tax brackets and rates for single filers as of 2022 are given below:

Marginal Tax Rate	For Single Filers	For Married Individuals Filing Joint Returns	For Heads of Households
10%	$\$0 - \$10,275$	$\$0 - \$20,550$	$\$0 - \$14,650$
12%	$\$10,275 - \$41,775$	$\$20,550 - \$83,550$	$\$14,650 - \$55,900$
22%	$\$41,775 - \$89,075$	$\$83,550 - \$178,150$	$\$55,900 - \$89,050$
24%	$\$89,075 - \$170,050$	$\$178,150 - \$340,100$	$\$89,050 - \$170,050$
32%	$\$170,050 - \$215,950$	$\$340,100 - \$431,900$	$\$170,050 - \$215,950$
35%	$\$215,950 - \$539,900$	$\$431,900 - \$647,850$	$\$215,950 - \$539,900$
37%	$\$539,900$ or more	$\$647,850$ or more	$\$539,900$ or more
Standard Deduction	$\mathbf{\$12,950}$	$\mathbf{\$25,900}$	$\mathbf{\$19,400}$

Question: So if your taxable income is $\$76,500$ and you file as a single filer, your tax bill will be 22% of that $\$76,500$, right?

Wrong. Your income is split among those brackets, and the money in each bracket is taxed at that bracket's marginal tax rate. Seems confusing. Here is a list of steps to follow to find the tax owed.

Why is the person in 22% tax bracket?

Since that person's taxable income $\$76,500$ falls in the single filer column in a 22% marginal rate row. We call the person whose taxable income falls in the 22% tax bracket. So that person needs to pay 10%, 12%, and 22% of his portion of the taxable income as shown in the following table.

The following calculation shows how to calculate the tax bill of the above single filer. That person's taxable income is in 22% tax bracket,

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$10,275 - \$0 = \$10,275$	$0.10 \times \$10,275 = \$1,027.50$
12%	$\$41,775 - \$10,275 = \$31,500$	$0.12 \times \$31,500 = \$3,780$
22%	$\$76,500 - \$41,775 = \$34,725$	$0.22 \times \$34,275 = \$7,639.50$

So the total tax bill will be

$$\$1,027.50 + \$3,780 + \$7,639.50 = \$12,447$$

We use the following steps to calculate a tax bill. It is important to understand how to calculate the marginal tax amount for each tax rate.

Step to Find the Tax Bill: 2021 Tax Rate

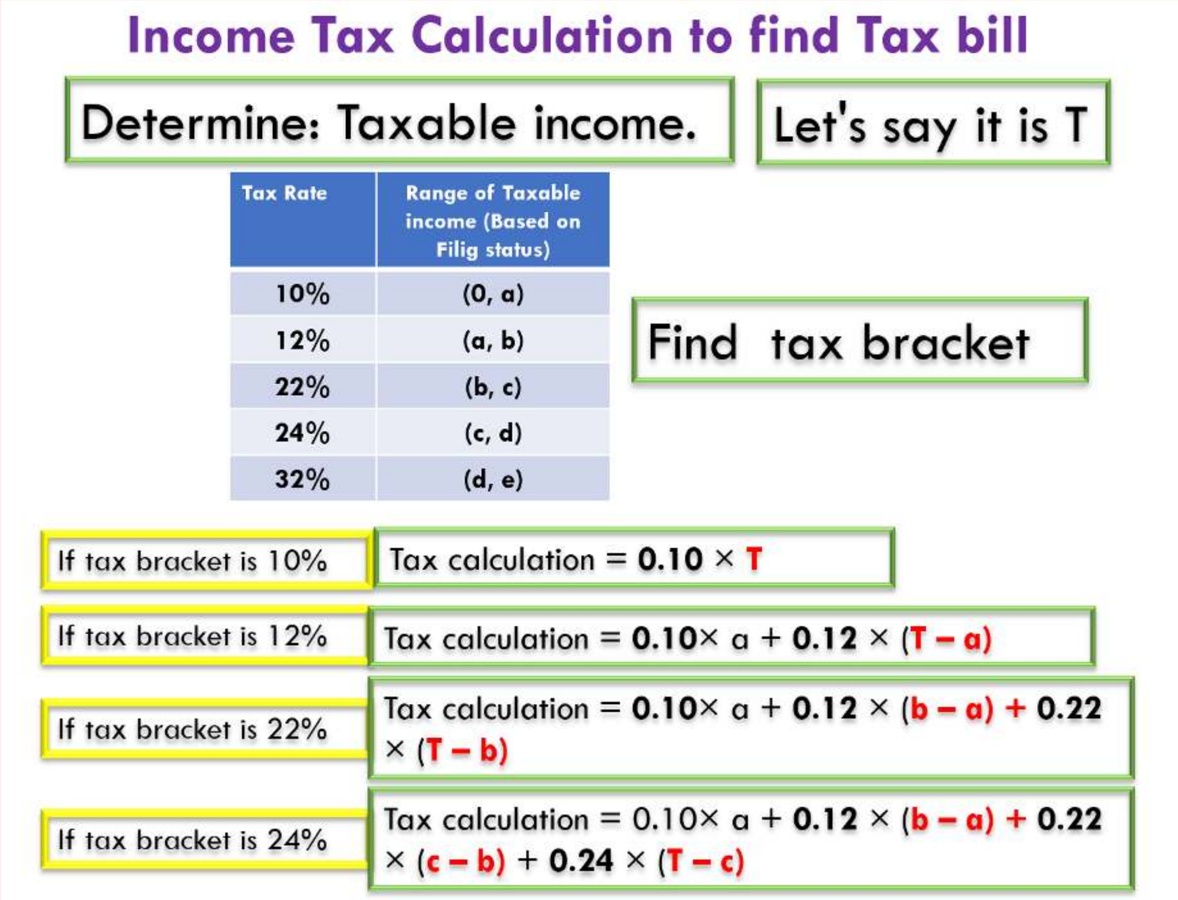

Figure $4.6.1$: Tax Calculation 2021 Tax Rate

There are various tax brackets, and the rates may change from one year to the next. The income limits may also change. For all examples going forward, we will use the single filer tax brackets, even if those brackets are not appropriate (e.g., married or head of household filers).

Example 4.6.6: Income Tax on Taxable Income

Use the 2022 Marginal Tax Rate Table to find the federal tax bill. Faith has a taxable income of $\$103,650$, and she is single.

Answer

$$0.10\times\$10,275+0.12\times(\$51,775-\$10,275)+0.22\times(\$89,075-\$41,775+0.24\times(\$103,650-\$89,075)=\$18,711.50 \tag{4.6.1}$$

Faith's taxable income falls in the marginal tax bracket of 24%. So she needs to pay 10%, 12%, 22%, and 24% of her portion of the taxable income as shown in the following table.

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$10,275-\$0=\$10,275$	$0.10\times\$10,275=\$1,027.50$
12%	$\$41,775-\$10,275=\$31,500$	$0.12\times\$31,500=\$3,780$
22%	$\$89,075-\$41,775=\$47,300$	$0.22\times\$47,300=\$10,406$
24%	$\$103,650-\$89,075=\$14,575$	$0.24\times14,575=\$3,498$

So Faith tax bill is

$$\$1,027.50+\$3,780+\$10,406+\$3,498=\$18,711.50$$

Your Turn 4.6.6: Determine Marginal Tax

Using the 2024 tax table, determine the marginal tax amount for each bracket and the total tax for a single person who has $177927 of taxable income.

Marginal Tax Rate **On Taxable Income**	**Filing Single**
10%	First $11,600
12%	$11,600 – $47,150
22%	$47,150 – $100,525
24%	$100,525 – $191,950
32%	$191,950 – $243,725
35%	$243,725 – $609,350
37%	Over $609,350

Determine the marginal tax amount for each bracket and calculate the total tax.

1. Marginal Tax amount of 10% = $
2. Marginal Tax amount of 12% =$
3. Marginal Tax amount of 22% =$
4. Marginal Tax amount of 24% =$

Total Tax $

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Tax Due a Refund or Owe Taxes

Remember that your employer will estimate how much tax you will owe and withhold it from your paychecks. This means you may have already paid some, if not all, and more of the income tax you owe.

If the total taxes owed after tax credit (if there is one) is more than the tax withholding (in W-2), the taxpayer is due a refund, and if the total taxes owed after tax credit (if there is one) is less than the tax withholding (in W-2), the taxpayer owes taxes.

Example 4.6.7: Finding Income Tax Owed

Emmanuel is preparing his taxes for the year 2022. His filing status is married jointly. His and his wife's income that is in W-2 for the year was $\$95,250$ He also has income from some freelance artwork he did, amounting to $\$7,500$, which is shown in the form 1099. Emmanuel also deposited $\$4,500$ into a tax-deferred retirement plan. He paid $\$7,920$ in mortgage interest for the year and $\$3,740$ in property taxes. He also qualified for $\$4,000$ in child tax credits. Based on this information, how much tax does Emmanuel owe, or how much does he get in a refund?

Answer

We need to know Emmanuel's taxable income, based on gross income, deductions, and exemptions.

His gross income is the amount from his W-2 and his 1099-MISC. Adding these gives a

$$\text{Gross income} = 95,250 + \$7,500 = \$102,750$$

$\$7,500$ is his adjustment because it is deposited in the tax-deferred retirement plan. So

$$\begin{aligned}\text{Adjusted Gross Income (AGI)} &= \text{Gross Income} - \text{adjustment} \\ &= \$102,750 - \$4,500 \\ &= \$98,250\end{aligned}$$

The amount $\$7,920$ for mortgage interest, $\$3,740$ for property taxes are itemized deduction.

$$\begin{aligned}\text{Itemize Deduction } &= \$7,920 + \$3,740 \\ &= \$11,660\end{aligned}$$

From the 2022 tax table, his standard deduction (for a married couple) is $\$25,900$, which is more than the itemized deduction. So we need to use the standard deduction to find the taxable income.

$$\begin{aligned}\text{Taxable Income} &= \$98,250 - \text{deduction} \\ &= \$98,250 - \$25,900 \\ &= \$72,350\end{aligned}$$

His taxable income belongs to the 12% marginal tax bracket.

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$20,550 - \$0 = \$20,550$	$0.10 \times \$20,550 = \$2,055$
12%	$\$72,350 - \$20,550 = \$51,800$	$0.12 \times \$51,800 = \$6,216$

Total tax computation $\$2,055 + \$6,216 = \$8,271$

Once his taxes are computed, he subtracts his tax credits. His only tax credit is $\$4,000$. The total he owes in taxes is

$$\text{Total tax owe } = \$8,271 - \$4,000 = \$4,271$$

So Emmanuel owe $\$4,271$ tax.

Your Turn $4.6.7$: Calculate Income Tax

Filing Status	Standard Deduction
Head of Household	$21,900
Single	$14,600
Married Filing Separately	$14,600
Married Filing Jointly	$29,200

Tax rate	Single	Head of household	Married filing jointly or qualified surviving spouse	Married fili separately
10%	$0 to $11,600	**$0–$16,550**	$0–$23,200	$0–$11,600
12%	$11,600 to $47,150	$16,550–$63,100	$23,200–$94,300	$11,600–$4
22%	$47,150 to $100,525	$63,100–$100,500	$94,300–$201,050	$47,150–$1
24%	$100,525 to $191,950	$100,500–$191,950	$201,050–$383,900	$100,525–$
32%	$191,500 to $243,725	$191,950–$243,700	$383,900–$487,450	$191,950–$

35%	$243,725 to $609,350	$243,700–$609,350	$487,450–$731,200	$243,725–$
37%	$609,350 or more	$609,350 or more	$731,200+ or more	$365,600 or

Tarana had $37,500 of income from wages and $2,000 of taxable interest. Tarana also made contributions of $3,500 to a tax-deferred retirement account. Tarana files as head of household.

What is Tarana's total income?

$

What is Tarana's adjusted gross income?

$

For Tarana's filing status, the standard deduction is $21,900. What is Tarana's taxable income?

$

Use the above tax table to find the income tax for Tarana filing as head of household. Round to the nearest dollar.

$

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Example 4.6.88: Refund or Tax Owe?

The Halls are a married couple with two children. Their total adjusted gross income is $181,770. They have following itemize deductions:

$$\begin{array}{l} \$9050 \text{ in property taxes} \\ \$9850 \text{ in charitable contributions} \\ \$910 \text{ in childcare expenses} \end{array} \tag{4.6.2}$$

In addition, they are eligible for a $2000 tax credit for each child. (For this problem, treat this credit as non-refundable.) Round all answers in this problem to the nearest cent, if necessary. To complete this problem, you will need to refer to the 2020 tax table.

2020 Marginal Tax Rates		
Rate	Single	Married Filing Jointly
10%	$0 – $9,875	$0 – $19,750
12%	$9,875 – $40,125	$19,750 – $80,250
22%	$40,125 – $85,525	$80,250 – $171,050
24%	$85,525 – $163,300	$171,050 – $326,600
32%	$163,300 – $207,350	$326,600 – $414,700
35%	$207,350 – $518,400	$414,700 – $622,050
37%	$518,400 or more	$622,050 or more

Standard Deductions	
Single	Married Filing Jointly
$\$12,400$	$\$24,800$

1. If they itemize their deductions, how much will the Hall' total deduction be?

2. Should they itemize their deductions or use the standard deduction?

3. What is the Hall' taxable income for 2020?

4. Calculate the Hall' total taxes owed, *before* tax credits.

5. Calculate the Hall' total taxes owed, *after* tax credits.

6. They have prepaid $\$22,253$ over the course of the year through withholding. Do the Hall owe taxes, or are they due a refund? How much do they owe, or what refund are they due?

Answer

1. Itemize deduction: $\$9,050+\$9,850+\$9,100=\$28,000$

2. They should use itemize deduction of $\$28,000$ since it is more then standard deduction $\$24,800$.

3. Taxable income is

$$\begin{aligned}\text{Taxable income} &= \text{AGI} - \text{deduction} \\ &= \$181,770-\$28,000 \\ &= \$153,770\end{aligned}$$

4. The Hall's taxable income belongs to the tax bracket of 22%.

Marginal Tax Rate	Taxable income that applies to the given marginal tax rate	Tax calculation
10%	$\$19,750-\$0=\$19,750$	$0.10\times\$19,750=\$1,975$
12%	$\$80,250-\$19,750=\$60,500$	$0.12\times\$60,500=\$7,260$
22%	$\$153,770-\$80,250=\$73,520$	$0.22\times\$73,520=\$16,174.40$

Calculate the Hall' total taxes owed, *before* tax credits, is $\$1,975+\$7,260+\$16,174.40=\$25,409.40$

5. Once his taxes are computed, he subtracts his tax credits. His only tax credit is $\$4,000$. Because $2\times\$2000$, since for each for each kid child tax credit is $\$2,000$.

$$\begin{aligned}\text{Total tax after tax credit } &= \$25,409.40-\$4,000 \\ &= \$21,409.40\end{aligned}$$

6. They have prepaid $\$22,253$ over the course of the year through withholding. That means they are already more than they are liable to. So they will get a refund of $\$22,253-21,409.40=\$843.60$

Your Turn 4.6.8: Tax Owe or Refund?

Tax Rates		
Rate	Single	Married Filing Jointly
10%	\$0 - \$9,875	\$0 - \$19,750
12%	\$9,875 - \$40,125	\$19,750 - \$80,250
22%	\$40,125 - \$85,525	\$80,250 - \$171,050
24%	\$85,525 - \$163,300	\$171,050 - \$326,600
32%	\$163,300 - \$207,350	\$326,600 - \$414,700
35%	\$207,350 - \$518,400	\$414,700 - \$622,050
37%	\$518,400 or more	\$622,050 or more

Standard Deductions	
Single	**Married Filing Jointly**
$12,400	$24,800

The Taylors are a married couple with three children. Their total adjusted gross income is $169,750.

If they itemize deductions, they can deduct:

- $8850 in charitable contributions
- $9700 in property taxes
- $6850 in business expenses

In addition, they are eligible for a $2000 tax credit for each child. (For this problem, treat this credit as non-refundable.)

Round all answers in this problem to the nearest cent, if necessary.

If they itemize their deductions, how much will the Taylors' total deduction be?

$

Should they itemize their deductions or use the standard deduction?

○ Itemized deductions

○ Standard deduction

○ It doesn't matter (the result will be the same)

What is the Taylors' taxable income for 2020?

$

Calculate the Taylors' total taxes owed, *before* tax credits.

$

Calculate the Taylors' total taxes owed, *after* tax credits.

$

They have prepaid $17,442 over the course of the year through withholding. Do the Taylors owe taxes, or are they due a refund?

○ Owe taxes

○ Due a refund

○ Neither

How much do they owe, or what refund are they due? (If neither, enter 0.)

$

Note: Correct answers shown for later questions are determined based on earlier answers.

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Your Turn 4.6.9: Check Your Understanding

Fill in the blanks below to produce a true statement.

1. FICA taxes are used to fund [Select an answer] and [Select an answer] services.

2. The portion of an employee's wages that an employer deducts and sends directly to the government as partial payment of the employee's income taxes is called a/an [Select an answer] .

3. Certain allowable deductions which can be subtraced from gross income are called a(an) [Select an answer] .

4. Is it true that you can subtract both the itemized deduction and the standard deduction during tax calculation?. [Select an answer] .

5. If the [Select an answer] credit is larger than the tax you owe, the extra amount is refunded to you.

6. If total taxes owe after tax credit (if there is) is more than tax withholding (in w2), tax payer [Select an answer] , and If total taxes owed after tax credit (if there is) is less than tax withholding (in w2), tax payer [Select an answer] .

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CHAPTER OVERVIEW

5: Voting Theory and Fairness Criteria

5.1: Voting Methods

Learning Objectives

1. Use the Plurality method to determine the winner of an election.
2. Use the Borda count method to determine the winner of an election.
3. Use the Instant Runoff Voting method to determine the winner of an election.
4. Use the Copland method to determine the winner of an election.

To begin, we will want more information than a traditional ballot typically provides. A traditional ballot usually asks you to pick your favorite from a list of choices. This ballot fails to provide any information on how a voter would rank the alternatives if their first choice were unsuccessful.

Preference ballot

A **preference ballot** is a type of voting system where voters rank candidates or options in order of preference, rather than selecting just one.

Example 5.1.1: Understanding Perference Ballot

A vacation club is trying to decide which destination to visit this year: Hawaii (H), Orlando (O), or Anaheim (A). Their votes are shown below:

	Bob	Ann	Marv	Alice	Eve	Omar	Lupe	Dave	Tish	Jim
1st choice	A	A	O	H	A	O	H	O	H	A
2nd choice	O	H	H	A	H	H	A	H	A	H
3rd choice	H	O	A	O	O	A	O	A	O	O

Answer

These individual ballots are typically combined into one **preference schedule (or preference table)**, which shows the number of voters in the top row that voted for each option:

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Notice that by totaling the vote counts across the top of the preference schedule, we can recover the total number of votes cast: 1 + 3 + 3 + 3 = 10 total votes.

Your Turn 5.1.1: Understanding Preference Table

Here is the preference schedule for a recent election among four candidates:

Number of voters	2	11	7	4	11	2	23
1st choice	F	D	X	X	K	F	X
2nd choice	D	F	F	D	D	X	K
3rd choice	K	K	K	F	X	D	F
4th choice	X	X	D	K	F	K	D

How many voters voted in this election?

How many first place votes are needed for a majority?

Which candidate/choice had the most first-place votes?

Which candidate/choice has the least first-place votes?

Which candidate/choice had the most last-place votes?

Which candidate/choice has the least last-place votes?

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The **Plurality Method**.

The voting method we're most familiar with in the United States is the **plurality method**.

How to find Winner by Plurality Method?

This method declares the choice with the most first-place votes as the winner. Ties are possible and must be settled through some sort of run-off vote.

This method is sometimes mistakenly referred to as the majority method or "majority rules," but it is not necessary for a choice to have gained a majority of votes to win.

A majority is over 50% of the total votes; a winner can have a **plurality** without having a majority.

Example $5.1.2$: Using Plularity Method

We had the preference table in our election, for example $5.1.1$, find the winner by the plurality method. Does that candidate have the majority of the first-place votes?

	1	3	3	3
1^{st} choice	A	A	O	H
2^{nd} choice	O	H	H	A
3^{rd} choice	H	O	A	O

Answer

For the plurality method, we only care about the first-choice options. Totaling them up:

Anaheim: $1+3=4$ first-choice votes

Orlando: 3 first-choice votes

Hawaii: 3 first-choice votes

Anaheim is the winner using the plurality voting method.

Notice that Anaheim won with 4 out of 10 votes, 40% of the votes, which is a plurality of the votes but not a majority.

Your Turn $5.1.2$: Plularity Method

A crafting club is selecting a new source for raw materials. Three options have been proposed. The preference schedule for the members of the club is:

Number of voters	19	21	25
1st choice	B	C	A
2nd choice	A	B	B
3rd choice	C	A	C

The club conducts a plurality vote. Source ______ wins with ______ votes.

If there is a Majority Winning Candidate, they must have ______ votes. There [Select an answer] a Majority Winning Candidate.

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Bora Count Method

Borda Count is another voting method, named for Jean-Charles de Borda, who developed the system in 1770.

How to Find Winner by Borda Count Method?

This method assigns points to candidates based on their ranking: 1 points for the last choice, 2 points for the second-to-last choice, and so on. The point values for all ballots are totaled, and the candidate with the largest point total is the winner.

Example 5.1.3:The Borda Count Method

A group of mathematicians is getting together for a conference. The members are coming from four cities: Seattle, Tacoma, Puyallup, and Olympia. Their approximate locations on a map are shown to the right.

The votes for where to hold the conference were:

	51	25	10	14
1^{st} choice	Seattle	Tacoma	Puyallup	Olympia
2^{nd} choice	Tacoma	Puyallup	Tacoma	Tacoma
3^{rd} choice	Olympia	Olympia	Olympia	Puyallup
4^{th} choice	Puyallup	Seattle	Seattle	Seattle

Answer

In each of the 51 ballots ranking Seattle first, Puyallup will be given 1 point, Olympia 2 points, Tacoma 3 points, and Seattle 4 points. Multiplying the points per vote times the number of votes allows us to calculate the points awarded:

	51	25	10	14
1^{st} choice 4 points	Seattle $4 \cdot 51 = 204$	Tacoma $4 \cdot 25 = 100$	Puyallup $4 \cdot 10 = 40$	Olympia $4 \cdot 14 = 56$
2^{nd} choice 3 points	Tacoma $3 \cdot 51 = 153$	Puyallup $3 \cdot 25 = 75$	Tacoma $3 \cdot 10 = 30$	Tacoma $3 \cdot 14 = 42$
3^{rd} choice 2 points	Olympia $2 \cdot 51 = 102$	Olympia $2 \cdot 25 = 50$	Olympia $2 \cdot 10 = 20$	Puyallup $2 \cdot 14 = 28$
4^{th} choice 1 point	Puyallup $1 \cdot 51 = 51$	Seattle $1 \cdot 25 = 25$	Seattle $1 \cdot 10 = 10$	Seattle $1 \cdot 14 = 14$

Adding up the points:

Seattle: $204 + 25 + 10 + 14 = 253$ points

Tacoma: $153 + 100 + 30 + 42 = 325$ points

Puyallup: $51 + 75 + 40 + 28 = 194$ points

Olympia: $102 + 50 + 20 + 56 = 228$ points

Under the Borda Count method, Tacoma is the winner of this vote.

Your Turn 5.1.3: Borda Count Method

Number of voters	12	7	17	8	13	20
1st choice	C	D	A	D	E	B
2nd choice	B	E	E	C	D	A
3rd choice	E	A	B	A	C	C
4th choice	A	B	C	B	B	D
5th choice	D	C	D	E	A	E

Find the points for each candidate and the winner using the Borda Count. Use the count that assigns 1 point to last place.

Points for candidate A:

Points for candidate B:

Points for candidate C:

Points for candidate D:

Points for candidate E:

The winner is
If there is a tie, enter a list, separated by commas, of all the candidates with the highest point total.

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The Instant Runoff Voting (Plurality with Elimination) Method

Instant Runoff Voting (IRV), also called Plurality with Elimination, is a modification of the plurality method that attempts to address the issue of insincere voting.

Insincere Voting

Situations like the one in example 6.1.4 in the previous section (see the table below), when there is more than one candidate that shares somewhat similar points of view, can lead to insincere voting. Insincere voting is when a person casts a ballot contrary to their actual preference for strategic purposes.

	342	214	298
1^{st} choice	Elle	Don	Key
2^{nd} choice	Don	Key	Don
3^{rd} choice	Key	Elle	Elle

In that case, the democratic leadership might realize that Don and Key will split the vote, and encourage voters to vote for Key by officially endorsing him. Not wanting to see their party lose the election, as happened in the scenario above, Don's supporters might insincerely vote for Key, effectively voting against Elle.

How to find the Winner by Instant Runoff Voting (IRV) Method?

In IRV (Plurality with Elimination), if the candidate or choice gets the majority of the first-place votes, that choice would be the winner. If not, the choice with the *least* first-place votes is then eliminated from the election, and any votes for that candidate are redistributed to the voter's next choice. This continues until a choice has a majority (over 50%).

This is similar to the idea of holding runoff elections, but since every voter's order of preference is recorded on the ballot, the runoff can be computed without requiring a second costly election.

This voting method is used in several political elections around the world, including the election of members of the Australian House of Representatives, and was used for county positions in Pierce County, Washington until it was eliminated by voters in 2009. A version of IRV is used by the International Olympic Committee to select host nations.

Example 5.1.4: Find the Winner Using IRV

Consider the preference schedule below, in which a company's advertising team is voting on five different advertising slogans: A, B, C, D, and E. The election result is summarized in the given preference table. Find the winner using IRV.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

Answer

If this were a plurality election, note that B would be the winner with 9 first-choice votes, compared to 6 for D, 4 for C, and 1 for E.

There are a total of $3+4+4+6+2+1=20$ votes. A majority would be 11 votes. No one yet has a majority, so we proceed to elimination rounds.

Round 1: We make our first elimination. Choice A has the fewest first-place votes (0 vote) , so we remove that choice

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C		D	C	E	
3rd choice		D	C			D
4th choice	D	B		E	C	B
5th choice	E	E	E	B	D	C

We then shift everyone's choices up to fill the gaps. There is still no choice with a majority, so we eliminate again.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	D	D	C	E	D
3rd choice	D	B	C	E	C	B
4th choice	E	E	E	B	D	C

Round 2: We make our second elimination. Choice E has the fewest first-place votes (1 vote) , so we remove that choice, shifting everyone's options to fill the gaps.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	D
2nd choice	C	D	D	C	C	B
3rd choice	D	B	C	B	D	C

Now B has 9 first-choice votes, C has 4 votes, and D has 7 votes. Still no majority, so we eliminate again.

Round 3: We make our third elimination. C has the fewest votes (4 votes).

	5	4	4	6	1
1st choice	B	D	B	D	D
2nd choice	D	B	D	B	B

D has gained a majority (11 votes) and is declared the winner under IRV.

Your Turn 5.1.4: IVR Method

We decide to find a winner using Instant Runoff Voting . Here are our initial results.

Round One:

Number of voters	9	6	4
1st choice	D	E	F
2nd choice	E	F	E
3rd choice	F	D	D

How may votes does each candidate have in the first place during the first round?

F:

E:

D:

Which candidate, if any, has the majority of first place votes?

☐ F ☐ E ☐ D ☐ No candidate

Which candidate, if any, will be eliminated?

☐ F ☐ D ☐ E ☐ No candidate

Round Two

How may votes in the first place does each candidate have during the second round?

If a candidate has already been eliminated, write DNE for that candidate's vote total. If a candidate won before this round was held, write DNE for all candidates' vote totals.

F: _____

E: _____

D: _____

Which candidate, if any, has the majority of first place votes?

☐ F ☐ E ☐ D ☐ No candidate

Who wins the election?

☐ E ☐ F ☐ D ☐ No winner

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Copland Method

So far none of our voting methods have satisfied the Condorcet Criterion. The Copeland Method specifically attempts to satisfy the Condorcet Criterion by looking at pairwise (one-to-one) comparisons.

How to Find the Winner by Copeland's (Pairwise Comparison Method) Method?

In this method, each pair of candidates is compared, using all preferences to determine which of the two is more preferred. The more preferred candidate is awarded 1 point. If there is a tie, each candidate is awarded $\frac{1}{2}$ point. After all pairwise comparisons are made, the candidate with the most points, and hence the most pairwise wins, is declared the winner.

Variations of Copeland's Method are used in many professional organizations, including the election of the Board of Trustees for the Wikimedia Foundation that runs Wikipedia.

Formula: Number of Comparisons

number of comparison for n candidate = $\frac{n(n-1)}{2}$

Example $5.1.5$: Copland Method

Consider our vacation group example from the beginning of the chapter. Determine the winner using Copeland's Method.

	1	3	3	3
1^{st} choice	A	A	O	H
2^{nd} choice	O	H	H	A
3^{rd} choice	H	O	A	O

Answer

We need to look at each choice and see which would win in a one-to-one comparison.

number of comparison for 3 candidate = $\frac{3(3-1)}{2} = 3$.

They are Hawaii vs Orlando, Hawaii vs Anaheim, and Orlando vs Anaheim.

You may recall we did this earlier when determining the Condorcet Winner. For example, comparing Hawaii vs Orlando, we see that 6 voters, those shaded below in the first table below, would prefer Hawaii to Orlando. Note that Hawaii doesn't have to be the voter's first choice – we're imagining that Anaheim wasn't an option. If it helps, you can imagine removing Anaheim, as in the second table below.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

	1	3	3	3
1st choice			O	H
2nd choice	O	H	H	
3rd choice	H	O		O

Based on this, in comparing Hawaii vs. Orlando, Hawaii wins and receives 1 point. Similarly, we compare Hawaii vs Anaheim, and Orlando vs Anaheim.

Hawaii vs Orlando:	$6(H)$ votes to $4(O)$ votes: Hawaii gets 1 point
Anaheim vs Orlando:	$7(A)$ votes to $3(O)$ votes: Anaheim gets 1 point
Hawaii vs Anaheim:	$6(H)$ votes to $4(A)$ votes: Hawaii gets 1 point

Total points for Hawaii is 2, Anaheim is 1, and Orlando is 0.

Hawaii is the winner under Copeland's Method, having earned the most points.

Notice this process is consistent with our determination of a Condorcet Winner.

Example 5.1.6: Copland Method

Consider the advertising group's vote we explored earlier. Determine the winner using Copeland's method.

	3	4	4	6	2	1
1st choice	B	C	B	D	B	E
2nd choice	C	A	D	C	E	A
3rd choice	A	D	C	A	A	D
4th choice	D	B	A	E	C	B
5th choice	E	E	E	B	D	C

Answer

With 5 candidates, there are 10 comparisons to make:

number of comparison for 5 candidate = $\frac{5(5-1)}{2} = 10$.

A vs B: 11 votes to 9 votes	A gets 1 point
A vs C: 3 votes to 17 votes	C gets 1 point
A vs D: 10 votes to 10 votes	A gets $\frac{1}{2}$ point, D gets $\frac{1}{2}$ point
A vs E: 17 votes to 3 votes	A gets 1 point
B vs C: 10 votes to 10 votes	B gets $\frac{1}{2}$ point, C gets $\frac{1}{2}$ point
B vs D: 9 votes to 11 votes	D gets 1 point
B vs E: 13 votes to 7 votes	B gets 1 point
C vs D: 9 votes to 11 votes	D gets 1 point
C vs E: 17 votes to 3 votes	C gets 1 point
D vs E: 17 votes to 3 votes	D gets 1 point

Totaling these up:

A gets $2\frac{1}{2}$ points

B gets $1\frac{1}{2}$ points

C gets $2\frac{1}{2}$ points

D gets $3\frac{1}{2}$ points

E gets 0 points

Using Copeland's Method, we declare D as the winner.

Notice that in this case, D is not a Condorcet Winner. While Copeland's method will also select a Condorcet Candidate as the winner, the method still works in cases where there is no Condorcet Winner.

Your Turn 5.1.6: Copland Method

Number of voters	33	12	35	37	16	28
1st choice	C	C	A	A	D	B
2nd choice	A	B	D	B	A	C
3rd choice	B	A	B	D	B	A
4th choice	D	D	C	C	C	D

Find the winner of the election using the Pairwise Comparison method. For each comparison, enter the number of times each candidate was preferred to the other.

A vs. B

Votes where A is preferred to B :

Votes where B is preferred to A :

A vs. C

Votes where A is preferred to C :

Votes where C is preferred to A :

A vs. D

Votes where A is preferred to D :

Votes where D is preferred to A :

B vs. C

Votes where B is preferred to C :

Votes where C is preferred to B :

B vs. D

Votes where B is preferred to D :

Votes where D is preferred to B :

C vs. D

Votes where C is preferred to D :

Votes where D is preferred to C :

Tally the results:

Points for A:

Points for B:

Points for C:

Points for D:

Who is the winner?

- ○ A
- ○ B
- ○ C
- ○ D

D

There is a tie

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[2] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See http://www.co.pierce.wa.us/xml/abtus...ec/summary.pdf

5.2: Fairness Criteria

Learning Objectives

1. Use the Condorcet criterion to determine the fairness of the voting system.
2. Use the Majority criterion to determine the fairness of the voting system.
3. Use the Monotonicity criterion to determine the fairness of the voting system.
4. Use the Independence of Irrelevant Alternatives (IIA) criterion to determine the fairness of the voting system.

Voting Fairness Criteria

Voting fairness criteria are rules used to judge whether a voting method produces fair and reasonable election results. In other words, they are standards used to decide whether an election method gives fair and logical outcomes. Sometimes, a voting method does not satisfy a fairness criterion. When this happens, we say the voting method **violates** that criterion, meaning the method fails to meet one of the rules that define a fair election outcome.

In other words, the election result produced by that voting method **contradicts what the fairness criterion states should happen**.

There are four fairness criteria.

1. Condorcet criterion (head-to-head)
2. Majority criterion
3. Monotonicity criterion
4. Independence of Irrelevant Alternatives (IIA) Criterion

What's Wrong With the Plurality Method?

The election from example 5.1.1 in the previous section may seem clean, but there is a problem lurking that arises whenever there are three or more choices. Looking back at our preference table, how would our members vote if they only had two choices? Let's compare the winner, Anaheim, to Orlando and Hawaii.

Anaheim vs Orlando: 7 out of the 10 would prefer Anaheim over Orlando

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Anaheim vs Hawaii: 6 out of 10 would prefer Hawaii over Anaheim

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

This doesn't seem right. Anaheim just won the election, yet 6 out of 10 voters, 60%, would have preferred Hawaii! That hardly seems fair. Marquis de Condorcet, a French philosopher, mathematician, and political scientist, wrote about how this could happen in 1785, and for him, we name our first **fairness criterion**.

Condorcet Criterion

If a choice is preferred in every one-to-one comparison with the other options, that choice should be the winner. We call this winner the **Condorcet Winner** or Condorcet Candidate. The Condorcet criterion is also called the **head-to-head** criterion.

Note: One-to-one comparison ≡ Head-to-head comparison.

Example 5.2.1: Find Condercet Winner

What choice is the Condorcet Winner in the following election?

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

Answer

To find the Condorcet choice, we compare every candidate to every other candidate. This is called a head-to-head comparison.

	1	3	3	3
1st choice	A	A	O	H
2nd choice	O	H	H	A
3rd choice	H	O	A	O

A vs H: 4 prefer A; 6 prefer H. H is preferred. In other words, Hawaii is the winner between Anaheim and Hawaii.

A vs O: 7 prefer A; 3 prefer O. A is preferred. In other words, Anaheim is the winner between Anaheim and Orlando.

O vs H: 4 prefer O; 6 prefer H. H is preferred. In other words, Hawaii is the winner between Orlando and Hawaii.

Hawaii is preferred (or you can say it wins) over every other candidate when we do a one-to-one comparison, so Hawaii is the Condorcet Winner.

When Does the Plurality Method Violate the Condorcet Criterion?

The plurality voting method meets (or satisfies) the Condorcet criterion if the winner according to this method is the same as the Condorcet winner. Otherwise, the voting method (Plurality) violates the criterion (Condorcet).

Example 5.2.2: The Plularity Method Violates the Condorcet Criterion

Consider a city council election in a district that is historically 60% Democratic voters and 40% Republican voters. Even though the city council is technically a nonpartisan office, people generally know the affiliations of the candidates. In this election, there are three candidates: Don and Key, both Democrats, and Elle, a Republican. A preference schedule for the votes looks as follows:

	342	214	298
1st choice	Elle	Don	Key
2nd choice	Don	Key	Don
3rd choice	Key	Elle	Elle

1. Who is the Condorcet candidate?
2. Under the Plurality method, who is the winner?
3. Does the plurality method violate the head-to-head criterion?

Answer

(a) We can see a total of $342+214+298=854$ voters participated in this election. Computing the percentage of first-place votes:

Don: $214/854=25.1\%$

Key: $298/854=34.9\%$

Elle: $342/854=40.0\%$

So in this election, the Democratic voters split their vote over the two Democratic candidates, allowing the Republican candidate Elle to win under the plurality method with 40% of the vote.

Analyzing this election closely, we see that it violates the Condorcet Criterion. Analyzing the one-to-one comparisons:

Elle vs Don: 342 prefer Elle; 512 prefer Don: Don is preferred

Elle vs Key: 342 prefer Elle; 512 prefer Key: Key is preferred

Don vs Key: 556 prefer Don; 298 prefer Key: Don is preferred

Don is the Condorcet winner, preferred in every one-to-one comparison with the other candidates.

(b) Let's count the most first-place votes

Don: 241; Key: 298; and Elle: 342. So the winner by the plurality method is Elle.

(c) YES. Since the winner using the plurality method (Elle) is not the same as the Condorcet winner (Don).

Your Turn 5.2.2: Plularity with Elimination and Head to Head

Four students are running for the position of student government president: Jeremiah (J), Gage (G), Francesca (F), and Latasha (L). A vote of all students was held and the results are summarized in the preference table below.

Number of Votes	28	30	23	43	43
First Choice	L	G	F	L	G
Second Choice	G	L	L	F	L
Third Choice	F	F	J	G	J
Fourth Choice	J	J	G	J	F

Fourth Choice	J	J	G	J	F

Using the plurality-with-elimination method, determine the winner of the election.

- ○ Francesca
- ○ Gage
- ○ Latasha
- ○ Jeremiah

Is the head-to-head criterion satisfied in this case? Explain why or why not.

- ○ Yes, since the candidate favored over the rest in a head-to-head comparison did not win the election when using the plurality-with-elimination method.
- ○ No, since the candidate favored over the rest in a head-to-head comparison won the election when using the plurality-with-elimination method.
- ○ No, since the candidate favored over the rest in a head-to-head comparison did not win the election when using the plurality-with-elimination method.
- ○ Yes, since the candidate favored over the rest in a head-to-head comparison won the election when using the plurality-with-elimination method.

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What's Wrong with Board Count?

You might have already noticed one potential flaw of the Borda Count from the previous example. In that example, Seattle had a majority of first-choice votes, yet lost the election! This seems odd and prompts our next fairness criterion:

Majority Criterion

If a choice has a majority of first-place votes, that choice should be the winner.

The election from example 5.2.1 using the Borda Count violates the Majority Criterion. Because the total number of voters is 100 and Seattle got 51, the majority of the first-place votes. But under the Borda Count method, Tacoma is the winner.

Notice also that this automatically means that the Condorcet Criterion will also be violated, as Seattle would have been preferred by 51% of voters in any head-to-head comparison.

Borda count is sometimes described as a consensus-based voting system, since it can sometimes choose a more broadly acceptable option over the one with majority support. In the example above, Tacoma is probably the best compromise location. This is a different approach from plurality and instant runoff voting, which focus on first-choice votes; Borda Count considers every voter's entire ranking to determine the outcome.

Because of this consensus behavior, Borda Count, or some variation, is commonly used in awarding sports awards. Variations are used to determine the Most Valuable Player in baseball, to rank teams in NCAA sports, and to award the Heisman trophy.

When Does the Borda Count Method Violate the Majority Criterion?

The Borda count voting method meets (or satisfies) the Majority criterion if the winner according to this method is the same as a candidate who got the majority of the first-place votes. Otherwise, the voting method (Borda count) violates the criterion (Majority).

Example 5.2.3: The Borda Count Violate Majority Criterion

Use the following preference table to answer the given questions.

Number of voters	16	20	11	13	7
1st choice	B	B	A	A	D
$2$and choice	A	A	C	D	C
3rd choice	C	D	D	C	A
4th choice	D	C	B	B	B

Answer the following question for the above preference table.

1. How many votes were cast?
2. How many votes are needed for a majority?
3. Does any candidate have a majority of first-place votes? If so, who is that candidate?
4. Find the winner under the Borda Count. Use the count that assigns 1 point to the last place.
5. Is the majority criterion violated by the Borda Count in this election?

Answer

1. Total number of voter are: $16+20+11+13+7=67$.
2. To get the majority, one should get more than $\frac{67}{2}=33.5$. So 34 or more should be needed in the first place to get a majority.
3. YES. B gets a majority. Since candidate B gets $16+20=36$, the most first-place votes, but this is a majority since it is more than 34.
4. Assign the points to candidates based on their ranking: 1 point for the last choice, 2 points for the second-to-last choice, and so on.

Number of voters	16	20	11	13	7
1st choice (4 points)	B 16× 4= 64	B 20× 4= 80	A 11× 4= 44	A 13× 4= 52	D 7× 4= 28
2nd choice (3 points)	A 16× 3= 48	A 20× 3= 60	C 11× 3= 33	D 13× 3= 39	C 7× 3= 21
3rd choice (2 points)	C 16× 2= 32	D 20× 2= 40	D 11× 2= 22	C 13× 2= 26	A 7× 2= 14

Number of voters	16	20	11	13	7
points)					
4th choice (1 points)	D $16\times 1 = 16$	C $20\times 1 = 20$	B $11\times 1 = 11$	B $13\times 1 = 13$	B $7\times 1 = 7$

Adding up the points:

A: $48+60+44+52+14=218$ points
B: $64+80+11+13+7=175$ points
C: $32+20+33+26+21=132$ points
D: $16+40+22+39+28=145$ points

5. The majority criterion is violated by the Borda Count in this election because the Borda Count winner (A) is not the same as the majority winner (B).

Your Turn 5.2.3: Borda Count and Majority

Number of voters	1	1	3	7	6	14
1st choice	B	A	D	C	E	B
2nd choice	D	C	E	B	A	C
3rd choice	A	E	C	D	D	A
4th choice	E	D	A	E	B	E
5th choice	C	B	B	A	C	D

Find the points for each candidate and the winner using the Borda Count. Use the count that assigns 1 point to last place.

Points for candidate A:

Points for candidate B:

Points for candidate C:

Points for candidate D:

Points for candidate E:

The winner is

Does this violate the Majority Criteria?

If there is a tie, enter a list, separated by commas, of all the candidates with the highest point total.

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What's Wrong with Plurality with the Elimination Method?

Let's return to our City Council Election

	342	214	298
1^{st} choice	Elle	Don	Key
2^{nd} choice	Don	Key	Don
3^{rd} choice	Key	Elle	Elle

In this election, Don has the smallest number of first-place votes, so Don is eliminated in the first round. The 214 people who voted for Don have their votes transferred to their second choice, Key.

	342	512
1^{st} choice	Elle	Key
2^{nd} choice	Key	Elle

So Key is the winner using the IRV method.

We can immediately notice that IRV violates the Condorcet Criterion in this election since we determined earlier that Don was the Condorcet winner.

On the other hand, the temptation has been removed for Don's supporters to vote for Key; they now know their vote will be transferred to Key, not simply discarded.

Monotonicity Criterion

If voters change their votes to increase their preference for a candidate, it should not harm that candidate's chances of winning.

When Does the IRV Method Satisfy the Monotonicity Criterion?

The IRV method meets (or satisfies) the monotonicity criterion if the winner, according to this method, is the same even after people increase their preference for the candidate who is more likely to win. Otherwise, the voting method (IRV) violates the criterion (Monotonicity).

Example 5.2.4: The IRV Satisfy the Monotonicity Criterion

[1]A class is voting to choose a new class mascot, and the three nominations are Kudu (K), Ferret (F), and Gopher (G). A vote of the entire class was held, and the results are summarized in the preference table below.

Number of Votes	37	40	34	36	32	44
First Choice	F	K	G	F	K	G
Second Choice	K	G	F	G	F	K
Third Choice	G	F	K	K	G	F

1. Using the plurality-with-elimination method, determine the winner of the team mascot election.
2. In the second election, suppose that the 36 students who initially voted for F, G, K (in that order) decided to change their votes to G, K, F (in that order). Using the plurality-with-elimination method and updated voting preferences, determine the winner of the team mascot election.
3. Determine if the monotonicity criterion is satisfied in this case. Explain why or why not.

Answer 1

Total number of voters $37+40+34+36+32+44=223$. The first-place votes must be 112 or higher to get a majority and win by the IRV method. Let's count the first-place votes.

F: $37+36=73$, K: $40+32=72$, and G: $34+44=78$. No one get more than or equal to 112. We make our first elimination. Choice K has the fewest first-place votes, so we remove that choice and then shift everyone's choices up to fill the gaps.

	37	40	34	36	32	44
1^{st} choice	F	G	G	F	F	G
2^{nd} choice	G	F	F	G	G	F

F: $37+32+36=105$, and G: $40+34+44=118$. The candidate G got 118 and it is the majority, so G is the winner by the IRV method.

Answer 2

In the second election, the 36 students who initially voted for F, G, K (in that order) decided to change their votes to G, K, F (in that order). Now we create a new preference table as given below.

Number of Votes	37	40	34	36	32	44
First Choice	F	K	G	**G**	K	G
Second Choice	K	G	F	**K**	F	K
Third Choice	G	F	K	**F**	G	F

F: 37, K: $40+32=72$, and G: $34+36+44=114$. The candidate G got 114 and it is the majority, so G is the winner by the IRV method in the second election.

Answer 3

Yes, the monotonicity criterion is satisfied because the winner of the second election was the same as the winner of the first election after the ballots were changed in favor of the first winner

In example $5.2.2$ above, the IRV method satisfies the monotonicity criterion, but that would not always be true. The example below shows that the IRV method could violate the monotonicity criterion.

Your Turn $5.2.4$: Monotonicity and Plurality with Elimination

A sports team is voting to choose a new class mascot and the four nominations are Pangolin (P), Zebra (Z), Meerkat (M), and Gopher (G). A vote of the entire team was held and the results are summarized in the preference table below.

Number of Votes	43	11	12	49
First Choice	Z	P	Z	M
Second Choice	G	G	M	Z
Third Choice	M	M	G	G
Fourth Choice	P	Z	P	P

Using the plurality-with-elimination method, determine the winner of team mascot election.

- ○ Zebra
- ○ Pangolin
- ○ Gopher
- ○ Meerkat

Meerkat

Suppose that the 12 team members who initially voted for Z, M, G, P (in that order) decided to change their votes to M, Z, P, G (in that order). Using the plurality-with-elimination method and updated voting preferences, determine the winner of team mascot election.

- Pangolin
- Gopher
- Meerkat
- Zebra

Determine if the monotonicity criterion is satisfied in this case. Explain why or why not.

- No, the monotonicity criterion is not satisfied because winner of the second election was the same as the winner of the first election, after the ballots were changed in favor of the first winner.
- Yes, the monotonicity criterion is satisfied because winner of the second election was the not the same as the winner of the first election, after the ballots were changed in favor of the first winner.
- No, the monotonicity criterion is not satisfied because winner of the second election was the not the same as the winner of the first election, after the ballots were changed in favor of the first winner.
- Yes, the monotonicity criterion is satisfied because winner of the second election was the same as the winner of the first election, after the ballots were changed in favor of the first winner.

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What's Wrong with Copland's Method?

As already noted, Copeland's Method does satisfy the Condorcet Criterion. It also satisfies the Majority Criterion and the Monotonicity Criterion. So is this the perfect method? Well, in a word, NO.

Example 5.2.5: Example that Leads to IIA Criterion

A committee is trying to award a scholarship to one of four students, Anna (A), Brian (B), Carlos (C), and Dimitry (D). The votes are shown below:

	5	5	6	4
1^{st} choice	D	A	C	B
2^{nd} choice	A	C	B	D
3^{rd} choice	C	B	D	A
4^{th} choice	B	D	A	C

Answer

Making the comparisons:

$$\begin{array}{ll} \text{A vs B: 10 votes to 10 votes} & \text{A gets } \frac{1}{2} \text{ point, B gets } \frac{1}{2} \text{ point} \\ \text{A vs C: 14 votes to 6 votes:} & \text{A gets 1 point} \\ \text{A vs D: 5 votes to 15 votes:} & \text{D gets 1 point} \\ \text{B vs C: 4 votes to 16 votes:} & \text{C gets 1 point} \\ \text{B vs D: 15 votes to 5 votes:} & \text{B gets 1 point} \\ \text{C vs D: 11 votes to 9 votes:} & \text{C gets 1 point} \end{array}$$

Totaling:

$$\begin{array}{ll} \text{A has } 1\frac{1}{2} \text{ points} & \text{B has } 1\frac{1}{2} \text{ points} \\ \text{C has 2 points} & \text{D has 1 point} \end{array}$$

So Carlos is awarded the scholarship. However, the committee then discovers that Dimitry was not eligible for the scholarship (he failed his last math class). Even though this seems like it shouldn't affect the outcome, the committee decides to recount the vote, removing Dimitry from consideration. This reduces the preference schedule to:

	5	5	6	4
1st choice	A	A	C	B
2nd choice	C	C	B	A
3rd choice	B	B	A	C

A vs B: 10 votes to 10 votes A gets $\frac{1}{2}$ point, B gets $\frac{1}{2}$ point
A vs C: 14 votes to 6 votes A gets 1 point
B vs C: 4 votes to 16 votes C gets 1 point

Totaling:

A has $1\frac{1}{2}$ points B has $\frac{1}{2}$ point
C has 1 point

Suddenly, Anna is the winner! This leads us to another fairness criterion.

The Independence of Irrelevant Alternatives (IIA) Criterion

If a non-winning choice is removed from the ballot, it should not change the election's winner.

If choice A is preferred over choice B, introducing or removing choice C should not cause B to be preferred over A.

This anecdote illustrating the IIA issue is attributed to Sidney Morgenbesser: After dinner, Sidney Morgenbesser decides to order dessert. The waitress tells him he has two choices: apple pie and blueberry pie. Sidney orders the apple pie. After a few minutes, the waitress returns and says that they also have cherry pie, at which point Morgenbesser says, "In that case, I'll have the blueberry pie."

Another disadvantage of Copeland's Method is that it is fairly easy for the election to end in a tie. For this reason, Copeland's method is usually the first part of a more advanced method that uses more sophisticated methods for breaking ties and determining the winner when there is no Condorcet Candidate.

When Does the Copland Method Violate the IIA Criterion?

The Copland voting method meets (or satisfies) the IIA criterion if the winner according to this method is the same as the winner after the non-winning choice is removed. Otherwise, the voting method (Copland) violates the criterion (IIA).

Your Turn 5.2.5: Copland and IAA

Four candidates are running for the mayor of a small town. The candidates are Ben (B), Cheick (C), Shawn (S) and Ermias (E). A vote of all town residents was held and the results are summarized in the preference table below.

Number of Votes	51	74	61	64
First Choice	E	S	B	E
Second Choice	B	C	C	B
Third Choice	S	E	S	C
Fourth Choice	C	B	E	S

Using the pairwise comparison method, determine winner of the mayoral race.

- ◯ Ben
- ◯ Cheick
- ◯ Shawn
- ◯ Ermias

Suppose that, prior to the announcement of the winner, Shawn and Cheick decide to drop from the mayoral race. Using the pairwise comparison method, determine winner of the mayoral race after Shawn and Cheick are no longer in the running.

- ◯ Ben
- ◯ Cheick
- ◯ Shawn
- ◯ Ermias

Determine if the irrelevant alternatives criterion is met in this case. Explain why or why not.

◯ Yes, the irrelevant alternatives criterion is satisfied, since the winner of the first election did not win the second election, after two of the initial non-winning candidates dropped from the race.

◯ Yes, the irrelevant alternatives criterion is satisfied, since the winner of the first election also won the second election, after two of the initial non-winning candidates dropped from the race.

◯ No, the irrelevant alternatives criterion is not satisfied, since the winner of the first election also won the second election, after two of the initial non-winning candidates dropped from the race.

◯ No, the irrelevant alternatives criterion is not satisfied, since the winner of the first election did not win the second election, after two of the initial non-winning candidates dropped from the race.

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Where is the Fair Method?

At this point, you're probably asking why we keep looking at method after method just to point out that they are not fully fair. We must be holding out on the perfect method.

Unfortunately, no. A mathematical economist, Kenneth Arrow, was able to prove in 1949 that there is <u>no</u> voting method that will satisfy all the fairness criteria we have discussed.

Arrow's Impossibility Theorem

Arrow's Impossibility Theorem states, roughly, that it is not possible for a voting method to satisfy every fairness criterion that we've discussed.

To see a very simple example of how difficult voting can be, consider the election below:

	5	5	5
1^{st} choice	A	C	B
2^{nd} choice	B	A	C
3^{rd} choice	C	B	A

Notice that in this election:

10 people prefer A to B

10 people prefer B to C

10 people prefer C to A

No matter whom we choose as the winner, $\frac{2}{3}$ of voters would prefer someone else! This scenario is dubbed **Condorcet's Voting Paradox**, and demonstrates how voting preferences are not transitive (just because A is preferred over B, and B over C, does not mean A is preferred over C). In this election, there is no fair resolution.

It is because of this impossibility of a totally fair method that Plurality, IRV, Borda Count, Copeland's Method, and dozens of variants are all still used. Usually, the decision of which method to use is based on what seems most fair for the situation in which it is being applied.

Summary of Violation of Fairness Criteria

Summary of Violations of Fairness Criteria

Plurality Borda Count	Plurality with Elimination (IRA)	Pairwise Comparisons (Copland)
Majority Criterion	*	*
Condorcet Criterion (Hea	Violation Possible	*

Borda Count / Plurality	Plurality with Elimination (IRA)	Pairwise Comparisons (Copland)
d to Head)		
Monotonicity Criterion	Violation Possible	*
IIA Criterion	Violation Possible	Violation Possible

Borda Count / Plurality	Plurality with Elimination (IRA)	Pairwise Comparisons (Copland)
ossible		

** The indicated voting method does not violate the indicated criterion in any election.*

Your Turn 5.2.6: Understaing Fairness Criteria

Which Criteria is the following?

"If a choice has a majority of first-place votes, that choice should be the winner".

Select an answer

Which Criteria is the following?

"If a candidate is favored when compared separately with every other candidate in an election, then that candidate should be declared the winner."

Select an answer

Which Criteria is the following?

"If voters change their votes to increase the preference for a candidate, it should not harm that candidate's chances of winning."

Select an answer

Which Criteria is the following?

"If a non-winning choice is removed from the ballot, it should not change the election's winner.If choice A is preferred over choice B, introducing or removing choice C should not cause B to be preferred over A."

Select an answer

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[2] This data is loosely based on the 2008 County Executive election in Pierce County, Washington. See http://www.co.pierce.wa.us/xml/abtus...ec/summary.pdf

CHAPTER OVERVIEW

6: Apportionment Method and Paradox

6.1: Apportionment Methods

Learning Objectives

1. Find the standard advisor and standard quota.
2. Apply Hamilton's method to solve the apportionment problem.
3. Apply Jefferson's method to solve the apportionment problem.
4. Apply Adam's method to solve the apportionment problem.
5. Apply Webster's method to solve the apportionment problem.

What is the Apportionment Problem?

Apportionment is the problem of dividing a fixed number of items among groups of varying sizes. In politics, this takes the form of allocating a limited number of representatives among voters. This problem, presumably, is older than the United States, but the best-known ways to solve it have their origins in the problem of assigning each state an appropriate number of representatives in the new Congress when the country was formed. States also face this apportionment problem in defining how to draw districts for state representatives. The apportionment problem also arises in various non-political contexts.

An **apportionment problem** is a mathematical method used to decide **how to fairly distribute a fixed number of items as a whole number (**you can't assign 2.4 items) (usually seats in a legislature) among groups (usually states or regions) **based on their population**.

- Assigning seats in the **U.S. House of Representatives**
- Dividing representatives in the student government
- Allocating funding to departments based on population or size
- Assigning voting power in committees

Standard Divisor and Standard Quota

The **standard divisor** is a concept used in **apportionment problems**, particularly when dividing the number of allocated items (seats or quotas) fairly among different groups based on population. The **standard divisor** is a key value used in apportionment problems to help decide how many seats each group should receive.

Formula: Standard Divisor

$$\text{Standard Divisor (SD)} = \frac{\text{Total Population}}{\text{Number of quota}}$$

Formula: Standard Quota of a Group

The standard quota for a particular group can be found by using the following formula

$$\text{Standard Quota for Particular Group} = \frac{\text{Population of that group}}{\text{Standard Divisor (SD)}}$$

Note: If we round the standard quota DOWN to the nearest whole number, we call it standard lower quota or (Just lower quota), and if we round the standard quota UP to the nearest whole number, we call it standard upper quota or (Just upper quota).

Example 6.1.1: Finding Standard Divisor and Standard Quota

Delaware has three counties: Kent, New Castle, and Sussex. The Delaware State House of Representatives has 41 members. Find the standard divisor, standard quota for each county, upper quota, and lower quota.

County	Population
Kent	$162,310$
New Castle	$538,479$
Sussex	$197,145$
Total	$\mathbf{897,934}$

Answer

First, we determine the standard divisor = $\frac{897,934}{41} = 21,900.82927$

Determine each county's standard quota by dividing the country's population by the standard divisor, and find the upper and lower quotas.

County	Population	Quota	Lower Qutoa	Upper Quota
Kent	$162,310$	7.4111	7	8
New Castle	$538,479$	24.5872	24	25
Sussex	$197,145$	9.0017	9	10
Total	$\mathbf{897,934}$		40	43

Your Turn 6.1.1: Standard Divisor and Standard Quota

Four states (A, B, C, and D) comprise a small country. The population of each state (in thousands) is given in the table below.

State	A	B	C	D	Total
Population (in thousands)	1300	390	1274	208	3172

According to the country's constitution, its congress shall have 26 seats, divided amongst the four states according to their respective populations.

Determine the standard divisor, in thousands, and the number of people represented by each seat.

The standard divisor is ______ and each seat represents ______ people.

Determine the standard quota, lower quota, and upper quota of each state. Fill in the blanks below. Round solutions to the nearest hundredth, if necessary.

State	A	B	C	D	Total
Population (in thousands)	1300	390	1274	208	3172
Standard Quota					
Lower Quota					
Upper Quota					

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As we see in example 6.1.1,, rounding the standard quota to the upper and lower quotas does not give the total number of representatives in the Delaware State House. Now, we will discuss how to solve the apportionment problem using various methods. The first one is the Hamilton Method.

Alexander Hamilton proposed the method that now bears his name. His method was approved by Congress in 1791, but was vetoed by President Washington. It was later adopted in 1852 and used through 1911. He begins by determining, to several decimal places, how many things each group should get. Since he was interested in the question of Congressional representation, we'll use the language of states and representatives, so he can determine how many representatives each state should get. He follows these steps:

Steps to Apportion by Hamilton's Method

There are five steps we follow when applying Hamilton's Method of apportionment:

Step 1: Find the standard divisor.

Step 2: Find each state's standard quota.

Step 3: Give each state the state's lower quota (with each state receiving at least one seat).

Step 4: Give each remaining seat one at a time to the states with ***the largest fractional parts*** of their standard quotas until no seats remain.

Step 5: Check the solution by confirming that the sum of the modified quotas equals the house size from step 4.

Note on rounding: Today, we have technological advantages that Hamilton (and the others) couldn't have imagined. Please take advantage of them, and keep them in several decimal places.

Example 6.1.2: Finding Standard Divisor and Standard Quota

Use Hamilton's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	**1,052,567**

Answer

1. The Standard divisor = $\frac{1,052,567}{75} = 14,034.22667$

2. Determine each county's quota by dividing its population by the divisor:

County	Population	Standard Quota
Bristol	49,875	3.5538
Kent	166,158	11.8395
Newport	82,888	5.9061
Providence	626,667	44.6528
Washington	126,979	9.0478
Total	**1,052,567**	

3. Remove the decimal part of each quota (That is, find the lower quota):

County	Population	Standard Quota	Initial (Lower Quota)
Bristol	49,875	3.5538	3
Kent	166,158	11.8395	11
Newport	82,888	5.9061	5
Providence	626,667	44.6528	44
Washington	126,979	9.0478	9
Total	**1,052,567**		**72**

4. We need 75 representatives and we only have 72, so we assign the remaining three, one each, to the three counties with the largest decimal parts, which are Newport, Kent, and Providence:

County	Population	Standard Quota	Initial (Lower Quota)	Final Quota
Bristol	49,875	3.5538	3	3
Kent	166,158	11.8395	11	$11+1=12$
Newport	82,888	5.9061	5	$5+1=6$
Providence	626,667	44.6528	44	$44+1=45$
Washington	126,979	9.0478	9	9
Total	**1,052,567**		**72**	**75**

Note that even though Bristol County's decimal part is greater than 0.5, it isn't big enough to get an additional representative, because three other counties have greater decimal parts.

Calculator for Hamilton's Method

Hamilton's Method

Province	A	B	C	D	E	F	Total
Population	251	379	154	228	195	217	1424
Number of seats: 100			Standard divisor: 14.24000			Reset	Islands
Exact quota	17.62640	26.61517	10.81461	16.01124	13.69382	15.23876	100
Lower quota	17	26	10	16	13	15	97
Frac. part	0.62640	0.61517	0.81461	0.01124	0.69382	0.23876	3
Surplus	1		1		1		3
Total	18	26	11	16	14	15	100

Your Turn 6.1.2: Hamilton Method

Five territories (A, B, C, D, and E) comprise a small country. The population of each territory (in thousands) is given in the table below.

Territory	A	B	C	D	E	Total
Population (in thousands)	2790	3060	1260	3870	2160	13140

According to the country's constitution, its congress shall have 90 seats, divided amongst the five territories according to their respective populations. Use Hamilton's method to determine each territory's apportionment of congressional seats.

Territory	A	B	C	D	E
Population (in thousands)	2790	3060	1260	3870	2160
Allocated Seats					

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Hamilton's method obeys something called the Quota Rule. The Quota Rule isn't a law of any sort, but just an idea that some people, including Hamilton, think is a good one.

Quota Rule

The Quota Rule says that the final number of representatives a state gets should be within one of that state's quota. Since we're dealing with whole numbers for our final answers, each state should either go up to the next whole number above its quota or down to the next whole number below its quota.

For example, let's say there are 100 seats to distribute and 5 states. If State A's population justifies exactly 9.4 seats, the Quota Rule says: State A should get **either** 9 **or** 10 seats. Getting **less than nine** or **more than** 10 would violate the rule. The Hamilton Method always satisfies the quota rule.

Jefferson's Method

Thomas Jefferson proposed a different method for apportionment. After Washington vetoed Hamilton's method, Jefferson's method was adopted and used in Congress from 1791 through 1842. Jefferson had political reasons for wanting his method to be used rather than Hamilton's. Primarily, his method favors larger states, and his own home state of Virginia was the largest in the country at the time. He would also argue that it's the ratio of people to representatives that is the critical thing, and apportionment methods should be based on that. However, the paradoxes we saw also provide mathematical reasons for concluding that Hamilton's method isn't so good, and while Jefferson's method might or might not be the best one to replace it, at least we should look for other possibilities.

The first steps of Jefferson's method are the same as Hamilton's method. He finds the same divisor and the same quota, and cuts off the decimal parts in the same way, giving a total number of representatives that is less than the required total. The difference is in how Jefferson resolves that difference. He says that since we ended up with a too small answer, our divisor must have been too big. He changes the divisor by making it smaller, finding new quotas with the new divisor, cutting off the decimal parts, and looking at the new total, until he finds a divisor that produces the required total.

Steps to Apportion by Jefferson's Method (Round Down)

Step 1: Find the standard divisor.

Step 2: Choose a modified divisor slightly less than the standard divisor.

Step 3: Find the states' **or group lower quota** and their sum (round down to the nearest whole number). Add all the lower quota numbers.

Step 4: If the total lower quota (after rounding) is **more** than available, **increase the divisor** (making quotas smaller). If the total is **less**, **decrease the divisor** (making quotas larger). Repeat until the sum of the lower quotas equals the total quota.

Step 5: Each group quota from step 4 is the final apportionment (or final quota).

Which Divisor is Higher? Standard or Modified?

In Jefferson's method, a modified divisor is less than the standard divisor.

Example 6.1.3: Jefferson's Method

Use Jefferson's method to apportion the 25 seats (senators) among six states of the small country.

Table $6.1.1$: Populations by State

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$

Answer

Standard divisor is $\frac{237,000}{25} = 9480$.

In Jefferson's method, the modified divisor is always smaller than the standard divisor. There is no formula to find the modified divisor, so we start by guessing.

Let's try the modified divisor, $d = 9000$.

Table $6.1.2$: Quotas for $d = 9000$

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$
Modified Quota $d = 9000$	2.67	6.22	3.11	1.89	7.22	5.22	
Lower Quota	2	6	3	1	7	5	24

The sum 24 is smaller than the given quota 25, so we must try again by making the modified divisor smaller than 9000. Let's try $d = 8000$.

Table $6.1.3$: Quotas for $d = 8000$

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$
Modified Quota $d = 8000$	3.00	7.00	3.50	2.13	8.13	5.88	
Lower Quota	3	7	3	2	8	5	28

This time, the sum of 28 is too big. Try again, making the modified divisor larger. We know the divisor must be between $d = 8000$ and $d = 9000$ so let's try $d = 8500$.

Table $6.1.4$: Quotas for $d = 8500$

State	A	B	C	D	E	F	Total
Population	$24,000$	$56,000$	$28,000$	$17,000$	$65,000$	$47,000$	$237,000$
Modified Quota $d = 8500$	2.82	6.59	3.29	2.00	7.65	5.53	
Lower Quota Final apportion	2	6	3	2	7	5	25

This time the sum is 25, so we are done. A gets two senators, B gets six senators, C gets three senators, D gets two senators, E gets seven senators, and F gets five senators.

Any modified divisor between $8,200$ and $8,500$ should work in this problem.

Note: You can double-check the above calculation using the following Jefferson Method Calculator.

Calculator for Jefferson's Method

Example 6.1.4: Jefferson's Method

Use Jefferson's method to apportion the 75 seats of Rhode Island's House of Representatives among its five counties.

County	Population
Bristol	49,875
Kent	166,158
Newport	82,888
Providence	626,667
Washington	126,979
Total	**1,052,567**

Answer

First, find the standard divisor. The Standard divisor = $\frac{1{,}052{,}567}{75} = 14{,}034.22667$

We need 75 representatives and only have 75, so we need to use a smaller divisor. Let's try 13,500.

County	Population	Modified Quota	Lower quota
Bristol	49,875	3.6944	3
Kent	166,158	12.3080	12
Newport	82,888	6.1399	6
Providence	626,667	46.4198	46
Washington	126,979	9.4059	9
Total	**1,052,567**		**76**

We've gone too far. We need a divisor that's greater than 13,500 but less than 14,034.22667. Let's try 13,700.

County	Population	Modified Quota	Lower quota (Final apportion)
Bristol	49,875	3.6405	3
Kent	166,158	12.1283	12
Newport	82,888	6.0502	6
Providence	626,667	45.7421	45
Washington	126,979	9.2685	9
Total	**1,052,567**		**75**

This works.

Any modified divisor between 13,650 and 13,800 should work in this problem.

Your Turn 6.1.4: Jefferson Method

Five territories (A, B, C, D, and E) comprise a small country. The population of each territory is given in the table below.

Territory	A	B	C	D	E	Total
Population	107,409	589,941	617,443	875,307	735,021	2,925,121

According to the country's constitution, its congress shall have 98 seats, divided amongst the five territories according to their respective populations. Use Jefferson's method to determine each territory's apportionment of congressional seats.

Territory	A	B	C	D	E
Population	107,409	589,941	617,443	875,307	735,021
Allocated Seats					

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Notice, in comparison to Hamilton's method, that although the results were the same, they came about in a different way, and the outcome was almost different. Providence County (the largest) almost went up to 46 representatives before Kent (which is much smaller) got to 12. Although that didn't happen here, it can. Divisor-adjusting methods, such as Jefferson's, are not guaranteed to follow the quota rule.

Adam's Method:

Adams's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, Adams used the upper quotas. Adam's method of apportionment is very similar to Jefferson's, but with a twist in how it rounds numbers. It is named after John Quincy Adams, the 6th President of the U.S.

Steps to Apportion by Adam's Method

Step 1: Find the standard divisor.

Step 2: Choose a modified divisor slightly greater than the standard divisor.

Step 3: Find the states' **or group upper quota** and their sum (round UP to the nearest whole number). Add all the upper quota numbers.

Step 4: If the total upper quota (after rounding) is **more** than available, **increase the divisor** (making quotas smaller). If the total is **less**, **decrease the divisor** (making quotas larger). Repeat until the sum of the upper quotas equals the total quota.

Step 5: Each group quota from step 4 is the final apportionment (or final quota).

Which Divisor is Higher? Standard or Modified?

Note: In Adam's method, a modified divisor is greater than the standard divisor.

Example 6.1.5: Using Adam's Method

In Adamstown, 42 new firefighters have just completed their training. They are to be assigned to the five firehouses in town in a manner proportional to the population in each fire district. The populations are listed in the following table.

Table $6.1.1$: Populations for the Fire Districts of Adamstown

District	A	B	C	D	E	Total
Population	$25,010$	$8,760$	$11,590$	$9,025$	$15,080$	$69,465$

Apportion the new firefighters to the firehouses using the Adams Method.

Answer

$$\text{Standard divisor} = \frac{69,465}{42} = 1653.92857$$

Adams's method always rounds up, making the sum of the upper quotas too large. First, choose **the modified divisor slightly larger than the standard divisor to see if that works.** The results are summarized below in Table $6.1.2$.

Guess #1 : $d = 1700$. The total is too still too large (43) so make the modified divisor larger. (Does not work)

Guess #2 : $d = 1800$. Now the total is too small (41) so make the modified divisor smaller. (Does not work)

Guess #3 : $d = 1750$. The total is too large again (43) so make the modified divisor larger. (Does not work)

Guess #4 : $d = 1775$. The sum is 42 so we are done. (Works)

Table $6.1.2$: Apportion by Adams's Method for Adamstown

District	A	B	C	D	E	Total
Population	$25,010$	$8,760$	$11,590$	$9,025$	$15,080$	$69,465$
$d = 1700$	14.712	5.153	6.818	5.309	8.871	
quota	15	6	7	6	9	43
$d = 1800$	13.894	4.86	6.439	5.014	8.378	
quota	14	5	7	6	9	41
$d = 1750$	14.291	5.006	6.623	5.157	8.617	
quota	15	6	7	6	9	43
$d = 1775$	14.090	4.935	6.530	5.085	8.496	
Final Allocation	15	5	7	6	9	42

Any modified divisor between 1755 and 1785 should work in this case.

Adam's Method Calculator

Your Turn 6.1.5: Adams Method

A local transit authority operates 41 subway trains that travel along six routes (A, B, C, D, E, and F). The average daily riders count for each route is given below.

Route	A	B	C	D	E	F
Average Daily Riders	1,022	1,986	1,605	1,235	1,177	1,077

Use Adams's method to determine the allocation of subway trains for each route.

Route	A	B	C	D	E	F
Average Daily Riders	1,022	1,986	1,605	1,235	1,177	1,077
Allocated Seats						

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Webster's Method

This method is named after Daniel Webster, an American statesman in the 19th century. Webster's method of apportionment is another method of apportionment that is based on a modified divisor. However, instead of basing the changes on the sum of the lower quotas, as Jefferson did, or the sum of the upper quotas, as Adams did, Webster used traditional rounding.

Steps to Apportion by Webster's Method (Normal Rounding)

Step 1: Find the standard divisor.

Step 2: Choose a modified divisor either slightly greater than or slightly less than the standard divisor. But you must start with a standard divisor. This could be a modified divisor.

Step 3: Find the states or group rounded quota (If the decimal is 0.5 or more, round UP. If it's less than 0.5, round DOWN). Add all rounded quota numbers.

Step 4: If the number of quotas (after rounding) is **more** than what is available, **increase the divisor** (making quotas smaller). If the total is **less**, **decrease the divisor** (making quotas larger). Repeat until the sum of the rounded quotas equals the total seats or quotas.

Step 5: Each group quota from step 4 is the final apportionment (or final quota).

Because some quotas will be rounded up and others will be rounded down, we cannot immediately determine whether the total number of seats is too large or too small. Unlike Jefferson's and Adam's methods, we do not know how to adjust the modified divisor. This forces us to use the standard divisor as the first modified divisor. So, the modified divisor in this method can be less than, greater than, or equal to the standard divisor.

Which Divisor is Higher? Standard or Modified?

The modified divisor in the Webster's Method can be less than, greater than, or equal to the standard divisor.

Example 6.1.6: Webester's Method

Use Webster's method to apportion the 25 seats among 6 states of the small country.

Table 6.1.1 : Population by States

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,00

Answer

First, find the standard divisor = $\frac{237,000}{25} = 9480$. and use it as a modified divisor to see if we are lucky.

Table 6.1.2 : Quotas for $d = 9480$

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
$d = 9480$	2.53	5.91	2.95	1.79	6.86	4.96	
Rounded Quota	3	6	3	2	7	5	26

Since the total of 26 seats is too big, we need to make the modified divisor larger. Try $d = 11,000$.

Table 6.1.3 : Quotas for $d = 11,000$

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
$d = 11,000$	2.18	5.09	2.55	1.55	5.91	4.27	
Rounded Quota	2	5	3	2	6	4	22

The total number of seats 22 is smaller than we hoped (25.) That means that $d = 11,000$ is much too big. We need to pick a new modified divisor between 9480 and 11,000. Try a divisor closer to 9480 such as $d = 10,000$.

Table 6.1.4 : Quotas for $d = 10,000$

State	A	B	C	D	E	F	Total
Population	24,000	56,000	28,000	17,000	65,000	47,000	237,000
$d = 10,000$	2.40	5.60	2.80	1.70	6.50	4.70	
Rounded Quota	2	6	3	2	7	5	25

Any modified divisor between 9,650 and 10,000 works here.

Note: You can double-check the above calculation using the following Webster's Method Calculator.

Webster Method Calculator

Your Turn 6.1.6: Webester's Method

The legislature in a state has 52 seats. Apportion these seats to the five counties below using Webster's method.

County	Population	Seats Received
Adams	388,000	
Grant	331,000	
Colton	491,000	
Davis	101,000	
Hayes	475,000	

Hayes | 475,000 |

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Example 6.1.7: Comparing all 4 Apportionment Methods

The following table summarizes the results of the Hamilton, Jefferson, Adams, and Webster methods when applied to the apportionment of 616 schools among five Hawaiian counties.

	Hawaii	Honolulu	Kalawao	Kauai	Maui
Under 19 years old	46,310	224,230	20	16,560	38,450
Hamilton	87	424	1	31	73
Jefferson	87	425	1	31	72
Adams	88	422	1	31	73
Webster	87	424	1	31	73

1. Do any of the apportionment methods result in the same apportionment? If so, which ones?
2. Which apportionment method would the citizens of the largest county likely favor most and least? Justify your answer.
3. As a group, which apportionment method would the citizens of the other four counties likely favor most and least? Justify your answer.

Answer

1. Yes, the Hamilton and Webster methods result in the same apportionment.
2. The largest county is Honolulu. The citizens would likely favor the Jefferson method of apportionment most since they received the most seats by that method. They would likely favor the Adams method of apportionment least because they received the least number of seats by that method.
3. As a group, the other four counties received 192 seats by either the Hamilton or Webster method, 193 seats by the Adams method, and 191 seats by the Jefferson method. They would likely favor the Adams method the most and favor the Jefferson method the least.

Your Turn 6.1.7: Definition of Apportionment Method

Answer the following questions.

How to find Standard Divisor?

Select an answer

Which apportion method we give remaining seat one at a time to the states with the largest fractional parts of their standard quotas until no seats remain?

Select an answer

In which method, the modified divisor is less than the standard divisor?

Select an answer

In which method, the modified divisor is larger than the standard divisor?

Select an answer

In which method, the modified divisor can be less than, greater than, or equal to the standard divisor?

Select an answer

In which method, we round the modified quota to the upper quota?

Select an answer

In which method, we use only standard divisor for apportion?

Select an answer

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6.2: Apportionment Paradoxes

Learning Objectives

1. Describe and illustrate the Alabama paradox.
2. Describe and illustrate the population paradox.
3. Describe and illustrate the new-states paradox.

Controversy

There are certain characteristics that they would reasonably expect from a fair apportionment.

- If the house size is increased, the state quotas should all increase or remain the same, but never decrease.
- If one state's population is growing more rapidly than another's, the faster-growing state should not lose a seat while the slower-growing state maintains or gains a seat.
- If there is a fixed number of seats, adding a new state should not cause an existing state to gain seats while others lose them.

However, apportionment methods are known to contradict these expectations. Before you decide on the right apportionment for Imaginarians, let's explore the **apportionment paradox**, a situation that occurs when an apportionment method produces results that seem to contradict reasonable expectations of fairness.

There is a lot that the founders of Imaginaria can learn from U.S. history. The constitution of the United States requires that the seats in the House of Representatives be apportioned according to the results of the census that occurs every decade, but the number of seats and the apportionment method is not stipulated. Over the years, several different apportionment methods and house sizes have been used and scrutinized for fairness. This scrutiny has led to the discovery of several of these apportionment paradoxes.

After seeing Hamilton's method, many people find that it makes sense, it's not that difficult to use (or, at least, the difficulty comes from the numbers that are involved and the amount of computation that's needed, not from the method), and they wonder why anyone would want another method. The problem is that Hamilton's method is subject to several paradoxes. Three of them happened, on separate occasions, when Hamilton's method was used to apportion the United States House of Representatives.

The Alabama Paradox

The **Alabama Paradox** is named for an incident that happened during the apportionment that took place after the 1880 census. (A similar incident happened ten years earlier involving the state of Rhode Island, but the paradox is named after Alabama.) The post-1880 apportionment had been completed, using Hamilton's method and the new population numbers from the census. Then it was decided that because of the country's growing population, the House of Representatives should be made larger. That meant that the apportionment would need to be done again, still using Hamilton's method and the same 1880 census numbers, but with more representatives. The assumption was that some states would gain another representative and others would stay with the same number they already had (since there weren't enough new representatives being added to give one more to every state). The paradox is that Alabama ended up *losing* a representative in the process, even though no populations were changed and the total number of representatives increased.

Alabama Paradox

Discovered in 1881.

	Seat 299	Seat 300
Alabama	9	7
Illinois	18	19
Texas	9	10

Figure 6.2.1: Illustration of Alabama Paradox

When Does the Alabama Paradox Occur?

The Alabama paradox occurs when an increase in the total number of items (quotas) to be apportioned results in a loss of an item for a group.

Example 6.2.1: Does Alabama Paradox Occur?

[1] Three states, A, B, and C, comprise a small country. The population of each state is given in the table below.

State	A	B	C	Total
Population	$11,251$	$18,245$	$16,468$	$45,964$

1. According to the country's constitution, its congress shall have 26 seats, divided amongst the three states according to their respective populations. Use Hamilton's method to determine each state's apportionment of congressional seats.
2. Use Hamilton's method to determine each state's apportionment of congressional seats if the number of seats in parliament is increased from 26 to 27.
3. Determine if the Alabama paradox occurred when the number of seats in Congress was increased from 26 to 27.

Answer

With 26 seats: Standard divisor = $\frac{45,964}{26} = 1767.84615$

Table $6.2.1$: Apportionment with 26 seats

State	A	B	C	Total
Population	$11,251$	$18,245$	$16,468$	$45,694$
Standard Quota	6.36424	10.32047	9.31529	
Lower Quota	6	10	9	25
Apportionment	$6+1=7$	10	9	26

With 27 seats: Standard divisor = $\frac{45,964}{27} = 1702.37037$

Table $6.2.2$: Apportionment with 27 seats

State	A	B	C	Total
Population	$11,251$	$18,245$	$16,468$	$45,694$
Standard Quota	6.60902	10.71741	9.67357	
Lower Quota	6	10	9	25
Apportionment	6	$10+1=11$	$9+1=10$	27

Yes, the Alabama occurred, since state A lost a seat in Congress.

State	With 26 seats	With 27 seats
A	7 →	6
B	10	11
C	9	10

Figure $6.2.2$: Illustration of Alabama Paradox

Your Turn 6.2.1: Alabama Paradox

The mathematics department has 23 peer tutors that are divided amongst sections of Calculus I, Trigonometry, and Intermediate Algebra. The total enrollment of each course is given in the table below.

Course	Calculus I	Trigonometry	Intermediate Algebra	Total
Enrollment	827	195	822	1844

Use Hamilton's method to determine the peer tutor apportionment of each class.

Course	Calculus I	Trigonometry	Intermediate Algebra	Total
Enrollment	827	195	822	1844
Number of Tutors				23

Use Hamilton's method to determine the peer tutor apportionment of each class if the number of peer tutors is increased from 23 to 24.

Course	Calculus I	Trigonometry	Intermediate Algebra	Total
Enrollment	827	195	822	1844
Number of Tutors				24

Determine if the Alabama paradox occurred when the number of peer tutors was increased from 23 to 24.

- ◯ No, the Alabama paradox did not occur, since no class lost a peer tutor.
- ◯ Yes, the Alabama paradox occurred, since Trigonometry lost a peer tutor.
- ◯ Yes, the Alabama paradox occurred, since Intermediate Algebra lost a peer tutor.
- ◯ Yes, the Alabama paradox occurred, since Calculus I lost a peer tutor.

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The New State Paradox

The **New States Paradox** happened when Oklahoma became a state in 1907. Oklahoma had enough population to qualify for five representatives in Congress. Those five representatives would need to come from somewhere, though, so five states, presumably, would lose one representative each. That happened, but another thing also happened: Maine gained a representative (from New York).

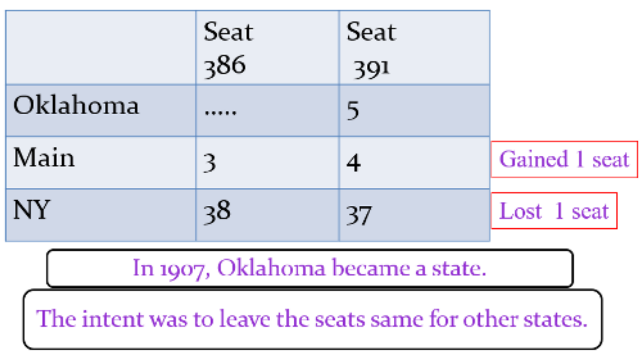

Figure 6.2.2: Illustration of New State Paradox

When Does the New State Paradox Occur?

The new state paradox occurs when adding a new group changes the apportionment of other groups.

✔ Example 6.2.2: Does New State Paradox Occur?

A small city is made up of three districts and governed by a committee with 100 members. District A has a population of 5,310. District B has a population of 1330, and District C has a population of 3308. The city annexes a small area, District D, with a population of 500. At the same time the number of committee members is increased by five. Hamilton's method was used to find the apportionment before and after the annexation.

Table 6.2.3 : Population Before the Annexation

State	A	B	C	Total
Population	5,310	1330	3,308	9,948

Table 6.2.4 : Population After the Annexation

State	A	B	C	D	Total
Population	5,310	1330	3,308	500	10,448

Answer

Before annexation: Standard divisor = $\frac{9948}{100} = 99.48$

Table 6.2.5 : Apportionment Before the Annexation

State	A	B	C	Total
Population	5,310	1330	3,308	9,948
Standard Quota	53.378	13.370	33.253	100.000
Lower Quota	53	13	33	99
Final apportionment	54	13	33	100

After annexation: Standard divisor = $\frac{10448}{105} = 99.505$

Table 6.2.6 : Apportionment After the Annexation

State	A	B	C	D	Total
Population	5,310	1330	3,308	500	10,448
Standard Quota	53.364	13.366	33.245	5.025	105.000
Lower Quota	53	13	33	5	104
Final apportionment	53	$13+1=14$	33	5	105

District D has a population of 500, so it should get five seats. When District D is added with its five seats, District A loses a seat, and District B gains a seat. This is an example of the New-States Paradox.

Your Turn 6.2.2: Population Paradox

Three voting districts (A, B, C, and D) comprise a city. The population of each voting district is given in the table below.

District	A	B	C	D	Total
Population	6,124	58,665	7,590	24,274	96,653

According to the city's bylaws, its city council shall have 24 seats, divided amongst the three voting districts according to their respective populations. Use Hamilton's method to determine each district's apportionment of city council seats.

District	A	B	C	D	Total
Population	6,124	58,665	7,590	24,274	96,653
Seats on City Council					24

Recent census data indicate that the population of each voting districts has changed. Determine the percent increase of the population of each voting districts. Round solutions to the nearest tenth, if necessary.

District	A	B	C	D	Total
Original Population	6,124	58,665	7,590	24,274	96,653
New Population	6,124	58,775	7,562	25,584	98,045
Percent Increase of Population					

Use Hamilton's method to recalculate the apportionment of the city council seats using the updated population data.

District	A	B	C	D	Total
New Population	6,124	58,775	7,562	25,584	98,045
Seats on City Council					24

Determine whether or not the population paradox occurred in this case.

◯ No, the population paradox did not occur.

◯ Yes, the population paradox occurred, since district A lost a seat on the city council to district B, even though district A grew at a faster rate than district B.

◯ Yes, the population paradox occurred, since district B lost a seat on the city council to district A, even though district B grew at a faster rate than district A.

◯ Yes, the population paradox occurred, since district B lost a seat on the city council to district D, even though county B grew at a faster rate than district D.

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The Population Paradox

The **Population Paradox** happened between the apportionments after the census of 1900 and of 1910. In those ten years, Virginia's population grew at an average annual rate of 1.07%, while Maine's grew at an average annual rate of 0.67%. Virginia started with more people, grew at a faster rate, grew by more people, and ended up with more people than Maine. By itself, that doesn't mean that Virginia should gain representatives or Maine shouldn't, because there are lots of other states involved. But Virginia ended up losing a representative *to Maine*.

When Does the Population Paradox Occur?

The population paradox occurs when group A loses an item(s) to group B, even though group A's population grew faster than group B's population.

Example 6.2.3: Does Population Paradox Occur?

A mom decides to split 11 candy bars among three children based on the number of minutes they spend on chores this week. The time spent by each of her children is given in the table. Near the end of the week, Mom reminds the children of the deal, and they each do some extra work. Abby does an extra two minutes, Bobby an additional 12 minutes, and Charley an extra 86 minutes.

Table 6.2.7 : Candy Bars Before the Extra Work

	Abby	Bobby	Charley	Total
Population (Before)	54	243	703	1,000
Population (After)	56	255	789	1100

1. Use Hamilton's method to apportion the candy bars before and after the extra work.
2. Does the population paradox occur? Why or why not?

Answer

Apportion using Hamilton before the extra work: Standard divisor = $\frac{1000}{11} = 90.90909$

Table 6.2.8 : Apportion of Candy Bars Before the Extra Work

	Abby	Bobby	Charley	Total
Population	54	243	703	1,000
Standard Quota	0.594	2.673	7.733	
Lower Quota	0	2	7	9
Apportionment	0	$2+1=3$	$7+1=8$	11

Apportion using Hamilton after extra work: Standard divisor = $\frac{1100}{11} = 100$

Table 6.2.9: Apportion of Candy Bars After the Extra Work

	Abby	Bobby	Charley	Total
Population	56	255	789	1,100
Standard Quota	0.560	2.550	7.890	
Lower Quota	0	2	7	9
Final Apportionment	$0+1=1$	2	$7+1=8$	11

Abby time increase by: $\frac{56-54}{54}=3.7\%$

Bobby time increase by: $\frac{255-243}{243}=4.9\%$

Charley time increase by: $\frac{789-703}{703}=12.2\%$

Abby's time increased by 3.7% while Bobby's time increased by 4.9%, more than Abby's. However, Abby gains a candy bar while Bobby loses one. This is an example of the Population Paradox.

✔ Example 6.2.4: Population Paradox

Suppose that 18 respirators are to be apportioned to three hospitals based on their capacities. The Hamilton method is used to allocate the respirators in 2020, then to reallocate based on new capacities in 2020. The results are shown in the table below. How does this demonstrate the population paradox?

Hospital / Capacity in 2020 / Respirators in 2020	Respirators in 2021	Growth Rate
A 825 8	9	$\frac{882-825}{825}=6.01\%$
B 613 6	6	$\frac{626-613}{613}=2.12\%$
C 239 2	3	$\frac{242-239}{239}=1.26\%$

Answer

Hospital B lost a respirator while hospital C gained one, even though hospital B had a higher growth rate (2.12%) than hospital C (1.26%.)

✎ Your Turn 6.2.4: New State Paradox

A college has three schools: the School of Languages, the School of Fine Arts, and the School of Science. The number of faculty in each school is given below.

School	Languages	Fine Arts	Science	Total
Number of Faculty	81	139	32	252

The shared governance model at the college includes a faculty senate with 52 seats, divided amongst the three schools according to their respective number of faculty. Use Hamilton's method to determine each school's apportionment of faculty senate seats.

School	Languages	Fine Arts	Science	Total
Number of Faculty	81	139	32	252
Allocated Seats				52

Suppose the college decides to add a new school: the School of History. The new school of History has 17 faculty and is to be apportioned 3 seats in the faculty senate. Use Hamilton's method to recalculate the apportionment for all four schools at the college.

School	Languages	Fine Arts	Science	History	Total
Number of Faculty	81	139	32	17	269
Allocated Seats					55

Determine if the new-states paradox occurred when the School of History was added to the faculty senate and the seats of the faculty senate were reapportioned.

- ◯ Yes; the new-states paradox occurred, since the school of Fine Arts lost a seat to the school of History.
- ◯ Yes; the new-states paradox occurred, since the school of History lost a seat to the school of Science.
- ◯ Yes; the new-states paradox occurred, since the school of Fine Arts lost a seat to the school of Science.
- ◯ Yes; the new-states paradox occurred, since the school of Science lost a seat to the school of Fine Arts.
- ◯ No; the new-states paradox did not occur.

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Quota Rule Violation

A small college has three departments. Department A has 98 faculty, Department B has 689 faculty, and Department C has 212 faculty. The college has a faculty senate with 100 representatives. Use Jefferson's method with a modified divisor of $d = 9.83$ to apportion the 100 representatives among the departments.

Apportion using Hamilton before the extra work: Standard divisor = $\frac{999}{100} = 9.99$

Table 6.2.10 : Quota Rule Violation

State	A	B	C	Total
Population	98	689	212	999
Standard Quota	9.810	68.969	21.221	100
$d = 9.83$	9.969	70.092	21.567	
quota	9	70	21	100

Department B has a standard quota of 68.969, so it should get either its lower quota 68 or its upper quota 69 seats. Using this method, District B received 70 seats, one more than its upper quota. This is a Quota Rule violation.

Is it Possible to Find the Perfect Apportionment Method?

In 1980, Michael Balinski (State University of New York at Stony Brook) and H. Peyton Young (Johns Hopkins University) proved that all apportionment methods violate the quota rule or suffer from one of the paradoxes. This means that it is impossible to find the "perfect" apportionment method. The methods and their potential flaws are listed in the following table.

Table 6.2.11 : Methods, Quota Rule Violations, and Paradoxes

			Paradoxes	
Method	**Quota Rule**	**Alabama**	**Population**	**New-States**
Hamilton	No violations	Yes	Yes	Yes
Jefferson	Upper-quota violations	No	No	No
Adams	Lower-quota violations	No	No	No
Webster	Lower- and upper-quota violations	No	No	No
Huntington-Hill	Lower- and upper-quota violations	No	No	No

Your Turn 6.2.5: Name Paradox

Name the Pardox.

"When an increase in the total number of items to be apportioned results in a loss of an item for a group".

Select an answer

Name the Paradox.

"If the addition of a new group changes the apportionment of other groups."

Select an answer

Name the Paradox.

"When group A loses an item(s) to group B, even though group A's population grew faster than group B's population."

Select an answer

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CHAPTER OVERVIEW

7: Graph Theory

7.1: Basic Graphs and Graphs Structure

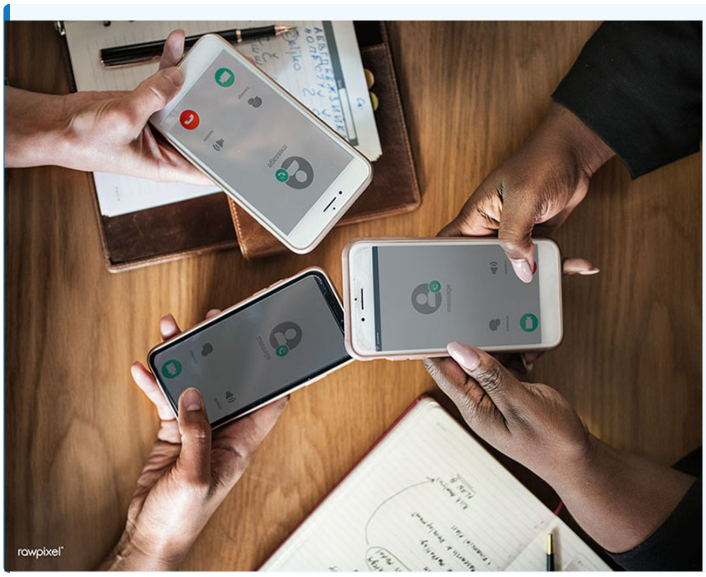


Figure 7.1.1: Cell phone networks connect individuals. (credit: "Business people using their phones" by Rawpixel Ltd./Flickr, CC BY 2.0)

Learning Objectives

1. Identify parts of a graph.
2. Model relationships with graphs.
3. Describe and identify walks, trails, paths, and circuits.

When you hear the word, *graph*, what comes to mind? You might think of the xy-coordinate system you learned about earlier in this course, or you might think of the line graphs and bar charts that are used to display data in news reports. The graphs we discuss in this chapter are probably very different from what you think of as a graph. They look like a bunch of dots connected by short-line segments. The dots represent a group of objects, and the line segments represent the connections, or relationships, between them. The objects might be bus stops, computers, Snapchat accounts, family members, or any other objects that have direct connections to each other. The connections can be physical or virtual, formal or casual, scientific or social. Regardless of the kind of connections, the purpose of the graph is to help us visualize the structure of the entire network to better understand the interactions of the objects within it.

Parts of a Graph

In a graph, the objects are represented with dots, and their connections are represented with lines like in Figure 7.1.2 displays a simple graph labeled *G* and a multigraph labeled *H*. The dots are called vertices; an individual dot is a **vertex**, which is one object of a set of objects, some of which may be connected. We often label vertices with letters. For example, Graph *G* has vertices *a*, *b*, *c*, and *d*, and Multigraph *H* has vertices *e*, *f*, *g*, and *h*. Each line segment or connection joining two vertices is referred to as an **edge.** *H* is considered a **multigraph** because it has a double edge between *f* and *h* and a double edge between *h* and *g*. Another reason *H* is called a multigraph is that it has a loop connecting vertex *e* to itself; a loop is an edge that joins a vertex to itself. Loops and double edges are not allowed in a simple graph.

To sum up, a simple graph is a collection of vertices and any edges that may connect them, such that every edge connects two vertices with no loops and no two vertices are joined by more than one edge. A **multigraph** is a graph in which loops or pairs of vertices are joined by more than one edge. In this chapter, most of our work will be with simple graphs, which we will call graphs for convenience.

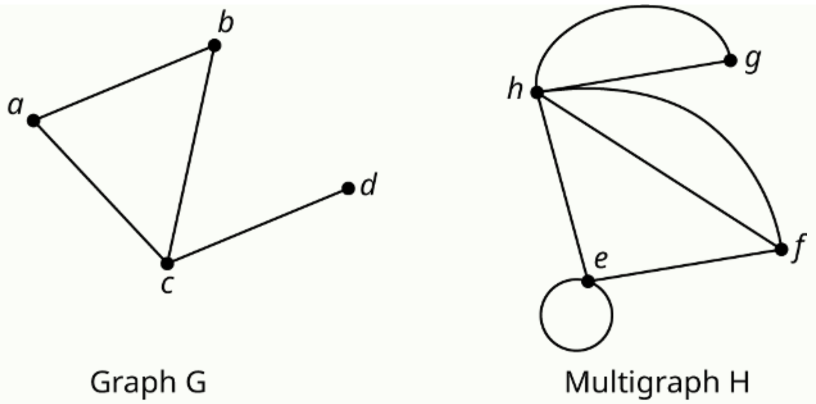


Figure 7.1.2: A Graph and a Multigraph

It is not necessary for the edges in a graph to be straight. In fact, you can draw an edge any way you want. Graph theory focuses on which vertices are connected, not how the connections are drawn. In a graph, each edge can be named by the two letters of the associated vertices. The four edges in Graph X in Figure 7.1.3 are ab, ac, ad, and ae. The order of the letters is not essential when you name the edge of a graph. For example, ab refers to the same edge as ba.

Checkpoint: Edge Could be Straight Line or Curve

A graph may have vertices that are not joined to other vertices by edges, such as vertex f in Graph X in Figure 7.1.3*, but any edge must have a vertex at each end.*

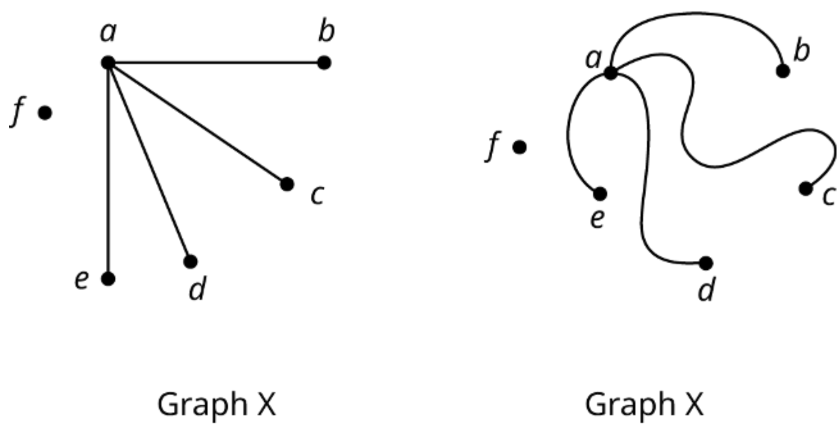


Figure 7.1.3 : Different Representations of the Same Graph

Example 7.1.1: Identifying Edges and Vertices

Name all the vertices and edges of graph F in Figure 7.1.4. Does this graph have a loop?

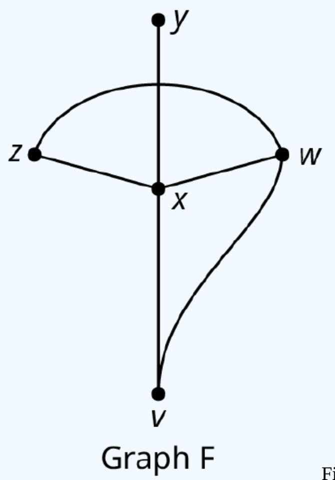


Figure 7.1.4: Graph F

Answer

The vertices are *v*, *w*, *x*, *y*, and *z*. The edges are *vw*, *vx*, *wx*, *wz*, *xy*, and *xz*. The graph has no loop.

Checkpoint

When listing the vertices and edges in a graph, work in alphabetical order to avoid accidentally listing the same item twice. When you are finished, count the number of vertices or edges you listed and compare that to the number of vertices or edges on the graph to ensure you didn't miss any.

Your Turn 7.1.1

Name all the vertices and edges of Graph *A*.

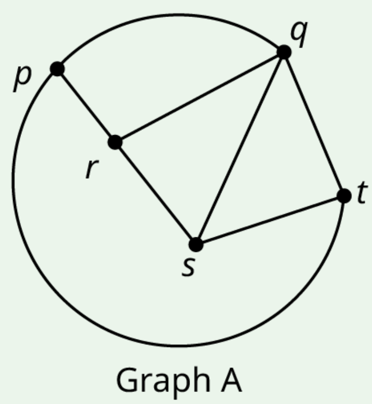


Figure 7.1.5: Graph A

Adjacent Vertices and Edges

Two **vertices** in a graph are called **adjacent** if they are connected directly by an **edge**. Two **edges** are called **adjacent** if they **share a common vertex**.

For example, in Figure 7.1.5 : Graph *A*, vertices **s** *and* **t** are adjacent, **s** and **r** are adjacent, **s** and **q** are adjacent, but vertices ***s*** and **p** are not, and also **t** and **p** are not adjacent.

Similarly, edges **st** and **sr** are adjacent, but **sr** and **tq** are not adjacent edges.

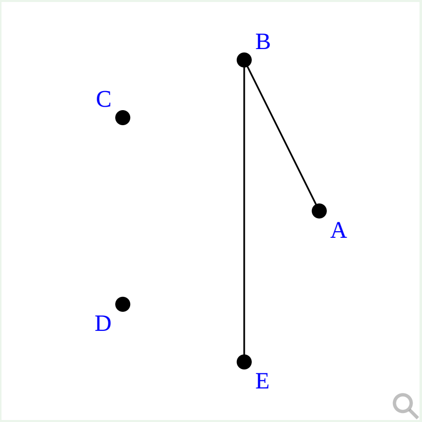
Example 7.1.2: Identifying Vertices That Are Not Adjacent

Name all the pairs of vertices of graph *F* in Figure 7.1.4 that are *not* adjacent.

Answer

The vertices not adjacent in graph *F* are *v* and *y*, *v* and *z*, *w* and *y*, and *y* and *z*.

Your Turn 7.1.2: Find Adjacent Vertex



Which vertices are **adjacent** to vertex A?

- ☐ B
- ☐ C
- ☐ D
- ☐ E
- ☐ None are adjacent to A

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People in Mathematics: Sergey Brin and Laurence Page

The “Google boys,” Sergey Brin and Laurence Page, transformed the World Wide Web in 1998 when they used the mathematics of graph theory to create an algorithm called Page Rank, which is known as the Google Search Engine today. The two computer scientists identified webpages as vertices and hyperlinks on those pages as edges because hyperlinks connect one website to the next. The number of edges influences the ranking of a website on the Google Search Engine because the websites with more links to “credible sources” are ranked higher. ("Page Rank: The Graph Theory-based Backbone of Google," September 20, 2011, Cornell University, Networks Blog.

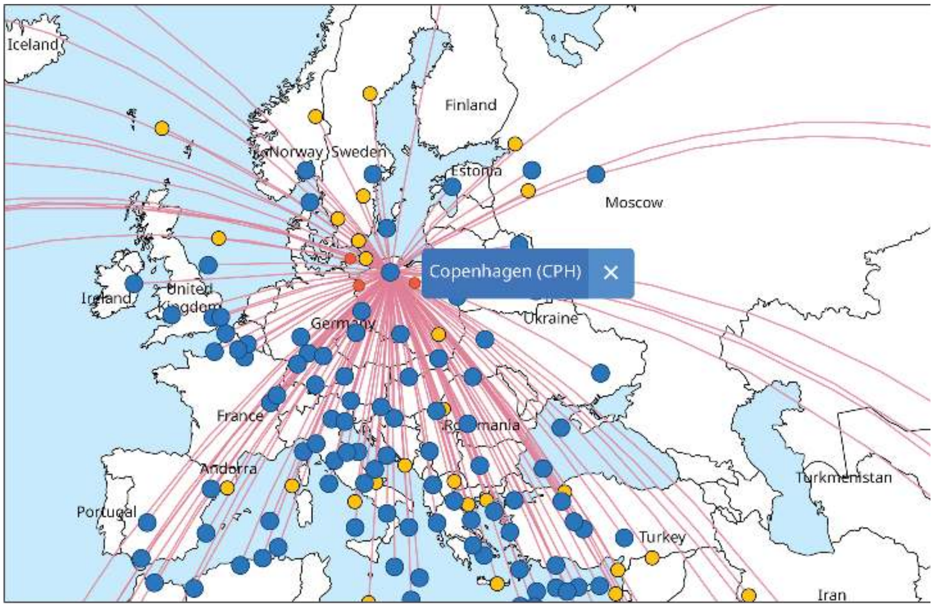

Figure 7.1.6: Commercial Airlines' Route Systems Create a Global Network.

When graphs are used to model and analyze real-world applications, the number of edges that meet at a particular vertex is important. For example, a graph may represent the direct flight connections for a particular airport as in Figure 7.1.7. Representing the connections with a graph rather than a map shifts the focus away from the relative positions and toward which airports are connected. In Figure 7.1.7, the vertices are the airports, and the edges are the direct flight paths. The number of flight connections between a particular airport and other South Florida airports is the number of edges meeting at a particular vertex. For example, Key West has direct flights to three of the five airports on the graph. In graph theory terms, we would say that vertex FYW has degree 3. The degree of a vertex is the number of edges that connect to that vertex.

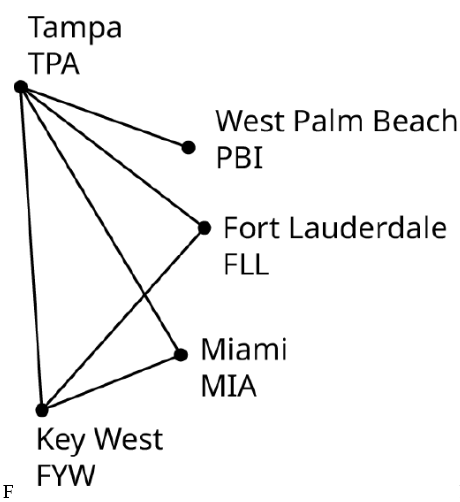

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Figure 7.1.7: Direct Flights between South Florida Airport.

Degree of a Vertex

The **degree** of a vertex is the **number of edges connected to it**. The **degree** of a vertex can be either **even** or **odd,** depending on how many edges are connected to it. If a graph contains a loop, that loop contributes two to the degree of the vertex.

Think about the intersection of two roads, Daniel Parkway and Summerline, near FSW. That intersection is a vertex, and its degree is four.

Example 7.1.3: Determining the Degree of a Vertex

Determine the degree of each vertex of Graph J in Figure 7.1.8. If Graph J represents direct flights between a set of airports, do any of the airports have direct flights to two or more of the other cities on the graph? Also, determine if the degree is even or odd.

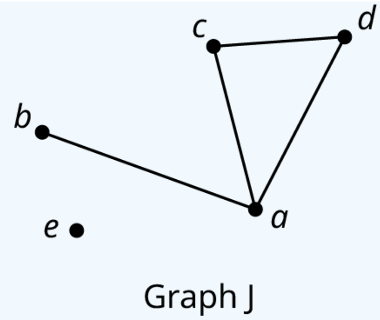


Figure $7.1.8$: Graph J

Answer

For each vertex, count the number of edges that meet at that vertex. This value is the degree of the vertex. In Figure $7.1.9$, the dashed edges indicate the edges that meet at the marked vertex.

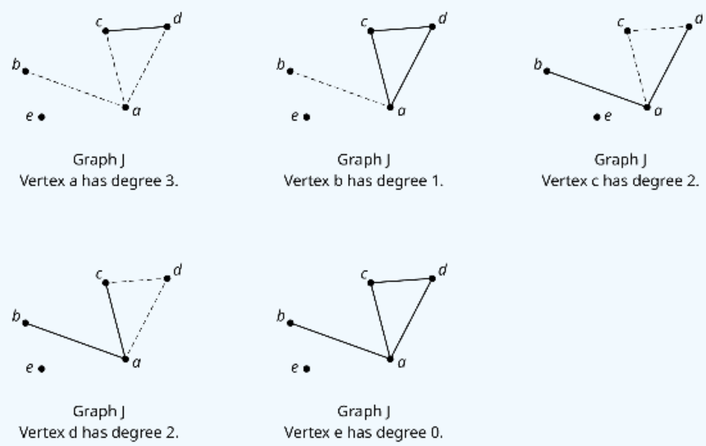
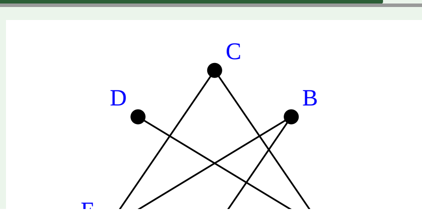
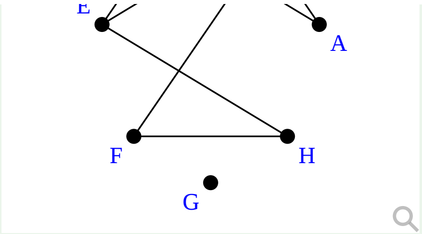


Figure $7.1.9$: Degrees of Vertices of Graph J *with their even and odd characteristic.*

The degree of a is 3, indicating that a is ODD.
The degree of b is 1, meaning that b is also ODD.
The degree of d is 2, which classifies d as EVEN.
The degree of e is 0, so e is also considered EVEN.
Finally, the degree of c is 2, making c EVEN.

Your Turn $7.1.3$: Even or Odd Vertices?





Which vertices are **even**?

- ☐ A
- ☐ B
- ☐ C
- ☐ D
- ☐ E
- ☐ F
- ☐ G
- ☐ H
- ☐ None are even

Which vertices are **odd**?

- ☐ A
- ☐ B
- ☐ C
- ☐ D
- ☐ E
- ☐ F
- ☐ G
- ☐ H
- ☐ None are odd

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Representation of Graph as Boundaries.

Graphs are also used to analyze regional boundaries. The states of Utah, Colorado, Arizona, and New Mexico all meet at a single point known as the "Four Corners," which is shown in the map in figure 7.1.10.

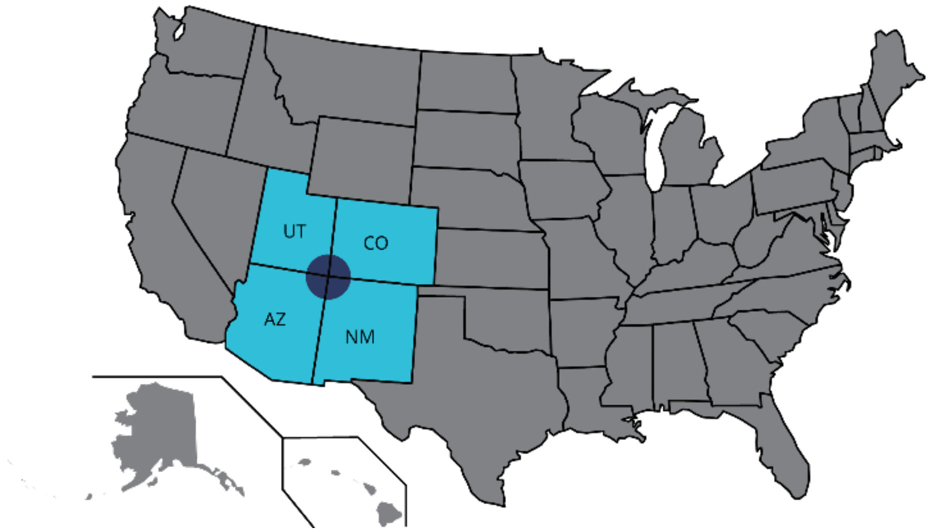


Figure 7.1.10 : Map of the Four Corners

In Figure 7.1.11, each vertex represents one of these states, and each edge represents a shared border. States like Utah and New Mexico that meet at only a single point are *not* considered to have a shared border. By representing this map as a graph, where the connections are shared borders, we shift our perspective from physical attributes such as shape, size, and distance toward the existence of the relationship of having a shared boundary.

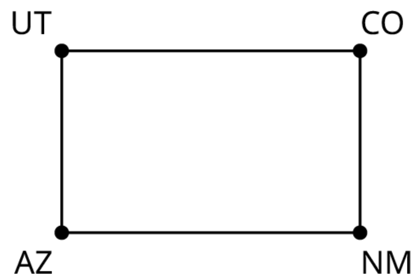


Figure 7.1.11: Graph of the Shared Boundaries of Four Corners States

Example 7.1.4: Graphing the Midwestern States

A map of the Midwest is given in Figure 7.1.12. Create a graph of the region in which each vertex represents a state and each edge represents a shared border.

Figure 7.1.12: Map of Midwestern States

Answer

For each state, draw and label a vertex. Draw edges between any two states that share a common land border as figure 7.1.14.

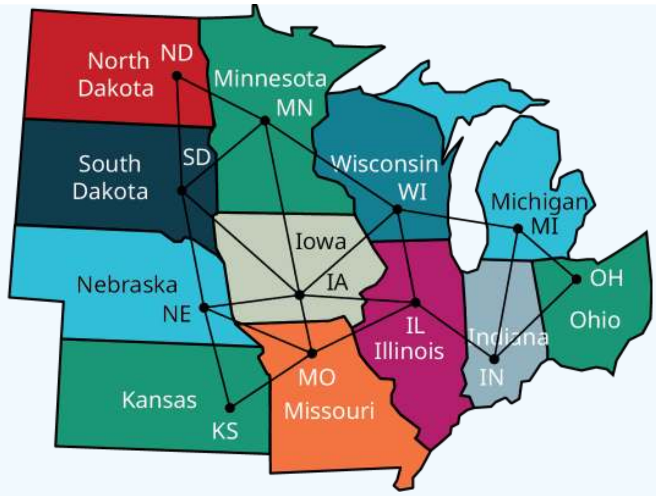


Figure 7.1.13: Edge Assigned to Each Pair of Midwestern States with Common Border

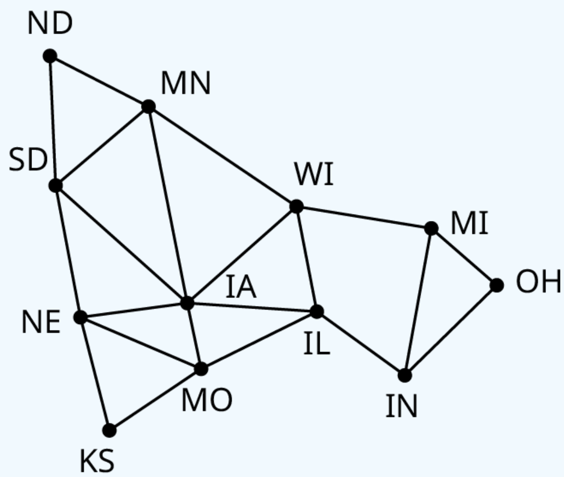
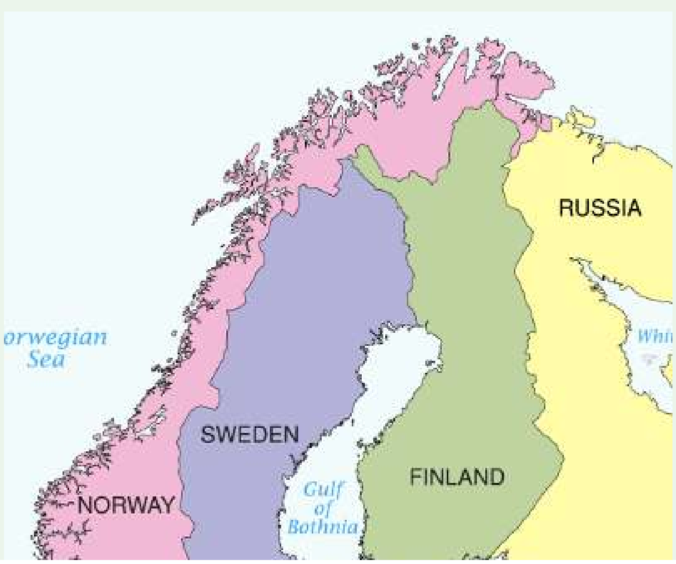
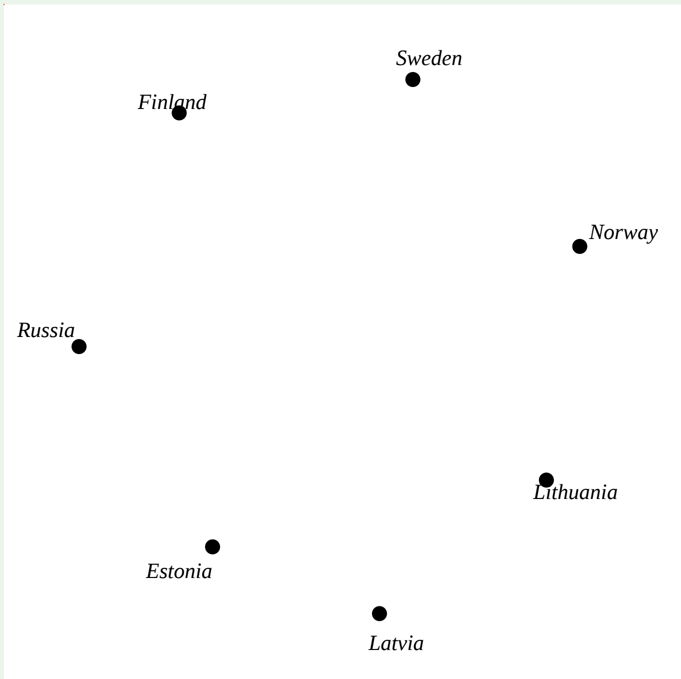


Figure 7.1.14: Final Graph Representing Common Boundaries between Midwestern States

Your Turn 7.1.4: Model the Map to Graph

The map below shows some countries in Northern Europe. Draw a graph that models the border relationships among the countries whose names appear in the map. Let the vertices represent the countries and the edges represent a common border between them. (That is, two vertices are connected by an edge when they share a border.)





Important Note: You may have difficulty drawing multiple edges to a vertex. When that happens, draw a line segment to a location ***near*** *the vertex, and then drag it to the desired vertex to make the connection.*

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Who Knew?: Using Graph Theory to Reduce Internet Fraud

Could graphs be used to reduce Internet fraud? At least one researcher thinks so. Graph theory is used every day to analyze our behavior, particularly on social network sites. Alex Buetel, a computer scientist from Carnegie Mellon University in Pittsburgh, Pennsylvania, published a research paper in 2016 that discussed the possibilities of distinguishing the normal interactions from those that might be fraudulent using graph theory. Buetel wrote, “To more effectively model and detect abnormal behavior, we model *how* fraudsters work, catching previously undetected fraud on Facebook, Twitter, and Tencent Weibo and improving classification accuracy by up to 68.” In the same paper, the researcher discusses how similar techniques can be used to model many other applications and even, “predict *why* you like a particular movie.” (Alex Beutel, "User Behavior Modeling with Large-Scale Graph Analysis," http://reports-archive.adm.cs.cmu.ed...-CS-16-105.pdf, May 2016, CMU-CS-16-105, Computer Science Department, School of Computer Science, Carnegie Mellon University, Pittsburgh, PA)

Completeness and Subgraphs

Suppose that there were five strangers in a room, A, B, C, D, and E, and each one would be introduced to each of the others. How many introductions are necessary? One way to begin to answer this question is to draw a graph with each vertex representing an individual in the room and each edge representing an introduction as in Figure 7.1.17.

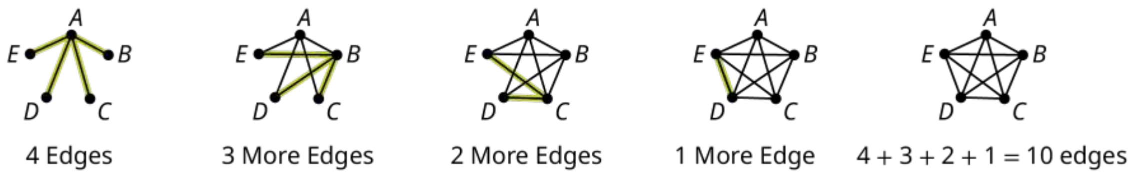

Figure 7.1.15: Model of Introductions between Five Strangers

Let's approach the problem by thinking about how many new people Person A would meet, then Person B, and so on, making sure not to repeat any introductions. The first graph in Figure 7.1.15 shows Person A meeting Persons B, C, D, and E, for a total of 4 introductions. The next graph shows that Person B still has to meet Persons C, D, and E, for a total of 3 more introductions. The next graph shows that Person C still has to meet Persons D and E, which is 2 more introductions. The next graph shows that Person D only remains to meet Person E, which is one more introduction. The final graph has $4+3+2+1=10$ edges representing 10 introductions.

Complete Graph

A complete graph is one in which an edge connects every pair of distinct vertices.

The last graph in Figure 7.1.17 is an example of a complete graph because an edge joins each pair of vertices. Another way of saying this is that the graph is complete because each vertex is adjacent to every other vertex. Figure 7.1.18 shows the complete graphs with three, four, five, and six vertices.

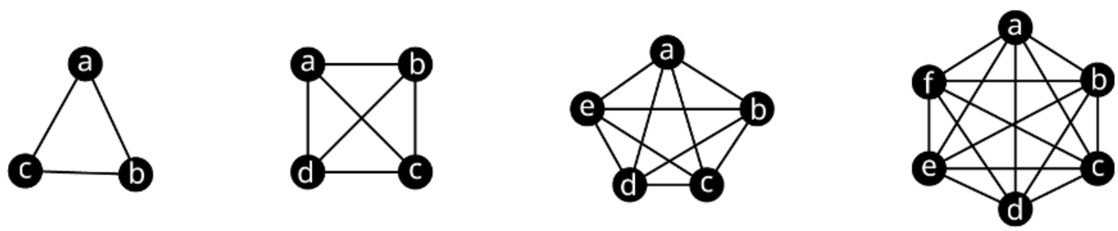

Figure 7.1.16: Complete Graphs with Up to Six Vertices

Subgraphs

Sometimes, a graph is a part of a larger graph. For example, the graph of South Florida Airports from Figure 7.1.7 is part of a larger graph that includes Orlando International Airport in Central Florida, which is shown in Figure 7.1.17. A subgraph is a smaller graph section comprising a subset of the vertices and edges from the original graph. Every edge and vertex of the subgraph must come from the original graph.

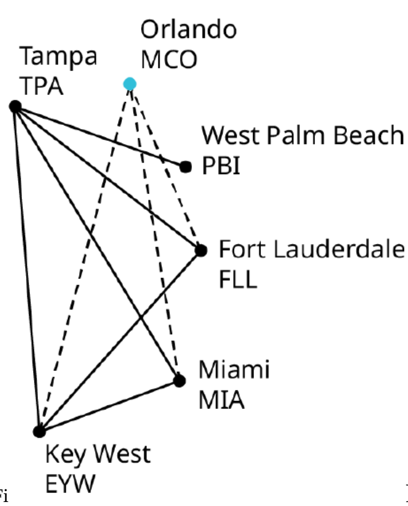

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Figure 7.1.17: Orlando and South Florida Airports

The graph in Figure 7.1.17 includes an additional vertex, MCO, and additional edges shown with dashed lines. The graph of direct flights between South Florida airports from Figure 7.1.7 is called a **subgraph** of the graph that also includes direct flights between Orlando and the same South Florida airports in Figure 7.1.17. In general terms, if Graph B consists entirely of a set of edges and vertices from a larger Graph A, then B is called a subgraph of A.

Example 7.1.5: Identifying a Subgraph

Graph G and four diagrams are given in Figure 7.1.18. Determine whether each diagram is or is not a subgraph of Graph G and explain why.

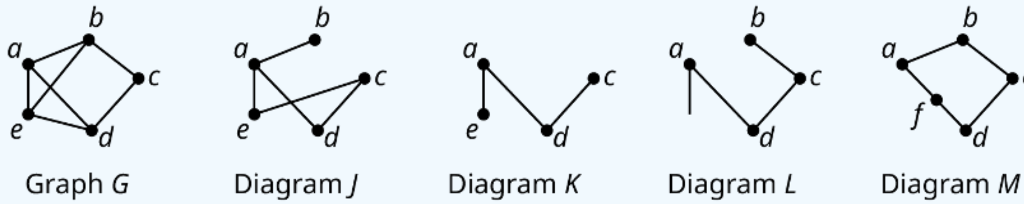


Figure 7.1.18: Graph G and Potential Subgraphs

Answer

Diagram J is not a subgraph of Graph G because edge ec is not in Graph G.

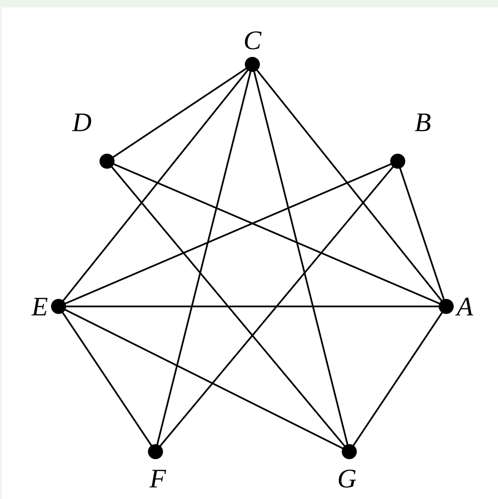
Diagram K is a subgraph of Graph G because all of its vertices and edges were part of Graph G.

Diagram L is not a graph at all because there is a line extending from vertex a that does not connect a to another vertex. So, Diagram L cannot be a subgraph.

Diagram M is not a subgraph of Graph G because it contains a vertex f, which is not part of G.

Your Turn 7.1.5: Complete Graph

Draw the edges needed in order to make the following graph complete.



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Recognizing Isomorphic (Equivalent) Graphs

Isomorphic graphs that represent the same pattern of connections can look very different despite having the same underlying structure. The edges can be stretched and twisted. The graph can be rotated or flipped. For example, in Figure 7.1.18, each diagram represents the same pattern of connections. The isomorphic graph is also called an equivalent graph.

Equivalent Graph (Isomorphic Graph)

Two graphs are said to be **isomorphic** if they are **structurally the same**, even if they look different in a drawing.

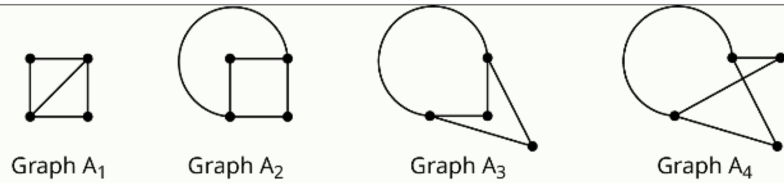


Figure 7.1.19: Four Representations of the Same Graph

Looking at Figure 7.1.19, how can we know that these graphs are isomorphic? We will start by checking for any obvious differences. Each of the graphs in Figure 7.1.19 has four vertices and five edges, so there are no differences. Next, we will focus on the degrees of the vertices, which have been labeled in Figure 7.1.20.

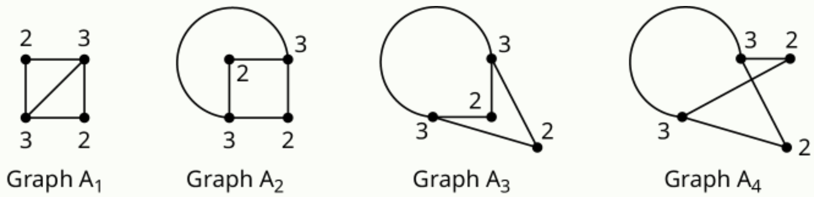


Figure 7.1.20: Graphs with Vertices of the Same Degrees

As shown in Figure 7.1.20, each graph has two vertices of degree 2 and two vertices of degree 3, so there are no differences.

Checkpoint

When you name isomorphisms, one way to check that your answer is reasonable is to make sure that the degrees of corresponding vertices are equal.

Example 7.1.6: Identifying Isomorphisms

Are the Graphs B_1 and B_2 in Figure 7.1.21 are isomorphic? Identify an isomorphism between them by listing corresponding pairs of vertices.

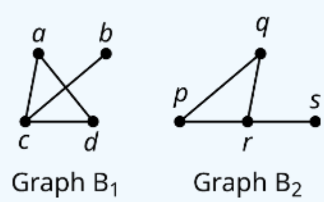


Figure 7.1.21: Graphs B1 and B2

Answer

Figure 7.1.22 shows how to transform Graph B_1 to get Graph B_2.

Vertex **b** corresponds to **s**, vertex **c** corresponds to **r,** vertex **q** corresponds to **a**, and vertex **d** corresponds to **p**.

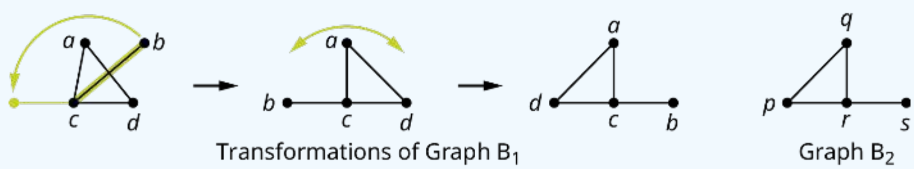
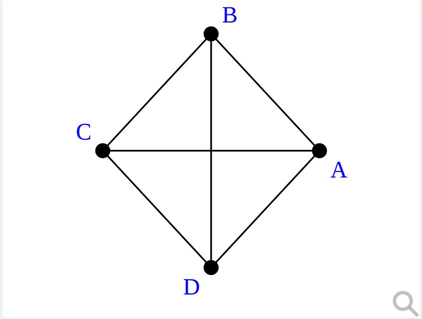


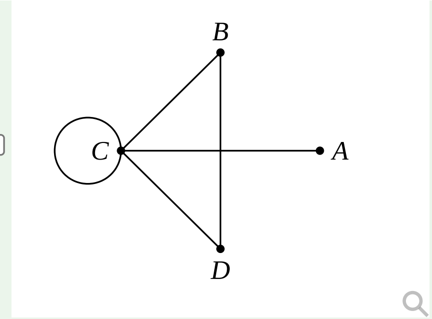
Figure 7.1.22: Transform Graph B_1 into Graph B_2

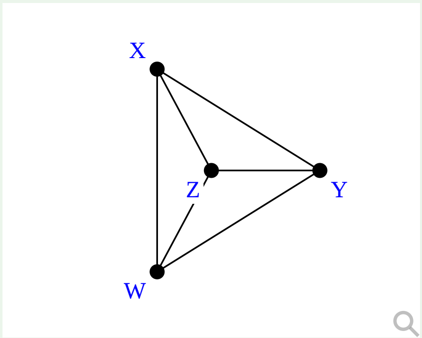
Your Turn 7.1.6: Equivalent Graphs

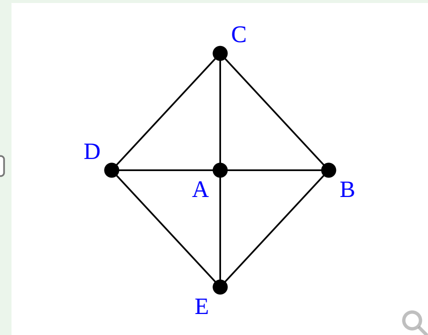
Graphs are **equivalent** if they have the same number of vertices and the same edge connections. The vertices do *not* need to have the same labels, and they do not have to be drawn in the same positions.

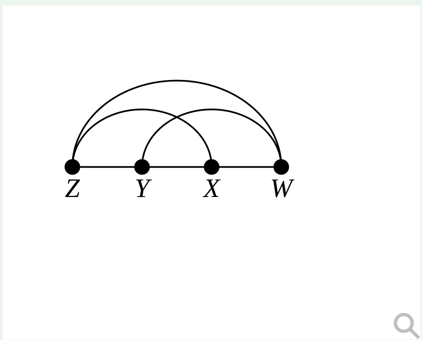
Select all the graphs that are equivalent.

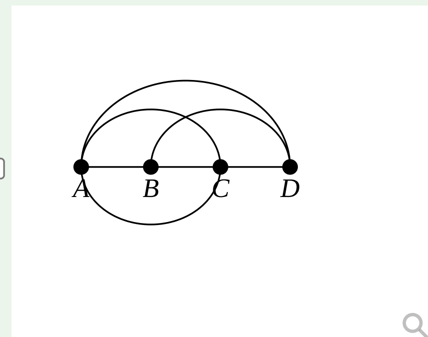












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Cycle

A cycle is a path that starts and ends at the same vertex, with **no repeated edges or vertices** (except the starting and ending vertex, which must be the same). All cycle are circuit but not all circuit are cycle.

Looking at Cycle to Check Isomorphism

Consider graph figure 7.1.23. The graph G contains quadrilateral cycle (b, f, d, c), but graph S contains a triangle cycle $(m, r, o.)$ Also, graph S contains a triangle cycle, but graph G has no triangle. This means that the graphs are not isomorphic.

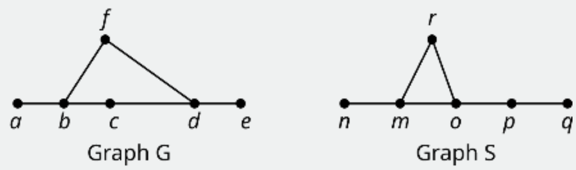


Figure 7.1.23: Graph G and Graph S

Walk, Path, Trails, Circuit, and Cycle

We know the basic parts of graphs, and we can distinguish one graph from another. It is time to really put our graphs to work for us. Many applications of graph theory involve navigating through a graph, very much like you would navigate through a maze. Imagine that you are at the entrance to a maze. Your goal is to get from one point to another as efficiently as possible. Maybe there are treasures hidden along the way that make straying from the shortest path worthwhile, or maybe you just need to get to the end fast. Either way, you definitely want to avoid any wrong turns that would cause unnecessary backtracking. Luckily, graph theory is here to help

Figure 7.1.24: Visitors navigate a garden maze. (credit: “Longleat Maze” by Niki Odolphie/Wikimedia, CC BY 2.0)

Definition: Walk, Trails, and Path

1. A walk is a sequence of vertices and edges where both edges and vertices can be repeated.
2. A trail is an open walk where no edge is repeated, though vertices may be repeated.
3. A path is a trail in which neither vertices nor edges are repeated.

A walk is the most basic way of navigating a graph, as it has no restrictions except for staying on it. We will call the walk by a different name when there are restrictions on which vertices or edges we can visit. For example, if we want to find a walk that avoids traveling the same edge twice, we will say we want to find a trail (or directed trail). If we want to find a walk that avoids visiting the same vertex twice, we will say we want to find a path (or directed path).

Walks, trails, and paths are all related.

1. All paths are trails, but trails that visit the same vertex twice are not paths.
2. All trails are walks, but walks in which an edge is visited twice would not be trails.

Since walks, trails, and paths are all related, closed walks, closed trails (circuits), and closed paths (directed cycles) are linked too.

1. All circuits are closed walks, but closed walks that visit the same edge twice are not circuits.
2. All directed cycles are circuits, but circuits in which a vertex is visited twice are not directed cycles.

We can visualize the relationship as in Figure 7.1.25 and Figure 7.1.26.

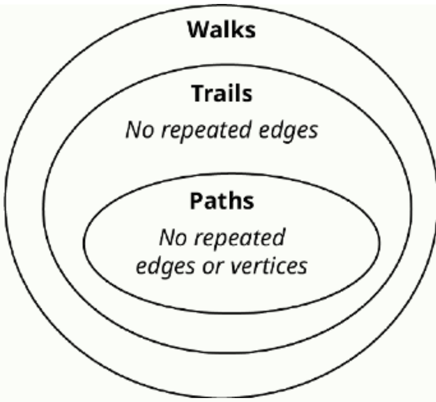


Figure 7.1.25: Walks, Trails, and Paths

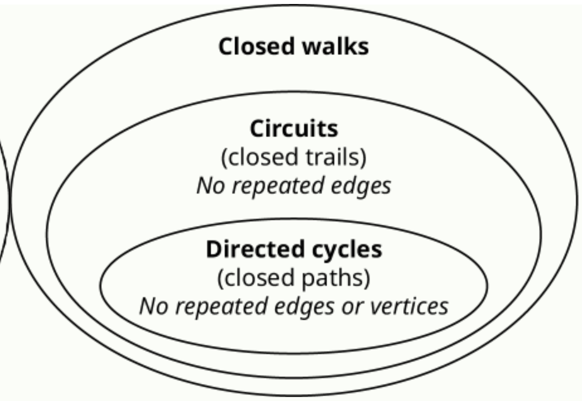
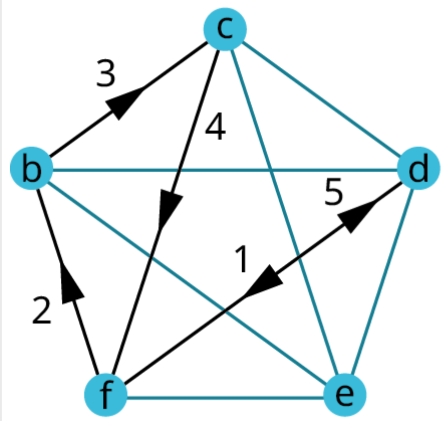
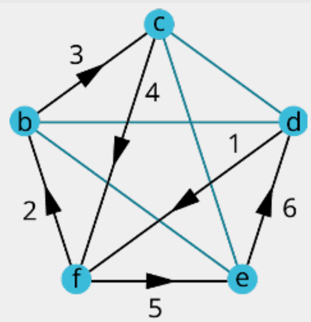
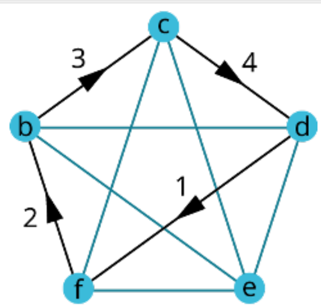


Figure 7.1.26: Closed Walks, Ccylce, and Circuits

Term	**Description**	**Edges Repeated?**	**Vertices Repeated?**
Walk	A sequence of edges and vertices	Yes	Yes
Trail	A walk with no repeated edges	No	Yes
Path	A trail with no repeated vertices	No	No
Circuit	A closed trail	No	Yes
Cycle	A closed path	No	No (except start/end)

In many applications of graph theory, such as creating efficient delivery routes that begin and end at the same location, this requirement is often necessary. When a walk, path, or trail ends at the same location or vertex where it began, we call it a closed path. Otherwise, we call it open (it does not begin and end at the same location or vertex). The following table provides examples of closed walks, trails, and paths.

Closed Walks, Trails, and Paths

DESCRIPTION	EXAMPLE	CHARACTERISTICS
A **closed walk** is a walk that begins and ends at the same vertex.	 $d \to f \to b \to c \to f \to d$	Alternating sequence of vertices and edges Begins and ends at the same vertex
A **closed trail** is a trail that begins and ends at the same vertex. **It is commonly called a circuit.**	 $d \to f \to b \to c \to f \to e \to d$	No repeated edges Begins and ends at the same vertex
A **closed path** is a path that begins and ends at the same vertex. It is also referred to as a directed cycle because it travels through a cyclic subgraph.	 $d \to f \to b \to c \to d$	No repeated edges or vertices Begins and ends at the same vertex

The same circuit can be named using any of its vertices as a starting point. For example, the circuit $d \to f \to b \to c \to d$ can also be referred to in the following ways.

$a \to b \to c \to d \to a$ is the same as

$$\begin{cases} b \to c \to d \to a \to b \\ c \to d \to a \to b \to c \\ d \to a \to b \to c \to d \end{cases}$$

Let's practice working with closed walks, circuits (closed trails), and directed cycles (closed paths). In the graph in Figure 7.1.8, the vertices are major central and south Florida airports. The edges are direct flights between them

✓ Example 7.1.7: Determining a Closed Walk, Circuit, or Directed Cycle

Suppose that you need to travel by air from Miami (MIA) to Orlando (MCO) and you are restricted to flights as represented on the graph. For the trip to Orlando, you decide to purchase tickets with a layover in Key West (EYW) as shown in Figure ✓ 7.1.27, but you still have to decide on the return trip. Determine if your round-trip itinerary is a closed walk, a circuit,/or a directed cycle based on the return trip described in each part.

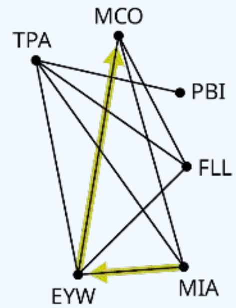


Figure 7.1.27: MIA to EYW to MCO

1. You returned to Miami (MIA) by reversing your route.
2. Your direct flight back left Orlando (MCO) but was diverted to Fort Lauderdale (FLL)! You flew to Tampa (TPA) from there before returning to Miami (MIA).

Answer

1. The whole trip was $MIA \rightarrow EYW \rightarrow MCO \rightarrow EYW \rightarrow MIA$. This is a closed walk because it begins and ends at the same vertex. It is not a circuit because it repeats edges. If it is not a circuit, then it cannot be a directed cycle.
2. The whole trip was $MIA \rightarrow EYW \rightarrow MCO \rightarrow FLL \rightarrow TPA \rightarrow MIA$. This is a closed walk because it begins and ends at the same vertex. It is a circuit because no edges were repeated. It is also a directed cycle because no vertices were repeated. So, it is all three!

✔ Example 7.1.8: Identifying Walks, Paths, and Trails

Consider each sequence of vertices from Graph in Figure 7.1.28. Determine if it is only a walk, both a walk and a path, both a walk and a trail, all three, or none of these.

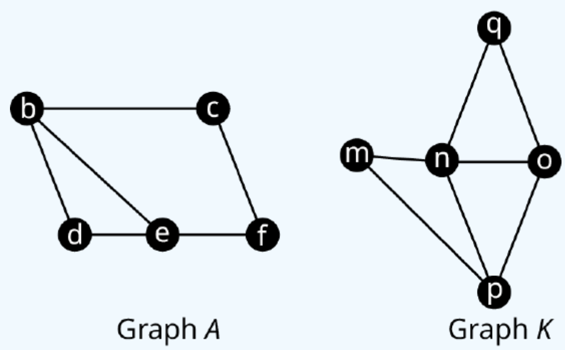


Figure 7.1.28

1. $b \rightarrow c \rightarrow d \rightarrow e \rightarrow f$
2. $c \rightarrow b \rightarrow d \rightarrow b \rightarrow e$
3. $c \rightarrow f \rightarrow e \rightarrow d \rightarrow b \rightarrow c$
4. $b \rightarrow e \rightarrow f \rightarrow c \rightarrow b \rightarrow d$

Answer

1. First, check to see if the sequence of vertices is a walk by making sure that the vertices are consecutive. As you can see in Figure 7.1.27, there is no edge between vertex c and vertex d.

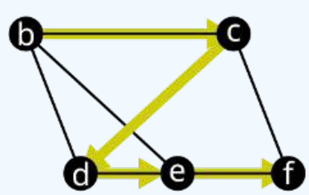


Figure 7.1.29

This means that the sequence is not a walk. If it is not a walk, then it can't be a path, and it cannot be a trail, so it is none of these.

2. First, check to see if the sequence is a walk. As you can see in Figure 7.1.30, the vertices are consecutive.

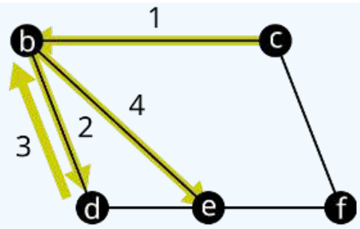


Figure 7.1.30

This means that the sequence is a walk. Since the vertex b is visited twice, this walk is not a path. Since edge semantics**bd** is traveled twice, this walk is not a trail. So, the sequence is only a walk.

3. First, check to see if the sequence is a walk. We can see in Figure 7.1.31 that the vertices are consecutive, which means it is a walk.

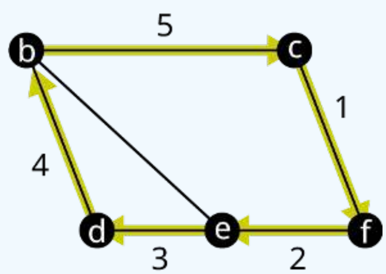


Figure 7.1.31

Next, check to see if any vertex is visited twice. Remember, we do **not** consider beginning and ending at the same vertex to be visiting a vertex twice. So, no vertex was visited twice. This means we have a walk that is also a path. The next check was to see if any edge was visited twice; none were. So, the sequence is a walk, a path, and a trail.

4. First, check to see if the sequence is a walk. We can see in Figure 7.1.32 that the vertices are consecutive, which means it is a walk.

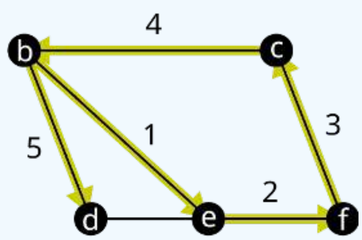
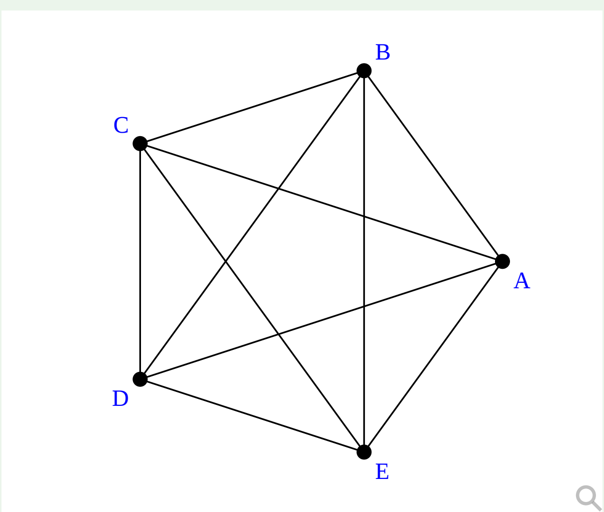


Figure 7.1.32

Next, check if any vertex has been visited twice. Since vertex b is visited twice, this is not a path. Finally, check to see if any edges are traveled twice. Since no edges are traveled twice, this is a trail. So, the sequence of vertices is a walk and a trail.

Your Turn 7.1.8: Walk or Path?



Using the graph above, determine if the following series of vertices gives you a walk, path, circuit, or none of these:

B→C→D→A→E.

- a walk, a path, and a circuit
- none of these
- a walk only
- a walk and a path only

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✔ Example 7.1.9: Naming a Walk Through A House

Figure \(\PageIndex{32\}) shows the floor plan of a house. Use the floor plan to answer each question.

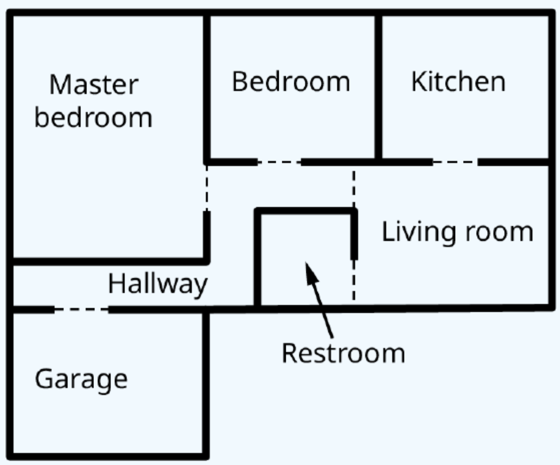


Figure 7.1.33: Floor Plan of a House

1. Draw a graph representing the floor plan in which each vertex represents a different room (or hallway), and edges represent doorways between rooms.
2. Name a walk through the house that begins in the living room, ends in the garage, and visits each room (or hallway) at least once.

Answer

1. We will need a vertex for each room, and it is convenient to label them according to the names of the rooms. Visualize the scenario in your head as shown in Figure 7.1.34. You don't have to write this step on your paper.

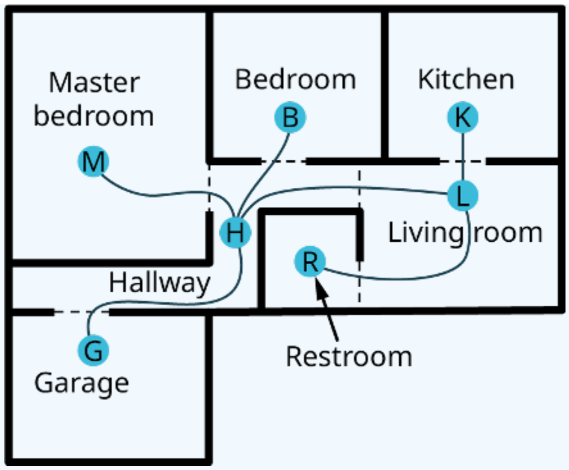


Figure 7.1.34: Assigning Vertices to Rooms

Draw a graph to represent the scenario. Start with the vertices. Then, connect those vertices that share a doorway in the floor plan, as shown in Figure 7.1.35.

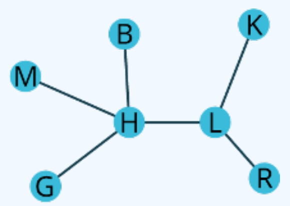


Figure 7.1.35: Graph of the Floor plan

Draw a walk that begins at vertex L, representing the living room, and ends at vertex G, representing the garage, making sure to visit every room at least once. There are many ways this can be done. You may want to number the edges to keep track of their order. One example is shown in Figure 7.1.36.

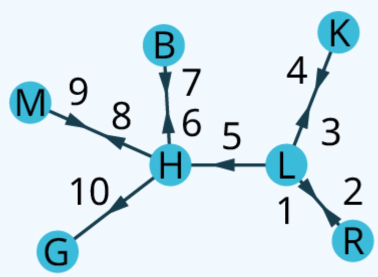


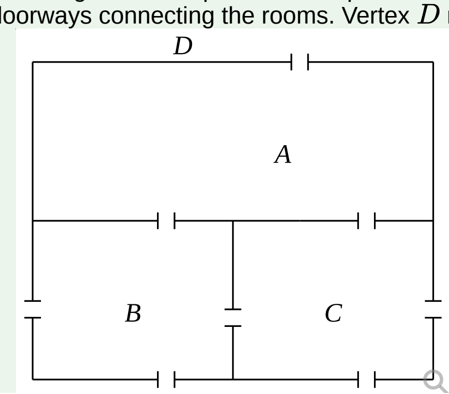
Figure 7.1.36: Draw the walk from L to G

2. Name the walk that you followed by listing the vertices in the order you visited them.

$L \rightarrow R \rightarrow L \rightarrow K \rightarrow L \rightarrow H \rightarrow B \rightarrow H \rightarrow M \rightarrow H \rightarrow G$

Your Turn 7.1.9: Draw a Graph to Model Given Floor plan

Draw a graph that models the connecting relationships in the floorplan below. The vertices represent the rooms and the edges represent doorways connecting the rooms. Vertex D represents the outdoors.



Is it possible to find a path through the house that uses each doorway once? If so, enter the sequence of rooms(vertices) visited, for example ABCDA. If it is not possible, enter DNE.

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Connectedness

Before we can discuss finding the best delivery route, we must ensure that a route exists. For example, suppose that you were tasked with visiting every airport on the graph in Figure 7.1.37 by plane. Could you accomplish that task by only taking direct flight paths between airports listed on this graph? In other words, are all the airports connected by paths? Or are some of the airports disconnected from the others?

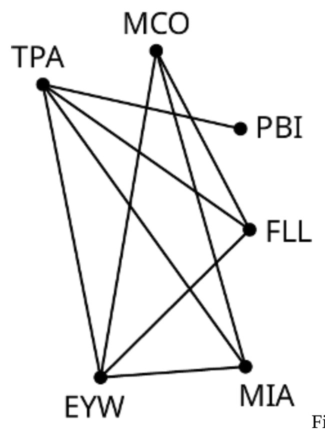


Figure 7.1.37: Major Central and South Florida Airports

In Figure 7.1.37, we can see TPA is adjacent to PBI, FLL, MIA, and EYW. Also, there is a path between TPA and MCO through FLL. This indicates there is a path between each pair of vertices. So, it is possible to travel to each of these airports only by taking direct flight paths and visiting no other airports.

Connectedness

A graph is **connected** if there is a path joining every pair of vertices on the graph. If graphs are not connected, it is called disconnected.

Let's take a closer look at graph X in Figure 7.1.38. Focus on vertex a. There is a path between vertices a and b, but there is no path between vertex a and vertex c. So, Graph X is disconnected. The graph Y is connected because of vertex g.

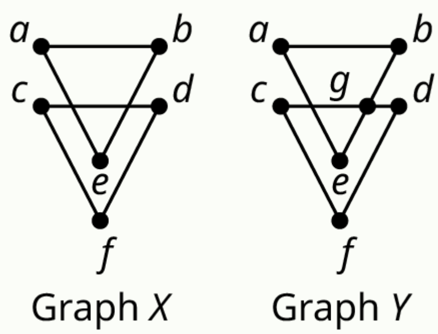


Figure 7.1.38: Connected vs. Disconnected

When you are working with a planar graph, you can also determine if a graph is connected by untangling it. If you draw it so that none of the edges are overlapping, as we did with Graph X in Figure 7.1.39, it is easier to see that the graph is disconnected.

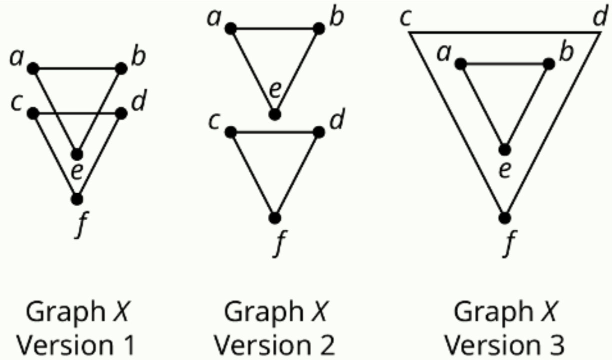


Figure 7.1.39: Untangling Graph X

Versions 2 and 3 of Graph X in Figure 7.1.439, each have the same number of vertices, number of edges, degrees of the vertices, and pairs of adjacent vertices as in version 1. In other words, each version retains the same structure as the original graph. Since versions 2 and 3 of Graph X, do not have overlapping edges, it is easier to identify pairs of vertices that do not have paths between them, and it is more obvious that Graph X is disconnected. In fact, there are two completely separate, disconnected subgraphs, one with the vertices in $\{a, b, e\}$, and the other with the vertices $\{c, d, f\}$

Example 7.1.10: Determining If a Graph Is Connected or Disconnected

Use Figure 7.1.41 to answer each question.

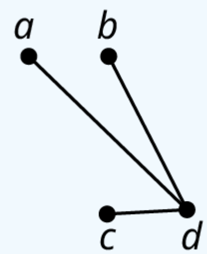


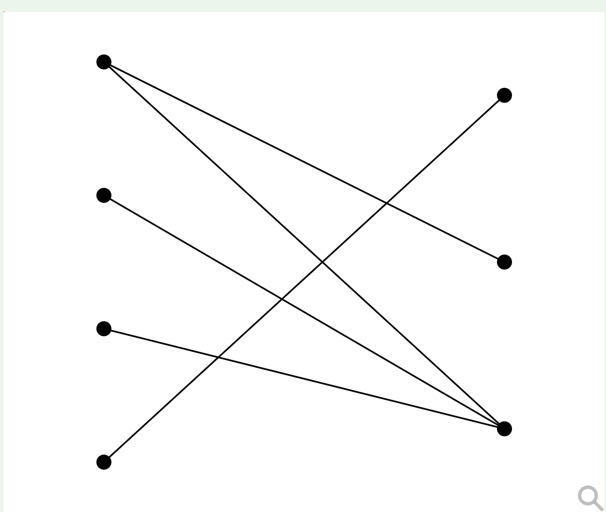
Figure 7.1.40: Graph E

1. Identify all the components of Graph E.
2. Determine whether the graph is connected or disconnected, and explain your reasoning.

Answer

1. There is only one component in Graph E. It has the vertices $\{a, b, c, d\}$.
2. The graph is connected because there is a path between vertex a and every other vertex. We can also see that Graph E is connected because it has only one component.

Your Turn 7.1.10: Is Graph Connected?



Is this graph connected?

- ◯ Not Connected
- ◯ Connected

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Definition: Bridge

A bridge is an edge that, if removed, would disconnect the graph, meaning there would be at least two separate parts that are no longer connected.

Example 7.1.11: Find Bridges

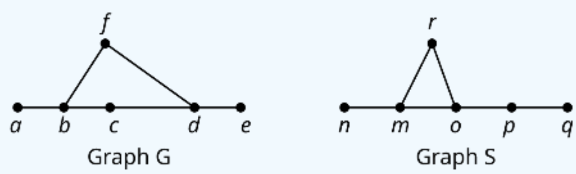


Figure 7.1.41

Find the bridges in **Graph G** and **Graph S**

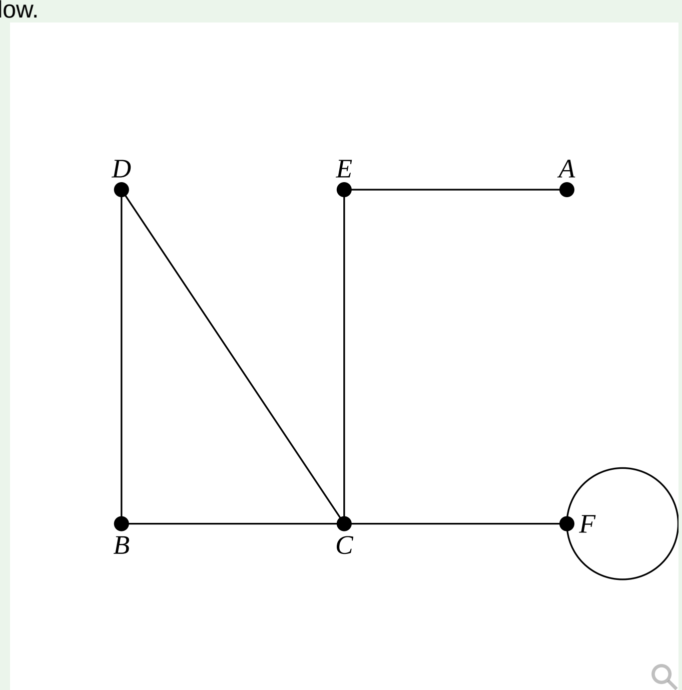
Answer

In G, bridges are **ab** and **de.**

In S, bridges are **nm, op, and pq.**

Your Turn 7.1.11: Idenfity Bridges

Consider the graph below.



Determine wether or not edge EC is a bridge. Justify the response.

Edge EC [Select an answer] since the graph is [Select an answer] when edge EC is included in the graph but is [Select an answer] when edge EC is excluded from the graph.

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Your Turn 7.1.12: Matching with Definition

Match each term with its definition.

- [-] Circuit
- [-] connected
- [-] walk
- [-] equivalent
- [-] trail
- [-] bridge
- [-] adjacent
- [-] complete graph
- [-] Degree of a vertex

a. A path that begins and ends at the same vertex
b. An open walk where no edge is repeated, though vertices may be repeated.
c. Two are structurally the same, even if they look different in a drawing.
d. A is a sequence of vertices and edges where both edges and vertices can be repeated.
e. Number of edges meeting at a vertex
f. A graph is one in which an edge connects every pair of distinct vertices.
g. An edge that, if removed, would disconnect the graph, meaning there would be at least two separate parts that are no longer connected.
h. A graph where there is a path joining every pair of vertices on the graph.
i. Two vertices in a simple graph are said to be adjacent iff they are the endpoints of the same edge

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7.2: Euler Circuits and Eulerization of Graph

Figure 7.2.1: Delivery trucks move goods from place to place. (credit: "Mack Midliner" by Jason Lawrence/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and identify Euler paths and Circuits.
2. Apply the Euler Circuit Theorem.
3. State the Chinese postman problem.
4. Evaluate Euler Circuits in real-world applications.

The delivery of goods is a huge part of our daily lives. From the factory to the distribution center to the local vendor or to your front door, nearly every product that you buy has been shipped multiple times to get to you. If the cost and time of that delivery is too great, you will not be able to afford the product. Delivery personnel have to leave from one location, deliver the goods to various places, and then return to their original location and do all of this in an organized way without losing money. How do delivery services find the most efficient delivery route? The answer lies in graph theory.

Origin of Euler Circuits and Terminology of Euler Path and Circuit

The city of Konigsberg, modern day Kaliningrad, Russia, has waterways that divide up the city. In the 1700s, the city had seven bridges over the various waterways. The map of those bridges is shown in Figure 7.2.2. The question as to whether it was possible to find a route that crossed each bridge exactly once and return to the starting point was known as the Konigsberg Bridge Problem. In 1735, one of the most influential mathematicians of all time, Leonard Euler, solved the problem using an area of mathematics he created himself: graph theory!

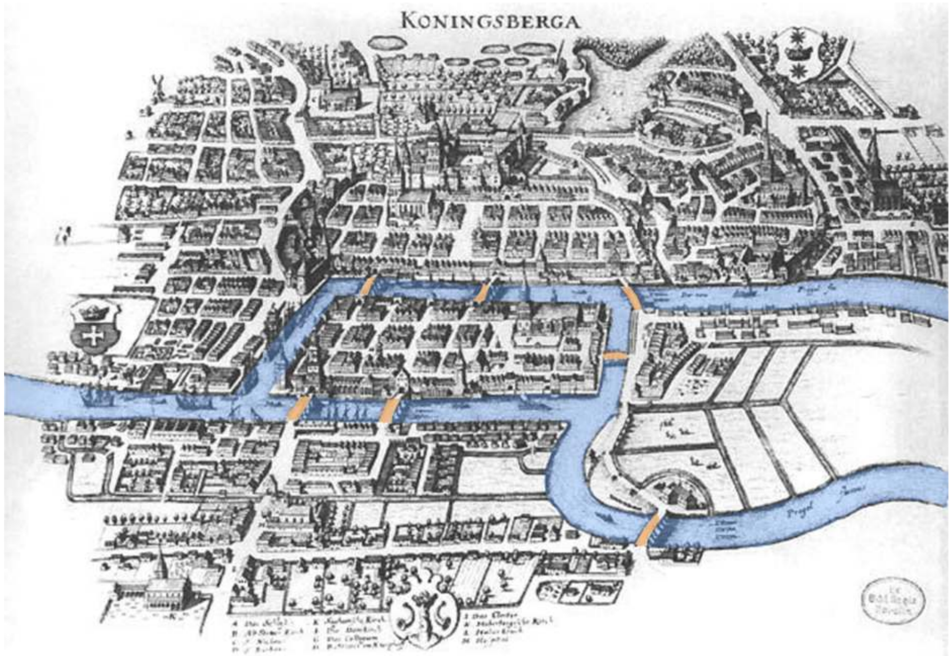

Figure $7.2.2$: Map of the Bridges of Konigsberg in 1700s (credit: “Konigsberg Bridge” by Merian Erben/Wikimedia Commons, Public Domain)

Euler drew a multigraph in which each vertex represented a land mass, and each edge represented a bridge connecting them, as shown in Figure $7.2.3$. Remember from Navigating Graphs that a circuit is a trail, so it never repeats an edge, and it is closed, so it begins and ends at the same vertex. Euler pointed out that the Konigsberg Bridge Problem was the same as asking this graph theory question: Is it possible to find a circuit that crosses every edge? Since then, circuits (or closed trails) that visit every edge in a graph exactly once have come to be known as Euler circuits in honor of Leonard Euler.

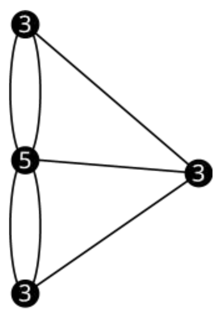

Figure $7.2.3$: Graph of Map of the Bridges of Konigsberg

In the first section, we created a graph of the Königsberg bridges and asked whether it was possible to walk across every bridge once. Because Euler first studied this question, these types of paths are named after him.

Euler Trail and Euler Circuit

Euler Trails start and end at two (distinct) vertices and traverse every edge exactly once.

Euler Circuit: Starts and ends at the same vertex and traverses every edge exactly once.
Note: In the previous section, we know a trail is not necessarily a path, but in this section. We will use the Euler trail as the Euler path.

Eulerian Graph

The graph is called Eulerian if it contains an Eulerian circuit.

Semi-Eulerian if it contains an Eulerian trail but *not* an Eulerian circuit.

Euler paths and Euler circuits are useful in many real-life situations where you need to travel through connections efficiently without repeating edges (roads, links, wires, routes, etc.). Here’s a clear and simple explanation of where they are used:

Euler's Path and Circuit Theorem

	Condition	Where to Start and End
Euler Circuit	All vertices are even	We can start at any even vertex and must end at the same vertex
Euler Path or Euler Trail (No Euler Circuit)	Exactly two odd vertices	We must start from one odd vertex and end at another odd vertex
No Euler Path and Euler Circuit	More than 2 odd vertices	

Your Turn \(\PageIndex{1 \): Euler Theorem

Consider a connected graph with two even vertices and nine odd vertices.

Determine whether or not the graph described above contains an Euler path or Euler circuit.

- ◯ The graph contains an Euler path but not an Euler circuit.
- ◯ The graph contains both an Euler path and Euler circuit.
- ◯ The graph contains an Euler circuit but not an Euler path.
- ◯ The graph contains neither an Euler path nor Euler circuit.

Justify the response given above.

- ◯ According to Euler's theorem, the above conclusion is true since the graph contains more than two odd vertices.
- ◯ According to Euler's theorem, the above conclusion is true since the graph contains more odd than even vertices.
- ◯ According to Euler's theorem, the above conclusion is true since the graph contains exactly two even vertices.
- ◯ According to Euler's theorem, the above conclusion is true since the graph contains exactly two odd vertices.
- ◯ According to Euler's theorem, the above conclusion is true since the graph contains no odd vertices.

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Euler Path and Euler Circuit

If we need a path that visits every edge in a graph, this is called an Euler path. Since trails are walks that do not repeat edges, an Euler trail visits every edge exactly once.

Example 7.2.2: Find a Euler Path

In the graph shown below, there are several Euler paths. Find one such path. What is the degree of the vertex that you start and end at to find such a path?

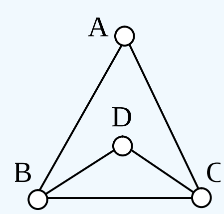


Figure 7.2.4

Answer

One such path is CABDCB. The path is shown in arrows to the right, with the order of edges numbered.

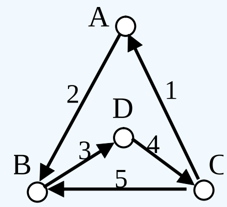


Figure $7.2.5$

You can write the above path in different ways

C, A, B, D, C, B

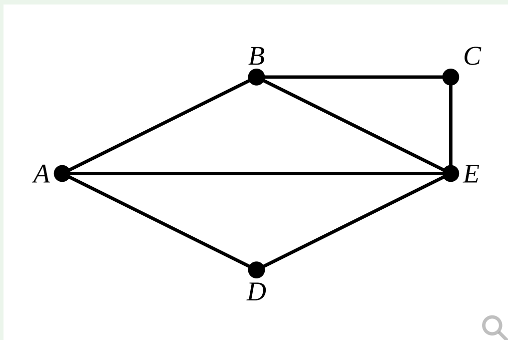
OR

$C \to A \to B \to D \to C \to B$

The degree of the vertex that we start is 3 (C: odd degree), and the degree of the vertex that we end is 3 (B: odd degree).

Your Turn $7.2.2$: Euler Path

Consider the graph below.



Explain why the above graph is guaranteed to contain at least one Euler path.

- ◯ The graph is guaranteed to contain at least one Euler path since the graph contains no odd vertices.
- ◯ The graph is guaranteed to contain at least one Euler path since all graphs must contain an Euler path.
- ◯ The graph is guaranteed to contain at least one Euler path since the graph contains exactly two even vertices.
- ◯ The graph is guaranteed to contain at least one Euler path since the graph contains more than two odd vertices.
- ◯ The graph is guaranteed to contain at least one Euler path since the graph contains exactly two odd vertices.

Starting at vertex A, find an Euler path for the graph whose fourth and seventh vertices are E and fifth vertex is D. Report the solution as a sequence of vertices (e.g., ABCDE or AECDCACBEA).

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Example $7.2.3$: Find a Euler Path

Does the graph below have an Euler Circuit? If so, find one.

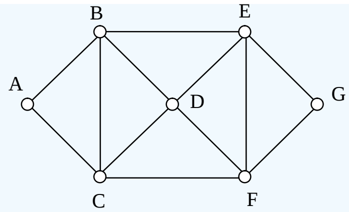


Figure 7.2.6

Answer

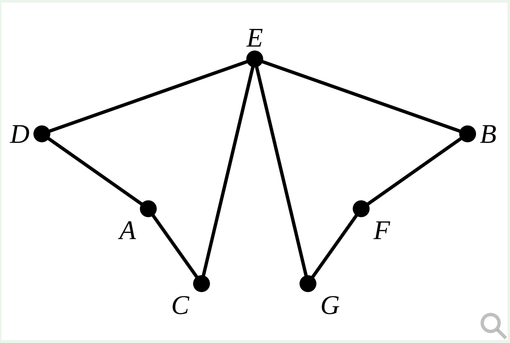
All vertices have even degrees, so this graph has an Euler Circuit (By Euler's Path and Circuit Theorem). There are several possibilities. One is ABEGFCDFEDBCA. We can write the same circuit another way, as follows

$A \to B \to E \to G \to F \to C \to D \to F \to E \to D \to B \to C \to A$

Since this graph has an Euler Circuit, we call this graph an **Eulerian graph.**

Your Turn 7.2.3: Euler Circuit

Consider the graph below.



Determine if the above graph contains an Euler circuit, an Euler path (but not an Euler circuit), or neither a Euler path nor Euler circuit.

- ○ The graph contains an Euler path, but not an Euler circuit.
- ○ The graph contains neither and Euler path nor Euler circuit.
- ○ The graph contains an Euler circuit.

If the graph contains an Euler path or circuit, find one such sequence whose first vertex is G and third vertex is C. Report the solution as a sequence of vertices (e.g., ABCDEFG or BAGFCEBDCBA). If no path or circuit exists, enter DNE.

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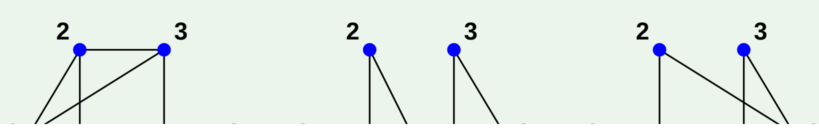
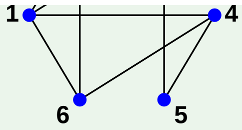
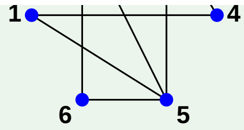
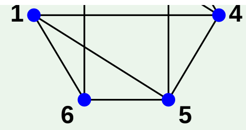
Your Turn 7.2.4: Euler Path, Euler Circuit or Neither?

For each graph below, decide if it contains an Euler circuit, an Euler trail, or neither.









Select an answer | Select an answer | Select an answer

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Here is a portion of a housing development. As part of her job, the development's lawn inspector has to walk down every street to ensure the homeowners' landscaping conforms to the community requirements.

Naturally, she wants to minimize the amount of walking she has to do. Is it possible for her to walk down every street in this development without doing any backtracking? While you might be able to answer that question just by looking at the picture for a while, it would be ideal to be able to answer the question for any picture regardless of its complexity job; the development's lawn inspector has to walk down every street in the development, making sure homeowners' landscaping conforms to the community requirements.

Figure 7.2.7: Portion of Housing Development

Why care if an Euler circuit exists? The lawn inspector is interested in walking as little as possible. The ideal situation would be a circuit that covers every street with no repeats. That's an Euler circuit! Luckily, Euler solved the question of whether or not an Euler path or circuit will exist.

Is there an Euler circuit on the housing development map Figure Figure 7.2.7, which is represented by the following graph in Figure 7.2.8. All the highlighted vertices have odd degrees (Figure 7.2.9.) Since there are more than two vertices with odd degrees, this graph has no Euler paths or circuits. Unfortunately, our lawn inspector will need to do some backtracking.

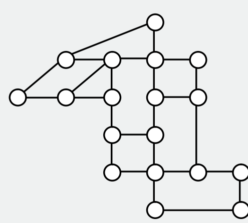


Figure 7.2.8

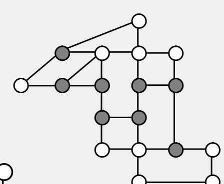


Figure 7.2.9

Now we know how to determine if a graph has an Euler circuit, but how do we find one if it does? While it is usually possible to find an Euler circuit just by pulling out your pencil and trying to find one, the more formal method is **Fleury's algorithm.**

Fleury's Algorithm: To Find the Euler Path and Euler Circuit

1. Start at any vertex if finding an Euler circuit. If finding an Euler path, start at one of the two vertices with odd degree.
2. Choose any edge leaving your current vertex, provided that deleting that edge will not separate the graph into two disconnected sets of edges.
3. Add that edge to your circuit and delete it from the graph.
4. Continue until you're done.

Example 7.2.5: Finding an Euler Circuit

Using Fleury's algorithm, starting at vertex A, let's find an Euler Circuit on this graph.

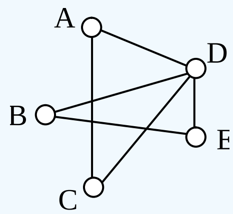


Figure 7.2.10

Answer

All vertices are even, so it has an Euler Circuit. In other words, it is an Eulerian graph.

Choose an edge AD.

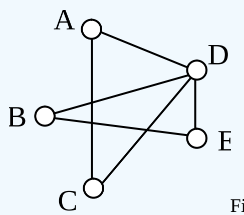


Figure 7.2.10

Delete AD: See Figure \(\PageIndex{11}. We can't choose DC since that would disconnect the graph.

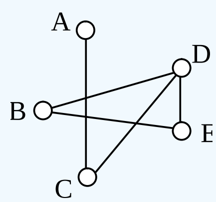


Figure 7.2.11

Choose DE. Delete DE. See figure 7.2.12. From E, there is only one option, so the rest of the circuit is determined.

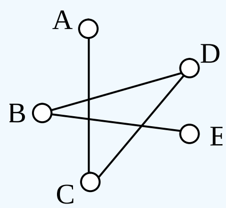


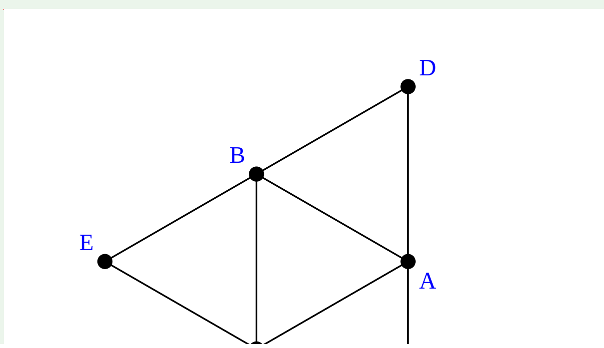
Figure 7.2.12

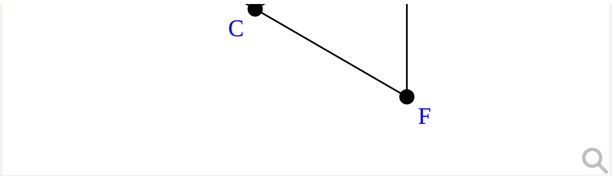
Circuit: ADEBDCA

or

$A \rightarrow D \rightarrow E \rightarrow B \rightarrow D \rightarrow C \rightarrow A$

Your Turn 7.2.5: Find Euler Circuit





Find any Euler circuit on the graph above using Fleury algorithm. Give your answer as a list of vertices, starting and ending at the same vertex. Example: ABCA

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Example 7.2.6: Finding an Euler Path using Fleury's Algorithm

Use Fleury's Algorithm to find an Euler trail for Graph J in Figure 7.2.13.

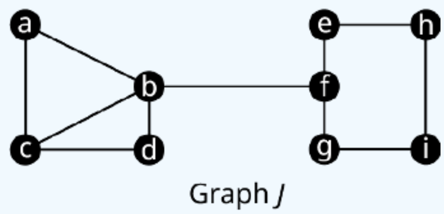


Figure 7.2.13: Graph J

Answer

Choose one of the two vertices of odd degree, c or f, as your starting vertex, since you must start from one odd vertex.

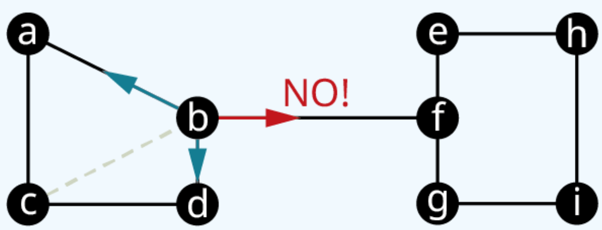


Figure 7.2.14: Graph J with cb Removed

The next choice is to remove edge ba, bd, or bf, as shown in Figure 7.2.14, but bf is not an option since it is a bridge. As shown in Figure 7.2.15, we will choose ba as the second edge removed.

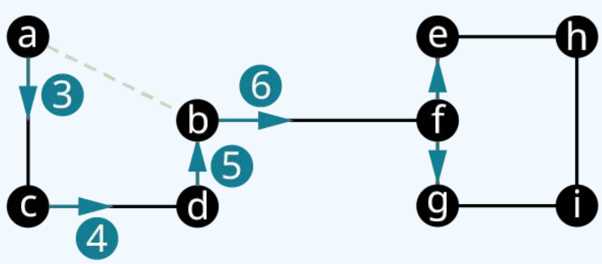


Figure 7.2.15: raph J with cb and ba Removed

For the third, fourth, fifth, sixth, and seventh edges. As shown in Figure 7.2.15, until we get to the seventh edge, there is only one option each time: ac, cd, db, and bf in that order. For the seventh edge, we must choose between fe and fg. Neither of these is a bridge. We choose fe. Figure 7.2.16 shows that ac, cd, db, bf, and fe have been removed.

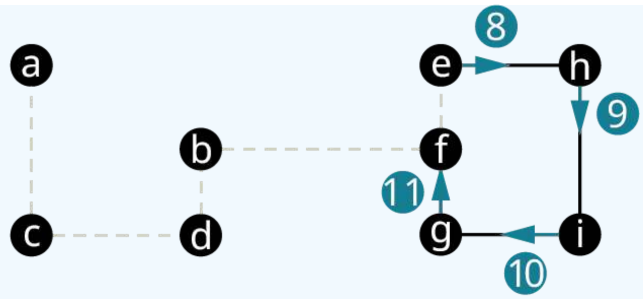


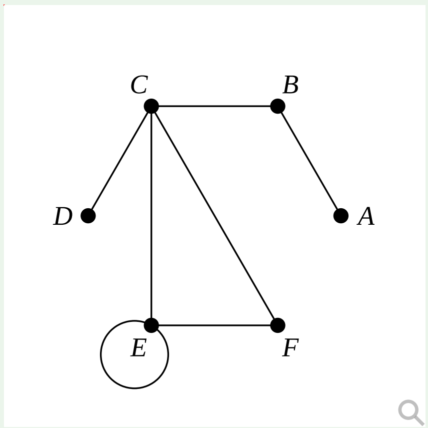
Figure 7.2.16: Graph J with Seven Edges Removed

Write out the Euler trail using the vertices in the sequence in which the edges were removed. We removed cb, ba, ac, cd, db, bf, fe, eh, hi, ig, and gf, in that order. The Euler trail is

$c \to b \to a \to c \to d \to b \to f \to e \to h \to i \to g \to f$

TIP! To avoid errors, count the number of edges in your graph and ensure that your Euler trail represents that number of edges.

Your Turn 7.2.6: Find Euler Circuit



Find an Euler path for the graph using Fleury algorithm. Enter your response as a sequence of vertices in the order they are visited, for example, ABCDEA.

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Chinese Postman Problem

Imagine you're a postman and must deliver mail by **walking along every street in a neighborhood at least once**, then return to your starting point. Naturally, you want to **walk the shortest possible distance**. How can you find the shortest closed path that covers every edge in a graph at least once? The name Chinese postman problem was coined in honor of the Chinese mathematician named Kwan Mei-Ko in 1960, who first studied the problem. This problem is essential in determining efficient routes for garbage trucks, school buses, parking meter checkers, and street sweepers.

Definition: Chinese Postman Problem

The task of finding the shortest circuit that visits every edge of a connected graph is often referred to as the **Chinese Postman problem**. In a given graph , the task is to find the shortest closed path that visits every edge at least once. The postman can traverse edges more than once if necessary, but each edge must be traversed at least once.

Example 7.2.7: Understanding Eulerian Graphs

A postal delivery person must deliver mail to every block on every street in a local subdivision. Figure $7.2.17$ is a map of the subdivision. Use the map to answer each question.

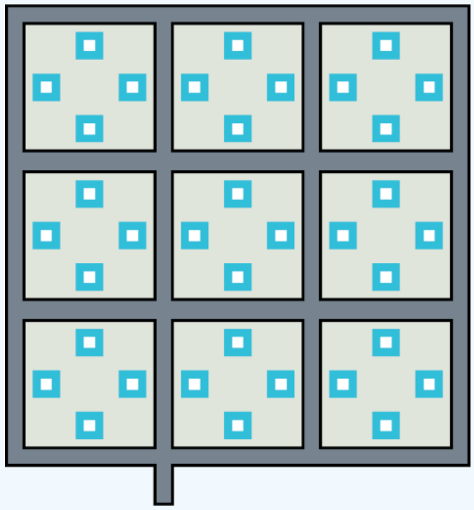


Figure $7.2.17$: Map of Subdivision

Draw a graph or multigraph to represent the subdivision in which the vertices represent the intersections and the edges represent the streets.

1. Is your graph connected? Explain how you know.
2. Determine the degrees of the vertices in the graph.
3. Is your graph an Eulerian graph?
4. Is it possible for the postal delivery person to visit each block on each street exactly once? Justify your answer.

Answer

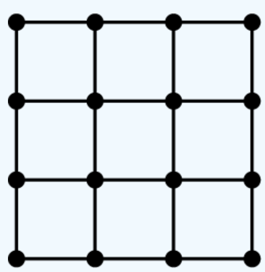


Figure $7.2.18$: Graph of Subdivision

The graph is show in Figure $7.2.18$.

1. The graph is connected. It has only one component and a path between each pair of vertices.
2. There are four corner vertices of degree 2, eight exterior vertices of degree 3, and four interior vertices of degree 3.
3. The graph is not Eulerian because it has vertices of odd degrees.
4. No. Since the graph is not Eulerian, there is no Euler circuit, meaning there is no route that would pass through every edge exactly once.

Your Turn 7.2.7

A pest control service has at least one customer on every block of every street or cul-de-sac in a neighborhood. Use the map of the neighborhood to answer each question.

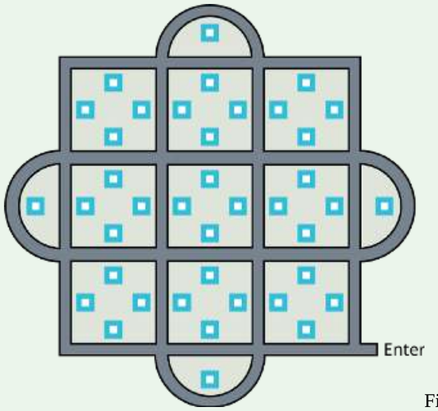


Figure \(\PageIndex{19}\): Map of Neighborhood

Draw a graph or multigraph of the neighborhood in which the vertices represent intersections, and the edges represent the streets between them.

1. Is your graph connected? Explain how you know.
2. Determine the degrees of the vertices in the graph.
3. Is your graph an Eulerian graph?
4. Can the postal delivery person visit each block on each street exactly once and start and end at the same position? Justify your answer.

Non-Eulerian Graph (Chinese Postman Problem)

Not every graph has an Euler path or circuit, yet our lawn inspector still needs to do her inspections. Her goal is to minimize the amount of walking she has to do. To do that, she will have to duplicate some edges in the graph until an Euler circuit exists. The problem of finding the optimal **Eulerization** is called the Chinese Postman Problem.

Eulerization

Eulerization is the process of adding edges to a graph to create an Euler circuit on a graph. To eulerize a graph, edges are duplicated to connect pairs of vertices with odd degrees. Connecting two odd-degree vertices increases the degree of each, giving them both even degrees. When two odd-degree vertices are not directly connected, we can duplicate all edges in a path connecting the two.

Note that we can only duplicate edges, not create edges where none existed. Duplicating edges would mean walking or driving down a road twice while creating an edge where there wasn't one before, which is akin to installing a new road!

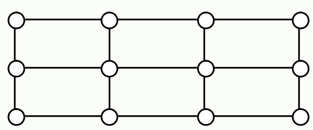


Figure $7.2.20$: Non Eulerian Graph

Three possible eulerizations for graph in figure $7.2.25$ are shown in figure $7.2.26$. Notice that in each of these cases, the vertices that started with odd degrees have even degrees after Eulerization, allowing for an Euler circuit.

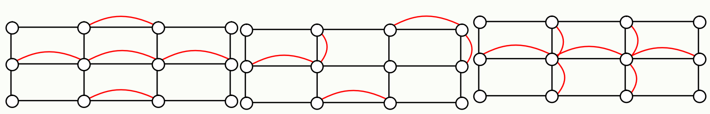


Figure $7.2.21$: Three Possible Eulerization

In the example above, you'll notice that the last Eulerization in figure $7.2.21$ required duplicating seven edges, while the first two only required duplicating five. If we were Eulerizing the graph to find a walking path, we would want the Eulerization with minimal duplications.

Example 7.2.8: Eulerized Map of Subdivision

Eulerized the graph of the Map of Subdivision given in figure $7.2.17$.

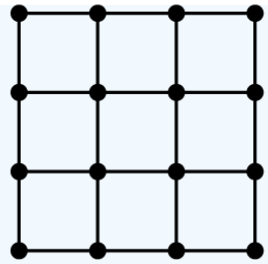


Figure 7.2.18: Graph of Subdivision

Answer

In Figure 7.2.22, the eight vertices of odd degrees in the graph of the subdivision are circled in green. We have added duplicate edges between the pairs of vertices, which changes the degrees of the vertices to even degrees, so the resulting multigraph has an Euler circuit. In other words, we have eulerized the graph shown below in figure 7.2.22.

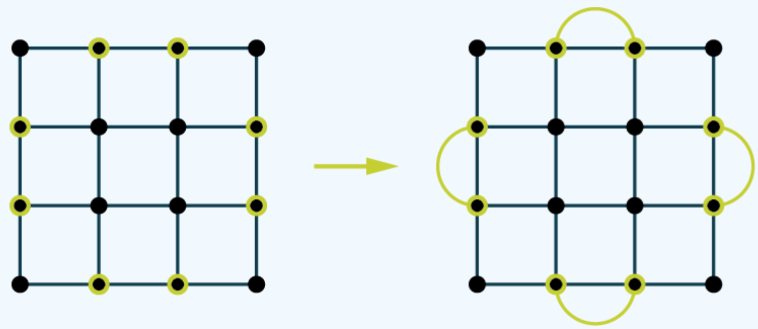
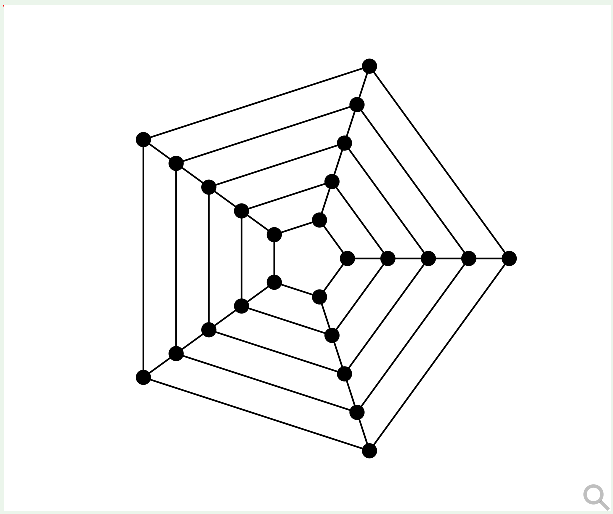


Figure 7.2.22: An Eulerized Graph

Your Turn 7.2.8: Eulerization of Graph



What is the minimum number of edges that would need to be duplicated to Eulerize this graph?

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Checkpoint

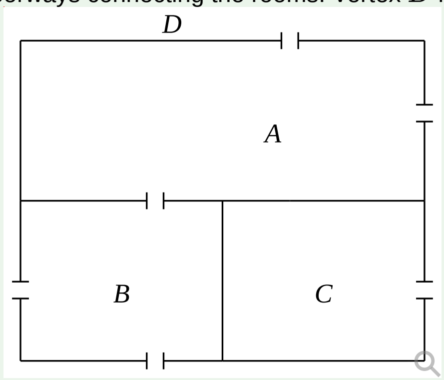
IMPORTANT! The duplicate edges that we add indicate places where the route will pass twice. An entirely new edge between two vertices that were not previously adjacent would indicate that our postal delivery person created a new road through someone's property! So, ***we can duplicate existing edges, but we cannot create new ones****.*

Checkpoint

For the following problem, the goal is achievable if this graph has an Euler path.

Your Turn 7.2.9: Understand Euler Path

Draw a graph that models the connecting relationships in the floorplan below. The vertices represent the rooms and the edges represent doorways connecting the rooms. Vertex D represents the outdoors.



Is it possible to find a path through the house that uses each doorway once? If so, enter the sequence of rooms(vertices) visited, for example ABCDA. If it is not possible, enter DNE.

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7.3: Hamiltonian Circuits and the Traveling Salesman Problem

Figure 7.3.1 : A school bus picks up children along a planned route. (credit: "Kids at School Bus Stop" by Ty Hatch/Flickr, CC BY 2.0)

Learning Objectives

1. Describe and evaluate Hamilton paths and circuits.
2. Evaluate Hamilton paths in real-world applications.
3. Apply brute force and the nearest neighbor methods to solve traveling salesperson problems.

In the United States, school buses carry 25 million children between school and home every day. The total distance they travel is around 6 billion kilometers per year. In the city of Boston, Massachusetts, the 2016 budget for running those buses was $120 million. In 2017, the city held a competition to find ways to cut costs and the Quantum Team from the MIT Operations Research Center came to the rescue, using a computer algorithm to identify the most efficient and least costly routes, which saved the city of Boston $5 million each year and even reduced daily CO_2 emissions by 9,000 kilograms! (*This* U.S. *city put an algorithm in charge of its school bus routes and saved $5 million*, Sean Fleming, World Economic Forum)

The problem the Quantum Team tackled involves graph theory. Imagine a graph in which the vertices are the bus depot, the school, and the bus stops along a particular route. The bus must start at the depot, visit every stop exactly once, and end at the school. The route is a special kind of path that visits every vertex exactly once. Can you guess what those paths are called?

Hamilton Paths and Hamilton Circuits

Definition: Hamiltonian Circuits and Paths

A **Hamiltonian circuit** is a circuit that visits every vertex once with no repeats. Being a circuit, it must start and end at the same vertex. A **Hamiltonian path** also visits every vertex once with no repeats, but does not have to start and end at the same vertex.

Hamiltonian circuits are named after William Rowan Hamilton, who studied them in the 1800s.

Just as circuits that visit each vertex in a graph exactly once are called Hamilton cycles (or Hamilton circuits), paths that visit each vertex on a graph exactly once are called Hamilton paths. As we explore Hamilton paths, you might find it helpful to refresh your memory about the relationships between walks, trails, and paths. We know that paths are walks that don't repeat vertices or edges. So, a Hamilton path visits every vertex without repeating any vertices or edges. Figure 7.3.2 shows a path from vertex A to vertex E and a Hamilton path from vertex A to vertex E.

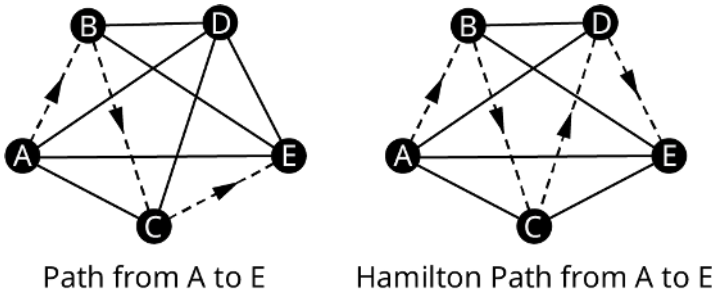


Figure 7.3.2: Path or Hamilton Path?

Example 7.3.1: Find the Hamiltonian Circuit and Path

Find a Hamilton circuit from A to A. There are several other Hamiltonian circuits possible on this graph. Can you find a Hamilton path from A to D?

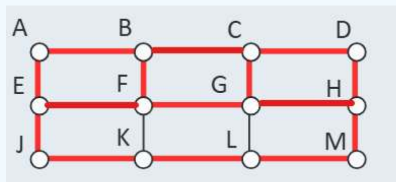


Figure 7.3.3: Copy and Paste Caption here. (Copyright; author via source)

Answer

For the Hamilton circuit, each vertex should be visited only once, and it should start and end at the same vertex:

ABFGCDHMLKJEA is a Hamilton circuit that goes from A to A.

Notice that the same circuit could be written in reverse order, or starting and ending at a different vertex.

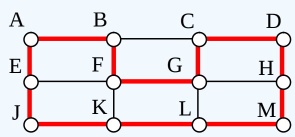
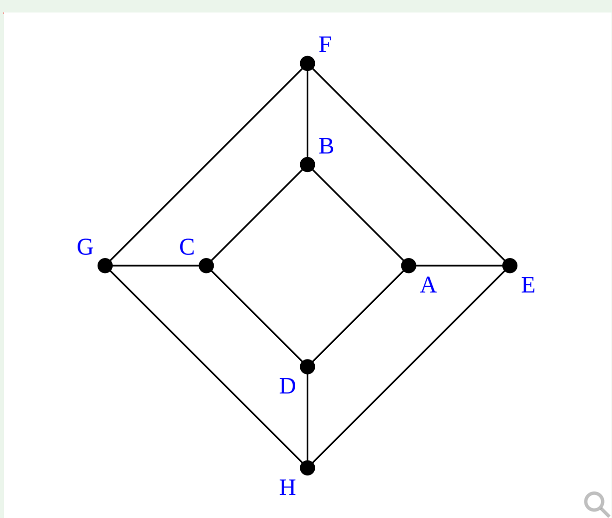


Figure 7.3.3A

AEJKFBCGLMHD is a Hamilton path from A to D.

Your Turn 7.3.1: Hamilton Circuit



Find any Hamiltonian circuit on the graph above. Give your answer as a list of vertices, starting and ending at the same vertex. Example: ABCA

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Unlike with Euler circuits, no nice theorem allows us to instantly determine whether or not a Hamiltonian circuit exists for all graphs. Just as circuits that visit each vertex in a graph exactly once are called Hamilton cycles (or Hamilton circuits), paths that visit each vertex on a graph exactly once are called Hamilton paths. As we explore Hamilton paths, you might find it helpful to refresh your memory about the relationships between walks, trails, and paths. We know that paths are walks that don't repeat vertices or edges. So, a Hamilton path visits every vertex without repeating any vertices or edges.

Example 7.3.2: Identifying Hamilton Paths

Which of the following sequences of vertices is a Hamilton path for Graph Q in Figure 7.3.4?

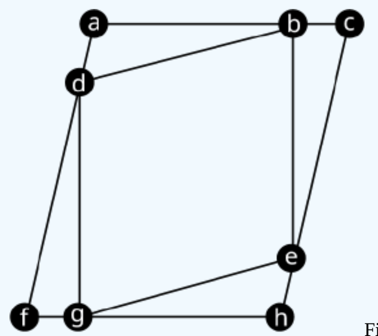


Figure 7.3.4: Graph Q

1. $a \to d \to b \to c \to e \to g \to f$
2. $c \to b \to e \to h \to g \to f \to d \to a$
3. $h \to e \to g \to d \to b \to e \to g \to f \to d \to a \to b \to c$

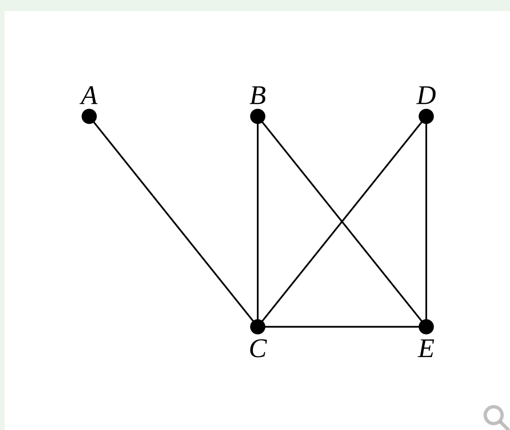
Answer

1. Sequence 1 is a path, because it is a walk that doesn't repeat any vertices or edges, but it is not a Hamilton path because it skips vertex h.
2. Sequence 2 is a path that visits each vertex exactly once; so, it is a Hamilton path.
3. Sequence 3 is a walk, but it is not a path because it visits vertices g, e, and b each more than once; so, it cannot be a Hamilton path. So, we can see that sequence 2 is only a Hamilton path.

Checkpoint

TIP! Since a Hamilton path visits each vertex exactly once, it must have the same number of vertices listed as appear in the graph.

Your Turn 7.3.2: Hamilton Path



Is a Hamilton circuit possible for this graph? If so, give your answer as a list of vertices, starting and ending at the same vertex. Example: ABCA If it is not possible, enter DNE.

Is a Hamilton path possible for the given graph? If so, give your answer as a list of vertices, for example, ABCD. If it is not possible, enter DNE.

Question Help: Video Video

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More on Finding Hamilton Circuit and Hamilton Path

Suppose you were visiting an aquarium with some friends. The map of the aquarium is given in 7.3.5. The letters represent the exhibits.

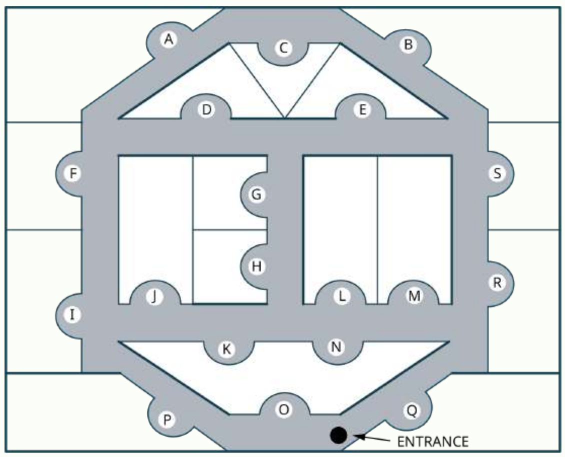


Figure 7.3.5: Map of Aquarium Exhibits

Figure 7.3.6 shows a graph of the aquarium in which each vertex represents an exhibit and each edge is a route between the pair of exhibits that doesn't bypass another exhibit. Suppose that, after exhibit O, we plan to visit.

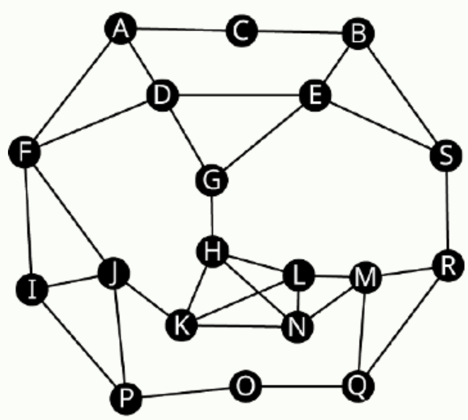


Figure 7.3.6: Graph of Aquarium

Let's see if we can plan a tour of the exhibits that visits each exhibit exactly once, beginning at exhibit O and ending at exhibit C. Suppose that, after exhibit O, we plan to visit exhibit Q and then exhibit M. After M, should we plan to visit N, L, or R? Take a look at Figure 7.3.7. If R is not chosen next, that will cause a problem later on. Do you see what it is?

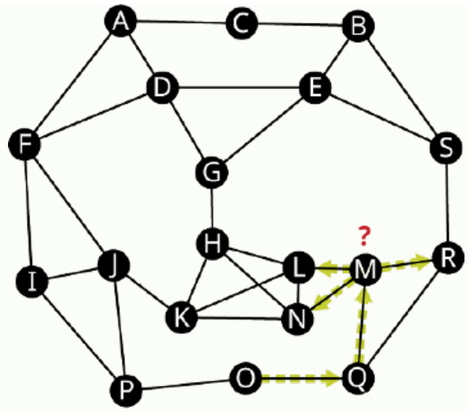


Figure 7.3.7 : Choosing Vertex *L*, *N*, or *R*

If *L* or *N* is chosen next, the only way to get to *R* later will be to go from *S* to *R*, and then we will not be able to continue without repeating a vertex. So, we will pick *R* next, and then the only option is *S*. After *S* we have another choice to make. As shown in Figure 7.3.8, the next choice is between *B* and *E*. Keeping in mind that the goal is to end at *C*, which would be the better choice?

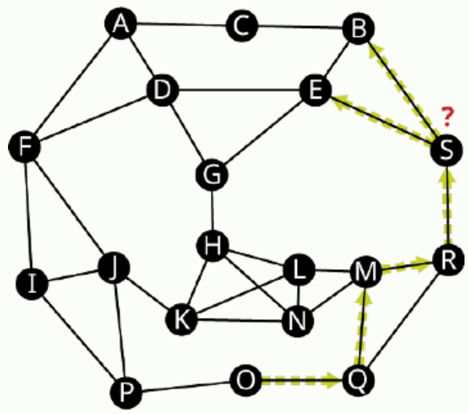


Figure 7.3.8 : Choosing Vertex *B* or *E*

If you said vertex *B*, you are right! Otherwise, we will not be able to visit *B* later. After *B*, the only option is *E*. Then we can choose either *D* or *G*. Either will work fine. Let's choose *G* as shown in Figure 7.3.9. After *G*, you must visit *H*, but should you visit *K* or *L* after that?

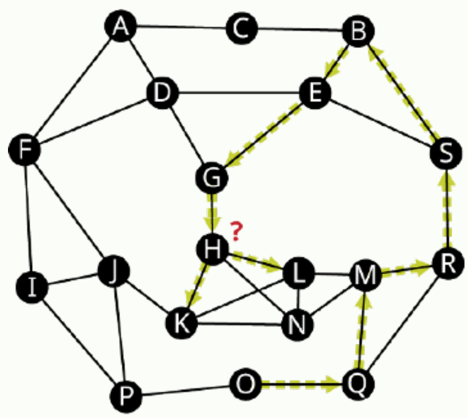


Figure 7.3.9 : Choose Vertex *L* or *K*

If you said to go to vertex *L* next, you are right! Otherwise, it will be impossible to visit *N* without repeating a vertex. So, next is *L*, then *N*, then *K*, and then at *J* you have another decision to make, we can see in Figure 7.3.10. Should you choose *F*, *I*, or *P* next?

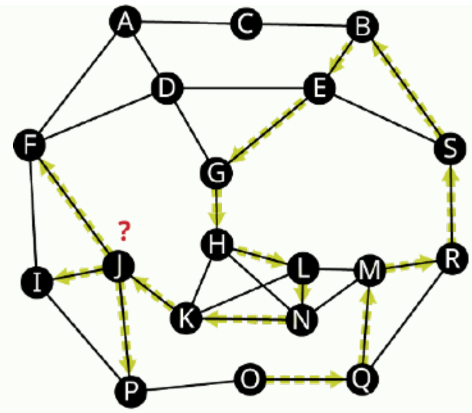


Figure 7.3.10 : Choose Vertex F, I, or P

If you said P, you are right! If you choose either of the other two vertices, you will not be able to visit P later without passing through another vertex twice. Once P is chosen, vertex I must be next, followed by F. Then you have to choose between A and D as shown in Figure 7.3.11.

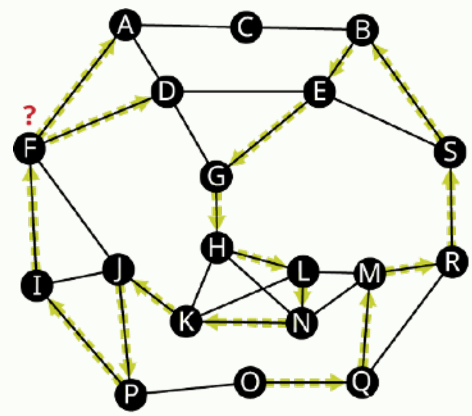


Figure
7.3.11 :
Choose Vertex A or D

In this case, we must go to D, then to A, so that we can visit C without backtracking. The complete Hamilton path is shown in Figure 7.3.12.

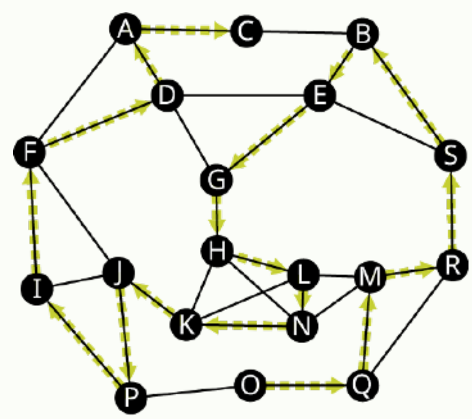


Figure 7.3.12 : Complete Hamilton Path from O to C

So, one Hamilton path that begins at O and ends at C is

$O \to Q \to M \to R \to S \to B \to E \to G \to H \to L \to N \to K \to J \to P \to I \to F \to D \to A \to C.$

There is no set sequence of steps that can be used to find a Hamilton path if it exists, but it does help to keep in mind where we are headed and avoid choices that will make returning to a particular vertex impossible without repeating vertices. Let’s practice finding Hamilton paths.

In the last section, we considered optimizing a walking route for a postal carrier. How is this different than the requirements of a package delivery driver? While the postal carrier needed to walk down every street (edge) to deliver the mail, the package delivery driver instead needs to visit every one of a set of delivery locations. Instead of looking for a circuit that covers every edge once, the package deliverer is interested in a circuit that visits every vertex once

The Traveling Salesperson Problem (TPS)

Now, let's focus our attention on the graph theory application known as the traveling salesperson problem (TSP), in which we must find the shortest route to visit a number of locations and return to the starting point.

With Hamiltonian circuits, our focus will not be on existence but on the question of optimization; given a graph where the edges have weights, can we find the optimal Hamiltonian circuit, the one with the lowest total weight?

This problem is known as the **Traveling Salesman Problem** (TSP) because it can be framed as follows: Suppose a salesman needs to give sales pitches in four cities. He looks up the airfares between each city, and puts the costs in a graph. In what order should he travel to visit each city once, then return home with the lowest cost?

We will consider some possible approaches to determining the lowest-cost Hamiltonian circuit. The first option that might come to mind is to just try all the different possible circuits.

Consider the following scenario. The U.S. Air Force officer stationed at Vandenberg Air Force Base must drive to visit three other California Air Force bases before returning to Vandenberg. The officer needed to visit each base once. We looked at the weighted graph in Figure $7.3.18$ representing the four U.S. Air Force bases: Vandenberg, Edwards, Los Angeles, and Beal, and the distances between them. There are different ways officers can visit three cities and return to Vandenberg, but we are interested in finding the optimal way (at least in terms of distance in this case).

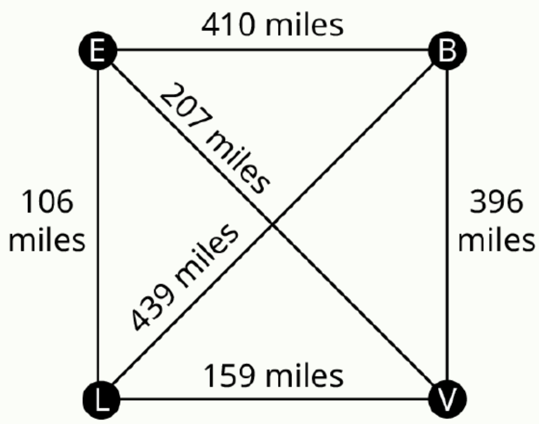


Figure $7.3.13$: Graph of Four California Air Force Bases

Consider another scenario. Suppose the salesman needs to give sales pitches in four cities. He looks up the airfares between each city listed in a graph. In what order should he travel to visit each city once, then return home with the lowest cost?

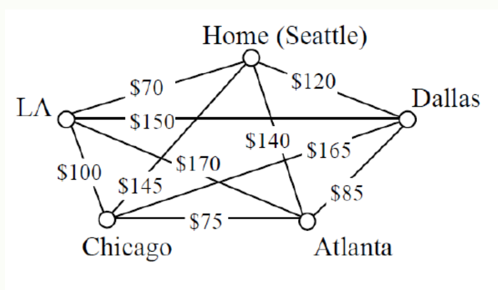
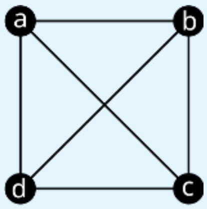
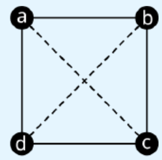
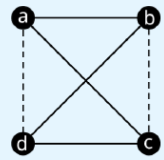
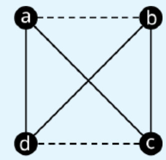
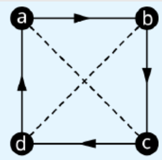
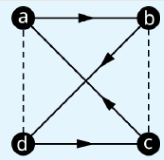
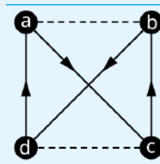
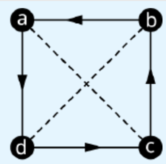
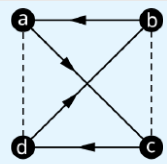
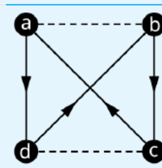


Figure $7.3.14$

In the first scenario, any route that visits each base and returns to the start would be a Hamilton cycle on the graph. If the officer wants to travel the shortest distance, this will correspond to a Hamilton cycle of lowest weight. We see in Table $7.3.1$ that there are six distinct Hamilton cycles (directed cycles) in a complete graph with four vertices, but some lie on the same cycle (undirected cycle) in the graph.

Table $7.3.1$: Hamilton Cycles in a Complete Graph with Four Vertices

Complete Graph	Cycle	Cycle	Cycle
			
Clockwise Hamilton Cycle	 $a \to b \to c \to d \to a$	 $a \to b \to d \to c \to a$	 $a \to c \to b \to d \to a$
Counterclockwise Hamilton Cycle	 $a \to d \to c \to b \to a$	 $a \to c \to d \to b \to a$	 $a \to d \to b \to c \to a$

Since the distance between bases is the same in either direction in Figure 7.3.15, it does not matter if the officer travels clockwise or counterclockwise. So, there are really only three possible distances as shown in Figure 7.3.15.

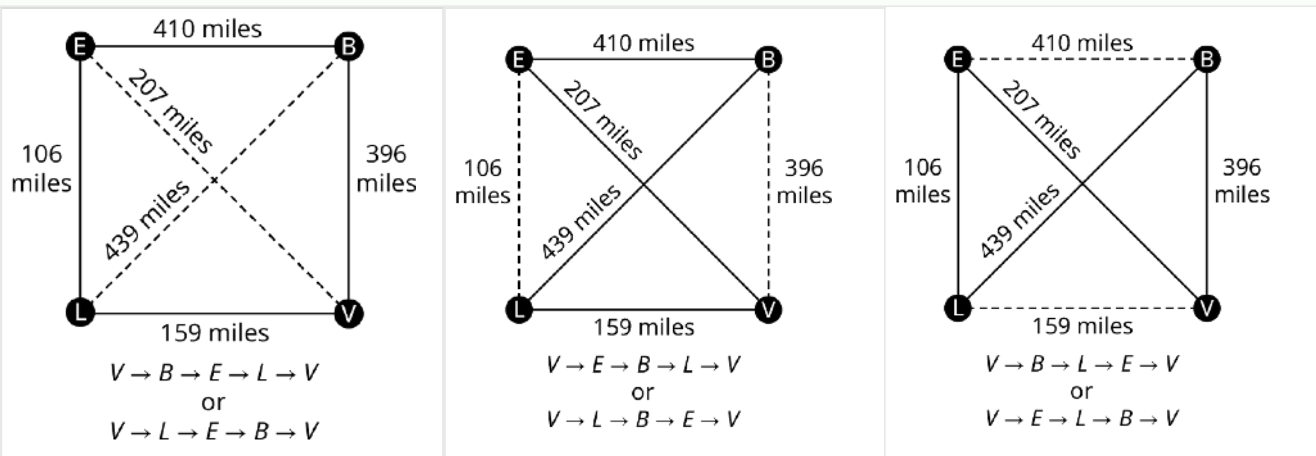


Figure 7.3.15 : Three Possible Distances

The possible distances are:

$$396 + 410 + 106 + 159 = 1071$$
$$207 + 410 + 439 + 159 = 1215$$
$$396 + 439 + 106 + 207 = 1148$$

So, a Hamilton cycle of least weight is $V \to B \to E \to L \to V$ (or the reverse direction). The officer should travel from Vandenberg to Beal to Edwards, to Los Angeles, and back to Vandenberg.

Finding Weights of All Hamilton Cycles in Complete Graphs

Notice that we listed all of the Hamilton cycles and found their weights when we solved the TSP for the officer from Vandenberg. This is a skill you will need to practice. To make sure you don't miss any, you can calculate the number of possible Hamilton cycles in a complete graph. It is also helpful to know that half of the directed cycles in a complete graph are the same cycle in the reverse direction, so you only have to calculate half the number of possible weights, and the rest are duplicates.

FORMULA: Number of Possible Hamilton Circuits

For n vertices in a complete graph, there will be $(n-1)! = 1 \times 2 \times 3 \times \ldots \times n$ routes (or Hamilton circuits).

Half of these are duplicates in reverse order, so there are $\frac{(n-1)!}{2}$ unique circuits.

The exclamation symbol, !, is read "factorial" and is shorthand for the product shown above.

Checkpoint

TIP! When listing all the distinct Hamilton cycles in a complete graph, you can start them all at any vertex you choose. Remember, the cycle a → b → c → a is the same cycle as b → c → a → b, so there is no need to list both.

Example 7.3.3: How Many Hamilton Circuit?

How many Hamilton circuits would a complete graph with eight vertices have?

Answer

A complete graph with 8 vertices would have

$$(8-1)! = 7! = 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 5040$$

We have 5040 possible Hamiltonian circuits. Half of the circuits are duplicates of other circuits but in reverse order, leaving 2520 unique Hamilton circuits.

Your Turn 7.3.3: Find Number of Hamilton Circuit

Consider a complete graph with 4 vertices.

Determine if the graph described above is guaranteed to contain a Hamilton circuit or not.

- ○ Yes, the graph is guaranteed to contain at least on Hamilton circuit.
- ○ No, the graph is not guaranteed to contain at least one Hamilton circuit.

Determine the number of Hamilton circuits contained in the graph described above.

- ◉ The number of Hamilton circuits contained in the graph above is ______
- ○ The above graph does not contain a Hamilton circuit.

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While this is a lot, it doesn't seem unreasonably huge. But consider what happens as the number of cities increases:

Cities	Hamilton Circuit	Unique Hamiltonian Circuits
9	$8! = 40,320$	$8!/2 = 20,160$
10	$9! = 362,880$	$9!/2 = 181,440$
11	$10! = 3,628,800$	$10!/2 = 1,814,400$
15	$14! = 8.71782912 \times 10^{10}$	$14!/2 = 43,589,145,600$
20	$19! = 6.40237371 \times 10^{15}$	$19!/2 = 60,822,550,204,416,000$

Example 7.3.4: Calculating Possible Weights of Hamilton Cycles

Suppose you have a complete weighted graph with vertices N, M, O, and P.

1. Use the formula $(n-1)!$ to calculate the number of distinct Hamilton cycles in the graph.
2. Use the formula $\frac{(n-1)!}{2}$ to calculate the greatest number of different weights possible for the Hamilton cycles.

3. Are all of the distinct Hamilton cycles listed here? How do you know?
Cycle 1: $N \to M \to O \to P \to N$
Cycle 2: $N \to M \to P \to O \to N$
Cycle 3: $N \to O \to M \to P \to N$
Cycle 4: $N \to O \to P \to M \to N$
Cycle 5: $N \to P \to M \to O \to N$
Cycle 6: $N \to P \to O \to M \to N$

4. Which pairs of cycles must have the same weights? How do you know?

Answer

1. There are 4 vertices; so, $n = 4$. This means there are

$$(4-1)! = 3! = 3 \times 2 \times 1 = 6$$

There are 6 distinct Hamilton cycles beginning at any given vertex.

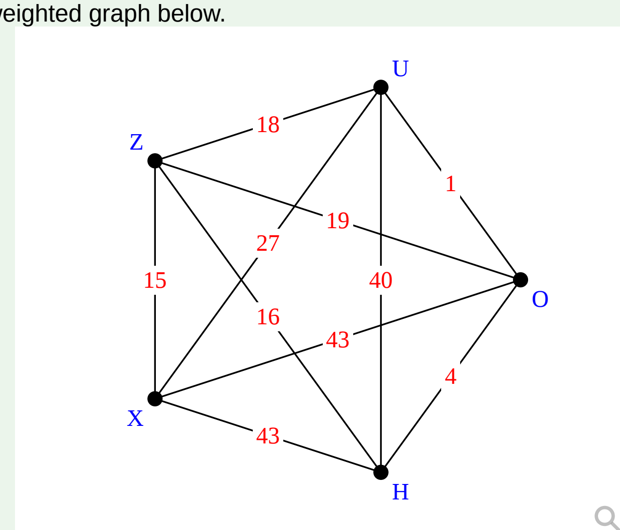
2. So there are $\frac{6}{3}$ possible weights to consider, which have different weights.
3. Yes, they are all distinct cycles, and there are 6 of them.
4. Cycles 1 and 6 have the same weight, cycles 2 and 4 have the same weight, and cycles 3 and 5 have the same weight, because these pairs follow the same route through the graph but in reverse.

Checkpoint

TIP! When listing the possible cycles, ignore the vertex where the cycle begins and ends, and focus on the ways to arrange the letters that represent the vertices in the middle. Using a systematic approach is best; for example, if you must arrange the letters M, O, and P, first list all those arrangements beginning with M, then beginning with O, and then beginning with P, as we did in Example 7.3.3.

Your Turn 7.3.4: Find the Weight of Circuit

Consider the complete, weighted graph below.



Determine the total weight of the Hamilton circuit given by OUZXHO.

The total weight of the Hamilton circuit given by OUZXHO is ______.

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The Brute Force Method

The method we have been using to find a Hamilton cycle of the least weight in a complete graph is a brute force algorithm, hence it is referred to as the brute force method. The steps in the brute force method are.

Step 1: Calculate the number of distinct Hamilton cycles and the number of possible weights.

Step 2: List all possible Hamilton cycles.

Step 3: Find the weight of each cycle.

Step 4: Identify the Hamilton cycle of the lowest weight.

✓ Example 7.3.5: Applying the Brute Force Method

On the next assignment, the air force officer must leave Travis Air Force Base, visit Beale, Edwards, and Vandenberg Air Force bases each exactly once, and return to Travis Air Force Base. **There is no need to visit the Los Angeles Air Force base.** Use Figure 7.3.16 to find the shortest route.

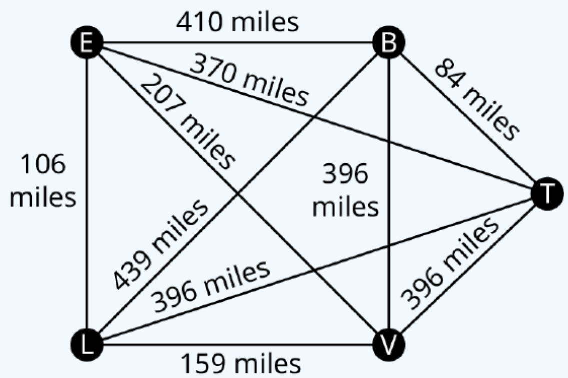


Figure 7.3.16 :Distances between Five California Air Force Bases

Answer

Step 1: Since there are 4 vertices, there will be afafa fadfad afadfa fadfa fadfa possible distances.

Step 2: List all the Hamilton cycles in the subgraph of the Figure 7.3.17.

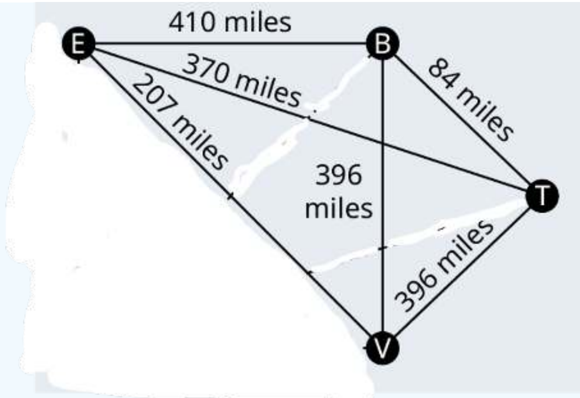


Figure 7.3.17 : S with Cities B, E, T, and V

To find the six cycles, focus on the three vertices in the middle, *B*, *E*, and *V*. The arrangements of these vertices are *BEV*, *BVE*, *EBV*, *EVB*, *VBE*, and *VEB*. These would correspond to the six cycles:

1: $T \to B \to E \to V \to T$

2: $T \to B \to V \to E \to T$

3: $T \to E \to B \to V \to T$

4: $T \to E \to V \to B \to T$

5: $T \to V \to B \to E \to T$

6: $T \to V \to E \to B \to T$

Step 3: Find the weight of each path. You can reduce your work by observing the cycles that are reverses of each other.

1: $84 + 410 + 207 + 396$
$= 1097$

2: $84 + 396 + 207 + 370$
$= 1071$

3: $370 + 410 + 396 + 396$
$= 1572$

4: Reverse of cycle 2 has 1071 weight.

5: Reverse of cycle 3 has 1572 weight.

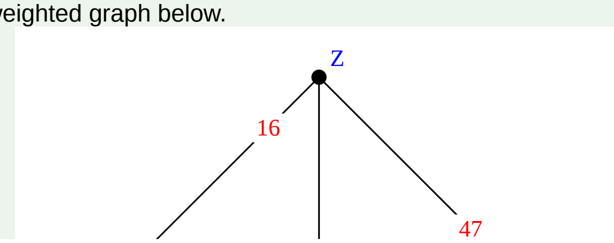
6: Reverse of cycle 1 has 1097 weight.

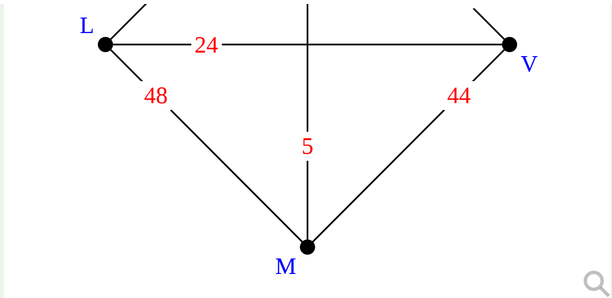
Step 4: Identify a Hamilton cycle of least weight.

The second path, $T \to B \to V \to E \to T$, and its reverse, $T \to E \to V \to B \to T$, have the least weight. The solution is that the officer should travel from Travis Air Force base to Beal Air Force Base, to Vandenberg Air Force base, to Edwards Air Force base, and return to Travis Air Force base, or the same route in reverse.

Your Turn 7.3.5: Apply Brute Force Method

Consider the complete, weighted graph below.





Determine the total weight of each Hamilton circuit described below.

Hamilton Circuit	Total Weight
VZLMV	
VZMLV	
VLZMV	
VLMZV	
VMLZV	
VMZLV	

Which Hamilton circuit(s) is/are the optimal solution(s)? Select all that apply.

- ☐ VMZLV
- ☐ VZMLV
- ☐ VLMZV
- ☐ VZLMV
- ☐ VLZMV
- ☐ VMLZV

Determine the weight of the optimal solution.

The weight of the optimal solution is ______.

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Now, suppose that the officer needed a cycle that visited all 5 of the Air Force bases in Example $7.3.16$. There would $(5-1)! = 4! = 4 \times 3 \times 2 \times 1 = 24$ different arrangement, so there are $\frac{24}{2} = 12$ distances compare using the brute force method. That's what we need to list 12 unique Hamilton circuits and find the weight of all of those circuits.

If you consider 10 Air Force bases, there would be

$(10-1)! = 9! = 9 \times 8 \times 7 \times 6 \times \cdots \times 1 = 362,880$ different distances (routes) to consider. And for 15 air force base, it must check about 14 trillion routes. This is impossible for humans—and even computers struggle as numbers grow. **There must be another way to solve the TPS problem.**

The Nearest Neighbor Method

The Brute force algorithm is optimal; it will always produce the Hamiltonian circuit with minimum weight. Is it efficient? To answer that question, we need to consider how many Hamiltonian circuits a graph could have. For simplicity, let's look at the worst-case possibility, where every vertex is connected to every other vertex. This is called a **complete graph.**

Suppose we had a complete graph with five vertices, like the air travel graph above. From Seattle, there are four cities we can visit first. There are three choices for each of those. There are two possible cities to visit next from each of those cities. There is then only one choice for the last city before returning home.

This can be shown visually:

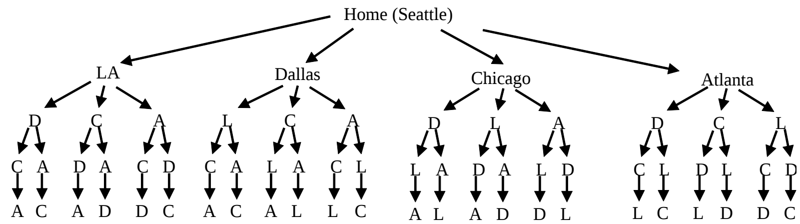


Figure 7.3.18

Counting the number of routes, we can see there are $4 \cdot 3 \cdot 2 \cdot 1 = 24$ routes. For six cities there would be $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ routes.

As you can see, the number of circuits is increasing rapidly. If a computer looked at one billion circuits a second, it would still take almost two years to examine all the possible circuits with only 20 cities! Certainly, Brute Force is **not** an efficient algorithm.

Unfortunately, no one has yet found an efficient *and* optimal algorithm to solve the TSP, and it is very unlikely anyone ever will. Since it is not practical to use brute force to solve the problem, we turn instead to **heuristic algorithms**; efficient algorithms that give approximate solutions. In other words, heuristic algorithms are fast, but may or may not produce the optimal circuit.

When the brute force method is impractical for solving a traveling salesperson problem, an alternative is a greedy algorithm known as the nearest neighbor method, which always visits the closest or least costly place first. This method finds a Hamilton cycle of relatively low weight in a complete graph in which, at each phase, the next vertex is chosen by comparing the edges between the current vertex and the remaining vertices to find the lowest weight. Since the nearest neighbor method is a greedy algorithm, it usually doesn't give the best solution, but it usually gives a solution that is "good enough." Most importantly, the number of steps will be the number of vertices. That's right! A problem with 10 vertices requires 10 steps, not $362,880$. Let's look at an example to see how it works.

Example: Suppose that a candidate for governor wants to hold rallies around the state. They plan to leave their home in city *A*, visit cities *B*, *C*, *D*, *E*, and *F* each once, and return home. The airfare between cities is indicated in the graph in Figure $7.3.19$. Let's help the candidate keep costs of travel down by applying the nearest neighbor method to find a Hamilton cycle that has a reasonably low weight

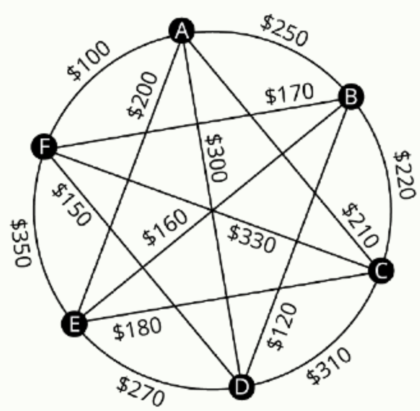


Figure $7.3.19$: Airfares between Cities *A*, *B*, *C*, *D*, *E*, and *F*

Begin by marking starting vertex as V_1 for "visited 1st." Then to compare the weights of the edges between A and vertices adjacent to A : $\$250, \$210, \$300, \$200,$ and $\$100$ as shown in Figure $7.3.20$. The lowest of these is $\$100$, which is the edge between A and F.

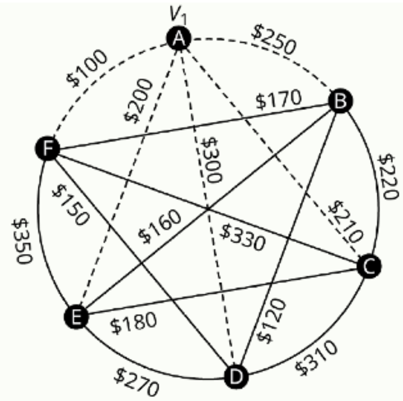


Figure 7.3.20 : Finding the Second Vertex

Mark F as V_2 for "visited 2nd" then compare the weights of the edges between F and the remaining vertices adjacent to F : $\$170, \$330, \$150$ and $\$350$ as shown in Figure 7.3.21. The lowest of these is $\$150$, which is the edge between F and D.

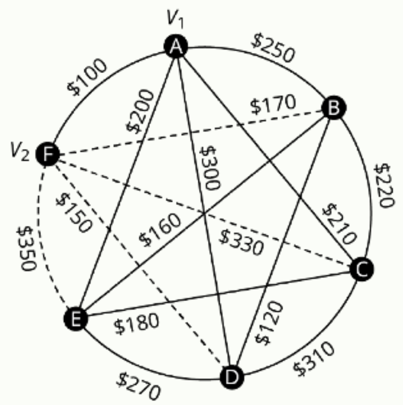


Figure 7.3.21 : Finding the Third Vertex

Mark D as V_3 for "visited 3rd." Next, compare the weights of the edges between D and the remaining vertices adjacent to D : $\$120, \$310$, and $\$270$ as shown in Figure 7.3.22. The lowest of these is $\$120$, which is the edge between D and B.

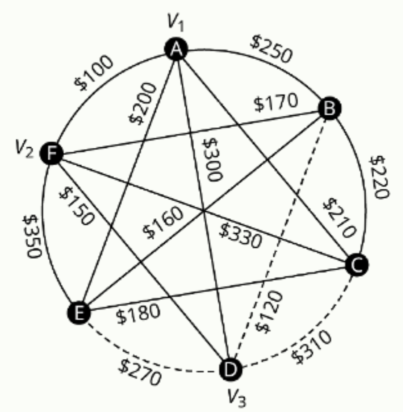


Figure 7.3.22 : Finding the Fourth Vertex

So, mark B as V_4 for "visited 4th." Finally, compare the weights of the edges between B and the remaining vertices adjacent to B : $\$160$ and $\$220$ as shown in Figure 7.3.23. The lower amount is $\$160$, which is the edge between B and E.

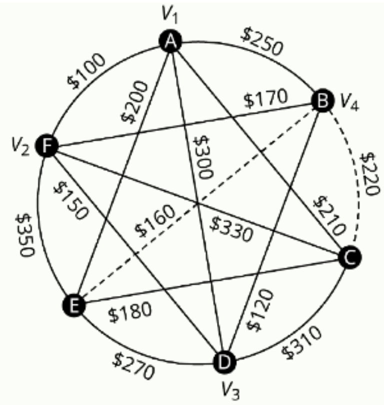


Figure 7.3.23 : Finding the Fifth Vertex

Now you can mark E as V_5 and mark the only remaining vertex, which is C, as V_6. This is shown in Figure 7.3.24. Make a note of the weight of the edge from E to C, which is $180, and from C back to A, which is $210.

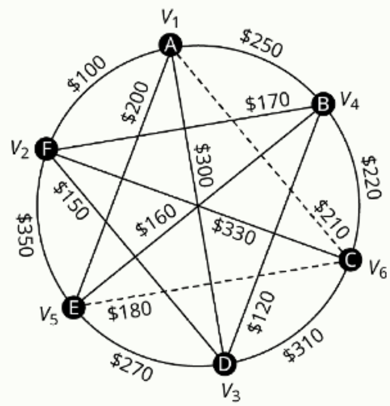


Figure 7.3.24 : Finding the Sixth Vertex

The Hamilton cycle we found is $A \to F \to D \to B \to E \to C \to A$. The weight of the circuit is

$$\$100 + \$150 + \$120 + \$160 + \$180 + \$210 = \$920$$

This may or may not be the route with the lowest cost, but there is a good chance it is very close since the weights are most of the lowest weights on the graph and we found it in six steps instead of finding 120 different Hamilton cycles and calculating 60 weights.

Let's summarize the procedure that we used.

Step 1: Select the starting vertex and label V_1 for the first visited vertex. Identify the edge of lowest weight between V_1 and the remaining vertices.

Step 2: Label the vertex at the end of the edge of lowest weight that you found in the previous step as V_n where the subscript n indicates the order the vertex is visited. Identify the edge of lowest weight between V_n and the vertices that remain to be visited.

Step 3: If vertices that have not been visited remain, repeat Step 2. Otherwise, a Hamilton cycle of low weight is $V_1 \to V_2 \to \cdots \to V_n \to V_1$

✔ Example 7.3.6: Using the Nearest Neighbor Method

Suppose that the candidate for governor wants to hold rallies around the state, but the time before the election is very limited. They would like to leave their home in city A, visit cities B, C, D, E, and F each once, and return home. The airfare between cities is not as important as the time of travel, which is indicated in Figure ✔ 7.3.25. Use the nearest neighbor method to find a route with relatively low travel time. What is the total travel time of the route that you found?

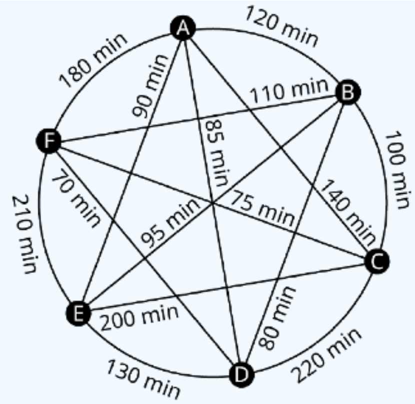


Figure $7.3.25$: Travel Times between Cities *A*, *B*, *C*, *D*, *E*, and *F*

Answer

Step 1: Label vertex A as V_1. The edge of lowest weight between *A* and the remaining vertices is 85 minutes between *A* and *D*.

Step 2: Label vertex V_2. The edge of lowest weight between *D* and the vertices that remain to be visited, *B*, *C*, *E*, and *F*, is 70 min between *D* and *F*.

Repeat Step 2: Label vertex V_3. The edge of lowest weight between *F* and the vertices that remain to be visited, *B*, *C*, and *E*, is 75 min between *F* and *C*.

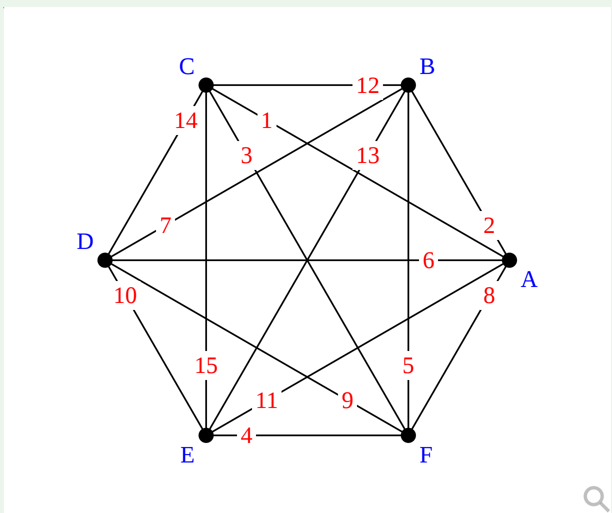
Repeat Step 2: Label vertex *C* as V_4. The edge of lowest weight between *C* and the vertices that remain to be visited, *B* and *E*, is 100 min between *C* and *B*.

Repeat Step 2: Label vertex V_5. The only vertex that remains to be visited is *E*. The weight of the edge between *B* and *E* is 95 min.

Step 3: A Hamilton cycle of low weight is A → D → F → C → B → E→ A. So, a relatively low travel time route is A to D to F to C to B to E to A. The total travel time of this route is

$85 + 70 + 75 + 100 + 95 + 90 = 515$ minutes.

Your Turn 7.3.6: Use the Nearest Neighbor Method



Apply the nearest neighbor algorithm to the graph above starting at vertex A. Give your answer as a list of vertices, starting and ending at vertex A. Example: ABCDEFA

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Your Turn 7.3.7: Fill in the Blank

Fill in the blank below to produce a true statement.

A path that travels through every vertex of graph exactly one time is called a/an [Select an answer] . Such a path that starts and ends at the same vertex and passes through all other vertices exactly once is called a/an [Select an answer] .

Fill in the blanks below to produce a true statement.

A method to determine the solution to the traveling salesman problem that involves listing all Hamilton circuits and identifying the circuit with the least total weight is called the [Select an answer] method.

Fill in the blanks below to produce a true statement.

A graph in which all edges are labeled with numbers is called a/an [Select an answer] graph. The numbers that label each edge of the graph are called edge [Select an answer] . The problem of determining a Hamilton circuit for which the sum of these numbers is a minimum is referred to as the [Select an answer] problem. Such a Hamilton circuit is called the [Select an answer] solution.

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7.4: Trees

Figure 7.4.1 : In graph theory, graphs known as trees have structures in common with live trees. (credit: “Row of trees in Roslev” by AKA CJ/Flickr, public Domain)

Learning Objectives

1. Describe and identify trees.
2. Determine a minimum spanning tree for a connected graph.
3. Solve application problems involving trees.

We saved the best for last! In this last section, we will discuss arguably the most fun kinds of graphs and trees. Have you ever researched your family tree? Family trees are a perfect example of the kind of trees we study in graph theory. One of the characteristics of a family tree graph is that it never loops back around because no one is their own grandparent!

What Is A Tree?

Whether we are talking about a family tree or a tree in a forest, none of the branches ever loops back around and rejoins the trunk. This means that a tree has no cyclic subgraphs or is acyclic. A tree also has only one component. So, a tree is a connected acyclic graph. Here are some graphs that have the same characteristics. Each of the graphs in figure 7.4.2 and figure 7.4.2 are a tree.

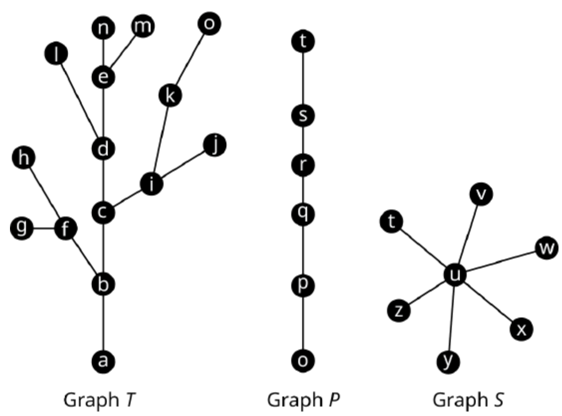


Figure 7.4.2 : Graphs T, P, and S

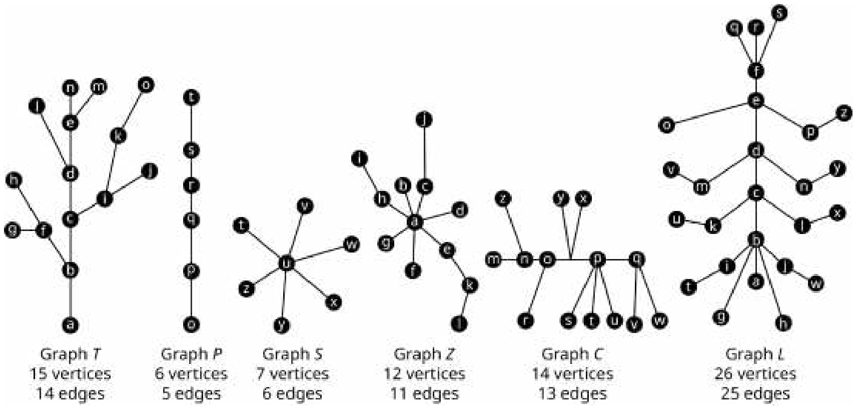


Figure 7.4.3 : Number of Vertices and Edges in Trees vs. Other Graph

Applications of Tree Graphs

Tree structures appear everywhere because they efficiently organize information. Here are the most common and important uses:

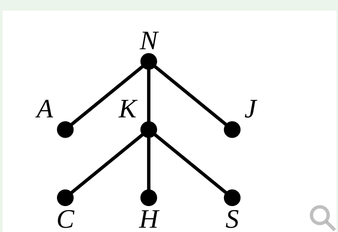
1. Folders and subfolders on a computer form a **tree**.
2. Communication networks
3. Shows relationships between generations.
4. Company Organization Chart

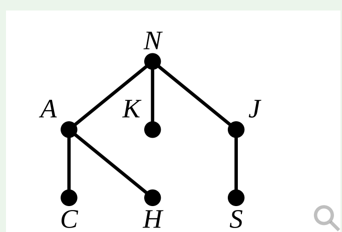
Your Turn 7.4.1: Family Tree

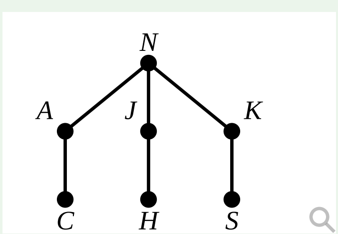
Consider the following family relationships.

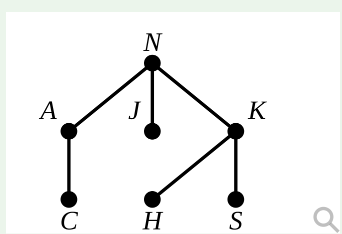
- Nathan has one child: Angelo, Jacy and Kaden.
- Angelo has one child: Christina.
- Jacy has no children.
- Kaden has two children: Heaven and Shawn.

Use graph vertices to individuals and graph edges to represent parent-child relationship and then create a graph to model the family relationships described above.

○ 

○ 

○ 

○ 

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Definition: Characteristic of Tree

1. There are **no loops** or closed paths.
2. Every pair of vertices is connected by **exactly one path**.
3. The number of edges is one less than the number of vertices. If tree has n vertices, It must have $n-1$ edges.
4. Every edge in a tree is a bridge.

Example 7.4.2: Identifying Trees

Identify any trees in Figure 7.4.4. If a graph is not a tree, explain how you know.

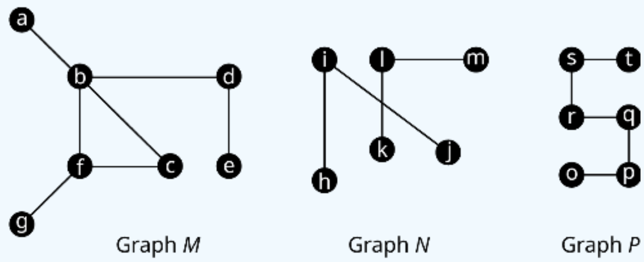


Figure 7.4.4 : Graphs *M*, *N*, and *P*

Answer

Graph *M* is not a tree because it contains the cycle (*b*, *c*, *f*).

Graph *N* is not a tree because it is not connected. It has two components, one with vertices *h*, *i*, *j*, and another with vertices *k*, *l*, *m*.

Graph *P* is a tree. It has no cycles, and it is connected.

Example 7.4.3: Exploring Characteristics of Trees

Use Graphs *I* and *J* in Figure 7.4.5 to answer each question.

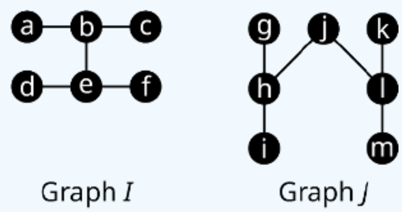


Figure 7.4.5 : Graphs *I* and *J*

1. Which vertices are in each of the components that remain when edge *be* is removed from Graph *I*?
2. Determine the number of edges and the number of vertices in Graph *J*. Explain how this confirms that Graph *J* is a tree.
3. What kind of cycle is created if edge *im* is added to Graph *J*?

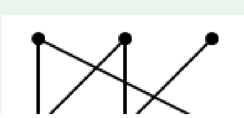
Answer

1. When edge *be* is removed, there are two components that remain. One component includes vertices *a*, *b*, and *c*. The other component includes vertices *d*, *e*, and *f*.
2. There are seven vertices and six edges in Graph *J*. This confirms that Graph *J* is a tree because the number of edges is one less than the number of vertices.
3. The pentagon (*i*, *h*, *j*, *l*, *m*) is created when edge *im* is added to Graph *J*.

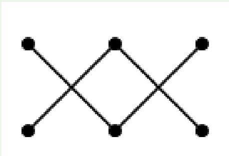
Your Turn 7.4.3: Identify Tree

Which of the following graphs are trees?

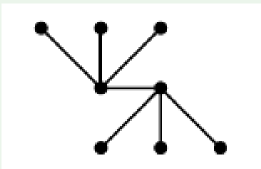
(a)



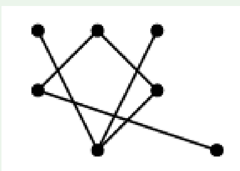
(b)



(c)



(d)



- [ ] a
- [ ] b
- [ ] c
- [ ] d

Question Help: Video

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Who Knew?: Graph Theory in the Movies

In the 1997 film *Good Will Hunting*, the main character, Will, played by Matt Damon, solves what is supposed to be an exceptionally difficult graph theory problem, “Draw all the homeomorphically irreducible trees of size $n = 10$.” That sounds terrifying! But don’t panic. Watch this great Numberphile video to see why this is actually a problem you can do at home!

Spanning Trees

Suppose that you planned to set up your own computer network with four devices. One option is to use a “mesh topology” like the one in Figure 7.4.9, in which each device is connected directly to every other device in the network.

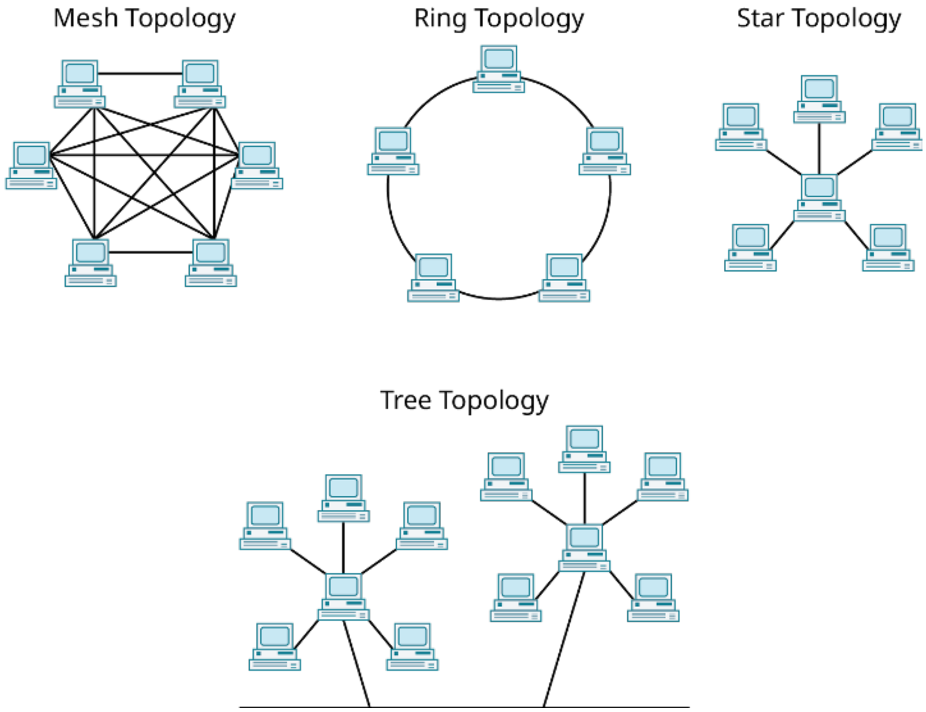


Figure 7.4.6 : Common Network Configurations

The mesh topology for four devices could be represented by the complete Graph A_1 in Figure 7.4.7 where the vertices represent the devices, and the edges represent network connections. However, the devices could be networked using fewer connections. Graphs A_2, A_3, and A_4 of Figure 7.4.7 show configurations in which three of the six edges have been removed. Each of the Graphs A_2, A_3 and A_4 in Figure 7.4.7 is a tree because it is connected and contains no cycles. Since Graphs A_2, A_3, and A_4 are also subgraphs of Graph A_1 that include every vertex of the original graph, they are also known as spanning trees.

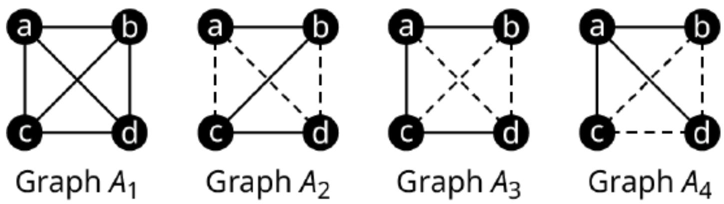


Figure 7.4.7 : Network Configurations for Four Devices

By definition, spanning trees must span the whole graph by visiting all the vertices. Since spanning trees are subgraphs, they may only have edges between vertices that were adjacent in the original graph. Since spanning trees are trees, they are connected, and they are acyclic.

So, when deciding whether a graph is a spanning tree, check the following characteristics:

- All vertices are included.
- No vertices are adjacent that were not adjacent in the original graph.
- The graph is connected.
- There are no cycles.

What is a **Spanning Tree**?

A spanning tree of a graph is a sub-graph that includes all the vertices of the original graph and has no cycles (because it is a tree).

Why use a Spanning Tree?

Because it connects all points with no redundancy and uses the least number of edges, it is used in:

1. Designing efficient communication networks
2. Reducing the cost of wiring or cabling
3. Optimizing routes in transportation

✔ Example 7.4.4: Identifying Spanning Trees

Use Figure 7.4.8 to determine which of graphs M_1, M_2, M_3, and M_4, are spanning trees of Q.

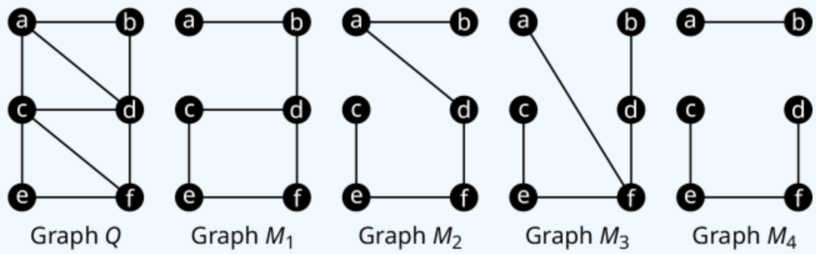


Figure 7.4.8 : Graphs Q, M_1, M_2, M_3, and M_4

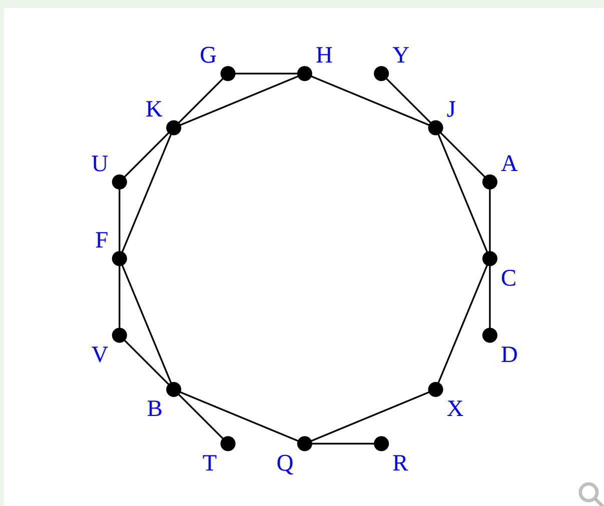
Answer

1. Graph M_1 is not a spanning tree of Graph Q because it has a cycle (c, d, f, e).
2. Graph M_2 is a spanning tree of Graph Q because it has all the original vertices, no vertices are adjacent in M_2 that weren't adjacent in Graph Q, Graph M_2 is connected, and it contains no cycles.
3. Graph M_3 is not a spanning tree of Graph Q because vertices a and f are adjacent in Graph M_3 but not in Graph Q.
4. Graph M_4 is not a spanning tree of Graph Q because it is not connected.

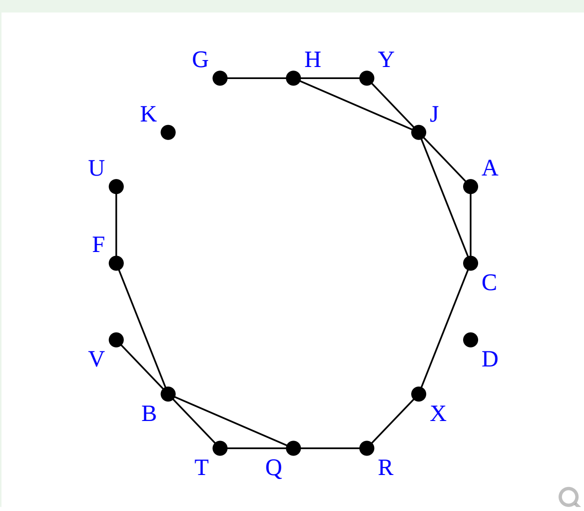
So, only graph M_2 is a spanning tree of Graph Q.

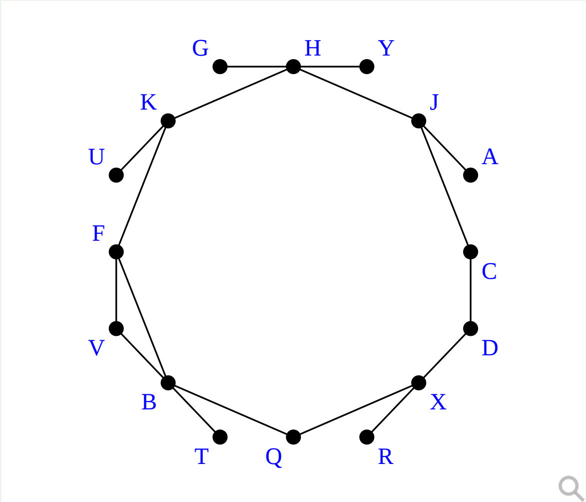
✎ Your Turn 7.4.4: Identify Spanning Tree

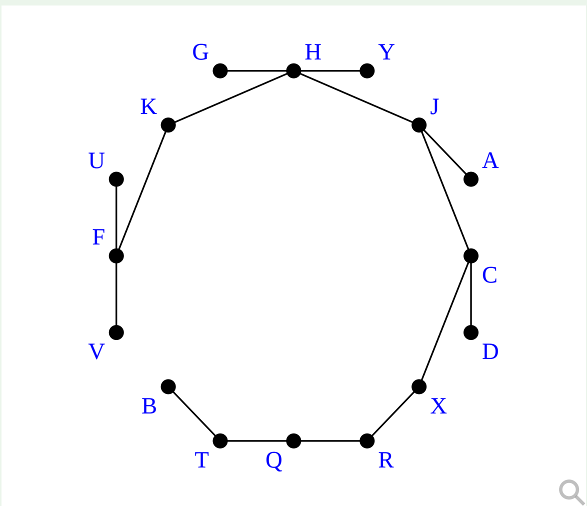
Consider the graph below.

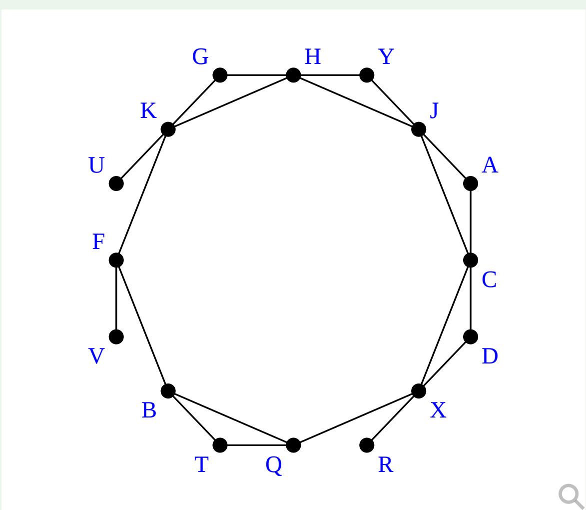


Determine a spanning tree for the above graph. Select the coorect answer below.









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Constructing a Spanning Tree Using Paths

Suppose that you wanted to find a spanning tree within a graph. One approach is to find paths within the graph. You can start at any vertex, go any direction, and create a path through the graph, stopping only when you can't continue without backtracking, as shown in Figure 7.4.9.

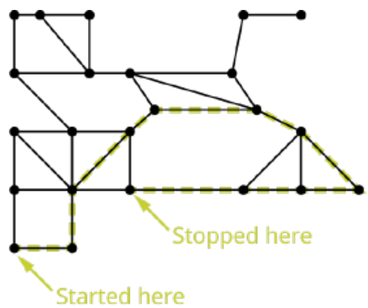


Figure 7.4.9 : First Phase to Construct a Spanning Tree

Once you have stopped, pick a vertex along the path you drew as a starting point for another path. Make sure to visit only the vertices you have not visited before, as shown in Figure 7.4.10.

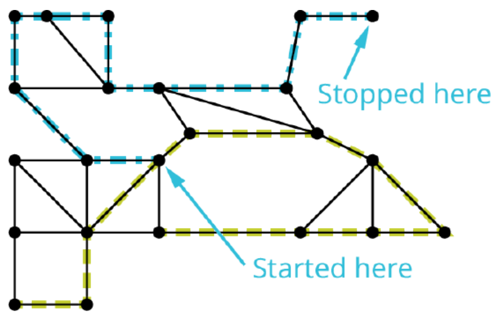


Figure 7.4.10 : Intermediate Phase to Construct a Spanning Tree

Repeat this process until all vertices have been visited as shown in Figure 7.4.11.

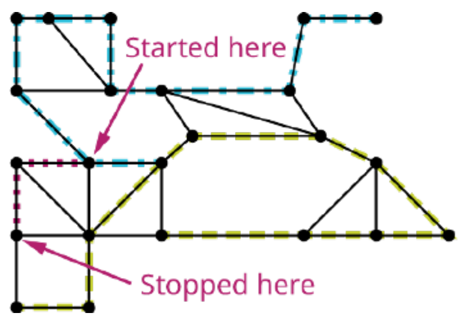


Figure 7.4.11 : Final Phase to Construct a Spanning Tree

The end result is a tree that spans the entire graph as shown in Figure 7.4.12.

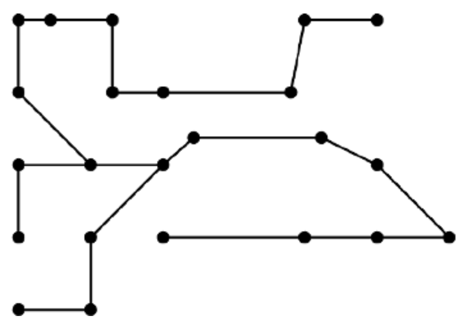


Figure 7.4.12 : The Resulting Spanning Tree

Notice that this subgraph is a tree because it is connected and acyclic. It also visits every vertex of the original graph, so it is a spanning tree. However, it is not the only spanning tree for this graph. By making different turns, we could create any number of distinct spanning trees.

Revealing Spanning Trees

Another approach to finding a spanning tree in a connected graph involves removing unwanted edges to reveal a spanning tree. Consider Graph D in Figure $7.4.13$.

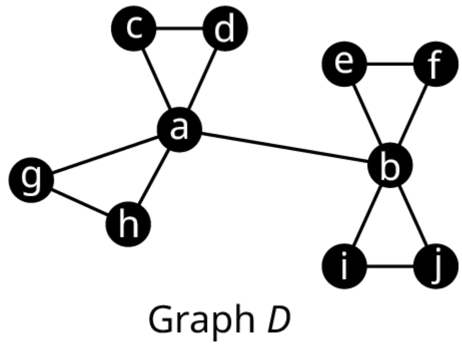


Figure $7.4.13$: Graph D

Graph D has 10 vertices. A spanning tree of Graph D must have 9 edges, because the number of edges is one less than the number of vertices in any tree. Graph D has 13 edges so 4 need to be removed. To determine which 4 edges to remove, remember that trees do not have cycles. There are four triangles in Graph D that we need to break up. We can accomplish this by removing 1 edge from each of the triangles. There are many ways this can be done. Two of these ways are shown in Figure \(\PageIndex{14}.\

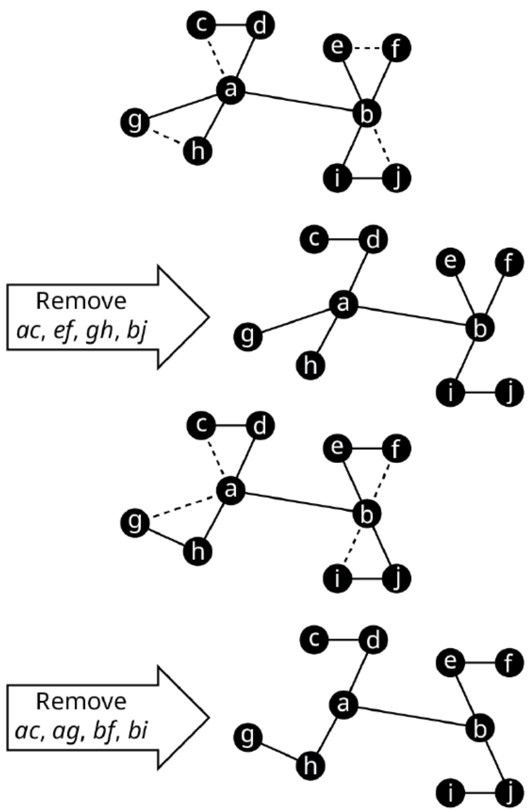


Figure $7.4.14$: Removing Four Edges from Graph D

✔ Example $7.4.5$: Removing Edges to Find Spanning Trees

Use the graph in Figure $7.4.15$ to answer each question.

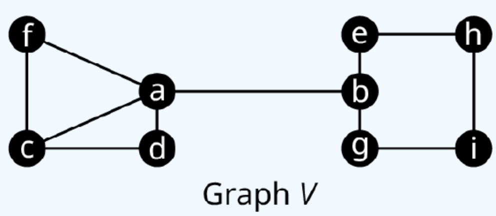


Figure $7.4.15$: Graph V

1. Determine the number of edges that must be removed to reveal a spanning tree.

2. Name all the undirected cycles in Graph V.

3. Find two distinct spanning trees of Graph V.

Answer

1. Graph V has nine vertices, so a spanning tree for the graph must have 8 edges. Since Graph V has 11 edges, 3 edges must be removed to reveal a spanning tree.

2. (a, c, d), (a, c, f), (a, d, c, f), and (b, e, h, i, g)

3. To find the first spanning tree, remove edge *ac*, which will break up both of the triangles, remove edge *cf*, which will break up the quadrilateral, and remove *be*, which will break up the pentagon, to give us the spanning tree shown in Figure $7.4.16$.

Figure $7.4.16$: Spanning Tree Formed Removing *ac*, *cf*, and *be*

To find another spanning tree, remove *ad*, which will break up (a, c, d) and (a, d, c, f), remove *af* to break up (a, c, f), and remove *hi* to break up (b, e, h, i, g). This will give us the spanning tree in Figure $7.4.17$.

Figure $7.4.17$: Spanning Tree Formed Removing *ad*, *af*, and *hi*

Your Turn 7.4.5: Constructing Spanning Trees

Construct two distinct spanning trees for the graph in Figure $7.4.18$.

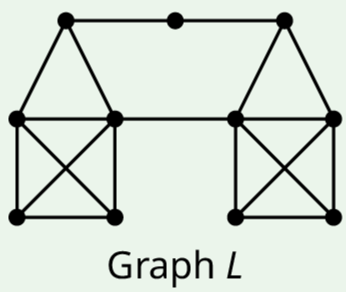


Figure $7.4.18$: Graph L

Answer

Two possible solutions are given in Figure $7.4.19$ and Figure $7.4.20$.

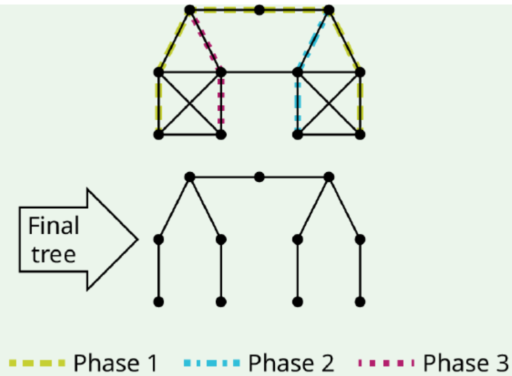


Figure 7.4.19 : First Spanning Tree for Graph L

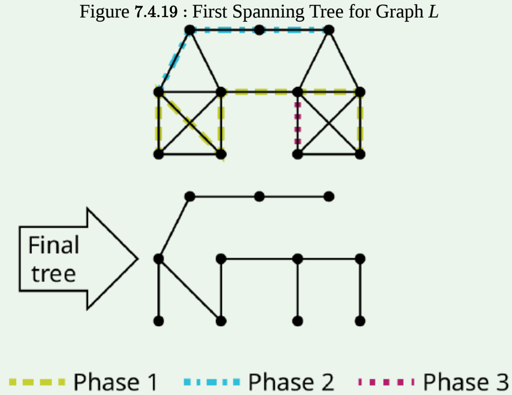


Figure 7.4.20 : Second Spanning Tree for Graph L

Minimum Spanning Tree (MST) and Kruskal's Algorithm

In many applications of spanning trees, the graphs are weighted, and we want to find the spanning tree of the least possible weight. For example, the graph might represent a computer network, and the weights might represent the cost involved in connecting two devices. So, finding a spanning tree with the lowest possible total weight, or minimum spanning tree, means saving money! The method that we will use to find a minimum spanning tree (MST) of a weighted graph is called Kruskal's algorithm. The steps for Kruskal's algorithm are:

Kruskal's Algorithm: To Find MST

Step 1: Choose any edge with the minimum weight of all edges.

Step 2: Choose another edge of minimum weight from the remaining edges. The second edge does not have to be connected to the first edge.

Step 3: Choose another edge of minimum weight from the remaining edges, but do not select any edge that creates a cycle in the subgraph you are creating.

Step 4: Repeat step 3 until all the vertices of the original graph are included and you have a spanning tree.

Where is MST used?

MSTs help design networks that connect all nodes at the minimum cost. Some examples include

1. Cable TV distribution networks, Internet wiring, fiber-optic networks, Telephone and communication networks
2. Electrical power grids
3. Cheapest road connections between cities, Railway routes, Pipeline networks (water, gas, oil)
4. Airline route optimization (connecting hubs with minimal cost)

Example 7.4.6 : Using Kruskal's Algorithm

A computer network will be set up with six devices. The vertices in the graph in Figure 7.4.21 represent the devices, and the edges represent the cost of a connection. Find the network configuration that will cost the least. What is the total cost?

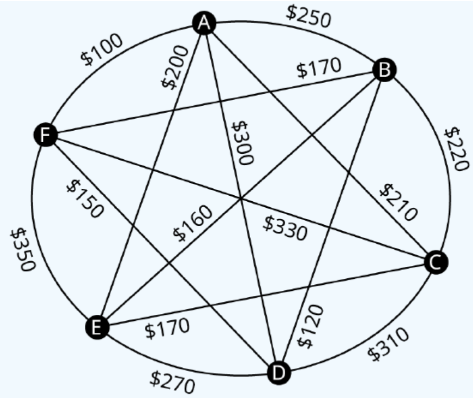


Figure 7.4.21 : Graph of Network Connection Costs

Answer

A minimum spanning tree will correspond to the network configuration of the least cost. We will use Kruskal's algorithm to find one. Since the graph has six vertices, the spanning tree will have six vertices and five edges.

Step 1: Choose an edge of least weight. We have sorted the weights into numerical order. The least is $\$100$. The only edge of this weight is edge *AF* as shown in Figure 7.4.22.

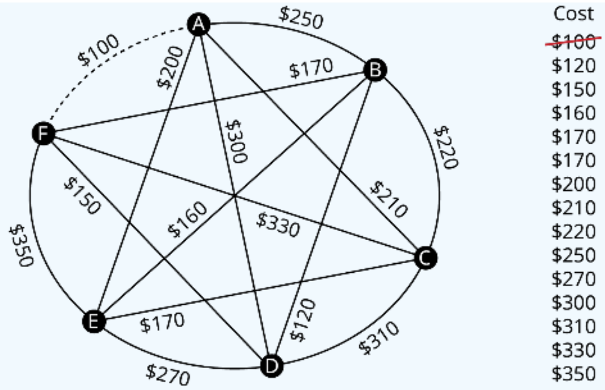


Figure 7.4.22 : Step 1 Select Edge *AF*

Step 2: Choose the edge of least weight of the remaining edges, which is *BD* with $120 Notice that the two selected edges do not need to be adjacent to each other as shown in Figure 7.4.23.

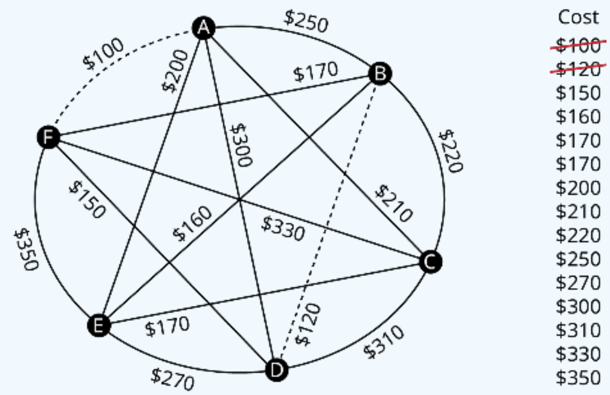


Figure 7.4.23 : Step 2 Select Edge *BD*

Step 3: Select the lowest weight edge of the remaining edges, as long as it does not result in a cycle. We select *DF* with $150 since it does not form a cycle as shown in Figure 7.4.24.

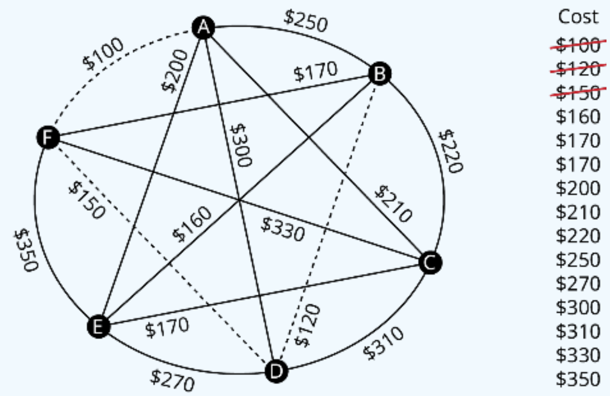


Figure 7.4.24: Step 3 Select Edge *DF*

Repeat Step 3: Select the lowest weight edge of the remaining edges, which is *BE* with $160 and it does not form a cycle as shown in Figure 7.4.25. This gives us four edges so we only need to repeat step 3 once more to get the fifth edge.

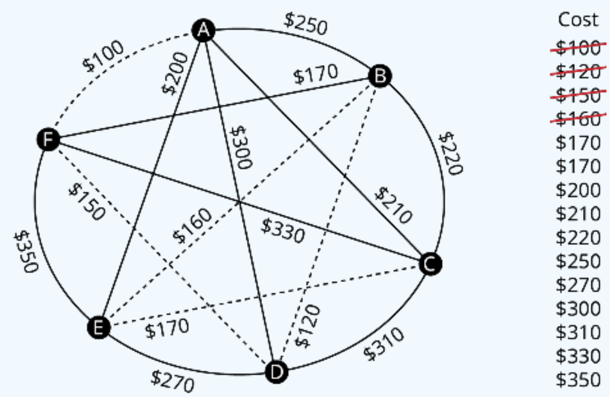


Figure 7.4.25 : Repeat Step 3 Select Edge *DF*

Repeat Step 3: The lowest weight of the remaining edges is $170. Both *BF* and *CE* have a weight of $170, but *BF* would create cycle (*b*, *d*, *f*) and there cannot be a cycle in a spanning tree as shown in Figure 7.4.26.

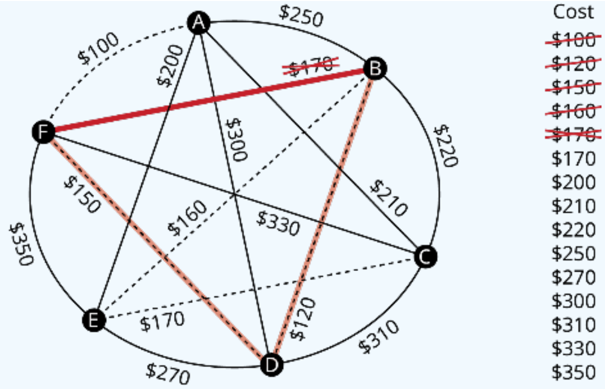


Figure 7.4.26 : Repeat Step 3 Do Not Select Edge *BF*

So, we will select *CE*, which will complete the spanning tree as shown in Figure 7.4.27.

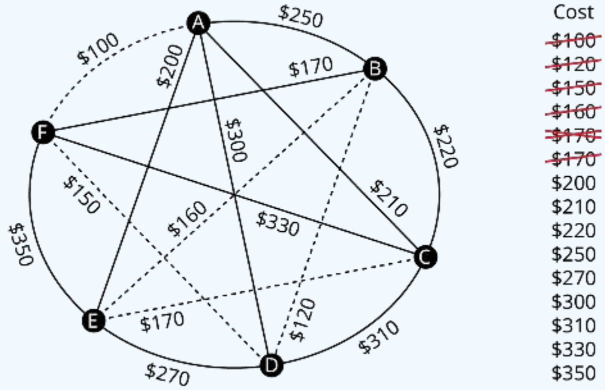


Figure 7.4.27 : Repeat Step 3 Select Edge *CE*

The minimum spanning tree is shown in Figure 7.4.28. This is the configuration of the network of least cost. The spanning tree has a total weight of $\$100 + \$120 + \$150 + \$160 + \$170 = \$700$, which is the total cost of this network configuration.

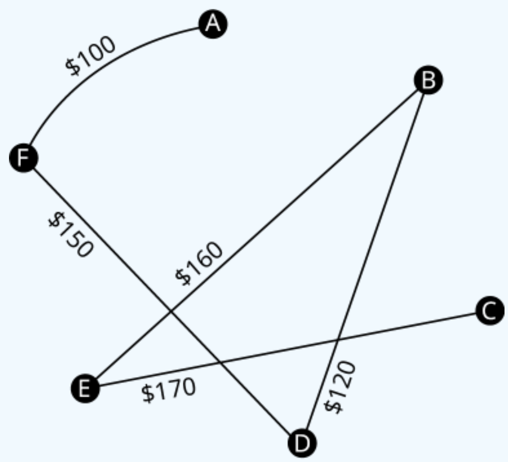
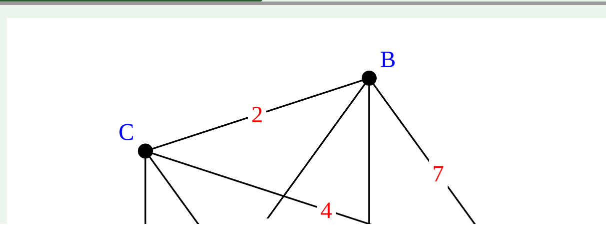
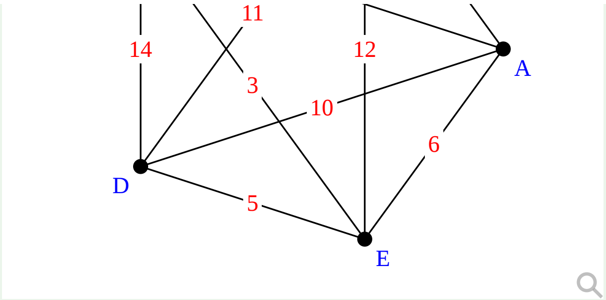


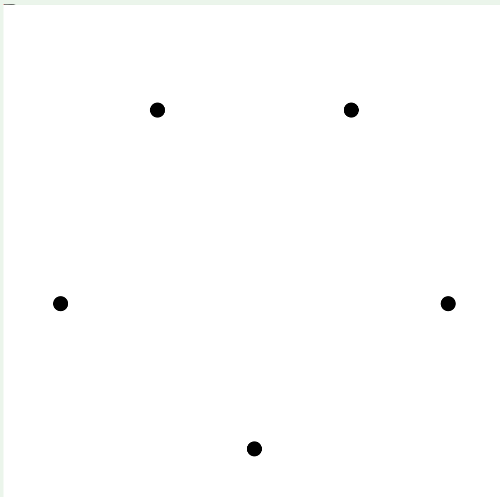
Figure 7.4.28 : Final Minimum Spanning Tree

Your Turn 7.4.6: Find MST





Find the minimum cost spanning tree on the graph above using Kruskal's algorithm. Draw the edges included in the minimum cost spanning tree on the graph below.



*Important Note: You may have difficulty drawing multiple edges to a vertex. When that happens, draw a line segment to a location **near** the vertex, and then drag it to the desired vertex to make the connection.*

What is the total weight of the minimum weight spanning tree?

Question Help: Video

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Your Turn 7.4.7: Definitions

Fill in the blanks below to produce a true statement.

A connected graph that has no circuits is called a/an [Select an answer] . For such a graph, every edge is a/an [Select an answer] and for such a graph with *n* vertices, the graph must contain [ ] edges.

Fill in the blank below to produce a true statement.

A subgraph that contains all of a connected graph's vertices, is connected, and contains no circuits is called a/an [Select an answer]

Fill in the blank below to produce a true statement.

A tree that is created from a weighted graph such that the tree is of minimum possible total weight is called a/an [Select an answer]

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Glossary

Sample Word 1 | Sample Definition 1

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