MAT1275 BASIC

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CHAPTER OVERVIEW

Front Matter

MAT1275basic

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Detailed Licensing

Licensing

A detailed breakdown of this resource's licensing can be found in ***Back Matter/Detailed Licensing***.

CHAPTER OVERVIEW

1: Linear Expressions and Equations

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1.2: Slope of a Line
1.3: Find the Equation of a Line
1.4: Chapter 1 Review Exercises

1.1: Graph Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Graph a linear equation by plotting points
- Graph vertical and horizontal lines
- Find the x- and y-intercepts
- Graph a line using the intercepts

Be Prepared

Before we get started, take this readiness quiz.

1. Evaluate $5x-4$ when $x=-1$.
2. Evaluate $3x-2y$ when $x=4$ and $y=-3$.
3. Solve for y: $8-3y=20$.

Linear Expressions and Linear Equations with One Variable

The expressions in the Be Prepared section are examples of what we call *linear expressions*.

Definition 1.1.1

1. An expression that can be written as

$$Ax+B$$

with A and B real numbers, $A\neq 0$, is called a **linear expression (with one variable)**, or more specifically, a **linear expression in x**.

2. An equation that can be written as

$$Ax+B=0$$

with A and B real numbers, $A\neq 0$, is called a **linear equation (with one variable)**, or more specifically, a **linear equation in x**.

3. A **solution** to a linear equation with one variable, say x, is a number, say a, that when substituted in for that variable yields a true statement. In this case we say that $x=a$ is a solution.

Be Prepared (3) can be written as $0=3y+12$ by adding $3y$ and subtracting 8 from both sides. So $8-3y=20$ is a linear equation in y. Notice that substituting $y=1$, for example, into the equation gives

$$\begin{aligned} 8-3(1) &= 20 \quad \text{or} \\ 5 &= 20, \end{aligned}$$

which is not true. However, substituting $y=-4$ into the equation gives

$$\begin{aligned} 8-3(-4) &= 20 \quad \text{or} \\ 20 &= 20, \end{aligned}$$

which is true. So $y=-4$ is a solution to $8-3y=20$, and $y=1$ is not. In solving Be Prepared 3, we found that $y=-4$ is the only solution.

Introduction to Linear Expressions and Linear Equations with Two Variables

In this section we will be looking at equations that have two variables, x and y, and more than one solution. We want to represent the solutions in a picture. Consider the equation $2x-3y=6$. The expression $2x-3y$ is an example of a linear expression with two variables. We can evaluate the expressions on both sides of the equal sign for any particular choice of x and y. For example, we can choose $x=3$ and $y=0$ and substitute them into the equation to get

$$\begin{aligned}2(3)-3(0)&=6 \quad \text{or}\\ 6&=6,\end{aligned}$$

which is a true statement. We can also choose $x=-3$ and $y=-4$ and substitute them into the equation to get

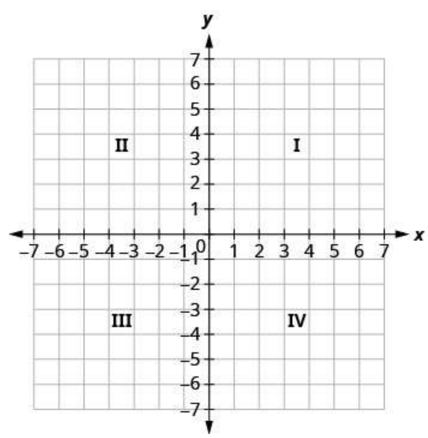
$$\begin{aligned}2(-3)-3(-4)&=6 \quad \text{or}\\ 6&=6,\end{aligned}$$

which is also true. We say that $x=3$ and $y=0$ is one solution, and $x=-3$ and $y=-4$ is another solution. We will actually see that equations like $2x-3y=6$ have infinitely many solutions. Next we introduce what is needed to make a picture of the solutions.

Plot Points on a Rectangular Coordinate System

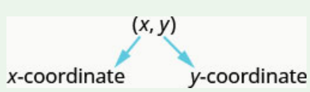
Just like maps use a grid system to identify locations, a grid system, or a rectangular coordinate system, is used in algebra to represent ordered pairs of numbers, and ultimately, to show a relationship between two variables. The rectangular coordinate system is also called the xy*-plane* or the "*coordinate plane*."

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the x*-axis*. The vertical number line is called the y*-axis* (note that in the context of an application these may take on different names). These axes divide a plane into four regions, called *quadrants*. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise.



Definition 1.1.2

An **ordered pair**, (x,y), gives the coordinates of a point in a rectangular coordinate system. The first number is the x-**coordinate**. The second number is the y-**coordinate**.



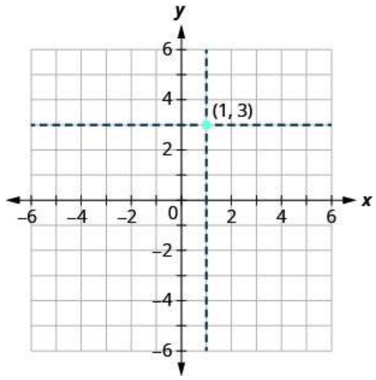
The phrase "ordered pair" means that the order is important. For example, $(2,5)$ and $(5,2)$ are different points.

What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0,0)$. The point $(0,0)$ has a special name.

Definition 1.1.3

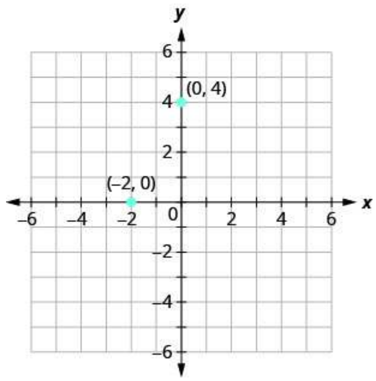
The point $(0,0)$ is called the **origin**. It is the point where the x-axis and y-axis intersect.

We use the coordinates to locate a point on the xy-plane. Let's plot the point $(1,3)$ as an example. First, locate the x-coordinate 1 on the x-axis and lightly sketch a vertical line through $x=1$. Then, locate the y-coordinate 3 on the y-axis and sketch a horizontal line through $y=3$. Now, find the point where these two lines meet -- that is the point with coordinates $(1,3)$.



Notice that the vertical line through $x=1$ and the horizontal line through $y=3$ are not part of the graph. We just used them to help us locate the point $(1,3)$.

When one of the coordinates is zero, the point lies on one of the axes. The graph below shows that the point $(0,4)$ is on the y-axis and the point $(-2,0)$ is on the x-axis.



Points on the x- or y-axis

1. Points with a y-coordinate equal to 0 are on the x-axis, and have the form $(p,0)$, where p is some real number.
2. Points with an x-coordinate equal to 0 are on the y-axis, and have the form $(0,q)$, where q is some real number.

Example 1.1.4

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-5,4)$

b. $(-3,-4)$

c. $(2,-3)$

d. $(0,-1)$

e. $\left(3,\frac{5}{2}\right)$

f. $(-2,3)$

Solution

The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate. To plot each point, sketch a vertical line through the x-coordinate and a horizontal line through the y-coordinate. Their intersection is the point.

a. Since the x-coordinate is -5, the point is to the left of the y-axis. Also, since the y-coordinate is 4, the point is above the x-axis. The point $(-5, 4)$ is in Quadrant II.

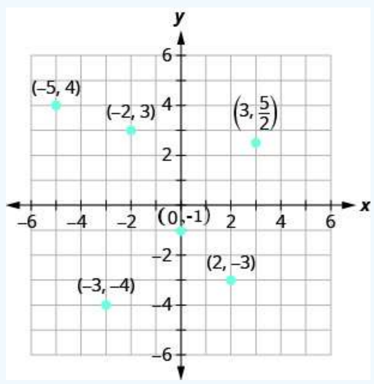
b. Since the x-coordinate is -3, the point is to the left of the y-axis. Also, since the y-coordinate is -4, the point is below the x-axis. The point $(-3, -4)$ is in Quadrant III.

c. Since the x-coordinate is 2, the point is to the right of the y-axis. Since the y-coordinate is -3, the point is below the x-axis. The point $(2, -3)$ is in Quadrant IV.

d. Since the x-coordinate is 0, the point whose coordinates are $(0, -1)$ is on the y-axis.

e. Since the x-coordinate is 3, the point is to the right of the y-axis. Since the y-coordinate is $\frac{5}{2}$, the point is above the x-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number, $2\frac{1}{2}$, or decimal, 2.5, so that we know $\frac{5}{2}$ is between 2 and 3.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.

f. Since the x-coordinate is -2, the point is to the left of the y-axis. Since the y-coordinate is 3, the point is above the x-axis. The point $(-2, 3)$ is in Quadrant II.



? Try It 1.1.5

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-2, 1)$

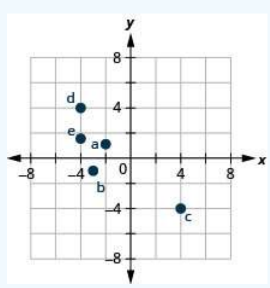
b. $(-3, -1)$

c. $(4, -4)$

d. $(-4, 4)$

e. $\left(-4, \frac{3}{2}\right)$

Answer



The points $(-2, 1)$, $(-4, 4)$, and $\left(-4, \frac{3}{2}\right)$ are in Quadrant II.

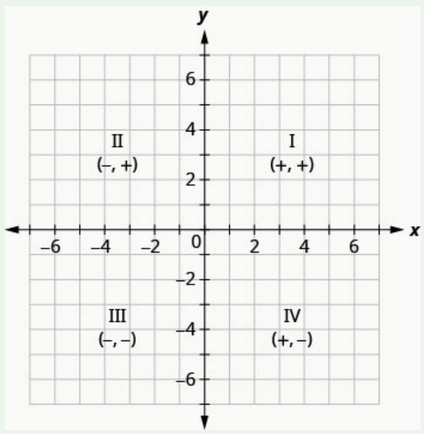
The point $(-3,-1)$ is in Quadrant III.

The point $(4,-4)$ is in Quadrant IV.

The signs of the x-coordinate and y-coordinate affect the location of the points. We may have noticed some patterns as we graphed the points in the previous example. We can summarize sign patterns of the quadrants in this way:

Quadrants

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$



Linear Expressions and Linear Equations with Two Variables

Up to now, all the equations we have solved were equations with just one variable. In almost every case, when we solved the equation we got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax+By=C$. An equation of this form is called a *linear equation with two variables*.

Definition 1.1.6

1. An expression that can be written as

$$Ax+By$$

with A and B real numbers, not both zero, is called a **linear expression (with two variables)**, or more specifically, a **linear expression in x and y.**

2. An equation that can be written as

$$Ax+By=C$$

with A and B real numbers, not both zero, is called a **linear equation (with two variables)**, or more specifically, a **linear equation in x and y.**

Here is an example of a linear equation with two variables, x and y.

$$x+4y=8$$

This equation is in the form $Ax+By=C$ with $A=1$, $B=4$, and $C=8$. The equation

$$y=-3x+5$$

is also a linear equation with two variables, x and y, but it does not appear to be in the form $Ax+By=C$. We can rewrite it in $Ax+By=C$ form in the following way.

	$y=-3x+5$
Add $3x$ to both sides.	$y+3x=-3x+5+3x$
Simplify.	$y+3x=5$
Use the Commutative Property to put it in $Ax+By=C$ form.	$3x+y=5$

By rewriting $y=-3x+5$ as $3x+y=5$, we can easily see that it is a linear equation with two variables because it is of the form $Ax+By=C$ with $A=3$, $B=1$, and $C=5$. When an equation is in the form $Ax+By=C$, we say it is in *standard form of a linear equation*.

Definition 1.1.7

A linear equation with two variables, x and y, is in **standard form** when it is written as $Ax+By=C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations with two variables have infinitely many solutions. For example, if $A \neq 0$, for every number that is substituted for x there is a corresponding y-value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Definition 1.1.8

An ordered pair (p, q) is a **solution of the linear equation** $Ax+By=C$, if the equation is a true statement when the x- and y-coordinates of the ordered pair, p and q, respectively, are substituted into the equation. We can also say in this case that $(x, y)=(p, q)$ is a solution, or $x=p$ and $y=q$ is a solution.

We can represent the solutions as points in the rectangular coordinate system. The points will line up perfectly in a straight line. We use a straight-edge to draw this line, and put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, “A picture is worth a thousand words.” The line (with the arrows) shows us *all* the solutions to that equation. Every point on the line corresponds a solution of the equation. And, every solution of this equation corresponds to a point on this line. This line is called the graph of the equation. Points *not* on the line do not correspond to solutions! We may say, as is common, then that the points on the line *are* the solutions.

Definition 1.1.9

The **graph of the linear equation** $Ax+By=C$ is the collection of all solutions (x, y).

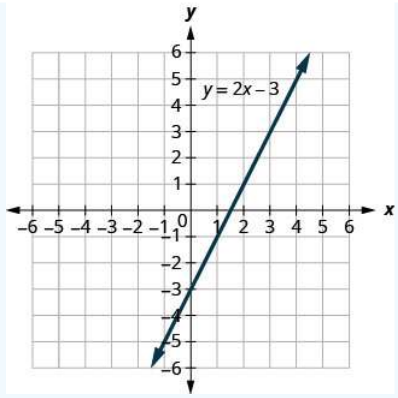
We can represent the graph on the coordinate plane. The **representation** is a straight line so that

- every solution of the equation is a point on this line, and
- every point on the line is a solution of the equation.

As universally accepted, a representation is also called a **graph of the linear equation**.

Example 1.1.10

The graph of $y=2x-3$ is shown below.



For each ordered pair,

$$A(0,-3), \qquad B(3,3), \qquad C(2,-3), \qquad \text{and} \qquad D(-1,-5),$$

decide:

a. is the ordered pair a solution to the equation?

b. is the point on the line?

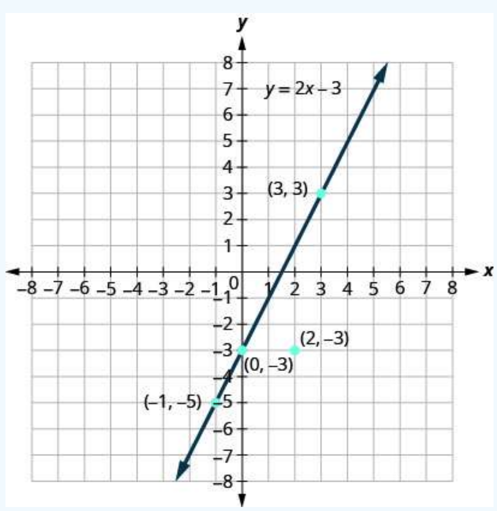
Solution

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

a.

Point	$A(0,-3)$	$B(3,3)$	$C(2,-3)$	$D(-1,-5)$
Write the equation of the line.	$y=3x-3$	$y=2x-3$	$y=2x-3$	$y=2x-3$
Substitute the x- and y-values.	$-3\overset{?}{=}2\cdot 0-3$	$3\overset{?}{=}2\cdot 3-3$	$-3\overset{?}{=}2\cdot 2-3$	$-5\overset{?}{=}2\cdot(-1)-3$
Simplify.	$-3\overset{?}{=}-3$	$3\overset{?}{=}3$	$-3\overset{?}{=}1$	$-5\overset{?}{=}5$
True or false?	True	True	False	True
Answer the question.	$(0,-3)$ is a solution	$(3,3)$ is a solution	$(2,-3)$ is not a solution	$(-1,-5)$ is a solution

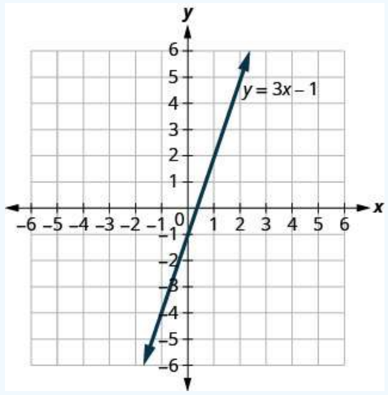
b. Plot the points $(0,-3)$, $(3,3)$, $(2,-3)$, and $(-1,-5)$.



The points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line.
The points that are solutions to $y = 2x - 3$ are on the line, but the points that are not solutions are not on the line.

? Try It 1.1.11

The graph of $y = 3x - 1$ is shown below.



For each ordered pair,

$$A(0, -1) \qquad \text{and} \qquad B(2, 5),$$

decide:

a. is the ordered pair a solution to the equation?
b. is the point on the line?

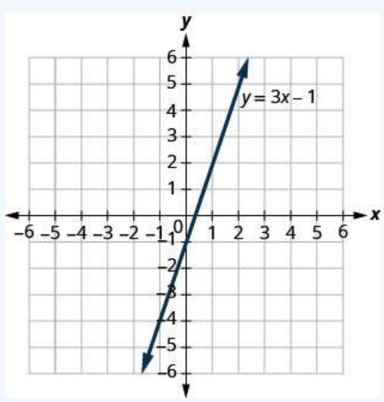
Answer

a. Both pairs are solutions.

b. Both pairs are on the line.

? Try It 1.1.12

The graph of $y = 3x - 1$ is shown below.



For each ordered pair,

$$A(3, -1) \qquad \text{and} \qquad B(-1, -4),$$

decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

Answer

a. A is a solution.

B is not a solution.

b. A is on the line.

B is not on the line.

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called *plotting points*. We find three points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation. While two points are enough to determine a line, using three points helps us detect errors.

Example 1.1.13

Graph the equation $y = 2x + 1$ by plotting points.

Solution

<table>
<tr>
<td>Step 1. Find three points whose coordinates are solutions to the equation.</td>
<td>You can choose any values for x or y.

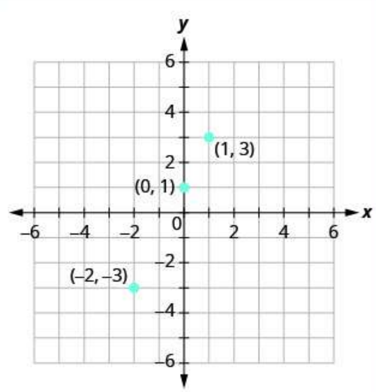
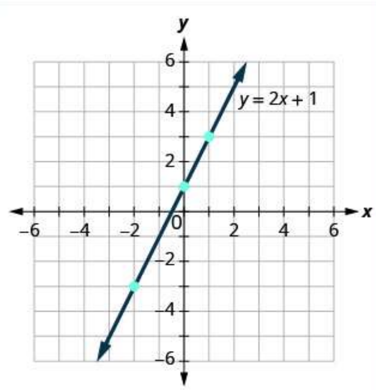
In this case, since y is isolated on the left side of the equation, it is easier to choose values for x.</td>
<td>$y = 2x + 1$

$x = 0$
$y = 2x + 1$
$y = 2 \cdot 0 + 1$
$y = 0 + 1$
$y = 1$

$x = 1$
$y = 2x + 1$
$y = 2 \cdot 1 + 1$
$y = 2 + 1$
$y = 3$

$x = -2$
$y = 2x + 1$
$y = 2(-2) + 1$
$y = -4 + 1$
$y = -3$</td>
</tr>
<tr>
<td>Organize the solutions in a table.</td>
<td>Put the three solutions in a table.</td>
<td>
<table>
<tr><th colspan="3">$y = 2x + 1$</th></tr>
<tr><th>x</th><th>y</th><th>(x, y)</th></tr>
<tr><td>0</td><td>1</td><td>(0, 1)</td></tr>
<tr><td>1</td><td>3</td><td>(1, 3)</td></tr>
<tr><td>–2</td><td>–3</td><td>(–2, –3)</td></tr>
</table>
</td>
</tr>
<tr>
<td>Step 2. Plot the points in a rectangular coordinate system.

Check that the points line up. If they do not, carefully check your work!</td>
<td>Plot:
(0, 1), (1, 3), (–2, –3).

Do the points line up?
Yes, the points line up.</td>
<td>
</td>
</tr>
<tr>
<td>Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.</td>
<td>This line is the graph of $y = 2x + 1$.</td>
<td>
</td>
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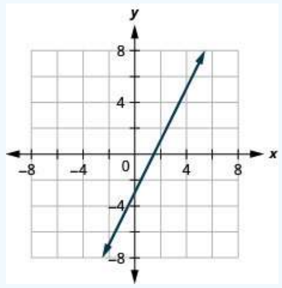
</table>

Try It 1.1.14

Graph the equation $y = 2x - 3$ by plotting points.

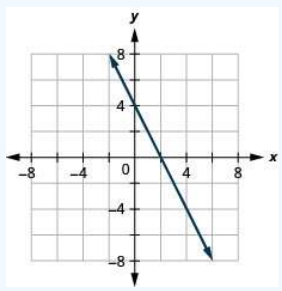
Answer



Try It 1.1.15

Graph the equation $y = -2x + 4$ by plotting points.

Answer



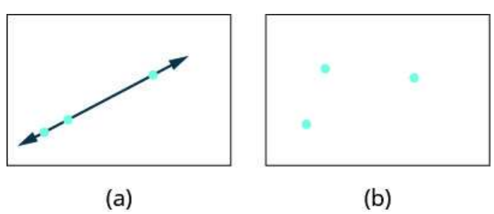
The steps to take when graphing a linear equation by plotting points are summarized here.

Graph a linear equation by plotting points

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If we only plot two points and one of them is incorrect, we can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If we use three points, and one is incorrect, the points will not line up. This tells us something is wrong and we need to check our work. Look at the difference between these illustrations.



When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the arithmetic is easier if we make "good" choices for the values of x. This way we will avoid fractional answers, which are hard to plot precisely.

Example 1.1.16

Graph the equation $y=\frac{1}{2}x+3$.

Solution

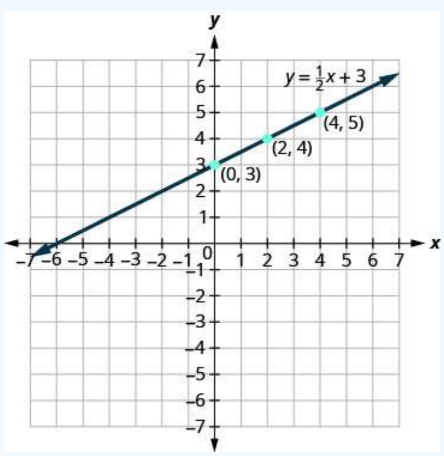
Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2 the multiplication by $\frac{1}{2}$ simplifies to a whole number.

Choose a value for x that is a multiple of 2.	$x=0$	$x=2$	$x=4$
Write the equation of the line.	$y=\frac{1}{2}x+3$	$y=\frac{1}{2}x+3$	$y=\frac{1}{2}x+3$
Substitute the x-value into the equation.	$y=\frac{1}{2}(0)+3$	$y=\frac{1}{2}(2)+3$	$y=\frac{1}{2}(4)+3$
Simplify.	$y=0+3$	$y=1+3$	$y=2+3$
Find y.	$y=3$	$y=4$	$y=5$

We organize the three solutions in a table.

$y=\frac{1}{2}x+3$		
x	y	(x,y)
0	3	$(0,3)$
2	4	$(2,4)$
4	5	$(4,5)$

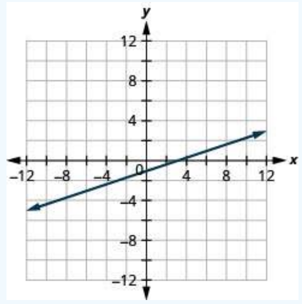
Plot the points, check that they line up, and draw the line.



Try It 1.1.17

Graph the equation $y=\frac{1}{3}x-1$.

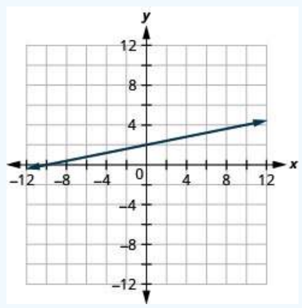
Answer



Try It 1.1.18

Graph the equation $y=\frac{1}{4}x+2$.

Answer



Graph Vertical and Horizontal Lines

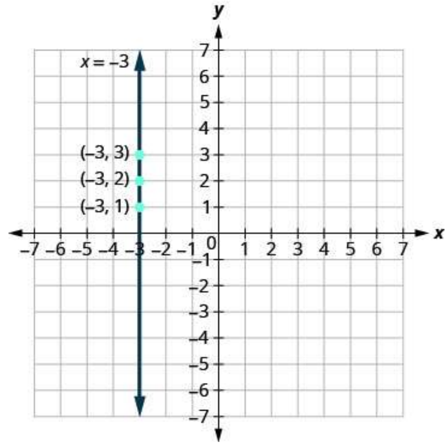
Some linear equations have only one variable. They may have just x and no y, or just y without an x. This changes how we make a table of values to get the points to plot.

Let's consider the equation $x=-3$. This equation has only one variable, x. The equation says that the x-coordinate of any solution is equal to -3, so its value does not depend on the y-coordinate. No matter what is the value of the y-coordinate, the value of the x-coordinate is always -3.

So to make a table of values, write -3 in for all the x-coordinates. Then choose any values for the y-coordinate. Since the x-coordinate does not depend on the y-coordinate, we can choose any numbers we like. But to fit the points on our coordinate graph, we'll use 1, 2, and 3 for the y-coordinates.

$x=-3$		
x	y	(x,y)
-3	1	$(-3,1)$
-3	2	$(-3,2)$
-3	3	$(-3,3)$

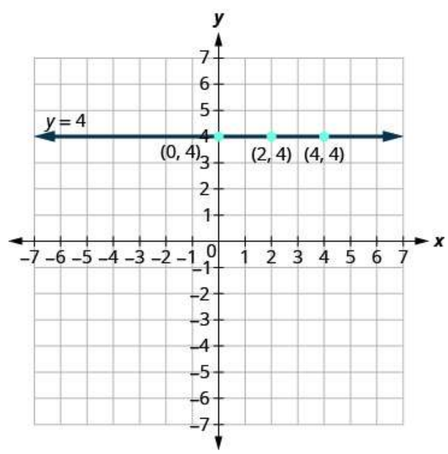
Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line**.



What if the equation has y but no x? Let's graph the equation $y = 4$. This time the y-coordinate of any solution is 4, so for this equation, the y-coordinate of the solution does not depend on the x-coordinate. Fill in 4 for all the y-coordinates in the table and then choose any values for x-coordinates. We will use 0, 2, and 4 for the x-coordinates.

$y = 4$		
x	y	(x, y)
0	4	$(0, 4)$
2	4	$(2, 4)$
4	4	$(4, 4)$

In this figure, we have graphed a **horizontal line** passing through the y-axis at 4.



Definition 1.1.19

1. A **vertical line** is the graph of an equation (with two variables x and y) of the form $x = a$.

The line passes through the x-axis at $(a, 0)$.

2. A **horizontal line** is the graph of an equation (with two variables x and y) of the form $y = b$.

The line passes through the y-axis at $(0, b)$.

? Example 1.1.20

Graph:

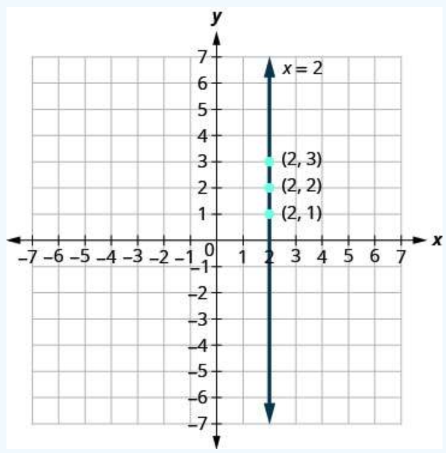
a. $x=2$

b. $y=-1$

Solution

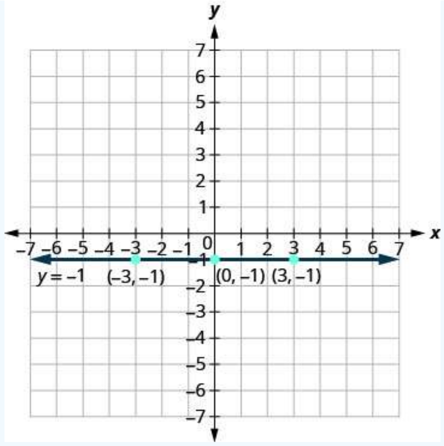
a. The equation has only one variable, x, and x is always equal to 2. We create a table where the x-coordinate is always 2 and then put in any values for the y-coordinate. The graph is a vertical line passing through the x-axis at 2.

$x=2$		
x	y	(x, y)
2	1	$(2,1)$
2	2	$(2,2)$
2	3	$(2,3)$



b. Similarly, the equation $y=-1$ has only one variable, y. The value of the y-coordinate of any solution is -1. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1.

$y=-1$		
x	y	(x, y)
0	-1	$(0,-1)$
3	-1	$(3,-1)$
-3	-1	$(-3,-1)$



Try It 1.1.21

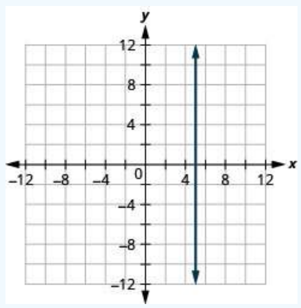
Graph:

a. $x = 5$

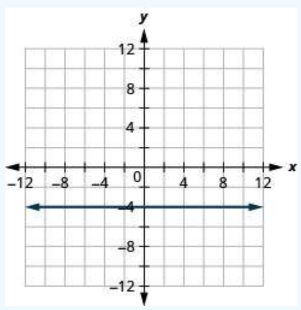
b. $y = -4$

Answer

a.



b.



Try It 1.1.22

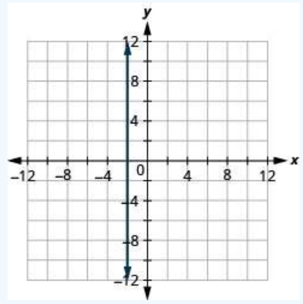
Graph:

a. $x = -2$

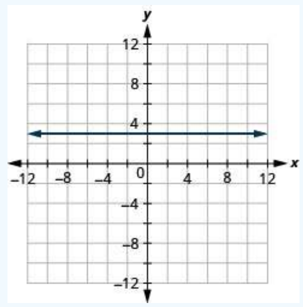
b. $y = 3$

Answer

a.



b.

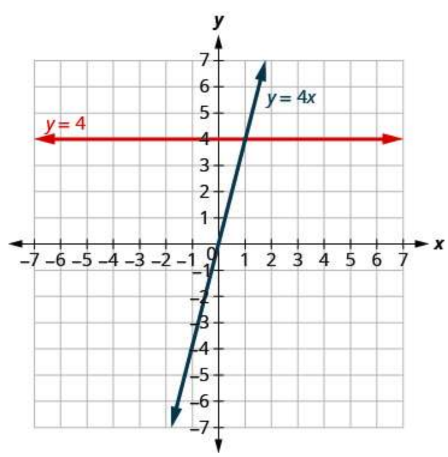


What is the difference between the equations $y=4x$ and $y=4$?

The equation $y=4x$ has both x and y. The value of the y-coordinate of a solution depends on the value of the x-coordinate, so the y-coordinate changes according to the value of the x-coordinate. The equation $y=4$ has only one variable. The value of y-coordinate of any solution is 4, it does not depend on the value of the x-coordinate.

$y = 4x$		
x	y	(x, y)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

$y = 4$		
x	y	(x, y)
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)



Notice, in the graph, the equation $y=4x$ gives a slanted line, while $y=4$ gives a horizontal line.

Example 1.1.23

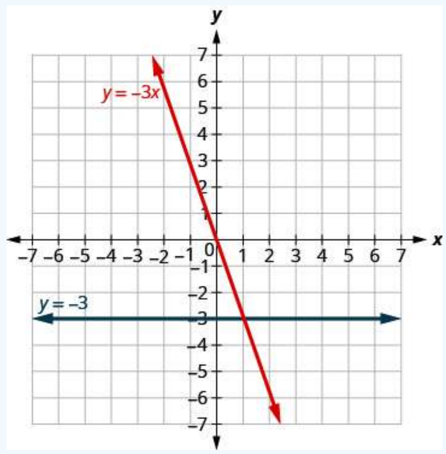
Graph $y=-3x$ and $y=-3$ in the same rectangular coordinate system.

Solution

We notice that the first equation has the variable x, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are shown.

$y=-3x$		
x	y	(x, y)
0	0	(0, 0)
1	−3	(1, −3)
2	−6	(2, −6)

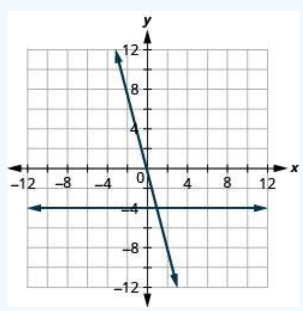
$y=-3$		
x	y	(x, y)
0	−3	(0, −3)
1	−3	(1, −3)
2	−3	(2, −3)



Try It 1.1.24

Graph $y=-4x$ and $y=-4$ in the same rectangular coordinate system.

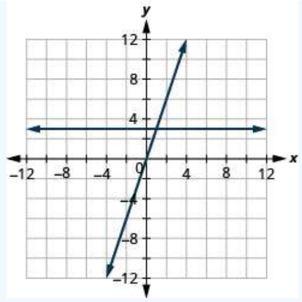
Answer



Try It 1.1.25

Graph $y=3$ and $y=3x$ in the same rectangular coordinate system.

Answer



Find x- and y-intercepts

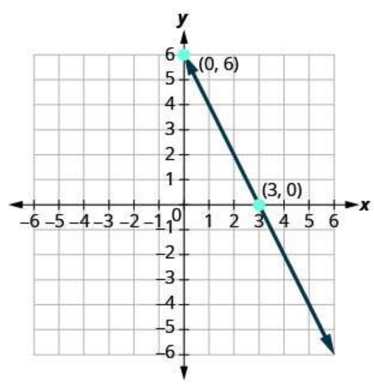
Every linear equation can be represented by a line. We have seen that when graphing a line by plotting points, we can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line intersects the x-axis and the y-axis. These points are called the *intercepts of a line*.

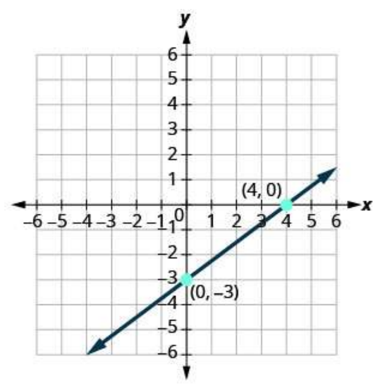
Definition 1.1.26

The points where a graph crosses the x-axis and the y-axis are called the **intercepts of the graph**.

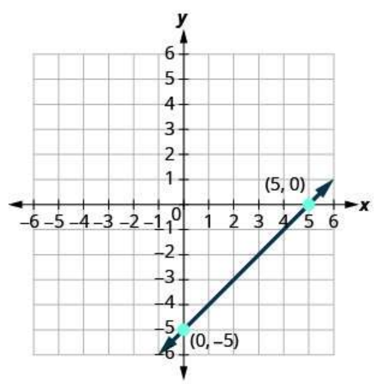
Let's look at the graphs of the lines.



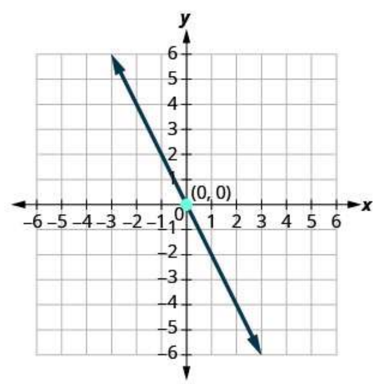
(a) $2x + y = 6$



(b) $3x - 4y = 12$



(c) $x - y = 5$



(d) $y = -2x$

First, notice where each of these lines crosses the x-axis. Now, let's look at the points where these lines cross the y-axis.

Figure	The line crosses the x-axis at	Ordered pair for this point	The line crosses the y-axis at	Ordered pair for this point
Figure (a)	3	$(3,0)$	6	$(0,6)$
Figure (b)	4	$(4,0)$	-3	$(0,-3)$
Figure (c)	5	$(5,0)$	-5	$(0,-5)$
Figure (d)	0	$(0,0)$	0	$(0,0)$
General Figure	a	$(a,0)$	b	$(0,b)$

Is there a pattern?

For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a,0)$ and is called the x-*intercept* of the line. The x-intercept occurs when y is zero.

In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0,b)$ and is called the y-*intercept* of the line. The y-intercept occurs when x is zero.

Definition 1.1.27

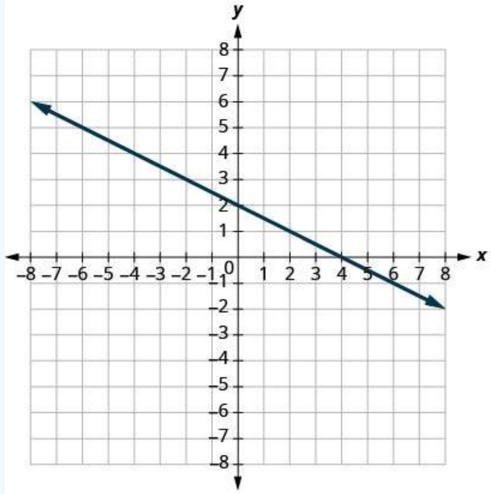
1. The x-**intercept** of a line is the point $(a,0)$ where the line crosses the x-axis.
2. The y-**intercept** of a line is the point $(0,b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

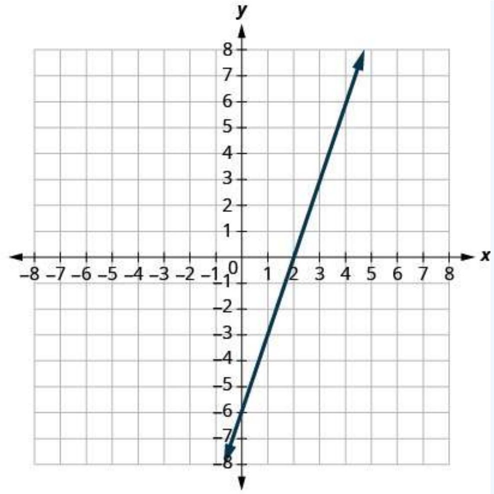
x	y
a	0
0	b

Example 1.1.28

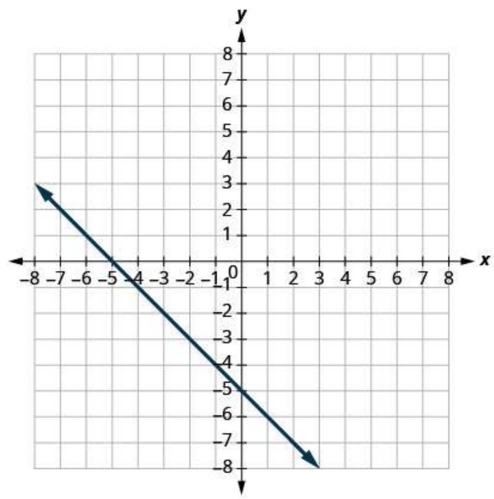
Find the x- and y-intercepts on each graph shown.



(a)



(b)



(c)

Solution

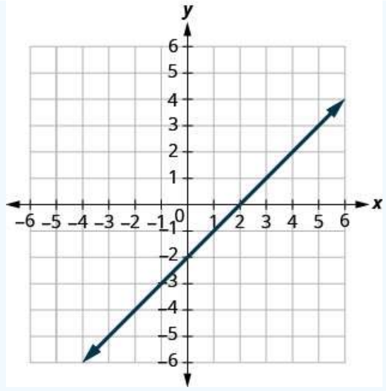
a. The graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$.
The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$.

b. The graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$.
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

c. The graph crosses the x-axis at the point $(-5, 0)$. The x-intercept is $(-5, 0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

? Try It 1.1.29

Find the x- and y-intercepts on the graph.

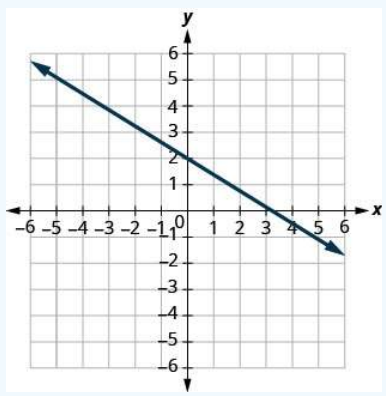


Answer

The x-intercept is $(2, 0)$.
The y-intercept is $(0, -2)$.

Try It 1.1.30

Find the x- and y-intercepts on the graph.



Answer

The x-intercept is $(3, 0)$.
The y-intercept is $(0, 2)$.

Recognizing that the x-intercept occurs when y is zero and that the y-intercept occurs when x is zero gives us a method to find the intercepts of a line from its equation. To find the x-intercept, let $y = 0$ and solve for x. To find the y-intercept, let $x = 0$ and solve for y.

Intercepts from the equation of a line

To find:

- the x-intercept of the line, let $y = 0$ and solve for x.
- the y-intercept of the line, let $x = 0$ and solve for y.

Example 1.1.31

Find the intercepts of the graph of $2x + y = 8$.

Solution

We will let $y=0$ to find the x-intercept, and let $x=0$ to find the y-intercept. We will fill in a table, which reminds us of what we need to find.

$2x+y=8$		
x	y	
	0	x-intercept
0		y-intercept

Let's find the x-intercept first.

$2x+y=8$	
To find the x-intercept, let $y=0$.	
Let $y=0$.	$2x+0=8$
Simplify.	$2x=8$
	$x=4$
Write the x-intercept as a point (x,y).	$(4,0)$

Now, the y-intercept.

$2x+y=8$	
To find the y-intercept, let $x=0$.	
Let $x=0$.	$2\cdot 0+y=8$
Simplify.	$y=8$
Write the y-intercept as a point (x,y).	$(0,8)$

The intercepts are the points $(4,0)$ and $(0,8)$ as shown in the table.

$2x+y=8$		
x	y	(x,y)
4	0	$(4,0)$
0	8	$(0,8)$

The x-intercept is $(4,0)$.

The y-intercept is $(0,8)$.

? Try It 1.1.32

Find the intercepts of the graph of $3x+y=12$.

Answer

The x-intercept is $(4,0)$.
The y-intercept is $(0,12)$.

Try It 1.1.33

Find the intercepts of the graph of $x+4y=8$.

Answer

The x-intercept is $(8,0)$.
The y-intercept is $(0,2)$.

Graph a Line Using the Intercepts

To graph a linear equation by plotting points, we need to find three points whose coordinates are solutions to the equation. We can use the x- and y- intercepts as two of our three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

Example 1.1.34

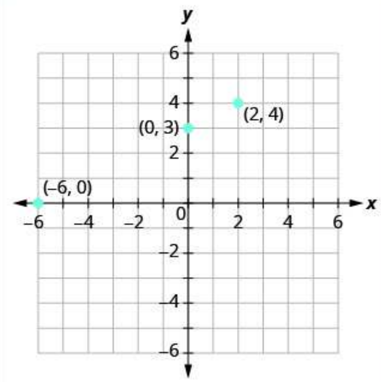
Graph $-x+2y=6$ using the intercepts.

Solution

Step 1. Find the x- and y-intercepts of the line. Let $y=0$ and solve for x.	Find the x-intercept.	Let $y=0$ $-x+2y=6$ $-x+2(0)=6$ $-x=6$ $x=-6$ The x-intercept is $(-6,0)$.
Let $x=0$ and solve for y.	Find the y-intercept.	Let $x=0$. $-x+2y=6$ $-0+2y=6$ $2y=6$ $y=3$ The y-intercept is $(0,3)$.
Step 2. Find another solution to the equation.	We'll use $x=2$.	Let $x=2$. $-x+2y=6$ $-2+2y=6$ $2y=8$ $y=4$ A third point is $(2,4)$.

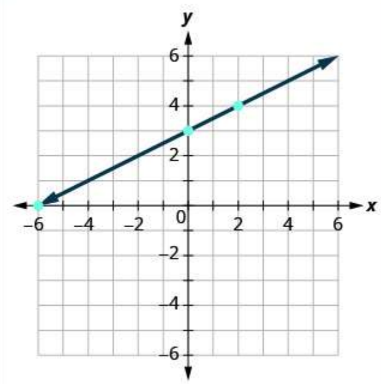
Step 3. Plot the three points. Check that the points line up.

x	y	(x, y)
−6	0	(−6, 0)
0	3	(0, 3)
2	4	(2, 4)



Step 4. Draw the line.

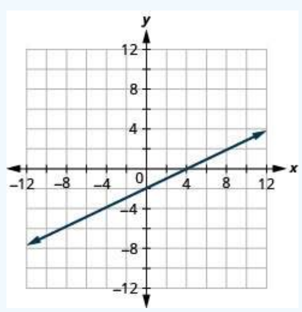
See the graph.



? Try It 1.1.35

Graph $x - 2y = 4$ using the intercepts.

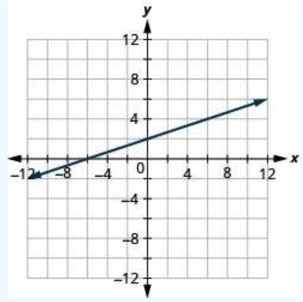
Answer



? Try It 1.1.36

Graph $-x + 3y = 6$ using the intercepts.

Answer



When the line passes through the origin, the x-intercept and the y-intercept are the same point.

Example 1.1.37

Graph $y = 5x$ using the intercepts.

Solution

$y = 5x$	
x-intercept	y-intercept
Let $y = 0$.	Let $x = 0$.
$0 = 5x$	$y = 5 \cdot 0$
$0 = x$	$y = 0$
$(0, 0)$	$(0, 0)$

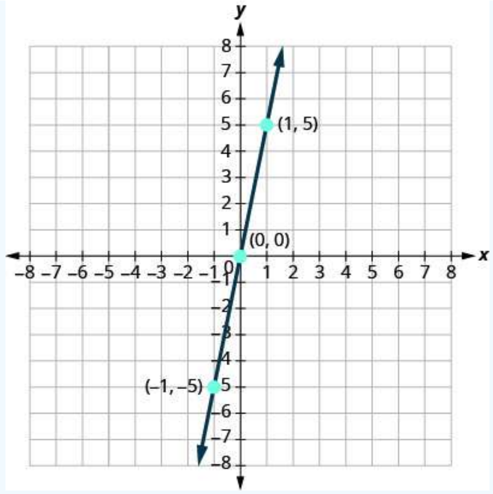
This line has only one intercept. It is the point $(0, 0)$.
To check accuracy, we need to plot three points. Since the x- and y-intercepts are the same point, we need *two* more points to graph the line.

$y = 5x$	
Let $x = 1$.	Let $x = -1$.
$y = 5 \cdot 1$	$y = 5 \cdot (-1)$
$y = 5$	$y = -5$

The resulting three points are summarized in the table.

$y = 5x$		
x	y	(x, y)
0	0	$(0, 0)$
1	5	$(1, 5)$
-1	-5	$(-1, -5)$

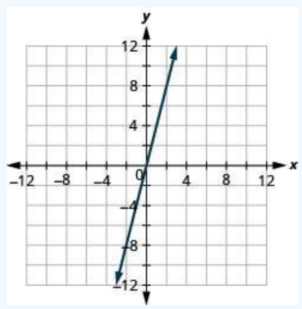
Plot the three points, check that they line up, and draw the line.



Try It 1.1.38

Graph $y = 4x$ using the intercepts.

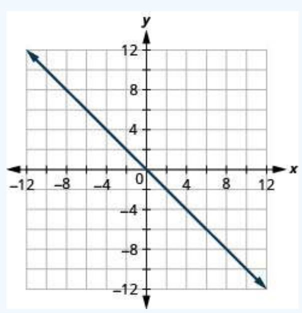
Answer



Try It 1.1.39

Graph $y = -x$ using the intercepts.

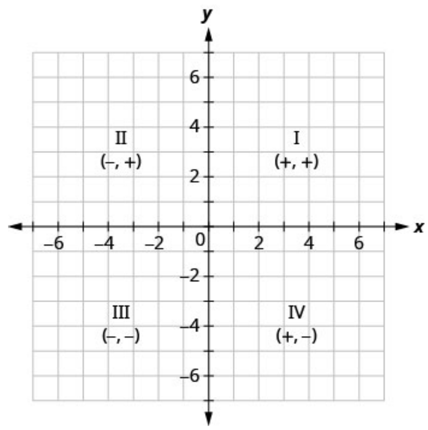
Answer



Key Concepts

- **Points on the Axes**
  - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
  - Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
- **Quadrant**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x, y)	(x, y)	(x, y)	(x, y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$



- **Graph of a Linear Equation:** The graph of a linear equation $Ax + By = C$ is a straight line.
  Every point on the line is a solution of the equation.
  Every solution of this equation is a point on this line.
- **How to graph a linear equation by plotting points.**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.
- **x-intercept and y-intercept of a Line**
  - The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.
  - The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

- **Find the x- and y-intercepts from the Equation of a Line**
  - Use the equation of the line. To find:
    the x-intercept of the line, let $y = 0$ and solve for x.
    the y-intercept of the line, let $x = 0$ and solve for y.
- **How to graph a linear equation using the intercepts.**
  1. Find the x- and y-intercepts of the line.
     Let $y = 0$ and solve for x.
     Let $x = 0$ and solve for y.
  2. Find a third solution to the equation.
  3. Plot the three points and check that they line up.
  4. Draw the line

Glossary

horizontal line

A horizontal line is the graph of an equation of the form $y = b$. The line passes through the y-axis at $(0, b)$.

intercepts of a line

The points where a line crosses the x-axis and the y-axis are called the intercepts of the line.

linear equation

An equation of the form $Ax + By = C$, where A and B are not both zero, is called a linear equation with two variables.

ordered pair

An ordered pair, (x, y) gives the coordinates of a point in a rectangular coordinate system. The first number is the x-coordinate. The second number is the y-coordinate.

origin

The point $(0, 0)$ is called the origin. It is the point where the x-axis and y-axis intersect.

solution of a linear equation with two variables

An ordered pair (x, y) is a solution of the linear equation $Ax + By = C$, if the equation is a true statement when the x- and y-values of the ordered pair are substituted into the equation.

standard form of a linear equation

A linear equation is in standard form when it is written $Ax + By = C$.

vertical line

A vertical line is the graph of an equation of the form $x = a$. The line passes through the x-axis at $(a, 0)$.

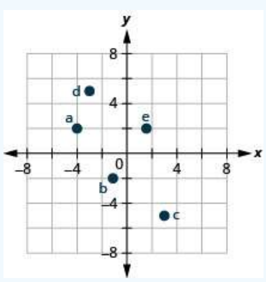
Practice Makes Perfect

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system and identify the quadrant in which the point is located.

1. a. $(-4, 2)$ b. $(-1, -2)$ c. $(3, -5)$ d. $(-3, 0)$
e. $\left(\frac{5}{3}, 2\right)$

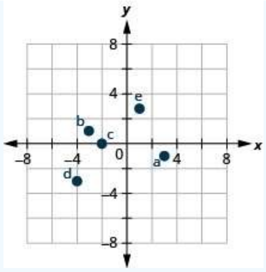
Answer



2. a. $(-2, -3)$ b. $(3, -3)$ c. $(-4, 1)$ d. $(4, -1)$
e. $\left(\frac{3}{2}, 1\right)$

3. a. $(3, -1)$ b. $(-3, 1)$ c. $(-2, 0)$ d. $(-4, -3)$
e. $\left(1, \frac{14}{5}\right)$

Answer



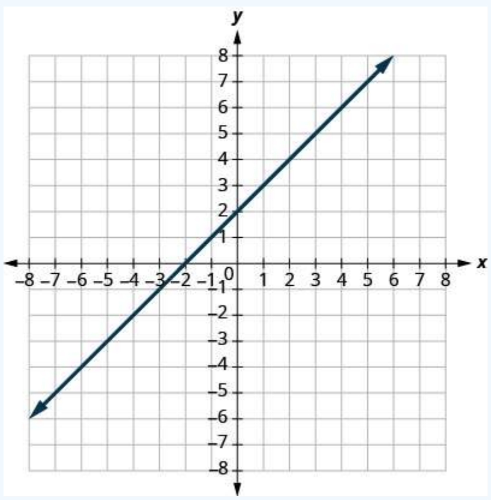
4. a. $(-1,1)$ b. $(-2,-1)$ c. $(2,0)$ d. $(1,-4)$
e. $\left(3,\dfrac{7}{2}\right)$

In the following exercises, for each ordered pair, decide

a. is the ordered pair a solution to the equation? b. is the point on the line?

5. $y=x+2$;

A: $(0,2)$; B: $(1,2)$; C: $(-1,1)$; D: $(-3,-1)$.

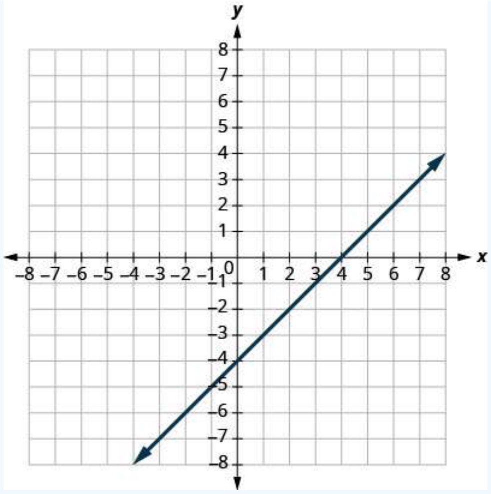


Answer

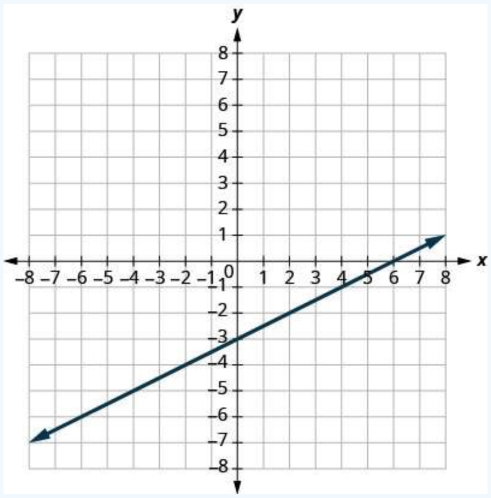
a. A: yes, B: no, C: yes, D: yes b. A: yes, B: no, C: yes, D: yes

6. $y=x-4$;

A: $(0,-4)$; B: $(3,-1)$; C: $(2,2)$; D: $(1,-5)$.



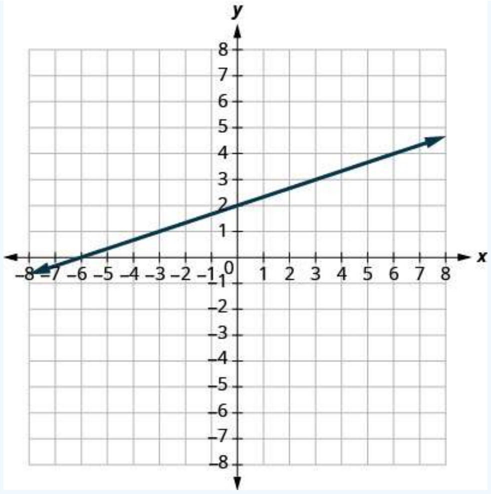
7. $y=\frac{1}{2}x-3$;
A: $(0,-3)$; B: $(2,-2)$; C: $(-2,-4)$; D: $(4,1)$.



Answer

a. A: yes, B: yes, C: yes, D: no b. A: yes, B: yes, C: yes, D: no

8. $y=\frac{1}{3}x+2$;
A: $(0,2)$; B: $(3,3)$; C: $(-3,2)$; D: $(-6,0)$.

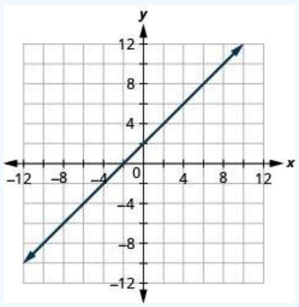


Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

9. $y=x+2$

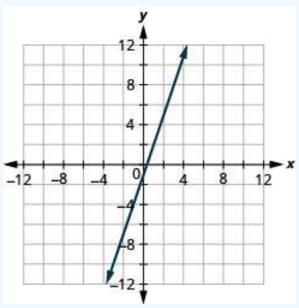
Answer



10. $y=x-3$

11. $y=3x-1$

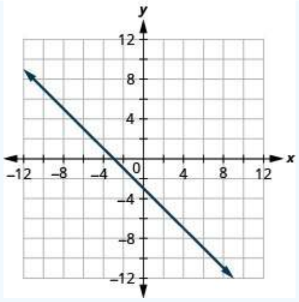
Answer



12. $y=-2x+2$

13. $y=-x-3$

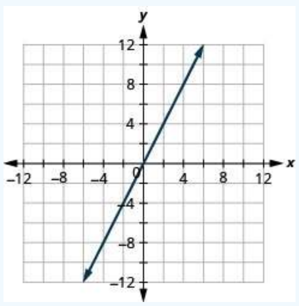
Answer



14. $y=-x-2$

15. $y=2x$

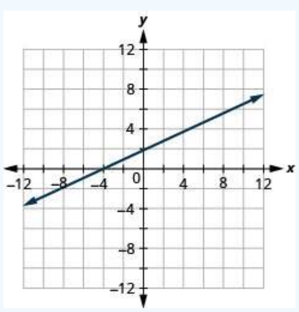
Answer



16. $y=-2x$

17. $y=\frac{1}{2}x+2$

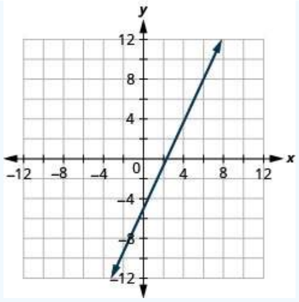
Answer



18. $y=\frac{1}{3}x-1$

19. $y=\frac{4}{3}x-5$

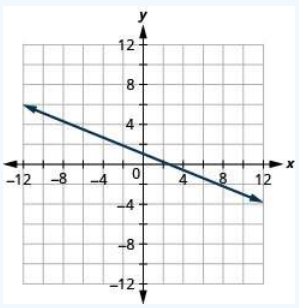
Answer



20. $y=\frac{3}{2}x-3$

21. $y=-\frac{2}{5}x+1$

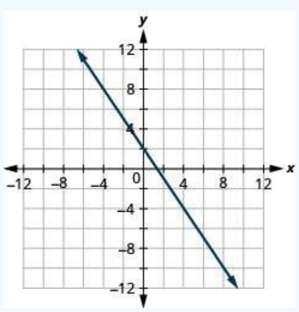
Answer



22. $y=-\frac{4}{5}x-1$

23. $y=-\frac{3}{2}x+2$

Answer



24. $y=-\frac{5}{3}x+4$

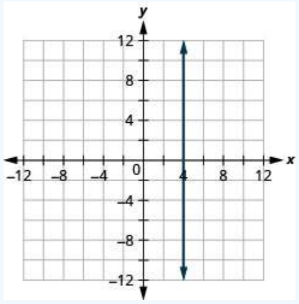
Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

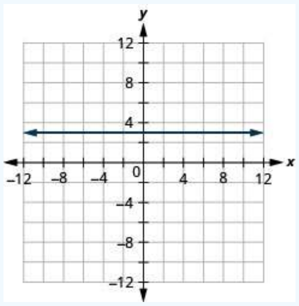
25. a. $x=4$ b. $y=3$

Answer

a.



b.

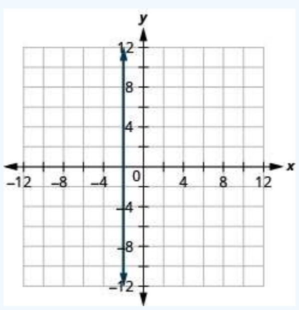


26. a. $x = 3$ b. $y = 1$

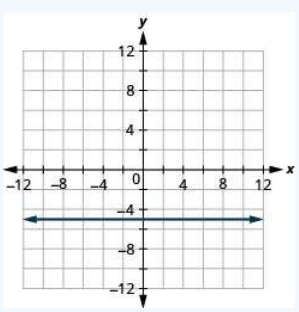
27. a. $x = -2$ b. $y = -5$

Answer

a.



b.

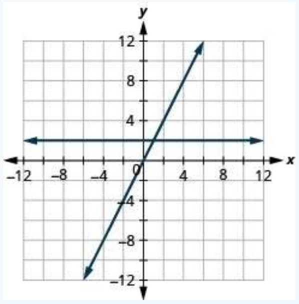


28. a. $x = -5$ b. $y = -2$

In the following exercises, graph each pair of equations in the same rectangular coordinate system.

29. $y=2x$ and $y=2$

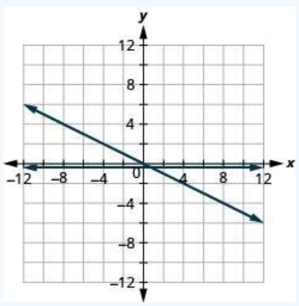
Answer



30. $y=5x$ and $y=5$

31. $y=-\frac{1}{2}x$ and $y=-\frac{1}{2}$

Answer

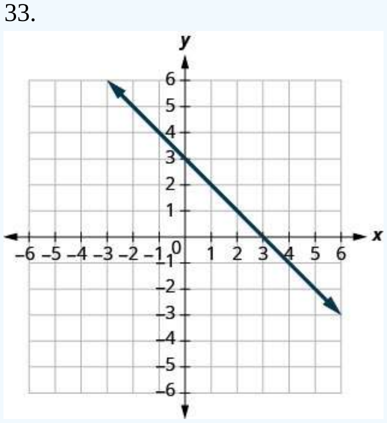


32. $y=-\frac{1}{3}x$ and $y=-\frac{1}{3}$

Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts on each graph.

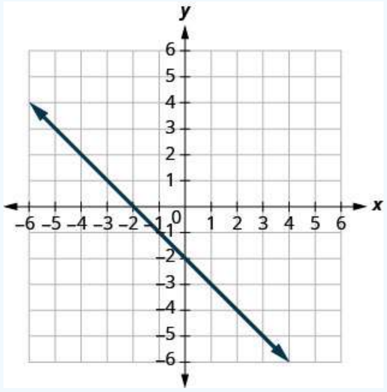
33.



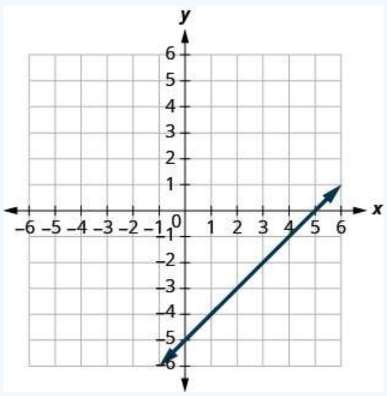
Answer

$(3,0),(0,3)$

34.



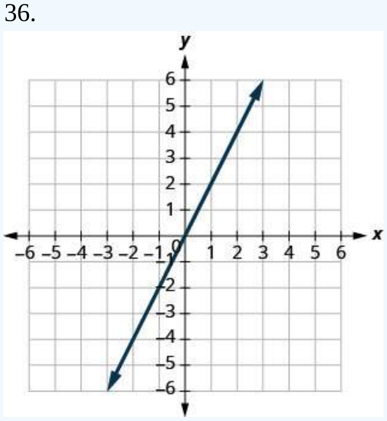
35.



Answer

$(5,0),(0,-5)$

36.



In the following exercises, find the intercepts for each equation.

37. $x-y=5$

Answer

x-int: $(5,0)$, y-int: $(0,-5)$

38. $x-y=-4$

39. $3x+y=6$

Answer

x-int: $(2,0)$, y-int: $(0,6)$

40. $x-2y=8$

41. $4x-y=8$

Answer

x-int: $(2,0)$, y-int: $(0,-8)$

42. $5x-y=5$

43. $2x+5y=10$

Answer

x-int: $(5,0)$, y-int: $(0,2)$

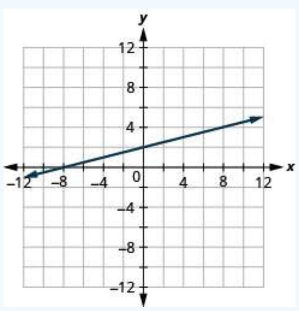
44. $3x-2y=12$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

45. $-x+4y=8$

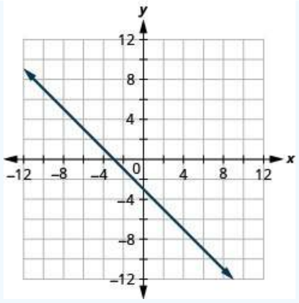
Answer



46. $x+2y=4$

47. $x+y=-3$

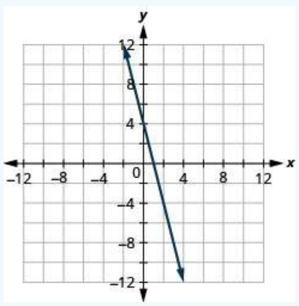
Answer



48. $x - y = -4$

49. $4x + y = 4$

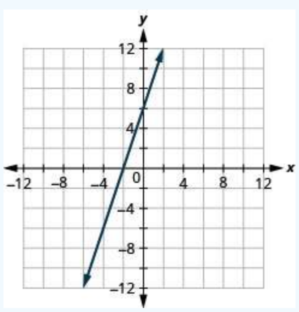
Answer



50. $3x + y = 3$

51. $3x - y = -6$

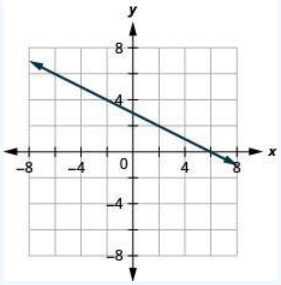
Answer



52. $2x - y = -8$

53. $2x + 4y = 12$

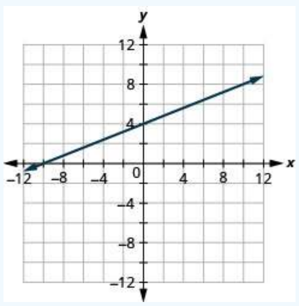
Answer



54. $3x - 2y = 6$

55. $2x - 5y = -20$

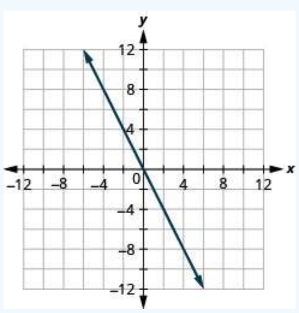
Answer



56. $3x - 4y = -12$

57. $y = -2x$

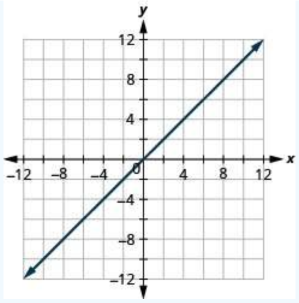
Answer



58. $y = 5x$

59. $y = x$

Answer



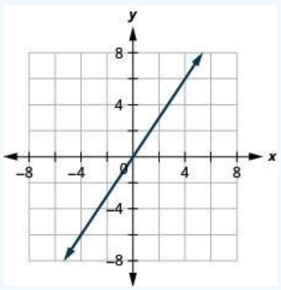
60. $y=-x$

Mixed Practice

In the following exercises, graph each equation.

61. $y=\frac{3}{2}x$

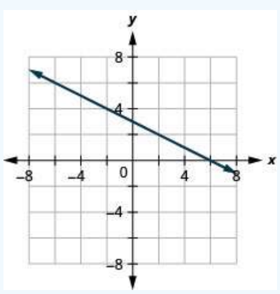
Answer



62. $y=-\frac{2}{3}x$

63. $y=-\frac{1}{2}x+3$

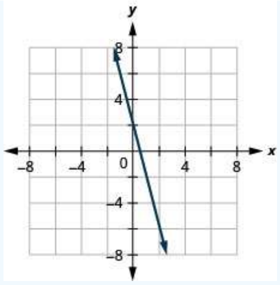
Answer



64. $y=\frac{1}{4}x-2$

65. $4x+y=2$

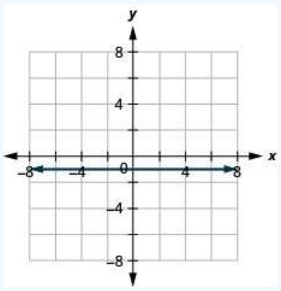
Answer



66. $5x+2y=10$

67. $y=-1$

Answer



68. $x=3$

Writing Exercises

69. Explain how you would choose three *x*-values to make a table to graph the line $y=\frac{1}{5}x-2$.

Answer

Answers will vary.

70. What is the difference between the equations of a vertical and a horizontal line?

71. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x+y=-4$? Why?

Answer

Answers will vary.

72. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y=\frac{2}{3}x-2$? Why?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
plot points on a rectangular coordinate system.			
graph a linear equation by plotting points.			
graph vertical and horizontal lines.			
find *x*- and *y*-intercepts.			
graph a line using the intercepts.			

b. If most of your checks were:

Confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

With some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

No, I don't get it. This is a warning sign and you must address it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

1.2: Slope of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope-intercept
- Use slopes to identify parallel and perpendicular lines

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\dfrac{1-4}{8-2}$.
2. Divide $\dfrac{0}{4}$ and $\dfrac{4}{0}$.
3. Simplify $\dfrac{15}{-3}$, $\dfrac{-15}{3}$, and $\dfrac{-15}{-3}$.

Find the Slope of a Line

When we graph linear equations, we may notice that some lines tilt up and some lines tilt down as they go from left to right. Some lines are very steep and some lines are flatter.

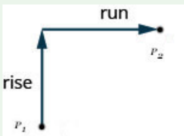
In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction, the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope.

We can assign a numerical value to the inclination of a line by finding the ratio of the rise and run. This is the slope.

Defintion 1.2.1

Given two points P_1 and P_2 on a line, the **rise** is the vertical "distance" and the **run** is the horizontal "distance" traveled and moving from P_1 to P_2, as shown in this illustration.



The "distance" is positive when we are moving up or to the right, and negative when we are moving down or to the left.

The **slope of a line** is $m = \dfrac{\text{rise}}{\text{run}}$.

To find the slope of a line, we locate any two points on the line, preferably whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

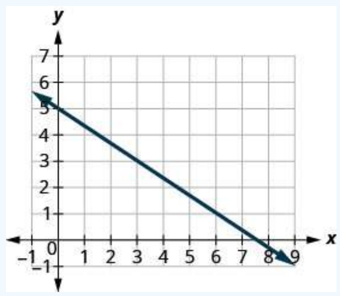
To find the slope of the line, we measure the "distance" along the vertical and horizontal sides of the triangle, that is, the rise and the run, respectively. The rise and the run can be positive, negative or zero.

Find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$

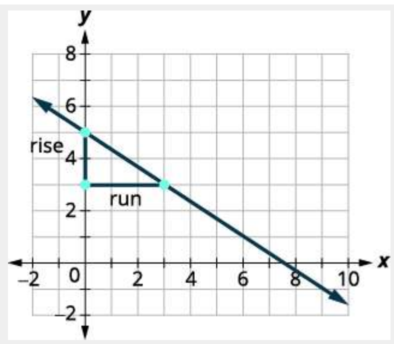
1. Locate two points on the line whose coordinates are integers.
2. Starting with one point, sketch a right triangle, going from the first point to the second point.
3. Determine the rise and the run on the legs of the triangle. This can be done by counting jumps vertically and horizontally when moving from the first point to the second.
4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

Example 1.2.2

Find the slope of the line shown.

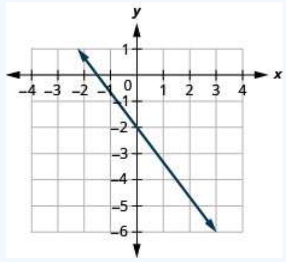


Solution

Find the slope of a line from its graph.	
Locate two points on the graph whose coordinates are integers.	$P_1 = (0, 5)$ and $P_2 = (3, 3)$
Starting at $(0, 5)$, sketch a right triangle to $(3, 3)$ as shown in this graph.	
Determine the rise. Since it goes down, it is negative.	The rise is -2.
Determine the run.	The run is 3.
Write the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m = \frac{-2}{3}$
Simplify.	$m = -\frac{2}{3}$
Answer the question.	The slope of the line is $-\frac{2}{3}$.
Interpret this as a rate of change.	As x increases by 3 units, y decreases by 2 units.

Try It 1.2.3

Find the slope of the line shown.

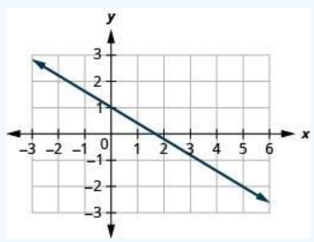


Answer

The slope of the line is $-\frac{4}{3}$.

Try It 1.2.4

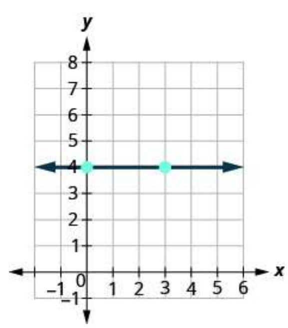
Find the slope of the line shown.



Answer

The slope of the line is $-\frac{3}{5}$.

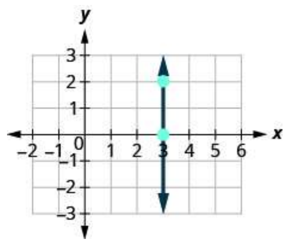
How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y = 4$, we could graph the line, find two points on it, and determine the rise and the run. Let's see what happens when we do this, as shown in the graph below.



Find the slope of a line from its graph.	
Locate two points on the graph whose coordinates are integers.	$P_1 = (0, 4)$ and $P_2 = (3, 4)$
Determine the rise.	The rise is 0.
Determine the run.	The run is 3.
Write the slope formula.	$m = \frac{\text{rise}}{\text{run}}$

Substitute the values of the rise and run.	$m=\frac{0}{3}$
Simplify.	$m=0$
Answer the question.	The slope of the horizontal line $y=4$ is 0.

Let's also consider a vertical line, the line $x=3$, as shown in the graph.



Find the slope of a line from its graph.	
Locate two points on the graph whose coordinates are integers.	$P_1=(3,0)$ and $P_2=(3,2)$
Determine the rise.	The rise is 2.
Determine the run.	The run is 0.
Write the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
Substitute the values of the rise and run.	$m=\frac{2}{0}$
Simplify.	m is undefined
Answer the question.	The slope of the vertical line $x=3$ is undefined.

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x=3$ is undefined.

All horizontal lines have slope 0. When the y-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the x-coordinates of a line are all the same, the run is 0.

Slope of a horizontal and vertical line

The **slope of a horizontal line**, $y=b$, is 0.

The **slope of a vertical line**, $x=a$, is undefined.

Example 1.2.5

Find the slope of each line:

a. $x=8$

b. $y=-5$

Solution

a. $x=8$
This is a vertical line. Its slope is undefined.

b. $y=-5$
This is a horizontal line. It has slope 0.

Try It 1.2.6

Find the slope of the line $x=-4$.

Answer

The slope of the line is undefined.

Try It 1.2.7

Find the slope of the line $y=7$.

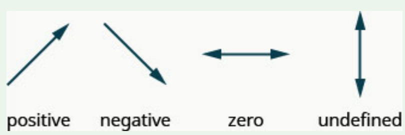
Answer

The slope of the line is 0.

How does the sign of the slope manifest itself in the line?

- When both the rise and the run are positive, the slope is positive. In this case, the line is "going up" from left to right.
- When both the rise and the run are negative, the slope is also positive. In this case, the line is "going up" from left to right.
- If the rise is positive and the run is negative, the slope is negative. In this case, the line is "going down" from left to right.
- If the rise is negative and the run is positive, the slope is negative. In this case, the line is "going down" from left to right.
- If the rise is zero, the slope is zero. In this case, the line is horizontal.
- If the run is zero, the slope is undefined. In this case, the line is vertical.

Quick guide to the slope of lines



Sometimes we'll need to find the slope of a line between two points when we don't have a graph to help determine the rise and the run. We could plot the points on grid paper, then count out the jumps to determine the rise and the run, but as we'll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

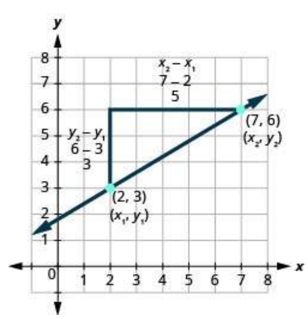
$$\left(x_1, y_1\right) \text{ read "} x \text{ sub } 1, y \text{ sub } 1 \text{."}$$

$$\left(x_2, y_2\right) \text{ read "} x \text{ sub } 2, y \text{ sub } 2 \text{."}$$

We will use $\left(x_1, y_1\right)$ to identify the first point and $\left(x_2, y_2\right)$ to identify the second point.

If we had more than two points, we could use $\left(x_3, y_3\right)$, $\left(x_4, y_4\right)$, and so on.

Let's see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2,3)$ and $(7,6)$, as shown in this graph.



Find the slope of the line given two points.	
Since we have two points, we will use subscript notation.	$(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (7, 6)$
Write the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Determine the rise and the run by counting jumps on the grid.	On the graph, we counted jumps and found a rise of 3 and a run of 5.
Substitute the values of the rise and the run.	$m = \frac{3}{5}$ Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.
We rewrite the rise and run by putting in the coordinates.	$m = \frac{6-3}{7-2}$ But 6 is y_2 the y-coordinate of the second point and 3 is y_1, the y-coordinate of the first point.
We can rewrite the slope using subscript notation.	$m = \frac{y_2 - y_1}{7-2}$ Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the first point.
So again we rewrite the slope using subscript notation.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Conclusion.	We've shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Definition 1.2.8

The **slope of the line** between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is

$$y \text{ of the second point minus } y \text{ of the first point}$$

$$\text{over}$$

$$x \text{ of the second point minus } x \text{ of the first point.}$$

Example 1.2.9

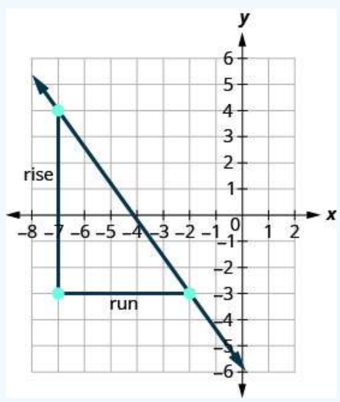
Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

Find the slope of the line given two points.	
Write the two points.	$P_1 = (-2, -3)$ and $P_2 = (-7, 4)$
Identify x_1 and y_1.	$P_1 = \underbrace{(-2, -3)}_{(x_1, y_1)}$ $x_1 = -2$, $y_1 = -3$

Find the slope of the line given two points.	
Write the two points.	$P_1=(-2,-3)$ and $P_2=(-7,4)$
Identify x_2 and y_2.	$P_2=\underbrace{(-7,4)}_{(x_2,y_2)}$ $x_2=-7,\ y_2=4$
Write the slope formula.	$m=\dfrac{y_2-y_1}{x_2-x_1}$
Substitute the values.	$m=\dfrac{4-(-3)}{-7-(-2)}$
Simplify.	$m=\dfrac{7}{-5}$
Simplify.	$m=-\dfrac{7}{5}$
Answer the question.	The slope of the line is $m=-\dfrac{7}{5}$.

Let's verify this slope on the graph shown.



$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{7}{-5} \\ &= -\frac{7}{5} \end{aligned}$$

? Try It 1.2.10

Use the slope formula to find the slope of the line through the points $(-3,4)$ and $(2,-1)$.

Answer

The slope of the line is -1.

? Try It 1.2.11

Use the slope formula to find the slope of the line through the points $(-2,6)$ and $(-3,-4)$.

Answer

The slope of the line is 10.

Graph a Line Given a Point and the Slope

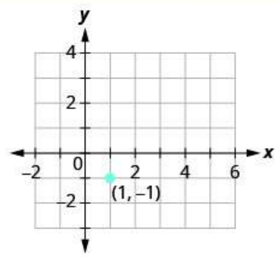
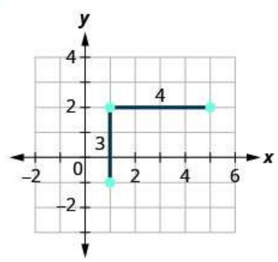
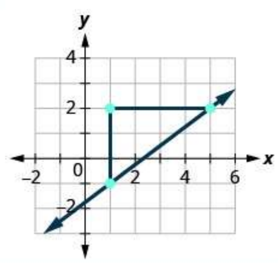
Up to now, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

? Example 1.2.12

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

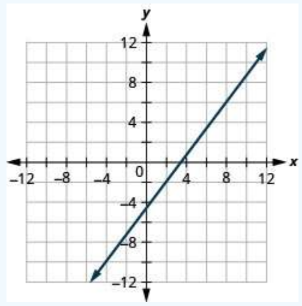
Step 1. Plot the given point.	Plot (1, –1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ $\text{rise} = 3$ $\text{run} = 4$
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

We can check our work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

? Try It 1.2.13

Graph the line passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.

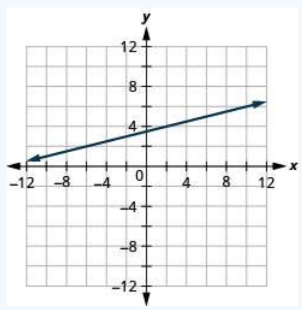
Answer



? Try It 1.2.14

Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Answer



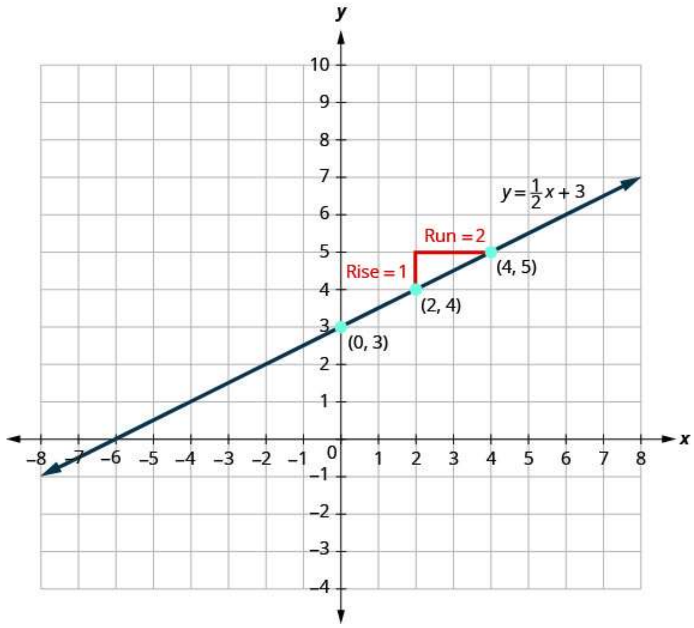
Graph a line given a point and the slope

1. Plot the given point.
2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
3. Starting at the given point, count jumps for the rise and run to mark the second point.
4. Draw a line passing through the points.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope-intercept form and its graph are related, we will have one more method we can use to graph lines.

Let's look at the graph of the equation $y = \frac{1}{2}x + 3$ and find its slope and y-intercept.



The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{1}{2} \end{aligned}$$

The y-intercept is $(0,3)$.

Look at the equation of this line.

$$y = \frac{1}{2}x + 3$$

Look at the slope and y-intercept.

$$\begin{aligned} \text{slope: } m &= \frac{1}{2} \\ y\text{-intercept: } &(0,3) \end{aligned}$$

When a linear equation is solved for y, the coefficient of the x term is the slope m and the constant term is the y-coordinate of the y-intercept, say b. We say that the equation $y = \frac{1}{2}x + 3$ is in slope-intercept form.

$$y = \underbrace{\frac{1}{2}}_{m} x + \underbrace{3}_{b}$$

$$y = mx + b$$

Slope-intercept form of an equation of a line

The **slope-intercept form** of an equation of a line with slope m and y-intercept, $(0,b)$, is $y = mx + b$.

Let's practice finding the values of the slope and y-intercept from the equation of a line.

Example 1.2.15

Identify the slope and y-intercept of the line from the equation:

a. $y = -\frac{4}{7}x - 2$

b. $x+3y=9$

Solution

a. We compare our equation to the slope-intercept form of the equation.

	$y=-\frac{4}{7}x-2$
Write the slope-intercept form of the equation of the line.	$y=mx+b$
Write the equation of the line. Note that it is in slope-intercept form.	$y=-\frac{4}{7}x-2$
Identify the slope.	The slope is $m=-\frac{4}{7}$.
Identify the y-intercept.	The y-intercept is $(0,-2)$.

b. When an equation of a line is not given in slope-intercept form, our first step will be to solve the equation for y.

	$x+3y=9$
Solve for y.	$x+3y=9$
Subtract x from each side.	$3y=-x+9$
Divide both sides by 3.	$\frac{3y}{3}=\frac{-x+9}{3}$
Simplify.	$y=-\frac{1}{3}x+3$
Write the slope-intercept form of the equation of the line.	$y=mx+b$
Write the equation of the line in slope-intercept form.	$y=-\frac{1}{3}x+3$
Identify the slope.	The slope is $m=-\frac{1}{3}$.
Identify the y-intercept.	The y-intercept is $(0,3)$.

? Try It 1.2.16

Identify the slope and y-intercept from the equation of the line.

a. $y=\frac{2}{5}x-1$

b. $x+4y=8$

Answer

a. The slope is $m=\frac{2}{5}$, and the y-intercept is $(0,-1)$.

b. The slope is $m=-\frac{1}{4}$, and the y-intercept is $(0,2)$.

? Try It 1.2.17

Identify the slope and y-intercept from the equation of the line.

a. $y=-\frac{4}{3}x+1$

b. $3x+2y=12$

Answer

a. The slope is $m=-\frac{4}{3}$, and the y-intercept is $(0,1)$.

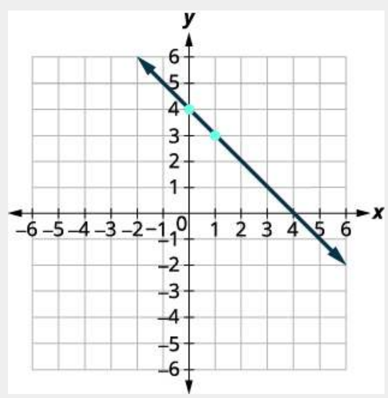
b. The slope is $m=-\frac{3}{2}$, and the y-intercept is $(0,6)$.

We have graphed a line using the slope and a point. Now that we know how to find the slope and y-intercept of a line from its equation, we can use the y-intercept as the point, and then count jumps determined by the slope from there to find a second point.

Example 1.2.18

Graph the line of the equation $y=-x+4$ using its slope and y-intercept.

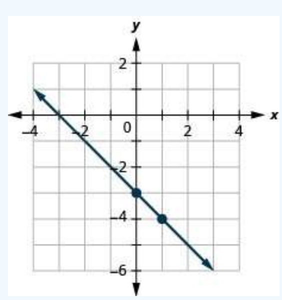
Solution

	$y=-x+4$
The equation is in slope–intercept form, $y=mx+b$.	$y=-x+4$
Identify the slope and y-intercept.	The slope is $m=-1$. The y-intercept is $(0,4)$.
Plot the y-intercept.	See the graph.
Rewrite the slope in the fraction form.	$m=\frac{-1}{1}$
Identify the rise and the run.	rise = -1 run =1
Count jumps using the rise and run to mark the second point. Draw the line as shown in the graph.	

Try It 1.2.19

Graph the line of the equation $y=-x-3$ using its slope and y-intercept.

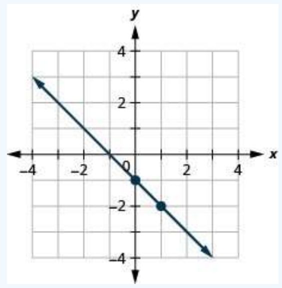
Answer



Try It 1.2.20

Graph the line of the equation $y = -x - 1$ using its slope and y-intercept.

Answer



Now that we have graphed lines by using the slope and y-intercept, let's summarize all the methods we have used to graph lines.

Methods to Graph Lines

Point Plotting	Slope–Intercept	Intercepts	Recognize Vertical and Horizontal Lines
x | y (table with three blank rows)	$y = mx + b$	x | y: 0 | ; | 0 ; (third row blank)	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope-intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this section, and the method we used to graph each of them.

$$\begin{array}{lll} & \textbf{Equation} & \textbf{Method} \\ \#1 & x = 2 & \text{Vertical line} \\ \#2 & y = -1 & \text{Horizontal line} \\ \#3 & -x + 2y = 6 & \text{Intercepts} \\ \#4 & 4x - 3y = 12 & \text{Intercepts} \\ \#5 & y = -x + 4 & \text{Slope-intercept} \end{array}$$

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is the same for every solution; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In Equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax + By = C$. We substituted $y = 0$ to find the x-intercept and $x = 0$ to find the y-intercept, and then found a third point by choosing another value for x or y.

Equation #5 is written in slope-intercept form. After identifying the slope and y-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Strategy for choosing the most convenient method to graph a line

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
  - $x = a$ is a vertical line passing through the x-axis at a.
  - $y = b$ is a horizontal line passing through the y-axis at b.
- If y is isolated on one side of the equation, in the form $y = mx + b$, graph by using the slope and y-intercept.
  - Identify the slope and y-intercept and then graph.
- If the equation is of the form $Ax + By = C$, find the intercepts.
  - Find the x- and y-intercepts, a third point, and then graph.

Example 1.2.21

Determine the most convenient method to graph each line:

a. $y = 5$

b. $4x - 5y = 20$

c. $x = -3$

d. $y = -\frac{5}{9}x + 8$

Solution

a. $y = 5$
This equation has only one variable, y. Its graph is a horizontal line crossing the y-axis at 5.

b. $4x - 5y = 20$
This equation is of the form $Ax + By = C$. The easiest way to graph it will be to find the intercepts and one more point.

c. $x = -3$
There is only one variable, x. The graph is a vertical line crossing the x-axis at -3.

d. $y = -\frac{5}{9}x + 8$
Since this equation is in $y = mx + b$ form, it will be easiest to graph this line by using the slope and y-intercepts.

Try It 1.2.22

Determine the most convenient method to graph each line:

a. $3x + 2y = 12$

b. $y = 4$

c. $y = \frac{1}{5}x - 4$

d. $x = -7$

Answer

a. intercepts

b. horizontal line

c. slope-intercept

d. vertical line

Try It 1.2.23

Determine the most convenient method to graph each line:

a. $x=6$

b. $y=-\frac{3}{4}x+1$

c. $y=-8$

d. $4x-3y=-1$

Answer

a. vertical line

b. slope-intercept

c. horizontal line

d. intercepts

Graph and Interpret Applications of Slope-Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so we can see how equations written in slope-intercept form relate to real-world situations.

Usually, when a linear equation models uses real-world data, different letters are used for the variables, instead of using only x and y. The variable names remind us of what quantities are being measured.

Also, we often will need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

Example 1.2.24

The equation $F=\frac{9}{5}C+32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

a. Find the Fahrenheit temperature for a Celsius temperature of 0.

b. Find the Fahrenheit temperature for a Celsius temperature of 20.

c. Interpret the slope and F-intercept of the equation.

d. Graph the equation.

Solution

a.

Find the Fahrenheit temperature for a Celsius temperature of 0.	
Write the conversion equation.	$F=\frac{9}{5}C+32$
Find F when $C=0$.	$F=\frac{9}{5}(0)+32$
Simplify.	$F=32$

Find the Fahrenheit temperature for a Celsius temperature of 0.	
Answer the question.	The Fahrenheit temperature for a Celsius temperature of 0 is 20.

b.

Find the Fahrenheit temperature for a Celsius temperature of 20.	
Write the conversion equation.	$F=\frac{9}{5}C+32$
Find F when $C=20$.	$F=\frac{9}{5}(20)+32$
Simplify.	$F=36+32$
Simplify.	$F=68$
Answer the question.	The Fahrenheit temperature for a Celsius temperature of 20 is 68.

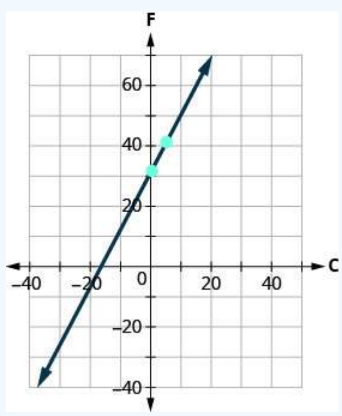
c.
Interpret the slope and F-intercept of the equation.
Even though this equation uses F and C, it is still in slope-intercept form.

$$\begin{aligned}y&=mx+b\\F&=mC+b\\F&=\frac{9}{5}C+32\end{aligned}$$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (F) increases 5 degrees.

The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

d. Graph the equation. We will need to use a larger scale than our usual. Start at the F-intercept $(0,32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in the graph.



? Try It 1.2.25

The equation $h = 2s + 50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

a. Estimate the height of a child who wears women's shoe size 0.

b. Estimate the height of a woman with shoe size 8.

c. Interpret the slope and h-intercept of the equation.

d. Graph the equation.

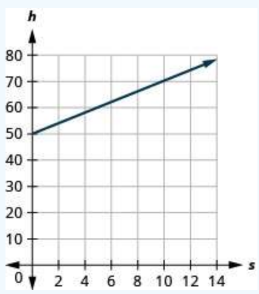
Answer

a. 50 inches

b. 66 inches

c. The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.

d.



? Try It 1.2.26

The equation $T = \frac{1}{4}n + 40$ is used to estimate the temperature in degrees Fahrenheit, T, based on the number of cricket chirps, n, in one minute.

a. Estimate the temperature when there are no chirps.

b. Estimate the temperature when the number of chirps in one minute is 100.

c. Interpret the slope and T-intercept of the equation.

d. Graph the equation.

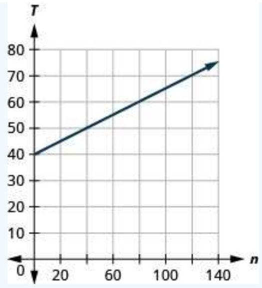
Answer

a. 40 degrees

b. 65 degrees

c. The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (T) increases 1 degree when the number of chirps, n, increases by 4. The T-intercept means that when the number of chirps is 0, the temperature is 40°.

d.



The cost of running some types business have two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

? Example 1.2.27

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, n, that he drives.

a. Find Sam's cost for a week when he drives 0 miles.

b. Find the cost for a week when he drives 250 miles.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Solution

a.

	Find Sam's cost for a week when he drives 0 miles.	
cost …	Write the equation that relates Sam's cost (C) per week when he drives n miles.	$C = 0.5n + 60$
cost …	Find C when $n = 0$.	$C = 0.5(0) + 60$
cost …	Simplify.	$C = 60$
cost …	Answer the question.	Sam's costs are \$60 when he drives 0 miles.

b.

	Find Sam's cost for a week when he drives 250 miles.	
cost …	Write the equation that relates Sam's cost (C) per week when he drives n miles.	$C = 0.5n + 60$
cost …	Find C when $n = 250$.	$C = 0.5(250) + 60$
cost …	Simplify.	$C = 185$
cost …	Answer the question.	Sam's costs are \$185 when he drives 250 miles.

c. Interpret the slope and C-intercept of the equation.

$$y = mx + b$$
$$C = 0.5n + 60$$

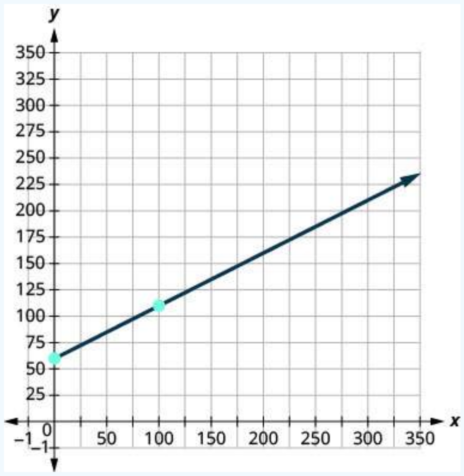
The slope, 0.5, means that the weekly cost, C, increases by $\$0.50$ when the number of miles driven, n, increases by 1.
The C-intercept means that when the number of miles driven is 0, the weekly cost is $\$60$.

d. Graph the equation. We will need to use a larger scale than our usual. Start at the C-intercept $(0, 60)$.

To count out the slope $m = 0.5$, we rewrite it as an equivalent fraction that will make our graphing easier.

	$m = 0.5$
Rewrite as a fraction.	$m = \frac{0.5}{1}$
Multiply numerator and denominator by 100.	$m = \frac{0.5(100)}{1(100)}$
Simplify.	$m = \frac{50}{100}$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be $(100, 110)$



Try It 1.2.28

Stella has a home business selling gourmet pizzas. The equation $C = 4p + 25$ models the relation between her weekly cost, C, in dollars and the number of pizzas, p, that she sells.

a. Find Stella's cost for a week when she sells no pizzas.

b. Find the cost for a week when she sells 15 pizzas.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

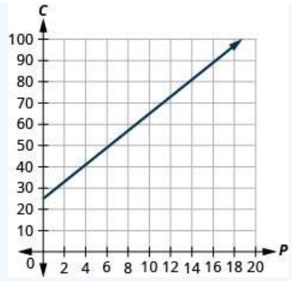
Answer

a. $\$25$

b. $\$85$

c. The slope, 4, means that the weekly cost, C, increases by $\$4$ when the number of pizzas sold, p, increases by 1. The C-intercept means that when the number of pizzas sold is 0, the weekly cost is $\$25$.

d.



? Try It 1.2.29

Loreen has a calligraphy business. The equation $C = 1.8n + 35$ models the relation between her weekly cost, C, in dollars and the number of wedding invitations, n, that she writes.

a. Find Loreen’s cost for a week when she writes no invitations.

b. Find the cost for a week when she writes 75 invitations.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

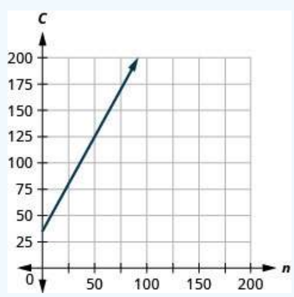
Answer

a. $\$35$

b. $\$170$

c. The slope, 1.8, means that the weekly cost, C, increases by $\$1.80$ when the number of invitations, n, increases by 1. The C-intercept means that when the number of invitations is 0, the weekly cost is $\$35$.

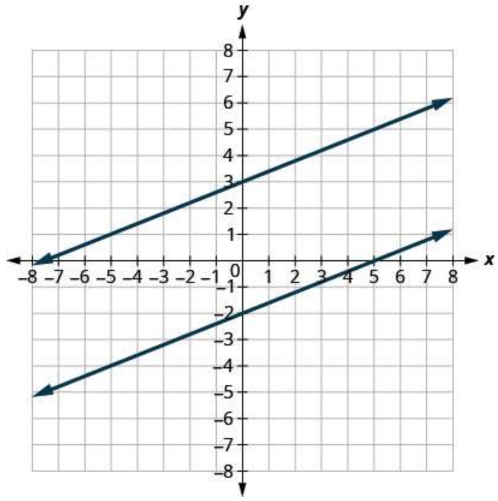
d.



Use Slopes to Identify Parallel and Perpendicular Lines

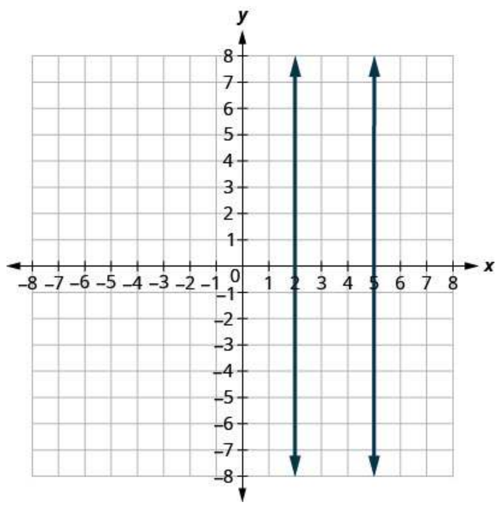
Two lines that don't intersect are called parallel. Parallel lines have the same steepness and never intersect.

We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different y-intercepts are called parallel lines.



Verify that both lines have the same slope, $m=\frac{2}{5}$, and different y-intercepts.

What about vertical lines? The slope of a vertical line is undefined but we see that vertical lines that have different x-intercepts are parallel, like the lines shown in this graph.



Definition 1.2.30

Parallel lines are lines in the same plane that do not intersect.

Parallel lines

- Parallel lines have the same slope and different y-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then $m_1=m_2$.
- Parallel vertical lines have different x-intercepts.

Since non-vertical parallel lines have the same slope and different y-intercepts, we can now just look at the slope-intercept form of the equations of lines and decide if the lines are parallel.

Example 1.2.31

Use slopes and y-intercepts to determine if the lines are parallel:

a. $3x-2y=6$ and $y=\frac{3}{2}x+1$

b. $y=2x-3$ and $-6x+3y=-9$

Solution

a.

	Verify that the lines are parallel.	
	$3x-2y=6$	$y=\frac{3}{2}x+1$
Is the equation in slope-intercept form, $y=mx+b$?	No	Yes
Solve the first equation for y and write it in slope-intercept form.	$-2y=-3x+6$ $\frac{-2y}{-2}=\frac{-3x+6}{-2}$ $y=\frac{3}{2}x-3$	
Now both equations are in slope-intercept form.	$y=\frac{3}{2}x-3$	$y=\frac{3}{2}x+1$
Identify the slope and y-intercept of both lines.	The slope is $m=\frac{3}{2}$. The y-intercept is $(0,-3)$.	The slope is $m=\frac{3}{2}$. The y-intercept is $(0,1)$.
Conclusion.	The lines have the same slope and different y-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.	

b.

	Verify that the lines are parallel.	
	$y=2x-3$	$-6x+3y=-9$
Is the equation in slope-intercept form, $y=mx+b$?	Yes	No
Solve the second equation for y and write it in slope-intercept form.		$3y=6x-9$ $\frac{3y}{3}=\frac{6x-9}{3}$ $y=2x-3$
Now both equations are in slope-intercept form.	$y=2x-3$	$y=2x-3$
Identify the slope and y-intercept of both lines.	The slope is $m=2$. The y-intercept is $(0,-3)$.	The slope is $m=2$. The y-intercept is $(0,-3)$.
Conclusion.	The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line and we say the lines are coincident. They are not parallel; they are the same line.	

? Try It 1.2.32

Use slopes and y-intercepts to determine if the lines are parallel:

a. $2x+5y=5$ and $y=-\frac{2}{5}x-4$

b. $y=-\frac{1}{2}x-1$ and $x+2y=-2$

Answer

a. The lines are parallel.

b. The lines are not parallel as the equations represent the same line.

Try It 1.2.33

Use slopes and y-intercepts to determine if the lines are parallel:

a. $4x - 3y = 6$ and $y = \frac{4}{3}x - 1$

b. $y = \frac{3}{4}x - 3$ and $3x - 4y = 12$

Answer

a. The lines are parallel.

b. The lines are not parallel as the equations represent the same line.

Example 1.2.34

Use slopes and y-intercepts to determine if the lines are parallel:

a. $y = -4$ and $y = 3$

b. $x = -2$ and $x = -5$

Solution

a. $y = -4$ and $y = 3$

We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0. Since the horizontal lines cross the y-axis at $y = -4$ and at $y = 3$, we know the y-intercepts are $(0, -4)$ and $(0, 3)$. The lines have the same slope and different y-intercepts and so they are parallel.

b. $x = -2$ and $x = -5$

We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined. Since the vertical lines cross the x-axis at $x = -2$ and $x = -5$, we know the y-intercepts are $(-2, 0)$ and $(-5, 0)$. The lines are vertical and have different x-intercepts and so they are parallel.

Try It 1.2.35

Use slopes and y-intercepts to determine if the lines are parallel:

a. $y = 8$ and $y = -6$

b. $x = 1$ and $x = -5$

Answer

a. The lines are parallel.

b. The lines are parallel.

Try It 1.2.36

Use slopes and y-intercepts to determine if the lines are parallel:

a. $y = 1$ and $y = -5$

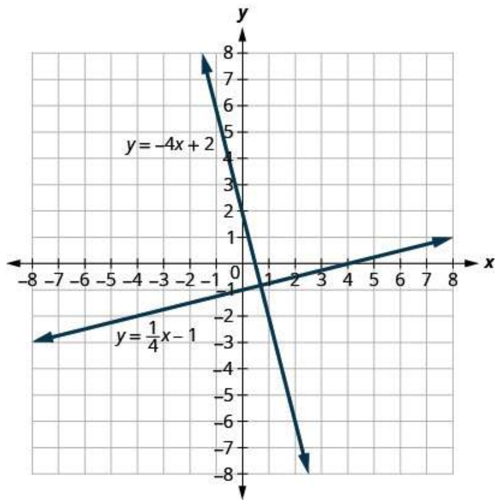
b. $x=8$ and $x=-6$

Answer

a. The lines are parallel.

b. The lines are parallel.

Let's look at the lines whose equations are $y=\frac{1}{4}x-1$ and $y=-4x+2$.



These lines lie in the same plane and intersect in right angles. We call these lines *perpendicular*.

If we look at the slope of the first line, $m_1=\frac{1}{4}$, and the slope of the second line, $m_2=-4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$\begin{aligned}&m_1m_2\\=&\frac{1}{4}(-4)\\=&-1\end{aligned}$$

This is always true for **perpendicular lines** and leads us to the following definition.

Definition 1.2.37

Perpendicular lines are lines in the same plane that intersect at a right angle.

Perpendicular lines

- If m_1 and m_2 are the slopes of two perpendicular lines, then their slopes are negative reciprocals of each other, $m_1=-\frac{1}{m_2}$. In other words, the product of their slopes is -1, that is, $m_1m_2=-1$.
- A vertical line and a horizontal line are always perpendicular to each other.

We were able to look at the slope-intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope-intercept form of the equation, and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

Example 1.2.38

Use slopes to determine if the lines are perpendicular:

a. $y=-5x-4$ and $x-5y=5$

b. $7x+2y=3$ and $2x+7y=5$

Solution

a.

	Verify that the lines are perpendicular.	
	$y=-5x-4$	$x-5y=5$
Is the equation in slope-intercept form, $y=mx+b$?	Yes	No
Solve the second equation for y and write it in slope-intercept form.		$\begin{aligned}-5y&=-x+5\\ y&=-\frac{1}{5}x-1\end{aligned}$
Now both equations are in slope-intercept form.	$y=-5x-4$	$y=-\frac{1}{5}x-1$
Identify the slope of each line.	$m_1=-5$	$m_2=\frac{1}{5}$
Conclusion.	The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes. Since $$m_1m_2=-5\cdot\frac{1}{5}=-1,$$ it checks.	

b.

	Verify that the lines are perpendicular.	
	$7x+2y=3$	$2x+7y=5$
Is the equation in slope-intercept form, $y=mx+b$?	No	No
Solve both equations for y and write it in slope-intercept form.	$\begin{aligned}2y&=-7x+3\\ y&=-\frac{7}{2}x+\frac{3}{2}\end{aligned}$	$\begin{aligned}7y&=-2x+5\\ y&=-\frac{2}{7}x+\frac{5}{7}\end{aligned}$
Now both equations are in slope-intercept form.	$y=-\frac{7}{2}x+\frac{3}{2}$	$y=-\frac{1}{5}x-1$
Identify the slope of each line.	$m_1=-\frac{7}{2}$	$m_2=-\frac{2}{7}$
Conclusion.	The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.	

Try It 1.2.39

Use slopes to determine if the lines are perpendicular:

a. $y=-3x+2$ and $x-3y=4$

b. $5x+4y=1$ and $4x+5y=3$

Answer

a. The lines are perpendicular.

b. The lines are not perpendicular.

? Try It 1.2.40

Use slopes to determine if the lines are perpendicular:

a. $y=2x-5$ and $x+2y=-6$

b. $2x-9y=3$ and $9x-2y=1$

Answer

a. The lines are perpendicular.

b. The lines are not perpendicular.

Key Concepts

- **Slope of a Line**
  - The slope of a line is $m=\frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change when moving from one point on the line to another on the line.
- **How to find the slope of a line from its graph using $m=\frac{\text{rise}}{\text{run}}$.**
  1. Locate two points on the line whose coordinates are integers.
  2. Starting with one point, sketch a right triangle, going from the first point to the second point.
  3. Count vertical and horizontal jumps needed when moving along the legs of the triangle to find the rise and the run.
  4. Take the ratio of rise to run to find the slope: $m=\frac{\text{rise}}{\text{run}}$.
- **Slope of a line between two points.**

The slope of the line between two points (x_1,y_1) and (x_2,y_2) is

$$m=\frac{y_2-y_1}{x_2-x_1}.$$

- **How to graph a line given a point and the slope.**
  1. Plot the given point.
  2. Use the slope formula $m=\frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  3. Starting at the given point, count jumps corresponding to the rise and run to mark the second point.
  4. Draw the line passing through the points.
- **Slope Intercept Form of an Equation of a Line**
  - The slope–intercept form of an equation of a line with slope m and y-intercept, $(0,b)$ is $y=mx+b$.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
$\begin{array}{c\|c} x & y \\ \hline & \\ \hline & \\ \hline & \\ \hline \end{array}$	$y = mx + b$	$\begin{array}{c\|c} x & y \\ \hline 0 & \\ \hline & 0 \\ \hline & \\ \hline \end{array}$	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

- **Parallel Lines**
  - Parallel lines are lines in the same plane that do not intersect.
  - Parallel lines have the same slope and different y-intercepts.
  - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
  - Parallel vertical lines have different x-intercepts.
- **Perpendicular Lines**
  - Perpendicular lines are lines in the same plane that intersect at a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$. Equivalently, the product of their slopes is -1, that is, $m_1 m_2 = -1$.
  - A vertical line and a horizontal line are always perpendicular to each other.

Glossary

parallel lines

Parallel lines are lines in the same plane that do not intersect.

perpendicular lines

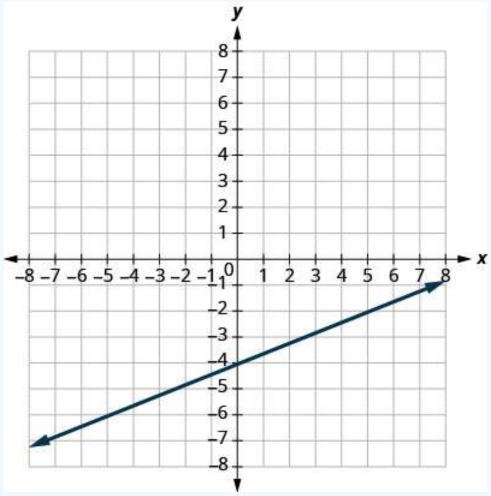
Perpendicular lines are lines in the same plane that form a right angle.

Practice Makes Perfect

Find the Slope of a Line

In the following exercises, find the slope of each line shown.

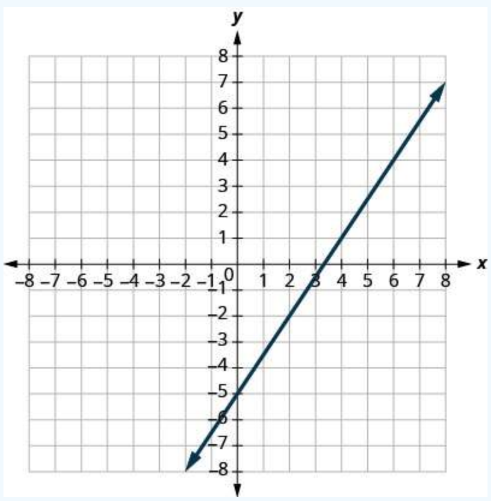
1.



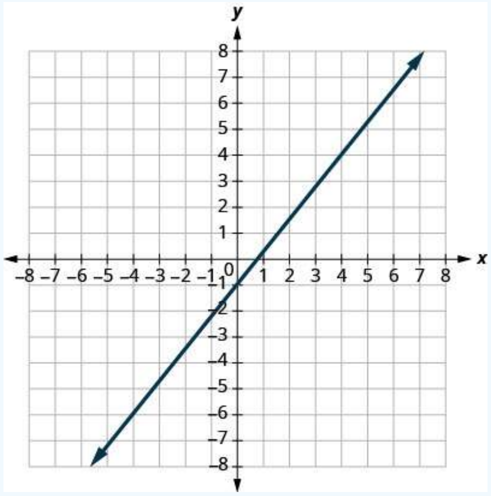
Answer

$$m = \frac{2}{5}$$

2.



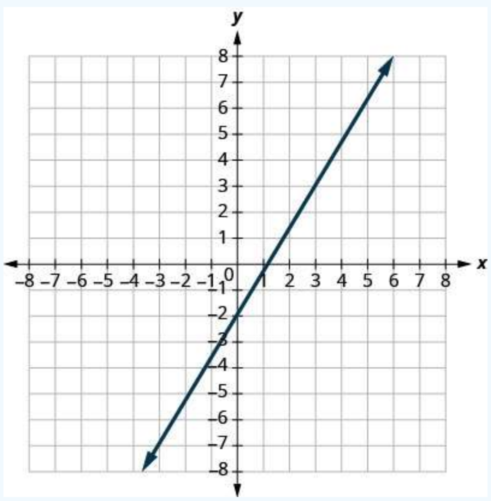
3.



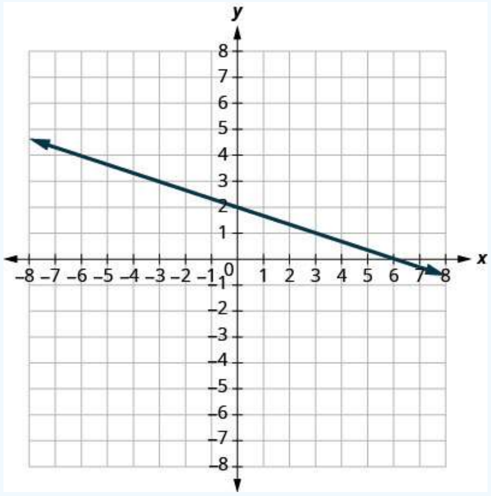
Answer

$$m = \frac{5}{4}$$

4.



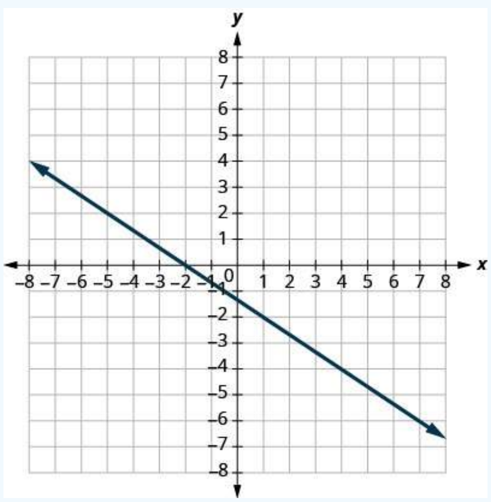
5.



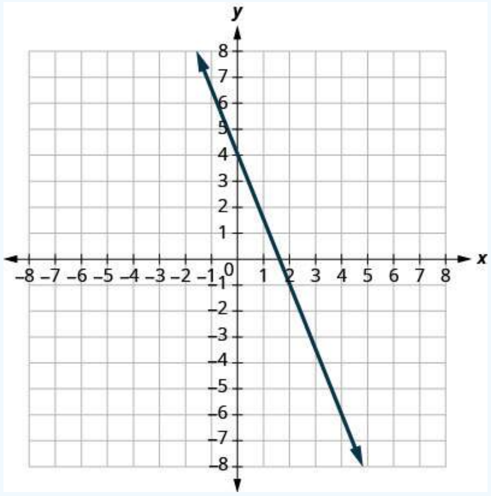
Answer

$$m = -\frac{1}{3}$$

6.



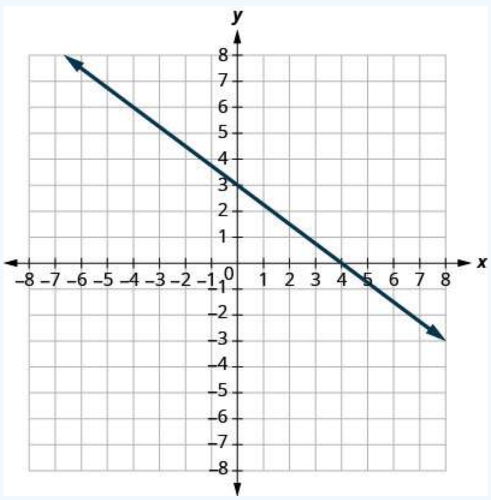
7.



Answer

$m = -\frac{5}{2}$

8.



In the following exercises, find the slope of each line.

9. $y = 3$

Answer

$m = 0$

10. $y = -2$

11. $x = -5$

Answer

undefined

12. $x = 4$

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

13. $(2, 5),\ (4, 0)$

Answer

$$m = -\frac{5}{2}$$

14. $(3, 6),\ (8, 0)$

15. $(-3, 3),\ (4, -5)$

Answer

$$m = -\frac{8}{7}$$

16. $(-2, 4),\ (3, -1)$

17. $(-1, -2),\ (2, 5)$

Answer

$$m = \frac{7}{3}$$

18. $(-2, -1),\ (6, 5)$

19. $(4, -5),\ (1, -2)$

Answer

$$m = -1$$

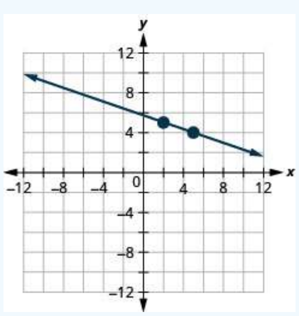
20. $(3, -6),\ (2, -2)$

Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

21. $(2, 5)$; $m = -\frac{1}{3}$

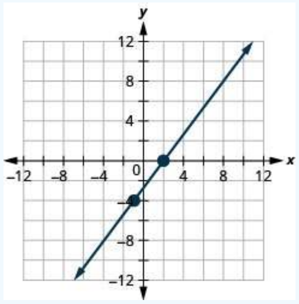
Answer



22. $(1,4)$; $m=-\frac{1}{2}$

23. $(-1,-4)$; $m=\frac{4}{3}$

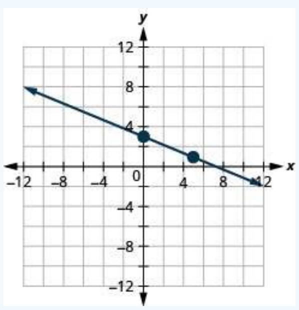
Answer



24. $(-3,-5)$; $m=\frac{3}{2}$

25. y-intercept: $(0,3)$; $m=-\frac{2}{5}$

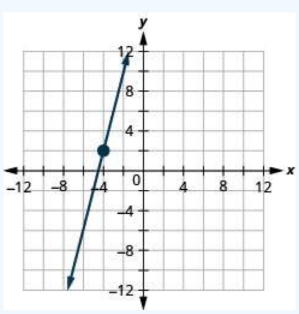
Answer



26. x-intercept: $(-2,0)$; $m=\frac{3}{4}$

27. $(-4,2)$; $m=4$

Answer



28. $(1,5)$; $m=-3$

Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and *y*-intercept of each line.

29. $y=-7x+3$

Answer

$m=-7;(0,3)$

30. $y=4x-10$

31. $3x+y=5$

Answer

$m=-3;(0,5)$

32. $4x+y=8$

33. $6x+4y=12$

Answer

$m=-\frac{3}{2};(0,3)$

34. $8x+3y=12$

35. $5x-2y=6$

Answer

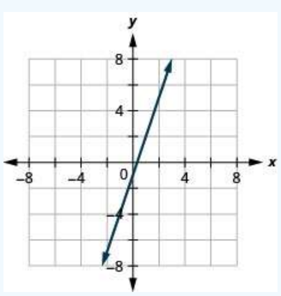
$m=\frac{5}{2};(0,-3)$

36. $7x-3y=9$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

37. $y=3x-1$

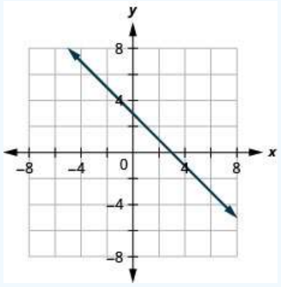
Answer



38. $y=2x-3$

39. $y=-x+3$

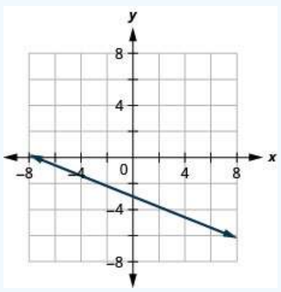
Answer



40. $y = -x - 4$

41. $y = -\frac{2}{5}x - 3$

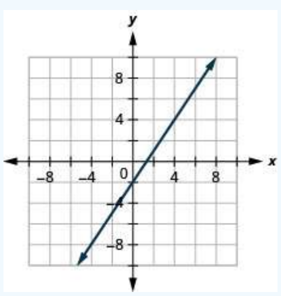
Answer



42. $y = -\frac{3}{5}x + 2$

43. $3x - 2y = 4$

Answer



44. $3x - 4y = 8$

Choose the Most Convenient Method to Graph a Line

In the following exercises, determine the most convenient method to graph each line.

45. $x = 2$

Answer

vertical line

46. $y=5$

47. $y=-3x+4$

Answer

slope-intercept

48. $x-y=5$

49. $x-y=1$

Answer

intercepts

50. $y=\frac{2}{3}x-1$

51. $3x-2y=-12$

Answer

intercepts

52. $2x-5y=-10$

Graph and Interpret Applications of Slope–Intercept

53. The equation $P=31+1.75w$ models the relation between the amount of Tuyet's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

a. Find Tuyet's payment for a month when 0 units of water are used.

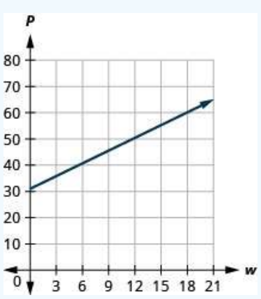
b. Find Tuyet's payment for a month when 12 units of water are used.

c. Interpret the slope and P-intercept of the equation.

d. Graph the equation.

Answer

a. $\$31$
b. $\$52$
c. The slope, 1.75, means that the payment, P, increases by $\$1.75$ when the number of units of water used, w, increases by 1. The P-intercept means that when the number units of water Tuyet used is 0, the payment is $\$31$.
d.



54. The equation $P = 28 + 2.54w$ models the relation between the amount of Randy's monthly water bill payment, P, in dollars, and the number of units of water, w, used.

a. Find the payment for a month when Randy used 0 units of water.

b. Find the payment for a month when Randy used 15 units of water.

c. Interpret the slope and P-intercept of the equation.

d. Graph the equation.

55. Bruce drives his car for his job. The equation $R = 0.575m + 42$ models the relation between the amount in dollars, R, that he is reimbursed and the number of miles, m, he drives in one day.

a. Find the amount Bruce is reimbursed on a day when he drives 0 miles.

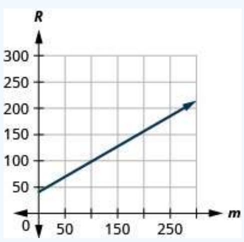
b. Find the amount Bruce is reimbursed on a day when he drives 220 miles.

c. Interpret the slope and R-intercept of the equation.

d. Graph the equation.

Answer

a. $\$42$
b. $\$168.50$
c. The slope, 0.575 means that the amount he is reimbursed, R, increases by $\$0.575$ when the number of miles driven, m, increases by 1. The R-intercept means that when the number miles driven is 0, the amount reimbursed is $\$42$.
d.



56. Janelle is planning to rent a car while on vacation. The equation $C = 0.32m + 15$ models the relation between the cost in dollars, C, per day and the number of miles, m, she drives in one day.

a. Find the cost if Janelle drives the car 0 miles one day.

b. Find the cost on a day when Janelle drives the car 400 miles.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

57. Cherie works in retail and her weekly salary includes commission for the amount she sells. The equation $S = 400 + 0.15c$ models the relation between her weekly salary, S, in dollars and the amount of her sales, c, in dollars.

a. Find Cherie's salary for a week when her sales were $\$0$.

b. Find Cherie's salary for a week when her sales were $\$3,600$.

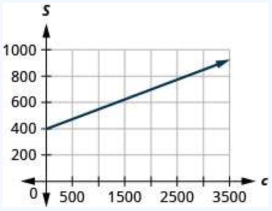
c. Interpret the slope and S-intercept of the equation.

d. Graph the equation.

Answer

a. $\$400$
b. $\$940$

c. The slope, 0.15, means that Cherie's salary, S, increases by $\$0.15$ for every $\$1$ increase in her sales. The S-intercept means that when her sales are $\$0$, her salary is $\$400$.
d.



58. Patel's weekly salary includes a base pay plus commission on his sales. The equation $S = 750 + 0.09c$ models the relation between his weekly salary, S, in dollars and the amount of his sales, c, in dollars.

a. Find Patel's salary for a week when his sales were 0.

b. Find Patel's salary for a week when his sales were $18,540$.

c. Interpret the slope and S-intercept of the equation.

d. Graph the equation.

59. Costa is planning a lunch banquet. The equation $C = 450 + 28g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

a. Find the cost if the number of guests is 40.

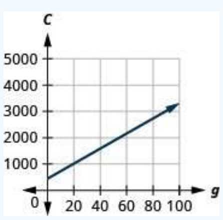
b. Find the cost if the number of guests is 80.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Answer

a. $\$1570$
b. $\$5690$
c. The slope gives the cost per guest. The slope, 28, means that the cost, C, increases by $\$28$ when the number of guests increases by 1. The C-intercept means that if the number of guests was 0, the cost would be $\$450$.
d.



60. Margie is planning a dinner banquet. The equation $C = 750 + 42g$ models the relation between the cost in dollars, C, of the banquet and the number of guests, g.

a. Find the cost if the number of guests is 50.

b. Find the cost if the number of guests is 100.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

61. $y = \frac{3}{4}x - 3$; $3x - 4y = -2$

Answer

parallel

62. $3x - 4y = -2$; $y = \frac{3}{4}x - 3$

63. $2x - 4y = 6$; $x - 2y = 3$

Answer

neither

64. $8x + 6y = 6$; $12x + 9y = 12$

65. $x = 5$; $x = -6$

Answer

parallel

66. $x = -3$; $x = -2$

67. $4x - 2y = 5$; $3x + 6y = 8$

Answer

perpendicular

68. $8x - 2y = 7$; $3x + 12y = 9$

69. $3x - 6y = 12$; $6x - 3y = 3$

Answer

neither

70. $9x - 5y = 4$; $5x + 9y = -1$

71. $7x - 4y = 8$; $4x + 7y = 14$

Answer

perpendicular

72. $5x - 2y = 11$; $5x - y = 7$

73. $3x - 2y = 8$; $2x + 3y = 6$

Answer

perpendicular

74. $2x+3y=5$; $3x-2y=7$

75. $3x-2y=1$; $2x-3y=2$

Answer

neither

76. $2x+4y=3$; $6x+3y=2$

77. $y=2$; $y=6$

Answer

parallel

78. $y=-1$; $y=2$

Writing Exercises

79. How does the graph of a line with slope $m=12$ differ from the graph of a line with slope $m=2$?

Answer

Answers will vary.

80. Why is the slope of a vertical line “undefined”?

81. Explain how you can graph a line given a point and its slope.

Answer

Answers will vary.

82. Explain in your own words how to decide which method to use to graph a line.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the slope of a line.			
graph a line given a point and the slope.			
graph a line using its slope and intercept.			
choose the most convenient method to graph a line.			
graph and interpret applications of slope-intercept.			
use slopes to identify parallel and perpendicular lines.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

1.3: Find the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $\frac{2}{5}(x+15)$.
2. Simplify $-3(x-(-2))$.
3. Solve for y: $y-3=-2(x+1)$.

How do online companies know that "we may also like" a particular item based on something we just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on our commuting time of an increase or decrease in gas prices? It's all mathematics.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. To create a mathematical model of a linear relation between two variables, we must be able to find an equation of a line. In this section, we will look at several ways to write an equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of a Line Given the Slope and y-intercept

If we have an equation of a line which is in slope-intercept form, $y=mx+b$, we can easily determine the line's slope and y-intercept. Now we will do the reverse - we will start with the slope and the y-intercept and find an equation of the line. A line has only one equation which is in slope-intercept form. So, if an equation is requested to be in slope-intercept form, we will write "the equation" instead of "an equation".

Example 1.3.1

Find the slope-intercept form of an equation of a line with slope -9 and y-intercept $(0,-4)$.

Solution

Since we are given the slope and y-intercept of the line, we can substitute the needed values into the slope-intercept form, $y=mx+b$.

Identify the slope m.	$m=-9$
Identify the y-intercept $(0,b)$ and b.	The y-intercept is $(0,-4)$, so $b=-4$.
Substitute the values into $y=mx+b$.	$y=mx+b$ $y=-9x+(-4)$
Simplify.	$y=-9x-4$
Answer the question in slope-intercept form.	The equation of the line is $y=-9x-4$.

Try It 1.3.2

Find the slope-intercept form of an equation of a line with slope $\frac{2}{5}$ and y-intercept $(0, 4)$.

Answer

The equation of the line is $y = \frac{2}{5}x + 4$.

Try It 1.3.3

Find the slope-intercept form of an equation of a line with slope -1 and y-intercept $(0, -3)$.

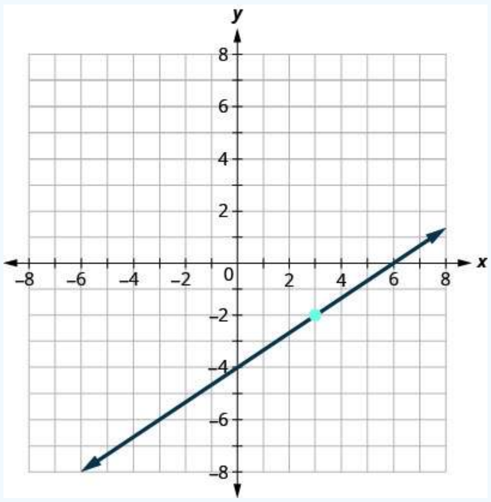
Answer

The equation of the line is $y = -x - 3$.

Sometimes, the slope and intercept need to be determined from the graph.

Example 1.3.4

Find the slope-intercept form of an equation of the line shown in the graph.



Solution

We need to find the slope and y-intercept of the line from the graph so that we can substitute the needed values into the slope-intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

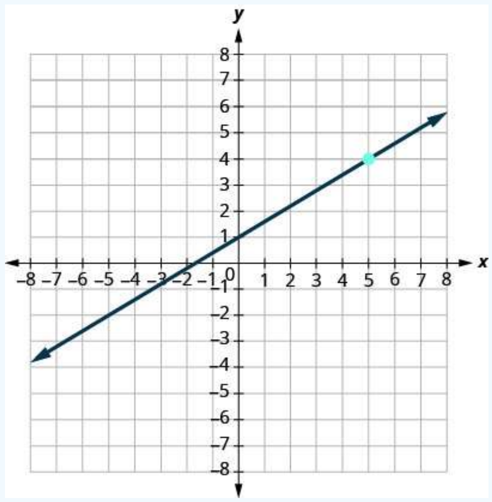
The y-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Pick two points on the line.	$(0, -4)$ and $(3, -2)$
Determine the rise and run.	The rise is 2, and the run is 3.
Substitute the rise and run into the slope formula, $m = \frac{\text{rise}}{\text{run}}$.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{2}{3}$
Find the y-intercept.	The y-intercept is $(0, -4)$.

Substitute the values into $y=mx+b$.	$y=mx+b$ $y=\frac{2}{3}x+(-4)$
Simplify.	$y=\frac{2}{3}x-4$
Answer the question.	The equation of the line is $y=\frac{2}{3}x-4$.

? Try It 1.3.5

Find the slope-intercept form of an equation of the line shown in the graph.

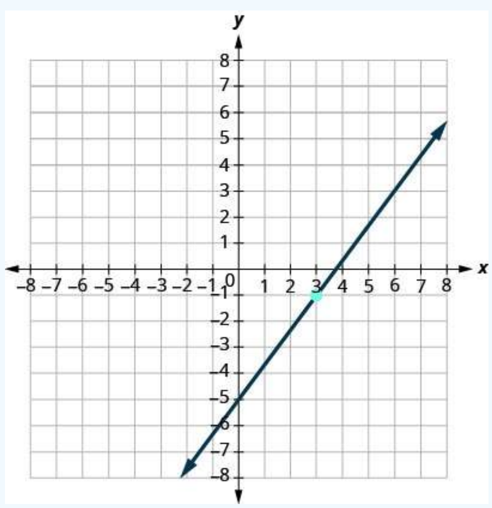


Answer

The equation of the line is $y=\frac{3}{5}x+1$.

? Try It 1.3.6

Find the slope-intercept form of an equation of the line shown in the graph.



Answer

The equation of the line is $y=\frac{4}{3}x-5$.

Find an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when we are given the slope and y-intercept or when we read them off a graph. But what happens when we have another point instead of the y-intercept?

We are going to use the slope formula to derive another form of an equation of the line. By 'another form' we mean that it is an equation that has the line as it's graph though the equation has a different structure, or looks different, from what we have seen before.

Suppose we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form. We will write our answer in slope-intercept form, or of the form $x=a$, since this gives a unique answer.

Write the slope formula based on two points, (x_1, y_1) and (x, y).	$m=\frac{y-y_1}{x-x_1}$
Multiply both sides of the equation by $x-x_1$.	$m(x-x_1)=\left(\frac{y-y_1}{x-x_1}\right)\cdot(x-x_1)$
Simplify.	$m(x-x_1)=y-y_1$
Rewrite the equation with the y terms on the left.	$y-y_1=m(x-x_1)$ This format is called the **point-slope form** of an equation of a line.

Point-slope form of an equation of a line

The **point-slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is

$$y-y_1=m(x-x_1).$$

We can use the point-slope form of an equation to find an equation of a line when we know the slope and at least one point.

How to Find an Equation of a Line Given a Point and the Slope

In the example below we will find an equation for a line with given attributes. In order to have a unique answer, we specify that the answer should be the equation which is in slope-intercept form, or in the form $x=a$.

Example 1.3.7

Find an equation of a line with slope $m=-\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form.

Solution

In order to use the point-slope form, we need the slope and one point.

Identify the slope.	$m=-\frac{1}{3}$
Identify the point, (x_1, y_1).	$\underbrace{(6,-4)}_{(x_1,y_1)}$
Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	$\begin{aligned} y-y_1&=m(x-x_1)\\ y-(-4)&=-\frac{1}{3}(x-6)\end{aligned}$
Simplify.	$y+4=-\frac{1}{3}x+2$

Write the equation in slope-intercept form, $y=mx+b$.	$y=-\frac{1}{3}x-2$
Answer the question in slope-intercept form, or in the form $x=a$.	The equation of the line is $y=-\frac{1}{3}x-2$.

Try It 1.3.8

Find an equation of a line with slope $m=-\frac{2}{5}$ that contains the point $(10,-5)$.

Answer

The equation of the line in slope-intercept form is $y=-\frac{2}{5}x-1$.

Try It 1.3.9

Find the equation of a line with slope $m=-\frac{3}{4}$ that contains the point $(4,-7)$.

Answer

The equation of the line in slope-intercept form is $y=-\frac{3}{4}x-4$.

We list the steps for easy reference.

To find an equation of a line given the slope and a point

1. Identify the slope.
2. Identify the point.
3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
4. Write the equation in slope-intercept form or in the form $x=a$.

Example 1.3.10

Find an equation of a horizontal line that contains the point $(-2,-6)$. Write the equation in slope-intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and a point into the point-slope form, $y-y_1=m(x-x_1)$. We need the slope and the point.

Identify the slope.	$m=0$
Identify the point.	$\underbrace{(-2,-6)}_{(x_1,y_1)}$
Substitute the values into $y-y_1=m(x-x_1)$.	$y-y_1=m(x-x_1)$ $y-(-6)=0(x-(-2))$
Simplify.	$y+6=0$
Solve for y.	$y=-6$
Write in slope-intercept form.	It is in y-form, but it could be written as $y=0x-6$.
Answer the question.	The equation of the horizontal line is $y=-6$ or $y=0x-6$.

Did we end up with the form of a horizontal line, $y = a$?

Try It 1.3.11

Find an equation of a horizontal line that contains the point $(-3, 8)$. Write the equation in slope-intercept form.

Answer

The equation is $y = 8$.

Try It 1.3.12

Find an equation of a horizontal line that contains the point $(-1, 4)$. Write the equation in slope-intercept form.

Answer

The equation is $y = 4$.

Find an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we will see how to find an equation of a line when just two points are given.

So far, we have two options for finding an equation of a line: by using the slope-intercept form or by using the point-slope form. When we start with two points, it makes more sense to use the point-slope form. But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find an equation. While our equation is an equation for the given line, we will then rewrite our equation in slope-intercept form, or in the form $x = a$ so that our answer is unique.

Example 1.3.13

Find an equation of the line that contains the points $(-3, -1)$ and $(2, -2)$. Write the equation in slope-intercept form.

Solution

In order to find the equation of the line using the point-slope form, we need the slope and the point.

Write the given points.	$(-3, -1)$ and $(2, -2)$
Choose one point and identify x_1 and y_1.	$\underbrace{(-3, -1)}_{(x_1, y_1)}$ $x_1 = -3$ $y_1 = -1$
Pick the other point and identify x_2 and y_2.	$\underbrace{(2, -2)}_{(x_2, y_2)}$ $x_2 = 2$ $y_2 = -2$
Find the slope, using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $m = \dfrac{-2 - (-1)}{2 - (-3)}$ $m = \dfrac{-1}{5}$ $m = -\dfrac{1}{5}$ The slope is $m = -\dfrac{1}{5}$.

Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	$$\begin{aligned} y-y_1 &= m(x-x_1) \\ y-(-1) &= -\frac{1}{5}(x-(-3)) \end{aligned}$$
Simplify to write the equation in slope-intercept form, $y=mx+b$.	$$\begin{aligned} y+1 &= -\frac{1}{5}(x+3) \\ y+1 &= -\frac{1}{5}x-\frac{3}{5} \\ y &= -\frac{1}{5}x-\frac{8}{5} \end{aligned}$$
Answer the question.	The equation of the line is $y=-\frac{1}{5}x-\frac{8}{5}$.

Try It 1.3.14

Find the equation of a line that contains the points $(-2,-4)$ and $(1,-3)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=\frac{1}{3}x-\frac{10}{3}$.

Try It 1.3.15

Find the equation of a line that contains the points $(-4,-3)$ and $(1,-5)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=-\frac{2}{5}x-\frac{23}{5}$.

The steps are summarized here.

To find an equation of a line given two points

1. Find the slope using the given points: $m=\frac{y_2-y_1}{x_2-x_1}$.
2. Choose one point.
3. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
4. Write the equation in slope-intercept form.

Example 1.3.16

Find an equation of a line that contains the points $(-3,5)$ and $(-3,4)$. Write the equation in slope-intercept form.

Solution

In order to find the equation of the line using the point-slope form, we need the slope and the point.

Write the given points.	$(-3,5)$ and $(-3,4)$
Choose one point, name it (x_1,y_1), and identify x_1 and y_1 .	$\underbrace{(-3,5)}_{(x_1,y_1)}$ $x_1=-3$ $y_1=5$

Pick the other point, name it (x_2, y_2), and identify x_2 and y_2.	$\underbrace{(-3,4)}_{(x_2,y_2)}$ $x_2 = -3$ $y_2 = 4$
Find the slope, using $m = \dfrac{y_2 - y_1}{x_2 - x_1}$.	$m = \dfrac{y_2 - y_1}{x_2 - x_1}$ $m = \dfrac{4-5}{-3-(-3)}$ $m = \dfrac{-1}{0}$ The slope is undefined.
Conclusion.	This tells us it is a vertical line. Both of our points have an x-coordinate of -2. So our equation of the line is $x = -2$. Since there is no y, we cannot write it in slope-intercept form.

We may want to sketch a graph using the two given points. Does the graph agree with our conclusion that this is a vertical line?

? Try It 1.3.17

Find the equation of a line that contains the points $(5,1)$ and $(5,-4)$.

Answer

The equation of the line is $x = 5$.

? Try It 1.3.18

Find the equation of a line that contains the points $(-4,4)$ and $(-4,3)$.

Answer

The equation of the line is $x = -4$.

We have seen that we can use the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

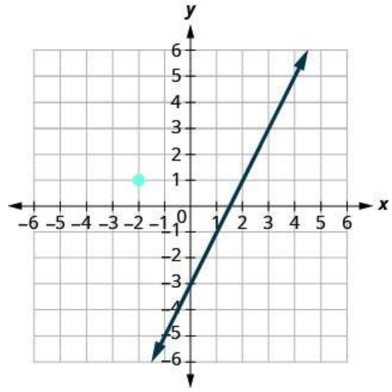
To write an equation of a line		
If given:	**Use:**	**Form:**
Slope and y-intercept	slope-intercept	$y = mx + b$
Slope and a point	point-slope	$y - y_1 = m(x - x_1)$
Two points	point-slope	$y - y_1 = m(x - x_1)$

Find an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope - just what we need to use the point-slope equation.

First, let's look at this graphically.

This graph shows $y = 2x - 3$. We want to graph a line parallel to this line and passing through the point $(-2,1)$.

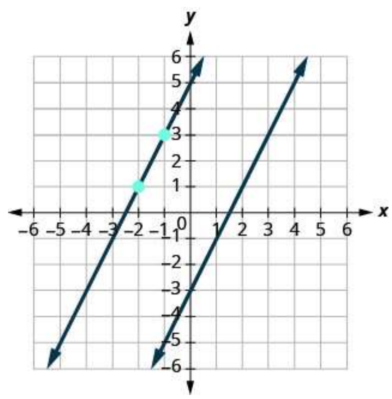


We know that parallel lines have the same slope. So the second line will have the same slope as $y = 2x - 3$. That slope is $m_{||} = 2$. We'll use the notation $m_{||}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $||$ looks like two parallel lines.)

The second line will pass through $(-2, 1)$ and have $m = 2$.

To graph the line, we start at $(-2, 1)$ and count out the rise and run.

With $m = 2$ (or $m = \frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through $(-2, 1)$?

We were asked to graph the line, now let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

? Example 1.3.19

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

	$y = 2x - 3$	
Find the slope of the given line.	The line is in slope-intercept form, $y = 2x - 3$.	$m = 2$
Find the slope of the parallel line.	Parallel lines have the same slopes.	$m_{\vert\vert} = 2$
Identify the point.	The given point is $(-2, 1)$.	$\underbrace{(-2, 1)}_{(x_1, y_1)}$

$y=2x-3$		
Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-1=2(x-(-2))$ $y-1=2(x+2)$ $y-1=2x+4$
Write the equation in slope-intercept form.		$y=2x+5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the y-intercept of the line? What is the slope?

Try It 1.3.20

Find an equation of a line parallel to the line $y=3x+1$ that contains the point $(4,2)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=3x-10$.

Try It 1.3.21

Find an equation of a line parallel to the line $y=\frac{1}{2}x-3$ that contains the point $(6,4)$. Write the equation in slope-intercept form.

Write the equation in slope-intercept form.

Answer

The equation of the line is $y=\frac{1}{2}x+1$.

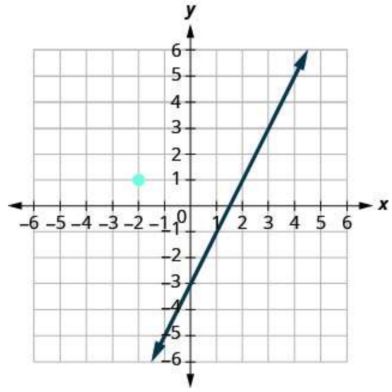
Find an equation of a line parallel to a given line

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
5. Write the equation in slope-intercept form.

Find an Equation of a Line Perpendicular to a Given Line

Now, let’s consider perpendicular lines. Suppose we need to find the line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y=2x-3$. Now, we want to graph a line perpendicular to this line and passing through $(-2,1)$.



We know that perpendicular lines have slopes that are negative reciprocals.

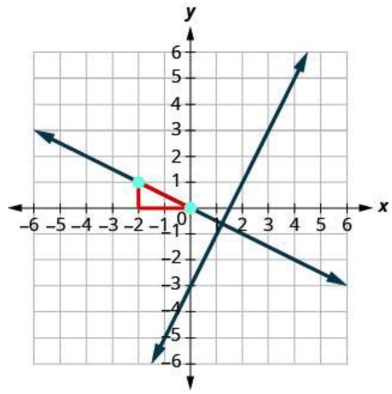
We will use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

$$y=2x-3 \qquad \text{perpendicular line}$$

$$m=2 \qquad m_{\perp}=-\frac{1}{2}$$

We now know the perpendicular line will pass through $(-2,1)$ with $m_{\perp}=-\frac{1}{2}$.

To graph the line, we will start at $(-2,1)$ and count out the rise -1 and the run 2. Then we draw the line.



Do the lines appear perpendicular? Does the second line pass through $(-2,1)$?

We were asked to graph the line, now, let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this Example we know one point, and can find the slope, so we will use the point-slope form.

Example 1.3.22

Find an equation of a line perpendicular to $y=2x-3$ that contains the point $(-2,1)$. Write the equation in slope-intercept form.

Solution

$y=2x-3$		
Find the slope of the given line.	The line is in slope-intercept form, $y=2x-3$.	$m=2$
Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative.	$m_{\perp}=-\frac{1}{2}$

	$y=2x-3$	
Identify the point.	The given point is $(-2,1)$.	$\underbrace{(-2,1)}_{(x_1,y_1)}$
Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.	Simplify.	$y-y_1=m(x-x_1)$ $y-1=-\frac{1}{2}(x-(-2))$ $y-1=-\frac{1}{2}(x+2)$ $y-1=-\frac{1}{2}x-1$
Write the equation in slope-intercept form.		$y=-\frac{1}{2}x$
Answer the question.	The equation of the line is $y=-\frac{1}{2}x$.	

Try It 1.3.23

Find an equation of a line perpendicular to the line $y=3x+1$ that contains the point $(4,2)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=-\frac{1}{3}x+\frac{10}{3}$.

Try It 1.3.24

Find an equation of a line perpendicular to the line $y=\frac{1}{2}x-3$ that contains the point $(6,4)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=-2x+16$.

Find an equation of a line perpendicular to a given line

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
5. Write the equation in slope-intercept form.

Example 1.3.25

Find an equation of a line perpendicular to $x=5$ that contains the point $(3,-2)$. Write the equation in slope-intercept form, or in the form $x=a$.

Solution

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x=5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_{\perp}=0$.

Identify the point.	$(3, -2)$
Identify the slope of the perpendicular line.	$m_{\perp} = 0$
Substitute the values into $y - y_1 = m(x - x_1)$.	$y - (-2) = 0(x - 3)$
Simplify.	$y + 2 = 0$
Write the equation in slope-intercept form.	$y = -2$
Answer the question.	The equation of the line is $y = -2$.

Sketch the graph of both lines. On the graph, do the lines appear to be perpendicular?

? Try It 1.3.26

Find an equation of a line that is perpendicular to the line $x = 4$ that contains the point $(4, -5)$. Write the equation in slope-intercept form, or in the form $x = a$.

Answer

The equation of the line is $y = -5$.

? Try It 1.3.27

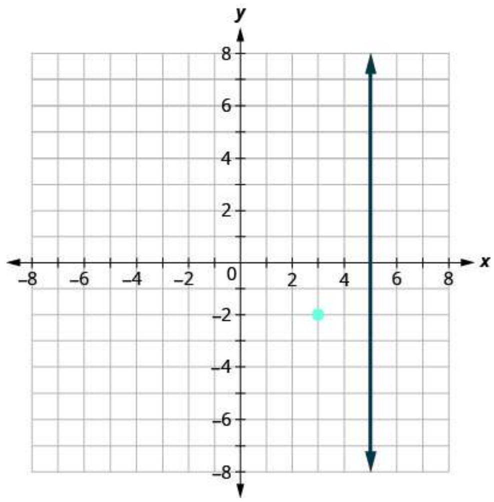
Find an equation of a line that is perpendicular to the line $x = 2$ that contains the point $(2, -1)$. Write the equation in slope-intercept form, or in the form $x = a$.

Answer

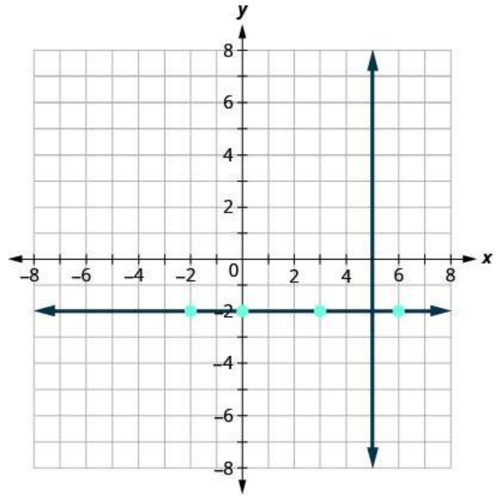
The equation of the line is $y = -1$.

In Example 1.3.25, we used the point-slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x = 5$ that contains the point $(3, -2)$. This graph shows us the line $x = 5$ and the point $(3, -2)$.



We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3, -2)$.



Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have y-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x=5$ is $y=-2$.

Example 1.3.28

Find an equation of a line that is perpendicular to $y=-3$ that contains the point $(-3,5)$. Write the equation in slope-intercept form, or in the form $x=a$.

Solution

The line $y=-3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x=a$. Since the perpendicular line is vertical and passes through $(-3,5)$, every point on it has an x-coordinate of -3. The equation of the perpendicular line is $x=-3$.

We may want to sketch the lines. Do they appear perpendicular?

Try It 1.3.29

Find an equation of a line that is perpendicular to the line $y=1$ that contains the point $(-5,1)$. Write the equation in slope-intercept form, or in the form $x=a$.

Answer

The equation of the line is $x=-5$.

Try It 1.3.30

Find an equation of a line that is perpendicular to the line $y=-5$ that contains the point $(-4,-5)$. Write the equation in slope-intercept form, or in the form $x=a$.

Answer

The equation of the line is $x=-4$.

Key Concepts

- **How to find an equation of a line given the slope and a point.**
  1. Identify the slope.
  2. Identify the point.

3. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
4. Write the equation in slope-intercept form.

- **How to find an equation of a line given two points.**

1. Find the slope using the given points. $m = \dfrac{y_2 - y_1}{x_2 - x_1}$
2. Choose one point.
3. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
4. Write the equation in slope-intercept form.

To Write an Equation of a Line

If given:	**Use:**	**Form:**
Slope and y-intercept	**slope-intercept**	$y = mx + b$
Slope and a point	**point-slope**	$y - y_1 = m(x - x_1)$
Two points	**point-slope**	$y - y_1 = m(x - x_1)$

- **How to find an equation of a line parallel to a given line.**

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point-slope form: $y - y_1 = m(x - x_1)$.
5. Write the equation in slope-intercept form

- **How to find an equation of a line perpendicular to a given line.**

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.
5. Write the equation in slope-intercept form.

Glossary

point-slope form

The point-slope form of an equation of a line with slope *m* and containing the point (x_1, y_1) is $y - y_1 = m(x - x_1)$.

Practice Makes Perfect

In the below, when we write "Write the equation in slope-intercept form" we mean "Write the equation in slope-intercept form or in the form $x = a$."

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and *y*-intercept. Write the equation in slope-intercept form.

1. slope 3 and y-intercept $(0, 5)$

Answer

$y = 3x + 5$

2. slope 8 and y-intercept $(0, -6)$

3. slope -3 and y-intercept $(0, -1)$

Answer

$y = -3x - 1$

4. slope -1 and y-intercept $(0, 3)$

5. slope $\frac{1}{5}$ and y-intercept $(0, -5)$

Answer

$y = \frac{1}{5}x - 5$

6. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

7. slope 0 and y-intercept $(0, -1)$

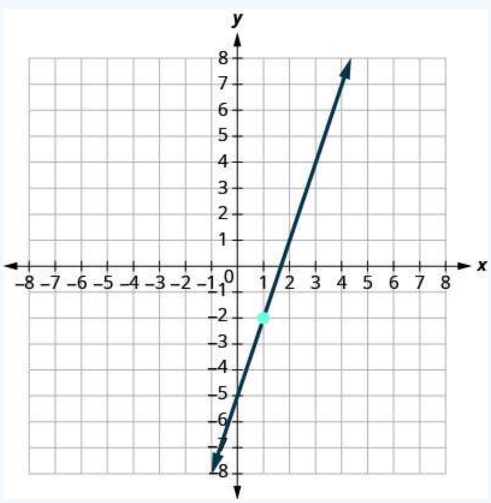
Answer

$y = -1$

8. slope -4 and y-intercept $(0, 0)$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope-intercept form.

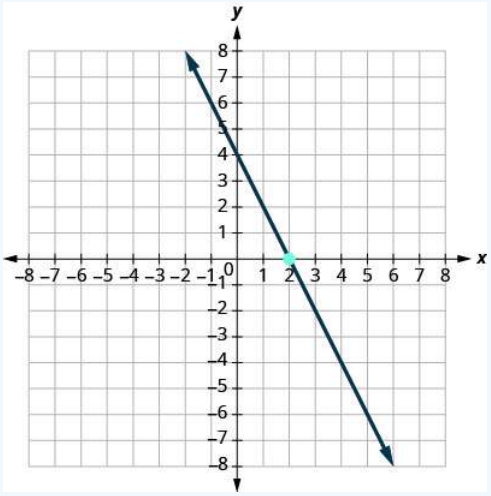
9.



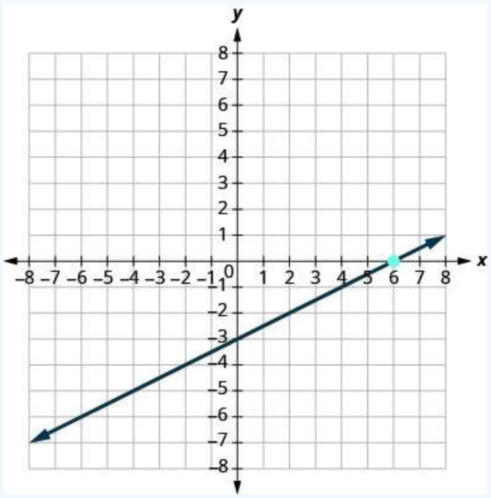
Answer

$y = 3x - 5$

10.



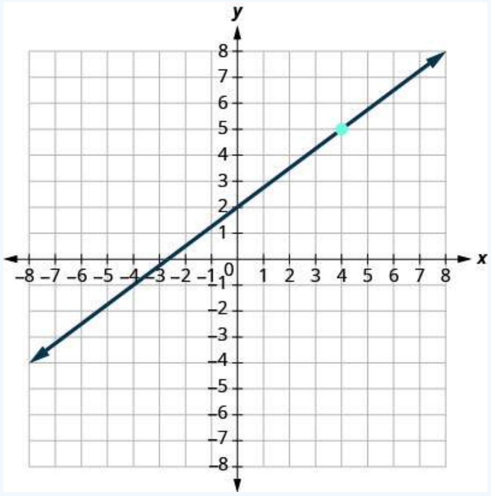
11.



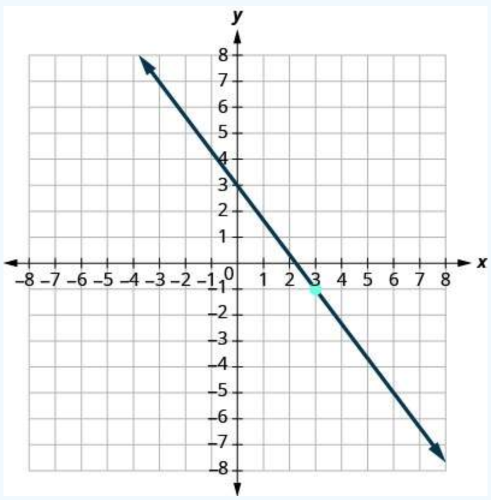
Answer

$$y=\frac{1}{2}x-3$$

12.



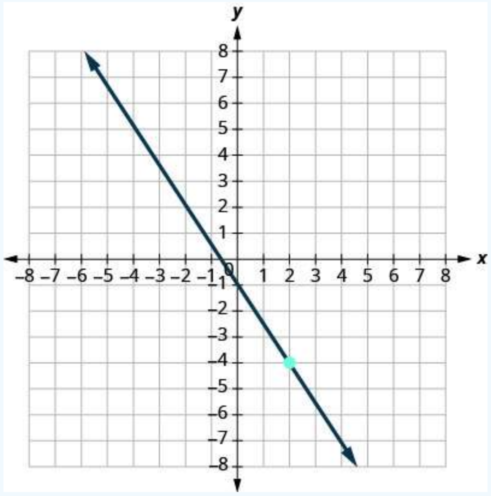
13.



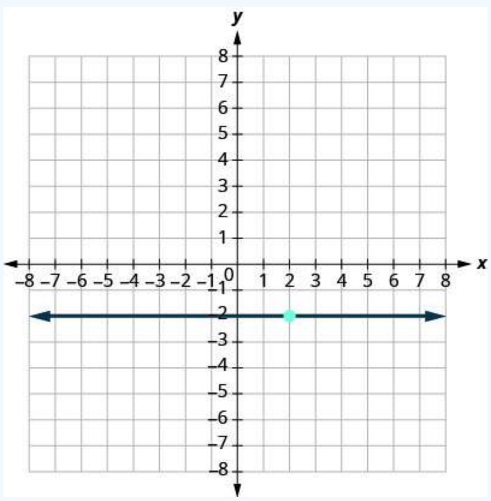
Answer

$$y=-\frac{4}{3}x+3$$

14.



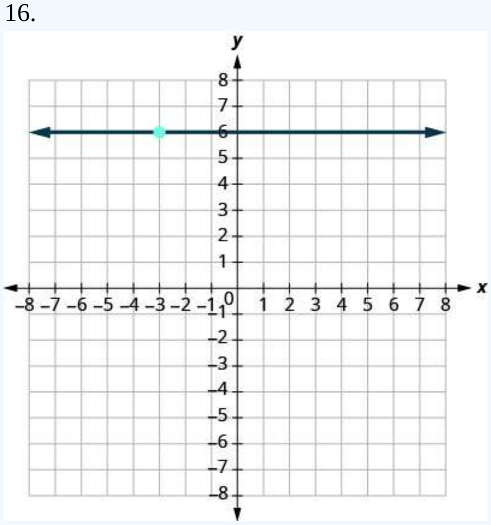
15.



Answer

$y = -2$

16.



Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope-intercept form.

17. $m=\frac{5}{8}$, point $(8,3)$

Answer

$y=\frac{5}{8}x-2$

18. $m=\frac{5}{6}$, point $(6,7)$

19. $m=-\frac{3}{5}$, point $(10,-5)$

Answer

$y=-\frac{3}{5}x+1$

20. $m=-\frac{3}{4}$, point $(8,-5)$

21. $m=-\frac{3}{2}$, point $(-4,-3)$

Answer

$y=-\frac{3}{2}x+9$

22. $m=-\frac{5}{2}$, point $(-8,-2)$

23. $m=-7$, point $(-1,-3)$

Answer

$y=-7x-10$

24. $m=-4$, point $(-2,-3)$

25. Horizontal line containing $(-2,5)$

Answer

$y=5$

26. Horizontal line containing $(-2,-3)$

27. Horizontal line containing $(-1,-7)$

Answer

$y=-7$

28. Horizontal line containing $(4,-8)$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope-intercept form.

29. $(2,6)$ and $(5,3)$

Answer

$y=-x+8$

30. $(4,3)$ and $(8,1)$

31. $(-3,-4)$ and $(5-2)$.

Answer

$y=\frac{1}{4}x-\frac{13}{4}$

32. $(-5,-3)$ and $(4,-6)$.

33. $(-1,3)$ and $(-6,-7)$.

Answer

$y=2x+5$

34. $(-2,8)$ and $(-4,-6)$.

35. $(0,4)$ and $(2,-3)$.

Answer

$y=-\frac{7}{2}x+4$

36. $(0,-2)$ and $(-5,-3)$.

37. $(7,2)$ and $(7,-2)$.

Answer

$x=7$

38. $(-2,1)$ and $(-2,-4)$.

39. $(3,-4)$ and $(5,-4)$.

Answer

$y=-4$

40. $(-6,-3)$ and $(-1,-3)$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope-intercept form.

41. line $y=4x+2$, point $(1,2)$

Answer

$y=4x-2$

42. line $y=-3x-1$, point $2,-3)$.

43. line $2x-y=6$, point $(3,0)$.

Answer

$y=2x-6$

44. line $2x+3y=6$, point $(0,5)$.

45. line $x=-4$, point $(-3,-5)$.

Answer

$x=-3$

46. line $x-2=0$, point $(1,-2)$

47. line $y=5$, point $(2,-2)$

Answer

$y=-2$

48. line $y+2=0$, point $(3,-3)$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope-intercept form.

49. line $y=-2x+3$, point $(2,2)$

Answer

$y=\frac{1}{2}x+1$

50. line $y=-x+5$, point $(3,3)$

51. line $y=\frac{3}{4}x-2$, point $(-3,4)$

Answer

$y=-\frac{4}{3}x$

52. line $y=\frac{2}{3}x-4$, point $(2,-4)$

53. line $2x-3y=8$, point $(4,-1)$

Answer

$$y=-\frac{3}{2}x+5$$

54. line $4x-3y=5$, point $(-3,2)$

55. line $2x+5y=6$, point $(0,0)$

Answer

$$y=\frac{5}{2}x$$

56. line $4x+5y=-3$, point $(0,0)$

57. line $x=3$, point $(3,4)$

Answer

$$y=4$$

58. line $x=-5$, point $(1,-2)$

59. line $x=7$, point $(-3,-4)$

Answer

$$y=-4$$

60. line $x=-1$, point $(-4,0)$

61. line $y-3=0$, point $(-2,-4)$

Answer

$$x=-2$$

62. line $y-6=0$, point $(-5,-3)$

63. line y-axis, point $(3,4)$

Answer

$$y=4$$

64. line y-axis, point $(2,1)$

Mixed Practice

In the following exercises, find the equation of each line. Write the equation in slope-intercept form.

65. Containing the points $(4,3)$ and $(8,1)$

Answer

$y=-\frac{1}{2}x+5$

66. Containing the points $(-2,0)$ and $(-3,-2)$

67. $m=\frac{1}{6}$, containing point $(6,1)$

Answer

$y=\frac{1}{6}x$

68. $m=\frac{5}{6}$, containing point $(6,7)$

69. Parallel to the line $4x+3y=6$, containing point $(0,-3)$

Answer

$y=-\frac{4}{3}x-3$

70. Parallel to the line $2x+3y=6$, containing point $(0,5)$

71. $m=-\frac{3}{4}$, containing point $(8,-5)$

Answer

$y=-\frac{3}{4}x+1$

72. $m=-\frac{3}{5}$, containing point $(10,-5)$

73. Perpendicular to the line $y-1=0$, point $(-2,6)$

Answer

$x=-2$

74. Perpendicular to the line *y*-axis, point $(-6,2)$

75. Parallel to the line $x=-3$, containing point $(-2,-1)$

Answer

$x=-2$

76. Parallel to the line $x=-4$, containing point $(-3,-5)$

77. Containing the points $(-3,-4)$ and $(2,-5)$

Answer

$$y=-\frac{1}{5}x-\frac{23}{5}$$

78. Containing the points $(-5,-3)$ and $(4,-6)$

79. Perpendicular to the line $x-2y=5$, point $(-2,2)$

Answer

$$y=-2x-2$$

80. Perpendicular to the line $4x+3y=1$, point $(0,0)$

Writing Exercises

81. Why are all horizontal lines parallel?

Answer

Answers will vary.

82. Explain in your own words why the slopes of two perpendicular lines must have opposite signs.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the equation of the line given the slope and *y*-intercept.			
find an equation of the line given the slope and a point.			
find an equation of the line given two points.			
find an equation of a line parallel to a given line.			
find an equation of a line perpendicular to a given line.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

1.4: Chapter 1 Review Exercises

Chapter Review Exercises

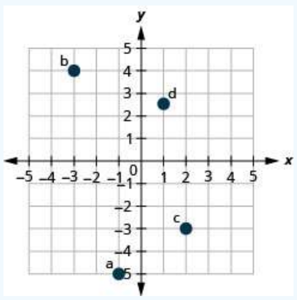
Graph Linear Equations in Two Variables

Plot Points in a Rectangular Coordinate System

In the following exercises, plot each point in a rectangular coordinate system.

1. ⓐ $(-1,-5)$
ⓑ $(-3,4)$
ⓒ $(2,-3)$
ⓓ $(1,\frac{5}{2})$

Answer



2. ⓐ $(-2,0)$
ⓑ $(0,-4)$
ⓒ $(0,5)$
ⓓ $(3,0)$

In the following exercises, determine which ordered pairs are solutions to the given equations.

3. $5x+y=10$;

ⓐ $(5,1)$
ⓑ $(2,0)$
ⓒ $(4,-10)$

Answer

ⓑ, ⓒ

4. $y=6x-2$;

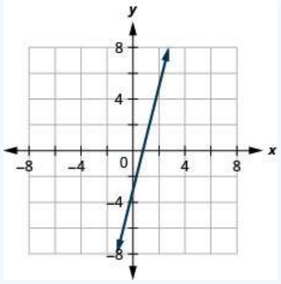
ⓐ $(1,4)$
ⓑ $(13,0)$
ⓒ $(6,-2)$

Graph a Linear Equation by Plotting Points

In the following exercises, graph by plotting points.

5. $y=4x-3$

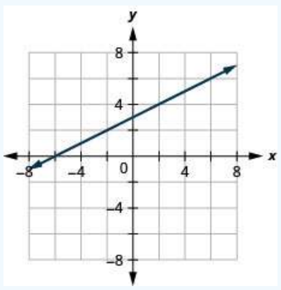
Answer



6. $y=-3x$

7. $y=\frac{1}{2}x+3$

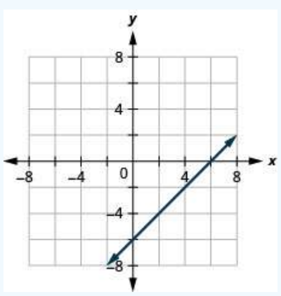
Answer



8. $y=-\frac{4}{5}|x-1$

9. $x-y=6$

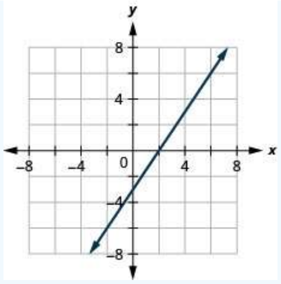
Answer



10. $2x+y=7$

11. $3x-2y=6$

Answer



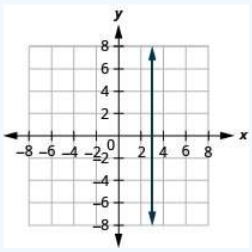
Graph Vertical and Horizontal lines

In the following exercises, graph each equation.

12. $y=-2$

13. $x=3$

Answer

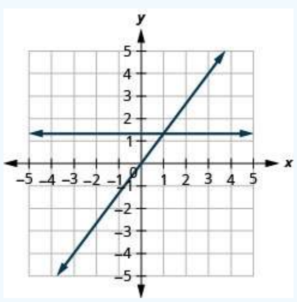


In the following exercises, graph each pair of equations in the same rectangular coordinate system.

14. $y=-2x$ and $y=-2$

15. $y=\frac{4}{3}x$ and $y=\frac{4}{3}$

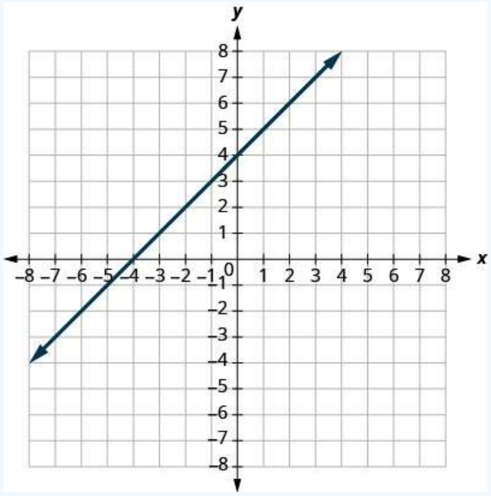
Answer



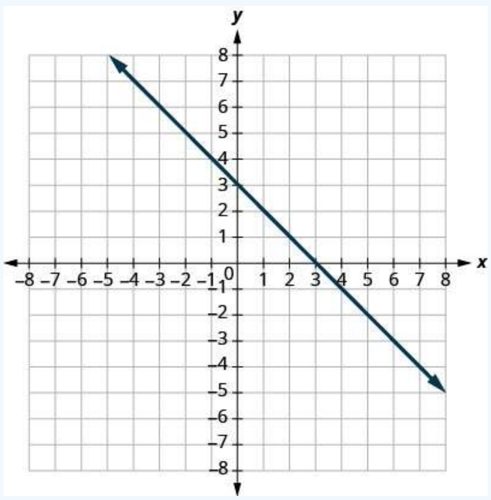
Find *x*- and *y*-Intercepts

In the following exercises, find the *x*- and *y*-intercepts.

16.



17.



Answer

$(0,3)(3,0)$

In the following exercises, find the intercepts of each equation.

18. $x-y=-1$

19. $x+2y=6$

Answer

$(6,0),\ (0,3)$

20. $2x+3y=12$

21. $y=\frac{3}{4}x-12$

Answer

$(16,0),\ (0,-12)$

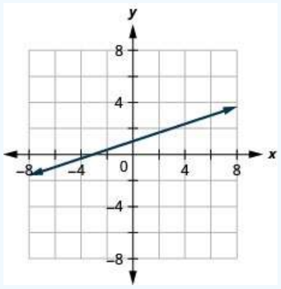
22. $y=3x$

Graph a Line Using the Intercepts

In the following exercises, graph using the intercepts.

23. $-x+3y=3$

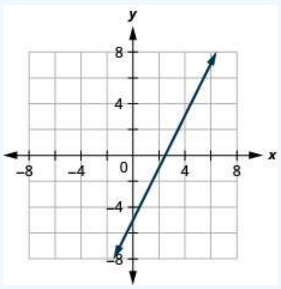
Answer



24. $x-y=4$

25. $2x-y=5$

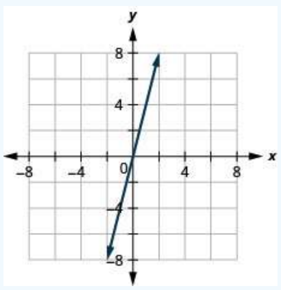
Answer



26. $2x-4y=8$

27. $y=4x$

Answer

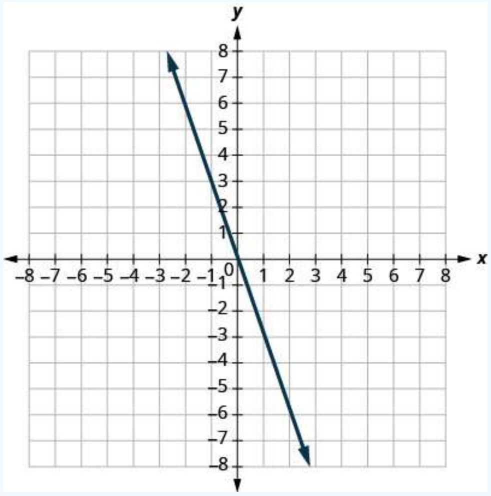


[Slope of a Line](https://math.libretexts.org)

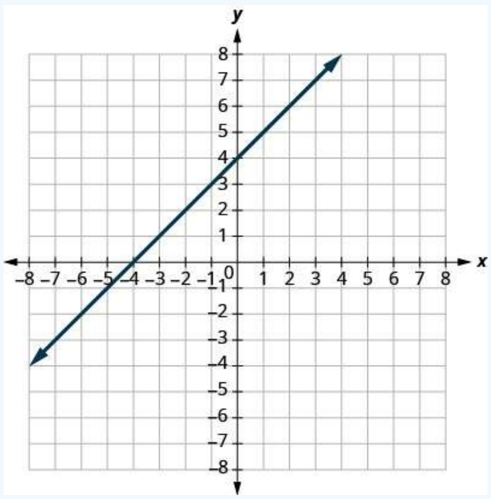
Find the Slope of a Line

In the following exercises, find the slope of each line shown.

28.



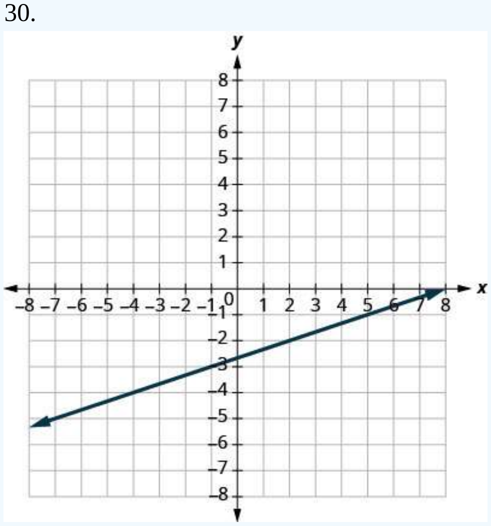
29.



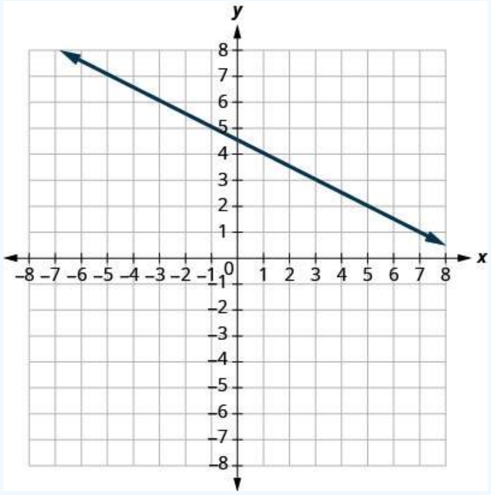
Answer

1

30.



31.



Answer

-12

In the following exercises, find the slope of each line.

32. $y = 2$

33. $x = 5$

Answer

undefined

34. $x = -3$

35. $y = -1$

Answer

0

Use the Slope Formula to find the Slope of a Line between Two Points

In the following exercises, use the slope formula to find the slope of the line between each pair of points.

36. $(-1, -1), (0, 5)$

37. $(3.5), (4, -1)$

Answer

-6

38. $(-5, -2), (3, 2)$

39. $(2, 1), (4, 6)$

Answer

52

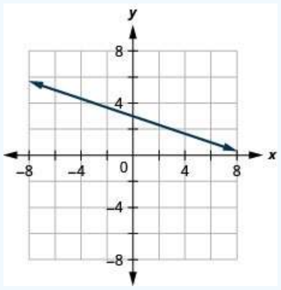
Graph a Line Given a Point and the Slope

In the following exercises, graph each line with the given point and slope.

40. $(2,-2);\ m=52$

41. $(-3,4);\ m=-13$

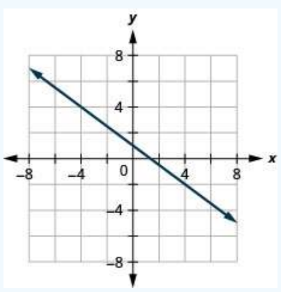
Answer



42. x-intercept $-4; m=3$

43. y-intercept $1; m=-34$

Answer



Graph a Line Using Its Slope and Intercept

In the following exercises, identify the slope and y-intercept of each line.

44. $y=-4x+9$

45. $y=53x-6$

Answer

$m=53;\ (0,-6)$

46. $5x+y=10$

47. $4x-5y=8$

Answer

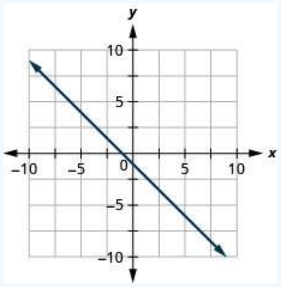
$m=45;\ (0,-85)$

In the following exercises, graph the line of each equation using its slope and *y*-intercept.

48. $y=2x+3$

49. $y=-x-1$

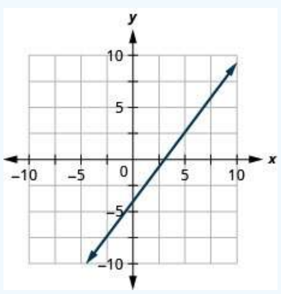
Answer



50. $y=-25x+3$

51. $4x-3y=12$

Answer



In the following exercises, determine the most convenient method to graph each line.

52. $x=5$

53. $y=-3$

Answer

horizontal line

54. $2x+y=5$

55. $x-y=2$

Answer

intercepts

56. $y=22x+2$

57. $y=34x-1$

Answer

plotting points

Graph and Interpret Applications of Slope-Intercept

58. Katherine is a private chef. The equation $C = 6.5m + 42$ models the relation between her weekly cost, *C*, in dollars and the number of meals, *m*, that she serves.

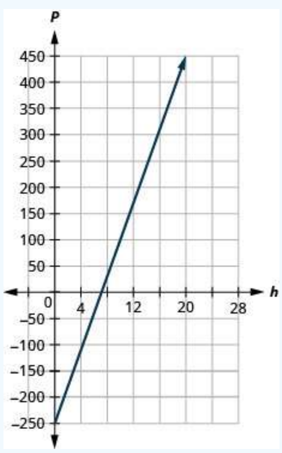
ⓐ Find Katherine's cost for a week when she serves no meals.
ⓑ Find the cost for a week when she serves 14 meals.
ⓒ Interpret the slope and *C*-intercept of the equation.
ⓓ Graph the equation.

59. Marjorie teaches piano. The equation $P = 35h - 250$ models the relation between her weekly profit, *P*, in dollars and the number of student lessons, *s*, that she teaches.

ⓐ Find Marjorie's profit for a week when she teaches no student lessons.
ⓑ Find the profit for a week when she teaches 20 student lessons.
ⓒ Interpret the slope and *P*-intercept of the equation.
ⓓ Graph the equation.

Answer

ⓐ $-\$250$
ⓑ $\$450$
ⓒ The slope, 35, means that Marjorie's weekly profit, *P*, increases by $35 for each additional student lesson she teaches. The *P*-intercept means that when the number of lessons is 0, Marjorie loses $250.
ⓓ



Use Slopes to Identify Parallel and Perpendicular Lines

In the following exercises, use slopes and y-intercepts to determine if the lines are parallel, perpendicular, or neither.

60. $4x - 3y = -1; \quad y = 43x - 3$

61. $y = 5x - 1; \quad 10x + 2y = 0$

Answer

neither

62. $3x - 2y = 5; \quad 2x + 3y = 6$

63. $2x - y = 8; \quad x - 2y = 4$

Answer

not parallel

Find the Equation of a Line

Find an Equation of the Line Given the Slope and *y*-Intercept

In the following exercises, find the equation of a line with given slope and y-intercept. Write the equation in slope–intercept form.

64. Slope $\frac{1}{3}$ and y-intercept $(0, -6)$

65. Slope -5 and y-intercept $(0, -3)$

Answer

$y = -5x - 3$

66. Slope 0 and y-intercept $(0, 4)$

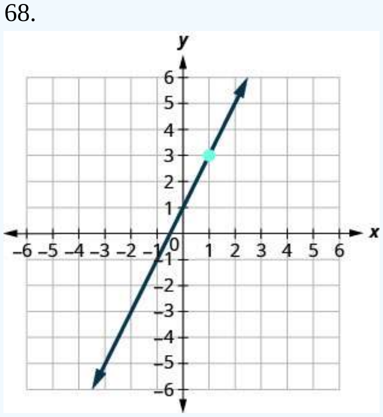
67. Slope -2 and y-intercept $(0, 0)$

Answer

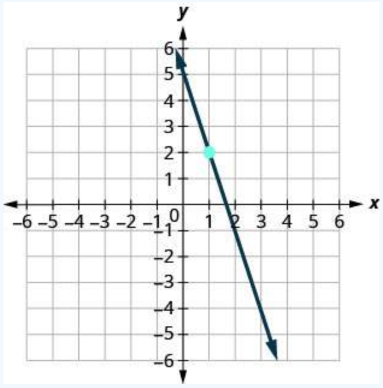
$y = -2x$

In the following exercises, find the equation of the line shown in each graph. Write the equation in slope–intercept form.

68.



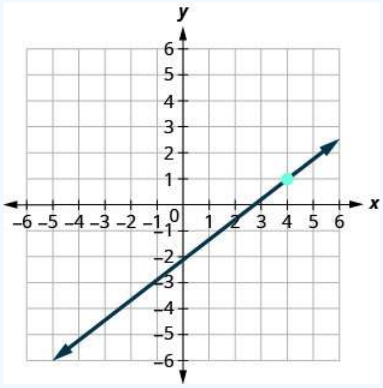
69.



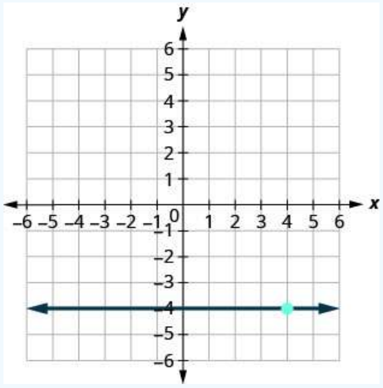
Answer

$y=-3x+5$

70.



71.



Answer

$y=-4$

Find an Equation of the Line Given the Slope and a Point

In the following exercises, find the equation of a line with given slope and containing the given point. Write the equation in slope–intercept form.

72. $m=-\frac{1}{4}$, point $(-8,3)$

73. $m = \frac{3}{5}$, point $(10, 6)$

Answer

$y = \frac{3}{5}x$

74. Horizontal line containing $(-2, 7)$

75. $m = -2$, point $(-1, -3)$

Answer

$y = -2x - 5$

Find an Equation of the Line Given Two Points

In the following exercises, find the equation of a line containing the given points. Write the equation in slope–intercept form.

76. $(2, 10)$ and $(-2, -2)$

77. $(7, 1)$ and $(5, 0)$

Answer

$y = \frac{1}{2}x - \frac{5}{2}$

78. $(3, 8)$ and $(3, -4)$

79. $(5, 2)$ and $(-1, 2)$

Answer

$y = 2$

Find an Equation of a Line Parallel to a Given Line

In the following exercises, find an equation of a line parallel to the given line and contains the given point. Write the equation in slope–intercept form.

80. line $y = -3x + 6$, point $(1, -5)$

81. line $2x + 5y = -10$, point $(10, 4)$

Answer

$y = -\frac{2}{5}x + 8$

82. line $x = 4$, point $(-2, -1)$

83. line $y = -5$, point $(-4, 3)$

Answer

$y = 3$

Find an Equation of a Line Perpendicular to a Given Line

In the following exercises, find an equation of a line perpendicular to the given line and contains the given point. Write the equation in slope–intercept form.

84. line $y=-\frac{4}{5}x+2$, point $(8,9)$

85. line $2x-3y=9$, point $(-4,0)$

Answer

$y=-\frac{3}{2}x-6$

86. line $y=3$, point $(-1,-3)$

87. line $x=-5$ point $(2,1)$

Answer

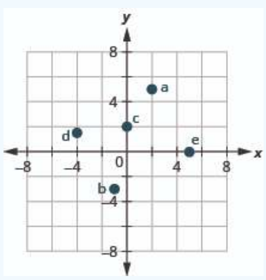
$y=1$

Practice Test

1. Plot each point in a rectangular coordinate system.

ⓐ $(2,5)$
ⓑ $(-1,-3)$
ⓒ $(0,2)$
ⓓ $(-4,32)$
ⓔ $(5,0)$

Answer

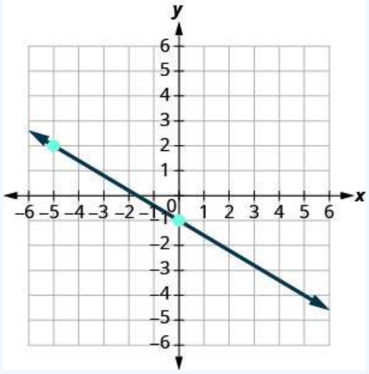


2. Which of the given ordered pairs are solutions to the equation $3x-y=6$?

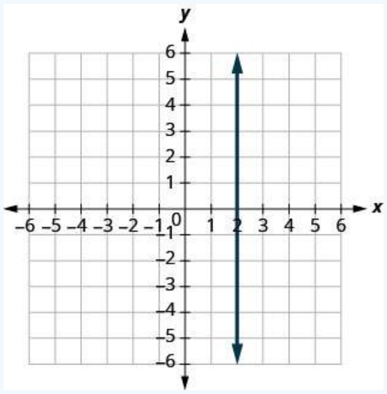
ⓐ $(3,3)$ ⓑ $(2,0)$ ⓒ $(4,-6)$

3. Find the slope of each line shown.

ⓐ



ⓑ



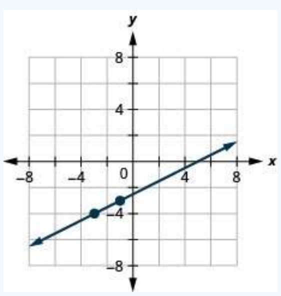
Answer

ⓐ $-\frac{3}{5}$ ⓑ undefined

4. Find the slope of the line between the points $(5, 2)$ and $(-1, -4)$.

5. Graph the line with slope $\frac{1}{2}$ containing the point $(-3, -4)$.

Answer

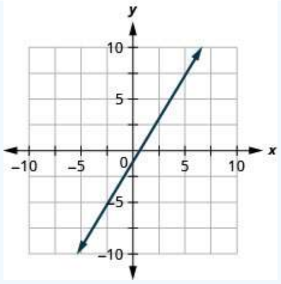


6. Find the intercepts of $4x + 2y = -8$ and graph.

Graph the line for each of the following equations.

7. $y = \frac{5}{3}x - 1$

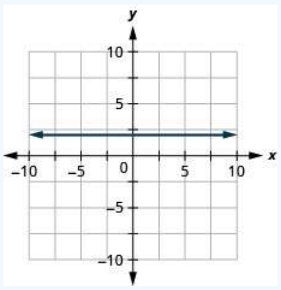
Answer



8. $y = -x$

9. $y = 2$

Answer



Find the equation of each line. Write the equation in slope-intercept form.

10. slope $-\frac{3}{4}$ and y-intercept $(0, -2)$

11. $m = 2$, point $(-3, -1)$

Answer

$y = 2x + 5$

12. containing $(10, 1)$ and $(6, -1)$

13. perpendicular to the line $y = \frac{5}{4}x + 2$, containing the point $(-10, 3)$

Answer

$y = -\frac{4}{5}x - 5$

CHAPTER OVERVIEW

2: Systems of Linear Equations

2.1: Solve Systems of Linear Equations with Two Variables
2.2: Solve Applications with Systems of Equations
2.3: Solve Mixture Applications with Systems of Equations
2.4: Solve Systems of Linear Equations with Three Variables
2.5: Solve Systems of Linear Equations Using Determinants
2.6: Chapter 2 Review Exercises

2.1: Solve Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Be Prepared

Before you get started, take this readiness quiz.

1. For the equation $y=\frac{2}{3}x-4$,

a. Is $(6,0)$ a solution?

b. Is $(-3,-2)$ a solution?

2. Find the slope and y-intercept of the line $3x-y=12$.
3. Find the x- and y-intercepts of the line $2x-3y=12$.
If you missed any of these problems, review [link].

Determine Whether an Ordered Pair is a Solution of a System of Equations

We have learned about solutions to linear equations with two variables. Now we will work with two or more linear equations grouped together, which is known as a *system of linear equations*.

Definition 2.1.1

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations with two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\left\{\begin{aligned} 2x+y&=7 \\ x-2y&=6 \end{aligned}\right.$$

A linear equation in two variables, such as $2x+y=7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (p,q) that when substituted in for (x,y) (p for x and q for y) make both equations true. These are called the *solutions of a system of equations*.

Definition 2.1.2

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (p,q). We say (x,y)=(p,q) is a solution to the system if when when we substitute p for x and q for y, both resulting equations are true.

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 2.1.3

Determine whether the ordered pair is a solution to the system $\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$.

a. $(-2,-1)$

b. $(-4,-3)$

Solution

a.

	$x-y=-1$	$2x-y=-5$
We substitute -2 for x and -1 for y in both equations.	$\begin{aligned} x-y&=-1 \\ -2-(-1)&\stackrel{?}{=}-1 \\ -1&\stackrel{?}{=}-1 \quad \text{True} \end{aligned}$	$\begin{aligned} 2x-y&=-5 \\ 2(-2)-(-1)&\stackrel{?}{=}-5 \\ -3&\stackrel{?}{=}-5 \quad \text{False} \end{aligned}$
Is $(-2,1)$ a solution to the equation?	Yes	No
Answer the question.	Since substituting in $(-2,-1)$ does not make both equations true, $(-2,-1)$ is not a solution.	

b.

	$x-y=-1$	$2x-y=-5$
We substitute -4 for x and -3 for y in both equations.	$\begin{aligned} x-y&=-1 \\ -4-(-3)&\stackrel{?}{=}-1 \\ -1&\stackrel{?}{=}-1 \quad \text{True} \end{aligned}$	$\begin{aligned} 2x-y&=-5 \\ 2(-4)-(-3)&\stackrel{?}{=}-5 \\ -5&\stackrel{?}{=}-5 \quad \text{True} \end{aligned}$
Is $(-4,-3)$ a solution to the equation?	Yes	Yes
Answer the question.	Since substituting in $(-4,-3)$ does make both equations true, $(-4,-3)$ is not a solution.	

Try It 2.1.4

Determine whether the ordered pair is a solution to the system $\begin{cases} 3x+y=0 \\ x+2y=-5 \end{cases}$.

a. $(1,-3)$

b. $(0,0)$

Answer

a. It is a solution.

b. It is not a solution.

Try It 2.1.5

Determine whether the ordered pair is a solution to the system $\begin{cases} x-3y=-8 \\ -3x-y=4 \end{cases}$.

a. $(2,-2)$

b. $(-2,2)$

Answer

a. It is not a solution.

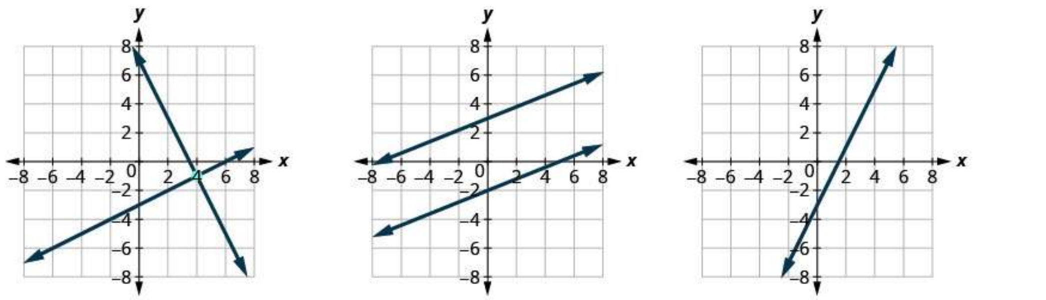
b. It is a solution.

Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we will use is graphing.

The graph of a linear equation is a line. With the representation of ordered pairs of numbers on the coordinate plane, we will call these ordered pairs points and imagine them on the coordinate plane. So, we will say, for example, that each point on the line *is* a solution to the equation instead of that each point on the line *represents* a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we will find the solution to the system.

When we solve a system of two linear equations represented by a graph of two lines in the same plane. There are three possible cases, as shown.



The lines intersect. Intersecting lines have one point in common. There is one solution to this system.

The lines are parallel. Parallel lines have no points in common. There is no solution to this system.

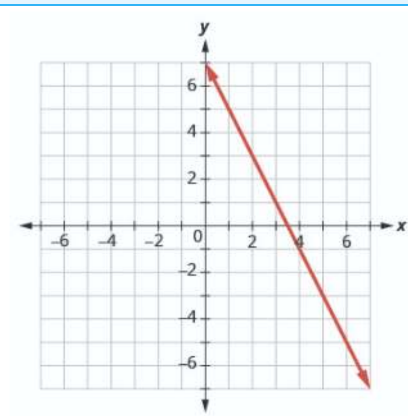
Both equations give the same line. Because we have just one line, there are infinitely many solutions.

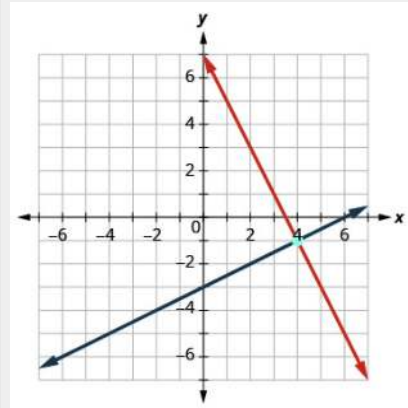
Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section we will decide which method was the most convenient way to solve this system.

Example 2.1.6

Solve the system $\begin{cases} 2x+y &= 7 \\ x-2y &= 6 \end{cases}$ by graphing.

Solution

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Graph the first equation.	To graph the first line, write the equation in slope-intercept form. $2x+y=7$ $y=-2x+7$ The slope is $m=-2$. The y-intercept is $(0,7)$, so $b=7$.	

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Graph the second equation on the same rectangular coordinate system.	To graph the second line, $x-2y=6$ use intercepts. The x-intercept is $(6,0)$. The y-intercept is $(0,-3)$.	
Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4,-1)$. $2x+y=7$ $x-2y=6$ $2(4)+(-1)\stackrel{?}{=}7$ $4-2(-1)\stackrel{?}{=}6$ $7\stackrel{?}{=}7$ $6\stackrel{?}{=}6$ True True Substituting $(4,-1)$ in for (x,y) makes both equations true.
Answer the question.		The solution is $(4,-1)$.

? Try It 2.1.7

Solve the system $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$ by graphing.

Answer

The solution is $(3,2)$.

? Try It 2.1.8

Solve the system $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$ by graphing.

Answer

The solution is $(2,3)$.

The steps to use to solve a system of linear equations by graphing are shown here.

Solve a system of linear equations by graphing

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.

4. Identify the solution to the system.
   - If the lines intersect, identify the point of intersection. This is the solution to the system.
   - If the lines are parallel, the system has no solution.
   - If the lines are the same, the system has an infinite number of solutions.
5. Check the solution in both equations.

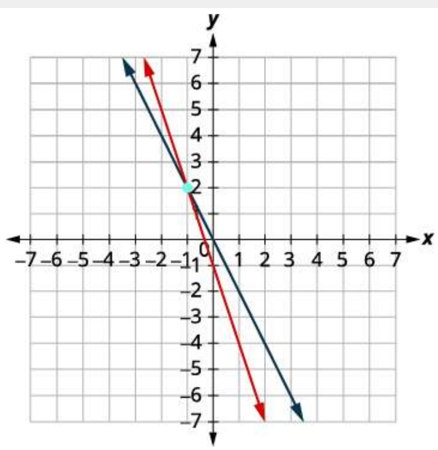
In the next example, we will first rewrite the equations into slope-intercept form as this will make it easy for us to quickly graph the lines.

Example 2.1.9

Solve the system $\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$ by graphing.

Solution

We will solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts.

	$\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$
Solve the first equation for y.	$3x+y=-1$ $y=-3x-1$
Find the slope and y-intercept.	The slope is $m=-3$. The y-intercept is $(0,-1)$.
Solve the second equation for y.	$2x+y=0$ $y=-2x$
Find the slope and y-intercept.	The slope is $m=-2$. The y-intercept is $(0,0)$.
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1,2)$.
Check the solution in both equations.	$3x+y=-1$ $2x+y=0$ $3(-1)+2\stackrel{?}{=}-1$ $2(-1)+2\stackrel{?}{=}0$ $-1\stackrel{?}{=}-1$ $0\stackrel{?}{=}0$ True True Substituting $(-1,2)$ in for (x,y) makes both equations true.
Answer the question.	The solution is $(-1,2)$.

? Try It 2.1.10

Solve the system $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$ by graphing.

Answer

The solution is $(3,4)$.

? Try It 2.1.11

Solve the system $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$ by graphing.

Answer

The solution is $(5,-4)$.

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

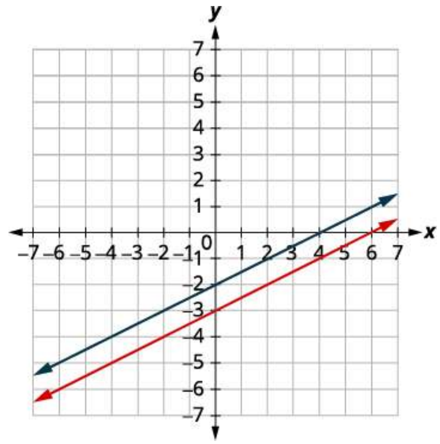
? Example 2.1.12

Solve the system $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$ by graphing.

Solution

	$\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$
To graph the first equation, we will use its slope and y-intercept.	$y=\frac{1}{2}x-3$ The slope is $m=\frac{1}{2}$. The y-intercept is $(0,-3)$.
To graph the second equation, we will use the intercepts.	$x-2y=4$
Find the x- and y-intercepts of the second equation.	see table below The x-intercept is $(4,0)$. The y-intercept is $(0,-2)$.

x	y	(x,y)
0	-2	$(0,-2)$
4	0	$(4,0)$

Graph the lines.	
Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true upon substitution.
Answer the question.	There is no solution to the system.

Try It 2.1.13

Solve the system $\begin{cases} y = -\frac{1}{4}x + 2 \\ x + 4y = 4 \end{cases}$ by graphing.

Answer

The system has no solution.

Try It 2.1.14

Solve the system $\begin{cases} y = 3x - 1 \\ 6x - 2y = 6 \end{cases}$ by graphing.

Answer

The system has no solution.

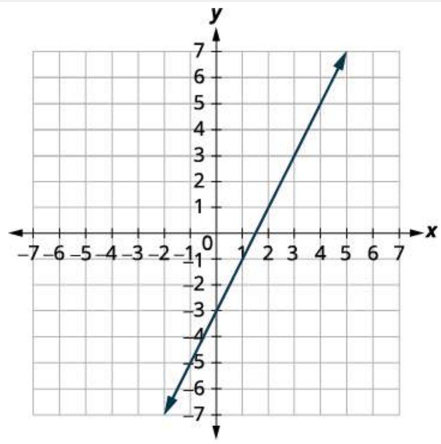
Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that, when substituted, make both equations true. Therefore there are infinitely many solutions to the system.

Example 2.1.15

Solve the system $\begin{cases} y = 2x - 3 \\ -6x + 3y = 9 \end{cases}$ by graphing.

Solution

	$\begin{cases} y = 2x - 3 \\ -6x + 3y = 9 \end{cases}$

Find the slope and y-intercept of the first equation.	$y=2x-3$ The slope is $m=2$. The y-intercept is $(0,-3)$.
To graph the second equation, we will use the intercepts.	$-6x+3y=9$
Find the x- and y-intercepts of the second equation.	x / y / (x,y): 0, -3, $(0,-3)$; $\frac{3}{2}$, 0, $\left(\frac{3}{2},0\right)$ The x-intercept is $\left(\frac{3}{2},0\right)$. The y-intercept is $(0,-3)$.
Graph the lines.	
Determine the points of intersection.	The lines are the same! Any point on the line is a point of intersection. Since every point on the line, when substituted, makes both equations true, there are infinitely many ordered pairs that make both equations true.
Answer the question.	There are infinitely many solutions to the system.

x	y	(x,y)
0	-3	$(0,-3)$
$\frac{3}{2}$	0	$\left(\frac{3}{2},0\right)$

If we write the second equation in slope-intercept form, we may recognize that the equations have the same slope and same y-intercept.

? Try It 2.1.16

Solve the system $\begin{cases} y=-3x-6 \\ 6x+2y=-12 \end{cases}$ by graphing.

Answer

The system has infinitely many solutions.

? Try It 2.1.17

Solve the system $\begin{cases} y=\frac{1}{2}x-4 \\ 2x-4y=16 \end{cases}$ by graphing.

Answer

The system has infinitely many solutions.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are *coincident*. Coincident lines have the same slope and same y-intercept.

Definition 2.1.18

Coincident lines have the same slope and same y-intercept.

The systems of equations in Example and Example each had two intersecting lines. Each system had one solution.

In Example, the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like Example, has no solution. We call a system of equations like this *inconsistent.* It has no solution.

Definition 2.1.19

A **consistent system of equations** is a system of equations with at least one solution.

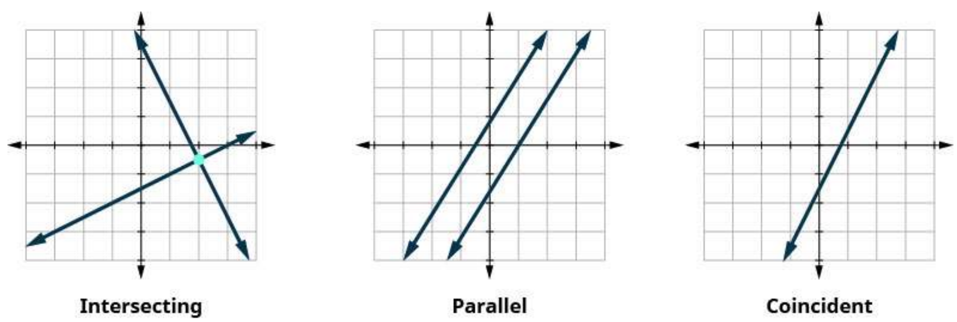
An **inconsistent system of equations** is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines. Otherwise, they each have their own different set of solutions and we say the two equations are *independent*. Intersecting lines and parallel lines are independent.

Definition 2.1.20

Two equations are **dependent** if they have the same set of solutions. Two equations are **independent** if their solutions differ.

Let's sum this up by looking at the graphs of the three types of systems. See below.



Lines	Intersecting	Parallel	Coincident
Number of solutions	One point	No solution	Infinitely many
Consistent/Inconsistent	Consistent	Inconsistent	Consistent
Dependent/Independent	Independent	Independent	Dependent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the *substitution method*.

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable - but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it is a solution to both equations.

Example 2.1.21

Solve the system $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ by substitution.

Solution

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Solve one of the equations for either variable.	We will solve the first equation for y.	$\begin{aligned} 2x+y&=7 \\ y&=7-2x \end{aligned}$
Substitute the expression from the previous step into the other equation.	We replace y in the second equation with the expression $7-2x$.	$\begin{aligned} x-2y&=6 \\ x-2(7-2x)&=6 \end{aligned}$
Solve the resulting equation.	Now we have an equation with just one variable. We know how to solve this.	$\begin{aligned} x-14+4x&=6 \\ 5x&=20 \\ x&=4 \end{aligned}$
Substitute the solution just found into one of the original equations to find the other variable.	We will use the first equation and replace x with 4.	$\begin{aligned} 2x+y&=7 \\ 2\cdot4+y&=7 \\ 8+y&=7 \\ y&=-1 \end{aligned}$
Write the solution as an ordered pair.	Write $x=4$ and $y=-1$ as an ordered pair.	$(4,-1)$
Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$ and $y=-1$ into both equations and make sure they are both true.	$\begin{aligned} 2x+y&=7 \\ 2(4)+(-1)&\stackrel{?}{=}7 \\ 7&\stackrel{?}{=}7 \\ &\text{True} \end{aligned}$ $\begin{aligned} x-2y&=6 \\ 4-2(-1)&\stackrel{?}{=}6 \\ 6&\stackrel{?}{=}6 \\ &\text{True} \end{aligned}$ Substituting $(4,-1)$ in for (x,y) makes both equations true.
Answer the question.		$(4,-1)$ is a solution to the system.

Try It 2.1.22

Solve the system $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$ by substitution.

Answer

The solution is $(6,1)$.

Try It 2.1.23

Solve the system $\begin{cases} 2x+y=-1 \\ 4x+3y=3 \end{cases}$ by substitution.

Answer

The solution is $(-3,5)$.

Solve a system of equations by substitution

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to **both** original equations. This step is added here in order to check for errors.

Be very careful with the signs in the next example.

Example 2.1.24

Solve the system $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$ by substitution.

Solution

We need to solve one equation for one variable. We will solve the first equation for y.

	$\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$
Solve the first equation for y.	$\begin{aligned} 4x+2y&=4 \\ 2y&=-4x+4 \\ y&=-2x+2 \end{aligned}$
Replace the y with $-2x+2$ in the second equation.	$\begin{aligned} 6x-y&=8 \\ 6x-(-2x+2)&=8 \end{aligned}$
Solve the equation for x.	$\begin{aligned} 6x+2x-2&=8 \\ 8x&=10 \\ x&=\frac{5}{4} \end{aligned}$
Substitute $x=\frac{5}{4}$ into $4x+2y=4$ to find y.	$\begin{aligned} 4x+2y&=4 \\ 4\cdot\frac{5}{4}+2y&=4 \\ 5+2y&=4 \\ 2y&=-1 \\ y&=-\frac{1}{2} \end{aligned}$
Write the solution as an ordered pair.	The ordered pair is $\left(\frac{5}{4},-\frac{1}{2}\right)$.

	$\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$
Check the ordered pair in both equations.	$\begin{aligned} 4x+2y&=4 \\ 4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)&\stackrel{?}{=}4 \\ 5-1&\stackrel{?}{=}4 \\ 4&\stackrel{?}{=}4 \\ &\text{True} \end{aligned}$ $\begin{aligned} 6x-y&=8 \\ 6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)&\stackrel{?}{=}8 \\ \frac{15}{4}+\frac{1}{2}&\stackrel{?}{=}8 \\ \frac{16}{2}&\stackrel{?}{=}8 \\ 4&\stackrel{?}{=}4 \\ &\text{True} \end{aligned}$ Substituting $\left(\frac{5}{4},-\frac{1}{2}\right)$ in for (x,y) makes both equations true.
Answer the question.	$\left(\frac{5}{4},-\frac{1}{2}\right)$ is a solution to the system.

Try It 2.1.25

Solve the system $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$ by substitution.

Answer

The solution is $\left(2,\frac{3}{2}\right)$.

Try It 2.1.26

Solve the system $\begin{cases} 4x-y=0 \\ 2x-3y=5 \end{cases}$ by substitution.

Answer

The solution is $\left(-\frac{1}{2},-2\right)$.

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the *Elimination Method*. When we solved a system by substitution, we started with two equations with two variables and reduced it to one equation with one variable. This is what we will do with the elimination method, too, but we will get there a different way.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when we add the same quantity to both sides of an equation, we still have equality. We will extend the Addition Property of Equality to say that when we add equal quantities to both sides of an equation, the results are equal. In other words, for any expressions a, b, c, and d,

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d. \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{cases} 3x+y=5 \\ \underline{2x-y=0} \end{cases}$$
$$5x=5$$

The y's add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} x+4y=2 \\ 2x+5y=-2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{cases} (-2)(x+4y)=(-2)2 \\ 2x+5y=-2 \end{cases}$$

Then rewrite the system of equations.

$$\begin{cases} -2x-8y=-4 \\ 2x+5y=-2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

$$\begin{cases} -2x-8y=-4 \\ \underline{2x+5y=-2} \end{cases}$$
$$-3y=-6$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we will see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Example 2.1.27

Solve the system $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ by elimination.

Solution

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax+By=C$. There are no fractions	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one of both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x+y=7$ by 2.	$\begin{cases} 2(2x+y) = 2(7) \\ x-2y=6 \end{cases}$
Add the equations resulting from the previous step to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x+2y=14 \\ \underline{x-2y=6} \\ 5x=20 \end{cases}$
Solve for the remaining variable.	Solve for x.	$x=4$
Substitute the solution from the previous step into one of the original equations. Then solve for the other variable.	Substitute $x=4$ into the second equation, $x-2y=6$. Then solve for y.	$x-2y=6$ $4-2y=6$ $-2y=2$ $y=-1$
Write the solution as an ordered pair.	Write it as (x,y).	$(4,-1)$
Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$, $y=-1$ into $2x+y=7$ and $x-2y=6$. Do they make both equations true?	$2x+y=7$ $\quad$ $x-2y=6$ $2(4)+(-1)\stackrel{?}{=}7$ $\quad$ $4-2(-1)\stackrel{?}{=}6$ $7\stackrel{?}{=}7$ $\quad$ $6\stackrel{?}{=}6$ True $\quad$ True Substituting $(x,y)=(4,-1)$ makes both equations true.
Answer the question.		The solution is $(4,-1)$

Try It 2.1.28

Solve the system $\begin{cases} 3x+y=5 \\ 2x-3y=7 \end{cases}$ by elimination.

Answer

The solution is $(2,-1)$.

Try It 2.1.29

Solve the system $\begin{cases} 4x+y=-5 \\ -2x-2y=-2 \end{cases}$ by elimination.

Answer

The solution is $(-2,3)$.

The steps are listed here for easy reference.

Solve a system of equations by elimination

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.

- Decide which variable you will eliminate.
- Multiply one or both equations by appropriate numbers so that the coefficients of that variable are opposites.

3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations. This step is included for the purpose of detecting errors.

Now we will do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example 2.1.30

Solve the system $\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$ by elimination.

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$\begin{cases} 4x - 3y = 9 \\ 7x + 2y = -6 \end{cases}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{cases} 2(4x - 3y) & = 2(9) \\ 3(7x + 2y) & = 3(-6) \end{cases}$
Simplify.	$\begin{cases} 8x - 6y & = 18 \\ 21x + 6y & = -18 \end{cases}$
Add the two equations to eliminate y.	$29x = 0$
Solve for x.	$x = 0$
Substitute $x = 0$ into one of the original equations.	$\begin{aligned} 7x + 2y &= -6 \\ 7(0) + 2y &= -6 \end{aligned}$
Simplify.	$2y = -6$
Solve for y.	$y = -3$
Check that the ordered pair is a solution to **both** original equations.	$\begin{aligned} 4x - 3y &= 9 \\ 4(0) - 3(-3) &\stackrel{?}{=} 9 \\ 9 &\stackrel{?}{=} 9 \\ &\text{True} \end{aligned}$ $\quad$ $\begin{aligned} 7x + 2y &= -6 \\ 7(0) + 2(-3) &\stackrel{?}{=} -6 \\ -6 &\stackrel{?}{=} -6 \\ &\text{True} \end{aligned}$ Substituting $(x, y) = (0, -3)$ makes both equations true.
Answer the question.	The solution is $(0, -3)$.

Try It 2.1.31

Solve the system $\begin{cases} 3x - 4y = -9 \\ 5x + 3y = 14 \end{cases}$ by elimination.

Answer

The solution is $(1, 3)$.

Try It 2.1.32

Solve the system $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$ by elimination.

Answer

The solution is $(4,-3)$.

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Example 2.1.33

Solve the system $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$ by elimination.

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$
To clear the fractions, multiply each equation by its LCD.	$\begin{cases} 2\left(x+\frac{1}{2}y\right)=2(6) \\ 6\left(\frac{3}{2}x+\frac{2}{3}y\right)=6\left(\frac{17}{2}\right) \end{cases}$
Simplify	$\begin{cases} 2x+y=12 \\ 9x+4y=51 \end{cases}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form. We can eliminate y by multiplying the top equation by -4.	$\begin{cases} -4(2x+y)=-4(12) \\ 9x+4y=51 \end{cases}$
Simplify.	$\begin{cases} -8x-4y=-48 \\ 9x+4y=51 \end{cases}$
Add.	$x=3$
Substitute $x=3$ into one of the original equations.	$x+\frac{1}{2}y=6$ $3+\frac{1}{2}y=6$
Simplify.	$\frac{1}{2}y=3$
Solve for y.	$y=6$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.

	$\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$
Check that the ordered pair is a solution to both original equations.	$x+\frac{1}{2}y=6$ $3+\frac{1}{2}(6)\stackrel{?}{=}6$ $3+3\stackrel{?}{=}6$ $6\stackrel{?}{=}6$ True $\frac{3}{2}x+\frac{2}{3}y\stackrel{?}{=}\frac{17}{2}$ $\frac{3}{2}(3)+\frac{2}{3}(6)\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+4\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+\frac{8}{2}\stackrel{?}{=}\frac{17}{2}$ $\frac{17}{2}\stackrel{?}{=}\frac{17}{2}$ True Substituting $(x,y)=(3,6)$ makes both equations true.
Answer the question.	The solution is $(3,6)$.

? Try It 2.1.34

Solve the system $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$ by elimination.

Answer

The solution is $(6,2)$.

? Try It 2.1.35

Solve the system $\begin{cases} x+\frac{3}{5}y=-\frac{1}{5} \\ -\frac{1}{2}x-\frac{2}{3}y=\frac{5}{6} \end{cases}$ by elimination.

Answer

The solution is $(1,-2)$.

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

? Example 2.1.36

Solve the system $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$ by elimination.

Solution

	$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$
Write the second equation in standard form.	$\begin{cases} 3x+4y=12 \\ \frac{3}{4}x+y=3 \end{cases}$
Clear the fractions by multiplying the second equation by 4.	$\begin{cases} 3x+4y=12 \\ 4(\frac{3}{4}x+y)=4(3) \end{cases}$
Simplify.	$\begin{cases} 3x+4y=12 \\ 3x+4y=12 \end{cases}$
To eliminate a variable, we multiply the second equation by -1 and simplify.	$\begin{cases} 3x+4y=12 \\ -3x-4y=-12 \end{cases}$
Add the equations.	$0=0$
Conclusion.	This is a true statement. The equations are consistent but dependent. Their graphs would be the same line.
Answer the question.	The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

? Try It 2.1.37

Solve the system $\begin{cases} 5x-3y=15 \\ 5y=-5+\frac{5}{3}x \end{cases}$ by elimination.

Answer

The system has infinitely many solutions.

? Try It 2.1.38

Solve the system $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$ by elimination.

Answer

The system has infinitely many solutions.

Choose the Most Convenient Method to Solve a System of Linear Equations

When we solve a system of linear equations in an application, we will not be told which method to use. We will need to make that decision ourselves. So we will want to choose the method that is easiest to do and minimizes our chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations		
Graphing	**Substitution**	**Elimination**
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations are in standard form.

Example 2.1.39

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a. $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$

b. $\begin{cases} 5x+6y=12 \\ y=\dfrac{2}{3}x-1 \end{cases}$

Solution

a.

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

b.

$$\begin{cases} 5x+6y=12 \\ y=\dfrac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient (especially since we see that the fraction will be reduced to an integer in the first step of simplification).

Try It 2.1.40

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a. $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$

b. $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

a. Since both equations are in standard form, using elimination will be most convenient.

b. Since one equation is already solved for x, using substitution will be most convenient.

Try It 2.1.41

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a. $\begin{cases} y=2x-1 \\ 3x-4y=-6 \end{cases}$

b. $\begin{cases} 6x-2y=12 \\ 3x+7y=-13 \end{cases}$

Answer

a. Since one equation is already solved for y, using substitution will be most convenient.

b. Since both equations are in standard form, using elimination will be most convenient.

Key Concepts

- **How to solve a system of linear equations by graphing.**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations. This step is included to make sure there was no error and that the intersection point was correctly identified.
- **How to solve a system of equations by substitution.**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.
  4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
  5. Write the solution as an ordered pair.
  6. In order to detect errors, check that the ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations by appropriate numbers so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.
  7. In order to detect errors, check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	**Substitution**	**Elimination**
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

Glossary

coincident lines

Coincident lines have the same slope and same y-intercept.

consistent and inconsistent systems

Consistent system of equations is a system of equations with at least one solution; inconsistent system of equations is a system of equations with no solution.

solutions of a system of equations

Solutions of a system of equations are the values of the variables that make *all* the equations true; solution is represented by an ordered pair (p, q).

system of linear equations

When two or more linear equations are grouped together, they form a system of linear equations.

Practice Makes Perfect

Determine Whether an Ordered Pair is a Solution of a System of Equations

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} 2x-6y=0 \\ 3x-4y=5 \end{cases}$

ⓐ $(3,1)$
ⓑ $(-3,4)$

Answer

ⓐ yes ⓑ no

2. $\begin{cases} -3x+y=8 \\ -x+2y=-9 \end{cases}$

ⓐ $(-5,-7)$
ⓑ $(-5,7)$

3. $\begin{cases} x+y=2 \\ y=\frac{3}{4}x \end{cases}$

ⓐ $(87,67)$
ⓑ $(1,34)$

Answer

ⓐ yes ⓑ no

4. $\begin{cases} 2x+3y=6 \\ y=\frac{2}{3}x+2 \end{cases}$

ⓐ $(-6,2)$
ⓑ $(-3,4)$

Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

5. $\begin{cases} 3x+y=-3 \\ 2x+3y=5 \end{cases}$

Answer

$(-3,2)$

6. $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

7. $\begin{cases} y=x+2 \\ y=-2x+2 \end{cases}$

Answer

$(0,2)$

8. $\begin{cases} y = x - 2 \\ y = -3x + 2 \end{cases}$

9. $\begin{cases} y = \frac{3}{2}x + 1 \\ y = -\frac{1}{2}x + 5 \end{cases}$

Answer

$(2, 4)$

10. $\begin{cases} y = \frac{2}{3}x - 2 \\ y = -\frac{1}{3}x - 5 \end{cases}$

11. $\begin{cases} x + y = -4 \\ -x + 2y = -2 \end{cases}$

Answer

$(-2, 2)$

12. $\begin{cases} -x + 3y = 3 \\ x + 3y = 3 \end{cases}$

13. $\begin{cases} -2x + 3y = 3 \\ x + 3y = 12 \end{cases}$

Answer

$(3, 3)$

14. $\begin{cases} 2x - y = 4 \\ 2x + 3y = 12 \end{cases}$

15. $\begin{cases} x + 3y = -6 \\ y = -\frac{4}{3}x + 4 \end{cases}$

Answer

$(6, -4)$

16. $\begin{cases} -x + 2y = -6 \\ y = -\frac{1}{2}x - 1 \end{cases}$

17. $\begin{cases} -2x + 4y = 4 \\ y = \frac{1}{2}x \end{cases}$

Answer

no solution

18. $\begin{cases} 3x+5y=10 \\ y=-\dfrac{3}{5}x+1 \end{cases}$

19. $\begin{cases} 4x-3y=8 \\ 8x-6y=14 \end{cases}$

Answer

no solution

20. $\begin{cases} x+3y=4 \\ -2x-6y=3 \end{cases}$

21. $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+6y=8\}$

22. $\begin{cases} 4x=3y+7 \\ 8x-6y=14 \end{cases}$

23. $\begin{cases} 2x+y=6 \\ -8x-4y=-24 \end{cases}$

Answer

infinite solutions with solution set: $\{(x,y)|2x+y=6\}$

24. $\begin{cases} 5x+2y=7 \\ -10x-4y=-14 \end{cases}$

Without graphing, determine the number of solutions and then classify the system of equations.

25. $\begin{cases} y=\dfrac{2}{3}x+1 \\ -2x+3y=5 \end{cases}$

Answer

1 point, consistent and independent

26. $\begin{cases} y=\dfrac{3}{2}x+1 \\ 2x-3y=7 \end{cases}$

27. $\begin{cases} 5x+3y=4 \\ 2x-3y=5 \end{cases}$

Answer

1 point, consistent and independent

28. $\begin{cases} y=-12x+5 \\ x+2y=10 \end{cases}$

29. $\begin{cases} 5x-2y=10 \\ y=52x-5 \end{cases}$

Answer

infinite solutions, consistent, dependent

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

30. $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

31. $\begin{cases} 2x+y=-2 \\ 3x-y=7 \end{cases}$

Answer

$(1,-4)$

32. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

33. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3,2)$

34. $\begin{cases} 5x-2y=-6 \\ y=3x+3 \end{cases}$

35. $\begin{cases} -2x+2y=6 \\ y=-3x+1 \end{cases}$

Answer

$(-1/2,5/2)$

36. $\begin{cases} 2x+5y=1 \\ y=\frac{1}{3}x-2 \end{cases}$

37. $\begin{cases} 3x+4y=1 \\ y=-\frac{2}{5}x+2 \end{cases}$

Answer

$(-5,4)$

38. $\begin{cases} 2x+y=5 \\ x-2y=-15 \end{cases}$

39. $\begin{cases} 4x+y=10 \\ x-2y=-20 \end{cases}$

Answer

$(0, 10)$

40. $\begin{cases} y=-2x-1 \\ y=-\frac{1}{3}x+4 \end{cases}$

41. $\begin{cases} y=x-6 \\ y=-\frac{3}{2}x+4 \end{cases}$

Answer

$(4, -2)$

42. $\begin{cases} x=2y \\ 4x-8y=0 \end{cases}$

43. $\begin{cases} 2x-16y=8 \\ -x-8y=-4 \end{cases}$

Answer

$(4, 0)$

44. $\begin{cases} y=\frac{7}{8}x+4 \\ -7x+8y=6 \end{cases}$

45. $\begin{cases} y=-\frac{2}{3}x+5 \\ 2x+3y=11 \end{cases}$

Answer

no solution

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination.

46. $\begin{cases} 5x+2y=2 \\ -3x-y=0 \end{cases}$

47. $\begin{cases} 6x-5y=-1 \\ 2x+y=13 \end{cases}$

Answer

$(4, 5)$

48. $\begin{cases} 2x-5y=7 \\ 3x-y=17 \end{cases}$

49. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$

Answer

$(7,12)$

50. $\begin{cases} 3x-5y=-9 \\ 5x+2y=16 \end{cases}$

51. $\begin{cases} 4x-3y=3 \\ 2x+5y=-31 \end{cases}$

Answer

$(-3,-5)$

52. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$

53. $\begin{cases} 11x+9y=-5 \\ 7x+5y=-1 \end{cases}$

Answer

$(2,-3)$

54. $\begin{cases} 3x+8y=67 \\ 5x+3y=60 \end{cases}$

55. $\begin{cases} 2x+9y=-4 \\ 3x+13y=-7 \end{cases}$

Answer

$(-11,2)$

56. $\begin{cases} \frac{1}{3}x-y=-3 \\ x+\frac{5}{2}y=2 \end{cases}$

57. $\begin{cases} x+\frac{1}{2}y=\frac{3}{2} \\ \frac{1}{5}x-\frac{1}{5}y=3 \end{cases}$

Answer

$(6/-9,24/7)$

58. $\begin{cases} x+\frac{1}{3}y=-1 \\ \frac{1}{3}x+\frac{1}{2}y=1 \end{cases}$

59. $\begin{cases} \frac{1}{3}x-y=-3 \\ \frac{2}{3}x+\frac{5}{2}y=3 \end{cases}$

Answer

$(-3,2)$

60. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

61. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x,y)|x-4y=-1\}$

62. $\begin{cases} -3x-y=8 \\ 6x+2y=-16 \end{cases}$

63. $\begin{cases} 4x+3y=2 \\ 20x+15y=10 \end{cases}$

Answer

infinitely many solutions with solution set: $\{(x,y)|4x+3y=2\}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

64.

ⓐ $\begin{cases} 8x-15y=-32 \\ 6x+3y=-5 \end{cases}$

ⓑ $\begin{cases} x=4y-3 \\ 4x-2y=-6 \end{cases}$

65.

ⓐ $\begin{cases} y=7x-5 \\ 3x-2y=16 \end{cases}$

ⓑ $\begin{cases} 12x-5y=-42 \\ 3x+7y=-15 \end{cases}$

Answer

ⓐ substitution ⓑ elimination

66.

ⓐ $\begin{cases} y = 4x + 95 \\ x - 2y = -21 \end{cases}$

ⓑ $\begin{cases} 9x - 4y = 24 \\ 3x + 5y = -14 \end{cases}$

67.

ⓐ $\begin{cases} 14x - 15y = -30 \\ 7x + 2y = 10 \end{cases}$

ⓑ $\begin{cases} x = 9y - 11 \\ 2x - 7y = -27 \end{cases}$

Answer

ⓐ elimination ⓑ substitution

Writing Exercises

68. In a system of linear equations, the two equations have the same intercepts. Describe the possible solutions to the system.

69. Solve the system of equations by substitution and explain all your steps in words: $\begin{cases} 3x + y = 1 \\ 2x = y - 8 \end{cases}$

Answer

Answers will vary.

70. Solve the system of equations by elimination and explain all your steps in words: $\begin{cases} 5x + 4y = 10 \\ 2x = 3y + 27 \end{cases}$

71. Solve the system of equations $\begin{cases} x + y = 10 \\ x - y = 6 \end{cases}$

ⓐ by graphing ⓑ by substitution
ⓒ Which method do you prefer? Why?

Answer

Answers will vary.

Self Check

After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered pair is a solution of a system of equations.			
solve a system of linear equations by graphing.			
solve a system of equations by substitution.			
solve a system of equations by elimination.			
choose the most convenient method to solve a system of linear equations.			

If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

2.2: Solve Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve direct translation applications
- Solve geometry applications
- Solve uniform motion applications

Be Prepared

Before you get started, take this readiness quiz.

1. The sum of twice a number and nine is 31. Find the number.
2. Twins Jon and Ron together earned \$96,000 last year. Ron earned \$8000 more than three times what Jon earned. How much did each of the twins earn?
3. An express train and a local train leave Pittsburgh to travel to Washington, D.C. The express train can make the trip in four hours and the local train takes five hours for the trip. The speed of the express train is 12 miles per hour faster than the speed of the local train. Find the speed of both trains.

Solve Direct Translation Applications

Systems of linear equations are very useful for solving applications. Some people find setting up word problems with two variables easier than setting them up with just one variable. To solve an application, we'll first translate the words into a system of linear equations. Then we will decide the most convenient method to use, and then solve the system.

Solve applications with systems of equations

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose variables to represent those quantities.
4. **Translate** into a system of equations.
5. **Solve** the system of equations using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We solved number problems with one variable earlier. Let's see how differently it works using two variables.

Example 2.2.1

Translate to a system of equations and then solve.

The sum of two numbers is zero. One number is nine less than the other. Find the numbers.

Solution

	Read the problem.	
oblem.	**Identify** what we are looking for.	We are looking for two numbers.
oblem.	**Name** what we are looking for.	Let $n =$ the first number and $m =$ the second number.
oblem.	**Translate** into a system of equations.	The sum of two numbers is zero: $n+m=0$. One number is nine less than the other: $n=m-9$. The system is $\begin{cases} n+m=0 \\ n=m-9 \end{cases}$.
oblem.	**Solve** the system of equations. We will use substitution since the second equation is solved for n.	$\begin{cases} n+m=0 \\ n=m-9 \end{cases}$

	Read the problem.	
oblem.	Substitute $m-9$ for n in the first equation.	$\begin{aligned} n+m &= 0 \\ m-9+m &= 0 \end{aligned}$
oblem.	Solve for m.	$\begin{aligned} 2m-9 &= 0 \\ 2m &= 9 \\ m &= \frac{9}{2} \end{aligned}$
oblem.	Substitute $m=\frac{9}{2}$ into the second equation and then solve for n.	$\begin{aligned} n &= m-9 \\ n &= \frac{9}{2}-9 \\ n &= \frac{9}{2}-\frac{18}{2} \\ n &= -\frac{9}{2} \end{aligned}$
oblem.	**Check** the answer in the problem.	$\begin{aligned} n+m &= 0 \\ -\frac{9}{2}+\frac{9}{2} &\stackrel{?}{=} 0 \\ 0 &\stackrel{?}{=} 0 \\ &\text{True} \end{aligned}$ $\begin{aligned} n &= m-9 \\ -\frac{9}{2} &\stackrel{?}{=} \frac{9}{2}-9 \\ -\frac{9}{2} &\stackrel{?}{=} -\frac{9}{2} \\ &\text{True} \end{aligned}$ Do these numbers make sense in the problem? We will leave this to you.
oblem.	**Answer** the question.	The numbers are $-\frac{9}{2}$ and $\frac{9}{2}$.

? Try It 2.2.2

Translate to a system of equations and then solve.

The sum of two numbers is 10. One number is 4 less than the other. Find the numbers.

Answer

The numbers are 3 and 7.

? Try It 2.2.3

Translate to a system of equations and then solve.

The sum of two numbers is -6. One number is 10 less than the other. Find the numbers.

Answer

The numbers are -8 and 2.

? Example 2.2.4

Translate to a system of equations and then solve.

Heather has been offered two options for her salary as a trainer at the gym. Option A would pay her $\$25,000$ plus $\$15$ for each training session. Option B would pay her $\$10,000+\$40$ for each training session. How many training sessions would make the salary options equal?

Solution

	Read the problem.	
oblem.	**Identify** what we are looking for.	We are looking for the number of training sessions that would make the pay equal.
oblem.	**Name** what we are looking for.	Let $s = \text{ Heather's salary and}$ $n = \text{ the number of training sessions}$
oblem.	**Translate** into a system of equations.	Option A would pay her $25,000$ plus 15 for each training session: $s = 25,000 + 15n$. Option B would pay her $10,000 + \$40$ for each training session: $s = 10,000 + 40n$. The system is $\begin{cases} s = 25,000 + 15n \\ s = 10,000 + 40n \end{cases}$.
oblem.	**Solve** the system of equations. We will use substitution since both equations are solved for s.	$\begin{cases} s = 25,000 + 15n \\ s = 10,000 + 40n \end{cases}$
oblem.	Substitute $25,000 + 15n$ for s in the second equation.	$s = 10,000 + 40n$ $25,000 + 15n = 10,000 + 40n$
oblem.	Solve for n	$25,000 = 10,000 + 25n$ $15,000 = 25n$ $600 = n$
oblem.	**Check** the answer.	Are the two options equal when $n = 600$? Option A: $s = 25,000 + 15n$ $s = 25,000 + 15(600)$ $s = 34,000$ Option B: $s = 10,000 + 40n$ $s = 10,000 + 40(600)$ $s = 34,000$ Yes, the two options are equal when $n = 600$. Are 600 training sessions a year reasonable? We will leave this to you.
oblem.	**Answer** the question.	The salary options would be equal for 600 training sessions.

? Try It 2.2.5

Translate to a system of equations and then solve.

Geraldine has been offered positions by two insurance companies. The first company pays a salary of $\$12,000$ plus a commission of $\$100$ for each policy sold. The second pays a salary of $\$20,000$ plus a commission of $\$50$ for each policy sold. How many policies would need to be sold to make the total pay the same?

Answer

To make the total pay the same 160 policies would need to be sold.

? Try It 2.2.6

Translate to a system of equations and then solve.

Kenneth currently sells suits for company A at a salary of $\$22,000$ plus a $\$10$ commission for each suit sold. Company B offers him a position with a salary of $\$28,000$ plus a $\$4$ commission for each suit sold. How many suits would Kenneth need to sell for the options to be equal?

Answer

1000suits would need to sell for the options to be equal.

As you solve each application, remember to analyze which method of solving the system of equations would be most convenient.

? Example 2.2.7

Translate to a system of equations and then solve.

When Jenna spent 10 minutes on the elliptical trainer and then did circuit training for 20 minutes, her fitness app says she burned 278 calories. When she spent 20 minutes on the elliptical trainer and 30 minutes circuit training she burned 473 calories. How many calories does she burn for each minute on the elliptical trainer? How many calories for each minute of circuit training?

Solution

	Read the problem.	
oblem.	**Identify** what we are looking for.	We are looking for the number of calories burned each minute on the elliptical trainer and each minute of circuit training.
oblem.	**Name** what we are looking for.	Let $e=$ number of calories burned per minute on the elliptical trair and $c=$ number of calories burned per minute while circuit training .
oblem.	**Translate** into a system of equations.	10 minutes on the elliptical and circuit training for 20 minutes, burned 278 calories: $10e+20c=278$. 20 minutes on the elliptical and 30 minutes of circuit training burned 473 calories: $20e+30c=473$. The system is $\begin{cases}10e+20c=278\\20e+30c=473\end{cases}$.
oblem.	**Solve** the system of equations. We will use the elimination method.	$\begin{cases}10e+20c=278\\20e+30c=473\end{cases}$
oblem.	Multiply the first equation by -2 to get opposite coefficients of e. Simplify.	$\begin{cases}(-2)(10e+20c)=(-2)(278)\\20e+30c=473\end{cases}$ $\begin{cases}-20e-40c=-556\\20e+30c=473\end{cases}$
oblem.	Add the equations.	$-10c=-83$
oblem.	Solve for c.	$c=8.3$
oblem.	Substitute $c=8.3$ into one of the original equations to solve for e.	$\begin{aligned}10e+20c&=278\\10e+20(8.3)&=278\\10e+166&=278\\10e&=112\\e&=11.2\end{aligned}$
oblem.	**Check** the answer in the problem.	$\begin{aligned}10e+20c&=278\\10(11.2)+20(8.3)&\stackrel{?}{=}278\\278&\stackrel{?}{=}278\\&\text{True}\\20e+30c&=473\\20(11.2)+30(8.3)&\stackrel{?}{=}473\\473&\stackrel{?}{=}473\\&\text{True}\end{aligned}$ Do the answer make sense? We will leave this to you.

	Read the problem.	
oblem.	**Answer** the question.	Jenna burns 8.3 calories per minute circuit training and 11.2 calories per minute while on the elliptical trainer.

Try It 2.2.8

Translate to a system of equations and then solve.

Mark went to the gym and did 40 minutes of Bikram hot yoga and 10 minutes of jumping jacks. He burned 510 calories. The next time he went to the gym, he did 30 minutes of Bikram hot yoga and 20 minutes of jumping jacks burning 470 calories. How many calories were burned for each minute of yoga? How many calories were burned for each minute of jumping jacks?

Answer

Mark burned 11 calories for each minute of yoga and 7 calories for each minute of jumping jacks.

Try It 2.2.9

Translate to a system of equations and then solve.

Erin spent 30 minutes on the rowing machine and 20 minutes lifting weights at the gym and burned 430 calories. During her next visit to the gym she spent 50 minutes on the rowing machine and 10 minutes lifting weights and burned 600 calories. How many calories did she burn for each minutes on the rowing machine? How many calories did she burn for each minute of weight lifting?

Answer

Erin burned 11 calories for each minute on the rowing machine and 5 calories for each minute of weight lifting.

Solve Geometry Applications

We will now solve geometry applications using systems of linear equations. We will need to add complementary angles and supplementary angles to our list some properties of angles.

The measures of two *complementary angles* add to 90 degrees. The measures of two *supplementary angles* add to 180 degrees.

Definition 2.2.10

Two angles are **complementary** if the sum of the measures of their angles is 90 degrees.

Two angles are **supplementary** if the sum of the measures of their angles is 180 degrees.

If two angles are complementary, we say that *one angle is the complement of the other.*

If two angles are supplementary, we say that *one angle is the supplement of the other.*

Example 2.2.11

Translate to a system of equations and then solve.

The difference of two complementary angles is 26 degrees. Find the measures of the angles.

Solution

	Read the problem.	
oblem.	**Identify** what we are looking for.	We are looking for the measure of each angle.
oblem.	**Name** what we are looking for.	Let $x =$ the measure of the first angle and $y =$ the measure of the second angle.

oblem.	**Read** the problem.	
oblem.	**Translate** into a system of equations.	The angles are complementary: $x+y=90.$ The difference of the two angles is 26 degrees: $x-y=26.$ The system is $\begin{cases}x+y=90\\x-y=26\end{cases}.$
oblem.	**Solve** the system of equations. We will use the elimination method.	$\begin{cases}x+y=90\\x-y=26\end{cases}$
oblem.	Add the equations and solve for x.	$2x=116$ $x=58$
oblem.	Substitute $x=58$ into the first equation.	$x+y=90$ $58+y=90$ $y=32$
oblem.	**Check** the answer in the problem.	$x+y=90$ $58+32\stackrel{?}{=}90$ $90\stackrel{?}{=}90$ True $x-y=26$ $58-32\stackrel{?}{=}26$ $26\stackrel{?}{=}26$ True Does the answer make sense? We will leave this to you.
oblem.	**Answer** the question.	The angle measures are 32 and 58 degrees.

? Try It 2.2.12

Translate to a system of equations and then solve.

The difference of two complementary angles is 20 degrees. Find the measures of the angles.

Answer

The angle measures are 35 and 55 degrees.

? Try It 2.2.13

Translate to a system of equations and then solve.

The difference of two complementary angles is 80 degrees. Find the measures of the angles.

Answer

The angle measures are 5 and 85 degrees.

In the next example, we remember that the measures of supplementary angles add to 180.

? Example 2.2.14

Translate to a system of equations and then solve.

Two angles are supplementary. The measure of the larger angle is twelve degrees less than five times the measure of the smaller angle. Find the measures of both angles.

Solution

Read the problem.	

	Read the problem.	
oblem.	**Identify** what we are looking for.	We are looking for measure of each angle.
oblem.	**Name** what we are looking for.	Let $x =$ the measure of the first angle and $y =$ the measure of the second angle.
oblem.	**Translate** into a system of equations.	The angles are supplementary: $x+y=180$. The larger angle is twelve less than five times the smaller angle: $y=5x-12$. The system is $\begin{cases} x+y=180 \\ y=5x-12 \end{cases}$.
oblem.	**Solve** the system of equations. We will use substitution since the second equation is solved for y.	$\begin{cases} x+y=180 \\ y=5x-12 \end{cases}$
oblem.	Substitute $5x-12$ for y in the first equation.	$x+y=180$ $x+(5x-12)=180$
oblem.	Solve for x.	$6x-12=180$ $6x=192$ $x=32$
oblem.	Substitute 32 for x in the second equation, then solve for y.	$y=5x-12$ $y=5(32)-12$ $y=160-12$ $y=148$
oblem.	**Check** the answer in the problem.	$x+y=180$ $32+148 \stackrel{?}{=} 180$ $180 \stackrel{?}{=} 180$ True $y=5x-12$ $148 \stackrel{?}{=} 5(32)-12$ $148 \stackrel{?}{=} 148$ True Does the answer make sense? We will leave this to you.
oblem.	**Answer** the question.	The angle measures are 32 and 148 degrees.

? Try It 2.2.15

Translate to a system of equations and then solve.

Two angles are supplementary. The measure of the larger angle is 12 degrees more than three times the smaller angle. Find the measures of the angles.

Answer

The angle measures are 42 and 138 degrees.

? Try It 2.2.16

Translate to a system of equations and then solve.

Two angles are supplementary. The measure of the larger angle is 18 less than twice the measure of the smaller angle. Find the measures of the angles.

Answer

The angle measures are 66 and 114 degrees.

Recall that the angles of a triangle add up to 180 degrees. A right triangle has one angle that is 90 degrees. What does that tell us about the other two angles? In the next example we will be finding the measures of the other two angles.

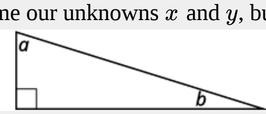
Example 2.2.17

Translate to a system of equations and then solve.

The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle. Find the measures of both angles.

Solution

We will draw and label a figure.

Read the problem.	
Identify what you are looking for.	We are looking for the measures of the angles.
Name what we are looking for.	Let $a =$ the measure of the first angle and $b =$ the measure of the second angle. Note that we don't have to name our unknowns x and y, but can pick letters we like. 
Translate into a system of equations.	The measure of one of the small angles of a right triangle is ten more than three times the measure of the other small angle: $a = 3b + 10$. The sum of the measures of the angles of a triangle is 180: $a + b + 90 = 180$. The system is $\begin{cases} a &= 3b + 10 \\ a + b + 90 &= 180 \end{cases}$.
Solve the system of equations. We will use substitution since the first equation is solved for a.	$\begin{cases} a &= 3b + 10 \\ a + b + 90 &= 180 \end{cases}$
Substitute $3b + 10$ for a in the second equation.	$a + b + 90 = 180$ $(3b + 10) + b + 90 = 180$
Solve for b.	$4b + 100 = 180$ $4b = 80$ $b = 20$
Substitute $b = 20$ into the first equation and then solve for a.	$a = 3b + 10$ $a = 3(20) + 10$ $a = 70$
Check the answer in the problem.	$a = 3b + 10$ $70 \stackrel{?}{=} 3(20) + 10$ $70 \stackrel{?}{=} 70$ True $a + b + 90 = 180$ $70 + 20 + 90 \stackrel{?}{=} 180$ $180 \stackrel{?}{=} 180$ True Does the answer make sense? We will leave this to you.
Answer the question.	The measures of the small angles are 20 and 70 degrees.

Try It 2.2.18

Translate to a system of equations and then solve.

The measure of one of the small angles of a right triangle is 2 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

The angle measures are 22 and 68 degrees.

Try It 2.2.19

Translate to a system of equations and then solve.

The measure of one of the small angles of a right triangle is 18 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

The angle measures are 36 and 54 degrees.

Often it is helpful when solving geometry applications to draw a picture to visualize the situation.

Example 2.2.20

Translate to a system of equations and then solve.

Randall has 125 feet of fencing to enclose the part of his backyard adjacent to his house. He will only need to fence around three sides, because the fourth side will be the wall of the house. He wants the length of the fenced yard (parallel to the house wall) to be 5 feet more than four times as long as the width. Find the length and the width.

Solution

	Read the problem.	
oblem.	**Identify** what you are looking for.	We are looking for the length and width. 
oblem.	**Name** what we are looking for.	Let $L =$ the length of the fenced yard and $W =$ the width of the fenced yard. Here we choose letters that will help remind us of their meaning.
oblem.	**Translate** into a system of equations.	One length and two widths equal 125: $L+2W=125.$ The length will be 5 feet more than four times the width: $L=4W+5.$ The system is $\begin{cases} L+2W &= 125 \\ L &= 4W+5 \end{cases}.$
oblem.	**Solve** the system. We will use substitution since the second equation is solved for L.	$\begin{cases} L+2W &= 125 \\ L &= 4W+5 \end{cases}$
oblem.	Substitute $L=4W+5$ into the first equation.	$\begin{aligned} L+2W &= 125 \\ (4W+5)+2W &= 125 \end{aligned}$

	Read the problem.	
oblem.	Solve for W.	$6W+5=125$ $6W=120$ $W=20$
oblem.	Substitute 20 for W in the second equation, then solve for L.	$L=4W+5$ $L=4(20)+5$ $L=85$
oblem.	**Check** the answer in the problem.	$L+2W=125$ $85+2(20)\stackrel{?}{=}125$ $125\stackrel{?}{=}125$ True $L=4W+5$ $85\stackrel{?}{=}4(20)+5$ $85\stackrel{?}{=}85$ True Does the answer make sense? We will leave this to you.
oblem.	**Answer** the question.	The length is 85 feet and the width is 20 feet.

? Try It 2.2.21

Translate to a system of equations and then solve.

Mario wants to put a fence around the pool in his backyard. Since one side is adjacent to the house, he will only need to fence three sides. There are two long sides and the one shorter side is parallel to the house. He needs 155 feet of fencing to enclose the pool. The length of the long side is 10 feet less than twice the width. Find the length and width of the pool area to be enclosed.

Answer

The length is 60 feet and the width is 35 feet.

? Try It 2.2.22

Translate to a system of equations and then solve.

Alexis wants to build a rectangular dog run in her yard adjacent to her neighbor's fence. She will use 136 feet of fencing to completely enclose the rectangular dog run. The length of the dog run along the neighbor's fence will be 16 feet less than twice the width. Find the length and width of the dog run.

Answer

The length is 60 feet and the width is 38 feet.

Solve Uniform Motion Applications

We used a table to organize the information in uniform motion problems when we introduced them earlier. We will continue using the table here. The basic equation was $D=rt$ where D is the distance traveled, r is the rate, and t is the time.

Our first example of a uniform motion application will be for a situation similar to some we have already seen, but now we can use two variables and two equations.

? Example 2.2.23

Translate to a system of equations and then solve.

Joni left St. Louis on the interstate, driving west towards Denver at a speed of 65 miles per hour. Half an hour later, Kelly left St. Louis on the same route as Joni, driving 78 miles per hour. How long will it take Kelly to catch up to Joni?

Solution

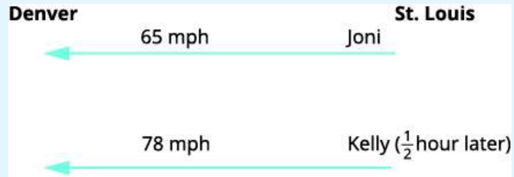
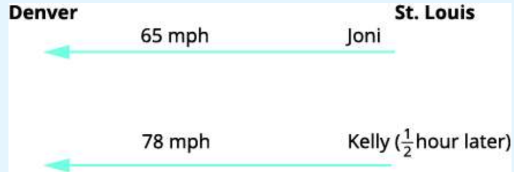
	Read the problem.	
	A diagram is useful in helping us visualize the situation.	
oble…	**Identify** what we are looking for.	We are looking for the length of time Kelly and Joni will each drive.
oble…	**Name** what we are looking for.	Let $k =$ the length of time Kelly will drive and $j =$ the length of time Joni will drive.
oble…	A chart will help us organize the data. We know the rates of both Joni and Kelly, and so we enter them in the chart.	See chart below. Since $\text{Distance} = \text{rate} \cdot \text{time}$, we can fill in the Distance column.
oble…	**Translate** into a system of equations.	To make the system of equations, we must recognize that Kelly and Joni will drive the same distance: $65j = 78k$. Also, since Kelly left later, her time will be $\frac{1}{2}$ hour less than Joni's time: $k = j - \frac{1}{2}$. The system is $\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$.
oble…	**Solve** the system. We will use substitution since the first equation is solved for k.	$\begin{cases} k = j - \frac{1}{2} \\ 65j = 78k \end{cases}$
oble…	Substitute $k = j - \frac{1}{2}$ into the second equation.	$\begin{cases} 65j = 78k \\ 65j = 78\left(j - \frac{1}{2}\right) \end{cases}$
oble…	Solve for j.	$\begin{cases} 65j = 78j - 39 \\ -13j = -39 \\ j = 3 \end{cases}$
oble…	To find Kelly's time, substitute $j = 3$ into the first equation, then solve for k.	$k = j - \frac{1}{2}$ $k = 3 - \frac{1}{2}$ $k = \frac{5}{2}$ $k = 2\frac{1}{2}$

Chart in the data row:

	Rate	· Time	= Distance
Joni	65	j	$65j$
Kelly	78	k	$78k$

	Read the problem.	
	A diagram is useful in helping us visualize the situation.	
oble…	**Check** the answer in the problem.	$k = j - \frac{1}{2}$ $2\frac{1}{2} \stackrel{?}{=} 3 - \frac{1}{2}$ $2\frac{1}{2} \stackrel{?}{=} 2\frac{1}{2}$ True $65j = 78k$ $65(3) \stackrel{?}{=} 78\left(\frac{5}{2}\right)$ $195 \stackrel{?}{=} 195$ True They will have traveled the same distance, 195 miles, when they meet. Does the answer make sense? We will leave this to you.
oble…	**Answer** the question.	Kelly will catch up to Joni in $2\frac{1}{2}$ hours. By then, Joni will have traveled 3 hours.

? Try It 2.2.24

Translate to a system of equations and then solve.

Mitchell left Detroit on the interstate driving south towards Orlando at a speed of 60 miles per hour. Clark left Detroit 1 hour later traveling at a speed of 75 miles per hour, following the same route as Mitchell. How long will it take Clark to catch Mitchell?

Answer

It will take Clark 4 hours to catch Mitchell.

? Try It 2.2.25

Translate to a system of equations and then solve.

Charlie left his mother's house traveling at an average speed of 36 miles per hour. His sister Sally left 15 minutes $\left(\frac{1}{4}\text{ hour}\right)$ later traveling the same route at an average speed of 42 miles per hour. How long before Sally catches up to Charlie?

Answer

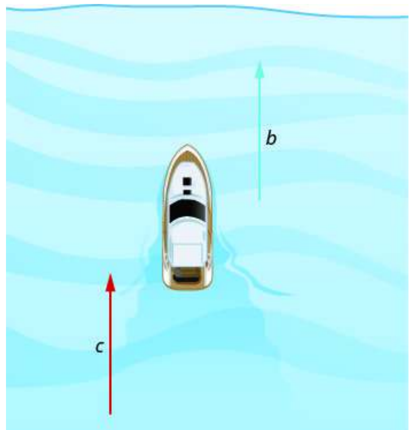
It will take Sally 112hours to catch up to Charlie.

Many real-world applications of uniform motion arise because of the effects of currents—of water or air—on the actual speed of a vehicle. Cross-country airplane flights in the United States generally take longer going west than going east because of the prevailing wind currents.

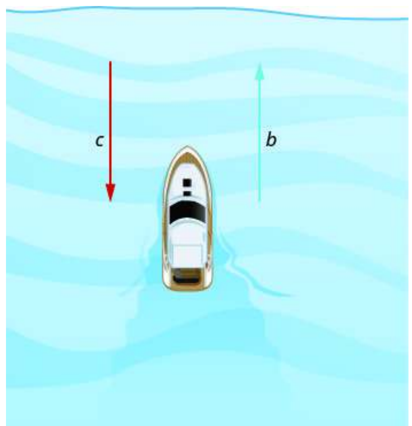
Let's take a look at a boat traveling on a river. Depending on which way the boat is going, the current of the water is either slowing it down or speeding it up.

The images below show how a river current affects the speed at which a boat is actually traveling. We will call the speed of the boat in still water b and the speed of the river current c.

The boat is going downstream, in the same direction as the river current. The current helps push the boat, so the boat's actual speed is faster than its speed in still water. The actual speed at which the boat is moving is $b+c$.



Now, the boat is going upstream, opposite to the river current. The current is going against the boat, so the boat's actual speed is slower than its speed in still water. The actual speed of the boat is $b-c$.



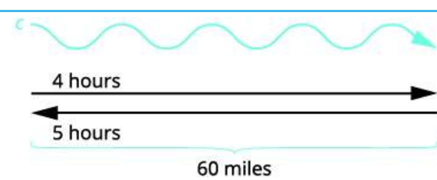
We will put some numbers to this situation in the next example.

? Example 2.2.26

Translate to a system of equations and then solve.

A river cruise ship sailed 60 miles downstream for 4 hours and then took 5 hours sailing upstream to return to the dock. Find the speed of the ship in still water and the speed of the river current.

Solution

	Read the problem.	
oblem.	This is a uniform motion problem and a picture will help us visualize the situation.	
oblem.	**Identify** what we are looking for.	We are looking for the speed of the ship in still water and the speed of the current.
oblem.	**Name** what we are looking for.	Let $s=$ the rate of the ship in still water and $c=$ the rate of the current.

<table>
<tr><td></td><td>Read the problem.</td><td></td></tr>
<tr><td>oblem.</td><td>A chart will help us organize the information.</td><td><table><tr><th></th><th colspan="3">Rate • Time = Distance</th></tr><tr><td>downstream</td><td>$s+c$</td><td>4</td><td>60</td></tr><tr><td>upstream</td><td>$s-c$</td><td>5</td><td>60</td></tr></table>$\text{Distance} = \text{rate} \cdot \text{time}$</td></tr>
<tr><td>oblem.</td><td>Translate into a system of equations.</td><td>The ship goes downstream and then upstream.
Going downstream, the current helps the ship and so the ship's actual rate is $s+c$.
Going upstream, the current slows the ship and so the actual rate is $s-c$.
Downstream it takes 4 hours. Upstream it takes 5 hours.
Each way the distance is 60 miles.
The system is $\left\{\begin{array}{l} 4(s+c)=60 \\ 5(s-c)=60 \end{array}\right.$.</td></tr>
<tr><td>oblem.</td><td>Solve the system of equations.
Distribute to put both equations in standard form, then solve by elimination.</td><td>$\left\{\begin{aligned} 4(s+c) &= 60 \\ 5(s-c) &= 60 \end{aligned}\right.$
$\left\{\begin{aligned} 4s+4c &= 60 \\ 5s-5c &= 60 \end{aligned}\right.$</td></tr>
<tr><td>oblem.</td><td>Multiply the top equation by 5 and the bottom equation by 4.</td><td>$\left\{\begin{aligned} 5(4s+4c) &= 5(60) \\ 4(5s-5c) &= 4(60) \end{aligned}\right.$
$\left\{\begin{aligned} 20s+20c &= 300 \\ 20s-20c &= 240 \end{aligned}\right.$</td></tr>
<tr><td>oblem.</td><td>Add the equations, then solve for s.</td><td>$\begin{aligned} 40s &= 540 \\ s &= 13.5 \end{aligned}$</td></tr>
<tr><td>oblem.</td><td>Substitute $s=13.5$ into of the original equations.</td><td>$\begin{aligned} 4(s+c) &= 60 \\ 4(13.5+c) &= 60 \\ 54+4c &= 60 \\ 4c &= 6 \\ c &= 1.5 \end{aligned}$</td></tr>
<tr><td>oblem.</td><td>Check the answer in the problem.</td><td>$\begin{aligned} 4(s+c) &= 60 \\ 4(13.5+1.5) &\stackrel{?}{=} 60 \\ 60 &\stackrel{?}{=} 60 \\ &\text{True} \end{aligned}$
The downstream rate would be $13.5+1.5=15$ mph. In 4 hours the ship would travel $15 \cdot 4 = 60$ miles.
$\begin{aligned} 5(s-c) &= 60 \\ 5(13.5-1.5) &\stackrel{?}{=} 60 \\ 60 &\stackrel{?}{=} 60 \\ &\text{True} \end{aligned}$
The upstream rate would be $13.5-1.5=12$ mph. In 5 hours the ship would travel $12 \cdot 5 = 60$ miles.
Does the answer make sense? We will leave this to you.</td></tr>
<tr><td>oblem.</td><td>Answer the question.</td><td>The rate of the ship is 13.5 mph and the rate of the current is 1.5 mph.</td></tr>
</table>

? Try It 2.2.27

Translate to a system of equations and then solve.

A Mississippi river boat cruise sailed 120 miles upstream for 12 hours and then took 10 hours to return to the dock. Find the speed of the river boat in still water and the speed of the river current.

Answer

The rate of the boat is 11 mph and the rate of the current is 1 mph.

Try It 2.2.28

Translate to a system of equations and then solve.

Jason paddled his canoe 24 miles upstream for 4 hours. It took him 3 hours to paddle back. Find the speed of the canoe in still water and the speed of the river current.

Answer

The speed of the canoe is 7 mph and the speed of the current is 1 mph.

Wind currents affect airplane speeds in the same way as water currents affect boat speeds. We will see this in the next example. A wind current in the same direction as the plane is flying is called a *tailwind*. A wind current blowing against the direction of the plane is called a *headwind*.

Example 2.2.29

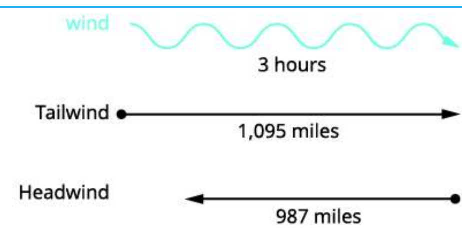
Translate to a system of equations and then solve.

A private jet can fly $1,095$ miles in three hours with a tailwind but only 987 miles in three hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Solution

Read the problem.

oblem. This is a uniform motion problem and a picture will help us visualize.



oblem. **Identify** what we are looking for.

We are looking for the speed of the jet in still air and the speed of the wind.

oblem. **Name** what we are looking for.

Let $j =$ the speed of the jet in still air and
$w =$ the speed of the wind.

oblem. A chart will help us organize the information.

	Rate	• Time	= Distance
tailwind	$j+w$	3	1095
headwind	$j-w$	3	987

$\text{Distance} = \text{Rate} \cdot \text{Time}$

oblem. **Translate** into a system of equations.

The jet makes two trips—one in a tailwind and one in a headwind.
In a tailwind, the wind helps the jet and so the rate is $j+w$.
In a headwind, the wind slows the jet and so the rate is $j-w$.
Each trip takes 3 hours. In a tailwind the jet flies $1,095$ miles. In a headwind the jet flies 987 miles.

The system is $\begin{cases} 3(j+w) = 1095 \\ 3(j-w) = 987 \end{cases}$.

	Read the problem.	
oblem.	**Solve** the system of equations. Distribute, then solve by elimination.	$\begin{cases} 3(j+w) &= 1095 \\ 3(j-w) &= 987 \end{cases}$ $\begin{cases} 3j+3w &= 1095 \\ 3j-3w &= 987 \end{cases}$
oblem.	Add and solve for j.	$6j = 2082$ $j = 347$
oblem.	Substitute $j = 347$ into one of the original equations,	$3(j+w) = 1095$ $3(347+w) = 1095$
oblem.	Solve for w.	$1041+3w = 1095$ $3w = 54$ $w = 18$
oblem.	**Check** the answer in the problem.	$3(j+w) = 1095$ $3(347+18) \stackrel{?}{=} 1095$ $1095 \stackrel{?}{=} 1095$ True With the tailwind, the actual rate of the jet would be $347+18=365$ mph. In 3 hours the jet would travel $365 \cdot 3 = 1,095$ miles $3(j-w) = 987$ $3(347-18) \stackrel{?}{=} 987$ $987 \stackrel{?}{=} 987$ True Going into the headwind, the jet's actual rate would be $347-18=329$ mph. In 3 hours the jet would travel $329 \cdot 3 = 987$ miles. Does the answer make sense? We will leave this to you.
oblem.	**Answer** the question.	The rate of the jet is 347 mph and the rate of the wind is 18 mph.

? Try It 2.2.30

Translate to a system of equations and then solve.

A small jet can fly $1,325$ miles in 5 hours with a tailwind but only $1,035$ miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 235 mph and the speed of the wind is 30 mph.

? Try It 2.2.31

Translate to a system of equations and then solve.

A commercial jet can fly $1,728$ miles in 4 hours with a tailwind but only $1,536$ miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

The speed of the jet is 408 mph and the speed of the wind is 24 mph.

Key Concepts

- **How To Solve Applications with Systems of Equations**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose variables to represent those quantities.
  4. **Translate** into a system of equations.
  5. **Solve** the system of equations using good algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Glossary

complementary angles

Two angles are complementary if the sum of the measures of their angles is 90 degrees.

supplementary angles

Two angles are supplementary if the sum of the measures of their angles is 180 degrees.

Practice Makes Perfect

Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

1. The sum of two number is 15. One number is 3 less than the other. Find the numbers.

2. The sum of two number is 30. One number is 4 less than the other. Find the numbers.

Answer

13 and 17

3. The sum of two number is −16. One number is 20 less than the other. Find the numbers.

4. The sum of two number is -26. One number is 12 less than the other. Find the numbers.

Answer

-7 and -19

5. The sum of two numbers is 65. Their difference is 25. Find the numbers.

6. The sum of two numbers is 37. Their difference is 9. Find the numbers.

Answer

14 and 23

7. The sum of two numbers is -27. Their difference is -59. Find the numbers.

8. The sum of two numbers is -45. Their difference is -89. Find the numbers.

Answer

22 and -67

9. Maxim has been offered positions by two car companies. The first company pays a salary of \$10,000 plus a commission of \$1000 for each car sold. The second pays a salary of \$20,000 plus a commission of \$500 for each car sold. How many cars would need to be sold to make the total pay the same?

10. Jackie has been offered positions by two cable companies. The first company pays a salary of $14,000 plus a commission of $100 for each cable package sold. The second pays a salary of $20,000 plus a commission of $25 for each cable package sold. How many cable packages would need to be sold to make the total pay the same?

Answer

Eighty cable packages would need to be sold to make the total pay the same.

11. Amara currently sells televisions for company A at a salary of $17,000 plus a $100 commission for each television she sells. Company B offers her a position with a salary of $29,000 plus a $20 commission for each television she sells. How televisions would Amara need to sell for the options to be equal?

12. Mitchell currently sells stoves for company A at a salary of $12,000 plus a $150 commission for each stove he sells. Company B offers him a position with a salary of $24,000 plus a $50 commission for each stove he sells. How many stoves would Mitchell need to sell for the options to be equal?

Answer

Mitchell would need to sell 120 stoves for the companies to be equal.

13. Two containers of gasoline hold a total of fifty gallons. The big container can hold ten gallons less than twice the small container. How many gallons does each container hold?

14. June needs 48 gallons of punch for a party and has two different coolers to carry it in. The bigger cooler is five times as large as the smaller cooler. How many gallons can each cooler hold?

Answer

8 and 40 gallons

15. Shelly spent 10 minutes jogging and 20 minutes cycling and burned 300 calories. The next day, Shelly swapped times, doing 20 minutes of jogging and 10 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

16. Drew burned 1800 calories Friday playing one hour of basketball and canoeing for two hours. Saturday he spent two hours playing basketball and three hours canoeing and burned 3200 calories. How many calories did he burn per hour when playing basketball? How many calories did he burn per hour when canoeing?

Answer

1000 calories playing basketball and 400 calories canoeing

17. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and thumb drive. Troy bought four notebooks and five thumb drives for $116. Lisa bought two notebooks and three thumb dives for $68. Find the cost of each notebook and each thumb drive.

18. Nancy bought seven pounds of oranges and three pounds of bananas for $17. Her husband later bought three pounds of oranges and six pounds of bananas for $12. What was the cost per pound of the oranges and the bananas?

Answer

Oranges cost $2 per pound and bananas cost $1 per pound

19. Andrea is buying some new shirts and sweaters. She is able to buy 3 shirts and 2 sweaters for $114 or she is able to buy 2 shirts and 4 sweaters for $164. How much does a shirt cost? How much does a sweater cost?

20. Peter is buying office supplies. He is able to buy 3 packages of paper and 4 staplers for $40 or he is able to buy 5 packages of paper and 6 staplers for $62. How much does a package of paper cost? How much does a stapler cost?

Answer

Package of paper $4, stapler $7

21. The total amount of sodium in 2 hot dogs and 3 cups of cottage cheese is 4720 mg. The total amount of sodium in 5 hot dogs and 2 cups of cottage cheese is 6300 mg. How much sodium is in a hot dog? How much sodium is in a cup of cottage cheese?

22. The total number of calories in 2 hot dogs and 3 cups of cottage cheese is 960 calories. The total number of calories in 5 hot dogs and 2 cups of cottage cheese is 1190 calories. How many calories are in a hot dog? How many calories are in a cup of cottage cheese?

Answer

Hot dog 150 calories, cup of cottage cheese 220 calories

23. Molly is making strawberry infused water. For each ounce of strawberry juice, she uses three times as many ounces of water as juice. How many ounces of strawberry juice and how many ounces of water does she need to make 64 ounces of strawberry infused water?

24. Owen is making lemonade from concentrate. The number of quarts of water he needs is 4 times the number of quarts of concentrate. How many quarts of water and how many quarts of concentrate does Owen need to make 100 quarts of lemonade?

Answer

Owen will need 80 quarts of water and 20 quarts of concentrate to make 100 quarts of lemonade.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

25. The difference of two complementary angles is 55 degrees. Find the measures of the angles.

26. The difference of two complementary angles is 17 degrees. Find the measures of the angles.

Answer

53.5degrees and 36.5degree.

27. Two angles are complementary. The measure of the larger angle is twelve less than twice the measure of the smaller angle. Find the measures of both angles.

28. Two angles are complementary. The measure of the larger angle is ten more than four times the measure of the smaller angle. Find the measures of both angles.

Answer

16 degrees and 74 degrees

29. The difference of two supplementary angles is 8 degrees. Find the measures of the angles.

30. The difference of two supplementary angles is 88 degrees. Find the measures of the angles.

Answer

134 degrees and 46 degrees

31. Two angles are supplementary. The measure of the larger angle is four more than three times the measure of the smaller angle. Find the measures of both angles.

32. Two angles are supplementary. The measure of the larger angle is five less than four times the measure of the smaller angle. Find the measures of both angles.

Answer

37 degrees and 143 degrees

33. The measure of one of the small angles of a right triangle is 14 more than 3 times the measure of the other small angle. Find the measure of both angles.

34. The measure of one of the small angles of a right triangle is 26 more than 3 times the measure of the other small angle. Find the measure of both angles.

Answer

16° and 74°

35. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

36. The measure of one of the small angles of a right triangle is 45 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

45° and 45°

37. Wayne is hanging a string of lights 45 feet long around the three sides of his patio, which is adjacent to his house. The length of his patio, the side along the house, is five feet longer than twice its width. Find the length and width of the patio.

38. Darrin is hanging 200 feet of Christmas garland on the three sides of fencing that enclose his front yard. The length is five feet less than three times the width. Find the length and width of the fencing.

Answer

Width is 41 feet and length is 118 feet.

39. A frame around a family portrait has a perimeter of 90 inches. The length is fifteen less than twice the width. Find the length and width of the frame.

40. The perimeter of a toddler play area is 100 feet. The length is ten more than three times the width. Find the length and width of the play area.

Answer

Width is 10 feet and length is 40 feet.

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

41. Sarah left Minneapolis heading east on the interstate at a speed of 60 mph. Her sister followed her on the same route, leaving two hours later and driving at a rate of 70 mph. How long will it take for Sarah's sister to catch up to Sarah?

42. College roommates John and David were driving home to the same town for the holidays. John drove 55 mph, and David, who left an hour later, drove 60 mph. How long will it take for David to catch up to John?

Answer

11 hours

43. At the end of spring break, Lucy left the beach and drove back towards home, driving at a rate of 40 mph. Lucy's friend left the beach for home 30 minutes (half an hour) later, and drove 50 mph. How long did it take Lucy's friend to catch up to Lucy?

44. Felecia left her home to visit her daughter driving 45 mph. Her husband waited for the dog sitter to arrive and left home twenty minutes (1/3 hour) later. He drove 55 mph to catch up to Felecia. How long before he reaches her?

Answer

1.5 hour

45. The Jones family took a 12-mile canoe ride down the Indian River in two hours. After lunch, the return trip back up the river took three hours. Find the rate of the canoe in still water and the rate of the current.

46. A motor boat travels 60 miles down a river in three hours but takes five hours to return upstream. Find the rate of the boat in still water and the rate of the current.

Answer

Boat rate is 16 mph and current rate is 4 mph.

47. A motor boat traveled 18 miles down a river in two hours but going back upstream, it took 4.54.5 hours due to the current. Find the rate of the motor boat in still water and the rate of the current. (Round to the nearest hundredth.)

48. A river cruise boat sailed 80 miles down the Mississippi River for four hours. It took five hours to return. Find the rate of the cruise boat in still water and the rate of the current.

Answer

Boat rate is 18 mph and current rate is 2 mph.

49. A small jet can fly 1072 miles in 4 hours with a tailwind but only 848 miles in 4 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

50. A small jet can fly 1435 miles in 5 hours with a tailwind but only 1,215 miles in 5 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 265 mph and wind speed is 22 mph.

51. A commercial jet can fly 868 miles in 2 hours with a tailwind but only 792 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

52. A commercial jet can fly 1,320 miles in 3 hours with a tailwind but only 1170 miles in 3 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

Jet rate is 415 mph and wind speed is 25 mph.

Writing Exercises

53. Write an application problem similar to Example 2.2.4. Then translate to a system of equations and solve it.\

54. Write a uniform motion problem similar to Example 2.2.23 that relates to where you live with your friends or family members. Then translate to a system of equations and solve it.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve direct translation applications.			
solve geometry applications.			
solve uniform motion applications.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

2.3: Solve Mixture Applications with Systems of Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve mixture applications
- Solve interest applications
- Solve applications of cost and revenue functions

Be Prepared

Before you get started, take this readiness quiz.

1. Multiply $4.025(1,562)$
2. Write 8.2% as a decimal.
3. Earl's dinner bill came to $\$32.50$ and he wanted to leave an 18% tip. How much should the tip be?

Solve Mixture Applications

Mixture applications involve combining two or more quantities. When we solved mixture applications with coins and tickets earlier, we started by creating a table so we could organize the information. For a coin example with nickels and dimes, the table looked like this:

	Number •	Value =	Total Value
nickels			
dimes			

Using one variable meant that we had to relate the number of nickels and the number of dimes. We had to decide if we were going to let n be the number of nickels and then write the number of dimes in terms of n, or if we would let d be the number of dimes and write the number of nickels in terms of d.

Now that we know how to solve systems of equations with two variables, we will just let n be the number of nickels and d be the number of dimes. We will write one equation based on the total value column, like we did before, and the other equation will come from the number column.

For the first example, we will do a ticket problem where the ticket prices are in whole dollars, so we won't need to use decimals just yet.

Example 2.3.1

Translate to a system of equations and solve.

A science center sold $1,363$ tickets on a busy weekend. The receipts totaled $\$12,146$. How many $\$12$ adult tickets and how many $\$7$ child tickets were sold?

Solution

Read the problem.
We will create a table to organize the information.

oble…	**Identify** what we are looking for.	We are looking for the number of adult tickets and the number of child tickets sold.
oble…	**Name** what we are looking for.	Let $a =$ the number of adult tickets and $c =$ the number of child tickets.

Read the problem.
We will create a table to organize the information.

oble… A table will help us organize the data.

We have two types of tickets, adult and child.
Write the total number of tickets sold at the bottom of the Number column: altogether $1,363$ were sold.
Write the value of each type of ticket in the Value column: the value of each adult ticket is \$12 and the value of each child ticket is \$7.
The number times the value gives the total value, so the total value of adult tickets is $a \cdot 12 = 12a$, and the total value of child tickets is $c \cdot 7 = 7c$.
Fill in the Total Value column: altogether the total value of the tickets was \$12,146.

Type	Number	•	Value	=	Total Value
adult	*a*		12		11*a*
child	*c*		7		8*c*
	1,363				12,146

oble… **Translate** into a system of equations.

The Number column and the Total value column give us the system of equations.

$$\begin{cases} a + c = 1,363 \\ 12a + 7c = 12,146 \end{cases}$$

oble… **Solve** the system.
We will use the elimination method to solve this system.

$$\begin{cases} a + c = 1,363 \\ 12a + 7c = 12,146 \end{cases}$$

oble… Multiply the first equation by -7.

$$\begin{cases} (-7)(a + c) = (-7)(1,363) \\ 12a + 7c = 12,146 \end{cases}$$

oble… Simplify.

$$\begin{cases} -7a - 7c = -9,541 \\ 12a + 7c = 12,146 \end{cases}$$

oble… Add, then solve for a.

$$\begin{aligned} 5a &= 2,605 \\ a &= 521 \end{aligned}$$

oble… Substitute $a = 521$ into the first original equation, then solve for c.

$$\begin{aligned} a + c &= 1,363 \\ 521 + c &= 1,363 \\ c &= 842 \end{aligned}$$

	Read the problem. We will create a table to organize the information.	
…oble…	**Check** the answer in the problem.	$$\begin{aligned} a+c &= 1,363 \\ 521+842 &\stackrel{?}{=} 1,363 \\ 1,363 &\stackrel{?}{=} 1,363 \\ &\text{True} \end{aligned}$$ 521 adult tickets and 842 children tickets give a total of $1,363$ tickets. $$\begin{aligned} 12a+7c &= 12,146 \\ 12(521)+7(842) &\stackrel{?}{=} 12,146 \\ 12,146 &\stackrel{?}{=} 12,146 \\ &\text{True} \end{aligned}$$ 521 adult tickets at \$12 per ticket make \$6, 252. 842 child tickets at \$7 per ticket make \$5, 894. The total receipts are \$12, 146. ✓ Do these numbers make sense in the problem? We will leave this to you.
…oble…	**Answer** the question.	The science center sold 521 adult tickets and 842 child tickets.

? Try It 2.3.2

Translate to a system of equations and solve.

The ticket office at the zoo sold 553 tickets one day. The receipts totaled $\$3,936$. How many \$9 adult tickets and how many \$6 child tickets were sold?

Answer

The ticket office sold 206 adult tickets and 347 children tickets.

? Try It 2.3.3

Translate to a system of equations and solve.

The box office at a movie theater sold 147 tickets for the evening show, and receipts totaled $\$1,302$. How many $\$11$ adult and how many $\$8$ child tickets were sold?

Answer

The box office sold 42 adult tickets and 105 children tickets.

In the next example, we will solve a coin problem. Now that we know how to work with systems of two variables, naming the variables in the 'number' column will be easy.

? Example 2.3.4

Translate to a system of equations and solve.

Juan has a pocketful of nickels and dimes. The total value of the coins is $\$8.10$. The number of dimes is 9 less than twice the number of nickels. How many nickels and how many dimes does Juan have?

Solution

Read the problem.
We will create a table to organize the information.

oble… **Identify** what we are looking for.

We are looking for the number of nickels and the number of dimes.

oble… **Name** what we are looking for.

Let $n = \text{the number of nickels}$ and
$d = \text{the number of dimes.}$

oble… A table will help us organize the data.

We have two types of coins, nickels and dimes.
Write n and d for the number of each type of coin.
Fill in the Value column with the value of each type of coin. The value of each nickel is $\$0.05$. The value of each dime is $\$0.10$.
The number times the value gives the total value, so, the total value of the nickels is $n(0.05) = 0.05n$ and the total value of dimes is $d(0.10) = 0.10d$. Altogether the total value of the coins is $\$8.10$.

Type	Number •	Value =	Total Value
nickels	n	0.05	$0.05n$
dimes	d	0.10	$0.10d$
			8.10

oble… **Translate** into a system of equations.

The Total Value column gives one equation: $0.05n + 0.10d = 8.10$.
We also know the number of dimes is 9 less than twice the number of nickels: $d = 2n - 9$.
Now we have the system to solve.

$$\begin{cases} 0.05n + 0.10d = 8.10 \\ d = 2n - 9 \end{cases}$$

oble… **Solve** the system of equations.
We will use the substitution method since the second equation is solved for d.

$$\begin{aligned} 0.05n + 0.10d &= 8.10 \\ d &= 2n - 9 \end{aligned}$$

oble… Substitute $d = 2n - 9$ into the first equation.

$$\begin{aligned} 0.05n + 0.10d &= 8.10 \\ 0.05n + 0.10(2n - 9) &= 8.10 \end{aligned}$$

oble… Simplify and solve for n.

$$\begin{aligned} 0.05n + 0.20n - 0.90 &= 8.10 \\ 0.25n - 0.90 &= 8.10 \\ 0.25n &= 9 \\ n &= 36 \end{aligned}$$

oble… To find the number of dimes, substitute $n = 36$ into the second equation.

$$\begin{aligned} d &= 2n - 9 \\ d &= 2(36) - 9 \\ d &= 63 \end{aligned}$$

	Read the problem. We will create a table to organize the information.	
oble…	**Check** the answer in the problem.	$\begin{aligned} 0.05n+0.10d &= 8.10 \\ 0.05(36)+0.10(63) &\stackrel{?}{=} 8.10 \\ 8.10 &\stackrel{?}{=} 8.10 \\ &\text{True} \end{aligned}$ 63 dimes at $\$0.10=\$6.30$ 36 nickels at $\$0.05=\$1.80$ Total $=\$8.10\checkmark$ $\begin{aligned} d &= 2n-9 \\ 63 &\stackrel{?}{=} 2(36)-9 \\ 63 &\stackrel{?}{=} 63 \\ &\text{True} \end{aligned}$ Do these numbers make sense in the problem? We will leave this to you.
oble…	**Answer** the question.	Juan has 36 nickels and 63 dimes.

Try It 2.3.5

Translate to a system of equations and solve.

Matilda has a handful of quarters and dimes, with a total value of $\$8.55$. The number of quarters is 3 more than twice the number of dimes. How many dimes and how many quarters does she have?

Answer

Matilda has 13 dimes and 29 quarters.

Try It 2.3.6

Translate to a system of equations and solve.

Priam has a collection of nickels and quarters, with a total value of $\$7.30$. The number of nickels is six less than three times the number of quarters. How many nickels and how many quarters does he have?

Answer

Priam has 19 quarters and 51 nickels.

Some mixture applications involve combining foods or drinks. Example situations might include combining raisins and nuts to make a trail mix or using two types of coffee beans to make a blend.

Example 2.3.7

Translate to a system of equations and solve.

Carson wants to make 20 pounds of trail mix using nuts and chocolate chips. His budget requires that the trail mix costs him $\$7.60$ per pound. Nuts cost $\$9.00$ per pound and chocolate chips cost $\$2.00$ per pound. How many pounds of nuts and how many pounds of chocolate chips should he use?

Solution

Read the problem. We will create a table to organize the information.	

	Read the problem. We will create a table to organize the information.	
...oble...	**Identify** what we are looking for.	We are looking for the number of pounds of nuts and the number of pounds of chocolate chips.
...oble...	**Name** what we are looking for.	Let $n=$ the number of pound of nuts and $c=$ the number of pounds of chips.
...oble...	A table will help us organize the data.	Carson will mix nuts and chocolate chips to get trail mix. Write in n and c for the number of pounds of nuts and chocolate chips. There will be 20 pounds of trail mix. Put the price per pound of each item in the Value column. Fill in the last column using $\text{Number} \cdot \text{Value} = \text{Total Value}$
...oble...	**Translate** into a system of equations.	We get the equations from the Number and Total Value columns. $\begin{cases} n+c=20 \\ 9n+2c=152 \end{cases}$
...oble...	**Solve** the system of equations We will use elimination to solve the system.	$\begin{cases} n+c=20 \\ 9n+2c=152 \end{cases}$
...oble...	Multiply the first equation by -2 to eliminate c.	$\begin{cases} -2(n+c)=-2(20) \\ 9n+2c=152 \end{cases}$
...oble...	Simplify.	$\begin{cases} -2n-2c=-40 \\ 9n+2c=152 \end{cases}$
...oble...	Add and solve for n.	$7n=112$ $n=16$
...oble...	To find the number of pounds of chocolate chips, substitute $n=16$ into the first equation, then solve for c.	$n+c=20$ $16+c=20$ $c=4$
...oble...	**Check** the answer in the problem.	$n+c=20$ $16+4\stackrel{?}{=}20$ $20\stackrel{?}{=}20$ True There are 20 pounds of trail mix. $9n+2c=152$ $9(16)+2(4)\stackrel{?}{=}152$ $152\stackrel{?}{=}152$ True 16 pounds of nuts at $\$9.00=\$144$ 4 pounds of chocolate chips at $\$2=\$8.00$ Total $=\$152$ ✓ Do these numbers make sense in the problem? We will leave this to you.

Table for "A table will help us organize the data":

Type	Number of pounds	•	Value	=	Total Value
Nuts	n		9.00		$9n$
Chocolate chips	c		2.00		$2c$
Trail mix	20		7.60		$7.60(20)=152$

	Read the problem. We will create a table to organize the information.	
oble…	**Answer** the question.	Carson should mix 16 pounds of nuts with 4 pounds of chocolate chips to create the trail mix.

? Try It 2.3.8

Translate to a system of equations and solve.

Greta wants to make 5 pounds of a nut mix using peanuts and cashews. Her budget requires the mixture to cost her $\$6$ per pound. Peanuts are $\$4$ per pound and cashews are $\$9$ per pound. How many pounds of peanuts and how many pounds of cashews should she use?

Answer

Greta should use 3 pounds of peanuts and 2 pounds of cashews.

? Try It 2.3.9

Translate to a system of equations and solve.

Sammy has most of the ingredients he needs to make a large batch of chili. The only items he lacks are beans and ground beef. He needs a total of 20 pounds combined of beans and ground beef and has a budget of $\$3$ per pound. The price of beans is $\$1$ per pound and the price of ground beef is $\$5$ per pound. How many pounds of beans and how many pounds of ground beef should he purchase?

Answer

Sammy should purchase 10 pounds of beans and 10 pounds of ground beef.

Another application of mixture problems relates to concentrated cleaning supplies, other chemicals, and mixed drinks. The concentration is given as a percent. For example, a 20% concentrated household cleanser means that 20% of the total amount is cleanser, and the rest is water. To make 35 ounces of a 20% concentration, we mix 7 ounces (20% of 35) of the cleanser with 28 ounces of water.

For these kinds of mixture problems, we will use “percent” instead of “value” for one of the columns in our table.

? Example 2.3.10

Translate to a system of equations and solve.

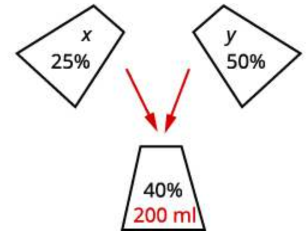
Sasheena is lab assistant at her community college. She needs to make 200 milliliters of a 40% solution of sulfuric acid for a lab experiment. The lab has only 25% and 50% solutions in the storeroom. How much should she mix of the 25% and the 50% solutions to make the 40% solution?

Solution

	Read the problem. A figure may help us visualize the situation, then we will create a table to organize the information.	
oble…	**Identify** what we are looking for.	We are looking for how much of each solution she needs.
oble…	**Name** what we are looking for.	Let $x =$ number of ml of 25% solution and $y =$ number of ml of 50% solution.

oble… A figure will help us visualize the situation.

Sasheena must mix some of the 25% solution and some of the 50% solution together to get 200 ml of the 40% solution.



oble… A table will help us organize the data.

She will mix x ml of 25% with y ml of 50% to get 200 ml of 40% solution.
We write the percents as decimals in the chart.
We multiply the number of units times the concentration to get the total amount of sulfuric acid in each solution.

Type	Number of units	•	Concentration %	=	Amount
25%	x		0.25		$0.25x$
50%	y		0.50		$0.50y$
40%	200		0.40		0.40(200)

oble… **Translate** into a system of equations.

We get the equations from the Number column and the Amount column.
Now we have the system.

$$\begin{cases} x + y = 200 \\ 0.25x + 0.50y = 0.40(200) \end{cases}$$

oble… **Solve** the system of equations
We will solve the system by elimination.

$$\begin{cases} x + y = 200 \\ 0.25x + 0.50y = 80 \end{cases}$$

oble… Multiply the first equation by -0.5 to eliminate y.

$$\begin{cases} -0.5(x + y) = -0.5(200) \\ 0.25x + 0.50y = 80 \end{cases}$$

oble… Simplify.

$$\begin{cases} -0.5x - 0.5y = -100 \\ 0.25x + 0.50y = 80 \end{cases}$$

oble… Add to solve for x.

$$\begin{aligned} -0.25x &= -20 \\ x &= 80 \end{aligned}$$

oble… To solve for y, substitute $x = 80$ into the first equation.

$$\begin{aligned} x + y &= 200 \\ 80 + y &= 200 \\ y &= 120 \end{aligned}$$

oble…	**Check** the answer in the problem.	$$\begin{aligned} x+y&=200\\ 80+120&\stackrel{?}{=}200\\ 200&\stackrel{?}{=}200\\ &\text{True}\end{aligned}$$ There are 200 ml of the 40% solution. $$\begin{aligned} 0.25x+0.50y&=80\\ 0.25(80)+0.50(120)&\stackrel{?}{=}80\\ 80&\stackrel{?}{=}80\\ &\text{True}\end{aligned}$$ 25% of 80 ml is 20 ml 50% of 120 ml is 60 ml Total = 80 ml ✓ Do these numbers make sense in the problem? We will leave this to you.
oble…	**Answer** the question.	Sasheena should mix 80 ml of the 25% solution with 120 ml of the 50% solution to get the 200 ml of the 40% solution.

? Try It 2.3.11

Translate to a system of equations and solve.

LeBron needs 150 milliliters of a 30% solution of sulfuric acid for a lab experiment but only has access to a 25% and a 50% solution. How much of the 25% and how much of the 50% solution should he mix to make the 30% solution?

Answer

LeBron should mix 120 ml of the 25% solution and 30 ml of the 50% solution.

? Try It 2.3.12

Translate to a system of equations and solve.

Anatole needs to make 250 milliliters of a 25% solution of hydrochloric acid for a lab experiment. The lab only has a 10% solution and a 40% solution in the storeroom. How much of the 10% and how much of the 40% solution should he mix to make the 25% solution?

Answer

Anatole should mix 125 ml of the 10% solution and 125 ml of the 40% solution.

Solve Interest Applications

The formula to model simple interest applications is $I=Prt$, where I is the interest, P is the principal, r is the rate, t is the time. In our work here, we will calculate the interest earned in one year, so t will be 1.

We modify the column titles in the mixture table to show the formula for interest, as we will see in the next example.

? Example 2.3.13

Translate to a system of equations and solve.

Adnan has $\$40,000$ to invest and hopes to earn 7.1% interest per year. He will put some of the money into a stock fund that earns 8% per year and the rest into bonds that earns 3% per year. How much money should he put into each fund?

Solution

oblem.	**Read** the problem.	
oblem.	**Identify** what we are looking for.	We are looking for the amount to invest in each fund.
oblem.	**Name** what we are looking for.	Let $s =$ the amount invested in stocks and $b =$ the amount invested in stocks.
oblem.	A chart will help us organize the information.	Write the interest rate as a decimal for each fund. Multiply $\text{Principal} \cdot \text{Rate} \cdot \text{Time}$ to find the interest.
oblem.	**Translate** into a system of equations.	We get our system of equations from the Principal column and the Interest column. $\begin{cases} s+b=40,000 \\ 0.08s+0.03b=0.071(40,000) \end{cases}$
oblem.	**Solve** the system of equations. We will solve by elimination.	$\begin{cases} s+b=40,000 \\ 0.08s+0.03b=0.071(40,000) \end{cases}$
oblem.	Multiply the top equation by -0.03.	$\begin{cases} -0.03(s+b)=-0.03(40,000) \\ 0.08s+0.03b=0.071(40,000) \end{cases}$
oblem.	Simplify.	$\begin{cases} -0.03s-0.03b=-1,200 \\ 0.08s+0.03b=2,840 \end{cases}$
oblem.	Add and solve for s.	$0.05s=1,640$ $s=32,800$
oblem.	To find b, substitute $s=32,800$ into the first equation.	$s+b=40,000$ $32,800+b=40,000$ $b=7,200$
oblem.	**Check** the answer in the problem.	We leave the check to you.
oblem.	**Answer** the question.	Adnan should invest $\$32,800$ in stock and $\$7,200$ in bonds.

Account	Principal •	Rate •	Time =	Interest
Stock fund	s	0.08	1	$0.08s$
Bonds	b	0.03	1	$0.03b$
Total	40,000	0.071		0.071(40,000)

Did you notice that the Principal column represents the total amount of money invested while the Interest column represents only the interest earned? Likewise, the first equation in our system, $s+b=40,000$, represents the total amount of money invested and the second equation, $0.08s+0.03b=0.071(40,000)$ represents the interest earned.

? Try It 2.3.14

Translate to a system of equations and solve.

Leon had $\$50,000$ to invest and hopes to earn 6.2% interest per year. He will put some of the money into a stock fund that earns 7% per year and the rest in to a savings account that earns 2% per year. How much money should he put into each fund?

Answer

Leon should put $\$42,000$ into the stock fund and $\$8,000$ into the savings account.

? Try It 2.3.15

Translate to a system of equations and solve.

Julius invested $\$7,000$ into two stock investments. One stock paid 11% interest and the other stock paid 13% interest. He earned 12.5% interest on the total investment. How much money did he put in each stock?

Answer

Julius put $\$1,750$at 11%interest and $\$5,250$at 13%interest.

The next example requires that we find the principal given the amount of interest earned.

Example 2.3.16

Translate to a system of equations and solve.

Rosie owes $\$21,540$ on her two student loans. The interest rate on her bank loan is 10.5 and the interest rate on the federal loan is 5.9. The total amount of interest she paid last year was $\$1,669.68$ What was the principal for each loan?

Solution

<table>
<tr><td></td><td>Read the problem.
A chart will help us organize the information.</td><td></td></tr>
<tr><td>oble…</td><td>Identify what we are looking for.</td><td>We are looking for the principal of each loan.</td></tr>
<tr><td>oble…</td><td>Name what we are looking for.</td><td>Let $b=$ the principal for the bank loan and
$f=$ the principal on the federal loan.</td></tr>
<tr><td>oble…</td><td></td><td>The total loans are $21,540.
Record the interest rates as decimals in the chart.
Multiply using the formula $I=Prt$ to get the Interest.
<table>
<tr><th>Account</th><th>Principal •</th><th>Rate •</th><th>Time =</th><th>Interest</th></tr>
<tr><td>Bank</td><td>b</td><td>0.105</td><td>1</td><td>0.105b</td></tr>
<tr><td>Federal</td><td>f</td><td>0.059</td><td>1</td><td>0.059f</td></tr>
<tr><td>Total</td><td>21,540</td><td></td><td></td><td>1669.68</td></tr>
</table></td></tr>
<tr><td>oble…</td><td>Translate into a system of equations.</td><td>The system of equations comes from the Principal column and the Interest column.
$\begin{cases} b+f=21,540 \\ 0.105b+0.059f=1669.68 \end{cases}$</td></tr>
<tr><td>oble…</td><td>Solve the system of equations.
We will use substitution to solve.</td><td>$\begin{cases} b+f=21,540 \\ 0.105b+0.059f=1669.68 \end{cases}$</td></tr>
<tr><td>oble…</td><td>Solve the first equation for b.</td><td>$\begin{aligned} b+f&=21,540 \\ b&=-f+21,540 \end{aligned}$</td></tr>
<tr><td>oble…</td><td>Substitute $b=-f+21,540$ into the second equation.</td><td>$\begin{aligned} 0.105b+0.059f&=1669.68 \\ 0.105(-f+21,540)+0.059f&=1669.68 \\ -0.105f+2261.70+0.059f&=1669.68 \end{aligned}$</td></tr>
<tr><td>oble…</td><td>Simplify and solve for f.</td><td>$\begin{aligned} -0.105f+2261.70+0.059f&=1669.68 \\ -0.046f+2261.70&=1669.68 \\ -0.046f&=-592.02 \\ f&=12,870 \end{aligned}$</td></tr>
<tr><td>oble…</td><td>To find b, substitute $f=12,870$ into the first equation.</td><td>$\begin{aligned} b+f&=21,540 \\ b+12,870&=21,540 \\ b&=8,670 \end{aligned}$</td></tr>
<tr><td>oble…</td><td>Check the answer in the problem.</td><td>We leave the check to you.</td></tr>
<tr><td>oble…</td><td>Answer the question.</td><td>The principal of the federal loan was $12, 870. and the principal for the bank loan was $8, 670.</td></tr>
</table>

Try It 2.3.17

Translate to a system of equations and solve.

Laura owes $\$18,000$ on her student loans. The interest rate on the bank loan is 2.5% and the interest rate on the federal loan is 6.9%. The total amount of interest she paid last year was $\$1,066$. What was the principal for each loan?

Answer

The principal of the bank loan was $\$4,000$ and the principal of the Federal loan was $\$14,000$.

Try It 2.3.18

Translate to a system of equations and solve.

Jill's Sandwich Shoppe owes $\$65,200$ on two business loans, one at 4.5% interest and the other at 7.2% interest. The total amount of interest owed last year was $\$3,582$. What was the principal for each loan?

Answer

The principal was $\$41,200$ at 4.5% interest, and the other one was $\$24,000$ at 7.2% interest.

Solve Applications of Cost and Revenue Functions

Suppose a company makes and sells x units of a product. The cost to the company is the total costs to produce x units. This is the cost to manufacture for each unit times x, the number of units manufactured, plus the fixed costs.

The *revenue* is the money the company brings in as a result of selling x units. This is the selling price of each unit times the number of units sold.

When the costs equal the revenue we say the business has reached the *break-even point*.

Cost and revenue functions

The **cost function** is the cost to manufacture each unit times x, the number of units manufactured, plus the fixed costs.

$$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$$

The **revenue function** is the selling price of each unit times x, the number of units sold.

$$R(x) = (\text{selling price per unit}) \cdot x$$

The **break-even point** is when the revenue equals the costs.

$$C(x) = R(x)$$

Example 2.3.19

The manufacturer of a weight training bench spends $\$105$ to build each bench and sells them for $\$245$. The manufacturer also has fixed costs each month of $\$7,000$.

a. Find the cost function C when x benches are manufactured.

b. Find the revenue function R when x benches are sold.

c. Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

d. Find the break-even point. Interpret what the break-even point means.

Solution

a. The manufacturer has $\$7,000$ of fixed costs no matter how many weight training benches it produces. In addition to the fixed costs, the manufacturer also spends $\$105$ to produce each bench. Suppose x benches are sold.

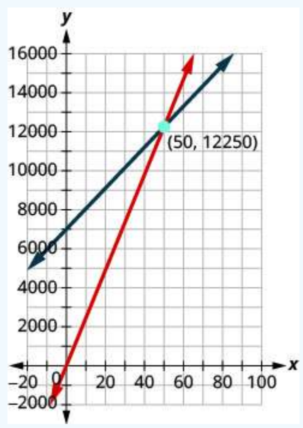
Write the general cost function formula.	$C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$
Substitute in the cost values.	$C(x) = 105x + 7000$
Answer the question.	The cost function is $C(x) = 105x + 7000$ when x benches are sold.

b. The manufacturer sells each weight training bench for \$245. We get the total revenue by multiplying the revenue per unit times the number of units sold.

Write the general revenue function.	$R(x) = (\text{selling price per unit}) \cdot x$
Substitute in the revenue per unit.	$R(x) = 245x$
Answer the question.	The revenue function is $R(x) = 245x$ when x benches are sold.

c. Essentially we have a system of linear equations. We will show the graph of the system as this helps make the idea of a break-even point more visual.

$$\begin{cases} C(x) = 105x + 7000 \\ R(x) = 245x \end{cases} \quad \text{or} \quad \begin{cases} y = 105x + 7000 \\ y = 245x \end{cases}$$



d. To find the actual value, we remember the break-even point occurs when costs equal revenue.

Write the break-even formula.	$C(x) = R(x)$
Substitute $C(x) = 105x + 7000$ and $R(x) = 245x$ into the formula.	$105x + 7000 = 245x$
Solve for x.	$7000 = 140x$ $50 = x$ $x = 50$

When 50 benches are sold, the costs equal the revenue.

$C(x) = 105x + 7000$	$R(x) = 245x$
$C(50) = 105(50) + 7000$	$R(50) = 245 \cdot 50$
$C(50) = 12,250$	$R(50) = 12,250$

When 50 benches are sold, the revenue and costs are both $\$12,250$ Notice this corresponds to the ordered pair $(50, 12250)$

Try It 2.3.20

The manufacturer of a weight training bench spends $\$15$ to build each bench and sells them for $\$32$. The manufacturer also has fixed costs each month of $\$25,500$.

a. Find the cost function C when x benches are manufactured.

b. Find the revenue function R when x benches are sold.

c. Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

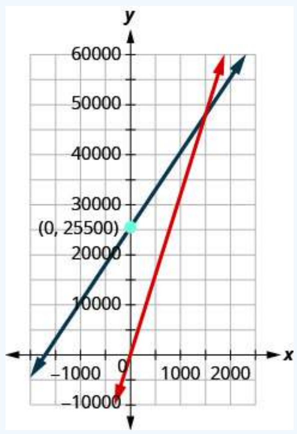
d. Find the break-even point. Interpret what the break-even point means.

Answer

a. The cost function is $C(x) = 15x + 25,500$.

b. The revenue function is $R(x) = 32x$.

c.



d. The break-even point is at $1,500$. When $1,500$ benches are sold, the cost and revenue will be both $48,000$

Try It 2.3.21

The manufacturer of a weight training bench spends $\$120$ to build each bench and sells them for $\$170$. The manufacturer also has fixed costs each month of $\$150,000$

a. Find the cost function C when x benches are manufactured.

b. Find the revenue function R when x benches are sold.

c. Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

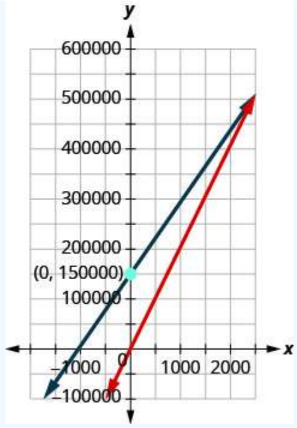
d. Find the break-even point. Interpret what the break-even point means.

Answer

a. The cost function is $C(x) = 120x + 150,000$.

b. The revenue function is $R(x) = 170x$.

c.



d. The break-even point is at $3,000$. When $3,000$ benches are sold, the revenue and costs are both $\$510,000$.

Access this online resource for additional instruction and practice with interest and mixtures.

- Interest and Mixtures

Key Concepts

- **Cost function:** The cost function is the cost to manufacture each unit times *x*, the number of units manufactured, plus the fixed costs.

  $C(x) = (\text{cost per unit}) \cdot x + \text{fixed costs}$
- **Revenue:** The revenue function is the selling price of each unit times *x*, the number of units sold.

  $R(x) = (\text{selling price per unit}) \cdot x$
- **Break-even point:** The break-even point is when the revenue equals the costs.

  $C(x) = R(x)$

Glossary

cost function

The cost function is the cost to manufacture each unit times xx, the number of units manufactured, plus the fixed costs; *C*(*x*) = (cost per unit)*x* + fixed costs.

revenue

The revenue is the selling price of each unit times *x*, the number of units sold; *R(x*) = (selling price per unit)*x*.

break-even point

The point at which the revenue equals the costs is the break-even point; C(x)=R(x).

Practice Makes Perfect

Solve Mixture Applications

In the following exercises, translate to a system of equations and solve.

1. Tickets to a Broadway show cost $35 for adults and $15 for children. The total receipts for 1650 tickets at one performance were $47,150. How many adult and how many child tickets were sold?

2. Tickets for the Cirque du Soleil show are $70 for adults and $50 for children. One evening performance had a total of 300 tickets sold and the receipts totaled $17,200. How many adult and how many child tickets were sold?

Answer

110 adult tickets, 190 child tickets

3. Tickets for an Amtrak train cost $10 for children and $22 for adults. Josie paid $1200 for a total of 72 tickets. How many children tickets and how many adult tickets did Josie buy?

4. Tickets for a Minnesota Twins baseball game are $69 for Main Level seats and $39 for Terrace Level seats. A group of sixteen friends went to the game and spent a total of $804 for the tickets. How many of Main Level and how many Terrace Level tickets did they buy?

Answer

6 good seats, 10 cheap seats

5. Tickets for a dance recital cost $15 for adults and $7 dollars for children. The dance company sold 253 tickets and the total receipts were $2771. How many adult tickets and how many child tickets were sold?

6. Tickets for the community fair cost $12 for adults and $5 dollars for children. On the first day of the fair, 312 tickets were sold for a total of $2204. How many adult tickets and how many child tickets were sold?

Answer

92 adult tickets, 220 children tickets

7. Brandon has a cup of quarters and dimes with a total value of $\$3.80$. The number of quarters is four less than twice the number of quarters. How many quarters and how many dimes does Brandon have?

8. Sherri saves nickels and dimes in a coin purse for her daughter. The total value of the coins in the purse is $\$0.95$. The number of nickels is two less than five times the number of dimes. How many nickels and how many dimes are in the coin purse?

Answer

13 nickels, 3 dimes

9. Peter has been saving his loose change for several days. When he counted his quarters and nickels, he found they had a total value $\$13.10$. The number of quarters was fifteen more than three times the number of dimes. How many quarters and how many dimes did Peter have?

10. Lucinda had a pocketful of dimes and quarters with a value of $\$6.20$. The number of dimes is eighteen more than three times the number of quarters. How many dimes and how many quarters does Lucinda have?

Answer

42 dimes, 8 quarters

11. A cashier has 30 bills, all of which are $10 or $20 bills. The total value of the money is $460. How many of each type of bill does the cashier have?

12. A cashier has 54 bills, all of which are $10 or $20 bills. The total value of the money is $910. How many of each type of bill does the cashier have?

Answer

17 $10 bills, 37 $20 bills

13. Marissa wants to blend candy selling for $\$1.80$ per pound with candy costing $\$1.20$ per pound to get a mixture that costs her $\$1.40$ per pound to make. She wants to make 90 pounds of the candy blend. How many pounds of each type of candy should she use?

14. How many pounds of nuts selling for $6 per pound and raisins selling for $3 per pound should Kurt combine to obtain 120 pounds of trail mix that cost him $5 per pound?

Answer

80 pounds nuts and 40 pounds raisins

15. Hannah has to make twenty-five gallons of punch for a potluck. The punch is made of soda and fruit drink. The cost of the soda is $\$1.79$ per gallon and the cost of the fruit drink is $\$2.49$ per gallon. Hannah's budget requires that the punch cost $\$2.21$ per gallon. How many gallons of soda and how many gallons of fruit drink does she need?

16. Joseph would like to make twelve pounds of a coffee blend at a cost of $6 per pound. He blends Ground Chicory at $5 a pound with Jamaican Blue Mountain at $9 per pound. How much of each type of coffee should he use?

Answer

9 pounds of Chicory coffee, 3 pounds of Jamaican Blue Mountain coffee

17. Julia and her husband own a coffee shop. They experimented with mixing a City Roast Columbian coffee that cost $7.80 per pound with French Roast Columbian coffee that cost $8.10 per pound to make a twenty-pound blend. Their blend should cost them $7.92 per pound. How much of each type of coffee should they buy?

18. Twelve-year old Melody wants to sell bags of mixed candy at her lemonade stand. She will mix M&M's that cost $4.89 per bag and Reese's Pieces that cost $3.79 per bag to get a total of twenty-five bags of mixed candy. Melody wants the bags of mixed candy to cost her $4.23 a bag to make. How many bags of M&M's and how many bags of Reese's Pieces should she use?

Answer

10 bags of M&M's, 15 bags of Reese's Pieces

19. Jotham needs 70 liters of a 50% solution of an alcohol solution. He has a 30% and an 80% solution available. How many liters of the 30% and how many liters of the 80% solutions should he mix to make the 50% solution?

20. Joy is preparing 15 liters of a 25% saline solution. She only has 40% and 10% solution in her lab. How many liters of the 40% and how many liters of the 10% should she mix to make the 25% solution?

Answer

7.5 liters of each solution

21. A scientist needs 65 liters of a 15% alcohol solution. She has available a 25% and a 12% solution. How many liters of the 25% and how many liters of the 12% solutions should she mix to make the 15% solution?

22. A scientist needs 120 milliliters of a 20% acid solution for an experiment. The lab has available a 25% and a 10% solution. How many liters of the 25% and how many liters of the 10% solutions should the scientist mix to make the 20% solution?

Answer

80 liters of the 25% solution and 40 liters of the 10% solution

23. A 40% antifreeze solution is to be mixed with a 70% antifreeze solution to get 240 liters of a 50% solution. How many liters of the 40% and how many liters of the 70% solutions will be used?

24. A 90% antifreeze solution is to be mixed with a 75% antifreeze solution to get 360 liters of an 85% solution. How many liters of the 90% and how many liters of the 75% solutions will be used?

Answer

240 liters of the 90% solution and 120 liters of the 75% solution

Solve Interest Applications

In the following exercises, translate to a system of equations and solve.

25. Hattie had $3000 to invest and wants to earn 10.6% interest per year. She will put some of the money into an account that earns 12% per year and the rest into an account that earns 10% per year. How much money should she put into each account?

26. Carol invested $2560 into two accounts. One account paid 8% interest and the other paid 6% interest. She earned 7.25% interest on the total investment. How much money did she put in each account?

Answer

$1600 at 8%, 960 at 6%

27. Sam invested $48,000, some at 6% interest and the rest at 10%. How much did he invest at each rate if he received $4000 in interest in one year?

28. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4500 in interest in one year?

Answer

$28,000 at 9%, $36,000 at 5.5%

29. After four years in college, Josie owes $65, 800 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $\$2878.50$ What is the amount of each loan?

30. Mark wants to invest $10,000 to pay for his daughter's wedding next year. He will invest some of the money in a short term CD that pays 12% interest and the rest in a money market savings account that pays 5% interest. How much should he invest at each rate if he wants to earn $1095 in interest in one year?

Answer

$8500 CD, $1500 savings account

31. A trust fund worth $25,000 is invested in two different portfolios. This year, one portfolio is expected to earn 5.25% interest and the other is expected to earn 4%. Plans are for the total interest on the fund to be $1150 in one year. How much money should be invested at each rate?

32. A business has two loans totaling $85,000. One loan has a rate of 6% and the other has a rate of 4.5% This year, the business expects to pay $4,650 in interest on the two loans. How much is each loan?

Answer

$55,000 on loan at 6% and $30,000 on loan at 4.5%

Solve Applications of Cost and Revenue Functions

33. The manufacturer of an energy drink spends \$1.20 to make each drink and sells them for \$2. The manufacturer also has fixed costs each month of \$8,000.

ⓐ Find the cost function C when x energy drinks are manufactured.

ⓑ Find the revenue function R when x drinks are sold.

ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.

ⓓ Find the break-even point. Interpret what the break-even point means.

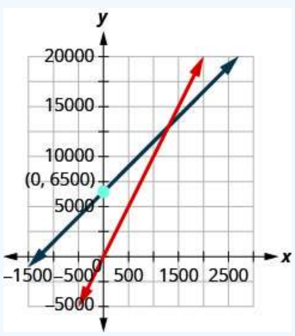
34. The manufacturer of a water bottle spends \$5 to build each bottle and sells them for \$10. The manufacturer also has fixed costs each month of \$6500. ⓐ Find the cost function C when x bottles are manufactured. ⓑ Find the revenue function R when x bottles are sold. ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid. ⓓ Find the break-even point. Interpret what the break-even point means.

Answer

ⓐ $C(x) = 5x + 6500$

ⓑ $R(x) = 10x$

ⓒ



ⓓ 1,500; when 1,500 water bottles are sold, the cost and the revenue equal \$15,000

Writing Exercises

35. Take a handful of two types of coins, and write a problem similar to Example relating the total number of coins and their total value. Set up a system of equations to describe your situation and then solve it.

36. In Example, we used elimination to solve the system of equations

$$\begin{cases} s + b = 40,000 \\ 0.08s + 0.03b = 0.071(40,000). \end{cases}$$

Could you have used substitution or elimination to solve this system? Why?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve mixture applications.			
solve interest applications.			

ⓑ What does this checklist tell you about your mastery of this section? What steps will you take to improve?

2.4: Solve Systems of Linear Equations with Three Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered triple is a solution of a system of three linear equations with three variables
- Solve a system of linear equations with three variables
- Solve applications using systems of linear equations with three variables

Be Prepared

Before you get started, take this readiness quiz.

1. Evaluate $5x-2y+3z$ when $x=-2, y=-4$, and $z=3$.
2. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution. $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$
3. Classify the equations as a conditional equation, an identity, or a contradiction and then state the solution. $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$

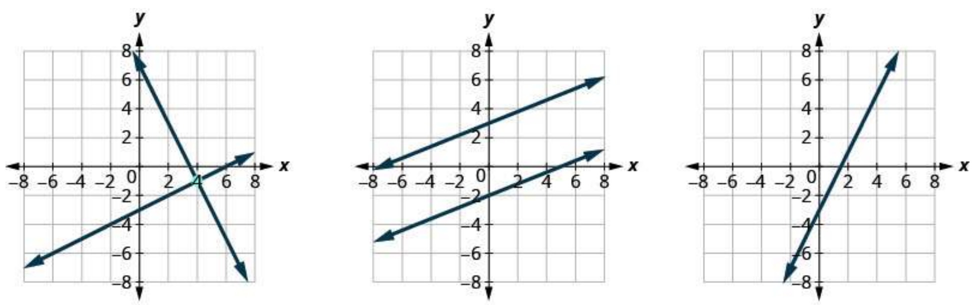
Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In this section, we will extend our work of solving a system of linear equations. So far we have worked with systems of equations with two equations and two variables. Now we will work with systems of three equations with three variables. But first let's review what we already know about solving equations and systems involving up to two variables.

We learned earlier that the graph of a linear equation, $Ax+By=C$, is a line. Each point on the line is a solution to the equation. For a system of two equations with two variables, we graph two lines. Then we can see that all the points that are solutions to each equation form a line. And, by finding what the lines have in common, we'll find the solution to the system.

Most linear equations in one variable have one solution, but we saw that some equations, called contradictions, have no solutions and for other equations, called identities, all numbers are solutions

We know when we solve a system of two linear equations represented by a graph of two lines in the same plane, there are three possible cases, as shown.



One solution

The lines intersect.
Intersecting lines have one point in common.
There is one solution to this system.

Consistent
Independent

No solution

The lines are parallel.
Parallel lines have no points in common.
There is no solution to this system.

Inconsistent

Infinitely many solutions

Both equations give the same line.
Because we have just one line, there are infinitely many solutions.

Consistent
Dependent

Similarly, for a linear equation with three variables $Ax+By+Cz=D$ every solution to the equation is an ordered triple, (p, q, r) that makes the equation true when substituted in for $(x, y, z$.

Definition 2.4.1

1. An expression that can be written as

$$Ax + By + Cz$$

with A, B, and C real numbers, not all zero, is called a **linear expression (with three variables)**, or more specifically, a **linear expression with x, y, and z.**

2. An equation that can be written as

$$Ax + By + Cz = D$$

with A, B, C, and D real numbers, not all zero, is called a **linear equation (with three variables)**, or more specifically, a **linear equation with x, y, and z.**

As before, triples of numbers can be graphed in three dimensional space, so we will call these triples points and imagine them in this space. All the points that are solutions to one equation form a plane in three-dimensional space. And, by finding what the planes have in common, we'll find the solution to the system.

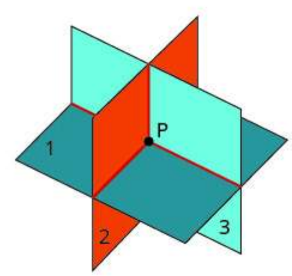
When we solve a system of three linear equations represented by a graph of three planes in space, there are three possible cases.

One solution

Consistent system and Independent equations

The 3 planes intersect.

The three intersecting planes have one point in common.

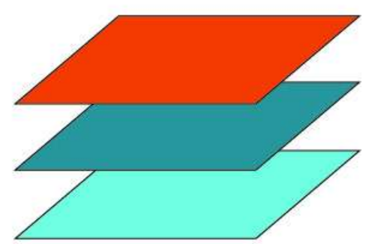


No solution

Inconsistent system

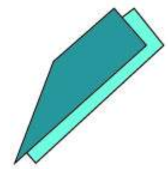
The planes are parallel.

Parallel planes have no points in common.



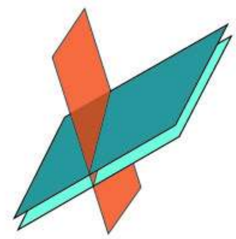
Two planes are coincident and parallel to the third plane.

The planes have no points in common.



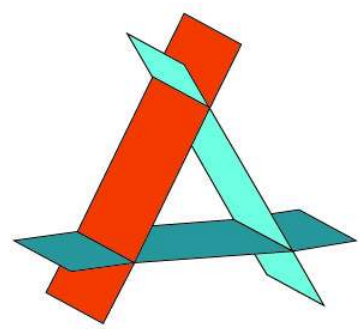
Two planes are parallel and each intersect the third plane.

The planes have no points in common.



Each plane intersects the other two, but all three share no points.

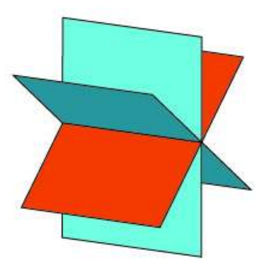
The planes have no points in common.



Infinitely many solutions
Consistent system and dependent equations

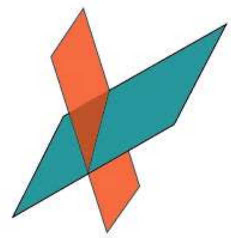
Three planes intersect in one line.

There is just one line, so there are infinitely many solutions.



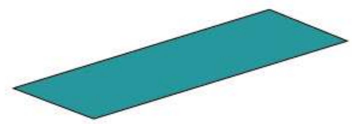
Two planes are coincident and intersect the third plane in a line.

There is just one line, so there are infinitely many solutions.



Three planes are coincident.

There is just one plane, so there are infinitely many solutions.



To solve a system of three linear equations, we want to find the values of the variables that are solutions to all three equations. In other words, we are looking for the ordered triple (x, y, z) that makes all three equations true. These are called the *solutions of the system of three linear equations with three variables.*

Definition 2.4.2

Solutions of a system of equations are the values of the variables that, when substituted, make all the equations true. A solution of a system of linear equations with three variables is represented by an ordered triple (p, q, r), so that substituting p for x, q for y. and r for z yields three true equations. We may also say $(x, y, z) = (p, q, r)$ is a solution.

To determine if an ordered triple is a solution to a system of three equations, we substitute the values of the variables into each equation. If the ordered triple makes all three equations true, it is a solution to the system.

Example 2.4.3

Determine whether the ordered triple is a solution to the system $\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$.

a. $(-2, -1, 3)$

b. $(-4, -3, 4)$

Solution

a.

$$\begin{cases} x - y + z = 2 \\ 2x - y - z = -6 \\ 2x + 2y + z = -3 \end{cases}$$

We substitute $x = -2$, $y = -1$ and $z = 3$ into all three equations.

$\begin{aligned} x - y + z &= 2 \\ -2 - (-1) + 3 &\stackrel{?}{=} 2 \\ 2 &\stackrel{?}{=} 2 \\ &\text{True} \end{aligned}$	$\begin{aligned} 2x - y - z &= -6 \\ 2(-2) - (-1) - 3 &\stackrel{?}{=} -6 \\ -6 &\stackrel{?}{=} -6 \\ &\text{True} \end{aligned}$	$\begin{aligned} 2x + 2y + z &= -3 \\ 2(-2) + 2(-1) + 3 &\stackrel{?}{=} -3 \\ -3 &\stackrel{?}{=} -3 \\ &\text{True} \end{aligned}$

$$\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$$

Answer: $(-2,-1,3)$ does make all three equations true, so $(-2,-1,3)$ is a solution.

b.

$$\begin{cases} x-y+z=2 \\ 2x-y-z=-6 \\ 2x+2y+z=-3 \end{cases}$$

We substitute $x=-4$, $y=-3$ and $z=4$ into all three equations.

$\begin{aligned} x-y+z &= 2 \\ -4-(-3)+4 &\stackrel{?}{=} 2 \\ 3 &\stackrel{?}{=} 2 \\ &\text{False} \end{aligned}$	$\begin{aligned} 2x-y-z &= -6 \\ 2(-4)-(-3)-4 &\stackrel{?}{=} -6 \\ -9 &\stackrel{?}{=} -6 \\ &\text{False} \end{aligned}$	$\begin{aligned} 2x+2y+z &= -3 \\ 2(-4)+2(-3)+4 &\stackrel{?}{=} -3 \\ -10 &\stackrel{?}{=} -3 \\ &\text{False} \end{aligned}$

Answer: $(-4,-3,4)$ does not make all three equations true, so $(-4,-3,4)$ is not a solution.

? Try It 2.4.4

Determine whether the ordered triple is a solution to the system $\begin{cases} 3x+y+z=2 \\ x+2y+z=-3 \\ 3x+y+2z=4 \end{cases}$.

a. $(1,-3,2)$

b. $(4,-1,-5)$

Answer

a. Yes, it is a solution.

b. No, it is not a solution.

? Try It 2.4.5

Determine whether the ordered triple is a solution to the system $\begin{cases} x-3y+z=-5 \\ -3x-y-z=1 \\ 2x-2y+3z=1 \end{cases}$.

a. $(2,-2,3)$

b. $(-2,2,3)$

Answer

a. No, it is not a solution.

b. Yes, it is a solution.

Solve a System of Linear Equations with Three Variables

To solve a system of linear equations with three variables, we basically use the same techniques we used with systems that had two variables. We start with two pairs of equations and in each pair we eliminate the same variable. This will then give us a system of equations with only two variables and then we know how to solve that system!

Next, we use the values of the two variables we just found to go back to the original equation and find the third variable. We write our answer as an ordered triple and then check our results.

Example 2.4.6

Solve the system $\begin{cases} x-2y+z=3 \\ 2x+y+z=4 \\ 3x+4y+3z=-1 \end{cases}$ by elimination.

Solution

	$\begin{cases} x-2y+z=3 \\ 2x+y+z=4 \\ 3x+4y+3z=-1 \end{cases}$	
Write the equations in standard form. • If any coefficients are fractions, clear them.	The equations are in standard form. There are no fractions.	$\begin{cases} x-2y+z &= 3 & (A) \\ 2x+y+z &= 4 & (B) \\ 3x+4y+3z &= -1 & (C) \end{cases}$
Eliminate the same variable from two equations. • Decide which variable you will eliminate. • Work with a pair of equations to eliminate the chosen variable. • Multiply one of both equations so that the coefficients of that variable are opposites. • Add the equations to eliminate one variable.	We can eliminate the y's from Equations A and B by multiplying Equation B by 2. Multiply. We add the x's, y's, and constants. The new equation, D, has only x and z. Add.	$\begin{cases} x-2y+z &= 3 & (A) \\ 2x+y+z &= 4 & (B) \end{cases}$ $\begin{cases} x-2y+z &= 3 & (A) \\ 2(2x+y+z) &= 2(4) & (B) \end{cases}$ $\begin{cases} x-2y+z &= 3 & (A) \\ 4x+2y+2z &= 8 & (B) \end{cases}$ $5x+3z=11 \quad (D)$
Repeat the previous step using two other equations and eliminate the same variable as in the previous step.	We can again eliminate the y's using the Equations (A) and (C) by multiplying Equation (A) by 2. Add the new equations and the result will be Equation (E).	$\begin{cases} x-2y+z &= 3 & (A) \\ 3x+4y+3z &= -1 & (C) \end{cases}$ $\begin{cases} 2(x-2y+z) &= 2(3) & (A) \\ 3x+4y+3z &= -1 & (C) \end{cases}$ $\begin{cases} 2x-4y+2z &= 6 & (A) \\ 3x+4y+3z &= -1 & (C) \end{cases}$ $5x+5z=5 \quad (E)$
The two new equations form a system of two equations with two variables. Solve this system.	Use Equations (D) and (E) to eliminate either x or z. Add and solve for z. Use either Equations (D) or (E) to solve for x using substitution.	$\begin{cases} 5x+3z &= 11 & (D) \\ 5x+5z &= 5 & (E) \end{cases}$ $\begin{cases} 5x+3z &= 11 & (D) \\ (-1)(5x+5z) &= (-1)(5) & (E) \end{cases}$ $\begin{cases} 5x+3z &= 11 & (D) \\ -5x-5z &= -5 & (E) \end{cases}$ $-2z=6$ $z=-3$ $5x+3z=11 \quad (D)$ $5x+3(-3)=11$ $5x-9=11$ $5x=20$ $x=4$
Use the values of the two variables found in the previous step to solve for the third variable.	Use an original equation to find y.	$2x+y+z=4 \quad (B)$ $2(4)+y+z=4$ $8+y-3=4$ $y=-1$

	$\begin{cases} x-2y+z=3 \\ 2x+y+z=4 \\ 3x+4y+3z=-1 \end{cases}$	
Write the solution as an ordered triple.	Write it as (x, y, z).	$(4, -1, -3)$
Check that the ordered triple is a solution to **all three original equations**. This step will help to detect errors.	Substitute $x=4,\ y=-1$ and $z=-3$ into all three equations. Do they make the equations true?	$x-2y+z=3$ $4-2(-1)+(-3) \stackrel{?}{=} 3$ $3 \stackrel{?}{=} 3$ True $2x+y+z=4$ $2(4)+(-1)+(-3) \stackrel{?}{=} 4$ $4 \stackrel{?}{=} 4$ True $3x+4y+3z=-1$ $3(4)+4(-1)+3(-3) \stackrel{?}{=} -1$ $-1 \stackrel{?}{=} -1$ True Substituting 4 into x, -1 into y, and -3 into for z makes both equations true.
Answer the question.	The solution is $(4, -1, -3)$.	

Try It 2.4.7

Solve the system $\begin{cases} 3x+y-z=2 \\ 2x-3y-2z=1 \\ 4x-y-3z=0 \end{cases}$ by elimination.

Answer

The solution is $(2, -1, 3)$.

Try It 2.4.8

Solve the system $\begin{cases} 4x+y+z=-1 \\ -2x-2y+z=2 \\ 2x+3y-z=1 \end{cases}$ by elimination.

Answer

The solution is $(-2, 3, 4)$.

The steps are summarized here.

Solve a system of linear equations with three variables (by elimination)

1. Write the equations in standard form.
   - If any coefficients are fractions, clear them.
2. Eliminate the same variable from two equations.
   - Decide which variable you will eliminate.
   - Work with a pair of equations to eliminate the chosen variable.
   - Multiply one or both equations so that the coefficients of that variable are opposites.

- Add the equations resulting from Step 2 to eliminate one variable.

3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
4. The two new equations form a system of two equations with two variables. Solve this system.
5. Use the values of the two variables found in Step 4 to solve for the third variable.
6. Write the solution as an ordered triple.
7. Check that the ordered triple is a solution to **all three** original equations (in order to detect errors).

Example 2.4.9

Solve the system $\begin{cases} 3x-4z=0 \\ 3y+2z=-3 \\ 2x+3y=-5 \end{cases}$.

Solution

	$\begin{cases} 3x-4z & =0 & (A) \\ 3y+2z & =-3 & (B) \\ 2x+3y & =-5 & (C) \end{cases}$
We can eliminate z from Equations (A) and (B) by multiplying Equation (B) by 2 and then adding the resulting equations.	$\begin{cases} 3x-4z & =0 & (A) \\ 2(3y+2z) & =2(-3) & (B) \end{cases}$ $\begin{cases} 3x-4z & =0 & (A) \\ 6y+4z & =-6 & (B) \end{cases}$
Add.	$3x+6y=-6 \quad (D)$
Notice that Equations (C) and (D) both have the variables x and y. We will solve this new system for x and y.	$\begin{cases} 2x+3y & =-5 & (C) \\ 3x+6y & =-6 & (D) \end{cases}$ $\begin{cases} (-2)(2x+3y) & =(-2)(-5) & (C) \\ 3x+6y & =-6 & (D) \end{cases}$ $\begin{cases} -4x-6y & =10 & (C) \\ 3x+6y & =-6 & (D) \end{cases}$
Add and solve for x.	$-x=4$ $x=-4$
To solve for y, we substitute $x=-4$ into Equation (C).	$2x+3y=-5 \quad (C)$ $2(-4)+3y=-5$ $-8+3y=-5$ $3y=3$ $y=1$
We now have $x=-4$ and $y=1$. We need to solve for z. We can substitute $x=-4$ into Equation (A) to find z.	$3x-4z=0 \quad (A)$ $3(-4)-4z=0$ $-12-4z=0$ $-4z=12$ $z=-3$
We write the solution as an ordered triple.	$(-4,1,-3)$

	$\begin{cases} 3x-4z &= 0 & (A) \\ 3y+2z &= -3 & (B) \\ 2x+3y &= -5 & (C) \end{cases}$
We check that the solution makes all three equations true.	$3x-4z=0 \quad (A)$ $3(-4)-4(-3) \stackrel{?}{=} 0$ $0 \stackrel{?}{=} 0$ True $3y+2z=-3 \quad (B)$ $3(1)+2(-3) \stackrel{?}{=} -3$ $-3 \stackrel{?}{=} -3$ True $2x+3y=-5 \quad (C)$ $2(-4)+3(1) \stackrel{?}{=} -5$ $-5 \stackrel{?}{=} -5$ True $(-4,1,-3)$ makes all three equations true.
Answer the question.	The solution is $(-4,1,-3)$.

Try It 2.4.10

Solve the system $\begin{cases} 3x-4z=-1 \\ 2y+3z=2 \\ 2x+3y=6 \end{cases}$.

Answer

The solution is $(-3,4,-2)$.

Try It 2.4.11

Solve the system $\begin{cases} 4x-3z=-5 \\ 3y+2z=7 \\ 3x+4y=6 \end{cases}$.

Answer

The solution is $(-2,3,-1)$.

When we solve a system and end up with no variables and a false statement, we know there are no solutions and that the system is inconsistent. The next example shows a system of equations that is inconsistent.

Example 2.4.12

Solve the system $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$.

Solution

	$\begin{cases} x+2y-3z &= -1 & (A) \\ x-3y+z &= 1 & (B) \\ 2x-y-2z &= 2 & (C) \end{cases}$

	$\begin{cases} x+2y-3z &= -1 & (A) \\ x-3y+z &= 1 & (B) \\ 2x-y-2z &= 2 & (C) \end{cases}$
Use Equations (A) and (B) to eliminate z.	$\begin{cases} x+2y-3z &= -1 & (A) \\ x-3y+z &= 1 & (B) \end{cases}$ $\begin{cases} x+2y-3z &= -1 & (A) \\ (3)(x-3y+z) &= (3)(1) & (B) \end{cases}$ $\begin{cases} x+2y-3z &= -1 & (A) \\ 3x-9y+3z &= 3 & (B) \end{cases}$
Add.	$4x-7y=2 \quad (D)$
Use Equations (B) and (C) to eliminate z again.	$\begin{cases} x-3y+z &= 1 & (B) \\ 2x-y-2z &= 2 & (C) \end{cases}$ $\begin{cases} (2)(x-3y+z) &= (2)(1) & (B) \\ 2x-y-2z &= 2 & (C) \end{cases}$ $\begin{cases} 2x-6y+2z &= 2 & (B) \\ 2x-y-2z &= 2 & (C) \end{cases}$
Add.	$4x-7y=4 \quad (E)$
Use Equations (D) and (E) to eliminate a variable.	$\begin{cases} 4x-7y &= 2 & (D) \\ 4x-7y &= 4 & (E) \end{cases}$ $\begin{cases} 4x-7y &= 2 & (D) \\ (-1)(4x-7y) &= (-1)(4) & (E) \end{cases}$ $\begin{cases} 4x-7y &= 2 & (D) \\ -4x+7y &= -4 & (E) \end{cases}$
Add.	$0=-2 \quad$ False
Answer the question.	We are left with a false statement and this tells us the system is inconsistent and has no solution.

? Try It 2.4.13

Solve the system $\begin{cases} x+2y+6z=5 \\ -x+y-2z=3 \\ x-4y-2z=1 \end{cases}$.

Answer

The system has no solution.

? Try It 2.4.14

Solve the system $\begin{cases} 2x-2y+3z=6 \\ 4x-3y+2z=0 \\ -2x+3y-7z=1 \end{cases}$.

Answer

The system has no solution.

When we solve a system and end up with no variables but a true statement, we know there are infinitely many solutions. The system is consistent with dependent equations. Our solution will show how two of the variables depend on the third.

Example 2.4.15

Solve the system $\begin{cases} x+2y-z=1 \\ 2x+7y+4z=11 \\ x+3y+z=4 \end{cases}$.

Solution

	$\begin{cases} x+2y-z &= 1 & (A) \\ 2x+7y+4z &= 11 & (B) \\ x+3y+z &= 4 & (C) \end{cases}$
Use Equations (A) and (C) to eliminate x.	$\begin{cases} x+2y-z &= 1 & (A) \\ x+3y+z &= 4 & (C) \end{cases}$ $\begin{cases} (-1)(x+2y-z) &= (-1)(1) & (A) \\ 2x+7y+4z &= 11 & (B) \\ x+3y+z &= 4 & (C) \end{cases}$ $\begin{cases} -x-2y+z &= -1 & (A) \\ 2x+7y+4z &= 11 & (B) \\ x+3y+z &= 4 & (C) \end{cases}$
Add.	$y+2z=3 \quad (D)$
Use Equations (A) and (B) to eliminate x again.	$\begin{cases} x+2y-z &= 1 & (A) \\ 2x+7y+4z &= 11 & (B) \end{cases}$ $\begin{cases} (-2)(x+2y-z) &= (-2)(1) & (A) \\ 2x+7y+4z &= 11 & (B) \end{cases}$ $\begin{cases} -2x-4y+2z &= -2 & (A) \\ 2x+7y+4z &= 11 & (B) \end{cases}$
Add.	$3y+6z=9 \quad (E)$
Use Equations (D) and (E) to eliminate y.	$\begin{cases} y+2z &= 3 & (D) \\ 3y+6z &= 9 & (E) \end{cases}$ $\begin{cases} (-3)(y+2z) &= (-3)(3) & (D) \\ 3y+6z &= 9 & (E) \end{cases}$ $\begin{cases} -3y-6z &= -9 & (D) \\ 3y+6z &= 9 & (E) \end{cases}$
Add.	$0=0$
Conclusion.	There are infinitely many solutions.
Represent the solution showing how x and y are dependent on z. Solve Equation (D) for y.	$y+2z=3 \quad (D)$ $y=-2z+3$
Substitute $y=-2z+3$ into Equation (A) and solve for x.	$\begin{aligned} x+2y-z &= 1 \quad (A) \\ x+2(-2z+3)-z &= 1 \\ x-4z+6-z &= 1 \\ x-5z+6 &= 1 \\ x &= 5z-5 \end{aligned}$
Answer the question.	The true statement $0=0$ tells us that this is a dependent system that has infinitely many solutions. The solutions are of the form (x,y,z) where $x=5z-5$, $y=-2z+3$, and z is any real number.

? Try It 2.4.16

Solve the system $\begin{cases} x+y-z=0 \\ 2x+4y-2z=6 \\ 3x+6y-3z=9 \end{cases}$.

Answer

The system has infinitely many solutions $(x, 3, z)$ where $x=z-3$ and z is any real number.

? Try It 2.4.17

Solve the system $\begin{cases} x-y-z=1 \\ -x+2y-3z=-4 \\ 3x-2y-7z=0 \end{cases}$.

Answer

The system has infinitely many solutions (x, y, z) where $x=5z-2$, $y=4z-3$, and z is any real number.

Solve Applications using Systems of Linear Equations with Three Variables

Applications that are modeled by a system of equations can be solved using the same techniques we used to solve the systems. Many of the application are just extensions to three variables of the types we have solved earlier.

? Example 2.4.18

The community college theater department sold three kinds of tickets to its latest play production. The adult tickets sold for \$15, the student tickets for \$10, and the child tickets for \$8. The theater department was thrilled to have sold 250 tickets and brought in \$2,825 in one night. The number of student tickets sold is twice the number of adult tickets sold. How many of each type did the department sell?

Solution

	We will use a chart to organize the information.	(chart below)
a ch…	Number of students is twice number of adults.	$y=2x$
a ch…	Rewrite the equation in standard form.	$y=2x$ $2x-y=0$
a ch…	Write the system of equations.	$\begin{cases} x+y+z&=250 & (A) \\ 15x+10y+8z&=2825 & (B) \\ -2x+y&=0 & (C) \end{cases}$
a ch…	Use Equations (A) and (B) to eliminate z.	$\begin{cases} x+y+z&=250 & (A) \\ 15x+10y+8z&=2825 & (B) \end{cases}$ $\begin{cases} (-8)(x+y+z)&=(-8)(250) & (A) \\ 15x+10y+8z&=2825 & (B) \end{cases}$ $\begin{cases} -8x-8y-8z&=-2000 & (A) \\ 15x+10y+8z&=2825 & (B) \end{cases}$
a ch…	Add.	$7x+2y=825 \quad (D)$

Type	Number	•	Value	=	Total Value
adult	x		15		$15x$
student	y		10		$10y$
child	z		8		$8z$
	250				2825

	We will use a chart to organize the information.	(table below)
a ch...	Use Equations (C) and (D) to eliminate y.	$\begin{cases} -2x+y &= 0 & (C) \\ 7x+2y &= 825 & (D) \end{cases}$ $\begin{cases} (-2)(-2x+y) &= (-2)(0) & (C) \\ 7x+2y &= 825 & (D) \end{cases}$ $\begin{cases} 4x-2y &= 0 & (C) \\ 7x+2y &= 825 & (D) \end{cases}$
a ch...	Add and solve for x	$11x=825$ $x=75 \text{ adult tickets}$
a ch...	Use Equation (C) to find y.	$-2x+y=0 \quad (C)$
a ch...	Substitute $x=75$.	$-2(75)+y=0$ $-150+y=0$ $y=150 \text{ student tickets}$
a ch...	Use Equation (A) to find z.	$x+y+z=250$
a ch...	Substitute in the values $x=75$ and $y=150$.	$75+150+z=250$ $225+z=250$ $z=25 \text{ child tickets}$
a ch...	Answer the question.	The theater department sold 75 adult tickets, 150 student tickets, and 25 child tickets.

Type	Number	• Value	= Total Value
adult	x	15	$15x$
student	y	10	$10y$
child	z	8	$8z$
	250		2825

Note that there are many ways of solving this but they all lead to the same solution.

? Try It 2.4.19

The community college fine arts department sold three kinds of tickets to its latest dance presentation. The adult tickets sold for $\$20$, the student tickets for $\$12$ and the child tickets for $\$10$. The fine arts department was thrilled to have sold 350 tickets and brought in $\$4,650$ in one night. The number of child tickets sold is the same as the number of adult tickets sold. How many of each type did the department sell?

Answer

The fine arts department sold 75 adult tickets, 200 student tickets, and 75 child tickets.

> **Try It 2.4.20**
>
> The community college soccer team sold three kinds of tickets to its latest game. The adult tickets sold for $\$10$, the student tickets for $\$8$ and the child tickets for $\$5$. The soccer team was thrilled to have sold 600 tickets and brought in $\$4,900$ for one game. The number of adult tickets is twice the number of child tickets. How many of each type did the soccer team sell?
>
> **Answer**
>
> The soccer team sold 200 adult tickets, 300 student tickets, and 100 child tickets.

Key Concepts

- **Linear Equation in Three Variables:** A linear equation with three variables, where A, B, C, and D are real numbers and A, B, and C are not all 0, is of the form

$$Ax+By+Cz=D$$

  Every solution to the equation is an ordered triple, (p,q,r) that makes the equations true when p, q, and r are substituted in for x, y, and z, respectively.
- **How to solve a system of linear equations with three variables.**
  1. Write the equations in standard form
     If any coefficients are fractions, clear them.
  2. Eliminate the same variable from two equations.
     Decide which variable you will eliminate.
     Work with a pair of equations to eliminate the chosen variable.
     Multiply one or both equations so that the coefficients of that variable are opposites.
     Add the equations resulting from Step 2 to eliminate one variable
  3. Repeat Step 2 using two other equations and eliminate the same variable as in Step 2.
  4. The two new equations form a system of two equations with two variables. Solve this system.
  5. Use the values of the two variables found in Step 4 to find the third variable.
  6. Write the solution as an ordered triple.
  7. Check that the ordered triple is a solution to **all three** original equations.

Glossary

solutions of a system of linear equations with three variables

The solutions of a system of equations are the values of the variables that make all the equations true; a solution is represented by an ordered triple (p,q,r).

Practice Makes Perfect

Determine Whether an Ordered Triple is a Solution of a System of Three Linear Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

1. $\begin{cases} 2x-6y+z=3 \\ 3x-4y-3z=2 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(3,1,3)$
ⓑ $(4,3,7)$

2. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

ⓐ $(-5,-7,4)$
ⓑ $(5,7,4)$

Answer

ⓐ no ⓑ yes

3. $\begin{cases} y-10z=-8 \\ 2x-y=2 \\ x-5z=3 \end{cases}$

ⓐ $(7,12,2)$
ⓑ $(2,2,1)$

4. $\begin{cases} x+3y-z=15 \\ y=\frac{2}{3}x-2 \\ x-3y+z=-2 \end{cases}$

ⓐ $\left(-6,5,\frac{1}{2}\right)$
ⓑ $\left(5,\frac{4}{3},-3\right)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

5. $\begin{cases} 5x+2y+z=5 \\ -3x-y+2z=6 \\ 2x+3y-3z=5 \end{cases}$

6. $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

Answer

$(4,5,2)$

7. $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

8. $\begin{cases} 5x-3y+2z=-5 \\ 2x-y-z=4 \\ 3x-2y+2z=-7 \end{cases}$

Answer

$(7,12,-2)$

9. $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

10. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

Answer

$(-3,-5,4)$

11. $\begin{cases} 3x+8y+2z=-5 \\ 2x+5y-3z=0 \\ x+2y-2z=-1 \end{cases}$

12. $\begin{cases} 11x+9y+2z=-9 \\ 7x+5y+3z=-7 \\ 4x+3y+z=-3 \end{cases}$

Answer

$(2,-3,-2)$

13. $\begin{cases} \frac{1}{3}x-y-z=1 \\ x+\frac{5}{2}y+z=-2 \\ 2x+2y+\frac{1}{2}z=-4 \end{cases}$

14. $\begin{cases} x+\frac{1}{2}y+\frac{1}{2}z=0 \\ \frac{1}{5}x-\frac{1}{5}y+z=0 \\ \frac{1}{3}x-\frac{1}{3}y+2z=-1 \end{cases}$

Answer

$(6,-9,-3)$

15. $\begin{cases} x+\frac{1}{3}y-2z=-1 \\ \frac{1}{3}x+y+\frac{1}{2}z=0 \\ \frac{1}{2}x+\frac{1}{3}y-\frac{1}{2}z=-1 \end{cases}$

16. $\begin{cases} \frac{1}{3}x-y+\frac{1}{2}z=4 \\ \frac{2}{3}x+\frac{5}{2}y-4z=0 \\ x-\frac{1}{2}y+\frac{3}{2}z=2 \end{cases}$

Answer

$(3,-4,-2)$

17. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

18. $\begin{cases} 2x+5y=4 \\ 3y-z=\dfrac{3}{4} \\ x+3z=-3 \end{cases}$

Answer

$(-3,2,3)$

19. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

20. $\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$

Answer

$(-2,0,-3)$

21. $\begin{cases} 4x-3y+2z=0 \\ -2x+3y-7z=1 \\ 2x-2y+3z=6 \end{cases}$

22. $\begin{cases} x-2y+2z=1 \\ -2x+y-z=2 \\ x-y+z=5 \end{cases}$

Answer

no solution

23. $\begin{cases} 2x+3y+z=1 \\ 2x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

24. $\begin{cases} x+4y+z=-8 \\ 4x-y+3z=9 \\ 2x+7y+z=0 \end{cases}$

Answer

$x=\dfrac{203}{16};\ y=-\dfrac{25}{16};\ z=-\dfrac{231}{16};$

25. $\begin{cases} x+2y+z=4 \\ x+y-2z=3 \\ -2x-3y+z=-7 \end{cases}$

26. $\begin{cases} x+y-2z=3 \\ -2x-3y+z=-7 \\ x+2y+z=4 \end{cases}$

Answer

(x, y, z) where $x=5z+2;\ y=-3z+1;\ z$ is any real number

27. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

28. $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

(x, y, z) where $x=5z-2;\ y=4z-3;\ z$ is any real number

Solve Applications using Systems of Linear Equations with Three Variables

In the following exercises, solve the given problem.

29. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is twice the measure of the first angle. The third angle is twelve more than the second. Find the measures of the three angles.

30. The sum of the measures of the angles of a triangle is 180. The sum of the measures of the second and third angles is three times the measure of the first angle. The third angle is fifteen more than the second. Find the measures of the three angles.

Answer

42, 50, 58

31. After watching a major musical production at the theater, the patrons can purchase souvenirs. If a family purchases 4 t-shirts, the video, and 1 stuffed animal, their total is $135.

A couple buys 2 t-shirts, the video, and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, the video, and 1 stuffed animal and their total is $85. What is the cost of each item?

32. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 cans of popcorn for a total sales of $250. What is the cost of each item?

Answer

$20, $5, $10

Writing Exercises

33. In your own words explain the steps to solve a system of linear equations with three variables by elimination.

34. How can you tell when a system of three linear equations with three variables has no solution? Infinitely many solutions?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine whether an ordered triple is a solution of a system of three linear equations with three variables.			
solve a system of linear equations with three variables.			
solve applications using systems of linear equations with three variables.			

ⓑ On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

2.5: Solve Systems of Linear Equations Using Determinants

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the determinant of a 2×2 matrix
- Evaluate the determinant of a 3×3 matrix
- Use Cramer's Rule to solve systems of equations
- Solve applications using determinants

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $5(-2)-(-4)(1)$.
2. Simplify: $-3(8-10)+(-2)(6-3)-4(-3-(-4))$.
3. Simplify: $\dfrac{-12}{-8}$.

In this section we will learn of another method to solve systems of linear equations called Cramer's rule. Before we can begin to use the rule, we need to learn some new definitions and notation.

Evaluate the Determinant of a 2×2 Matrix

If a **matrix** (a rectangular array of numbers) has the same number of rows and columns, we call it a **square matrix**. Each square matrix has a real number associated with it called its **determinant**. To find the determinant of the square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, we first write it as $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$. To get the real number value of the determinate we subtract the products of the diagonals, as shown.

$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Definition 2.5.1

The **determinant** of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where a, b, c, and d are real numbers, is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 2.5.2

Evaluate the determinate of ⓐ $\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$ ⓑ $\begin{bmatrix} -3 & -4 \\ -2 & 0 \end{bmatrix}$.

Solution

ⓐ

	$\begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix}$
Write the determinant.	$\begin{vmatrix} 4 & -2 \\ 3 & -1 \end{vmatrix}$

Subtract the products of the diagonals.	$4(-1)-3(-2)$
Simplify.	$-4+6$
Simplify.	2

ⓑ

	$\begin{bmatrix} -3 & -4 \\ -2 & 0 \end{bmatrix}$
Write the determinant.	$\begin{vmatrix} -3 & -4 \\ -2 & 0 \end{vmatrix}$
Subtract the products of the diagonals.	$-3(0)-(-2)(-4)$
Simplify.	$0-8$
Simplify.	-8

Try It 2.5.3

Evaluate the determinate of ⓐ $\begin{bmatrix} 5 & -3 \\ 2 & -4 \end{bmatrix}$ ⓑ $\begin{bmatrix} -4 & -6 \\ 0 & 7 \end{bmatrix}$.

Answer

ⓐ -14; ⓑ -28

Example 2.5.4

Evaluate the determinate of ⓐ $\begin{bmatrix} -1 & 3 \\ -2 & 4 \end{bmatrix}$ ⓑ $\begin{bmatrix} -7 & -3 \\ -5 & 0 \end{bmatrix}$.

Answer

ⓐ 2 ⓑ -15

Evaluate the Determinant of a 3×3 Matrix

To evaluate the determinant of a 3×3 matrix, we have to be able to evaluate the **minor of an entry** in the determinant. The minor of an entry is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

Definition 2.5.5

The **minor of an entry** in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

To find the minor of entry a_1, we eliminate the row and column which contain it. So we eliminate the first row and first column. Then we write the 2×2 determinant that remains.

$\begin{vmatrix} \cancel{a_1} & \cancel{b_1} & \cancel{c_1} \\ \cancel{a_2} & b_2 & c_2 \\ \cancel{a_3} & b_3 & c_3 \end{vmatrix}$ minor of a_1 $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$

To find the minor of entry b_2, we eliminate the row and column that contain it. So we eliminate the 2^{nd} row and 2^{nd} column. Then we write the 2×2 determinant that remains.

$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ minor of b_2 $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$

Example 2.5.6

For the determinant $\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_1 ⓑ b_3 ⓒ c_2.

Solution

ⓐ

	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Eliminate the row and column that contains a_1.	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of a_1 $\begin{vmatrix} 0 & -3 \\ -4 & 2 \end{vmatrix}$
Evaluate.	$0(2) - (-3)(-4)$
Simplify.	-12

ⓑ

Eliminate the row and column that contains b_3.	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of b_3 $\begin{vmatrix} 4 & 3 \\ 1 & -3 \end{vmatrix}$
Evaluate.	$4(-3) - (1)(3)$
Simplify.	-15

ⓒ

	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Eliminate the row and column that contains c_2.	$\begin{vmatrix} 4 & -2 & 3 \\ 1 & 0 & -3 \\ -2 & -4 & 2 \end{vmatrix}$
Write the 2×2 determinant that remains.	minor of c_2 $\begin{vmatrix} 4 & -2 \\ -2 & 4 \end{vmatrix}$
Evaluate.	$4(4) - (-2)(-2)$
Simplify.	12

? Try it 2.5.7

For the determinant $\begin{vmatrix} 1 & -1 & 4 \\ 0 & 2 & -1 \\ -2 & -3 & 3 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_1 ⓑ b_2 ⓒ c_3.

Answer

ⓐ 3 ⓑ 11 ⓒ 2

? Try it 2.5.8

For the determinant $\begin{vmatrix} -2 & -1 & 0 \\ 3 & 0 & -1 \\ -1 & -2 & 3 \end{vmatrix}$, find and then evaluate the minor of ⓐ a_2 ⓑ b_3 ⓒ c_2.

Answer

ⓐ -3 ⓑ 2 ⓒ 3

We are now ready to evaluate a 3×3 determinant. To do this we expand by minors, which allows us to evaluate the 3×3 determinant using 2×2 determinants—which we already know how to evaluate!

To evaluate a 3×3 determinant by expanding by minors along the first row, we use the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

Remember, to find the minor of an entry we eliminate the row and column that contains the entry.

Definition 2.5.9

To evaluate a 3×3 determinant by **expanding by minors along the first row**, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1 \underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1 \underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

? Example 2.5.10

Evaluate the determinant $\begin{vmatrix} 2 & -3 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & -2 \end{vmatrix}$ by expanding by minors along the first row.

Solution

	$\begin{vmatrix} 2 & -3 & -1 \\ 3 & 2 & 0 \\ -1 & -1 & -2 \end{vmatrix}$
Expand by minors along the first row	$2\underbrace{\begin{vmatrix} 2 & 0 \\ -1 & -2 \end{vmatrix}}_{\text{minor of } 2} - (-3)\underbrace{\begin{vmatrix} 3 & 0 \\ -1 & -2 \end{vmatrix}}_{\text{minor of } -3} + (-1)\underbrace{\begin{vmatrix} 3 & 2 \\ -1 & -1 \end{vmatrix}}_{\text{minor of } -1}$
Evaluate each determinant.	$2(-4-0)+3(-6-0)-1(-3-(-2))$
Simplify.	$2(-4)+3(-6)-1(-1)$

Simplify.	$-8-18+1$
Simplify.	-25

? Try It 2.5.11

Evaluate the determinant $\begin{vmatrix} 3 & -2 & 4 \\ 0 & -1 & -2 \\ 2 & 3 & -1 \end{vmatrix}$, by expanding by minors along the first row.

Answer

37

? Try It 2.5.12

Evaluate the determinant $\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}$, by expanding by minors along the first row.

Answer

7

To evaluate a 3×3 determinant we can expand by minors using any row or column. Choosing a row or column other than the first row sometimes makes the work easier.

When we expand by any row or column, we must be careful about the sign of the terms in the expansion. To determine the sign of the terms, we use the following sign pattern chart.

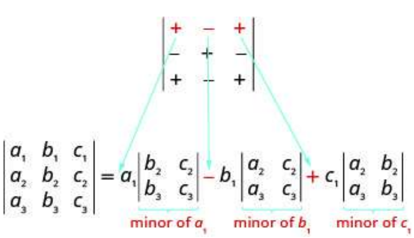
$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

SIGN PATTERN

When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

Notice that the sign pattern in the first row matches the signs between the terms in the expansion by the first row.



Since we can expand by any row or column, how do we decide which row or column to use? Usually we try to pick a row or column that will make our calculation easier. If the determinant contains a 0, using the row or column that contains the 0 will make the calculations easier.

Example 2.5.13

Evaluate the determinant $\begin{vmatrix} 4 & -1 & -3 \\ 3 & 0 & 2 \\ 5 & -4 & -3 \end{vmatrix}$ by expanding by minors.

Solution

To expand by minors, we look for a row or column that will make our calculations easier. Since 0 is in the second row and second column, expanding by either of those is a good choice. Since the second row has fewer negatives than the second column, we will expand by the second row.

	$\begin{vmatrix} 4 & -1 & -3 \\ 3 & 0 & 2 \\ 5 & -4 & -3 \end{vmatrix}$
Expand using the second row.	
Be careful of the signs.	$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$ $\quad -3\underbrace{\begin{vmatrix} -1 & -3 \\ -4 & -3 \end{vmatrix}}_{\text{minor of 3}} + (0)\underbrace{\begin{vmatrix} 4 & -3 \\ 5 & -3 \end{vmatrix}}_{\text{minor of 0}} - 2\underbrace{\begin{vmatrix} 4 & -1 \\ 5 & -4 \end{vmatrix}}_{\text{minor of 2}}$
Evaluate each determinant.	$-3(3-12)+0(-12-(-15))-2(-16-(-5))$
Simplify.	$-3(-9)+0-2(-11)$
Simplify.	$27+0+22$
Add.	49

Try It 2.5.14

Evaluate the determinant $\begin{vmatrix} 2 & -1 & -3 \\ 0 & 3 & -4 \\ 3 & -4 & -3 \end{vmatrix}$ by expanding by minors.

Answer

-11

Try It 2.5.15

Evaluate the determinant $\begin{vmatrix} -2 & -1 & -3 \\ -1 & 2 & 2 \\ 4 & -4 & 0 \end{vmatrix}$ by expanding by minors.

Answer

8

Use Cramer's Rule to Solve Systems of Equations

Cramer's Rule is a method of solving systems of equations using determinants. It can be derived by solving the general form of the systems of equations by elimination. Here we will demonstrate the rule for both systems of two equations with two variables and for systems of three equations with three variables.

Let's start with the systems of two equations with two variables.

CRAMER'S RULE FOR SOLVING A SYSTEM OF TWO EQUATIONS

For the system of equations $\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$, the solution (x, y) can be determined by

$x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$

where $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

$D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the y coefficients with the constants.

Notice that to form the determinant D, we use take the coefficients of the variables.

$\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$ $D = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$

Coefficients Coefficient of x Coefficient of y

Notice that to form the determinant D_x and D_y, we substitute the constants for the coefficients of the variable we are finding.

$\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$ $D_x = \begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ $\begin{cases} a_1x + b_1y = k_1 \\ a_2x + b_2y = k_2 \end{cases}$ $D_y = \begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$

Constants Coefficients of y Coefficients of x Constants

Example 2.5.16: How to Solve a System of Equations Using Cramer's Rule

Solve using Cramer's Rule: $\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$

Solution

Step 1. Evaluate the determinant D, using the coefficients of the variables.		$\begin{cases} 2x + y = -4 \\ 3x - 2y = -6 \end{cases}$ $D = \begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}$ $D = -4 - 3$ $D = -7$
Step 2. Evaluate the determinant D_x. Use the constants in place of the x coefficients.	We replace the coefficients of x, 2 and 3, with the constants, –4 and –6.	$D_x = \begin{vmatrix} -4 & 1 \\ -6 & -2 \end{vmatrix}$ $D_x = 8 - (-6)$ $D_x = 14$
Step 3. Evaluate the determinant D_y. Use the constants in place of the y coefficients.	We replace the coefficients of y, 1 and 2, with the constants, –4 and –6.	$D_y = \begin{vmatrix} 2 & -4 \\ 3 & -6 \end{vmatrix}$ $D_y = -12 - (-12)$ $D_y = 0$
Step 4. Find x and y.	Substitute in the values of D, D_x and D_y.	$x = \frac{D_x}{D}$ and $y = \frac{D_y}{D}$ $x = \frac{14}{-7}$ and $y = \frac{0}{-7}$ $x = -2$ and $y = 0$

Step 5. Write the solution as an ordered pair.	The ordered pair is (*x*, *y*).	(–2, 0)
Step 6. Check that the ordered pair is a solution to **both** original equations.	Substitute *x* = –2, *y* = 0 into both equations and make sure they are both true.	(–2, 0) is the solution to the system.

? Try It 2.5.17

Solve using Cramer's rule: $\begin{cases} 3x+y=-3 \\ 2x+3y=6 \end{cases}$

Answer

$(-\frac{15}{7}, \frac{24}{7})$

? Try It 2.5.18

Solve using Cramer's rule: $\begin{cases} -x+y=2 \\ 2x+y=-4 \end{cases}$

Answer

$(-2, 0)$

SOLVE A SYSTEM OF TWO EQUATIONS USING CRAMER'S RULE.

1. Evaluate the determinant \(*D*\), using the coefficients of the variables.
2. Evaluate the determinant D_x. Use the constants in place of the *x* coefficients.
3. Evaluate the determinant D_y. Use the constants in place of the *y* coefficients.
4. Find x and y. $x=\frac{D_x}{D}$, $y=\frac{D_y}{D}$
5. Write the solution as an ordered pair.
6. Check that the ordered pair is a solution to both original equations.

To solve a system of three equations with three variables with Cramer's Rule, we basically do what we did for a system of two equations. However, we now have to solve for three variables to get the solution. The determinants are also going to be 3×3 which will make our work more interesting!

CRAMER'S RULE FOR SOLVING A SYSTEM OF THREE EQUATIONS

For the system of equations $\begin{cases} a_1x+b_1y+c_1z=k_1 \\ a_2x+b_2y+c_2z=k_2 \\ a_3x+b_3y+c_3z=k_3 \end{cases}$, the solution (x, y, z) can be determined by

$$x=\frac{D_x}{D},\ y=\frac{D_y}{D} \text{ and } z=\frac{D_z}{D}$$

where $D=\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ use the coefficients of the variables.

$D_x=\begin{vmatrix} k_1 & b_1 & c_1 \\ k_2 & b_2 & c_2 \\ k_3 & b_3 & c_3 \end{vmatrix}$ replace the x coefficients with the constants.

$D_y=\begin{vmatrix} a_1 & k_1 & c_1 \\ a_2 & k_2 & c_2 \\ a_3 & k_3 & c_3 \end{vmatrix}$ replace the y coefficients with the constants.

$D_z=\begin{vmatrix} a_1 & b_1 & k_1 \\ a_2 & b_2 & k_2 \\ a_3 & b_3 & k_3 \end{vmatrix}$ replace the z coefficients with the constants.

Example 2.5.19

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

Answer

Evaluate the determinant \(D\).	$D=\begin{vmatrix} 3 & -5 & 4 \\ 5 & 2 & 1 \\ 2 & 3 & -2 \end{vmatrix}$
Expand by minors using column 1.	
Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D=3\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}-5\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix}+2\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix}$
Evaluate the determinants.	$D=3(-4-3)-5(10-12)+2(-5-8)$
Simplify.	$D=3(-7)-5(-2)+2(-13)$
Simplify.	$D=-21+10-26$
Simplify.	$D=-37$
Evaluate the determinant D_x. Use the constants to replace the coefficients of x.	$D_x=\begin{vmatrix} 5 & -5 & 4 \\ 0 & 2 & 1 \\ 3 & 3 & -2 \end{vmatrix}$
Expand by minors using column 1.	$D_x=5\begin{vmatrix} 2 & 1 \\ 3 & -2 \end{vmatrix}-0\begin{vmatrix} -5 & 4 \\ 3 & -2 \end{vmatrix}+3\begin{vmatrix} -5 & 4 \\ 2 & 1 \end{vmatrix}$
Evaluate the determinants.	$D_x=5(-4-3)-0(10-12)+3(-5-8)$
Simplify.	$D_x=5(-7)-0+3(-13)$
Simplify.	$D_x=-74$
Evaluate the determinant D_y. Use the constants to replace the coefficients of y.	$D_y=\begin{vmatrix} 3 & 5 & 4 \\ 5 & 0 & 1 \\ 2 & 3 & -2 \end{vmatrix}$

Expand by minors using column 2. Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D_y = -5\begin{vmatrix} 5 & 1 \\ 2 & -2 \end{vmatrix} + 0\begin{vmatrix} 5 & 4 \\ 3 & -2 \end{vmatrix} - 3\begin{vmatrix} 3 & 4 \\ 5 & 1 \end{vmatrix}$
Evaluate the determinants.	$D_y = -5(-10-2) + 0(-10-12) - 3(3-20)$
Simplify.	$D_y = -5(-12) + 0 - 3(-17)$
Simplify.	$D_y = 60 + 0 + 51$
Simplify.	$D_y = 111$
Evaluate the determinant D_z. Use the constants to replace the coefficients of z.	$D_z = \begin{vmatrix} 3 & -5 & 5 \\ 5 & 2 & 0 \\ 2 & 3 & 3 \end{vmatrix}$
Expand by minors using column 3. Be careful of the signs. $\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$	$D_z = 5\begin{vmatrix} 5 & 2 \\ 2 & 3 \end{vmatrix} - 0\begin{vmatrix} 3 & -5 \\ 2 & 3 \end{vmatrix} + 3\begin{vmatrix} 3 & -5 \\ 5 & 2 \end{vmatrix}$
Evaluate the determinants.	$D_z = 5(15-4) - 0(9-(-10)) + 3(6-(-25))$
Simplify.	$D_z = 5(11) - 0 + 3(31)$
Simplify.	$D_z = 55 - 0 + 93$
Simplify.	$D_z = 148$
Find x, y, and z.	$x = \frac{D_x}{D}, y = \frac{D_y}{D}$ and $z = \frac{D_z}{D}$
Substitute in the values.	$x = \frac{-74}{-37}, y = \frac{111}{-37}$ and $z = \frac{148}{-37}$
Simplify.	$x = 2, y = -3$ and $z = -4$
Write the solution as an ordered triple.	$(2, -3, -4)$
Check that the ordered triple is a solution to **all three** original equations.	We leave the check to you.
	The solution is $(2, -3, -4)$.

Try It 2.5.20

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x + 8y + 2z = -5 \\ 2x + 5y - 3z = 0 \\ x + 2y - 2z = -1 \end{cases}$

Answer

$(-9, 3, -1)$

Try It 2.5.21

Solve the system of equations using Cramer's Rule: $\begin{cases} 3x + y - 6z = -3 \\ 2x + 6y + 3z = 0 \\ 3x + 2y - 3z = -6 \end{cases}$

Answer

$(-6, 3, -2)$

Cramer's rule does not work when the value of the D determinant is 0, as this would mean we would be dividing by 0. But when $D = 0$, the system is either inconsistent or dependent.

When the value of $D = 0$ and D_x, D_y and D are all zero, the system is consistent and dependent and there are infinitely many solutions.

When the value of $D = 0$ and D_x, D_y and D_z are not all zero, the system is inconsistent and there is no solution.

DEPENDENT AND INCONSISTENT SYSTEMS OF EQUATIONS

For any system of equations, where the **value of the determinant** $D = 0$,

Value of determinants	Type of system	Solution
$D = 0$ and D_x, D_y and D_z are all zero	consistent and dependent	infinitely many solutions
$D = 0$ and D_x, D_y and D_z are not all zero	inconsistent	no solution

In the next example, we will use the values of the determinants to find the solution of the system.

Example 2.5.22

Solve the system of equations using Cramer's rule : $\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$

Solution

$$\begin{cases} x + 3y = 4 \\ -2x - 6y = 3 \end{cases}$$

Evaluate the determinantD,using the coefficients of the variables.

$$D = \begin{vmatrix} 1 & 3 \\ -2 & -6 \end{vmatrix}$$

$$D = -6 - (-6)$$

$$D = 0$$

We cannot use Cramer's Rule to solve this system. But by looking at the value of the determinants D_x and D_y, we can determine whether the system is dependent or inconsistent.

Evaluate the determinant D_x.

$$D_x = \begin{vmatrix} 4 & 3 \\ 3 & -6 \end{vmatrix}$$

$$D_x = -24 - 9$$

$$D_x = 15$$

Since all the determinants are not zero, the system is inconsistent. There is no solution.

Try It 2.5.23

Solve the system of equations using Cramer's rule: $\begin{cases} 4x - 3y = 8 \\ 8x - 6y = 14 \end{cases}$

Answer

no solution

> **Try It 2.5.24**
>
> Solve the system of equations using Cramer's rule: $\begin{cases} x=-3y+4 \\ 2x+6y=8 \end{cases}$
>
> **Answer**
>
> infinite solutions

Access these online resources for additional instruction and practice with solving systems of linear inequalities by graphing.

- Solving Systems of Linear Inequalities by Graphing
- Systems of Linear Inequalities

Key Concepts

- **Determinant:** The determinant of any square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, where *a*, *b*, *c*, and *d* are real numbers, is

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad-bc$$

- **Expanding by Minors along the First Row to Evaluate a 3 × 3 Determinant:** To evaluate a 3×3 determinant by expanding by minors along the first row, the following pattern:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\underbrace{\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}}_{\text{minor of } a_1} - b_1\underbrace{\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}}_{\text{minor of } b_1} + c_1\underbrace{\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}}_{\text{minor of } c_1}$$

- **Sign Pattern:** When expanding by minors using a row or column, the sign of the terms in the expansion follow the following pattern.

$$\begin{vmatrix} + & - & + \\ - & + & - \\ + & - & + \end{vmatrix}$$

- **Cramer's Rule:** For the system of equations $\begin{cases} a_1x+b_1y=k_1 \\ a_2x+b_2y=k_2 \end{cases}$, the solution (x,y) can be determined by

  $x=\dfrac{D_x}{D}$ and $y=\dfrac{D_y}{D}$

  where $D=\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ use the coefficients of the variables.

  $D_x=\begin{vmatrix} k_1 & b_1 \\ k_2 & b_2 \end{vmatrix}$ replace the *x* coefficients with the constants.

  $D_y=\begin{vmatrix} a_1 & k_1 \\ a_2 & k_2 \end{vmatrix}$ replace the *y* coefficients with the constants.

  Notice that to form the determinant D, we use take the coefficients of the variables.
- **How to solve a system of two equations using Cramer's rule.**
  1. Evaluate the determinant D, using the coefficients of the variables.
  2. Evaluate the determinant D_x. Use the constants in place of the *x* coefficients.
  3. Evaluate the determinant D_y. Use the constants in place of the *y* coefficients.
  4. Find x and y. $x=\dfrac{D_x}{D}$, $y=\dfrac{D_y}{D}$.
  5. Write the solution as an ordered pair.
  6. Check that the ordered pair is a solution to **both** original equations.
  7. **Dependent and Inconsistent Systems of Equations:** For any system of equations, where the **value of the determinant** $D=0$,

Value of determinants	Type of system	Solution
$D=0$ and $D_x,\ D_y$ and D_z are all zero	consistent and dependent	infinitely many solutions
$D=0$ and $D_x,\ D_y$ and D_z are not all zero	inconsistent	no solution

8. **Test for Collinear Points:** Three points (x_1,y_1), (x_2,y_2), and (x_3,y_3) are collinear if and only if

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} = 0$$

Glossary

determinant

Each square matrix has a real number associated with it called its determinant.

minor of an entry in a 3×3 determinant

The minor of an entry in a 3×3 determinant is the 2×2 determinant found by eliminating the row and column in the 3×3 determinant that contains the entry.

square matrix

A square matrix is a matrix with the same number of rows and columns.

Practice Makes Perfect

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercises, evaluate the determinant of each square matrix.

1. $\begin{bmatrix} 6 & -2 \\ 3 & -1 \end{bmatrix}$

2. $\begin{bmatrix} -4 & 8 \\ -3 & 5 \end{bmatrix}$

Answer

4

3. $\begin{bmatrix} -3 & 5 \\ 0 & -4 \end{bmatrix}$

4. $\begin{bmatrix} -2 & 0 \\ 7 & -5 \end{bmatrix}$

Answer

10

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercises, find and then evaluate the indicated minors.5.

5. $\begin{vmatrix} 3 & -1 & 4 \\ -1 & 0 & -2 \\ -4 & 1 & 5 \end{vmatrix}$

Find the minor ⓐ a_1 ⓑ b_2 ⓒ c_3

6. $\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$

Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

Answer

ⓐ 6 ⓑ −14 ⓒ −6

7. $\begin{vmatrix} 2 & -3 & -4 \\ -1 & 2 & -3 \\ 0 & -1 & -2 \end{vmatrix}$

Find the minor ⓐ a_2 ⓑ b_2 ⓒ c_2

8. $\begin{vmatrix} -2 & -2 & 3 \\ 1 & -3 & 0 \\ -2 & 3 & -2 \end{vmatrix}$

Find the minor ⓐ a_3 ⓑ b_3 ⓒ c_3

Answer

ⓐ 9 ⓑ −3 ⓒ 8

In the following exercises, evaluate each determinant by expanding by minors along the first row.

9. $\begin{vmatrix} -2 & 3 & -1 \\ -1 & 2 & -2 \\ 3 & 1 & -3 \end{vmatrix}$

10. $\begin{vmatrix} 4 & -1 & -2 \\ -3 & -2 & 1 \\ -2 & -5 & 7 \end{vmatrix}$

Answer

-77

11. $\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$

12. $\begin{vmatrix} 1 & 3 & -2 \\ 5 & -6 & 4 \\ 0 & -2 & -1 \end{vmatrix}$

Answer

49

In the following exercises, evaluate each determinant by expanding by minors.

13. $\begin{vmatrix} -5 & -1 & -4 \\ 4 & 0 & -3 \\ 2 & -2 & 6 \end{vmatrix}$

14. $\begin{vmatrix} 4 & -1 & 3 \\ 3 & -2 & 2 \\ -1 & 0 & 4 \end{vmatrix}$

Answer

-24

15. $\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$

16. $\begin{vmatrix} 2 & -4 & -3 \\ 5 & -1 & -4 \\ 3 & 2 & 0 \end{vmatrix}$

Answer

25

Use Cramer’s Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer’s Rule.

17. $\begin{cases} -2x+3y=3 \\ x+3y=12 \end{cases}$

18. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

Answer

$(7,6)$

19. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

20. $\begin{cases} 2x+y=-4 \\ 3x-2y=-6 \end{cases}$

Answer

$(-2,0)$

21. $\begin{cases} x-2y=-5 \\ 2x-3y=-4 \end{cases}$

22. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3,2)$

23. $\begin{cases} 5x-3y=-1 \\ 2x-y=2 \end{cases}$

24. $\begin{cases} 3x+8y=-3 \\ 2x+5y=-3 \end{cases}$

Answer

$(-9,3)$

25. $\begin{cases} 6x-5y+2z=3 \\ 2x+y-4z=5 \\ 3x-3y+z=-1 \end{cases}$

26. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

Answer

$(-3,-5,4)$

27. $\begin{cases} 2x-5y+3z=8 \\ 3x-y+4z=7 \\ x+3y+2z=-3 \end{cases}$

28. $\begin{cases} 11x+9y+2z=-9 \\ 7x+5y+3z=-7 \\ 4x+3y+z=-3 \end{cases}$

Answer

$(2,-3,-2)$

29. $\begin{cases} x+2z=0 \\ 4y+3z=-2 \\ 2x-5y=3 \end{cases}$

30. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

Answer

$(-3,2,3)$

31. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

32. $\begin{cases} 3x-z=-3 \\ 5y+2z=-6 \\ 4x+3y=-8 \end{cases}$

Answer

$(-2,0,-3)$

33. $\begin{cases} 2x+y=3 \\ 6x+3y=9 \end{cases}$

34. $\begin{cases} x-4y=-1 \\ -3x+12y=3 \end{cases}$

Answer

infinitely many solutions

35. $\begin{cases} -3x-y=4 \\ 6x+2y=-16 \end{cases}$

36. $\begin{cases} 4x+3y=2 \\ 20x+15y=5 \end{cases}$

Answer

inconsistent

37. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

38. $\begin{cases} 2x+3y+z=1 \\ 2x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

Answer

inconsistent

39. $\begin{cases} 3x+4y-3z=-2 \\ 2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

40. $\begin{cases} x-2y+3z=1 \\ x+y-3z=7 \\ 3x-4y+5z=7 \end{cases}$

Answer

infinitely many solutions

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

41. $(0,1)$, $(2,0)$, and $(-2,2)$.

42. $(0,-5)$, $(-2,-2)$, and $(2,-8)$.

Answer

yes

43. $(4, -3)$, $(6, -4)$, and $(2, -2)$.

44. $(-2, 1)$, $(-4, 4)$, and $(0, -2)$.

Answer

no

Writing Exercises

45. Explain the difference between a square matrix and its determinant. Give an example of each.

46. Explain what is meant by the minor of an entry in a square matrix.

Answer

Answers will vary.

47. Explain how to decide which row or column you will use to expand a 3×3 determinant.

48. Explain the steps for solving a system of equations using Cramer's rule.

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the determinant of a 2 × 2 matrix.			
evaluate the determinant of a 3 × 3 matrix.			
use Cramer's rule to solve systems of equations.			
solve applications using determinants.			

ⓑ After reviewing this checklist, what will you do to become confident for all objectives?

2.6: Chapter 2 Review Exercises

Chapter Review Exercises

Solve Systems of Linear Equations with Two Variables

Determine Whether an Ordered Pair is a Solution of a System of Equations.

In the following exercises, determine if the following points are solutions to the given system of equations.

1. $\begin{cases} x+3y=-9 \\ 2x-4y=12 \end{cases}$

ⓐ $(-3,-2)$
ⓑ $(0,-3)$

2. $\begin{cases} x+y=8 \\ y=x-4 \end{cases}$

ⓐ $(6,2)$
ⓑ $(9,-1)$

Answer

ⓐ yes ⓑ no

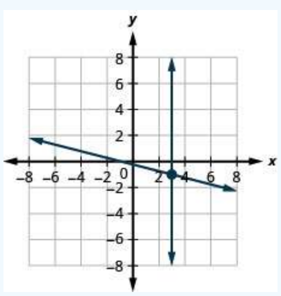
Solve a System of Linear Equations by Graphing

In the following exercises, solve the following systems of equations by graphing.

3. $\begin{cases} 3x+y=6 \\ x+3y=-6 \end{cases}$

4. $\begin{cases} x+4y=-1 \\ x=3 \end{cases}$

Answer

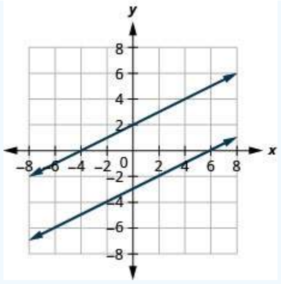


$(3,-1)$

5. $\begin{cases} 2x-y=5 \\ 4x-2y=10 \end{cases}$

6. $\begin{cases} -x+2y=4 \\ y=\frac{1}{2}x-3 \end{cases}$

Answer



no solution

In the following exercises, without graphing determine the number of solutions and then classify the system of equations.

7. $\begin{cases} y=\frac{2}{5}x+2 \\ -2x+5y=10 \end{cases}$

8. $\begin{cases} 3x+2y=6 \\ y=-3x+4 \end{cases}$

Answer

one solution, consistent system, independent equations

9. $\begin{cases} 5x-4y=0 \\ y=\frac{5}{4}x-5 \end{cases}$

Solve a System of Equations by Substitution

In the following exercises, solve the systems of equations by substitution.

10. $\begin{cases} 3x-2y=2 \\ y=\frac{1}{2}x+3 \end{cases}$

Answer

$(4,5)$

11. $\begin{cases} x-y=0 \\ 2x+5y=-14 \end{cases}$

12. $\begin{cases} y=-2x+7 \\ y=\frac{2}{3}x-1 \end{cases}$

Answer

$(3,1)$

13. $\begin{cases} y=-5x \\ 5x+y=6 \end{cases}$

14. $\begin{cases} y=-\frac{1}{3}x+2 \\ x+3y=6 \end{cases}$

Answer

infinitely many solutions

Solve a System of Equations by Elimination

In the following exercises, solve the systems of equations by elimination

15. $\begin{cases} x+y=12 \\ x-y=-10 \end{cases}$

16. $\begin{cases} 3x-8y=20 \\ x+3y=1 \end{cases}$

Answer

$(4,-1)$

17. $\begin{cases} 9x+4y=2 \\ 5x+3y=5 \end{cases}$

18. $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$

Answer

$(6,2)$

19. $\begin{cases} -x+3y=8 \\ 2x-6y=-20 \end{cases}$

Choose the Most Convenient Method to Solve a System of Linear Equations

In the following exercises, decide whether it would be more convenient to solve the system of equations by substitution or elimination.

20. $\begin{cases} 6x-5y=27 \\ 3x+10y=-24 \end{cases}$

Answer

elimination

21. $\begin{cases} y=3x-9 \\ 4x-5y=23 \end{cases}$

Solve Applications with Systems of Equations

Solve Direct Translation Applications

In the following exercises, translate to a system of equations and solve.

22. Mollie wants to plant 200 bulbs in her garden, all irises and tulips. She wants to plant three times as many tulips as irises. How many irises and how many tulips should she plant?

Answer

50 irises and 150 tulips

23. Ashanti has been offered positions by two phone companies. The first company pays a salary of $22,000 plus a commission of $100 for each contract sold. The second pays a salary of $28,000 plus a commission of $25 for each contract sold. How many contract would need to be sold to make the total pay the same?

24. Leroy spent 20 minutes jogging and 40 minutes cycling and burned 600 calories. The next day, Leroy swapped times, doing 40 minutes of jogging and 20 minutes of cycling and burned the same number of calories. How many calories were burned for each minute of jogging and how many for each minute of cycling?

Answer

10 calories jogging and 10 calories cycling

25. Troy and Lisa were shopping for school supplies. Each purchased different quantities of the same notebook and calculator. Troy bought four notebooks and five calculators for $116. Lisa bought two notebooks and three calculators for $68. Find the cost of each notebook and each thumb drive.

Solve Geometry Applications

In the following exercises, translate to a system of equations and solve.

26. The difference of two supplementary angles is 58 degrees. Find the measures of the angles.

Answer

119, 61

27. Two angles are complementary. The measure of the larger angle is five more than four times the measure of the smaller angle. Find the measures of both angles.

28. The measure of one of the small angles of a right triangle is 15 less than twice the measure of the other small angle. Find the measure of both angles.

Answer

35° and 55°

29. Becca is hanging a 28 foot floral garland on the two sides and top of a pergola to prepare for a wedding. The height is four feet less than the width. Find the height and width of the pergola.

30. The perimeter of a city rectangular park is 1428 feet. The length is 78 feet more than twice the width. Find the length and width of the park.

Answer

the length is 450 feet, the width is 264 feet

Solve Uniform Motion Applications

In the following exercises, translate to a system of equations and solve.

31. Sheila and Lenore were driving to their grandmother's house. Lenore left one hour after Sheila. Sheila drove at a rate of 45 mph, and Lenore drove at a rate of 60 mph. How long will it take for Lenore to catch up to Sheila?

32. Bob left home, riding his bike at a rate of 10 miles per hour to go to the lake. Cheryl, his wife, left 45 minutes (34(34 hour) later, driving her car at a rate of 25 miles per hour. How long will it take Cheryl to catch up to Bob?

Answer

12 an hour

33. Marcus can drive his boat 36 miles down the river in three hours but takes four hours to return upstream. Find the rate of the boat in still water and the rate of the current.

34. A passenger jet can fly 804 miles in 2 hours with a tailwind but only 776 miles in 2 hours into a headwind. Find the speed of the jet in still air and the speed of the wind.

Answer

the rate of the jet is 395 mph, the rate of the wind is 7 mph

Solve Mixture Applications with Systems of Equations

Solve Mixture Applications with Systems of Equations

For the following exercises, translate to a system of equations and solve.

35. Lynn paid a total of $2,780 for 261 tickets to the theater. Student tickets cost $10 and adult tickets cost $15. How many student tickets and how many adult tickets did Lynn buy?

36. Priam has dimes and pennies in a cup holder in his car. The total value of the coins is $4.21. The number of dimes is three less than four times the number of pennies. How many dimes and how many pennies are in the cup?

Answer

41 dimes and 11 pennies

37. Yumi wants to make 12 cups of party mix using candies and nuts. Her budget requires the party mix to cost her $1.29 per cup. The candies are $2.49 per cup and the nuts are $0.69 per cup. How many cups of candies and how many cups of nuts should she use?

38. A scientist needs 70 liters of a 40% solution of alcohol. He has a 30% and a 60% solution available. How many liters of the 30% and how many liters of the 60% solutions should he mix to make the 40% solution?

Answer

$46\frac{2}{3}$ liters of 30% solution, $23\frac{1}{3}$ liters of 60% solution

Solve Interest Applications

For the following exercises, translate to a system of equations and solve.

39. Jack has $12,000 to invest and wants to earn 7.5% interest per year. He will put some of the money into a savings account that earns 4% per year and the rest into CD account that earns 9% per year. How much money should he put into each account?

40. When she graduates college, Linda will owe $43,000 in student loans. The interest rate on the federal loans is 4.5% and the rate on the private bank loans is 2%. The total interest she owes for one year was $1,585. What is the amount of each loan?

Answer

$29,000 for the federal loan, $14,000 for the private loan

Solve Systems of Equations with Three Variables

Solve Systems of Equations with Three Variables

In the following exercises, determine whether the ordered triple is a solution to the system.

41. $\begin{cases} 3x-4y-3z=2 \\ 2x-6y+z=3 \\ 2x+3y-2z=3 \end{cases}$

ⓐ $(2,3,-1)$
ⓑ $(3,1,3)$

42. $\begin{cases} y=\frac{2}{3}x-2 \\ x+3y-z=15 \\ x-3y+z=-2 \end{cases}$

ⓐ $(-6,5,\frac{1}{2})$
ⓑ $(5,\frac{4}{3},-3)$

Answer

ⓐ no ⓑ yes

Solve a System of Linear Equations with Three Variables

In the following exercises, solve the system of equations.

43. $\begin{cases} 3x-5y+4z=5 \\ 5x+2y+z=0 \\ 2x+3y-2z=3 \end{cases}$

44. $\begin{cases} x+\frac{5}{2}y+z=-2 \\ 2x+2y+\frac{1}{2}z=-4 \\ \frac{1}{3}x-y-z=1 \end{cases}$

Answer

$(-3,2,-4)$

45. $\begin{cases} 5x+3y=-6 \\ 2y+3z=-1 \\ 7x+z=1 \end{cases}$

46. $\begin{cases} 2x+3y+z=12 \\ x+y+z=9 \\ 3x+4y+2z=20 \end{cases}$

Answer

no solution

47. $\begin{cases} -x-3y+2z=14 \\ -x+2y-3z=-4 \\ 3x+y-2z=6 \end{cases}$

Solve Applications using Systems of Linear Equations with Three Variables

48. After attending a major league baseball game, the patrons often purchase souvenirs. If a family purchases 4 t-shirts, a cap and 1 stuffed animal their total is $135. A couple buys 2 t-shirts, a cap and 3 stuffed animals for their nieces and spends $115. Another couple buys 2 t-shirts, a cap and 1 stuffed animal and their total is $85. What is the cost of each item?

Answer

$25, 20, 15$

Solve Systems of Equations Using Matrices

Write the Augmented Matrix for a System of Equations.

Write each system of linear equations as an augmented matrix.

49. $\begin{cases} 3x - y = -1 \\ -2x + 2y = 5 \end{cases}$

50. $\begin{cases} 4x + 3y = -2 \\ x - 2y - 3z = 7 \\ 2x - y + 2z = -6 \end{cases}$

Answer

$$\begin{bmatrix} 4 & 3 & 0 & -2 \\ 1 & -2 & -3 & 7 \\ 2 & -1 & 2 & -6 \end{bmatrix}$$

Write the system of equations that that corresponds to the augmented matrix.

51. $\left[\begin{array}{cc|c} 2 & -4 & -2 \\ 3 & -3 & -1 \end{array}\right]$

52. $\left[\begin{array}{ccc|c} 1 & 0 & -3 & -1 \\ 1 & -2 & 0 & -2 \\ 0 & -1 & 2 & 3 \end{array}\right]$

Answer

$$\begin{cases} x - 3z = -1 \\ x - 2y = -27 \\ -y + 2z = 3 \end{cases}$$

In the following exercises, perform the indicated operations on the augmented matrices.

53. $\left[\begin{array}{cc|c} 4 & -6 & -3 \\ 3 & 2 & 1 \end{array}\right]$

ⓐ Interchange rows 2 and 1.
ⓑ Multiply row 1 by 4.
ⓒ Multiply row 2 by 3 and add to row 1.

54. $\left[\begin{array}{ccc|c} 1 & -3 & -2 & 4 \\ 2 & 2 & -1 & -3 \\ 4 & -2 & -3 & -1 \end{array}\right]$

ⓐ Interchange rows 2 and 3.
ⓑ Multiply row 1 by 2.
ⓒ Multiply row 3 by −2−2 and add to row 2.

Answer

ⓐ $\begin{bmatrix} 1 & -3 & -2 & 4 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓑ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 2 & 2 & -1 & -3 \end{bmatrix}$

ⓒ $\begin{bmatrix} 2 & -6 & -4 & 8 \\ 4 & -2 & -3 & -1 \\ 0 & -6 & -1 & 5 \end{bmatrix}$

Solve Systems of Equations Using Matrices

In the following exercises, solve each system of equations using a matrix.

55. $\begin{cases} 4x+y=6 \\ x-y=4 \end{cases}$

56. $\begin{cases} 2x-y+3z=-3 \\ -x+2y-z=10 \\ x+y+z=5 \end{cases}$

Answer

$(-2,5,-2)$

57. $\begin{cases} 2y+3z=-1 \\ 5x+3y=-6 \\ 7x+z=1 \end{cases}$

58. $\begin{cases} x+2y-3z=-1 \\ x-3y+z=1 \\ 2x-y-2z=2 \end{cases}$

Answer

no solution

59. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

Solve Systems of Equations Using Determinants

Evaluate the Determinant of a 2 × 2 Matrix

In the following exercise, evaluate the determinant of the square matrix.

60. $\begin{bmatrix} 8 & -4 \\ 5 & -3 \end{bmatrix}$

Answer

-4

Evaluate the Determinant of a 3 × 3 Matrix

In the following exercise, find and then evaluate the indicated minors.

61. $\begin{vmatrix} -1 & -3 & 2 \\ 4 & -2 & -1 \\ -2 & 0 & -3 \end{vmatrix}$; Find the minor ⓐ a_1 ⓑ b_1 ⓒ c_2

In the following exercise, evaluate each determinant by expanding by minors along the first row.

62. $\begin{vmatrix} -2 & -3 & -4 \\ 5 & -6 & 7 \\ -1 & 2 & 0 \end{vmatrix}$

Answer

21In the following exercise, evaluate each determinant by expanding by minors.

63. $\begin{vmatrix} 3 & 5 & 4 \\ -1 & 3 & 0 \\ -2 & 6 & 1 \end{vmatrix}$

Use Cramer's Rule to Solve Systems of Equations

In the following exercises, solve each system of equations using Cramer's rule

64. $\begin{cases} x-3y=-9 \\ 2x+5y=4 \end{cases}$

Answer

$(-3,2)$

65. $\begin{cases} 4x-3y+z=7 \\ 2x-5y-4z=3 \\ 3x-2y-2z=-7 \end{cases}$

66. $\begin{cases} 2x+5y=4 \\ 3y-z=3 \\ 4x+3z=-3 \end{cases}$

Answer

$(-3,2,3)$

67. $\begin{cases} x+y-3z=-1 \\ y-z=0 \\ -x+2y=1 \end{cases}$

68. $\begin{cases} 3x+4y-3z=-2 \\ 2x+3y-z=-1 \\ 2x+y-2z=6 \end{cases}$

Answer

inconsistent

Solve Applications Using Determinants

In the following exercises, determine whether the given points are collinear.

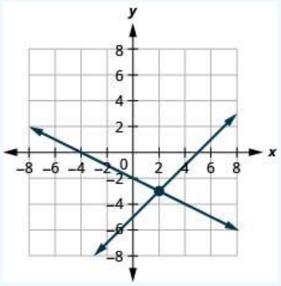
69. $(0,2)$, $(-1,-1)$, and $(-2,4)$

Chapter Practice Test

In the following exercises, solve the following systems by graphing.

1. $\begin{cases} x-y=5 \\ x+2y=-4 \end{cases}$

Answer



$(2,-3)$

2. $\begin{cases} x-y>-2 \\ y\leq 3x+1 \end{cases}$

In the following exercises, solve each system of equations. Use either substitution or elimination.

3. $\begin{cases} x+4y=6 \\ -2x+y=-3 \end{cases}$

Answer

$(2,1)$

4. $\begin{cases} -3x+4y=2 \\ 5x-5y=-23 \end{cases}$

5. $\begin{cases} x+y-z=-1 \\ 2x-y+2z=8 \\ -3x+2y+z=-9 \end{cases}$

Answer

$(2,-2,1)$

Solve the system of equations using a matrix.

6. $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$

7. $\begin{cases} -3x+y+z=-4 \\ -x+2y-2z=1 \\ 2x-y-z=-1 \end{cases}$

Answer

$(5, 7, 4)$

Solve using Cramer's rule.

8. $\begin{cases} 3x + y = -3 \\ 2x + 3y = 6 \end{cases}$

9. Evaluate the determinant by expanding by minors:

$$\begin{vmatrix} 3 & -2 & -2 \\ 2 & -1 & 4 \\ -1 & 0 & -3 \end{vmatrix}$$

Answer

99

In the following exercises, translate to a system of equations and solve.

10. Greg is paddling his canoe upstream, against the current, to a fishing spot 10 miles away. If he paddles upstream for 2.5 hours and his return trip takes 1.25 hours, find the speed of the current and his paddling speed in still water.

11. A pharmacist needs 20 liters of a 2% saline solution. He has a 1% and a 5% solution available. How many liters of the 1% and how many liters of the 5% solutions should she mix to make the 2% solution?

Answer

15 liters of 1% solution, 5 liters of 5% solution

12. Arnold invested $64,000, some at 5.5% interest and the rest at 9%. How much did he invest at each rate if he received $4,500 in interest in one year?

13. The church youth group is selling snacks to raise money to attend their convention. Amy sold 2 pounds of candy, 3 boxes of cookies and 1 can of popcorn for a total sales of $65. Brian sold 4 pounds of candy, 6 boxes of cookies and 3 cans of popcorn for a total sales of $140. Paulina sold 8 pounds of candy, 8 boxes of cookies and 5 can of popcorn for a total sales of $250. What is the cost of each item?

Answer

The candy cost $20; the cookies cost $5; and the popcorn cost $10.

14. The manufacturer of a granola bar spends $1.20 to make each bar and sells them for $2. The manufacturer also has fixed costs each month of $8,000.

ⓐ Find the cost function *C* when *x* granola bars are manufactured
ⓑ Find the revenue function *R* when *x* granola bars are sold.
ⓒ Show the break-even point by graphing both the Revenue and Cost functions on the same grid.
ⓓ Find the break-even point. Interpret what the break-even point means.

15. Translate to a system of inequalities and solve.

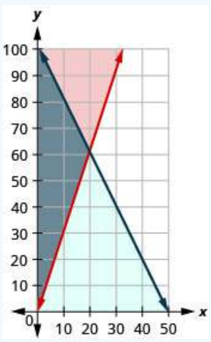
Andi wants to spend no more than $50 on Halloween treats. She wants to buy candy bars that cost $1 each and lollipops that cost $0.50 each, and she wants the number of lollipops to be at least three times the number of candy bars.

ⓐ Write a system of inequalities to model this situation.
ⓑ Graph the system.
ⓒ Can she buy 20 candy bars and 40 lollipops?

Answer

ⓐ $\begin{cases} C \geq 0 \\ L \geq 0 \\ C + 0.5L \leq 50 \\ L \geq 3C \end{cases}$

ⓑ



ⓒ no
ⓓ yes

CHAPTER OVERVIEW

3: Polynomials and Integer Exponents

In this chapter you will investigate polynomials and polynomial functions and learn how to perform mathematical operations on them.

3.1: Polynomials Review
3.2: Properties of Integer Exponents
3.3: Chapter 3 Review Exercises

3.1: Polynomials Review

Learning Objectives

By the end of this section, you will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Evaluate a polynomial function for a given value

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $3x^2+3x+1+8x^2+5x+5$.
2. Subtract $(5n+8)-(2n-1)$.
3. Evaluate $4xy^2$ when $x=-2x$ and $y=5$.

Determine the Degree of Polynomials

We have learned that a *term* is a constant or the product of a constant and one or more variables. A **monomial** is an algebraic expression with one term. When it is of the form ax^m, where a is a constant, x is the variable, and m is a positive integer, it is called a monomial in one variable. Some examples of monomials in one variable are $2x^5$ and $-3x^{10}$. Monomials can also have more than one variable such as $-4a^2b^3c^2$.

Definition 3.1.1

A **monomial** is an algebraic expression with one term. A monomial in one variable is a term of the form ax^m, where a is a constant and m is a positive integer.

A monomial, or two or more monomials combined by addition or subtraction, is a **polynomial**. Some polynomials have special names, based on the number of terms. A monomial is a polynomial with exactly one term. A binomial has exactly two terms, and a **trinomial** has exactly three terms. There are no special names for polynomials with more than three terms.

Definition 3.1.2

- A monomial, or two or more algebraic terms combined by addition or subtraction is a **polynomial**.
- A polynomial with exactly one term is called a **monomial**.
- A polynomial with exactly two terms is called a **binomial**.
- A polynomial with exactly three terms is called a **trinomial**.

Here are some examples of polynomials.

Polynomial	$y+1$	$4a^2-7ab+2b^2$	$4x^4+x^3+8x^2-9x+1$	0
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7ba+7b$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$9m^2+2mn-8n^2$	$6k^4-k^3+8k$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the "family" of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable. A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0.

Definition 3.1.3

- The **degree of a term** is the sum of the exponents of its variables.
- The **degree of a constant** is 0.
- The **degree of a polynomial** is the highest degree among all its terms.

Let’s see how this works by looking at several polynomials. We’ll take it step by step, starting with monomials, and then progressing to polynomials with more terms. Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of b is 2. The sum of the exponents, 1+2,1+2, is 3 so the degree is 3.

	$8\ a\ b^2$
exponents	1 2
degree of monomial	3 (1 + 2)

Here are some additional examples.

Monomial	14	$8ab^2$	$-9x^3y^5$	$-13a$
Degree	0	3	8	1

Binomial	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Degree of each term	$1,\ 0$	$2,\ 1$	$4,\ 0$	$3,\ 2$
Degree of the polynomial	1	2	4	3

Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of each term	$2,\ 1,\ 0$	$2,\ 2,\ 2$	$4,\ 5,\ 6$	$4,\ 2,\ 0$
Degree of the polynomial	2	2	6	4

Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$	
Degree of each term	$1,\ 0$	$2,\ 1,\ 0$	$4,\ 3,\ 2,\ 1,\ 0$	
Degree of the polynomial	1	2	4	

Working with polynomials is easier when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first.

Example 3.1.4

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $7y^2-5y+3$

b. $-2a^4b^2$

c. $3x^5-4x^3-6x^2+x-8$

d. $2y-8xy^3$

e. 15

Solution

	Polynomial	Number of terms	Type	Degree of terms	Degree of polynomial
a.	$7y^2 - 5y + 3$	3	Trinomial	2, 1, 0	2
b.	$-2a^4b^2$	1	Monomial	4, 2	6
c.	$3x^5 - 4x^3 - 6x^2 + x - 8$	5	Polynomial	5, 3, 2, 1, 0	5
d.	$2y - 8xy^3$	2	Binomial	1, 4	4
e.	15	1	Monomial	0	0

Try It 3.1.5

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. -5

b. $8y^3 - 7y^2 - y - 3$

c. $-3x^2y - 5xy + 9xy^3$

d. $81m^2 - 4n^2$

e. $-3x^6y^3z$

Answer a

It is a monomial of degree 0.

Answer b

It is a polynomial of degree 3.

Answer c

It is a trinomial of degree 3.

Answer d

It is a binomial of degree 2.

Answer e

It is a monomial of degree 10.

Try It 3.1.6

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $64k^3 - 8$

b. $9m^3 + 4m^2 - 2$

c. 56

d. $8a^4 - 7a^3b - 6a^2b^2 - 4ab^3 + 7b^4$

e. $-p^4q^3$

Answer

a. It is a binomial of degree 3.

b. It is a trinomial of degree 3.

c. It is a monomial of degree 0.

d. It is a polynomial of degree 4.

e. It is a monomial of degree 7.

Add and Subtract Polynomials

We have learned how to simplify expressions by combining like terms. Remember, like terms must have the same variables with the same exponents. Since monomials are terms, adding and subtracting monomials is the same as combining like terms. If the monomials are like terms, we just combine them by adding or subtracting the coefficients.

Example 3.1.7

Add or subtract:

a. $25y^2+15y^2$

b. $16pq^3-(-7pq^3)$

Solution

a.

	$25y^2+15y^2$
Combine like terms.	$=40y^2$

b.

	$16pq^3-(-7pq^3)$
Combine like terms.	$=23pq^3)$

Try It 3.1.8

Add or subtract:

a. $12q^2+9q^2$

b. $8mn^3-(-5mn^3)$

Answer

a. $21q^2$

b. $13mn^3$

Try It 3.1.9

Add or subtract:

a. $-15c^2+8c^2$

b. $-15y^2z^3-(-5y^2z^3)$

Answer

a. $-7c^2$

b. $-10y^2z^3$

Remember that like terms must have the same variables with the same exponents.

Example 3.1.10

Simplify:

a. $a^2+7b^2-6a^2$

b. $u^2v+5u^2-3v^2$

Solution

a.

	$a^2+7b^2-6a^2$
Combine like terms.	$=-5a^2+7b^2$

b.

	$u^2v+5u^2-3v^2$
Combine like terms.	There are no like terms to combine. In this case, the polynomial is unchanged. $u^2v+5u^2-3v^2$

Try It 3.1.11

Add:

a. $8y^2+3z^2-3y^2$

b. $m^2n^2-8m^2+4n^2$

Answer

a. $5y^2+3z^2$
b. $m^2n^2-8m^2+4n^2$

Try It 3.1.12

Add:

a. $3m^2+n^2-7m^2$

b. $pq^2-6p-5q^2$

Answer

a. $-4m^2+n^2$
b. $pq^2-6p-5q^2$

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 3.1.13

Find the sum $(7y^2-2y+9)\ +\ (4y^2-8y-7)$.

Solution

	$(7y^2-2y+9)\ +\ (4y^2-8y-7)$
Identify like terms.	$=(\underline{\underline{7y^2}}-\underline{2y}+9)+(\underline{\underline{4y^2}}-\underline{8y}-7)$
Rewrite without the parentheses, rearranging to get the like terms together.	$=\underline{\underline{7y^2+4y^2}}\ -\underline{2y-8y}\ +9-7$
Combine like terms.	$=11y^2-10y+2$

Try It 3.1.14

Find the sum $(7x^2-4x+5)\ +\ (x^2-7x+3)$.

Answer

$8x^2-11x+8$

Try It 3.1.15

Find the sum $(14y^2+6y-4)\ +\ (3y^2+8y+5)$.

Answer

$17y^2+14y+1$

Be careful with the signs as you distribute while subtracting the polynomials in the next example.

Example 3.1.16

Find the difference $(9w^2-7w+5)\ -\ (2w^2-4)$.

Solution

	$(9w^2-7w+5)\ -\ (2w^2-4)$
Distribute and identify like terms.	$=\underline{\underline{9w^2}}-\underline{7w}+5-\underline{\underline{2w^2}}+4$
Rearrange the terms.	$=\underline{\underline{9w^2-2w^2}}\ -\underline{7w}+5+4$
Combine like terms.	$=7w^2-7w+9$

Try It 3.1.17

Find the difference $(8x^2+3x-19)\ -\ (7x^2-14)$.

Answer

x^2+3x-5

Try It 3.1.18

Find the difference $(9b^2 - 5b - 4) - (3b^2 - 5b - 7)$.

Answer

$6b^2 + 3$

Example 3.1.19

Subtract $p^2 + 10pq - 2q^2$ from $p^2 + q^2$.

Solution

	$(p^2 + q^2) - (p^2 + 10pq - 2q^2)$
Distribute and identify like terms.	$= \underline{\underline{p^2}} + \underline{q^2} - \underline{\underline{p^2}} - 10pq + \underline{2q^2}$
Rearrange the terms, putting like terms together.	$= \underline{\underline{p^2 - p^2}} - 10pq + \underline{q^2 + 2q^2}$
Combine like terms.	$= -10pq + 3q^2$

Try It 3.1.20

Subtract $a^2 + 5ab - 6b^2$ from $a^2 + b^2$.

Answer

$-5ab + 7b^2$

Try It 3.1.21

Subtract $m^2 - 7mn - 3n^2$ from $m^2 + n^2$.

Answer

$7mn + 4n^2$

Example 3.1.22

Find the sum $(u^2 - 6uv + 5v^2) + (3u^2 + 2uv)$.

Solution

	$(u^2 - 6uv + 5v^2) + (3u^2 + 2uv)$
Distribute and identify like terms.	$= \underline{\underline{u^2}} - \underline{6uv} + 5v^2 + \underline{\underline{3u^2}} + \underline{2uv}$
Rearrange the terms to put like terms together.	$= \underline{\underline{u^2}} + \underline{\underline{3u^2}} - \underline{6uv} + \underline{2uv} + 5v^2$
Combine like terms.	$= 4u^2 - 4uv + 5v^2$

Try It 3.1.23

Find the sum $(3x^2 - 4xy + 5y^2) + (2x^2 - xy)$.

Answer

$5x^2 - 5xy + 5y^2$

Try It 3.1.24

Find the sum $(2x^2-3xy-2y^2)\ +\ (5x^2-3xy)$.

Answer

$7x^2-6xy-2y^2$

When we add and subtract more than two polynomials, the process is the same.

Example 3.1.25

Simplify $(a^3-a^2b)\ -\ (ab^2+b^3)\ +\ (a^2b+ab^2)$.

Solution

	$(a^3-a^2b)\ -\ (ab^2+b^3)\ +\ (a^2b+ab^2)$
Distribute.	$=a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Rearrange the terms to put like terms together.	$=a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	$=a^3-b^3$

Try It 3.1.26

Simplify $(x^3-x^2y)\ -\ (xy^2+y^3)\ +\ (x^2y+xy^2)$.

Answer

x^3+y^3

Try It 3.1.27

Simplify $(p^3-p^2q)\ +\ (pq^2+q^3)\ -\ (p^2q+pq^2)$.

Answer

$p^3-3p^2q+q^3$

Evaluate a Polynomial

Example 3.1.28

For the polynomial $5x^2-8x+4$ evaluate where:

a. $x=4$

b. $x=-2$

c. $x=0$

Solution

a.

	$5x^2-8x+4$
Substitute 4 for x	$5(4)^2-8(4)+4$
Simplify the exponents.	$=5\cdot 16-8(4)+4$

	$5x^2-8x+4$
Multiply.	$=80-32+4$
Simplify.	$=52$

b.

	$5x^2-8x+4$
To find $f(-2)$, substitute -2 for x.	$5(-2)^2-8(-2)+4$
Simplify the exponents.	$=5\cdot 4-8(-2)+4$
Multiply.	$=20+16+4$
Simplify.	$=40$

c.

	$5x^2-8x+4$
To find $f(0)$, substitute 0 for x.	$5(0)^2-8(0)+4$
Simplify the exponents.	$=5\cdot 0-8(0)+4$
Multiply.	$=0+0+4$
Simplify.	$=4$

? Try It 3.1.29

For the polynomial $3x^2+2x-15$, evaluate at

a. $x=3$

b. $x=-5$

c. $x=0$

Answer

a. 18

b. 50

c. -15

? Try It 3.1.30

For the polynomial $5x^2-x-4$, evaluate at

a. $x=-2$

b. $x=-1$

c. $x=0$

Answer

a. 20

b. 2

c. -4

Polynomials similar to the one in the next example are used in many fields to model the height of an object at some time after it is projected into the air. The polynomial in the next example is used specifically to model the height of an object which is dropped from 250 ft.

Example 3.1.31

The polynomial $-16t^2+250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t=2$ seconds.

Solution

If we call the height h, then the solutions to the equation $h=-16t^2+250$ are (t,h) where h is the height in feet of the ball at time t seconds.

	$-16t^2+250$
To find the height at 2 seconds, we substitute 2 for t.	$h=-16(2)^2+250$
Simplify.	$=-16\cdot 4+250$
Simplify.	$=-64+250$
Simplify.	$=186$
Answer the question.	After 2 seconds the height of the ball is 186 feet. That is, from finding the solution $(t,h)=(2,186)$, we conclude that the height of the ball after 2 seconds is 186 feet.

Note that in the above example, the interpretation of the polynomial leads us to write the equation that relates height and time that we claim is true. So its solutions are of interest. We are asked, in particular, about the height after 2 seconds, so we proceed to find a solution where $t=2$. Our answer is the corresponding h-coordinate.

Try It 3.1.32

The polynomial $-16t^2+150$ gives the height of a stone t seconds after it is dropped from a 150-foot tall cliff. Find the height after $t=0$ seconds (the initial height of the object).

Answer

The height is 150 feet.

Try It 3.1.33

The polynomial $-16t^2+175$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t=3$ seconds.

Answer

The height is 31 feet.

Key Concepts

- **Monomial**
  - A **monomial** is an algebraic expression with one term.
  - A monomial in one variable is a term of the form ax^m where a is a constant and m is a whole number.
- **Polynomials**
  - **Polynomial**—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
  - **Monomial** —A polynomial with exactly one term is called a monomial.

- **Binomial** — A polynomial with exactly two terms is called a binomial.
- **Trinomial** —A polynomial with exactly three terms is called a trinomial.

- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.

Glossary

binomial

A binomial is a polynomial with exactly two terms.

degree of a constant

The degree of any constant is 0.

degree of a polynomial

The degree of a polynomial is the highest degree of all its terms.

degree of a term

The degree of a term is the sum of the exponents of its variables.

monomial

A monomial is an algebraic expression with one term. A monomial in one variable is a term of the form axm,axm, where *a* is a constant and *m* is a whole number.

polynomial

A monomial or two or more monomials combined by addition or subtraction is a polynomial.

standard form of a polynomial

A polynomial is in standard form when the terms of a polynomial are written in descending order of degrees.

trinomial

A trinomial is a polynomial with exactly three terms.

3.2: Properties of Integer Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the properties for exponents
- Use the definition of a negative exponent

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $(-2)(-2)(-2)$.

2. Simplify $\frac{8x}{24y}$.

3. Name the decimal $(-2.6)(4.21)$.

Simplify Expressions Using the Properties for Exponents

Remember that a positive integer exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the positive integer *exponent* m tells us how many times we use the *base* a as a factor.

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{m factors}}$$

For example,

$$(-9)^5 = \underbrace{(-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9)}_{\text{5 factors}}$$

Let's review the vocabulary for expressions with exponents.

Definition 3.2.1

$$a^m = \underbrace{a \cdot a \cdot a \cdots a}_{m\text{ factors}}$$

This is read a to the m^{th} **power**.

In the expression a^m with positive integer m and $a \neq 0$, the **exponent** m tells us how many times we use the **base** a as a factor.

When we combine like terms by adding and subtracting, we need to have the same base with the same exponent. But when we multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

First, we will look at an example that leads to the *Product Property for Positive Integer Exponents*.

	x^2x^3
What does this mean?	$= \underbrace{x \cdot x}_{\text{2 factors}} \cdot \underbrace{x \cdot x \cdot x}_{\text{3 factors}}$ $= \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{\text{5 factors}}$

	x^2x^3
	$= x^5$

The base stayed the same and we added the exponents.

Product Property for Positive Integer Exponents

If a is a real number and m and n are positive integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Example 3.2.2

Simplify each expression:

a. y^5y^6

b. $2^x \cdot 2^{3x}$

c. $2a^7 \cdot 3a$

d. $d^4d^5d^2$

Solution

a.

	y^5y^6
Use the Product Property, $a^m a^n = a^{m+n}$.	$= y^{5+6}$
Simplify.	$= y^{11}$

b.

	$2^x \cdot 2^{3x}$
Use the Product Property, $a^m a^n = a^{m+n}$.	$= 2^{x+3x}$
Simplify.	$= 2^{4x}$

c.

	$2a^7 \cdot 3a$
Rewrite, $a = a^1$.	$= 2a^7 \cdot 3a^1$
Use the Commutative Property and use the Product Property, $a^m a^n = a^{m+n}$.	$= 2 \cdot 3a^{7+1}$
Simplify.	$= 6a^8$

d.

	$d^4d^5d^2$
Add the exponents, since the bases are the same.	$= d^{4+5+2}$
Simplify.	$= d^{11}$

Try It 3.2.3

Simplify each expression:

a. $b^9 b^8$

b. $4^{2x} \cdot 4^x$

c. $3p^5 \cdot 4p$

d. $x^6 x^4 x^8$

Answer

a. b^{17}

b. 4^{3x}

c. $12p^6$

d. x^{18}

Try It 3.2.4

Simplify each expression:

a. $x^{12} x^4$

b. $10 \cdot 10^x$

c. $2z \cdot 6z^7$

d. $b^5 b^9 b^5$

Answer

a. x^{16}

b. 10^{x+1}

c. $12z^8$

d. b^{19}

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$=\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$=\frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$=\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$=\frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	$=x^3$		$=\frac{1}{x}$
Note.	$x^3 = x^{5-2}$		$\frac{1}{x} = \frac{1}{x^1} = \frac{1}{x^{3-2}}$

Here we see how to use the initial exponents to arrive at the simplified expression. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator--notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, and so we need a 1 in the numerator: $\frac{\cancel{x}}{\cancel{x}} = 1$. This leads to the *Quotient Property for Positive Integer Exponents*.

Quotient Property for Positive Integer Exponents

If a is a real number, $a \neq 0$, and m and n are distinct positive integers, then

$$\frac{a^m}{a^n} = \begin{cases} a^{m-n} & \text{if } m > n \\ \dfrac{1}{a^{n-m}} & \text{if } n > m. \end{cases}$$

Example 3.2.5

Simplify each expression:

a. $\frac{x^9}{x^7}$

b. $\frac{3^{10}}{3^2}$

c. $\frac{b^8}{b^{12}}$

d. $\frac{7^3}{7^5}$

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

	$\frac{x^9}{x^7}$
Since $9 > 7$, there are more factors of x in the numerator. Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, for $m > n$.	$= x^{9-7}$
Simplify.	$= x^2$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

b.

	$\frac{3^{10}}{3^2}$
Since $10 > 2$, there are more factors of 3 in the numerator. Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, for $m > n$.	$= 3^{10-2}$
Simplify.	$= 3^8$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

c.

	$\frac{b^8}{b^{12}}$

	$\frac{b^8}{b^{12}}$
Since $12>8$, there are more factors of b in the denominator. Use the Quotient Property, $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$, for $n>m$.	$=\frac{1}{b^{12-8}}$
Simplify.	$=\frac{1}{b^4}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

d.

	$\frac{7^3}{7^5}$
Since $5>3$, there are more factors of 3 in the denominator. Use the Quotient Property, $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$ for $n>m$.	$=\frac{1}{7^{5-3}}$
Simplify.	$=\frac{1}{7^2}$
Simplify.	$=\frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Try It 3.2.6

Simplify each expression:

a. $\frac{x^{15}}{x^{10}}$

b. $\frac{6^{14}}{6^5}$

c. $\frac{x^{18}}{x^{22}}$

d. $\frac{12^{15}}{12^{30}}$

Answer

a. x^5

b. 6^9

c. $\frac{1}{x^4}$

d. $\frac{1}{12^{15}}$

Try It 3.2.7

Simplify each expression:

a. $\frac{y^{43}}{y^{37}}$

b. $\dfrac{10^{15}}{10^{7}}$

c. $\dfrac{m^{7}}{m^{15}}$

d. $\dfrac{9^{8}}{9^{19}}$

Answer

a. y^{6}

b. 10^{8}

c. $\dfrac{1}{m^{8}}$

d. $\dfrac{1}{9^{11}}$

Note that, while so far an exponent that is not a positive integer has no meaning, we see that blindly applying the above properties for such exponents leads to a couple definitions.

	1
For m a positive integer (so that x^m has meaning, $x \neq 0$).	$=\dfrac{x^m}{x^m}$
Blindly follow the quotient rule that we know to be true in another case.	$=x^{m-m}$
Reduce the fraction on the left.	$=x^0$
Conclude.	$x^0=1$

So **if** the Quotient Property is also to hold for the exponent zero we **must** define, for $x \neq 0$,

$$x^0=1.$$

Similarly,

Consider this expression where m is a positive integer (so that x^m has meaning).	$x^{-m}x^m$
Blindly applying the product property.	$=x^{-m+m}$
Simplify exponent.	$=x^0$
Using our new definition: $x^0=1$.	$=1$
Draw conclusion.	$x^{-m}x^m=1$
Divide both sides by x^m.	$\dfrac{x^{-m}x^m}{x^m}=\dfrac{1}{x^m}$
Simplify.	$x^{-m}=\dfrac{1}{x^m}$

So, **if** the product property of exponents holds also for (so far undefined) negative integers, we **must** define, for m a negative integer,

$$x^{-m}=\frac{1}{x^m}.$$

Also, since taking the reciprocal of both sides preserves the equality we also have, equivalently,

$$\frac{1}{x^{-m}} = x^m.$$

We could also write these two statements above simultaneously as

$$x^{-m} = \frac{1}{x^m}, m \text{ any integer.}$$

So, we **must** define

Definition 3.2.8

For a any non-zero real number

$$a^0 = 1$$

and for m any positive integer

$$a^{-m} = \frac{1}{a^m} \text{ or, equivalently, } \frac{1}{a^{-m}} = a^m.$$

In the above, the base can be anything (x can be anything) which we know to be different from zero, and in this text, we assume any variable that we raise to the zero power is not zero.

Example 3.2.9

Simplify each expression:

a. 9^0

b. n^0

c. $(-4a^2b)^0$

d. -3^0

Solution

The definition says any non-zero number raised to the zero power is 1.

a. Use the definition of the zero exponent. $9^0 = 1$

b. Use the definition of the zero exponent. $n^0 = 1$

To simplify the expression n raised to the zero power we just use the definition of the zero exponent. The result is 1.

c. Anything raised to the power zero is 1. Here the base is $-4a^2b$, so $(-4a^2b)^0 = 1$

d. Anything raised to the power zero is 1. Here the base is 3, so this is the opposite of 3^0, or, the opposite of 1. So, $-3^0 = -1$

Try It 3.2.10

Simplify each expression:

a. 11^0

b. q^0

c. $(-12p^3q^2)^0$

d. -7^0

Answer

a. 1

b. 1

c. 1

d. -1

? Try It 3.2.11

Simplify each expression:

a. 23^0

b. r^0

c. $(2st^5)^0$

d. $-s^0$

Answer

a. 1

b. 1

c. 1

d. -1

? Example 3.2.12

Simplify each expression. Write your answer using positive exponents.

a. x^{-5}

b. 10^{-3}

c. $\frac{1}{y^{-4}}$

d. $\frac{1}{3^{-2}}$

Solution

a.

	x^{-5}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{x^5}$

b.

	10^{-3}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{10^3}$
Simplify.	$= \frac{1}{1000}$

c.

	$\frac{1}{y^{-4}}$

	$\dfrac{1}{y^{-4}}$
Use the definition of a negative exponent, $\dfrac{1}{a^{-n}} = a^n$.	$= y^4$

d.

	$\dfrac{1}{3^{-2}}$
Use the definition of a negative exponent, $\dfrac{1}{a^{-n}} = a^n$.	$= 3^2$
Simplify.	$= 9$

? Try It 3.2.13

Simplify each expression. Write your answer using positive exponents.

a. z^{-3}

b. 10^{-7}

c. $\dfrac{1}{p^{-8}}$

d. $\dfrac{1}{4^{-3}}$

Answer

a. $\dfrac{1}{z^3}$

b. $\dfrac{1}{10,000,000}$

c. p^8

d. 64

? Try It 3.2.14

Simplify each expression. Write your answer using positive exponents.

a. n^{-2}

b. 10^{-4}

c. $\dfrac{1}{q^{-7}}$

d. $\dfrac{1}{2^{-4}}$

Answer

a. $\dfrac{1}{n^2}$

b. $\dfrac{1}{10,000}$

c. q^7

d. 16

Properties of Negative Exponents

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in *simplest form*. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$=\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$=\frac{16}{9}$
But we know that	$\frac{16}{9}=\left(\frac{4}{3}\right)^2$
This tells us that	$\left(\frac{3}{4}\right)^{-2}=\left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Integer Exponent Property*.

Quotient to a Negative Integer Exponent Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^n.$$

Example $3.2.15$

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{5}{7}\right)^{-2}$

b. $\left(-\frac{x}{y}\right)^{-3}$

Solution

a.

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(\frac{7}{5}\right)^{2}$
Simplify.	$=\frac{49}{25}$

b.

	$\left(-\frac{x}{y}\right)^{-3}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(-\frac{y}{x}\right)^{3}$
Simplify.	$=-\frac{y^3}{x^3}$

? Try It 3.2.16

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{2}{3}\right)^{-4}$

b. $\left(-\frac{m}{n}\right)^{-2}$

Answer

a. $\frac{81}{16}$

b. $\frac{n^2}{m^2}$

? Try It 3.2.17

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{3}{5}\right)^{-3}$

b. $\left(-\frac{a}{b}\right)^{-4}$

Answer

a. $\frac{125}{27}$

b. $\frac{b^4}{a^4}$

We would like to verify that the properties of positive integer exponents can be extended to all integer exponents. We will do some examples. Let m and n be non-negative integers, $m > n$. Let's simplify $x^m x^{-n}$.

	$x^m x^{-n}$
Use the definition: $x^{-n} = \frac{1}{x^n}$.	$= x^m \cdot \frac{1}{x^n}$
Multiply fractions.	$= \frac{x^m}{x^n}$
Use the Quotient Property of exponents.	x^{m-n} or $x^{m+(-n)}$
Conclude.	$x^m x^{-n} = x^{m-n} = x^{m+(-n)}$

So, the product property holds in this case.

And if $m < n$,

	$x^m x^{-n}$
Use the definition: $x^{-n} = \frac{1}{x^n}$.	$= x^m \cdot \frac{1}{x^n}$
Multiply fractions.	$= \frac{x^m}{x^n}$
Use the Quotient Property of exponents.	$= \frac{1}{x^{n-m}}$
Use second variation of the definition of negative exponents.	$= x^{-(n-m)}$
Simplify.	x^{m-n} or $x^{m+(-n)}$
Conclude.	$x^m x^{-n} = x^{m-n} = x^{m+(-n)}$

So the Product Property for Positive Integer Exponents holds in this case, too. We can in a similar way check other combinations to see that the Product Property for Positive Integer Exponents holds for all integers. The Product Property for Integer Exponents follows directly

$$\frac{x^m}{x^n} = x^m \cdot \frac{1}{x^n} = x^m \cdot x^{-n} = x^{m-n}$$

and

$$\frac{x^m}{x^n} = \frac{1}{x^{-m}} \cdot \frac{1}{x^n} = \frac{1}{x^{-m} \cdot x^n} = \frac{1}{x^{-m+n}} \quad \text{or} \quad \frac{1}{x^{n-m}}.$$

So, both the Product Property and the Quotient Property hold for all integer exponents.

Product Property for Integer Exponents

If a is a real number and m and n are integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

> **Quotient Property for Integer Exponents**
>
> If a is a real number, $a \neq 0$, and m and n are integers, then
>
> $$\frac{a^m}{a^n} = a^{m-n}$$
>
> and
>
> $$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$
>
> To divide with like bases, subtract the exponents as above.

We will now use the product property with expressions that have negative exponents.

> **Example 3.2.18**
>
> Simplify each expression:
>
> **a.** $z^{-5}z^{-3}$
>
> **b.** $(m^4n^{-3})(m^{-5}n^{-2})$
>
> **c.** $(2x^{-6}y^8)(-5x^5y^{-3})$
>
> **Solution**
>
> **a.**
>
> | | $z^{-5}z^{-3}$ |
> |---|---|
> | Add the exponents, since the bases are the same. | $=z^{-5-3}$ |
> | Simplify. | $=z^{-8}$ |
> | Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$. | $=\frac{1}{z^8}$ |
>
> **b.**
>
> | | $(m^4n^{-3})(m^{-5}n^{-2})$ |
> |---|---|
> | Use the Commutative Property to get like bases together. | $=m^4m^{-5}n^{-2}n^{-3}$ |
> | Add the exponents for each base. | $=m^{-1}n^{-5}$ |
> | Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$. | $=\frac{1}{m^1}\cdot\frac{1}{n^5}$ |
> | Simplify. | $=\frac{1}{mn^5}$ |
>
> **c.**
>
> | | $(2x^{-6}y^8)(-5x^5y^{-3})$ |
> |---|---|
> | Rewrite with the like bases together. | $=2(-5)(x^{-6}x^5)(y^8y^{-3})$ |
> | Multiply the coefficients and add the exponents of each variable. | $=-10x^{-1}y^5$ |
> | Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$. | $=-10\frac{1}{x}y^5$ |
> | Simplify. | $=-\frac{10y^5}{x}$ |

Try It 3.2.19

Simplify each expression:

a. $z^{-4}z^{-5}$

b. $(p^6q^{-2})(p^{-9}q^{-1})$

c. $(3u^{-5}v^7)(-4u^4v^{-2})$

Answer

a. $\frac{1}{z^9}$

b. $\frac{1}{p^3q^3}$

c. $-\frac{12v^5}{u}$

Try It 3.2.20

Simplify each expression:

a. $c^{-8}c^{-7}$

b. $(r^5s^{-3})(r^{-7}s^{-5})$

c. $(-6c^{-6}d^4)(-5c^{-2}d^{-1})$

Answer

a. $\frac{1}{c^{15}}$

b. $\frac{1}{r^2s^8}$

c. $\frac{30d^3}{c^8}$

Now let’s look at an exponential expression that contains a power raised to a power. Let's see if we can discover a general property.

	$(x^2)^3$
What does this mean?	$=x^2x^2x^2$
How many factors altogether?	$=\underbrace{x\cdot x}_{2\ factors}\cdot\underbrace{x\cdot x}_{2\text{ factors}}\cdot\underbrace{x\cdot x}_{2\text{ factors}}$ $=\underbrace{x\cdot x\cdot x\cdot x\cdot x\cdot x}_{6\text{ factors}}$
So we have	$=x^6$

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2\cdot3}$ or x^6. We can also see that

$$(x^{-2})^3=\left(\frac{1}{x^2}\right)^3=\frac{1}{x^2}\cdot\frac{1}{x^2}\cdot\frac{1}{x^2}=\frac{1}{x^6}=x^{-6}$$

so that $(x^{-2})^3=x^{-6}$. In these examples we multiplied the exponents.

We can check various combinations of signs of the exponents which leads us to the *Power Property for Integer Exponents.*

Power Property for Integer Exponents

If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}.$$

To raise a power to a power, multiply the exponents.

Example 3.2.21

Simplify each expression:

a. $(y^5)^9$

b. $(4^{-4})^7$

c. $(y^3)^6(y^5)^4$

Solution

a.

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{mn}$.	$y^{5\cdot 9}$
Simplify.	y^{45}

b.

	$(4^{-4})^7$
Use the power property.	$=4^{-4\cdot 7}$
Simplify.	$=4^{-28}$

c.

	$(y^3)^6(y^5)^4$
Use the power property.	$=y^{18}y^{20}$
Add the exponents.	$=y^{38}$

Try It 3.2.22

Simplify each expression:

a. $(b^7)^5$

b. $(5^4)^{-3}$

c. $(a^4)^5(a^7)^4$

Answer

a. b^{35}

b. 5^{-12}

c. a^{48}

Try It 3.2.23

Simplify each expression:

a. $(z^6)^9$

b. $(3^{-7})^7$

c. $(q^4)^5(q^3)^3$

Answer

a. z^{54}

b. 3^{-49}

c. q^{29}

We will now look at an expression containing a product that is raised to a power. Can we find this pattern?

	$(2x)^3$
What does this mean?	$=2x\cdot 2x\cdot 2x$
We group the like factors together.	$=2\cdot 2\cdot 2xxx$
How many factors of 2 and of x?	$=2^3x^3$

Notice that each factor was raised to the power and $(2x)^3$ is 2^3x^3.

The exponent applies to each of the factors! We can say that the exponent distributes over multiplication. If we were to check various examples with exponents which are negative or zero, then we would find the same pattern emerges. This leads to the *Product to a Power Property for Integer Exponents.*

Product to a Power Property for Integer Exponents

If a and b are real numbers and m is an integer, then

$$(ab)^m=a^mb^m.$$

To raise a product to a power, raise each factor to that power.

Example 3.2.24

Simplify each expression:

a. $(-3mn)^3$

b. $(6k^3)^{-2}$

c. $(5x^{-3})^2$

Solution

a.

	$(-3mn)^3$
Use Power of a Product Property, $(ab)^m=a^mb^m$.	$=(-3)^3m^3n^3$
Simplify.	$=-27m^3n^3$

b.

	$(6k^3)^{-2}$
Use the Power of a Product Property, $(ab)^m = a^m b^m$.	$= 6^{-2}(k^3)^{-2}$
Use the Power Property, $(a^m)^n = a^{mn}$.	$= 6^{-2}k^{-6}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{6^2} \cdot \frac{1}{k^6}$
Simplify.	$= \frac{1}{36k^6}$

c.

	$(5x^{-3})^2$
Use the power of a product property, $(ab)^m = a^m b^m$.	$= 5^2(x^{-3})^2$
Simplify.	$= 25x^{-6}$
Rewrite x^{-6} using $a^{-n} = \frac{1}{a^n}$.	$= 25\frac{1}{x^6}$
Simplify.	$= \frac{25}{x^6}$

? Try It 3.2.25

Simplify each expression:

a. $(2wx)^5$

b. $(2b^3)^{-4}$

c. $(8a^{-4})^2$

Answer

a. $32w^5x^5$

b. $\frac{1}{16b^{12}}$

c. $\frac{64}{a^8}$

? Try It 3.2.26

Simplify each expression:

a. $(-3y)^3$

b. $(-4x^4)^{-2}$

c. $(2c^{-4})^3$

Answer

a. $-27y^3$

b. $\frac{1}{16x^8}$

c. $8c^{12}$

Now we will look at an example that will lead us to the quotient to a power property.

	$\left(\frac{x}{y}\right)^3$
This means	$=\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$=\frac{xxx}{yyy}$
Write with exponents.	$=\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

This leads to the *Quotient to a Power Property for Integer Exponents*. We can say that the exponent distributes over division as well as multiplication.

Quotient to a Power Property for Integer Exponents

If a and b are real numbers, $b\neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}.$$

To raise a fraction to a power, raise the numerator and denominator to that power.

Example 3.2.27

Simplify each expression:

a. $\left(\frac{b}{3}\right)^4$

b. $\left(\frac{k}{j}\right)^{-3}$

c. $\left(\frac{2xy^2}{z}\right)^3$

d. $\left(\frac{4p^{-3}}{q^2}\right)^2$

Solution

a.

	$\left(\frac{b}{3}\right)^4$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$=\frac{b^4}{3^4}$

	$\left(\frac{b}{3}\right)^4$
Simplify.	$=\frac{b^4}{81}$

b.

	$\left(\frac{k}{j}\right)^{-3}$
Raise the numerator and denominator to the power.	$=\frac{k^{-3}}{j^{-3}}$
Use the definition of negative exponent, $a^{-n}=\frac{1}{a^n}$ and $\frac{1}{a^{-n}}=a^n$.	$=\frac{1}{k^3}\cdot\frac{j^3}{1}$
Multiply.	$=\frac{j^3}{k^3}$

c.

	$\left(\frac{2xy^2}{z}\right)^3$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$=\frac{(2xy^2)^3}{z^3}$
Use the Product to a Power Property, $(ab)^m=a^mb^m$, and the Power Property, $(a^m)^n=a^{mn}$.	$=\frac{8x^3y^6}{z^3}$

d.

	$\left(\frac{4p^{-3}}{q^2}\right)^2$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$=\frac{(4p^{-3})^2}{(q^2)^2}$
Use the Product to a Power Property, $(ab)^m=a^mb^m$.	$=\frac{16p^{-6}}{q^4}$
Use the definition of negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=\frac{16}{q^4}\cdot\frac{1}{p^6}$
Simplify.	$=\frac{16}{p^6q^4}$

? Try It 3.2.28

Simplify each expression:

a. $\left(\frac{p}{10}\right)^4$

b. $\left(\frac{m}{n}\right)^{-7}$

c. $\left(\frac{3ab^3}{c^2}\right)^4$

d. $\left(\frac{3x^{-2}}{y^3}\right)^3$

Answer

a. $\frac{p^4}{10000}$

b. $\frac{n^7}{m^7}$

c. $\frac{81a^4b^{12}}{c^8}$

d. $\frac{27}{x^6y^9}$

? Try It 3.2.29

Simplify each expression:

a. $\left(\frac{-2}{q}\right)^3$

b. $\left(\frac{w}{x}\right)^{-4}$

c. $\left(\frac{xy^3}{3z^2}\right)^2$

d. $\left(\frac{2m^{-2}}{n^{-2}}\right)^3$

Answer

a. $\frac{-8}{q^3}$

b. $\frac{x^4}{w^4}$

c. $\frac{x^2y^6}{9z^4}$

d. $\frac{8n^6}{m^6}$

We now have several properties for exponents. Let's summarize them and then we'll do some more examples that use more than one of the properties.

Summary of Integer Exponent Definitions and Properties

If a and b are real numbers, and m and n are integers, then

Definition	**Description**
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

Property	**Description**

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Example 3.2.30

Simplify each expression by applying several properties:

a. $(3x^2y)^4(2xy^2)^3$

b. $\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$

c. $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$

Solution

a.

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$=(3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$=(81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$=81 \cdot 8x^8x^3y^4y^6$
Multiply the constants and add the exponents.	$=648x^{11}y^{10}$

b.

	$\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$=\frac{x^{12}x^{-10}}{x^{30}}$
Add the exponents in the numerator.	$=\frac{x^2}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$	$=\frac{1}{x^{28}}$

c.

	$\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$
Simplify inside the parentheses first.	$=\left(\frac{2y^4}{x^2}\right)^2\left(\frac{12y^4}{x^2}\right)^{-1}$

	$\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$=\frac{(2y^4)^2}{(x^2)^2}\frac{(12y^4)^{-1}}{(x^2)^{-1}}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$=\frac{4y^8}{x^4}\cdot\frac{12^{-1}y^{-4}}{x^{-2}}$
Simplify.	$=\frac{4y^4}{12x^2}$
Simplify.	$=\frac{y^4}{3x^2}$

Try It 3.2.31

Simplify each expression:

a. $(c^4d^2)^5(3cd^5)^4$

b. $\frac{(a^{-2})^3(a^2)^4}{(a^4)^5}$

c. $\left(\frac{3xy^2}{x^2y^{-3}}\right)^2$

Answer

a. $81c^{24}d^{30}$

b. $\frac{1}{a^{18}}$

c. $\frac{9y^{10}}{x^2}$

Try It 3.2.32

Simplify each expression:

a. $(a^3b^2)^6(4ab^3)^4$

b. $\frac{(p^{-3})^4(p^5)^3}{(p^7)^6}$

c. $\left(\frac{4x^3y^2}{x^2y^{-1}}\right)^2\left(\frac{8xy^{-3}}{x^2y}\right)^{-1}$

Answer

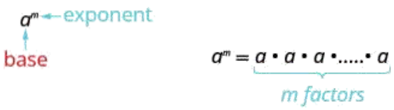
a. $256a^{22}b^{24}$

b. $\frac{1}{p^{39}}$

c. $2x^3y^{10}$

Key Concepts

- **Exponential Notation**



This is read a to the m^{th} power.
In the expression a^m, the *exponent* m (when positive) tells us how many times we use the *base* a as a factor.
- **Zero Exponent (Definition)** If a is a non-zero number, then $a^0 = 1$.
- **Negative Exponent (Definition)** If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$ or, equivalently, $\frac{1}{a^{-n}} = a^n$.
- **Product Property for Exponents**
If a is a real number and m and n are integers, then

$$a^m a^n = a^{m+n}$$

To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
If a is a real number, $a \neq 0$, and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

- **Quotient to a Negative Exponent Property**
If a and b are real numbers, $a \neq 0$, $b \neq 0$ and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

- **Power Property for Exponents**
If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}$$

To raise a power to a power, multiply the exponents.
- **Product to a Power Property for Exponents**
If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

To raise a product to a power, raise each factor to that power.
- **Quotient to a Power Property for Exponents**
If a and b are real numbers, $b \neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
If a and b are real numbers, and m and n are integers, then
-
Property	**Description**
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

-
Property	**Description**
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$

Property	Description
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Glossary

Product Property

According to the Product Property, a to the m times a to the n equals a to the m plus n.

Power Property

According to the Power Property, a to the m to the n equals a to the m times n.

Product to a Power

According to the Product to a Power Property, a times b in parentheses to the m equals a to the m times b to the m.

Quotient Property

According to the Quotient Property, a to the m divided by a to the n equals a to the m minus n as long as a is not zero.

Quotient to a Power Property

According to the Quotient to a Power Property, a divided by b in parentheses to the power of m is equal to a to the m divided by b to the m as long as b is not zero.

Definition of Negative Exponents

According to the definition of Negative Exponents, a to the negative n equals 1 divided by a to the n and 1 divided by a to the negative n equals a to the n.

Quotient to a Negative Exponent

Raising a quotient to a negative exponent occurs when a divided by b in parentheses to the power of negative n equals b divided by a in parentheses to the power of n.

Definition of the Zero Exponent

Raising a non-zero base to the power zero is one.

Practice Makes Perfect

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

1. ⓐ $d^3 \cdot d^6$ ⓑ $4^{5x} \cdot 4^{9x}$ ⓒ $2y \cdot 4y^3$ ⓓ $w \cdot w^2 \cdot w^3$

Answer

ⓐ d^9 ⓑ 4^{14x} ⓒ $8y^4$ ⓓ w^6

2. ⓐ $x^4 \cdot x^2$ ⓑ $8^{9x} \cdot 8^3$ ⓒ $3z^{25} \cdot 5z^8$ ⓓ $y \cdot y^3 \cdot y^5$

3. ⓐ $n^{19} \cdot n^{12}$ ⓑ $3^x \cdot 3^6$ ⓒ $7w^5 \cdot 8w$ ⓓ $a^4 \cdot a^3 \cdot a^9$

Answer

ⓐ n^{31} ⓑ 3^{x+6} ⓒ $56w^{6}$
ⓓ a^{16}

4. ⓐ $q^{27} \cdot q^{15}$ ⓑ $5^{x} \cdot 5^{4x}$ ⓒ $9u^{41} \cdot 7u^{53}$
ⓓ $c^{5} \cdot c^{11} \cdot c^{2}$

5. $m^{x} \cdot m^{3}$

Answer

m^{x+3}

6. $n^{y} \cdot n^{2}$

7. $y^{a} \cdot y^{b}$

Answer

y^{a+b}

8. $x^{p} \cdot x^{q}$

9. ⓐ $\frac{x^{18}}{x^{3}}$ ⓑ $\frac{5^{12}}{5^{3}}$ ⓒ $\frac{q^{18}}{q^{36}}$ ⓓ $\frac{10^{2}}{10^{3}}$

Answer

ⓐ x^{15} ⓑ 5^{9} ⓒ $\frac{1}{q^{18}}$ ⓓ $\frac{1}{10}$

10. ⓐ $\frac{y^{20}}{y^{10}}$ ⓑ $\frac{7^{16}}{7^{2}}$ ⓒ $\frac{t^{10}}{t^{40}}$ ⓓ $\frac{8^{3}}{8^{5}}$

11. ⓐ $\frac{p^{21}}{p^{7}}$ ⓑ $\frac{4^{16}}{4^{4}}$ ⓒ $\frac{b}{b^{9}}$ ⓓ $\frac{4}{4^{6}}$

Answer

ⓐ p^{14} ⓑ 4^{12} ⓒ $\frac{1}{b^{8}}$ ⓓ $\frac{1}{4^{5}}$

12. ⓐ $\frac{u^{24}}{u^{3}}$ ⓑ $\frac{9^{15}}{9^{5}}$ ⓒ $\frac{x}{x^{7}}$ ⓓ $\frac{10}{10^{3}}$

13. ⓐ 20^{0} ⓑ b^{0}

Answer

ⓐ 1 ⓑ 1

14. ⓐ 13^{0} ⓑ k^{0}

15. ⓐ -27^{0} ⓑ $-(27^{0})$

Answer

ⓐ -1 ⓑ -1

16. ⓐ -15^0 ⓑ $-(15^0)$

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression. Write your answer using positive exponents.

17. ⓐ a^{-2} ⓑ 10^{-3} ⓒ $\frac{1}{c^{-5}}$ ⓓ $\frac{1}{3^{-2}}$

Answer

ⓐ $\frac{1}{a^2}$ ⓑ $\frac{1}{1000}$ ⓒ c^5 ⓓ 9

18. ⓐ b^{-4} ⓑ 10^{-2} ⓒ $\frac{1}{c^{-5}}$ ⓓ $\frac{1}{5^{-2}}$

19. ⓐ r^{-3} ⓑ 10^{-5} ⓒ $\frac{1}{q^{-10}}$ ⓓ $\frac{1}{10^{-3}}$

Answer

ⓐ $\frac{1}{r3}$ ⓑ $\frac{1}{100,000}$ ⓒ q^{10} ⓓ $1,000$

20. ⓐ s^{-8} ⓑ 10^{-2} ⓒ $\frac{1}{t^{-9}}$ ⓓ $\frac{1}{10^{-4}}$

21. ⓐ $\left(\frac{5}{8}\right)^{-2}$ ⓑ $\left(-\frac{b}{a}\right)^{-2}$

Answer

ⓐ $\frac{64}{25}$ ⓑ $\frac{a^2}{b^2}$

22. ⓐ $\left(\frac{3}{10}\right)^{-2}$ ⓑ $\left(-\frac{2}{z}\right)^{-3}$

23. ⓐ $\left(\frac{4}{9}\right)^{-3}$ ⓑ $\left(-\frac{u}{v}\right)^{-5}$

Answer

ⓐ $\frac{729}{64}$ ⓑ $-\frac{v^5}{u^5}$

24. ⓐ $\left(\frac{7}{2}\right)^{-3}$ ⓑ $\left(-\frac{3}{x}\right)^{-3}$

25. ⓐ $(-5)^{-2}$ ⓑ -5^{-2} ⓒ $\left(-\frac{1}{5}\right)^{-2}$ ⓓ $-\left(\frac{1}{5}\right)^{-2}$

Answer

ⓐ $\frac{1}{25}$ ⓑ $\frac{1}{25}$ ⓒ 25 ⓓ -25

26. ⓐ -5^{-3} ⓑ $\left(-\frac{1}{5}\right)^{-3}$ ⓒ $-\left(\frac{1}{5}\right)^{-3}$ ⓓ $(-5)^{-3}$

27. ⓐ $3\cdot 5^{-1}$ ⓑ $(3\cdot 5)^{-1}$

Answer

ⓐ $\frac{3}{5}$ ⓑ $\frac{1}{15}$

28. ⓐ $3\cdot 4^{-2}$ ⓑ $(3\cdot 4)^{-2}$

In the following exercises, simplify each expression using the Product Property. Write your answer using positive exponents.

29. ⓐ b^4b^{-8} ⓑ $(w^4x^{-5})(w^{-2}x^{-4}))$ ⓒ $(-6c^{-3}d^9)(2c^4d^{-5})$

Answer

ⓐ $\frac{1}{b^4}$ ⓑ $\frac{w^2}{x^9}$ ⓒ $-12cd^4$

30. ⓐ $s^3\cdot s^{-7}$ ⓑ $(m^3n^{-3})(m^5n^{-1})$
ⓒ $(-2j^{-5}k^8)(7j^2k^{-3})$

31. ⓐ $a^3\cdot a^{-3}$ ⓑ $(uv^{-2})(u^{-5}v^{-3})$
ⓒ $(-4r^{-2}s^{-8})(9r^4s^3)$

Answer

ⓐ 1 ⓑ $\frac{1}{u^4v^5}$ ⓒ $-36\frac{r^2}{j^5}$

32. ⓐ $y^5\cdot y^{-5}$ ⓑ $(pq^{-4})(p^{-6}q^{-3})$
ⓒ $(-5m^4n^6)(8m^{-5}n^{-3})$

33. $p^5\cdot p^{-2}\cdot p^{-4}$

Answer

$\frac{1}{p}$

34. $x^4\cdot x^{-2}\cdot x^{-3}$

In the following exercises, simplify each expression using the Power Property. Write your answer using positive exponents.

35. ⓐ $(m^4)^2$ ⓑ $(10^3)^6$ ⓒ $(x^3)^{-4}$

Answer

ⓐ m^8 ⓑ 10^{18} ⓒ $\frac{1}{x^{12}}$

36. ⓐ $(b^2)^7$ ⓑ $(3^8)^2$ ⓒ $(k^2)^{-5}$

37. ⓐ $(y^3)^x$ ⓑ $(5^x)^x$ ⓒ $(q^6)^{-8}$

Answer

ⓐ y^{3x} ⓑ 5^{xy} ⓒ $\frac{1}{q^{48}}$

38. ⓐ $(x^2)^y$ ⓑ $(7^a)^b$ ⓒ $(a^9)^{-10}$

In the following exercises, simplify each expression using the Product to a Power Property. Write your answer using positive exponents.

39. ⓐ $(-3xy)^2$ ⓑ $(6a)^0$ ⓒ $(5x^2)^{-2}$ ⓓ $(-4y^{-3})^2$

Answer

ⓐ $9x^2y^2$ ⓑ 1 ⓒ $\frac{1}{25x^4}$ ⓓ $\frac{16}{y^6}$

40. ⓐ $(-4ab)^2$ ⓑ $(5x)^0$ ⓒ $(4y^3)^{-3}$ ⓓ $(-7y^{-3})^2$

41. ⓐ $(-5ab)^3$ ⓑ $(-4pq)^0$ ⓒ $(-6x^3)^{-2}$ ⓓ $(3y^{-4})^2$

Answer

ⓐ $-125a^3b^3$ ⓑ 1 ⓒ $\frac{1}{36x^6}$ ⓓ $\frac{9}{y^8}$

42. ⓐ $(-3xyz)^4$ ⓑ $(-7mn)^0$ ⓒ $(-3x^3)^{-2}$
ⓓ $(2y^{-5})^2$

In the following exercises, simplify each expression using the Quotient to a Power Property. Write your answer using positive exponents.

43. ⓐ $(p^2)^5$ ⓑ $\left(\frac{x}{y}\right)^{-6}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{4p^{-3}}{q^2}\right)^2$

Answer

ⓐ $\frac{p^5}{32}$ ⓑ $\frac{y^6}{x^6}$ ⓒ $\frac{8x^3y^6}{z^3}$
ⓓ $\frac{16}{p^6q^4}$

44. ⓐ $\left(\frac{x}{3}\right)^4$ ⓑ $\left(\frac{a}{b}\right)^{-5}$ ⓒ $\left(\frac{2xy^2}{z}\right)^3$ ⓓ $\left(\frac{x^3y}{z^4}\right)^2$

45. ⓐ $\left(\frac{a}{3b}\right)^4$ ⓑ $\left(\frac{5}{4m}\right)^{-2}$ ⓒ $\left(\frac{3a^{-2}b^3}{c^2}\right)^{-2}$ ⓓ $\left(\frac{p^{-1}q^4}{r^{-4}}\right)^2$

Answer

ⓐ $\frac{a^4}{81b^4}$ ⓑ $\frac{16m^2}{25}$ ⓒ $\frac{a^4c^4}{9b^6}$ ⓓ $\frac{q^8r^8}{p^2}$

46. ⓐ $\left(\frac{x^2}{y}\right)^3$ ⓑ $\left(\frac{10}{3q}\right)^{-4}$ ⓒ $\left(\frac{2x^3y^4}{3z^2}\right)^5$ ⓓ $\left(\frac{5a^3b^{-1}}{2c^4}\right)^{-3}$

In the following exercises, simplify each expression by applying several properties. Write your answer using positive exponents.

47. ⓐ $(5t^2)^3(3t)^2$ ⓑ $\frac{(t^2)^5(t^{-4})^2}{(t^3)^7}$ ⓒ $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$

Answer

ⓐ $1125t^8$ ⓑ $\frac{1}{t^{19}}$ ⓒ $\frac{y^4}{3x^2}$

48. ⓐ $(10k^4)^3(5k^6)^2$ ⓑ $\frac{(q^3)^6(q^{-2})^3}{(q^4)^8}$

49. ⓐ $(m^2n)^2(2mn^5)^4$ ⓑ $\frac{(-2p^{-2})^4(3p^4)^2}{(-6p^3)^2}$

Answer

ⓐ $16m^8n^{22}$ ⓑ $\frac{4}{p^6}$

50. ⓐ $(3pq^4)^2(6p^6q)^2$ ⓑ $\frac{(-2k^{-3})^2(6k^2)^4}{(9k^4)^2}$

Mixed Practice

In the following exercises, simplify each expression. Write your answer using positive exponents.

51. ⓐ $7n^{-1}$ ⓑ $(7n)^{-1}$ ⓒ $(-7n)^{-1}$

Answer

ⓐ $\frac{7}{n}$ ⓑ $\frac{1}{7n}$ ⓒ $-\frac{1}{7n}$

52. ⓐ $6r^{-1}$ ⓑ $(6r)^{-1}$ ⓒ $(-6r)^{-1}$

53. ⓐ $(3p)^{-2}$ ⓑ $3p^{-2}$ ⓒ $-3p^{-2}$

Answer

ⓐ $\frac{1}{9p^2}$ ⓑ $\frac{3}{p^2}$ ⓒ $-\frac{3}{p^2}$

54. ⓐ $(2q)^{-4}$ ⓑ $2q^{-4}$ ⓒ $-2q^{-4}$

55. $(x^2)^4\cdot(x^3)^2$

Answer

x^{14}

56. $(y^4)^3 \cdot (y^5)^2$

57. $(a^2)^6 \cdot (a^3)^8$

Answer

a^{30}

58. $(b^7)^5 \cdot (b^2)^6$

59. $(2m^6)^3$

Answer

$2m^{18}$

60. $(3y^2)^4$

61. $(10x^2y)^3$

Answer

$1,000x^6y^3$

62. $(2mn^4)^5$

63. $(-2a^3b^2)^4$

Answer

$16a^{12}b^8$

64. $(-10u^2v^4)^3$

65. $\left(\frac{2}{3}x^2y\right)^3$

Answer

$\frac{8}{27}x^6y^3$

66. $\left(\frac{7}{9}pq^4\right)^2$

67. $(8a^3)^2(2a)^4$

Answer

$1,024a^{10}$

68. $(5r^2)^3(3r)^2$

69. $(10p^4)^3(5p^6)^2$

Answer

$25,000p^{24}$

70. $(4x^3)^3(2x^5)^4$

71. $\left(\frac{1}{2}x^2y^3\right)^4(4x^5y^3)^2$

Answer

$x^{18}y^{18}$

72. $\left(\frac{1}{3}m^3n^2\right)^4(9m^8n^3)^2$

73. $(3m^2n)^2(2mn^5)^4$

Answer

$144m^8n^{22}$

74. $(2pq^4)^3(5p^6q)^2$

75. ⓐ $(3x)^2(5x)$ ⓑ $(2y)3(6y)$

Answer

ⓐ $45x^3$ ⓑ $48y^4$

76. ⓐ $\left(\frac{1}{2}y^2\right)^3\left(\frac{2}{3}y\right)^2$ ⓑ $\left(\frac{1}{2}j^2\right)^5\left(\frac{2}{5}j^3\right)^2$

77. ⓐ $(2r^{-2})^3(4^{-1}r)^2$ ⓑ $(3x^{-3})^3(3^{-1}x^5)^4$

Answer

ⓐ $12r^4$ ⓑ $13x^{11}$

78. $\left(\frac{k^{-2}k^8}{k^3}\right)^2$

79. $\left(\frac{j^{-2}j^5}{j^4}\right)^3$

Answer

$\frac{1}{j^3}$

80. $\frac{(-4m^{-3})^2(5m^4)^3}{(-10m^6)^3}$

81. $\frac{(-10n^{-2})^3(4n^5)^2}{(2n^8)^2}$

Answer

$$-\frac{4000}{n^{12}}$$

Writing Exercises

82. Use the Product Property for Exponents to explain why $x \cdot x = x^2$.

83. Jennifer thinks the quotient $\frac{a^{24}}{a^6}$ simplifies to a^4. What is wrong with her reasoning?

Answer

Answers will vary.

84. Explain why $-5^3 = (-5)^3$ but $-5^4 \neq (-5)^4$.

85. When you convert a number from decimal notation to scientific notation, how do you know if the exponent will be positive or negative?

Answer

Answers will vary.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No- I dont' get it
Simplify expressions using the properties for exponents			
Use the definition of negative exponents			

ⓑ After reviewing this checklist, what will you do to become confident for all goals?

3.3: Chapter 3 Review Exercises

Chapter Review Exercises

Add and Subtract Polynomials

Determine the Degree of Polynomials

In the following exercises, determine the type of polynomial.

1. $16x^2 - 40x - 25$

2. $5m + 9$

Answer

binomial

3. -15

4. $y^2 + 6y^3 + 9y^4$

Answer

other polynomial

Add and Subtract Polynomials

In the following exercises, add or subtract the polynomials.

5. $4p + 11p$

6. $-8y^3 - 5y^3$

Answer

$-13y^3$

7. $(4a^2 + 9a - 11) + (6a^2 - 5a + 10)$

8. $(8m^2 + 12m - 5) - (2m^2 - 7m - 1)$

Answer

$6m^2 + 19m - 4$

9. $(y^2 - 3y + 12) + (5y^2 - 9)$

10. $(5u^2 + 8u) - (4u - 7)$

Answer

$5u^2 + 4u + 7$

11. Find the sum of $8q^3 - 27$ and $q^2 + 6q - 2$.

12. Find the difference of $x^2 + 6x + 8$ and $x^2 - 8x + 15$.

Answer

$2x^2 - 2x + 23$

In the following exercises, simplify.

13. $17mn^2 - (-9mn^2) + 3mn^2$

14. $18a - 7b - 21a$

Answer

$-7b - 3a$

15. $2pq^2 - 5p - 3q^2$

16. $(6a^2 + 7) + (2a^2 - 5a - 9)$

Answer

$8a^2 - 5a - 2$

17. $(3p^2 - 4p - 9) + (5p^2 + 14)$

18. $(7m^2 - 2m - 5) - (4m^2 + m - 8)$

Answer

$-3m + 3$

19. $(7b^2 - 4b + 3) - (8b^2 - 5b - 7)$

20. Subtract $(8y^2 - y + 9)$ from $(11y^2 - 9y - 5)$

Answer

$3y^2 - 8y - 14$

21. Find the difference of $(z^2 - 4z - 12)$ and $(3z^2 + 2z - 11)$

22. $(x^3 - x^2y) - (4xy^2 - y^3) + (3x^2y - xy^2)$

Answer

$x^3 + 2x^2y - 4xy^2$

23. $(x^3 - 2x^2y) - (xy^2 - 3y^3) - (x^2y - 4xy^2)$

Evaluate a Polynomial Function for a Given Value of the Variable

In the following exercises, find the function values for each polynomial function.

24. For the function $f(x) = 7x^2 - 3x + 5$ find:
a. $f(5)$ b. $f(-2)$ c. $f(0)$

Answer

a. 165 b. 39 c. 5

25. For the function $g(x)=15-16x^2$, find:
a. $g(-1)$ b. $g(0)$ c. $g(2)$

26. A pair of glasses is dropped off a bridge 640 feet above a river. The polynomial function $h(t)=-16t^2+640$ gives the height of the glasses t seconds after they were dropped. Find the height of the glasses when $t=6$.

Answer

The height is 64 feet.

27. A manufacturer of the latest soccer shoes has found that the revenue received from selling the shoes at a cost of p dollars each is given by the polynomial $R(p)=-5p^2+360p$. Find the revenue received when $p=110$ dollars.

Add and Subtract Polynomial Functions

In the following exercises, find a. $(f+g)(x)$ b. $(f+g)(3)$ c. $(f-g)(x$ d. $(f-g)(-2)$

28. $f(x)=2x^2-4x-7$ and $g(x)=2x^2-x+5$

Answer

a. $(f+g)(x)=4x^2-5x-2$
b. $(f+g)(3)=19$
c. $(f-g)(x)=-3x-12$
d. $(f-g)(-2)=-6$

29. $f(x)=4x^3-3x^2+x-1$ and $g(x)=8x^3-1$

Properties of Exponents and Scientific Notation

Simplify Expressions Using the Properties for Exponents

In the following exercises, simplify each expression using the properties for exponents.

30. $p^3\cdot p^{10}$

Answer

p^{13}

31. $2\cdot 2^6$

32. $a\cdot a^2\cdot a^3$

Answer

a^6

33. $x\cdot x^8$

34. $y^a\cdot y^b$

Answer

y^{a+b}

35. $\dfrac{2^8}{2^2}$

36. $\dfrac{a^6}{a}$

Answer

a^5

37. $\dfrac{n^3}{n^{12}}$

38. $\dfrac{1}{x^5}$

Answer

$\dfrac{1}{x^4}$

39. 3^0

40. y^0

Answer

1

41. $(14t)^0$

42. $12a^0 - 15b^0$

Answer

-3

Use the Definition of a Negative Exponent

In the following exercises, simplify each expression.

43. 6^{-2}

44. $(-10)^{-3}$

Answer

$-\dfrac{1}{1000}$

45. $5 \cdot 2^{-4}$

46. $(8n)^{-1}$

Answer

$\dfrac{1}{8n}$

47. y^{-5}

48. 10^{-3}

Answer

$\frac{1}{1000}$

49. $\frac{1}{a^{-4}}$

50. $\frac{1}{6^{-2}}$

Answer

36

51. -5^{-3}

52. $\left(-\frac{1}{5}\right)^{-3}$

Answer

$-\frac{1}{25}$

53. $-(12)^{-3}$

54. $(-5)^{-3}$

Answer

$-\frac{1}{125}$

55. $\left(\frac{5}{9}\right)^{-2}$

56. $\left(-\frac{3}{x}\right)^{-3}$

Answer

$\frac{x^3}{27}$

In the following exercises, simplify each expression using the Product Property.

57. $(y^4)^3$

58. $(3^2)^5$

Answer

3^{10}

59. $(a^{10})^y$

60. $x^{-3} \cdot x^9$

Answer

x^5

61. $r^{-5} \cdot r^{-4}$

62. $(uv^{-3})(u^{-4}v^{-2})$

Answer

$\frac{1}{u^3v^5}$

63. $(m^5)^{-1}$

64. $p^5 \cdot p^{-2} \cdot p^{-4}$

Answer

$\frac{1}{m^5}$

In the following exercises, simplify each expression using the Power Property.

65. $(k-2)^{-3}$

66. $\frac{q^4}{q^{20}}$

Answer

$\frac{1}{q^{16}}$

67. $\frac{b^8}{b^{-2}}$

68. $\frac{n^{-3}}{n^{-5}}$

Answer

n^2

In the following exercises, simplify each expression using the Product to a Power Property.

69. $(-5ab)^3$

70. $(-4pq)^0$

Answer

1

71. $(-6x^3)^{-2}$

72. $(3y^{-4})^2$

Answer

$\frac{9}{y^8}$

In the following exercises, simplify each expression using the Quotient to a Power Property.

73. $\left(\frac{3}{5x}\right)^{-2}$

74. $\left(\frac{3xy^2}{z}\right)^4$

Answer

$\frac{81x^4y^8}{z^4}$

75. $(4p-3q^2)^2$

In the following exercises, simplify each expression by applying several properties.

76. $(x^2y)^2(3xy^5)^3$

Answer

$27x^7y^{17}$

77. $(-3a^{-2})^4(2a^4)^2(-6a^2)^3$

78. $\left(\frac{3xy^3}{4x^4y^{-2}}\right)^2\left(\frac{6xy^4}{8x^3y^{-2}}\right)^{-1}$

Answer

$\frac{3y^4}{4x^4}$

In the following exercises, write each number in scientific notation.

79. 2.568

80. $5,300,000$

Answer

5.3×10^6

81. 0.00814

In the following exercises, convert each number to decimal form.

82. 2.9×10^{4}

Answer

$29,000$

83. 3.75×10^{-1}

84. 9.413×10^{-5}

Answer

0.00009413

In the following exercises, multiply or divide as indicated. Write your answer in decimal form.

85. $(3 \times 10^{7})(2 \times 10^{-4})$

86. $(1.5 \times 10^{-3})(4.8 \times 10^{-1})$

Answer

0.00072

87. $\dfrac{6 \times 10^{9}}{2 \times 10^{-1}}$

88. $\dfrac{9 \times 10^{-3}}{1 \times 10^{-6}}$

Answer

$9,000$

Multiply Polynomials

Multiply Monomials

In the following exercises, multiply the monomials.

89. $(-6p^{4})(9p)$

90. $\left(\frac{1}{3}c^{2}\right)(30c^{8})$

Answer

$10c^{10}$

91. $(8x^{2}y^{5})(7xy^{6})$

92. $\left(\frac{2}{3}m^{3}n^{6}\right)\left(\frac{1}{6}m^{4}n^{4}\right)$

Answer

$\dfrac{m^{7}n^{10}}{9}$

Multiply a Polynomial by a Monomial

In the following exercises, multiply.

93. $7(10-x)$

94. $a^2(a^2-9a-36)$

Answer

$a^4-9a^3-36a^2$

95. $-5y(125y^3-1)$

96. $(4n-5)(2n^3)$

Answer

$8n^4-10n^3$

Multiply a Binomial by a Binomial

In the following exercises, multiply the binomials using:

a. the Distributive Property b. the FOIL method c. the Vertical Method.

97. $(a+5)(a+2)$

98. $(y-4)(y+12)$

Answer

$y^2+8y-48$

99. $(3x+1)(2x-7)$

100. $(6p-11)(3p-10)$

Answer

$18p^2-93p+110$

In the following exercises, multiply the binomials. Use any method.

101. $(n+8)(n+1)$

102. $(k+6)(k-9)$

Answer

$k^2-3k-54$

103. $(5u-3)(u+8)$

104. $(2y-9)(5y-7)$

Answer

$10y^2-59y+63$

105. $(p+4)(p+7)$

106. $(x-8)(x+9)$

Answer

x^2+x-72

107. $(3c+1)(9c-4)$

108. $(10a-1)(3a-3)$

Answer

$30a^2-33a+3$

Multiply a Polynomial by a Polynomial

In the following exercises, multiply using a. the Distributive Property b. the Vertical Method.

109. $(x+1)(x^2-3x-21)$

110. $(5b-2)(3b^2+b-9)$

Answer

$15b^3-b^2-47b+18$

In the following exercises, multiply. Use either method.

111. $(m+6)(m^2-7m-30)$

112. $(4y-1)(6y^2-12y+5)$

Answer

$24y^2-54y^2+32y-5$

Multiply Special Products

In the following exercises, square each binomial using the Binomial Squares Pattern.

113. $(2x-y)^2$

114. $(x+\frac{3}{4})^2$

Answer

$x^2+\frac{3}{2}x+\frac{9}{16}$

115. $(8p^3-3)^2$

116. $(5p+7q)^2$

Answer

$25p^2 + 70pq + 49q^2$

In the following exercises, multiply each pair of conjugates using the Product of Conjugates.

117. $(3y+5)(3y-5)$

118. $(6x+y)(6x-y)$

Answer

$36x^2 - y^2$

119. $(a+\frac{2}{3}b)(a-\frac{2}{3}b)$

120. $(12x^3 - 7y^2)(12x^3 + 7y^2)$

Answer

$144x^6 - 49y^4$

121. $(13a^2 - 8b4)(13a^2 + 8b^4)$

Divide Monomials

Divide Monomials

In the following exercises, divide the monomials.

122. $72p^{12} \div 8p^3$

Answer

$9p^9$

123. $-26a^8 \div (2a^2)$

124. $\frac{45y^6}{-15y^{10}}$

Answer

$-3y^4$

125. $\frac{-30x^8}{-36x^9}$

126. $\frac{28a^9b}{7a^4b^3}$

Answer

$\frac{4a^5}{b^2}$

127. $\frac{11u^6v^3}{55u^2v^8}$

128. $\frac{(5m^9n^3)(8m^3n^2)}{(10mn^4)(m^2n^5)}$

Answer

$\frac{4m^9}{n^4}$

129. $\frac{(42r^2s^4)(54rs^2)}{(6rs^3)(9s)}$

Divide a Polynomial by a Monomial

In the following exercises, divide each polynomial by the monomial

130. $(54y^4-24y^3)\div(-6y^2)$

Answer

$-9y^2+4y$

131. $\frac{63x^3y^2-99x^2y^3-45x^4y^3}{9x^2y^2}$

132. $\frac{12x^2+4x-3}{-4x}$

Answer

$-3x-1+\frac{3}{4x}$

Divide Polynomials using Long Division

In the following exercises, divide each polynomial by the binomial.

133. $(4x^2-21x-18)\div(x-6)$

134. $(y^2+2y+18)\div(y+5)$

Answer

$y-3+\frac{33}{q+6}$

135. $(n^3-2n^2-6n+27)\div(n+3)$

136. $(a^3-1)\div(a+1)$

Answer

a^2+a+1

Divide Polynomials using Synthetic Division

In the following exercises, use synthetic Division to find the quotient and remainder.

137. $x^3-3x^2-4x+12$ is divided by $x+2$

138. $2x^3-11x^2+11x+12$ is divided by $x-3$

Answer

$2x^2-5x-4;\ 0$

139. $x^4+x^2+6x-10$ is divided by $x+2$

Divide Polynomial Functions

In the following exercises, divide.

140. For functions $f(x)=x^2-15x+45$ and $g(x)=x-9$, find a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(-2)$

Answer

a. $\left(\frac{f}{g}\right)(x)=x-6$
b. $\left(\frac{f}{g}\right)(-2)=-8$

141. For functions $f(x)=x^3+x^2-7x+2$ and $g(x)=x-2$, find a. $\left(\frac{f}{g}\right)(x)$

b. $\left(\frac{f}{g}\right)(3)$

Use the Remainder and Factor Theorem

In the following exercises, use the Remainder Theorem to find the remainder.

142. $f(x)=x^3-4x-9$ is divided by $x+2$

Answer

-9

143. $f(x)=2x^3-6x-24$ divided by $x-3$

In the following exercises, use the Factor Theorem to determine if $x-c$ is a factor of the polynomial function.

144. Determine whether $x-2$ is a factor of x^3-7x^2+7x-6

Answer

no

145. Determine whether $x-3$ is a factor of $x^3-7x^2+11x+3$

Chapter Practice Test

1. For the polynomial $8y^4-3y^2+1$

a. Is it a monomial, binomial, or trinomial? b. What is its degree?

Answer

a. trinomial b. 4

2. $(5a^2+2a-12)(9a^2+8a-4)$

3. $(10x^2-3x+5)-(4x^2-6)$

Answer

$6x^2-3x+11$

4. $\left(-\frac{3}{4}\right)^3$

5. $x^{-3}x^4$

Answer

x

6. $5^6 5^8$

7. $(47a^{18}b^{23}c^5)^0$

Answer

1

8. 4^{-1}

9. $(2y)^{-3}$

Answer

$\frac{1}{8y^3}$

10. $p^{-3}\cdot p^{-8}$

11. $\frac{x^4}{x^{-5}}$

Answer

x^9

12. $(3x^{-3})^2$

13. $\frac{24r^3s}{6r^2s^7}$

Answer

$\frac{4r}{s^6}$

14. $(x4y9x-3)2$

15. $(8xy^3)(-6x^4y^6)$

Answer

$-48x^5y^9$

16. $4u(u^2-9u+1)$

17. $(m+3)(7m-2)$

Answer

$21m^2-19m-6$

18. $(n-8)(n^2-4n+11)$

19. $(4x-3)^2$

Answer

$16x^2-24x+9$

20. $(5x+2y)(5x-2y)$

21. $(15xy^3-35x^2y)\div 5xy$

Answer

$3y^2-7x$

22. $(3x^3-10x^2+7x+10)\div(3x+2)$

23. Use the Factor Theorem to determine if $x+3$ a factor of $x^3+8x^2+21x+18$.

Answer

yes

24. a. Convert 112,000 to scientific notation.
b. Convert 5.25×10^{-4} to decimal form.

In the following exercises, simplify and write your answer in decimal form.

25. $(2.4\times 10^8)(2\times 10^{-5})$

Answer

4.4×10^3

26. $\dfrac{9\times 10^4}{3\times 10^{-1}}$

27. For the function $f(x)=6x^2-3x-9$ find:
a. $f(3)$ b. $f(-2)$ c. $f(0)$

Answer

a. 36 b. 21 c. -9

28. For $f(x)=2x^2-3x-5$ and $g(x)=3x^2-4x+1$, find
a. $(f+g)(x)$ b. $(f+g)(1)$
c. $(f-g)(x)$ d. $(f-g)(-2)$

29. For functions $f(x)=3x^2-23x-36$ and $g(x)=x-9$, find
a. $\left(\frac{f}{g}\right)(x)$ b. $\left(\frac{f}{g}\right)(3)$

Answer

a. $\left(\frac{f}{g}\right)(x)=3x+4$
b. $\left(\frac{f}{g}\right)(3)=13$

30. A hiker drops a pebble from a bridge 240 feet above a canyon. The function $h(t)=-16t^2+240$ gives the height of the pebble t seconds after it was dropped. Find the height when $t=3$.

CHAPTER OVERVIEW

4: Factoring

4.1: Greatest Common Factor and Factor by Grouping
4.2: Factor Trinomials
4.3: Factor Special Products
4.4: General Strategy for Factoring Polynomials
4.5: Polynomial Equations
4.6: Chapter 4 Review Exercises

4.1: Greatest Common Factor and Factor by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more polynomials
- Factor the greatest common factor from a polynomial
- Factor by grouping

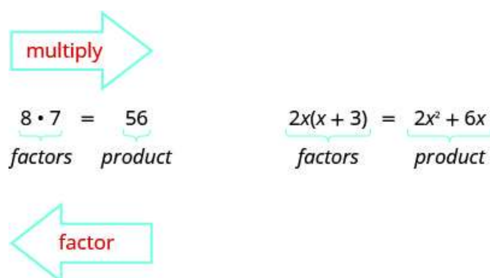
Be Prepared

Before you get started, take this readiness quiz.

1. Factor 56 into primes.
2. Find the least common multiple (LCM) of 18 and 24.
3. Multiply $-3a(7a+8b)$.

Find the Greatest Common Factor of a Polynomial

Earlier we multiplied factors together to get a *product*. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called *factoring*.



Above, we have recalled how to factor numbers to find the least common multiple (LCM) of two or more numbers. Our goal is to be able to factor polynomials. The first step is to find the greatest common factor of a given polynomial. The method we use is similar to what we can use to find the LCM.

Definition 4.1.1

1. If we write a polynomial P as a product of polynomials, we say that we have **factored** P.
2. A polynomial F is a **factor** of P if we can write P as $P=F\cdot G$ for some polynomial G.
3. The **greatest common factor (GCF) of two or more monomials** is a monomial F that satisfies the following conditions:
   - F is a factor of all the monomials, that is, F is a common factor, and
   - any other common factor of all the monomials is a factor of F.
4. The **greatest common factor (GCF) of a polynomial** is the GCF of its terms.

While we say *the* greatest common factor, there are actually two: one with a positive coefficient and one with a negative coefficient.

Factor as a Noun and a Verb

We use "factor" as both a noun and a verb:

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a+3$

We summarize the steps we use to find the greatest common factor.

Find the Greatest Common Factor (GCF) of a Polynomial

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors.
3. Bring down the common factors that all polynomials share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three monomials.

Example 4.1.2

Find the greatest common factor of $21x^3$, $9x^2$, and $15x$.

Solution

	$21x^3, 9x^2, 15x$
Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot \quad 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot \quad x \cdot x$ $15x = 3 \cdot \quad 5 \cdot \quad x$ $\text{GCF} = 3 \cdot \quad x$
Multiply the factors.	$\text{GCF} = 3x$
Answer the question.	The GCF of $21x^3$, $9x^2$, and $15x$ is $3x$.

Try It 4.1.3

Find the greatest common factor of $25m^4$, $35m^3$, and $20m^2$.

Answer

The GCF is $5m^2$.

Try It 4.1.4

Find the greatest common factor of $14x^3$, $70x^2$, and $105x$.

Answer

The GCF is $7x$.

Factor the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with a product of its factors, $3x(x + 5)$. To do this we apply the Distributive Property “in reverse.”

We state the Distributive Property here just as you saw it in earlier chapters and “in reverse.”

Distributive property

If a, b, and c are real numbers, then

$$a(b+c) = ab+ac$$

and

$$ab+ac = a(b+c).$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to (partially) factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

Example 4.1.5

Factor $8m^3 - 12m^2n + 20mn^2$.

Solution

	$8m^3 - 12m^2n + 20mn^2$	
Find the GCF of all the termsof the polynomial.	Find the GCF of $8m^3$, $12m^2n$, and $20mn^2$.	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, and $20mn^2$ as products of their GCF, $4m$.	$8m^3 - 12m^2n + 20mn^2$ $= 4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Use the "reverse" Distributive Property to factor the expression.		$= 4m(2m^2 - 3mn + 5n^2)$
Check by multiplying the factors.		$4m(2m^2 - 3mn + 5n^2)$ $= 4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$ $= 8m^3 - 12m^2n + 20n^2 \quad \checkmark$

Try It 4.1.6

Factor $9xy^2 + 6x^2y^2 + 21y^3$.

Answer

$3y^2(3x + 2x^2 + 7y)$

Try It 4.1.7

Factor $3p^3 - 6p^2q + 9pq^3$.

Answer

$3p(p^2 - 2pq + 3q^3)$

Factor the Greatest Common Factor from a polynomial

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the “reverse” Distributive Property to factor the polynomial.
4. Check by multiplying the factors.

Example 4.1.8

Factor out the GCF of $5x^3 - 25x^2$.

Solution

	$5x^3 - 25x^2$
Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $\text{GCF} = 5 \cdot x \cdot x$ The GCF is $5x^2$.
Rewrite each term.	$5x^3 - 25x^2$ $= 5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$= 5x^2(x-5)$
Check.	$5x^2(x-5)$ $= 5x^2 \cdot x - 5x^2 \cdot 5$ $= 5x^3 - 25x^2 \quad \checkmark$

Try It 4.1.9

Factor out the GCF of $2x^3 + 12x^2$.

Answer

$2x^2(x+6)$

Try It 4.1.10

Factor out the GCF of $6y^3 - 15y^2$.

Answer

$3y^2(2y-5)$

Example 4.1.11

Factor out the GCF of $8x^3y - 10x^2y^2 + 12xy^3$.

Solution

	$8x^3y - 10x^2y^2 + 12xy^3$
The GCF of $8x^3y,\ -10x^2y^2,$ and $12xy^3$ is $2xy$.	$8x^3y = 2 \cdot 2 \cdot 2 \cdot x \cdot x \cdot x \cdot y$ $10x^2y^2 = 2 \cdot 5 \cdot x \cdot x \cdot y \cdot y$ $12xy^3 = 2 \cdot 2 \cdot 3 \cdot x \cdot y \cdot y \cdot y$ $\text{GCF} = 2 \cdot x \cdot y$ The GCF is $2xy$.
Rewrite each term using the GCF, $2xy$.	$8x^3y - 10x^2y^2 + 12xy^3$ $= 2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$= 2xy(4x^2 - 5xy + 6y^2)$
Check.	$2xy(4x^2 - 5xy + 6y^2)$ $= 2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$ $= 8x^3y - 10x^2y^2 + 12xy^3 \quad \checkmark$

Try It 4.1.12

Factor out the GCF of $15x^3y - 3x^2y^2 + 6xy^3$.

Answer

$3xy(5x^2 - xy + 2y^2)$

Try It 4.1.13

Factor out the GCF of $8a^3b + 2a^2b^2 - 6ab^3$.

Answer

$2ab(4a^2 + ab - 3b^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example 4.1.14

Factor out the GCF of $-4a^3 + 36a^2 - 8a$.

Solution

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3 + 36a^2 - 8a$
Rewrite each term using the GCF, $-4a$.	$= -4a \cdot a^2 - 4a \cdot (-9a) - 4a \cdot 2$
Factor the GCF.	$= -4a(a^2 - 9a + 2)$
Check.	$-4a(a^2 - 9a + 2)$ $= -4a \cdot a^2 - (-4a) \cdot 9a + (-4a) \cdot 2$ $= -4a^3 + 36a^2 - 8a \quad \checkmark$

Try It 4.1.15

Factor out the GCF of $-4b^3 + 16b^2 - 8b$.

Answer

$-4b(b^2 - 4b + 2)$

Try It 4.1.16

Factor out the GCF of $-7a^3 + 21a^2 - 14a$.

Answer

$-7a(a^2 - 3a + 2)$

In the next example, we extend the idea of factoring out the GCF to a binomial.

Example 4.1.17

Factor out the common factor of the two terms of $3y(y+7) - 4(y+7)$.

Solution

	$3y(y+7)-4(y+7)$
The binomial $y+7$ is a common factor of the two terms.	$=3y(y+7)-4(y+7)$
Factor $(y+7)$.	$=(y+7)(3y-4)$
Check.	$3y(y+7)-4(y+7)$ $=3y\cdot y+3y\cdot 7-4y-4\cdot 7$ $=3y^2+21y-4y-28$ $=3y^2+17y-28$ $(y+7)(3y-4)$ $=y\cdot 3y+y(-4)+7\cdot 3y+7\cdot(-4)$ $=3y^2-4y+21y-28$ $=3y^2+17y-28\quad\checkmark$

? Try It 4.1.18

Factor out the common factor of the two terms of $4m(m+3)-7(m+3)$.

Answer

$(m+3)(4m-7)$

? Try It 4.1.19

Factor out the common factor of the two terms of $8n(n-4)+5(n-4)$.

Answer

$(n-4)(8n+5)$

Factor by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the **GCF** in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored.

? Example 4.1.20

Factor by grouping: $xy+3y+2x+6$.

Solution

	$xy+3y+2x+6$	
Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$ $=\underbrace{xy+3y}+\underbrace{2x+6}$
Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$=y(x+3)+\underbrace{2x+6}$ $=y(x+3)+2(x+3)$
Factor the common factor from the expression.	Notice that each terms has a common factor of $x+3$.	$=y(x+3)+2(x+3)$ $=(x+3)(y+2)$

	$xy+3y+2x+6$	
Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $=xy+2x+3y+6$ $=xy+3y+2x+6$ ✓

Try It 4.1.21

Factor by grouping: $xy+8y+3x+24$.

Answer

$(x+8)(y+3)$

Try It 4.1.22

Factor by grouping: $ab+7b+8a+56$.

Answer

$(a+7)(b+8)$

Factor by Grouping

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the polynomial.
4. Check by multiplying the factors.

Example 4.1.23

Factor by grouping:

a. $x^2+3x-2x-6$

b. $6x^2-3x-4x+2$

Solution

a.

	$x^2+3x-2x-6$
There is no GCF in all four terms.	$=x^2+3x-2x-6$
Separate into two parts.	$=\underbrace{x^2+3x}\ \underbrace{-2x-6}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$=x(x+3)-2(x+3)$
Factor out the common factor, $x+3$.	$=(x+3)(x-2)$
Check.	$(x+3)(x-2)$ $=x^2-2x+3x-6$ $=x^2+x-6$

b.

	$6x^2-3x-4x+2$
There is no GCF in all four terms.	$=6x^2-3x-4x+2$
Separate into two parts.	$=\underbrace{6x^2-3x}\ \underbrace{-4x+2}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$=3x(2x-1)-2(2x-1)$
Factor out the common factor, $2x-1$.	$=(2x-1)(3x-2)$
Check.	$(2x-1)(3x-2)$ $=6x^2-4x-3x+2$ $=6x^2-3x-4x+2$

? Try It 4.1.24

Factor by grouping:

a. $x^2+2x-5x-10$

b. $20x^2-16x-15x+12$

Answer

a. $(x-5)(x+2)$
b. $(5x-4)(4x-3)$

? Try It 4.1.25

Factor by grouping:

a. $y^2+4y-7y-28$

b. $42m^2-18m-35m+15$

Answer

a. $(y+4)(y-7)$
b. $(7m-3)(6m-5)$

Key Concepts

- **Factor as a Noun and a Verb:** We use "factor" as both a noun and a verb.

$$\begin{array}{ll}\text{Noun:} & 7 \text{ is a factor of } 14\\ \text{Verb:} & \text{factor } 3 \text{ from } 3a+3\end{array}$$

- **How to find the greatest common factor (GCF) of two polynomials.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all polynomials share.
  4. Multiply the factors.
- **Distributive Property:** If a, b and c are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

  The form on the left is used to multiply. The form on the right is used to factor.
- **How to factor the greatest common factor from a polynomial.**

1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the “reverse” Distributive Property to factor the polynomial.
4. Check by multiplying the factors.

- **How to factor by grouping.**
  1. Group terms with common factors.
  2. Factor out the common factor in each group.
  3. Factor the common factor from the polnomial.
  4. Check by multiplying the factors.

Glossary

factoring a polynomial

Writing a polynomial as a product is called factoring.

greatest common factor

The greatest common factor (GCF) of two or more polynomials is the largest polynomial that is a factor of all the expressions.

Practice Makes Perfect

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $10p^3q, 12pq^2$

Answer

$2pq$

2. $8a^2b^3, 10ab^2$

3. $12m^2n^3, 30m^5n^3$

Answer

$6m^2n^3$

4. $28x^2y^4, 42x^4y^4$

5. $10a^3, 12a^2, 14a$

Answer

$2a$

6. $20y^3, 28y^2, 40y$

7. $35x^3y^2, 10x^4y, 5x^5y^3$

Answer

$5x^3y$

8. $27p^2q^3, 45p^3q^4, 9p^4q^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

9. $6m+9$

Answer

$3(2m+3)$

10. $14p+35$

11. $9n-63$

Answer

$9(n-7)$

12. $45b-18$

13. $3x^2+6x-9$

Answer

$3(x^2+2x-3)$

14. $4y^2+8y-4$

15. $8p^2+4p+2$

Answer

$2(4p^2+2p+1)$

16. $10q^2+14q+20$

17. $8y^3+16y^2$

Answer

$8y^2(y+2)$

18. $12x^3-10x$

19. $5x^3-15x^2+20x$

Answer

$5x(x^2-3x+4)$

20. $8m^2-40m+16$

21. $24x^3-12x^2+15x$

Answer

$3x(8x^2-4x+5)$

22. $24y^3-18y^2-30y$

23. $12xy^2 + 18x^2y^2 - 30y^3$

Answer

$6y^2(2x + 3x^2 - 5y)$

24. $21pq^2 + 35p^2q^2 - 28q^3$

25. $20x^3y - 4x^2y^2 + 12xy^3$

Answer

$4xy(5x^2 - xy + 3y^2)$

26. $24a^3b + 6a^2b^2 - 18ab^3$

27. $-2x - 4$

Answer

$-2(x + 4)$

28. $-3b + 12$

29. $-2x^3 + 18x^2 - 8x$

Answer

$-2x(x^2 - 9x + 4)$

30. $-5y^3 + 35y^2 - 15y$

31. $-4p^3q - 12p^2q^2 + 16pq^2$

Answer

$-4pq(p^2 + 3pq - 4q)$

32. $-6a^3b - 12a^2b^2 + 18ab^2$

33. $5x(x + 1) + 3(x + 1)$

Answer

$(x + 1)(5x + 3)$

34. $2x(x - 1) + 9(x - 1)$

35. $3b(b - 2) - 13(b - 2)$

Answer

$(b - 2)(3b - 13)$

36. $6m(m - 5) - 7(m - 5)$

Factor by Grouping

In the following exercises, factor by grouping.

37. $ab+5a+3b+15$

Answer

$(b+5)(a+3)$

38. $cd+6c+4d+24$

39. $8y^2+y+40y+5$

Answer

$(y+5)(8y+1)$

40. $6y^2+7y+24y+28$

41. $uv-9u+2v-18uv$

Answer

$(u+2)(v-9)$

42. $pq-10p+8q-80$

43. $u^2-u+6u-6$

Answer

$(u-1)(u+6)$

44. $x^2-x+4x-4$

45. $9p^2-15p+12p-20$

Answer

$(3p-5)(3p+4)$

46. $16q^2+20q-28q-35$

47. $mn-6m-4n+24$

Answer

$(n-6)(m-4)$

48. $r^2-3r-r+3$

49. $2x^2-14x-5x+35$

Answer

$(x-7)(2x-5)$

50. $4x^2-36x-3x+27$

Mixed Practice

In the following exercises, factor.

51. $-18xy^2-27x^2y$

Answer

$-9xy(3x+2y)$

52. $-4x^3y^5-x^2y^3+12xy^4$

53. $3x^3-7x^2+6x-14$

Answer

$(x^2+2)(3x-7)$

54. x^3+x^2-x-1

55. $x^2+xy+5x+5y$

Answer

$(x+y)(x+5)$

56. $5x^3-3x^2+5x-3$

Writing Exercises

57. What does it mean to say a polynomial is in factored form?

Answer

Answers will vary.

58. How do you check result after factoring a polynomial?

59. The greatest common factor of 36 and 60 is 12. Explain what this means.

Answer

Answers will vary.

60. What is the GCF of $y^4,\ y^5$, and y^{10}? Write a general rule that tells you how to find the GCF of $y^a,\ y^b$, and y^c.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
find the greatest common factor of two or more expressions.			
factor the greatest common factor from a polynomial.			
factor by grouping.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

F

4.2: Factor Trinomials

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form ax^2+bx+c using trial and error
- Factor trinomials of the form ax^2+bx+c using the 'ac' method
- Factor using substitution

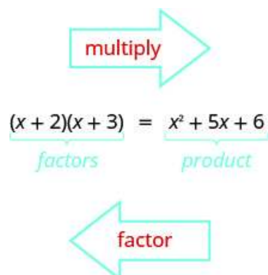
Be Prepared

Before you get started, take this readiness quiz.

1. Find all the factors of 72.
2. Find the product $(3y+4)(2y+5)$.
3. Simplify $-9(6)$ and $-9(-6)$.

Factor Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using **FOIL**. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.



To figure out how we would factor a **trinomial** of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\underbrace{x^2+(m+n)x+mn}_{x^2+bx+c}$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

Example 4.2.1

Factor $x^2+11x+24$.

Solution

	$x^2+11x+24$

		$x^2+11x+24$	
24\)	Write the factors as two binomials with first terms x.	Write two sets of parentheses and put x as the first term.	$x^2+11x+24$ $=\underbrace{(x+\square)}_{(x+m)}\underbrace{(x+\square)}_{(x+n)}$
24\)	Identify b and c.	$\underbrace{x^2+11x+24}_{ax^2+bx+c}$	$b=11$ $c=24$
24\)	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn &= c \\ m+n &= b \end{cases}$	Find two numbers m and n that multiply to 24 and add to 11. $\begin{cases} mn &= 24 \\ m+n &= 11 \end{cases}$	Factors of 24 / Sum of factors: $1, 24$ — $1+24=25$ $2, 12$ — $2+12=14$ $3, 8$ — $3+8=11^{*}$ $4, 6$ — $4+6=10$ $m=3$ $n=8$
24\)	Use m and n as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$\underbrace{(x+3)}_{(x+m)}\underbrace{(x+8)}_{(x+n)}$
24\)	Check by multiplying the factors.		$(x+3)(x+8)$ $=x^2+8x+3x+24$ $=x^2+11x+24$ ✓
24\)	Answer the question.		The factorization is $(x+3)(x+8)$.

? Try It 4.2.2

Factor $q^2+10q+24$.

Answer

The factorization is $(q+4)(q+6)$.

? Try It 4.2.3

Factor $t^2+14t+24$.

Answer

The factorization is $(t+2)(t+12)$.

Let's summarize the steps we used to find the factors.

Factoring x^2+bx+c

1. Write the factors as two binomials with first terms x. $\quad x^2+bx+c$ $\quad (x+\square)(x+\square)$
2. Find two numbers m and n that
   - multiply to c: $\quad mn=c$
   - add to b: $\quad m+n=b$
3. Use m and n as the last terms of the factors. $\quad (x+m)(x+n)$
4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

Example 4.2.4

Factor $y^2 - 11y + 28$.

Solution

		$y^2 - 11y + 28$
28\)	Write the factors as two binomials with first terms y.	$y^2 - 11y + 28$ $= \underbrace{(y + \square)}_{(y+m)} \underbrace{(y + \square)}_{(y+n)}$
28\)	Identify b and c.	$\underbrace{y^2 - 11y + 28}_{ay^2+by+c}$ $b = -11$ $c = 28$
28\)	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn & = c \\ m+n & = b \end{cases}$	Find m and n such that $\begin{cases} mn & = 28 \\ m+n & = -11 \end{cases}$ Factors of 28 / Sum of factors $-1, -28$ / $-1 + (-28) = -29$ $-2, -14$ / $-2 + (-14) = -16$ $-4, -7$ / $-4 + (-7) = -11^*$ $m = -4$ $n = -7$
28\)	Use m and n as the last terms of the factors.	$\underbrace{(y-4)}_{(y+m)} \underbrace{(y-7)}_{(y+n)}$
28\)	Check.	$(y-4)(y-7)$ $= y^2 - 7y - 4y + 28$ $= y^2 - 11y + 28$ ✓
28\)	Answer the question.	The factorization is $(y-4)(y-7)$.

Try It 4.2.5

Factor $u^2 - 9u + 18$.

Answer

The factorization is $(u-3)(u-6)$.

Try It 4.2.6

Factor $y^2 - 16y + 63$.

Answer

The factorization is $(y-7)(y-9)$.

Now, what if the last term in the trinomial is negative? Think about **FOIL**. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

Example 4.2.7

Factor $2x+x^2-48$.

Solution

		$2x+x^2-48$
	First we put the terms in decreasing degree order.	$x^2+2x-48$
the …	Write the factors as two binomials with first terms y.	$x^2+2x-48$ $=\underbrace{(x+\square)}_{(x+m)}\underbrace{(x+\square)}_{(x+n)}$
the …	Identify b and c.	$\underbrace{x^2+2x-48}_{ax^2+bx+c}$ $b=2$ $c=-48$
the …	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn & =c \\ m+n & =b \end{cases}$	Find m and n such that $\begin{cases} mn & =-48 \\ m+n & =2 \end{cases}$ (see table below) $m=-6$ $n=8$
the …	Use m and n as the last terms of the factors.	$\underbrace{(x-6)}_{(x+m)}\underbrace{(x+8)}_{(x+n)}$
the …	Check.	$(x-6)(x+8)$ $=x^2+8x-6x-48$ $=x^2+2x-48$ ✓
the …	Answer the question.	The factorization is $(x-6)(x+8)$.

Factors of -48	Sum of factors
$-1, 48$	$-1+48=47$
$-2, 24$	$-2+24=22$
$-3, 16$	$-3+16=13$
$-4, 12$	$-4+12=8$
$-6, 8$	$-6+8=2^{*}$

Try It 4.2.8

Factor $9m+m^2+18$.

Answer

The factorization is $(m+3)(m+6)$.

Try It 4.2.9

Factor $-7n+12+n^2$.

Answer

The factorization is $(n-3)(n-4)$.

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, xx. The y^2 in the last term means that the second terms of the binomial factors must each contain y. To get the coefficients b and c, you use the same process summarized in How To Factor trinomials.

Example 4.2.10

Factor $r^2-8rs-9s^2$.

Solution

		$r^2-8rs-9s^2$
-9\)	Write the factors as two binomials with r in the first term of each binomial and s in the second term.	$r^2-8rs-9s^2$ $=\underbrace{(r+\square s)}_{(r+ms)}\underbrace{(r+\square s)}_{(r+ns)}$
-9\)	Identify b and c.	$\underbrace{r^2-8rs-9s^2}_{ar^2+brs+cs^2}$ $b=-8$ $c=-9$
-9\)	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn &= c \\ m+n &= b \end{cases}$	Find m and n such that $\begin{cases} mn &= -9 \\ m+n &= -8 \end{cases}$ Factors of -9 / Sum of factors: $1,-9$: $1+(-9)=-8^{*}$ $-1,9$: $-1+9=8$ $-3,3$: $-3+3=0$ $m=1$ $n=-9$
-9\)	Use m and n as the last terms of the factors.	$\underbrace{(r+s)}_{(r+ms)}\underbrace{(r-9s)}_{(r+ns)}$
-9\)	Check.	$(r+s)(r-9s)$ $=r^2-9rs+rs-9s^2$ $=r^2-8rs-9s^2$ ✓
-9\)	Answer the question.	The factorization is $(r+s)(r-9s)$.

Try It 4.2.11

Factor $a^2-11ab+10b^2$.

Answer

The factorization is $(a-b)(a-10b)$.

Try It 4.2.12

Factor $m^2 - 13mn + 12n^2$.

Answer

The factorization is $(m-n)(m-12n)$

Some trinomials are prime. The only way to be certain a trinomial is **prime** is to list all the possibilities and show that none of them work.

Example 4.2.13

Factor $u^2 - 9uv - 12v^2$.

Solution

	$u^2 - 9uv - 12v^2$
-12\) Write the factors as two binomials with u in the first term of each binomial and v in the second term.	$u^2 - 9uv - 12v^2$ $= \underbrace{(u + \square v)}_{(u+mv)} \underbrace{(u + \square v)}_{(u+nv)}$
-12\) Identify b and c.	$\underbrace{u^2 - 9uv - 12v^2}_{au^2+buv+cn^2}$ $b = -9$ $c = -12$
-12\) Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn & = c \\ m+n & = b \end{cases}$	Find m and n such that $\begin{cases} mn & = -12 \\ m+n & = -9 \end{cases}$ Factors of -12 / Sum of factors: $1, -12$: $1 + (-12) = -11$ $-1, 12$: $-1 + 12 = 11$ $2, -6$: $2 + (-6) = -4$ $-2, 6$: $-2 + 6 = 4$ $3, -4$: $3 + (-4) = -1$ $-3, 4$: $-3 + 4 = 1$ Note there are no factor pairs that give us -9 as a sum.
-12\) Answer the question.	The trinomial is prime.

Try It 4.2.14

Factor $x^2 - 7xy - 10y^2$.

Answer

The trinomial is prime.

Try It 4.2.15

Factor $p^2 + 15pq + 20q^2$.

Answer

The trinomial is prime.

Let’s summarize the method we just developed to factor trinomials of the form x^2+bx+c .

Strategy for factoring trinomials of the form x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

		x^2+bx+c	
		$(x+m)(x+n)$	
	When c is positive, m and n have the same sign.		
	b positive		b negative
	m, n positive		m, n negative
	x^2+5x+6		
	$(x+2)(x+3)$		$(x-4)(x-2)$
	same signs		same signs
	When c is negative, m and n have the opposite sign.		
	$(x+4)(x-3)$		$(x-5)(x+3)$
	opposite signs		opposite signs

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of \(b\

Factor Trinomials of the form $ax^2 + bx + c$ using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c .

Remember to always check for a **GCF** first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we’ve used so far. Let’s do an example to see how this works.

Example 4.2.16

Factor $4x^3+16x^2-20x$ completely.

Solution

		$4x^3+16x^2-20x$
-5\)	What is the greatest common factor?	The GCF is $4x$.
-5\)	Factor it.	$4x(x^2+4x-5)$
-5\)	Is the GCF multiplied by a binomial, a trinomial, or a polynomial with more than three terms?	$=\underbrace{4x}_{\text{GCF}}\underbrace{(x^2+4x-5)}_{\text{trinomial}}$ It is a trinomial.
-5\)	“Undo FOIL.” Write the trinomial as factors of two binomials with first terms x.	$=\underbrace{4x}_{\text{GCF}}\underbrace{(x^2+4x-5)}_{(x+\square)(x+\square)}$

<table>
<tr><td>-5\)</td><td>Identify b and c.</td><td>$= \underbrace{4x}_{\text{GCF}}\underbrace{(x^2+4x-5)}_{ax^2+bx+c}$
$b=4$
$c=-5$</td></tr>
<tr><td>-5\)</td><td>Find two numbers m and n that multiply to c and add to b.
$\begin{cases} mn &= c \\ m+n &= b \end{cases}$</td><td>Find m and n such that
$\begin{cases} mn &= -5 \\ m+n &= 4 \end{cases}$
<table><tr><th>Factors of -5</th><th>Sum of factors</th></tr><tr><td>$-1, 5$
$1, -5$</td><td>$-1+5=4^*$
$1+(-5)=-4$</td></tr></table>
$m=-1$
$n=4$</td></tr>
<tr><td>-5\)</td><td>Use m and n as the last terms of the factors.</td><td>$\underbrace{4x}_{\text{GCF}}\underbrace{(x-1)(x+4)}_{(x+m)(x+n)}$</td></tr>
<tr><td>-5\)</td><td>Check.</td><td>$4x(x-1)(x+5)$
$=4x(x^2+5x-x-5$
$=4x(x^2+4x-5)$
$=4x^3+16x^2-20x \quad \checkmark$</td></tr>
<tr><td>-5\)</td><td>Answer the question.</td><td>The complete factorization is $4x(x-1)(x+5)$.</td></tr>
</table>

? Try It 4.2.17

Factor $5x^3+15x^2-20x$ completely.

Answer

The complete factorization is $5x(x-1)(x+4)$.

? Try It 4.2.18

Factor $6y^3+18y^2-60y$ completely.

Answer

The complete factorization is $6y(y-2)(y+5)$.

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2+5x+2$.

From our earlier work, we expect this will factor into two binomials.

$$3x^2+5x+2$$

$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

$$3x^2 + 5x + 2$$
$$1x,\ 3x$$
$$(x \quad)(3x \quad)$$

Check: Does $1x \cdot 3x = 3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write $1, 2$ or $2, 1$.

$$\begin{array}{ccc} 3x^2 + 5x + 2 & & 3x^2 + 5x + 2 \\ 1x,\ 3x \qquad 1,\ 2 & & 1x,\ 3x \qquad 1,\ 2 \\ (x+1)(3x+2) & \text{or} & (x+2)(3x+1) \end{array}$$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$\begin{array}{ccc} 3x^2 + 5x + 2 & & 3x^2 + 5x + 2 \\ 1x,\ 3x \qquad 1,\ 2 & & 1x,\ 3x \qquad 1,\ 2 \\ (x+1)(3x+2) & \text{or} & (x+2)(3x+1) \\ 3x & & 6x \\ \underline{2x} & & \underline{1x} \\ 5x & & 7x \end{array}$$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x+1)(3x+2)$$

$$3x^2 + 2x + 3x + 2$$

$$3x^2 + 5x + 2 \checkmark$$

Our result of the factoring is:

$$3x^2 + 5x + 2$$

$$(x+1)(3x+2)$$

Example 4.2.19

Factor $3y^2 + 22y + 7$ completely using trial and error.

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2 + 22y + 7$
Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y, 3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2 + 22y + 7$ $1y,\ 3y$ $3y^2 + 22y + 7$ $1y,\ 3y$ $(y \quad)(3y \quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2 + 22y + 7$ $1y,\ 3y \qquad 1,\ 7$ $(y \quad)(3y \quad)$

Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2 + 22y + 7$ $1y, 3y \quad 1, 7$ $(y+1)(3y+7)$ $3y$ $7y$ $10y$ No! We need $22y$ $3y^2 + 22y + 7$ $1y, 3y \quad 1, 7$ $(y+7)(3y+1)$ $21y$ $+y$ $22y$	$3y^2 + 22y + 7$ Possible factors: $(y+1)(3y+7)$ — Product: $3y^2 + 10y + 7$ Possible factors: $(y+7)(3y+1)$ — Product: $3y^2 + 22y + 7$
Step 6. Check by multiplying.		$(y+7)(3y+1)$ $3y^2 + 22y + 7$ ✓

? Try It 4.2.20

Factor $2a^2 + 5a + 3$ completely using trial and error.

Answer

The complete factorization is $(a+1)(2a+3)$.

? Try It 4.2.21

Factor $4b^2 + 5b + 1$ completely using trial and error.

Answer

The complete factorization is $(b+1)(4b+1)$.

Factor Trinomials of the Form $ax^2 + bx + c$ Using Trial and Error

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.
6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

? Example 4.2.22

Factor $6b^2 - 13b + 5$ completely using trial and error.

Solution

The trinomial is already in descending order.	$6b^2 - 13b + 5$
Find the factors of the first term.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $2b \cdot 3b$

Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2 - 13b + 5$ $1b \cdot 6b$ $\quad -1, -5$ $2b \cdot 3b$

Consider all the combinations of factors.

$6b^2 - 13b + 5$	
Possible factors	Product
$(b-1)(6b-5)$	$6b^2 - 11b + 5$
$(b-5)(6b-1)$	$6b^2 - 31b + 5$
$(2b-1)(3b-5)$	$6b^2 - 13b + 5^*$
$(2b-5)(3b-1)$	$6b^2 - 17b + 5$

The correct factors are those whose product is the original trinomial. $\quad (2b-1)(3b-5)$

Check by multiplying:

$$\begin{gathered}(2b-1)(3b-5)\\ 6b^2 - 10b - 3b + 5\\ 6b^2 - 13b + 5 \checkmark\end{gathered}$$

? Try It 4.2.23

Factor $8x^2 - 13x + 3$ completely using trial and error.

Answer

The complete factorization is $(2x-3)(4x-1)$.

? Try It 4.2.24

Factor $10y^2 - 37y + 7$ completely using trial and error.

Answer

The complete factorization is $(2y-7)(5y-1)$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

? Example 4.2.25

Factor $18x^2 - 37xy + 15y^2$ completely using trial and error.

Solution

The trinomial is already in descending order.	$18x^2 - 37xy + 15y^2$
Find the factors of the first term.	$18x^2 - 37xy + 15y^2$ $1x \cdot 18x$ $2x \cdot 9x$ $3x \cdot 6x$

Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2-37xy+15y^2$ $1x \cdot 18x$ $-1,-15$ $2x \cdot 9x$ $-3,-5$ $3x \cdot 6x$

Consider all the combinations of factors.

$18x^2-37xy+15y^2$	
Possible factors	**Product**
$(x-1y)(18x-15y)$	Not an option
$(x-15y)(18x-1y)$	$18x^2-271xy+15y^2$
$(x-3y)(18x-5y)$	$18x^2-59xy+15y^2$
$(x-5y)(18x-3y)$	Not an option
$(2x-1y)(9x-15y)$	Not an option
$(2x-15y)(9x-1y)$	$18x^2-137xy+15y^2$
$(2x-3y)(9x-5y)$	$18x^2-37xy+15y^2$*
$(2x-5y)(9x-3y)$	Not an option
$(3x-1y)(6x-15y)$	Not an option
$(3x-15y)(6x-1y)$	Not an option
$(3x-3y)(6x-5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial.	$(2x-3y)(9x-5y)$
Check by multiplying:	

$$\begin{gathered}(2x-3y)(9x-5y)\\ 18x^2-10xy-27xy+15y^2\\ 18x^2-37xy+15y^2\checkmark\end{gathered}$$

? Try It 4.2.26

Factor $18x^2-3xy-10y^2$ completely using trial and error.

Answer

The complete factorization is $(3x+2y)(6x-5y)$.

? Try It 4.2.27

Factor $30x^2-53xy-21y^2$ completely using trial and error.

Answer

The complete factorization is $(3x+y)(10x-21y)$.

Don’t forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

? Example 4.2.28

Factor $-10y^4 - 55y^3 - 60y^2$ completely using trial and error.

Solution

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$
Factor the trinomial.	$-5y^2\left(\underset{y\cdot 2y}{2y^2} + 11y + \underset{\substack{1\cdot 12\\ 2\cdot 6\\ 3\cdot 4}}{12}\right)$

Consider all the combinations.

$2y^2 + 11y + 12$	
Possible factors	**Product**
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2 + 25y + 12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2 + 11y + 12$*

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. Remember to include the factor -5^y2.	$-5y^2(y+4)(2y+3)$

Check by multiplying:

$$-5y^2(y+4)(2y+3)$$
$$-5y^2(2y^2+8y+3y+12)$$
$$-10y^4-55y^3-60y^2\checkmark$$

? Try It 4.2.29

Factor $15n^3 - 85n^2 + 100n$ completely using trial and error.

Answer

The complete factorization is $5n(n-4)(3n-5)$.

? Try It 4.2.30

Factor $56q^3 + 320q^2 - 96q$ completely using trial and error.

Answer

The complete factorization is $8q(q+6)(7q-2)$.

Factor Trinomials of the Form $ax^2 + bx + c$ using the "ac" Method

Another way to factor trinomials of the form $ax^2 + bx + c$ is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 4.2.31

Factor using the “ac” method: $6x^2+7x+2$.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2+7x+2$
Step 2. Find the product *ac*.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2+7x+2$
Step 3. Find two numbers *m* and *n* that: Multiply to *ac*. $m \cdot n = a \cdot c$ Add to *b*. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3+4=7$	
Step 4. Split the middle term using *m* and *n*. ax^2+bx+c $ax^2 + \overbrace{mx+nx}^{bx} + c$	Rewrite $7x$ as $3x+4x$. It would also give the same result if we used $4x+3x$. Notice that $6x^2+3x+4x+2$ is equal to $6x^2+7x+2$. We just split the middle term to get a more useful form.	$6x^2+7x+2$ $\underbrace{6x^2+3x}+\underbrace{4x+2}$
Step 5. Factor by grouping.		$3x(2x+1)+2(2x+1)$ $(2x+1)(3x+2)$
Step 6. Check by multiplying the factors.		$(2x+1)(3x+2)$ $6x^2+4x+3x+2$ $6x^2+7x+2$ ✓

Try It 4.2.32

Factor using the “ac” method: $6x^2+13x+2$.

Answer

$(x+2)(6x+1)$

Try It 4.2.33

Factor using the “ac” method: $4y^2+8y+3$.

Answer

$(2y+1)(2y+3)$

The “ac” method is summarized here.

Factor Trinomials of the Form $ax^2 + bx + c$ Using the "ac" Method

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:

Multiply to ac	$mn = ac$
Add to b	$m + n = b$
	$ax^2 + bx + c$

4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!

Example 4.2.34

Factor using the "“ac” method: $10y^2 - 55y + 70$.

Solution

Is there a greatest common factor?		
Yes. The GCF is 5.		$10y^2 - 55y + 70$
Factor it.		$5(2y^2 - 11y + 14)$
The trinomial inside the parentheses has a leading coefficient that is not 1.		$5(\overset{ax^2}{2y^2} \overset{+\ bx}{- 11y} \overset{+\ c}{+ 14})$
Find the product ac.	$ac = 28$	
Find two numbers that multiply to ac	$(-4)(-7) = 28$	
and add to b.	$-4(-7) = -11$	
Split the middle term.		$5(2y^2 - 11y + 14)$
		$5(\underbrace{2y^2 - 7y}\ \underbrace{- 4y + 14})$
Factor the trinomial by grouping.		
		$5(y - 2)(2y - 7)$
Check by multiplying all three factors. $$5(y-2)(2y-7)$$ $$5(2y^2 - 7y - 4y + 14)$$ $$5(2y^2 - 11y + 14)$$ $$10y^2 - 55y + 70 \checkmark$$		

Try It 4.2.35

Factor using the “ac” method: $16x^2 - 32x + 12$.

Answer

$4(2x - 3)(2x - 1)$

Try It 4.2.36

Factor using the "ac" method: $18w^2-39w+18$.

Answer

$3(3w-2)(2w-3)$

Factor Using Substitution (optional)

Sometimes a trinomial does not appear to be in the ax^2+bx+c form. However, we can often make a thoughtful substitution that will allow us to make it fit the ax^2+bx+c form. This is called **factoring by substitution**. It is standard to use u for the substitution.

In the ax^2+bx+c, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Example 4.2.37

Factor by substitution: x^4-4x^2-5.

Solution

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x^2)^2=x^4$). If we let $u=x^2$, we can put our trinomial in the ax^2+bx+c form we need to factor it.

	x^4-4x^2-5
Rewrite the trinomial to prepare for the substitution.	$(x^2)^2-4(x^2)-5$
Let $u=x^2$ and substitute.	u^2-4u-5
Factor the trinomial.	$(u+1)(u-5)$
Replace u with x^2.	$(x^2+1)(x^2-5)$
Check: $(x^2+1)(x^2-5)$ $x^4-5x^2+x^2-5$ x^4-4x^2-5✓	

Try It 4.2.38

Factor by substitution: h^4+4h^2-12.

Answer

$(h^2-2)(h^2+6)$

Try It 4.2.39

Factor by substitution: y^4-y^2-20.

Answer

$(y^2+4)(y^2-5)$

Sometimes the expression to be substituted is not a monomial.

Example 4.2.40

Factor by substitution: $(x-2)^2+7(x-2)+12$.

Solution

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$u^2+7u+12$
Factor the trinomial.	$(u+3)(u+4)$
Replace u with $x-2$.	$((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

Try It 4.2.41

Factor by substitution: $(x-5)^2+6(x-5)+8$.

Answer

$(x-3)(x-1)$

Try It 4.2.42

Factor by substitution: $(y-4)^2+8(y-4)+15$.

Answer

$(y-1)(y+1)$

Key Concepts

- **How to factor trinomials of the form x^2+bx+c .**
  1. Write the factors as two binomials with first terms x. $\quad l)x^2+bx+c$ $\quad (x\quad)(x\quad)$
  2. Find two numbers m and n that
     multiply to $\quad c,\ m\cdot n=c$
     add to $\quad b,\ m+n=b$
  3. Use m and n as the last terms of the factors. $\quad (x+m)(x+n)$
  4. Check by multiplying the factors.
- **Strategy for Factoring Trinomials of the Form x^2+bx+c** : When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

  For trinomials of the form: $x^2+bx+c=(x+m)(x+n)$

  When c is positive, m and n must have the same sign (and this will be the sign of b).

Examples: $x^2+5x+6=(x+2)(x+3)$, $x^2-6x+8=(x-4)(x-2)$

When c is negative, m and n have opposite signs. The larger of m and n will have the sign of b.

Examples: $x^2+x-12=(x+4)(x-3)$, $x^2-2x-15=(x-5)(x+3)$

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

- **How to factor trinomials of the form ax^2+bx+c using trial and error.**
  1. Write the trinomial in descending order of degrees as needed.
  2. Factor any GCF.
  3. Find all the factor pairs of the first term.
  4. Find all the factor pairs of the third term.
  5. Test all the possible combinations of the factors until the correct product is found.
  6. Check by multiplying.
- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**
  1. Factor any GCF.
  2. Find the product ac.
  3. Find two numbers m and n that:
     $$\begin{array}{ll} \text{Multiply to } ac. & m \cdot n = a \cdot c \\ \text{Add to } b. & m+n=b \\ & ax^2+bx+c \end{array}$$
  4. Split the middle term using m and n. $\quad ax^2+mx+nx+c$
  5. Factor by grouping.
  6. Check by multiplying the factors.

Practice Makes Perfect

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

1. $p^2+11p+30$

Answer

$(p+5)(p+6)$

2. $w^2+10w+21$

3. $n^2+19n+48$

Answer

$(n+3)(n+16)$

4. $b^2+14b+48$

5. $a^2+25a+100$

Answer

$(a+5)(a+20)$

6. $u^2+101u+100$

7. $x^2 - 8x + 12$

Answer

$(x-2)(x-6)$

8. $q^2 - 13q + 36$

9. $y^2 - 18y + 45$

Answer

$(y-3)(y-15)$

10. $m^2 - 13m + 30$

11. $x^2 - 8x + 7$

Answer

$(x-1)(x-7)$

12. $y^2 - 5y + 6$

13. $5p - 6 + p^2$

Answer

$(p-1)(p+6)$

14. $6n - 7 + n^2$

15. $8 - 6x + x^2$

Answer

$(x-4)(x-2)$

16. $7x + x^2 + 6$

17. $x^2 - 12 - 11x$

Answer

$(x-12)(x+1)$

18. $-11 - 10x + x^2$

In the following exercises, factor each trinomial of the form $x^2 + bxy + cy^2$.

19. $x^2 - 2xy - 80y^2$

Answer

$(x+8y)(x-10y)$

20. $p^2 - 8pq - 65q^2$

21. $m^2-64mn-65n^2$

Answer

$(m+n)(m-65n)$

22. $p^2-2pq-35q^2$

23. $a^2+5ab-24b^2$

Answer

$(a+8b)(a-3b)$

24. $r^2+3rs-28s^2$

25. $x^2-3xy-14y^2$

Answer

Prime

26. $u^2-8uv-24v^2$

27. $m^2-5mn+30n^2$

Answer

Prime

28. $c^2-7cd+18d^2$

Factor Trinomials of the Form ax^2+bx+c Using Trial and Error

In the following exercises, factor completely using trial and error.

29. p^3-8p^2-20p

Answer

$p(p-10)(p+2)$

30. q^3-5q^2-24q

31. $3m^3-21m^2+30m$

Answer

$3m(m-5)(m-2)$

32. $11n^3-55n^2+44n$

33. $5x^4+10x^3-75x^2$

Answer

$5x^2(x-3)(x+5)$

34. $6y^4+12y^3-48y^2$

35. $2t^2+7t+5$

Answer

$(2t+5)(t+1)$

36. $5y^2+16y+11$

37. $11x^2+34x+3$

Answer

$(11x+1)(x+3)$

38. $7b^2+50b+7$

39. $4w^2-5w+1$

Answer

$(4w-1)(w-1)$

40. $5x^2-17x+6$

41. $4q^2-7q-2$

Answer

$(4q+1)(q-2)$

42. $10y^2-53y-111$

43. $6p^2-19pq+10q^2$

Answer

$(2p-5q)(3p-2q)$

44. $21m^2-29mn+10n^2$

45. $4a^2+17ab-15b^2$

Answer

$(4a-3b)(a+5b)$

46. $6u^2+5uv-14v^2$

47. $-16x^2-32x-16$

Answer

$-16(x+1)(x+1)$

48. $-81a^2+153a+18$

49. $-30q^3-140q^2-80q$

Answer

$-10q(3q+2)(q+4)$

50. $-5y^3-30y^2+35y$

Factor Trinomials of the Form ax^2+bx+c using the 'ac' Method

In the following exercises, factor using the 'ac' method.

51. $5n^2+21n+4$

Answer

$(5n+1)(n+4)$

52. $8w^2+25w+3$

53. $4k^2-16k+15$

Answer

$(2k-3)(2k-5)$

54. $5s^2-9s+4$

55. $6y^2+y-15$

Answer

$(3y+5)(2y-3)$

56. $6p^2+p-22$

57. $2n^2-27n-45$

Answer

$(2n+3)(n-15)$

58. $12z^2-41z-11$

59. $60y^2+290y-50$

Answer

$10(6y-1)(y+5)$

60. $6u^2-46u-16$

61. $48z^3-102z^2-45z$

Answer

$3z(8z+3)(2z-5)$

62. $90n^3+42n^2-216n$

63. $16s^2+40s+24$

Answer

$8(2s+3)(s+1)$

64. $24p^2+160p+96$

65. $48y^2+12y-36$

Answer

$12(4y-3)(y+1)$

66. $30x^2+105x-60$

Factor Using Substitution

In the following exercises, factor using substitution.

67. x^4-x^2-12

Answer

$(x^2+3)(x^2-4)$

68. x^4+2x^2-8

69. x^4-3x^2-28

Answer

$(x^2-7)(x^2+4)$

70. x^4-13x^2-30

71. $(x-3)^2-5(x-3)-36$

Answer

$(x-12)(x+1)$

72. $(x-2)^2-3(x-2)-54$

73. $(3y-2)^2-(3y-2)-2$

Answer

$(3y-4)(3y-1)$

74. $(5y-1)^2-3(5y-1)-18$

Mixed Practice

In the following exercises, factor each expression using any method.

75. $u^2-12u+36$

Answer

$(u-6)(u-6)$

76. $x^2-14x-32$

77. $r^2-20rs+64s^2$

Answer

$(r-4s)(r-16s)$

78. $q^2-29qr-96r^2$

79. $12y^2-29y+14$

Answer

$(4y-7)(3y-2)$

80. $12x^2+36y-24z$

81. $6n^2+5n-4$

Answer

$(2n-1)(3n+4)$

82. $3q^2+6q+2$

83. $13z^2+39z-26$

Answer

$13(z^2+3z-2)$

84. $5r^2+25r+30$

85. $3p^2+21p$

Answer

$3p(p+7)$

86. $7x^2-21x$

87. $6r^2+30r+36$

Answer

$6(r+2)(r+3)$

88. $18m^2+15m+3$

89. $24n^2+20n+4$

Answer

$4(2n+1)(3n+1)$

90. $4a^2+5a+2$

91. x^4-4x^2-12

Answer

$(x^2+2)(x^2-6)$

92. x^4-7x^2-8

93. $(x+3)^2-9(x+3)-36$

Answer

$(x-9)(x+6)$

94. $(x+2)^2-25(x+2)-54$

Writing Exercises

95. Many trinomials of the form x^2+bx+c factor into the product of two binomials $(x+m)(x+n)$. Explain how you find the values of m and n.

Answer

Answers will vary.

96. Tommy factored x^2-x-20 as $(x+5)(x-4)$. Sara factored it as $(x+4)(x-5)$. Ernesto factored it as $(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.

97. List, in order, all the steps you take when using the “ac” method to factor a trinomial of the form ax^2+bx+c.

Answer

Answers will vary.

98. How is the “ac” method similar to the “undo FOIL” method? How is it different?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
factor trinomials of the form $x^2 + bx + c$			
factor trinomials of the form $ax^2 + bx + c$ using trial and error.			
factor trinomials of the form $ax^2 + bx + c$ with using the "ac" method.			
factor using substitution.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

4.3: Factor Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $(3x^2)^3$.

2. Multiply: $(m+4)^2$.

3. Multiply: $(x-3)(x+3)$.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If we recognize these kinds of polynomials, we can use the special products patterns to factor them much more quickly.

Factor Differences of Squares

One special product we are familiar with is the Product of Conjugates pattern. We use this to multiply two binomials that were conjugates. Here's an example:

$$\begin{array}{c} (a\;-\;b)\;(a\;+\;b) \\ (3x-4)(3x+4) \\ \\ (a)^2\;-\;(b)^2 \\ (3x)^2-(4)^2 \\ \\ 9x^2-16 \end{array}$$

A difference of squares factors to a product of conjugates (in this context, $a-b$ is said to be conjugate to $a+b$, and vice versa.)

Difference of Squares Pattern

If a and b are real numbers,

$$a^2-b^2=(a-b)(a+b) \qquad \underbrace{a^2}_{\text{squares}} \overset{\text{difference}}{-} \underbrace{b^2}_{\text{squares}} = \underbrace{(a-b)(a+b)}_{\text{conjugates}}$$

Remember, "difference" refers to subtraction. So, to use this pattern we must make sure we have a binomial in which two squares are being subtracted.

Example 4.3.1

Factor $64y^2-1$.

Solution

Step 1. Does the binomial fit the pattern?		$64y^2 - 1$
• Is this a difference?	Yes	$64y^2 - 1$
• Are the first and last terms perfect squares?	Yes	
Step 2. Write them as squares.	Write them as x^2 and 2^2.	$\overset{a^2}{(8y)^2} - \overset{b^2}{1^2}$
Step 3. Write the product of conjugates.		$\overset{(a\ -\ b)\ (a\ +\ b)}{(8y-1)(8y+1)}$
Step 4. Check.		$(8y-1)(8y+1)$ $64y^2 - 1$ ✓

? Try It 4.3.2

Factor $121m^2 - 1$.

Answer

$(11m-1)(11m+1)$

? Try It 4.3.3

Factor $81y^2 - 1$.

Answer

$(9y-1)(9y+1)$

Factor Difference of Squares

Step 1.	Does the binomial fit the pattern?	$a^2 - b^2$
	Is this a difference?	____$-$____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression $a^2 + b^2$ can not be factored any further!

The next example shows variables in both terms.

? Example 4.3.4

Factor $144x^2 - 49y^2$.

Solution

$$144x^2 - 49y^2$$

Is this a difference of squares? Yes.	$(12x)^2 - (7y)^2$
Factor as the product of conjugates.	$(12x - 7y)(12x + 7y)$
Check by multiplying.	$(12x - 7y)(12x + 7y)$

Check by multiplying.

$$\begin{gathered}(12x - 7y)(12x + 7y) \\ 144x^2 - 49y^2 \checkmark\end{gathered}$$

? Try It 4.3.5

Factor $196m^2 - 25n^2$.

Answer

$(16m - 5n)(16m + 5n)$

? Try It 4.3.6

Factor $121p^2 - 9q^2$.

Answer

$(11p - 3q)(11p + 3q)$

As always, we should look for a common factor first whenever we have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and we won't recognize the perfect squares until we factor the GCF.

Also, to completely factor the binomial in the next example, we'll factor a difference of squares twice!

? Example 4.3.7

Factor $48x^4y^2 - 243y^2$.

Solution

$$48x^4y^2 - 243y^2$$

Is there a GCF? Yes, $3y^2$—factor it out!	$3y^2(16x^4 - 81)$
Is the binomial a difference of squares? Yes.	$3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the first binomial is also a difference of squares!	$3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$

The last factor, the sum of squares, cannot be factored.

Check by multiplying:

$$3y^2(2x - 3)(2x + 3)(4x^2 + 9)$$

$$\begin{gathered}3y^2(4x^2 - 9)(4x^2 + 9) \\ 3y^2(16x^4 - 81) \\ 48x^4y^2 - 243y^2 \checkmark\end{gathered}$$

Try It 4.3.8

Factor $2x^4y^2 - 32y^2$.

Answer

$2y^2(x-2)(x+2)(x^2+4)$

Try It 4.3.9

Factor $7a^4c^2 - 7b^4c^2$.

Answer

$7c^2(a-b)(a+b)(a^2+b^2)$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

Example 4.3.10

Factor $x^2 - 6x + 9 - y^2$.

Solution

Notice that the first three terms form a perfect square trinomial.

Factor by grouping the first three terms.	
Use the perfect square trinomial pattern.	$(x-3)^2 - y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	$\overset{a^2}{(x-3)^2}\overset{-b^2}{-y^2}$
Factor as the product of conjugates.	$\overset{(a}{((x-3)}\overset{-b)}{-y)}\overset{(x}{((x-3)}\overset{+b)}{+y)}$
	$(x-3-y)(x-3+y)$

we may want to rewrite the solution as $(x-y-3)(x+y-3)$.

Try It 4.3.11

Factor $x^2 - 10x + 25 - y^2$.

Answer

$(x-5-y)(x-5+y)$

Try It 4.3.12

Factor $x^2 + 6x + 9 - 4y^2$.

Answer

$(x+3-2y)(x+3+2y)$

Factor Perfect Square Trinomials (optional discussion)

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\left(\overset{a}{3x} \overset{+}{+} \overset{b}{4}\right)^2$$

$$\overset{a^2}{(3x)^2} + \overset{2\cdot a\cdot b}{2(3x\cdot 4)} + \overset{b^2}{4^2}$$

$$9x^2 + 24x + 16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, we will start with a perfect square trinomial and factor it into its as many factors as possible. We could factor this **trinomial** using the methods described in the last section, since it is of the form ax^2+bx+c. But if we recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, we will save ourselves a lot of work. Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomial Pattern

If a and b are real numbers

$$a^2+2ab+b^2=(a+b)^2,$$
$$a^2-2ab+b^2=(a-b)^2.$$

To make use of this pattern, we have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, we can easily write the factors.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2-2ab+b^2$, which factors to $(a-b)^2$.

The steps are summarized here.

Factor Perfect Square Trinomials

Step 1.	Does the trinomial fit the pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \qquad (b)^2$	$(a)^2 \qquad (b)^2$
	Check the middle term. Is it $2ab$?	$\searrow 2\cdot a\cdot b \swarrow$	$\searrow 2\cdot a\cdot b \swarrow$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, we will recognize a perfect square trinomial.

The above is for consideration, but we will not give any examples of this here.

Key Concepts

- **Difference of Squares Pattern:** If a, b are real numbers,

$$a^2-b^2=(a-b)(a+b) \qquad \underbrace{a^2}_{\text{squares}} \overset{\text{difference}}{-} b^2 = \underbrace{(a-b)(a+b)}_{\text{conjugates}}$$

- **How to factor differences of squares**

Step 1.	Does the binomial fit the pattern?	$a^2 - b^2$
	Is this a difference?	$\_\_\_\_ - \_\_\_\_$
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2 - (b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

- **Perfect Square Trinomials Pattern:** If a and b are real numbers,

$$a^2 + 2ab + b^2 = (a+b)^2$$
$$a^2 - 2ab + b^2 = (a-b)^2$$

- **How to factor perfect square trinomials**

Step 1.	Does the trinomial fit the pattern?	$a^2 + 2ab + b^2$	$a^2 - 2ab + b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
	Check the middle term. Is it $2ab$?	$2 \cdot a \cdot b$	$2 \cdot a \cdot b$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

Practice Makes Perfect

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

1. $25v^2 - 1$

Answer

$(5v-1)(5v+1)$

2. $169q^2 - 1$

3. $4 - 49x^2$

Answer

$(7x-2)(7x+2)$

4. $121 - 25s^2$

5. $6p^2q^2 - 54p^2$

Answer

$6p^2(q-3)(q+3)$

6. $98r^3 - 72r$

7. $24p^2 + 54$

Answer

$6(4p^2 + 9)$

8. $20b^2 + 140$

9. $121x^2 - 144y^2$

Answer

$(11x - 12y)(11x + 12y)$

10. $49x^2 - 81y^2$

11. $169c^2 - 36d^2$

Answer

$(13c - 6d)(13c + 6d)$

12. $36p^2 - 49q^2$

13. $16z^4 - 1$

Answer

$(2z - 1)(2z + 1)(4z^2 + 1)$

14. $m^4 - n^4$

15. $162a^4b^2 - 32b^2$

Answer

$2b^2(3a - 2)(3a + 2)(9a^2 + 4)$

16. $48m^4n^2 - 243n^2$

17. $x^2 - 16x + 64 - y^2$

Answer

$(x - 8 - y)(x - 8 + y)$

18. $p^2 + 14p + 49 - q^2$

19. $a^2 + 6a + 9 - 9b^2$

Answer

$(a + 3 - 3b)(a + 3 + 3b)$

20. $m^2 - 6m + 9 - 16n^2$

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

21. $16y^2 + 24y + 9$

Answer

$(4y+3)^2$

22. $25v^2+20v+4$

23. $36s^2+84s+49$

Answer

$(6s+7)^2$

24. $49s^2+154s+121$

25. $100x^2-20x+1$

Answer

$(10x-1)^2$

26. $64z^2-16z+1$

27. $25n^2-120n+144$

Answer

$(5n-12)^2$

28. $4p^2-52p+169$

29. $49x^2+28xy+4y^2$

Answer

$(7x+2y)^2$

30. $25r^2+60rs+36s^2$

31. $100y^2-52y+1$

Answer

$(50y-1)(2y-1)$

32. $64m^2-34m+1$

33. $10jk^2+80jk+160j$

Answer

$10j(k+4)^2$

34. $64x^2y-96xy+36y$

35. $75u^4-30u^3v+3u^2v^2$

Answer

$3u^2(5u-v)^2$

36. $90p^4+300p^4q+250p^2q^2$

Mixed Practice

In the following exercises, factor completely.

37. $64a^2-25$

Answer

$(8a-5)(8a+5)$

38. $121x^2-144$

39. $27q^2-3$

Answer

$3(3q-1)(3q+1)$

40. $4p^2-100$

41. $16x^2-72x+81$

Answer

$(4x-9)^2$

42. $36y^2+12y+1$

43. $8p^2+2$

Answer

$2(4p^2+1)$

44. $81x^2+169$

45. $45n^2+60n+20$

Answer

$5(3n+2)^2$

46. $x^2-10x+25-y^2$

Answer

$(x+y-5)(x-y-5)$

47. $x^2+12x+36-y^2$

Writing Exercises

48. Why was it important to practice using the binomial squares pattern in the chapter on multiplying polynomials?

Answer

Answers will vary.

49. How do you recognize the binomial squares pattern?

50. Explain why $n^2+25 \neq (n+5)^2$. Use algebra, words, or pictures.

Answer

Answers will vary.

51. Maribel factored $y^2-30y+81$ as $(y-9)^2$. Was she right or wrong? How do you know?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

4.4: General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

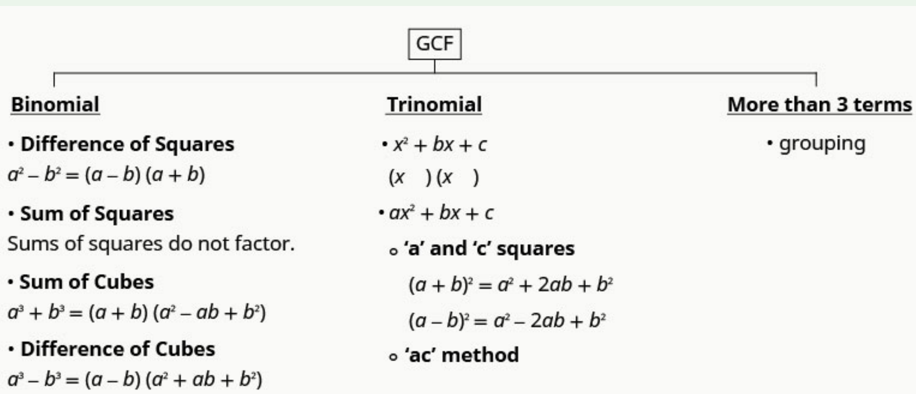
- Recognize and use the appropriate method to factor a polynomial completely

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials

This chart shows the general strategies for factoring polynomials. It shows ways to find GCF of binomials, trinomials and polynomials with more than 3 terms. For binomials, we have difference of squares: a squared minus b squared equals a minus b, a plus b; sum of squares do not factor; sub of cubes: a cubed plus b cubed equals open parentheses a plus b close parentheses open parentheses a squared minus ab plus b squared close parentheses; difference of cubes: a cubed minus b cubed equals open parentheses a minus b close parentheses open parentheses a squared plus ab plus b squared close parentheses. For trinomials, we have x squared plus bx plus c where we put x as a term in each factor and we have a squared plus bx plus c. Here, if a and c are squares, we have a plus b whole squared equals a squared plus $2ab$ plus b squared and a minus b whole squared equals a squared minus $2ab$ plus b squared. If a and c are not squares, we use the "ac" method. For polynomials with more than 3 terms, we use grouping.



General Strategy for Factoring Polynomials

1. Is there a greatest common factor?
   Factor it out.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   If it is a binomial:
   - Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
   - Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.

   If it is a trinomial:
   - Is it of the form $x^2 + bx + c$? Undo FOIL.
   - Is it of the form $ax^2 + bx + c$?
     If a and c are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.

If it has more than three terms:

- Use the grouping method.

3. Check.
   Is it factored completely?
   Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are **prime**!

Example 4.4.1

Factor completely: $7x^3-21x^2-70x$.

Solution

$7x^3-21x^2-70x$	
Is there a GCF? Yes, $7x$.	
Factor out the GCF.	$7x(x^2-3x-10)$
In the parentheses, is it a binomial, trinomial, or are there more terms?	
Trinomial with leading coefficient 1.	
"Undo" FOIL.	$7x(x\quad)(x\quad)$
	$7x(x+2)(x-5)$
Is the expression factored completely? Yes.	
Neither binomial can be factored.	
Check your answer.	
Multiply.	

$$7x(x+2)(x-5)$$
$$7x(x^2-5x+2x-10)$$
$$7x(x^2-3x-10)$$
$$7x^3-21x^2-70x\checkmark$$

Try It 4.4.2

Factor completely: $8y^3+16y^2-24y$.

Answer

$8y(y-1)(y+3)$

Try It 4.4.3

Factor completely: $5y^3-15y^2-270y$.

Answer

$5y(y-9)(y+6)$

Be careful when you are asked to factor a binomial as there are several options!

Example 4.4.4

Factor completely: $24y^2 - 150$

Solution

	$24y^2 - 150$
Is there a GCF? Yes, 6.	
Factor out the GCF.	$6(4y^2 - 25)$
In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial.	
Is it a sum? No.	
Is it a difference? Of squares or cubes? Yes, squares.	$6((2y)^2 - (5)^2)$
Write as a product of conjugates.	$6(2y - 5)(2y + 5)$

Is the expression factored completely?
Neither binomial can be factored.

Check:

Multiply.

$6(2y - 5)(2y + 5)$

$6(4y^2 - 25)$
$24y^2 - 150 \checkmark$

Try It 4.4.5

Factor completely: $16x^3 - 36x$.

Answer

$4x(2x - 3)(2x + 3)$

Try It 4.4.6

Factor completely: $27y^2 - 48$.

Answer

$3(3y - 4)(3y + 4)$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

Example 4.4.7

Factor completely: $4a^2 - 12ab + 9b^2$.

Solution

$4a^2 - 12ab + 9b^2$

Is there a GCF? No.

Is it a binomial, trinomial, or are there more terms?

Trinomial with $a \neq 1$. But the first term is a perfect square.

Is the last term a perfect square? Yes. $(2a)^2 - 12ab + (3b)^2$

Does it fit the pattern, $a^2 - 2ab + b^2$? Yes. $(2a)^2 - \underbrace{12ab}_{-2(2a)(3b)} + (3b)^2$

Write it as a square. $(2a - 3b)^2$

Is the expression factored completely? Yes.

The binomial cannot be factored.

Check your answer.

Multiply.

$$(2a - 3b)^2$$
$$(2a)^2 - 2 \cdot 2a \cdot 3b + (3b)^2$$
$$4a^2 - 12ab + 9b^2 \checkmark$$

Try It 4.4.8

Factor completely: $4x^2 + 20xy + 25y^2$.

Answer

$(2x + 5y)^2$

Try It 4.4.9

Factor completely: $9x^2 - 24xy + 16y^2$.

Answer

$(3x - 4y)^2$

Remember, sums of squares do not factor, but sums of cubes do!

Example 4.4.10

Factor completely $12x^3y^2 + 75xy^2$.

Solution

	$12x^3y^2 + 75xy^2$
Is there a GCF? Yes, $3xy^2$.	
Factor out the GCF.	$3xy^2(4x^2+25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial.	
Is it a sum? Of squares? Yes.	Sums of squares are prime.

Is the expression factored completely? Yes.

Check:

Multiply.

$$3xy^2(4x^2+25)$$
$$12x^3y^2 + 75xy^2 \checkmark$$

? Try It 4.4.11

Factor completely: $50x^3y + 72xy$.

Answer

$2xy(25x^2+36)$

? Try It 4.4.12

Factor completely: $27xy^3 + 48xy$.

Answer

$3xy(9y^2+16)$

When using the sum or difference of cubes pattern, being careful with the signs.

? Example 4.4.13

Factor completely: $24x^3 + 81y^3$.

Solution

Is there a GCF? Yes, 3.	$24x^3 + 81y^3$
Factor it out.	$3(8x^3 + 27y^3)$
In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial.	
Is it a sum or difference? Sum.	
Of squares or cubes? Sum of cubes.	$3\left(\overset{a^3}{(2x)^3} \overset{+}{+} \overset{b^3}{(3y)^3}\right)$
Write it using the sum of cubes pattern.	$3\left(\overset{a}{2x} \overset{+}{+} \overset{b}{3y}\right)\left(\overset{a^2}{(2x)^2} \overset{-}{-} \overset{ab}{2x \cdot 3y} \overset{+}{+} \overset{b^2}{(3y)^2}\right)$
Is the expression factored completely? Yes.	$3(2x+3y)(4x^2-6xy+9y^2)$

Check by multiplying.	

Try It 4.4.14

Factor completely: $250m^3+432n^3$.

Answer

$2(5m+6n)(25m^2-30mn+36n^2)$

Try It 4.4.15

Factor completely: $2p^3+54q^3$.

Answer

$2(p+3q)(p^2-3pq+9q^2)$

Example 4.4.16

Factor completely: $3x^5y-48xy$.

Solution

	$3x^5y-48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4-16)$
Is the binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$3xy\left((x^2)^2-(4)2\right)$
Factor it as a product of conjugates	$3xy(x^2-4)(x^2+4)$
The first binomial is again a difference of squares.	$3xy\left((x)^2-(2)^2\right)(x^2+4)$
Factor it as a product of conjugates.	$3xy(x-2)(x+2)(x^2+4)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	
$3xy(x-2)(x+2)(x^2+4)$	
$3xy(x^2-4)(x^2+4)$	
$3xy(x^4-16)$	
$3x^5y-48xy\checkmark$	

Try It 4.4.17

Factor completely: $4a^5b-64ab$.

Answer

$4ab(a^2+4)(a-2)(a+2)$

Try It 4.4.18

Factor completely: $7xy^5-7xy$.

Answer

$7xy(y^2+1)(y-1)(y+1)$

Example 4.4.19

Factor completely: $4x^2+8bx-4ax-8ab$.

Solution

	$4x^2+8bx-4ax-8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x^2+2bx-ax-2ab)$
There are four terms. Use grouping.	$4[x(x+2b)-a(x+2b)]4(x+2b)(x-a)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	

$$4(x+2b)(x-a)$$
$$4(x^2-ax+2bx-2ab)$$
$$4x^2+8bx-4ax-8ab\checkmark$$

Try It 4.4.20

Factor completely: $6x^2-12xc+6bx-12bc$.

Answer

$6(x+b)(x-2c)$

Try It 4.4.21

Factor completely: $16x^2+24xy-4x-6y$.

Answer

$2(4x-1)(2x+3y)$

Taking out the complete GCF in the first step will always make your work easier.

Example 4.4.22

Factor completely: $40x^2y+44xy-24y$.

Solution

	$40x^2y+44xy-24y$
Is there a GCF? Factor out the GCF, $4y$.	$4y(10x^2+11x-6)$
Factor the trinomial with $a\neq 1$.	$4y(10x^2+11x-6)$
	$4y(5x-2)(2x+3)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	

$$4y(5x-2)(2x+3)$$
$$4y(10x^2+11x-6)$$
$$40x^2y+44xy-24y\checkmark$$

Try It 4.4.23

Factor completely: $4p^2q - 16pq + 12q$.

Answer

$4q(p-3)(p-1)$

Try It 4.4.24

Factor completely: $6pq^2 - 9pq - 6p$.

Answer

$3p(2q+1)(q-2)$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Example 4.4.25

Factor completely: $9x^2 - 12xy + 4y^2 - 49$.

Solution

	$9x^2 - 12xy + 4y^2 - 49$
Is there a GCF? No.	
With more than 3 terms, use grouping. Last 2 terms have no GCF. Try grouping first 3 terms.	$9x^2 - 12xy + 4y^2 - 49$
Factor the trinomial with $a \neq 1$. But the first term is a perfect square.	
Is the last term of the trinomial a perfect square? Yes.	$(3x)^2 - 12xy + (2y)^2 - 49$
Does the trinomial fit the pattern, $a^2 - 2ab + b^2$? Yes.	$(3x)^2 - 12xy + (2y)^2 - 49$ $\searrow -2(3x)(2y)) \swarrow$
Write the trinomial as a square.	$(3x-2y)^2 - 49$
Is this binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$(3x-2y)^2 - 72$
Write it as a product of conjugates.	$((3x-2y)-7)((3x-2y)+7)$ $(3x-2y-7)(3x-2y+7)$
Is the expression factored completely? Yes.	
Check your answer.	
Multiply.	

$$(3x-2y-7)(3x-2y+7)$$
$$9x^2 - 6xy - 21x - 6xy + 4y^2 + 14y + 21x - 14y - 49$$
$$9x^2 - 12xy + 4y^2 - 49 \checkmark$$

Try It 4.4.26

Factor completely: $4x^2 - 12xy + 9y^2 - 25$.

Answer

$(2x-3y-5)(2x-3y+5)$

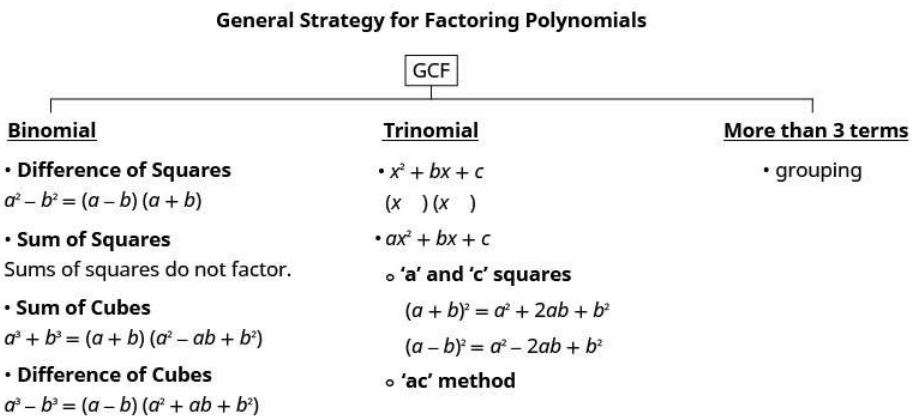
Try It 4.4.27

Factor completely: $16x^2 - 24xy + 9y^2 - 64$.

Answer

$(4x - 3y - 8)(4x - 3y + 8)$

Key Concepts



- **How to use a general strategy for factoring polynomials.**
  1. Is there a greatest common factor?
     Factor it out.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     If it is a binomial:
     Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
     Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.
     If it is a trinomial:
     Is it of the form $x^2 + bx + c$? Undo FOIL.
     Is it of the form $ax^2 + bx + c$?
     If *a* and *c* are squares, check if it fits the trinomial square pattern.
     Use the trial and error or "ac" method.
     If it has more than three terms:
     Use the grouping method.
  3. Check.
     Is it factored completely?
     Do the factors multiply back to the original polynomial?

Practice Makes Perfect

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

1. $2n^2 + 13n - 7$

Answer

$(2n - 1)(n + 7)$

2. $8x^2 - 9x - 3$

3. $a^5 + 9a^3$

Answer

$a^3(a^2 + 9)$

4. $75m^3 + 12m$

5. $121r^2 - s^2$

Answer

$(11r - s)(11r + s)$

6. $49b^2 - 36a^2$

7. $8m^2 - 32$

Answer

$8(m - 2)(m + 2)$

8. $36q^2 - 100$

9. $25w^2 - 60w + 36$

Answer

$(5w - 6)^2$

10. $49b^2 - 112b + 64$

11. $m^2 + 14mn + 49n^2$

Answer

$(m + 7n)^2$

12. $64x^2 + 16xy + y^2$

13. $7b^2 + 7b - 42$

Answer

$7(b + 3)(b - 2)$

14. $30n^2 + 30n + 72$

15. $3x^4y - 81xy$

Answer

$3xy(x - 3)(x^2 + 3x + 9)$

16. $4x^5y-32x^2y$

17. k^4-16

Answer

$(k-2)(k+2)(k^2+4)$

18. m^4-81

19. $5x5y^2-80xy^2$

Answer

$5xy^2(x^2+4)(x+2)(x-2)$

20. $48x^5y^2-243xy^2$

21. $15pq-15p+12q-12$

Answer

$3(5p+4)(q-1)$

22. $12ab-6a+10b-5$

23. $4x^2+40x+84$

Answer

$4(x+3)(x+7)$

24. $5q^2-15q-90$

25. $4u^5v+4u^2v^3$

Answer

$u^2(u+1)(u^2-u+1)$

26. $5m^4n+320mn^4$

27. $4c^2+20cd+81d^2$

Answer

prime

28. $25x^2+35xy+49y^2$

29. $10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

30. $3v^4-768$

31. $36x^2y+15xy-6y$

Answer

$3y(3x+2)(4x-1)$

32. $60x^2y-75xy+30y$

33. $8x^3-27y^3$

Answer

$(2x-3y)(4x^2+6xy+9y^2)$

34. $64x^3+125y^3$

35. y^6-1

Answer

$(y+1)(y-1)(y^2-y+1)$

36. y^6+1

37. $9x^2-6xy+y^2-49$

Answer

$(3x-y+7)(3x-y-7)$

38. $16x^2-24xy+9y^2-64$

39. $(3x+1)^2-6(3x-1)+9$

Answer

$(3x-2)2$

40. $(4x-5)^2-7(4x-5)+12$

Writing Exercises

41. Explain what it mean to factor a polynomial completely.

Answer

Answers will vary.

42. The difference of squares y^4-625 can be factored as $(y^2-25)(y^2+25)$. But it is not completely factored. What more must be done to completely factor.

43. Of all the factoring methods covered in this chapter (GCF, grouping, undo FOIL, ‘ac’ method, special products) which is the easiest for you? Which is the hardest? Explain your answers.

Answer

Answers will vary.

44. Create three factoring problems that would be good test questions to measure your knowledge of factoring. Show the solutions.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize and use the appropriate method to factor a polynomial completely.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

4.5: Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Use the Zero Product Property
- Solve quadratic equations by factoring
- Solve polynomial equations
- Solve applications modeled by polynomial equations

Be Prepared

Before you get started, take this readiness quiz.

1. Solve: $5y-3=0$.
2. Factor completely: n^3-9n^2-22n .
3. If $f(x)=8x-16$, find $f(3)$ and solve $f(x)=0$.

Polynomial Equations

We have spent considerable time learning how to factor polynomials. We will now look at polynomial equations and solve them using factoring, if possible.

Definition 4.5.1

1. A **polynomial equation** is an equation that contains a polynomial expression.
2. The **solution of a polynomial equation** is an assignment of the values to the variables that make the equation true; so substituting the values for the variables gives a true statement.
3. The **degree of the polynomial equation** is the highest degree among all the terms appearing in the equation.

We have already solved polynomial equations of **degree one**. Polynomial equations of degree one are linear equations of the form $ax+b=c$.

We are now going to solve polynomial equations of **degree two**. A polynomial equation of degree two is called a **quadratic equation**. Listed below are some examples of quadratic equations:

$$x^2+5x+6=0, \qquad 3y^2=-4y+10, \qquad 64u^2-81=0, \qquad n(n+1)=42.$$

The last equation doesn't appear to have the variable squared, but when we simplify the expression on the left we will get n^2+n .

The general form of a quadratic equation is $ax^2+bx+c=0$, with $a\neq 0$. (If $a=0$, then $0\cdot x^2=0$ and we are left with no quadratic term.)

Definition 4.5.2

An equation of the form $ax^2+bx+c=0$ is called a **quadratic equation** where $a, b,$ and c are real numbers and $a\neq 0$.

Polynomial vs Polynomial Equation

A polynomial and a polynomial equation are not the same. Here are some examples.

Polynomial(s)	Polynomial Equation
$2x+1$, 0	$2x+1=0$
x^2, $x-2$	$x^2=x-2$
$3x^2+5x-1$, 0	$3x^2+5x-1=0$

Note that $x^2+5x=x(x+5)$ is not a polynomial equation that we aim to solve. This is an identity of polynomials that was developed in the process of factoring the GCF out.

It does not make sense to "solve a polynomial." We can only solve equations. For example, we cannot solve $2x+1$ as there is no statement to assess. Whereas $2x+1=0$ is either true or false for a particular value of x. Below we check whether or not some values of x are solutions to the equation $2x+1=0$.

x	True or False	Solution or Not a Solution
5	$2\cdot 5+1=0$ is false	Not a Solution
0	$2\cdot 0+1=0$ is false	Not a Solution
$-\frac{1}{2}$	$2\cdot\left(-\frac{1}{2}\right)+1=0$ is true	Solution

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Use the Zero Product Property

We will first solve some quadratic equations by using the **Zero Product Property**. The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero. The only way to get a product equal to zero is to multiply by zero itself.

Zero Product Property

If $ab=0$, then $a=0$ or $b=0$.

We will now use the Zero Product Property to solve a **quadratic equation**.

Example 4.5.3

Solve $(5n-2)(6n-1)=0$.

Solution

Step 1. Set each factor equal to zero.	The product equals zero, so at least one factor must equal zero.	$(5n-2)(6n-1)=0$ $5n-2=0$ or $6n-1=0$
Step 2. Solve the linear equations.	Solve each equation.	$n=\frac{2}{5}$ $\quad n=\frac{1}{6}$

Step 3. Check.	Substitute each solution separately into the original equation.	$n=\frac{2}{5}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{2}{5}-2\right)\left(6\cdot\frac{2}{5}-1\right)\stackrel{?}{=}0$ $(2-2)\left(\frac{12}{5}-1\right)\stackrel{?}{=}0$ $0\cdot\frac{7}{5}\stackrel{?}{=}0$ $0=0\checkmark$ $n=\frac{1}{6}$ $(5n-2)(6n-1)=0$ $\left(5\cdot\frac{1}{6}-2\right)\left(6\cdot\frac{1}{6}-1\right)\stackrel{?}{=}0$ $\left(\frac{5}{6}-\frac{12}{6}\right)(1-1)\stackrel{?}{=}0$ $\left(-\frac{7}{6}\right)(0)\stackrel{?}{=}0$ $0=0\checkmark$

Try It 4.5.4

Solve $(3m-2)(2m+1)=0$.

Answer

$$m=\frac{2}{3},\ m=-\frac{1}{2}$$

Try It 4.5.5

Solve $(4p+3)(4p-3)=0$.

Answer

$$p=-\frac{3}{4},\ p=\frac{3}{4}$$

Use the Zero Product Property

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Optional: Check answer to see if a mistake has been made.

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So be sure to start with the quadratic equation in **standard form**, $ax^2+bx+c=0$. Then factor the expression on the left.

Example 4.5.6

Solve $2y^2=13y+45$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.	Write the equation in standard form.	$2y^2 = 13y + 45$ $2y^2 - 13y - 45 = 0$
Step 2. Factor the quadratic expression.	Factor $2y^2 - 13y + 45$ $(2y + 5)(y - 9)$	$(2y + 5)(y - 9) = 0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y + 5 = 0 \quad y - 9 = 0$
Step 4. Solve the linear equations.		$y = -\frac{5}{2} \qquad y = 9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y = -\frac{5}{2}$ $2y^2 = 13y + 45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right) + 45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right) + \frac{90}{2}$ $\frac{25}{2} = \frac{25}{2}$ ✓ $y = 9$ $2y^2 = 13y + 45$ $2(9)^2 \stackrel{?}{=} 13(9) + 45$ $2(81) \stackrel{?}{=} 117 + 45$ $162 = 162$ ✓

? Try It 4.5.7

Solve $3c^2 = 10c - 8$.

Answer

$$c = 2,\ c = \frac{4}{3}$$

? Try It 4.5.8

Solve $2d^2 - 5d = 3$.

Answer

$$d = 3,\ d = -\frac{1}{2}$$

Solve a Quadratic Equation by Factoring

1. Write the quadratic equation in standard form, $ax^2 + bx + c = 0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Optional: Check answer by substituting each solution separately into the original equation to see if a mistake has been made.

Before we factor, we must make sure the **quadratic equation** is in **standard form**.

Solving quadratic equations by factoring will make use of all the factoring techniques we have learned in this chapter! Do we recognize the special product pattern in the next example?

Example 4.5.9

Solve $169q^2 = 49$.

Solution

	$169q^2 = 49$
Write the quadratic equation in standard form.	$169q^2 - 49 = 0$
Factor. It is a difference of squares.	$(13q-7)(13q+7) = 0$
Use the Zero Product Property to set each factor to 0.	
	$13q-7=0 \quad 13q+7=0$
Solve each equation.	$13q=7 \quad 13q=-7$
	$q=\frac{7}{13} \quad q=-\frac{7}{13}$
Check the answers.	The check is left as an exercise.

Try It 4.5.10

Solve $25p^2 = 49$.

Answer

$p=\frac{7}{5}, p=-\frac{7}{5}$

Try It 4.5.11

Solve $36x^2 = 121$.

Answer

$x=\frac{11}{6}, x=-\frac{11}{6}$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the **Zero Product Property**, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 4.5.12

Solve $(3x-8)(x-1) = 3x$.

Solution

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
	$3x-2=0 \quad x-4=0$
Use the Zero Product Property to set each factor to 0.	$3x=2 \quad x=4$
Solve each equation.	$x=\frac{2}{3}$
Check the answers.	The check is left as an exercise.

Try It 4.5.13

Solve $(2m+1)(m+3)=12m$.

Answer

$$m=1,\ m=\frac{3}{2}$$

Try It 4.5.14

Solve $(k+1)(k-1)=8$.

Answer

$$k=3,\ k=-3$$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 4.5.15

Solve $3x^2=12x+63$.

Solution

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq0 \quad x=7 \quad x=-3$
Check the answers.	The check is left as an exercise.

Try It 4.5.16

Solve $18a^2-30=-33a$.

Answer

$$a=-\frac{5}{2},a=\frac{2}{3}$$

Try It 4.5.17

Solve $123b=-6-60b^2$.

Answer

$$b=-2,\ b=-\frac{1}{20}$$

The **Zero Product Property** also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

Example 4.5.18

Solve $9m^3+100m=60m^2$.

Solution

	$9m^3+100m=60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3-60m^2+100m=0$
Factor the greatest common factor first.	$m(9m^2-60m+100)=0$
Factor the trinomial.	$m(3m-10)^2=0$
Use the Zero Product Property to set each factor to 0.	$m=0 \quad 3m-10=0$
Solve each equation.	$m=0 \quad m=\frac{10}{3}$
Check your answers.	The check is left to you.

Try It 4.5.19

Solve $8x^3=24x^2-18x$.

Answer

$x=0,\ x=\frac{3}{2}$

Try It 4.5.20

Solve $16y^2=32y^3+2y$.

Answer

$y=0,\ y=14$

Solve Equations with Polynomials

As our study of polynomials continues, it will often be important to know when a polynomial equals another polynomial. Our work with the **Zero Product Property** will be help us find these answers.

Solve Applications Modeled by Polynomial Equations

The problem-solving strategy we used earlier for applications that translate to linear equations will work just as well for applications that translate to polynomial equations. We will copy the problem-solving strategy here so we can use it for reference.

Use a Problem Solving Strategy to Solve Word Problems

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using appropriate algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We will start with a number problem to get practice translating words into a polynomial equation.

Example 4.5.21

The product of two consecutive odd integers is 323. Find the integers.

Solution

Step 1. Read the problem.	
Step 2. Identify what we are looking for.	We are looking for two consecutive integers.
Step 3. Name what we are looking for.	Let $n =$ the first integer.
	$n+2 =$ next consecutive odd integer
Step 4. Translate into an equation. Restate the problem in a sentence.	The product of the two consecutive odd integers is 323.
	$n(n+2) = 323$
Step 5. Solve the equation. $n^2+2n=323$	
Bring all the terms to one side.	$n^2+2n-323=0$
Factor the trinomial.	$(n-17)(n+19)=0$
Use the Zero Product Property.	$n-17=0 \quad n+19=0$
Solve the equations.	$n=17 \quad n=-19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n=17$ then the next odd integer is	If the first integer is $n=-19$ then the next odd integer is
$n+2$	$n+2$
$17+2$	$-19+2$
19	-17
$17, 19$	$-17, -19$

Step 6. Check the answer.

The results are consecutive odd integers

$17,\ 19$ and $-19,\ -17$.

$17 \cdot 19 = 323 \checkmark \qquad -19(-17) = 323 \checkmark$

Both pairs of consecutive integers are solutions.

Step 7. Answer the question — The consecutive integers are $17, 19$ and $-19, -17$.

Try It 4.5.22

The product of two consecutive odd integers is 255. Find the integers.

Answer

$-15, -17$ and $15, 17$

Try It 4.5.23

The product of two consecutive odd integers is 483 Find the integers.

Answer

$-23, -21$ and $21, 23$

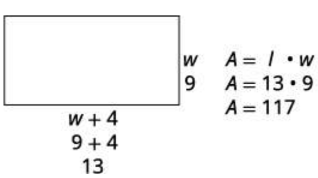
Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

Example 4.5.24

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom.
The length is four feet more than the width.	$w + 4 =$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A = l \cdot w$
Substitute in the variables.	$117 = (w + 4)w$
Step 5. Solve the equation Distribute first.	$117 = w^2 + 4w$
Get zero on one side.	$117 = w^2 + 4w$
Factor the trinomial.	$0 = w^2 + 4w - 117$
Use the Zero Product Property.	$0 = (w^2 + 13)(w - 9)$
Solve each equation.	$0 = w + 13 \quad 0 = w - 9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	$\cancel{w = -13} \quad w = 9$
	$w = 9$ The width is 9 feet.
Find the value of the length.	$w + 4$ $9 + 4$ 13 The length is 13 feet.
Step 6. Check the answer. Does the answer make sense?  Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

Try It 4.5.25

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

The width is 5 feet and length is 6 feet.

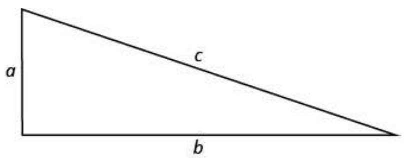
Try It 4.5.26

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

The length of the patio is 12 feet and the width 15 feet.

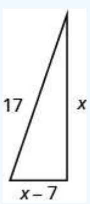
In the next example, we will use the Pythagorean Theorem $(a^2 + b^2 = c^2)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

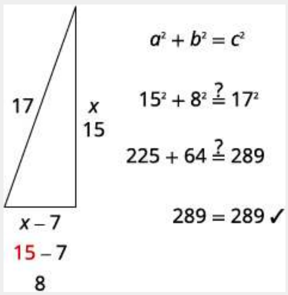
Example 4.5.27

A boat’s sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Solution

Step 1. Read the problem	
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x =$ length of a side of the sail. $x - 7 =$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Substitute in the variables.	$x^2 + (x-7)^2 = 17^2$
Step 5. Solve the equation Simplify.	$x^2 + x^2 - 14x + 49 = 289$
	$2x^2 - 14x + 49 = 289$
It is a quadratic equation, so get zero on one side.	$2x^2 - 14x - 240 = 0$
Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x-15)(x+8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$

Solve.	$2 \neq 0 \quad x=15 \quad x=-8$
Since x is a side of the triangle, $x=-8$ does not make sense.	$2 \neq 0 \quad x=15 \quad \cancel{x=-8}$
Find the length of the other side.	
If the length of one side is then the length of the other side is	$x=15$ $x-7$ $15-7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense?  	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

Try It 4.5.28

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

Answer

5 feet and 12 feet

Try It 4.5.29

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

Access this online resource for additional instruction and practice with quadratic equations.

- Beginning Algebra & Solving Quadratics with the Zero Property

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2+bx+c=0$ is called a quadratic equation.

$$a, b, c \text{ are real numbers and } a \neq 0$$

- **Zero Product Property:** If $a \cdot b=0$, then either $a=0$ or $b=0$ or both.
- **How to use the Zero Product Property**

1. Set each factor equal to zero.
2. Solve the linear equations.
3. Check.

- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **Zero of a Polynomial:** For any polynomial f, if $f(x)=0$, then x is a zero of the polynomial.
- **How to use a problem solving strategy to solve word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
  5. **Solve** the equation using appropriate algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

Practice Makes Perfect

Use the Zero Product Property

In the following exercises, solve.

1. $(3a-10)(2a-7)=0$

Answer

$$a=\frac{10}{3},\ a=\frac{7}{2}$$

2. $(5b+1)(6b+1)=0$

3. $6m(12m-5)=0$

Answer

$$m=0,\ m=\frac{5}{12}$$

4. $2x(6x-3)=0$

5. $(2x-1)^2=0$

Answer

$$x=\frac{1}{2}$$

6. $(3y+5)^2=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

7. $5a^2-26a=24$

Answer

$$a=-\frac{4}{5},\ a=6$$

8. $4b^2+7b=-3$

9. $4m^2=17m-15$

Answer

$$m=\frac{5}{4},\ m=3$$

10. $n^2=5-6n$

11. $7a^2+14a=7a$

Answer

$$a=-1,\ a=0$$

12. $12b^2-15b=-9b$

13. $49m^2=144$

Answer

$$m=\frac{12}{7},\ m=-\frac{12}{7}$$

14. $625=x^2$

15. $16y^2=81$

Answer

$$y=-\frac{9}{4},\ y=\frac{9}{4}$$

16. $64p^2=225$

17. $121n^2=36$

Answer

$$n=-\frac{6}{11},\ n=\frac{6}{11}$$

18. $100y^2=9$

19. $(x+6)(x-3)=-8$

Answer

$$x=2,\ x=-5$$

20. $(p-5)(p+3)=-7$

21. $(2x+1)(x-3)=-4x$

Answer

$x=\frac{3}{2},\ x=-1$

22. $(y-3)(y+2)=4y$

23. $(3x-2)(x+4)=12x$

Answer

$x=\frac{3}{2},\ x=-1$

24. $(2y-3)(3y-1)=8y$

25. $20x^2-60x=-45$

Answer

$x=-\frac{2}{3}$

26. $3y^2-18y=-27$

27. $15x^2-10x=40$

Answer

$x=2,\ x=-\frac{4}{3}$

28. $14y^2-77y=-35$

29. $18x^2-9=-21x$

Answer

$x=-\frac{3}{2},\ x=\frac{1}{3}$

30. $16y^2+12=-32y$

31. $16p^3=24p^2-9p$

Answer

$p=0,\ p=\frac{3}{4}$

32. $m^3-2m^2=-m$

33. $2x^3+72x=24x^2$

Answer

$x=0,\ x=6$

34. $3y^3+48y=24y^2$

35. $36x^3+24x^2=-4x$

Answer

$x=0,\ x=\frac{1}{3}$

36. $2y^3+2y^2=12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

37. For the function, $f(x)=x^2-8x+8$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x=2$ or $x=6$ ⓑ $(2,-4)$ $(6,-4)$

38. For the function, $f(x)=x^2+11x+20$, ⓐ find when $f(x)=-8$ ⓑ Use this information to find two points that lie on the graph of the function.

39. For the function, $f(x)=8x^2-18x+5$, ⓐ find when $f(x)=-4$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x=\frac{3}{2}$ or $x=\frac{3}{4}$
ⓑ $(\frac{3}{2},-4)$ $(\frac{3}{4},-4)$

40. For the function, $f(x)=18x^2+15x-10$, ⓐ find when $f(x)=15$ ⓑ Use this information to find two points that lie on the graph of the function.

In the following exercises, for each function, find: ⓐ the zeros of the function ⓑ the x-intercepts of the graph of the function ⓒ the y-intercept of the graph of the function.

41. $f(x)=9x^2-4$

Answer

ⓐ $x=\frac{2}{3}$ or $x=-\frac{2}{3}$
ⓑ $(\frac{2}{3},0)$, $(-\frac{2}{3},0)$
ⓒ $(0,-4)$

42. $f(x)=25x^2-49$

43. $f(x)=6x^2-7x-5$

Answer

ⓐ $x=\frac{5}{3}$ or $x=-\frac{1}{2}$
ⓑ $(\frac{5}{3},0)$, $(-\frac{1}{2},0)$
ⓒ $(0,-5)$

44. $f(x)=12x^2-11x+2$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

45. The product of two consecutive odd integers is 143. Find the integers.

Answer

$-13,\ -11$ and $11,\ 13$

46. The product of two consecutive odd integers is 195. Find the integers.

47. The product of two consecutive even integers is 168. Find the integers.

Answer

$-14,\ -12$ and $12,\ 14$

48. The product of two consecutive even integers is 288. Find the integers.

49. The area of a rectangular carpet is 28 square feet. The length is three feet more than the width. Find the length and the width of the carpet.

Answer

-4 and 7

50. A rectangular retaining wall has area 15 square feet. The height of the wall is two feet less than its length. Find the height and the length of the wall.

51. The area of a bulletin board is 55 square feet. The length is four feet less than three times the width. Find the length and the width of the a bulletin board.

Answer

$5,\ 11$

52. A rectangular carport has area 150 square feet. The height of the carport is five feet less than twice its length. Find the height and the length of the carport.

53. A pennant is shaped like a right triangle, with hypotenuse 10 feet. The length of one side of the pennant is two feet longer than the length of the other side. Find the length of the two sides of the pennant.

Answer

$6,\ 8$

54. A stained glass window is shaped like a right triangle. The hypotenuse is 15 feet. One leg is three more than the other. Find the lengths of the legs.

55. A reflecting pool is shaped like a right triangle, with one leg along the wall of a building. The hypotenuse is 9 feet longer than the side along the building. The third side is 7 feet longer than the side along the building. Find the lengths of all three sides of the reflecting pool.

Answer

$8,\ 15,\ 17$

56. A goat enclosure is in the shape of a right triangle. One leg of the enclosure is built against the side of the barn. The other leg is 4 feet more than the leg against the barn. The hypotenuse is 8 feet more than the leg along the barn. Find the three sides of the goat enclosure.

57. Juli is going to launch a model rocket in her back yard. When she launches the rocket, the function $h(t)=-16t^2+32t$ models the height, h, of the rocket above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the rocket will hit the ground. ⓑ the time the rocket will be 16 feet above the ground.

Answer

ⓐ 0, 2 ⓑ 1

58. Gianna is going to throw a ball from the top floor of her middle school. When she throws the ball from 48 feet above the ground, the function $h(t)=-16t^2+32t+48$ models the height, h, of the ball above the ground as a function of time, t. Find:

ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 48 feet above the ground. ⓒ the height the ball will be at $t=1$ seconds which is when the ball will be at its highest point.

Writing Exercises

59. Explain how you solve a quadratic equation. How many answers do you expect to get for a quadratic equation?

Answer

Answers will vary.

60. Give an example of a quadratic equation that has a GCF and none of the solutions to the equation is zero.

Self Check

ⓐ After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations by using the Zero Product Property.			
solve quadratic equations by factoring.			
solve applications modeled by quadratic equations.			

ⓑ Overall, after looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

4.6: Chapter 4 Review Exercises

Chapter Review Exercises

Greatest Common Factor and Factor by Grouping

Find the Greatest Common Factor of Two or More Expressions

In the following exercises, find the greatest common factor.

1. $12a^2b^3,\ 15ab^2$

Answer

$3ab^2$

2. $12m^2n^3, 42m^5n^3$

3. $15y^3,\ 21y^2,\ 30y$

Answer

$3y$

4. $45x^3y^2,\ 15x^4y,\ 10x^5y^3$

Factor the Greatest Common Factor from a Polynomial

In the following exercises, factor the greatest common factor from each polynomial.

5. $35y+84$

Answer

$7(5y+12)$

6. $6y^2+12y-6$

7. $18x^3-15x$

Answer

$3x(6x^2-5)$

8. $15m^4+6m^2n$

9. $4x^3-12x^2+16x$

Answer

$4x(x^2-3x+4)$

10. $-3x+24$

11. $-3x^3+27x^2-12x$

Answer

$-3x(x^2-9x+4)$

12. $3x(x-1)+5(x-1)$

Factor by Grouping

In the following exercises, factor by grouping.

13. $ax-ay+bx-by$

Answer

$(a+b)(x-y)$

14. $x^2y-xy^2+2x-2y$

15. $x^2+7x-3x-21$

Answer

$(x-3)(x+7)$

16. $4x^2-16x+3x-12$

17. m^3+m^2+m+1

Answer

$(m^2+1)(m+1)$

18. $5x-5y-y+x$

Factor Trinomials

Factor Trinomials of the Form x^2+bx+c

In the following exercises, factor each trinomial of the form x^2+bx+c .

19. $a^2+14a+33$

Answer

$(a+3)(a+11)$

20. $k^2-16k+60$

21. $m^2+3m-54$

Answer

$(m+9)(m-6)$

22. $x^2-3x-10$

In the following examples, factor each trinomial of the form $x^2+bxy+cy^2$.

23. $x^2+12xy+35y^2$

Answer

$(x+5y)(x+7y)$

24. $r^2+3rs-28s^2$

25. $a^2+4ab-21b^2$

Answer

$(a+7b)(a-3b)$

26. $p^2-5pq-36q^2$

27. $m^2-5mn+30n^2$

Answer

Prime

Factor Trinomials of the Form $ax^2+bx+cax^2+bx+c$ Using Trial and Error

In the following exercises, factor completely using trial and error.

28. x^3+5x^2-24x

29. $3y^3-21y^2+30y$

Answer

$3y(y-5)(y-2)$

30. $5x^4+10x^3-75x^2$

31. $5y^2+14y+9$

Answer

$(5y+9)(y+1)$

32. $8x^2+25x+3$

33. $10y^2-53y-11$

Answer

$(5y+1)(2y-11)$

34. $6p^2-19pq+10q^2$

35. $-81a^2+153a+18$

Answer

$-9(9a-1)(a+2)$

Factor Trinomials of the Form $ax^2+bx+cax^2+bx+c$ using the 'ac' Method

In the following exercises, factor.

36. $2x^2 + 9x + 4$

37. $18a^2 - 9a + 1$

Answer

$(3a - 1)(6a - 1)$

38. $15p^2 + 2p - 8$

39. $15x^2 + 6x - 2$

Answer

$(3x - 1)(5x + 2)$

40. $8a^2 + 32a + 24$

41. $3x^2 + 3x - 36$

Answer

$3(x + 4)(x - 3)$

42. $48y^2 + 12y - 36$

43. $18a^2 - 57a - 21$

Answer

$3(2a - 7)(3a + 1)$

44. $3n^4 - 12n^3 - 96n^2$

Factor using substitution

In the following exercises, factor using substitution.

45. $x^4 - 13x^2 - 30$

Answer

$(x^2 - 15)(x^2 + 2)$

46. $(x - 3)^2 - 5(x - 3) - 36$

Factor Special Products

Factor Perfect Square Trinomials

In the following exercises, factor completely using the perfect square trinomials pattern.

47. $25x^2 + 30x + 9$

Answer

$(5x + 3)^2$

48. $36a^2 - 84ab + 49b^2$

49. $40x^2 + 360x + 810$

Answer

$10(2x+9)^2$

50. $5k^3 - 70k^2 + 245k$

51. $75u^4 - 30u^3v + 3u^2v^2$

Answer

$3u^2(5u-v)^2$

Factor Differences of Squares

In the following exercises, factor completely using the difference of squares pattern, if possible.

52. $81r^2 - 25$

53. $169m^2 - n^2$

Answer

$(13m+n)(13m-n)$

54. $25p^2 - 1$

55. $9 - 121y^2$

Answer

$(3+11y)(3-11y)$

56. $20x^2 - 125$

57. $169n^3 - n$

Answer

$n(13n+1)(13n-1)$

58. $6p^2q^2 - 54p^2$

59. $24p^2 + 54$

Answer

$6(4p^2+9)$

60. $49x^2 - 81y^2$

61. $16z^4 - 1$

Answer

$(2z-1)(2z+1)(4z^2+1)$

62. $48m^4n^2-243n^2$

63. $a^2+6a+9-9b^2$

Answer

$(a+3-3b)(a+3+3b)$

64. $x^2-16x+64-y^2$

Factor Sums and Differences of Cubes

In the following exercises, factor completely using the sums and differences of cubes pattern, if possible.

65. a^3-125

Answer

$(a-5)(a^2+5a+25)$

66. b^3-216

67. $2m^3+54$

Answer

$2(m+3)(m^2-3m+9)$

68. $81m^3+3$

General Strategy for Factoring Polynomials

Recognize and Use the Appropriate Method to Factor a Polynomial Completely

In the following exercises, factor completely.

69. $24x^3+44x^2$

Answer

$4x^2(6x+11)$

70. $24a^4-9a^3$

71. $16n^2-56mn+49m^2$

Answer

$(4n-7m)^2$

72. $6a^2-25a-9$

73. $5u^4-45u^2$

Answer

$5u^2(u+3)(u-3)$

74. n^4-81

75. $64j^2+225$

Answer

prime

76. $5x^2+5x-60$

77. b^3-64

Answer

$(b-4)(b^2+4b+16)$

78. m^3+125

79. $2b^2-2bc+5cb-5c^2$

Answer

$(2b+5c)(b-c)$

80. $48x^5y^2-243xy^2$

81. $5q^2-15q-90$

Answer

$5(q+3)(q-6)$

82. $4u^5v+4u^2v^3$

83. $10m^4-6250$

Answer

$10(m-5)(m+5)(m^2+25)$

84. $60x^2y-75xy+30y$

85. $16x^2-24xy+9y^2-64$

Answer

$(4x-3y+8)(4x-3y-8)$

Polynomial Equations

Use the Zero Product Property

In the following exercises, solve.

86. $(a-3)(a+7)=0$

87. $(5b+1)(6b+1)=0$

Answer

$b=-\frac{1}{5},\ b=-\frac{1}{6}$

88. $6m(12m-5)=0$

$(2x-1)^2=0$

Answer

$x=\frac{1}{2}$

89. $3m(2m-5)(m+6)=0$

Solve Quadratic Equations by Factoring

In the following exercises, solve.

90. $x^2+9x+20=0$

Answer

$x=-4,\ x=-5$

91. $y^2-y-72=0$

92. $2p^2-11p=40$

Answer

$p=-\frac{5}{2}, p=8$

93. $q^3+3q^2+2q=0$

94. $144m^2-25=0$

Answer

$m=\frac{5}{12},\ m=-\frac{5}{12}$

95. $4n^2=36$

96. $(x+6)(x-3)=-8$

Answer

$x=2,\ x=-5$

97. $(3x-2)(x+4)=12$

98. $16p^3=24p^2+9p$

Answer

$p=0,\ p=\frac{3}{4}$

99. $2y^3+2y^2=12y$

Solve Equations with Polynomial Functions

In the following exercises, solve.

100. For the function, $f(x)=x^2+11x+20$, ⓐ find when $f(x)=-8$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x=-7$ or \$x=-4$
ⓑ $(-7,-8)\ (-4,-8)$

101. For the function, $f(x)=9x^2-18x+5$, ⓐ find when $f(x)=-3$ ⓑ Use this information to find two points that lie on the graph of the function.

In each function, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒ the *y*-intercept of the graph of the function.

102. $f(x)=64x^2-49$

Answer

ⓐ $x=\frac{7}{8}$ or $x=-\frac{7}{8}$
ⓑ $(\frac{7}{8},0),\ (-\frac{7}{8},0)$ ⓒ $(0,-49)$

103. $f(x)=6x^2-13x-5$

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve.

104. The product of two consecutive numbers is 399. Find the numbers.

Answer

The numbers are -21 and -19 or 19 and 21.

105. The area of a rectangular shaped patio 432 square feet. The length of the patio is 6 feet more than its width. Find the length and width.

106. A ladder leans against the wall of a building. The length of the ladder is 9 feet longer than the distance of the bottom of the ladder from the building. The distance of the top of the ladder reaches up the side of the building is 7 feet longer than the distance of the bottom of the ladder from the building. Find the lengths of all three sides of the triangle formed by the ladder leaning against the building.

Answer

The lengths are 8, 15, and 17 ft.

107. Shruti is going to throw a ball from the top of a cliff. When she throws the ball from 80 feet above the ground, the function $h(t)=-16t^2+64t+80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the

zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 80 feet above the ground. ⓒ the height the ball will be at $t=2$ seconds which is when the ball will be at its highest point.

Chapter Practice Test

In the following exercises, factor completely.

108. $80a^2+120a^3$

Answer

$40a^2(2+3a)$

109. $5m(m-1)+3(m-1)$

110. $x^2+13x+36$

Answer

$(x+7)(x+6)$

111. $p^2+pq-12q^2$

112. $xy-8y+7x-56$

Answer

$(x-8)(y+7)$

113. $40r^2+810$

$9s^2-12s+4$

Answer

$(3s-2)^2$

114. $6x^2-11x-10$

115. $3x^2-75y^2$

Answer

$3(x+5y)(x-5y)$

116. $6u^2+3u-18$

117. x^3+125

Answer

$(x+5)(x^2-5x+25)$

118. $32x^5y^2-162xy^2$

119. $6x^4-19x^2+15$

Answer

$(3x^2 - 5)(2x^2 - 3)$

120. $3x^3 - 36x^2 + 108x$

In the following exercises, solve

121.$5a^2 + 26a = 24$

Answer

$a = \frac{4}{5},\ a = -6$

122. The product of two consecutive integers is 156. Find the integers.

123. The area of a rectangular place mat is 168 square inches. Its length is two inches longer than the width. Find the length and width of the placemat.

Answer

The width is 12 inches and the length is 14 inches.

124. Jing is going to throw a ball from the balcony of her condo. When she throws the ball from 80 feet above the ground, the function $h(t) = -16t^2 + 64t + 80$ models the height, *h*, of the ball above the ground as a function of time, *t*. Find: ⓐ the zeros of this function which tells us when the ball will hit the ground. ⓑ the time(s) the ball will be 128 feet above the ground. ⓒ the height the ball will be at $t = 4$ seconds.

125. For the function, $f(x) = x^2 - 7x + 5$, ⓐ find when $f(x) = -7$ ⓑ Use this information to find two points that lie on the graph of the function.

Answer

ⓐ $x = 3$ or $x = 4$ ⓑ $(3, -7)$ $(4, -7)$

126. For the function $f(x) = 25x^2 - 81$, find: ⓐ the zeros of the function ⓑ the *x*-intercepts of the graph of the function ⓒthe *y*-intercept of the graph of the function.

Glossary

degree of the polynomial equation
The degree of the polynomial equation is the degree of the polynomial.

polynomial equation
A polynomial equation is an equation that contains a polynomial expression.

quadratic equation
Polynomial equations of degree two are called quadratic equations.

zero of the function
A value of xx where the function is 0, is called a zero of the function.

Zero Product Property
The Zero Product Property says that if the product of two quantities is zero, then at least one of the quantities is zero.

CHAPTER OVERVIEW

5: Rational Expressions

5.1: Multiply and Divide Rational Expressions
5.2: Add and Subtract Rational Expressions
5.3: Simplify Complex Rational Expressions
5.4: Solve Rational Equations
5.5: Applications with Rational Equations
5.6: Chapter 5 Review Exercises

5.1: Multiply and Divide Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions
- Multiply and divide rational functions

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $\frac{90y}{15y^2}$.
2. Multiply: $\frac{14}{15} \cdot \frac{6}{35}$.
3. Divide: $\frac{12}{10} \div \frac{8}{25}$.

Above are examples of some properties of fractions and their operations. Recall that these fractions where the numerators and denominators are integers are called rational numbers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Definition 5.1.1

A **rational expression** is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Determine the Values for Which a Rational Expression is Undefined

If the denominator is zero, the rational expression is undefined. The numerator of a rational expression may be 0—but not the denominator.

When we work with a numerical fraction, it is easy to avoid dividing by zero because we can see the number in the denominator. In order to avoid dividing by zero in a rational expression, we must not allow values of the variable that will make the denominator be zero.

So before we begin any operation with a rational expression, we examine it first to find the values that would make the denominator zero. That way, when we solve a rational equation for example, we will know whether the algebraic solutions we find are allowed or not.

> ✎ Determine the values for which a rational expression is undefined
>
> 1. Set the denominator equal to zero.
> 2. Solve the equation.

> **? Example 5.1.2**
>
> Determine the value for which each rational expression is undefined:
>
> **a.** $\dfrac{8a^2b}{3c}$
>
> **b.** $\dfrac{4b-3}{2b+5}$
>
> **c.** $\dfrac{x+4}{x^2+5x+6}$.
>
> **Solution**
>
> The expression will be undefined when the denominator is zero.
>
> **a.**
>
> | | |
> |---|---|
> | | $\dfrac{8a^2b}{3c}$ |
> | Set the denominator equal to zero and solve for the variable. | $3c=0$ |
> | | $c=0$ |
> | | $\dfrac{8a^2b}{3c}$ is undefined for $c=0$ |
>
> **b.**
>
> | | |
> |---|---|
> | | $\dfrac{4b-3}{2b+5}$ |
> | Set the denominator equal to zero and solve for the variable. | $2b+5=0$ |
> | | $2b=-5$ |
> | | $b=-\dfrac{5}{2}$ |
> | | $\dfrac{4b-3}{2b+5}$ is undefined for $b=-\dfrac{5}{2}$ |
>
> **c.**
>
> | | |
> |---|---|
> | | $\dfrac{x+4}{x^2+5x+6}$ |
> | Set the denominator equal to zero and solve for the variable. | $x^2+5x+6=0$ |
> | | $(x+2)(x+3)=0$ |
> | | $x+2=0$ or $x+3=0$ |
> | | $x=-2$ or $x=-3$ |
> | | $\dfrac{x+4}{x^2+5x+6}$ is undefined for $x=-2$ or $x=-3$ |

Try It 5.1.3

Determine the value for which each rational expression is undefined.

a. $\frac{3y^2}{8x}$

b. $\frac{8n-5}{3n+1}$

c. $\frac{a+10}{a^2+4a+3}$

Answer

a. $x=0$

b. $n=-\frac{1}{3}$

c. $a=-1, a=-3$

Try It 5.1.4

Determine the value for which each rational expression is undefined.

a. $\frac{4p}{5q}$

b. $\frac{y-1}{3y+2}$

c. $\frac{m-5}{m^2+m-6}$

Answer

a. $q=0$

b. $y=-\frac{2}{3}$

c. $m=2, m=-3$

Simplify Rational Expressions

A fraction is considered simplified if there are no common factors, other than 1 and -1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1 and -1, in its numerator and denominator.

Simplified Rational Expression

A rational expression is considered simplified if there are no common factors other than 1 and -1 in its numerator and denominator. There should be no more "$-$"s than necessary.

For example,

$$\frac{x+2}{x+3} \text{ is simplified because there are no common factors of } x+2 \text{ and } x+3.$$
$$\frac{2x}{3x} \text{ is not simplified because x is a common factor of } 2x \text{ and } 3x.$$

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

Equivalent Fractions Property

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0, c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

$$\frac{2 \cdot \cancel{3} \cdot \cancel{7}}{\cancel{3} \cdot 5 \cdot \cancel{7}}$$

$$\frac{2}{5}$$

We removed the common factors 3 and 7. They are factors of the product.

$$\frac{3x\cancel{(x-9)}}{5\cancel{(x-9)}} \quad \text{where } x \neq 9$$

$$\frac{3x}{5}$$

We removed the common factor $(x-9)$. It is a factor of the product.

$$\frac{x+5}{x}$$

NO COMMON FACTORS

While there is an x in both the numerator and denominator, the x in the numerator is a term of a sum!

Removing the x's from $\frac{x+5}{x}$ would be like cancelling the 2's in the fraction $\frac{2+5}{2}$!

How to Simplify a Rational Expression

Example 5.1.5

Simplify $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\frac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator completely.	$=\frac{(x+2)(x+3)}{(x+2)(x+6)}$
Simplify by dividing out common factors.	$=\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $=\frac{x+3}{x+6}$ $x \neq 2$ and $x \neq -6$

Try It 5.1.6

Simplify $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}$ with $x \neq 2$ and $x \neq 1$

Try It 5.1.7

Simplify $\dfrac{x^2-3x-10}{x^2+x-2}$.

Answer

$\dfrac{x-5}{x-1}$ with $x \neq -2$ and $x \neq 1$

We now summarize the steps you should follow to simplify rational expressions.

Simplify a Rational Expression

1. Factor the numerator and denominator completely.
2. Simplify by dividing out common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Example 5.1.8

Simplify $\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

	$\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$
Factor the numerator and denominator, first factoring out the GCF.	$=\dfrac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$ $=\dfrac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$
Remove the common factors of $a-2b$ and 3.	$=\dfrac{\cancel{3}(a-2b)\ \cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\ \cancel{(a-2b)}}$ $=\dfrac{a-2b}{2(a+2b)}$

Try It 5.1.9

Simplify $\dfrac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Answer

$\dfrac{2(x-3y)}{3(x+3y)}$

Try It 5.1.10

Simplify $\dfrac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Answer

$\dfrac{5(x-y)}{2(x+5y)}$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a=-1\cdot a$.

The numerical fraction, say $\dfrac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\dfrac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

$$\begin{array}{ll} \text{Let's look at the expression } b-a. & b-a \\ \text{Rewrite.} & -a+b \\ \text{Factor out } -1. & -1(a-b) \end{array}$$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\dfrac{a-b}{b-a}$ simplifies to -1.

Opposites in a Rational Expression

The **opposite** of $a-b$ is $b-a$.

$$\frac{a-b}{b-a}=-1 \quad a\neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a\neq -b$, then $\dfrac{a+b}{b+a}=1$.

Example 5.1.11

Simplify $\dfrac{x^2-4x-32}{64-x^2}$.

Solution

	$\dfrac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$=\dfrac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$=(-1)\dfrac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$=-\dfrac{x+4}{x+8}$

Try It 5.1.12

Simplify $\dfrac{x^2-4x-52}{5-x^2}$

Answer

$-\dfrac{x+1}{x+5}$

Try It 5.1.13

Simplify $\dfrac{x^2+x-2}{1-x^2}$.

Answer

$-\dfrac{x+2}{x+1}$

Multiply Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$, $x \neq 3$, and $x \neq 4$.

Example 5.1.14

Simplify $\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\dfrac{2x}{x^2-7x+12} \cdot \dfrac{x^2-9}{6x^2}$ $= \dfrac{2x}{(x-3)(x-4)} \cdot \dfrac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$= \dfrac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$

Step 3. Reduce the fraction.	Recognize factors common to the numerator and denominator and reduce the fraction. "Divide out" the common factors (divide numerator and denominator by those common factors). Leave the result in factored form.	$\frac{\cancel{2}\,\cancel{x}\,\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot \cancel{x}\cdot x\,\cancel{(x-3)}(x-4)}$ $=\frac{x+3}{3x(x-4)}$

Try It 5.1.15

Simplify $\frac{5x}{x^2+5x+6}\cdot\frac{x^2-4}{10x}$.

Answer

$\frac{x-2}{2(x+3)}$

Try It 5.1.16

Simplify $\frac{9x^2}{x^2+11x+30}]\cdot\frac{x^2-36}{3x^2}$.

Answer

$\frac{3(x-6)}{x+5}$

Multiply Rational Expressions

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out the common factors.

Example 5.1.17

Multiply $\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.
$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.
$$\frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)}$$

Simplify.
$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.
$$\frac{(a-3)(a+5)}{(a-5)^2}$$

Try It 5.1.18

Simplify $\frac{2x^2+5x-12}{x^2-16}\cdot\frac{x^2-8x+16}{2x^2-13x+15}$.

Answer

$\frac{x-4}{x-5}$

Try It 5.1.19

Simplify $\frac{4b^2+7b-2}{1-b^2}\cdot\frac{b^2-2b+1}{4b^2+15b-4}$.

Answer

$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$

Divide Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Division of Rational Expressions

If p, q, r, and s are polynomials where $q\neq 0$, $r\neq 0$, $s\neq 0$, then

$$\frac{p}{q}\div\frac{r}{s}=\frac{p}{q}\cdot\frac{s}{r}.$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

? Example 5.1.20

Divide $\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\frac{p^3+q^3}{2p^2+2pq+2q^2} \div \frac{p^2-q^2}{6}$ $\frac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \frac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\frac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \frac{2 \cdot 3}{(p-q)(p+q)}$
Step 3. Multiply the numerators and denominators.	Multiply the numerators and multiply the denominators.	$\frac{(p+q)(p^2-pq+q^2)2 \cdot 3}{2(p^2+pq+q^2)(p-q)(p+q)}$
Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\frac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\frac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

? Try It 5.1.21

Simplify $\frac{x^3-8}{3x^2-6x+12} \div \frac{x^2-4}{6}$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

? Try It 5.1.22

Simplify $\frac{2z^2}{z^2-1} \div \frac{z^3-z^2+z}{z^3+1}$.

Answer

$$\frac{2z}{z-1}$$

Division of Rational Expressions

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

A complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Recall that a fraction bar means division. A complex fraction is another way of writing division of two fractions.

Example 5.1.23

Divide $\dfrac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\dfrac{\dfrac{6x^2-7x+2}{4x-8}}{\dfrac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign.	$\dfrac{6x^2-7x+2}{4x-8} \div \dfrac{2x^2-7x+3}{x^2-5x+6}$
Rewrite as product of first times reciprocal of second.	$\dfrac{6x^2-7x+2}{4x-8} \cdot \dfrac{x^2-5x+6}{2x^2-7x+3}$
Factor the numerators and the denominators, and then multiply.	$\dfrac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$
Simplify by dividing out common factors.	$\dfrac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$
Simplify.	$\dfrac{3x-2}{4}$

Try It 5.1.24

Simplify $\dfrac{\dfrac{3x^2+7x+2}{4x+24}}{\dfrac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\dfrac{x+2}{4}$

Try It 5.1.25

Simplify $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

Answer

$\dfrac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

Example 5.1.26

Perform the indicated operations: $\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$.

Solution

	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \div \frac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\frac{3x-6}{4x-4} \cdot \frac{x^2+2x-3}{x^2-3x-10} \cdot \frac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\frac{3(x-2)}{4(x-1)} \cdot \frac{(x+3)(x-1)}{(x+2)(x-5)} \cdot \frac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	$\frac{3 \cdot \cancel{8}(x-2)(x+3)\cancel{(x-1)}\cancel{(x+2)}}{\cancel{4} \cdot 2\cancel{(x-1)}\cancel{(x+2)}(x-5)(x+6)}$
Simplify.	$\frac{3(x-2)(x+3)}{(x-5)(x+6)}$

Try It 5.1.27

Perform the indicated operations: $\frac{4m+4}{3m-15} \cdot \frac{m^2-3m-10}{m^2-4m-32} \div \frac{12m-36}{6m-48}$.

Answer

$$\frac{2(m+1)(m+2)}{3(m+4)(m-3)}$$

Try It 5.1.28

Perform the indicated operations: $\frac{2n^2+10n}{n-1} \div \frac{n^2+10n+24}{n^2+8n-9} \cdot \frac{n+4}{8n^2+12n}$.

Answer

$$\frac{(n+5)(n+9)}{2(n+6)(2n+3)}$$

Key Concepts

- **Determine the values for which a rational expression is undefined.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b \neq 0$, $c \neq 0$, then

  $$\frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}.$$
- **How to simplify a rational expression.**
  1. Factor the numerator and denominator completely.

2. Simplify by dividing out common factors.

- **Opposites in a Rational Expression**
  The opposite of $a-b$ is $b-a$.

  $$\frac{a-b}{b-a}=-1 \qquad a\neq b$$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q\neq 0$, $s\neq 0$, then

  $$\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$$
- **How to multiply rational expressions.**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.
- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q\neq 0$, $r\neq 0$, $s\neq 0$, then

  $$\frac{p}{q}\div\frac{r}{s}=\frac{p}{q}\cdot\frac{s}{r}$$
- **How to divide rational expressions.**
  1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  2. Factor the numerators and denominators completely.
  3. Multiply the numerators and denominators together.
  4. Simplify by dividing out common factors.

Glossary

rational expression

A rational expression is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q\neq 0$.

simplified rational expression

A simplified rational expression has no common factors, other than 1 and -1, in its numerator and denominator.

Practice Makes Perfect

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1. a. $\frac{2x^2}{z}$ b. $\frac{4p-1}{6p-5}$ c. $\frac{n-3}{n^2+2n-8}$

Answer

a. $z=0$
b. $p=\frac{5}{6}$
c. $n=-4, n=2$

2. a. $\frac{10m}{11n}$ b. $\frac{6y+13}{4y-9}$ c. $\frac{b-8}{b^2-36}$

3. a. $\frac{4x^2y}{3y}$ b. $\frac{3x-2}{2x+1}$ c. $\frac{u-1}{u^2-3u-28}$

Answer

a. $y=0$
b. $x=-\frac{1}{2}$
c. $u=-4, u=7$

4. a. $\frac{5pq^2}{9q}$ b. $\frac{7a-4}{3a+5}$ c. $\frac{1}{x^2-4}$

Simplify Rational Expressions

In the following exercises, simplify each rational expression.

5. $-\frac{44}{55}$

Answer

$-\frac{4}{5}$

6. $\frac{56}{63}$

7. $\frac{8m^3n}{12mn^2}$

Answer

$\frac{2m^2}{3n}$

8. $\frac{36v^3w^2}{27vw^3}$

9. $\frac{8n-96}{3n-36}$

Answer

$\frac{8}{3}$

10. $\frac{12p-240}{5p-100}$

11. $\frac{x^2+4x-5}{x^2-2x+1}$

Answer

$\frac{x+5}{x-1}$

12. $\dfrac{y^2+3y-4}{y^2-6y+5}$

13. $\dfrac{a^2-4}{a^2+6a-16}$

Answer

$\dfrac{a+2}{a+8}$

14. $\dfrac{y^2-2y-3}{y^2-9}$

15. $\dfrac{p^3+3p^2+4p+12}{p^2+p-6}$

Answer

$\dfrac{p^2+4}{p-2}$

16. $\dfrac{x^3-2x^2-25x+50}{x^2-25}$

17. $\dfrac{8b^2-32b}{2b^2-6b-80}$

Answer

$\dfrac{4b(b-4)}{(b+5)(b-8)}$

18. $\dfrac{-5c^2-10c}{-10c^2+30c+100}$

19. $\dfrac{3m^2+30mn+75n^2}{4m^2-100n^2}$

Answer

$\dfrac{3(m+5n)}{4(m-5n)}$

20. $\dfrac{5r^2+30rs-35s^2}{r^2-49s^2}$

21. $\dfrac{a-5}{5-a}$

Answer

-1

22. $\dfrac{5-d}{d-5}$

23. $\dfrac{20-5y}{y^2-16}$

Answer

$\dfrac{-5}{y+4}$

24. $\dfrac{4v-32}{64-v^2}$

25. $\dfrac{w^3+21}{6w^2-36}$

Answer

$\dfrac{w^2-6w+3}{6w-6}$

26. $\dfrac{v^3+125}{v^2-25}$

27. $\dfrac{z^2-9z+20}{16-z^2}$

Answer

$\dfrac{-z-5}{4+z}$

28. $\dfrac{a^2-5z-36}{81-a^2}$

Multiply Rational Expressions

In the following exercises, multiply the rational expressions.

29. $\dfrac{12}{16}\cdot\dfrac{4}{10}$

Answer

$\dfrac{3}{10}$

30. $\dfrac{3}{25}\cdot\dfrac{16}{24}$

31. $\dfrac{5x^2y^4}{12xy^3}\cdot\dfrac{6x^2}{20y^2}$

Answer

$\dfrac{x^3}{8y}$

32. $\dfrac{12a^3b}{b^2}\cdot\dfrac{2ab^2}{9b^3}$

33. $\frac{5p^2}{p^2-5p-36}\cdot\frac{p^2-16}{10p}$

Answer

$\frac{p(p-4)}{2(p-9)}$

34. $\frac{3q^2}{q^2+q-6}\cdot\frac{q^2-9}{9q}$

35. $\frac{2y^2-10y}{y^2}+\frac{10y+2}{5}\cdot\frac{y+5}{6y}$

Answer

$\frac{y-5}{3(y+5)}$

36. $\frac{z^2+3z}{z^2-3z-4}\cdot\frac{z-4}{z^2}$

37. $\frac{28-4b}{3b-3}\cdot\frac{b^2+8b-9}{b^2-49}$

Answer

$\frac{-4(b+9)}{3(b+7)}$

38. $\frac{72m-12m^2}{8m+32}\cdot\frac{m^2+10m+2}{4m^2-36}$

39. $\frac{5c^2+9c+2}{c^2-25}\cdot\frac{c^2+10c+25}{3c^2-14c-5}$

Answer

$\frac{(c+2)(c+2)}{(c-2)(c-3)}$

40. $\frac{2d^2+d-3}{d^2-16}\cdot\frac{d^2-8d+16}{2d^2-9d-18}$

41. $\frac{2m^2-3m-2}{2m2+7m+3}\cdot\frac{3m^2-14m+15}{3m^2+17m-20}$

Answer

$\frac{(m-3)(m-2)}{(m+4)(m+3)}$

42. $\frac{2n^2-3n-14}{25-n^2}\cdot\frac{n^2-10n+25}{2n^2-13n+21}$

Divide Rational Expressions

In the following exercises, divide the rational expressions.

43. $\frac{v-5}{11-v} \div \frac{v^2-25}{v-11}$

Answer

$-\frac{1}{v+5}$

44. $\frac{10+w}{w-8} \div \frac{100-w^2}{8-w}$

45. $\frac{3s^2}{s^2-16} \div \frac{s^3-4s^2+16s}{s^3-64}$

Answer

$\frac{3s}{s+4}$

46. $\frac{r^2-9}{15} \div \frac{r^3-27}{5r^2+15r+45}$

47. $\frac{p^3+q^3}{3p^2+3pq+3q^2} \div \frac{p^2-q^2}{12}$

Answer

$\frac{4(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

48. $\frac{v^3-8w^3}{2v^2+4vw+8w^2} \div \frac{v^2-4w^2}{4}$

49. $\frac{x^2+3x-10}{4x} \div (2x^2+20x+50)$

Answer

$\frac{x-2}{8x}$

50. $\frac{2y^2-10yz-48z^2}{2y-1} \div (4y^2-32yz)$

51. $\frac{\frac{2a^2-a-21}{5a+20}}{\frac{a^2+7a+12}{a^2+8a+16}}$

Answer

$\frac{2a-7}{5}$

52. $\dfrac{\dfrac{3b^2+2b-8}{12b+18}}{\dfrac{3b^2+2b-8}{2b^2-7b-15}}$

53. $\dfrac{\dfrac{12c^2-12}{2c^2-3c+1}}{\dfrac{4c+4}{6c^2-13c+5}}$

Answer

$3(3c-5)$

54. $\dfrac{\dfrac{4d^2+7d-2}{35d+10}}{\dfrac{d^2-4}{7d^2-12d-4}}$

For the following exercises, perform the indicated operations.

55. $\dfrac{10m^2+80m}{3m-9}\cdot\dfrac{m^2+4m-21}{m^2-9m+20}\div\dfrac{5m^2+10m}{2m-10}$

Answer

$\dfrac{4(m+8)(m+7)}{3(m-4)(m+2)}$

56. $\dfrac{4n^2+32n}{3n+2}\cdot\dfrac{3n^2-n-2}{n^2+n-30}\div\dfrac{108n^2-24n}{n+6}$

57. $\dfrac{12p^2+3p}{p+3}\div\dfrac{p^2+2p-63}{p^2-p-12}\cdot\dfrac{p-7}{9p^3-9p^2}$

Answer

$\dfrac{(4p+1)(p-4)}{3p(p+9)(p-1)}$

58. $\dfrac{6q+3}{9q^2-9q}\div\dfrac{q^2+14q+33}{q^2+4q-5}\cdot\dfrac{4q^2+12q}{12q+6}$

Writing Exercises

59. Explain how you find the values of x for which the rational expression $\dfrac{x^2-x-20}{x^2-4}$ is undefined.

Answer

Answers will vary.

60. Explain all the steps you take to simplify the rational expression $\dfrac{p^2+4p-21}{9-p^2}$.

61. a. Multiply $\frac{7}{4} \cdot \frac{9}{10}$ and explain all your steps.

b. Multiply $\frac{n}{n-3} \cdot \frac{9}{n+3}$ and explain all your steps.

c. Evaluate your answer to part b. when $n = 7$. Did you get the same answer you got in part a.? Why or why not?

Answer

Answers will vary.

62. a. Divide $\frac{24}{5} \div 6$ and explain all your steps.

b. Divide $\frac{x^2-1}{x} \div (x+1)$ and explain all your steps.

c. Evaluate your answer to part b. when $x = 5$. Did you get the same answer you got in part a.? Why or why not?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
determine the values for which a rational expression is undefined.			
simplify rational expressions.			
multiply rational expressions.			
divide rational expressions.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved your goals in this section! Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific!

…with some help. This must be addressed quickly as topics you do not master become potholes in your road to success. Math is sequential - every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is critical and you must not ignore it. You need to get help immediately or you will quickly be overwhelmed. See your instructor as soon as possible to discuss your situation. Together you can come up with a plan to get you the help you need.

5.2: Add and Subtract Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Be Prepared

Before you get started, take this readiness quiz.

1. Add: $\frac{7}{10}+\frac{8}{15}$.
2. Subtract: $\frac{3x}{4}-\frac{8}{9}$.
3. Subtract: $6(2x+1)-4(x-5)$.

Add and Subtract Rational Expressions with a Common Denominator

What is the first step we take when we add numerical fractions? We check if they have a common denominator. If they do, we add the numerators and place the sum over the common denominator. If they do not have a common denominator, we find one before we add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, we add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction

If p, q, and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}.$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after subtracting the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 5.2.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

$$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator. $\frac{11x+28+x^2}{x+4}$

Write the degrees in descending order. $\frac{x^2+11x+28}{x+4}$

Factor the numerator. $\frac{(x+4)(x+7)}{x+4}$

Simplify by removing common factors. $\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$

Simplify. $x+7$

The expression simplifies to $x+7$ but the original expression had a denominator of $x+4$ so $x\neq -4$.

? Try It 5.2.2

Simplify $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

? Try It 5.2.3

Simplify $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, we subtract the numerators and place the difference over the common denominator. Be careful of the signs when subtracting a binomial or trinomial.

? Example 5.2.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\dfrac{5x^2-7x+3}{x^2-3x+18}-\dfrac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\dfrac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\dfrac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\dfrac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\dfrac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\dfrac{(x-2)\ \cancel{(x-6)}}{(x+3)\ \cancel{(x-6)}}$
	$(x-2)(x+3)$

Try It 5.2.5

Subtract: $\dfrac{4x^2-11x+8}{x^2-3x+2}-\dfrac{3x^2+x-3}{x^2-3x+2}$.

Answer

$$\frac{x-11}{x-2}$$

Try It 5.2.6

Subtract: $\dfrac{6x^2-x+20}{x^2-81}-\dfrac{5x^2+11x-7}{x^2-81}$.

Answer

$$\frac{x-3}{x+9}$$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works.

	$\frac{7}{d}+\frac{5}{-d}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{7}{d}+\frac{(-1)5}{(-1)(-d)}$
The denominators are the same.	$\frac{7}{d}+\frac{-5}{d}$

Simplify.	$\frac{2}{d}$

Be careful with the signs as we work with the opposites when the fractions are being subtracted.

Example 5.2.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-1(3m+2)}{-1(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2}{m^2-1}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m-1)(m+1)}$
Simplify by dividing out common factors.	$\frac{\cancel{(m-1)}(m-2)}{\cancel{(m-1)}(m+1)}$
Simplify.	$\frac{m-2}{m+1}$

Try It 5.2.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$$\frac{y+3}{y+2}$$

Try It 5.2.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$$\frac{3n-2}{n-1}$$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes, lining up any common primes in columns. Then we "brought down" one prime from each column. Finally, we multiplied the factors to find the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number. We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}$$

$$\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}$$

$$\frac{21}{36}+\frac{10}{36}$$

$$\begin{array}{l} 12=2\cdot 2\cdot 3 \\ 18=2\cdot \, 3\cdot 3 \\ \hline \text{LCD}=2\cdot 2\cdot 3\cdot 3 \\ \text{LCD}=36 \end{array}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Find the Least Common Denominator (LCD) of Rational Expressions

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

Example 5.2.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	
Factor each denominator completely, lining up common factors. Bring down the columns.	$\begin{array}{l} x^2-2x-3=(x+1)(x-3) \\ x^2+4x+3=(x+1)(x+3) \\ \hline \text{LCD}=(x+1)(x-3)(x+3) \end{array}$
Write the LCD as the product of the factors.	The LCD is $(x+1)(x-3)(x+3)$.

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x+1)(x-3)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}, \frac{3x^2-9x}{(x+1)(x+3)(x-3)}$

Try It 5.2.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$
b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

Try It 5.2.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}, \frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$
b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example 5.2.13

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

Step 1. Determine if the expressions have a common denominator. • Yes–go to step 2. • No–Rewrite each rational expression with the LCD. ◦ Find the LCD. ◦ Rewrite each rational expression as an equivalent rational expression with the LCD.	No. Find the LCD of $(x-3)$ and $(x-2)$. Change into equivalent rational expressions with the LCD, $(x-3)$ and $(x-2)$. Keep the denominators factored!	$x-3 : (x-3)$ $x-2 \quad : (x-2)$ LCD : $(x-3)(x-2)$ $\frac{3}{x-3}+\frac{2}{x-2}$ $\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}$ $\frac{3x-6}{(x-3)(x-2)}+\frac{2x-6}{(x-2)(x-3)}$
Step 2. Add or subtract the rational expressions.	Add the numerators and place the sum over the common denominator.	$\frac{3x-6+2x-6}{(x-3)(x-2)}$
Step 3. Simplify, if possible.	Because $5x-12$ cannot be factored, the answer is simplified.	$\frac{5x-12}{(x-3)(x-2)}$

Try It 5.2.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Try It 5.2.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

Add or Subtract Rational Expressions

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after combining the numerators.

Example 5.2.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $\begin{array}{r} x^2-2x-3=(x+1)(x-3) \\ \underline{x^2+4x+3=(x+1)\quad(x+3)} \\ LCD=(x+1)(x-3)(x+3) \end{array}$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8(x+3)}{(x+1)(x-3)(x+3)}+\frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x+1)(x-3)(x+3)}+\frac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$\frac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$\frac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

Try It 5.2.17

Add: $\dfrac{1}{m^2-m-2}+\dfrac{5m}{m^2+3m+2}$.

Answer

$$\frac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

Try It 5.2.18

Add: $\dfrac{2n}{n^2-3n-10}+\dfrac{6}{n^2+5n+6}$.

Answer

$$\frac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

Example 5.2.19

Subtract: $\dfrac{8y}{y^2-16}-\dfrac{4}{y-4}$.

Solution

	$\frac{8y}{y^2-16}-\frac{4}{y-4}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	
Find the LCD. $\begin{aligned} y^2-16&=(y-4)(y+4)\\ \underline{y-4}&\underline{=y-4}\\ LCD&=(y-4)(y+4)\end{aligned}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{(y-4)(y+4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$\frac{8y}{(y-4)(y+4)}-\frac{4y+16}{(y-4)(y+4)}$

Subtract the rational expressions.	$\frac{8y-4y-16}{(y-4)(y+4)}$
Simplify the numerator.	$\frac{4y-16}{(y-4)(y+4)}$
Factor the numerator to look for common factors.	$\frac{4(y-4)}{(y-4)(y+4)}$
Remove common factors	$\frac{4\cancel{(y-4)}}{\cancel{(y-4)}(y+4)}$
Simplify.	$\frac{4}{(y+4)}$

Try It 5.2.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Try It 5.2.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

Example 5.2.22

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(-1)(2-n)}$
Write $(-1)(2-n)$ as $n-2$.	$\frac{-3n-9}{(n-2)(n+3)}-\frac{(-1)(n+3)}{(n-2)}$
Simplify. Remember, $a-(-b)=a+b$.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)}{(n-2)}$
Do the rational expressions have a common denominator? No.	

Find the LCD. $\begin{array}{r} n^2+n-6=(n-2)(n+3) \\ \underline{n-2=(n-2)} \\ LCD= \quad (n-2)(n+3) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{(n+3)(n+3)}{(n-2)(n+3)}$
Simplify the numerators.	$\frac{-3n-9}{(n-2)(n+3)}+\frac{n^2+6n+9}{(n-2)(n+3)}$
Add the rational expressions.	$\frac{-3n-9+n^2+6n+9}{(n-2)(n+3)}$
Simplify the numerator.	$\frac{n^2+3n}{(n-2)(n+3)}$
Factor the numerator to look for common factors.	$\frac{n\cancel{(n+3)}}{(n-2)\cancel{(n+3)}}$
Simplify.	$\frac{n}{(n-2)}$

Try It 5.2.23

Subtract: $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Answer

$$\frac{5x+1}{(x-6)(x+1)}$$

Try It 5.2.24

Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Answer

$$\frac{y+3}{y+4}$$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example 5.2.25

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$
Factor the denominators.	$\frac{4}{(a+1)(a+5)}-\frac{3}{(a+2)(a+5)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{r} a^2+6a+5=(a+1)(a+5) \\ \underline{a^2+7a+10=(a+5)(a+2)} \\ LCD=(a+1)(a+5)(a+2) \end{array}$	

Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{4(a+2)}{(a+1)(a+5)(a+2)}-\frac{3(a+1)}{(a+2)(a+5)(a+1)}$
Simplify the numerators.	$\frac{4a+8}{(a+1)(a+5)(a+2)}-\frac{3a+3}{(a+2)(a+5)(a+1)}$
Subtract the rational expressions.	$\frac{4a+8-(3a+3)}{(a+1)(a+5)(a+2)}$
Simplify the numerator.	$\frac{4a+8-3a+3}{(a+1)(a+5)(a+2)}$
	$\frac{a+5}{(a+1)(a+5)(a+2)}$
Look for common factors.	$\frac{\cancel{(a+5)}}{(a+1)\cancel{(a+5)}(a+2)}$
Simplify.	$\frac{1}{(a+1)(a+2)}$

? Try It 5.2.26

Subtract: $\frac{3}{b^2-4b-5}-\frac{2}{b^2-6b+5}$.

Answer

$$\frac{1}{(b+1)(b-1)}$$

? Try It 5.2.27

Subtract: $\frac{4}{x^2-4}-\frac{3}{x^2-x-2}$.

Answer

$$\frac{1}{(x+2)(x+1)}$$

Access this online resource for additional instruction and practice with adding and subtracting rational expressions.

- Add and Subtract Rational Expressions- Unlike Denominators

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.

- Find the LCD.
- Rewrite each rational expression as an equivalent rational expression with the LCD.

2. Add or subtract the rational expressions.
3. Simplify, if possible.

Practice Makes Perfect

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, add.

1. $\frac{2}{15}+\frac{7}{15}$

Answer

$\frac{3}{5}$

2. $\frac{7}{24}+\frac{11}{24}$

3. $\frac{3c}{4c-5}+\frac{5}{4c-5}$

Answer

$\frac{3c+5}{4c-5}$

4. $\frac{7m}{2m+n}+\frac{4}{2m+n}$

5. $\frac{2r^2}{2r-1}+\frac{15r-8}{2r-1}$

Answer

$r+8$

6. $\frac{3s^2}{3s-2}+\frac{13s-10}{3s-2}$

7. $\frac{2w^2}{w^2-16}+\frac{8w}{w^2-16}$

Answer

$\frac{2w}{w-4}$

8. $\frac{7x^2}{x^2-9}+\frac{21x}{x^2-9}$

In the following exercises, subtract.

9. $\frac{9a^2}{3a-7}-\frac{49}{3a-7}$

Answer

$3a+7$

10. $\dfrac{25b^2}{5b-6}-\dfrac{36}{5b-6}$

11. $\dfrac{3m^2}{6m-30}-\dfrac{21m-30}{6m-30}$

Answer

$\dfrac{m-2}{2}$

12. $\dfrac{2n^2}{4n-32}-\dfrac{18n-16}{4n-32}$

13. $\dfrac{6p^2+3p+4}{p^2+4p-5}-\dfrac{5p^2+p+7}{p^2+4p-5}$

Answer

$\dfrac{p+3}{p+5}$

14. $\dfrac{5q^2+3q-9}{q^2+6q+8}-\dfrac{4q^2+9q+7}{q^2+6q+8}$

15. $\dfrac{5r^2+7r-33}{r^2-49}-\dfrac{4r^2+5r+30}{r^2-49}$

Answer

$\dfrac{r+9}{r+7}$

16. $\dfrac{7t^2-t-4}{t^2-25}-\dfrac{6t^2+12t-4}{4t^2-25}$

Add and Subtract Rational Expressions whose Denominators are Opposites

In the following exercises, add or subtract.

17. $\dfrac{10v}{2v-1}+\dfrac{2v+4}{1-2v}$

Answer

4

18. $\dfrac{20w}{5w-2}+\dfrac{5w+6}{2-5w}$

19. $\dfrac{10x^2+16x-7}{8x-3}+\dfrac{2x^2+3x-1}{3-8x}$

Answer

$x+2$

20. $\frac{6y^2+2y-11}{3y-7}+\frac{3y^2-3y+17}{7-3y}$

21. $\frac{z^2+6z}{z^2-25}-\frac{3z+20}{25-z^2}$

Answer

$\frac{z+4}{z-5}$

22. $\frac{a^2+3a}{a^2-9}-\frac{3a-27}{9-a^2}$

23. $\frac{2b^2+30b-13}{b^2-49}-\frac{2b^2-5b-8}{49-b^2}$

Answer

$\frac{4b-3}{b-7}$

24. $\frac{c^2+5c-10}{c^2-16}-\frac{c^2-8c-10}{16-c^2}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, a. find the LCD for the given rational expressions b. rewrite them as equivalent rational expressions with the lowest common denominator.

25. $\frac{5}{x^2-2x-8}, \frac{2x}{x^2-x-12}$

Answer

a. $(x+2)(x-4)(x+3)$
b. $\frac{5x+15}{(x+2)(x-4)(x+3)}$,
$\frac{2x^2+4x}{(x+2)(x-4)(x+3)}$

26. $\frac{8}{y^2+12y+35}, \frac{3y}{y^2+y-42}$

27. $\frac{9}{z^2+2z-8}, \frac{4z}{z^2-4}$

Answer

a. $(z-2)(z+4)(z-4)$
b. $\frac{9z-36}{(z-2)(z+4)(z-4)}$,
$\frac{4z^2-8z}{(z-2)(z+4)(z-4)}$

28. $\frac{6}{a^2+14a+45}, \frac{5a}{a^2-81}$

29. $\frac{4}{b^2+6b+9}, \frac{2b}{b^2-2b-15}$

Answer

a. $(b+3)(b+3)(b-5)$
b. $\frac{4b-20}{(b+3)(b+3)(b-5)}$,
$\frac{2b^2+6b}{(b+3)(b+3)(b-5)}$

30. $\frac{5}{c^2-4c+4}, \frac{3c}{c^2-7c+10}$

31. $\frac{2}{3d^2+14d-5}, \frac{5d}{3d^2-19d+6}$

Answer

a. $(d+5)(3d-1)(d-6)$
b. $\frac{2d-12}{(d+5)(3d-1)(d-6)}$,
$\frac{5d^2+25d}{(d+5)(3d-1)(d-6)}$

32. $\frac{3}{5m^2-3m-2}, \frac{6m}{5m^2+17m+6}$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

33. $\frac{7}{10x^2y}+\frac{4}{15xy^2}$

Answer

$\frac{21y+8x}{30x^2y^2}$

34. $\frac{1}{12a^3b^2}+\frac{5}{9a^2b^3}$

35. $\frac{3}{r+4}+\frac{2}{r-5}$

Answer

$\frac{5r-7}{(r+4)(r-5)}$

36. $\frac{4}{s-7}+\frac{5}{s+3}$

37. $\frac{5}{3w-2}+\frac{2}{w+1}$

Answer

$$\frac{11w+1}{(3w-2)(w+1)}$$

38. $\frac{4}{2x+5}+\frac{2}{x-1}$

39. $\frac{2y}{y+3}+\frac{3}{y-1}$

Answer

$$\frac{2y^2+y+9}{(y+3)(y-1)}$$

40. $\frac{3z}{z-2}+\frac{1}{z+5}$

41. $\frac{5b}{a^2b-2a^2}+\frac{2b}{b^2-4}$

Answer

$$\frac{b(5b+10+2a^2)}{a^2(b-2)(b+2)}$$

42. $\frac{4}{cd+3c}+\frac{1}{d^2-9}$

43. $\frac{-3m}{3m-3}+\frac{5m}{m^2+3m-4}$

Answer

$$-\frac{m}{m+4}$$

44. $\frac{8}{4n+4}+\frac{6}{n^2-n-2}$

45. $\frac{3r}{r^2+7r+6}+\frac{9}{r^2+4r+3}$

Answer

$$\frac{3(r^2+6r+18)}{(r+1)(r+6)(r+3)}$$

46. $\frac{2s}{s^2+2s-8}+\frac{4}{s^2+3s-10}$

47. $\frac{t}{t-6}-\frac{t-2}{t+6}$

Answer

$$\frac{2(7t-6)}{(t-6)(t+6)}$$

48. $\frac{x-3}{x+6}-\frac{x}{x+3}$

49. $\frac{5a}{a+3}-\frac{a+2}{a+6}$

Answer

$$\frac{4a^2+25a-6}{(a+3)(a+6)}$$

50. $\frac{3b}{b-2}-\frac{b-6}{b-8}$

51. $\frac{6}{m+6}-\frac{12m}{m^2-36}$

Answer

$$\frac{-6}{m-6}$$

52. $\frac{4}{n+4}-\frac{8n}{n^2-16}$

53. $\frac{-9p-17}{p^2-4p-21}-\frac{p+1}{7-p}$

Answer

$$\frac{p+2}{p+3}$$

54. $\frac{-13q-8}{q^2+2q-24}-\frac{q+2}{4-q}$

55. $\frac{-2r-16}{r^2+6r-16}-\frac{5}{2-r}$

Answer

$$\frac{3}{r-2}$$

56. $\frac{2t-30}{t^2+6t-27}-\frac{2}{3-t}$

57. $\frac{2x+7}{10x-1}+3$

Answer

$$\frac{4(8x+1)}{10x-1}$$

58. $\frac{8y-4}{5y+2}-6$

59. $\frac{3}{x^2-3x-4}-\frac{2}{x^2-5x+4}$

Answer

$$\frac{x-5}{(x-4)(x+1)(x-1)}$$

60. $\frac{4}{x^2-6x+5}-\frac{3}{x^2-7x+10}$

61. $\frac{5}{x^2+8x-9}-\frac{4}{x^2+10x+9}$

Answer

$$\frac{1}{(x-1)(x+1)}$$

62. $\frac{3}{2x^2+5x+2}-\frac{1}{2x^2+3x+1}$

63. $\frac{5a}{a-2}+\frac{9}{a}-\frac{2a+18}{a^2-2a}$

Answer

$$\frac{5a^2+7a-36}{a(a-2)}$$

64. $\frac{2b}{b-5}+\frac{3}{2b}-\frac{2b-15}{2b^2-10b}$

65. $\frac{c}{c+2}+\frac{5}{c-2}-\frac{10c}{c^2-4}$

Answer

$$\frac{c-5}{c+2}$$

66. $\frac{6d}{d-5}+\frac{1}{d+4}+\frac{7d-5}{d^2-d-20}$

67. $\frac{3d}{d+2}+\frac{4}{d}-\frac{d+8}{d^2+2d}$

Answer

$$\frac{3(d+1)}{d+2}$$

68. $\frac{2q}{q+5}+\frac{3}{q-3}-\frac{13q+15}{q^2+2q-15}$

Writing Exercises

69. Donald thinks that $\frac{3}{x}+\frac{4}{x}$ is $\frac{7}{2x}$. Is Donald correct? Explain.

Answer

Answers will vary.

70. Explain how you find the Least Common Denominator of x^2+5x+4 and x^2-16.

71. Felipe thinks $\frac{1}{x}+\frac{1}{y}$ is $\frac{2}{x+y}$.

a. Choose numerical values for *x* and *y* and evaluate $\frac{1}{x}+\frac{1}{y}$.

b. Evaluate $\frac{2}{x+y}$ for the same values of x and y you used in part a..

c. Explain why Felipe is wrong.

d. Find the correct expression for $1x+1y$.

Answer

a. Answers will vary.
b. Answers will vary.
c. Answers will vary.
d. $\frac{x+y}{x}$

72. Simplify the expression $\frac{4}{n^2+6n+9}-\frac{1}{n^2-9}$ and explain all your steps.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No, I don't get it!
add and subtract rational expressions with a common denominator.			
add and subtract rational expressions whose denominators are opposites			
find the least common denominator of rational expressions			
add and subtract rational expressions with unlike denominators			

b. After reviewing this checklist, what will you do to become confident for all objectives?

5.3: Simplify Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.
2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.
3. Solve: $\frac{1}{2x}+\frac{1}{4}=\frac{1}{8}$.

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \quad \text{and} \quad \frac{\frac{x}{2}}{\frac{xy}{6}}.$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Definition 5.3.1

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are some complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}, \quad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \quad \text{and} \quad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter:

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}.$$

We noted that fraction bars tell us to divide, and rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right)\div\left(\frac{2x^2-8x+3}{x^2-5x+6}\right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example 5.3.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}.$$

Solution

	$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$
Rewrite the complex fraction as division.	$=\frac{6}{x-4} \div \frac{3}{x^2-16}$
Rewrite as the product of first times the reciprocal of the second.	$=\frac{6}{x-4} \cdot \frac{x^2-16}{3}$
Factor.	$=\frac{3 \cdot 2}{x-4} \cdot \frac{(x-4)(x+4)}{3}$
Multiply.	$=\frac{3 \cdot 2(x-4)(x+4)}{3(x-4)}$
Remove common factors.	$=\frac{\cancel{3} \cdot 2\,\cancel{(x-4)}\,(x+4)}{\cancel{3}\,\cancel{(x-4)}}$
Simplify.	$=2(x+4)$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Try It 5.3.3

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}.$$

Answer

$$\frac{2}{3(x-1)}$$

Try It 5.3.4

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}.$$

Answer

$$\frac{1}{2(x-3)}$$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example 5.3.5

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}.$$

Solution

	$\dfrac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.	$=\dfrac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$=\dfrac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$ $=\dfrac{\frac{3}{6}}{\frac{1}{6}}$
Rewrite the complex rational expression as a division problem.	$=\frac{3}{6}\div\frac{1}{6}$
Multiply the first by the reciprocal of the second.	$=\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	$=3$

Try It 5.3.6

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$$

Answer

$$\frac{14}{11}$$

Try It 5.3.7

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$$

Answer

$\frac{10}{23}$

We follow the same procedure when the complex rational expression contains variables.

Example 5.3.8

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}.$$

Solution

	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$	
Simplify the numerator.	We will simplify the sum in the numerator and the difference in the denominator.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find the LCD in the numerator and denominator.	$=\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Simplify.	$=\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Add the fractions in the numerator and subtract the fractions in the denominator. We now have just one rational expression in the numerator and one in the denominator.	$=\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$=\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Divide the expressions.	Multiply the first by the reciprocal of the second.	$=\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$=\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$=\frac{\cancel{xy}\,\cancel{(y+x)}}{\cancel{xy}(x-y)\,\cancel{(x+y)}}$
	Simplify.	$=\frac{1}{x-y}$

Try It 5.3.9

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}.$$

Answer

$$\frac{y+x}{y-x}$$

Try It 5.3.10

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}.$$

Answer

$$\frac{ab}{b-a}$$

We summarize the steps here.

How to Simplify a Complex Rational Expression by Writing It as Division

1. Rewrite the complex rational expression as a division problem.
2. Divide the expressions.

Example 5.3.11

Simplify the complex rational expression by writing it as division:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}.$$

Solution

	$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator. Find common denominators for the numerator and denominator.	$=\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$=\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$

	$\dfrac{n-\dfrac{4n}{n+5}}{\dfrac{1}{n+5}+\dfrac{1}{n-5}}$
Subtract the rational expressions in the numerator and add in the denominator.	$=\dfrac{\dfrac{n^2+5n-4n}{n+5}}{\dfrac{n-5+n+5}{(n+5)(n-5)}}$
Simplify. (We now have one rational expression over one rational expression.)	$=\dfrac{\dfrac{n^2+n}{n+5}}{\dfrac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$=\dfrac{n^2+n}{n+5}\div\dfrac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$=\dfrac{n^2+n}{n+5}\cdot\dfrac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$=\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$=\dfrac{\cancel{n}(n+1)\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\cancel{n}}$
Simplify.	$=\dfrac{(n+1)(n-5)}{2}$

Try It 5.3.12

Simplify the complex rational expression by writing it as division:

$$\dfrac{b-\dfrac{3b}{b+5}}{\dfrac{2}{b+5}+\dfrac{1}{b-5}}.$$

Answer

$$\dfrac{b(b+2)(b-5)}{3b-5}$$

Try It 5.3.13

Simplify the complex rational expression by writing it as division:

$$\dfrac{1-\dfrac{3}{c+4}}{\dfrac{1}{c+4}+\dfrac{c}{3}}.$$

Answer

$$\dfrac{3}{c+3}$$

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.4.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

Example 5.3.14

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}.$$

Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6. Clear the fractions by multiplying the numerator and denominator by that LCD.	$=\frac{{\color{red}6}\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{{\color{red}6}\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$=\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$=\frac{2+1}{3-2}$ $=\frac{3}{1}$ $=3$

Try It 5.3.15

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}.$$

Answer

$\frac{7}{3}$

Try It 5.3.16

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}.$$

Answer

$\frac{10}{3}$

We will use the same example as in Example 7.4.3. Decide which method works better for you.

Example 5.3.17

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}.$$

Solution

		$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Find the LCD of all fractions in the is complex rational expression.	The LCD of all the fractions is xy.	$=\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$=\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Simplify the expression.	Distribute.	$=\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$ $=\frac{y+x}{x^2-y^2}$
	Simplify.	$=\frac{\cancel{(y+x)}}{(x-y)\cancel{(x+y)}}$
	Remove common factors.	$=\frac{1}{x-y}$

Try It 5.3.18

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}.$$

Answer

$$\frac{b+a}{a^2+b^2}$$

Try It 5.3.19

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}.$$

Answer

$$\frac{y-x}{xy}$$

How to Simplify a Complex Rational Expression by Using the LCD

1. Multiply the numerator and denominator by the LCD of all rational expressions.
2. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 5.3.20

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Solution

	$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is: $x^2-36=(x+6)(x-6)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Distribute in the denominator.	$=\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$=\frac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$=\frac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$=\frac{2(x-6)}{4x+20}$
Factor the denominator.	$=\frac{2(x-6)}{4(x+5)}$
Remove common factors.	$=\frac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$
Simplify.	$=\frac{x-6}{2(x+5)}$ Notice that there are no more factors common to the numerator and denominator.

Try It 5.3.21

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}.$$

Answer

$\frac{3(x-2)}{5x+7}$

Try It 5.3.22

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}.$$

Answer

$\frac{x+21}{6x-43}$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example 5.3.23

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}.$$

Solution

	$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is $(m-3)(m-4)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$=\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$
Simplify.	$=\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$=\frac{4}{3m-12-2m+6}$

	$\dfrac{\dfrac{4}{m^2-7m+12}}{\dfrac{3}{m-3}-\dfrac{2}{m-4}}$
Combine like terms.	$=\dfrac{4}{m-6}$

Try It 5.3.24

Simplify the complex rational expression by using the LCD:

$$\frac{\dfrac{3}{x^2+7x+10}}{\dfrac{4}{x+2}+\dfrac{1}{x+5}}.$$

Answer

$\dfrac{3}{5x+22}$

Try It 5.3.25

Simplify the complex rational expression by using the LCD:

$$\frac{\dfrac{4y}{y+5}+\dfrac{2}{y+6}}{\dfrac{3y}{y^2+11y+30}}.$$

Answer

$\dfrac{2\left(2y^2+13y+5\right)}{3y}$

Example 5.3.26

Simplify the complex rational expression by using the LCD:

$$\frac{\dfrac{y}{y+1}}{1+\dfrac{1}{y-1}}.$$

Solution

	$\dfrac{\dfrac{y}{y+1}}{1+\dfrac{1}{y-1}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is $(y+1)(y-1)$.
Multiply the numerator and denominator by the LCD.	$=\dfrac{(y+1)(y-1)\dfrac{y}{y+1}}{(y+1)(y-1)\left(1+\dfrac{1}{y-1}\right)}$

	$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$
Distribute in the denominator and simplify.	$$=\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$$
Simplify.	$$=\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$$
Simplify the denominator and leave the numerator factored.	$$=\frac{y(y-1)}{y^2-1+y+1}$$ $$=\frac{y(y-1)}{y^2+y}$$
Factor the denominator and remove factors common with the numerator.	$$=\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$$
Simplify.	$$=\frac{y-1}{y+1}$$

Try It 5.3.27

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}.$$

Answer

$$\frac{x}{x+4}$$

Try It 5.3.28

Simplify the complex rational expression by using the LCD:

$$\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}.$$

Answer

$$\frac{x(x+1)}{3(x-1)}$$

Key Concepts

- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.

3. Simplify the expression.

Glossary

complex rational expression

A complex rational expression is a rational expression in which the numerator and/or denominator contains a rational expression.

Practice Makes Perfect

Simplify a Complex Rational Expression by Writing it as Division

In the following exercises, simplify each complex rational expression by writing it as division.

1. $\dfrac{\dfrac{2a}{a+4}}{\dfrac{4a^2}{a^2-16}}$

Answer

$\dfrac{a-4}{2a}$

2. $\dfrac{\dfrac{3b}{b-5}}{\dfrac{b^2}{b^2-25}}$

3. $\dfrac{\dfrac{5}{c^2+5c-14}}{\dfrac{10}{c+7}}$

Answer

$\dfrac{1}{2(c-2)}$

4. $\dfrac{\dfrac{8}{d^2+9d+18}}{\dfrac{12}{d+6}}$

5. $\dfrac{\frac{1}{2}+\frac{5}{6}}{\frac{2}{3}+\frac{7}{9}}$

Answer

$\dfrac{12}{13}$

6. $\dfrac{\frac{1}{2}+\frac{3}{4}}{\frac{3}{5}+\frac{7}{10}}$

7. $\dfrac{\frac{2}{3}-\frac{1}{9}}{\frac{3}{4}+\frac{5}{6}}$

Answer

$\frac{20}{57}$

8. $\dfrac{\frac{1}{2}-\frac{1}{6}}{\frac{2}{3}+\frac{3}{4}}$

9. $\dfrac{\frac{n}{m}+\frac{1}{n}}{\frac{1}{n}-\frac{n}{m}}$

Answer

$\frac{n^2+m}{m-n^2}$

10. $\dfrac{\frac{1}{p}+\frac{p}{q}}{\frac{q}{p}-\frac{1}{q}}$

11. $\dfrac{\frac{1}{r}+\frac{1}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$

Answer

$\frac{rt}{t-r}$

12. $\dfrac{\frac{2}{v}+\frac{2}{w}}{\frac{1}{v^2}-\frac{1}{w^2}}$

13. $\dfrac{x-\frac{2x}{x+3}}{\frac{1}{x+3}+\frac{1}{x-3}}$

Answer

$\frac{(x+1)(x-3)}{2}$

14. $\dfrac{y-\frac{2y}{y-4}}{\frac{2}{y-4}+\frac{2}{y+4}}$

15. $\dfrac{2-\frac{2}{a+3}}{\frac{1}{a+3}+\frac{a}{2}}$

Answer

$$\frac{4}{a+1}$$

16. $\dfrac{4+\dfrac{4}{b-5}}{\dfrac{1}{b-5}+\dfrac{b}{4}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify each complex rational expression by using the LCD.

17. $\dfrac{\dfrac{1}{3}+\dfrac{1}{8}}{\dfrac{1}{4}+\dfrac{1}{12}}$

Answer

$$\frac{11}{8}$$

18. $\dfrac{\dfrac{1}{4}+\dfrac{1}{9}}{\dfrac{1}{6}+\dfrac{1}{12}}$

19. $\dfrac{\dfrac{5}{6}+\dfrac{2}{9}}{\dfrac{7}{18}-\dfrac{1}{3}}$

Answer

$$19$$

20. $\dfrac{\dfrac{1}{6}+\dfrac{4}{15}}{\dfrac{3}{5}-\dfrac{1}{2}}$

21. $\dfrac{\dfrac{c}{d}+\dfrac{1}{d}}{\dfrac{1}{d}-\dfrac{d}{c}}$

Answer

$$\frac{c^2+c}{c-d^2}$$

22. $\dfrac{\dfrac{1}{m}+\dfrac{m}{n}}{\dfrac{n}{m}-\dfrac{1}{n}}$

23. $\dfrac{\dfrac{1}{p}+\dfrac{1}{q}}{\dfrac{1}{p^2}-\dfrac{1}{q^2}}$

Answer

$$\frac{pq}{q-p}$$

24. $$\frac{\frac{2}{r}+\frac{2}{t}}{\frac{1}{r^2}-\frac{1}{t^2}}$$

25. $$\frac{\frac{2}{x+5}}{\frac{3}{x-5}+\frac{1}{x^2-25}}$$

Answer

$$\frac{2x-10}{3x+16}$$

26. $$\frac{\frac{5}{y-4}}{\frac{3}{y+4}+\frac{2}{y^2-16}}$$

27. $$\frac{\frac{5}{z^2-64}+\frac{3}{z+8}}{\frac{1}{z+8}+\frac{2}{z-8}}$$

Answer

$$\frac{3z-19}{3z+8}$$

28. $$\frac{\frac{3}{s+6}+\frac{5}{s-6}}{\frac{1}{s^2-36}+\frac{4}{s+6}}$$

29. $$\frac{\frac{4}{a^2-2a-15}}{\frac{1}{a-5}+\frac{2}{a+3}}$$

Answer

$$\frac{4}{3a-7}$$

30. $$\frac{\frac{5}{b^2-6b-27}}{\frac{3}{b-9}+\frac{1}{b+3}}$$

31. $$\frac{\frac{5}{c+2}-\frac{3}{c+7}}{\frac{5c}{c^2+9c+14}}$$

Answer

$$\frac{2c+29}{5c}$$

32. $\dfrac{\dfrac{6}{d-4}-\dfrac{2}{d+7}}{\dfrac{2d}{d^2+3d-28}}$

33. $\dfrac{2+\dfrac{1}{p-3}}{\dfrac{5}{p-3}}$

Answer

$$\frac{2p-5}{5}$$

34. $\dfrac{\dfrac{n}{n-2}}{3+\dfrac{5}{n-2}}$

35. $\dfrac{\dfrac{m}{m+5}}{4+\dfrac{1}{m-5}}$

Answer

$$\frac{m(m-5)}{(4m-19)(m+5)}$$

36. $\dfrac{7+\dfrac{2}{q-2}}{\dfrac{1}{q+2}}$

In the following exercises, simplify each complex rational expression using either method.

37. $\dfrac{\dfrac{3}{4}-\dfrac{2}{7}}{\dfrac{1}{2}+\dfrac{5}{14}}$

Answer

$$\frac{13}{24}$$

38. $\dfrac{\dfrac{v}{w}+\dfrac{1}{v}}{\dfrac{1}{v}-\dfrac{v}{w}}$

39. $\dfrac{\dfrac{2}{a+4}}{\dfrac{1}{a^2-16}}$

Answer

$$2(a-4)$$

40. $\dfrac{\dfrac{3}{b^2-3b-40}}{\dfrac{5}{b+5}-\dfrac{2}{b-8}}$

41. $\dfrac{\dfrac{3}{m}+\dfrac{3}{n}}{\dfrac{1}{m^2}-\dfrac{1}{n^2}}$

Answer

$$\frac{3mn}{n-m}$$

42. $\dfrac{\dfrac{2}{r-9}}{\dfrac{1}{r+9}+\dfrac{3}{r^2-81}}$

43. $\dfrac{x-\dfrac{3x}{x+2}}{\dfrac{3}{x+2}+\dfrac{3}{x-2}}$

Answer

$$\frac{(x-1)(x-2)}{6}$$

44. $\dfrac{\dfrac{y}{y+3}}{2+\dfrac{1}{y-3}}$

Writing Exercises

45. In this section, you learned to simplify the complex fraction $\dfrac{\dfrac{3}{x+2}}{\dfrac{x}{x^2-4}}$ two ways: rewriting it as a division problem or multiplying the numerator and denominator by the LCD. Which method do you prefer? Why?

Answer

Answers will vary.

46. Efraim wants to start simplifying the complex fraction $\dfrac{\dfrac{1}{a}+\dfrac{1}{b}}{\dfrac{1}{a}-\dfrac{1}{b}}$ by cancelling the variables from the numerator and denominator, $\dfrac{\dfrac{1}{\cancel{a}}+\dfrac{1}{\cancel{b}}}{\dfrac{1}{\cancel{a}}-\dfrac{1}{\cancel{b}}}$. Explain what is wrong with Efraim's plan.

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5.4: Solve Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Be Prepared

Before you get started, take this readiness quiz.

1. Solve: $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$
2. Solve: $n^2-5n-36=0$
3. Solve the formula $5x+2y=10$ for y

After defining the terms 'expression' and 'equation' earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve a **rational equation**.

Definition 5.4.1

A **rational equation** is an equation that contains a rational expression.

Let's recall the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

Sometimes we have used equal signs during a simplification of an expression. These equal signs implicitly mean that equation holds for all values of the variables except those that leave an expression undefined. In the case of the equation, these equalities say something about the variable, and for 'most' values of the variable the equation is false. We will seek the values which make the equality true, .i.e., we seek to solve these equations.

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to "clear" the fractions.

We will use the same strategy to solve rational equations. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve. But because the original equation may have a variable in a denominator, we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard.

An algebraic solution to a rational equation that would cause any of the rational expressions to be undefined is called an **extraneous solution to a rational equation**.

Definition 5.4.2

An **extraneous solution to a rational equation** is an algebraic solution to an equation that is equivalent to the original except for a certain finite number of values that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x \neq c$ next to the equation.

Example 5.4.3

Solve:

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

Solution

Step 1. Note any value of the variable that would make any denominator zero.

If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$$

Step 2. Find the least common denominator of all denominators in the equation.

Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$

The LCD is $6x$.

Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.

Multiply both sides of the equation by the LCD, $6x$.

$$\color{red}{6x} \cdot\left(\frac{1}{x}+\frac{1}{3}\right)=\color{red}{6x} \cdot\left(\frac{5}{6}\right)$$

Use the Distributive Property.

$$\color{red}{6x} \cdot \frac{1}{x}+\color{red}{6x} \cdot \frac{1}{3}=\color{red}{6x} \cdot\left(\frac{5}{6}\right)$$

Simplify - and notice, no more fractions!

$$6+2x=5x$$

Step 4. Solve the resulting equation.

Simplify.

$$\begin{aligned} 6&=3x \\ 2&=x \end{aligned}$$

Step 5. Check.

If any values found in Step 1 are algebraic solutions, discard them. Check any remaining solutions in the original equation. (This is not stricly necessary, but is helpful in case an error was made).

We did not get 0 as an algebraic solution.

$$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$$

We substitute $x=2$ into the original equation.

$$\begin{aligned}\frac{1}{2}+\frac{1}{3} &\stackrel{?}{=} \frac{5}{6}\\ \frac{3}{6}+\frac{2}{6} &\stackrel{?}{=} \frac{5}{6}\\ \frac{5}{6} &= \frac{5}{6}\surd\end{aligned}$$

The solution is $x=2$

Try It 5.4.4

Solve:

$$\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$$

Answer

$$y=-\frac{7}{15}$$

Try It 5.4.5

Solve:

$$\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$$

Answer

$$x=\frac{13}{15}$$

The steps of this method are shown.

How to solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 5.4.6

Solve:

$$1-\frac{5}{y}=-\frac{6}{y^2}$$

Solution

Note any value of the variable that would make any denominator zero.

$$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$$

Find the least common denominator of all denominators in the equation. The LCD is y^2.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Distribute.

$$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$$

Multiply.

$$y^2-5y=-6$$

Solve the resulting equation. First write the quadratic equation in standard form.

$$y^2-5y+6=0$$

Factor.

$$(y-2)(y-3)=0$$

Use the Zero Product Property.

$$y-2=0 \text{ or } y-3=0$$

Solve.

$$y=2 \text{ or } y=3$$

Check. We did not get 0 as an algebraic solution.

Check $y=2$ and $y=3$ in the original equation.

$$\begin{array}{rcl} 1-\frac{5}{y}=-\frac{6}{y^2} & \qquad & 1-\frac{5}{y}=-\frac{6}{y^2} \\ 1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2} & & 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2} \\ 1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} & & 1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9} \\ \frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4} & & \frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9} \\ -\frac{3}{2}\stackrel{?}{=}-\frac{6}{4} & & -\frac{2}{3}\stackrel{?}{=}-\frac{6}{9} \\ -\frac{3}{2}=-\frac{3}{2}\checkmark & & -\frac{2}{3}=-\frac{2}{3}\checkmark \end{array}$$

The solution is $y=2, y=3$

Try It 5.4.7

Solve:

$$1-\frac{2}{x}=\frac{15}{x^2}$$

Answer

$x=-3, x=5$

? Try It 5.4.8

Solve:

$$1-\frac{4}{y}=\frac{12}{y^2}$$

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 5.4.9

Solve:

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$$

Solution

Note any value of the variable that would make any denominator zero.

$$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x \neq -2, x \neq 2$$

Find the least common denominator of all denominators in the equation. The LCD is $(x+2)(x-2)$.

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Distribute.

$$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$$

Remove common factors.

$$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\ \cancel{(x-2)}\frac{4}{\cancel{x-2}}=\ \cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$$

Simplify.

$$2(x-2)+4(x+2)=x-1$$

Distribute.

$$2x-4+4x+8=x-1$$

Solve.

$$\begin{aligned} 6x+4 &= x-1 \\ 5x &= -5 \\ x &= -1 \end{aligned}$$

Check: We did not get 2 or −2 as algebraic solutions.

Check $x=-1$ in the original equation.

$$\begin{aligned}
\frac{2}{x+2}+\frac{4}{x-2} &= \frac{x-1}{x^2-4} \\
\frac{2}{(-1)+2}+\frac{4}{(-1)-2} &\stackrel{?}{=} \frac{(-1)-1}{(-1)^2-4} \\
\frac{2}{1}+\frac{4}{-3} &\stackrel{?}{=} \frac{-2}{-3} \\
\frac{6}{3}-\frac{4}{3} &\stackrel{?}{=} \frac{2}{3} \\
\frac{2}{3} &= \frac{2}{3}\surd
\end{aligned}$$

The solution is $x=-1$.

Try It 5.4.10

Solve:

$$\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$$

Answer

$x=\frac{2}{3}$

Try It 5.4.11

Solve:

$$\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$$

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example 5.4.12

Solve:

$$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$$

Solution

Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.

$$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m\neq 4, m\neq 1$$

Find the least common denominator of all denominators in the equation. The LCD is $(m-4)(m-1)$

Clear the fractions by multiplying both sides of the equation by the LCD.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$$

Distribute.

$$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-1}$$

Remove common factors.

$$(m-4)\cancel{(m-1)}\left(\frac{m+11}{(m-4)\cancel{(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\ \cancel{(m-1)}\frac{3}{\cancel{m-1}}$$

Simplify.

$$m+11=5(m-1)-3(m-4)$$

Solve the resulting equation.

$$\begin{aligned}m+11&=5m-5-3m+12\\4&=m\end{aligned}$$

Check. The only algebraic solution was 4, but we said that 4 would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Try It 5.4.13

Solve:

$$\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$$

Answer

There is no solution.

Try It 5.4.14

Solve:

$$\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$$

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

Example 5.4.15

Solve:

$$\frac{y}{y+6}=\frac{72}{y^2-36}+4$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$$

Find the least common denominator. The LCD is $(y-6)(y+6)$

Clear the fractions.

$$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$$

Simplify.

$$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$$

Simplify.

$$y(y-6)=72+4\left(y^2-36\right)$$

Solve the resulting equation.

$$\begin{aligned} y^2-6y &= 72+4y^2-144 \\ 0 &= 3y^2+6y-72 \\ 0 &= 3\left(y^2+2y-24\right) \\ 0 &= 3(y+6)(y-4) \\ y &= -6, y=4 \end{aligned}$$

Check.

$y=-6$ is an extraneous solution. Check $y=4$ in the original equation.

$$\begin{aligned} \frac{y}{y+6} &= \frac{72}{y^2-36}+4 \\ \frac{4}{4+6} &\stackrel{?}{=} \frac{72}{4^2-36}+4 \\ \frac{4}{10} &\stackrel{?}{=} \frac{72}{-20}+4 \\ \frac{4}{10} &\stackrel{?}{=} -\frac{36}{10}+\frac{40}{10} \\ \frac{4}{10} &= \frac{4}{10}\checkmark \end{aligned}$$

The solution is $y=4$.

? Try It 5.4.16

Solve:

$$\frac{x}{x+4}=\frac{32}{x^2-16}+5$$

Answer

$x=3$

? Try It 5.4.17

Solve:

$$\frac{y}{y+8}=\frac{128}{y^2-64}+9$$

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

Example 5.4.18

Solve:

$$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$$

Solution

We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$$

Note any value of the variable that would make any denominator zero.

$$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$$

Find the least common denominator. The LCD is $12(x-1)(x+1)$.

Clear the fractions.

$$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$$

Simplify.

$$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$$

Simplify.

$$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$$

Solve the resulting equation.

$$\begin{aligned} 6x^2+6x-8x+8 &= 5x^2-2x+9 \\ x^2-1 &= 0 \\ (x-1)(x+1) &= 0 \\ x &= 1 \text{ or } x=-1 \end{aligned}$$

Check.

$x=1$ and $x=-1$ are extraneous solutions.

The equation has no solution.

Try It 5.4.19

Solve:

$$\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$$

Answer

There is no solution.

Try It 5.4.20

Solve:

$$\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-16}{8z^2+2z-64}$$

Answer

There is no solution.

Example 5.4.21

Solve:

$$\frac{4}{3x^2-10x+3} + \frac{3}{3x^2+2x-1} = \frac{2}{x^2-2x-3}$$

Solution

Factor all the denominators, so we can note any value of the variable that would make any denominator zero.

$$\frac{4}{(3x-1)(x-3)} + \frac{3}{(3x-1)(x+1)} = \frac{2}{(x-3)(x+1)}, x \neq -1, x \neq \frac{1}{3}, x \neq 3$$

Find the least common denominator. The LCD is $(3x-1)(x+1)(x-3)$.

Clear the fractions.

$$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)} + \frac{3}{(3x-1)(x+1)}\right) = (3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$$

Simplify.

$$4(x+1)+3(x-3) = 2(3x-1)$$

Distribute.

$$4x+4+3x-9 = 6x-2$$

Simplify.

$$7x-5 = 6x-2$$

$$x = 3$$

The only algebraic solution was $x=3$, but we said that $x=3$ would make a denominator equal to zero. The algebraic solution is an extraneous solution.

There is no solution to this equation.

Try It 5.4.22

Solve:

$$\frac{15}{x^2+x-6} - \frac{3}{x-2} = \frac{2}{x+3}$$

Answer

There is no solution.

Try It 5.4.23

Solve:

$$\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$$

Answer

There is no solution.

Solve a Rational Equation for a Specific Variable (Optional)

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} \\ m\left(x-x_1\right) &= \left(\frac{y-y_1}{x-x_1}\right)\left(x-x_1\right) \quad \text{Multiply both sides of the equation by } x-x_1. \\ m\left(x-x_1\right) &= y-y_1 \quad \text{Simplify.} \\ y-y_1 &= m\left(x-x_1\right) \quad \text{Rewrite the equation with the y terms on the left.} \end{aligned}$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 5.4.24

Solve: $m=\dfrac{y-2}{x-3}$ for y.

Solution

$$m=\frac{y-2}{x-3}$$

Note any value of the variable that would make any denominator zero.

$$m=\frac{y-2}{x-3}, x\neq 3$$

Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.

$$(x-3)m=(x-3)\left(\frac{y-2}{x-3}\right)$$

Simplify.

$$xm-3m=y-2$$

Isolate the term with y.

$$xm-3m+2=y$$

Try It 5.4.25

Solve: $m=\dfrac{y-5}{x-4}$ for y.

Answer

$y = mx - 4m + 5$

Try It 5.4.26

Solve: $m = \frac{y-1}{x+5}$ for y.

Answer

$y = mx + 5m + 1$

Remember to multiply both sides by the LCD in the next example.

Example 5.4.27

Solve: $\frac{1}{c} + \frac{1}{m} = 1$ for c

Solution

$$\frac{1}{c} + \frac{1}{m} = 1 \text{ for } c$$

Note any value of the variable that would make any denominator zero.

$$\frac{1}{c} + \frac{1}{m} = 1, c \neq 0, m \neq 0$$

Clear the fractions by multiplying both sides of the equations by the LCD, cm.

$$cm\left(\frac{1}{c} + \frac{1}{m}\right) = cm(1)$$

Distribute.

$$cm\left(\frac{1}{c}\right) + cm\frac{1}{m} = cm(1)$$

Simplify.

$$m + c = cm$$

Collect the terms with c to the right.

$$m = cm - c$$

Factor the expression on the right.

$$m = c(m-1)$$

To isolate c, divide both sides by $m-1$.

$$\frac{m}{m-1} = \frac{c(m-1)}{m-1}$$

Simplify by removing common factors.

$$\frac{m}{m-1} = c$$

Notice that even though we excluded $c = 0$ and $m = 0$ from the original equation, we must also now state that $m \neq 1$.

? Try It 5.4.28

Solve: $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

? Try It 5.4.29

Solve: $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y

Answer

$$y=\frac{3x}{x+6}$$

Practice Makes Perfect

Solve Rational Equations

In the following exercises, solve each rational equation.

1. $\frac{1}{a}+\frac{2}{5}=\frac{1}{2}$

Answer

$$a=10$$

2. $\frac{6}{3}-\frac{2}{d}=\frac{4}{9}$

3. $\frac{4}{5}+\frac{1}{4}=\frac{2}{v}$

Answer

$$v=\frac{40}{21}$$

4. $\frac{3}{8}+\frac{2}{y}=\frac{1}{4}$

5. $1-\frac{2}{m}=\frac{8}{m^2}$

Answer

$$m=-2,\ m=4$$

6. $1+\frac{4}{n}=\frac{21}{n^2}$

7. $1+\frac{9}{p}=\frac{-20}{p^2}$

Answer

$p=-5,\ p=-4$

8. $1-\frac{7}{q}=\frac{-6}{q^2}$

9. $\frac{5}{3v-2}=\frac{7}{4v}$

Answer

$v=14$

10. $\frac{8}{2w+1}=\frac{3}{w}$

11. $\frac{3}{x+4}+\frac{7}{x-4}=\frac{8}{x^2-16}$

Answer

$x=-\frac{4}{5}$

12. $\frac{5}{y-9}+\frac{1}{y+9}=\frac{18}{y^2-81}$

13. $\frac{8}{z-10}-\frac{7}{z+10}=\frac{5}{z^2-100}$

Answer

$z=-145$

14. $\frac{9}{a+11}-\frac{6}{a-11}=\frac{6}{a^2-121}$

15. $\frac{-10}{q-2}-\frac{7}{q+4}=1$

Answer

$q=-18,\ q=-1$

16. $\frac{2}{s+7}-\frac{3}{s-3}=1$

17. $\frac{v-10}{v^2-5v+4}=\frac{3}{v-1}-\frac{6}{v-4}$

Answer

no solution

18. $\frac{w+8}{w^2-11w+28}=\frac{5}{w-7}+\frac{2}{w-4}$

19. $\dfrac{x-10}{x^2+8x+12}=\dfrac{3}{x+2}+\dfrac{4}{x+6}$

Answer

no solution

20. $\dfrac{y-5}{y^2-4y-5}=\dfrac{1}{y+1}+\dfrac{1}{y-5}$

21. $\dfrac{b+3}{3b}+\dfrac{b}{24}=\dfrac{1}{b}$

Answer

$b=-8$

22. $\dfrac{c+3}{12c}+\dfrac{c}{36}=\dfrac{1}{4c}$

23. $\dfrac{d}{d+3}=\dfrac{18}{d^2-9}+4$

Answer

$d=2$

24. $\dfrac{m}{m+5}=\dfrac{50}{m^2-25}+6$

25. $\dfrac{n}{n+2}-3=\dfrac{8}{n^2-4}$

Answer

$m=1$

26. $\dfrac{p}{p+7}-8=\dfrac{98}{p^2-49}$

27. $\dfrac{q}{3q-9}-\dfrac{3}{4q+12}=\dfrac{7q^2+6q+63}{24q^2-216}$

Answer

no solution

28. $\dfrac{r}{3r-15}-\dfrac{1}{4r+20}=\dfrac{3r^2+17r+40}{12r^2-300}$

29. $\dfrac{s}{2s+6}-\dfrac{2}{5s+5}=\dfrac{5s^2-3s-7}{10s^2+40s+30}$

Answer

$s=\dfrac{5}{4}$

30. $\frac{t}{6t-12}-\frac{5}{2t+10}=\frac{t^2-23t+70}{12t^2+36t-120}$

31. $\frac{2}{x^2+2x-8}-\frac{1}{x^2+9x+20}=\frac{4}{x^2+3x-10}$

Answer

$x=-\frac{4}{3}$

32. $\frac{5}{x^2+4x+3}+\frac{2}{x^2+x-6}=\frac{3}{x^2-x-2}$

33. $\frac{3}{x^2-5x-6}+\frac{3}{x^2-7x+6}=\frac{6}{x^2-1}$

Answer

no solution

34. $\frac{2}{x^2+2x-3}+\frac{3}{x^2+4x+3}=\frac{6}{x^2-1}$

Solve a Rational Equation for a Specific Variable

In the following exercises, solve:

35. $\frac{c}{r}=2\pi$ for r

Answer

$r=\frac{C}{2\pi}$

36. $\frac{I}{r}=P$ for r

37. $\frac{v+3}{w-1}=\frac{1}{2}$ for w

Answer

$w=2v+7$

38. $\frac{x+5}{2-y}=\frac{4}{3}$ for y

39. $a=\frac{b+3}{c-2}$ for c

Answer

$c=\frac{b+3+2a}{a}$

40. $m=\frac{n}{2-n}$ for n

41. $\dfrac{1}{p}+\dfrac{2}{q}=4$ for p

Answer

$$p=\frac{q}{4q-2}$$

42. $\dfrac{3}{s}+\dfrac{1}{t}=2$ for s

43. $\dfrac{2}{v}+\dfrac{1}{5}=\dfrac{3}{w}$ for w

Answer

$$w=\frac{15v}{10+v}$$

44. $\dfrac{6}{x}+\dfrac{2}{3}=\dfrac{1}{y}$ for y

45. $\dfrac{m+3}{n-2}=\dfrac{4}{5}$ for n

Answer

$$n=\frac{5m+23}{4}$$

46. $r=\dfrac{s}{3-t}$ for t

47. $\dfrac{E}{e}=m^2$ for c

Answer

$$c=\frac{E}{m^2}$$

48. $\dfrac{R}{T}=W$ for T

49. $\dfrac{3}{x}-\dfrac{5}{y}=\dfrac{1}{4}$ for y

Answer

$$y=\frac{20x}{12-x}$$

50. $c=\dfrac{2}{a}+\dfrac{b}{5}$ for a

Writing Exercises

51. Your class mate is having trouble in this section. Write down the steps you would use to explain how to solve a rational equation.

Answer

Answers will vary.

52. Alek thinks the equation $\frac{y}{y+6}=\frac{72}{y^2-36}+4$ has two solutions, $y=-6$ and $y=4$. Explain why Alek is wrong.

5.5: Applications with Rational Equations

Learning Objectives

- Solve proportions
- Solve similar figure applications
- Solve uniform motion applications
- Solve work applications
- Solve direct variation problems
- Solve inverse variation problems

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$.
2. Solve $n^2-5n-36=0$.
3. Solve the formula $5x+2y=10$ for y.

Solve Proportions

When two rational expressions are equal, the equation relating them is called a **proportion**.

Proportion

A proportion is an equation of the form $\frac{a}{b}=\frac{c}{d}$, where $b\neq 0, d\neq 0$.

The proportion is read "a is to b as c is to d."

The equation $\frac{1}{2}=\frac{4}{8}$ is a proportion because the two fractions are equal. The proportion $\frac{1}{2}=\frac{4}{8}$ is read "1 is to 2 as 4 is to 8."

Since a proportion is an equation with rational expressions, we will solve proportions the same way we solved rational equations. We'll multiply both sides of the equation by the LCD to clear the fractions and then solve the resulting equation.

Example 5.5.1

Solve: $\frac{n}{n+14}=\frac{5}{7}$.

Solution

$$\frac{n}{n+14}=\frac{5}{7}, \quad n\neq -14$$

Multiply both sides by LCD.

$$7(n+14)\left(\frac{n}{n+14}\right)=7(n+14)\left(\frac{5}{7}\right)$$

Remove common factors on each side.

$$7n=5(n+14)$$

Simplify.

$$7n=5n+70$$

Solve for n.

$$2n = 70$$
$$n = 35$$

Check.

$$\frac{n}{n+14} = \frac{5}{7}$$

Substitute $n = 35$

$$\frac{35}{35+14} \stackrel{?}{=} \frac{5}{7}$$

Simplify.

$$\frac{35}{49} \stackrel{?}{=} \frac{5}{7}$$

Show common factors.

$$\frac{5 \cdot 7}{7 \cdot 7} \stackrel{?}{=} \frac{5}{7}$$

Simplify.

$$\frac{5}{7} = \frac{5}{7} \checkmark$$

? Try It 5.5.2

Solve the proportion: $\frac{y}{y+55} = \frac{3}{8}$.

Answer

$y = 33$

? Try It 5.5.3

Solve the proportion: $\frac{z}{z-84} = -\frac{1}{5}$.

Answer

$z = 14$

Notice in the last example that when we cleared the fractions by multiplying by the LCD, the result is the same as if we had cross-multiplied.

$$\begin{array}{rlrl} \frac{n}{n+14} &= \frac{5}{7} & \frac{n}{n+14} &= \frac{5}{7} \\ 7(n+14)\left(\frac{n}{n+14}\right) &= 7(n+14)\left(\frac{5}{7}\right) & \frac{n}{n+14} &= \frac{5}{7} \\ 7n &= 5(n+14) & 7n &= 5(n+14) \end{array}$$

For any proportion, $\frac{a}{b} = \frac{c}{d}$, we get the same result when we clear the fractions by multiplying by the LCD as when we cross-multiply.

$$\begin{array}{cc} \dfrac{a}{b}=\dfrac{c}{d} & \dfrac{a}{b}=\dfrac{c}{d} \\ bd\left(\dfrac{a}{b}=\dfrac{c}{d}\right)bd & \dfrac{a}{b}=\dfrac{c}{d} \\ ad=bc & ad=bc \end{array}$$

To solve applications with proportions, we will follow our usual strategy for solving applications But when we set up the proportion, we must make sure to have the units correct—the units in the numerators must match each other and the units in the denominators must also match each other.

Example $5.5.4$

When pediatricians prescribe acetaminophen to children, they prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of the child's weight. If Zoe weighs 80 pounds, how many milliliters of acetaminophen will her doctor prescribe?

Solution

Identify what we are asked to find, and choose a variable to represent it.

How many ml of acetaminophen will the doctor prescribe?

Let $a = ml$ of acetaminophen.

Write a sentence that gives the information to find it.

If 5 ml is prescribed for every 25 pounds, how much will be prescribed for 80 pounds?

Translate into a proportion—be careful of the units.

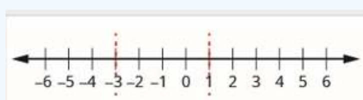
Step 1. Write the inequality as one quotient on the left and zero on the right. Our inequality is in this form.

$$\frac{x-1}{x+3} \geq 0$$

Step 2. Determine the critical points-the points where the rational expression will be zero or undefined.

The rational expression will be zero when the numerator is zero. Since $x-1=0$ when $x=1$, then 1 is a critical point. The rational expression will be undefined when the denominator is zero. Since $x+3=0$ when $x=-3$, then -3 is a critical point.

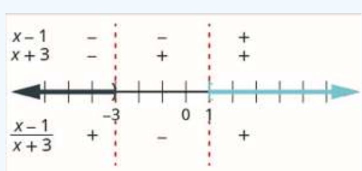
Step 3. Use the critical points to divide the number line into intervals.



Step 4. Above the number line show the sign of each factor of the rational expression in each interval. Below the number line show the sign of the quotient.

Use values in each interval to determine the value of each factor in the interval. In the interval (-3,1), zero is a good value to test. For example, when $x=0$ then $x-1=-1$ and $x+3=3$ The factor $x-1$ is marked negative and $x+3$ marked positive. Since a negative divided by a positive is negative, the quotient is marked negative in that interval.

Step 5. Determine the intervals where the inequality is correct. Write the solution in interval notation.



We want the quotient to be greater than or equal to zero, so the numbers in the intervals $(-\infty, -3)$ and $(1, \infty)$ are solutions. Since 3 must be excluded since it makes the rational expression 0, we cannot include it in the solution. We can include 1 in our solution.

$$(-\infty, -3) \cup [1, \infty)$$

Multiply both sides by the LCD, 400. Remove common factors on each side. Simplify, but don't multiply on the left. Notice what the next step will be.

$$16 \cdot 5 = 5a$$

Solve for a.

$$\frac{16 \cdot 5}{5} = \frac{5a}{5}$$
$$16 = a$$

Check. Is the answer reasonable? Write a complete sentence.

The pediatrician would prescribe 16 ml of acetaminophen to Zoe.

Try It 5.5.5

Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Emilia, who weighs 60 pounds?

Answer

The pediatrician will prescribe 12 ml of acetaminophen to Emilia.

Try It 5.5.6

For every 1 kilogram (kg) of a child's weight, pediatricians prescribe 15 milligrams (mg) of a fever reducer. If Isabella weighs 12 kg, how many milligrams of the fever reducer will the pediatrician prescribe?

Answer

The pediatrician will prescribe 180 mg of fever reducer to Isabella.

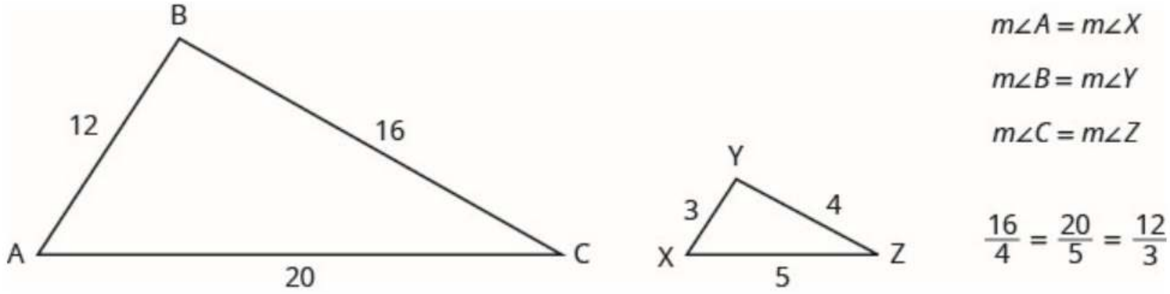
Solve similar figure applications

When you shrink or enlarge a photo on a phone or tablet, figure out a distance on a map, or use a pattern to build a bookcase or sew a dress, you are working with similar figures. If two figures have exactly the same shape, but different sizes, they are said to be similar. One is a scale model of the other. All their corresponding angles have the same measures and their corresponding sides have the same ratio.

Similar Figures

Two figures are similar if the measures of their corresponding angles are equal and their corresponding sides have the same ratio.

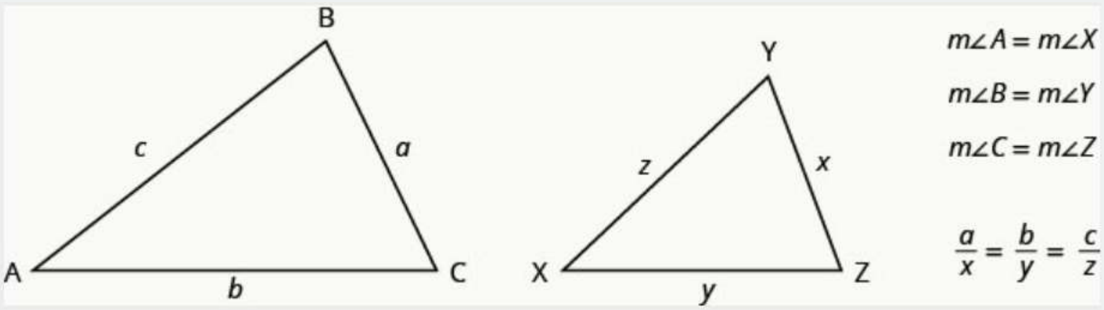
For example, the two triangles in Figure below are similar. Each side of ΔABC is four times the length of the corresponding side of ΔXYZ.



This is summed up in the Property of Similar Triangles.

Property of Similar Triangles

If ΔABC is similar to ΔXYZ, then their corresponding angle measure are equal and their corresponding sides have the same ratio.



Solve Uniform Motion Applications

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate · Time = Distance		

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

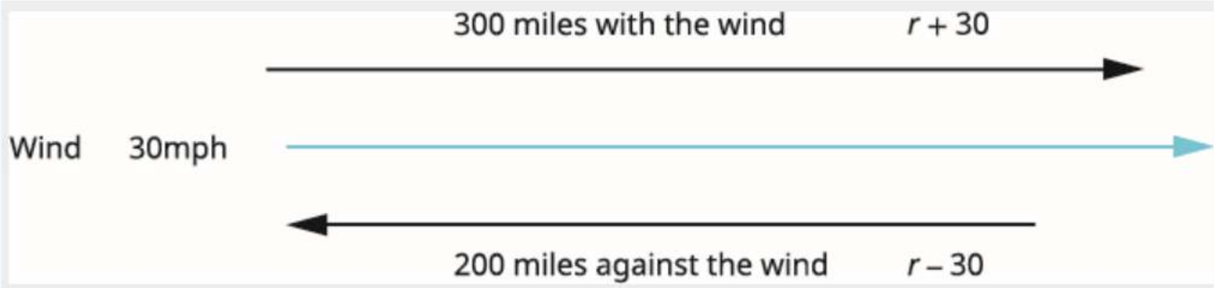
We have also explained how flying with or against the wind affects the speed of a plane. We will revisit that idea in the next example.

Example 5.5.7

An airplane can fly 200 miles into a 30 mph headwind in the same amount of time it takes to fly 300 miles with a 30 mph tailwind. What is the speed of the airplane?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for the speed of the airplane. Let r = the speed of the airplane.

When the plane flies with the wind, the wind increases its speed and so the rate is $r+30$.

When the plane flies against the wind, the wind decreases its speed and the rate is $r-30$.

Write in the rates. Write in the distances. Since $D=r\cdot t$, we solve for t and get $t=\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.

	Rate · Time = Distance		
Headwind	$r-30$	$\frac{200}{r-30}$	200
Tailwind	$r+30$	$\frac{300}{r+30}$	300

We know the times are equal and so we write our equation.

$$\frac{200}{r-30}=\frac{300}{r+30}$$

We multiply both sides by the LCD.

$$(r+30)(r-30)\left(\frac{200}{r-30}\right)=(r+30)(r-30)\left(\frac{300}{r+30}\right)$$

Simplify and solve.

$$\begin{aligned}(r+30)(200) &= (r-30)300\\ 200r+6000 &= 300r-9000\\ 15000 &= 100r\end{aligned}$$

Check.

Is 150mph a reasonable speed for an airplane? Yes. If the plane is traveling 150mph and the wind is 30mph,

$$\text{Tailwind}\quad 150+30=180\text{mph}\ \frac{300}{180}=\frac{5}{3}\text{ hours}$$

$$\text{Headwind}\ 150-30=120\text{mph}\frac{200}{120}=\frac{5}{3}\text{ hours}$$

The times are equal, so it checks. The plane was traveling 150mph.

? Try It 5.5.8

Link can ride his bike 20 miles into a 3 mph headwind in the same amount of time he can ride 30 miles with a 3 mph tailwind. What is Link's biking speed?

Answer

Link’s biking speed is 15 mph.

? Try It 5.5.9

Danica can sail her boat 5 miles into a 7 mph headwind in the same amount of time she can sail 12 miles with a 7 mph tailwind. What is the speed of Danica’s boat without a wind?

Answer

The speed of Danica’s boat is 17 mph.

In the next example, we will know the total time resulting from traveling different distances at different speeds.

✓ Example 5.5.10

Jazmine trained for 3 hours on Saturday. She ran 8 miles and then biked 24 miles. Her biking speed is 4 mph faster than her running speed. What is her running speed?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

We fill in the chart to organize the information. We are looking for Jazmine’s running speed. Let r = Jazmine’s running speed.

Her biking speed is 4 miles faster than her running speed. $r+4$ = her biking speed

The distances are given, enter them into the chart. Since $D=r\cdot t$, we solve for t and get $t=\frac{D}{r}$.We divide the distance by the rate in each row, and place the expression in the time column.

	Rate · Time = Distance		
Run	r	$\frac{8}{r}$	8
Bike	$r+4$	$\frac{24}{r+4}$	24
		3	

Write a word sentence: Her time plus the time biking is 3 hours.

Translate the sentence to get the equation.

$$\frac{8}{r}+\frac{24}{r+4}=3$$

Solve.

$$\begin{aligned} r(r+4)\left(\frac{8}{r}+\frac{24}{r+4}\right) &= 3\cdot r(r+4) \\ 8(r+4)+24r &= 3r(r+4) \\ 8r+32+24r &= 3r^2+12r \\ 32+32r &= 3r^2+12r \\ 0 &= 3r^2-20r-32 \\ 0 &= (3r+4)(r-8) \end{aligned}$$

$$(3r+4)=0 \qquad (r-8)=0$$

$$r=\cancel{\frac{4}{3}} \qquad\qquad r=8$$

Check.

A negative speed does not make sense in this problem, so $r=8$ is the solution.

Is 8 mph a reasonable running speed? Yes.

If Jazmine's running rate is 4, then her biking rate, $r+4$, which is $8+4=12$.

$$\text{Run 8mph} \quad \frac{\text{8miles}}{\text{8mph}}=1 \text{ hour}$$

$$\text{Bike 12 mph} \quad \frac{\text{24 miles}}{\text{12mph}}=2 \text{ hours}$$

Total 3 hours.

Jazmine's running speed is 8 mph.

Try It 5.5.11

Dennis went cross-country skiing for 6 hours on Saturday. He skied 20 mile uphill and then 20 miles back downhill, returning to his starting point. His uphill speed was 5 mph slower than his downhill speed. What was Dennis' speed going uphill and his speed going downhill?

Answer

Dennis's uphill speed was 10 mph and his downhill speed was 5 mph.

Try It 5.5.12

Joon drove 4 hours to his home, driving 208 miles on the interstate and 40 miles on country roads. If he drove 15 mph faster on the interstate than on the country roads, what was his rate on the country roads?

Answer

Joon's rate on the country roads is 50 mph.

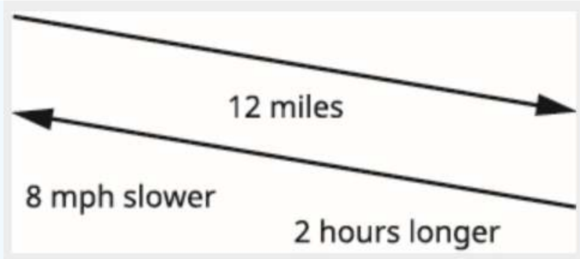
Once again, we will use the uniform motion formula solved for the variable t.

Example 5.5.13

Hamilton rode his bike downhill 12 miles on the river trail from his house to the ocean and then rode uphill to return home. His uphill speed was 8 miles per hour slower than his downhill speed. It took him 2 hours longer to get home than it took him to get to the ocean. Find Hamilton's downhill speed.

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.



We fill in the chart to organize the information.

We are looking for Hamilton's downhill speed. Let h= Hamilton's downhill speed.

His uphill speed is 8 miles per hour slower. $h-8$ = Hamilton's uphill speed.

Enter the rates into the chart.

The distance is the same in both directions: 12 miles.

Since $D=r \cdot t$, we solve for t and get $t=\frac{D}{r}$. We divide the distance by the rate in each row, and place the expression in the time column.

	Rate · Time = Distance		
Downhill	h	$\frac{12}{h}$	12
Uphill	$h-8$	$\frac{12}{h-8}$	12

Write a word sentence about the line: He took 2 hours longer uphill than downhill. The uphill time is 2 more than the downhill time.

Translate the sentence to get the equation.

$$\frac{12}{h-8}=\frac{12}{h}+2$$

Solve.

$$\begin{aligned} h(h-8)\left(\frac{12}{h-8}\right) &= h(h-8)\left(\frac{12}{h}+2\right) \\ 12h &= 12(h-8)+2h(h-8) \\ 12h &= 12h-96+2h^2-16h \\ 0 &= 2h^2-16h-96 \\ 0 &= 2\left(h^2-8h-48\right) \\ 0 &= 2(h-12)(h+4) \end{aligned}$$

$$\begin{array}{ll} h-12=0 & h+4=0 \\ h=12 & h=\cancel{-4} \end{array}$$

Check. Is $12 \mathrm{mph}$ a reasonable speed for biking downhill? Yes.

$$\text{Downhill } 12 \mathrm{mph} \quad \frac{12 \text{ miles}}{12 \mathrm{mph}}=1 \text{ hour}$$

$$\text{Uphill } 12-8=4 \mathrm{mph} \quad \frac{12 \text{ miles}}{4 \mathrm{mph}}=3 \text{ hours}$$

The uphill time is 2 hours more that the downhill time.

Hamilton's downhill speed is $12 \mathrm{mph}$.

Try It 5.5.14

Kayla rode her bike 75 miles home from college one weekend and then rode the bus back to college. It took her 2 hours less to ride back to college on the bus than it took her to ride home on her bike, and the average speed of the bus was 10 miles per hour faster than Kayla's biking speed. Find Kayla's biking speed.

Answer

Kayla's biking speed was 15 mph.

Try It 5.5.15

Victoria jogs 12 miles to the park along a flat trail and then returns by jogging on an 20 mile hilly trail. She jogs 1 mile per hour slower on the hilly trail than on the flat trail, and her return trip takes her two hours longer. Find her rate of jogging on the flat trail.

Answer

Victoria jogged 6 mph on the flat trail.

Solve Work Applications

The weekly gossip magazine has a big story about the Princess' baby and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours to do the job and Press #2 takes 12 hours to do the job. How long will it take the printer to get the magazine printed with both presses running together?

This is a typical 'work' application. There are three quantities involved here—the time it would take each of the two presses to do the job alone and the time it would take for them to do the job together.

If Press #1 can complete the job in 6 hours, in one hour it would complete $\frac{1}{6}$ of the job.

If Press #2 can complete the job in 12 hours, in one hour it would complete $\frac{1}{12}$ of the job.

We will let t be the number of hours it would take the presses to print the magazines with both presses running together. So in 1 hour working together they have completed $\frac{1}{t}$ of the job.

We can model this with the word equation and then translate to a rational equation. To find the time it would take the presses to complete the job if they worked together, we solve for t.

Follow the steps to organize the information. We are looking for how many hours it would take to complete the job with both presses running together.

Step 1: Let t = the number of hours needed to complete the job together.

Step 2: Enter the hours per job for Press #1, Press #2, and when they work together.

If a job on Press #1 takes 6 hours, then in 1 hour $\frac{1}{6}$ of the job is completed.

Similarly find the part of the job completed/hours for Press #2 and when they both together.

	Number of hours to complete the job.	Part of job completed/hour
Press #1	6	$\frac{1}{6}$
Press #2	12	$\frac{1}{12}$
Together	t	$\frac{1}{t}$

Write a word sentence. The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together.

Step 3: Translate into an equation.

$$\underbrace{\begin{array}{c}\text{Work completed by}\\ \text{Press }\#1+\text{Press }\#2=\text{Together}\end{array}}$$

$$\frac{1}{6} \quad + \quad \frac{1}{12} \quad = \quad \frac{1}{t}$$

Step 4: Solve. Simplify.

$$\frac{1}{6}+\frac{1}{12}=\frac{1}{t}$$

Multiply by the LCD, $12t$ and simplify.

$$\begin{aligned}12t\left(\frac{1}{6}+\frac{1}{12}\right)&=12t\left(\frac{1}{t}\right)\\ 2t+t&=12\\ 3t&=12\\ t&=4\end{aligned}$$

When both presses are running it takes 4 hours to do the job.

Keep in mind, it should take less time for two presses to complete a job working together than for either press to do it alone.

Example 5.5.16

Suppose Pete can paint a room in 10 hours. If he works at a steady pace, in 1 hour he would paint $\frac{1}{10}$ of the room. If Alicia would take 8 hours to paint the same room, then in 1 hour she would paint $\frac{1}{8}$ of the room. How long would it take Pete and Alicia to paint the room if they worked together (and didn't interfere with each other's progress)?

Solution

This is a 'work' application. The steps below will help us organize the information. We are looking for the numbers of hours it will take them to paint the room together.

In one hour Pete did $\frac{1}{10}$ of the job. Alicia did $\frac{1}{8}$ of the job. And together they did $\frac{1}{t}$ of the job.

Step 1: Let t be the number of hours needed to paint the room together.

Step 2: Enter the hours per job for Pete, Alicia, and when they work together. In 1 hour working together, they have completed $\frac{1}{t}$ of the job. Similarly, find the part of the job completed/hour by Pete and then by Alicia.

	Number of hours to complete the job.	Part of job completed/hour
Pete	10	$\frac{1}{10}$
Alicia	8	$\frac{1}{8}$
Together	t	$\frac{1}{t}$

Write a word sentence. The work completed by Pete plus the work completed by Alicia equals the total work completed.

Step 3: Translate into an equation.

$$\underbrace{\text{Pete } + \text{Alicia } = \text{Together}}_{}^{\text{Work completed by}}$$

$$\frac{1}{10} \quad + \quad \frac{1}{8} \quad = \quad \frac{1}{t}$$

Step 4: Simplify. Solve.

Multiply by the LCD, $40t$.

$$40t\left(\frac{1}{10}+\frac{1}{8}\right)=40t\left(\frac{1}{t}\right)$$

Distribute.

$$40t\cdot\frac{1}{10}+40t\cdot\frac{1}{8}=40t\left(\frac{1}{t}\right)$$

Simplify and solve.

$$\begin{aligned} 4t+5t&=40 \\ 9t&=40 \\ t&=\frac{40}{9} \end{aligned}$$

We'll write as a mixed number so that we can convert it to hours and minutes.

$$t=4\frac{4}{9}\text{ hours}$$

Remember, 1 hour = 60 minutes.

$$t=4\text{ hours }+\frac{4}{9}(60\text{ minutes })$$

Multiply, and then round to the nearest minute.

$$t=4\text{ hours }+27\text{ minutes}$$

It would take Pete and Alica about 4 hours and 27 minutes to paint the room.

? Try It 5.5.17

One gardener can mow a golf course in 4 hours, while another gardener can mow the same golf course in 6 hours. How long would it take if the two gardeners worked together to mow the golf course?

Answer

When the two gardeners work together it takes 2 hours and 24 minutes.

? Try It 5.5.18

Daria can weed the garden in 7 hours, while her mother can do it in 3. How long will it take the two of them working together?

Answer

When Daria and her mother work together it takes 2 hours and 6 minutes.

✓ Example 5.5.19

Ra’shon can clean the house in 7 hours. When his sister helps him it takes 3 hours. How long does it take his sister when she cleans the house alone?

Solution

This is a work problem. The steps below will help us organize the information. We are looking for how many hours it would take Ra’shon’s sister to complete the job by herself.

Step 1: Let s be the number of hours Ra’shon’s sister takes to clean the house alone.

Step 2: Enter the hours per job for Ra’shon, his sister, and when they work together. If Ra’shon takes 7 hours, then in 1 hour $\frac{1}{s}$ of the job is completed. If Ra’shon’s sister takes s hours, then in 1 hour $\frac{1}{s}$ of the job is completed.

	Number of hours to complete the job.	Part of job completed/hour
Ra’shon	7	$\frac{1}{7}$
His sister	s	$\frac{1}{s}$
Together	3	$\frac{1}{3}$

Write a word sentence. The part completed by Ra’shon plus the part by his sister equals the amount completed together.

Step 3: Translate to an equation.

$$\underbrace{\begin{array}{c}\text{Work completed by}\\ \text{Ra'shon } + \text{His sister } = \text{Together}\end{array}}$$
$$\frac{1}{7} \quad + \quad \frac{1}{s} \quad = \quad \frac{1}{3}$$

Step 4: Simplify. Solve.

$$\frac{1}{7}+\frac{1}{5}=\frac{1}{3}$$

Multiply by the LCD, 21s.

$$\begin{aligned}21s\left(\frac{1}{7}+\frac{1}{s}\right) &= \left(\frac{1}{3}\right)21s\\ 3s+21 &= 7s\end{aligned}$$

Simplify.

$$\begin{aligned}-4s &= -21\\ s &= \frac{-21}{-4}=\frac{21}{4}\end{aligned}$$

Write as a mixed number to convert it to hours and minutes.

$$s=5\frac{1}{4}\text{ hours}$$

There are 60 minutes in 1 hour.

$$\begin{aligned}s &= 5\text{ hours }+\frac{1}{4}(60\text{ minutes })\\ s &= 5\text{ hours }+15\text{ minutes}\end{aligned}$$

It would take Ra’shon’s sister 5 hours and 15 minutes to clean the house alone.

Try It 5.5.20

Alice can paint a room in 6 hours. If Kristina helps her it takes them 4 hours to paint the room. How long would it take Kristina to paint the room by herself?

Answer

Kristina can paint the room in 12 hours.

Try It 5.5.21

Tracy can lay a slab of concrete in 3 hours, with Jordan's help they can do it in 2 hours. If Jordan works alone, how long will it take?

Answer

It will take Jordan 6 hours.

Solve Direct Variation Problems

When two quantities are related by a proportion, we say they are proportional to each other. Another way to express this relation is to talk about the variation of the two quantities. We will discuss direct variation and inverse variation in this section.

Lindsay gets paid $15 per hour at her job. If we let s be her salary and h be the number of hours she has worked, we could model this situation with the equation

$$s=15h$$

Lindsay's salary is the product of a constant, 15, and the number of hours she works. We say that Lindsay's salary varies directly with the number of hours she works. Two variables vary directly if one is the product of a constant and the other.

Direct Variation

For any two variables x and y, y varies directly with x if

$$y=kx\text{, where }k\neq 0$$

The constant k is called the constant of variation.

In applications using direct variation, generally we will know values of one pair of the variables and will be asked to find the equation that relates x and y. Then we can use that equation to find values of y for other values of x.

We'll list the steps here.

How to solve direct variation problems

Step 1. Write the formula for direct variation.

Step 2. Substitute the given values for the variables.

Step 3. Solve for the constant of variation.

Step 4. Write the equation that relates x and y using the constant of variation.

Now we'll solve an application of direct variation.

Example 5.5.22

When Raoul runs on the treadmill at the gym, the number of calories, c, he burns varies directly with the number of minutes, m, he uses the treadmill. He burned 315 calories when he used the treadmill for 18 minutes.

a. Write the equation that relates c and m.
b. How many calories would he burn if he ran on the treadmill for 25 minutes?

Solution

a.

The number of calories, c, varies directly with the number of minutes, m, on the treadmill, and $c = 315$ when $m = 18$.

Write the formula for direct variation.

$$y = kx$$

We will use c in place of y and m in place of x.

$$c = km$$

Substitute the given values for the variables.

$$315 = k \cdot 18$$

Solve for the constant of variation.

$$\begin{aligned} \frac{315}{18} &= \frac{k \cdot 18}{18} \\ 17.5 &= k \end{aligned}$$

Write the equation that relates c and m.

$$c = km$$

Substitute in the constant of variation.

$$c = 17.5m$$

b.

Find c when $m = 25$.

Write the equation that relates c and m.

$$c = 17.5m$$

Substitute the given value for m.

$$c = 17.5(25)$$

Simplify.

$$c = 437.5$$

Raoul would burn 437.5 calories if he used the treadmill for 25 minutes.

? Try It 5.5.23

The number of calories, c, burned varies directly with the amount of time, t, spent exercising. Arnold burned 312 calories in 65 minutes exercising.

a. Write the equation that relates c and t.
b. How many calories would he burn if he exercises for 90 minutes?

Answer

a. $c = 4.8t$
b. He would burn 432 calories.

> **Try It 5.5.24**
>
> The distance a moving body travels, d, varies directly with time, t, it moves. A train travels 100 miles in 2 hours.
>
> a. Write the equation that relates d and t.
> b. How many miles would it travel in 5 hours?
>
> **Answer**
>
> a. $d=50t$
> b. It would travel 250 miles.

Solve Inverse Variation Problems

Many applications involve two variable that vary inversely. As one variable increases, the other decreases. The equation that relates them is $y=\frac{k}{x}$.

> **Inverse Variation**
>
> For any two variables x and y, y varies inversely with x if
>
> $$y=\frac{k}{x}, \text{ where } k\neq 0$$
>
> The constant k is called the constant of variation.

The word 'inverse' in inverse variation refers to the multiplicative inverse. The multiplicative inverse of x is $\frac{1}{x}$.

We solve inverse variation problems in the same way we solved direct variation problems. Only the general form of the equation has changed. We will copy the procedure box here and just change 'direct' to 'inverse'.

> **How to solve inverse variation problems**
>
> **Step 1**. Write the formula for inverse variation.
>
> **Step 2**. Substitute the given values for the variables.
>
> **Step 3**. Solve for the constant of variation.
>
> **Step 4**. Write the equation that relates x and y using the constant of variation.

> **Example 5.5.25**
>
> The frequency of a guitar string varies inversely with its length. A 26 in.-long string has a frequency of 440 vibrations per second.
>
> a. Write the equation of variation.
> b. How many vibrations per second will there be if the string's length is reduced to 20 inches by putting a finger on a fret?
>
> **Solution**
>
> a.
>
> The frequency varies inversely with the length.
>
> Name the variables. Let f = frequency. L = length
>
> Write the formula for inverse variation.
>
> $$y=\frac{k}{x}$$
>
> We will use f in place of y and L in place of x.

$$f=\frac{k}{L}$$

$$f=440 \text{ when } L=26$$

Substitute the given values for the variables.

$$440=\frac{k}{26}$$

Solve for the constant of variation.

$$26(440)=26\left(\frac{k}{26}\right)$$
$$11,440=k$$

Write the equation that relates f and L.

$$f=\frac{k}{L}$$

Substitute the constant of variation.

$$f=\frac{11,440}{L}$$

b.

Find f when $L=20$.

Write the equation that relates f and L.

$$f=\frac{11,440}{L}$$

Substitute the given value forL.

$$f=\frac{11,440}{20}$$

Simplify.

$$f=572$$

A 20"-guitar string has frequency 572 vibrations per second.

? Try It 5.5.26

The number of hours it takes for ice to melt varies inversely with the air temperature. Suppose a block of ice melts in 2 hours when the temperature is 65 degrees Celsius.

a. Write the equation of variation.
b. How many hours would it take for the same block of ice to melt if the temperature was 78 degrees?

Answer

a. $h=\frac{130}{t}$
b. $1\frac{2}{3}$ hours

Try It 5.5.27

Xander's new business found that the daily demand for its product was inversely proportional to the price, p. When the price is \$5, the demand is 700 units.

a. Write the equation of variation.
b. What is the demand if the price is raised to \$7?

Answer

a. $x=\frac{3500}{p}$
b. 500 units

Practice Makes Perfect

Solve Proportions

In the following exercises, solve each proportion.

1. $\frac{x}{56}=\frac{7}{8}$

Answer

$x=49$

2. $\frac{56}{72}=\frac{y}{9}$

3. $\frac{98}{154}=\frac{-7}{p}$

Answer

$p=-11$

4. $\frac{72}{156}=\frac{-6}{q}$

5. $\frac{a}{a+12}=\frac{4}{7}$

Answer

$a=16$

6. $\frac{b}{b-16}=\frac{11}{9}$

7. $\frac{m+90}{25}=\frac{m+30}{15}$

Answer

$m=60$

8. $\frac{n+10}{4}=\frac{40-n}{6}$

9. $\frac{2p+4}{8}=\frac{p+18}{6}$

Answer

$p=30$

10. $\frac{q-2}{2}=\frac{2q-7}{18}$

In the following exercises, solve.

11. Kevin wants to keep his heart rate at 160 beats per minute while training. During his workout he counts 27 beats in 10 seconds.
    a. How many beats per minute is this?
    b. Has Kevin met his target heart rate?

Answer

a. 162 beats per minute

b. yes

11. Jesse's car gets 30 miles per gallon of gas.
    a. If Las Vegas is 285 miles away, how many gallons of gas are needed to get there and then home?
    b. If gas is \$3.09 per gallon, what is the total cost of the gas for the trip?
12. Pediatricians prescribe 5 milliliters (ml) of acetaminophen for every 25 pounds of a child's weight. How many milliliters of acetaminophen will the doctor prescribe for Jocelyn, who weighs 45 pounds?

Answer

9 ml

14. A veterinarian prescribed Sunny, a 65-pound dog, an antibacterial medicine in case an infection emerges after her teeth were cleaned. If the dosage is 5 mg for every pound, how much medicine was Sunny given?
15. A new energy drink advertises 106 calories for 8 ounces. How many calories are in 12 ounces of the drink?

Answer

159 calories

16. One 12-ounce can of soda has 150 calories. If Josiah drinks the big 32-ounce size from the local mini-mart, how many calories does he get?
17. Kyra is traveling to Canada and will change \$250 US dollars into Canadian dollars. At the current exchange rate, \$1 US is equal to \$1.3 Canadian. How many Canadian dollars will she get for her trip?

Answer

325 Canadian Dollars

18. Maurice is traveling to Mexico and needs to exchange \$450 into Mexican pesos. If each dollar is worth 12.29 pesos, how many pesos will he get for his trip?
19. Ronald needs a morning breakfast drink that will give him at least 390 calories. Orange juice has 130 calories in one cup. How many cups does he need to drink to reach his calorie goal?

Answer

3 cups

20. Sonya drinks a 32-ounce energy drink containing 80 calories per 12 ounce. How many calories did she drink?
21. Phil wants to fertilize his lawn. Each bag of fertilizer covers about 4,000 square feet of lawn. Phil's lawn is approximately 13,500 square feet. How many bags of fertilizer will he have to buy?

Answer

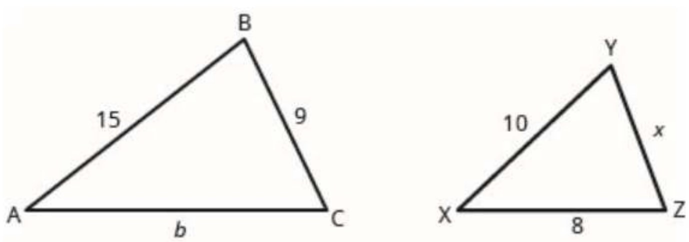
4 bags

22. An oatmeal cookie recipe calls for $\frac{1}{2}$ cup of butter to make 4 dozen cookies. Hilda needs to make 10 dozen cookies for the bake sale. How many cups of butter will she need?

Solve Similar Figure Applications

In the following exercises, the triangles are similar. Find the length of the indicated side.

23.

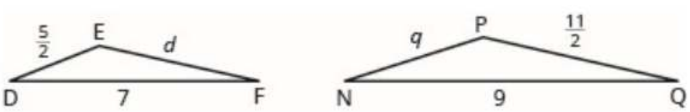


a. side x
b. side b

Answer

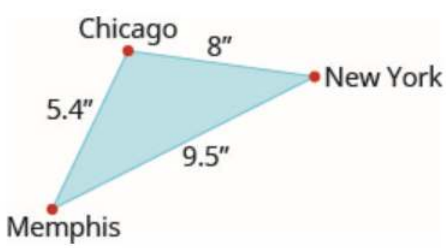
a. 6
b. 12

24.



a. side d
b. side q

In the following exercises, use the map shown. On the map, New York City, Chicago, and Memphis form a triangle. The actual distance from New York to Chicago is 800 miles.



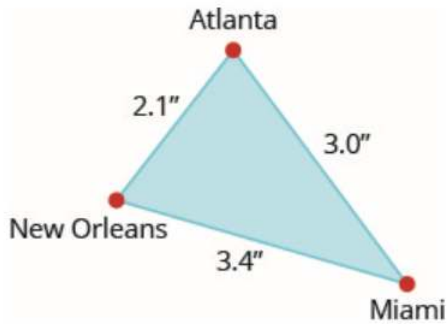
25. Find the actual distance from New York to Memphis.

Answer

950 miles

26. Find the actual distance from Chicago to Memphis.

In the following exercises, use the map shown. On the map, Atlanta, Miami, and New Orleans form a triangle. The actual distance from Atlanta to New Orleans is 420 miles.



27. Find the actual distance from New Orleans to Miami.

Answer

680 miles

28. Find the actual distance from Atlanta to Miami.

In the following exercises, answer each question.

29. A 2-foot-tall dog casts a 3-foot shadow at the same time a cat casts a one foot shadow. How tall is the cat?

Answer

$\frac{2}{3}$ foot (8 in.)

30. Larry and Tom were standing next to each other in the backyard when Tom challenged Larry to guess how tall he was. Larry knew his own height is 6.5 feet and when they measured their shadows, Larry's shadow was 8 feet and Tom's was 7.75 feet long. What is Tom's height?
31. The tower portion of a windmill is 212 feet tall. A six foot tall person standing next to the tower casts a seven-foot shadow. How long is the windmill's shadow?

Answer

247.3 feet

32. The height of the Statue of Liberty is 305 feet. Nikia, who is standing next to the statue, casts a 6-foot shadow and she is 5 feet tall. How long should the shadow of the statue be?

Solve Uniform Motion Applications

In the following exercises, solve the application problem provided.

33. Mary takes a sightseeing tour on a helicopter that can fly 450 miles against a 35-mph headwind in the same amount of time it can travel 702 miles with a 35-mph tailwind. Find the speed of the helicopter.

Answer

160 mph

34. A private jet can fly 1,210 miles against a 25-mph headwind in the same amount of time it can fly 1694 miles with a 25-mph tailwind. Find the speed of the jet.
35. A boat travels 140 miles downstream in the same time as it travels 92 miles upstream. The speed of the current is 6mph. What is the speed of the boat?

Answer

29 mph

36. Darrin can skateboard 2 miles against a 4-mph wind in the same amount of time he skateboards 6 miles with a 4-mph wind. Find the speed Darrin skateboards with no wind.
37. Jane spent 2 hours exploring a mountain with a dirt bike. First, she rode 40 miles uphill. After she reached the peak she rode for 12 miles along the summit. While going uphill, she went 5 mph slower than when she was on the summit. What was her rate along the summit?

Answer

30 mph

38. Laney wanted to lose some weight so she planned a day of exercising. She spent a total of 2 hours riding her bike and jogging. She biked for 12 miles and jogged for 6 miles. Her rate for jogging was 10 mph less than biking rate. What was her rate when jogging?
39. Byron wanted to try out different water craft. He went 62 miles downstream in a motor boat and 27 miles downstream on a jet ski. His speed on the jet ski was 10 mph faster than in the motor boat. Bill spent a total of 4 hours on the water. What was his rate of speed in the motor boat?

Answer

20 mph

40. Nancy took a 3-hour drive. She went 50 miles before she got caught in a storm. Then she drove 68 miles at 9 mph less than she had driven when the weather was good. What was her speed driving in the storm?
41. Chester rode his bike uphill 24 miles and then back downhill at 2 mph faster than his uphill. If it took him 2 hours longer to ride uphill than downhill, what was his uphill rate?

Answer

4 mph

42. Matthew jogged to his friend's house 12 miles away and then got a ride back home. It took him 2 hours longer to jog there than ride back. His jogging rate was 25 mph slower than the rate when he was riding. What was his jogging rate?
43. Hudson travels 1080 miles in a jet and then 240 miles by car to get to a business meeting. The jet goes 300 mph faster than the rate of the car, and the car ride takes 1 hour longer than the jet. What is the speed of the car?

Answer

60 mph

44. Nathan walked on an asphalt pathway for 12 miles. He walked the 12 miles back to his car on a gravel road through the forest. On the asphalt he walked 2 miles per hour faster than on the gravel. The walk on the gravel took one hour longer than the walk on the asphalt. How fast did he walk on the gravel.
45. John can fly his airplane 2800 miles with a wind speed of 50 mph in the same time he can travel 2400 miles against the wind. If the speed of the wind is 50 mph, find the speed of his airplane.

Answer

650 mph

46. Jim's speedboat can travel 20 miles upstream against a 3-mph current in the same amount of time it travels 22 miles downstream with a 3-mph current speed . Find the speed of the Jim's boat.
47. Hazel needs to get to her granddaughter's house by taking an airplane and a rental car. She travels 900 miles by plane and 250 miles by car. The plane travels 250 mph faster than the car. If she drives the rental car for 2 hours more than she rode the plane, find the speed of the car.

Answer

50 mph

48. Stu trained for 3 hours yesterday. He ran 14 miles and then biked 40 miles. His biking speed is 6 mph faster than his running speed. What is his running speed?
49. When driving the 9-hour trip home, Sharon drove 390 miles on the interstate and 150 miles on country roads. Her speed on the interstate was 15 more than on country roads. What was her speed on country roads?

Answer

50 mph

50. Two sisters like to compete on their bike rides. Tamara can go 4 mph faster than her sister, Samantha. If it takes Samantha 1 hours longer than Tamara to go 80 miles, how fast can Samantha ride her bike?
51. Dana enjoys taking her dog for a walk, but sometimes her dog gets away, and she has to run after him. Dana walked her dog for 7 miles but then had to run for 1 mile, spending a total time of 2.5 hours with her dog. Her running speed was 3 mph faster than her walking speed. Find her walking speed.

Answer

4.2 mph

52. Ken and Joe leave their apartment to go to a football game 45 miles away. Ken drives his car 30 mph faster Joe can ride his bike. If it takes Joe 2 hours longer than Ken to get to the game, what is Joe's speed?

Solve Work Applications

53. Mike, an experienced bricklayer, can build a wall in 3 hours, while his son, who is learning, can do the job in 6 hours. How long does it take for them to build a wall together?

Answer

2 hours

54. It takes Sam 4 hours to rake the front lawn while his brother, Dave, can rake the lawn in 2 hours. How long will it take them to rake the lawn working together?
55. Mia can clean her apartment in 6 hours while her roommate can clean the apartment in 5 hours. If they work together, how long would it take them to clean the apartment?

Answer

2 hours and 44 minutes

56. Brian can lay a slab of concrete in 6 hours, while Greg can do it in 4 hours. If Brian and Greg work together, how long will it take?
57. Josephine can correct her students test papers in 5 hours, but if her teacher's assistant helps, it would take them 3 hours. How long would it take the assistant to do it alone?

Answer

7 hours and 30 minutes

58. Washing his dad's car alone, eight year old Levi takes 2.5 hours. If his dad helps him, then it takes 1 hour. How long does it take Levi's dad to wash the car by himself?
59. At the end of the day Dodie can clean her hair salon in 15 minutes. Ann, who works with her, can clean the salon in 30 minutes. How long would it take them to clean the shop if they work together?

Answer

10 min

60. Ronald can shovel the driveway in 4 hours, but if his brother Donald helps it would take 2 hours. How long would it take Donald to shovel the driveway alone?

Solve Direct Variation Problems

In the following exercises, solve.

61. If y varies directly as x and $y = 14$, when $x = 3$. Find the equation that relates x and y.

Answer

$y = \frac{14}{3}x$

62. If a varies directly as b and $a = 16$, when $b = 4$. Find the equation that relates a and b.
63. If p varies directly as q and $p = 9$, when $q = 3$. Find the equation that relates p and q.

Answer

$p = 3.2q$

64. If v varies directly as w and $v = 8$, when $w = 12$. Find the equation that relates v and w.
65. The price, P, that Eric pays for gas varies directly with the number of gallons, g, he buys. It costs him $50 to buy 20 gallons of gas.
    a. Write the equation that relates P and g.
    b. How much would 33 gallons cost Eric?

Answer

a. $P = 2.5g$
b. $\$82.50$

66. Joseph is traveling on a road trip. The distance, d, he travels before stopping for lunch varies directly with the speed, v, he travels. He can travel 120 miles at a speed of 60 mph.
    a. Write the equation that relates d and v.
    b. How far would he travel before stopping for lunch at a rate of 65 mph?
67. The mass of a liquid varies directly with its volume. A liquid with mass 16 kilograms has a volume of 2 liters.
    a. Write the equation that relates the mass to the volume.
    b. What is the volume of this liquid if its mass is 128 kilograms?

Answer

a. $m = 8v$
b. 16 liters

68. The length that a spring stretches varies directly with a weight placed at the end of the spring. When Sarah placed a 10-pound watermelon on a hanging scale, the spring stretched 5 inches.
    a. Write the equation that relates the length of the spring to the weight.
    b. What weight of watermelon would stretch the spring 6 inches?
69. The maximum load a beam will support varies directly with the square of the diagonal of the beam's cross-section. A beam with diagonal 6 inch will support a maximum load of 108 pounds.
    a. Write the equation that relates the load to the diagonal of the cross-section.
    b. What load will a beam with a 10-inch diagonal support?

Answer

a. $L = 3d^2$
b. 300 pounds

70. The area of a circle varies directly as the square of the radius. A circular pizza with a radius of 6 inches has an area of 113.04 square inches.
    a. Write the equation that relates the area to the radius.
    b. What is the area of a personal pizza with a radius 4 inches?

Solve Inverse Variation Problems

In the following exercises, solve.

71. If y varies inversely with x and $y=5$, when $x=4$. Find the equation that relates x and y.

Answer

$$y=\frac{20}{x}$$

72. If p varies inversely with q and $p=2$, when $q=1$. Find the equation that relates p and q.
73. If v varies inversely with w and $v=6$, when $w=12$. Find the equation that relates v and w.

Answer

$$v=\frac{3}{w}$$

74. If a varies inversely with b and $a=12$, when $b=13$. Find the equation that relates a and b.

In the following exercises, write an inverse variation equation to solve the following problems.

75. The fuel consumption (mpg) of a car varies inversely with its weight. A Toyota Corolla weighs 2800 pounds getting 33 mpg on the highway.
    a. Write the equation that relates the mpg to the car's weight.
    b. What would the fuel consumption be for a Toyota Sequoia that weighs 5500 pounds?

Answer

a. $g=\dfrac{92,400}{w}$
b. 16.8 mpg

76. A car's value varies inversely with its age. Jackie bought a 10-year-old car for $2,400.
    a. Write the equation that relates the car's value to its age.
    b. What will be the value of Jackie's car when it is 15 years old?
77. The time required to empty a tank varies inversely as the rate of pumping. It took Ada 5 hours to pump her flooded basement using a pump that was rated at 200 gpm (gallons per minute).
    a. Write the equation that relates the number of hours to the pump rate.
    b. How long would it take Ada to pump her basement if she used a pump rated at 400 gpm?

Answer

1. $t=\dfrac{1000}{r}$
2. 2.5 hours

78. On a string instrument, the length of a string varies inversely as the frequency of its vibrations. An 11-inch string on a violin has a frequency of 400 cycles per second.
    a. Write the equation that relates the string length to its frequency.
    b. What is the frequency of a 10 inch string?
79. Paul, a dentist, determined that the number of cavities that develops in his patient's mouth each year varies inversely to the number of minutes spent brushing each night. His patient, Lori, had four cavities when brushing her teeth 30 seconds (0.5 minutes) each night.

a. Write the equation that relates the number of cavities to the time spent brushing.
b. How many cavities would Paul expect Lori to have if she had brushed her teeth for 2 minutes each night?

Answer

a. $c = \frac{2}{t}$
b. 1 cavity

80. Boyle's law states that if the temperature of a gas stays constant, then the pressure varies inversely to the volume of the gas. Braydon, a scuba diver, has a tank that holds 6 liters of air under a pressure of 220 psi.
    a. Write the equation that relates pressure to volume.
    b. If the pressure increases to 330 psi, how much air can Braydon's tank hold?
81. The cost of a ride service varies directly with the distance traveled. It costs \$35 for a ride from the city center to the airport, 14 miles away.
    a. Write the equation that relates the cost, c, with the number of miles, m.
    b. What would it cost to travel 22 miles with this service?

Answer

a. $c = 2.5m$
b. $\$55$

82. The number of hours it takes Jack to drive from Boston to Bangor is inversely proportional to his average driving speed. When he drives at an average speed of 40 miles per hour, it takes him 6 hours for the trip.
    a. Write the equation that relates the number of hours, h, with the speed, s.
    b. How long would the trip take if his average speed was 75 miles per hour?

Writing Exercises

83. Marisol solves the proportion $\frac{144}{a} = \frac{9}{4}$ by 'cross multiplying,' so her first step looks like $4 \cdot 144 = 9 \cdot a$. Explain how this differs from the method of solution shown in Example 7.6.2.

Answer

Answers will vary.

84. Paula and Yuki are roommates. It takes Paula 3 hours to clean their apartment. It takes Yuki 4 hours to clean the apartment. The equation $\frac{1}{3} + \frac{1}{4} = \frac{1}{t}$ can be used to find t, the number of hours it would take both of them, working together, to clean their apartment. Explain how this equation models the situation.
85. In your own words, explain the difference between direct variation and inverse variation.

Answer

Answers will vary.

86. Make up an example from your life experience of inverse variation.

5.6: Chapter 5 Review Exercises

Simplify, Multiply, and Divide Rational Expressions

Determine the Values for Which a Rational Expression is Undefined

In the following exercises, determine the values for which the rational expression is undefined.

1. $\frac{5a+3}{3a-2}$

Answer

$a \neq \frac{2}{3}$

2. $\frac{b-7}{b^2-25}$

3. $\frac{5x^2y^2}{8y}$

Answer

$y \neq 0$

4. $\frac{x-3}{x^2-x-30}$

Simplify Rational Expressions

In the following exercises, simplify.

5. $\frac{18}{24}$

Answer

$\frac{3}{4}$

6. $\frac{9m^4}{18mn^3}$

7. $\frac{x^2+7x+12}{x^2+8x+16}$

Answer

$\frac{x+3}{x+4}$

8. $\frac{7v-35}{25-v^2}$

Multiply Rational Expressions

In the following exercises, multiply.

9. $\frac{5}{8} \cdot \frac{4}{15}$

Answer

$\frac{1}{6}$

10. $\frac{3xy^2}{8y^3} \cdot \frac{16y^2}{24x}$

11. $\frac{72x-12x^2}{8x+32} \cdot \frac{x^2+10x+24}{x^2-36}$

Answer

$\frac{-3x}{2}$

12. $\frac{6y^2-2y-10}{9-y^2} \cdot \frac{y^2-6y+9}{6y^2+29y-20}$

Divide Rational Expressions

In the following exercises, divide.

13. $\frac{x^2-4x-12}{x^2+8x+12} \div \frac{x^2-36}{3x}$

Answer

$\frac{3x}{(x+6)(x+6)}$

14. $\frac{y^2-16}{4} \div \frac{y^3-64}{2y^2+8y+32}$

15. $\frac{11+w}{w-9} \div \frac{121-w^2}{9-w}$

Answer

$\frac{1}{11+w}$

16. $\frac{3y^2-12y-63}{4y+3} \div \left(6y^2-42y\right)$

17. $\frac{\frac{c^2-64}{3c^2+26c+16}}{\frac{c^2-4c-32}{15c+10}}$

Answer

$\frac{5}{c+4}$

18. $\dfrac{8a^2+16a}{a-4} \cdot \dfrac{a^2+2a-24}{a^2+7a+10} \div \dfrac{2a^2-6a}{a+5}$

Multiply and Divide Rational Functions

19. Find $R(x)=f(x)\cdot g(x)$ where $f(x)=\dfrac{9x^2+9x}{x^2-3x-4}$ and $g(x)=\dfrac{x^2-16}{3x^2+12x}$.

Answer

$R(x)=3$

20. Find $R(x)=\dfrac{f(x)}{g(x)}$ where $f(x)=\dfrac{27x^2}{3x-21}$ and $g(x)=\dfrac{9x^2+54x}{x^2-x-42}$.

Add and Subtract Rational Expressions

Add and Subtract Rational Expressions with a Common Denominator

In the following exercises, perform the indicated operations.

21. $\dfrac{7}{15}+\dfrac{8}{15}$

Answer

1

22. $\dfrac{4a^2}{2a-1}-\dfrac{1}{2a-1}$

23. $\dfrac{y^2+10y}{y+5}+\dfrac{25}{y+5}$

Answer

$y+5$

24. $\dfrac{7x^2}{x^2-9}+\dfrac{21x}{x^2-9}$

25. $\dfrac{x^2}{x-7}-\dfrac{3x+28}{x-7}$

Answer

$x+4$

26. $\dfrac{y^2}{y+11}-\dfrac{121}{y+11}$

27. $\dfrac{4q^2-q+3}{q^2+6q+5}-\dfrac{3q^2-q-6}{q^2+6q+5}$

Answer

$\dfrac{q-3}{q+5}$

28. $\dfrac{5t+4t+3}{t^2-25}-\dfrac{4t^2-8t-32}{t^2-25}$

Add and Subtract Rational Expressions Whose Denominators Are Opposites

In the following exercises, add and subtract.

29. $\dfrac{18w}{6w-1}+\dfrac{3w-2}{1-6w}$

Answer

$$\frac{15w+2}{6w-1}$$

30. $\dfrac{a^2+3a}{a^2-4}-\dfrac{3a-8}{4-a^2}$

31. $\dfrac{2b^2+3b-15}{b^2-49}-\dfrac{b^2+16b-1}{49-b^2}$

Answer

$$\frac{3b-2}{b+7}$$

32. $\dfrac{8y^2-10y+7}{2y-5}+\dfrac{2y^2+7y+2}{5-2y}$

Find the Least Common Denominator of Rational Expressions

In the following exercises, find the LCD.

33. $\dfrac{7}{a^2-3a-10},\dfrac{3a}{a^2-a-20}$

Answer

$$(a+2)(a-5)(a+4)$$

34. $\dfrac{6}{n^2-4},\dfrac{2n}{n^2-4n+4}$

35. $\dfrac{5}{3p^2+17p-6},\dfrac{2m}{3p^2-23p-8}$

Answer

$$(3p+1)(p+6)(p+8)$$

Add and Subtract Rational Expressions with Unlike Denominators

In the following exercises, perform the indicated operations.

36. $\dfrac{7}{5a}+\dfrac{3}{2b}$

37. $\dfrac{2}{c-2}+\dfrac{9}{c+3}$

Answer

$$\frac{11c-12}{(c-2)(c+3)}$$

38. $\frac{3x}{x^2-9}+\frac{5}{x^2+6x+9}$

39. $\frac{2x}{x^2+10x+24}+\frac{3x}{x^2+8x+16}$

Answer

$$\frac{5x^2+26x}{(x+4)(x+4)(x+6)}$$

40. $\frac{5q}{p^2q-p^2}+\frac{4q}{q^2-1}$

41. $\frac{3y}{y+2}-\frac{y+2}{y+8}$

Answer

$$\frac{2\left(y^2+10y-2\right)}{(y+2)(y+8)}$$

42. $\frac{-3w-15}{w^2+w-20}-\frac{w+2}{4-w}$

43. $\frac{7m+3}{m+2}-5$

Answer

$$\frac{2m-7}{m+2}$$

44. $\frac{n}{n+3}+\frac{2}{n-3}-\frac{n-9}{n^2-9}$

45. $\frac{8a}{a^2-64}-\frac{4}{a+8}$

Answer

$$\frac{4}{a-8}$$

46. $\frac{5}{12x^2y}+\frac{7}{20xy^3}$

Add and Subtract Rational Functions

In the following exercises, find $R(x)=f(x)+g(x)$ where $f(x)$ and $g(x)$ are given.

47. $f(x)=\frac{2x^2+12x-11}{x^2+3x-10}, g(x)=\frac{x+1}{2-x}$

Answer

$$R(x)=\frac{x+8}{x+5}$$

48. $f(x)=\frac{-4x+31}{x^2+x-30}, g(x)=\frac{5}{x+6}$

In the following exercises, find $R(x)=f(x)-g(x)$ where $f(x)$ and $g(x)$ are given.

49. $f(x)=\frac{4x}{x^2-121}, g(x)=\frac{2}{x-11}$

Answer

$$R(x)=\frac{2}{x+11}$$

50. $f(x)=\frac{7}{x+6}, g(x)=\frac{14x}{x^2-36}$

Simplify Complex Rational Expressions

Simplify a Complex Rational Expression by Writing It as Division

In the following exercises, simplify.

51. $\dfrac{\frac{7x}{x+2}}{\frac{14x^2}{x^2-4}}$

Answer

$$\frac{x-2}{2x}$$

52. $\dfrac{\frac{2}{5}+\frac{5}{6}}{\frac{1}{3}+\frac{1}{4}}$

53. $\dfrac{x-\frac{3x}{x+5}}{\frac{1}{x+5}+\frac{1}{x-5}}$

Answer

$$\frac{(x-8)(x-5)}{2}$$

54. $\dfrac{\frac{2}{m}+\frac{m}{n}}{\frac{n}{m}-\frac{1}{n}}$

Simplify a Complex Rational Expression by Using the LCD

In the following exercises, simplify.

55. $\dfrac{\frac{1}{3}+\frac{1}{8}}{\frac{1}{4}+\frac{1}{12}}$

Answer

$\dfrac{11}{8}$

56. $\dfrac{\frac{3}{a^2}-\frac{1}{b}}{\frac{1}{a}+\frac{1}{b^2}}$

57. $\dfrac{\frac{2}{z^2-49}+\frac{1}{z+7}}{\frac{9}{z+7}+\frac{12}{z-7}}$

Answer

$\dfrac{z-5}{21z+21}$

58. $\dfrac{\frac{3}{y^2-4y-32}}{\frac{2}{y-8}+\frac{1}{y+4}}$

Solve Rational Equations

Solve Rational Equations

In the following exercises, solve.

59. $\frac{1}{2}+\frac{2}{3}=\frac{1}{x}$

Answer

$x=\frac{6}{7}$

60. $1-\frac{2}{m}=\frac{8}{m^2}$

61. $\frac{1}{b-2}+\frac{1}{b+2}=\frac{3}{b^2-4}$

Answer

$b=\frac{3}{2}$

62. $\frac{3}{q+8}-\frac{2}{q-2}=1$

63. $\frac{v-15}{v^{2}-9v+18}=\frac{4}{v-3}+\frac{2}{v-6}$

Answer

no solution

64. $\frac{z}{12}+\frac{z+3}{3z}=\frac{1}{z}$

Solve Rational Equations that Involve Functions

65. For rational function, $f(x)=\frac{x+2}{x^{2}-6x+8}$,

a. Find the domain of the function
b. Solve $f(x)=1$
c. Find the points on the graph at this function value.

Answer

a. The domain is all real numbers except $x\neq 2$ and $x\neq 4$
b. $x=1, x=6$
c. $(1,1),(6,1)$

66. For rational function, $f(x)=\frac{2-x}{x^{2}+7x+10}$,

a. Solve $f(x)=2$
b. Find the points on the graph at this function value.

Solve a Rational Equation for a Specific Variable

In the following exercises, solve for the indicated variable.

67. $\frac{V}{l}=hw$ for l

Answer

$l=\frac{V}{hw}$

68. $\frac{1}{x}-\frac{2}{y}=5$ for y

69. $x=\frac{y+5}{z-7}$ for z

Answer

$z=\frac{y+5+7x}{x}$

70. $P=\frac{k}{V}$ for V

Solve Proportions

In the following exercises, solve.

71. $\dfrac{x}{4}=\dfrac{3}{5}$

Answer

$x=\dfrac{12}{5}$

72. $\dfrac{3}{y}=\dfrac{9}{5}$

73. $\dfrac{s}{s+20}=\dfrac{3}{7}$

Answer

$s=15$

74. $\dfrac{t-3}{5}=\dfrac{t+2}{9}$

Solve Applications Using Proportions

In the following exercises, solve.

75. Rachael had a 21-ounce strawberry shake that has 739 calories. How many calories are there in a 32-ounce shake?

Answer

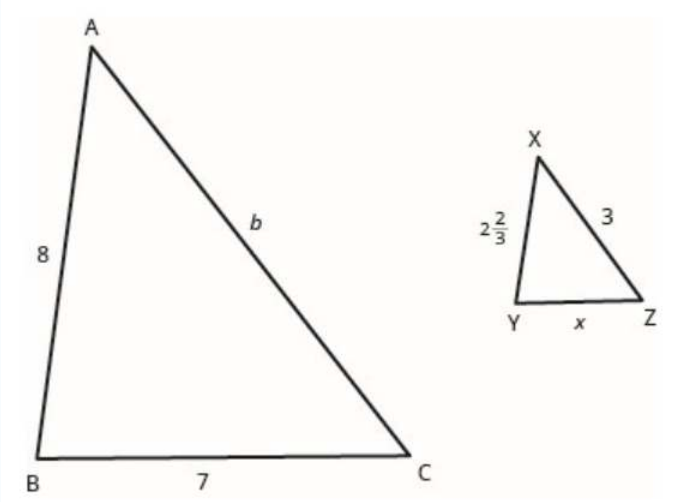
1161 calories

76. Leo went to Mexico over Christmas break and changed $525 dollars into Mexican pesos. At that time, the exchange rate had $1 US is equal to 16.25 Mexican pesos. How many Mexican pesos did he get for his trip?

Solve Similar Figure Applications

In the following exercises, solve.

77. ΔABC is similar to ΔXYZ. The lengths of two sides of each triangle are given in the figure. Find the lengths of the third sides.

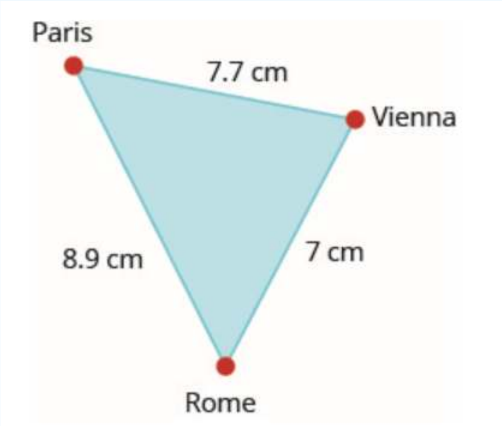


Answer

$b=9;\ x=2\frac{1}{3}$

78. On a map of Europe, Paris, Rome, and Vienna form a triangle whose sides are shown in the figure below. If the actual distance from Rome to Vienna is 700 miles, find the distance from

a. Paris to Rome
b. Paris to Vienna



79. Francesca is 5.75 feet tall. Late one afternoon, her shadow was 8 feet long. At the same time, the shadow of a nearby tree was 32 feet long. Find the height of the tree.

Answer

23 feet

80. The height of a lighthouse in Pensacola, Florida is 150 feet. Standing next to the statue, 5.5-foot-tall Natasha cast a 1.1-foot shadow. How long would the shadow of the lighthouse be?

Solve Uniform Motion Applications

In the following exercises, solve.

81. When making the 5-hour drive home from visiting her parents, Lolo ran into bad weather. She was able to drive 176 miles while the weather was good, but then driving 10 mph slower, went 81 miles when it turned bad. How fast did she drive when the weather was bad?

Answer

45 mph

82. Mark is riding on a plane that can fly 490 miles with a tailwind of 20 mph in the same time that it can fly 350 miles against a tailwind of 20 mph. What is the speed of the plane?

83. Josue can ride his bicycle 8 mph faster than Arjun can ride his bike. It takes Luke 3 hours longer than Josue to ride 48 miles. How fast can John ride his bike?

Answer

16 mph

84. Curtis was training for a triathlon. He ran 8 kilometers and biked 32 kilometers in a total of 3 hours. His running speed was 8 kilometers per hour less than his biking speed. What was his running speed?

Solve Work Applications

In the following exercises, solve.

85. Brandy can frame a room in 1 hour, while Jake takes 4 hours. How long could they frame a room working together?

Answer

$\frac{4}{5}$ hour

86. Prem takes 3 hours to mow the lawn while her cousin, Barb, takes 2 hours. How long will it take them working together?

87. Jeffrey can paint a house in 6 days, but if he gets a helper he can do it in 4 days. How long would it take the helper to paint the house alone?

Answer

12 days

88. Marta and Deb work together writing a book that takes them 90 days. If Sue worked alone it would take her 120 days. How long would it take Deb to write the book alone?

Solve Direct Variation Problems

In the following exercises, solve.

89. If y varies directly as x when $y=9$ and $x=3$, find x when $y=21$.

Answer

7

90. If y varies inversely as x when $y=20$ and $x=2$ find y when $x=4$.

91. Vanessa is traveling to see her fiancé. The distance, d, varies directly with the speed, v, she drives. If she travels 258 miles driving 60 mph, how far would she travel going 70 mph?

Answer

301 mph

92. If the cost of a pizza varies directly with its diameter, and if an 8” diameter pizza costs $12, how much would a 6” diameter pizza cost?

93. The distance to stop a car varies directly with the square of its speed. It takes 200 feet to stop a car going 50 mph. How many feet would it take to stop a car going 60 mph?

Answer

288 feet

Solve Inverse Variation Problems

In the following exercises, solve.

94. If m varies inversely with the square of n, when $m=4$ and $n=6$ find m when $n=2$.

95. The number of tickets for a music fundraiser varies inversely with the price of the tickets. If Madelyn has just enough money to purchase 12 tickets for $6, how many tickets can Madelyn afford to buy if the price increased to $8?

Answer

97 tickets

96. On a string instrument, the length of a string varies inversely with the frequency of its vibrations. If an 11-inch string on a violin has a frequency of 360 cycles per second, what frequency does a 12-inch string have?

Solve Rational Inequalities

Solve Rational Inequalities

In the following exercises, solve each rational inequality and write the solution in interval notation.

97. $\dfrac{x-3}{x+4} \leq 0$

Answer

$(-4,3]$

98. $\dfrac{5x}{x-2} > 1$

99. $\dfrac{3x-2}{x-4} \leq 2$

Answer

$[-6,4)$

100. $\dfrac{1}{x^2-4x-12} < 0$

101. $\dfrac{1}{2} - \dfrac{4}{x^2} \geq \dfrac{1}{x}$

Answer

$(-\infty, -2] \cup [4, \infty)$

102. $\dfrac{4}{x-2} < \dfrac{3}{x+1}$

Solve an Inequality with Rational Functions

In the following exercises, solve each rational function inequality and write the solution in interval notation

103. Given the function, $R(x) = \dfrac{x-5}{x-2}$, find the values of x that make the function greater than or equal to 0.

Answer

$(-\infty, 2) \cup [5, \infty)$

104. Given the function, $R(x) = \dfrac{x+1}{x+3}$, find the values of x that make the function greater than or equal to 0.

105. The function $C(x) = 150x + 100,000$ represents the cost to produce x, number of items. Find

a. The average cost function, $c(x)$
b. How many items should be produced so that the average cost is less than $160.

Answer

a. $c(x) = \dfrac{150x + 100000}{x}$
b. More than 10,000 items must be produced to keep the average cost below $160 per item.

106. Tillman is starting his own business by selling tacos at the beach. Accounting for the cost of his food truck and ingredients for the tacos, the function $C(x) = 2x + 6,000$ represents the cost for Tillman to produce x, tacos. Find

a. The average cost function, $c(x)$ for Tillman's Tacos
b. How many tacos should Tillman produce so that the average cost is less than $4.

Practice Test

In the following exercises, simplify.

1. $\dfrac{4a^2b}{12ab^2}$

Answer

$\dfrac{a}{3b}$

2. $\dfrac{6x-18}{x^2-9}$

In the following exercises, perform the indicated operation and simplify.

3. $\dfrac{4x}{x+2} \cdot \dfrac{x^2+5x+6}{12x^2}$

Answer

$$\frac{x+3}{3x}$$

4. $\frac{2y^2}{y^2-1} \div \frac{y^3-y^2+y}{y^3-1}$

5. $\frac{6x^2-x+20}{x^2-81} - \frac{5x^2+11x-7}{x^2-81}$

Answer

$$\frac{x-3}{x+9}$$

6. $\frac{-3a}{3a-3} + \frac{5a}{a^2+3a-4}$

7. $\frac{2n^2+8n-1}{n^2-1} - \frac{n^2-7n-1}{1-n^2}$

Answer

$$\frac{3n-2}{n-1}$$

8. $\frac{10x^2+16x-7}{8x-3} + \frac{2x^2+3x-1}{3-8x}$

9. $\frac{\frac{1}{m}-\frac{1}{n}}{\frac{1}{n}+\frac{1}{m}}$

Answer

$$\frac{n-m}{m+n}$$

In the following exercises, solve each equation.

10. $\frac{1}{x} + \frac{3}{4} = \frac{5}{8}$

11. $\frac{1}{z-5} + \frac{1}{z+5} = \frac{1}{z^2-25}$

Answer

$$z = \frac{1}{2}$$

12. $\frac{z}{2z+8} - \frac{3}{4z-8} = \frac{3z^2-16z-16}{8z^2+2z-64}$

In the following exercises, solve each rational inequality and write the solution in interval notation.

13. $\frac{6x}{x-6} \leq 2$

Answer

$[-3, 6)$

14. $\dfrac{2x+3}{x-6} > 1$

15. $\dfrac{1}{2} + \dfrac{12}{x^2} \geq \dfrac{5}{x}$

Answer

$(-\infty, 0) \cup (0, 4] \cup [6, \infty)$

In the following exercises, find $R(x)$ given $f(x) = \dfrac{x-4}{x^2-3x-10}$ and $g(x) = \dfrac{x-5}{x^2-2x-8}$.

16. $R(x) = f(x) - g(x)$

17. $R(x) = f(x) \cdot g(x)$

Answer

$$R(x) = \frac{1}{(x+2)(x+2)}$$

18. $R(x) = f(x) \div g(x)$

19. Given the function, $R(x) = \dfrac{2}{2x^2+x-15}$, find the values of x that make the function less than or equal to 0.

Answer

$(2, 5]$

In the following exercises, solve.

20. If y varies directly with x, and $x = 5$ when $y = 30$, find x when $y = 42$.

21. If y varies inversely with the square of x and $x = 3$ when $y = 9$, find y when $x = 4$.

Answer

$$y = \frac{81}{16}$$

22. Matheus can ride his bike for 30 miles with the wind in the same amount of time that he can go 21 miles against the wind. If the wind's speed is 6 mph, what is Matheus' speed on his bike?

23. Oliver can split a truckload of logs in 8 hours, but working with his dad they can get it done in 3 hours. How long would it take Oliver's dad working alone to split the logs?

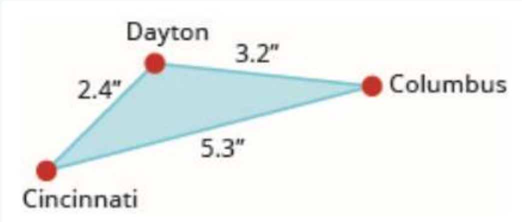
Answer

Oliver's dad would take $4\frac{4}{5}$ hours to split the logs himself.

24. The volume of a gas in a container varies inversely with the pressure on the gas. If a container of nitrogen has a volume of 29.5 liters with 2000 psi, what is the volume if the tank has a 14.7 psi rating? Round to the nearest whole number.

25.

The cities of Dayton, Columbus, and Cincinnati form a triangle in southern Ohio. The diagram gives the map distances between these cities in inches.



The actual distance from Dayton to Cincinnati is 48 miles. What is the actual distance between Dayton and Columbus?

Answer

The distance between Dayton and Columbus is 64 miles.

CHAPTER OVERVIEW

6: Roots and Radicals

6.1: Simplify Expressions with Square Roots
6.2: Simplify Radical Expressions
6.3: Simplify Rational Exponents
6.4: Add, Subtract, and Multiply Radical Expressions
6.5: Divide Radical Expressions
6.6: Solve Radical Equations
6.7: Complex Numbers
6.8: Chapter 6 Review Exercises

6.1: Simplify Expressions with Square Roots

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify

a. $(-9)^2$

b. -9^2

2. Round 3.846 to the nearest hundredth.

3. Simplify

a. $x^3 \cdot x^3$

b. $y^2 \cdot y^2$

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number n is multiplied by itself, we write n^2 and read it 'n squared'. This number is called the **square** of n, and n is called the **square root**. For example,

$$13^2 \text{ is read "13 squared"}$$

$$169 \text{ is called the square of } 13\text{, since } 13^2=169$$

$$13 \text{ is a square root of } 169$$

Definition 6.1.1

Square

$$\text{If } n^2=m\text{, then } m \text{ is the } \textbf{square} \text{ of } n.$$

Square Root

$$\text{If } n^2=m\text{, then } n \text{ is a } \textbf{square root} \text{ of } m.$$

Notice $(-13)^2=169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of m. The positive square root is also called the **principal square root**.

We also use the radical sign for the square root of zero. Because $0^2=0$, $\sqrt{0}=0$. Notice that zero has only one square root.

Definition 6.1.2

$\sqrt{m}$ is read "**the square root of m.**"

If $n^2=m$, then $n=\sqrt{m}$, for $n\geq 0$.

$$\text{radical sign} \longrightarrow \sqrt{m} \longleftarrow \text{radicand}$$

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169}=13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169}=-13$.

Example 6.1.3

Simplify:

a. $\sqrt{144}$

b. $-\sqrt{289}$

Solution

a.

$$\sqrt{144}$$

Since $12^2=144$, and $12 \geq 0$

$$12$$

b.

$$-\sqrt{289}$$

Since $17^2=289$, $17 \geq 0$, and the negative is in front of the radical sign.

$$-17$$

Try It 6.1.4

Simplify:

a. $-\sqrt{64}$

b. $\sqrt{225}$

Answer

a. -8

b. 15

Try It 6.1.5

Simplify:

a. $\sqrt{100}$

b. $-\sqrt{121}$

Answer

a. 10

b. -11

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?

$$(\_\_\_)^2=-49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

Example 6.1.6

Simplify:

a. $\sqrt{-196}$

b. $-\sqrt{64}$

Solution

a.

$$\sqrt{-196}$$

There is no real number whose square is -196.

$$\sqrt{-196} \text{ is not a real number.}$$

b.

$$-\sqrt{64}$$

The negative is in front of the radical.

$$-8$$

Exercise 6.1.7

Simplify:

a. $\sqrt{-169}$

b. $-\sqrt{81}$

Answer

a. not a real number

b. -9

Exercise 6.1.8

Simplify:

a. $-\sqrt{49}$

b. $\sqrt{-121}$

Answer

a. -7

b. not a real number

Properties of $\sqrt{a}$

When

- $a \geq 0$, then $\sqrt{a}$ is a real number.
- $a < 0$, then $\sqrt{a}$ is not a real number.

Simplify Variable Expressions with Square Roots

Note, for example,

$$\sqrt{4^2} = \sqrt{16} = 4$$

but,

$$\sqrt{(-4)^2} = \sqrt{16} = 4,$$

So that the result is positive.

How can we make sure the square root of -5 squared is 5? We can use the absolute value. $|-5| = 5:$

$$\sqrt{a^2} = |a|. \tag{6.1.1}$$

This guarantees the principal root is positive.

Summary

We have

$$\sqrt{a^2} = |a|$$

Example 6.1.9

Simplify $\sqrt{x^2}$.

Solution

We use the absolute value to be sure to get the positive root.

$$\sqrt{x^2} = |x|$$

Try It 6.1.10

Simplify $\sqrt{b^2}$.

Answer

$|b|$

What about square roots of higher powers of variables? The power property of exponents says $(a^m)^n = a^{m\cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root.

$$\sqrt{a^{2m}} = \sqrt{(a^m)^2}$$

Since 2 is even, $\sqrt[2]{x^2} = |x|$. So

$$\sqrt{a^{2m}} = |a^m|. \tag{6.1.2}$$

We apply this concept in the next example.

Example 6.1.11

Simplify:

a. $\sqrt{x^6}$

b. $\sqrt{y^{16}}$

Solution

a.

$$\sqrt{x^6}$$

Since $\left(x^3\right)^2 = x^6$, this is equal to

$$\sqrt{\left(x^3\right)^2}.$$

Since $\sqrt{a^2} = |a|$, this is equal to

$$\left|x^3\right|$$

b.

$$\sqrt{y^{16}}$$

Since $\left(y^8\right)^2 = y^{16}$, this is equal to

$$\sqrt{\left(y^8\right)^2}.$$

Since $\sqrt{a^2} = |a|$, this is equal to

$$y^8$$

In this case the absolute value sign is not needed as y^8 is positive.

? Try It 6.1.12

Simplify:

a. $\sqrt{y^{18}}$

b. $\sqrt{z^{12}}$

Answer

a. $\left|y^9\right|$

b. z^6

? Exercise 6.1.13

Simplify:

a. $\sqrt{m^4}$

b. $\sqrt{b^{10}}$

Answer

a. m^2

b. $\left|b^5\right|$

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}} = |a^m|$ works in much the same way.

$$\sqrt{16r^{22}} = 4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^2 = 16r^{22}.$$

But notice $\sqrt{25u^8} = 5u^4$ and no absolute value sign is needed as u^4 is always non-negative.

Example 6.1.14

Simplify:

a. $\sqrt{16n^2}$

b. $-\sqrt{81c^2}$

Solution

a.

$$\sqrt{16n^2}$$

Since $(4n)^2 = 16n^2$, this is equal to

$$\sqrt{(4n)^2}.$$

Since $\sqrt{a^2} = |a|$, this is equal to

$$4|n|.$$

b.

$$-\sqrt{81c^2}$$

Since $(9c)^2 = 81c^2$, this is equal to

$$-\sqrt{(9c)^2}.$$

Since $\sqrt{a^2} = |a|$, this is then equal to

$$-9|c|.$$

Try It 6.1.15

Simplify:

a. $\sqrt{64x^2}$

b. $-\sqrt{100p^2}$

Answer

a. $8|x|$

b. $-10|p|$

Exercise 6.1.16

Simplify:

a. $\sqrt{169y^2}$

b. $-\sqrt{121y^2}$

Answer

a. $13|y|$

b. $-11|y|$

The next examples have two variables.

Example 6.1.17

Simplify:

a. $\sqrt{36x^2y^2}$

b. $\sqrt{121a^6b^8}$

Solution

a.

$$\sqrt{36x^2y^2}$$

Since $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

Take the square root.

$$6|xy|$$

b.

$$\sqrt{121a^6b^8}$$

Since $\left(11a^3b^4\right)^2 = 121a^6b^8$

$$\sqrt{\left(11a^3b^4\right)^2}$$

Take the square root.

$$11\left|a^3\right|b^4$$

Try It 6.1.18

Simplify:

a. $\sqrt{100a^2b^2}$

b. $\sqrt{144p^{12}q^{20}}$

Answer

a. $10|ab|$

b. $12p^6q^{10}$

Try It 6.1.19

Simplify:

a. $\sqrt{225m^2n^2}$

b. $\sqrt{169x^{10}y^{14}}$

Answer

a. $15|mn|$

b. $13\left|x^5y^7\right|$

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'

- If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

Figure 8.1.1

- The square root of m, $\sqrt{m}$, is a positive number whose square is m.

- **Properties of $\sqrt{a}$**
  - $a \geq 0$, then $\sqrt{a}$ is a real number
  - $a < 0$, then $\sqrt{a}$ is not a real number
- **Simplifying Odd and Even Roots**
  - $\sqrt{a^2} = |a|$. We must use the absolute value signs when we take a square root of an expression with a variable in the radical.

Glossary

square of a number

If $n^2 = m$, then m is the square of n.

square root of a number

If $n^2 = m$, then n is a square root of m.

Practice makes perfect

Simplifying Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{64}$

b. $-\sqrt{81}$

Answer

a. 8

b. -9

2. a. $\sqrt{169}$

b. $-\sqrt{100}$

3. a. $\sqrt{196}$

b. $-\sqrt{1}$

Answer

a. 14

b. -1

4. a. $\sqrt{144}$

b. $-\sqrt{121}$

5. a. $\sqrt{\frac{4}{9}}$

b. $-\sqrt{0.01}$

Answer

a. $\frac{2}{3}$

b. -0.1

6. a. $\sqrt{\frac{64}{121}}$

b. $-\sqrt{0.16}$

7. a. $\sqrt{-121}$

b. $-\sqrt{289}$

Answer

a. not a real number

b. -17

8. a. $-\sqrt{400}$

b. $\sqrt{-36}$

9. a. $-\sqrt{225}$

b. $\sqrt{-9}$

Answer

a. -15

b. not a real number

10. a. $\sqrt{-49}$

b. $-\sqrt{256}$

11. $\sqrt{70}$

Answer

$8<\sqrt{70}<9$

12. $\sqrt{55}$

13. $\sqrt{200}$

Answer

$14<\sqrt{200}<15$

14. $\sqrt{172}$

In the following exercises, approximate each root and round to two decimal places.

15. $\sqrt{19}$

Answer

≈ 4.36

16. $\sqrt{21}$

17. $\sqrt{53}$

Answer

≈ 7.28

18. $\sqrt{47}$

Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

19. a. $\sqrt{x^{6}}$

b. $\sqrt{y^{16}}$

Answer

a. $|x^{3}|$

b. y^{8}

20. a. $\sqrt{a^{14}}$

b. $\sqrt{w^{24}}$

21. a. $\sqrt{x^{24}}$

b. $\sqrt{y^{22}}$

Answer

a. x^{12}

b. $|y^{11}|$

22. a. $\sqrt{a^{12}}$

b. $\sqrt{b^{26}}$

23. a. $\sqrt{49x^{2}}$

b. $-\sqrt{81x^{18}}$

Answer

a. $7|x|$

b. $-9|x^{9}|$

24. a. $\sqrt{100y^{2}}$

b. $-\sqrt{100m^{32}}$

25. a. $\sqrt{121m^{20}}$

b. $-\sqrt{64a^{2}}$

Answer

a. $11m^{10}$

b. $-8|a|$

26. a. $\sqrt{81x^{36}}$

b. $-\sqrt{25x^{2}}$

27. a. $\sqrt[4]{16x^{8}}$

b. $\sqrt[6]{64y^{12}}$

Answer

a. $2x^{2}$

b. $2y^{2}$

28. a. $\sqrt{144x^{2}y^{2}}$

b. $\sqrt{169w^{8}y^{10}}$

Answer

a. $12|xy|$

b. $13w^{4}\left|y^{5}\right|$

29. a. $\sqrt{196a^{2}b^{2}}$

b. $\sqrt{81p^{24}q^{6}}$

30. a. $\sqrt{121a^{2}b^{2}}$

b. $\sqrt{9c^{8}d^{12}}$

Answer

a. $11|ab|$

b. $3c^{4}d^{6}$

31. a. $\sqrt{225x^{2}y^{2}z^{2}}$

b. $\sqrt{36r^{6}s^{20}}$

Writing Exercises

32. Why is there no real number equal to $\sqrt{-64}$?

Answer

Since the square of any real number is positive, it's not possible for a real number to square to -64.

33. What is the difference between 9^2 and $\sqrt{9}$?

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with roots.			
estimate and approximate roots.			
simplify variable expressions with roots.			

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

6.2: Simplify Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\dfrac{x^9}{x^4}$.
2. Simplify $\dfrac{y^3}{y^{11}}$.
3. Simplify $\left(n^2\right)^6$.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A **radical expression**, $\sqrt{a}$, is considered simplified if it has no factors of the form m^2. So, to simplify a radical expression, we look for any factors in the radicand that are squares.

Definition 6.2.1

For non-negative integers a and m,

$$\boldsymbol{\sqrt{a}} \textbf{ is considered simplified} \text{ if } a \text{ has no factors of the form } m^2.$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that

$$(ab)^n = a^n b^n. \tag{6.2.1}$$

The corresponding of **Product Property of Roots** says that

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}. \tag{6.2.2}$$

To see why this is true we note that

$$(\sqrt{ab})^2 = ab \tag{6.2.3}$$

since $\sqrt{ab}$ is the non-negative quantity you square to get ab by definition.

Also,

$$(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2 = ab, \tag{6.2.4}$$

where the first equality follows from the product property of exponents and the second by the definition of the square root (as above).

So, the left and the right hand sides, being both non-negative, are square roots of ab, and therefore are equal.

It may not come as a surprise that due to this property, it is written that $\sqrt{a} = a^{\frac{1}{2}}$ and the properties of exponents can be shown to be extended to the exponents obtained this way.

Fact 6.2.2

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

Note that you can also read the equality: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

We use the Product Property of Roots to remove all perfect square factors from a square root.

Example 6.2.3

Simplify $\sqrt{98}$.

Solution

Find the largest factor in the radicand that is a perfect power of the index.	We see that 49 is the largest factor of 98 that has a power of 2.	$\sqrt{98}$
Rewrite the radicand as a product of two factors, using that factor.	In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{49 \cdot 2}$
Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Simplify the root of the perfect power.		$7\sqrt{2}$

Try It 6.2.4

Simplify $\sqrt{48}$.

Answer

$4\sqrt{3}$

Try It 6.2.5

Simplify $\sqrt{45}$.

Answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index (which we will discuss later). The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

Simplify a Radical Expression Using the Product Property

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares.

Example 6.2.6

Simplify $\sqrt{500}$.

Solution

$$\sqrt{500}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{100 \cdot 5}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{100} \cdot \sqrt{5}$$

Simplify.

$$10\sqrt{5}$$

Try It 6.2.7

Simplify $\sqrt{288}$.

Answer

$12\sqrt{2}$

Try It 6.2.8

Simplify $\sqrt{432}$.

Answer

$12\sqrt{3}$

The next example is much like the previous examples, but with variables. Don’t forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example 6.2.9

Simplify $\sqrt{x^3}$.

Solution

$$\sqrt{x^3}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{x^2 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{x^2} \cdot \sqrt{x}$$

Simplify.

$$|x|\sqrt{x}$$

? Try It 6.2.10

Simplify $\sqrt{b^5}$.

Answer

$b^2\sqrt{b}$

? Try It 6.2.11

Simplify $\sqrt{p^9}$.

Answer

$p^4\sqrt{p}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

? Example 6.2.12

Simplify $\sqrt{72n^7}$.

Solution

$$\sqrt{72n^7}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6\cdot 2n}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6}\cdot\sqrt{2n}$$

Simplify.

$$6\left|n^3\right|\sqrt{2n}$$

? Try It 6.2.13

Simplify $\sqrt{32y^5}$.

Answer

$4y^2\sqrt{2y}$

? Try It 6.2.14

Simplify $\sqrt{75a^9}$.

Answer

$5a^4\sqrt{3a}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

Example 6.2.15

Simplify $\sqrt{63u^3v^5}$.

Answer

$$\sqrt{63u^3v^5}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9u^2v^4 \cdot 7uv}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$$

Rewrite the first radicand as $\left(3uv^2\right)^2$.

$$\sqrt{\left(3uv^2\right)^2} \cdot \sqrt{7uv}$$

Simplify.

$$3|u|v^2\sqrt{7uv}$$

Try It 6.2.16

Simplify $\sqrt{98a^7b^5}$.

Answer

$7\left|a^3\right|b^2\sqrt{2ab}$

Try It 6.2.17

Simplify $\sqrt{180m^9n^{11}}$.

Answer

$6m^4\left|n^5\right|\sqrt{5mn}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect square. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

Example 6.2.18

Simplify $\sqrt{\frac{45}{80}}$.

Solution

$$\sqrt{\frac{45}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt{\frac{9}{16}}$$

Simplify. Note $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

$$\frac{3}{4}$$

Try It 6.2.19

Simplify $\sqrt{\frac{75}{48}}$.

Answer

$\frac{5}{4}$

Try It 6.2.20

Simplify $\sqrt{\frac{98}{162}}$.

Answer

$\frac{7}{9}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the **Quotient Property** to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example 6.2.21

Simplify $\sqrt{\frac{m^6}{m^4}}$.

Solution

$$\sqrt{\frac{m^6}{m^4}}$$

Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.

$$\sqrt{m^2}$$

Simplify.

$$|m|$$

Try It 6.2.22

Simplify $\sqrt{\frac{a^8}{a^6}}$.

Answer

$|a|$

Try It 6.2.23

Simplify $\sqrt{\frac{x^{14}}{x^{10}}}$.

Answer

x^2

Remember the **Quotient to a Power Property**? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Quotient Property of Radical Expressions

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Example 6.2.24

Simplify $\sqrt{\frac{27m^3}{196}}$.

Solution

Simplify the fraction in the radicand, if possible.

$\frac{27m^3}{196}$ cannot be simplified.

$$\sqrt{\frac{27m^3}{196}}$$

Use the Quotient Property to rewrite the radical as the quotient of two radicals.

We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.

$$\frac{\sqrt{27m^3}}{\sqrt{196}}$$

Simplify the radicals in the numerator and the denominator.

$9m^2$ and 196 are perfect squares.

$$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$$

$$\frac{3m\sqrt{3m}}{14}$$

Try It 6.2.25

Simplify $\sqrt{\frac{24p^3}{49}}$.

Answer

$$\frac{2|p|\sqrt{6p}}{7}$$

Try It 6.2.26

Simplify $\sqrt{\frac{48x^5}{100}}$.

Answer

$$\frac{2x^2\sqrt{3x}}{5}$$

Simplify a Square Root Using the Quotient Property

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

Example 6.2.27

Simplify $\sqrt{\frac{45x^5}{y^4}}$.

Solution

$$\sqrt{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt{5x}}{y^2}$$

Try It 6.2.28

Simplify $\sqrt{\frac{80m^3}{n^6}}$.

Answer

$$\frac{4|m|\sqrt{5m}}{|n^3|}$$

Try It 6.2.29

Simplify $\sqrt{\frac{54u^7}{v^8}}$.

Answer

$\frac{3u^3\sqrt{6u}}{v^4}$

Be sure to simplify the fraction in the radicand first, if possible.

Example 6.2.30

Simplify $\sqrt{\frac{18p^5q^7}{32pq^2}}$.

Solution

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt{\frac{9p^4q^5}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$$

Simplify.

$$\frac{3p^2q^2\sqrt{q}}{4}$$

Try It 6.2.31

Simplify $\sqrt{\frac{50x^5y^3}{72x^4y}}$.

Answer

$\frac{5|y|\sqrt{x}}{6}$

Try It 6.2.32

Simplify $\sqrt{\frac{48m^7n^2}{100m^5n^8}}$.

Answer

$$\frac{2|m|\sqrt{3}}{5\left|n^3\right|}$$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the **Quotient Property** again, to combine them into one radical. We will then look to see if we can simplify the expression.

Example 6.2.33

Simplify $\frac{\sqrt{48a^7}}{\sqrt{3a}}$.

Solution

$$\frac{\sqrt{48a^7}}{\sqrt{3a}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

Simplify.

$$4\left|a^3\right|$$

Try It 6.2.34

Simplify $\frac{\sqrt{98z^5}}{\sqrt{2z}}$.

Answer

$7z^2$

Try It 6.2.35

Simplify $\frac{\sqrt{128m^9}}{\sqrt{2m}}$.

Answer

$8m^4$

Key Concepts

- **Simplified Radical Expression**
  - For real numbers a, m and $n \geq 2$
    $\sqrt{a}$ is considered simplified if a has no factors of m^2
- **Product Property of n^{th} Roots**
  - For any real numbers, $\sqrt{a}$ and $\sqrt{b}$, and for any integer $n \geq 2$
    $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{a}\sqrt{b} = \sqrt{ab}$
- **How to simplify a radical expression using the Product Property**

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

- **Quotient Property of Radical Expressions**
  - If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.
- **How to simplify a radical expression using the Quotient Property.**

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

Practice Makes Perfect

? Use the product property to simplify radical expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{27}$
2. $\sqrt{80}$
3. $\sqrt{125}$
4. $\sqrt{96}$
5. $\sqrt{147}$
6. $\sqrt{450}$
7. $\sqrt{800}$
8. $\sqrt{675}$
9. a. $\sqrt[4]{32}$
   b. $\sqrt[5]{64}$
10. a. $\sqrt[3]{625}$
    b. $\sqrt[6]{128}$
11. a. $\sqrt[5]{64}$
    b. $\sqrt[3]{256}$
12. a. $\sqrt[4]{3125}$
    b. $\sqrt[3]{81}$

Answer

1. $3\sqrt{3}$

3. $5\sqrt{5}$

5. $7\sqrt{3}$

7. $20\sqrt{2}$

9.

a. $2\sqrt[4]{2}$
b. $2\sqrt[5]{2}$

11.

a. $2\sqrt[5]{2}$
b. $4\sqrt[3]{4}$

? Use the product property to simplify radical expressions

In the following exercises, simplify using absolute value signs as needed.

13. a. $\sqrt{y^{11}}$
b. $\sqrt[3]{r^5}$
c. $\sqrt[4]{s^{10}}$
14. a. $\sqrt{m^{13}}$
b. $\sqrt[5]{u^7}$
c. $\sqrt[6]{v^{11}}$
15. a. $\sqrt{n^{21}}$
b. $\sqrt[3]{q^8}$
c. $\sqrt[8]{n^{10}}$
16. a. $\sqrt{r^{25}}$
b. $\sqrt[5]{p^8}$
c. $\sqrt[4]{m^5}$
17. a. $\sqrt{125r^{13}}$
b. $\sqrt[3]{108x^5}$
c. $\sqrt[4]{48y^6}$
18. a. $\sqrt{80s^{15}}$
b. $\sqrt[5]{96a^7}$
c. $\sqrt[6]{128b^7}$
19. a. $\sqrt{242m^{23}}$
b. $\sqrt[4]{405m10}$
c. $\sqrt[5]{160n^8}$
20. a. $\sqrt{175n^{13}}$
b. $\sqrt[5]{512p^5}$
c. $\sqrt[4]{324q^7}$
21. a. $\sqrt{147m^7n^{11}}$
b. $\sqrt[3]{48x^6y^7}$
c. $\sqrt[4]{32x^5y^4}$
22. a. $\sqrt{96r^3s^3}$
b. $\sqrt[3]{80x^7y^6}$
c. $\sqrt[4]{80x^8y^9}$
23. a. $\sqrt{192q^3r^7}$
b. $\sqrt[3]{54m^9n^{10}}$
c. $\sqrt[4]{81a^9b^8}$
24. a. $\sqrt{150m^9n^3}$
b. $\sqrt[3]{81p^7q^8}$
c. $\sqrt[4]{162c^{11}d^{12}}$
25. a. $\sqrt[3]{-864}$
b. $\sqrt[4]{-256}$
26. a. $\sqrt[5]{-486}$
b. $\sqrt[6]{-64}$
27. a. $\sqrt[5]{-32}$
b. $\sqrt[8]{-1}$
28. a. $\sqrt[3]{-8}$
b. $\sqrt[4]{-16}$
29. a. $5+\sqrt{12}$

b. $\dfrac{10-\sqrt{24}}{2}$

30. a. $8+\sqrt{96}$
    b. $\dfrac{8-\sqrt{80}}{4}$
31. a. $1+\sqrt{45}$
    b. $\dfrac{3+\sqrt{90}}{3}$
32. a. $3+\sqrt{125}$
    b. $\dfrac{15+\sqrt{75}}{5}$

Answer

13.

a. $\left|y^5\right|\sqrt{y}$
b. $r\sqrt[3]{r^2}$
c. $s^2\sqrt[4]{s^2}$

a. $n^{10}\sqrt{n}$
b. $q^2\sqrt[3]{q^2}$
c. $|n|\sqrt[8]{n^2}$

17.

a. $5r^6\sqrt{5r}$
b. $3x\sqrt[3]{4x^2}$
c. $2|y|\sqrt[4]{3y^2}$

19.

a. $11\left|m^{11}\right|\sqrt{2m}$
b. $3m^2\sqrt[4]{5m^2}$
c. $2n\sqrt[5]{5n^3}$

21.

a. $7\left|m^3n^5\right|\sqrt{3mn}$
b. $2x^2y^2\sqrt[3]{6y}$
c. $2|xy|\sqrt[4]{2x}$

23.

a. $8\left|qr^3\right|\sqrt{3qr}$
b. $3m^3n^3\sqrt[3]{2n}$
c. $3a^2b^2\sqrt[4]{a}$

25.

a. $-6\sqrt[3]{4}$
b. not real

27.

a. -2
b. not real

29.

a. $5+2\sqrt{3}$
b. $5-\sqrt{6}$

31.

a. $1+3\sqrt{5}$
b. $1+\sqrt{10}$

? Use the quotient property to simplify radical expressions

In the following exercises, use the Quotient Property to simplify square roots.

33. a. $\sqrt{\frac{45}{80}}$
    b. $\sqrt[3]{\frac{8}{27}}$
    c. $\sqrt[4]{\frac{1}{81}}$
34. a. $\sqrt{\frac{72}{98}}$
    b. $\sqrt[3]{\frac{24}{81}}$
    c. $\sqrt[4]{\frac{6}{96}}$
35. a. $\sqrt{\frac{100}{36}}$
    b. $\sqrt[3]{\frac{81}{375}}$
    c. $\sqrt[4]{\frac{1}{256}}$
36. a. $\sqrt{\frac{121}{16}}$
    b. $\sqrt[3]{\frac{16}{250}}$
    c. $\sqrt[4]{\frac{32}{162}}$
37. a. $\sqrt{\frac{x^{10}}{x^6}}$
    b. $\sqrt[3]{\frac{p^{11}}{p^2}}$
    c. $\sqrt[4]{\frac{q^{17}}{q^{13}}}$
38. a. $\sqrt{\frac{p^{20}}{p^{10}}}$
    b. $\sqrt[5]{\frac{d^{12}}{d^7}}$
    c. $\sqrt[8]{\frac{m^{12}}{m^4}}$
39. a. $\sqrt{\frac{y^4}{y^8}}$
    b. $\sqrt[5]{\frac{u^{21}}{u^{11}}}$
    c. $\sqrt[6]{\frac{v^{30}}{v^{12}}}$

40. a. $\sqrt{\frac{q^{8}}{q^{14}}}$

b. $\sqrt[3]{\frac{r^{14}}{r^{5}}}$

c. $\sqrt[4]{\frac{c^{21}}{c^{9}}}$

41. $\sqrt{\frac{96x^{7}}{121}}$
42. $\sqrt{\frac{108y^{4}}{49}}$
43. $\sqrt{\frac{300m^{5}}{64}}$
44. $\sqrt{\frac{125n^{7}}{169}}$
45. $\sqrt{\frac{98r^{5}}{100}}$
46. $\sqrt{\frac{180s^{10}}{144}}$
47. $\sqrt{\frac{28q^{6}}{225}}$
48. $\sqrt{\frac{150r^{3}}{256}}$
49. a. $\sqrt{\frac{75r^{9}}{s^{8}}}$

b. $\sqrt[3]{\frac{54a^{8}}{b^{3}}}$

c. $\sqrt[4]{\frac{64c^{5}}{d^{4}}}$

50. a. $\sqrt{\frac{72x^{5}}{y^{6}}}$

b. $\sqrt[5]{\frac{96r^{11}}{s^{5}}}$

c. $\sqrt[6]{\frac{128u^{7}}{v^{12}}}$

51. a. $\sqrt{\frac{28p^{7}}{q^{2}}}$

b. $\sqrt[3]{\frac{81s^{8}}{t^{3}}}$

c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$

52. a. $\sqrt{\frac{45r^{3}}{s^{10}}}$

b. $\sqrt[3]{\frac{625u^{10}}{v^{3}}}$

c. $\sqrt[4]{\frac{729c^{21}}{d^{8}}}$

53. a. $\sqrt{\frac{32x^{5}y^{3}}{18x^{3}y}}$

b. $\sqrt[3]{\dfrac{5x^6y^9}{40x^5y^3}}$
c. $\sqrt[4]{\dfrac{5a^8b^6}{80a^3b^2}}$

54. a. $\sqrt{\dfrac{75r^6s^8}{48rs^4}}$
b. $\sqrt[3]{\dfrac{24x^8y^4}{81x^2y}}$
c. $\sqrt[4]{\dfrac{32m^9n^2}{162mn^2}}$

55. a. $\sqrt{\dfrac{27p^2q}{108p^4q^3}}$
b. $\sqrt[3]{\dfrac{16c^5d^7}{250c^2d^2}}$
c. $\sqrt[6]{\dfrac{2m^9n^7}{128m^3n}}$

56. a. $\sqrt{\dfrac{50r^5s^2}{128r^2s^6}}$
b. $\sqrt[3]{\dfrac{24m^9n^7}{375m^4n}}$
c. $\sqrt[4]{\dfrac{81m^2n^8}{256m^1n^2}}$

57. a. $\dfrac{\sqrt{45p^9}}{\sqrt{5q^2}}$
b. $\dfrac{\sqrt[4]{64}}{\sqrt[4]{2}}$
c. $\dfrac{\sqrt[5]{128x^8}}{\sqrt[5]{2x^2}}$

58. a. $\dfrac{\sqrt{80q^5}}{\sqrt{5q}}$
b. $\dfrac{\sqrt[3]{-625}}{\sqrt[3]{5}}$
c. $\dfrac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

59. a. $\dfrac{\sqrt{50m^7}}{\sqrt{2m}}$
b. $\sqrt[3]{\dfrac{1250}{2}}$
c. $\sqrt[4]{\dfrac{486y^9}{2y^3}}$

60. a. $\dfrac{\sqrt{72n^{11}}}{\sqrt{2n}}$
b. $\sqrt[3]{\dfrac{162}{6}}$
c. $\sqrt[4]{\dfrac{160r^{10}}{5r^3}}$

Answer

33.

a. $\frac{3}{4}$
b. $\frac{2}{3}$
c. $\frac{1}{3}$

35.

a. $\frac{5}{3}$
b. $\frac{3}{5}$
c. $\frac{1}{4}$

37.

a. x^2
b. p^3
c. $|q|$

39.

a. $\frac{1}{y^2}$
b. u^2
c. $|v^3|$

41. $\frac{4\left|x^3\right|\sqrt{6x}}{11}$

43. $\frac{10m^2\sqrt{3m}}{8}$

45. $\frac{7r^2\sqrt{2r}}{10}$

47. $\frac{2\left|q^3\right|\sqrt{7}}{15}$

49.

a. $\frac{5r^4\sqrt{3r}}{s^4}$
b. $\frac{3a^2\sqrt[3]{2a^2}}{|b|}$
c. $\frac{2|c|\sqrt[4]{4c}}{|d|}$

51.

a. $\frac{2\left|p^3\right|\sqrt{7p}}{|q|}$
b. $\frac{3s^2\sqrt[3]{3s^2}}{t}$
c. $\frac{2\left|p^3\right|\sqrt[4]{4p^3}}{\left|q^3\right|}$

53.

a. $\frac{4|xy|}{3}$

b. $\dfrac{y^2\sqrt[3]{x}}{2}$
c. $\dfrac{|ab|\sqrt[4]{a}}{4}$

55.

a. $\dfrac{1}{2|pq|}$
b. $\dfrac{2cd\sqrt[5]{2d^2}}{5}$
c. $\dfrac{|mn|\sqrt[6]{2}}{2}$

57.

a. $\dfrac{3p^4\sqrt{p}}{|q|}$
b. $2\sqrt[4]{2}$
c. $2x\sqrt[5]{2x}$

59.

a. $5\,|m^3|$
b. $5\sqrt[3]{5}$
c. $3|y|\sqrt[4]{3y^2}$

? Writing exercises

61. Explain why $\sqrt{x^4}=x^2$. Then explain why $\sqrt{x^{16}}=x^8$.
62. Explain why $7+\sqrt{9}$ is not equal to $\sqrt{7+9}$.
63. Explain how you know that $\sqrt[5]{x^{10}}=x^2$.
64. Explain why $\sqrt[4]{-64}$ is not a real number but $\sqrt[3]{-64}$ is.

Answer

61. Answers may vary

63. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the Product Property to simplify radical expressions.			
use the Quotient Property to simplify radical expressions.			

Figure 8.2.1

b. After reviewing this checklist, what will you do to become confident for all objectives?

6.3: Simplify Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Be Prepared

Before you get started, take this readiness quiz.

1. Add $\frac{7}{15}+\frac{5}{12}$.
2. Simplify $(4x^2y^5)^3$.
3. Simplify 5^{-3}.

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{mn}$ when m and n are integers. Let's assume we are now not limited to integers.

Suppose we want to find a number p such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of p.

$$(8^p)^3 = 8$$

Multiple the exponents on the left.

$$8^{3p} = 8$$

Write the exponent 1 on the right.

$$8^{3p} = 8^1$$

Since the bases are the same, the exponents must be equal.

$$3p = 1$$

Solve for p.

$$p = \frac{1}{3}$$

So $\left(8^{\frac{1}{3}}\right)^3 = 8$. But we know also $(\sqrt[3]{8})^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent n to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Definition 6.3.1

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

In this case, we say that n is the **index of the radical**.

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example 6.3.2

Write as a radical expression:

a. $x^{\frac{1}{2}}$

b. $y^{\frac{1}{3}}$

c. $z^{\frac{1}{4}}$

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

a.

$$x^{\frac{1}{2}}$$

The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.

$$\sqrt{x}$$

b.

$$y^{\frac{1}{3}}$$

The denominator of the exponent is 3, so the index is 3.

$$\sqrt[3]{y}$$

c.

$$z^{\frac{1}{4}}$$

The denominator of the exponent is 4, so the index is 4.

$$\sqrt[4]{z}$$

Try It 6.3.3

Write as a radical expression:

a. $t^{\frac{1}{2}}$

b. $m^{\frac{1}{3}}$

c. $r^{\frac{1}{4}}$

Answer

a. $\sqrt{t}$

b. $\sqrt[3]{m}$

c. $\sqrt[4]{r}$

Try It 6.3.4

Write as a radical expression:

a. $b^{\frac{1}{6}}$

b. $z^{\frac{1}{5}}$

c. $p^{\frac{1}{4}}$

Answer

a. $\sqrt[6]{b}$

b. $\sqrt[5]{z}$

c. $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

Example 6.3.5

Write with a rational exponent:

$$\sqrt{5y} \tag{6.3.1}$$

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$

$$\sqrt{5y}$$

No index is shown, so it is 2.

The denominator of the exponent will be 2.

Put parentheses around the entire expression $5y$.

$$(5y)^{\frac{1}{2}}$$

Try It 6.3.6

Write with a rational exponent:

$$\sqrt{10m} \tag{6.3.2}$$

Answer

$(10m)^{\frac{1}{2}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example 6.3.7

Simplify:

a. $25^{\frac{1}{2}}$

b. $64^{\frac{1}{3}}$

c. $256^{\frac{1}{4}}$

Solution

a.

$$25^{\frac{1}{2}}$$

Rewrite as a square root.

$$\sqrt{25}$$

Simplify.

$$5$$

b.

$$64^{\frac{1}{3}}$$

Rewrite as a cube root.

$$\sqrt[3]{64}$$

Recognize 64 is a perfect cube.

$$\sqrt[3]{4^3}$$

Simplify.

$$4$$

c.

$$256^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{256}$$

Recognize 256 is a perfect fourth power.

$$\sqrt[4]{4^4}$$

Simplify.

$$4$$

Try It 6.3.8

Simplify:

a. $36^{\frac{1}{2}}$

b. $8^{\frac{1}{3}}$

c. $16^{\frac{1}{4}}$

Answer

a. 6

b. 2

c. 2

Try It 6.3.9

Simplify:

a. $100^{\frac{1}{2}}$

b. $27^{\frac{1}{3}}$

c. $81^{\frac{1}{4}}$

Answer

a. 10

b. 3

c. 3

Be careful of the placement of the negative signs in the next example. We will need to use the definition $a^{-n} = \dfrac{1}{a^n}$ in one case.

Example 6.3.10

Simplify:

a. $(-16)^{\frac{1}{4}}$

b. $-16^{\frac{1}{4}}$

c. $(16)^{-\frac{1}{4}}$

Solution

a.

$$(-16)^{\frac{1}{4}}$$

Rewrite as a fourth root.

$$\sqrt[4]{-16}$$
$$\sqrt[4]{(-2)^4}$$

Simplify.

No real solution

b.

$$-16^{\frac{1}{4}}$$

The exponent only applies to the 16. Rewrite as a fourth root.

$$-\sqrt[4]{16}$$

Rewrite 16 as 2^4

$$-\sqrt[4]{2^4}$$

Simplify.

$$-2$$

c.

$$(16)^{-\frac{1}{4}}$$

Rewrite using the defintion $a^{-n} = \dfrac{1}{a^n}$.

$$\frac{1}{(16)^{\frac{1}{4}}}$$

Rewrite as a fourth root.

$$\frac{1}{\sqrt[4]{16}}$$

Rewrite 16 as 2^4.

$$\frac{1}{\sqrt[4]{2^4}}$$

Simplify.

$$\frac{1}{2}$$

Try It 6.3.11

Simplify:

a. $(-64)^{-\frac{1}{2}}$

b. $-64^{\frac{1}{2}}$

c. $(64)^{-\frac{1}{2}}$

Answer

a. No real solution

b. -8

c. $\frac{1}{8}$

Try It 6.3.12

Simplify:

a. $(-256)^{\frac{1}{4}}$

b. $-256^{\frac{1}{4}}$

c. $(256)^{-\frac{1}{4}}$

Answer

a. No real solution

b. -4

c. $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^m=(\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}}=(a^m)^{\frac{1}{n}}=\sqrt[n]{a^m}$$

This leads us to the following defintion.

Definition 6.3.13

For any positive integers m and n,

$$a^{\frac{m}{n}}=(\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}}=\sqrt[n]{a^m}.$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

Example 6.3.14

Write with a rational exponent:

a. $\sqrt{y^3}$

b. $(\sqrt[3]{2x})^4$

c. $\sqrt{\left(\frac{3a}{4b}\right)^3}$

Solution

We want to use $a^{\frac{m}{n}}=\sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$

a.

	$\sqrt{y^3}$
When the index is missing, it is implicitly 2.	$\sqrt[2]{y^3}$
$m=3, n=2$ in $\sqrt[n]{a^m}=a^{\frac{m}{n}}$	$y^{\frac{3}{2}}$

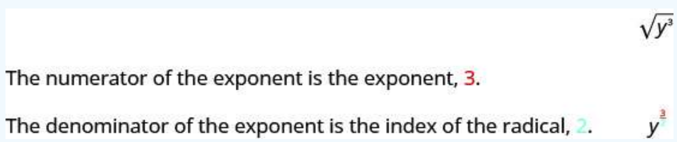


Figure 8.3.1

b.

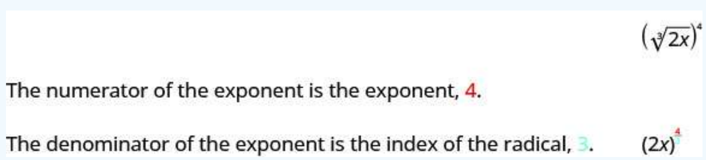


Figure 8.3.2

c.

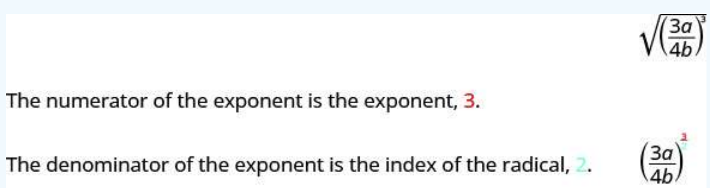


Figure 8.3.3

Try It 6.3.15

Write with a rational exponent:

a. $\sqrt{x^5}$

b. $(\sqrt[4]{3y})^3$

c. $\sqrt{\left(\frac{2m}{3n}\right)^5}$

Answer

a. $x^{\frac{5}{2}}$

b. $(3y)^{\frac{3}{4}}$

c. $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

Try It 6.3.16

Write with a rational exponent:

a. $\sqrt[5]{a^2}$

b. $\left(\sqrt[3]{5ab}\right)^5$

c. $\sqrt{\left(\frac{7xy}{z}\right)^3}$

Answer

a. $a^{\frac{2}{5}}$

b. $(5ab)^{\frac{5}{3}}$

c. $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

Example 6.3.17

Simplify:

a. $125^{\frac{2}{3}}$

b. $16^{-\frac{3}{2}}$

c. $32^{-\frac{2}{5}}$

Solution

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

a.

$$125^{\frac{2}{3}}$$

The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.

$$\left(\sqrt[3]{125}\right)^2$$

Simplify.

$$(5)^2$$

$$25$$

b. We will rewrite each expression first using $a^{-n} = \frac{1}{a^n}$ and then change to radical form.

$$16^{-\frac{3}{2}}$$

Rewrite using $a^{-n}=\frac{1}{a^n}$.

$$\frac{1}{16^{\frac{3}{2}}}$$

Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.

$$\frac{1}{\left(\sqrt{16}\right)^3}$$

Simplify.

$$\frac{1}{4^3}$$

$$\frac{1}{64}$$

c.

$$32^{-\frac{2}{5}}$$

Rewrite using $a^{-n}=\frac{1}{a^n}$.

$$\frac{1}{32^{\frac{2}{5}}}$$

Change to radical form.

$$\frac{1}{(\sqrt[5]{32})^2}$$

Rewrite the radicand as a power.

$$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$$

Simplify.

$$\frac{1}{2^2}$$

$$\frac{1}{4}$$

? Try It 6.3.18

Simplify:

a. $27^{\frac{2}{3}}$

b. $81^{-\frac{3}{2}}$

c. $16^{-\frac{3}{4}}$

Answer

a. 9

b. $\frac{1}{729}$

c. $\frac{1}{8}$

Try It 6.3.19

Simplify:

a. $4^{\frac{3}{2}}$

b. $27^{-\frac{2}{3}}$

c. $625^{-\frac{3}{4}}$

Answer

a. 8

b. $\frac{1}{9}$

c. $\frac{1}{125}$

Example 6.3.20

Simplify:

a. $-25^{\frac{3}{2}}$

b. $-25^{-\frac{3}{2}}$

c. $(-25)^{\frac{3}{2}}$

Solution

a.

$$-25^{\frac{3}{2}}$$

Rewrite in radical form.

$$-(\sqrt{25})^3$$

Simplify the radical.

$$-(5)^3$$

Simplify.

$$-125$$

b.

$$-25^{-\frac{3}{2}}$$

Rewrite using $a^{-n}=\frac{1}{a^n}$.

$$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$$

Rewrite in radical form.

$$-\left(\frac{1}{(\sqrt{25})^3}\right)$$

Simplify the radical.

$$-\left(\frac{1}{5^3}\right)$$

Simplify.

$$-\frac{1}{125}$$

c.

$$(-25)^{\frac{3}{2}}$$

Rewrite in radical form.

$$(\sqrt{-25})^3$$

There is no real number whose square root is -25.

Not a real number.

? Try It 6.3.21

Simplify:

a. $-16^{\frac{3}{2}}$

b. $-16^{-\frac{3}{2}}$

c. $(-16)^{-\frac{3}{2}}$

Answer

a. -64

b. $-\frac{1}{64}$

c. Not a real number

? Try It 6.3.22

Simplify:

a. $-81^{\frac{3}{2}}$

b. $-81^{-\frac{3}{2}}$

c. $(-81)^{-\frac{3}{2}}$

Answer

a. -729

b. $-\frac{1}{729}$

c. Not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used for integers also apply to rational exponents. We will not show this here, but doing some examples will convince you. For example,

$$16^{\frac{1}{2}} \cdot 16^{\frac{3}{4}} = 4 \cdot 8 = 32$$

and

$$16^{\frac{1}{2}+\frac{3}{4}} = 16^{\frac{5}{4}} = 32.$$

So $16^{\frac{1}{2}} \cdot 16^{\frac{3}{4}} = 16^{\frac{1}{2}+\frac{3}{4}}$.

We will list the Properties of Exponents below to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property

$$a^m \cdot a^n = a^{m+n}$$

Power Property

$$(a^m)^n = a^{mn}$$

Product to a Power

$$(ab)^m = a^m b^m$$

Quotient Property

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

Zero Exponent Definition

$$a^0 = 1, a \neq 0$$

Quotient to a Power Property

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Negative Exponent Property

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

We will apply these properties in the next example.

Example 6.3.23

Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\dfrac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution

a. The Product Property tells us that when we multiple the same base, we add the exponents.

$$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$$

The bases are the same, so we add the exponents.

$$x^{\frac{1}{2}+\frac{5}{6}}$$

Add the fractions.

$$x^{\frac{8}{6}}$$

Simplify the exponent.

$$x^{\frac{4}{3}}$$

b. The Power Property tells us that when we raise a power to a power, we multiple the exponents.

$$\left(z^9\right)^{\frac{2}{3}}$$

To raise a power to a power, we multiple the exponents.

$$z^{9\cdot\frac{2}{3}}$$

Simplify.

$$z^6$$

c. The Quotient Property tells us that when we divide with the same base, we subtract the exponents.

$$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$$

To divide with the same base, we subtract the exponents.

$$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$$

Simplify.

$$\frac{1}{x^{\frac{4}{3}}}$$

Alternatively:

? Try It 6.3.24

Simplify:

a. $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$

b. $\left(x^6\right)^{\frac{4}{3}}$

c. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

Answer

a. $x^{\frac{3}{2}}$

b. x^8

c. $\frac{1}{x}$

? Try It 6.3.25

Simplify:

a. $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$

b. $\left(m^9\right)^{\frac{2}{9}}$

c. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

Answer

a. $y^{\frac{11}{8}}$

b. m^2

c. $\frac{1}{d}$

Key Concepts

- **Rational Exponent $a^{\frac{1}{n}}$**
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent $a^{\frac{m}{n}}$**
  - For any positive integers m and n,
    $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If a, b are real numbers and m, n are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{mn}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, a \neq 0$

Practice Makes Perfect

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $x^{\frac{1}{2}}$ b. $y^{\frac{1}{3}}$ c. $z^{\frac{1}{4}}$
2. a. $r^{\frac{1}{2}}$ b. $s^{\frac{1}{3}}$ c. $t^{\frac{1}{4}}$
3. a. $u^{\frac{1}{5}}$ b. $v^{\frac{1}{9}}$ c. $w^{\frac{1}{20}}$
4. a. $g^{\frac{1}{7}}$ b. $h^{\frac{1}{5}}$ c. $j^{\frac{1}{25}}$

Answer

1. a. $\sqrt{x}$ b. $\sqrt[3]{y}$ c. $\sqrt[4]{z}$

3. a. $\sqrt[5]{u}$ b. $\sqrt[9]{v}$ c. $\sqrt[20]{w}$

Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

5. a. $\sqrt[7]{x}$ b. $\sqrt[9]{y}$ c. $\sqrt[5]{f}$
6. a. $\sqrt[8]{4}$ b. $\sqrt[10]{s}$ c. $\sqrt[4]{t}$
7. a. $\sqrt[3]{7c}$ b. $\sqrt[7]{12d}$ c. $2\sqrt[4]{6b}$

8. a. $\sqrt[4]{5x}$ b. $\sqrt[8]{9y}$ c. $7\sqrt[5]{3z}$
9. a. $\sqrt{21p}$ b. $\sqrt[4]{8q}$ c. $4\sqrt[6]{36r}$
10. a. $\sqrt[3]{25a}$ b. $\sqrt{3b}$ c. $\sqrt[8]{40c}$

Answer

5. a. $x^{\frac{1}{7}}$ b. $y^{\frac{1}{9}}$ c. $f^{\frac{1}{5}}$

7. a. $(7c)^{\frac{1}{4}}$ b. $(12d)^{\frac{1}{7}}$ c. $2(6b)^{\frac{1}{4}}$

9. a. $(21p)^{\frac{1}{2}}$ b. $(8q)^{\frac{1}{4}}$ c. $4(36r)^{\frac{1}{6}}$

? Simplify expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

11. a. $81^{\frac{1}{2}}$ b. $125^{\frac{1}{3}}$ c. $64^{\frac{1}{2}}$
12. a. $625^{\frac{1}{4}}$ b. $243^{\frac{1}{5}}$ c. $32^{\frac{1}{5}}$
13. a. $16^{\frac{1}{4}}$ b. $16^{\frac{1}{2}}$ c. $625^{\frac{1}{4}}$
14. a. $64^{\frac{1}{3}}$ b. $32^{\frac{1}{5}}$ c. $81^{\frac{1}{4}}$
15. a. $(-216)^{\frac{1}{3}}$ b. $-216^{\frac{1}{3}}$ c. $(216)^{-\frac{1}{3}}$
16. a. $(-1000)^{\frac{1}{3}}$ b. $-1000^{\frac{1}{3}}$ c. $(1000)^{-\frac{1}{3}}$
17. a. $(-81)^{\frac{1}{4}}$ b. $-81^{\frac{1}{4}}$ c. $(81)^{-\frac{1}{4}}$
18. a. $(-49)^{\frac{1}{2}}$. b. $-49^{\frac{1}{2}}$ c. $(49)^{-\frac{1}{2}}$
19. a. $(-36)^{\frac{1}{2}}$ b. $-36^{\frac{1}{2}}$ c. $(36)^{-\frac{1}{2}}$
20. a. $(-16)^{\frac{1}{4}}$. b. $-16^{\frac{1}{4}}$ c. $16^{-\frac{1}{4}}$
21. a. $(-100)^{\frac{1}{2}}$ b. $-100^{\frac{1}{2}}$ c. $(100)^{-\frac{1}{2}}$
22. a. $(-32)^{\frac{1}{5}}$ b. $(243)^{-\frac{1}{5}}$ c. $-125^{\frac{1}{3}}$

Answer

11. a. 9 b. 5 c. 8

13. a. 2 b. 4 c. 5

15. a. -6 b. -6 c. $\frac{1}{6}$

17. a. not real b. -3 c. $\frac{1}{3}$

19. a. not real b. -6 c. $\frac{1}{6}$

21. a. not real b. -10 c. $\frac{1}{10}$

? Simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

23. a. $\sqrt{m^5}$ b. $(\sqrt[3]{3y})^7$ c. $\sqrt[5]{\left(\frac{4x}{5y}\right)^3}$

24. a. $\sqrt[4]{r^7}$ b. $(\sqrt[5]{2pq})^3$ c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

25. a. $\sqrt[5]{u^2}$ b. $(\sqrt[3]{6x})^5$ c. $\sqrt[4]{\left(\dfrac{18a}{5b}\right)^7}$

26. a. $\sqrt[3]{a}$ b. $(\sqrt[4]{21v})^3$ c. $\sqrt[4]{\left(\dfrac{2xy}{5z}\right)^2}$

Answer

23. a. $m^{\frac{5}{2}}$ b. $(3y)^{\frac{7}{3}}$ c. $\left(\dfrac{4x}{5y}\right)^{\frac{3}{5}}$

25. a. $u^{\frac{2}{5}}$ b. $(6x)^{\frac{5}{3}}$ c. $\left(\dfrac{18a}{5b}\right)^{\frac{7}{4}}$

? Simplify expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

27. a. $64^{\frac{5}{2}}$ b. $81^{\frac{-3}{2}}$ c. $(-27)^{\frac{2}{3}}$
28. a. $25^{\frac{3}{2}}$ b. $9^{-\frac{3}{2}}$ c. $(-64)^{\frac{2}{3}}$
29. a. $32^{\frac{2}{5}}$ b. $27^{-\frac{2}{3}}$ c. $(-25)^{\frac{1}{2}}$
30. a. $100^{\frac{3}{2}}$ b. $49^{-\frac{5}{2}}$ c. $(-100)^{\frac{3}{2}}$
31. a. $-9^{\frac{3}{2}}$ b. $-9^{-\frac{3}{2}}$ c. $(-9)^{\frac{3}{2}}$
32. a. $-64^{\frac{3}{2}}$ b. $-64^{-\frac{3}{2}}$ c. $(-64)^{\frac{3}{2}}$

Answer

27. a. $32,768$ b. $\dfrac{1}{729}$ c. 9

29. a. 4 b. $\dfrac{1}{9}$ c. not real

31. a. -27 b. $-\dfrac{1}{27}$ c. not real

? Use the laws of exponents to simplify expressions with rational exponents

In the following exercises, simplify. Assume all variables are positive.

33. a. $c^{\frac{1}{4}} \cdot c^{\frac{5}{8}}$ b. $\left(p^{12}\right)^{\frac{3}{4}}$ c. $\dfrac{r^{\frac{4}{5}}}{r^{\frac{9}{5}}}$
34. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$ b. $\left(b^{15}\right)^{\frac{3}{5}}$ c. $\dfrac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
35. a. $y^{\frac{1}{2}} \cdot y^{\frac{3}{4}}$ b. $\left(x^{12}\right)^{\frac{2}{3}}$ c. $\dfrac{m^{\frac{5}{8}}}{m^{\frac{13}{8}}}$
36. a. $q^{\frac{2}{3}} \cdot q^{\frac{5}{6}}$ b. $\left(h^{6}\right)^{\frac{4}{3}}$ c. $\dfrac{n^{\frac{3}{5}}}{n^{\frac{8}{5}}}$
37. a. $\left(27q^{\frac{3}{2}}\right)^{\frac{4}{3}}$ b. $\left(a^{\frac{1}{3}}b^{\frac{2}{3}}\right)^{\frac{3}{2}}$
38. a. $\left(64s^{\frac{3}{7}}\right)^{\frac{1}{6}}$ b. $\left(m^{\frac{4}{3}}n^{\frac{1}{2}}\right)^{\frac{3}{4}}$
39. a. $\left(16u^{\frac{1}{3}}\right)^{\frac{3}{4}}$ b. $\left(4p^{\frac{1}{3}}q^{\frac{1}{2}}\right)^{\frac{3}{2}}$

40. a. $\left(625n^{\frac{8}{3}}\right)^{\frac{3}{4}}$ b. $\left(9x^{\frac{2}{5}}y^{\frac{3}{5}}\right)^{\frac{5}{2}}$

41. a. $\dfrac{r^{\frac{5}{2}}\cdot r^{-\frac{1}{2}}}{r^{-\frac{3}{2}}}$ b. $\left(\dfrac{36s^{\frac{1}{5}}t^{-\frac{3}{2}}}{s^{-\frac{9}{5}}t^{\frac{1}{2}}}\right)^{\frac{1}{2}}$

42. a. $\dfrac{a^{\frac{3}{4}}\cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$ b. $\left(\dfrac{27b^{\frac{2}{3}}c^{-\frac{5}{2}}}{b^{-\frac{7}{3}}c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

43. a. $\dfrac{c^{\frac{5}{3}}\cdot c^{-\frac{1}{3}}}{c^{-\frac{2}{3}}}$ b. $\left(\dfrac{8x^{\frac{5}{3}}y^{-\frac{1}{2}}}{27x^{-\frac{4}{3}}y^{\frac{5}{2}}}\right)^{\frac{1}{3}}$

44. a. $\dfrac{m^{\frac{7}{4}}\cdot m^{-\frac{5}{4}}}{m^{-\frac{2}{4}}}$ b. $\left(\dfrac{16m^{\frac{1}{5}}n^{\frac{3}{2}}}{81m^{\frac{9}{5}}n^{-\frac{1}{2}}}\right)^{\frac{1}{4}}$

Answer

33. a. $c^{\frac{7}{8}}$ b. p^9 c. $\dfrac{1}{r}$

35. a. $y^{\frac{5}{4}}$ b. x^8 c. $\dfrac{1}{m}$

37. a. $81q^2$ b. $a\dfrac{1}{2}b$

39. a. $8u^{\frac{1}{4}}$ b. $8p^{\frac{1}{2}}q^{\frac{3}{4}}$

41. a. $r^{\frac{7}{2}}$ b. $\dfrac{6s}{t}$

43. a. c^2 b. $\dfrac{2x}{3y}$

? Writing exercises

45. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.

46. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.

Answer

45. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
simplify expressions with $a^{\frac{1}{n}}$.			
simplify expressions with $a^{\frac{m}{n}}$.			
use the Laws of Exponents to simply expressions with rational exponents.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

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6.4: Add, Subtract, and Multiply Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Be Prepared

Before you get started, take this readiness quiz.

1. Add $3x^2+9x-5-(x^2-2x+3)$.
2. Simplify $(2+a)(4-a)$.
3. Simplify $(9-5y)^2$.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Definition 6.4.1

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$. Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Let's think about adding like terms with variables as we do the next few examples. When we have like radicals, we just add or subtract the coefficients. When the radicals are not like, we cannot combine the terms.

Example 6.4.2

Simplify $2\sqrt{2}-7\sqrt{2}$.

Solution

	$2\sqrt{2}-7\sqrt{2}$
Since the radicals are like, we subtract the coefficients.	$=-5\sqrt{2}$

Try It 6.4.3

Simplify $8\sqrt{2}-9\sqrt{2}$.

Answer

$-\sqrt{2}$

Try It 6.4.4

Simplify $5\sqrt{3}-9\sqrt{3}$.

Answer

$-4\sqrt{3}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

Example 6.4.5

Simplify $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$.

Solution

	$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
Since the radicals are like, we combine them.	$=0\sqrt{5n}$
Simplify.	$=0$

Try It 6.4.6

Simplify $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$.

Answer

$-2\sqrt{7x}$

Try It 6.4.7

Simplify $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$.

Answer

$-\sqrt{3y}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Example 6.4.8

Simplify $\sqrt{20}+3\sqrt{5}$.

Solution

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$=\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$
Simplify.	$=2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$=5\sqrt{5}$

Try It 6.4.9

Simplify $\sqrt{18}+6\sqrt{2}$.

Answer

$9\sqrt{2}$

Try It 6.4.10

Simplify $\sqrt{27}+4\sqrt{3}$.

Answer

$7\sqrt{3}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

Example 6.4.11

Simplify $9\sqrt{50m^2}-6\sqrt{48m^2}$.

Solution

	$9\sqrt{50m^2}-6\sqrt{48m^2}$
Simplify the radicals.	$=9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$
Simplify,	$=9\cdot 5m\cdot\sqrt{2}-6\cdot 4m\cdot\sqrt{3}$
The radicals are not like and so cannot be combined.	$=45m\sqrt{2}-24m\sqrt{3}$

Try It 6.4.12

Simplify $\sqrt{32m^7}-\sqrt{50m^7}$.

Answer

$-m^3\sqrt{2m}$

Try It 6.4.13

Simplify $\sqrt{27p^3}-\sqrt{48p^3}$.

Answer

$-p\sqrt{3p}$

Multiply Radical Expressions

We have used the **Product Property of Roots** to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots ‘in reverse’ to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Product Property of Roots

For any real numbers, $\sqrt{a}$ and $\sqrt{b}$, we have

$$\sqrt{ab}=\sqrt{a}\cdot\sqrt{b} \quad \text{and} \quad \sqrt{a}\cdot\sqrt{b}=\sqrt{ab}.$$

When we multiply two radicals, they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example 6.4.14

Simplify $(6\sqrt{2})(3\sqrt{10})$.

Solution

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$= 18\sqrt{20}$
Simplify the radical.	$= 18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$= 18 \cdot 2 \cdot \sqrt{5}$
Simplify.	$= 36\sqrt{5}$

Try It 6.4.15

Simplify $(3\sqrt{2})(2\sqrt{30})$.

Answer

$12\sqrt{15}$

Try It 6.4.16

Simplify $(3\sqrt{3})(3\sqrt{6})$.

Answer

$27\sqrt{2}$

We follow the same procedures when there are variables in the radicands.

Example 6.4.17

Simplify $\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$.

Solution

	$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
Multiply.	$= 40\sqrt{18p^4}$
Simplify the radical.	$= 40\sqrt{9p^4} \cdot \sqrt{2}$
Simplify.	$= 40 \cdot 3p^2 \cdot \sqrt{2}$
Simplify.	$= 120p^2\sqrt{2}$

Try It 6.4.18

Simplify $\left(6\sqrt{6x^2}\right)\left(2\sqrt{30x^4}\right)$.

Answer

$72x^3\sqrt{5}$

Try It 6.4.19

Simplify $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$.

Answer

$144y^2\sqrt{5y}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the **Distributive Property** to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Example 6.4.20

Simplify $\sqrt{6}(\sqrt{2}+\sqrt{18})$.

Solution

	$\sqrt{6}(\sqrt{2}+\sqrt{18})$
Multiply.	$=\sqrt{12}+\sqrt{108}$
Simplify.	$=\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$
Simplify.	$=2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$=8\sqrt{3}$

Try It 6.4.21

Simplify $\sqrt{6}(1+3\sqrt{6})$.

Answer

$18+\sqrt{6}$

Try It 6.4.22

Simplify $\sqrt{8}(2-5\sqrt{8})$.

Answer

$-40+4\sqrt{2}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example 6.4.23

Simplify $(3-2\sqrt{7})(4-2\sqrt{7})$.

Solution

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply.	$=12-6\sqrt{7}-8\sqrt{7}+4\left(\sqrt{7}\right)^2$

Simplify.	$=12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$ $=12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$=40-14\sqrt{7}$

Try It 6.4.24

Simplify $(6-3\sqrt{7})(3+4\sqrt{7})$.

Answer

$-66+15\sqrt{7}$

Try It 6.4.25

Simplify $(2-3\sqrt{11})(4-\sqrt{11})$.

Answer

$41-14\sqrt{11}$

Example 6.4.26

Simplify $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$.

Solution

	$(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$
Multiply.	$=3(\sqrt{2})^2+12\sqrt{2}\sqrt{5}-\sqrt{5}\sqrt{2}-4(\sqrt{5})^2$
Simplify.	$=6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$=-14+11\sqrt{10}$

Try It 6.4.27

Simplify $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$.

Answer

$1+9\sqrt{21}$

Try It 6.4.28

Simplify $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$.

Answer

$-12-20\sqrt{3}$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares

Product of Conjugates

$(a+b)(a-b)=a^2-b^2$

We will use the special product formulas in the next few examples. We will start with the **Product of Binomial Squares Pattern**.

Example 6.4.29

Simplify:

a. $(2+\sqrt{3})^2$

b. $(4-2\sqrt{5})^2$

Solution

a.

	$\underbrace{(2+\sqrt{3})^2}_{(a+b)^2}$
Multiply using the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$, or FOIL $(a+b)(a+b)$.	$=\underbrace{2^2+2\cdot 2\sqrt{3}+\left(\sqrt{3}\right)^2}_{a^2+2ab+b^2}$
Simplify.	$=4+4\sqrt{3}+3$
Combine like terms.	$=7+4\sqrt{3}$

b.

	$\underbrace{(4-2\sqrt{5})^2}_{(a-b)^2}$
Multiple, using the Product of Binomial Squares Pattern, $(a-b)^2=a^2-2ab+b^2$, or FOIL $(a-b)(a-b)$.	$=\underbrace{4^2+2\cdot 4\cdot 2\sqrt{5}+\left(2\sqrt{5}\right)^2}_{a^2-2ab+b^2}$
Simplify.	$=16-16\sqrt{5}+4\cdot 5$ $=16-16\sqrt{5}+20$
Combine like terms.	$=36-16\sqrt{5}$

Try It 6.4.30

Simplify:

a. $(10+\sqrt{2})^2$

b. $(1+3\sqrt{6})^2$

Answer

a. $102+20\sqrt{2}$

b. $55+6\sqrt{6}$

Try It 6.4.31

Simplify:

a. $(6-\sqrt{5})^2$

b. $(9-2\sqrt{10})^2$

Answer

a. $41-12\sqrt{5}$

b. $121-36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Example 6.4.32

Simplify $(5-2\sqrt{3})(5+2\sqrt{3})$.

Solution

	$\underbrace{(5-2\sqrt{3})(5+2\sqrt{3})}_{(a-b)(a+b)}$
Multiply using the Product of Conjugates Pattern.	$=\underbrace{5^2-\left(2\sqrt{3}\right)^2}_{a^2-b^2}$
Simplify.	$=25-4\cdot 3$
Simplify.	$=13$

Try It 6.4.33

Simplify $(3-2\sqrt{5})(3+2\sqrt{5})$.

Answer

-11

Try It 6.4.34

Simplify $(4+5\sqrt{7})(4-5\sqrt{7})$.

Answer

-159

Key Concepts

- **Product Property of Roots**
  - For any real numbers, $\sqrt[n]{a}$ and $\sqrt[n]{b}$, and for any integer $n\geq 2$ $\sqrt[n]{ab}=\sqrt[n]{a}\cdot\sqrt[n]{b}$ and $\sqrt[n]{a}\cdot\sqrt[n]{b}=\sqrt[n]{ab}$
- **Special Products**

Practice Makes Perfect

❓ Add and subtract radical expressions

In the following exercises, simplify. Assume all variables are greater than or equal to zero so that absolute values are not needed.

1. a. $8\sqrt{2}-5\sqrt{2}$ b. $5\sqrt[3]{m}+2\sqrt[3]{m}$ c. $8\sqrt[4]{m}-2\sqrt[4]{n}$
2. a. $7\sqrt{2}-3\sqrt{2}$ b. $7\sqrt[3]{p}+2\sqrt[3]{p}$ c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
3. a. $3\sqrt{5}+6\sqrt{5}$ b. $9\sqrt[3]{a}+3\sqrt[3]{a}$ c. $5\sqrt[4]{2z}+\sqrt[4]{2z}$
4. a. $4\sqrt{5}+8\sqrt{5}$ b. $\sqrt[3]{m}-4\sqrt[3]{m}$ c. $\sqrt{n}+3\sqrt{n}$
5. a. $3\sqrt{2a}-4\sqrt{2a}+5\sqrt{2a}$ b. $5\sqrt[4]{3ab}-3\sqrt[4]{3ab}-2\sqrt[4]{3ab}$
6. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$ b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
7. a. $8\sqrt{3c}+2\sqrt{3c}-9\sqrt{3c}$ b. $2\sqrt[3]{4pq}-5\sqrt[3]{4pq}+4\sqrt[3]{4pq}$
8. a. $3\sqrt{5d}+8\sqrt{5d}-11\sqrt{5d}$ b. $11\sqrt[3]{2rs}-9\sqrt[3]{2rs}+3\sqrt[3]{2rs}$
9. a. $\sqrt{27}-\sqrt{75}$ b. $\sqrt[3]{40}-\sqrt[3]{320}$ c. $\frac{1}{2}\sqrt[4]{32}+\frac{2}{3}\sqrt[4]{162}$
10. a. $\sqrt{72}-\sqrt{98}$ b. $\sqrt[3]{24}+\sqrt[3]{81}$ c. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$
11. a. $\sqrt{48}+\sqrt{27}$ b. $\sqrt[3]{54}+\sqrt[3]{128}$ c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
12. a. $\sqrt{45}+\sqrt{80}$ b. $\sqrt[3]{81}-\sqrt[3]{192}$ c. $\frac{5}{2}\sqrt[4]{80}+\frac{7}{3}\sqrt[4]{405}$
13. a. $\sqrt{72a^5}-\sqrt{50a^5}$ b. $9\sqrt[4]{80p^4}-6\sqrt[4]{405p^4}$
14. a. $\sqrt{48b^5}-\sqrt{75b^5}$ b. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$
15. a. $\sqrt{80c^7}-\sqrt{20c^7}$ b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
16. a. $\sqrt{96d^9}-\sqrt{24d^9}$ b. $5\sqrt[4]{243s^6}+2\sqrt[4]{3s^6}$
17. $3\sqrt{128y^2}+4y\sqrt{162}-8\sqrt{98y^2}$
18. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1. a. $3\sqrt{2}$ b. $7\sqrt[3]{m}$ c. $6\sqrt[4]{m}$
3. a. $9\sqrt{5}$ b. $12\sqrt[3]{a}$ c. $6\sqrt[4]{2z}$
5. a. $4\sqrt{2a}$ b. 0
7. a. $\sqrt{3c}$ b. $\sqrt[3]{4pq}$
9. a. $-2\sqrt{3}$ b. $-2\sqrt[3]{5}$ c. $3\sqrt[4]{2}$
11. a. $7\sqrt{3}$ b. $7\sqrt[3]{2}$ c. $3\sqrt[4]{5}$

13. a. $a^2\sqrt{2a}$ b. 0

15. a. $2c^3\sqrt{5c}$ b. $14r^2\sqrt[4]{2r^2}$

17. $4y\sqrt{2}$

? Multiply radical expressions

In the following exercises, simplify.

19. a. $(-2\sqrt{3})(3\sqrt{18})$

    b. $(8\sqrt[3]{4})(-4\sqrt[3]{18})$

    c.

20. a. $(-4\sqrt{5})(5\sqrt{10})$

    b. $(-2\sqrt[3]{9})(7\sqrt[3]{9})$

21. a. $(5\sqrt{6})(-\sqrt{12})$

    b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$

    c.

22. a. $(-2\sqrt{7})(-2\sqrt{14})$

    b. $(-3\sqrt[4]{8})(-5\sqrt[4]{6})$

23. a. $\left(4\sqrt{12z^3}\right)\left(3\sqrt{9z}\right)$

    b. $\left(5\sqrt[3]{3x^3}\right)\left(3\sqrt[3]{18x^3}\right)$

    c.

24. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$

    b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

25. a. $\left(-2\sqrt{7z^3}\right)\left(3\sqrt{14z^8}\right)$

    b. $\left(2\sqrt[4]{8y^2}\right)\left(-2\sqrt[4]{12y^3}\right)$

26. a. $\left(4\sqrt{2k^5}\right)\left(-3\sqrt{32k^6}\right)$

    b. $\left(-\sqrt[4]{6b^3}\right)\left(3\sqrt[4]{8b^3}\right)$

Answer

19.

a. $-18\sqrt{6}$

b. $-64\sqrt[3]{9}$

c.

21.

a. $-30\sqrt{2}$

b. $6\sqrt[4]{2}$

c.

23.

a. $72z^2\sqrt{3}$

b. $45x^2\sqrt[3]{2}$

c.

25.

a. $-42z^5\sqrt{2z}$

b. $-8y\sqrt[4]{6y}$

Use polynomial multiplication to multiply radical expressions

In the following exercises, multiply.

27. a. $\sqrt{7}(5+2\sqrt{7})$

b. $\sqrt[3]{6}(4+\sqrt[3]{18})$

28. a. $\sqrt{11}(8+4\sqrt{11})$

b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$

29. a. $\sqrt{11}(-3+4\sqrt{11})$

b. $\sqrt[4]{3}(\sqrt[4]{54}+\sqrt[4]{18})$

30. a. $\sqrt{2}(-5+9\sqrt{2})$

b. $\sqrt[4]{2}(\sqrt[4]{12}+\sqrt[4]{24})$

31. $(7+\sqrt{3})(9-\sqrt{3})$

32. $(8-\sqrt{2})(3+\sqrt{2})$

33. a. $(9-3\sqrt{2})(6+4\sqrt{2})$

b. $(\sqrt[3]{x}-3)(\sqrt[3]{x}+1)$

34. a. $(3-2\sqrt{7})(5-4\sqrt{7})$

b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$

35. a. $(1+3\sqrt{10})(5-2\sqrt{10})$

b. $(2\sqrt[3]{x}+6)(\sqrt[3]{x}+1)$

36. a. $(7-2\sqrt{5})(4+9\sqrt{5})$

b. $(3\sqrt[3]{x}+2)(\sqrt[3]{x}-2)$

37. $(\sqrt{3}+\sqrt{10})(\sqrt{3}+2\sqrt{10})$

38. $(\sqrt{11}+\sqrt{5})(\sqrt{11}+6\sqrt{5})$

39. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$

40. $(4\sqrt{6}+7\sqrt{13})(8\sqrt{6}-3\sqrt{13})$

41. a. $(3+\sqrt{5})^2$

b. $(2-5\sqrt{3})^2$

42. a. $(4+\sqrt{11})^2$

b. $(3-2\sqrt{5})^2$

43. a. $(9-\sqrt{6})^2$

b. $(10+3\sqrt{7})^2$

44. a. $(5-\sqrt{10})^2$

b. $(8+3\sqrt{2})^2$

c.

45. $(4+\sqrt{2})(4-\sqrt{2})$

46. $(7+\sqrt{10})(7-\sqrt{10})$

47. $(4+9\sqrt{3})(4-9\sqrt{3})$

48. $(1+8\sqrt{2})(1-8\sqrt{2})$

49. $(12-5\sqrt{5})(12+5\sqrt{5})$

50. $(9-4\sqrt{3})(9+4\sqrt{3})$

51. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

52. $(\sqrt[3]{4x}+3)(\sqrt[3]{4x}-3)$

Answer

27.

a. $14+5\sqrt{7}$

b. $4\sqrt[3]{6}+3\sqrt[3]{4}$

29.

a. $44-3\sqrt{11}$

b. $3\sqrt[4]{2}+\sqrt[4]{54}$

31. $60+2\sqrt{3}$

33.

a. $30+18\sqrt{2}$

b. $\sqrt[3]{x^2}-2\sqrt[3]{x}-3$

35.

a. $-54+13\sqrt{10}$

b. $2\sqrt[3]{x^2}+8\sqrt[3]{x}+6$

37. $23+3\sqrt{30}$

39. $-439-2\sqrt{77}$

41.

a. $14+6\sqrt{5}$

b. $79-20\sqrt{3}$

43.

a. $87-18\sqrt{6}$

b. $163+60\sqrt{7}$

45. 14

47. -227

49. 19

51. $\sqrt[3]{9x^2}-4$

? Mixed practice

53. $\frac{2}{3}\sqrt{27}+\frac{3}{4}\sqrt{48}$

54. $\sqrt{175k^4}-\sqrt{63k^4}$

55. $\frac{5}{6}\sqrt{162}+\frac{3}{16}\sqrt{128}$

56. $\sqrt[3]{24}+\sqrt[3]{81}$

57. $\frac{1}{2}\sqrt[4]{80}-\frac{2}{3}\sqrt[4]{405}$

58. $8\sqrt[4]{13}-4\sqrt[4]{13}-3\sqrt[4]{13}$

59. $5\sqrt{12c^4}-3\sqrt{27c^6}$

60. $\sqrt{80a^5}-\sqrt{45a^5}$

61. $\frac{3}{5}\sqrt{75}-\frac{1}{4}\sqrt{48}$

62. $21\sqrt[3]{9}-2\sqrt[3]{9}$

63. $8\sqrt[3]{64q^6}-3\sqrt[3]{125q^6}$

64. $11\sqrt{11}-10\sqrt{11}$

65. $\sqrt{3}\cdot\sqrt{21}$

66. $(4\sqrt{6})(-\sqrt{18})$

67. $(7\sqrt[3]{4})(-3\sqrt[3]{18})$

68. $\left(4\sqrt{12x^5}\right)\left(2\sqrt{6x^3}\right)$

69. $(\sqrt{29})^2$

70. $(-4\sqrt{17})(-3\sqrt{17})$

71. $(-4+\sqrt{17})(-3+\sqrt{17})$

72. $\left(3\sqrt[4]{8a^2}\right)\left(\sqrt[4]{12a^3}\right)$

73. $(6-3\sqrt{2})^2$

74. $\sqrt{3}(4-3\sqrt{3})$

75. $\sqrt[3]{3}(2\sqrt[3]{9}+\sqrt[3]{18})$

76. $(\sqrt{6}+\sqrt{3})(\sqrt{6}+6\sqrt{3})$

Answer

53. $5\sqrt{3}$
55. $9\sqrt{2}$
57. $-\sqrt[4]{5}$
59. $10c^2\sqrt{3}-9c^3\sqrt{3}$
61. $2\sqrt{3}$
63. $17q^2$
65. $3\sqrt{7}$

67. $-42\sqrt[3]{9}$

69. 29

71. $29-7\sqrt{17}$

73. $72-36\sqrt{2}$

75. $6+3\sqrt[3]{2}$

Writing exercises

77. Explain when a radical expression is in simplest form.
78. Explain the process for determining whether two radicals are like or unlike. Make sure your answer makes sense for radicals containing both numbers and variables.
79. a. Explain why $(-\sqrt{n})^2$ is always non-negative, for $n \geq 0$.
    b. Explain why $-(\sqrt{n})^2$ is always non-positive, for $n \geq 0$.
80. Use the binomial square pattern to simplify $(3+\sqrt{2})^2$. Explain all your steps.

Answer

77. Answers will vary

79. Answers will vary

Additional Exercises

81. Simplify:

a. $(8+\sqrt{a})(8-\sqrt{a})$

b. $(x+\sqrt{2})(x+\sqrt{6})$

c. $(\sqrt{5}-\sqrt{y})^2$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
add and subtract radical expressions.			
multiply radical expressions.			
use polynomial multiplication to multiply radical expressions.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

6.5: Divide Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\frac{30}{48}$.
2. Simplify $x^2 \cdot x^4$.
3. Multiply $(7+3x)(7-3x)$.

Divide Radical Expressions

We have used the **Quotient Property of Radical Expressions** to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals. We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Quotient Property of Radical Expressions

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers with $b \neq 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example 6.5.1

Simplify $\frac{\sqrt{72x^3}}{\sqrt{162x}}$.

Solution

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property,	$=\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$=\sqrt{\frac{\cancel{18} \cdot 4 \cdot x^2 \cdot \cancel{x}}{\cancel{18} \cdot 9 \cdot \cancel{x}}}$
Simplify.	$=\sqrt{\frac{4x^2}{9}}$
Simplify the radical.	$=\frac{2x}{3}$

Try It 6.5.2

Simplify $\frac{\sqrt{50s^3}}{\sqrt{128s}}$.

Answer

$\frac{5s}{8}$

Try It 6.5.3

Simplify $\frac{\sqrt{75q^5}}{\sqrt{108q}}$.

Answer

$\frac{5q^2}{6}$

Example 6.5.4

Simplify $\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$.

Solution

	$\frac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property.	$=\sqrt{\frac{147ab^8}{3a^3b^4}}$
Remove common factors in the fraction.	$=\sqrt{\frac{49b^4}{a^2}}$
Simplify the radical.	$=\frac{7b^2}{a}$

Try It 6.5.5

Simplify $\frac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$.

Answer

$\frac{9x^2}{y^2}$

Try It 6.5.6

Simplify $\frac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$.

Answer

$\frac{10n^3}{m}$

Example 6.5.7

Simplify $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

Solution

	$\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$
Rewrite using the quotient property.	$=\sqrt{\frac{54x^5y^3}{3x^2y}}$
Remove common factors in the fraction.	$=\sqrt{18x^3y^2}$
Rewrite the radicand as a product using the largest perfect square factor.	$=\sqrt{9x^2y^2\cdot 2x}$
Rewrite the radical as the product of two radicals.	$=\sqrt{9x^2y^2}\cdot\sqrt{2x}$
Simplify.	$=3xy\sqrt{2x}$

Try It 6.5.8

Simplify $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

Answer

$4xy\sqrt{2x}$

Try It 6.5.9

Simplify $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

Answer

$4ab\sqrt{3b}$

Rationalize a One-Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process! For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator. This process is still used today, and is useful in other areas of mathematics too.

Definition 6.5.10

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator should still be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a **radical expression** is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

Simplified Radical Expressions

A radical expression is considered **simplified** if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example 6.5.11

Simplify:

a. $\dfrac{4}{\sqrt{3}}$

b. $\sqrt{\dfrac{3}{20}}$

c. $\dfrac{3}{\sqrt{6x}}$

Solution

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

	$\dfrac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$=\dfrac{4\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{3}}$
Simplify.	$=\dfrac{4\sqrt{3}}{3}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\dfrac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$=\dfrac{\sqrt{3}}{\sqrt{20}}$
Simplify the denominator.	$=\dfrac{\sqrt{3}}{2\sqrt{5}}$

Multiply the numerator and denominator by $\sqrt{5}$.	$=\dfrac{\sqrt{3}\cdot\sqrt{5}}{2\sqrt{5}\cdot\sqrt{5}}$
Simplify.	$=\dfrac{\sqrt{15}}{2\cdot 5}$
Simplify.	$=\dfrac{\sqrt{15}}{10}$

c.

	$\dfrac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$=\dfrac{3\cdot\sqrt{6x}}{\sqrt{6x}\cdot\sqrt{6x}}$
Simplify.	$=\dfrac{3\sqrt{6x}}{6x}$
Simplify.	$=\dfrac{\sqrt{6x}}{2x}$

? Try It 6.5.12

Simplify:

a. $\dfrac{5}{\sqrt{3}}$

b. $\sqrt{\dfrac{3}{32}}$

c. $\dfrac{2}{\sqrt{2x}}$

Answer

a. $\dfrac{5\sqrt{3}}{3}$

b. $\dfrac{\sqrt{6}}{8}$

c. $\dfrac{\sqrt{2x}}{x}$

? Try It 6.5.13

Simplify:

a. $\dfrac{6}{\sqrt{5}}$

b. $\sqrt{\dfrac{7}{18}}$

c. $\dfrac{5}{\sqrt{5x}}$

Answer

a. $\dfrac{6\sqrt{5}}{5}$

b. $\frac{\sqrt{14}}{6}$

c. $\frac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

Rationalize a Two-Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the **Product of Conjugates Pattern** to **rationalize the denominator**.

$$\begin{array}{ll} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ =a^2-b^2 & =2^2-(\sqrt{5})^2 \\ & =4-5 \\ & =-1 \end{array}$$

When we multiple a binomial that includes a square root by its conjugate, the product has no square roots.

Example 6.5.14

Simplify $\frac{5}{2-\sqrt{3}}$.

Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$=\frac{5\,(2+\sqrt{3})}{(2-\sqrt{3})\,(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$=\frac{5\,(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$=\frac{5\,(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$=\frac{5\,(2+\sqrt{3})}{1}$
Simplify.	$=5\,(2+\sqrt{3})$

Try It 6.5.15

Simplify $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

Try It 6.5.16

Simplify $\frac{2}{4-\sqrt{6}}$.

Answer

$$\frac{4+\sqrt{6}}{5}$$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

Example 6.5.17

Simplify $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$=\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}-\sqrt{6}\right)\left(\sqrt{u}+\sqrt{6}\right)}$
Multiply the conjugates in the denominator.	$=\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{\left(\sqrt{u}\right)^2-\left(\sqrt{6}\right)^2}$
Simplify the denominator.	$=\frac{\sqrt{3}\left(\sqrt{u}+\sqrt{6}\right)}{u-6}$

Try It 6.5.18

Simplify $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$$

Try It 6.5.19

Simplify $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Answer

$$\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Example 6.5.20

Simplify $\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution

	$\frac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$=\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$=\frac{(\sqrt{x}+\sqrt{7})(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$=\frac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Try It 6.5.21

Simplify $\frac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Answer

$$\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$$

Try It 6.5.22

Simplify $\frac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

Answer

$$\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$$

Key Concepts

- **Quotient Property of Radical Expressions**
    - If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, $b \neq 0$, and then, $\sqrt{\frac{a}{b}}=\frac{\sqrt{a}}{\sqrt{b}}$.
- **Simplified Radical Expressions**
    - A radical expression is considered simplified if there are:
        - no factors in the radicand that have perfect powers of the index
        - no fractions in the radicand
        - no radicals in the denominator of a fraction

Glossary

rationalizing the denominator

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Practice Makes Perfect

? Divide square roots

In the following exercises, simplify.

1. $\dfrac{\sqrt{128}}{\sqrt{72}}$
2. $\dfrac{\sqrt{48}}{\sqrt{75}}$
3. $\dfrac{\sqrt{200m^5}}{\sqrt{98m}}$
4. $\dfrac{\sqrt{108n^7}}{\sqrt{243n^3}}$
5. $\dfrac{\sqrt{75r^3}}{\sqrt{108r^7}}$
6. $\dfrac{\sqrt{196q}}{\sqrt{484q^5}}$
7. $\dfrac{\sqrt{108p^5q^2}}{\sqrt{3p^3q^6}}$
8. $\dfrac{\sqrt{98rs^{10}}}{\sqrt{2r^3s^4}}$
9. $\dfrac{\sqrt{320mn^{-5}}}{\sqrt{45m-7n^3}}$
10. $\dfrac{\sqrt{810c^{-3}d^7}}{\sqrt{1000cd}}$
11. $\dfrac{\sqrt{56x^5y^4}}{\sqrt{2xy^3}}$
12. $\dfrac{\sqrt{72a^3b^6}}{\sqrt{3ab^3}}$

Answer

1. $\dfrac{4}{3}$
3. $\dfrac{10m^2}{7}$
5. $\dfrac{5}{6r^2}$
7. $\dfrac{6p}{q^2}$
9. $\dfrac{8m^4}{3n^4}$

11. $4x^4\sqrt{7y}$

Rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

12. a. $\frac{10}{\sqrt{6}}$ b. $\sqrt{\frac{4}{27}}$ c. $\frac{10}{\sqrt{5x}}$

13. a. $\frac{8}{\sqrt{3}}$ b. $\sqrt{\frac{7}{40}}$ c. $\frac{8}{\sqrt{2y}}$

14. a. $\frac{6}{\sqrt{7}}$ b. $\sqrt{\frac{8}{45}}$ c. $\frac{12}{\sqrt{3p}}$

15. a. $\frac{4}{\sqrt{5}}$ b. $\sqrt{\frac{27}{80}}$ c. $\frac{18}{\sqrt{6q}}$

Answer

12. a. $\frac{5\sqrt{6}}{3}$ b. $\frac{2\sqrt{3}}{9}$ c. $\frac{2\sqrt{5x}}{x}$

14. a. $\frac{6\sqrt{7}}{7}$ b. $\frac{2\sqrt{10}}{15}$ c. $\frac{4\sqrt{3p}}{p}$

Rationalize a Two Term Denominator

In the following exercises, simplify.

16. $\frac{8}{1-\sqrt{5}}$

17. $\frac{7}{2-\sqrt{6}}$

18. $\frac{6}{3-\sqrt{7}}$

19. $\frac{5}{4-\sqrt{11}}$

20. $\frac{\sqrt{3}}{\sqrt{m}-\sqrt{5}}$

21. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$

22. $\frac{\sqrt{2}}{\sqrt{x}-\sqrt{6}}$

23. $\frac{\sqrt{7}}{\sqrt{y}+\sqrt{3}}$

24. $\frac{\sqrt{r}+\sqrt{5}}{\sqrt{r}-\sqrt{5}}$

25. $\frac{\sqrt{s}-\sqrt{6}}{\sqrt{s}+\sqrt{6}}$

26. $\dfrac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

27. $\dfrac{\sqrt{m}-\sqrt{3}}{\sqrt{m}+\sqrt{3}}$

Answer

16. $-2(1+\sqrt{5})$

18. $3(3+\sqrt{7})$

20. $\dfrac{\sqrt{3}(\sqrt{m}+\sqrt{5})}{m-5}$

22. $\dfrac{\sqrt{2}(\sqrt{x}+\sqrt{6})}{x-6}$

24. $\dfrac{(\sqrt{r}+\sqrt{5})^2}{r-5}$

26. $\dfrac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

? Writing Exercises

28. a. Simplify $\sqrt{\dfrac{27}{3}}$ and explain all your steps.

b. Simplify $\sqrt{\dfrac{27}{5}}$ and explain all your steps.

c. Why are the two methods of simplifying square roots different?

29. Explain what is meant by the word rationalize in the phrase, "rationalize a denominator."

30. Explain why multiplying $\sqrt{2x}-3$ by its conjugate results in an epression with no radicals.

Answer

28. Answers will vary.

30. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
divide radical expressions.			
rationalize a one-term denominator.			
rationalize a two-term denominator.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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6.6: Solve Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $(y-3)^2$.
2. Solve $2x-5=0$.
3. Solve $n^2-6n+8=0$.

Solve Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition 6.6.1

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the radical, our strategy will be to raise both sides of the equation to the power of the index. This will eliminate the radical.

Our focus is on the index 2. Solving radical equations with square roots by squaring both sides may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution as we did when we solved rational equations.

In the next example, we will see how to solve a radical equation. Our strategy is based on squaring a square root. This will eliminate the radical.

For $a \geq 0$, $(\sqrt{a})^2 = a$.

Example 6.6.2

Solve $\sqrt{5n-4}-9=0$.

Solution

	$\sqrt{5n-4}-9=0$
Isolate the radical on one side of the equation.	$\sqrt{5n-4}-9+9=0+9$
Simplify.	$\sqrt{5n-4}=9$
Square both sides of the equation. Remember that $(\sqrt{a})^2=a$.	$(\sqrt{5n-4})^2=(9)^2$ $5n-4=81$
Solve the new equation.	$5n-4+4=81+4$ $5n=85$ $n=\frac{85}{5}$ $n=17$

	$\sqrt{5n-4}-9=0$
Check the answer in the original equation.	$\sqrt{5n-4}-9=0$ $\sqrt{5\cdot 17-4}-9\stackrel{?}{=}0$ $\sqrt{85-4}-9\stackrel{?}{=}0$ $\sqrt{81}-9\stackrel{?}{=}0$ $9-9\stackrel{?}{=}0$ True
	The solution is $n=17$.

Try It 6.6.3

Solve $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Try It 6.6.4

Solve $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Square Root

1. Isolate the radical on one side of the equation.
2. Square both sides of the equation.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a square root equal to a negative number, that equation will have no solution.

Example 6.6.5

Solve $\sqrt{9k-2}+1=0$.

Solution

	$\sqrt{9k-2}+1=0$
Isolate the radical on one side of the equation.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Try It 6.6.6

Solve $\sqrt{2r-3}+5=0$.

Answer

no solution

Try It 6.6.7

Solve $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

Don't forget the middle term!

Example 6.6.8

Solve $\sqrt{p-1}+1=p$.

Solution

	$\sqrt{p-1}+1=p$
To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$ $(p-1)(p-2)=0$
Use the Zero Product Property.	$p-1=0 \quad \text{or} \quad p-2=0$
Solve each equation.	$p=1 \quad \text{or} \quad p=2$

Check the answers.	$p=1 \quad \sqrt{p-1}+1 = p$ $\sqrt{1-1}+1 \stackrel{?}{=} 1$ $0+1 \stackrel{?}{=} 1$ $1 \stackrel{?}{=} 1$ True $p=2 \quad \sqrt{p-1}+1 = p$ $\sqrt{2-1}+1 \stackrel{?}{=} 2$ $\sqrt{1}+1 \stackrel{?}{=} 2$ $2 \stackrel{?}{=} 2$ True
	The solutions are $p=1$ or $p=2$.

Try It 6.6.9

Solve $\sqrt{x-2}+2=x$.

Answer

$x=2$ or $x=3$

Try It 6.6.10

Solve $\sqrt{y-5}+5=y$.

Answer

$y=5$ or $y=6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

Example 6.6.11

Solve $\sqrt{r+4}-r+2=0$.

Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	
If it is a quadratic equation, so get zero on one side.	
Factor the right side.	$0=r(r-5)$
Use the Zero Product Property.	$0=r$ or $0=r-5$
Solve the equation.	$r=0 \quad r=5$

Check your answer.	$r=0 \quad \sqrt{r+4}-r+2 = 0$ $\sqrt{0+4}-0+2 \stackrel{?}{=} 0$ $\sqrt{4}+2 \stackrel{?}{=} 0$ $4 \stackrel{?}{=} 0$ False $r=5 \quad \sqrt{r+4}-r+2 = 0$ $\sqrt{5+4}-5+2 \stackrel{?}{=} 0$ $\sqrt{9}-3 \stackrel{?}{=} 0$ $0 \stackrel{?}{=} 0$ True
	$r=0$ is an extraneous solution. The solution is $r=5$.

? Try It 6.6.12

Solve $\sqrt{m+9}-m+3=0$.

Answer

$m=7$

? Try It 6.6.13

Solve $\sqrt{n+1}-n+1=0$.

Answer

$n=3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

? Example 6.6.14

Solve $3\sqrt{3x-5}-8=4$.

Solution

	$3\sqrt{3x-5}-8=4$
Isolate the radical term.	$3\sqrt{3x-5}=12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5}=4$
Square both sides of the equation.	$(\sqrt{3x-5})^2=(4)^2$
Simplify, then solve the new equation.	$3x-5=16$
	$3x=21$
Solve the equation.	$x=7$

Check the answer.	$x=7 \quad 3\sqrt{3x-5}-8 = 4$ $3\sqrt{3(7)-5}-8 \stackrel{?}{=} 4$ $3\sqrt{21-5}-8 \stackrel{?}{=} 4$ $3\sqrt{16}-8 \stackrel{?}{=} 4$ $3(4)-8 \stackrel{?}{=} 4$ $4 \stackrel{?}{=} 4$ True
	The solution is $x=7$.

? Try It 6.6.15

Solve $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

? Try It 6.6.16

Solve $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

? Example 6.6.17

Solve $\sqrt{4x-3}=\sqrt{3x+2}$.

Solution

The radical terms are isolated.	$\sqrt{4x-3}=\sqrt{3x+2}$
Square both sides of the equation.	$(\sqrt{4x-3})^2=(\sqrt{3x+2})^2$
Simplify, then solve the new equation.	$4x-3=3x+2$ $4x-3-3x+3=3x+2-3x+3$ $x=5$
Check the answer.	$x=5 \quad \sqrt{4x-3} = \sqrt{3x+2}$ $\sqrt{4\cdot 5-3} \stackrel{?}{=} \sqrt{3\cdot 5+2}$ $\sqrt{17} \stackrel{?}{=} \sqrt{17}$ True
	The solution is $x=5$.

Try It 6.6.18

Solve $\sqrt{5x-4}=\sqrt{2x+5}$.

Answer

$x=3$

Try It 6.6.19

Solve $\sqrt{7x+1}=\sqrt{2x-5}$.

Answer

There is no real solution.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and square both sides of the equation again.

Example 6.6.20

Solve $\sqrt{m}+1=\sqrt{m+9}$.

Solution

Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$m+2\sqrt{m}+1=m+9$ $2\sqrt{m}=8$ $\sqrt{m}=4$ $(\sqrt{m})^2=(4)^2$ $m=16$
Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5$ The solution is $m=16$.

Try It 6.6.21

Solve $3-\sqrt{x}=\sqrt{x-3}$.

Answer

$x=4$

Try It 6.6.22

Solve $\sqrt{x}+2=\sqrt{x+16}$.

Answer

$x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 6.6.23

Solve $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution

Try It 6.6.24

Solve $\sqrt{x-1}+2=\sqrt{2x+6}$.

Answer

$x=5$

Try It 6.6.25

Solve $\sqrt{x}+2=\sqrt{3x+4}$.

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications with Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.

4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula

$$t=\frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h=64$ into the formula.

	$t=\frac{\sqrt{h}}{4}$
$h=64$	$t=\frac{\sqrt{64}}{4}$
Take the square root of 64.	$t=\frac{8}{4}$
Simplify the fraction.	$t=2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 6.6.26

Marissa dropped her sunglasses from a bridge 400 feet above a river. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the sunglasses to reach the river.

Solution

Read the problem.	
Identify what we are looking.	The time it takes for the sunglasses to reach the river.
Name what we are looking.	Let $t=$ time.
Translate into an equation by writing the appropriate formula. Substitute in the given information.	$t=\frac{\sqrt{h}}{4}$, and $h=400$ $t=\frac{\sqrt{400}}{4}$
Solve the equation.	$t=\frac{20}{4}$
	$t=5$
Check the answer in the problem and make sure it makes sense.	$5\stackrel{?}{=}\frac{\sqrt{400}}{4}$ $5\stackrel{?}{=}\frac{20}{4}$ $5=5\checkmark$

Does 5 seconds seem like a reasonable length of time?	Yes.
Answer the equation.	It will take 5 seconds for the sunglasses to reach the river.

Try It 6.6.27

A helicopter dropped a rescue package from a height of $1,296$ feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Try It 6.6.28

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula

$$s=\sqrt{24d}.$$

Example 6.6.29

After a car accident, the skid marks for one car measured 190 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Read the problem.	
Identify what we are looking for.	The speed of a car.
Name what we are looking for.	Let $s=\sqrt{24d}$ the speed.
Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s=\sqrt{24d}$, and $d=190$ $s=\sqrt{24(190)}$
Solve the equation.	$s=\sqrt{4,560}$
	$s=67.52777...$
Round to 1 decimal place.	$s\approx 67.5$

	$67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277... ✓$
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Try It 6.6.30

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. Use the formula $s = \sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7feet

Try It 6.6.31

The skid marks of a vehicle involved in an accident were 122 feet long. Use the formula $s = \sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1feet

Key Concepts

- **Binomial Squares**
  $(a+b)^2 = a^2 + 2ab + b^2$
  $(a-b)^2 = a^2 - 2ab + b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t = \frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**

- If the length of the skid marks is d feet, then the speed, s, of the car before the brakes were applied can be found by using the formula $s=\sqrt{24d}$.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

Practice Makes Perfect

Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{5x-6}=8$

2. $\sqrt{4x-3}=7$

3. $\sqrt{5x+1}=-3$

4. $\sqrt{3y-4}=-2$

7. $\sqrt{2m-3}-5=0$

8. $\sqrt{2n-1}-3=0$

9. $\sqrt{6v-2}-10=0$

10. $\sqrt{12u+1}-11=0$

11. $\sqrt{4m+2}+2=6$

12. $\sqrt{6n+1}+4=8$

13. $\sqrt{2u-3}+2=0$

14. $\sqrt{5v-2}+5=0$

15. $\sqrt{u-3}+3=u$

16. $\sqrt{v-10}+10=v$

17. $\sqrt{r-1}=r-1$

18. $\sqrt{s-8}=s-8$

23. $(6x+1)^{\frac{1}{2}}-3=4$

24. $(3x-2)^{\frac{1}{2}}+1=6$

29. $\sqrt{x+1}-x+1=0$

30. $\sqrt{y+4}-y+2=0$

31. $\sqrt{z+100}-z=-10$

32. $\sqrt{w+25}-w=-5$

33. $3\sqrt{2x-3}-20=7$

34. $2\sqrt{5x+1}-8=0$

35. $2\sqrt{8r+1}-8=2$

36. $3\sqrt{7y+1}-10=8$

Answer

1. $m=14$

3. no solution

7. $m=14$

9. $v=17$

11. $m=\frac{7}{2}$

13. no solution

15. $u=3, u=4$

17. $r=1, r=2$

23. $x=8$

29. $x=3$

31. $z=21$

33. $x=42$

35. $r=3$

? Solve Radical Equations with Two Radicals

In the following exercises, solve.

37. $\sqrt{3u+7}=\sqrt{5u+1}$

38. $\sqrt{4v+1}=\sqrt{3v+3}$

39. $\sqrt{8+2r}=\sqrt{3r+10}$

40. $\sqrt{10+2c}=\sqrt{4c+16}$

Optional

41. $\sqrt{a}+2=\sqrt{a+4}$

42. $\sqrt{r}+6=\sqrt{r+8}$

Answer

37. $u=3$

39. $r=-2$

41. $a=0$

? Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

43. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answer to the nearest tenth of a foot.
44. **Landscaping** Vince wants to make a square patio in his yard. He has enough concrete to pave an area of 130 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his patio. Round your answer to the nearest tenth of a foot.
45. **Gravity** A hang glider dropped his cell phone from a height of 350 feet. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the cell phone to reach the ground.
46. **Gravity** A construction worker dropped a hammer while building the Grand Canyon skywalk, 4000 feet above the Colorado River. Use the formula $t=\frac{\sqrt{h}}{4}$ to find how many seconds it took for the hammer to reach the river.
47. **Accident investigation** The skid marks for a car involved in an accident measured 216 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

48. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

43. 8.7 feet

45. 4.7 seconds

47. 72 feet

? Writing Exercises

49. Explain why an equation of the form $\sqrt{x}+1=0$ has no solution.
50. a. Solve the equation $\sqrt{r+4}-r+2=0$.
    b. Explain why one of the "solutions" that was found was not actually a solution to the equation.

Answer

49. Answers will vary.

? Additional exercise

51. Solve for x: $\sqrt{x^2+5x-7}=x+4$.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve radical equations.			
solve radical equations with two radicals.			
use radicals in applications.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

6.7: Complex Numbers

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Be Prepared

Before you get started, take this readiness quiz.

1. Given the numbers $-4, -\sqrt{7}, 0.\bar{5}, \dfrac{7}{3}, 3, \sqrt{81}$, list the
   a. rational numbers
   b. irrational numbers
   c. real numbers
2. Multiply: $(x-3)(2x+5)$.
3. Rationalize the denominator: $\dfrac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2=-1$. Since all real numbers squared are positive numbers, there is no real number that equals -1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as the number whose square is -1.

Definition 6.7.1

The **imaginary unit** i is the number whose square is -1.\

We will use the imaginary unit to simplify the square roots of negative numbers.

Definition 6.7.2

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b}=\sqrt{b}i.$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b}=i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b}=\sqrt{b}i$ is considered standard form.

Example 6.7.3

Write each expression in terms of i and simplify is possible:

a. $\sqrt{-25}$
b. $\sqrt{-7}$
c. $\sqrt{-12}$

Solution:

a.

$$\sqrt{-25}$$

Use the definition of the square root of negative numbers.

$$\sqrt{25} i$$

Simplify.

$$5 i$$

b.

$$\sqrt{-7}$$

Use the definition of the square root of negative numbers.

$$\sqrt{7} i$$

Simplify.

Be careful that it is clear that i is not under the radical sign.

c.

$$\sqrt{-12}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12} i$$

Simplify $\sqrt{12}$.

$$2 \sqrt{3} i$$

Try It 6.7.4

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-81}$
b. $\sqrt{-5}$
c. $\sqrt{-18}$

Answer

a. $9 i$
b. $\sqrt{5} i$
c. $3 \sqrt{2} i$

Try It 6.7.5

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-36}$
b. $\sqrt{-3}$
c. $\sqrt{-27}$

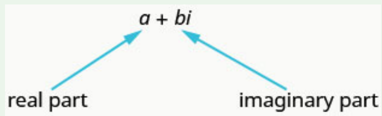
Answer

a. $6i$
b. $\sqrt{3}i$
c. $3\sqrt{3}i$

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a+bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Definition 6.7.6

A **complex number** is of the form $a+bi$, where a and b are real numbers.



A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0\text{R}$	$0+bi$ bi	Pure imaginary numbe4

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

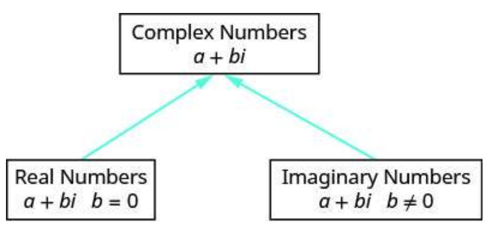


Figure 8.8.2

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example 6.7.7

Add: $\sqrt{-12}+\sqrt{-27}$.

Solution:

$$\sqrt{-12}+\sqrt{-27}$$

Use the definition of the square root of negative numbers.

$$\sqrt{12}i+\sqrt{27}i$$

Simplify the square roots.

$$2\sqrt{3}i+3\sqrt{3}i$$

Add.

$$5\sqrt{3}i$$

Try It 6.7.8

Add: $\sqrt{-8}+\sqrt{-32}$.

Answer

$6\sqrt{2}i$

Try It 6.7.9

Add: $\sqrt{-27}+\sqrt{-48}$.

Answer

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example 6.7.10

Simplify:

a. $(4-3i)+(5+6i)$
b. $(2-5i)-(5-2i)$

Solution:

a.

Use the Associative Property to put the real parts and the imaginary parts together.

Simplify.

$$9+3i$$

b.

Distribute.

Use the Associative Property to put the real parts and the imaginary parts together.

Simplify.

Try It 6.7.11

Simplify:

a. $(2+7i)+(4-2i)$
b. $(8-4i)-(2-i)$

Answer

a. $6+5i$
b. $6-3i$

Try It 6.7.12

Simplify:

a. $(3-2i)+(-5-4i)$
b. $(4+3i)-(2-6i)$

Answer

a. $-2-6i$
b. $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Example 6.7.13

Multiply: $2i(7-5i)$.

Solution:

Distribute.

Simplify i^2.

$$14i-10(-1)$$

Multiply.

$$14i+10$$

Write in standard form.

$$10+14i$$

Try It 6.7.14

Multiply: $4i(5-3i)$.

Answer

$12+20i$

Try It 6.7.15

Multiply: $-3i(2+4i)$.

Answer

$12+6i$

In the next example, we multiply the binomials using the **Distributive Property** or **FOIL**.

Example 6.7.16

Multiply: $(3+2i)(4-3i)$.

Solution:

Use FOIL.

Simplify i^2 and combine like terms.

$$12-i-6(-1)$$

Multiply.

$$12-i+6$$

Combine the real parts.

$$18-i$$

Try It 6.7.17

Multiple: $(5-3i)(-1-2i)$.

Answer

$-11-7i$

Try It 6.7.18

Multiple: $(-4-3i)(2+i)$.

Answer

$-5-10i$

In the next example, we could use FOIL or the **Product of Binomial Squares Pattern**.

Example 6.7.19

Multiply: $(3+2i)^2$.

Solution:

	$\begin{pmatrix} a+b \\ 3+2i \end{pmatrix}^2$
Use the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$.	$\begin{array}{c} a^2+2\quad a\quad b+\ b^2 \\ 3^2+2\cdot3\cdot2i+(2i)^2 \end{array}$
Simplify.	$9+12i+4i^2$
Simplify i^2.	$9+12i+4(-1)$
Simplify.	$5+12i$

Try It 6.7.20

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Answer

$-21-20i$

Try It 6.7.21

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Answer

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$.This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

Example 6.7.22

Multiply: $\sqrt{-36}\cdot\sqrt{-4}$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$\sqrt{-36}\cdot\sqrt{-4}$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

$$\sqrt{36}i\cdot\sqrt{4}i$$

Simplify.

$$6i\cdot 2i$$

Multiply.

$$12i^2$$

Simplify i^2 and multiply.

$$-12$$

Try It 6.7.23

Multiply: $\sqrt{-49}\cdot\sqrt{-4}$.

Answer

-14

Try It 6.7.24

Multiply: $\sqrt{-36}\cdot\sqrt{-81}$.

Answer

-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Example 6.7.25

Multiply: $(3-\sqrt{-12})(5+\sqrt{-27})$.

Solution:

To multiply square roots of negative numbers, we first write them as complex numbers.

$$(3-\sqrt{-12})(5+\sqrt{-27})$$

Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.

Use FOIL.

Combine like terms and simplify i^2.

Multiply and combine like terms.

$33-\sqrt{3}i$

Try It 6.7.26

Multiply: $(4-\sqrt{-12})(3-\sqrt{-48})$.

Answer

$-12-22\sqrt{3}i$

Try It 6.7.27

Multiply: $(-2+\sqrt{-8})(3-\sqrt{-18})$.

Answer

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Definition 6.7.28

A **complex conjugate pair** is of the form $a+bi, a+bi$.

We will multiply a complex conjugate pair in the next example.

Example 6.7.29

Multiply: $(3-2i)(3+2i)$.

Solution:

Use FOIL

Combine like terms and simplify i^2.

$$9-4(-1)$$

Multiply and combine like terms.

$$13$$

Try It 6.7.30

Multiply: $(4-3i) \cdot (4+3i)$.

Answer

25

Try It 6.7.31

Multiply: $(-2+5i) \cdot (-2-5i)$.

Answer

29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$ .The result is called a **difference of squares**. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the **Product of Conjugates Pattern**.

$$\left(\overset{a}{3}-\overset{b}{2i}\right)\left(\overset{a}{3}+\overset{b}{2i}\right)$$

$$\overset{a^2}{(3)^2}-\overset{b^2}{(2i)^2}$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Notice this is the same result we found in Example 8.8.9.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

Product of Complex Conjugates

If a and b are real numbers, then

$$(a-bi)(a+bi)=a^2+b^2.$$

Example 6.7.32

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution:

Table 8.8.3

	$(a-b)(a+b)$ $(8-2i)(8+2i)$

Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	a^2+b^2 8^2+2^2
Simplify the squares.	$64+4$
Add.	68

Try It 6.7.33

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Answer

109

Try It 6.7.34

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Answer

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example 6.7.35

Divide: $\frac{4+3i}{3-4i}$.

Solution:

Table 8.8.4

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	
Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	
Step 3: Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. Combine like terms. Simplify. Write the result in standard form.	$\frac{12+16i+9i+12i^2}{9+16}$ $\frac{12+25i-12}{25}$ $\frac{25i}{25}$ i

Try It 6.7.36

Divide: $\frac{2+5i}{5-2i}$.

Answer

i

Try It 6.7.37

Divide: $\frac{1+6i}{6-i}$.

Answer

i

We summarize the steps here.

How to Divide Complex Numbers

1. Write both the numerator and denominator in standard form.
2. Multiply both the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

Example 6.7.38

Divide, writing the answers in standard form: $\frac{-3}{5+2i}$.

Solution:

$$\frac{-3}{5+2i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.

$$\frac{-15+6i}{5^2+2^2}$$

Simplify.

$$\frac{-15+6i}{29}$$

Write in standard form.

$$-\frac{15}{29}+\frac{6}{29}i$$

Try It 6.7.39

Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Answer

$\frac{4}{17}+\frac{16}{17}i$

Try It 6.7.40

Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Answer

$\frac{2}{5}+\frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

Example 6.7.41

Divide: $\frac{5+3i}{4i}$.

Solution:

$$\frac{5+3i}{4i}$$

Write the denominator in standard form.

$$\frac{5+3i}{0+4i}$$

Multiply the numerator and denominator by the complex conjugate of the denominator.

Simplify.

Multiply.

Simplify the i^2.

Rewrite in standard form.

$$\frac{12}{16}-\frac{20}{16}i$$

Simplify the fractions.

$$\frac{3}{4}-\frac{5}{4}i$$

Try It 6.7.42

Divide: $\frac{3+3i}{2i}$.

Answer

$\frac{3}{2}-\frac{3}{2}i$

Try It 6.7.43

Divide: $\frac{2+4i}{5i}$.

Answer

$\frac{4}{5}-\frac{2}{5}i$

Simplify Powers of i

The powers of i make an interesting pattern that will help us simplify higher powers of i. Let's evaluate the powers of i to see the pattern.

$$\begin{array}{cccc} i^1 & i^2 & i^3 & i^4 \\ i & -1 & i^2 \cdot i & i^2 \cdot i^2 \\ & & -1 \cdot i & (-1)(-1) \\ & & -i & 1 \end{array}$$

$$\begin{array}{cccc} i^5 & i^6 & i^7 & i^8 \\ i^4\cdot i & i^4\cdot i^2 & i^4\cdot i^3 & i^4\cdot i^4 \\ 1\cdot i & 1\cdot i^2 & 1\cdot i^3 & 1\cdot 1 \\ i & i^2 & i^3 & 1 \\ & -1 & -i & \end{array}$$

We summarize this now.

$$\begin{array}{ll} i^1=i & i^5=i \\ i^2=-1 & i^6=-1 \\ i^3=-i & i^7=-i \\ i^4=1 & i^8=1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4=1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n=\left(i^4\right)^q\cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57=4\cdot 14+1$. So we write $i^{57}=\left(1^4\right)^{14}\cdot i^1$ and then simplify from there.

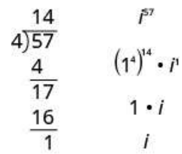


Figure 8.8.13

Example 6.7.44

Simplify: i^{86}.

Solution:

$$i^{86}$$

Divide 86 by 4 and rewrite i^{86} in the $i^n=\left(i^4\right)^q\cdot i^r$ form.

$$\left(1^4\right)^{21}\cdot i^2$$

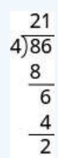


Figure 8.8.14

Simplify.

$$(1)^{21}\cdot(-1)$$

Simplify.

$$-1$$

Try It 6.7.45

Simplify: i^{75}.

Answer

-1

Try It 6.7.46

Simplify: i^{92}.

Answer

1

Key Concepts

- **Square Root of a Negative Number**
  - If b is a positive real number, then \(\sqrt{-b}=\sqrt{b} i\

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

- - A complex number is in **standard form** when written as *a* + *bi*, where *a*, *b* are real numbers.

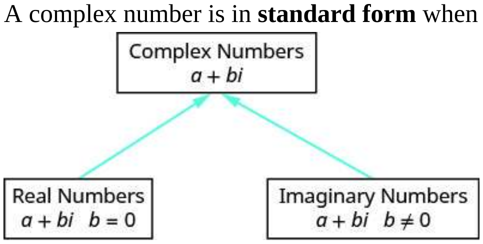


Figure 8.8.2

- **Product of Complex Conjugates**
  - If a, b are real numbers, then
    $(a-bi)(a+bi)=a^2+b^2$
- **How to Divide Complex Numbers**
  1. Write both the numerator and denominator in standard form.
  2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  3. Simplify and write the result in standard form.

Glossary

complex conjugate pair

A complex conjugate pair is of the form $a+bi, a-bi$.

complex number

A complex number is of the form $a+bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit

The imaginary unit i is the number whose square is -1. $i^2=-1$ or $i=\sqrt{-1}$.

standard form

A complex number is in standard form when written as $a+bi$, where a, b are real numbers.

Practice Makes Perfect

❓ Evaluate a Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-16}$ b. $\sqrt{-11}$ c. $\sqrt{-8}$
2. a. $\sqrt{-121}$ b. $\sqrt{-1}$ c. $\sqrt{-20}$
3. a. $\sqrt{-100}$ b. $\sqrt{-13}$ c. $\sqrt{-45}$
4. a. $\sqrt{-49}$ b. $\sqrt{-15}$ c. $\sqrt{-75}$

Answer

1. a. $4i$ b. $i\sqrt{11}$ c. $2i\sqrt{2}$

3. a. $10i$ b. $i\sqrt{13}$ c. $3i\sqrt{5}$

❓ Add or Subtract Complex Numbers

In the following exercises, add or subtract, putting the answer in $a+bi$ form.

5. $\sqrt{-75}+\sqrt{-48}$

6. $\sqrt{-12}+\sqrt{-75}$

7. $\sqrt{-50}+\sqrt{-18}$

8. $\sqrt{-72}+\sqrt{-8}$

9. $(1+3i)+(7+4i)$

10. $(6+2i)+(3-4i)$

11. $(8-i)+(6+3i)$

12. $(7-4i)+(-2-6i)$

13. $(1-4i)-(3-6i)$

14. $(8-4i)-(3+7i)$

15. $(6+i)-(-2-4i)$

16. $(-2+5i)-(-5+6i)$

17. $(5-\sqrt{-36})+(2-\sqrt{-49})$

18. $(-3+\sqrt{-64})+(5-\sqrt{-16})$

19. $(-7-\sqrt{-50})-(-32-\sqrt{-18})$

20. $(-5+\sqrt{-27})-(-4-\sqrt{-48})$

Answer

5. $0+\left(9\sqrt{3}\right)i$

7. $0+\left(8\sqrt{2}\right)i$

9. $8+7i$

11. $14+2i$

13. $-2+2i$

15. $8+5i$

17. $7-13i$

19. $25-\left(2\sqrt{2}\right)i$

? Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

21. $4i(5-3i)$

22. $2i(-3+4i)$

23. $-6i(-3-2i)$

24. $-i(6+5i)$

25. $(4+3i)(-5+6i)$

26. $(-2-5i)(-4+3i)$

27. $(-3+3i)(-2-7i)$

28. $(-6-2i)(-3-5i)$

Answer

21. $12+20i$

23. $-12+18i$

25. $-38+9i$

27. $27+15i$

? Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern, putting the answer in $a+bi$ form.

29. $(3+4i)^2$

30. $(-1+5i)^2$

31. $(-2-3i)^2$

32. $(-6-5i)^2$

Answer

29. $-7+24i$

31. $-5-12i$

? Multiply Complex Numbers

In the following exercises, multiply, putting the answer in $a+bi$ form.

33. $\sqrt{-25}\cdot\sqrt{-36}$

34. $\sqrt{-4}\cdot\sqrt{-16}$

35. $\sqrt{-9}\cdot\sqrt{-100}$

36. $\sqrt{-64}\cdot\sqrt{-9}$

37. $(-2-\sqrt{-27})(4-\sqrt{-48})$

38. $(5-\sqrt{-12})(-3+\sqrt{-75})$

39. $(2+\sqrt{-8})(-4+\sqrt{-18})$

40. $(5+\sqrt{-18})(-2-\sqrt{-50})$

41. $(2-i)(2+i)$

42. $(4-5i)(4+5i)$

43. $(7-2i)(7+2i)$

44. $(-3-8i)(-3+8i)$

Answer

33. $30i=0+30i$

35. $-30=-30+0i$

37. $-44+\left(4\sqrt{3}\right)i$

39. $-20-\left(2\sqrt{2}\right)i$

41. $5=5+0i$

43. $53=53+0i$

Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

45. $(7-i)(7+i)$

46. $(6-5i)(6+5i)$

47. $(9-2i)(9+2i)$

48. $(-3-4i)(-3+4i)$

Answer

45. 50

47. 85

Divide Complex Numbers

In the following exercises, divide, putting the answer in $a+bi$ form.

49. $\dfrac{3+4i}{4-3i}$

50. $\dfrac{5-2i}{2+5i}$

51. $\dfrac{2+i}{3-4i}$

52. $\dfrac{3-2i}{6+i}$

53. $\dfrac{3}{2-3i}$

54. $\dfrac{2}{4-5i}$

55. $\dfrac{-4}{3-2i}$

56. $\dfrac{-1}{3+2i}$

57. $\dfrac{1+4i}{3i}$

58. $\dfrac{4+3i}{7i}$

59. $\dfrac{-2-3i}{4i}$

60. $\dfrac{-3-5i}{2i}$

Answer

49. $i=0+i$

51. $\dfrac{2}{25}+\dfrac{11}{25}i$

53. $\dfrac{6}{13}+\dfrac{9}{13}i$

55. $-\dfrac{12}{13}-\dfrac{8}{13}i$

57. $\dfrac{4}{3}-\dfrac{1}{3}i$

59. $-\dfrac{3}{4}+\dfrac{1}{2}i$

? Simplify Powers of i

In the following exercises, simplify.

61. i^{41}

62. i^{39}

63. i^{66}

64. i^{48}

65. i^{128}

66. i^{162}

67. i^{137}

68. i^{255}

Answer

61. $i^{41}=i^{40}\cdot i=\left(i^4\right)^{10}\cdot i=i$

63. $i^{66}=i^{64}\cdot i^2=\left(i^4\right)^{16}\cdot(-1)=-1$

65. $i^{128}=\left(i^4\right)^{32}=1$

67. $i^{137}=i^{136}\cdot i=\left(i^4\right)^{34}\cdot i=1\cdot i=i$

? Writing Exercises

69. Explain the relationship between real numbers and complex numbers.

70. Aniket multiplied as follows and he got the wrong answer. What is wrong with his reasoning?

$\sqrt{-7}\cdot\sqrt{-7}$

$\sqrt{49}$

7

71. Why is $\sqrt{-64}=8i$ but $\sqrt[3]{-64}=-4$?

72. Explain how dividing complex numbers is similar to rationalizing a denominator.

Answer

69. Answers may vary

71. Answers may vary

Self Check

a. After completing the Try It s, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate the square root of a negative number.			
add or subtract complex numbers.			
multiply complex numbers.			
divide complex numbers.			
simplify powers of *i*.			

b. On a scale of 1-10, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

6.8: Chapter 6 Review Exercises

Chapter Review Exercises

Simplify Expressions with Roots

Exercise 6.8.1 Simplify Expressions with Roots

In the following exercises, simplify.

1. a. $\sqrt{225}$
   b. $-\sqrt{16}$
2. a. $-\sqrt{169}$
   b. $\sqrt{-8}$
3. a. $\sqrt[3]{8}$
   b. $\sqrt[4]{81}$
   c. $\sqrt[5]{243}$
4. a. $\sqrt[3]{-512}$
   b. $\sqrt[4]{-81}$
   c. $\sqrt[5]{-1}$

Answer

1.

a. 15
b. -4

3.

a. 2
b. 3
c. 3

Exercise 6.8.2 Estimate and Approximate Roots

In the following exercises, estimate each root between two consecutive whole numbers.

1. a. $\sqrt{68}$
   b. $\sqrt[3]{84}$

Answer

1.

a. $8<\sqrt{68}<9$
b. $4<\sqrt[3]{84}<5$

Exercise 6.8.3 Estimate and Approximate Roots

In the following exercises, approximate each root and round to two decimal places.

1. a. $\sqrt{37}$
   b. $\sqrt[3]{84}$
   c. $\sqrt[4]{125}$

Answer

1. Solve for yourself

? Exercise 6.8.4 Simplify Variable Expressions with Roots

In the following exercises, simplify using absolute values as necessary.

1. a. $\sqrt[3]{a^3}$
   b. $\sqrt[7]{b^7}$
2. a. $\sqrt{a^{14}}$
   b. $\sqrt{w^{24}}$
3. a. $\sqrt[4]{m^8}$
   b. $\sqrt[5]{n^{20}}$
4. a. $\sqrt{121m^{20}}$
   b. $-\sqrt{64a^2}$
5. a. $\sqrt[3]{216a^6}$
   b. $\sqrt[5]{32b^{20}}$
6. a. $\sqrt{144x^2y^2}$
   b. $\sqrt{169w^8y^{10}}$
   c. $\sqrt[3]{8a^{51}b^6}$

Answer

1.

   a. a
   b. $|b|$

3.

   a. m^2
   b. n^4

5.

   a. $6a^2$
   b. $2b^4$

Simplify Radical Expressions

? Exercise 6.8.5 Use the Product Property to Simplify Radical Expressions

In the following exercises, use the Product Property to simplify radical expressions.

1. $\sqrt{125}$
2. $\sqrt{675}$
3. a. $\sqrt[3]{625}$
   b. $\sqrt[6]{128}$

Answer

1. $5\sqrt{5}$

3.

   a. $5\sqrt[3]{5}$
   b. $2\sqrt[6]{2}$

Exercise 6.8.6 Use the Product Property to Simplify Radical Expressions

In the following exercises, simplify using absolute value signs as needed.

1. a. $\sqrt{a^{23}}$
   b. $\sqrt[3]{b^8}$
   c. $\sqrt[8]{c^{13}}$
2. a. $\sqrt{80s^{15}}$
   b. $\sqrt[5]{96a^7}$
   c. $\sqrt[6]{128b^7}$
3. a. $\sqrt{96r^3s^3}$
   b. $\sqrt[3]{80x^7y^6}$
   c. $\sqrt[4]{80x^8y^9}$
4. a. $\sqrt[5]{-32}$
   b. $\sqrt[8]{-1}$
5. a. $8+\sqrt{96}$
   b. $\frac{2+\sqrt{40}}{2}$

Answer

2.

a. $4\left|s^7\right|\sqrt{5s}$
b. $2a\sqrt[5]{3a^2}$
c. $2|b|\sqrt[6]{2b}$

4.

a. -2
b. not real

Exercise 6.8.7 Use the Quotient Property to Simplify Radical Expressions

In the following exercises, use the Quotient Property to simplify square roots.

1. a. $\sqrt{\frac{72}{98}}$
   b. $\sqrt[3]{\frac{24}{81}}$
   c. $\sqrt[4]{\frac{6}{96}}$
2. a. $\sqrt{\frac{y^4}{y^8}}$
   b. $\sqrt[5]{\frac{u^{21}}{u^{11}}}$
   c. $\sqrt[6]{\frac{v^{30}}{v^{12}}}$
3. $\sqrt{\frac{300m^5}{64}}$
4. a. $\sqrt{\frac{28p^7}{q^2}}$
   b. $\sqrt[3]{\frac{81s^8}{t^3}}$
   c. $\sqrt[4]{\frac{64p^{15}}{q^{12}}}$
5. a. $\sqrt{\frac{27p^2q}{108p^4q^3}}$

b. $\sqrt[3]{\frac{16c^5d^7}{250c^2d^2}}$
c. $\sqrt[6]{\frac{2m^9n^7}{128m^3n}}$

6. a. $\frac{\sqrt{80q^5}}{\sqrt{5q}}$
b. $\frac{\sqrt[3]{-625}}{\sqrt[3]{5}}$
c. $\frac{\sqrt[4]{80m^7}}{\sqrt[4]{5m}}$

Answer

1.
a. $\frac{6}{7}$
b. $\frac{2}{3}$
c. $\frac{1}{2}$

3. $\frac{10m^2\sqrt{3m}}{8}$

5.
a. $\frac{1}{2|pq|}$
b. $\frac{2cd\sqrt[5]{2d^2}}{5}$
c. $\frac{|mn|\sqrt[6]{2}}{2}$

Simplify Rational Exponents

Exercise 6.8.8 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write as a radical expression.

1. a. $r^{\frac{1}{2}}$
b. $s^{\frac{1}{3}}$
c. $t^{\frac{1}{4}}$

Answer

1.
a. $\sqrt{r}$
b. $\sqrt[3]{s}$
c. $\sqrt[4]{t}$

Exercise 6.8.9 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt{21p}$
b. $\sqrt[4]{8q}$
c. $4\sqrt[6]{36r}$

Answer

1. Solve for yourself

Exercise 6.8.10 Simplify Expressions with $a^{\frac{1}{n}}$

In the following exercises, simplify.

1. a. $625^{\frac{1}{4}}$
   b. $243^{\frac{1}{5}}$
   c. $32^{\frac{1}{5}}$
2. a. $(-1,000)^{\frac{1}{3}}$
   b. $-1,000^{\frac{1}{3}}$
   c. $(1,000)^{-\frac{1}{3}}$
3. a. $(-32)^{\frac{1}{5}}$
   b. $(243)^{-\frac{1}{5}}$
   c. $-125^{\frac{1}{3}}$

Answer

1.

a. 5
b. 3
c. 2

3.

a. -2
b. $\frac{1}{3}$
c. -5

Exercise 6.8.11 Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, write with a rational exponent.

1. a. $\sqrt[4]{r^7}$
   b. $(\sqrt[5]{2pq})^3$
   c. $\sqrt[4]{\left(\frac{12m}{7n}\right)^3}$

Answer

1. Solve for yourself

Exercise 6.8.12 Simplify Expressions with $a^{\frac{m}{n}}$

In the following exercises, simplify.

1. a. $25^{\frac{3}{2}}$
   b. $9^{-\frac{3}{2}}$
   c. $(-64)^{\frac{2}{3}}$
2. a. $-64^{\frac{3}{2}}$
   b. $-64^{-\frac{3}{2}}$
   c. $(-64)^{\frac{3}{2}}$

Answer

1.

a. 125
b. $\frac{1}{27}$
c. 16

Exercise 6.8.13 Use the Laws of Exponents to Simplify Expressions with Rational Exponents

In the following exercises, simplify.

1. a. $6^{\frac{5}{2}} \cdot 6^{\frac{1}{2}}$
   b. $\left(b^{15}\right)^{\frac{3}{5}}$
   c. $\frac{w^{\frac{2}{7}}}{w^{\frac{9}{7}}}$
2. a. $\frac{a^{\frac{3}{4}} \cdot a^{-\frac{1}{4}}}{a^{-\frac{10}{4}}}$
   b. $\left(\frac{27 b^{\frac{2}{3}} c^{-\frac{5}{2}}}{b^{-\frac{7}{3}} c^{\frac{1}{2}}}\right)^{\frac{1}{3}}$

Answer

1.

a. 6^3
b. b^9
c. $\frac{1}{w}$

Add, Subtract and Multiply Radical Expressions

Exercise 6.8.14 add and Subtract Radical Expressions

In the following exercises, simplify.

1. a. $7\sqrt{2}-3\sqrt{2}$
   b. $7\sqrt[3]{p}+2\sqrt[3]{p}$
   c. $5\sqrt[3]{x}-3\sqrt[3]{x}$
2. a. $\sqrt{11b}-5\sqrt{11b}+3\sqrt{11b}$
   b. $8\sqrt[4]{11cd}+5\sqrt[4]{11cd}-9\sqrt[4]{11cd}$
3. a. $\sqrt{48}+\sqrt{27}$
   b. $\sqrt[3]{54}+\sqrt[3]{128}$
   c. $6\sqrt[4]{5}-\frac{3}{2}\sqrt[4]{320}$
4. a. $\sqrt{80c^7}-\sqrt{20c^7}$
   b. $2\sqrt[4]{162r^{10}}+4\sqrt[4]{32r^{10}}$
5. $3\sqrt{75y^2}+8y\sqrt{48}-\sqrt{300y^2}$

Answer

1.

a. $4\sqrt{2}$
b. $9\sqrt[3]{p}$
c. $2\sqrt[3]{x}$

3.

a. $7\sqrt{3}$
b. $7\sqrt[3]{2}$

c. $3\sqrt[4]{5}$

5. $37y\sqrt{3}$

❓ Exercise 6.8.15 Multiply Radical Expressions

In the following exercises, simplify.

1. a. $(5\sqrt{6})(-\sqrt{12})$
   b. $(-2\sqrt[4]{18})(-\sqrt[4]{9})$
2. a. $\left(3\sqrt{2x^3}\right)\left(7\sqrt{18x^2}\right)$
   b. $\left(-6\sqrt[3]{20a^2}\right)\left(-2\sqrt[3]{16a^3}\right)$

Answer

2.

a. $126x^2\sqrt{2}$
b. $48a\sqrt[3]{a^2}$

❓ Exercise 6.8.16 Use Polynomial Multiplication to Multiply Radical Expressions

In the following exercises, multiply.

1. a. $\sqrt{11}(8+4\sqrt{11})$
   b. $\sqrt[3]{3}(\sqrt[3]{9}+\sqrt[3]{18})$
2. a. $(3-2\sqrt{7})(5-4\sqrt{7})$
   b. $(\sqrt[3]{x}-5)(\sqrt[3]{x}-3)$
3. $(2\sqrt{7}-5\sqrt{11})(4\sqrt{7}+9\sqrt{11})$
4. a. $(4+\sqrt{11})^2$
   b. $(3-2\sqrt{5})^2$
5. $(7+\sqrt{10})(7-\sqrt{10})$
6. $(\sqrt[3]{3x}+2)(\sqrt[3]{3x}-2)$

Answer

2.

a. $71-22\sqrt{7}$
b. $\sqrt[3]{x^2}-8\sqrt[3]{x}+15$

4.

a. $27+8\sqrt{11}$
b. $29-12\sqrt{5}$

6. $\sqrt[3]{9x^2}-4$

Divide Radical Expressions

❓ Exercise 6.8.17 Divide Square Roots

In the following exercises, simplify.

1. a. $\frac{\sqrt{48}}{\sqrt{75}}$
   b. $\frac{\sqrt[3]{81}}{\sqrt[3]{24}}$
2. a. $\frac{\sqrt{320mn^{-5}}}{\sqrt{45m^{-7}n^3}}$

b. $\frac{\sqrt[3]{16x^4y^{-2}}}{\sqrt[3]{-54x^{-2}y^4}}$

Answer

2.

a. $\frac{8m^4}{3n^4}$

b. $-\frac{x^2}{2y^2}$

Exercise 6.8.18 rationalize a One Term Denominator

In the following exercises, rationalize the denominator.

1. a. $\frac{8}{\sqrt{3}}$
   b. $\sqrt{\frac{7}{40}}$
   c. $\frac{8}{\sqrt{2y}}$
2. a. $\frac{1}{\sqrt[3]{11}}$
   b. $\sqrt[3]{\frac{7}{54}}$
   c. $\frac{3}{\sqrt[3]{3x^2}}$
3. a. $\frac{1}{\sqrt[4]{4}}$
   b. $\sqrt[4]{\frac{9}{32}}$
   c. $\frac{6}{\sqrt[4]{9x^3}}$

Answer

2.

a. $\frac{\sqrt[3]{121}}{11}$

b. $\frac{\sqrt[3]{28}}{6}$

c. $\frac{\sqrt[3]{9x}}{x}$

Exercise 6.8.19 Rationalize a Two Term Denominator

In the following exercises, simplify.

1. $\frac{7}{2-\sqrt{6}}$
2. $\frac{\sqrt{5}}{\sqrt{n}-\sqrt{7}}$
3. $\frac{\sqrt{x}+\sqrt{8}}{\sqrt{x}-\sqrt{8}}$

Answer

1. $-\frac{7(2+\sqrt{6})}{2}$

3. $\frac{(\sqrt{x}+2\sqrt{2})^2}{x-8}$

Solve Radical Equations

Exercise 6.8.20 Solve Radical Equations

In the following exercises, solve.

1. $\sqrt{4x-3}=7$
2. $\sqrt{5x+1}=-3$
3. $\sqrt[3]{4x-1}=3$
4. $\sqrt{u-3}+3=u$
5. $\sqrt[3]{4x+5}-2=-5$
6. $(8x+5)^{\frac{1}{3}}+2=-1$
7. $\sqrt{y+4}-y+2=0$
8. $2\sqrt{8r+1}-8=2$

Answer

2. no solution

4. $u=3, u=4$

6. $x=-4$

8. $r=3$

Exercise 6.8.21 Solve Radical Equations with Two Radicals

In the following exercises, solve.

1. $\sqrt{10+2c}=\sqrt{4c+16}$
2. $\sqrt[3]{2x^2+9x-18}=\sqrt[3]{x^2+3x-2}$
3. $\sqrt{r}+6=\sqrt{r+8}$
4. $\sqrt{x+1}-\sqrt{x-2}=1$

Answer

2. $x=-8, x=2$

4. $x=3$

Exercise 6.8.22 Use Radicals in Applications

In the following exercises, solve. Round approximations to one decimal place.

1. **Landscaping** Reed wants to have a square garden plot in his backyard. He has enough compost to cover an area of 75 square feet. Use the formula $s=\sqrt{A}$ to find the length of each side of his garden. Round your answers to th nearest tenth of a foot.
2. **Accident investigation** An accident investigator measured the skid marks of one of the vehicles involved in an accident. The length of the skid marks was 175 feet. Use the formula $s=\sqrt{24d}$ to find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

2. 64.8 feet

Use Radicals in Functions

? Exercise 6.8.23 Evaluate a Radical Function

In the following exercises, evaluate each function.

1. $g(x)=\sqrt{6x+1}$, find
   a. $g(4)$
   b. $g(8)$
2. $G(x)=\sqrt{5x-1}$, find
   a. $G(5)$
   b. $G(2)$
3. $h(x)=\sqrt[3]{x^2-4}$, find
   a. $h(-2)$
   b. $h(6)$
4. For the function $g(x)=\sqrt[4]{4-4x}$, find
   a. $g(1)$
   b. $g(-3)$

Answer

2.

a. $G(5)=2\sqrt{6}$
b. $G(2)=3$

4.

a. $g(1)=0$
b. $g(-3)=2$

? Exercise 6.8.24 Find the Domain of a Radical Function

In the following exercises, find the domain of the function and write the domain in interval notation.

1. $g(x)=\sqrt{2-3x}$
2. $F(x)=\sqrt{\frac{x+3}{x-2}}$
3. $f(x)=\sqrt[3]{4x^2-16}$
4. $F(x)=\sqrt[4]{10-7x}$

Answer

2. $(2,\infty)$

4. $\left[\frac{7}{10},\infty\right)$

? Exercise 6.8.25 graph Radical Functions

In the following exercises,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $g(x)=\sqrt{x+4}$
2. $g(x)=2\sqrt{x}$
3. $f(x)=\sqrt[3]{x-1}$
4. $f(x)=\sqrt[3]{x}+3$

Answer

2.

a. domain: $[0, \infty)$

b.

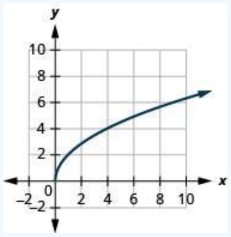


Figure 8.E.1

c. range: $[0, \infty)$

4.

a. domain: $(-\infty, \infty)$

b.

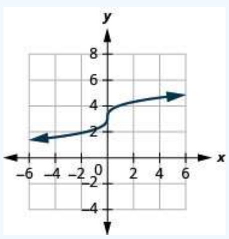


Figure 8.E.2

c. range: $(-\infty, \infty)$

Use the Complex Number System

Exercise 6.8.26 evaluate the Square Root of a Negative Number

In the following exercises, write each expression in terms of i and simplify if possible.

1. a. $\sqrt{-100}$
   b. $\sqrt{-13}$
   c. $\sqrt{-45}$

Answer

Solve for yourself

Exercise 6.8.27 Add or Subtract Complex Numbers

In the following exercises, add or subtract.

1. $\sqrt{-50}+\sqrt{-18}$
2. $(8-i)+(6+3i)$
3. $(6+i)-(-2-4i)$
4. $(-7-\sqrt{-50})-(-32-\sqrt{-18})$

Answer

1. $8\sqrt{2}i$

3. $8+5i$

Exercise 6.8.28 Multiply Complex Numbers

In the following exercises, multiply.

1. $(-2-5i)(-4+3i)$
2. $-6i(-3-2i)$
3. $\sqrt{-4}\cdot\sqrt{-16}$
4. $(5-\sqrt{-12})(-3+\sqrt{-75})$

Answer

1. $23+14i$

3. -6

Exercise 6.8.29 Multiply Complex Numbers

In the following exercises, multiply using the Product of Binomial Squares Pattern.

1. $(-2-3i)^2$

Answer

1. $-5-12i$

Exercise 6.8.30 Multiply Complex Numbers

In the following exercises, multiply using the Product of Complex Conjugates Pattern.

1. $(9-2i)(9+2i)$

Answer

Solve for yourself

Exercise 6.8.31 divide Complex Numbers

In the following exercises, divide.

1. $\frac{2+i}{3-4i}$
2. $\frac{-4}{3-2i}$

Answer

1. $\frac{2}{25}+\frac{11}{25}i$

Exercise 6.8.32 Simplify Powers of i

In the following exercises, simplify.

1. i^{48}
2. i^{255}

Answer

1. 1

Practice Test

Exercise 6.8.33

In the following exercises, simplify using absolute values as necessary.

1. $\sqrt[3]{125x^9}$
2. $\sqrt{169x^8y^6}$
3. $\sqrt[3]{72x^8y^4}$
4. $\sqrt{\frac{45x^3y^4}{180x^5y^2}}$

Answer

1. $5x^3$

3. $2x^2y\sqrt[3]{9x^2y}$

Exercise 6.8.34

In the following exercises, simplify. Assume all variables are positive.

1. a. $216^{-\frac{1}{4}}$
   b. $-49^{\frac{3}{2}}$
2. $\sqrt{-45}$
3. $\frac{x^{-\frac{1}{4}}\cdot x^{\frac{5}{4}}}{x^{-\frac{3}{4}}}$
4. $\left(\frac{8x^{\frac{2}{3}}y^{-\frac{5}{2}}}{x^{-\frac{7}{3}}y^{\frac{1}{2}}}\right)^{\frac{1}{3}}$
5. $\sqrt{48x^5}-\sqrt{75x^5}$
6. $\sqrt{27x^2}-4x\sqrt{12}+\sqrt{108x^2}$
7. $2\sqrt{12x^5}\cdot 3\sqrt{6x^3}$
8. $\sqrt[3]{4}(\sqrt[3]{16}-\sqrt[3]{6})$
9. $(4-3\sqrt{3})(5+2\sqrt{3})$
10. $\frac{\sqrt[3]{128}}{\sqrt[3]{54}}$
11. $\frac{\sqrt{245xy^{-4}}}{\sqrt{45x^4y^3}}$
12. $\frac{1}{\sqrt[3]{5}}$
13. $\frac{3}{2+\sqrt{3}}$
14. $\sqrt{-4}\cdot\sqrt{-9}$
15. $-4i(-2-3i)$
16. $\frac{4+i}{3-2i}$
17. i^{172}

Answer

1.
   a. $\frac{1}{4}$
   b. -343

3. $x^{\frac{7}{4}}$

5. $-x^2\sqrt{3x}$

7. $36x^4\sqrt{2}$

9. $2-7\sqrt{3}$

11. $\frac{7x^5}{3y^7}$

13. $3(2-\sqrt{3})$

15. $-12+8i$

17. $-i$

Exercise 6.8.35

In the following exercises, solve.

1. $\sqrt{2x+5}+8=6$
2. $\sqrt{x+5}+1=x$
3. $\sqrt[3]{2x^2-6x-23}=\sqrt[3]{x^2-3x+5}$

Answer

2. $x=4$

Exercise 6.8.36

In the following exercise,

a. find the domain of the function
b. graph the function
c. use the graph to determine the range

1. $g(x)=\sqrt{x+2}$

Answer

1.

a. domain: $[-2,\infty)$
b.

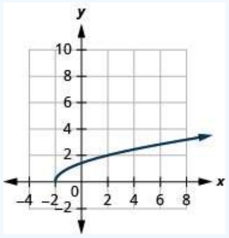


Figure 8.E.3

c. range: $[0,\infty)$

CHAPTER OVERVIEW

7: Quadratic Equations

7.1: Solve Quadratic Equations Using the Square Root Property
7.2: Solve Quadratic Equations Completing the Square
7.3: Solve Quadratic Equations Using the Quadratic Formula
7.4: Applications of Quadratic Equations
7.5: Graph Quadratic Equations Using Properties
7.6: Graph Quadratic Equations Using Transformations
7.7: Chapter 7 Review Exercises

7.1: Solve Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\sqrt{128}$.
2. Simplify $\sqrt{\frac{32}{5}}$.
3. Factor $9x^2 - 12x + 4$.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the Form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the difference of squares.	$(x-3)(x+3) = 0$
Use the Zero Produce Property.	$x - 3 = 0 \quad \text{or} \quad x - 3 = 0$
Solve each equation.	$x = 3 \quad \text{or} \quad x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad .$$

These two solutions are often written

$$x = \pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2 = k$, the principal square root of k and its opposite. We could also write the solution as $x = \pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2 = 9$ again, this time using the Square Root Property.

	$x^2 = 9$
Use the Square Root Property.	$x = \pm\sqrt{9}$
Simplify.	$x = \pm 3$
Rewrite to show two solutions.	$x = 3 \quad \text{or} \quad x = -3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2 = 7$.

	$x^2 = 7$
Use the Square Root Property.	$x = \pm\sqrt{7}$
Rewrite to show two solutions.	$x = \sqrt{7} \quad \text{or} \quad x = -\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example 7.1.1

Solve $x^2 - 50 = 0$.

Solution

		$x^2 - 50 = 0$
Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2 = 50$
Use the Square Root Property.	Remember to write the $\pm$ symbol or list the solutions.	$x = \pm\sqrt{50}$
Simplify the radical.		$x = \pm\sqrt{25} \cdot \sqrt{2}$ $x = \pm 5\sqrt{2}$
Rewrite to show two solutions.		$x = 5\sqrt{2} \quad \text{or} \quad x = -5\sqrt{2}$
Check the solutions in order to detect errors.		

Try It 7.1.2

Solve $x^2 - 48 = 0$.

Answer

$x = 4\sqrt{3} \quad \text{or} \quad x = -4\sqrt{3}$

Try It 7.1.3

Solve $y^2 - 27 = 0$.

Answer

$y = 3\sqrt{3} \quad \text{or} \quad y = -3\sqrt{3}$

The steps to take to use the **Square Root Property** to solve a quadratic equation are listed here.

Solve a Quadratic Equation Using the Square Root Property

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions in order to detect errors.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example 7.1.4

Solve $3z^2 = 108$.

Solution

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6 \quad \text{or} \quad z = -6$
Check the solutions.	$3z^2 = 108$, $3(6)^2 \stackrel{?}{=} 108$, $3(36) \stackrel{?}{=} 108$, $108 = 108$ ✓ $3z^2 = 108$, $3(-6)^2 \stackrel{?}{=} 108$, $3(36) \stackrel{?}{=} 108$, $108 = 108$ ✓

Try It 7.1.5

Solve $2x^2 = 98$.

Answer

$x = 7 \quad \text{or} \quad x = -7$

Try It 7.1.6

Solve $5m^2 = 80$.

Answer

$m = 4 \quad \text{or} \quad m = -4$

The Square Root Property states 'If $x^2 = k$,' What will happen if $k < 0$? This will be the case in the next example.

Example 7.1.7

Solve $x^2 + 72 = 0$.

Solution

	$x^2 + 72 = 0$
Isolate the quadratic term.	$x^2 = -72$
Use the Square Root Property.	$x = \pm\sqrt{-72}$
Simplify using complex numbers.	$x = \pm\sqrt{72}i$
Simplify the radical.	$x = \pm 6\sqrt{2}i$
Rewrite to show two solutions	$x = 6\sqrt{2}i \quad \text{or} \quad x = -6\sqrt{2}i$
Check the solutions.	$x^2 + 72 = 0$ $(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $6^2(\sqrt{2})^2 i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0 \checkmark$ $x^2 + 72 = 0$ $(6\sqrt{2}i)^2 + 72 \stackrel{?}{=} 0$ $(-6)^2(\sqrt{2})^2 i^2 + 72 \stackrel{?}{=} 0$ $36 \cdot 2 \cdot (-1) + 72 \stackrel{?}{=} 0$ $0 = 0 \checkmark$

Try It 7.1.8

Solve $c^2 + 12 = 0$.

Answer

$c = 2\sqrt{3}i \quad \text{or} \quad c = -2\sqrt{3}i$

Try It 7.1.9

Solve $q^2 + 24 = 0$.

Answer

$c = 2\sqrt{6}i \quad \text{or} \quad c = -2\sqrt{6}i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

Example 7.1.10

Solve $\frac{2}{3}u^2 + 5 = 17$.

Solution

	$\frac{2}{3}u^2 + 5 = 17$
Isolate the quadratic term.	$\frac{2}{3}u^2 = 12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2} \cdot \frac{2}{3}u^2 = \frac{3}{2} \cdot 12$

	$\frac{2}{3}u^2+5=17$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2}$ or $u=-3\sqrt{2}$
Check.	$\frac{2}{3}u^2+5=17$ $\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$ $\frac{2}{3}u^2+5=17$ $\frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$ $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$ $12+5\stackrel{?}{=}17$ $17=17\checkmark$

? Try It 7.1.11

Solve $\frac{1}{2}x^2+4=24$.

Answer

$x=2\sqrt{10}$ or $x=-2\sqrt{10}$

? Try It 7.1.12

Solve $\frac{3}{4}y^2-3=18$.

Answer

$y=2\sqrt{7}$ or $y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must **rationalize the denominator**.

? Example 7.1.13

Solve $2x^2-8=41$.

Solution

	$2x^2-8=41$
Isolate the quadratic term.	$2x^2=49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2}=\frac{49}{2}$
Simplify.	$x^2=\frac{49}{2}$
Use the Square Root Property.	$x=\pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x=\pm\frac{\sqrt{49}}{\sqrt{2}}$

	$2x^2-8=41$
Rationalize the denominator.	$x=\pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$x=\pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x=\frac{7\sqrt{2}}{2}$ or $x=-\frac{7\sqrt{2}}{2}$
Check: We leave te check for you.	

Try It 7.1.14

Solve $5r^2-2=34$.

Answer

$$r=\frac{6\sqrt{5}}{5} \quad \text{or} \quad r=-\frac{6\sqrt{5}}{5}$$

Try It 7.1.15

Solve $3t^2+6=70$.

Answer

$$t=\frac{8\sqrt{3}}{3} \quad \text{or} \quad t=-\frac{8\sqrt{3}}{3}$$

Solve Quadratic Equation of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the **Square Root Property** to solve an equation of the form $a(x-h)^2=k$ as well. Notice that the quadratic term, x, in the original form $ax^2=k$ is replaced with $(x-h)$.

$$ax^2=k \qquad a(x-h)^2=k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x-h)^2$.

Example 7.1.16

Solve $4(y-7)^2=48$.

Solution

	$4(y-7)^2=48$
Divide both sides by the coefficient 4.	$(y-7)^2=12$
Use the Square Root Property on the binomial.	$y-7=\pm\sqrt{12}$
Simplify the radical.	$y-7=\pm 2\sqrt{3}$
Solve for y.	$y=7\pm 2\sqrt{3}$
Rewrite to show two solutions.	$y=7+2\sqrt{3}$ or $y=7-2\sqrt{3}$

	$4(y-7)^2=48$
Check.	$4(y-7)^2=48$ $4(7+2\sqrt{3}-7)^2\stackrel{?}{=}48$ $4(2\sqrt{3})^2\stackrel{?}{=}48$ $4(12)\stackrel{?}{=}48$ $48=48\checkmark$ $4(y-7)^2=48$ $4(7-2\sqrt{3}-7)^2\stackrel{?}{=}48$ $4(-2\sqrt{3})^2\stackrel{?}{=}48$ $4(12)\stackrel{?}{=}48$ $48=48\checkmark$

? Try It 7.1.17

Solve $3(a-3)^2=54$.

Answer

$a=3+3\sqrt{2} \quad$ or $\quad a=3-3\sqrt{2}$

? Try It 7.1.18

Solve $2(b+2)^2=80$.

Answer

$b=-2+2\sqrt{10} \quad$ or $\quad b=-2-2\sqrt{10}$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

? Example 7.1.19

Solve $\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$.

Solution

	$\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$
Rewrite the radical as a fraction of square roots.	$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{\sqrt{9}}$
Simplify the radical.	$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3}$
Solve for x.	$x=\frac{1}{3}\pm\frac{\sqrt{5}}{3}$
Rewrite to show two solutions.	$x=\frac{1}{3}+\frac{\sqrt{5}}{3} \quad$ or $\quad x=\frac{1}{3}-\frac{\sqrt{5}}{3}$
Check.	We leave the check for you.

? Try It 7.1.20

Solve $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Answer

$$x=\frac{1}{2}+\frac{\sqrt{5}}{2} \quad \text{or} \quad x=\frac{1}{2}-\frac{\sqrt{5}}{2}$$

Try It 7.1.21

Solve $\left(y+\frac{3}{4}\right)^{2}=\frac{7}{16}$.

Answer

$$y=-\frac{3}{4}+\frac{\sqrt{7}}{4} \quad \text{or} \quad y=-\frac{3}{4}-\frac{\sqrt{7}}{4}$$

We will start the solution to the next example by isolating the binomial term.

Example 7.1.22

Solve $2(x-2)^{2}+3=57$.

Solution

	$2(x-2)^{2}+3=57$
Subtract 3 from both sides to isolate the binomial term.	$2(x-2)^{2}=54$
Divide both sides by 2.	$(x-2)^{2}=27$
Use the Square Root Property.	$x-2=\pm\sqrt{27}$
Simplify the radical.	$x-2=\pm 3\sqrt{3}$
Solve for x.	$x=2\pm 3\sqrt{3}$
Rewrite to show two solutions.	$x=2+3\sqrt{3} \quad \text{or} \quad x=2-3\sqrt{3}$
Check.	We leave the check for you.

Try It 7.1.23

Solve $5(a-5)^{2}+4=104$.

Answer

$$a=5+2\sqrt{5} \quad \text{or} \quad a=5-2\sqrt{5}$$

Try It 7.1.24

Solve $3(b+3)^{2}-8=88$.

Answer

$$b=-3+4\sqrt{2} \quad \text{or} \quad b=-3-4\sqrt{2}$$

Sometimes the solutions are complex numbers.

Example 7.1.25

Solve $(2x-3)^2=-12$.

Solution

	$(2x-3)^2=-12$
Use the Square Root Property.	$2x-3=\pm\sqrt{-12}$
Simplify the radical.	$2x-3=\pm 2\sqrt{3}i$
Add 3 to both sides.	$2x=3\pm 2\sqrt{3}i$
Divide both sides by 2.	$x=\frac{3\pm 2\sqrt{3}i}{2}$
Rewrite in standard form.	$x=\frac{3}{2}\pm\frac{2\sqrt{3}i}{2}$
Simplify.	$x=\frac{3}{2}\pm\sqrt{3}i$
Rewrite to show two solutions.	$x=\frac{3}{2}+\sqrt{3}i \quad \text{or} \quad x=\frac{3}{2}-\sqrt{3}i$
Check.	We leave the check for you.

Try It 7.1.26

Solve $(3r+4)^2=-8$.

Answer

$$r=-\frac{4}{3}+\frac{2\sqrt{2}i}{3} \quad \text{or} \quad r=-\frac{4}{3}-\frac{2\sqrt{2}i}{3}$$

Try It 7.1.27

Solve $(2t-8)^2=-10$.

Answer

$$t=4+\frac{\sqrt{10}i}{2} \quad \text{or} \quad t=4-\frac{\sqrt{10}i}{2}$$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 7.1.28

Solve $4n^2+4n+1=16$.

Answer

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$

	$4n^2+4n+1=16$
Simplify the radical.	$2n+1=\pm 4$
Solve for n.	$2n=-1\pm 4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm 4}{2}$
Simplify.	$n=\frac{-1\pm 4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2}$ or $n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2}$ or $n=-\frac{5}{2}$
Check.	$4n^2+4n+1=16$ $4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $9+6+1\stackrel{?}{=}16$ $16=16\checkmark$ $4n^2+4n+1=16$ $4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $25-10+1\stackrel{?}{=}16$ $16=16\checkmark$

? Try It 7.1.29

Solve $9m^2-12m+4=25$.

Answer

$$m=\frac{7}{3} \quad \text{or} \quad m=-1$$

? Try It 7.1.30

Solve $16n^2+40n+25=4$.

Answer

$$n=-\frac{3}{4} \quad \text{or} \quad n=-\frac{7}{4}$$

Key Concepts

- Square Root Property
  - If $x^2=k$, then $x=\sqrt{k}$ or $x=-\sqrt{k}$or $x=\pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  1. Isolate the quadratic term and make its coefficient one.
  2. Use Square Root Property.
  3. Simplify the radical.
  4. Check the solutions.

Practice Makes Perfect

? Solve Quadratic Equations of the Form $ax^2=k$ Using the Square Root Property

In the following exercises, solve each equation.

1. $a^2=49$

2. $b^2 = 144$

3. $r^2 - 24 = 0$

4. $t^2 - 75 = 0$

5. $u^2 - 300 = 0$

6. $v^2 - 80 = 0$

7. $4m^2 = 36$

8. $3n^2 = 48$

9. $\dfrac{4}{3}x^2 = 48$

10. $\dfrac{5}{3}y^2 = 60$

11. $x^2 + 25 = 0$

12. $y^2 + 64 = 0$

13. $x^2 + 63 = 0$

14. $y^2 + 45 = 0$

15. $\dfrac{4}{3}x^2 + 2 = 110$

16. $\dfrac{2}{3}y^2 - 8 = -2$

17. $\dfrac{2}{5}a^2 + 3 = 11$

18. $\dfrac{3}{2}b^2 - 7 = 41$

19. $7p^2 + 10 = 26$

20. $2q^2 + 5 = 30$

21. $5y^2 - 7 = 25$

22. $3x^2 - 8 = 46$

Answer

1. $a = \pm 7$
3. $r = \pm 2\sqrt{6}$
5. $u = \pm 10\sqrt{3}$
7. $m = \pm 3$
9. $x = \pm 6$
11. $x = \pm 5i$
13. $x = \pm 3\sqrt{7}i$
15. $x = \pm 9$
17. $a = \pm 2\sqrt{5}$
19. $p = \pm\dfrac{4\sqrt{7}}{7}$
21. $y = \pm\dfrac{4\sqrt{10}}{5}$

Solve Quadratic Equations of the Form $a(x-h)^2=k$ Using the Square Root Property

In the following exercises, solve each equation.

23. $(u-6)^2=64$

24. $(v+10)^2=121$

25. $(m-6)^2=20$

26. $(n+5)^2=32$

27. $\left(r-\frac{1}{2}\right)^2=\frac{3}{4}$

28. $\left(x+\frac{1}{5}\right)^2=\frac{7}{25}$

29. $\left(y+\frac{2}{3}\right)^2=\frac{8}{81}$

30. $\left(t-\frac{5}{6}\right)^2=\frac{11}{25}$

31. $(a-7)^2+5=55$

32. $(b-1)^2-9=39$

33. $4(x+3)^2-5=27$

34. $5(x+3)^2-7=68$

35. $(5c+1)^2=-27$

36. $(8d-6)^2=-24$

37. $(4x-3)^2+11=-17$

38. $(2y+1)^2-5=-23$

39. $m^2-4m+4=8$

40. $n^2+8n+16=27$

41. $x^2-6x+9=12$

42. $y^2+12y+36=32$

43. $25x^2-30x+9=36$

44. $9y^2+12y+4=9$

45. $36x^2-24x+4=81$

46. $64x^2+144x+81=25$

Answer

23. $u=14, u=-2$

25. $m=6\pm2\sqrt{5}$

27. $r=\frac{1}{2}\pm\frac{\sqrt{3}}{2}$

29. $y=-\frac{2}{3}\pm\frac{2\sqrt{2}}{9}$

31. $a=7\pm5\sqrt{2}$

33. $x=-3\pm2\sqrt{2}$

35. $c=-\frac{1}{5}\pm\frac{3\sqrt{3}}{5}i$

37. $x=\frac{3}{4}\pm\frac{\sqrt{7}}{2}i$

39. $m=2\pm2\sqrt{2}$

41. $x=3+2\sqrt{3}, x=3-2\sqrt{3}$

43. $x=-\frac{3}{5}, x=\frac{9}{5}$

45. $x=-\frac{7}{6}, x=\frac{11}{6}$

? Mixed Practice

In the following exercises, solve using the Square Root Property.

47. $2r^2=32$

48. $4t^2=16$

49. $(a-4)^2=28$

50. $(b+7)^2=8$

51. $9w^2-24w+16=1$

52. $4z^2+4z+1=49$

53. $a^2-18=0$

54. $b^2-108=0$

55. $\left(p-\frac{1}{3}\right)^2=\frac{7}{9}$

56. $\left(q-\frac{3}{5}\right)^2=\frac{3}{4}$

57. $m^2+12=0$

58. $n^2+48=0$

59. $u^2-14u+49=72$

60. $v^2+18v+81=50$

61. $(m-4)^2+3=15$

62. $(n-7)^2-8=64$

63. $(x+5)^2=4$

64. $(y-4)^2=64$

65. $6c^2+4=29$

66. $2d^2-4=77$

67. $(x-6)^2+7=3$

68. $(y-4)^2+10=9$

Answer

47. $r=\pm4$

49. $a=4\pm2\sqrt{7}$

51. $w=1, w=\frac{5}{3}$

53. $a=\pm 3\sqrt{2}$

55. $p=\frac{1}{3}\pm\frac{\sqrt{7}}{3}$

57. $m=\pm 2\sqrt{2i}$

59. $u=7\pm 6\sqrt{2}$

61. $m=4\pm 2\sqrt{3}$

63. $x=-3, x=-7$

65. $c=\pm\frac{5\sqrt{6}}{6}$

67. $x=6\pm 2i$

? Writing exercises

69. In your own words, explain the Square Root Property.

70. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2=16$.

Answer

69. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations of the form $ax^2=k$ using the square root property.			
solve quadratic equations of the form $a(x-h)^2=k$ using the square root property.			

Choose how would you respond to the statement "I can solve quadratic equations of the form a times the square of x minus h equals k using the Square Root Property." "Confidently," "with some help," or "No, I don't get it."

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

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7.2: Solve Quadratic Equations Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square

Be Prepared

Before you get started, take this readiness quiz.

1. Expand $(x+9)^2$.
2. Factor $y^2-14y+49$.
3. Factor $5n^2+40n+80$.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.

$$\begin{aligned}(y-7)^2 &= 12\\ y-7 &= \pm\sqrt{12}\\ y-7 &= \pm 2\sqrt{3}\\ y &= 7\pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the Square Root Property.

$$\begin{aligned}x^2-10x+25 &= 18\\ (x-5)^2 &= 18\\ x-5 &= \pm 3\sqrt{2}\\ x &= 5\pm 3\sqrt{2}\end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$\begin{array}{ll}(x+9)^2 & (y-7)^2\\ =(x+9)(x+9) & =(y-7)(y-7)\\ =x^2+9x+9x+81 & =y^2-7y-7y+49\\ =x^2+18x+81 & =y^2-14y+49\end{array}$$

We restate the patterns here for reference.

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2,$$

and

$$(a-b)^2=a^2-2ab+b^2.$$

We can use this pattern to “make” a perfect square.

We will start with the expression x^2+6x. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern, $a^2+2ab+b^2=(a+b)^2$.

$$\underbrace{x^2+6x+\ldots}_{a^2+2ab+b^2}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of x^2+6x is a square, x^2. This tells us that $a=x$.

$$\underbrace{x^2+2xb+b^2}_{a^2+2ab+b^2}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b=3$.

$$\underbrace{x^2+2\cdot 3x+\ldots}_{a^2+2ab+b^2}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2=9$.

$$\underbrace{x^2+6x+9}_{a^2+2ab+b^2}$$

We can now factor.

$$\underbrace{(x+3)^2}_{(a+b)^2}$$

So we found that adding 9 to x^2+6x ‘completes the square’, and we write it as $(x+3)^2$.

Complete a Square of x^2+bx

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx.
4. Factor the perfect square trinomial, writing it as a binomial squared.

Example 7.2.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. x^2-26x

b. y^2-9y

c. $n^2+\frac{1}{2}n$

Solution

a.

	$\underbrace{x^2-26x}_{x^2+bx}$
The coefficient of x is -26.	$b=-26$
Find $\left(\frac{1}{2}b\right)^2$.	$\left(\frac{1}{2}\cdot(-26)\right)^2$ $=(-13)^2$ $=169$

	$\underbrace{x^2-26x}_{x^2+bx}$
Add 169 to the binomial to complete the square.	$x^2-26x+169$
Factor the perfect square trinomial, writing it as a binomial squared.	$=(x-13)^2$

b.

	$\underbrace{y^2-9y}_{x^2+bx}$
The coefficient of y is -9.	$b=-9$
Find $\left(\frac{1}{2}b\right)^2$.	$\left(\frac{1}{2}\cdot(-9)\right)^2$ $=\left(-\frac{9}{2}\right)^2$ $=\frac{81}{4}$
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2-9y+\frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$=\left(y-\frac{9}{2}\right)^2$

c.

	$\underbrace{n^2+\frac{1}{2}n}_{x^2+bx}$
The coefficient of n is $\frac{1}{2}$.	$b=\frac{1}{2}$
Find $\left(\frac{1}{2}b\right)^2$.	$\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $=\left(\frac{1}{4}\right)^2$ $=\frac{1}{16}$
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2+\frac{1}{2}n+\frac{1}{16}$
Rewrite as a binomial square.	$=\left(n+\frac{1}{4}\right)^2$

? Try It 7.2.2

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. a^2-20a

b. m^2-5m

c. $p^2+\frac{1}{4}p$

Answer

a. $(a-10)^2$

b. $\left(m-\frac{5}{2}\right)^2$

c. $\left(p+\frac{1}{8}\right)^2$

Try It 7.2.3

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. b^2-4b

b. n^2+13n

c. $q^2-\frac{2}{3}q$

Answer

a. $(b-2)^2$

b. $\left(n+\frac{13}{2}\right)^2$

c. $\left(q-\frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a **quadratic equation** by **completing the square** too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2+6x=40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+\cdots=40+\cdots$
Add 9 to both sides to complete the square.	$x^2+6x+9=40+9$
Rewrite it as a binomial square.	$(x+3)^2=49$

Now the equation is in the form to solve using the **Square Root Property**! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example 7.2.4

Solve by completing the square: $x^2+8x=48$.

Solution

		$x^2+8x=48$
Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2+bx \quad c$ $x^2+8x=48$
Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	$b=8$ Take half of 8 and square it. $4^2=16$ Add 16 to BOTH sides of the equation.	$x^2+8x+\underline{\quad\left(\frac{1}{2}\cdot 8\right)^2\quad}=48$ $x^2+8x+16=48+16$

		$x^2+8x=48$
Factor the perfect square trinomial as a binomial square.	$x^2+8x+16=(x+4)^2$ Add the terms on the right.	$(x+4)^2=64$
Use the Square Root Property.		$x+4=\pm\sqrt{64}$
Simplify the radical.		$x+4=\pm 8$
Solve the two resulting equations.		$\begin{aligned} x+4&=8 \\ x&=4 \end{aligned} \qquad \begin{aligned} x+4&=-8 \\ x&=-12 \end{aligned}$
Check the solutions.	Put each answer in the original equation to check. Substitute $x=4$ and $x=-12$.	$\begin{aligned} x^2+8x&=48 \\ x=\color{red}{4}:\quad (\color{red}{4})^2+8(\color{red}{4})&\stackrel{?}{=}48 \\ 16+32&\stackrel{?}{=}48 \\ 48&\stackrel{?}{=}48 \text{ True} \end{aligned}$ $\begin{aligned} x^2+8x&=48 \\ x=\color{red}{-12}:\quad (\color{red}{-12})^2+8(\color{red}{-12})&\stackrel{?}{=}48 \\ 144-96&\stackrel{?}{=}48 \\ 48&\stackrel{?}{=}48 \text{ Tr} \end{aligned}$
		The solutions are $x=4$ or $x=-12$.

Try It 7.2.5

Solve by completing the square: $x^2+4x=5$.

Answer

$x=-5$ or $x=1$

Try It 7.2.6

Solve by completing the square: $y^2-10y=-9$.

Answer

$y=1$ or $y=9$

The steps to solve a quadratic equation by completing the square are listed here.

Solve a Quadratic Equation of the Form $x^2+bx+c=0$ by Completing the Square

1. Isolate the variable terms on one side and the constant terms on the other.
2. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
4. Use the Square Root Property.
5. Simplify the radical and then solve the two resulting equations.
6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Example 7.2.7

Solve by completing the square: $x^2+4x=-21$.

Solution

	$x^2+4x=-21$
The variable terms are on the left side.	$x^2+bx \quad c$ $x^2+4x=-21$
Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2=4$	$x^2+4x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot 4\right)^2}=-21$
Add 4 to both sides.	$x^2+4x+4=-21+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2=-17$
Use the Square Root Property.	$x+2=\pm\sqrt{-17}$
Simplifying using complex numbers.	$x+2=\pm\sqrt{17}i$
Subtract 2 from each side.	$x=-2\pm\sqrt{17}i$
Rewrite to show two solutions.	$x=-2+\sqrt{17}i \quad$ or $\quad x=-2-\sqrt{17}i$
Check.	We leave the check to you.
	The solutions are $x=-2+\sqrt{17}i \quad$ or $\quad x=-2-\sqrt{17}i$.

Try It 7.2.8

Solve by completing the square: $y^2-10y=-35$.

Answer

$y=5+\sqrt{10}i \quad$ or $\quad y=5-\sqrt{15}i$

Try It 7.2.9

Solve by completing the square: $z^2+8z=-19$.

Answer

$z=-4+\sqrt{3}i \quad$ or $\quad z=-4-\sqrt{3}i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Example 7.2.10

Solve by completing the square: $y^2-18y=-6$.

Solution

	$y^2-18y=-6$
The variable terms are on the left side.	$x^2+bx \quad c$ $y^2-18y=-6$
Take half of -18 and square it. $\left(\frac{1}{2}(-18)\right)^2=81$	$y^2-18y+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(-18)\right)^2}=-6$
Add 81 to both sides.	$y^2-18y+81=-6+81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2=75$

Use the Square Root Property.	$y-9=\pm\sqrt{75}$
Simplify the radical.	$y-9=\pm 5\sqrt{3}$
Solve for y.	$y=9\pm 5\sqrt{3}$
Rewrite to show two solutions.	$y=9+5\sqrt{3} \quad \text{or} \quad y=9-5\sqrt{3}$
Check.	$y^2-18y=-6$ $(9+5\sqrt{3})^2-18(9+5\sqrt{3})\overset{?}{=}-6$ $81+90\sqrt{3}+75-162+90\sqrt{3}\overset{?}{=}-6$ $-6=-6\checkmark$ $y^2-18y=-6$ $(9-5\sqrt{3})^2-18(9-5\sqrt{3})\overset{?}{=}-6$ $81+90\sqrt{3}+75-162+90\sqrt{3}\overset{?}{=}-6$ $-6=-6\checkmark$

Another way to check this would be to use a calculator. Evaluate y^2-18y for both solutions. The answer should be -6.

Try It 7.2.11

Solve by completing the square: $x^2-16x=-16$.

Answer

$x=8+4\sqrt{3} \quad \text{or} \quad x=8-4\sqrt{3}$

Try It 7.2.12

Solve by completing the square: $y^2+8y=11$.

Answer

$y=-4+3\sqrt{3} \quad \text{or} \quad y=-4-3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example 7.2.13

Solve by completing the square: $x^2+10x+4=15$.

Solution

	$x^2+10x+4=15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2+10x=11$
Take half of 10 and square it. $\left(\frac{1}{2}(10)\right)^2=25$	$x^2-10x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(10)\right)^2}=11$
Add 25 to both sides.	$x^2+10x+25=11+25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2=36$
Use the Square Root Property.	$x+5=\pm\sqrt{36}$
Simplify the radical.	$x+5=\pm 6$
Solve for x.	$x=-5\pm 6$
Rewrite to show two solutions.	$x=-5+6 \quad \text{or} \quad x=-5-6$
Solve the equations.	$x=1 \quad \text{or} \quad x=-11$

	$x^2+10x+4=15$
Check.	$x^2+10x+4=15$ $(1)^2+10(1)+4\stackrel{?}{=}15$ $1+10+4\stackrel{?}{=}15$ $15=15\checkmark$ $x^2+10x+4=15$ $(-11)^2+10(-11)+4\stackrel{?}{=}15$ $121+110+4\stackrel{?}{=}15$ $15=15\checkmark$
	The solutions are $x=1$ or $x=-11$.

Try It 7.2.14

Solve by completing the square: $a^2+4a+9=30$.

Answer

$a=-7$ or $a=3$

Try It 7.2.15

Solve by completing the square: $b^2+8b-4=16$.

Answer

$b=-10$ or $b=2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Example 7.2.16

Solve by completing the square: $n^2=3n+11$.

Answer

	$n^2=3n+11$
Subtract $3n$ to get the variable terms on the left side.	$n^2-3n=11$
Take half of -3 and square it.	$\left(\frac{1}{2}(-3)\right)^2=\frac{9}{4}$
	$n^2-18y+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(-18)\right)^2}=-6$
Add $\frac{9}{4}$ to both sides.	$n^2-3n+\frac{9}{4}=11+\frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n-\frac{3}{2}\right)^2=\frac{44}{4}+\frac{9}{4}$
Add the fractions on the right side.	$\left(n-\frac{3}{2}\right)^2=\frac{53}{4}$
Use the Square Root Property.	$n-\frac{3}{2}=\pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n-\frac{3}{2}=\pm\frac{\sqrt{53}}{2}$
Solve for n.	$n=\frac{3}{2}\pm\frac{\sqrt{53}}{2}$

	$n^2=3n+11$
Rewrite to show two solutions.	$n=\frac{3}{2}+\frac{\sqrt{53}}{2}$ or $n=\frac{3}{2}-\frac{\sqrt{53}}{2}$
Check.	We leave the check for you!

Try It 7.2.17

Solve by completing the square: $p^2=5p+9$.

Answer

$$p=\frac{5}{2}+\frac{\sqrt{61}}{2} \quad \text{or} \quad p=\frac{5}{2}-\frac{\sqrt{61}}{2}$$

Try It 7.2.18

Solve by completing the square: $q^2=7q-3$.

Answer

$$q=\frac{7}{2}+\frac{\sqrt{37}}{2} \quad \text{or} \quad q=\frac{7}{2}-\frac{\sqrt{37}}{2}$$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the **Zero Product Property** since it says "If $a\cdot b=0$, then $a=0$ or $b=0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

Example 7.2.19

Solve by completing the square: $(x-3)(x+5)=9$.

Solution

	$(x-3)(x+5)=9$
We multiple the binomials on the left.	$x^2+2x-15=9$
Add 15 to isolate the constant terms on the right.	$x^2+2x=24$
	$x^2+2x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(2)\right)^2}=24$
Take half of 2 and square it.	$\left(\frac{1}{2}\cdot(2)\right)^2=1$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for x.	$x=-1\pm 5$
Rewrite to show two solutions.	$x=-1+5$ or $x=-1-6$
Simplify.	$x=4$ or $x=-6$
Check.	We leave the check for you!

Try It 7.2.20

Solve by completing the square: $(c-2)(c+8)=11$.

Answer

$c=-9 \quad$ or $\quad c=3$

Try It 7.2.21

Solve by completing the square: $(d-7)(d+3)=56$.

Answer

$d=11 \quad$ or \quadd $=-7$

Solve Quadratic Equations of the Form $ax^2+bx+c=0$ by Completing the Square

The process of **completing the square** works best when the coefficient of x^2 is 1, so the left side of the equation is of the form x^2+bx+c . If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example 7.2.22

Solve by completing the square: $3x^2-12x-15=0$.

Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2-12x-15=0$
Factor out the greatest common factor.	$3(x^2-4x-5)=0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2-4x-5)}{3}=\frac{0}{3}$
Simplify.	$x^2-4x-5=0$
Add 5 to get the constant terms on the right side.	$x^2-4x=5$
Take half of 4 and square it.	$\left(\frac{1}{2}(-4)\right)^2=4$
	$x^2-4x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(4)\right)^2}=5$
Add 4 to both sides.	$x^2-4x+4=5+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-2)^2=9$
Use the Square Root Property.	$x-2=\pm\sqrt{9}$ \
Solve for x.	$x-2=\pm 3$
Rewrite to show two solutions.	$x=2+3 \quad$ or $\quad x=2-3$
Simplify.	$x=5 \quad$ or $\quad x=-1$

Check.	$x = 5$ $3x^2 - 12x - 15 = 0$ $3(5)^2 - 12(5) - 15 \stackrel{?}{=} 0$ $75 - 60 - 15 \stackrel{?}{=} 0$ $0 = 0 ✓$ $x = -1$ $3x^2 - 12x - 15 = 0$ $3(-1)^2 - 12(-1) - 15 \stackrel{?}{=} 0$ $3 + 12 - 15 \stackrel{?}{=} 0$ $0 = 0 ✓$

? Try It 7.2.23

Solve by completing the square: $2m^2 + 16m + 14 = 0$.

Answer

$m = -7 \quad \text{or} \quad m = -1$

? Try It 7.2.24

Solve by completing the square: $4n^2 - 24n - 56 = 8$.

Answer

$n = -2 \quad \text{or} \quad n = 8$

To complete the square, the coefficient of the x^2 must be 1. When the **leading coefficient** is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

? Example 7.2.25

Solve by completing the square: $2x^2 - 3x = 20$.

Solution

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$
	$x^2 - \frac{3}{2}x + \backslash(x^2 - 4x + \underbrace{\ldots}_{\left(\frac{1}{2}\cdot(4)\right)^2} = 5$ $x^2 - \frac{3}{2}x + \underset{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2}{\underline{\qquad\qquad}} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$

Use the Square Root Property.	$x-\frac{3}{4}=\pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x-\frac{3}{4}=\pm\frac{13}{4}$
Solve for x.	$x=\frac{3}{4}\pm\frac{13}{4}$
Rewrite to show two solutions.	$x=\frac{3}{4}+\frac{13}{4}\quad\text{or}\quad x=\frac{3}{4}\pm\frac{13}{4}$
Simplify.	$x=4\quad\text{or}\quad x=-\frac{5}{2}$
Check.	We leave the check for you!

Try It 7.2.26

Solve by completing the square: $3r^2-2r=21$.

Answer

$$r=-\frac{7}{3}\quad\text{or}\quad r=3$$

Try It 7.2.27

Solve by completing the square: $4t^2+2t=20$.

Answer

$$t=-\frac{5}{2}\quad\text{or}\quad t=2$$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a **quadratic equation** by completing the square to include equations of the form $ax^2+bx+c=0$.

Solve a Quadratic Equation of the Form $ax^2+bx+c=0$ by Completing the Square

1. Divide by aa to make the coefficient of x^2 term 1.
2. Isolate the variable terms on one side and the constant terms on the other.
3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
5. Use the Square Root Property.
6. Simplify the radical and then solve the two resulting equations.
7. Check the solutions.

Example 7.2.28

Solve by completing the square: $3x^2+2x=4$.

Solution

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$

	$3x^2+2x=4$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	
	$x^2+\frac{2}{3}x+\underset{\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2}{\underline{\qquad}}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for x.	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\quad x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$
Check.	We leave the check for you!

? Try It 7.2.29

Solve by completing the square: $4x^2+3x=2$.

Answer

$$x=-\frac{3}{8}+\frac{\sqrt{41}}{8}\quad\text{or}\quad x=-\frac{3}{8}-\frac{\sqrt{41}}{8}$$

? Try It 7.2.30

Solve by completing the square: $3y^2-10y=-5$.

Answer

$$y=\frac{5}{3}+\frac{\sqrt{10}}{3}\quad\text{or}\quad y=\frac{5}{3}-\frac{\sqrt{10}}{3}$$

Key Concepts

- Binomial Squares Pattern
  If a and b are real numbers,

$$(a+b)^2=a^2+2ab+b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2=a^2-2ab+b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx
4. Rewrite the trinomial as a binomial square

- How to solve a quadratic equation of the form $ax^2+bx+c=0$ by completing the square.
  1. Divide by a to make the coefficient of x^2 term 1.
  2. Isolate the variable terms on one side and the constant terms on the other.
  3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
  4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
  5. Use the Square Root Property.
  6. Simplify the radical and then solve the two resulting equations.
  7. Check the solutions.

Practice Makes Perfect

? Complete the Square of a Binomial Expression

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. a. m^2-24m
   b. x^2-11x
   c. $p^2-\frac{1}{3}p$
2. a. n^2-16n
   b. y^2+15y
   c. $q^2+\frac{3}{4}q$
3. a. p^2-22p
   b. y^2+5y
   c. $m^2+\frac{2}{5}m$
4. a. q^2-6q
   b. x^2-7x
   c. $n^2-\frac{2}{3}n$

Answer

1. a. $(m-12)^2$ b. $\left(x-\frac{11}{2}\right)^2$ c. $\left(p-\frac{1}{6}\right)^2$

3. a. $(p-11)^2$ b. $\left(y+\frac{5}{2}\right)^2$ c. $\left(m+\frac{1}{5}\right)^2$

? Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In the following exercises, solve by completing the square.

5. $u^2+2u=3$

6. $z^2+12z=-11$

7. $x^2-20x=21$

8. $y^2-2y=8$

9. $m^2+4m=-44$

10. $n^2 - 2n = -3$

11. $r^2 + 6r = -11$

12. $t^2 - 14t = -50$

13. $a^2 - 10a = -5$

14. $b^2 + 6b = 41$

15. $x^2 + 5x = 2$

16. $y^2 - 3y = 2$

17. $u^2 - 14u + 12 = -1$

18. $z^2 + 2z - 5 = 2$

19. $r^2 - 4r - 3 = 9$

20. $t^2 - 10t - 6 = 5$

21. $v^2 = 9v + 2$

22. $w^2 = 5w - 1$

23. $x^2 - 5 = 10x$

24. $y^2 - 14 = 6y$

25. $(x+6)(x-2) = 9$

26. $(y+9)(y+7) = 80$

27. $(x+2)(x+4) = 3$

28. $(x-2)(x-6) = 5$

Answer

5. $u = -3, u = 1$

7. $x = -1, x = 21$

9. $m = -2 \pm 2\sqrt{10}i$

11. $r = -3 \pm \sqrt{2}i$

13. $a = 5 \pm 2\sqrt{5}$

15. $x = -\frac{5}{2} \pm \frac{\sqrt{33}}{2}$

17. $u = 1, u = 13$

19. $r = -2, r = 6$

21. $v = \frac{9}{2} \pm \frac{\sqrt{89}}{2}$

23. $x = 5 \pm \sqrt{30}$

25. $x = -7, x = 3$

27. $x = -5, x = -1$

? Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

29. $3m^2 + 30m - 27 = 6$

30. $2x^2 - 14x + 12 = 0$

31. $2n^2 + 4n = 26$

32. $5x^2+20x=15$

33. $2c^2+c=6$

34. $3d^2-4d=15$

35. $2x^2+7x-15=0$

36. $3x^2-14x+8=0$

37. $2p^2+7p=14$

38. $3q^2-5q=9$

39. $5x^2-3x=-10$

40. $7x^2+4x=-3$

Answer

29. $m=-11, m=1$

31. $n=1\pm\sqrt{14}$

33. $c=-2, c=\frac{3}{2}$

35. $x=-5, x=\frac{3}{2}$

37. $p=-\frac{7}{4}\pm\frac{\sqrt{161}}{4}$

39. $x=\frac{3}{10}\pm\frac{\sqrt{191}}{10}i$

? Writing exercises

41. Solve the equation $x^2+10x=-25$

a. by using the Square Root Property
b. by Completing the Square
c. Which method do you prefer? Why?

42. Solve the equation $y^2+8y=48$ by completing the square and explain all your steps.

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
complete the square of a binomial expression.			
solve quadratic equations of the form $x^2+bx+c=0$ by completing the square.			
solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

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7.3: Solve Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Be Prepared

Before you get started, take this readiness quiz.

1. Evaluate $b^2 - 4ab$ when $a = 3$ and $b = -2$.
2. Simplify $\sqrt{108}$.
3. Simplify $\sqrt{50}$.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the exercise set, you may have been wondering 'isn't there an easier way to do this?' The answer is 'yes'. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable 'in general', so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2 + bx + c = 0, \quad a \neq 0$
Isolate the variable terms on one side.	$ax^2 + bx \quad = -c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a} + \frac{b}{a}x \quad = -\frac{c}{a}$
Simplify.	$x^2 + \frac{b}{a}x \quad = -\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2} \cdot \frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\frac{b}{a}\right)^2 = \frac{b^2}{4a^2}$	$x^2 + \frac{b}{a}x + {\color{red}\frac{b^2}{4a^2}} = -\frac{c}{a} + {\color{red}\frac{b^2}{4a^2}}$
The left side is a perfect square, factor it.	$\left(x + \frac{b}{2a}\right)^2 = -\frac{c}{a} + \frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c \cdot {\color{red}4a}}{a \cdot {\color{red}4a}}$
Simplify.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x + \frac{b}{2a}\right)^2 = \frac{b^2 - 4ac}{4a^2}$
Use the square root property.	$x + \frac{b}{2a} = \pm\sqrt{\frac{b^2 - 4ac}{4a^2}}$

Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula." We have just solved all quadratic equations!

Definition 7.3.1

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the **Quadratic Formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the **Quadratic Formula**, we substitute the values of $a, b,$ and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula is an equation. Make sure you use both sides of the equation. This equation requires an understanding of the meaning of $a, b,$ and c.

Example 7.3.2

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution

Step 1: Write the quadratic equation in standard form. Identify the a, b, c values.	This equation is in standard form.	
Step 2: Write the quadratic formula. Then substitute in the values of a, b, c.	Substitute in $a=2, b=9, c=-5$	$x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		
Step 4: Check the solutions to detect errors.	Put each answer in the original equation to check. Substitute $x=\color{red}{\frac{1}{2}}$ and $x=\color{red}{-5}$.	$\begin{aligned} 2x^2+9x-5&=0\\ 2\left(\color{red}{\frac{1}{2}}\right)^2+9\cdot\color{red}{\frac{1}{2}}-5&\stackrel{?}{=}0\\ 2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5&\stackrel{?}{=}0\\ 2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5&\stackrel{?}{=}0\\ \frac{1}{2}+\frac{9}{2}-5&\stackrel{?}{=}0\\ \frac{10}{2}-5&\stackrel{?}{=}0\\ 5-5&\stackrel{?}{=}0\\ 0&=0\\ 2x^2+9x-5&=0\\ 2(\color{red}{-5})^2+9(\color{red}{-5})-5&\stackrel{?}{=}0\\ 2\cdot 25-45-5&\stackrel{?}{=}0\\ 50-45-5&\stackrel{?}{=}0\\ 0&=0 \end{aligned}$

Note:
Step 4 is not strictly necessary since, provided that we didn't make a mistake, the Quadratic Formula tells us what the solutions are!

Try It 7.3.3

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$$y=1, y=\frac{2}{3}$$

Try It 7.3.4

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$$z=1, z=-\frac{3}{2}$$

HowTo: Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions to detect errors. This step is only to guard against mistakes!

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 7.3.5

Solve by using the Quadratic Formula: $x^2-6x=-5$.

Solution:

Write the equation in standard form by adding 5 to each side.	
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2-6x+5=0$
Identify the values of a, b, c.	$a=1$, $b=-6$, $c=5$
Write the Quadratic Formula.	
Then substitute in the values of a, b, c.	
Simplify.	$x=\frac{6\pm\sqrt{36-20}}{2}$ $x=\frac{6\pm\sqrt{16}}{2}$ $x=\frac{6\pm 4}{2}$
Rewrite to show two solutions.	$x=\frac{6+4}{2}, \quad x=\frac{6-4}{2}$
Simplify.	$x=\frac{10}{2}, \quad x=\frac{2}{2}$

	$x=5, \quad x=1$
Check: $x^2-6x+5=0$; $5^2-6\cdot5+5\stackrel{?}{=}0$; $25-30+5\stackrel{?}{=}0$; $0=0$ ✓ $x^2-6x+5=0$; $1^2-6\cdot1+5\stackrel{?}{=}0$; $1-6+5\stackrel{?}{=}0$; $0=0$ ✓	

Try It 7.3.6

Solve by using the Quadratic Formula: $a^2-2a=15$.

Answer

$a=-3, a=5$

Try It 7.3.7

Solve by using the Quadratic Formula: $b^2+24=-10b$.

Answer

$b=-6, b=-4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer should have the radical in its simplified form.

Example 7.3.8

Solve by using the Quadratic Formula: $2x^2+10x+11=0$.

Solution:

	$2x^2+10x+11=0$
This equation is in standard form.	$\begin{array}{cccc} ax^2 & +\ bx & +\ c & =0 \\ 2x^2 & +\ 10x & +\ 11 & =0 \end{array}$
Identify the values of a, b and c.	$a=2, b=10, c=11$
Write the Quadratic Formula.	$x=\dfrac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x=\dfrac{-(10)\pm\sqrt{(10)^2-4\cdot2\cdot(11)}}{2\cdot2}$
Simplify.	$x=\dfrac{-10\pm\sqrt{100-88}}{4}$
	$x=\dfrac{-10\pm\sqrt{12}}{4}$
Simplify the radical.	$x=\dfrac{-10\pm2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x=\dfrac{2(-5\pm\sqrt{3})}{4}$ $x=\dfrac{2(-5\pm\sqrt{3})}{4}$
Remove the common factors.	$x=\dfrac{-5\pm\sqrt{3}}{2}$

Rewrite to show two solutions.	$x=\frac{-5+\sqrt{3}}{2}, \quad x=\frac{-5-\sqrt{3}}{2}$
Check: We leave the check for you!	

Try It 7.3.9

Solve by using the Quadratic Formula: $3m^2+12m+7=0$.

Answer

$$m=\frac{-6+\sqrt{15}}{3}, m=\frac{-6-\sqrt{15}}{3}$$

Try It 7.3.10

Solve by using the Quadratic Formula: $5n^2+4n-4=0$.

Answer

$$n=\frac{-2+2\sqrt{6}}{5}, n=\frac{-2-2\sqrt{6}}{5}$$

When we substitute a, b, and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

Example 7.3.11

Solve by using the Quadratic Formula: $3p^2+2p+9=0$.

Solution:

	$3p^2+2p+9=0$
This equation is in standard form.	$\begin{array}{l} ax^2 + bx + c = 0 \\ 3p^2+2p+9=0 \end{array}$
Identify the values of a, b, c.	$a=3, b=2, c=9$
Write the Quadratic Formula.	$p=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$p=\frac{-(2) \pm \sqrt{(2)^2-4 \cdot 3 \cdot (9)}}{2 \cdot 3}$
Simplify.	$p=\frac{-2 \pm \sqrt{4-108}}{6}$
	$p=\frac{-2 \pm \sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2 \pm \sqrt{104}\, i}{6}$
Simplify the radical.	$p=\frac{-2 \pm 2\sqrt{26}\, i}{6}$
Factor the common factor in the numerator.	$p=\frac{2(-1 \pm \sqrt{26}\, i)}{6}$
Remove the common factors.	$p=\frac{-1 \pm \sqrt{26}\, i}{3}$

Rewrite in standard $a+bi$ form.	$p=-\frac{1}{3}\pm\frac{\sqrt{26}\,i}{3}$
Write as two solutions.	$p=-\frac{1}{3}+\frac{\sqrt{26}\,i}{3},\quad p=-\frac{1}{3}-\frac{\sqrt{26}\,i}{3}$

Try It 7.3.12

Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

Answer

$$a=\frac{1}{4}+\frac{\sqrt{31}}{4}i,\quad a=\frac{1}{4}-\frac{\sqrt{31}}{4}i$$

Try It 7.3.13

Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Answer

$$b=-\frac{1}{5}+\frac{\sqrt{19}}{5}i,\quad b=-\frac{1}{5}-\frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example 7.3.14

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution:

Our first step is to get the equation in standard form.

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2+6x+4=0$
Identify the values of a, b, c.	$a=1,\ b=6,\ c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$x=\frac{-(6)\pm\sqrt{(6)^2-4\cdot 1\cdot(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$
	$x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm 2\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm 2\sqrt{5}$
Write as two solutions.	$x=-3+2\sqrt{5},\quad x=-3-2\sqrt{5}$

Check: We leave the check for you!	

Try It 7.3.15

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$x=-1+\sqrt{6}, x=-1-\sqrt{6}$

Try It 7.3.16

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

Answer

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example 7.3.17

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$6\left(\frac{1}{2}u^2+\frac{2}{3}u\right)=6\left(\frac{1}{3}\right)$
Multiply.	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$ax^2+bx+c=0$ $3u^2+4u-2=0$
Identify the values of a, b, and c.	$a=3, b=4, c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$u=\frac{-(4)\pm\sqrt{(4)^2-4\cdot 3\cdot(-2)}}{2\cdot 3}$
Simplify.	$u=\frac{-4\pm\sqrt{16+24}}{6}$
	$u=\frac{-4\pm\sqrt{40}}{6}$
Simplify the radical.	$u=\frac{-4\pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u=\frac{2(-2\pm\sqrt{10})}{6}$
Remove the common factors.	$u=\frac{-2\pm\sqrt{10}}{3}$

Rewrite to show two solutions.	$u=\frac{-2+\sqrt{10}}{3}, \quad u=\frac{-2-\sqrt{10}}{3}$
Check: We leave the check for you!	

? Try It 7.3.18

Solve by using the Quadratic Formula: $\frac{1}{4}c^2-\frac{1}{3}c=\frac{1}{12}$.

Answer

$$c=\frac{2+\sqrt{7}}{3}, \quad c=\frac{2-\sqrt{7}}{3}$$

? Try It 7.3.19

Solve by using the Quadratic Formula: $\frac{1}{9}d^2-\frac{1}{2}d=-\frac{1}{3}$.

Answer

$$d=\frac{9+\sqrt{33}}{4}, d=\frac{9-\sqrt{33}}{4}$$

Think about the equation $(x-3)^2=0$. We know from the **Zero Product Property** that this equation has only one solution, $x=3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

✓ Example 7.3.20

Solve by using the Quadratic Formula: $4x^2-20x=-25$.

Solution:

	$4x^2-20x=-25$
Add 25 to get the equation in standard form.	$\begin{array}{l} ax^2+bx+c=0 \\ 4x^2-20x+25=0 \end{array}$
Identify the values of a, b, and c.	$a=4, b=-20, c=25$
Write the quadratic formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x=\frac{-(-20)\pm\sqrt{(-20)^2-4\cdot 4\cdot(25)}}{2\cdot 4}$
Simplify.	$x=\frac{20\pm\sqrt{400-400}}{8}$
	$x=\frac{20\pm\sqrt{0}}{8}$
Simplify the radical.	$x=\frac{20}{8}$
Simplify the fraction.	$x=\frac{5}{2}$

Check: We leave the check for you!	

Did you recognize that $4x^2-20x+25$ is a perfect square trinomial. It is equivalent to $(2x-5)^2$? If you solve $4x^2-20x+25=0$ by factoring and then using the Square Root Property, do you get the same result?

Try It 7.3.21

Solve by using the Quadratic Formula: $r^2+10r+25=0$.

Answer

$r=-5$

Try It 7.3.22

Solve by using the Quadratic Formula: $25t^2-40t=-16$.

Answer

$t=\frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 7.3.23

If $ax^2+bx+c=0$, the quantity

$$b^2-4ac$$

is called the **discriminant**. It is the radicand in the quadratic formula.

Let’s look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant b^2-4ac	Value of the Discriminant	Number and Type of Solutions
$2x^2+9x-5=0$	$9^2-4\cdot 2(-5)$ 121	$+$	2 real
	0	0	1 real
$3p^2+2p+9=0$	-104	$-$	2 complex

When the discriminant is positive, the quadratic equation has **2 real solutions**.	$x = \frac{-b \pm \sqrt{+}}{2a}$
When the discriminant is zero, the quadratic equation has **1 real solution**.	$x = \frac{-b \pm \sqrt{0}}{2a}$
When the discriminant is negative, the quadratic equation has **2 complex solutions**.	$x = \frac{-b \pm \sqrt{-}}{2a}$

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 7.3.24

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$

b. $5n^2 + n + 4 = 0$

c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

Substitute in the values of a, b, and c.

Simplify.

$$\begin{gathered} 49 + 108 \\ 157 \end{gathered}$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2 + n + 4 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 5, \quad b = 1, \quad c = 4$$

Write the discriminant.

Substitute in the values of a, b, and c.

Simplify.

$$\begin{gathered} 1 - 80 \\ -79 \end{gathered}$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

The equation is in standard form, identify a, b, and c.

$a=9, \quad b=-6, \quad c=1$

Write the discriminant.

Substitute in the values of a, b, and c.

Simplify.

$$\begin{gathered}36-36 \\ 0\end{gathered}$$

Since the discriminant is 0, there is 1 real solution to the equation.

? Try It 7.3.25

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$

b. $5z^2+6z-2=0$

c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions

b. 2 real solutions

c. 1 real solution

? Try It 7.3.26

Determine the number and type of solutions to each quadratic equation.

a. $b^2+7b-13=0$

b. $5a^2-6a+10=0$

c. $4r^2-20r+25=0$

Answer

a. 2 real solutions

b. 2 complex solutions

c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2=k$ or $a(x-h)^2=k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x-h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 7.3.27

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2 = 17$
b. $4x^2 - 12x + 9 = 0$
c. $8u^2 + 6u = 11$

Solution:

a.

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

b.

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$8u^2 + 6u = 11$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Try It 7.3.28

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Try It 7.3.29

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$

b. $4b^2+4b+1=0$
c. $5c^2=125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions to detect errors.
- Using the Discriminant, b^2-4ac, to Determine the Number and Type of Solutions of a Quadratic Equation
- For a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$,
  - If $b^2-4ac>0$, the equation has 2 real solutions.
  - If $b^2-4ac=0$, the equation has 1 real solution.
  - If $b^2-4ac<0$, the equation has 2 complex solutions.

Practice Makes Perfect

? Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4m^2+m-3=0$

2. $4n^2-9n+5=0$

3. $2p^2-7p+3=0$

4. $3q^2+8q-3=0$

5. $p^2+7p+12=0$

6. $q^2+3q-18=0$

7. $r^2-8r=33$

8. $t^2+13t=-40$

9. $3u^2+7u-2=0$

10. $2p^2+8p+5=0$

11. $2a^2-6a+3=0$

12. $5b^2+2b-4=0$

13. $x^2+8x-4=0$

14. $y^2+4y-4=0$

15. $3y^2+5y-2=0$

16. $6x^2+2x-20=0$

17. $2x^2+3x+3=0$

18. $2x^2-x+1=0$

19. $8x^2-6x+2=0$

20. $8x^2-4x+1=0$

21. $(v+1)(v-5)-4=0$

22. $(x+1)(x-3)=2$

23. $(y+4)(y-7)=18$

24. $(x+2)(x+6)=21$

25. $\frac{1}{4}m^2+\frac{1}{12}m=\frac{1}{3}$

26. $\frac{1}{3}n^2+n=-\frac{1}{2}$

27. $\frac{3}{4}b^2+\frac{1}{2}b=\frac{3}{8}$

28. $\frac{1}{9}c^2+\frac{2}{3}c=3$

29. $16c^2+24c+9=0$

30. $25d^2-60d+36=0$

31. $25q^2+30q+9=0$

32. $16y^2+8y+1=0$

Answer

1. $m=-1, m=\frac{3}{4}$

3. $p=\frac{1}{3}, p=2$

5. $p=-4, p=-3$

7. $r=-3, r=11$

9. $u=\frac{-7\pm\sqrt{73}}{6}$

11. $a=\frac{3\pm\sqrt{3}}{2}$

13. $x=-4\pm2\sqrt{5}$

15. $y=-\frac{2}{3}, y=-1$

17. $x=-\frac{3}{4}\pm\frac{\sqrt{15}}{4}i$

19. $x=\frac{3}{8}\pm\frac{\sqrt{7}}{8}i$

21. $v=2\pm2\sqrt{2}$

23. $y=-4, y=7$

25. $m=1, m=\frac{-4}{3}$

27. $b=\frac{-2\pm\sqrt{22}}{6}$

29. $c=-\frac{3}{4}$

31. $q=-\frac{3}{5}$

? Use the Discriminant to Predict the Number of Real Solutions of a Quadratic Equation

In the following exercises, determine the number of real solutions for each quadratic equation.

33. a. $4x^2-5x+16=0$
 b. $36y^2+36y+9=0$
 c. $6m^2+3m-5=0$
34. a. $9v^2-15v+25=0$
 b. $100w^2+60w+9=0$
 c. $5c^2+7c-10=0$
35. a. $r^2+12r+36=0$
 b. $8t^2-11t+5=0$
 c. $3v^2-5v-1=0$
36. a. $25p^2+10p+1=0$
 b. $7q^2-3q-6=0$
 c. $7y^2+2y+8=0$

Answer

33. a. no real solutions b. 1 c. 2

35. a. 1 b. no real solutions c. 2

? Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

37. a. $x^2-5x-24=0$
 b. $(y+5)^2=12$
 c. $14m^2+3m=11$
38. a. $(8v+3)^2=81$
 b. $w^2-9w-22=0$
 c. $4n^2-10=6$
39. a. $6a^2+14=20$
 b. $\left(x-\frac{1}{4}\right)^2=\frac{5}{16}$
 c. $y^2-2y=8$
40. a. $8b^2+15b=4$
 b. $\frac{5}{9}v^2-\frac{2}{3}v=1$
 c. $\left(w+\frac{4}{3}\right)^2=\frac{2}{9}$

Answer

37. a. Factor b. Square Root c. Quadratic Formula

39. a. Quadratic Formula b. Square Root c. Factor

Writing Exercises

41. Solve the equation $x^2 + 10x = 120$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?
42. Solve the equation $12y^2 + 23y = 24$
    a. by completing the square
    b. using the Quadratic Formula
    c. Which method do you prefer? Why?

Answer

41. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve quadratic equations using the quadratic formula.			
use the discriminant to predict the number of solutions of a quadratic equation.			
identify the most appropriate method to use to solve a quadratic equation.			

b. What does this checklist tell you about your mastery of this section? What steps will you take to improve?

7.4: Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Be Prepared

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
2. Solve $\dfrac{2}{x+1}+\dfrac{1}{x-1}=\dfrac{1}{x^2-1}$.
3. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As we solve each equation, we choose the method that is most convenient for us to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Use a Problem-Solving Strategy

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved a number of applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

$$\begin{array}{rl} & \text{Consecutive even integers} \\ & 64, 66, 68 \\ n & 1^{\text{st}} \text{ even integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer} \end{array}$$

$$\begin{array}{rl} & \text{Consecutive odd integers} \\ & 77, 79, 81 \\ n & 1^{\text{st}} \text{ odd integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer} \end{array}$$

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as we name the variables.

Example 7.4.1

The product of two consecutive odd integers is 195. Find the integers.

Solution

Read the problem.	
Identify what we are looking for.	We are looking for two consecutive odd integers.
Name what we are looking for.	Let $n =$ the first odd integer. Then (n+2=\) the next odd integer.
Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation.	$n(n+2)=195$
Solve the equation. Distribute.	$n^2+2n=195$
Write the equation in standard form.	$n^2+2n-195=0$
Factor.	$(n+15)(n-13)=0$
Use the Zero Product Property.	$n+15=0 \quad \text{or} \quad n-13=0$
Solve each equation.	$n=-15 \quad \text{or} \quad n=13$
There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.	$\begin{array}{cc} \text{First odd integer: } n=-15 & \text{First odd integer: } n=13 \\ \text{Next odd integer} & \text{Next odd integer} \\ n+2 & n+2 \\ =-15+2 & =13+2 \\ =-13 & =15 \end{array}$
Check the answer. Do these pairs work? Are they consecutive odd integers?	• $-15, -13$ are consecutive odd integers. Is their product 195? Yes: $-13(-15)=195$ • 13, 15 are consecutive odd integers. Is their product 195? Yes: $13 \cdot 15=195$
Answer the question.	Two consecutive odd integers whose product is 195 are $-15, -13$ and 13, 15.

Try It 7.4.2

The product of two consecutive odd integers is 99. Find the integers.

Answer

The two consecutive odd integers whose product is 99 are $9, 11$, and $-9, -11$.

Try It 7.4.3

The product of two consecutive even integers is 168. Find the integers.

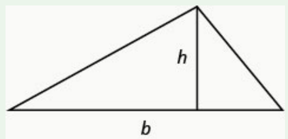
Answer

The two consecutive even integers whose product is 128 are $12, 14$ and $-12, -14$.

We will use the formula for the area of a triangle to solve the next example.

Area of a Triangle

For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

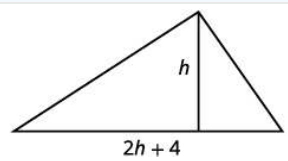


Recall that when we solve geometric applications, it is helpful to draw the figure.

Example 7.4.4

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution

Read the problem. Draw a picture.	
Identify what we are looking for.	We are looking for the base and height.
Name what we are looking for.	Let $h =$ the height of the triangle. Then $2h + 4 =$ the base of the triangle.
Translate into an equation. We know the area. Write the formula for the area of a triangle.	$A = \frac{1}{2}bh$
Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$h^2 + 2h - 120 = 0$
Factor.	$(h - 10)(h + 12) = 0$
Use the Zero Product Property.	$h - 10 = 0 \quad h + 12 = 0$
Simplify.	$h = 10, \quad h = \cancel{-12}$
	Since h is the height of a window, a value of $h = -12$ does not make sense. The height of the triangle $h = 10$. The base of the triangle $2h + 4$. $2 \cdot 10 + 4$ 24

Check the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Answer the question.	The height of the triangular window is 10 feet and the base is 24 feet.

? Try It 7.4.5

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

Answer

The height of the triangle is 12 inches and the base is 76 inches.

? Try It 7.4.6

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

Answer

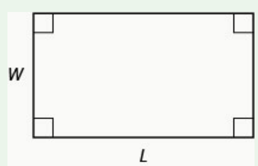
The height of the triangle is 11 feet and the base is 20 feet.

In the two preceding examples, the number in the radical in the **Quadratic Formula** was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

Area of a Rectangle

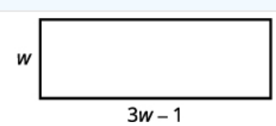
For a rectangle with length, L, and width, W, the area, A, is given by the formula $A = LW$.



? Example 7.4.7

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution

Read the problem. Draw a picture.	
Identify what we are looking for.	We are looking for the length and width.
Name what we are looking for.	Let $W =$ the width of the rectangle. Then $3W - 1 =$ the length of the rectangle

Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Solve the equation. Substitute in the values.	$150 = (3W - 1)W$
Distribute.	$150 = 3W^2 - W$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\underbrace{3W^2 - W - 150 = 0}_{ax^2+bx+c=0}$
Identify the a, b, c values.	$a = 3$ $b = -1$ $c = -150$
Write the Quadratic Formula.	$W = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$W = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$W = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $W = \frac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$W = \frac{1 + \sqrt{1801}}{6}, \quad \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2, \quad \cancel{w \approx -6.9}$ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Check the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Answer the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

? Try It 7.4.8

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

Answer

The length of the garden is approximately 18 feet and the width 11 feet.

? Try It 7.4.9

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot?

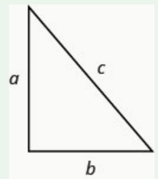
Answer

The length of the tablecloth is approximately 11.8feet and the width 6.8 feet.

The **Pythagorean Theorem** gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Pythagorean Theorem

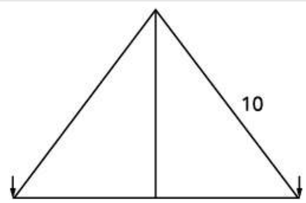
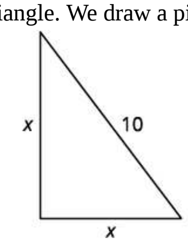
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2+b^2=c^2$.



Example 7.4.10

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution

Read the problem. Draw a picture.	
Identify what we are looking for.	We are looking for the height of the pole.
Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x=$ the height of the pole. Then $x=$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them. 
Translate into an equation. We can use the Pythagorean Theorem to solve for x. Write the Pythagorean Theorem.	$a^2+b^2=c^2$
Solve the equation. Substitute.	$x^2+x^2=10^2$
Simplify.	$2x^2=100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2}=\frac{100}{2}$
Simplify.	$x^2=50$
Use the Square Root Property.	$x=\pm\sqrt{50}$
Simplify the radical.	$x=\pm 5\sqrt{2}$
Rewrite to show two solutions.	$x=5\sqrt{2}$ or $x=\cancel{-5\sqrt{2}}$

	If we approximate $x = 5\sqrt{2}$ to the nearest tenth with a calculator, we find $x \approx 7.1$.
Check the answer. Check on your own in the Pythagorean Theorem.	
Answer the question.	The pole should be about 7.1 feet tall.

? Try It 7.4.11

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

Answer

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

? Try It 7.4.12

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Answer

The distance between the opposite corners is approximately 7.2 feet.

OPTIONAL APPLICATIONS

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Projectile Motion

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula

$$h = -16t^2 + v_0 t\ .$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

? Example 7.4.13

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution

Read the problem.	
Identify what we are looking for.	We are looking for the number of seconds, which is time.
Name what we are looking for.	Let $t =$ the number of seconds.
Translate into an equation. Use the formula.	$h = -16t^2 + v_0 t$ \)
Solve the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\underbrace{16t^2 - 130t + 260 = 0}_{ax^2+bx+c=0}$

Identify the values of a, b, c.	$a=16$ $b=-130$ $c=260$
Write the Quadratic Formula.	$t=\frac{-b \pm \sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c.	$t=\frac{-(-130) \pm \sqrt{(-130)^2-4 \cdot 16 \cdot (260)}}{2 \cdot 16}$ $t=\frac{-(-130) \pm \sqrt{(-130)^2-4 \cdot 16 \cdot 260}}{2 \cdot 16}$
Simplify.	$t=\frac{130 \pm \sqrt{16,900-16,640}}{32}$ $t=\frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t=\frac{130+\sqrt{260}}{32}, \quad t=\frac{130-\sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds $t \approx 3.6$ seconds
Check the answer. The check is left to you.	
Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Try It 7.4.14

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h=-16t^2+v_0t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

Answer

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

Try It 7.4.15

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h=-16t^2+v_0t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

Answer

The ball will reach 48 feet on its way up after approximately .6 second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D=rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate	• Time	= Distance

Figure 9.5.29

The formula $D=rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t=\frac{D}{r}$.

Some uniform motion problems are also modeled by quadratic equations.

Example 7.4.16

Professor Smith just returned from a conference that was $2,000$ miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

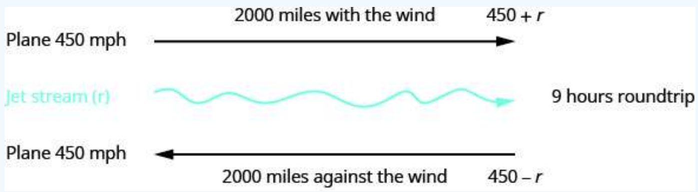


Figure 9.5.30

We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Write in the rates.
Write in the distances.
Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.

Type	Rate	·	Time	=	Distance
Headwind	$450-r$		$\frac{2000}{450-r}$		2000
Tailwind	$450+r$		$\frac{2000}{450+r}$		2000
			9		

We know the times add to 9 and so we write our equation.	$\frac{2000}{450-r} + \frac{2000}{450+r} = 9$
We multiply both sides by the LCD.	$(450-r)(450+r)\left(\frac{2000}{450-r} + \frac{2000}{450+r}\right) = 9(450-r)(450+r)$
Simplify.	$2000(450+r) + 2000(450-r) = 9(450-r)(450+r)$
Factor the $2,000$.	$2000(450+r+450-r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ \ $50 = r$ The speed of the jet stream is 50 mph.

Check:	Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450+50=500\text{mph} \quad \frac{2000}{500}=4 \text{ hours}$ Headwind $450-50=400\text{mph} \quad \frac{2000}{400}=5 \text{ hours}$ The times add to 9 hours, so it checks.
Answer the question.	The speed of the jet stream was 50 mph.

? Try It 7.4.17

MaryAnne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 100 mph.

? Try It 7.4.18

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

? Example 7.4.19

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Let $x=$ the number of hours for Press #2 to complete the job. Enter the hours per job for Press #1, Press #2, and when they work together.	see table below
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together. Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$

	Number of hours needed to complete the job.	Part of job completed/hour
Press #1	$x+12$	$\frac{1}{x+12}$
Press #2	x	$\frac{1}{x}$
Together	8	$\frac{1}{8}$

Solve.	$\frac{1}{x+12}+\frac{1}{x}=\frac{1}{8}$
Multiply by the LCD, $8x(x+12)$.	$8x(x+12)\left(\frac{1}{x+12}+\frac{1}{x}\right)=\left(\frac{1}{8}\right)8x(x+12)$
Simplify.	$8x+8(x+12)=x(x+12)$ $8x+8x+96=x^2+12x$ Figure 9.5.37 $0=x^2-4x-96$ Figure 9.5.38
Solve.	$0=(x-12)(x+8)$ $x-12=0, x+8=0$ Figure 9.5.40 $x=12, \cancel{x=-8}$ hours Figure 9.5.41
Since the idea of negative hours does not make sense, we use the values $x=12$.	$12+12$ 12 24 hours 12 hours Figure 9.5.43
Write our sentence answer.	Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

? Try It 7.4.20

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Answer

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

? Try It 7.4.21

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

Answer

The red hose take 6 hours and the green hose take 3 hours alone.

Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
  5. **Solve** the equation using good algebra techniques.

6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

- Area of a Triangle
  - For a triangle with base, b, and height, h, the area, A, is given by the formula $A=\frac{1}{2}bh$.

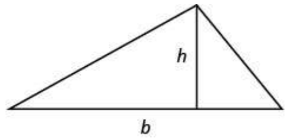


Figure 9.5.1

- Area of a Rectangle
  - For a rectangle with length,L, and width, W, the area, A, is given by the formula $A=LW$.

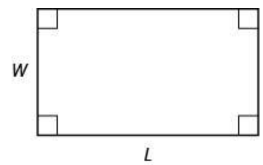


Figure 9.5.3

- Pythagorean Theorem
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2+b^2=c^2$.

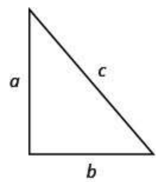


Figure 9.5.16

- Projectile motion
  - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h=-16t^2+v_0t$.

Practice Makes Perfect

Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method.

1. The product of two consecutive odd numbers is 255. Find the numbers.
2. The product of two consecutive even numbers is 360. Find the numbers.
3. The product of two consecutive even numbers is 624. Find the numbers.
4. The product of two consecutive odd numbers is $1,023$. Find the numbers.
5. The product of two consecutive odd numbers is 483. Find the numbers.
6. The product of two consecutive even numbers is 528. Find the numbers.

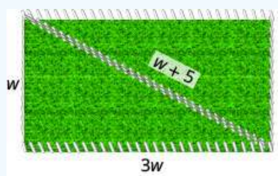
Answer

1. Two consecutive odd numbers whose product is 255 are 15 and 17, and -15 and -17.
3. The first and second consecutive odd numbers are 24 and 26, and -26 and -24.
5. Two consecutive odd numbers whose product is 483 are 21 and 23, and -21 and -23.

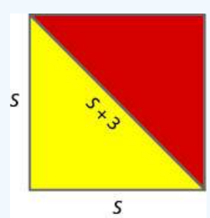
Solve Applications Modeled by Quadratic Equations

In the following exercises, solve using any method. Round your answers to the nearest tenth, if needed.

7. A triangle with area 45 square inches has a height that is two less than four times the base Find the base and height of the triangle.
8. The base of a triangle is six more than twice the height. The area of the triangle is 88 square yards. Find the base and height of the triangle.
9. The area of a triangular flower bed in the park has an area of 120 square feet. The base is 4 feet longer that twice the height. What are the base and height of the triangle?
10. A triangular banner for the basketball championship hangs in the gym. It has an area of 75 square feet. What is the length of the base and height , if the base is two-thirds of the height?
11. The length of a rectangular driveway is five feet more than three times the width. The area is 50 square feet. Find the length and width of the driveway.
12. A rectangular lawn has area 140 square yards. Its length is six yards less than twice its width. What are the length and width of the lawn?
13. A rectangular table for the dining room has a surface area of 24 square feet. The length is two more feet than twice the width of the table. Find the length and width of the table.
14. The new computer has a surface area of 168 square inches. If the the width is 5.5 inches less that the length, what are the dimensions of the computer?
15. The hypotenuse of a right triangle is twice the length of one of its legs. The length of the other leg is three feet. Find the lengths of the three sides of the triangle.
16. The hypotenuse of a right triangle is 10 cm long. One of the triangle's legs is three times as the length of the other leg . Round to the nearest tenth. Find the lengths of the three sides of the triangle.
17. A rectangular garden will be divided into two plots by fencing it on the diagonal. The diagonal distance from one corner of the garden to the opposite corner is five yards longer than the width of the garden. The length of the garden is three times the width. Find the length of the diagonal of the garden.



18. Nautical flags are used to represent letters of the alphabet. The flag for the letter, O consists of a yellow right triangle and a red right triangle which are sewn together along their hypotenuse to form a square. The hypotenuse of the two triangles is three inches longer than a side of the flag. Find the length of the side of the flag.



19. Gerry plans to place a 25-foot ladder against the side of his house to clean his gutters. The bottom of the ladder will be 5 feet from the house.How for up the side of the house will the ladder reach?

20. John has a 10-foot piece of rope that he wants to use to support his 8-foot tree. How far from the base of the tree should he secure the rope?

21. An arrow is shot vertically upward at a rate of $v_0 = 220$ feet per second. Use the projectile formula $h = -16t^2 + v_0 t$, to determine when the height of the arrow will be 400 feet.

22. A firework rocket is shot upward at a rate of 640 ft/sec. Use the projectile formula $h = -16t^2 + v_0 t$ to determine when the height of the firework rocket will be 1200 feet.

23. A bullet is fired straight up from a BB gun with initial velocity 1120 feet per second at an initial height of 8 feet. Use the formula $h=-16t^2+v_0t+8$ to determine how many seconds it will take for the bullet to hit the ground. (That is, when will $h=0$?)

24. A stone is dropped from a 196-foot platform. Use the formula $h=-16t^2+v_0t+196$ to determine how many seconds it will take for the stone to hit the ground. (Since the stone is dropped, $v_0=0$.)

25. The businessman took a small airplane for a quick flight up the coast for a lunch meeting and then returned home. The plane flew a total of 4 hours and each way the trip was 200 miles. What was the speed of the wind that affected the plane which was flying at a speed of 120 mph?

26. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 300 miles. If the plane was flying at 125 mph, what was the speed of the wind that affected the plane?

27. Roy kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

28. Rick paddled up the river, spent the night camping, and then paddled back. He spent 10 hours paddling and the campground was 24 miles away. If Rick kayaked at a speed of 5 mph, what was the speed of the current?

29. Two painters can paint a room in 2 hours if they work together. The less experienced painter takes 3 hours more than the more experienced painter to finish the job. How long does it take for each painter to paint the room individually?

30. Two gardeners can do the weekly yard maintenance in 8 minutes if they work together. The older gardener takes 12 minutes more than the younger gardener to finish the job by himself. How long does it take for each gardener to do the weekly yard maintenance individually?

31. It takes two hours for two machines to manufacture $10,000$ parts. If Machine #1 can do the job alone in one hour less than Machine #2 can do the job, how long does it take for each machine to manufacture $10,000$ parts alone?

32. Sully is having a party and wants to fill his swimming pool. If he only uses his hose it takes 2 hours more than if he only uses his neighbor's hose. If he uses both hoses together, the pool fills in 4 hours. How long does it take for each hose to fill the pool?

Answer

7. The width of the triangle is 5 inches and the height is 18 inches.
9. The base is 24 feet and the height of the triangle is 10 feet.
11. The length of the driveway is 15.0 feet and the width is 3.3 feet.
13. The length of table is 8 feet and the width is 3 feet.
15. The length of the legs of the right triangle are 3.2 and 9.6 cm.
17. The length of the diagonal fencing is 7.3 yards.
19. The ladder will reach 24.5 feet on the side of the house.
21. The arrow will reach 400 feet on its way up in 2.2 seconds and on the way down in 11.6 seconds.
23. The bullet will take 70 seconds to hit the ground.
25. The speed of the wind was 49 mph.
27. The speed of the current was 4.3 mph.
29. The less experienced painter takes 6 hours and the experienced painter takes 3 hours to do the job alone.
31. Machine #1 takes 3.6 hours and Machine #2 takes 4.6 hours to do the job alone.

? Writing exercises

33. Make up a problem involving the product of two consecutive odd integers.

a. Start by choosing two consecutive odd integers. What are your integers?
b. What is the product of your integers?

c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
d. Did you get the numbers you started with?

34. Make up a problem involving the product of two consecutive even integers.
    a. Start by choosing two consecutive even integers. What are your integers?
    b. What is the product of your integers?
    c. Solve the equation $n(n+2)=p$, where p is the product you found in part (b).
    d. Did you get the numbers you started with?

Answer

33. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve applications of the quadratic formula.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

7.5: Graph Quadratic Equations Using Properties

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic equation in 2-variables
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic equations using properties
- Solve maximum and minimum applications

Be Prepared

Before you get started, take this readiness quiz.

1. Graph the equation $y=x^2$ by plotting points.
2. Solve: $2x^2+3x-2=0$.
3. Evaluate $-\frac{b}{2a}$ when $a=3$ and $b=-6$.

Recognize the Graph of a Quadratic Equation (in 2 variables)

Previously we very briefly looked at the equation $y=x^2$. It was one of the first non-linear equations we looked at. Now we will graph equations of the form $y=ax^2+bx+c$ if $a\neq 0$. We call the kind of expression on the right hand side a quadratic expression.

Definition 7.5.1

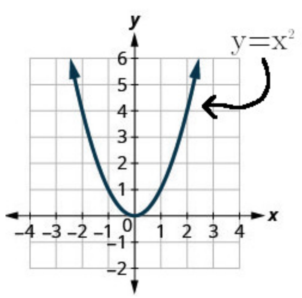
Let a, b, and c be real numbers with $a\neq 0$. We call

$$ax^2+bx+c\ ,$$

a **quadratic expression** (in x). We will call an equation $y=ax^2+bx+c$ a **quadratic equation (in two variables)**, or more specifically, an **equation quadratic in x.**

Recall that the graph of an equation is a picture of the solutions we get by using an identification of solutions of the equation with points on a coordinate plane. This is ideal in numerous ways, for example, our drawing instruments are not precise or our space is limited (which it always is!). In what follows (as we did with lines before) we will be happy to capture certain features of the graph.

We graphed the quadratic equation $y=x^2$ by plotting points.



x	$y=x^2$	(x,y)
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

We call this figure a **parabola**. Parabolas are figures that can be squeezed vertically and horizontally, flipped or rotated, and translated so as to lay directly on the one above. We will look at a few special types. Let’s practice graphing a parabola by plotting a few points.

Example 7.5.2

Graph $y = x^2 - 1$.

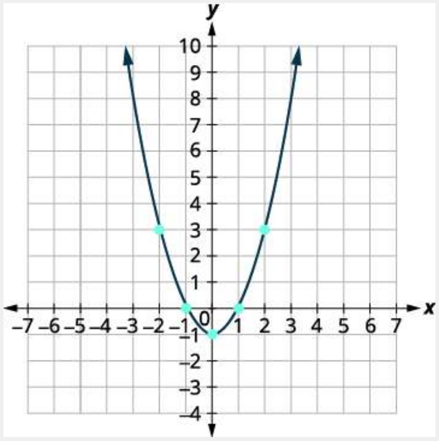
Solution

We will graph the equation by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find y. Record the values of the ordered pairs in the chart.

$y = x^2 - 1$	
x	y
0	–1
1	0
–1	0
2	3
–2	3

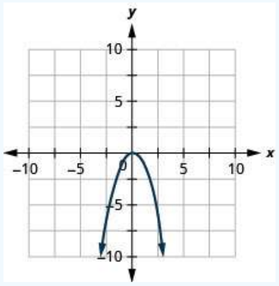
Plot the points, and then connect them with a smooth curve. The result will be the graph of the equation $y = x^2 - 1$.



Try It 7.5.3

Graph $y = -x^2$.

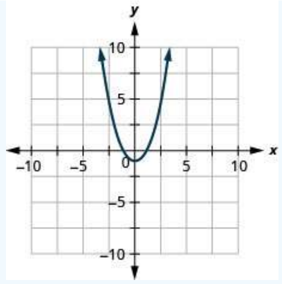
Answer



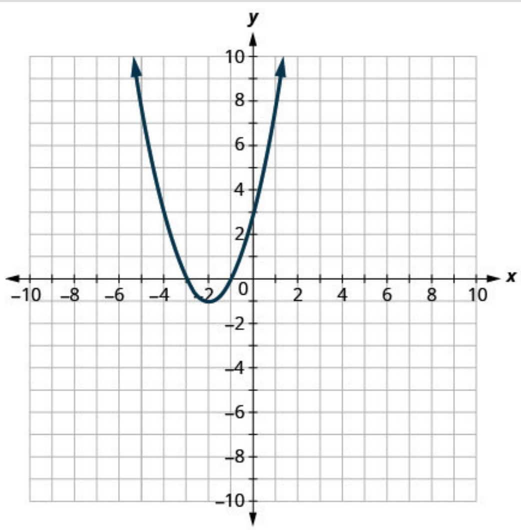
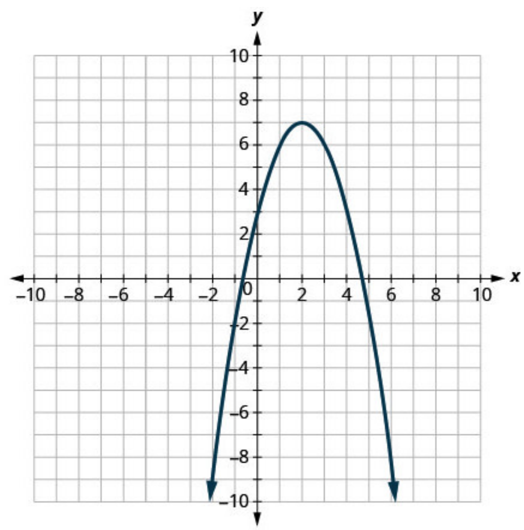
Try It 7.5.4

Graph $y = x^2 + 1$.

Answer



All graphs of quadratic equations of the form $y = ax^2 + bx + c$ are parabolas that open upward or downward.

	
$y = ax^2 + bx + c$ $y = x^2 + 4x + 3$ $a > 0$ opens upward	$y = ax^2 + bx + c$ $y = -x^2 + 4x + 3$ $a < 0$ opens downward

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in *Figure 9.6.6*). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Definition 7.5.5

Parabola Orientation

For the graph of the quadratic equation $y = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward 
- $a < 0$, the parabola opens downward 

Example 7.5.6

Determine whether each parabola described by the following opens upward or downward:

a. $y = -3x^2 + 2x - 4$

b. $y = 6x^2 + 7x - 9$

Solution

a. Find the value of a.

Comparing $y = ax^2 + bx + c$ to $y = -3x^2 + 2x - 4$, we see that $a = -3$, which is negative. The associated parabola opens downward.

b. Find the value of a.

Comparing $y = ax^2 + bx + c$ to $y = 6x^2 + 7x - 9$, we see that $a = 6$, which is positive. The associated parabola opens upward.

? Try It 7.5.7

Determine whether each parabola described by the following opens upward or downward:

a. $y = 2x^2 + 5x - 2$

b. $y = -3x^2 - 4x + 7$

Answer

a. up

b. down

? Try It 7.5.8

Determine whether each parabola described by the following opens upward or downward:

a. $y = -2x^2 - 2x - 3$

b. $y = 5x^2 - 2x - 1$

Answer

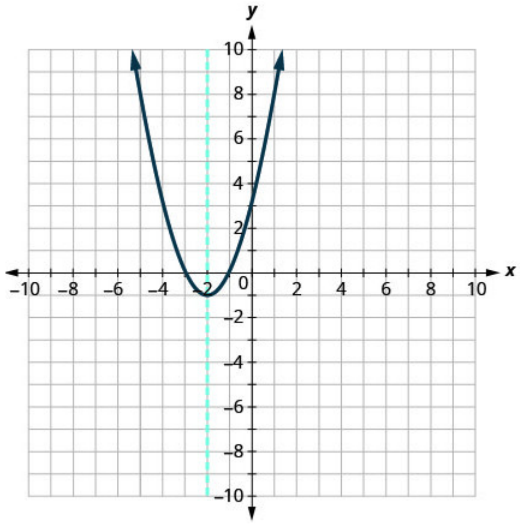
a. down

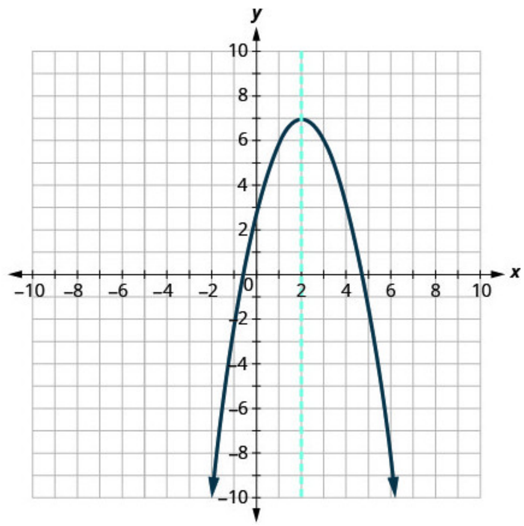
b. up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The ‘fold line’ is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.





$y=x^2+4x+3$	$y=-x^2+4x+3$

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $y=ax^2+bx+c$ is $x=-\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x=-\frac{b}{2a}$.

$y=ax^2+bx+c$	$y=ax^2+bx+c$
$y=x^2+4x+3$	$y=-x^2+4x+3$
$x=-\frac{b}{2a}$	$x=-\frac{b}{2a}$
$x=-\frac{4}{2\cdot 1}$	$x=-\frac{4}{2(-1)}$
$x=-2$	$x=2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic equation.

$y=x^2+4x+3$	$y=-x^2+4x+3$
axis of symmetry is $x=-2$	axis of symmetry is $x=2$
vertex is $\boxed{(-2, \underline{\quad})}$	vertex is $\boxed{(2, \underline{\quad})}$
$y=x^2+4x+3$	$y=-x^2+4x+3$
$y=(-2)^2+4(-2)+3$	$y=-(2)^2+4(2)+3$
$y=-1$	$y=7$
vertex is $(-2,-1)$	vertex is $(2,7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the equation $y=ax^2+bx+c$ is a parabola where:

- the axis of symmetry is the vertical line $x=-\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
- the y-coordinate of the vertex is found by substituting $x=-\frac{b}{2a}$ into the quadratic equation.

Example 7.5.9

For the graph of $y=3x^2-6x+2$ find:

a. the axis of symmetry

b. the vertex

Solution

a.

	$a=3, b=-6, c=2$
The axis of symmetry is the vertical line $x=-\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x=-\frac{-6}{2\cdot 3}$
Simplify.	$x=1$
	The axis of symmetry is the line $x=1$.

b.

The vertex is a point on the line of symmetry, so its x-coordinate will be $x=1$. Find the y-coordinate so that $(1, y$ is a solution.	$y=3(1)^2-6(1)+2$
Simplify.	$y=3\cdot 1-6+2=-1$
The result is the y-coordinate.	$y=-1$
	$(1,-1)$ is the solution of the equation on the axis of symmetry. The vertex is $(1,-1)$.

Try It 7.5.10

For the graph of $y=2x^2-8x+1$ find:

a. the axis of symmetry

b. the vertex

Answer

a. $x=2$

b. $(2,-7)$

? Try It 7.5.11

For the graph of $y=2x^2-4x-3$ find:

a. the axis of symmetry

b. the vertex

Answer

a. $x=1$

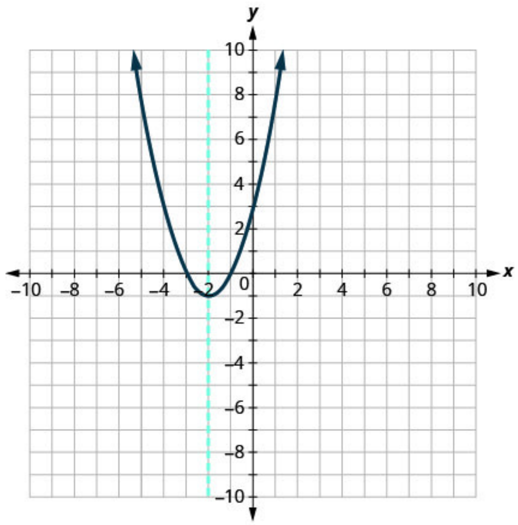
b. $(1,-5)$

Find the Intercepts of a Parabola

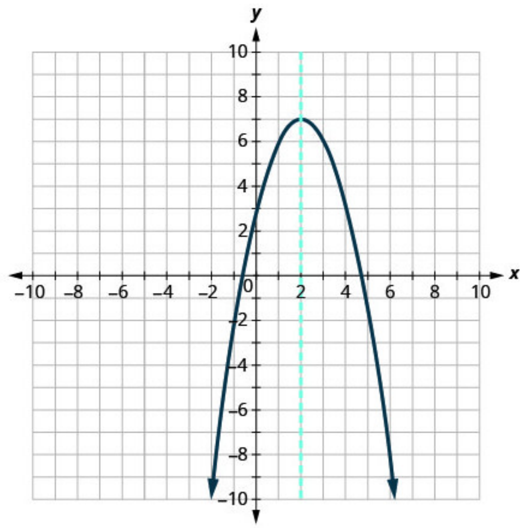
When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x=0$ into the equation.

Let's find the y-intercepts of the two parabolas shown in *Figure 9.6.20*.



$y=x^2+4x+3$
Look for a solution of the form $(0,b)$.
$b=0^2+4\cdot 0+3$
$b=3$
The y-intercept is $(0,3)$.



$y=x^2+4x+3$
Look for a solution of the form $(0,b)$.
$b=-0^2+4\cdot 0+3$
$b=3$
The y-intercept is $(0,3)$.

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x)=0$. In other words, we will need to solve the equation $0=ax^2+bx+c$ for x.

$$y=ax^2+bx+c$$
$$0=ax^2+bx+c$$

Solving quadratic equations like this (one variable) is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose equation is $y=x^2+4x+3$.

	$y = x^2 + 4x + 3$
Let $y = 0$.	$0 = x^2 + 4x + 3$
Factor.	$0 = (x+1)(x+3)$
Use the Zero Product Property.	$x + 1 = 0 \quad x + 3 = 0$
Solve.	
	The x-intercepts are $(-1, 0)$ and $(-3, 0)$.

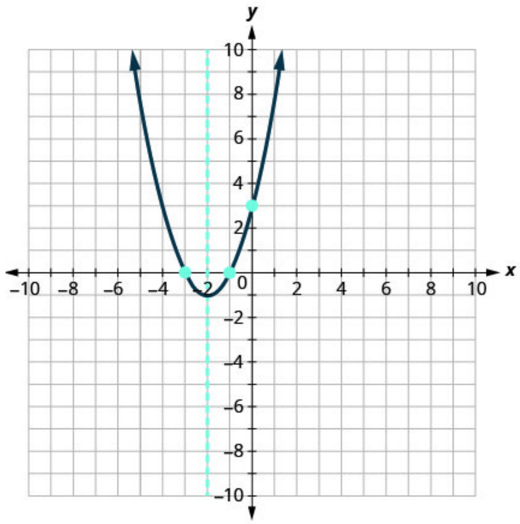
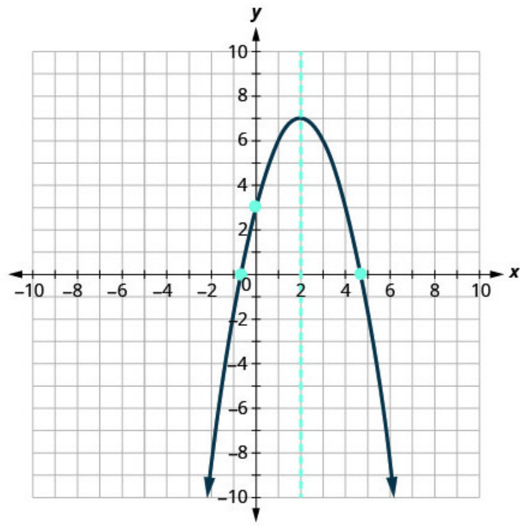
Now we will find the x-intercepts of the parabola whose equation is $y = -x^2 + 4x + 3$.

	$y = -x^2 + 4x + 3$
Let $y = 0$.	$0 = -x^2 + 4x + 3$
This quadratic does not factor, so we use the Quadratic Formula.	$a = -1, b = 4, c = 3$
Simplify.	
	$x = \frac{-2(2 \pm \sqrt{7})}{-2}$
	$x = 2 \pm \sqrt{7}$
	The x-intercepts are $(2 + \sqrt{7}, 0)$ and $(2 - \sqrt{7}, 0)$.

We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2 + \sqrt{7}, 0) \approx (4.6, 0) \quad (2 - \sqrt{7}, 0) \approx (-0.6, 0)$$

Do these results agree with our graphs? See *Figure 9.6.34*

	
$y=x^2+4x+3$ y-intercept is $(0,3)$ x-intercepts are $(-1,0)$ and $(-3,0)$	$y=-x^2+4x+3$ y-intercept is $(0,3)$ x-intercepts are $(2+\sqrt{7},0)\approx(4.6,0)$ and $(2-\sqrt{7},0)\approx(-0.6,0)$

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose equation is $y=ax^2+bx+c$:

y-intercept

Let $x=0$ and solve for y.

x-intercepts

Let $y=0$ and solve for x

Example 7.5.12

Find the intercepts of the parabola whose equation is $y=x^2-2x-8$.

Solution

To find the y-intercept, let $x=0$ and solve for y.	
	$y=-8$
	When $x=0$, then $y=-8$. The y-intercept is the point $(0,-8)$.
To find the x-intercept, let $y=0$ and solve for x.	
Solve by factoring.	$0=(x-4)(x+2)$
	$4=x \quad -2=x$

	When $y=0$, then $x=4$ or $x=-2$. The x-intercepts are the points $(4,0)$ and $(-2,0)$.

Try It 7.5.13

Find the intercepts of the parabola whose equation is $y=x^2+2x-8$.

Answer

y -intercept: $(0,-8)$

x -intercepts: $(-4,0),(2,0)$

Try It 7.5.14

Find the intercepts of the parabola whose equation is $y=x^2-4x-12$.

Answer

y -intercept: $(0,-12)$

x -intercepts: $(-2,0),(6,0)$

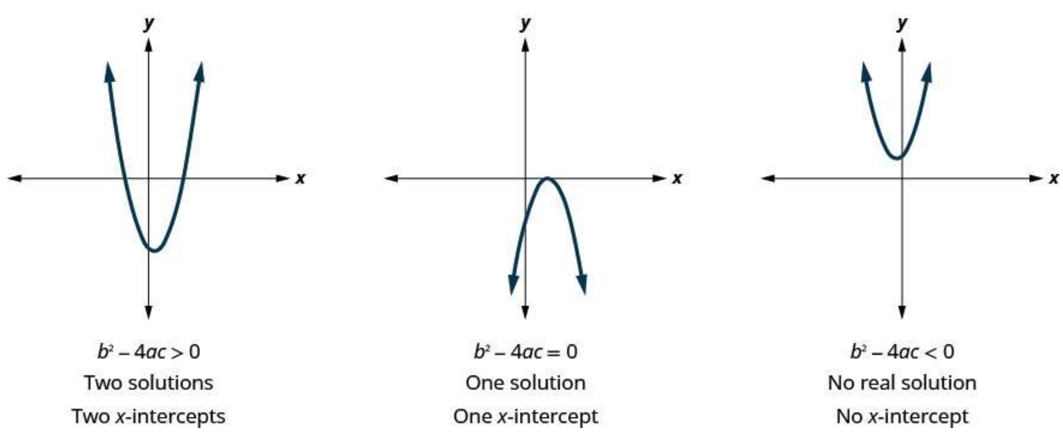
In this chapter, we have been solving quadratic equations of the form $ax^2+bx+c=0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic equations of the form $y=ax^2+bx+c$. The graphs of these equations are parabolas. The x-intercepts of the parabolas occur where $y=0$.

The solutions of the quadratic equation resulting from replacing y with 0 are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have $2, 1$, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the **discriminant** to determine the number of solutions of a quadratic function of the form $ax^2+bx+c=0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



Before you to find the values of the x-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example 7.5.15

Find the intercepts of the parabola for the equation $y=5x^2+x+4$.

Solution

		$y=5x^2+x+4$.	
To find the y-intercept, let $x=0$ and solve for y.		$y=50^2+0+4$.	
		$y=4$	
		When $x=0$, then $y=4$. The y-intercept is the point $(0,4)$.	
To find the x-intercept, let $f(x)=0$ and solve for x.		$0=5x^2+x+4$.	
Find the value of the discriminant to predict the number of solutions which is also the number of x-intercepts.		$a=5, b=1, c=4$ b^2-4ac $=1^2-4\cdot5\cdot4=-79$	
		Since the value of the discriminant is negative, there is no real solution to the equation. There are no x-intercepts.	

Try It 7.5.16

Find the intercepts of the parabola whose equation is $y=3x^2+4x+4$.

Answer

y-intercept: $(0,4)$

no x-intercept

Try It 7.5.17

Find the intercepts of the parabola whose equation is $y=x^2-4x-5$.

Answer

y-intercept: $(0,-5)$

x-intercepts: $(-1,0)$, $(5,0)$

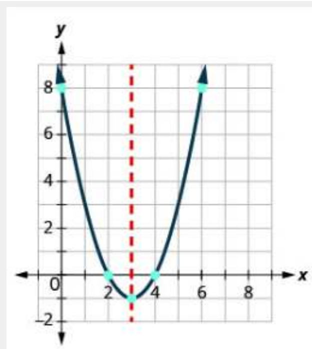
Graph Quadratic Equations Using Properties

Now we have all the pieces we need in order to graph a quadratic equation. We just need to put them together. In the next example we will see how to do this.

Example 7.5.18

Graph $y=x^2-6x+8$ by using its properties.

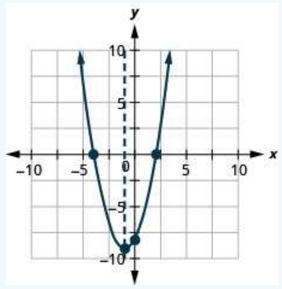
Solution

Determine whether the parabola opens upward or downward.	Look a in the equation $y=x^2-6x+8$ Since a is positive, the parabola opens upward.	$y=x^2-6x+8$ $a=1, b=-6, c=8$ **The parabola opens upward.**
Find the axis of symmetry.	$y=x^2-6x+8$ The axis of symmetry is the line $x=-\frac{b}{2a}$.	Axis of Symmetry $x=-\frac{b}{2a}$ $x=-\frac{(-6)}{2\cdot 1}$ $x=3$ **The axis of symmetry is the line** $x=3$**.**
Find the vertex.	The vertex is on the axis of symmetry. Substitute $x=3$ into the function.	Vertex $y=x^2-6x+8$ the y-coordinate of the vertex $=(3)^2-6(3)+8$ $=-1$ **The vertex is** $(3,-1)$**.**
Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find solutions of the form $(0,y)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x=3$. A point 3 units to the right of the axis of symmetry has $x=6$.	y-intercept The y-coordinate of the y-intercept $=(0)^2-6(0)+8$ $=8$ **The y-intercept is** $(0,8)$**.** The point symmetric to the y-intercept: **The point is** $(6,8)$**.**
Find the x-intercepts. Find additional points if needed.	We look for solutions of the form $(x,0)$. We can solve this quadratic equation by factoring.	x-intercepts We substitute 0 for y in the equation: $0=x^2-6x+8$ $0=(x-2)(x-4)$ $x=2 or x=4$ **The x-intercepts are** $(2,0)$ **and** $(4,0)$**.**
Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Try It 7.5.19

Graph $y=x^2+2x-8$ by using its properties.

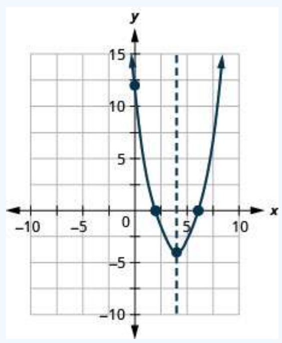
Answer



Try It 7.5.20

Graph $y=x^2-8x+12$ by using its properties.

Answer



We list the steps to take in order to graph a quadratic function here.

To Graph a Quadratic Function Using Properties

1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. Find additional points if needed.
6. Graph the parabola.

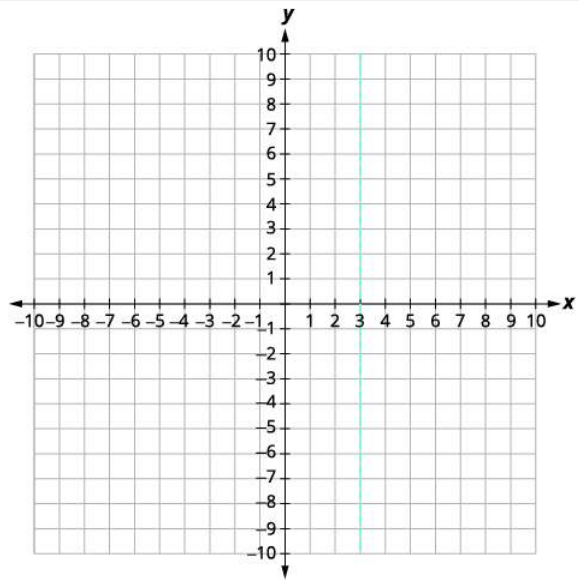
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

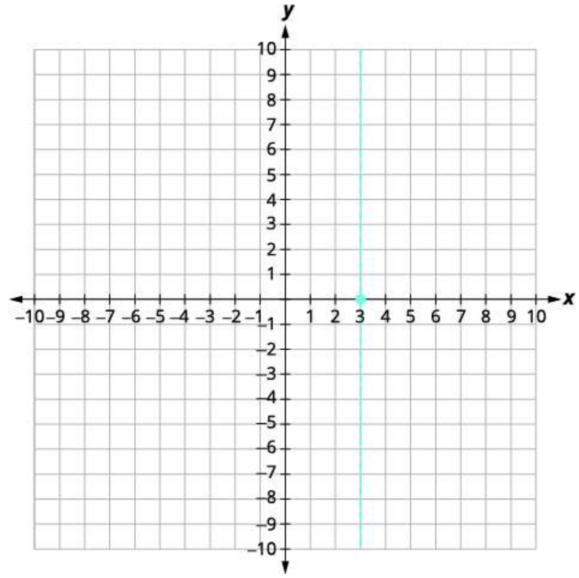
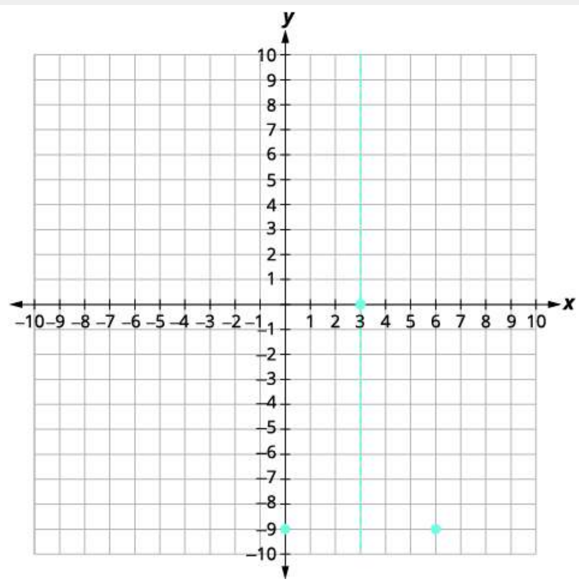
Example 7.5.21

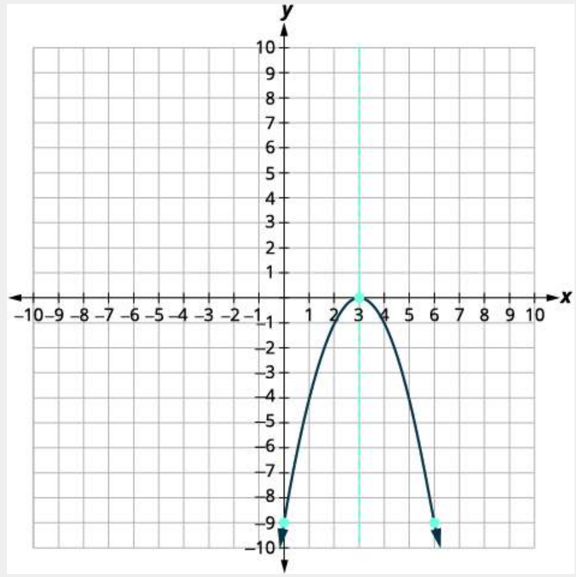
Graph $y=x^2+6x-9$ by using its properties.

Solution

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{6}{2(-1)}$
	$x=3$
	The axis of symmetry is $x=3$. The vertex is on the line $x=3$.
	
Find a solution of the form $(3, y)$.	$y=-x^2+6x-9$
	$y=-3^2+6\cdot 3-9$
	$y=-9+18-9$
	$y=0$
	The vertex is $(3, 0)$.

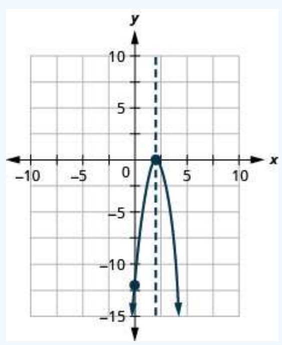
	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
	
The y-intercept occurs when $x = 0$. Find a solution of the form $(0, y)$.	$y = -x^2 + 6x - 9$
xSubstitute $x = 0$.	$y = -0^2 + 6 \cdot 0 - 9$
Simplify.	$y = -9$
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $y = 0$. So we look for a solution of the form $x, 0)$	$y = -x^2 + 6x - 9$
Find $y = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$x = 3$

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Connect the points to graph the parabola.	

? Try It 7.5.22

Graph $y = 3x^2 + 12x - 12$ by using its properties.

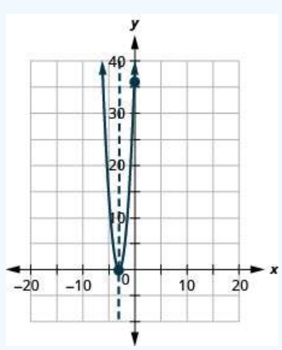
Answer



? Try It 7.5.23

Graph $y = 4x^2 + 24x + 36$ by using its properties.

Answer



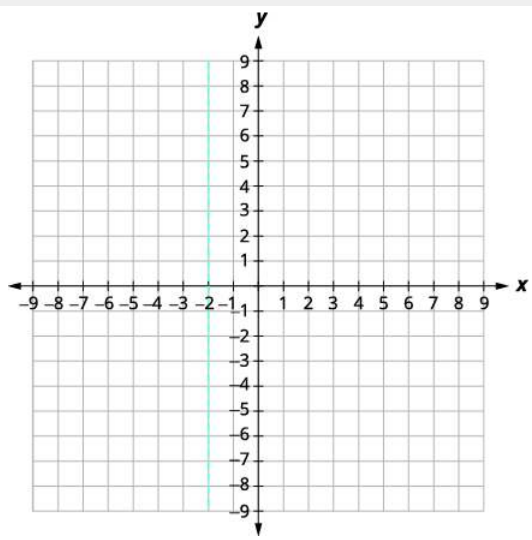
For the graph of $y=-x^2+6x-9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0=-x^2+6x-9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

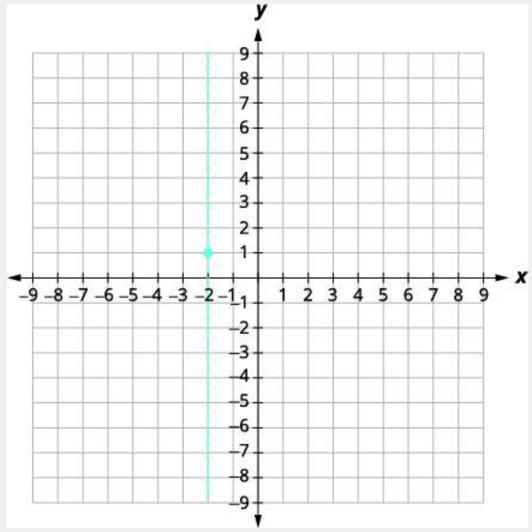
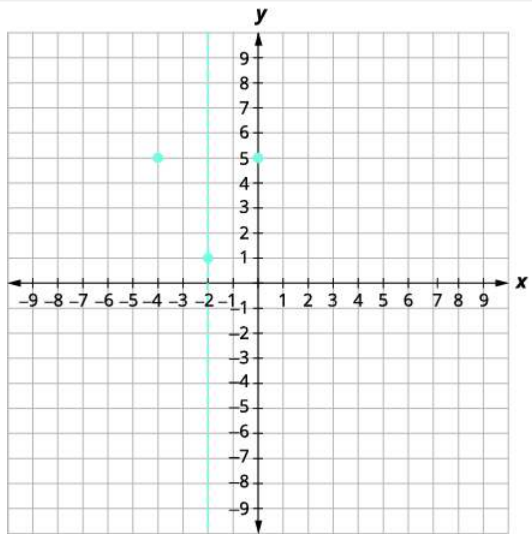
How many x-intercepts would you expect to see on the graph of $f(x)=x^2+4x+5$?

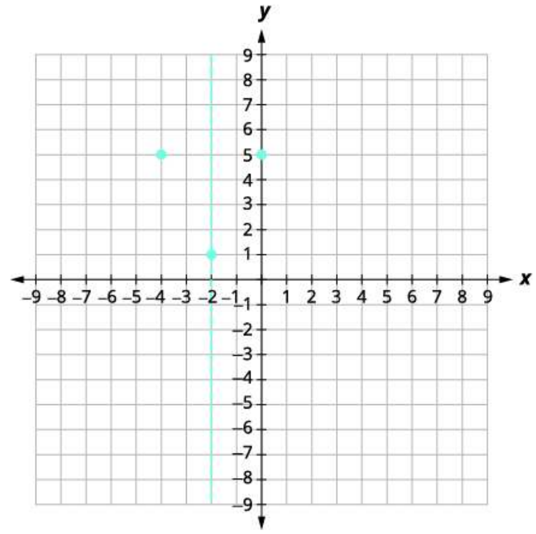
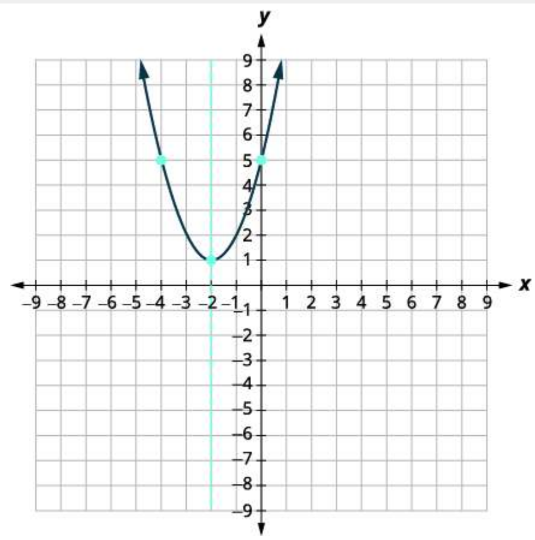
? Example 7.5.24

Graph $y=x^2+4x+5$ by using its properties.

Solution

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{4}{(2)1}$
	$x=-2$
	The equation of the axis of symmetry is $x=-2$.
	
The vertex is on the line $x=-2$.	
Find y when $x=-2$.	$y=x^2+4x+5$
	$y=(-2)^2+4(-2)+5$
	$y=4-8+5$
	$y=1$
	The vertex is $(-2,1)$.

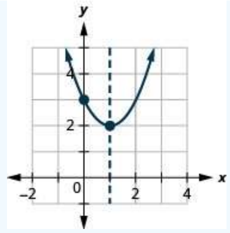
	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
	
The y-intercept occurs when $x=0$.	$y=x^2+4x+5$
Find the solution of the form $(0,y)$.	$y=0^2+4(0)+5$
Simplify.	$y=5$
	The y-intercept is $(0,5)$.
The point $(-4,5)$ is two units to the left of the line of symmetry. The point to units to the right of the line of symmetry is \((0,5)\.	
	Point symmetric to the y-intercept is $(-4,5)$.

	$f(x) = ax^2 + bx + c$ $f(x) = -x^2 + 6x - 9$
The x-intercept occurs when $y = 0$.	
Find the solution of the form $(x, 0)$.	$0 = x^2 + 4x + 5$
Test the discriminant.	$a = 1, b = 4, c = 5$
	$b^2 - 4ac$
	$4^2 - 4 \cdot 1 \cdot 5$
	$16 - 20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy. For example, you could try to find a solution of the form $(-1, y)$ which would lead to a solution $(-1, 2$.	

? Try It 7.5.25

Graph $y = x^2 - 2x + 3$ by using its properties.

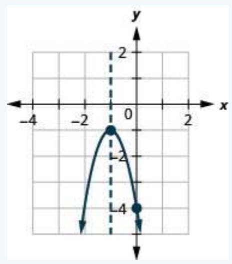
Answer



Try It 7.5.26

Graph $y=-3x^2-6x-4$ by using its properties.

Answer



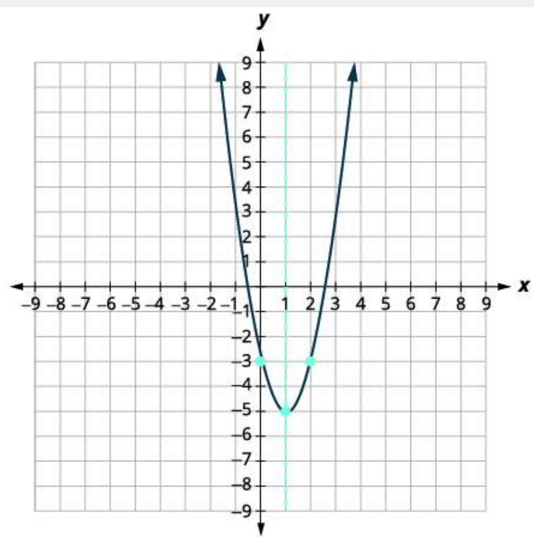
Finding the y-intercept of the graph of an equation that looks like $y=ax^2+bx+c$ is easy, isn't it? Finding the x-intercepts of such a graph can be more challenging. Sometimes we need to use the **Quadratic Formula**!

Example 7.5.27

Graph $y=2x^2-4x-3$ by using its properties.

Solution

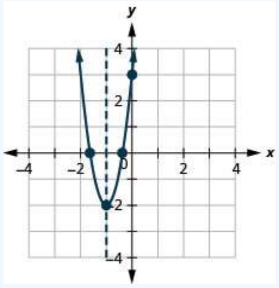
	$f(x) = ax^2 + bx + c$ $f(x) = 2x^2 - 4x - 3$
Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{-4}{2\cdot 2}$
	$x=1$
	The equation of the axis of symmetry is $x=1$.
The vertex is on the line $x=1$.	
Find a solution of the form $(1,y)$.	$y=21^2-4\cdot 1-3$
	$y=2-4-3$
	\(y=-5)
	The vertex is $(1,-5)$.
The y-intercept occurs when $x=0$.	
Find a solution of the form $(0,y)$.	$y=20^2-4\cdot 0-3$
Simplify.	$y=-3$

	$f(x) = ax^2 + bx + c$ $f(x) = 2x^2 - 4x - 3$
	The y-intercept is $(0, -3)$.
The point $(0, -3)$ is one unit to the left of the line of symmetry.	Point symmetric to the y-intercept is $(2, -3)$
The point one unit to the right of the line of symmetry is $(2, 3)$.	
The x-intercept occurs when $y = 0$.	
Find a solution of the form $(x, 0)$.	$0 = 2x^2 - 4x - 3$
Use the Quadratic Formula.	
Substitute in the values of a, b and c.	$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(2)(3)}}{2(2)}$
Simplify.	
Simplify inside the radical.	$x = \frac{4 \pm \sqrt{40}}{4}$
Simplify the radical.	$x = \frac{4 \pm 2\sqrt{10}}{4}$
Factor the GCF.	$x = \frac{2(2 \pm \sqrt{10})}{4}$
Remove common factors.	$x = \frac{2 \pm \sqrt{10}}{2}$
Write as two equations.	$x = \frac{2 + \sqrt{10}}{2}, \quad x = \frac{2 - \sqrt{10}}{2}$
Approximate the values.	$x \approx 2.5, \quad x \approx -0.6$
	The approximate values of the x-intercepts are $(2.5, 0)$ and $(-0.6, 0)$.
Graph the parabola using the points found.	

Try It 7.5.28

Graph $y=5x^2+10x+3$ by using its properties.

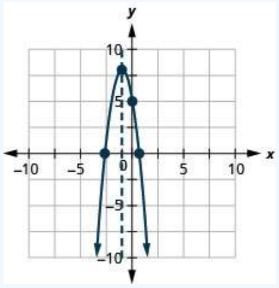
Answer



Try It 7.5.29

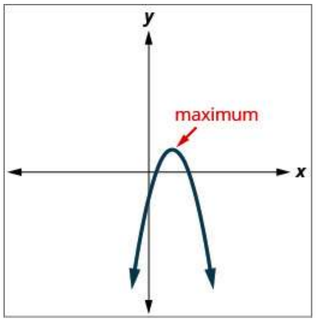
Graph $y=-3x^2-6x+5$ by using its properties.

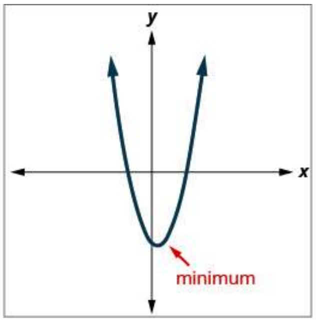
Answer



Solve Maximum and Minimum Applications

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic expression. Given a quadratic expression, with variable x, say, we consider the equation in two variables \y\) equals the given expression. The y-coordinate of the vertex is the **minimum** value of that expression if the graph of the equation is a parabola that opens upward. It is the **maximum** value of the expression if the graph of the equation is a parabola that opens downward.





Optional: Minimum or Maximum Values of a Quadratic Expression

Minimum and Maximum Values

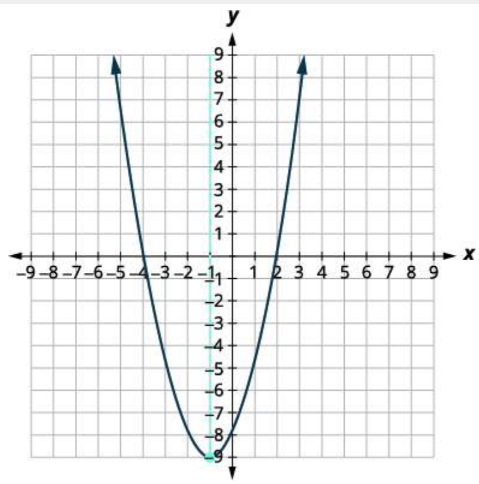
The y-coordinate of the vertex of the graph of a quadratic equation is the

- *minimum* value of the quadratic expression if the parabola opens *upward*.
- *maximum* value of the quadratic expression if the parabola opens *downward*.

Example 7.5.30

Find the minimum or maximum value of the quadratic expression $x^2 + 2x - 8$.

Solution

Consider the equation	$y = x^2 + 2x - 8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x = -\frac{b}{2a}$
	$x = -\frac{2}{2 \times 1}$
	$x = -1$
	The equation of the axis of symmetry is $x = -1$.
The vertex is on the line $x = -1$.	$y = x^2 + 2x - 8$
Find the y-coordinate of the vertex.	$y = (-1)^2 + 2(-1) - 8$
	$y = 1 - 2 - 8$
	$y = -9$
	The vertex is $(-1, -9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x = -1$.	
	

Show the graph to verify the result.

The minimum value of the expression is -9 and it occurs when $x = -1$.

Try It 7.5.31

Find the maximum or minimum value of the quadratic expression $x^2 - 8x + 12$.

Answer

The minimum value of the quadratic expression is -4 and it occurs when $x=4$.

Try It 7.5.32

Find the maximum or minimum value of the quadratic expression $-4x^2+16x-11$.

Answer

The maximum value of the quadratic expression is 5 and it occurs when $x=2$.

We have used the formula

$$h=-16t^2+v_0t$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds (assuming the initial shot occurs at height 0). The solutions to this equation in the context of the application are precisely the pairs (t,h) so that h is the height of the object at time t.

This formula is a quadratic equation in t, so its graph is a parabola. By solving for the coordinates of the vertex (t,h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

Example 7.5.33

The quadratic expression $-16t^2+176t+4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

a. How many seconds will it take the volleyball to reach its maximum height?

b. Find the maximum height of the volleyball.

Solution

Let us consider the equation $h=-16t^2+176t+4$ (here we have used h for 'height' instead of y). So again, the solutions of the equation are exactly the ordered pairs (t,h) so that h is the height at time t.

Since a is negative, the parabola opens downward. The quadratic function has a maximum.

a. Find the equation of the axis of symmetry.

$$t=-\frac{b}{2a}$$

$$t=-\frac{176}{2(-16)}$$

$$t=5.5$$

The equation of the axis of symmetry is $t=5.5$.

The vertex is on the line $t=5.5$.

The maximum occurs when $t=5.5$ seconds.

b. Find the height at time $t=5.5$. This is the same as searching for a solution of the form $(h,5.5)$.

Use a calculator to simplify.

$h=488$

The vertex is $(5.5,488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic expression.

The maximum value of the quadratic expression is then 488 feet and it occurs when $t=5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

Exercise 7.5.34

Solve, rounding answers to the nearest tenth.

The quadratic function $h=-16t^2+128t+32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Answer

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

Exercise 7.5.35

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h=-16t^2+208t$. When will the rocket reach its maximum height? What will be the maximum height?

Answer

It will take 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Key Concepts

- Parabola Orientation
  - For the graph of the quadratic equation $y=ax^2+bx+c$, if
    - $a>0$, the parabola opens upward.
    - $a<0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the equation $y=ax^2+bx+c$ is a parabola where:
  - the axis of symmetry is the vertical line $x=-\frac{b}{2a}$.
  - the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
  - the y-coordinate of the vertex is found by substituting $x=-\frac{b}{2a}$ into the quadratic equation.
- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose equation is $y=ax^2+bx+c$:
    - **y-intercept**
      - Let $x=0$ and solve for y.
    - **x-intercepts**
      - Let \(y=0\) and solve for x.
- How to graph a quadratic equation using properties.
  1. Determine whether the parabola opens upward or downward.
  2. Find the equation of the axis of symmetry.
  3. Find the vertex.
  4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  5. Find the x-intercepts. Find additional points if needed.
  6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Expression
  - The y-coordinate of the vertex of the graph of a quadratic equation formed by setting the expression (in x) equal to y is the
  - *minimum* value of the quadratic expression if the parabola opens *upward*.
  - *maximum* value of the quadratic expression if the parabola opens *downward*.

Glossary

quadratic expression (in x)

An expression of the form ax^2+bx+c , where $a, b,$ and c are real numbers and $a \neq 0$, is called a quadratic expression (in x).

quadratic equation (in x)

An equation which can be written in the form $y=ax^2+bx+c$, where $a, b,$ and c are real numbers and $a \neq 0$ is a quadratic equation (in x)

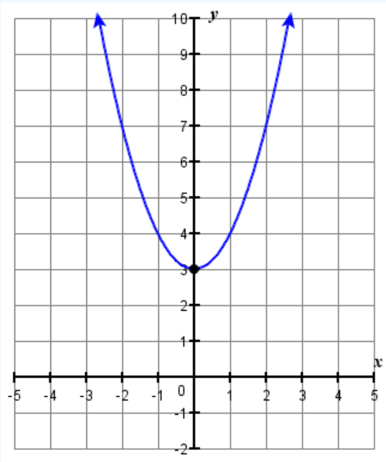
Practice Makes Perfect

In the following exercises, graph the equation by plotting points.

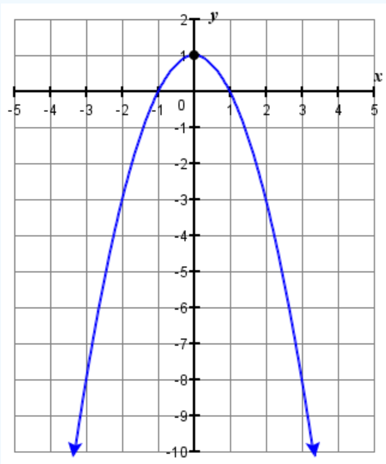
1. $y=x^2+3$
2. $y=x^2-3$
3. $y=-x^2+1$
4. $y=-x^2-1$

Answer

1.



3.



For each of the following exercises, determine if the parabola opens up or down.

5. a. $y=-2x^2-6x-7$ b. $y=6x^2+2x+3$

6. a. $y=4x^2+x-4$ b. $y=-9x^2-24x-16$

7. a. $y=-3x^2+5x-1$ b. $y=2x^2-4x+5$

8. a. $y=x^2+3x-4$ b. $y=-4x^2-12x-9$

Answer

5. a. down b. up

7. a. down b. up

In the following equations, find the a) axis of symmetry and b) the vertex.

9. $y=x^2+8x-1$

10. $y=x^2+10x+25$

11. $y=-x^2+2x+5$

12. $y=-2x^2-8x-3$

Answer

9. a. Axis of symmetry: $x=-4$ b. Vertex: $(-4,-17)$

11. a. Axis of symmetry: $x=1$ b. Vertex: $(1,2)$

In the following exercises, find the intercepts of the parabola whose equation is given.

13. $y=x^2+7x+6$

14. $y=x^2+10x-11$

15. $y=x^2+8x+12$

16. $y=x^2+5x+6$

17. $y=-x^2+8x-19$

18. $y=-3x^2+x-1$

19. $y=x^2+6x+13$

20. $y=x^2+8x+12$

21. $y=4x^2-20x+25$

22. $y=-x^2-14x-49$

23. $y=-x^2-6x-9$

24. $y=4x^2+4x+1$

Answer

13. y-intercept: $(0,6)$; x-intercept(s): $(-1,0),(-6,0)$

15. y-intercept: $(0,12)$; x-intercept(s): $(-2,0),(-6,0)$

17. y-intercept: $(0,-19)$; x-intercept(s): none

19. y-intercept: $(0,13)$; x-intercept(s): none

21. y-intercept: $(0,-16)$; x-intercept(s): $(\frac{5}{2},0)$

23. y-intercept: $(0,9)$; x-intercept(s): $(-3,0)$

In the following exercises, graph the equation by using its properties.

25. $y=x^2+6x+5$

26. $y=x^2+4x-12$

27. $y=x^2+4x+3$

28. $y=x^2-6x+8$

29. $y=9x^2+12x+4$

30. $y=-x^2+8x-16$

31. $y=-x^2+2x-7$

32. $y=5x^2+2$

33. $y=2x^2-4x+1$

34. $y=3x^2-6x-1$

35. $y=2x^2-4x+2$

36. $y=-4x^2-6x-2$

37. $y=-x^2-4x+2$

38. $y=x^2+6x+8$

39. $y=5x^2-10x+8$

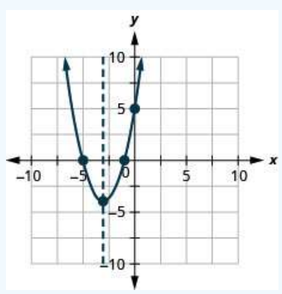
40. $y=-16x^2+24x-9$

41. $y=3x^2+18x+20$

42. $y=-2x^2+8x-10$

Answer

25.



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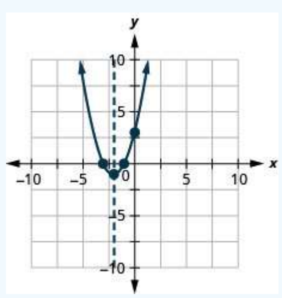

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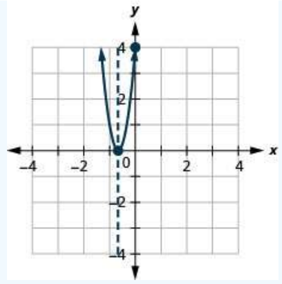
y
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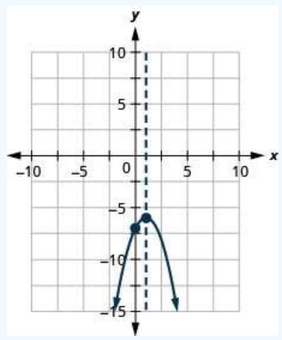
2
−4
−2
0
2
4
−2
−4

31.


y
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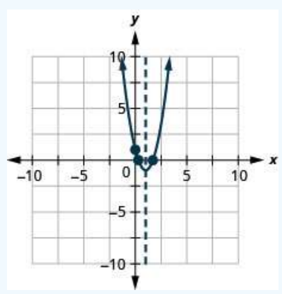
−10
−15

33.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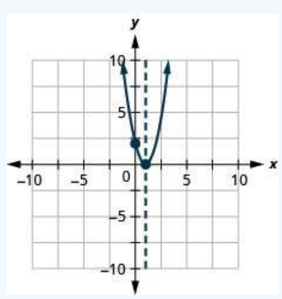
35.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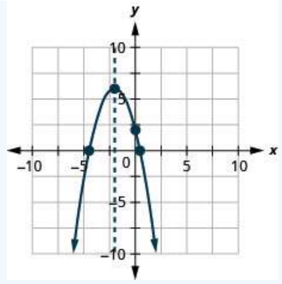
−5
−10

37.



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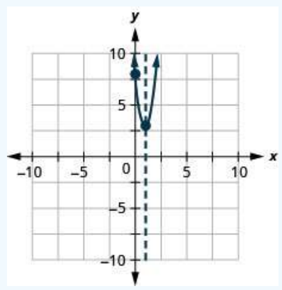

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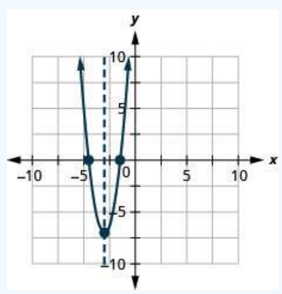


In the following exercises, find the maximum or minimum value of each function.

43. $y=2x^2+x-1$

44. $y=-4x^2+12x-5$

45. $y=x^2-6x+15$

46. $y=-x^2+4x-5$

47. $y=-9x^2+16$

48. $y=4x^2-49$

Answer

43. The minimum value is $-\frac{9}{8}$ when $x=-\frac{1}{4}$.

45. The maximum value is 6 when $x=3$.

47. The maximum value is 16 when $x=0$.

In the following exercises, solve. Round answers to the nearest tenth.

49. An arrow is shot vertically upward from a platform 45 feet high at a rate of 168 ft/sec. Use the quadratic equation $h=-16t^2+168t+45$ find how long it will take the arrow to reach its maximum height, and then find the maximum height.

50. A stone is thrown vertically upward from a platform that is 20 feet height at a rate of 160 ft/sec. Use the quadratic equation $h=-16t^2+160t+20$ to find how long it will take the stone to reach its maximum height, and then find the maximum height.

51. A ball is thrown vertically upward from the ground with an initial velocity of 109 ft/sec. Use the quadratic equation $h=-16t^2+109t+0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

52. A ball is thrown vertically upward from the ground with an initial velocity of 122 ft/sec. Use the quadratic equation $h=-16t^2+122t+0$ to find how long it will take for the ball to reach its maximum height, and then find the maximum height.

53. A computer store owner estimates that by charging x dollars each for a certain computer, he can sell $40-x$ computers each week. The quadratic equation $R=-x^2+40x$ is used to relate the revenue, R, received when the selling price of a computer is x, Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

54. A retailer who sells backpacks estimates that by selling them for x dollars each, he will be able to sell $100-x$ backpacks a month. The quadratic equation $R=-x^2+100x$ is used to relate the revenue, R, received when the selling price of a backpack is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue.

55. A retailer who sells fashion boots estimates that by selling them for x dollars each, he will be able to sell $70-x$ boots a week. The quadratic equation $R=-x^2+70x$ relates the revenue, R, to the sale price, x. Find the revenue received when the average selling price of a pair of fashion boots is x. Find the selling price that will give him the maximum revenue, and then find the amount of the maximum revenue per day.

56. A cell phone company estimates that by charging x dollars each for a certain cell phone, they can sell $8-x$ cell phones per day. Use the quadratic equation $R=-x^2+8x$ to find the revenue received per day when the selling price of a cell phone is x. Find the selling price that will give them the maximum revenue per day, and then find the amount of the maximum revenue.

57. A rancher is going to fence three sides of a corral next to a river. He needs to maximize the corral area using 240 feet of fencing. The quadratic equation $A=x(240-2x)$ gives the area of the corral, A, for the length, x, of the corral along the river. Find the length of the corral along the river that will give the maximum area, and then find the maximum area of the corral.

58. A veterinarian is enclosing a rectangular outdoor running area against his building for the dogs he cares for. He needs to maximize the area using 100 feet of fencing. The quadratic equation $A=x(100-2x)$ gives the area, A, of the dog run for the length, x, of the building that will border the dog run. Find the length of the building that should border the dog run to give the maximum area, and then find the maximum area of the dog run.

59. A land owner is planning to build a fenced in rectangular patio behind his garage, using his garage as one of the “walls.” He wants to maximize the area using 80 feet of fencing. The quadratic equation $A=x(80-2x)$ gives the area of the patio, where x is the width of one side. Find the maximum area of the patio.

60. A family of three young children just moved into a house with a yard that is not fenced in. The previous owner gave them 300 feet of fencing to use to enclose part of their backyard. Use the quadratic equation $A=x(300-2x)$ to determine the maximum area of the fenced in yard.

Answer

49. In 5.3 sec the arrow will reach maximum height of 486 ft.

51. In 3.4 seconds the ball will reach its maximum height of 185.6 feet.

53. 20 computers will give the maximum of $\$400$ in receipts.

55. He will be able to sell 35 pairs of boots at the maximum revenue of $\$1,225$.

57. The length of the side along the river of the corral is 120 feet and the maximum area is $7,200$ square feet.

59. The maximum area of the patio is 800 feet.

Writing Exercises

61. How do the graphs of the equations $y=x^2$ and $y=x^2-1$ differ? We graphed them at the start of this section. What is the difference between their graphs? How are their graphs the same?

62. Explain the process of finding the vertex of a parabola.

63. Explain how to find the intercepts of a parabola.

64. How can you use the discriminant when you are graphing a quadratic equation?

Answer

61. Answers will vary.

63. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
recognize the graph of a quadratic equation.			
find the axis of symmetry and vertex of a parabola.			
find the intercepts of a parabola.			
graph quadratic equations in two variables.			
solve maximum and minimum applications.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

7.6: Graph Quadratic Equations Using Transformations

Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic expressions of the form $y=x^2+k$
- Graph quadratic expressions of the form $y=(x-h)^2$
- Graph quadratic expressions of the form $y=ax^2$
- Graph quadratic expressions using transformations
- Find a quadratic expressions from its graph

Be Prepared

Before you get started, take this readiness quiz.

1. Graph the expressions $y=x^2$ by plotting points.
2. Factor completely: $y^2-14y+49$.
3. Factor completely: $2x^2-16x+32$.

Graph Quadratic Expressions of the Form $y=x^2+k$

In the last section, we learned how to graph quadratic expressions using their properties. Another method involves starting with the basic graph of $y=x^2$ and 'moving' it according to information given in the equation. We call this graphing quadratic equations using transformations.

In the first example, we will graph the quadratic equation $y=x^2$ by plotting points. Then we will see what effect adding a constant, k, to the equation will have on the graph of the new equation $y=x^2+k$.

Example 7.6.1

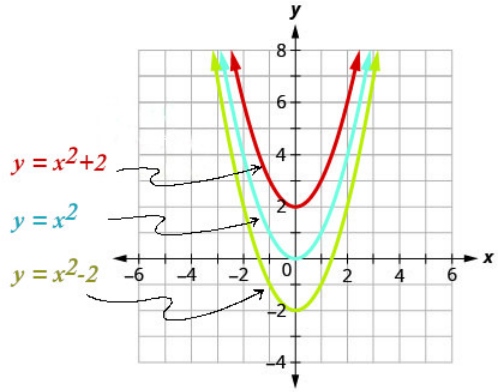
Graph $y=x^2$, $y=x^2+2$, and $y=x^2-2$ on the same rectangular coordinate system. Describe what effect adding a constant to the right side of the equation has on the basic parabola.

Solution:

Plotting points will help us see the effect of the constants on the basic $y=x^2$ graph. We fill in the chart for all three equations.

x	$y=x^2$	(x,y)	$y=x^2+2$	(x,y)	$y=x^2-2$	(x,y)
–3	9	(–3, 9)	9 + 2	(–3, 11)	9 – 2	(–3, 7)
–2	4	(–2, 4)	4 + 2	(–2, 6)	4 – 2	(–2, 2)
–1	1	(–1, 1)	1 + 2	(–1, 3)	1 – 2	(–1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 – 2	(0, –2)
1	1	(1, 1)	1 + 2	(1, 3)	1 – 2	(1, –1)
2	4	(2, 4)	4 + 2	(2, 6)	4 – 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 – 2	(3, 7)

The y-coordinates of the solutions to $y=x^2+2$ are two more than the y-coordinates of the solutions to $y=x^2$ values. Also, the y-coordinates of the solutions to $y=x^2-2$ values are two less than the y-coordinates of the solutions to $y=x^2$ values. Now we will graph all three equations on the same rectangular coordinate system.



The graph of $y = x^2+2$ is the same as the graph of $y = x^2$ but shifted up 2 units.
The graph of $y = x^2-2$ is the same as the graph of $y = x^2$ but shifted down 2 units.

The graph of $y=x^2+2$ is the same as the graph of $y=x^2$ but shifted up 2 units.

The graph of $y=x^2-2$ is the same as the graph of $y=x^2$ but shifted down 2 units.

? Try It 7.6.2

a. Graph $y=x^2, y=x^2+1,$ and $y=x^2-1$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the equation has on the basic parabola.

Answer

a.

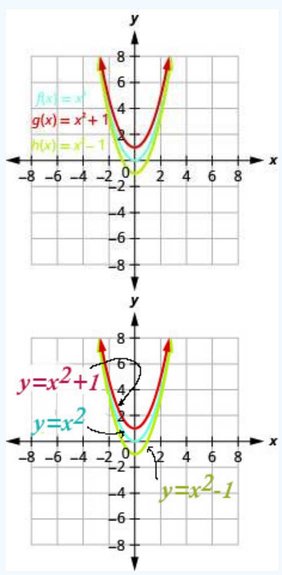


Figure 9.7.3

b. The graph of $y=x^2+1$ is the same as the graph of $y=x^2$ but shifted up 1 unit. The graph of $y=x^2-1$ is the same as the graph of $y=x^2$ but shifted down 1 unit.

Try It 7.6.3

a. Graph $y=x^2, y=x^2+6,$ and $y=x^2-6$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the equation has on the basic parabola.

Answer

a.

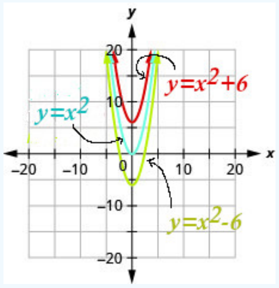


Figure 9.7.4

b. The graph of $y=x^2+6$ is the same as the graph of $y=x^2$ but shifted up 6 units. The graph of $y=x^2-6$ is the same as the graph of $y=x^2$ but shifted down 6 units.

The last example shows us that to graph a quadratic equation of the form $y=x^2+k$, we take the basic parabola graph of $y=x^2$ and vertically shift it up $(k>0)$ or shift it down $(k<0)$.

This transformation is called a vertical shift.

Graph a Quadratic equation of the Form $y=x^2+k$ Using a Vertical Shift

The graph of $y=x^2+k$ shifts the graph of $y=x^2$ vertically k units.

- If $k>0$, shift the parabola vertically up k units.
- If $k<0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph equations of the form $y=x^2+k$. We just start with the basic parabola of $y=x^2$ and then shift it up or down.

It may be helpful to practice sketching $y=x^2$ quickly. We know the values and can sketch the graph from there.

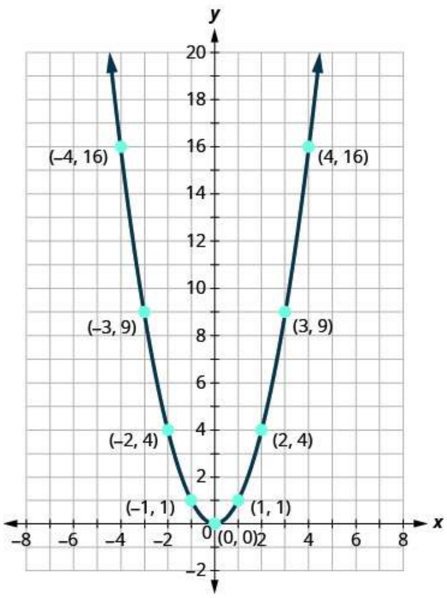


Figure 9.7.5

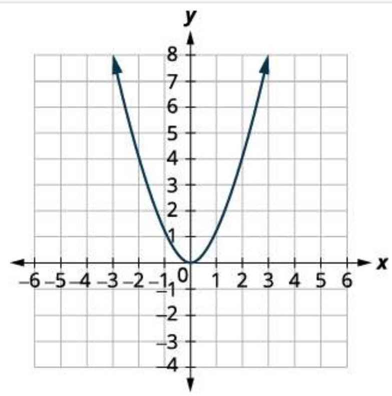
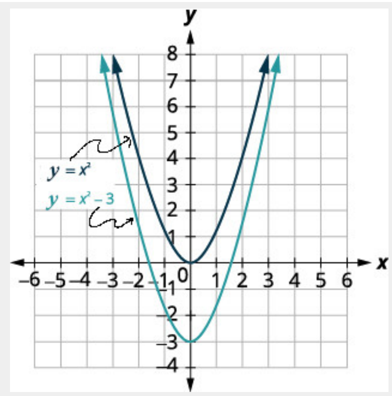
Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

Example 7.6.4

Graph $y=x^2-3$ using a vertical shift.

Solution:

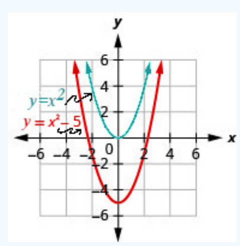
Table 9.7.1

We first draw the graph of $y=x^2$ on the grid.	
Determine k.	$y=x^2+k$ $y=x^2-3$
	$k=-3$
Shift the graph $y=x^2$ down 3.	

Try It 7.6.5

Graph $y=x^2-5$ using a vertical shift.

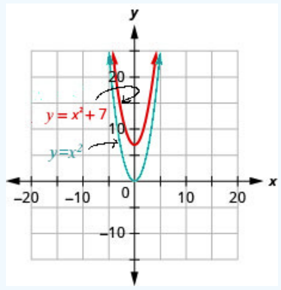
Answer



Try It 7.6.6

Graph $y=x^2+7$ using a vertical shift.

Answer



Graph Quadratic equations of the Form $y=(x-h)^2$

In the first example, we graphed the quadratic equation $y=x^2$ by plotting points and then saw the effect of adding a constant k to the right side of the equation (or, equivalently, subtracting k from y) had on the resulting graph of the new equation $y=x^2+k$.

We will now explore the effect of subtracting a constant, h, from x has on the resulting graph of the new equation $y=(x-h)^2$.

Example 7.6.7

Graph $y=x^2, y=(x-1)^2,$ and $y=(x+1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the equation has on the basic parabola.

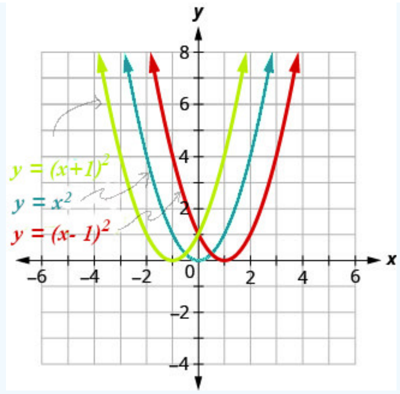
Solution:

Plotting points will help us see the effect of the constants on the basic $y=x^2$ graph. We fill in the chart for all three equations.

x	$y=x^2$	(x,y)	$y=(x-1)^2$	(x,y)	$y=(x+1)^2$	(x,y)
–3	9	(–3, 9)	16	(–3, 16)	4	(–3, 4)
–2	4	(–2, 4)	9	(–2, 9)	1	(–2, 1)
–1	1	(–1, 1)	4	(–1, 4)	0	(–1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

Figure 9.7.12

The y-coordinates share the common numbers $0, 1, 4, 9,$ and 16, but are shifted, in that they correspond to different x-coordinates in the different cases.



The graph of $y=(x-1)^2$ is the same as the graph of $y=x^2$ but shifted right 1 unit.
The graph of $y=(x+1)^2$ is the same as the graph of $y=x^2$ but shifted left 1 unit.

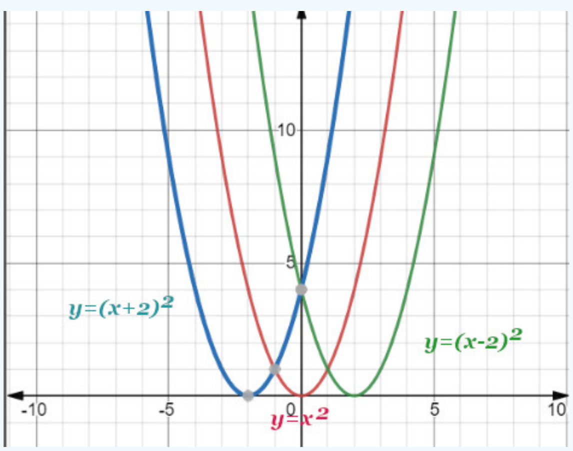
$y=(x-1)^2$ 1 unit $\longrightarrow$ $y=(x+1)^2$ $\longleftarrow$ 1 unit

? Try It 7.6.8

a. Graph $y=x^2, y=(x+2)^2,$ and $y=(x-2)^2$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the x variable has on the basic parabola.

Answer

a.



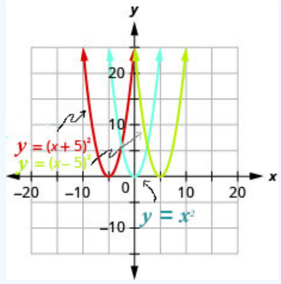
b. The graph of $y=(x+2)^2$ is the same as the graph of $y=x^2$ but shifted left 2 units. The graph of $y=(x-2)^2$ is the same as the graph of $y=x^2$ but shifted right 2 units.

? Try It 7.6.9

a. Graph $y=x^2, y=x^2+5,$ and $y=x^2-5$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the equation has on the basic parabola.

Answer

a.



b. The graph of $y=(x+5)^2$ is the same as the graph of $y=x^2$ but shifted left 5 units. The graph of $y=(x-5)^2$ is the same as the graph of $y=x^2$ but shifted right 5 units.

The last example shows us that to graph a quadratic equation of the form $y=(x-h)^2$, we take the basic parabola graph of $y=x^2$ and shift it left $(h>0)$ or shift it right $(h<0)$.

This transformation is called a horizontal shift.

Graph a Quadratic equation of the Form $y=(x-h)^2$ Using a Horizontal Shift

The graph of $y=(x-h)^2$ shifts the graph of $y=x^2$ horizontally h units.

- If $h>0$, shift the parabola horizontally left h units.
- If $h<0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, h, it is easy to graph equations of the form $y=(x-h)^2$. We just start with the basic parabola of $y=x^2$ and then shift it left or right.

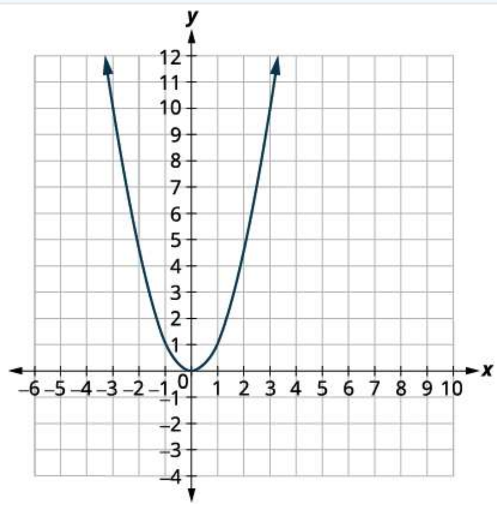
The next example will require a horizontal shift.

Example 7.6.10

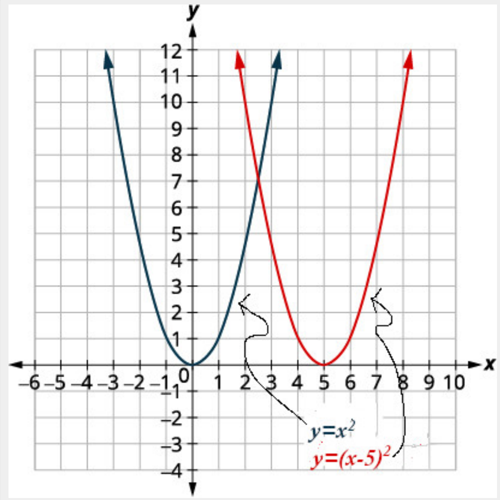
Graph $y=(x-5)^2$ using a horizontal shift.

Solution:

Table 9.7.2

We first draw the graph of $y=x^2$ on the grid.	
Determine h.	$y=(x-h)^2$ $y=(x-5)^2$
	$h=5$

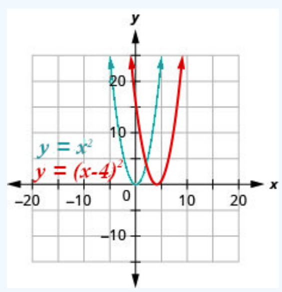
Shift the graph $y=x^2$ to the right 5 units.



? Try It 7.6.11

Graph $y=(x-4)^2$ using a horizontal shift.

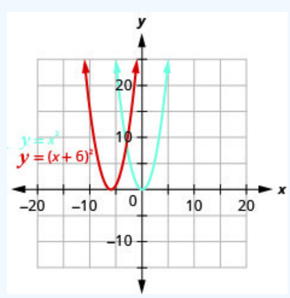
Answer



? Try It 7.6.12

Graph $y=(x+6)^2$ using a horizontal shift.

Answer



(Note that you may also view the consideration of the equation $y=x^2+k$ as $y-k=x^2$ and think of this as a shift along the y-axis. Viewing it this way allows for a similar approach to the transformation of graphs of equations formed by adding (or subtracting) constants to (or from) variables.)

Now that we know the effect of the constants h and k, we will graph a quadratic equation of the form $y=(x-h)^2+k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

Example 7.6.13

Graph $y=(x+1)^2-2$ using transformations.

Solution:

This equation will involve two transformations and we need a plan.

Let’s first identify the constants h, k.

$$y=(x+1)^2-2$$
$$y=(x-h)^2+k$$

$$y=(x-(-1))^2-2$$

$$h=-1, k=-2$$

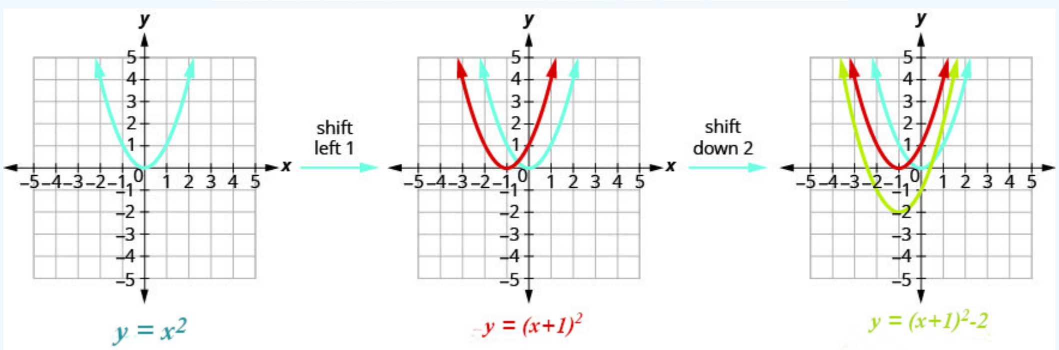
The h constant gives us a horizontal shift and the k gives us a vertical shift.

$y=x^2$	→	$y=(x+1)^2$	→	$y=(x+1)^2-2$
	$h=-1$		$k=-2$	
	Shift left 1 unit		Shift down 2 units	

We first draw the graph of $y=x^2$ on the grid.

To graph $y=(x+1)^2$, shift the graph $y=x^2$ to the left 1 unit.

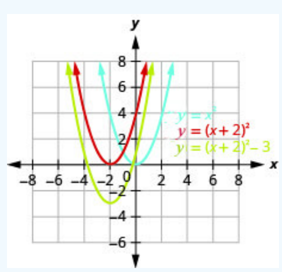
To graph $y=(x+1)^2-2$, shift the graph $y=(x+1)^2$ down 2 units.



Try It 7.6.14

Graph $y=(x+2)^2-3$ using transformations.

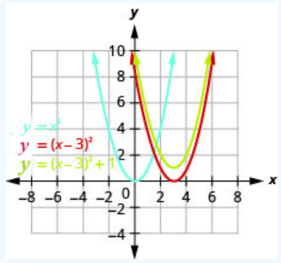
Answer



Try It 7.6.15

Graph $y=(x-3)^2+1$ using transformations.

Answer



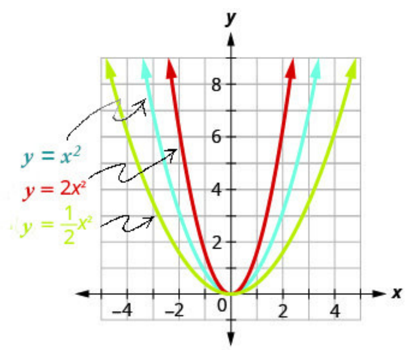
Graph Quadratic Equations of the Form $y=ax^2$

So far we graphed the quadratic equation $y=x^2$ and then saw the effect of including a constant h or k in the equation had on the resulting graph of the new equation. We will now explore the effect of the coefficient a on the resulting graph of the new equation $y=ax^2$.

Let's look at the quadratic functions $y=x^2$, $y=2x^2$ and $y=\frac{1}{2}x^2$.

x	$y=x^2$	(x,y)	$y=2x^2$	(x,y)	$y=\frac{1}{2}x^2$	(x,y)
−2	4	(−2, 4)	2 • 4	(−2, 8)	$\frac{1}{2}\cdot 4$	(−2, 2)
−1	1	(−1, 1)	2 • 1	(−1, 2)	$\frac{1}{2}\cdot 1$	$\left(-1, \frac{1}{2}\right)$
0	0	(0, 0)	2 • 0	(0, 0)	$\frac{1}{2}\cdot 0$	(0, 0)
1	1	(1, 1)	2 • 1	(1, 2)	$\frac{1}{2}\cdot 1$	$\left(1, \frac{1}{2}\right)$
2	4	(2, 4)	2 • 4	(2, 8)	$\frac{1}{2}\cdot 4$	(2, 2)

If we graph these equations, we can see the effect of the constant a, assuming $a>0$.



The graph of the equation $y=2x^2$ is "skinnier" than the graph of $y=x^2$.

The graph of the equation $y=\frac{1}{2}x^2$ is "wider" than the graph of $y=x^2$.

To graph a equation with constant a it is easiest to choose a few points on $y=x^2$ and multiply the y-coordinates by a.

Graph of a Quadratic equation of the Form $y=ax^2$

The coefficient a in the equation $y=ax^2$ affects the graph of $y=x^2$ by stretching or compressing it.

- If $0<|a|<1$, the graph of $y=ax^2$ will be "wider" than the graph of $y=x^2$.

- If $|a| > 1$, the graph of $y = ax^2$ will be "skinnier" than the graph of $y = x^2$.

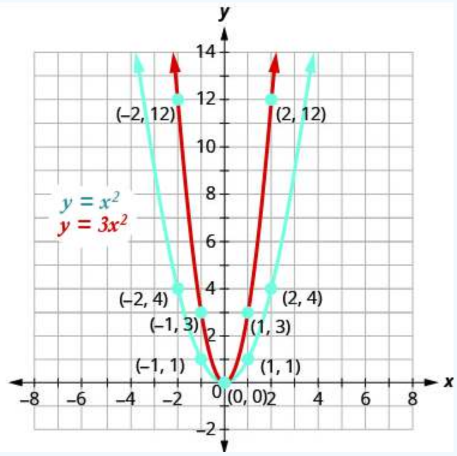
Example 7.6.16

Graph $y = 3x^2$.

Solution:

We will graph the equations $y = x^2$ and $y = 3x^2$ on the same grid. We will choose a few points on $y = x^2$ and then multiply the y-values by 3 to get the points for $y = 3x^2$.

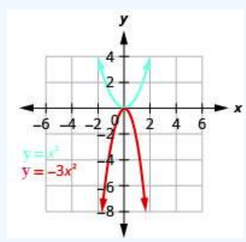
	$y = x^2$	$y = 3x^2$	
x	(x, y)	(x, y)	
–2	(–2, 4)	(–2, 12)	$3 \cdot 4 = 12$
–1	(–1, 1)	(–1, 3)	$3 \cdot 1 = 3$
0	(0, 0)	(0, 0)	$3 \cdot 0 = 0$
1	(1, 1)	(1, 3)	$3 \cdot 1 = 3$
2	(2, 4)	(2, 12)	$3 \cdot 4 = 12$



Try It 7.6.17

Graph $y = -3x^2$.

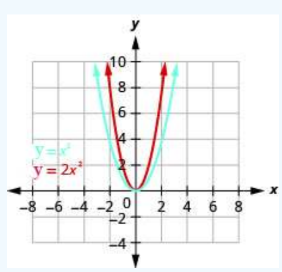
Answer



Try It 7.6.18

Graph $y = 2x^2$.

Answer



Graph Quadratic equations Using Transformations

We have learned how the constants a, h, and k in the equations, $y=x^2+k, y=(x-h)^2$, and $y=ax^2$ affect their graphs. We can now put this together and graph quadratic equations $y=ax^2+bx+c$ by first putting them into the form $y=a(x-h)^2+k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the expression to complete the square. We cannot add the number to 'both sides' (both sides of what?) as we did when we completed the square with quadratic equations. Note that we could also add and subtract a number from the same side in the case of the quadratic equation as well.

Quadratic Equation	Quadratic Expression
$x^2+8x+6=0$	x^2+8x+6
$x^2+8x=-6$	x^2+8x+6
$x^2+8x+16=-6+16$ -- add 16 to both sides	$x^2+8x+16-16+6$ --add and subtract 16 from the expression
$(x+4)^2=10$	$(x+4)^2-10$

When we complete the square in a equation with a coefficient of x^2 that is not one, we have to factor that coefficient from just the x-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the x-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

Example 7.6.19

Rewrite $y=-3x^2-6x-1$ in the $y=a(x-h)^2+k$ form by completing the square.

Solution:

	$y=-3x^2-6x-1$
Separate the x terms from the constant.	$y=-3x^2-6x \quad -1$
Factor the coefficient of x^2, -3.	$y=-3(x^2+2x) \quad -1$
Prepare to complete the square.	$y=-3(x^2+2x \qquad) \quad -1$
Take half of 2 and then square it to complete the square $(\frac{1}{2}\cdot 2)^2=1$	
The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3. We must then add 3 to not change the value of the equation.	$y=-3((x^2+2x+1-1))-1$ $y=-3(x^2+2x+1)+3-1$
Rewrite the trinomial as a square and subtract the constants.	$y=-3(x+1)^2+2$
The equation is now in the $y=a(x-h)^2+k$ form.	$y=a(x-h)^2+k$ $y=-3(x+1)^2+2$

Try It 7.6.20

Rewrite $y=-4x^2-8x+1$ in the $y=a(x-h)^2+k$ form by completing the square.

Answer

$y=-4(x+1)^2+5$

Try It 7.6.21

Rewrite $y=2x^2-8x+3$ in the $y=a(x-h)^2+k$ form by completing the square.

Answer

$y=2(x-2)^2-5$

Once we put the equation into the $y=(x-h)^2+k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

Example 7.6.22

Graph $y=x^2+6x+5$ by using transformations.

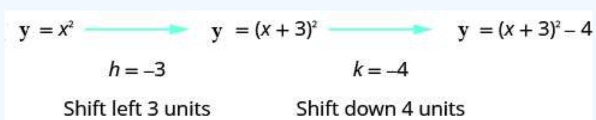
Solution:

Step 1: Rewrite the equation in $y=a(x-h)^2+k$ vertex form by completing the square.

	$y=x^2+6x+5$
Separate the x terms from the constant.	$y=x^2+6x+5$
Take half of 6 and then square it to complete the square. $(\frac{1}{2}\cdot 6)^2=9$	
We both add 9 and subtract 9 to not change the value of the equation.	$y=x^2+6x+9-9+5$
Rewrite the trinomial as a square and subtract the constants.	$y=(x+3)^2-4$
The equation is now in the $y=(x-h)^2+k$ form.	$y=(x-h)^2+k$ $y=(x+3)^2-4$

Step 2: Graph the equation using transformations.

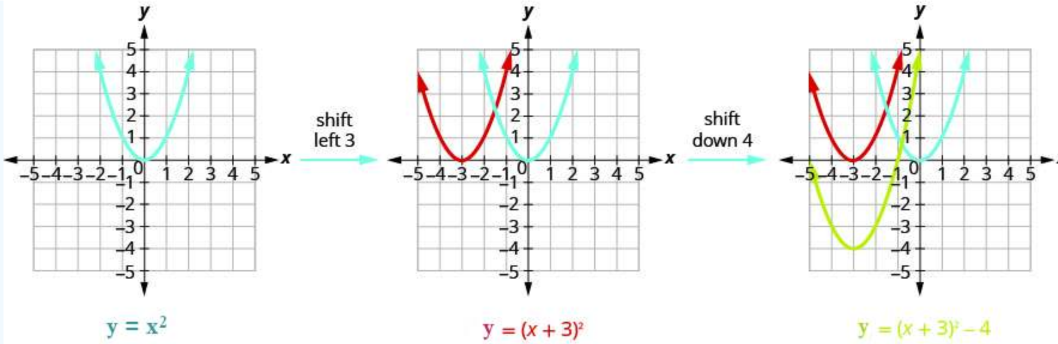
Looking at the h, k values, we see the graph will take the graph of $y=x^2$ and shift it to the left 3 units and down 4 units.



We first draw the graph of $y=x^2$ on the grid.

To graph $y=(x+3)^2$, shift the graph $y=x^2$ to the left 3 units.

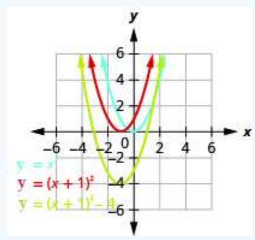
To graph $y=(x+3)^2-4$, shift the graph $y=(x+3)^2$ down 4 units.



? Try It 7.6.23

Graph $y=x^2+2x-3$ by using transformations.

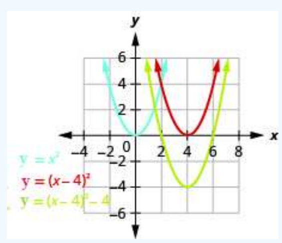
Answer



? Try It 7.6.24

Graph $y=x^2-8x+12$ by using transformations.

Answer



We list the steps to take a graph a quadratic equation using transformations here.

Graph a Quadratic equation Using Transformations

1. Rewrite the equation in $y=a(x-h)^2+k$ form by completing the square.
2. Graph the equation using transformations.

✓ Example 7.6.25

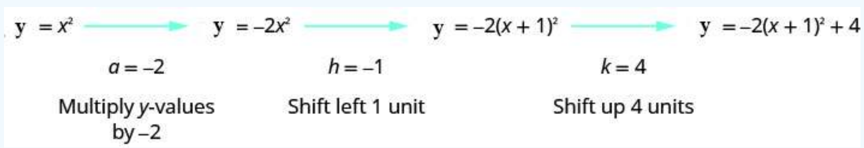
Graph $y=-2x^2-4x+2$ by using transformations.

Solution:

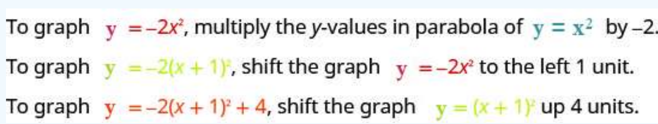
Step 1: Rewrite the equation in $y=a(x-h)^2+k$ vertex form by completing the square.

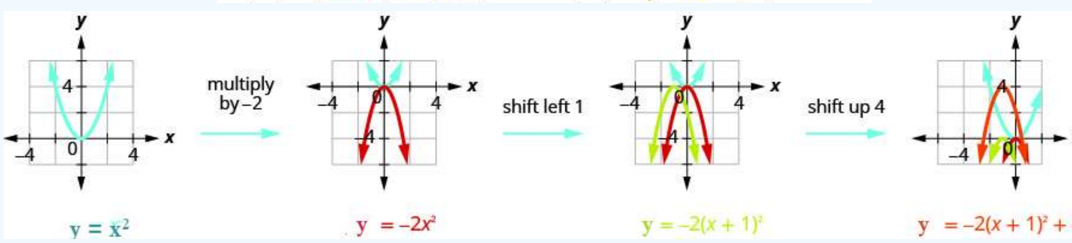
	$y=-2x^2-4x+2$
Separate the x terms from the constant.	$y=-2x^2-4x \quad +2$
We need the coefficient of x^2 to be one. We factor -2 from the x-terms.	$y=-2(x^2+2x \quad)+2$
Take half of 2 and then square it to complete the square. $(\frac{1}{2}\cdot 2)^2=1$	
We add 1 to complete the square in the parentheses, but the parentheses is multiplied by -2. So we are really adding -2. To not change the value of the equation we add 2.	$y=-2((x^2+2x+1)-1)+2$ $y=-2(x^2+2x+1)+2+2$
Rewrite the trinomial as a square ad subtract the constants.	$y=-2(x+1)^2+4$
The equation is now in the $y=a(x-h)^2+k$ form.	$y=a(x-h)^2+k$ $y=-2(x+1)^2+4$

Step 2: Graph the equation using transformations.



We first draw the graph of $y=x^2$ on the grid.

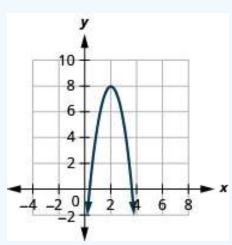




? Try It 7.6.26

Graph $y=-3x^2+12x-4$ by using transformations.

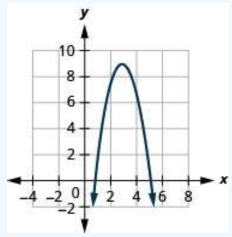
Answer



Try It 7.6.27

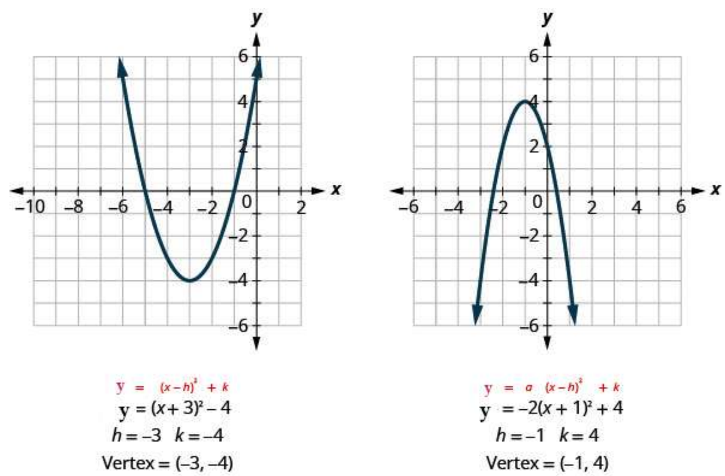
Graph $y=-2x^2+12x-9$ by using transformations.

Answer



Now that we have completed the square to put a quadratic equation into $y=a(x-h)^2+k$ form, we can also use this technique to graph the equation using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants h and k.



In each case, the vertex is (h,k). Also the **axis of symmetry** is the line $x=h$.

We rewrite our steps for graphing a quadratic equation using properties for when the equation is in $y=a(x-h)^2+k$ form.

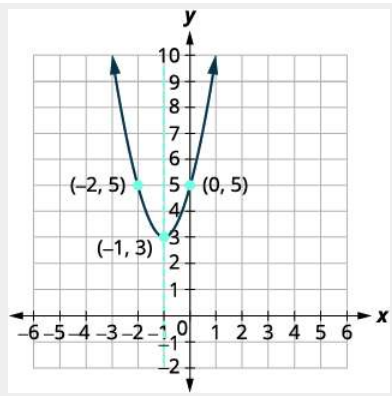
Graph a Quadratic equation in the Form $y=a(x-h)^2+k$ Using Properties

1. Rewrite the equation $y=a(x-h)^2+k$ form.
2. Determine whether the parabola opens upward, $a>0$, or downward, $a<0$.
3. Find the axis of symmetry, $x=h$.
4. Find the vertex, (h,k).
5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
6. Find the x-intercepts.
7. Graph the parabola.

Example 7.6.28

a. Rewrite $y=2x^2+4x+5$ in $y=a(x-h)^2+k$ form
b. Graph the equation using properties

Solution:

Rewrite the equation in $y=a(x-h)^2+k$ form by completing the square.	$y=2x^2+4x+5$
	$y=2\left(x^2+2x\right)+5$
	$y=2\left(x^2+2x+1\right)+5-2$
	$y=2(x+1)^2+3$
Identify the constants a, h, k.	
Since $a=2$, the parabola opens upward.	
The axis of symmetry is $x=h$.	The axis of symmetry is $x=-1$.
The vertex is (h, k).	The vertex is $(-1,3)$.
Find the y-intercept by finding $f(0)$.	$f(0)=2\cdot 0^2+4\cdot 0+5$
	$f(0)=5$
	y-intercept $(0,5)$
Find the point symmetric to $(0,5)$ across the axis of symmetry.	$(-2,5)$
Find the x-intercepts.	The discriminant is negative, so there are no x-intercepts. Graph the parabola.
	

? Try It 7.6.29

a. Rewrite $y=3x^2-6x+5$ in $y=a(x-h)^2+k$ form
b. Graph the equation using properties

Answer

a. $y=3(x-1)^2+2$
b.

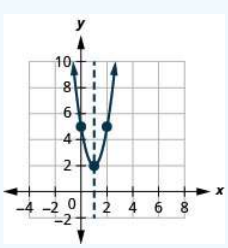


Figure 9.7.66

Try It 7.6.30

a. Rewrite $y=-2x^2+8x-7$ in $y=a(x-h)^2+k$ form
b. Graph the equation using properties

Answer

a. $y=-2(x-2)^2+1$
b.

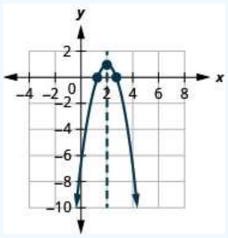


Figure 9.7.67

Challenge section:

Find a Quadratic Equation from its Graph

So far we have started with a equation and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the equation.

Example 7.6.31

Determine the quadratic equation whose graph is shown.

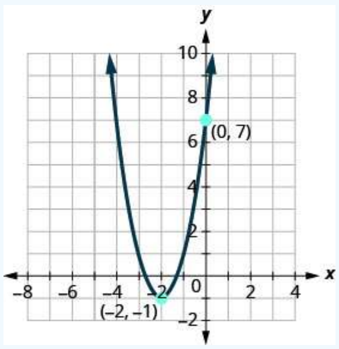


Figure 9.7.68

Solution:

Since it is quadratic, we start with the $y=a(x-h)^2+k$ form.

The vertex, (h,k), is $(-2,-1)$ so $h=-2$ and $k=-1$.

$$y=a(x-(-2))^2-1$$

To find a, we use the y-intercept, $(0,7)$.

So $f(0)=7$.

$$7=a(0+2)^2-1$$

Solve for a.

$$\begin{aligned}7&=4a-1\\8&=4a\\2&=a\end{aligned}$$

Write the equation.

$$y=a(x-h)^2+k$$

Substitute in $h=-2, k=-1$ and $a=2$.

$$y=2(x+2)^2-1$$

Try It 7.6.32

Write the quadratic equation in $y=a(x-h)^2+k$ form whose graph is shown.

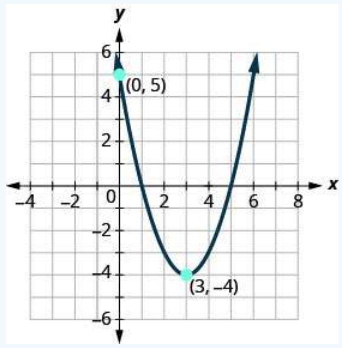


Figure 9.7.69

Answer

$y=(x-3)^2-4$

Try It 7.6.33

Determine the quadratic equation whose graph is shown.

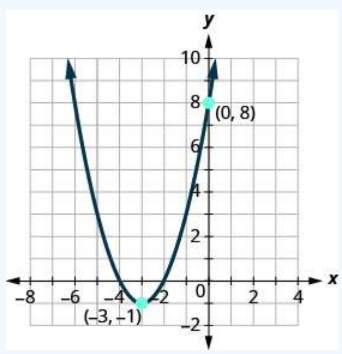


Figure 9.7.70

Answer

$y=(x+3)^2-1$

Key Concepts

- Graph a Quadratic equation of the form $y=x^2+k$ Using a Vertical Shift
  - The graph of $y=x^2+k$ shifts the graph of $y=x^2$ vertically k units.
    - If $k>0$, shift the parabola vertically up k units.
    - If $k<0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic equation of the form $y=(x-h)^2$ Using a Horizontal Shift
  - The graph of $y=(x-h)^2$ shifts the graph of $y=x^2$ horizontally h units.
    - If $h>0$, shift the parabola horizontally left h units.

- If $h < 0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic equation of the form $y = ax^2$
  - The coefficient a in the equation $y = ax^2$ affects the graph of $y = x^2$ by stretching or compressing it. If $0 < |a| < 1$, then the graph of $y = ax^2$ will be “wider” than the graph of $y = x^2$. If $|a| > 1$, then the graph of $y = ax^2$ will be “skinnier” than the graph of $y = x^2$.
- How to graph a quadratic equation using transformations
  1. Rewrite the equation in $y = a(x-h)^2 + k$ form by completing the square.
  2. Graph the equation using transformations.
- Graph a quadratic equation in the vertex form $y = a(x-h)^2 + k$ using properties
  1. Rewrite the equation in $y = a(x-h)^2 + k$ form.
  2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
  3. Find the axis of symmetry, $x = h$.
  4. Find the vertex, (h, k).
  5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  6. Find the x-intercepts, if possible.
  7. Graph the parabola.

Practice Makes Perfect

? Graph Quadratic equations of the Form $y = x^2 = k$

In the following exercises,

a. Graph the quadratic equations on the same rectangular coordinate system
b. Describe what effect adding a constant, k, to the equation has on the basic parabola.

1. $y = x^2, y = x^2 + 4,$ and $y = x^2 - 4$
2. $y = x^2, y = x^2 + 7,$ and $y = x^2 - 7$

Answer

1.

a. Use DESMOS
b. The graph of $y = x^2 + 4$ is the same as the graph of $y = x^2$ but shifted up 4 units. The graph of $y = x^2 - 4$ is the same as the graph of $y = x^2$ but shift down 4 units.

? Graph Quadratic equations of the Form $y = x^2 = k$

In the following exercises, graph each equation using a vertical shift.

3. $y = x^2 + 3$
4. $y = x^2 - 7$
5. $y = x^2 + 2$
6. $y = x^2 + 5$
7. $y = x^2 - 4$
8. $y = x^2 - 5$

Answer

3.

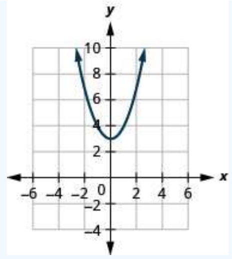


Figure 9.7.72

5.

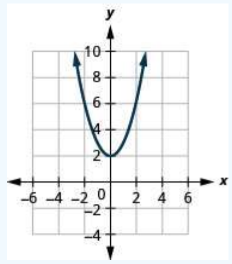


Figure 9.7.73

7.

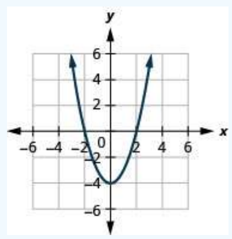


Figure 9.7.74

Graph Quadratic equations of the Form $y=(x-h)^2$

In the following exercises,

a. Graph the quadratic equations on the same rectangular coordinate system
b. Describe what effect adding a constant, h, inside the parentheses has

9. $y=x^2, y=(x-3)^2,$ and $y=(x+3)^2$
10. $y=x^2, y=(x+4)^2,$ and $y=(x-4)^2$

Answer

9.

a. Use DESMOS
b. The graph of $y=(x-3)^2$ is the same as the graph of $y=x^2$ but shifted right 3 units. The graph of $y=(x+3)^2$ is the same as the graph of $y=x^2$ but shifted left 3 units.

Graph Quadratic equations of the Form $y=(x-h)^2$

In the following exercises, graph each equation using a horizontal shift.

11. $y=(x-2)^2$
12. $y=(x-1)^2$
13. $y=(x+5)^2$

14. $y=(x+3)^2$
15. $y=(x-5)^2$
16. $y=(x+2)^2$

Answer

11.

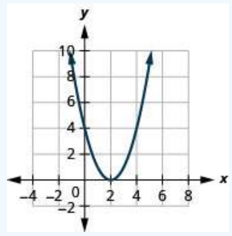


Figure 9.7.76

13.

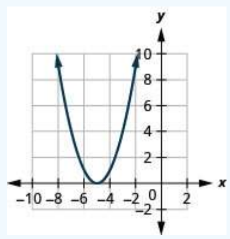


Figure 9.7.77

15.

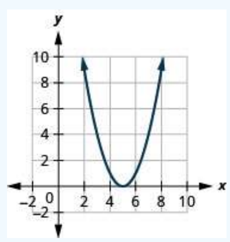


Figure 9.7.78

Graph Quadratic equations of the Form $y=(x-h)^2$

In the following exercises, graph each equation using transformations.

17. $y=(x+2)^2+1$
18. $y=(x+4)^2+2$
19. $y=(x-1)^2+5$
20. $y=(x-3)^2+4$
21. $y=(x+3)^2-1$
22. $y=(x+5)^2-2$
23. $y=(x-4)^2-3$
24. $y=(x-6)^2-2$

Answer

17.

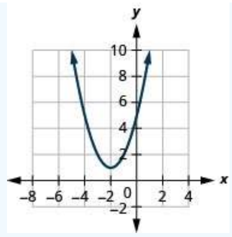


Figure 9.7.79

19.

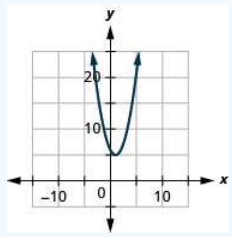


Figure 9.7.80

21.

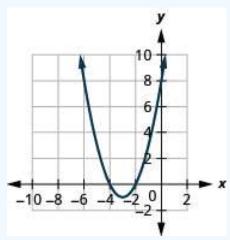


Figure 9.7.81

23.

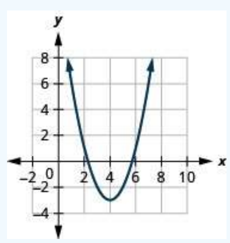


Figure 9.7.82

? Graph Quadratic equations of the Form $y = ax^2$

In the following exercises, graph each equation.

25. $y = -2x^2$
26. $y = 4x^2$
27. $y = -4x^2$
28. $y = -x^2$
29. $y = \frac{1}{2}x^2$
30. $y = \frac{1}{3}x^2$
31. $y = \frac{1}{4}x^2$
32. $y = -\frac{1}{2}x^2$

Answer

25.

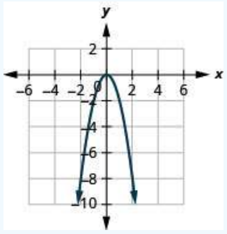


Figure 9.7.83

27.

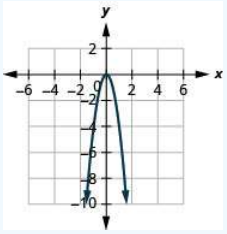


Figure 9.7.84

29.

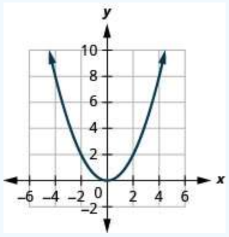


Figure 9.7.85

31.

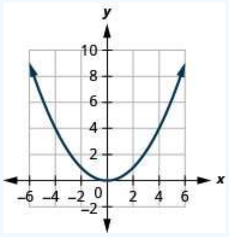


Figure 9.7.86

? Graph Quadratic equations Using Transformations

In the following exercises, rewrite each equation in the $y=a(x-h)^2+k$ form by completing the square.

33. $y=-3x^2-12x-5$
34. $y=2x^2-12x+7$
35. $y=3x^2+6x-1$
36. $y=-4x^2-16x-9$

Answer

33. $y=-3(x+2)^2+7$

35. $y=3(x+1)^2-4$

Graph Quadratic equations Using Transformations

In the following exercises,

a. Rewrite each equation in $y=a(x-h)^2+k$ form
b. Graph it by using transformations

37. $y=x^2+6x+5$
38. $(x)=x^2+4x-12$
39. $y=x^2+4x-12$
40. $y=x^2-6x+8$
41. $y=x^2-6x+15$
42. $y=x^2+8x+10$
43. $y=-x^2+8x-16$
44. $y=-x^2+2x-7$
45. $y=-x^2-4x+2$
46. $y=-x^2+4x-5$
47. $y=5x^2-10x+8$
48. $y=3x^2+18x+20$
49. $y=2x^2-4x+1$
50. $y=3x^2-6x-1$
51. $y=-2x^2+8x-10$
52. $y=-3x^2+6x+1$

Answer

37.

a. y=(x+3)^{2}-4
b.

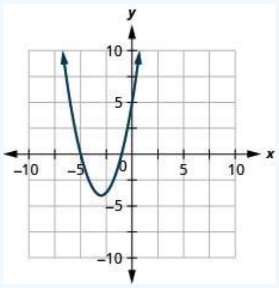


Figure 9.7.87

39.

a. $y=(x+2)^2-1$
b.

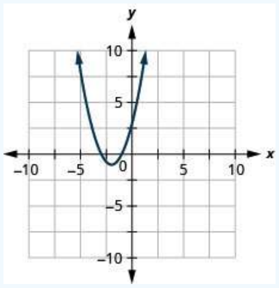


Figure 9.7.88

41.

a. $y=(x-3)^2+6$
b.

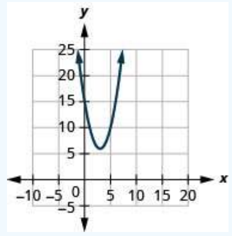


Figure 9.7.89

43.

a. $y=-(x-4)^2+0$
b.

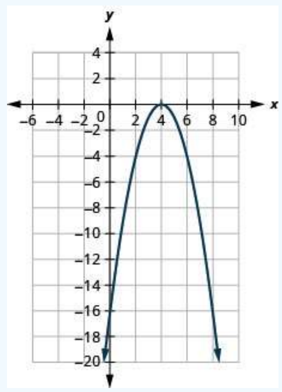


Figure 9.7.90

45.

a. $y=-(x+2)^2+6$
b.

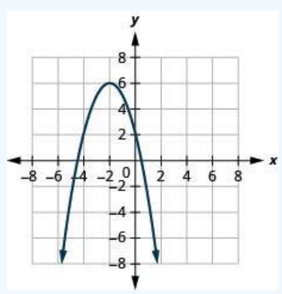


Figure 9.7.91

47.

a. $y=5(x-1)^2+3$
b.

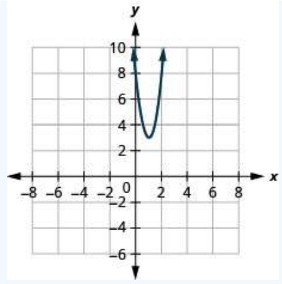


Figure 9.7.92

49.

a. $y=2(x-1)^2-1$
b.

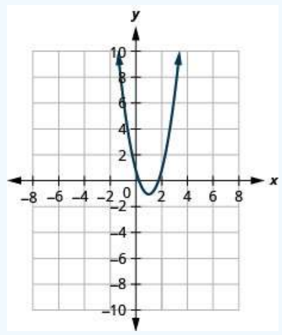


Figure 9.7.93

51.

a. $y=-2(x-2)^2-2$
b.

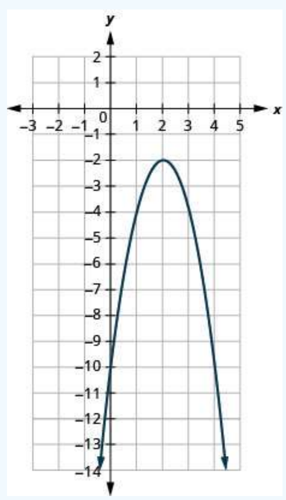


Figure 9.7.94

? Graph Quadratic equations Using Transformations

In the following exercises,

a. Rewrite each equation in $y=a(x-h)^2+k$ form
b. Graph it using properties

53. $y=2x^2+4x+6$
54. $y=3x^2-12x+7$

55. $y=-x^2+2x-4$
56. $y=-2x^2-4x-5$

Answer

53.

a. $y=2(x+1)^2+4$
b.

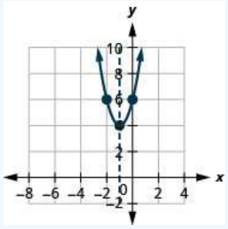


Figure 9.7.95

55.

a. $y=-(x-1)^2-3$
b.

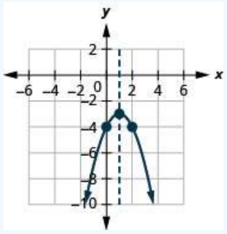


Figure 9.7.96

? Matching

In the following exercises, match the graphs to one of the following equations:

a. $y=x^2+4$
b. $y=x^2-4$
c. $y=(x+4)^2$
d. $y=(x-4)^2$
e. $y=(x+4)^2-4$
f. $y=(x+4)^2+4$
g. $y=(x-4)^2-4$
h. $y=(x-4)^2+4$

57.

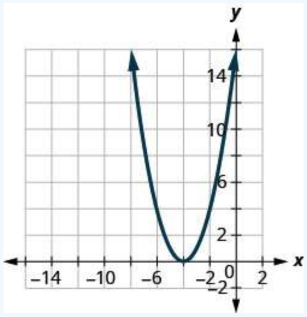


Figure 9.7.97

58.

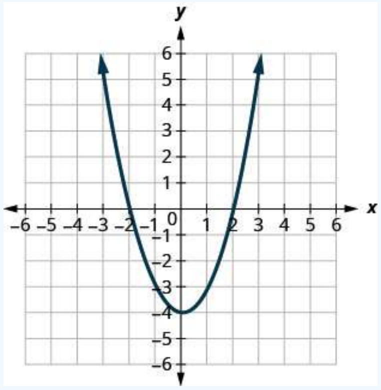


Figure 9.7.98

59.

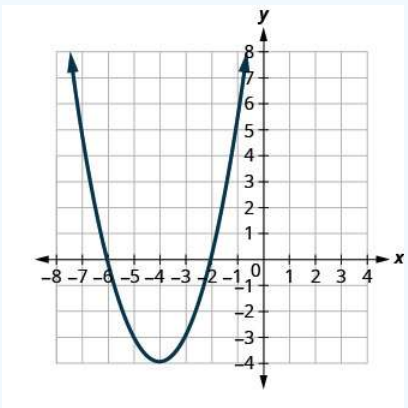


Figure 9.7.99

60.

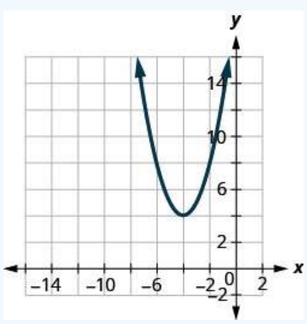


Figure 9.7.100

61.

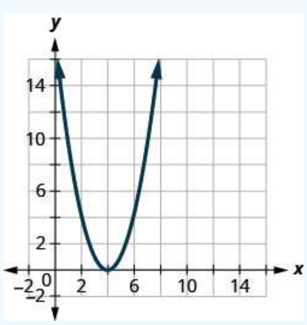


Figure 9.7.101

62.

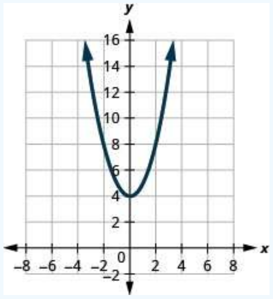


Figure 9.7.102

63.

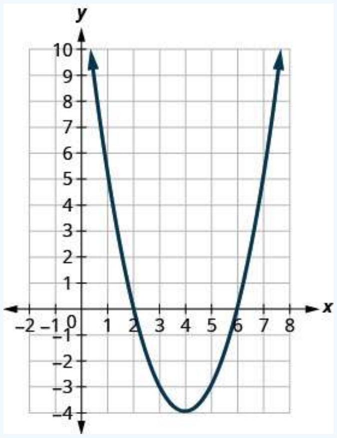


Figure 9.7.103

64.

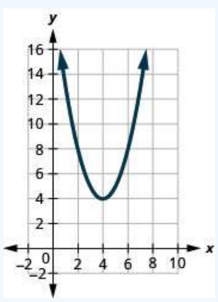


Figure 9.7.104

Answer

57. c

59. e

61. d

63. g

? Find a Quadratic equation from its Graph

In the following exercises, write the quadratic equation in $y = a(x-h)^2 + k$ form whose graph is shown.

65.

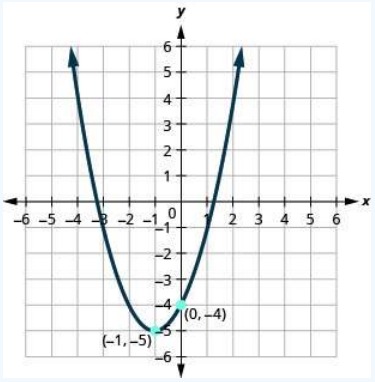


Figure 9.7.105

66.

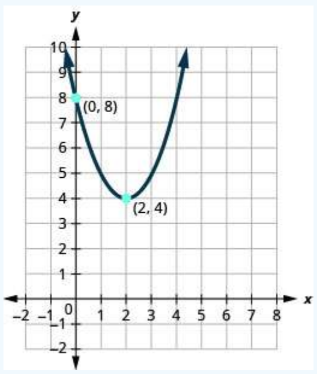


Figure 9.7.106

67.

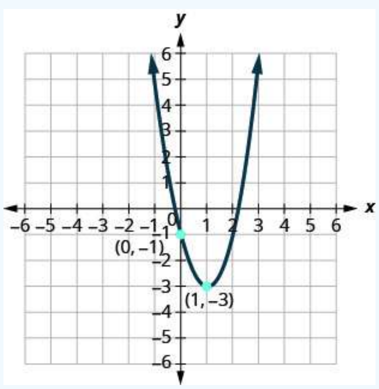


Figure 9.7.107

68.

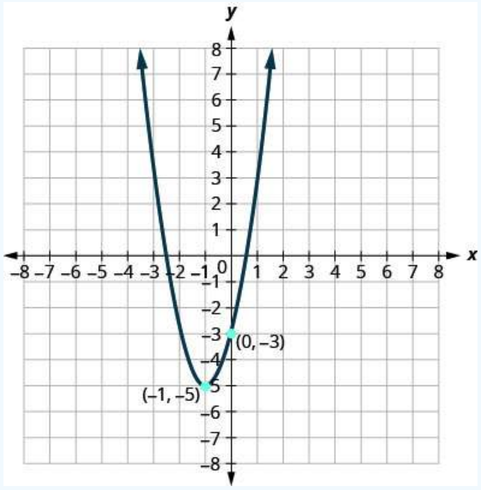


Figure 9.7.108

Answer

65. $y=(x+1)^2-5$

67. $y=2(x-1)^2-3$

? Writing Exercise

69. Graph the quadratic equation $y=x^2+4x+5$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?
70. Graph the quadratic equation $y=2x^2-4x-3$ first using the properties as we did in the last section and then graph it using transformations. Which method do you prefer? Why?

Answer

69. Answers may vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No--I don't get it!
graph quadratic equations of the form $y=x^2+k$.			
graph quadratic equations of the form $y=(x-h)^2$.			
graph quadratic equations of the form $y=ax^2$.			
graph quadratic equations using transformations.			
find a quadratic equation from its graph.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

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7.7: Chapter 7 Review Exercises

Chapter Review Exercises

Solve Quadratic Equations Using the Square Root Property

Exercise 7.7.1 Solve Quadratic Equations of the Form $ax^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $y^2 = 144$
2. $n^2 - 80 = 0$
3. $4a^2 = 100$
4. $2b^2 = 72$
5. $r^2 + 32 = 0$
6. $t^2 + 18 = 0$
7. $\frac{2}{3}w^2 - 20 = 30$
8. $5c^2 + 3 = 19$

Answer

1. $y = \pm 12$

3. $a = \pm 5$

5. $r = \pm 4\sqrt{2}i$

7. $w = \pm 5\sqrt{3}$

Exercise 7.7.2 Solve Quadratic Equations of the Form $a(x-h)^2 = k$ Using the Square Root Property

In the following exercises, solve using the Square Root Property.

1. $(p-5)^2 + 3 = 19$
2. $(u+1)^2 = 45$
3. $\left(x - \frac{1}{4}\right)^2 = \frac{3}{16}$
4. $\left(y - \frac{2}{3}\right)^2 = \frac{2}{9}$
5. $(n-4)^2 - 50 = 150$
6. $(4c-1)^2 = -18$
7. $n^2 + 10n + 25 = 12$
8. $64a^2 + 48a + 9 = 81$

Answer

1. $p = -1, 9$

3. $x = \frac{1}{4} \pm \frac{\sqrt{3}}{4}$

5. $n = 4 \pm 10\sqrt{2}$

7. $n = -5 \pm 2\sqrt{3}$

Solve Quadratic Equations by Completing the Square

Exercise 7.7.3 Solve Quadratic Equations Using Completing the Square

In the following exercises, complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

1. $x^2 + 22x$

2. $m^2 - 8m$
3. $a^2 - 3a$
4. $b^2 + 13b$

Answer

1. $(x+11)^2$

3. $\left(a - \frac{3}{2}\right)^2$

Exercise 7.7.4 Solve Quadratic Equations Using Completing the Square

In the following exercises, solve by completing the square.

1. $d^2 + 14d = -13$
2. $y^2 - 6y = 36$
3. $m^2 + 6m = -109$
4. $t^2 - 12t = -40$
5. $v^2 - 14v = -31$
6. $w^2 - 20w = 100$
7. $m^2 + 10m - 4 = -13$
8. $n^2 - 6n + 11 = 34$
9. $a^2 = 3a + 8$
10. $b^2 = 11b - 5$
11. $(u+8)(u+4) = 14$
12. $(z-10)(z+2) = 28$

Answer

1. $d = -13, -1$

3. $m = -3 \pm 10i$

5. $v = 7 \pm 3\sqrt{2}$

7. $m = -9, -1$

9. $a = \frac{3}{2} \pm \frac{\sqrt{41}}{2}$

11. $u = -6 \pm 2\sqrt{2}$

Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

Exercise 7.7.5 Solve Quadratic Equations of the Form $ax^2 + bx + c = 0$ by Completing the Square

In the following exercises, solve by completing the square.

1. $3p^2 - 18p + 15 = 15$
2. $5q^2 + 70q + 20 = 0$
3. $4y^2 - 6y = 4$
4. $2x^2 + 2x = 4$
5. $3c^2 + 2c = 9$
6. $4d^2 - 2d = 8$
7. $2x^2 + 6x = -5$
8. $2x^2 + 4x = -5$

Answer

1. $p = 0, 6$

3. $y=-\frac{1}{2}, 2$
5. $c=-\frac{1}{3} \pm \frac{2\sqrt{7}}{3}$
7. $x=\frac{3}{2} \pm \frac{1}{2} i$

? Exercise 7.7.6 Solve Quadratic Equations Using the Quadratic Formula

In the following exercises, solve by using the Quadratic Formula.

1. $4x^2-5x+1=0$
2. $7y^2+4y-3=0$
3. $r^2-r-42=0$
4. $t^2+13t+22=0$
5. $4v^2+v-5=0$
6. $2w^2+9w+2=0$
7. $3m^2+8m+2=0$
8. $5n^2+2n-1=0$
9. $6a^2-5a+2=0$
10. $4b^2-b+8=0$
11. $u(u-10)+3=0$
12. $5z(z-2)=3$
13. $\frac{1}{8}p^2-\frac{1}{5}p=-\frac{1}{20}$
14. $\frac{2}{5}q^2+\frac{3}{10}q=\frac{1}{10}$
15. $4c^2+4c+1=0$
16. $9d^2-12d=-4$

Answer

1. $x=\frac{1}{4}, 1$
3. $r=-6, 7$
5. $v=\frac{-1 \pm \sqrt{21}}{8}$
7. $m=\frac{-4 \pm \sqrt{10}}{3}$
9. $a=\frac{5}{12} \pm \frac{\sqrt{23}}{12} i$
11. $u=5 \pm \sqrt{21}$
13. $p=\frac{4 \pm \sqrt{5}}{5}$
15. $c=-\frac{1}{2}$

? Exercise 7.7.7 Use the Discriminant to Predict the Number of Solutions of a Quadratic Equation

In the following exercises, determine the number of solutions for each quadratic equation.

1. a. $9x^2-6x+1=0$
   b. $3y^2-8y+1=0$
   c. $7m^2+12m+4=0$
   d. $5n^2-n+1=0$
2. a. $5x^2-7x-8=0$
   b. $7x^2-10x+5=0$
   c. $25x^2-90x+81=0$
   d. $15x^2-8x+4=0$

Answer

1.

a. 1
b. 2
c. 2
d. 2

? Exercise 7.7.8 Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

In the following exercises, identify the most appropriate method (Factoring, Square Root, or Quadratic Formula) to use to solve each quadratic equation. Do not solve.

1. a. $16r^2-8r+1=0$
   b. $5t^2-8t+3=9$
   c. $3(c+2)^2=15$
2. a. $4d^2+10d-5=21$
   b. $25x^2-60x+36=0$
   c. $6(5v-7)^2=150$

Answer

1.

a. Factor
b. Quadratic Formula
c. Square Root

Solve Equations in Quadratic Form

? Exercise 7.7.9 Solve Equations in Quadratic Form

In the following exercises, solve.

1. $x^4-14x^2+24=0$
2. $x^4+4x^2-32=0$
3. $4x^4-5x^2+1=0$
4. $(2y+3)^2+3(2y+3)-28=0$
5. $x+3\sqrt{x}-28=0$
6. $6x+5\sqrt{x}-6=0$
7. $x^{\frac{2}{3}}-10x^{\frac{1}{3}}+24=0$
8. $x+7x^{\frac{1}{2}}+6=0$
9. $8x^{-2}-2x^{-1}-3=0$

Answer

1. $x=\pm\sqrt{2}, x=\pm2\sqrt{3}$

3. $x=\pm1, x=\pm\frac{1}{2}$

5. $x=16$

7. $x=64, x=216$

9. $x=-2, x=\frac{4}{3}$

Exercise 7.7.10 Solve Applications Modeled by Quadratic Equations

In the following exercises, solve by using the method of factoring, the square root principle, or the Quadratic Formula. Round your answers to the nearest tenth, if needed.

1. Find two consecutive odd numbers whose product is 323.
2. Find two consecutive even numbers whose product is 624.
3. A triangular banner has an area of 351 square centimeters. The length of the base is two centimeters longer than four times the height. Find the height and length of the base.
4. Julius built a triangular display case for his coin collection. The height of the display case is six inches less than twice the width of the base. The area of the of the back of the case is 70 square inches. Find the height and width of the case.
5. A tile mosaic in the shape of a right triangle is used as the corner of a rectangular pathway. The hypotenuse of the mosaic is 5 feet. One side of the mosaic is twice as long as the other side. What are the lengths of the sides? Round to the nearest tenth.

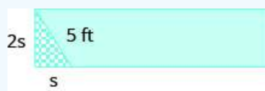


Figure 9.E.1

6. A rectangular piece of plywood has a diagonal which measures two feet more than the width. The length of the plywood is twice the width. What is the length of the plywood's diagonal? Round to the nearest tenth.

7. The front walk from the street to Pam's house has an area of 250 square feet. Its length is two less than four times its width. Find the length and width of the sidewalk. Round to the nearest tenth.

8. For Sophia's graduation party, several tables of the same width will be arranged end to end to give serving table with a total area of 75 square feet. The total length of the tables will be two more than three times the width. Find the length and width of the serving table so Sophia can purchase the correct size tablecloth . Round answer to the nearest tenth.

9. A ball is thrown vertically in the air with a velocity of 160 ft/sec. Use the formula $h = -16t^2 + v_0 t$ to determine when the ball will be 384 feet from the ground. Round to the nearest tenth.

10. The couple took a small airplane for a quick flight up to the wine country for a romantic dinner and then returned home. The plane flew a total of 5 hours and each way the trip was 360 miles. If the plane was flying at 150 mph, what was the speed of the wind that affected the plane?

11. Ezra kayaked up the river and then back in a total time of 6 hours. The trip was 4 miles each way and the current was difficult. If Roy kayaked at a speed of 5 mph, what was the speed of the current?

12. Two handymen can do a home repair in 2 hours if they work together. One of the men takes 3 hours more than the other man to finish the job by himself. How long does it take for each handyman to do the home repair individually?

Answer

2. Two consecutive even numbers whose product is 624 are 24 and 26, and -24 and -26.

4. The height is 14 inches and the width is 10 inches.

6. The length of the diagonal is 3.6 feet.

8. The width of the serving table is 4.7 feet and the length is 16.1 feet.

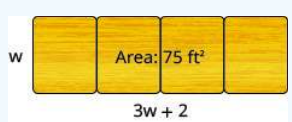


Figure 9.E.2

10. The speed of the wind was 30 mph.

12. One man takes 3 hours and the other man 6 hours to finish the repair alone.

Graph Quadratic equations Using Properties

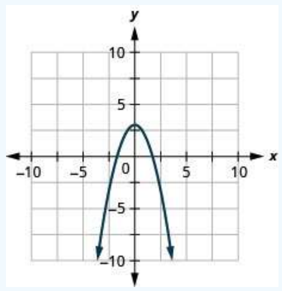
Exercise 7.7.11 Recognize the Graph of a Quadratic equation

In the following exercises, graph by plotting point.

1. Graph $y=x^2-2$
2. Graph $y=-x^2+3$

Answer

2.



Exercise 7.7.12 Recognize the Graph of a Quadratic equation

In the following exercises, determine if the following parabolas open up or down.

1. a. $y=-3x^2+3x-1$
   b. $y=5x^2+6x+3$
2. a. $y=x^2+8x-1$
   b. $y=-4x^2-7x+1$

Answer

2.

a. Up
b. Down

Exercise 7.7.13 Find the Axis of Symmetry and Vertex of a Parabola

In the following exercises, find

a. The equation of the axis of symmetry
b. The vertex
   1. $y=-x^2+6x+8$
   2. $y=2x^2-8x+1$

Answer

2. $x=2$; $(2,-7)$

Exercise 7.7.14 Find the Intercepts of a Parabola

In the following exercises, find the x- and y-intercepts.

1. $y=x^2-4x+5$
2. $y=x^2-8x+15$

3. $y = x^2 - 4x + 10$
4. $y = -5x^2 - 30x - 46$
5. $y = 16x^2 - 8x + 1$
6. $y = x^2 + 16x + 64$

Answer

2. $y : (0, 15)$
   $x : (3, 0), (5, 0)$
4. $y : (0, -46)$
   $x :$ none
6. $y : (0, -64)$
   $x : (-8, 0)$

Graph Quadratic equations Using Properties

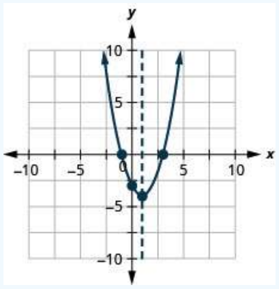
Exercise 7.7.15 Graph Quadratic equations Using Properties

In the following exercises, graph by using its properties.

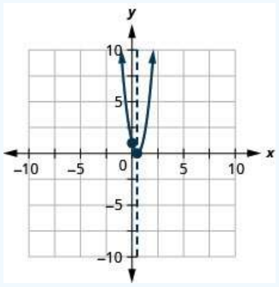
1. $y = x^2 + 8x + 15$
2. $y = x^2 - 2x - 3$
3. $y = -x^2 + 8x - 16$
4. $y = 4x^2 - 4x + 1$
5. $y = x^2 + 6x + 13$
6. $y = -2x^2 - 8x - 12$

Answer

2.



4.



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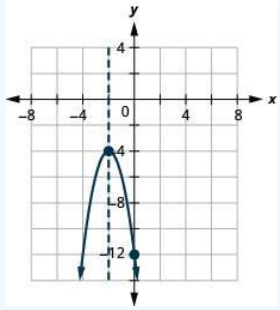


? Exercise 7.7.16 Solve Maximum and Minimum Applications

In the following exercises, find the minimum or maximum value.

1. $y = 7x^2 + 14x + 6$
2. $y = -3x^2 + 12x - 10$

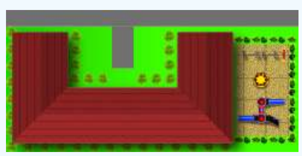
Answer

2. The maximum value is 2 when $x = 2$.

? Exercise 7.7.17 Solve Maximum and Minimum Applications

In the following exercises, solve. Rounding answers to the nearest tenth.

1. A ball is thrown upward from the ground with an initial velocity of 112 ft/sec. Use the quadratic equation $h = -16t^2 + 112t$ to find how long it will take the ball to reach maximum height, and then find the maximum height.
2. A daycare facility is enclosing a rectangular area along the side of their building for the children to play outdoors. They need to maximize the area using 180 feet of fencing on three sides of the yard. The quadratic equation $A = -2x^2 + 180x$ gives the area, A, of the yard for the length, x, of the building that will border the yard. Find the length of the building that should border the yard to maximize the area, and then find the maximum area.



Answer

2. The length adjacent to the building is 90 feet giving a maximum area of $4,050$ square feet.

Graph Quadratic equations Using Transformations

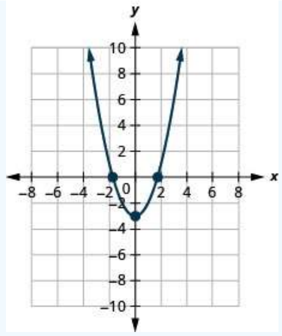
? Exercise 7.7.18 Graph Quadratic equations of the Form $y = x^2 + k$

In the following exercises, graph each equation using a vertical shift.

1. $y = x^2 + 4$
2. $y = x^2 - 3$

Answer

2.



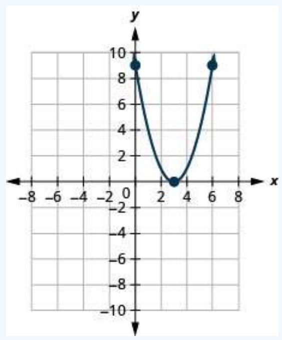
? Exercise 7.7.19 Graph Quadratic equations of the Form $y = x^2 + k$

In the following exercises, graph each equation using a horizontal shift.

1. $y = (x+1)^2$
2. $y = (x-3)^2$

Answer

2.



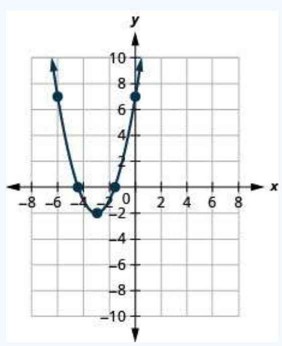
? Exercise 7.7.20 Graph Quadratic equations of the Form $y = x^2 + k$

In the following exercises, graph each equation using transformations.

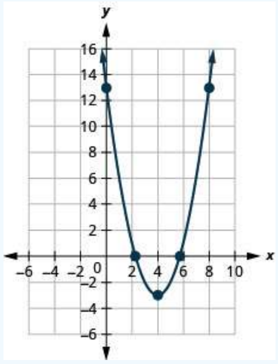
1. $y = (x+2)^2 + 3$
2. $y = (x+3)^2 - 2$
3. $y = (x-1)^2 + 4$
4. $y = (x-4)^2 - 3$

Answer

2.



4.



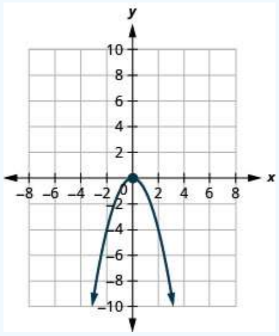
Exercise 7.7.21 Graph Quadratic equations of the Form $y = ax^2$

In the following exercises, graph each equation.

1. $y = 2x^2$
2. $y = -x^2$
3. $y = \frac{1}{2}x^2$

Answer

2.



Exercise 7.7.22 Graph Quadratic equations Using Transformations

In the following exercises, rewrite each equation in the $y = a(x-h)^2 + k$ form by completing the square.

1. $y = 2x^2 - 4x - 4$
2. $y = 3x^2 + 12x + 8$

Answer

1. $y = 2(x-1)^2 - 6$

Exercise 7.7.23 Graph Quadratic equations Using Transformations

In the following exercises,

a. Rewrite each equation in $y = a(x-h)^2 + k$ form
b. Graph it by using transformations

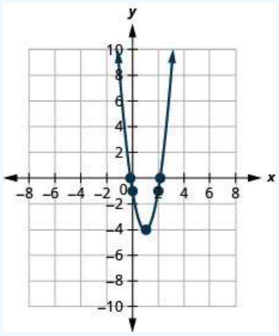
1. $y = 3x^2 - 6x - 1$
2. $y = -2x^2 - 12x - 5$
3. $y = 2x^2 + 4x + 6$

4. $y = 3x^2 - 12x + 7$

Answer

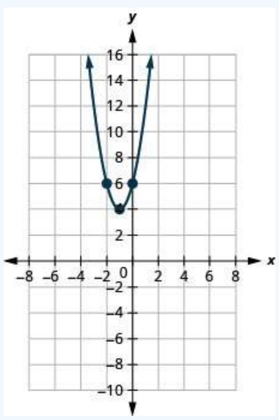
1.

a. $y = 3(x-1)^2 - 4$
b.



3.

a. $y = 2(x+1)^2 + 4$
b.



Exercise 7.7.24 Graph Quadratic equations Using Transformations

In the following exercises,

a. Rewrite each equation in $y = a(x-h)^2 + k$ form
b. Graph it using properties

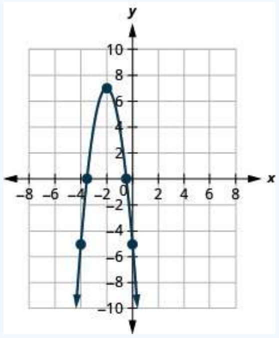
1. $y = -3x^2 - 12x - 5$
2. $y = 2x^2 - 12x + 7$

Answer

1.

a. $y = -3(x+2)^2 + 7$

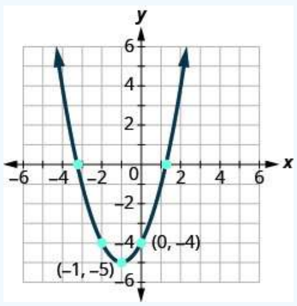
b.



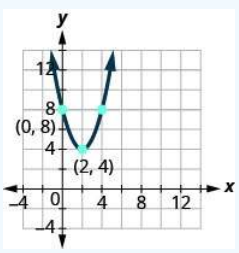
Exercise 7.7.25 Find a Quadratic equation From its Graph

In the following exercises, write the quadratic equation in $y=a(x-h)^2+k$ form.

1.



2.



Answer

1. $y=(x+1)^2-5$

Practice Test

Exercise 7.7.28

1. Use the Square Root Property to solve the quadratic equation $3(w+5)^2=27$.
2. Use Completing the Square to solve the quadratic equation $a^2-8a+7=23$.
3. Use the Quadratic Formula to solve the quadratic equation $2m^2-5m+3=0$.

Answer

1. $w=-2, w=-8$

3. $m=1, m=\frac{3}{2}$

Exercise 7.7.29

Solve the following quadratic equations. Use any method.

1. $2x(3x-2)-1=0$
2. $\frac{9}{4}y^2-3y+1=0$

Answer

2. $y=\frac{2}{3}$

Exercise 7.7.30

Use the discriminant to determine the number and type of solutions of each quadratic equation.

1. $6p^2-13p+7=0$
2. $3q^2-10q+12=0$

Answer

2. 2 complex

Exercise 7.7.31

Solve each equation.

1. $4x^4-17x^2+4=0$
2. $y^{\frac{2}{3}}+2y^{\frac{1}{3}}-3=0$

Answer

2. $y=1, y=-27$

Exercise 7.7.32

For each parabola, find

a. Which direction it opens
b. The equation of the axis of symmetry
c. The vertex
d. The x-and y-intercepts
e. The maximum or minimum value

1. $y=3x^2+6x+8$
2. $y=-x^2-8x+16$

Answer

2.

a. down
b. $x=-4$
c. $(-4,0)$
d. $y:(0,16); x:(-4,0)$
e. minimum value of -4 when $x=0$

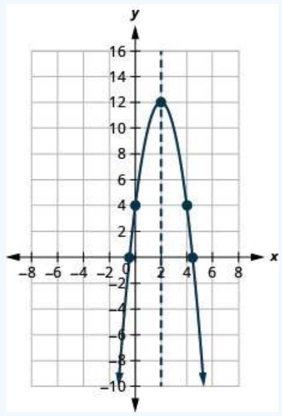
Exercise 7.7.33

Graph each quadratic equation using intercepts, the vertex, and the equation of the axis of symmetry.

1. $y=x^2+6x+9$
2. $y=-2x^2+8x+4$

Answer

2.



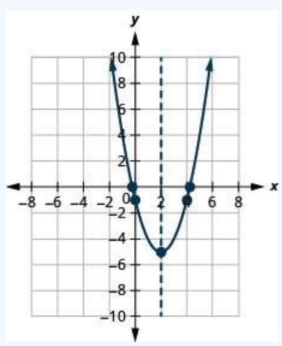
Exercise 7.7.34

In the following exercises, graph each equation using transformations.

1. $y=(x+3)^2+2$
2. $y=x^2-4x-1$

Answer

2.



Exercise 7.7.36

Model the situation with a quadratic equation and solve by any method.

1. Find two consecutive even numbers whose product is 360.
2. The length of a diagonal of a rectangle is three more than the width. The length of the rectangle is three times the width. Find the length of the diagonal. (Round to the nearest tenth.)

Answer

2. A water balloon is launched upward at the rate of 86 ft/sec. Using the formula $h=-16t^2+86t$ find how long it will take the balloon to reach the maximum height, and then find the maximum height. Round to the nearest tenth.

CHAPTER OVERVIEW

8: Conics

In this chapter, you will learn about conics, including circles, parabolas, ellipses, and hyperbolas. Then you will use what you learn to investigate systems of nonlinear equations.

Topic hierarchy

8.1: More Parabolas

8.2: Distance and Midpoint Formulas and Circles

8.3: Solve Systems of Nonlinear Equations

8.4: Chapter 8 Review Exercises

8.1: More Parabolas

Learning Objectives

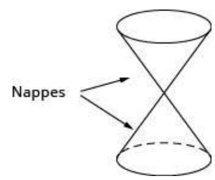
By the end of this section, you will be able to:

- Graph vertical parabolas
- Graph horizontal parabolas
- Solve applications with parabolas

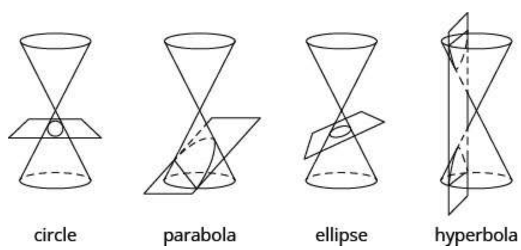
Before you get started, take this readiness quiz.

1. Graph: $y=-3x^2+12x-12$.
2. Solve by completing the square: $x^2-6x+6=0$.
3. Write in standard form: $y=3x^2-6x+5$.

In this chapter we will be looking at the conic sections, usually called the conics, and their properties. The conics are curves that result from a plane intersecting a double cone—two cones placed point-to-point. Each half of a double cone is called a nappe.



There are four conics—the **circle**, **parabola**, **ellipse**, and **hyperbola**. The next figure shows how the plane intersecting the double cone results in each curve.



Each of the curves has many applications that affect your daily life, from your cell phone to acoustics and navigation systems. In this section we will look at the properties of a parabola

Graph Vertical Parabolas

The next conic section we will look at is a **parabola**.

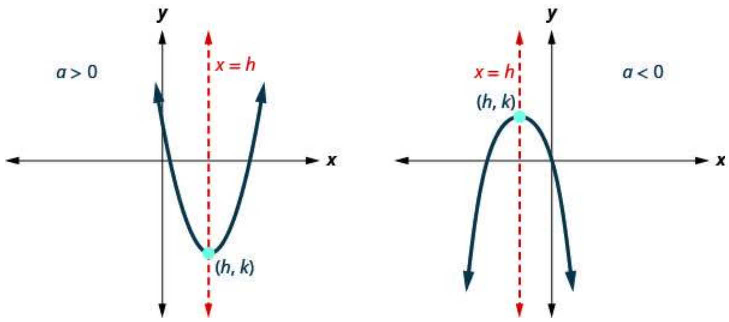
Previously, we learned to graph vertical parabolas from the general form or the standard form using properties. Those methods will also work here. We will summarize the properties here.

Vertical Parabolas

	General form $y=ax^2+bx+c$	**Standard Form** $y=a(x-h)^2+k$
Orientation	$a>0$ up; $a<0$ down	$a>0$ up; $a<0$ down
Axis of Symmetry	$x=-\frac{b}{2a}$	$x=h$

	General form $y=ax^2+bx+c$	Standard Form $y=a(x-h)^2+k$
Vertex	Substitute $x=-\frac{b}{2a}$ and solve for y.	(h,k)
y-intercept	Let $x=0$	Let $x=0$
x-intercepts	Let $y=0$	Let $y=0$

The graphs show what the parabolas look like when they open up or down. Their position in relation to the x- or y-axis is merely an example.



To graph a parabola from these forms, we used the following steps.

Note

How to Graph Vertical Parabolas $y=ax^2+bx+c$ or $y=a(x-h)^2+k$ **using Properties**.

Step 1: Determine whether the parabola opens upward or downward.

Step 2. Find the axis of symmetry.

Step 3. Find the vertex.

Step 4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.

Step 5. Find the x-intercepts.

Step 6. Graph the parabola.

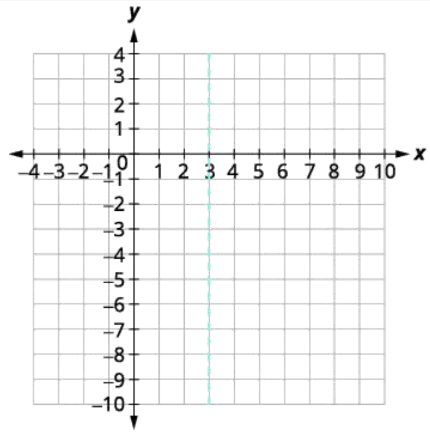
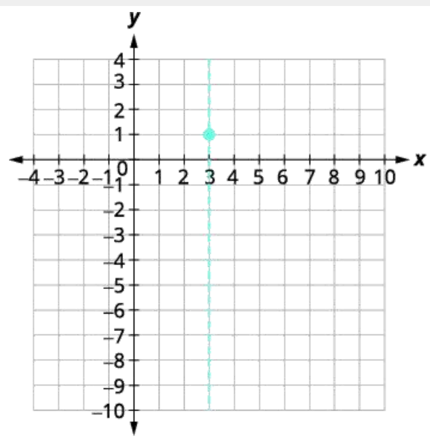
The next example reviews the method of graphing a parabola from the general form of its equation.

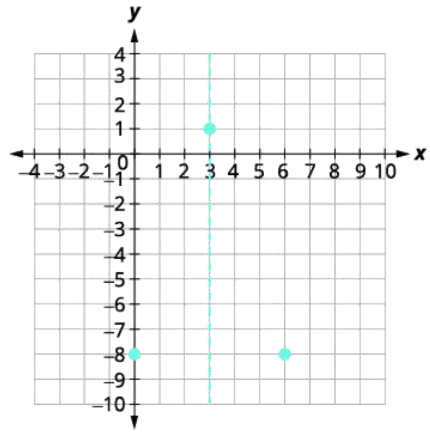
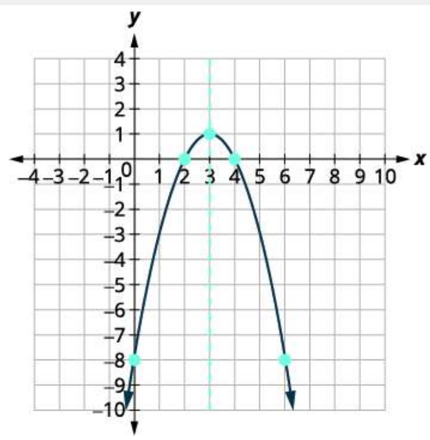
Example 8.1.1

Graph $y=-x^2+6x-8$ by using properties.

Solution:

	$\overset{y=-x^2+6x-8}{y=ax^2+bx+c}$
Since a is -1, the parabola opens downward.	
	
To find the axis of symmetry, find $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$

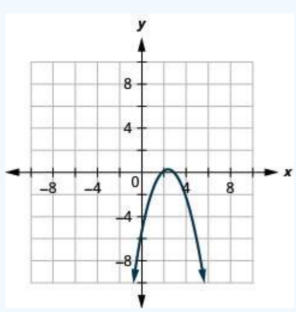
	$x = -\frac{6}{2(-1)}$ $x = 3$
Identify the axis of symmetry	The axis of symmetry is $x = 3$.
	
Find the vertex. Note the vertex is on the line $x = 3$.	
Let $x = 3$ and find y so that $(3, y)$ is on the graph, i.e., is a solution to the equation.	$y = -x^2 + 6x - 8$ $y = -(3)^2 + 6(3) - 8$ $y = -9 + 18 - 8$ $y = 1$
	The vertex is $(3, 1)$.
	
The y-intercept occurs when $x = 0$. Substitute $x = 0$.	$y = -x^2 + 6x - 8$ $y = -(0)^2 + 6(0) - 8$ $y = -8$
	The y-intercept is $(0, -8)$.
The point $(0, -8)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -8)$.	Point symmetric to the y-intercept is $(6, -8)$.

	
The x-intercept occurs when $y=0$. Let $y=0$.	$0=-x^2+6x-8$
Factor the GCF.	$0=-(x^2-6x+8)$
Factor the trinomial.	$0=-(x-4)(x-2)$
Solve for x.	$x-4=0$ or $x-2=0$ $x=4$ or $x=2$
	The x-intercepts are $(4,0)$, $(2,0)$.
Graph the parabola.	

Try It 8.1.2

Graph $y=-x^2+5x-6$ by using properties.

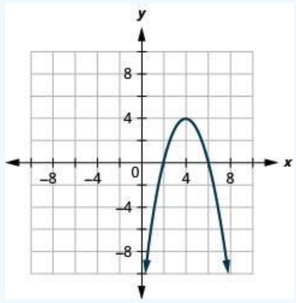
Answer



? Try It 8.1.3

Graph $y=-x^2+8x-12$ by using properties.

Answer



The next example reviews the method of graphing a **parabola** from the standard form of its equation, $y=a(x-h)^2+k$.

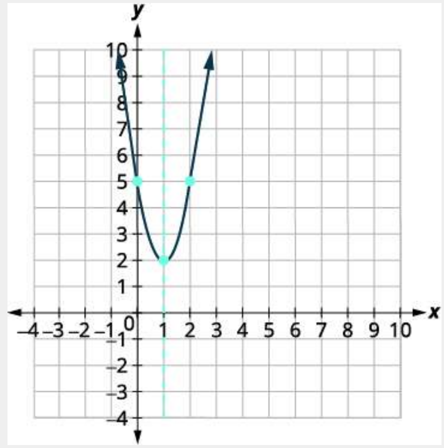
✓ Example 8.1.4

Write $y=3x^2-6x+5$ in standard form and then use properties of standard form to graph the equation.

Solution:

Rewrite the equation in $y=a(x-h)^2+k$ form by completing the square.	
	$y=3(x-1)^2+2$
Identify the constants a,h,k.	$a=3, h=1, k=2$
Since $a=2$, the parabola opens upward.	
	
The axis of symmetry is $x=h$.	The axis of symmetry is $x=1$.
The vertex is (h,k).	The vertex is $(1,2)$.
Find the y-intercept by substituting $x=0$,	$y=3(x-1)^2+2$ $y=3\cdot 0^2-6\cdot 0+5$
	$y=5$
	y-intercept $(0,5)$
Find the point symmetric to $(0,5)$ across the axis of symmetry.	$(2,5)$
Find the x-intercepts.	
	The square root of a negative number tells us the solutions are complex numbers. So there are no x-intercepts.

Graph the parabola.

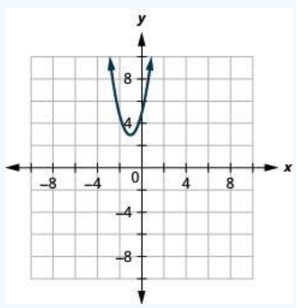


Try It 8.1.5

a. Write $y=2x^{2}+4x+5$ in standard form and
b. use properties of standard form to graph the equation.

Answer

a. $y=2(x+1)^{2}+3$
b.

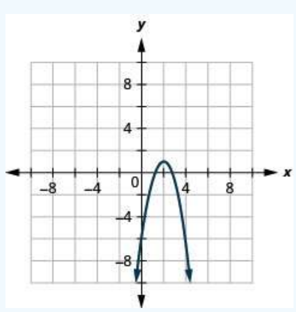


Try It 8.1.6

a. Write $y=-2x^{2}+8x-7$ in standard form and
b. use properties of standard form to graph the equation.

Answer

a. $y=-2(x-2)^{2}+1$
b.



Graph Horizontal Parabolas

Our work so far has only dealt with parabolas that open up or down. We are now going to look at horizontal parabolas. These parabolas open either to the left or to the right. If we interchange the x and y in our previous equations for parabolas, we get the equations for the parabolas that open to the left or to the right.

Horizontal Parabolas

	General form $x=ay^2+by+c$	Standard form $x=a(y-k)^2+h$
Orientation	$a>0$ right; $a<0$ left	$a>0$ right; $a<0$ left
Axis of Symmetry	$y=-\frac{b}{2a}$	$y=k$
Vertex	Substitute $y=-\frac{b}{2a}$ and solve for x.	(h,k)
x-intercepts	Let $x=0$	Let $x=0$
y-intercept	Let $y=0$	Let $y=0$

The graphs show what the parabolas look like when they to the left or to the right. Their position in relation to the x- or y-axis is merely an example.

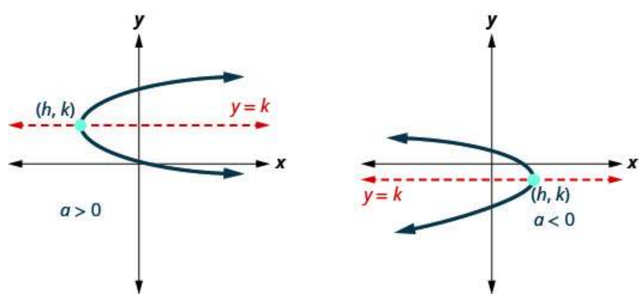


Figure 11.2.30

To graph a **parabola** that opens to the left or to the right is basically the same as what we did for parabolas that open up or down, with the reversal of the x and y variables. The graph is then the one we would get by graphing the equation with the x and y exchanged, and exchanging the x and y coordinates (e.g., if (2,3) is on the graph of the equation with the exchange, then (3,2) is on the graph of the given equation).

Graph Horizontal Parabolas $x=ay^2+by+c$ or $x=a(y-h)^2+k$ using Properties

- Step 1: Determine whether the parabola opens to the left or to the right.
- Step 2: Find the axis of symmetry.
- Step 3: Find the vertex.
- Step 4: Find the x-intercept. Find the point symmetric to the x-intercept across the axis of symmetry.
- Step 5: Find the y-intercepts.
- Step 6: Graph the parabola.

Example 8.1.7

Graph $x=2y^2$ by using properties.

Solution:

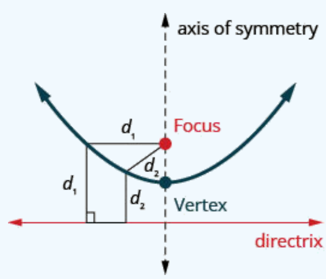
	$\overset{x=2y^2}{x = ay^2 + by + c}$
Since $a = 2$, the parabola opens to the right.	
	
To find the axis of symmetry, find $y = -\frac{b}{2a}$	$y = -\frac{b}{2a}$ $y = -\frac{0}{2(2)}$ $y = 0$ The axis of symmetry is $y = 0$.
The vertex is on the line $y = 0$. Let $y = 0$.	$x = 2y^2$ $x = 2 \cdot (0)^2$ $x = 0$ The vertex is $(0, 0)$.

Since the vertex is $(0, 0)$, both the x- and y-intercepts are the point $(0, 0)$. To graph the parabola we need more points. In this case it is easiest to choose values of y.

y	x	(x, y)
1	$x = 2 \cdot 1^2 = 2$	$(1, 2)$
2	$x = 2 \cdot (2)^2 = 2 \cdot 4 = 8$	$(8, 2)$

We also plot the points symmetric to $(2, 1)$ and $(8, 2)$ across the y-axis, the points $(2, -1)$, $(8, -2)$.

Graph the parabola.



? Try It 8.1.8

Graph $x = y^2$ by using properties.

Answer

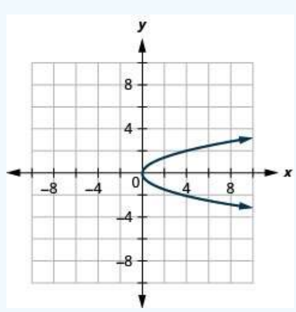


Figure 11.2.40

Try It 8.1.9

Graph $x=-y^2$ by using properties.

Answer

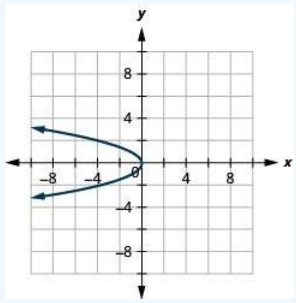


Figure 11.2.41

In the next example, the vertex is not the origin.

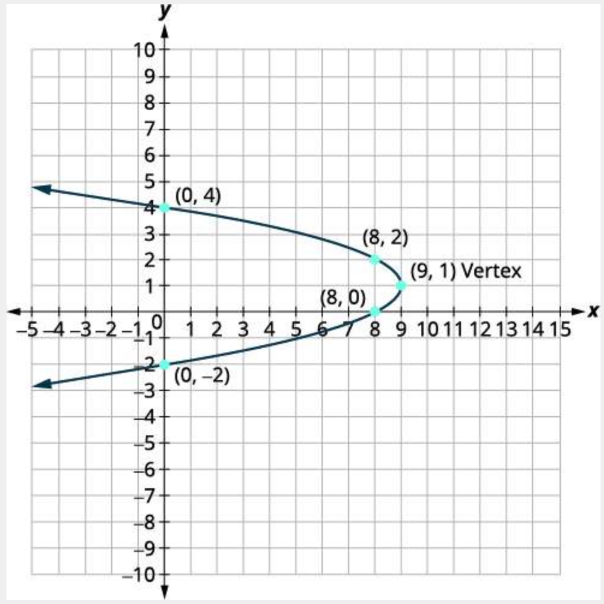
Example 8.1.10

Graph $x=-y^2+2y+8$ by using properties.

Solution:

Table 11.2.6

	$\overset{x=-y^2+2y+8}{x=ay^2+by+c}$
Since $a=-1$, the parabola opens to the left.	
	
To find the axis of symmetry, find $y=-\frac{b}{2a}$	$y=-\frac{b}{2a}$ $y=-\frac{2}{2(-1)}$ $y=1$ The axis of symmetry is $y=1$.
The vertex is on the line $y=1$. Let $y=1$.	$x=-y^2+2y+8$ $x=-(1)^2+2(1)+8$ $x=9$ The vertex is $(9,1)$.
The x-intercept occurs when $y=0$.	$x=-y^2+2y+8$ $x=-(0)^2+2(0)+8$ $x=8$ The x-intercept is $(8,0)$.
The point $(8,0)$ is one unit below the line of symmetry. The symmetric point one unit above the line of symmetry is $(8,2)$	Symmetric point is $(8,2)$.
The y-intercept occurs when $x=0$. Substitute $x=0$.	$x=-y^2+2y+8$ $0=-y^2+2y+8$

Solve.	$y^2-2y-8=0$ $(y-4)(y+2)=0$ $y-4=0$ or $y+2=0$ $y=4$ or $y=-2$ The y -intercepts are $(0,4)$ and $(0,-2)$.
Connect the points to graph the parabola.	

Try It 8.1.11

Graph $x=-y^2-4y+12$ by using properties.

Answer

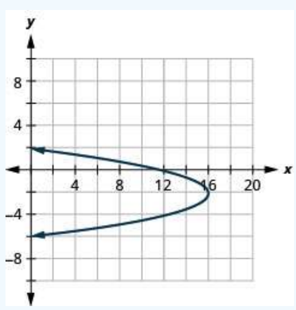


Figure 11.2.58

Try It 8.1.12

Graph $x=-y^2+2y-3$ by using properties.

Answer

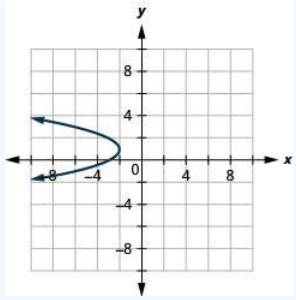


Figure 11.2.59

We will use the properties of the parabola in the next example.

Example 8.1.13

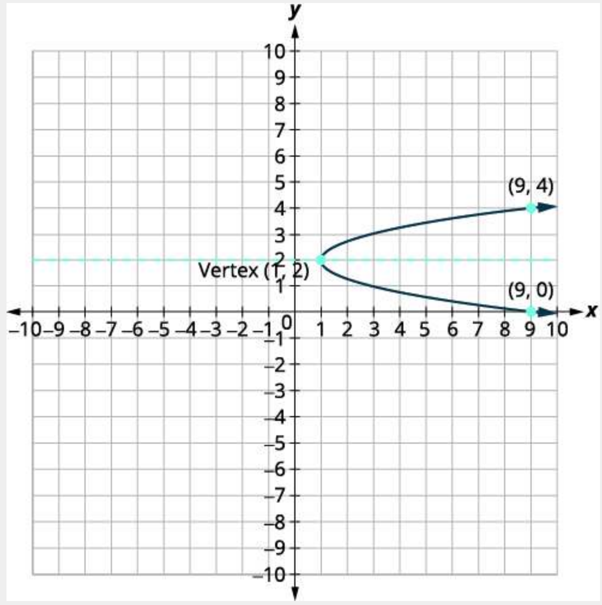
Graph $x=2(y-2)^2+1$ using properties.

Solution:

Table 11.2.7

	$\overset{x=2(y-2)^2+1}{x=a(y-k)^2+h}$
Identify the constants a, h, k.	$a=2, h=1, k=2$
Since $a=2$, the parabola opens to the right.	
	
The axis of symmetry is $y=k$.	The axis of symmetry is $y=2$.
The vertex is (h, k).	The vertex is $(1, 2)$.
Find the x-intercept by substituting $y=0$.	$x=2(y-2)^2+1$ $x=2(0-2)^2+1$ $x=9$
	The x-intercept is $(9, 0)$.
Find the point symmetric to $(9, 0)$ across the axis of symmetry.	$(9, 4)$
Find the y-intercepts. Let $x=0$.	
	A square cannot be negative, so there is no real solution. So there are no y-intercepts.

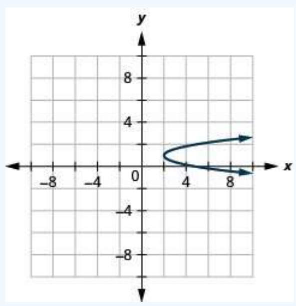
Graph the parabola.



? Try It 8.1.14

Graph $x=3(y-1)^2+2$ using properties.

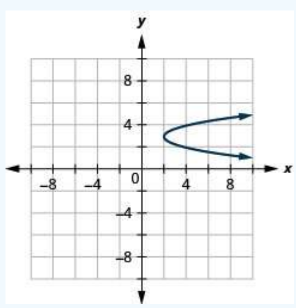
Answer



? Try It 8.1.15

Graph $x=2(y-3)^2+2$ using properties.

Answer

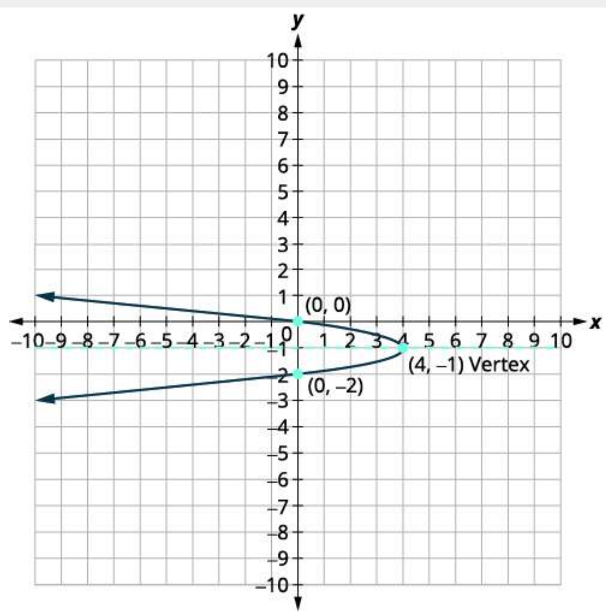


In the next example, we notice the a is negative and so the parabola opens to the left.

Example 8.1.16

Graph $x=-4(y+1)^2+4$ using properties.

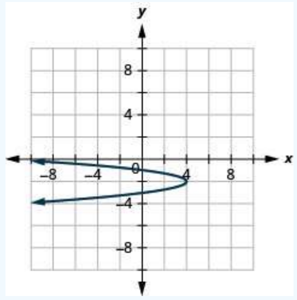
Solution:

	$\overset{x=-4(y+1)^2+4}{x=a(y-k)^2+h}$
Identify the constants a, h, k.	$a=-4, h=4, k=-1$
Since $a=-4$, the parabola opens to the left.	
	
The axis of symmetry is $y=k$.	The axis of symmetry is $y=-1$.
The vertex is (h,k).	The vertex is $(4,-1)$.
Find the x-intercept by substituting $y=0$.	$x=-4(y+1)^2+4$ $x=-4(0+1)^2+4$ $x=0$
	The x-intercept is $(0,0)$.
Find the point symmetric to $(0,0)$ across the axis of symmetry.	$(0,-2)$
Find the y-intercepts.	$x=-4(y+1)^2+4$
Let $x=0$.	
	$y=-1+1 \quad y=-1-1$
	$y=0 \qquad y=-2$
	The y-intercepts are $(0,0)$ and $(0,-2)$.
Graph the parabola.	

Try It 8.1.17

Graph $x=-4(y+2)^2+4$ using properties.

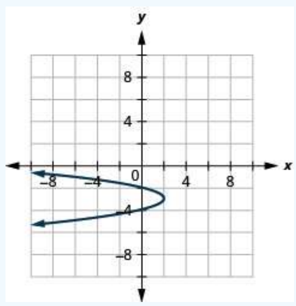
Answer



Try It 8.1.18

Graph $x=-2(y+3)^2+2$ using properties.

Answer



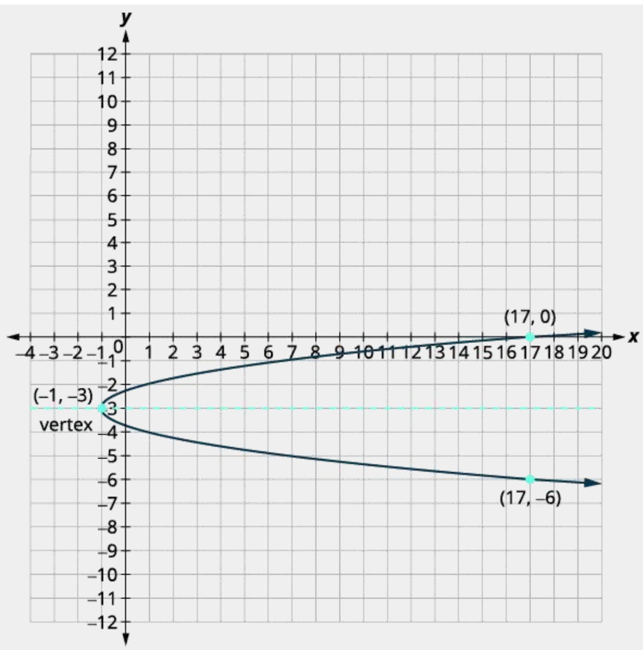
The next example requires that we first put the equation in standard form and then use the properties.

Example 8.1.19

Write $x=2y^2+12y+17$ in standard form and then use the properties of the standard form to graph the equation.

Solution:

	$x=2y^2+12y+17$
Rewrite the equation in $x=a(y-k)^2+h$ form by completing the square.	$x=2(y^2+6y)+17$ $x=2((y^2+6y+9)-9)+17$ $x=2(y^2+6y+9)-18+17$ $x=2(y+3)^2-1$
Identify the constants a, h, k.	$\overset{x=2(y+3)^2-1}{x=a(y-k)^2+h}$ $a=2, h=-1, k=-3$
Since $a=2$, the parabola opens to the right.	

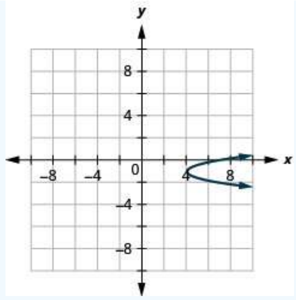
	
The axis of symmetry is $y=k$.	The axis of symmetry is $y=-3$.
The vertex is (h,k).	The vertex is $(-1,-3)$.
Find the x-intercept by substituting $y=0$.	$x=2(y+3)^2-1$ $x=2(0+3)^2-1$ $x=17$
	The x-intercept is $(17,0)$.
Find the point symmetric to $(17,0)$ across the axis of symmetry.	$(17,-6)$
Find the y-intercepts. Let $x=0$.	$0=2(y+3)^2-1$ $(y+3)^2=\frac{1}{2}$ $y+3=\pm\frac{1}{\sqrt{2}}=\pm\frac{\sqrt{2}}{2}$ $y=-3+\frac{\sqrt{2}}{2}\quad y=-3-\frac{\sqrt{2}}{2}$
	The y-intercepts are $\left(0,-3+\frac{\sqrt{2}}{2}\right),\left(0,-3-\frac{\sqrt{2}}{2}\right)$.
Approximate so that the points can be located on the graph.	The y intercepts are approximately $(0,-2.3)$ and $(0,-3.7)$
Graph the parabola.	

? Try It 8.1.20

a. Write $x=3y^2+6y+7$ in standard form and
b. Use properties of the standard form to graph the equation.

Answer

a. $x=3(y+1)^2+4$
b.

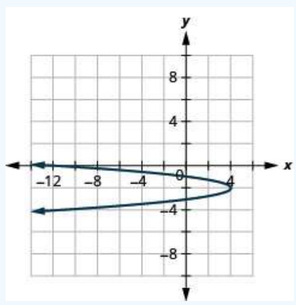


? Try It 8.1.21

a. Write $x=-4y^2-16y-12$ in standard form and
b. Use properties of the standard form to graph the equation.

Answer

a. $x=-4(y+2)^2+4$
b.



Another Look at Parabolas

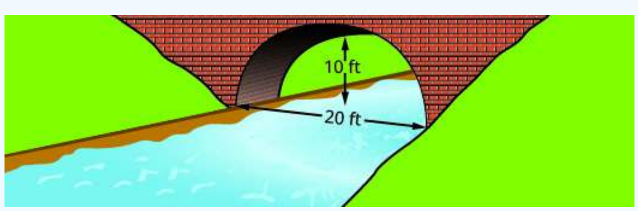
We have learned here how to graph both vertical and horizontal parabolas. We started with the basic graphs of $y=x^2$ and $x=y^2$. We then were able to graph $y=(x-h)^2+k$ and $x=(y-k)^2+h$. If we subtract the k on both sides of the first equation and h on both sides of the second, we have equivalent equations (they have the same solutions) $y-k=(x-h)^2$ and $x-h=(y-k)^2$. The vertex of the parabolas are the 'simplest' solution that results in $0=0$ when being substituted, namely, (h,k). So, the parabolas can be graphed by translating the basic form so that the vertex sits at (h,k). Viewing parabolas in this way will lead more naturally into the next section.

Solve Applications with Parabolas

Many architectural designs incorporate parabolas. It is not uncommon for bridges to be constructed using parabolas as we will see in the next example.

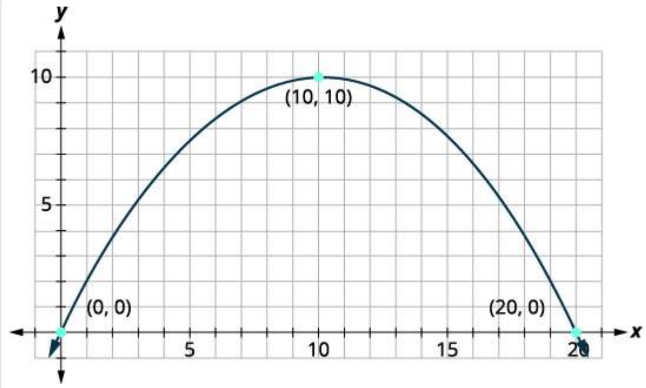
✓ Example 8.1.22

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



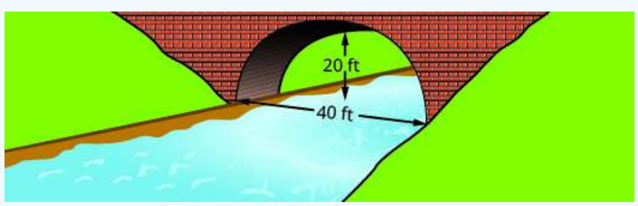
Solution:

We will first set up a coordinate system and draw the parabola. The graph will give us the information we need to write the equation of the graph in the standard form $y=a(x-h)^2+k$.

Let the lower left side of the bridge be the origin of the coordinate grid at the point $(0,0)$. Since the base is 20 feet wide the point $(20,0)$ represents the lower right side. The bridge is 10 feet high at the highest point. The highest point is the vertex of the parabola so the y-coordinate of the vertex will be 10. Since the bridge is symmetric, the vertex must fall halfway between the left most point, $(0,0)$, and the rightmost point $(20,0)$. From this we know that the x-coordinate of the vertex will also be 10.	
Identify the vertex, (h,k).	$(h,k)=(10,10)$
	$h=10, \quad k=10$
Substitute the values into the standard form. The value of a is still unknown. To find the value of a use one of the other points on the parabola.	$y=a(x-h)^2+k$ $y=a(x-10)^2+10$ $(x,y)=(0,0)$
Substitute the values of the other point into the equation.	$y=a(x-10)^2+10$ $0=a(0-10)^2+10$
Solve for a.	
	$y=a(x-10)^2+10$
Substitute the value for a into the equation.	$y=-\frac{1}{10}(x-10)^2+10$

? Try It 8.1.23

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.

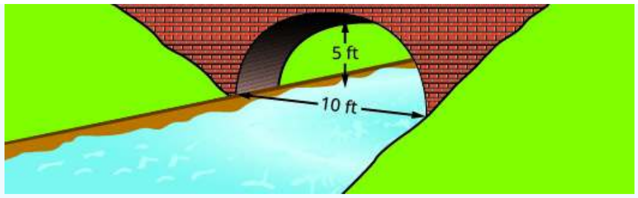


Answer

$$y=-\frac{1}{20}(x-20)^2+20$$

? Try It 8.1.24

Find the equation of the parabolic arch formed in the foundation of the bridge shown. Write the equation in standard form.



Answer

$$y=-\frac{1}{5}x^2+2xy=-\frac{1}{5}(x-5)^2+5$$

Key Concepts

- **Parabola:** Parabolas are examples of conic sections. The graphs of the equations of the form $y=ax^2+bx+c$ and $x=ay^2+by+c$ are examples of what are called parabolas.

Vertical Parabolas

Table 11.2.1

	General form $y=ax^2+bx+c$	**Standard Form** $y=a(x-h)^2+k$
Orientation	$a>0$ up; $a<0$ down	$a>0$ up; $a<0$ down
Axis of Symmetry	$x=-\frac{b}{2a}$	$x=h$
Vertex	Substitute $x=-\frac{b}{2a}$ and solve for y.	(h,k)
y-intercept	Let $x=0$	Let $x=0$
x-intercepts	Let $y=0$	Let $y=0$

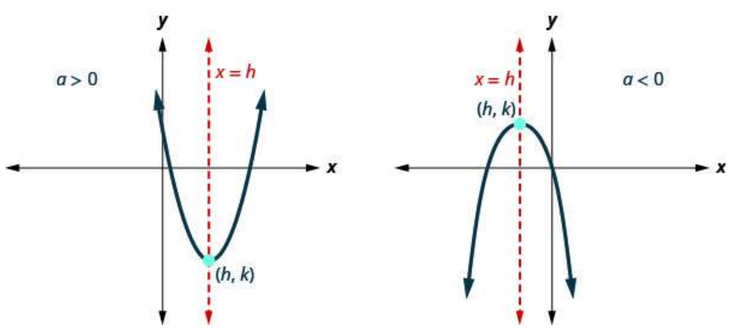


Figure 11.2.3

- **How to graph vertical parabolas** $y=ax^2+bx+c$ or $y=a(x-h)^2+k$ **using properties.**

1. Determine whether the parabola opens upward or downward.
2. Find the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts.
6. Graph the parabola.

Horizontal Parabolas

Table 11.2.4

	General form $x=ay^2+by+c$	Standard form $x=a(y-k)^2+h$
Orientation	$a>0$ right; $a<0$ left	$a>0$ right; $a<0$ left
Axis of Symmetry	$y=-\frac{b}{2a}$	$y=k$
Vertex	Substitute $y=-\frac{b}{2a}$ and solve for x.	(h,k)
x-intercepts	Let $x=0$	Let $x=0$
y-intercept	Let $y=0$	Let $y=0$

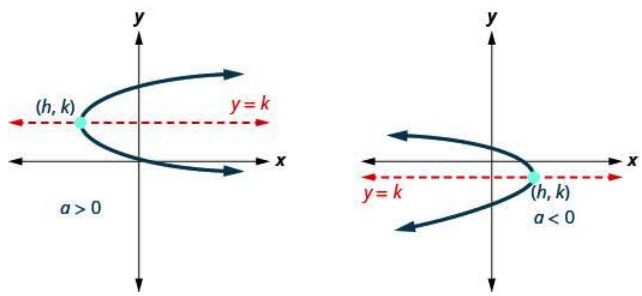


Figure 11.2.30

- **How to graph horizontal parabolas** $x=ay^2+by+c$ or $x=a(y-k)^2+h$ **using properties.**

1. Determine whether the parabola opens to the left or to the right.
2. Find the axis of symmetry.
3. Find the vertex.
4. Find the x-intercept. Find the point symmetric to the x-intercept across the axis of symmetry.
5. Find the y-intercepts.
6. Graph the parabola.

Glossary

parabola

A parabola is one of the conic sections. Though we do not discuss this here, it can be geometrically described: all points in a plane that are the same distance from a fixed point and a fixed line. The graphs of equations of the form $y=ax^2+bx+c$ and $x=ay^2+by+c$, $a\neq 0$ are examples of parabolas.

Practice Makes Perfect

? Graph Vertical Parabolas

In the following exercises, graph each equation by using properties.

1. $y=-x^2+4x-3$
2. $y=-x^2+8x-15$
3. $y=6x^2+2x-1$
4. $y=8x^2-10x+3$

Answer

1.

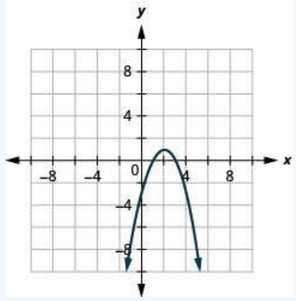


Figure 11.2.83

3.

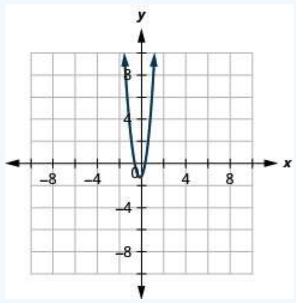


Figure 11.2.84

? Graph Vertical Parabolas

In the following exercises,

a. Write the equation in standard form and
b. Use properties of the standard form to graph the equation.

5. $y=-x^2+2x-4$
6. $y=2x^2+4x+6$
7. $y=-2x^2-4x-5$
8. $y=3x^2-12x+7$

Answer

5.

a. $y=-(x-1)^2-3$
b.

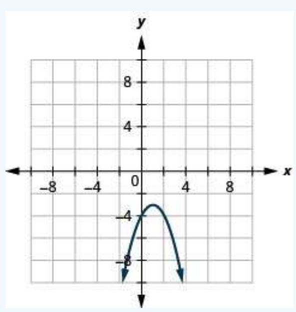


Figure 11.2.85

7.

a. $y=-2(x+1)^2-3$
b.

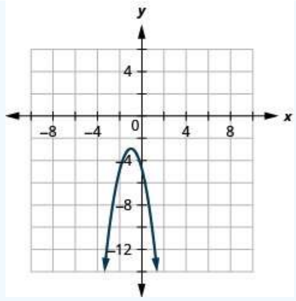


Figure 11.2.86

? Graph Horizontal Parabolas

In the following exercises, graph each equation by using properties.

9. $x=-2y^2$
10. $x=3y^2$
11. $x=4y^2$
12. $x=-4y^2$
13. $x=-y^2-2y+3$
14. $x=-y^2-4y+5$
15. $x=y^2+6y+8$
16. $x=y^2-4y-12$
17. $x=(y-2)^2+3$
18. $x=(y-1)^2+4$
19. $x=-(y-1)^2+2$
20. $x=-(y-4)^2+3$
21. $x=(y+2)^2+1$
22. $x=(y+1)^2+2$
23. $x=-(y+3)^2+2$
24. $x=-(y+4)^2+3$
25. $x=-3(y-2)^2+3$
26. $x=-2(y-1)^2+2$
27. $x=4(y+1)^2-4$
28. $x=2(y+4)^2-2$

Answer

9.

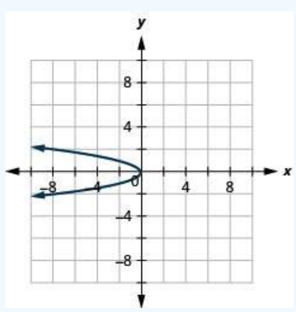


Figure 11.2.87

11.

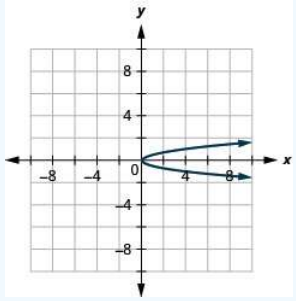


Figure 11.2.88

13.

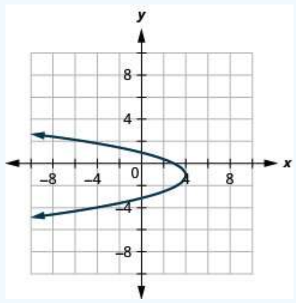


Figure 11.2.89

15.

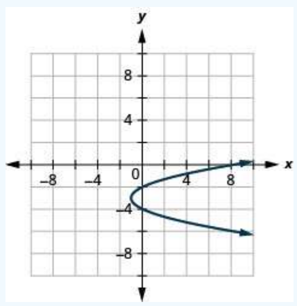


Figure 11.2.90

17.

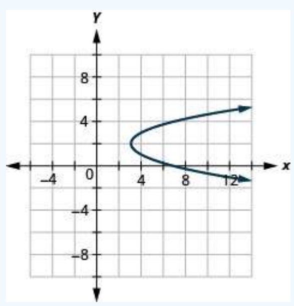


Figure 11.2.91

19.

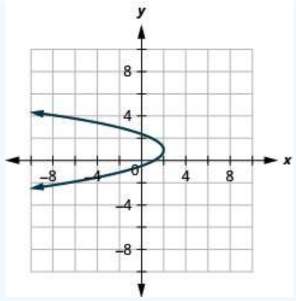


Figure 11.2.92

21.

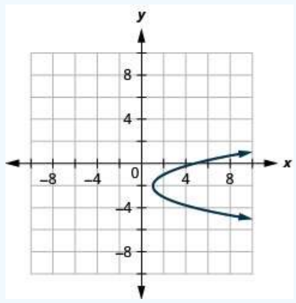


Figure 11.2.93

23.

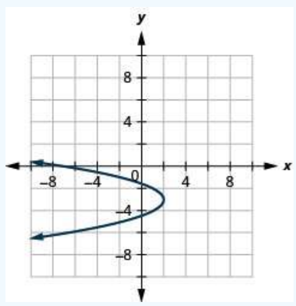


Figure 11.2.94

25.

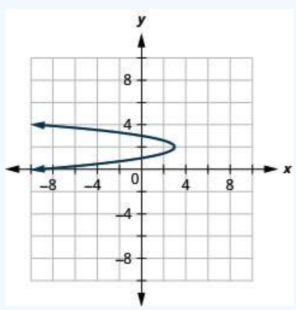


Figure 11.2.95

27.

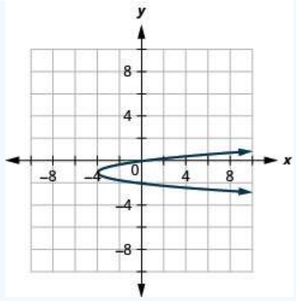


Figure 11.2.96

Graph Horizontal Parabolas

In the following exercises,

a. Write the equation in standard form and
b. Use properties of the standard form to graph the equation.

29. $x=y^2+4y-5$
30. $x=y^2+2y-3$
31. $x=-2y^2-12y-16$
32. $x=-3y^2-6y-5$

Answer

29.

a. $x=(y+2)^2-9$
b.

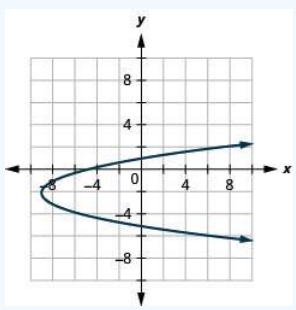


Figure 11.2.97

31.

a. $x=-2(y+3)^2+2$
b.

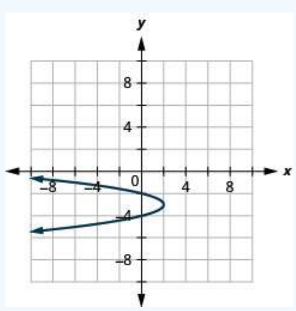


Figure 11.2.98

? Solve Applications with Parabolas

Write the equation in standard form of the parabolic arch formed in the foundation of the bridge shown. Use the lower left side of the bridge as the origin $(0, 0)$.

33.

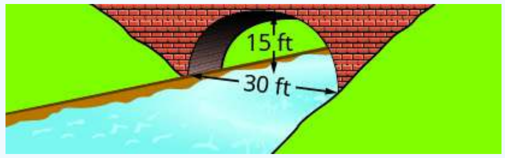


Figure 11.2.105

34.

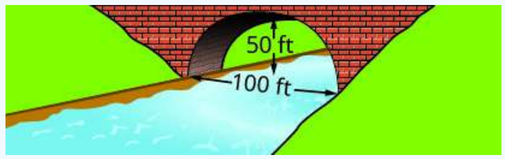


Figure 11.2.106

35.

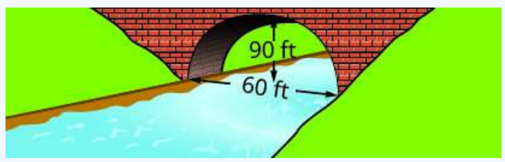


Figure 11.2.107

36.

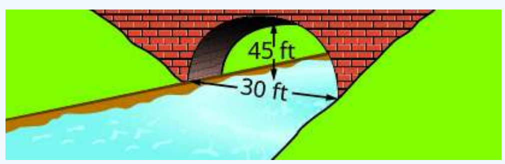


Figure 11.2.108

Answer

33. $y = -\frac{1}{15}(x-15)^2 + 15$

35. $y = -\frac{1}{10}(x-30)^2 + 90$

? Writing Exercises

37. In your own words, define a parabola.
38. Is the parabola $y = x^2$ a function? Is the parabola $x = y^2$ a function? Explain why or why not.
39. Write the equation of a parabola that opens up or down in standard form and the equation of a parabola that opens left or right in standard form. Provide a sketch of the parabola for each one, label the vertex and axis of symmetry.
40. Explain in your own words, how you can tell from its equation whether a parabola opens up, down, left or right.

Answer

37. Answers may vary

39. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
graph vertical parabolas.			
graph horizontal parabolas.			
solve applications with parabolas.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

8.2: Distance and Midpoint Formulas and Circles

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance Formula
- Use the Midpoint Formula
- Write the equation of a circle in standard form
- Graph a circle

Be Prepared

Before you get started, take this readiness quiz.

1. Find the length of the hypotenuse of a right triangle whose legs are 12 and 16 inches.
2. Factor: $x^2-18x+81$.
3. Solve by completing the square: $x^2-12x-12=0$.

Here we will discuss the next conic section: the circle. We need some preliminary discussions.

The Distance Formula

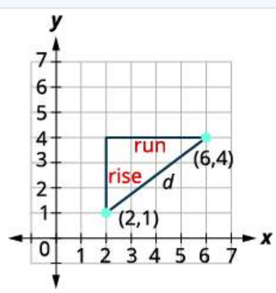
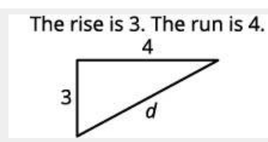
We have used the Pythagorean Theorem to find the lengths of the sides of a right triangle. Here we will use this theorem again to find distances on the rectangular coordinate system. By finding distance on the rectangular coordinate system, we can make a connection between the geometry of a conic and algebra—which opens up a world of opportunities for application.

Our first step is to develop a formula to find distances between points on the rectangular coordinate system. We will plot the points and create a right triangle much as we did when we found slope of a line. We then take it one step further and use the Pythagorean Theorem to find the length of the hypotenuse of the triangle—which is the distance between the points.

Example 8.2.1

Use the rectangular coordinate system to find the distance between the points $(6,4)$ and $(2,1)$.

Solution

Plot the two points. Connect the two points with a line. Draw a right triangle as if you were going to find slope.	
Find the length of each leg.	
Use the Pythagorean Theorem to find d, the distance between the two points.	$a^2+b^2=c^2$, where c is the length of the hypotenuse (opposite the right angle) and a and b are the lengths of the other two sides.
Substitute in the values.	$3^2+4^2=d^2$

Simplify.	$9 + 16 = d^2$
	$25 = d^2$
Use the Square Root Property.	$d = 5 \quad \text{or,} \quad \cancel{d = -5}$
Since distance, d is positive, we can eliminate $d = -5$.	The distance between the points $(6, 4)$ and $(2, 1)$ is 5.

? Try It 8.2.2

Use the rectangular coordinate system to find the distance between the points $(6, 1)$ and $(2, -2)$.

Answer

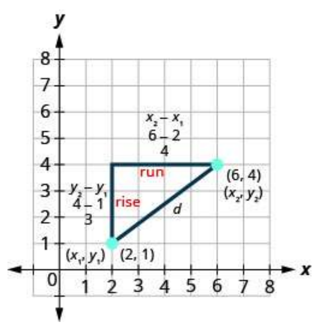
$d = 5$

? Try It 8.2.3

Use the rectangular coordinate system to find the distance between the points $(5, 3)$ and $(-3, -3)$.

Answer

$d = 10$



The method we used in the last example leads us to the formula to find the distance between the two points (x_1, y_1) and (x_2, y_2).

When we found the length of the horizontal leg we subtracted $6 - 2$ which is $x_2 - x_1$.

When we found the length of the vertical leg we subtracted $4 - 1$ which is $y_2 - y_1$.

If the triangle had been in a different position, we may have subtracted $x_1 - x_2$ or $y_1 - y_2$. The expressions $x_2 - x_1$ and $x_1 - x_2$ vary only in the sign of the resulting number. To get the positive value-since distance is positive- we can use absolute value. So to generalize we will say $|x_2 - x_1|$ and $|y_2 - y_1|$.

In the Pythagorean Theorem, we substitute the general expressions $|x_2 - x_1|$ and $|y_2 - y_1|$ rather than the numbers.

	$a^2 + b^2 = c^2$
Substitute in the values.	$(\lvert x_2 - x_1 \rvert)^2 + (\lvert y_2 - y_1 \rvert)^2 = d^2$
Squaring the expressions makes them positive so eliminate the absolute value.	$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2$
Use the Square Root Property.	$d = \pm\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Distance is positive, so eliminate the negative solution.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$

This is the Distance Formula we use to find the distance d between the two points (x_1, y_1) and (x_2, y_2).

The Distance Formula

The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

Example 8.2.4

Use the Distance Formula to find the distance between the points $(-5, -3)$ and $(7, 2)$.

Solution

	$(-5, -3)$ and $(7, 2)$
Write the Distance Formula.	$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$
Label the points.	$\overset{(-5,-3)}{(x_1, y_1)}$ and $\overset{(7,2)}{(x_2, y_2)}$
Substitute.	$d = \sqrt{(7-(-5))^2 + (2-(-3))^2}$
Simplify.	$d = \sqrt{12^2 + 5^2}$ $d = \sqrt{144 + 25}$ $d = \sqrt{169}$ $d = 13$

Try It 8.2.5

Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(5, 7)$.

Answer

$d = 15$

Try It 8.2.6

Use the Distance Formula to find the distance between the points $(-2, -5)$ and $(-14, -10)$.

Answer

$d = 13$

Example 8.2.7

Use the Distance Formula to find the distance between the points $(10, -4)$ and $(-1, 5)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution

Write the Distance Formula.	$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$
Label the points,	$\overset{(10,-4)}{(x_1, y_1)}$ and $\overset{(-1,5)}{(x_2, y_2)}$
and substitute.	$d = \sqrt{(-1-10)^2 + (5-(-4))^2}$
Simplify.	$d = \sqrt{(-11)^2 + 9^2}$ $d = \sqrt{121 + 81}$ $d = \sqrt{202}$

Since 202 is not a perfect square, we can leave the answer in exact form or if we need to, we can find a decimal approximation.	$d=\sqrt{202}$, $d\approx 14.2$

? Try It 8.2.8

Use the Distance Formula to find the distance between the points $(-4,-5)$ and $(3,4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Answer

$d=\sqrt{130}, d\approx 11.4$

? Try It 8.2.9

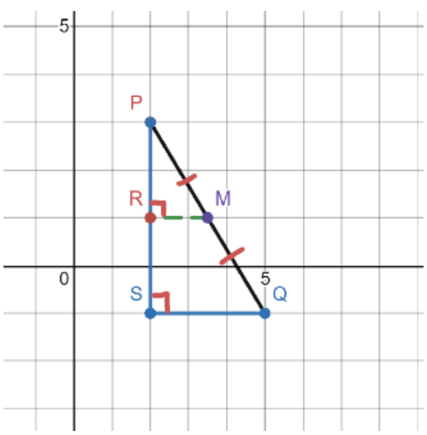
Use the Distance Formula to find the distance between the points $(-2,-5)$ and $(-3,-4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Answer

$d=\sqrt{2}, d\approx 1.4$

Use the Midpoint Formula

It is often useful to be able to find the midpoint of a segment, i.e., the point that divides a segment in half. For example, if you have the endpoints of the diameter of a circle, you may want to find the center of the circle which is the midpoint of the diameter. To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints. To see why this is the case, consider two points, P and Q, together with the midpoint M of the segment PQ and the triangle formed below:



The triangle $\triangle PQS$ is similar to the triangle $\triangle PMR$ because $\angle PRM$ and $\angle PSQ$ both right triangles and the two triangles share the angle P. Since M is the midpoint the length of PQ, $|PQ|$, is twice the length of PM, $|PM|$, i.e.,

$$|PM|=\frac{1}{2}\cdot|PQ|. \tag{8.2.1}$$

When two triangles are similar, the ratios of corresponding sides are equal. So,

$$\frac{|PR|}{|PS|}=\frac{|PM|}{|PQ|}=\frac{|PQ|}{2|PQ|}=\frac{1}{2}. \tag{8.2.2}$$

So, we see that

$$|PR| = \frac{1}{2} \cdot |PS|. \tag{8.2.3}$$

We see then that R is the midpoint of PS, and since the segment is vertical we can calculate the y-coordinate of the midpoint R by calculating the average of the y coordinates of P and S. But the y-coordinates of $P, M, andQ$ are the y-coordinates of P, R, and S, respectively, so we can calculate the y-coordinate of M by calculating the average of the y coordinates of P and Q. A similar story unfolds for the x coordinates.

The Midpoint Formula

The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

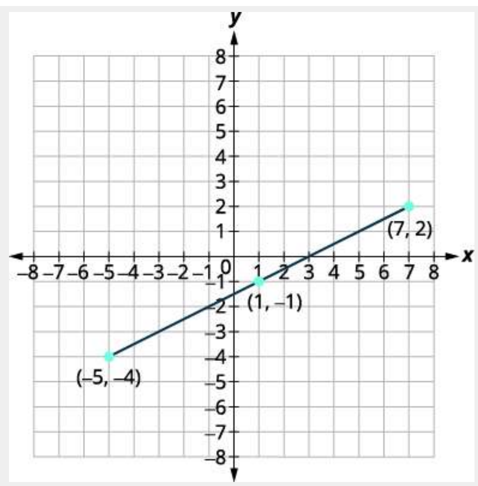
$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.

Example 8.2.10

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-5, -4)$ and $(7, 2)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

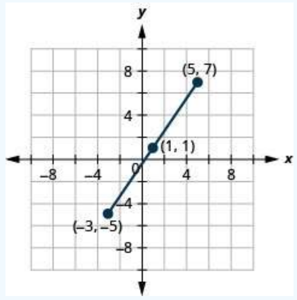
Solution:

Write the Midpoint Formula.	$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$
Label the points,	$\overset{(-5,-4)}{(x_1, y_1)}$ and $\overset{(7,2)}{(x_2, y_2)}$
and substitute.	$\left(\frac{-5 + 7}{2}, \frac{-4 + 2}{2}\right)$
Simplify.	$\left(\frac{2}{2}, \frac{-2}{2}\right)$
	$(1, -1)$ The midpoint of the segment is the point $(1, -1)$.
Plot the endpoints and midpoint.	

Try It 8.2.11

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-3,-5)$ and $(5,7)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

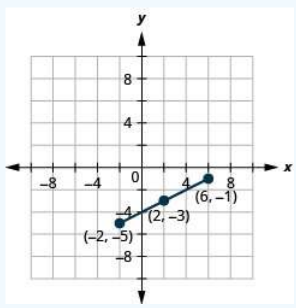
Answer



Try It 8.2.12

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-2,-5)$ and $(6,-1)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Answer



Both the Distance Formula and the Midpoint Formula depend on two points, (x_1,y_1) and (x_2,y_2). It is easy to confuse which formula requires addition and which subtraction of the coordinates. If we remember where the formulas come from, is may be easier to remember the formulas.

Distance Formula	**Midpoint Formula**
$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$	$\left(\frac{x_1+x_2}{2},\frac{y_1+y_2}{2}\right)$
Subtract the coordinates.	Add the coordinates.

Finding the Perpendicular Bisector of a Line Segment

Definition 8.2.13

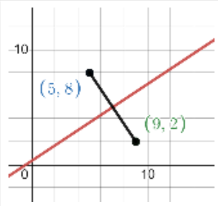
The **Perpendicular Bisector** of a line segment PQ is the line that is perpendicular to PQ and passes through though the midpoint of PQ.

Example 8.2.14

Find the perpendicular bisector of the line segment whose endpoints are $5,8$ and $9,2$.

Solution

A line is determined by a point on the line (in this case the midpoint of the given segment) and the slope.

The perpendicular bisector passes through the midpoint of the given line segment. So we must calculate the midpoint by averaging the x-coordinates, and averaging the y-coordinates.	$M=\left(\frac{5+9}{2}, \frac{8+2}{2}\right)$
Simplify.	$M=(7,5)$
We now need to find the slope of the line. To do this, we first find the slope of the given segment.	$m_{\text{segment}}=\frac{2-8}{9-5}=\frac{-6}{4}=-\frac{3}{2}$
The slope of a line perpendicular to this segment has a slope that is the negative reciprocal of the slope of this segment. We calculate the slope of the desired line.	$m_{\text{p.b.}}=-\frac{1}{m_{\text{segment}}}=-\frac{1}{-\frac{3}{2}}=\frac{2}{3}$
Use the slope obtained together with the midpoint to find an equation of the perpendicular bisector.	$M=(7,5), \quad m=\frac{2}{3}$ $y-5=\frac{2}{3}(x-7)$
If desired, write in standard form (or any desired form).	$3(y-5)=2(x-7)$ $3y-15=2x-14$ $2x-3y=-1$
We could check our answer by graphing which will help to detect major errors.	 The line appears to be perpendicular to the line segment and also pass through the midpoint of the line segment.

? Try It 8.2.15

Find the perpendicular bisector of the line segment connecting the points $(-3,5)$ and $(5,-7)$.

Answer

$$y+1=\frac{2}{3}(x-1)$$

or

$$2x-3y=5$$

? Try It 8.2.16

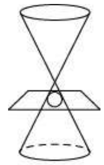
Find the perpendicular bisector of the line segment connecting the points $\left(-2, \frac{7}{2}\right)$ and $\left(-5,-\frac{5}{2}\right)$.

Answer

$$y-\frac{1}{2}=-\frac{1}{2}\left(x+\frac{7}{2}\right)$$

Write the Equation of a Circle in Standard Form

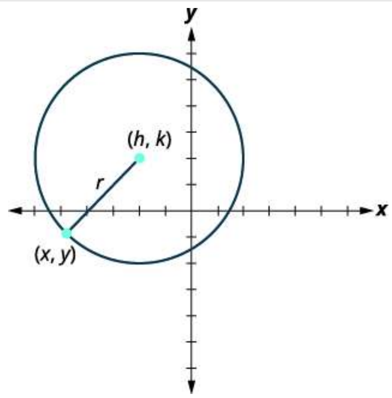
As we mentioned, our goal is to connect the geometry of a conic with algebra. By using the coordinate plane, we are able to do this easily.



We define a **circle** as all points in a plane that are a fixed distance from a given point in the plane. The given point is called the *center,* (h,k), and the fixed distance is called the *radius,* r, of the circle.

Definition 8.2.17

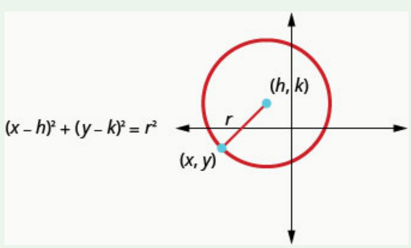
A **circle** is all points in a plane that are a fixed distance from a given point in the plane. The given point is called the **center**, (h,k), and the fixed distance is called the **radius**, r, of the circle.

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, (h,k),to a point on the circle, (x,y).	
To derive the equation of a circle, we can use the distance formula with the points (h,k), (x,y) and the distance, r.	$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Substitute the values.	$r=\sqrt{(x-h)^2+(y-k)^2}$
Square both sides.	$r^2=(x-h)^2+(y-k)^2$

This is the standard form of the equation of a circle with center, (h,k), and radius, r.

Definition 8.2.18

The **standard form** of the equation of a circle with center, (h,k), and radius, r, is



Example 8.2.19

Write the standard form of the equation of the circle with radius 3 and center $(0,0)$.

Solution:

Use the standard form of the equation of a circle.	$(x-h)^2+(y-k)^2=r^2$
Identify the center and radius.	$\overset{(0,0)}{(h,k)}$ and $\overset{3}{r}$
Substitute in the values $r=3$, $h=0$, and $k=0$.	$(x-0)^2+(y-0)^2=3^2$
Simplify.	$x^2+y^2=9$

Try It 8.2.20

Write the standard form of the equation of the circle with a radius of 6 and center $(0,0)$.

Answer

$x^2+y^2=36$

Try It 8.2.21

Write the standard form of the equation of the circle with a radius of 8 and center $(0,0)$.

Answer

$x^2+y^2=64$

In the last example, the center was $(0,0)$. Notice what happened to the equation. Whenever the center is $(0,0)$, the standard form becomes $x^2+y^2=r^2$.

Example 8.2.22

Write the standard form of the equation of the circle with radius 2 and center $(-1,3)$.

Solution:

Use the standard form of the equation of a circle.	$(x-h)^2+(y-k)^2=r^2$
Identify the center and radius.	$\overset{(-1,3)}{(h,k)}$ and $\overset{2}{r}$
Substitute in the values.	$(x-(-1))^2+(y-3)^2=2^2$
Simplify.	$(x+1)^2+(y-3)^2=4$

Try It 8.2.23

Write the standard form of the equation of the circle with a radius of 7 and center $(2,-4)$.

Answer

$(x-2)^2+(y+4)^2=49$

Try It 8.2.24

Write the standard form of the equation of the circle with a radius of 9 and center $(-3,-5)$.

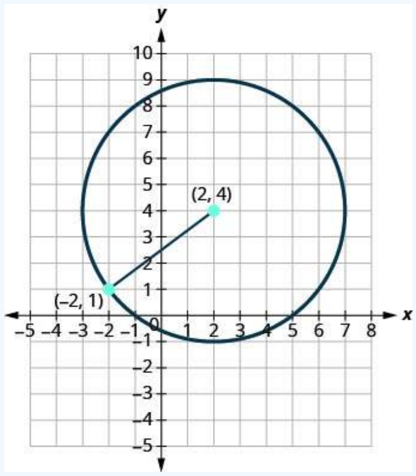
Answer

$(x+3)^2+(y+5)^2=81$

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

Example 8.2.25

Write the standard form of the equation of the circle with center $(2,4)$ that also contains the point $(-2,1)$.



Solution:

The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We will use the center $(2,4)$ and point $(-2,1)$ Use the Distance Formula to find the radius.	$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Label the points,	$\overset{(2,4)}{(x_1,y_1)}$ and $\overset{(-2,1)}{(x_2,y_2)}$
and substitute the values to find r.	$r=\sqrt{(-2-2)^2+(1-4)^2}$
Simplify.	$r=\sqrt{(-4)^2+(-3)^2}$ $r=\sqrt{16+9}$ $r=\sqrt{25}$ $r=5$
Now that we know the radius, $r=5$, and the center, $(2,4)$, we can use the standard form of the equation of a circle to find the equation.	$(x-h)^2+(y-k)^2=r^2$
Substitute in the values.	$(x-2)^2+(y-4)^2=5^2$
Simplify.	$(x-2)^2+(y-4)^2=25$

Try It 8.2.26

Write the standard form of the equation of the circle with center $(2,1)$ that also contains the point $(-2,-2)$.

Answer

$(x-2)^2+(y-1)^2=25$

Try It 8.2.27

Write the standard form of the equation of the circle with center $(7,1)$ that also contains the point $(-1,-5)$.

Answer

$(x-7)^2+(y-1)^2=100$

Graph a Circle

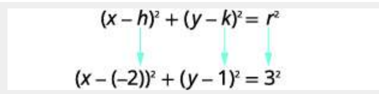
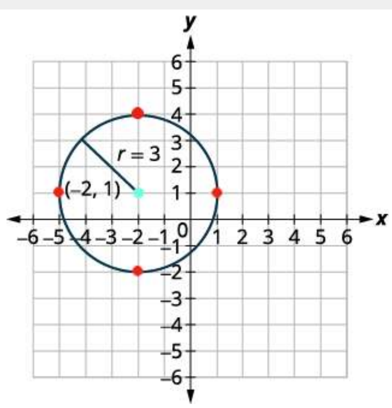
Any equation of the form $(x-h)^2+(y-k)^2=r^2$ is the standard form of the equation of a **circle** with center, (h,k), and radius, r. We can then graph the circle on a rectangular coordinate system.

Note that the standard form calls for subtraction from x and y. In the next example, the equation has $x+2$, so we need to rewrite the addition as subtraction of a negative.

Example 8.2.28

Find the center and radius, then graph the circle: $(x+2)^2+(y-1)^2=9$.

Solution:

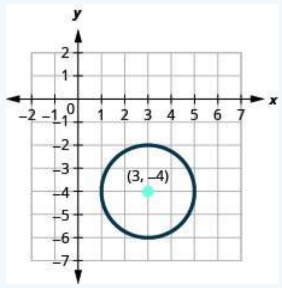
	$(x+2)^2+(y-1)^2=9$
Use the standard form of the equation of a circle. Identify the center, (h,k) and radius, r.	 $(x-h)^2+(y-k)^2=r^2$ $(x-(-2))^2+(y-1)^2=3^2$
	Center: $(-2,1)$ radius: 3
Use four points (N,S,E, and W of center) to guide the sketch by adding and subtracting the radius to the y-coordinate and the x-coordinate in turn.	Center: $(-2,1)$ $(-2,1+3)=(-2,4)$ $(-2,1-3)=(-2,-2)$ $(-2+3,1)=(1,1)$ $(-2-3,1)=(-5,1)$
Graph the circle.	

Try It 8.2.29

a. Find the center and radius, then
b. Find the 4 points N, S, E, and W of the center. Graph the circle: $(x-3)^2+(y+4)^2=4$.

Answer

a. The circle is centered at $(3,-4)$ with a radius of 2.
b. $(3,-2)$, $(3,-6)$, $(5,-4)$, $(1,-4)$

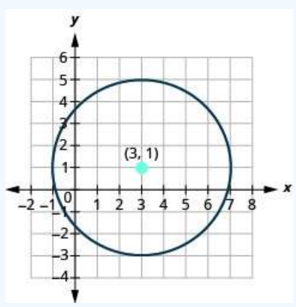


Try It 8.2.30

a. Find the center and radius, then
b. Find the four points N, S, E, and W of center. Graph the circle: $(x-3)^2+(y-1)^2=16$.

Answer

a. The circle is centered at $(3,1)$ with a radius of 4.
b. $(3,5)$, $(3,-3)$, $(7,1)$, $(-1,1)$



If we expand the equation from Example 11.24, we see a different form:

	$(x+2)^2+(y-1)^2=9$
Square the binomials.	$x^2+4x+4+y^2-2y+1=9$
Arrange the terms in descending degree order, and get zero on the right	$x^2+y^2+4x-2y-4=0.$

This form of the equation is called the general form of the equation of the **circle**.

Definition 8.2.31

The **general form of the equation of a circle** is

$$x^2+y^2+ax+by+c=0$$

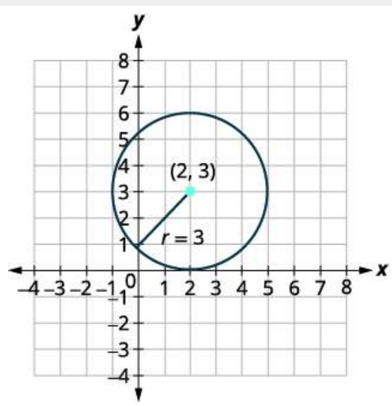
If we are given an equation in general form, we can change it to standard form by completing the squares in both x and y. Then we can graph the circle using its center and radius.

Example 8.2.32

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2-4x-6y+4=0$

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

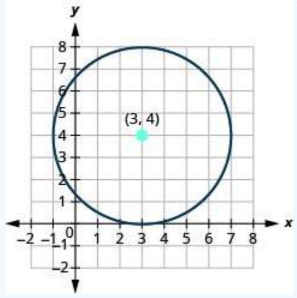
	$x^2+y^2-4x-6y+4=0$
Group the x-terms and y-terms. Collect the constants on the right side.	$x^2-4x+y^2-6y=-4$
Complete the squares.	$x^2-4x+4-4+y^2-6y+9-9=-4$ $x^2-4x+4+y^2-6y+9=-4+4+9$
Rewrite as binomial squares.	$(x-2)^2+(y-3)^2=9$
Identify the center and radius.	Center: $(2,3)$ radius: 3
Identify four points on the circle to guide the sketch.	$(2,6)$, $(2,0)$, $(5,3)$, $(-1,3)$
Graph the circle.	 Figure 11.1.31

Try It 8.2.33

a. Find the center and radius, then
b. Identify four points to guide the sketch of the graph. Graph the circle: $x^2+y^2-6x-8y+9=0$.

Answer

a. The circle is centered at $(3,4)$ with a radius of 4.
b. $(3,8)$, $(3,0)$, $(7,4)$, $(-1,4)$

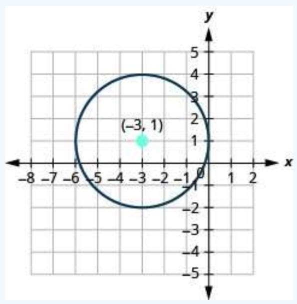


? Try It 8.2.34

a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 + 6x - 2y + 1 = 0$

Answer

a. The circle is centered at $(-3, 1)$ with a radius of 3.
b. $(-3, 4)$, $(-3, -2)$, $(0, 1)$, $(-6, 1)$



In the next example, there is a y-term and a y^2-term. But notice that there is no x-term, only an x^2-term. We have seen this before and know that it means h is 0. We will need to complete the square for the y terms, but not for the x terms.

✔ Example 8.2.35

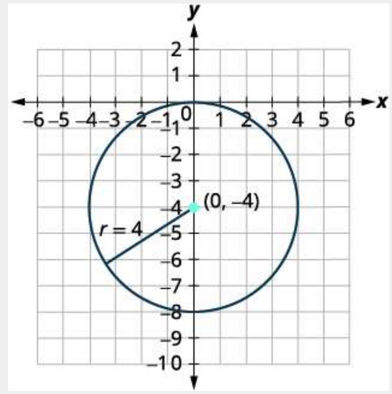
a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 + 8y = 0$

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

	$x^2 + y^2 + 8y = 0$
Group the x-terms and y-terms.	$x^2 + y^2 + 8y = 0$
There are no constants to collect on the right side.	
Complete the square for $y^2 + 8y$.	$x^2 + y^2 + 8y + 16 - 16 = 0$ $x^2 + y^2 + 8y + 16 = 16$
Rewrite as binomial squares.	$(x - 0)^2 + (y + 4)^2 = 16$
Identify the center and radius.	Center: $(0, -4)$ radius: 4
Identify four points to guide our sketch.	$(0, 0)$, $(0, -16)$, $(4, 0)$, $(-4, 0)$

Graph the circle.

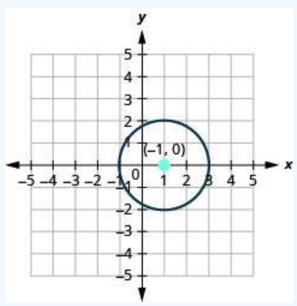


Try It 8.2.36

a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 - 2x - 3 = 0$.

Answer

a. The circle is centered at $(-1, 0)$ with a radius of 2.
b. $(-1, 2)$, $(-1, -2)$, $(1, 0)$, $(-3, 0)$

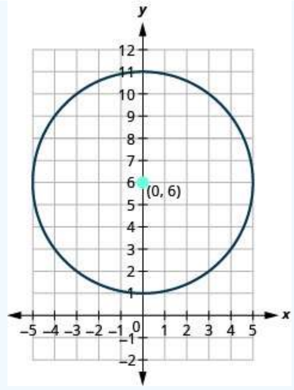


Try It 8.2.37

a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 - 12y + 11 = 0$.

Answer

a. The circle is centered at $(0, 6)$ with a radius of 5.
b. $(0, 11)$, $(0, 1)$, $(5, 6)$, $(-5, 6)$



Key Concepts

- **Distance Formula:** The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{\left(x_2 - x_1\right)^2 + \left(y_2 - y_1\right)^2}$$

- **Midpoint Formula:** The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.
- **Circle:** A circle is all points in a plane that are a fixed distance from a fixed point in the plane. The given point is called the *center,* (h, k), and the fixed distance is called the *radius,* r, of the circle.
- **Standard Form of the Equation a Circle:** The standard form of the equation of a circle with center, (h, k), and radius, r, is

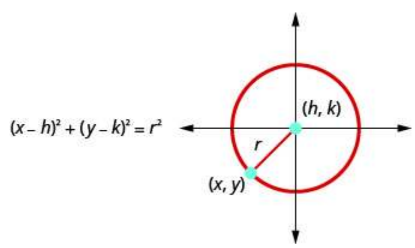


Figure 11.1.41

- **General Form of the Equation of a Circle:** The general form of the equation of a circle is

$$x^2 + y^2 + ax + by + c = 0$$

Glossary

circle

A circle is all points in a plane that are a fixed distance from a fixed point in the plane.

Practice Makes Perfect

Use the Distance Formula

In the following exercises, find the distance between the points. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

1. $(2, 0)$ and $(5, 4)$
2. $(-4, -3)$ and $(2, 5)$
3. $(-4, -3)$ and $(8, 2)$

4. $(-7,-3)$ and $(8,5)$
5. $(-1,4)$ and $(2,0)$
6. $(-1,3)$ and $(5,-5)$
7. $(1,-4)$ and $(6,8)$
8. $(-8,-2)$ and $(7,6)$
9. $(-3,-5)$ and $(0,1)$
10. $(-1,-2)$ and $(-3,4)$
11. $(3,-1)$ and $(1,7)$
12. $(-4,-5)$ and $(7,4)$

Answer

1. $d=5$

3. 13

5. 5

7. 13

9. $d=3\sqrt{5}, d\approx 6.7$

11. $d=\sqrt{68}, d\approx 8.2$

Use the Midpoint Formula

In the following exercises,

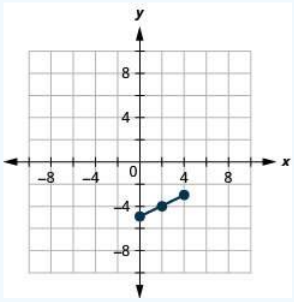
a. find the midpoint of the line segments whose endpoints are given and
b. plot the endpoints and the midpoint on a rectangular coordinate system.

13. $(0,-5)$ and $(4,-3)$
14. $(-2,-6)$ and $(6,-2)$
15. $(3,-1)$ and $(4,-2)$
16. $(-3,-3)$ and $(6,-1)$

Answer

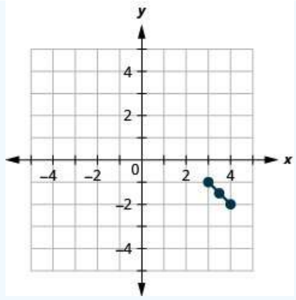
13.

a. Midpoint: $(2,-4)$
b.



15.

a. Midpoint: $\left(3\frac{1}{2},-1\frac{1}{2}\right)$
b.



? Write an equation for the perpendicular bisector of the line segment with the given endpoints.

17. $(-9, 9), (25, -25)$

18. $(0.02, -3.5), (1.06, -11.7)$

? Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center $(0, 0)$.

19. Radius: 7
20. Radius: 9
21. Radius: $\sqrt{2}$
22. Radius: $\sqrt{5}$

Answer

19. $x^2 + y^2 = 49$

21. $x^2 + y^2 = 2$

? Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center

23. Radius: 1, center: $(3, 5)$
24. Radius: 10, center: $(-2, 6)$
25. Radius: 2.5, center: $(1.5, -3.5)$
26. Radius: 1.5, center: $(-5.5, -6.5)$

Answer

23. $(x - 3)^2 + (y - 5)^2 = 1$

25. $(x - 1.5)^2 + (y + 3.5)^2 = 6.25$

? Write the Equation of a Circle in Standard Form

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

27. Center $(3, -2)$ with point $(3, 6)$
28. Center $(6, -6)$ with point $(2, -3)$
29. Center $(4, 4)$ with point $(2, 2)$
30. Center $(-5, 6)$ with point $(-2, 3)$

Answer

27. $(x-3)^2+(y+2)^2=64$

29. $(x-4)^2+(y-4)^2=8$

? Graph a Circle

In the following exercises,

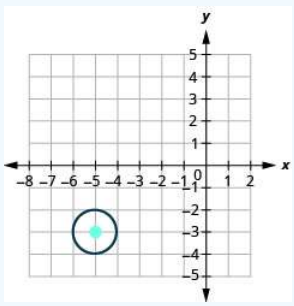
a. find the center and radius, then
b. graph each circle.

31. $(x+5)^2+(y+3)^2=1$
32. $(x-2)^2+(y-3)^2=9$
33. $(x-4)^2+(y+2)^2=16$
34. $(x+2)^2+(y-5)^2=4$
35. $x^2+(y+2)^2=25$
36. $(x-1)^2+y^2=36$
37. $(x-1.5)^2+(y+2.5)^2=0.25$
38. $(x-1)^2+(y-3)^2=\frac{9}{4}$
39. $x^2+y^2=64$
40. $x^2+y^2=49$

Answer

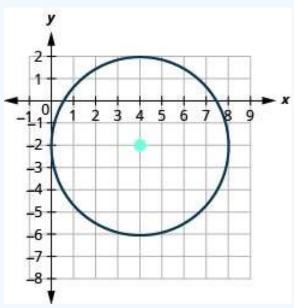
31.

a. The circle is centered at $(-5,-3)$ with a radius of 1.
b.



33.

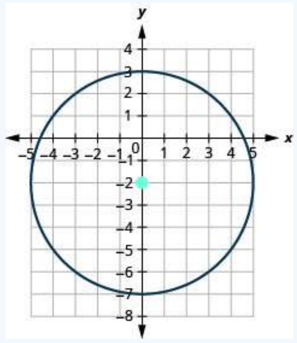
a. The circle is centered at $(4,-2)$ with a radius of 4.
b.



35.

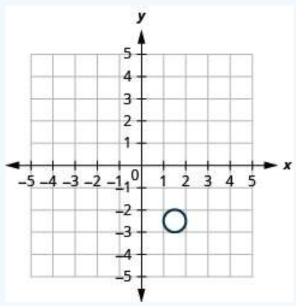
a. The circle is centered at $(0,-2)$ with a radius of 5.

b.



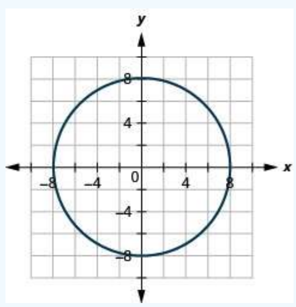
37.

a. The circle is centered at $(1.5, 2.5)$with a radius of 0.5.
b.



39.

a. The circle is centered at $(0, 0)$ with a radius of 8.
b.



? Graph a Circle

In the following exercises,

a. identify the center and radius and
b. graph.

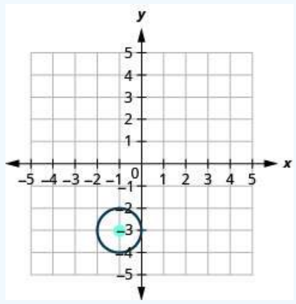
41. $x^2 + y^2 + 2x + 6y + 9 = 0$
42. $x^2 + y^2 - 6x - 8y = 0$
43. $x^2 + y^2 - 4x + 10y - 7 = 0$
44. $x^2 + y^2 + 12x - 14y + 21 = 0$
45. $x^2 + y^2 + 6y + 5 = 0$

46. $x^2+y^2-10y=0$
47. $x^2+y^2+4x=0$
48. $x^2+y^2-14x+13=0$

Answer

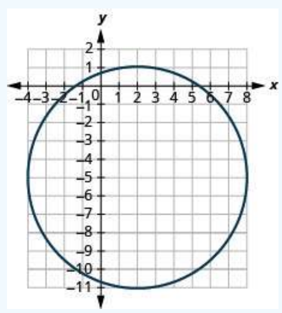
41.

a. Center: $(-1,-3)$, radius: 1
b.



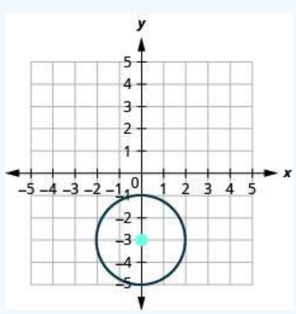
43.

a. Center: $(2,-5)$, radius: 6
b.



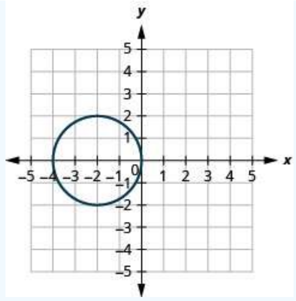
45.

a. Center: $(0,-3)$, radius: 2
b.



47.

a. Center: $(-2,0)$, radius: -2
b.

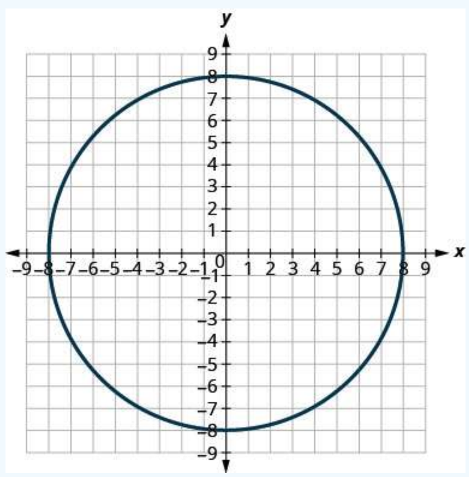


Mixed Practice

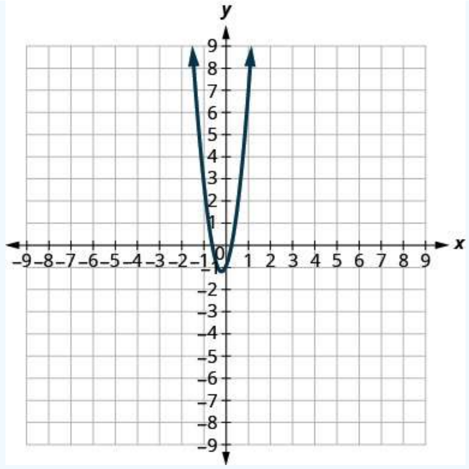
In the following exercises, match each graph to one of the following equations:

a. $x^2+y^2=64$
b. $x^2+y^2=49$
c. $(x+5)^2+(y+2)^2=4$
d. $(x-2)^2+(y-3)^2=9$
e. $y=-x^2+8x-15$
f. $y=6x^2+2x-1$

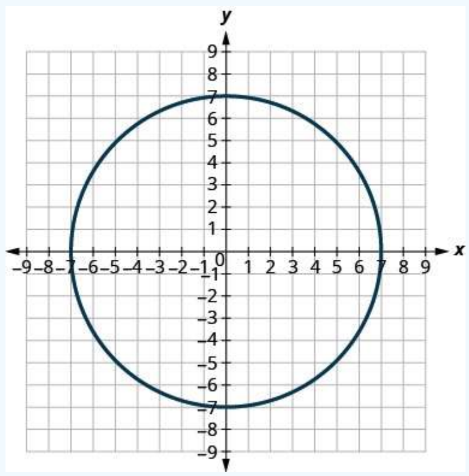
49.



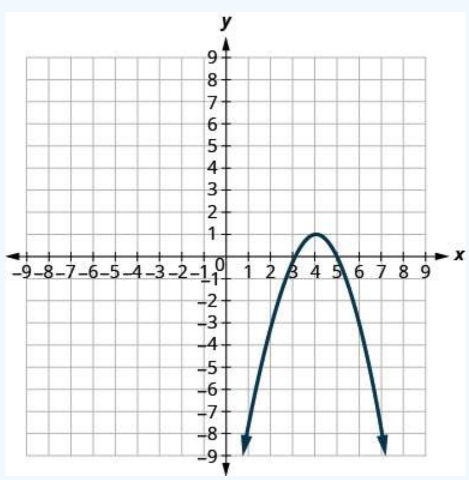
50.



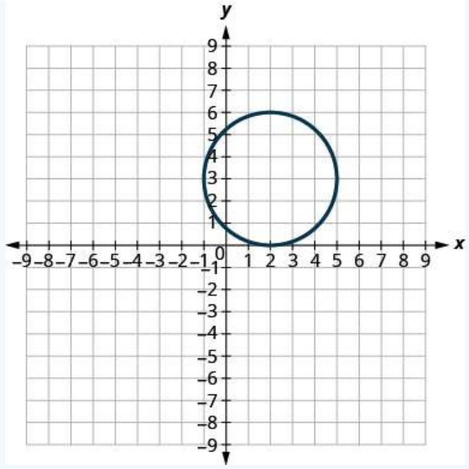
51.



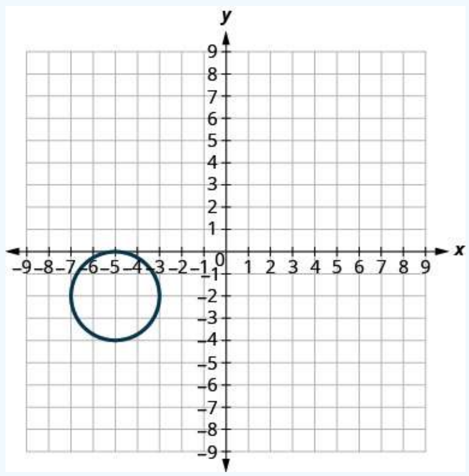
52.



53.



54.



Answer

49. a

51. b

53. d

? Writing Exercises

53. Explain the relationship between the distance formula and the equation of a circle.
54. Is a circle a function? Explain why or why not.
55. In your own words, state the definition of a circle.
56. In your own words, explain the steps you would take to change the general form of the equation of a circle to the standard form.

Answer

53. Answers will vary.

55. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance formula.			
use the midpoint formula.			
write the equation of a circle in standard form.			
graph a circle.			

Figure 11.1.53

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

8.3: Solve Systems of Nonlinear Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of nonlinear equations using graphing
- Solve a system of nonlinear equations using substitution
- Solve a system of nonlinear equations using elimination
- Use a system of nonlinear equations to solve applications

Be Prepared

Before you get started, take this readiness quiz.

1. Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$.
2. Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$
3. Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Solve a System of Nonlinear Equations using Graphing

We learned how to solve systems of linear equations with two variables by graphing, substitution and elimination. We will be using these same methods as we look at nonlinear systems of equations with two equations and two variables. A **system of nonlinear equations** is a system where at least one of the equations is not linear.

For example each of the following systems is a **system of nonlinear equations**.

$$\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$$

$$\begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases}$$

$$\begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$$

Definition 8.3.1

A **system of nonlinear equations** is a system where at least one of the equations is not linear.

Just as with systems of linear equations, a solution of a nonlinear system is an ordered pair that makes both equations true. In a nonlinear system, there may be more than one solution. We will see this as we solve a system of nonlinear equations by graphing.

When we solved systems of linear equations, the solution of the system was the point of intersection of the two lines. With systems of nonlinear equations, the graphs may be circles, parabolas or hyperbolas and there may be several points of intersection, and so several solutions. Once you identify the graphs, visualize the different ways the graphs could intersect and so how many solutions there might be.

To solve systems of nonlinear equations by graphing, we use basically the same steps as with systems of linear equations modified slightly for nonlinear equations. The steps are listed below for reference.

Solve a System of Nonlinear Equations by Graphing

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Graph the first equation.
3. Graph the second equation on the same rectangular coordinate system.

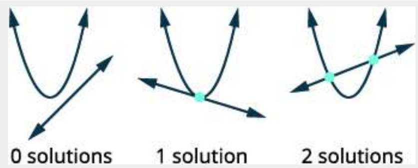
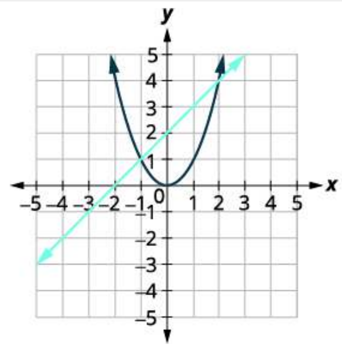
4. Determine whether the graphs intersect.
5. Identify the points of intersection.
6. Check that each ordered pair is a solution to both original equations.

Solve a System of Nonlinear Equations by Graphing.

Example 8.3.2

Solve the system by graphing: $\begin{cases} x-y=-2 \\ y=x^2 \end{cases}$

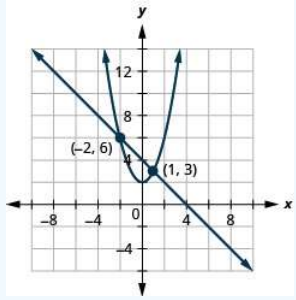
Solution

Identify each graph.	$\begin{cases} x-y=-2 & \text{line} \\ y=x^2 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
Graph the line, $x-y=-2$. Slope-intercept form $y=x+2$. Graph the parabola, $y=x^2$.	
Identify the points of intersection.	The points of intersection appear to be $(2,3)$ and $(-1,1)$.
Check to make sure each solution makes both equations true. $(2,4)$ $(-1,1)$	
	The solutions are $(2,4)$ and $(-1,1)$.

Try It 8.3.3

Solve the system by graphing: $\begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$.

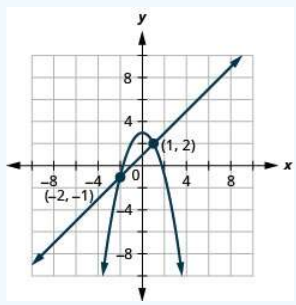
Answer



? Try It 8.3.4

Solve the system by graphing: $\begin{cases} x-y=-1 \\ y=-x^2+3 \end{cases}$

Answer



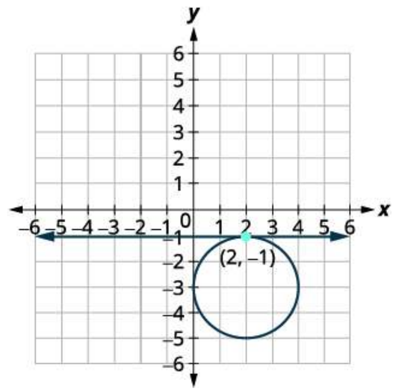
To identify the graph of each equation, keep in mind the characteristics of the x^2 and y^2 terms of each conic.

? Example 8.3.5

Solve the system by graphing: $\begin{cases} y=-1 \\ (x-2)^2+(y+3)^2=4 \end{cases}$.

Solution

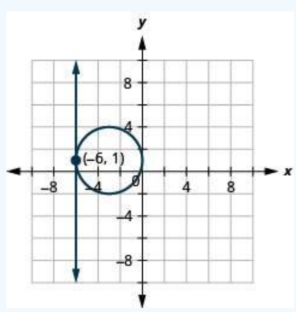
Identify each graph.	$\begin{cases} y=-1 & \text{line} \\ (x-2)^2+(y+3)^2=4 & \text{circle} \end{cases}$
Sketch the possible options for the intersection of a circle and a line.	

Graph the circle, $(x-2)^2+(y+3)^2=4$ Center: $(2,-3)$ radius: 2 Graph the line, $y=-1$. It is a horizontal line.	
Identify the points of intersection.	The point of intersection appears to be$(2,-1)$.
Check to make sure the solution makes both equations true.	$(2,-1)$ $(x-2)^2+(y+3)^2=4 \qquad y=-1$ $(2-2)^2+(-1+3)^2 \stackrel{?}{=} 4 \quad -1=-1$ $(0)^2+(2)^2 \stackrel{?}{=} 4$ $4=4$
	The solution is $(2,-1)$

? Try It 8.3.6

Solve the system by graphing: $\begin{cases} x=-6 \\ (x+3)^2+(y-1)^2=9 \end{cases}$

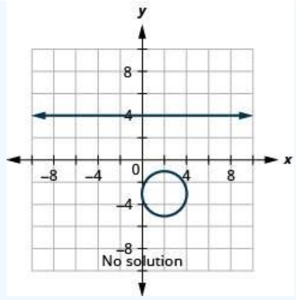
Answer



? Try It 8.3.7

Solve the system by graphing: $\begin{cases} y=4 \\ (x-2)^2+(y+3)^2=4 \end{cases}$

Answer



Solve a System of Nonlinear Equations Using Substitution

The graphing method works well when the points of intersection are integers and so easy to read off the graph. But more often it is difficult to read the coordinates of the points of intersection. The substitution method is an algebraic method that will work well in many situations. It works especially well when it is easy to solve one of the equations for one of the variables.

The substitution method is very similar to the substitution method that we used for systems of linear equations. The steps are listed below for reference.

Solve a System of Nonlinear Equations by Substitution

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Solve one of the equations for either variable.
3. Substitute the expression from Step 2 into the other equation.
4. Solve the resulting equation.
5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
6. Write each solution as an ordered pair.
7. Check that each ordered pair is a solution to **both** original equations.

Example 8.3.8

Solve the system by using substitution: $\begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases}$

Solution

Identify each graph.	$\begin{cases} 9x^2+y^2=9 & \text{ellipse} \\ y=3x-3 & \text{line} \end{cases}$
Sketch the possible options for intersection of an ellipse and a line.	
The equation $y=3x-3$ is solved for y.	$y=3x-3$
	$9x^2+y^2=9$
Substitute $3x-3$ for y in the first equation.	$9x^2+(3x-3)^2=9$
Solve the equation for x.	$9x^2+9x^2-18x+9=9$
	$18x^2-18x=0$ $18x(x-1)=0$ $x=0 \quad x=1$
Substitute $x=0$ and $x=1$ into $y=3x-3$ to find y-.	$y=3x-3 \qquad y=3x-3$
	$y=3\cdot 0-3 \qquad y=3\cdot 1-3$ $y=-3 \qquad y=0$

	The ordered pairs are $(0, -3), (1, 0)$.
Check **both** ordered pairs in **both** equations.	$(0, -3)$ $9x^2 + y^2 = 9 \qquad y = 3x - 3$ $9 \cdot 0^2 + (-3)^2 \stackrel{?}{=} 9 \qquad -3 \stackrel{?}{=} 3 \cdot 0 - 3$ $0 + 9 \stackrel{?}{=} 9 \qquad -3 \stackrel{?}{=} 0 - 3$ $9 = 9 \qquad -3 = -3$ $(1, 0)$ $9x^2 + y^2 = 9 \qquad y = 3x - 3$ $9 \cdot 1^2 + (0)^2 \stackrel{?}{=} 9 \qquad 0 \stackrel{?}{=} 3 \cdot 1 - 3$ $9 + 0 \stackrel{?}{=} 9 \qquad 0 \stackrel{?}{=} 3 - 3$ $9 = 9 \qquad 0 = 0$
	The solutions are $(0, -3), (1, 0)$.

? Try It 8.3.9

Solve the system by using substitution: $\begin{cases} x^2 + 9y^2 = 9 \\ y = \frac{1}{3}x - 3 \end{cases}$

Answer

No solution

? Try It 8.3.10

Solve the system by using substitution: $\begin{cases} 4x^2 + y^2 = 4 \\ y = x + 2 \end{cases}$

Answer

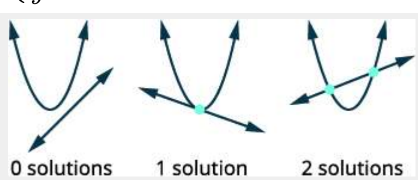
$\left(-\frac{4}{5}, \frac{6}{5}\right), (0, 2)$

So far, each system of nonlinear equations has had at least one solution. The next example will show another option.

? Example 8.3.11

Solve the system by using substitution: $\begin{cases} x^2 - y = 0 \\ y = x - 2 \end{cases}$

Solution

Identify each graph.	$\begin{cases} x^2 - y = 0 & \text{parabola} \\ y = x - 2 & \text{line} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
The equation $y = x - 2$ is solved for y.	$y = x - 2$
	$x^2 - y = 0$

Substitute $x-2$ for y in the first equation.	$x^2-(x-2)=0$
Solve the equation for x.	$x^2-x+2=0$
This doesn't factor easily, so we can check the discriminant.	
	The discriminant is negative, so there is no real solution. The system has no solution.

Try It 8.3.12

Solve the system by using substitution: $\begin{cases} x^2-y=0 \\ y=2x-3 \end{cases}$

Answer

No solution

Try It 8.3.13

Solve the system by using substitution: $\begin{cases} y^2-x=0 \\ y=3x-2 \end{cases}$

Answer

$\left(\frac{4}{9},-\frac{2}{3}\right),(1,1)$

Solve a System of Nonlinear Equations Using Elimination

When we studied systems of linear equations, we used the method of elimination to solve the system. We can also use elimination to solve systems of nonlinear equations. It works well when the equations have both variables squared. When using elimination, we try to make the coefficients of one variable to be opposites, so when we add the equations together, that variable is eliminated.

The elimination method is very similar to the elimination method that we used for systems of linear equations. The steps are listed for reference.

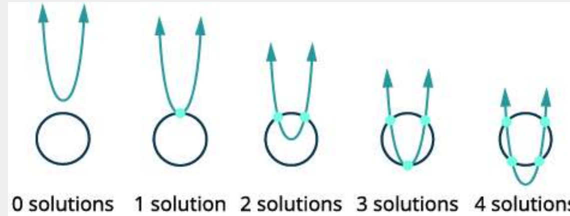
Solve a System of Equations by Elimination

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Write both equations in standard form.
3. Make the coefficients of one variable opposites.
   Decide which variable you will eliminate.
   Multiply one or both equations so that the coefficients of that variable are opposites.
4. Add the equations resulting from Step 3 to eliminate one variable.
5. Solve for the remaining variable.
6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
7. Write each solution as an ordered pair.
8. Check that each ordered pair is a solution to **both** original equations.

Example 8.3.14

Solve the system by elimination: $\begin{cases} x^2+y^2=4 \\ x^2-y=4 \end{cases}$

Solution

Identify each graph.	$\begin{cases} x^2+y^2=4 & \text{circle} \\ x^2-y\ \ =4 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a circle and a parabola.	
Both equations are in standard form.	$\begin{cases} x^2+y^2=4 \\ x^2-y\ \ =4 \end{cases}$
To get opposite coefficients of x^2, we will multiply the second equation by -1.	$\begin{cases} x^2+y^2=4 \\ -1(x^2-y)\ \ =-1(4) \end{cases}$
Simplify.	$\begin{cases} x^2+y^2=4 \\ -x^2+y\ \ =4 \end{cases}$
Add the two equation to eliminate x^2/	$\begin{cases} x^2+y^2=4 \\ -x^2+y\ \ =4 \end{cases}$ $\overline{y^2+y\ \ =0}$
Solve for y.	$y(y+1)=0$
	$y=0 \quad y+1=0$ $y=-1$
Substitute $y=0$ and $y=-1$ into one of the original equations. Then solve for x.	$y=0 \quad y=-1$
	$\begin{aligned} x^2-y&=4 \\ x^2-0&=4 \\ x^2&=4 \\ x&=\pm 2 \end{aligned} \qquad \begin{aligned} x^2-y&=4 \\ x^2-(-1)&=4 \\ x^2&=3 \\ x&=\pm\sqrt{3} \end{aligned}$
Write each solution as an ordered pair.	The ordered pairs are $(-2,0)(2,0)$. $(\sqrt{3},-1)(-\sqrt{3},-1)$
Check that each ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,0)$, $(2,0)$, $(\sqrt{3},-1)$, and $(-\sqrt{3},-1)$.

? Try It 8.3.15

Solve the system by elimination: $\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$

Answer

$(-3,0),(3,0),(-2\sqrt{2},-1),(2\sqrt{2},-1)$

? Try It 8.3.16

Solve the system by elimination: $\begin{cases} x^2+y^2=1 \\ -x+y^2=1 \end{cases}$

Answer

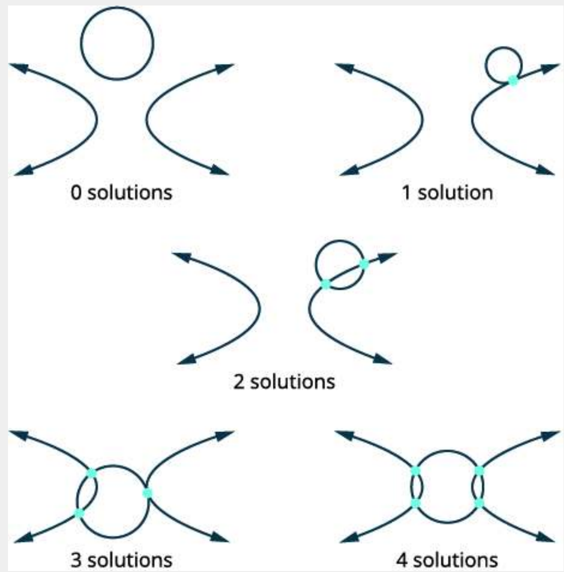
$(-1,0),(0,1),(0,-1)$

There are also four options when we consider a circle and a hyperbola.

Example 8.3.17

Solve the system by elimination: $\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$

Solution

Identify each graph.	$\begin{cases} x^2+y^2=7 & \text{circle} \\ x^2-y^2=1 & \text{hyperbola} \end{cases}$
Sketch the possible options for intersection of a circle and hyperbola.	
Both equations are in standard form.	$\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$
The coefficients of y^2 are opposite, so we will add the equations.	$\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$ $2x^2=8$
Simplify.	$x^2=4$ $x=\pm 2$ $x=2 \quad x=-2$
Substitue $x=2$ and $x=-2$ into one of the original equations. Then solve for y.	$\begin{array}{rl} x^2+y^2=7 & x^2+y^2=7 \\ 2^2+y^2=7 & (-2)^2+y^2=7 \\ 4+y^2=7 & 4+y^2=7 \\ y^2=3 & y^2=3 \\ y=\pm\sqrt{3} & y=\pm\sqrt{3} \end{array}$
Write each solution as an ordered pair.	The ordered pairs are $(-2, \sqrt{3})$, $(-2, -\sqrt{3})$, $(2, \sqrt{3})$, and $(2, -\sqrt{3})$.
Check that the ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2, \sqrt{3})$, $(-2, -\sqrt{3})$, $(2, \sqrt{3})$, and $(2, -\sqrt{3})$.

Try It 8.3.18

Solve the system by elimination: $\begin{cases} x^2+y^2=25 \\ y^2-x^2=7 \end{cases}$

Answer

$(-3,-4),(-3,4),(3,-4),(3,4)$

Try It 8.3.19

Solve the system by elimination: $\begin{cases} x^2+y^2=4 \\ x^2-y^2=4 \end{cases}$

Answer

$(-2,0),(2,0)$

Use a System of Nonlinear Equations to Solve Applications

Systems of nonlinear equations can be used to model and solve many applications. We will look at an everyday geometric situation as our example.

Example 8.3.20

The difference of the squares of two numbers is 15. The sum of the numbers is 5. Find the numbers.

Solution

Identify what we are looking for.	Two different numbers.
Define the variables.	x=first number y=second number
Translate the information into a system of equations.	
First sentence.	The difference of the squares of two numbers is 15.
	$x^2-y^2=15$
Second sentence.	The sum of the numbers is 5.
	$x+y=5$
Solve the system by substitution.	$\begin{cases} x^2-y^2=15 \\ x+y=5 \end{cases}$
Solve the second equation for x.	$x=5-y$
Substitute x into the first equation.	$x^2-y^2=15$
	$(5-y)^2-y^2=15$
Expand and simplify.	$(25-10y+y^2)-y^2=15$
	$25-10y+y^2-y^2=15$ $25-10y=15$
Solve for y.	$-10y=-10$
	$y=1$
Substitute back into the second equation.	$x+y=5$

	$x+1=5$ $x=4$
	The numbers are 1 and 4.

Try It 8.3.21

The difference of the squares of two numbers is -20. The sum of the numbers is 10. Find the numbers.

Answer

4 and 6

Try It 8.3.22

The difference of the squares of two numbers is 35. The sum of the numbers is -1. Find the numbers.

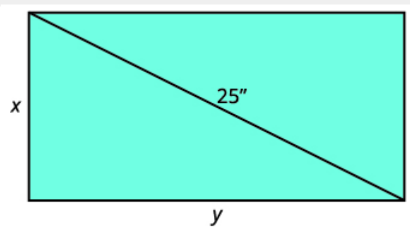
Answer

-18 and 17

Example 8.3.23

Myra purchased a small 25” TV for her kitchen. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 300 square inches. What are the length and width of the TV screen?

Solution

Identify what we are looking for.	The length and width of the rectangle.
Define the variables.	Let x= width of the rectangle y=length of the rectangle
Draw a diagram to help visualize the situation.	
	Area is 300 square inches.
Translate the information into a system of equations.	The diagonal of the right triangle is 25 inches.
	$x^2+y^2=25^2$ $x^2+y^2=625$
	The area of the rectangle is 300 square inches.
	$x \cdot y = 300$ $\begin{cases} x^2+y^2=625 \\ x \cdot y=300 \end{cases}$
Solve the system using substitution.	$xy=300$
Solve the second equation for x.	$x=\dfrac{300}{y}$
Substitute x into the first equation.	$x^2+y^2=625$

	$\left(\frac{300}{y}\right)^2 + y^2 = 625$
Simplify.	$\frac{90000}{y^2} + y^2 = 625$
Multiply by y^2 to clear the fractions.	$90000 + y^4 = 625y^2$
Put in standard form.	$y^4 - 625y^2 + 90000 = 0$
Solve by factoring.	$(y^2 - 225)(y^2 - 400) = 0$
	$y^2 - 225 = 0 \quad y^2 - 400 = 0$
	$y^2 = 225 \quad y^2 = 400$ $y = \pm 15 \quad y = \pm 20$
Since y is a side of the rectangle, we discard the negative values.	$y = 15 \quad y = 20$
Substitute back into the second equation.	$x \cdot y = 300 \quad x \cdot y = 300$
	$x \cdot 15 = 300 \quad x \cdot 20 = 300$ $x = 20 \quad x = 15$
	If the length is 15 inches, the width is 20 inches.
	If the length is 20 inches, the width is 15 inches.

? Try It 8.3.24

Edgar purchased a small 20” TV for his garage. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 192 square inches. What are the length and width of the TV screen?

Answer

If the length is 12 inches, the width is 16 inches. If the length is 16 inches, the width is 12 inches.

? Try It 8.3.25

The Harper family purchased a small microwave for their family room. The diagonal of the door measures 15 inches. The door also has an area of 108 square inches. What are the length and width of the microwave door?

Answer

If the length is 12 inches, the width is 9 inches. If the length is 9 inches, the width is 12 inches.

Key Concepts

- **How to solve a system of nonlinear equations by graphing.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Graph the first equation.
  3. Graph the second equation on the same rectangular coordinate system.
  4. Determine whether the graphs intersect.
  5. Identify the points of intersection.
  6. Check that each ordered pair is a solution to both original equations.
- **How to solve a system of nonlinear equations by substitution.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Solve one of the equations for either variable.
  3. Substitute the expression from Step 2 into the other equation.
  4. Solve the resulting equation.

5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
6. Write each solution as an ordered pair.
7. Check that each ordered pair is a solution to **both** original equations.

- **How to solve a system of equations by elimination.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Write both equations in standard form.
  3. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  4. Add the equations resulting from Step 3 to eliminate one variable.
  5. Solve for the remaining variable.
  6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
  7. Write each solution as an ordered pair.
  8. Check that each ordered pair is a solution to **both** original equations.

Practice Makes Perfect

Note that answers to even exercises are provided.

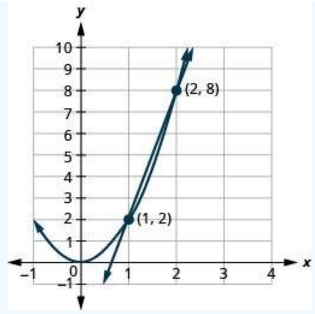
? Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

1. $\begin{cases} y=2x+2 \\ y=-x^2+2 \end{cases}$
2. $\begin{cases} y=6x-4 \\ y=2x^2 \end{cases}$
3. $\begin{cases} x+y=2 \\ x=y^2 \end{cases}$
4. $\begin{cases} x-y=-2 \\ x=y^2 \end{cases}$
5. $\begin{cases} y=\dfrac{3}{2}x+3 \\ y=-x^2+2 \end{cases}$
6. $\begin{cases} y=x-1 \\ y=x^2+1 \end{cases}$
7. $\begin{cases} x=-2 \\ x^2+y^2=4 \end{cases}$
8. $\begin{cases} y=-4 \\ x^2+y^2=16 \end{cases}$
9. $\begin{cases} x=2 \\ (x+2)^2+(y+3)^2=16 \end{cases}$
10. $\begin{cases} y=-1 \\ (x-2)^2+(y-4)^2=25 \end{cases}$
11. $\begin{cases} y=-2x+4 \\ y=\sqrt{x}+1 \end{cases}$
12. $\begin{cases} y=-\dfrac{1}{2}x+2 \\ y=\sqrt{x}-2 \end{cases}$

Answer

2.



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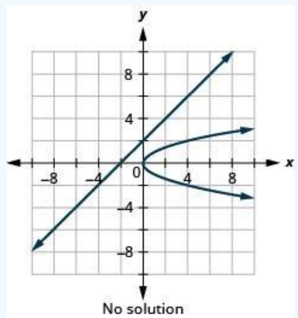

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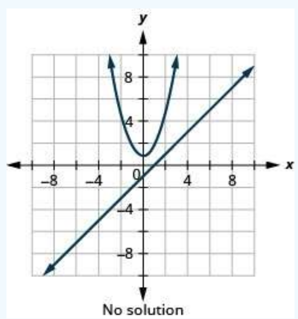

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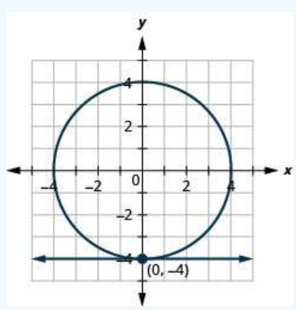

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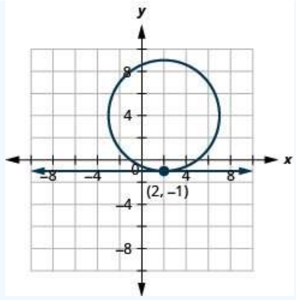

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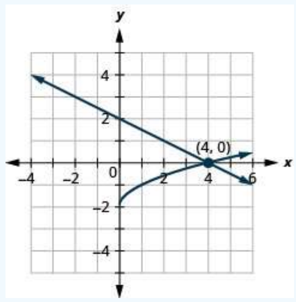


? Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

13. $\begin{cases} x^2+4y^2=4 \\ y=\dfrac{1}{2}x-1 \end{cases}$

14. $\begin{cases} 9x^2+y^2=9 \\ y=3x+3 \end{cases}$

15. $\begin{cases} 9x^2+y^2=9 \\ y=x+3 \end{cases}$

16. $\begin{cases} 9x^2+4y^2=36 \\ x=2 \end{cases}$

17. $\begin{cases} 4x^2+y^2=4 \\ y=4 \end{cases}$

18. $\begin{cases} x^2+y^2=169 \\ x=12 \end{cases}$

19. $\begin{cases} 3x^2-y=0 \\ y=2x-1 \end{cases}$

20. $\begin{cases} 2y^2-x=0 \\ y=x+1 \end{cases}$

21. $\begin{cases} y=x^2+3 \\ y=x+3 \end{cases}$

22. $\begin{cases} y=x^2-4 \\ y=x-4 \end{cases}$

23. $\begin{cases} x^2+y^2=25 \\ x-y=1 \end{cases}$

24. $\begin{cases} x^2+y^2=25 \\ 2x+y=10 \end{cases}$

Answer

14. $(-1,0),(0,3)$

16. $(2,0)$

18. $(12,-5),(12,5)$

20. No solution

22. $(0,-4),(1,-3)$

24. $(3,4),(5,0)$

? Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

25. $\begin{cases} x^2+y^2=16 \\ x^2-2y=8 \end{cases}$

26. $\begin{cases} x^2+y^2=16 \\ x^2-y=4 \end{cases}$

27. $\begin{cases} x^2+y^2=4 \\ x^2+2y=1 \end{cases}$

28. $\begin{cases} x^2+y^2=4 \\ x^2-y=2 \end{cases}$

29. $\begin{cases} x^2+y^2=9 \\ x^2-y=3 \end{cases}$

30. $\begin{cases} x^2+y^2=4 \\ y^2-x=2 \end{cases}$

31. $\begin{cases} x^2+y^2=25 \\ 2x^2-3y^2=5 \end{cases}$

32. $\begin{cases} x^2+y^2=20 \\ x^2-y^2=-12 \end{cases}$

33. $\begin{cases} x^2+y^2=13 \\ x^2-y^2=5 \end{cases}$

34. $\begin{cases} x^2+y^2=16 \\ x^2-y^2=16 \end{cases}$

35. $\begin{cases} 4x^2+9y^2=36 \\ 2x^2-9y^2=18 \end{cases}$

36. $\begin{cases} x^2-y^2=3 \\ 2x^2+y^2=6 \end{cases}$

37. $\begin{cases} 4x^2-y^2=4 \\ 4x^2+y^2=4 \end{cases}$

38. $\begin{cases} x^2-y^2=-5 \\ 3x^2+2y^2=30 \end{cases}$

39. $\begin{cases} x^2-y^2=1 \\ x^2-2y=4 \end{cases}$

40. $\begin{cases} 2x^2+y^2=11 \\ x^2+3y^2=28 \end{cases}$

Answer

26. $(0,-4),(-\sqrt{7},3),(\sqrt{7},3)$

28. $(0,-2),(-\sqrt{3},1),(\sqrt{3},1)$

30. $(-2,0),(1,-\sqrt{3}),(1,\sqrt{3})$

32. $(-2,-4),(-2,4),(2,-4),(2,4)$

34. $(-4,0),(4,0)$

36. $(-\sqrt{3},0),(\sqrt{3},0)$

38. $(-2,-3),(-2,3),(2,-3),(2,3)$

40. $(-1,-3),(-1,3),(1,-3),(1,3)$

Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

41. The sum of two numbers is -6 and the product is 8. Find the numbers.
42. The sum of two numbers is 11 and the product is -42. Find the numbers.
43. The sum of the squares of two numbers is 65. The difference of the number is 3. Find the numbers.
44. The sum of the squares of two numbers is 113. The difference of the number is 1. Find the numbers.
45. The difference of the squares of two numbers is 15. The difference of twice the square of the first number and the square of the second number is 30. Find the numbers.
46. The difference of the squares of two numbers is 20. The difference of the square of the first number and twice the square of the second number is 4. Find the numbers.
47. The perimeter of a rectangle is 32 inches and its area is 63 square inches. Find the length and width of the rectangle.
48. The perimeter of a rectangle is 52 cm and its area is 165 cm^2. Find the length and width of the rectangle.
49. Dion purchased a new microwave. The diagonal of the door measures 17 inches. The door also has an area of 120 square inches. What are the length and width of the microwave door?
50. Jules purchased a microwave for his kitchen. The diagonal of the front of the microwave measures 26 inches. The front also has an area of 240 square inches. What are the length and width of the microwave?
51. Roman found a widescreen TV on sale, but isn't sure if it will fit his entertainment center. The TV is 60". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1728 square inches. His entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Roman's entertainment center?
52. Donnette found a widescreen TV at a garage sale, but isn't sure if it will fit her entertainment center. The TV is 50". The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1200 square inches. Her entertainment center has an insert for the TV with a length of 38 inches and width of 27 inches. What are the length and width of the TV screen and will it fit Donnette's entertainment center?

Answer

42. -3 and 14

44. -7 and -8 or 8 and 7

46. -6 and -4 or -6 and 4 or 6 and -4 or 6 and 4

48. If the length is 11 cm, the width is 15 cm. If the length is 15 cm, the width is 11 cm.

50. If the length is 10 inches, the width is 24 inches. If the length is 24 inches, the width is 10 inches.

52. The length is 40 inches and the width is 30 inches. The TV will not fit Donnette's entertainment center.

Writing Exercises

53. In your own words, explain the advantages and disadvantages of solving a system of equations by graphing.
54. Explain in your own words how to solve a system of equations using substitution.
55. Explain in your own words how to solve a system of equations using elimination.
56. A circle and a parabola can intersect in ways that would result in $0, 1, 2, 3,$ or 4 solutions. Draw a sketch of each of the possibilities.

Answer

54. Answers may vary

56. Answers may vary

? Additional Exercise

57. Solve for x:

$$\begin{cases} x^2 - y^2 = -4 \\ y = 2\sqrt{x} \end{cases}$$

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve a system of nonlinear equations using graphing.			
solve a system of nonlinear equations using substitution.			
solve a system of nonlinear equations using elimination.			
use a system of nonlinear equations to solve applications.			

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

8.4: Chapter 8 Review Exercises

Chapter Review Exercises

Distance and Midpoint Formulas; Circles

? Exercise 8.4.1 Use the Distance Formula

In the following exercises, find the distance between the points. Round to the nearest tenth if needed.

1. $(-5,1)$ and $(-1,4)$
2. $(-2,5)$ and $(1,5)$
3. $(8,2)$ and $(-7,-3)$
4. $(1,-4)$ and $(5,-5)$

Answer

2. $d=3$

4. $d=\sqrt{17}, d\approx 4.1$

? Exercise 8.4.2 Use the Midpoint Formula

In the following exercises, find the midpoint of the line segments whose endpoints are given.

1. $(-2,-6)$ and $(-4,-2)$
2. $(3,7)$ and $(5,1)$
3. $(-8,-10)$ and $(9,5)$
4. $(-3,2)$ and $(6,-9)$

Answer

2. $(4,4)$

4. $\left(\frac{3}{2},-\frac{7}{2}\right)$

? Exercise 8.4.3 Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given information.

1. radius is 15 and center is $(0,0)$
2. radius is $\sqrt{7}$ and center is $(0,0)$
3. radius is 9 and center is $(-3,5)$
4. radius is 7 and center is $(-2,-5)$
5. center is $(3,6)$ and a point on the circle is $(3,-2)$
6. center is $(2,2)$ and a point on the circle is $(4,4)$

Answer

2. $x^2+y^2=7$

4. $(x+2)^2+(y+5)^2=49$

6. $(x-2)^2+(y-2)^2=8$

Exercise 8.4.4 Graph a Circle

In the following exercises,

a. Find the center and radius, then
b. Graph each circle.

1. $2x^2 + 2y^2 = 450$
2. $3x^2 + 3y^2 = 432$
3. $(x+3)^2 + (y-5)^2 = 81$
4. $(x+2)^2 + (y+5)^2 = 49$
5. $x^2 + y^2 - 6x - 12y - 19 = 0$
6. $x^2 + y^2 - 4y - 60 = 0$

Answer

2.

a. radius: 12, center: $(0, 0)$
b.

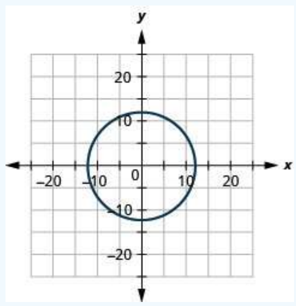


Figure 11.E.1

4.

a. radius: 7, center: $(-2, -5)$
b.

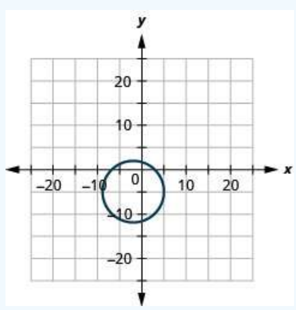


Figure 11.E.2

6.

a. radius: 8, center: $(0, 2)$
b.

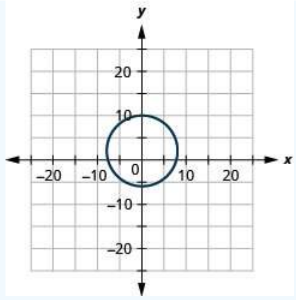


Figure 11.E.3

Parabolas

Exercise 8.4.5 Graph Vertical Parabolas

In the following exercises, graph each equation by using its properties.

1. $y=x^2+4x-3$
2. $y=2x^2+10x+7$
3. $y=-6x^2+12x-1$
4. $y=-x^2+10x$

Answer

2.

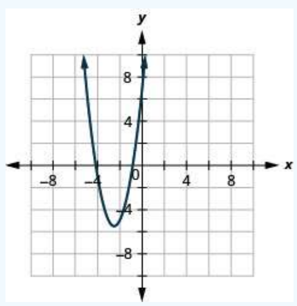


Figure 11.E.4

4.

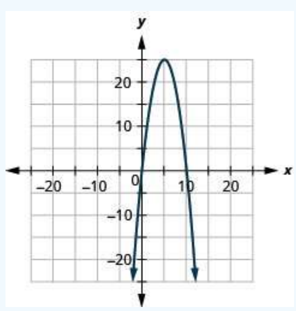


Figure 11.E.5

Exercise 8.4.6 Graph Vertical Parabolas

In the following exercises,

a. Write the equation in standard form, then
b. Use properties of the standard form to graph the equation.

1. $y=x^2+4x+7$

2. $y=2x^2-4x-2$
3. $y=-3x^2-18x-29$
4. $y=-x^2+12x-35$

Answer

2.

a. $y=2(x-1)^2-4$
b.

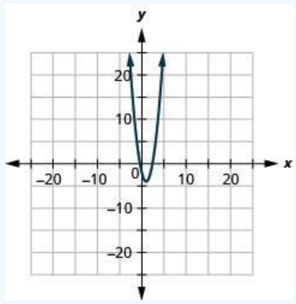


Figure 11.E.6

4.

a. $y=-(x-6)^2+1$
b.

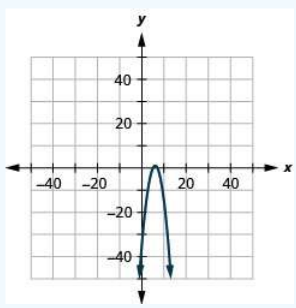


Figure 11.E.7

? Exercise 8.4.7 Graph Horizontal Parabolas

In the following exercises, graph each equation by using its properties.

1. $x=2y^2$
2. $x=2y^2+4y+6$
3. $x=-y^2+2y-4$
4. $x=-3y^2$

Answer

2.

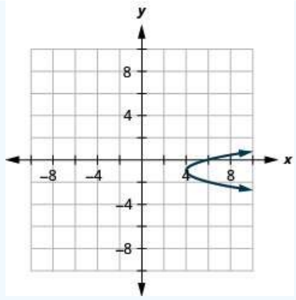


Figure 11.E.8

4.

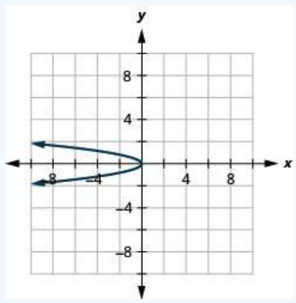


Figure 11.E.9

? Exercise 8.4.8 Graph Horizontal Parabolas

In the following exercises,

a. Write the equation in standard form, then
b. Use properties of the standard form to graph the equation.

1. $x=4y^2+8y$
2. $x=y^2+4y+5$
3. $x=-y^2-6y-7$
4. $x=-2y^2+4y$

Answer

2.

a. $x=(y+2)^2+1$
b.

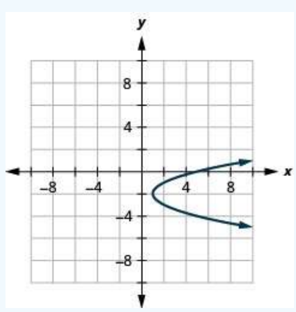


Figure 11.E.10

4.

a. $x=-2(y-1)^2+2$
b.

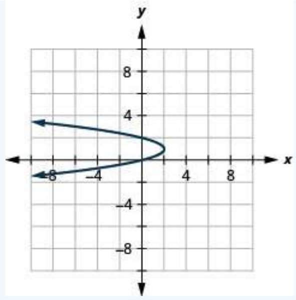


Figure 11.E.11

? Exercise 8.4.9 Solve Applications with Parabolas

In the following exercises, create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in standard form.

1.

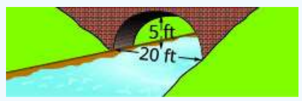


Figure 11.E.12

2.

Figure 11.E.13

Answer

2. $y=-\frac{1}{9}x^2+\frac{10}{3}x$

Ellipses

? Exercise 8.4.10 Graph an Ellipse with Center at the Origin

In the following exercises, graph each ellipse.

1. $\frac{x^2}{36}+\frac{y^2}{25}=1$
2. $\frac{x^2}{4}+\frac{y^2}{81}=1$
3. $49x^2+64y^2=3136$
4. $9x^2+y^2=9$

Answer

2.

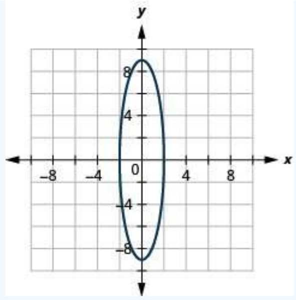


Figure 11.E.14

4.

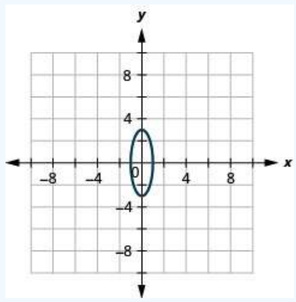


Figure 11.E.15

? Exercise 8.4.11 Find the Equation of an Ellipse with Center at the Origin

In the following exercises, find the equation of the ellipse shown in the graph.

1.

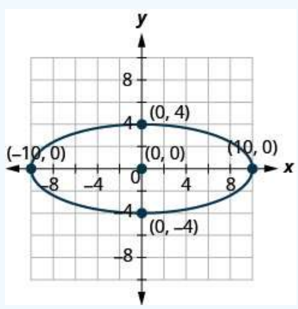


Figure 11.E.16

2.

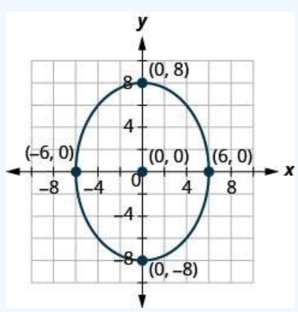


Figure 11.E.17

Answer

2. $\frac{x^2}{36}+\frac{y^2}{64}=1$

Exercise 8.4.12 Graph an Ellipse with Center Not at the Origin

In the following exercises, graph each ellipse.

1. $\frac{(x-1)^2}{25}+\frac{(y-6)^2}{4}=1$
2. $\frac{(x+4)^2}{16}+\frac{(y+1)^2}{9}=1$
3. $\frac{(x-5)^2}{16}+\frac{(y+3)^2}{36}=1$
4. $\frac{(x+3)^2}{9}+\frac{(y-2)^2}{25}=1$

Answer

2.

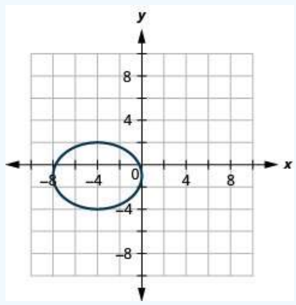


Figure 11.E.18

4.

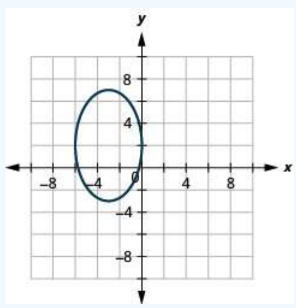


Figure 11.E.19

Exercise 8.4.13 Graph an Ellipse with Center Not at the Origin

In the following exercises,

a. Write the equation in standard form and
b. Graph.

1. $x^2+y^2+12x+40y+120=0$
2. $25x^2+4y^2-150x-56y+321=0$
3. $25x^2+4y^2+150x+125=0$
4. $4x^2+9y^2-126x+405=0$

Answer

2.

a. $\frac{(x-3)^2}{4}+\frac{(y-7)^2}{25}=1$

b.

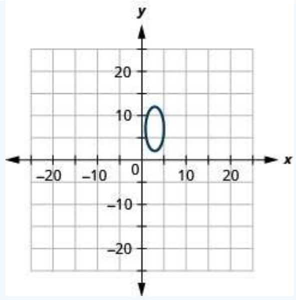


Figure 11.E.20

4.

a. $\frac{x^2}{9}+\frac{(y-7)^2}{4}=1$
b.

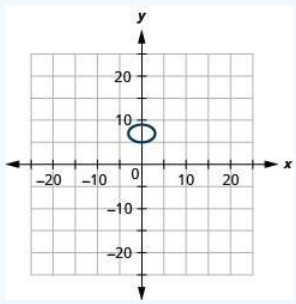


Figure 11.E.21

? Exercise 8.4.14 Solve Applications with Ellipses

In the following exercises, write the equation of the ellipse described.

1. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 10 AU and the furthest is approximately 90 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.

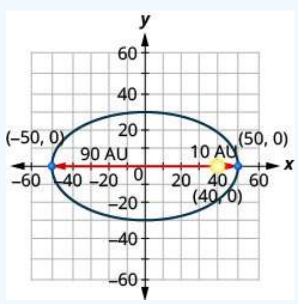


Figure 11.E.22

Answer

1. Solve

Hyperbolas

Exercise 8.4.15 Graph a Hyperbola with Center at $(0, 0)$

In the following exercises, graph.

1. $\frac{x^2}{25} - \frac{y^2}{9} = 1$
2. $\frac{y^2}{49} - \frac{x^2}{16} = 1$
3. $9y^2 - 16x^2 = 144$
4. $16x^2 - 4y^2 = 64$

Answer

1.

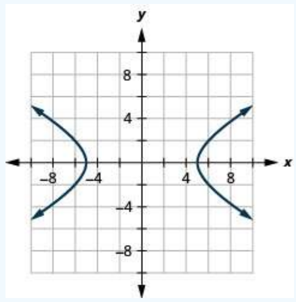


Figure 11.E.23

3.

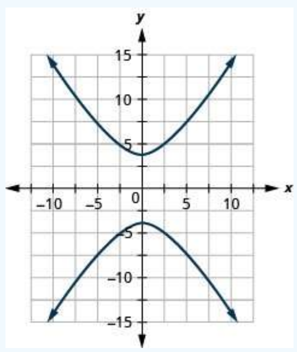


Figure 11.E.24

Exercise 8.4.16 Graph a Hyperbola with Center at (h, k)

In the following exercises, graph.

1. $\frac{(x+1)^2}{4} - \frac{(y+1)^2}{9} = 1$
2. $\frac{(x-2)^2}{4} - \frac{(y-3)^2}{16} = 1$
3. $\frac{(y+2)^2}{9} - \frac{(x+1)^2}{9} = 1$
4. $\frac{(y-1)^2}{25} - \frac{(x-2)^2}{9} = 1$

Answer

1.

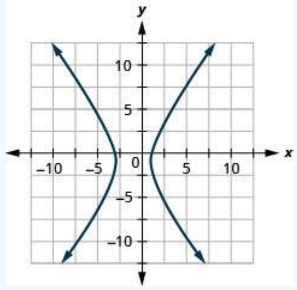


Figure 11.E.25

3.

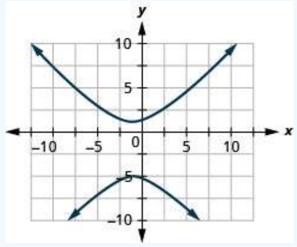


Figure 11.E.26

? Exercise 8.4.17 Graph a Hyperbola with Center at (h, k)

In the following exercises,

a. Write the equation in standard form and
b. Graph.

1. $4x^2 - 16y^2 + 8x + 96y - 204 = 0$
2. $16x^2 - 4y^2 - 64x - 24y - 36 = 0$
3. $4y^2 - 16x^2 + 32x - 8y - 76 = 0$
4. $36y^2 - 16x^2 - 96x + 216y - 396 = 0$

Answer

1.

a. $\frac{(x+1)^2}{16} - \frac{(y-3)^2}{4} = 1$
b.

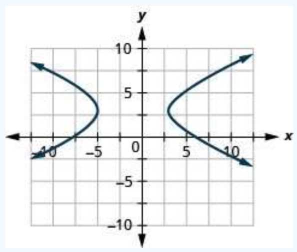


Figure 11.E.27

3.

a. $\frac{(y-1)^2}{16} - \frac{(x-1)^2}{4} = 1$
b.

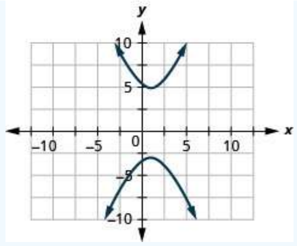


Figure 11.E.28

? Exercise 8.4.18 Identify the Graph of Each Equation as a Circle, Parabola, Ellipse, or Hyperbola

In the following exercises, identify the type of graph.

1. a. $16y^2-9x^2-36x-96y-36=0$
   b. $x^2+y^2-4x+10y-7=0$
   c. $y=x^2-2x+3$
   d. $25x^2+9y^2=225$
2. a. $x^2+y^2+4x-10y+25=0$
   b. $y^2-x^2-4y+2x-6=0$
   c. $x=-y^2-2y+3$
   d. $16x^2+9y^2=144$

Answer

1.

   a. Hyperbola
   b. Circle
   c. Parabola
   d. Ellipse

Solve Systems of Nonlinear Equations

? Exercise 8.4.19 Solve a System of Nonlinear Equations Using Graphing

In the following exercises, solve the system of equations by using graphing.

1. $\begin{cases} 3x^2-y=0 \\ y=2x-1 \end{cases}$
2. $\begin{cases} y=x^2-4 \\ y=x-4 \end{cases}$
3. $\begin{cases} x^2+y^2=169 \\ x=12 \end{cases}$
4. $\begin{cases} x^2+y^2=25 \\ y=-5 \end{cases}$

Answer

1.

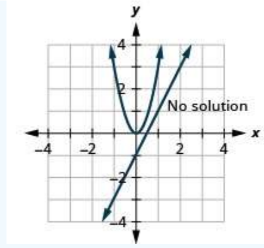


Figure 11.E.29

3.

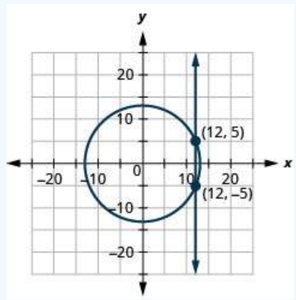


Figure 11.E.30

❓ Exercise 8.4.20 Solve a System of Nonlinear Equations Using Substitution

In the following exercises, solve the system of equations by using substitution.

1. $\begin{cases} y = x^2 + 3 \\ y = -2x + 2 \end{cases}$
2. $\begin{cases} x^2 + y^2 = 4 \\ x - y = 4 \end{cases}$
3. $\begin{cases} 9x^2 + 4y^2 = 36 \\ y - x = 5 \end{cases}$
4. $\begin{cases} x^2 + 4y^2 = 4 \\ 2x - y = 1 \end{cases}$

Answer

1. $(-1, 4)$

3. No solution

❓ Exercise 8.4.21 Solve a System of Nonlinear Equations Using Elimination

In the following exercises, solve the system of equations by using elimination.

1. $\begin{cases} x^2 + y^2 = 16 \\ x^2 - 2y - 1 = 0 \end{cases}$
2. $\begin{cases} x^2 - y^2 = 5 \\ -2x^2 - 3y^2 = -30 \end{cases}$
3. $\begin{cases} 4x^2 + 9y^2 = 36 \\ 3y^2 - 4x = 12 \end{cases}$
4. $\begin{cases} x^2 + y^2 = 14 \\ x^2 - y^2 = 16 \end{cases}$

Answer

1. $(-\sqrt{7}, 3), (\sqrt{7}, 3)$

3. $(-3,0),(0,-2),(0,2)$

Exercise 8.4.22 Use a System of Nonlinear Equations to Solve Applications

In the following exercises, solve the problem using a system of equations.

1. The sum of the squares of two numbers is 25. The difference of the numbers is 1. Find the numbers.
2. The difference of the squares of two numbers is 45. The difference of the square of the first number and twice the square of the second number is 9. Find the numbers.
3. The perimeter of a rectangle is 58 meters and its area is 210 square meters. Find the length and width of the rectangle.
4. Colton purchased a larger microwave for his kitchen. The diagonal of the front of the microwave measures 34 inches. The front also has an area of 480 square inches. What are the length and width of the microwave?

Answer

1. -3 and -4 or 4 and 3

3. If the length is 14 inches, the width is 15 inches. If the length is 15 inches, the width is 14 inches.

Practice Test

Exercise 8.4.23

In the following exercises, find the distance between the points and the midpoint of the line segment with the given endpoints. Round to the nearest tenth as needed.

1. $(-4,-3)$ and $(-10,-11)$
2. $(6,8)$ and $(-5,-3)$

Answer

1. distance: $10,$ midpoint: $(-7,-7)$

Exercise 8.4.24

In the following exercises, write the standard form of the equation of the circle with the given information.

1. radius is 11 and center is $(0,0)$
2. radius is 12 and center is $(10,-2)$
3. center is $(-2,3)$ and a point on the circle is $(2,-3)$
4. Find the equation of the ellipse shown in the graph.

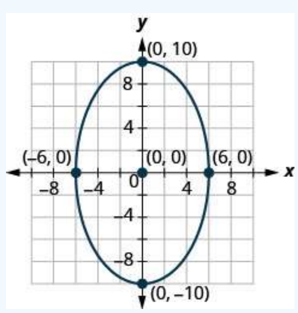


Figure 11.E.31

Answer

1. $x^2+y^2=121$

3. $(x+2)^2+(y-3)^2=52$

Exercise 8.4.25

In the following exercises,

a. Identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola, and
b. Graph the equation.

1. $4x^2 + 49y^2 = 196$
2. $y = 3(x-2)^2 - 2$
3. $3x^2 + 3y^2 = 27$
4. $\frac{y^2}{100} - \frac{x^2}{36} = 1$
5. $\frac{x^2}{16} + \frac{y^2}{81} = 1$
6. $x = 2y^2 + 10y + 7$
7. $64x^2 - 9y^2 = 576$

Answer

1.

a. Ellipse
b.

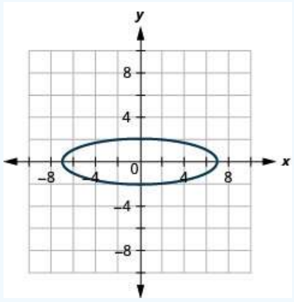


Figure 11.E.32

3.

a. Circle
b.

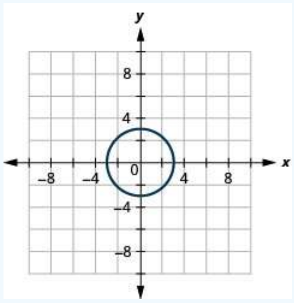


Figure 11.E.33

5.

a. Ellipse
b.

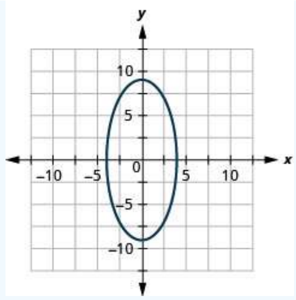


Figure 11.E.34

7.

a. Hyperbola
b.

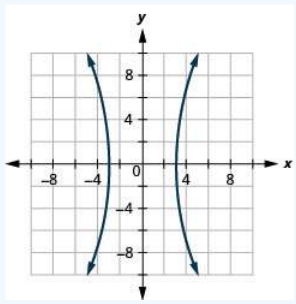


Figure 11.E.35

❓ Exercise 8.4.26

In the following exercises,

a. Identify the type of graph of each equation as a circle, parabola, ellipse, or hyperbola,
b. Write the equation in standard form, and
c. Graph the equation.

1. $25x^2+64y^2+200x-256y-944=0$
2. $x^2+y^2+10x+6y+30=0$
3. $x=-y^2+2y-4$
4. $9x^2-25y^2-36x-50y-214=0$
5. $y=x^2+6x+8$
6. Solve the nonlinear system of equations by graphing: $\begin{cases}3y^2-x=0\\ y=-2x-1\end{cases}$.
7. Solve the nonlinear system of equations using substitution: $\begin{cases}x^2+y^2=8\\ y=-x-4\end{cases}$.
8. Solve the nonlinear system of equations using elimination: $\begin{cases}x^2+9y^2=9\\ 2x^2-9y^2=18\end{cases}$
9. Create the equation of the parabolic arch formed in the foundation of the bridge shown. Give the answer in $y=ax^2+bx+c$ form.

Figure 11.E.36

10. A comet moves in an elliptical orbit around a sun. The closest the comet gets to the sun is approximately 20 AU and the furthest is approximately 70 AU. The sun is one of the foci of the elliptical orbit. Letting the ellipse center at the origin and

labeling the axes in AU, the orbit will look like the figure below. Use the graph to write an equation for the elliptical orbit of the comet.

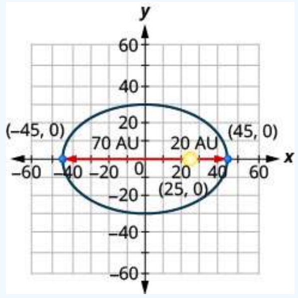


Figure 11.E.37

11. The sum of two numbers is 22 and the product is -240. Find the numbers.

12. For her birthday, Olive’s grandparents bought her a new widescreen TV. Before opening it she wants to make sure it will fit her entertainment center. The TV is 55”. The size of a TV is measured on the diagonal of the screen and a widescreen has a length that is larger than the width. The screen also has an area of 1452 square inches. Her entertainment center has an insert for the TV with a length of 50 inches and width of 40 inches. What are the length and width of the TV screen and will it fit Olive’s entertainment center?

Answer

2.

a. Circle
b. $(x+5)^2+(y+3)^2=4$
c.

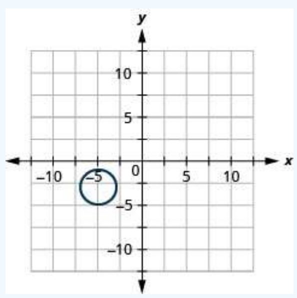


Figure 11.E.38

4.

a. Hyperbola
b. $\frac{(x-2)^2}{25}-\frac{(y+1)^2}{9}=1$
c.

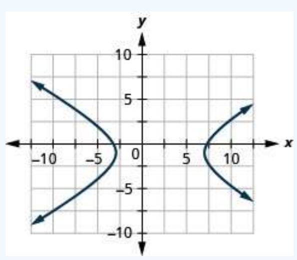


Figure 11.E.39

6. No solution

8. $(0,-3),(0,3)$

10. $\frac{x^2}{2025}+\frac{y^2}{1400}=1$

12. The length is 44 inches and the width is 33 inches. The TV will fit Olive’s entertainment center.

Glossary

system of nonlinear equations

A system of nonlinear equations is a system where at least one of the equations is not linear.

CHAPTER OVERVIEW

9: Exponential and Logarithmic Expressions and Equations

9.1: Evaluate Exponential Expressions and Graph Basic Exponential Equations

9.2: Evaluate Logarithms and Graph Basic Logarithmic Equations

9.3: Use the Properties of Logarithms

9.4: Solve Exponential and Logarithmic Equations

9.5: Chapter 9 Review Exercises

9.1: Evaluate Exponential Expressions and Graph Basic Exponential Equations

Learning Objectives

By the end of this section, you will be able to:

- Graph basic exponential equations
- Solve exponential equations
- Use exponential models in applications

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $\left(\frac{x^3}{x^2}\right)$.
2. Evaluate: a. 2^0 b. $\left(\frac{1}{3}\right)^0$.
3. Evaluate: a. 2^{-1} b. $\left(\frac{1}{3}\right)^{-1}$.

Basic Exponential Expressions

The expressions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential expressions.

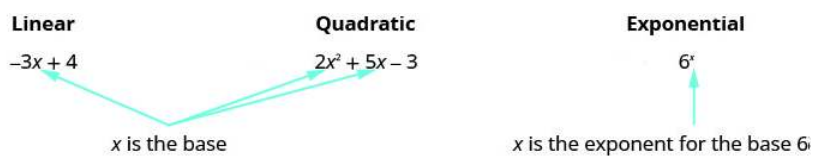
An **exponential expression** is an expression of the form a^x where $a>0$ and $a\neq 1$.

Definition 9.1.1

An **exponential expression**, where $a>0$ and $a\neq 1$, is an expression of the form

$$a^x$$

Notice that in this expression, the variable is the exponent. In our expressions so far, the variables were the base.



Our definition says $a\neq 1$. If we let $a=1$, then a^x becomes 1^x. But we know $1^x=1$ for all real numbers.

Our definition also says $a>0$. If we let a base be negative, say -4, then $(-4)^x$ is not a real number when $x=\frac{1}{2}$.

In fact, $(-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a>0$.

Graphing a few exponential equations will be illuminating.

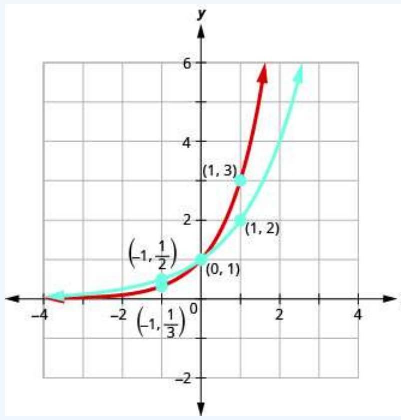
Example 9.1.2

On the same coordinate system graph $y=2^x$ and $y=3^x$.

Solution:

We will use point plotting to graph the equations.

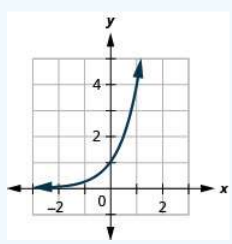
x	$y = 2^x$	(x,y)	$y = 3^x$	(x,y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$\left(-2, \frac{1}{4}\right)$	$3^{-2} = \frac{1}{3^2} = \frac{1}{9}$	$\left(-2, \frac{1}{9}\right)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$\left(-1, \frac{1}{2}\right)$	$3^{-1} = \frac{1}{3^1} = \frac{1}{3}$	$\left(-1, \frac{1}{3}\right)$
0	$2^0 = 1$	$(0, 1)$	$3^0 = 1$	$(0, 1)$
1	$2^1 = 2$	$(1, 2)$	$3^1 = 3$	$(1, 3)$
2	$2^2 = 4$	$(2, 4)$	$3^2 = 9$	$(2, 9)$
3	$2^3 = 8$	$(3, 8)$	$3^3 = 27$	$(3, 27)$



❓ Try It 9.1.3

Graph: $y = 4^x$.

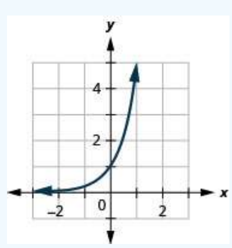
Answer



❓ Try It 9.1.4

Graph: $y = 5^x$

Answer



If we look at the graphs from the previous Example 9.1.2 and Try Its 9.1.3 and 9.1.4, we can identify some of the properties of exponential expressions.

The graphs of $y=2^{x}$ and $y=3^{x}$, as well as the graphs of $y=4^{x}$ and $y=5^{x}$, all have the same basic shape. This is the shape we expect of the graph of $y=a^{x}$ where $a>1$.

We notice, that for each equation, the graph contains the point $(0,1)$. This make sense because $a^{0}=1$ for any a.

The graph of each equation, $y=a^{x}$ also contains the point $(1, a)$. The graph of $y=2^{x}$ contained $(1,2)$ and the graph of $y=3^{x}$ contained $(1,3)$. This makes sense as $a^{1}=a$.

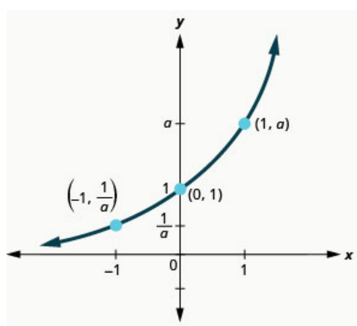
Notice too, the graph of each equation $y=a^{x}$ also contains the point $(-1, \frac{1}{a})$. The graph of $y=2^{x}$ contained $(-1, \frac{1}{2})$ and the graph of $y=3^{x}$ contained $(-1, \frac{1}{3})$.This makes sense as $a^{-1}=\frac{1}{a}$.

Notice, that the expression a^{x}, $a>1$ makes sense for any value of x.

And, look at each graph. The graph never hits the x-axis which means $a^{x}>0$

Whenever a graph of an equation approaches a line but never touches it, we call that line an **asymptote**. For the exponential equations we are looking at, the graph approaches the x-axis very closely but will never cross it, we call the line $y=0$, the x-axis, a horizontal asymptote. Recognizing that these all have the asymptote $y=0$ can be helpful when graphing.

The following graph summarizes the situation when $a>1$.



Our definition of an exponential expression a^{x} says $a>0$, but the examples and discussion so far has been about equations where $a>1$. What happens when $0<a<1$ The next example will explore this possibility.

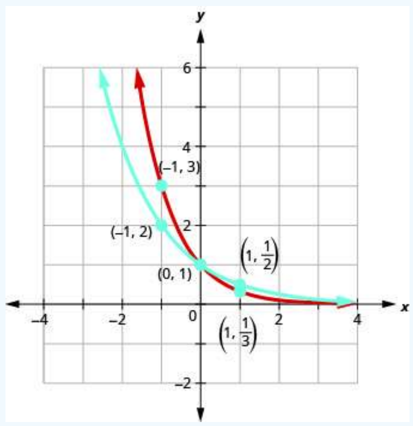
Example 9.1.5

On the same coordinate system, graph $y=\left(\frac{1}{2}\right)^{x}$ and $y=\left(\frac{1}{3}\right)^{x}$.

Solution:

We will use point plotting to graph the equations.

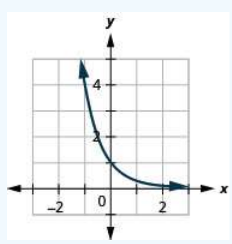
x	$y=\left(\frac{1}{2}\right)^x$	(x,y)	$y=\left(\frac{1}{3}\right)^x$	(x,y)
−2	$\left(\frac{1}{2}\right)^{-2}=2^2=4$	(−2, 4)	$\left(\frac{1}{3}\right)^{-2}=3^2=9$	(−2, 9)
−1	$\left(\frac{1}{2}\right)^{-1}=2^1=2$	(−1, 2)	$\left(\frac{1}{3}\right)^{-1}=3^1=3$	(−1, 3)
0	$\left(\frac{1}{2}\right)^{0}=1$	(0, 1)	$\left(\frac{1}{3}\right)^{0}=1$	(0, 1)
1	$\left(\frac{1}{2}\right)^{1}=\frac{1}{2}$	$\left(1,\frac{1}{2}\right)$	$\left(\frac{1}{3}\right)^{1}=\frac{1}{3}$	$\left(1,\frac{1}{3}\right)$
2	$\left(\frac{1}{2}\right)^{2}=\frac{1}{4}$	$\left(2,\frac{1}{4}\right)$	$\left(\frac{1}{3}\right)^{2}=\frac{1}{9}$	$\left(2,\frac{1}{9}\right)$
3	$\left(\frac{1}{2}\right)^{3}=\frac{1}{8}$	$\left(3,\frac{1}{8}\right)$	$\left(\frac{1}{3}\right)^{3}=\frac{1}{27}$	$\left(3,\frac{1}{27}\right)$



? Try It 9.1.6

Graph: $y=\left(\frac{1}{4}\right)^x$.

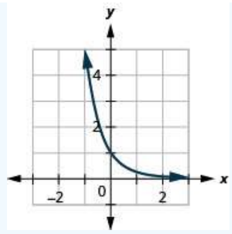
Answer



? Try It 9.1.7

Graph: $y=\left(\frac{1}{5}\right)^x$.

Answer



Now let's look at the graphs from the previous Example 10.2.5 and Try Its 10.2.6 and 10.2.7 so we can now identify some of the properties of exponential expressions where $0<a<1$.

The graphs of $y=\left(\frac{1}{2}\right)^{x}$ and $y=\left(\frac{1}{3}\right)^{x}$ as well as the graphs of $y=\left(\frac{1}{4}\right)^{x}$ and $y=\left(\frac{1}{5}\right)^{x}$ all have the same basic shape. While this is the shape we expect from an exponential equation of the form $y=a^{x}$, where $0<a<1$, the graphs go down from left to right while the previous graphs, when $a>1$, went from up from left to right.

We notice that for each equation, the graph still contains the point $(0,1)$. This make sense because $a^{0}=1$ for any a.

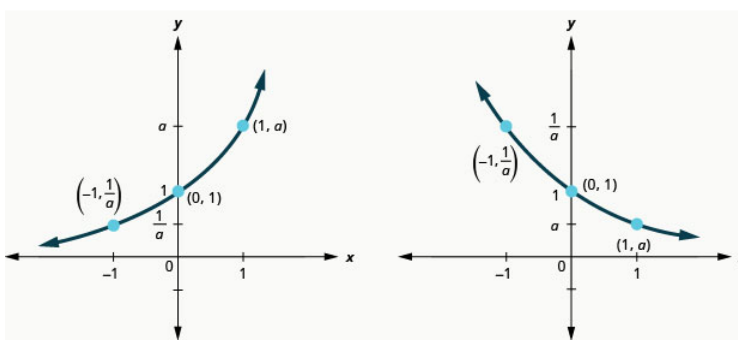
As before, the graph of each equation, $f(x)=a^{x}$, also contains the point $(1,a)$. The graph of $y=\left(\frac{1}{2}\right)^{x}$ contained $\left(1,\frac{1}{2}\right)$ and the graph of $y=\left(\frac{1}{3}\right)^{x}$ contained $\left(1,\frac{1}{3}\right)$. This makes sense as $a^{1}=a$.

Notice too that the graph of each equation, $y=a^{x}$, also contains the point $\left(-1,\frac{1}{a}\right)$. The graph of $y=\left(\frac{1}{2}\right)^{x}$ contained $(-1,2)$ and the graph of $y=\left(\frac{1}{3}\right)^{x}$ contained $(-1,3)$. This makes sense as $a^{-1}=\frac{1}{a}$.

Notice, that the expression a^{x}, $0<a<1$ makes sense for any value of x.

Again, the graph never hits the x-axis. The range is all positive numbers. We write the range in interval notation as $(0,\infty)$.

We will summarize these properties in the chart below. Which also includes when $a>1$.



Note 9.1.8

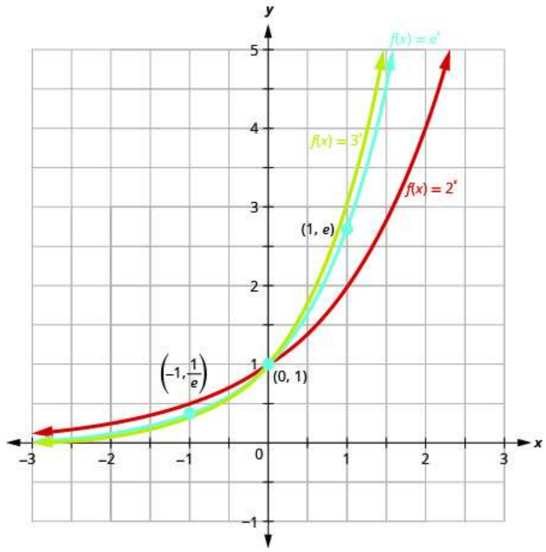
Natural Base e

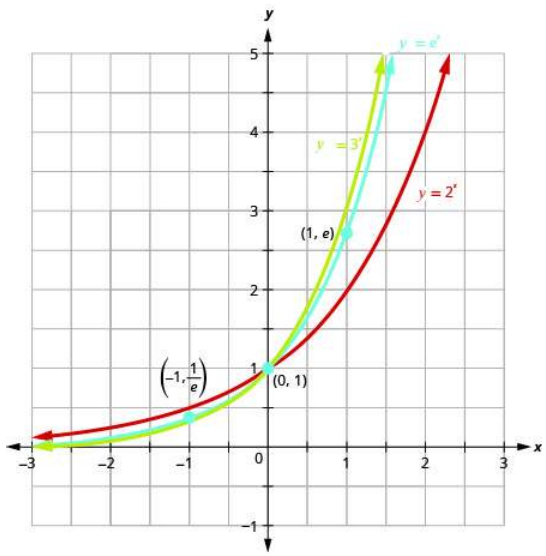
In applications, it happens that different bases are convenient for writing relevant expressions. There is one number which is written "e" . We will not delve into the details of this number except to say that this number is irrational and

$$e\approx 2.718281827.$$

Most calculators will include this particular base on a button often labeled e^{x}.

Let's graph the equation $y=e^{x}$ on the same coordinate system as $y=2^{x}$ and $y=3^{x}$.





Notice that the graph of $y = e^x$ is "between" the graphs of $y = 2^x$ and $y = 3^x$.Does this make sense as $2 < e < 3$?

Solve Exponential Equations

Equations that include an exponential expression a^x are called exponential equations. To solve them we use a property that says as long as $a > 0$ and $a \neq 1$, if $a^x = a^y$ then it is true that $x = y$. In other words, in an exponential equation, if the bases are equal then the exponents are equal. To see why this is true consider the graph of $y = 2^x$ and see that if the two y-coordinates on the graph are the same, then the x-coordinates must also agree.

Definition 9.1.9

One-to-One Property of Exponential Equations

For $a > 0$ and $a \neq 1$,

$$\text{If } a^x = a^y\text{, then } x = y.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Example 9.1.10

Solve: $3^{2x-5}=27$.

Solution:

Write both sides of the equation with the same base.	Since the left side has base 3, we write the right side with base 3. $27=3^3$	$3^{2x-5}=27$ $3^{2x-5}=3^3$
Write a new equation by setting the exponents equal.	Since the bases are the same, the exponents must be equal.	$2x-5=3$
Solve the equation.	Add 5 to each side. Divide by 2.	$2x=8$ $x=4$
Check the solution (to ensure there are no errors).	Substitute $x=4$ into the original equation.	$3^{2x-5}=27$ $3^{2\cdot 4-5}\stackrel{?}{=}27$ $3^3\stackrel{?}{=}27$ $27\stackrel{?}{=}27$ True

Try It 9.1.11

Solve: $3^{3x-2}=81$.

Answer

$x=2$

Try It 9.1.12

Solve: $7^{x-3}=7$.

Answer

$x=4$

The steps are summarized below.

How to Solve an Exponential Equation

1. Write both sides of the equation with the same base, if possible.
2. Write a new equation by setting the exponents equal.
3. Solve the equation.
4. Check the solution to make sure no error has been made.

In the next example, we will use our properties on exponents.

Example 9.1.13

Solve $\dfrac{e^{x^2}}{e^3}=e^{2x}$.

Solution:

	$\dfrac{e^{x^2}}{e^3}=e^{2x}$

Use the Property of Exponents: $\frac{a^m}{a^n}=a^{m-n}$.	$e^{x^2-3}=e^{2x}$
Write a new equation by setting the exponents equal.	$x^2-3=2x$
Solve the equation.	
	$(x-3)(x+1)=0$
	$x=3$ or $x=-1$
Check the solutions.	$x=3 \quad x=-1$ $\frac{e^{x^2}}{e^3}=e^{2x} \quad \frac{e^{x^2}}{e^3}=e^{2x}$ $\frac{e^{3^2}}{e^3}\stackrel{?}{=}e^{2\cdot 3} \quad \frac{e^{(-1)^2}}{e^3}\stackrel{?}{=}e^{2\cdot(-1)}$ $\frac{e^9}{e^3}\stackrel{?}{=}e^6 \quad \frac{e^1}{e^3}\stackrel{?}{=}e^{-2}$ $e^6\stackrel{?}{=}e^6$ True $\quad e^{-2}\stackrel{?}{=}e^{-2}$ True The solutions are $x=3$ and $x=-1$.

Try It 9.1.14

Solve: $\frac{e^{x^2}}{e^x}=e^2$.

Answer

$x=-1, x=2$

Try It 9.1.15

Solve: $\frac{e^{x^2}}{e^x}=e^6$.

Answer

$x=-2, x=3$

Use Exponential Models in Applications

Exponential equations model many situations. If you own a bank account, you have experienced the use of an exponential equation. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, P, is invested at an interest rate, r, for t years, the new balance, A, will depend on how often the interest is compounded, i.e., how often the interest is calculated and then added to the new balance. If the interest is compounded n times a year we use the formula $A=P\left(1+\frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A=Pe^{rt}$. These are the formulas for **compound interest**.

Definitions and Formulae 9.1.16

Compound Interest is interest that accumulates on the interest earned.

The **principal** is the amount invested.

Interest is compounded when the interest is calculated and then added to the new balance.

For a **principal**, P, invested at an interest **rate**, r, for t years, the new balance, A, is:

$$A = P\left(1+\frac{r}{n}\right)^{nt} \quad \text{when compounded } n \text{ times a year.}$$
$$A = Pe^{rt} \quad \text{when continuously.}$$

To see why the first formula works, work out some examples by writing down how the compounding works in detail. As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example 9.1.17

A total of $\$10,000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Solution:

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.

$$\begin{aligned} A &= ? \\ P &= \$10,000 \\ r &= 0.05 \\ t &= 18 \text{ years} \end{aligned}$$

a. For quarterly compounding, $n = 4$. There are 4 quarters in a year.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1+\frac{0.05}{4}\right)^{4\cdot 18}$$

Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.

$$A = \$24,459.20$$

b. For monthly compounding, $n = 12$.There are 12 months in a year.

$$A = P\left(1+\frac{r}{n}\right)^{nt}$$

Substitute the values in the formula.

$$A = 10,000\left(1+\frac{0.05}{12}\right)^{12\cdot 18}$$

Compute the amount.

$$A = \$24,550.08$$

c. For compounding continuously,

$$A = Pe^{rt}$$

Substitute the values in the formula.

$$A = 10,000e^{0.05\cdot 18}$$

Compute the amount.

$$A = \$24,596.03$$

Try It 9.1.18

Angela invested $\$15,000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$22,332.96$
b. $\$22,362.49$
c. $\$22,377.37$

Try It 9.1.19

Allan invested $\$10,000$ in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

Answer

a. $\$21,071.81$
b. $\$21,137.04$
c. $\$21,170.00$

Other topics that are modeled by exponential equations involve growth and decay. Both also use the formula $A = Pe^{rt}$ we used for the growth of money. For growth and decay, generally we use A_0, as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate of growth and **exponential decay** has a negative rate of growth.

Definition 9.1.20

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount, A, is:

$$A = A_0e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example 9.1.21

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match--the rate is per hour and the time is in hours.

$$\begin{aligned} A &= ? \\ A_0 &= 100 \\ r &= 0.25/\text{ hour} \\ t &= 24 \text{ hours} \end{aligned}$$

Substitute the values in the formula: $A = A_0 e^{rt}$.

$$A = 100e^{0.25 \cdot 24}$$

Compute the amount.

$$A = 40,342.88$$

Round to the nearest whole virus.

$$A = 40,343$$

The researcher will find $40,343$ viruses.

Try It 9.1.22

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Answer

She will find 166 bacteria.

Try It 9.1.23

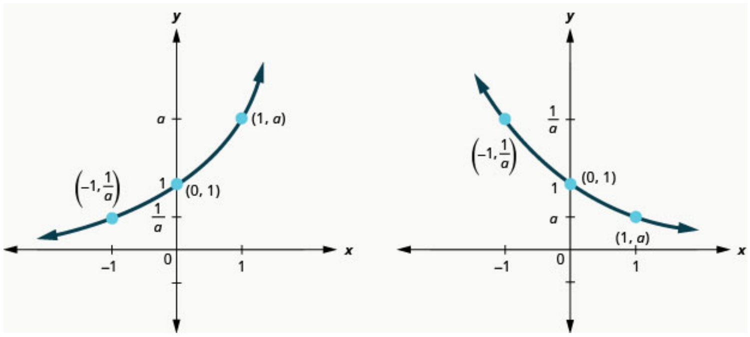
Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Answer

She will find $1,102$ viruses.

Key Concepts

- **Properties of the Graph of $y = a^x$:**



- **One-to-One Property of Exponential Equations:**
  For $a > 0$ and $a \neq 1$, if $a^x = a^y$ then $x = y$.
- **How to Solve an Exponential Equation**
  1. Write both sides of the equation with the same base, if possible.
  2. Write a new equation by setting the exponents equal.
  3. Solve the equation.
  4. Check the solution to ensure there are no errors.
- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is

  $A = P\left(1 + \frac{r}{n}\right)^{nt}$ when compounded n times a year.

  $A = Pe^{rt}$ when compounded continuously.

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A = A_0 e^{rt}$.

Glossary

asymptote

A line which a graph of an equation approaches closely but never touches.

exponential expression

An exponential expression, where $a > 0$ and $a \neq 1$, is an expression of the form a^x.

natural base

The number e is an irrational number that appears in many applications. $e \approx 2.718281827\ldots$

Practice Makes Perfect

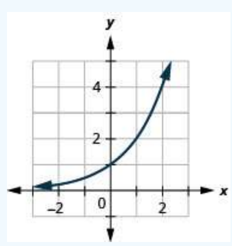
Graph Basic Exponential equations

In the following exercises, graph each exponential function.

1. $y = 2^x$
2. $y = 3^x$
3. $y = 6^x$
4. $y = 7^x$
5. $y = (1.5)^x$
6. $y = (2.5)^x$
7. $y = \left(\frac{1}{2}\right)^x$
8. $y = \left(\frac{1}{3}\right)^x$
9. $y = \left(\frac{1}{6}\right)^x$
10. $y = \left(\frac{1}{7}\right)^x$
11. $y = (0.4)^x$
12. $y = (0.6)^x$

Answer

1.



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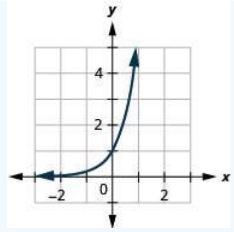

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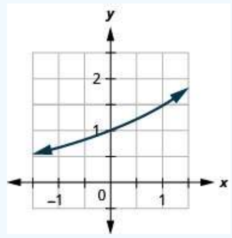

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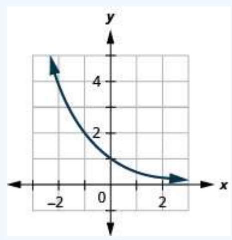
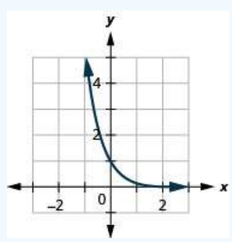


9.



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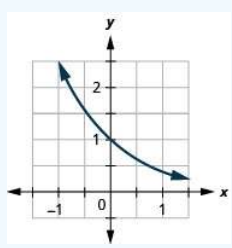


? Solve Exponential Equations

In the following exercises, solve each equation.

13. $2^{3x-8}=16$

14. $2^{2x-3}=32$
15. $3^{x+3}=9$
16. $3^{x^2}=81$
17. $4^{x^2}=4$
18. $4^x=32$
19. $4^{x+2}=64$
20. $4^{x+3}=16$
21. $2^{x^2+2x}=\frac{1}{2}$
22. $3^{x^2-2x}=\frac{1}{3}$
23. $e^{3x}\cdot e^4=e^{10}$
24. $e^{2x}\cdot e^3=e^9$
25. $\frac{e^{x^2}}{e^2}=e^x$
26. $\frac{e^{x^2}}{e^3}=e^{2x}$

Answer

13. $x=4$

15. $x=-1$

17. $x=-1, x=1$

19. $x=1$

21. $x=-1$

23. $x=2$

25. $x=-1, x=2$

Use Exponential Models in Applications

In the following exercises, use an exponential model to solve.

27. Edgar accumulated $\$5,000$ in credit card debt. If the interest rate is 20% per year, and he does not make any payments for 2 years, how much will he owe on this debt in 2 years by each method of compounding?
    a. compound quarterly
    b. compound monthly
    c. compound continuously
28. Cynthia invested $\$12,000$ in a savings account. If the interest rate is 6%, how much will be in the account in 10 years by each method of compounding?
    a. compound quarterly
    b. compound monthly
    c. compound continuously
29. Rochelle deposits $\$5,000$ in an IRA. What will be the value of her investment in 25 years if the investment is earning 8% per year and is compounded continuously?
30. Nazerhy deposits $\$8,000$ in a certificate of deposit. The annual interest rate is 6% and the interest will be compounded quarterly. How much will the certificate be worth in 10 years?
31. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. He starts his experiment with 100 of the bacteria that grows at a rate of 6% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?
32. A biologist is observing the growth pattern of a virus. She starts with 50 of the virus that grows at a rate of 20% per hour. She will check on the virus in 24 hours. How many viruses will she find?

33. In the last ten years the population of Indonesia has grown at a rate of 1.12% per year to $258,316,051$ If this rate continues, what will be the population in 10 more years?
34. In the last ten years the population of Brazil has grown at a rate of 0.9% per year to $205,823,665$ If this rate continues, what will be the population in 10 more years?

Answer

27.

a. $\$7,387.28$
b. $\$7,434.57$
c. $\$7,459.12$

29. $\$36,945.28$

31. 223 bacteria

33. $288,929,825$

Writing Exercises

35. Explain how you can distinguish between exponential expressions and polynomial expressions.
36. Compare and contrast the graphs of $y=x^2$ and $y=2^x$.
37. What happens to an exponential function as the values of x decreases? Will the graph ever cross the x-axis? Explain.

Answer

35. Answers will vary

37. Answers will vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
evaluate exponential expressions			
graph basic exponential equations			
solve exponential equations			
Use exponential models in applications			

b. After reviewing this checklist, what will you do to become confident for all objectives?

9.2: Evaluate Logarithms and Graph Basic Logarithmic Equations

Learning Objectives

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Evaluate logarithmic expressions
- Graph basic logarithmic equations
- Solve logarithmic equations
- Use logarithmic models in applications

Be Prepared

Before you get started, take this readiness quiz.

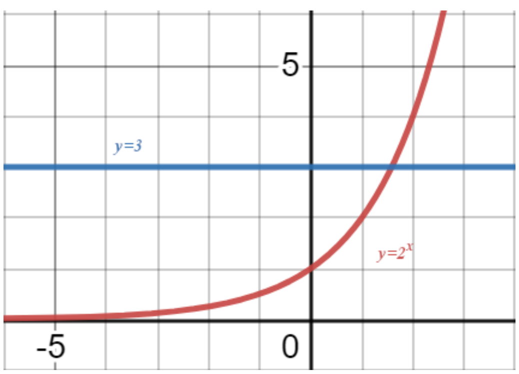
1. Solve: $x^2 = 81$.
2. Evaluate: 3^{-2}.
3. Solve: $2^4 = 3x - 5$.

We have spent some time solving many basic equations. It works well to 'undo' an operation with another operation. Subtracting 'undoes' addition, multiplication 'undoes' division, taking the square root 'undoes' squaring.

Because of the ono-to-one property of exponential expressions (for $a \neq 1$ and $a > 0$, if $a^x = a^y$ then $x = y$), the equation $a^x = b$ has at most one solution. Also, it has a solution (as long as $b > 0$ since $a^x > 0$ for all x). To see why this is so consider this (non-linear) system of equations

$$\begin{cases} y = b \\ y = a^x \end{cases}. \tag{9.2.1}$$

The solution of this is also a solution to the equation $a^x = b$ and graphically it is seen as the intersection of the two curves, and they can be seen to intersect as long as $b > 0$. Below is an example of such a situation.



We can approximate the solution by trial and error with a calculator. But this is time consuming. Sometimes, as we have seen, we can solve these equations by hand, and fortunately, in other cases, our calculator can provide us with an approximation (after a little work, depending on the calculator).

We use the standard notation for the solution to $a^x = b$. The solution is written $x = \log_a b$.

Definition 9.2.1

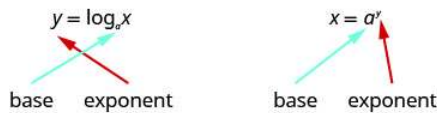
The solution to the equation $x = a^y$ is written

$\log_a x$ and is called the **logarithm** of x with **base** a, where $a > 0, x > 0$, and $a \neq 1$.

$$y = \log_a x \text{ is equivalent to } x = a^y$$

Convert Between Exponential and Logarithmic Form

Since the equations $y = \log_a x$ and $x = a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See the figure below. Notice the positions of the exponent and base.



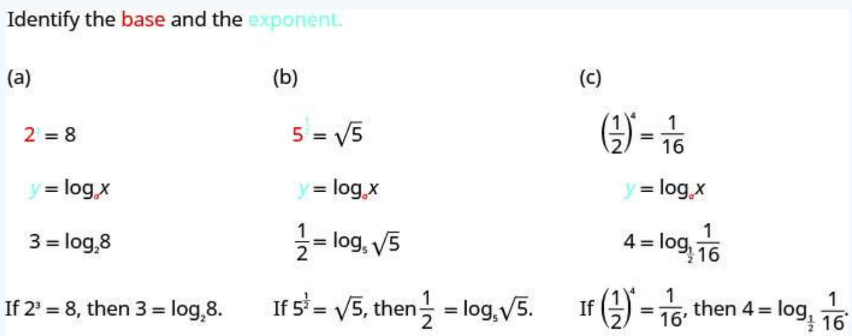
If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

✔ Example $9.2.2$

Convert to logarithmic form:

a. $2^3 = 8$
b. $5^{\frac{1}{2}} = \sqrt{5}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{16}$

Solution:



? Try It $9.2.3$

Convert to logarithmic form:

a. $3^2 = 9$
b. $7^{\frac{1}{2}} = \sqrt{7}$
c. $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Answer

a. $\log_3 9 = 2$
b. $\log_7 \sqrt{7} = \frac{1}{2}$
c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Try It 9.2.4

Convert to logarithmic form:

a. $4^3 = 64$
b. $4^{\frac{1}{3}} = \sqrt[3]{4}$
c. $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Answer

a. $\log_4 64 = 3$
b. $\log_4 \sqrt[3]{4} = \frac{1}{3}$
c. $\log_{\frac{1}{2}} \frac{1}{32} = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example 9.2.5

Convert to exponential form:

a. $2 = \log_8 64$
b. $0 = \log_4 1$
c. $-3 = \log_{10} \frac{1}{1000}$

Solution:

Identify the base and the exponent.

(a)	(b)	(c)
$2 = \log_8 64$	$0 = \log_4 1$	$-3 = \log_{10} \frac{1}{1000}$
$x = a$	$x = a$	$x = a$
$64 = 8^2$	$1 = 4^0$	$\frac{1}{1000} = \log^{-3}$
If $2 = \log_8 64$, then $64 = 8^2$.	If $0 = \log_4 1$, then $1 = 4^0$.	If $-3 = \log_{10} \frac{1}{1000}$, then $\frac{1}{1000} = 10^{-3}$.

Try It 9.2.6

Convert to exponential form:

a. $3 = \log_4 64$
b. $0 = \log_x 1$
c. $-2 = \log_{10} \frac{1}{100}$

Answer

a. $64 = 4^3$
b. $1 = x^0$
c. $\frac{1}{100} = 10^{-2}$

Try It 9.2.7

Convert to exponential form:

a. $3=\log_3 27$
b. $0=\log_x 1$
c. $-1=\log_{10}\frac{1}{10}$

Answer

a. $27=3^3$
b. $1=x^0$
c. $\frac{1}{10}=10^{-1}$

Evaluate Logarithmic Expressions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example 9.2.8

Find the value of x:

a. $\log_x 36=2$

b. $\log_4 x=3$

c. $\log_{\frac{1}{2}}\frac{1}{8}=x$

Solution:

a.

$$\log_x 36=2$$

Convert to exponential form.

$$x^2=36$$

Solve the quadratic.

$$x=6, \quad x=\cancel{-6}$$

The base of a logarithm must be positive, so we eliminate $x=-6$.

$$x=6 \quad \text{Therefore, } \log_6 36=2$$

b.

$$\log_4 x=3$$

Convert to exponential form.

$$4^3=x$$

Simplify.

$$x=64 \quad \text{Therefore , } \log_4 64=3$$

c.

$$\log_{\frac{1}{2}}\frac{1}{8}=x$$

Convert to exponential form.

$$\left(\frac{1}{2}\right)^x = \frac{1}{8}$$

Rewrite $\frac{1}{8}$ as $\left(\frac{1}{2}\right)^3$.

$$\left(\frac{1}{2}\right)^x = \left(\frac{1}{2}\right)^3$$

With the same base, the exponents must be equal.

$$x = 3 \quad \text{Therefore } \log_{\frac{1}{2}} \frac{1}{8} = 3$$

? Try It 9.2.9

Find the value of x:

a. $\log_x 64 = 2$

b. $\log_5 x = 3$

c. $\log_{\frac{1}{2}} \frac{1}{4} = x$

Answer

a. $x = 8$

b. $x = 125$

c. $x = 2$

? Try It 9.2.10

Find the value of x:

a. $\log_x 81 = 2$

b. $\log_3 x = 5$

c. $\log_{\frac{1}{3}} \frac{1}{27} = x$

Answer

a. $x = 9$

b. $x = 243$

c. $x = 3$

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $\log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example 9.2.11

Find the exact value of each logarithm without using a calculator:

a. $\log_5 25$

b. $\log_9 3$

c. $\log_2 \frac{1}{16}$

Solution:

a.

$$\log_5 25$$

5 to what power will be 25?

$$\log_5 25 = 2$$

Or

Set the expression equal to x.

$$\log_5 25 = x$$

Change to exponential form.

$$5^x = 25$$

Rewrite 25 as 5^2.

$$5^x = 5^2$$

With the same base the exponents must be equal.

$$x = 2 \quad \text{Therefore}, \log_5 25 = 2.$$

b.

$$\log_9 3$$

Set the expression equal to x.

$$\log_9 3 = x$$

Change to exponential form.

$$9^x = 3$$

Rewrite 9 as 3^2.

$$\left(3^2\right)^x = 3^1$$

Simplify the exponents.

$$3^{2x} = 3^1$$

With the same base the exponents must be equal.

$$2x = 1$$

Solve the equation.

$$x = \frac{1}{2} \quad \text{Therefore}, \log_9 3 = \frac{1}{2}.$$

c.

$$\log_2 \frac{1}{16}$$

Set the expression equal to x.

$$\log_2 \frac{1}{16} = x$$

Change to exponential form.

$$2^x = \frac{1}{16}$$

Rewrite 16 as 2^4.

$$2^x = \frac{1}{2^4}$$

$$2^x = 2^{-4}$$

With the same base the exponents must be equal.

So, $x = -4$ and therefore $\log_2 \frac{1}{16} = -4$

? Try It 9.2.12

Find the exact value of each logarithm without using a calculator:

a. $\log_{12} 144$

b. $\log_4 2$

c. $\log_2 \frac{1}{32}$

Answer

a. 2

b. $\frac{1}{2}$

c. -5

? Try It 9.2.13

Find the exact value of each logarithm without using a calculator:

a. $\log_9 81$

b. $\log_8 2$

c. $\log_3 \frac{1}{9}$

Answer

a. 2

b. $\frac{1}{3}$

c. -2

Optional: Graph Basic Logarithmic Equations

To graph a logarithmic equation $y = log_a x$, it is easiest to convert the equation to its exponential form, $x = a^y$. Generally, when we look for ordered pairs for the graph of an equation, we usually choose an x-value and then determine its corresponding y-value. In this case you may find it easier to choose y-values and then determine its corresponding x-value.

✓ Example 9.2.14

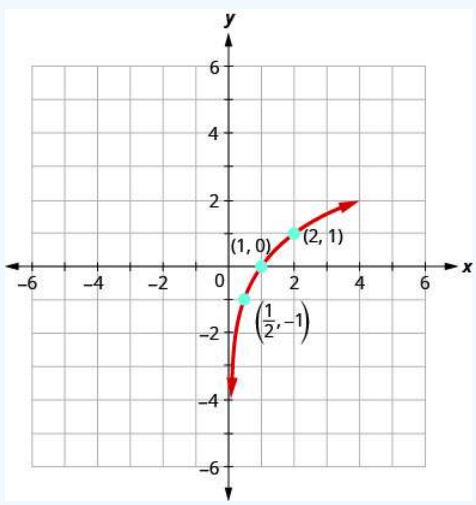
Graph $y = \log_2 x$.

Solution:

To graph the equation, we will first rewrite the logarithmic equation, $y = \log_2 x$, in exponential form, $2^y = x$. We can see that we have already graphed this (found the solutions to this equation, or rather the equation $2^x = y$ from which we get the solutions by exchanging the order of the $x-$coordinate and the $y-$coordinate of each solution.)

Alternatively, we can use point plotting to graph the equation. It will be easier to start with values of y and then get x.

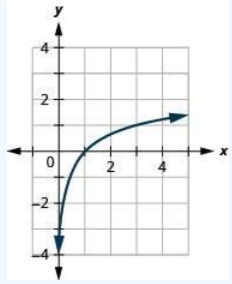
y	$2^y = x$	(x, y)
-2	$2^{-2} = \frac{1}{2^2} = \frac{1}{4}$	$(\frac{1}{4}, 2)$
-1	$2^{-1} = \frac{1}{2^1} = \frac{1}{2}$	$(\frac{1}{2}, -1)$
0	$2^0 = 1$	$(1, 0)$
1	$2^1 = 2$	$(2, 1)$
2	$2^2 = 4$	$(4, 2)$
3	$2^3 = 8$	$(8, 3)$



? Try It 9.2.15

Graph: $y = \log_3 x$.

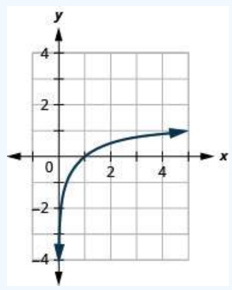
Answer



? Try It 9.2.16

Graph: $y = \log_5 x$.

Answer



✓ Example 9.2.17

Graph $y = \log_{\frac{1}{3}} x$.

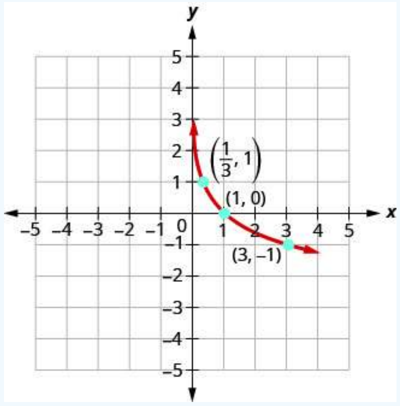
Solution:

To graph the equation, we will first rewrite the logarithmic equation, $y = \log_{\frac{1}{3}} x$, in exponential form, $\left(\frac{1}{3}\right)^y = x$.

We will use point plotting to graph the equation. It will be easier to start with values of y and then get x.

y	$\left(\frac{1}{3}\right)^y = x$	(x, y)
-2	$\left(\frac{1}{3}\right)^{-2} = 3^2 = 9$	$(9, -2)$
-1	$\left(\frac{1}{3}\right)^{-1} = 3^1 = 3$	$(3, -1)$
0	$\left(\frac{1}{3}\right)^0 = 1$	$(1, 0)$
1	$\left(\frac{1}{3}\right)^1 = \frac{1}{3}$	$\left(\frac{1}{3}, 1\right)$
2	$\left(\frac{1}{3}\right)^2 = \frac{1}{9}$	$\left(\frac{1}{9}, 2\right)$

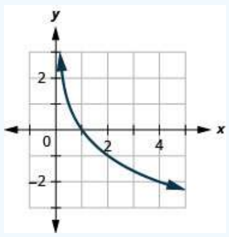
y	$\left(\frac{1}{3}\right)^{y}=x$	(x,y)
3	$\left(\frac{1}{3}\right)^{3}=\frac{1}{27}$	$\left(\frac{1}{27},3\right)$



Try It 9.2.18

Graph: $y=\log_{\frac{1}{2}} x$.

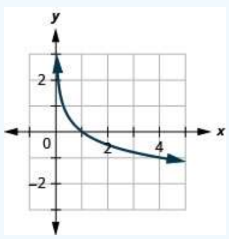
Answer



Try It 9.2.19

Graph: $y=\log_{\frac{1}{4}} x$.

Answer



Solve Logarithmic Equations

When we talked about exponential expressions, we introduced the number e. Just as e was a base for an exponential expression, it can be used as a base for logarithms too. The logarithm with base e is called the **natural logarithm**. The expression $\log_e x$ is generally written $\ln x$ and we read it as “el en of x.”

Definition 9.2.20

$\log_e x = \ln x$ is the **natural logarithm** where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

When the base of the logarithm is 10, we call it the **common logarithmic** and the base is not shown. If the base a of a logarithm is not shown, we assume it is 10. Note that in some texts it is assumed to be e in this case.

Definition 9.2.21

$\log_{10} x = \log x$ is the **common logarithm** where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

To solve logarithmic equations, one strategy is to change the equation to exponential form and then solve the exponential equation as we did before. As we solve basic logarithmic equations, $y = log_a x$, we need to remember that for the base a, $a > 0$ and $a \neq 1$. Also, $x > 0$ since $a^y > 0$. Just as with radical equations, we must check our solutions to eliminate any extraneous solutions.

Example 9.2.22

Solve:

a. $\log_a 49 = 2$

b. $\ln x = 3$

Solution:

a. $\log_a 49 = 2$

Rewrite in exponential form.

$$a^2 = 49$$

Solve the equation using the square root property.

$$a = \pm 7$$

The base cannot be negative, so we eliminate $a = -7$.

$$a = 7, \quad \cancel{a = -7}$$

Check. $a = 7$

$$\begin{aligned} \log_a 49 &= 2 \\ \log_7 49 &\stackrel{?}{=} 2 \\ 7^2 &\stackrel{?}{=} 49 \\ 49 &\stackrel{?}{=} 49 \text{ True} \end{aligned}$$

b.

$$\ln x = 3$$

Rewrite in exponential form.

$$e^3 = x$$

Check. $x = e^3$

$$\ln x = 3$$
$$\ln e^3 \stackrel{?}{=} 3$$
$$e^3 \stackrel{?}{=} e^3 \text{ True}$$

? Try It 9.2.23

Solve:

a. $\log_a 121 = 2$

b. $\ln x = 7$

Answer

a. $a = 11$

b. $x = e^7$

? Try It 9.2.24

Solve:

a. $\log_a 64 = 3$

b. $\ln x = 9$

Answer

a. $a = 4$

b. $x = e^9$

Example 9.2.25

Solve:

a. $\log_2(3x - 5) = 4$

b. $\ln e^{2x} = 4$

Solution:

a.

$$\log_2(3x - 5) = 4$$

Rewrite in exponential form.

$$2^4 = 3x - 5$$

Simplify.

$$16 = 3x - 5$$

Solve the equation.

$$21 = 3x$$
$$7 = x$$

Check. $x = 7$

$$\begin{aligned} \log_2(3x-5) &= 4 \\ \log_2(3\cdot 7-5) &\stackrel{?}{=} 4 \\ \log_2(16) &\stackrel{?}{=} 4 \\ 2^4 &\stackrel{?}{=} 16 \\ 16 &= 16 \end{aligned}$$

b.

$$\ln e^{2x} = 4$$

Rewrite in exponential form.

$$e^4 = e^{2x}$$

Since the bases are the same the exponents are equal.

$$4 = 2x$$

Solve the equation.

$$2 = x$$

Check. $x = 2$

$$\begin{aligned} \ln e^{2x} &= 4 \\ \ln e^{2\cdot 2} &\stackrel{?}{=} 4 \\ \ln e^4 &\stackrel{?}{=} 4 \\ e^4 &\stackrel{?}{=} e^4 \text{ True} \end{aligned}$$

Try It 9.2.26

Solve:

a. $\log_2(5x-1) = 6$

b. $\ln e^{3x} = 6$

Answer

a. $x = 13$

b. $x = 2$

Try It 9.2.27

Solve:

a. $\log_3(4x+3) = 3$

b. $\ln e^{4x} = 4$

Answer

a. $x = 6$

b. $x = 1$

Use Logarithmic Models in Applications

There are many applications that are modeled by logarithmic equations. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Definition 9.2.28

Decibel Level of Sound

The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$D = 10\log\left(\frac{I}{10^{-12}}\right)$$

Example 9.2.29

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$D = 10\log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$D = 10\log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$D = 10\log(10^{10})$
Since $\log 10^{10} = 10$.	$D = 10 \cdot 10$
Multiply.	$D = 100$
	The decibel level of music coming through earphones is 100 dB.

Try It 9.2.30

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Answer

The quiet dishwashers have a decibel level of 50 dB.

Try It 9.2.31

What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Answer

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $R = \log I$, where I is the intensity of the shock wave. This model provides a way to measure **earthquake intensity**.

Definition 9.2.32: Earthquake Intensity

The magnitude R of an earthquake is measured by$R = \log I$, where I is the intensity of its shock wave.

Example 9.2.33

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused \$108 million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

Convert the magnitudes to intensities.

$$R = \log I$$

1906 earthquake

$$7.8 = \log I$$

Convert to exponential form.

$$I = 10^{7.8}$$

2014 earthquake

$$5.1 = \log I$$

Convert to exponential form.

$$I = 10^{5.1}$$

Form a ratio of the intensities.

$$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$$

Substitute in the values.

$$\frac{10^{7.8}}{10^{5.1}}$$

Divide by subtracting the exponents.

$$10^{2.7}$$

Evaluate.

$$501$$

The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

? Try It 9.2.34

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

? Try It 9.2.35

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the earthquake in Chile was about $1,259$ times the intensity of the earthquake in Los Angeles.

Key Concepts

- **Evaluation of logarithms**
- **(Optional) Basic shape of the graph of** $y = \log_a x$ for various a values.

- **Decibel Level of Sound:** The loudness level, D, measured in decibels, of a sound of intensity, I, measured in watts per square inch is $D = 10 \log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude R of an earthquake is measured by $R = \log I$, where I is the intensity of its shock wave.

Glossary

common logarithm

$\log_{10} x = \log x$ is the common logarithm, where $x > 0$.

$$y = \log x \text{ is equivalent to } x = 10^y$$

logarithm

For $a > 0, a \neq 1$ and $x > 0$, the solution to the equation $a^y = x$ is denoted $\log_a x$ is called the logarithm of x with base a. $y = \log_a x$ is equivalent to $x = a^y$

natural logarithm

$\log_e x = \ln x$ is the natural logarithm, where $x > 0$.

$$y = \ln x \text{ is equivalent to } x = e^y$$

Practice Makes Perfect

Note that even answers are provided in this section.

Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

1. $4^2 = 16$
2. $2^5 = 32$
3. $3^3 = 27$
4. $5^3 = 125$
5. $10^3 = 1000$
6. $10^{-2} = \frac{1}{100}$
7. $x^{\frac{1}{2}} = \sqrt{3}$
8. $x^{\frac{1}{3}} = \sqrt[3]{6}$
9. $32^x = \sqrt[4]{32}$
10. $17^x = \sqrt[5]{17}$
11. $\left(\frac{1}{4}\right)^2 = \frac{1}{16}$
12. $\left(\frac{1}{3}\right)^4 = \frac{1}{81}$
13. $3^{-2} = \frac{1}{9}$
14. $4^{-3} = \frac{1}{64}$
15. $e^x = 6$
16. $e^3 = x$

Answer

2. $\log_2 32 = 5$

4. $\log_5 125 = 3$

6. $\log \frac{1}{100} = -2$

8. $\log_x \sqrt[3]{6} = \frac{1}{3}$

10. $\log_{17} \sqrt[5]{17} = x$

12. $\log_{\frac{1}{3}} \frac{1}{81} = 4$

14. $\log_4 \frac{1}{64} = -3$

16. $\ln x = 3$

Convert Between Exponential and Logarithmic Form

In the following exercises, convert each logarithmic equation to exponential form.

17. $3 = \log_4 64$
18. $6 = \log_2 64$
19. $4 = \log_x 81$
20. $5 = \log_x 32$
21. $0 = \log_{12} 1$
22. $0 = \log_7 1$
23. $1 = \log_3 3$
24. $1 = \log_9 9$
25. $-4 = \log_{10} \frac{1}{10,000}$
26. $3 = \log_{10} 1,000$
27. $5 = \log_e x$
28. $x = \log_e 43$

Answer

18. $64 = 2^6$

20. $32 = x^5$

22. $1 = 7^0$

24. $9 = 9^1$

26. $1,000 = 10^3$

28. $43 = e^x$

Evaluate Logarithms

In the following exercises, find the value of x in each logarithmic equation.

29. $\log_x 49 = 2$
30. $\log_x 121 = 2$
31. $\log_x 27 = 3$
32. $\log_x 64 = 3$
33. $\log_3 x = 4$
34. $\log_5 x = 3$
35. $\log_2 x = -6$
36. $\log_3 x = -5$

37. $\log_{\frac{1}{4}} \frac{1}{16} = x$
38. $\log_{\frac{1}{3}} \frac{1}{9} = x$
39. $\log_{\frac{1}{4}} 64 = x$
40. $\log_{\frac{1}{9}} 81 = x$

Answer

30. $x = 11$

32. $x = 4$

34. $x = 125$

36. $x = \frac{1}{243}$

38. $x = 2$

40. $x = -2$

? Evaluate Logarithms

In the following exercises, find the exact value of each logarithm without using a calculator.

41. $\log_7 49$
42. $\log_6 36$
43. $\log_4 1$
44. $\log_5 1$
45. $\log_{16} 4$
46. $\log_{27} 3$
47. $\log_{\frac{1}{2}} 2$
48. $\log_{\frac{1}{2}} 4$
49. $\log_2 \frac{1}{16}$
50. $\log_3 \frac{1}{27}$
51. $\log_4 \frac{1}{16}$
52. $\log_9 \frac{1}{81}$

Answer

42. 2

44. 0

46. $\frac{1}{3}$

48. -2

50. -3

52. -2

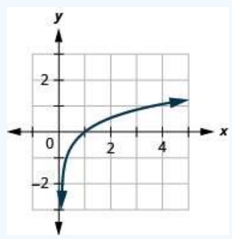
(Optional) Graph Logarithmic Equations

In the following exercises, graph each logarithmic function.

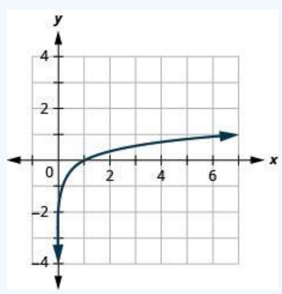
53. $y=\log_2 x$
54. $y=\log_4 x$
55. $y=\log_6 x$
56. $y=\log_7 x$
57. $y=\log_{1.5} x$
58. $y=\log_{2.5} x$
59. $y=\log_{\frac{1}{3}} x$
60. $y=\log_{\frac{1}{5}} x$
61. $y=\log_{0.4} x$
62. $y=\log_{0.6} x$

Answer

54.



56.



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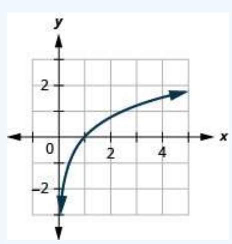
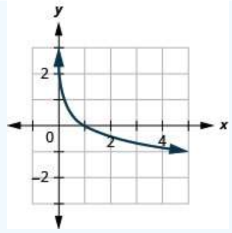
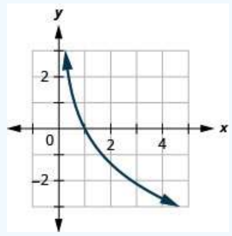


60.



62.



? Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

63. $\log_a 16 = 2$
64. $\log_a 81 = 2$
65. $\log_a 8 = 3$
66. $\log_a 27 = 3$
67. $\log_a 32 = 2$
68. $\log_a 24 = 3$
69. $\ln x = 5$
70. $\ln x = 4$
71. $\log_2(5x+1) = 4$
72. $\log_2(6x+2) = 5$
73. $\log_3(4x-3) = 2$
74. $\log_3(5x-4) = 4$
75. $\log_4(5x+6) = 3$
76. $\log_4(3x-2) = 2$
77. $\ln e^{4x} = 8$
78. $\ln e^{2x} = 6$
79. $\log x^2 = 2$
80. $\log\left(x^2-25\right) = 2$
81. $\log_2\left(x^2-4\right) = 5$
82. $\log_3\left(x^2+2\right) = 3$

Answer

64. $a = 9$

66. $a = 3$

68. $a = \sqrt[3]{24}$

70. $x = e^4$

72. $x = 5$

74. $x = 17$

76. $x=6$

78. $x=3$

80. $x=-5\sqrt{5}, x=5\sqrt{5}$

82. $x=-5, x=5$

? Use Logarithmic Models in Applications

In the following exercises, use a logarithmic model to solve.

83. What is the decibel level of normal conversation with intensity 10^{-6} watts per square inch?
84. What is the decibel level of a whisper with intensity 10^{-10} watts per square inch?
85. What is the decibel level of the noise from a motorcycle with intensity 10^{-2} watts per square inch?
86. What is the decibel level of the sound of a garbage disposal with intensity 10^{-2} watts per square inch?
87. In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2010, Haiti also experienced an intense earthquake which measured 7.0 on the Richter scale. Compare the intensities of the two earthquakes.
88. The Los Angeles area experiences many earthquakes. In 1994, the Northridge earthquake measured magnitude of 6.7 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

84. A whisper has a decibel level of 20 dB.

86. The sound of a garbage disposal has a decibel level of 100 dB.

88. The intensity of the 1994 Northridge earthquake in the Los Angeles area was about 40 times the intensity of the 2014 earthquake.

? Writing Exercises

89. Explain how to change an equation from logarithmic form to exponential form.
90. Explain the difference between common logarithms and natural logarithms.
91. Explain why $\log_a a^x = x$.
92. Explain how to find the $\log_7 32$ on your calculator.

Answer

90. Answers may vary

92. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No- I don't get it!
Convert between exponential and logarithmic form.			
Evaluate logarithms.			
Graph basic logarithmic equations.			
Solve logarithm equations.			
Use logarithmic models in applications.			

b. After reviewing this checklist, what will you do to become confident for all objectives?

9.3: Use the Properties of Logarithms

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Be Prepared

Before you get started, take this readiness quiz.

1. Evaluate

a. a^0

b. a^1

2. Write $\sqrt[3]{x^2y}$ with a rational exponent.

3. Round $2.5646415$to three decimal places.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic expressions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $a^0=1$, we can convert this to logarithmic form and get $\log_a 1=0$. Also, since $a^1=a$, we get $\log_a a=1$.

Properties of Logarithms

$$\log_a 1=0$$
$$\log_a a=1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example $9.3.1$

Evaluate using the properties of logarithms:

a. $\log_8 1$

b. $\log_6 6$

Solution

a.

	$\log_8 1$
Use the property, $\log_a 1=0$.	$=0$

b.

	$\log_6 6$
Use the property, $\log_a a=1$.	$=1$

Try It 9.3.2

Evaluate using the properties of logarithms:

a. $\log_{13} 1$

b. $\log_9 9$

Answer

a. 0

b. 1

Try It 9.3.3

Evaluate using the properties of logarithms:

a. $\log_5 1$

b. $\log_7 7$

Answer

a. 0

b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $a^{\log_a x} = x$ converts to the logarithmic equation $\log_a x = \log_a x$, which is a true statement for positive values for x only.

The logarithmic equation $\log_a a^x = x$ converts to the exponential equation $a^x = a^x$, which is also a true statement.

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power.

Inverse Properties of Logarithms

For $a > 0$, $x > 0$ and $a \neq 1$,

$$a^{\log_a x} = x \quad \text{and} \quad \log_a a^x = x.$$

In the next example, we apply the inverse properties of logarithms.

Example 9.3.4

Evaluate using the properties of logarithms:

a. $4^{\log_4 9}$

b. $\log_3 3^5$

Solution

a.

	$4^{\log_4 9}$
Use the property, $a^{\log_a x} = x$.	$= 9$

b.

	$\log_3 3^5$
Use the property, $\log_a a^x = x$.	$= 5$

Try It 9.3.5

Evaluate using the properties of logarithms:

a. $5^{\log_5 15}$

b. $\log_7 7^4$

Answer

a. 15

b. 4

Try It 9.3.6

Evaluate using the properties of logarithms:

a. $2^{\log_2 8}$

b. $\log_2 2^{15}$

Answer

a. 8

b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential expressions and logarithmic expressions are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $a^m a^n = a^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_a(MN) = \log_a M + \log_a N$, tells us that to take the log of a product, we add the log of the factors.

Product Property of Logarithms

If $M > 0, N > 0, a > 0$ and $a \neq 1$, then

$$\log_a(MN) = \log_a M + \log_a N.$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Example 9.3.7

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3(7x)$

b. $\log_4(64xy)$

Solution

a.

	$\log_3(7x)$

	$\log_3(7x)$
Use the Product Property, $\log_a(MN) = \log_a M + \log_a N$.	$= \log_3 7 + \log_3 x$

b.

	$\log_4(64xy)$
Use the Product Property, $\log_a(MN) = \log_a M + \log_a N$.	$= \log_4 64 + \log_4 x + \log_4 y$
Simplify by evaluating $\log_4 64$.	$= 3 + \log_4 x + \log_4 y$

Try It 9.3.8

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3(3x)$

b. $\log_2(8xy)$

Answer

a. $1 + \log_3 x$

b. $3 + \log_2 x + \log_2 y$

Try It 9.3.9

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_9(9x)$

b. $\log_3(27xy)$

Answer

a. $1 + \log_9 x$

b. $3 + \log_3 x + \log_3 y$

Similarly, in the Quotient Property of Exponents, $\frac{a^m}{a^n} = a^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_a \frac{M}{N} = \log_a M - \log_a N$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Quotient Property of Logarithms

If $M > 0$, $N > 0$, $a > 0$ and $a \neq 1$, then

$$\log_a \frac{M}{N} = \log_a M - \log_a N.$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_a M = \log_a N \neq \log_a(M - N)$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

Example 9.3.10

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5 \frac{5}{7}$
b. $\log \frac{x}{100}$

Solution

a.

	$\log_5 \frac{5}{7}$
Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.	$= \log_5 5 - \log_5 7$
Simplify.	$= 1 - \log_5 7$

b.

	$\log \frac{x}{100}$
Use the Quotient Property, $\log_a \frac{M}{N} = \log_a M - \log_a N$.	$= \log x - \log 100$
Simplify.	$= \log x - 2$

Try It 9.3.11

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4 \frac{3}{4}$

b. $\log \frac{x}{1000}$

Answer

a. $\log_4 3 - 1$

b. $\log x - 3$

Try It 9.3.12

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_2 \frac{5}{4}$

b. $\log \frac{10}{y}$

Answer

a. $\log_2 5 - 2$

b. $1 - \log y$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_a M^p = p \log_a M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Power Property of Logarithms

If $M>0, \mathrm{a}>0, \mathrm{a} \neq 1$ and p is any real number then,

$$\log_a M^p = p\log_a M.$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

Example 9.3.13

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5 4^3$

b. $\log x^{10}$

Solution

a.

	$\log_5 4^3$
Use the Power Property, $\log_a M^p = p\log_a M$.	$=3\color{red}{\backslash(}\log_5 4\backslash)$

b.

	$\log x^{10}$
Use the Power Property, $\log_a M^p = p\log_a M$.	$=10\log x$

Try It 9.3.14

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7 5^4$

b. $\log x^{100}$

Answer

a. $4\log_7 5$

b. $100\cdot\log x$

Try It 9.3.15

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_2 3^7$

b. $\log x^{20}$

Answer

a. $7\log_2 3$

b. $20\cdot\log x$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M>0, \mathrm{a}>0, \mathrm{a} \neq 1$ and p is any real number then,

Property	**Base a**	**Base e**
	$\log_a 1=0$	$\ln 1=0$
	$\log_a a=1$	$\ln e=1$
Inverse Properties	$a^{\log_a x}=x$ $\log_a a^x=x$	$e^{\ln x}=x$ $\ln e^x=x$
Product Property of Logarithms	$\log_a(M \cdot N)=\log_a M+\log_a N$	$\ln(M \cdot N)=\ln M+\ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N}=\log_a M-\log_a N$	$\ln \frac{M}{N}=\ln M-\ln N$
Power Property of Logarithms	$\log_a M^p=p \log_a M$	$\ln M^p=p \ln M$

Now that we have the properties we can use them to "expand" a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 9.3.16

Use the Properties of Logarithms to expand the logarithm $\log_4\left(2x^3y^2\right)$. Simplify, if possible.

Solution

	$\log_4\left(2x^3y^2\right)$
Use the Product Property, $\log_a M \cdot N=\log_a M+\log_a N$.	
Use the Power Property, $\log_a M^p=p \log_a M$, on the last two terms.	
Simplify.	

Try It 9.3.17

Use the Properties of Logarithms to expand the logarithm $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

Answer

$\log_2 5+4 \log_2 x+2 \log_2 y$

Try It 9.3.18

Use the Properties of Logarithms to expand the logarithm $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

Answer

$\log_3 7+5 \log_3 x+3 \log_3 y$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

Example 9.3.19

Use the Properties of Logarithms to expand the logarithm $\log_2 \sqrt[4]{\frac{x^3}{3y^2 z}}$. Simplify, if possible.

Solution

	$\log_2 \sqrt[4]{\frac{x^3}{3y^2 z}}$
Rewrite the radical with a rational exponent.	$= \log_2 \left(\frac{x^3}{3y^2 z}\right)^{\frac{1}{4}}$
Use the Power Property, $\log_a M^p = p \log_a M$.	$= \frac{1}{4} \log_2 \left(\frac{x^3}{3y^2 z}\right)$
Use the Quotient Property, $\log_a M \cdot N = \log_a M - \log_a N$.	$= \frac{1}{4}\left(\log_2 (x^3) - \log_2 (3y^2 z)\right)$
Use the Product Property, $\log_a M \cdot N = \log_a M + \log_a N$, in the second term.	$= \frac{1}{4}\left(\log_2 (x^3) - \left(\log_2 3 + \log_2 y^2 + \log_2 z\right)\right)$
Use the Power Property, $\log_a M^p = p \log_a M$, inside the parentheses.	$= \frac{1}{4}(3 \log_2 x - (\log_2 3 + 2 \log_2 y + \log_2 z))$
Simplify by distributing.	

Try It 9.3.20

Use the Properties of Logarithms to expand the logarithm $\log_4 \sqrt[5]{\frac{x^4}{2y^3 z^2}}$. Simplify, if possible.

Answer

$$\frac{1}{5}\left(4 \log_4 x - \frac{1}{2} - 3 \log_4 y - 2 \log_4 z\right)$$

Try It 9.3.21

Use the Properties of Logarithms to expand the logarithm $\log_3 \sqrt[3]{\frac{x^2}{5yz}}$. Simplify, if possible.

Answer

$$\frac{1}{3}(2 \log_3 x - \log_3 5 - \log_3 y - \log_3 z)$$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

Example 9.3.22

Use the Properties of Logarithms to condense the logarithm $\log_4 3 + \log_4 x - \log_4 y$. Simplify, if possible.

Solution

	$\log_4 3 + \log_4 x - \log_4 y$
The log expressions all have the same base, 4.	
The first two terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M : N$.	
Since the logs are subtracted, we use the Quotient Property, $\log_a M - \log_a N = \log_a \frac{M}{N}$.	

Try It 9.3.23

Use the Properties of Logarithms to condense the logarithm $\log_2 5 + \log_2 x - \log_2 y$. Simplify, if possible.

Answer

$$\log_2 \frac{5x}{y}$$

Try It 9.3.24

Use the Properties of Logarithms to condense the logarithm $\log_3 6 - \log_3 x - \log_3 y$. Simplify, if possible.

Answer

$$\log_3 \frac{6}{xy}$$

Example 9.3.25

Use the Properties of Logarithms to condense the logarithm $2\log_3 x + 4\log_3 (x+1)$. Simplify, if possible.

Solution

	$2\log_3 x + 4\log_3 (x+1)$
The log expressions have the same base, 3.	$= 2\log_3 x + 4\log_3 (x+1)$
Use the Power Property, $\log_a M + \log_a N = \log_a M \cdot N$.	$= \log_3 x^2 + \log_3 (x+1)^4$
The terms are added, so we use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.	$= \log_3 x^2 (x+1)^4$

Try It 9.3.26

Use the Properties of Logarithms to condense the logarithm $3\log_2 x + 2\log_2 (x-1)$. Simplify, if possible.

Answer

$$\log_2 x^3 (x-1)^2$$

Try It 9.3.27

Use the Properties of Logarithms to condense the logarithm $2 \log x+2 \log (x+1)$. Simplify, if possible.

Answer

$\log x^{2}(x+1)^{2}$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the **Change-of-Base Formula**. We will show how this is derived.

$$\begin{array}{ll}
\text{Suppose we want to evaluate } \log_a M & \log_a M \\
\text{Let } y=\log_a M. & y=\log_a M \\
\text{Rewrite the epression in exponential form.} & a^{y}=M \\
\text{Take the } \log_b \text{ of each side.} & \log_b a^{y}=\log_b M \\
\text{Use the Power Property.} & y \log_b a=\log_b M \\
\text{Solve for } y. & y=\dfrac{\log_b M}{\log_b a} \\
\text{Substiture } y=\log_a M. & \log_a M=\dfrac{\log_b M}{\log_b a}
\end{array}$$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b>0, b \neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Change-of-Base Formula

For any logarithmic bases a, b and $M>0$,

$$\begin{array}{lll}
\log_a M=\dfrac{\log_b M}{\log_b a} & \log_a M=\dfrac{\log M}{\log a} & \log_a M=\dfrac{\ln M}{\ln a} \\
\text{new base } b & \text{new base } 10 & \text{new base } e
\end{array}$$

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

Example 9.3.28

Rounding to three decimal places, approximate $\log_4 35$.

Solution

	$\log_4 35$
Use the Change-of-Base Formula, \log _{a} M=\dfrac{\log _{b} M}{\log _{b} a}.	
Identify a and M. Choose 10 for b.	$=\dfrac{\log 35}{\log 4}$
Enter the expression $\dfrac{\log 35}{\log 4}$ in the calculator using the log button for base 10. Round to three decimal places.	≈ 2.565

Try It 9.3.29

Rounding to three decimal places, approximate $\log_3 42$.

Answer

3.402

Try It 9.3.30

Rounding to three decimal places, approximate $\log_5 46$.

Answer

2.379

Key Concepts

- $\log_a 1 = 0 \quad \log_a a = 1$
- **Inverse Properties of Logarithms**
  - For $a > 0, x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \quad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M > 0, N > 0, a > 0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

  The logarithm of a product is the sum of the logarithms.

- **Quotient Property of Logarithms**
  - If $M > 0, N > 0, \mathrm{a} > 0$ and $a \neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

  The logarithm of a quotient is the difference of the logarithms.

- **Power Property of Logarithms**
  - If $M > 0, a > 0, a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

  The log of a number raised to a power is the product of the power times the log of the number.

- **Properties of Logarithms Summary**
  If $M > 0, a > 0, a \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
  For any logarithmic bases a and b, and $M > 0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$
$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

Practice Makes Perfect

Note that answers to even-numbered questions are provided.

Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

1. a. $\log_4 1$
   b. $\log_8 8$
2. a. $\log_{12} 1$
   b. $\ln e$
3. a. $3^{\log_3 6}$
   b. $\log_2 2^7$
4. a. $5^{\log_5 10}$
   b. $\log_4 4^{10}$
5. a. $8^{\log_8 7}$
   b. $\log_6 6^{-2}$
6. a. $6^{\log_6 15}$
   b. $\log_8 8^{-4}$
7. a. $10^{\log \sqrt{5}}$
   b. $\log 10^{-2}$
8. a. $10^{\log \sqrt{3}}$
   b. $\log 10^{-1}$
9. a. $e^{\ln 4}$
   b. $\ln e^2$
10. a. $e^{\ln 3}$
    b. $\ln e^7$

Answer

2.

a. 0
b. 1

4.

a. 10
b. 10

6.

a. 15
b. -4

8.

a. $\sqrt{3}$
b. -1

10.

a. 3
b. 7

? Use the Properties of Logarithms

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

11. $\log_4 6x$
12. $\log_5 8y$
13. $\log_2 32xy$
14. $\log_3 81xy$
15. $\log 100x$
16. $\log 1000y$

Answer

12. $\log_5 8+\log_5 y$

14. $4+\log_3 x+\log_3 y$

16. $3+\log y$

? Use the Properties of Logarithms

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

17. $\log_3 \frac{3}{8}$
18. $\log_6 \frac{5}{6}$
19. $\log_4 \frac{16}{y}$
20. $\log_5 \frac{125}{x}$
21. $\log \frac{x}{10}$
22. $\log \frac{10,000}{y}$
23. $\ln \frac{e^3}{3}$
24. $\ln \frac{e^4}{16}$

Answer

18. $\log_6 5-1$

20. $3-\log_5 x$

22. $4-\log y$

24. $4-\ln 16$

? Use the Properties of Logarithms

In the following exercises, use the Power Property of Logarithms to expand each. Simplify if possible.

25. $\log_3 x^2$
26. $\log_2 x^5$

27. $\log x^{-2}$
28. $\log x^{-3}$
29. $\log_4 \sqrt{x}$
30. $\log_5 \sqrt[3]{x}$
31. $\ln x^{\sqrt{3}}$
32. $\ln x^{\sqrt[3]{4}}$

Answer

26. $5 \log_2 x$

28. $-3 \log x$

30. $\frac{1}{3} \log_5 x$

32. $\sqrt[3]{4} \ln x$

Use the Properties of Logarithms

In the following exercises, use the Properties of Logarithms to expand the logarithm. Simplify if possible.

33. $\log_5 \left(4x^6y^4\right)$
34. $\log_2 \left(3x^5y^3\right)$
35. $\log_3 \left(\sqrt{2}x^2\right)$
36. $\log_5 \left(\sqrt[4]{21}y^3\right)$
37. $\log_3 \frac{xy^2}{z^2}$
38. $\log_5 \frac{4ab^3c^4}{d^2}$
39. $\log_4 \frac{\sqrt{x}}{16y^4}$
40. $\log_3 \frac{\sqrt[3]{x^2}}{27y^4}$
41. $\log_2 \frac{\sqrt{2x+y^2}}{z^2}$
42. $\log_3 \frac{\sqrt{3x+2y^2}}{5z^2}$
43. $\log_2 \sqrt[4]{\frac{5x^3}{2y^2z^4}}$
44. $\log_5 \sqrt[3]{\frac{3x^2}{4y^3z}}$

Answer

34. $\log_2 3 + 5 \log_2 x + 3 \log_2 y$

36. $\frac{1}{4} \log_5 21 + 3 \log_5 y$

38. $\log_5 4 + \log_5 a + 3 \log_5 b + 4 \log_5 c - 2 \log_5 d$

40. $\frac{2}{3} \log_3 x - 3 - 4 \log_3 y$

42. $\frac{1}{2} \log_3 \left(3x + 2y^2\right) - \log_3 5 - 2 \log_3 z$

44. $\frac{1}{3} (\log_5 3 + 2 \log_5 x - \log_5 4 - 3 \log_5 y - \log_5 z)$

Use the Properties of Logarithms

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

45. $\log_6 4 + \log_6 9$
46. $\log 4 + \log 25$
47. $\log_2 80 - \log_2 5$
48. $\log_3 36 - \log_3 4$
49. $\log_3 4 + \log_3(x+1)$
50. $\log_2 5 - \log_2(x-1)$
51. $\log_7 3 + \log_7 x - \log_7 y$
52. $\log_5 2 - \log_5 x - \log_5 y$
53. $4\log_2 x + 6\log_2 y$
54. $6\log_3 x + 9\log_3 y$
55. $\log_3\left(x^2-1\right) - 2\log_3(x-1)$
56. $\log\left(x^2+2x+1\right) - 2\log(x+1)$
57. $4\log x - 2\log y - 3\log z$
58. $3\ln x + 4\ln y - 2\ln z$
59. $\frac{1}{3}\log x - 3\log(x+1)$
60. $2\log(2x+3) + \frac{1}{2}\log(x+1)$

Answer

46. 2

48. 2

50. $\log_2 \frac{5}{x-1}$

52. $\log_5 \frac{2}{xy}$

54. $\log_3 x^6 y^9$

56. 0

58. $\ln \frac{x^3 y^4}{z^2}$

60. $\log(2x+3)^2 \cdot \sqrt{x+1}$

Use the Change-of-Base Formula

In the following exercises, use the Change-of-Base Formula, rounding to three decimal places, to approximate each logarithm.

61. $\log_3 42$
62. $\log_5 46$
63. $\log_{12} 87$
64. $\log_{15} 93$
65. $\log_{\sqrt{2}} 17$
66. $\log_{\sqrt{3}} 21$

Answer

62. 2.379

64. 1.674

66. 5.542

? Writing Exercises

67. Write the Product Property in your own words. Does it apply to each of the following? $\log_a 5x, \log_a (5+x)$. Why or why not?
68. Write the Power Property in your own words. Does it apply to each of the following? $\log_a x^p$, $(\log_a x)^r$. Why or why not?
69. Use an example to show that $\log(a+b) \neq \log a + \log b$?
70. Explain how to find the value of $\log_7 15$ using your calculator.

Answer

68. Answers may vary

70. Answers may vary

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the properties of logarithms.			
use the Change of Base Formula.			

Figure 10.4.5

b. On a scale of $1-10$, how would you rate your mastery of this section in light of your responses on the checklist? How can you improve this?

9.4: Solve Exponential and Logarithmic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve logarithmic equations using the properties of logarithms
- Solve exponential equations using logarithms
- Use exponential models in applications

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $x^2 = 16$.
2. Solve $x^2 - 5x + 6 = 0$.
3. Solve $x(x+6) = 2x + 5$.

Solve Logarithmic Equations Using the Properties of Logarithms

In the section on logarithms, we solved some equations by rewriting the equation in exponential form. Now that we have the properties of logarithms, we have additional methods we can use to solve logarithmic equations.

If our equation has two logarithms we can use a property that says that if $\log_a M = \log_a N$ then it is true that $M = N$. This is the **One-to-One Property of Logarithmic Equations**.

One-to-One Property of Logarithmic Equations

For $M > 0, N > 0, a > 0$, and $a \neq 1$ is any real number,

$$\text{if } \log_a M = \log_a N, \text{ then } M = N.$$

To use this property, we must be certain that both sides of the equation are written with the same base.

Remember that logarithms are defined only for positive real numbers. Check your results in the original equation. You may have obtained a result that gives a logarithm of zero or a negative number.

Example 9.4.1

Solve $2\log_5 x = \log_5 81$.

Solution

	$2\log_5 x = \log_5 81$
Use the Power Property.	$\log_5 x^2 = \log_5 81$
Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.	$x^2 = 81$
Solve using the Square Root Property.	$x = \pm 9$
We eliminate $x = -9$ as we cannot take the logarithm of a negative number.	$x = 9 \quad \text{or} \quad \cancel{x = -9}$
Check. $x = 9$	$2\log_5 x = \log_5 81$ $2\log_5 9 \stackrel{?}{=} \log_5 81$ $\log_5 9^2 \stackrel{?}{=} \log_5 81$ $\log_5 81 = \log_5 81$

Try It 9.4.2

Solve $2\log_3 x = \log_3 36$.

Answer

$x = 6$

Try It 9.4.3

Solve $3\log x = \log 64$

Answer

$x = 4$

Another strategy to use to solve logarithmic equations is to condense sums or differences into a single logarithm.

Example 9.4.4

Solve $\log_3 x + \log_3(x-8) = 2$.

Solution

	$\log_3 x + \log_3(x-8) = 2$
Use the Product Property, $\log_a M + \log_a N = \log_a M \cdot N$.	$\log_3 x(x-8) = 2$
Rewrite in exponential form.	$3^2 = x(x-8)$
Simplify.	
Subtract 9 from each side.	
Factor.	$0 = (x-9)(x+1)$
Use the Zero-Product-Property	$x-9=0 \quad \text{or} \quad x+1=0$
Solve each equation.	$x = 9, \quad \cancel{x=-1}$
Check.	$x=-1$ $\log_3 x + \log_3(x-8) = 2$ $\log_3(-1) + \log_3(-1-8) \stackrel{?}{=} 2$ We cannot take the log of a negative number. $x = 9$ $\log_3 x + \log_3(x-8) = 2$ $\log_3 9 + \log_3(9-8) \stackrel{?}{=} 2$ $2+0 \stackrel{?}{=} 2$ $2 = 2$

Try It 9.4.5

Solve $\log_2 x + \log_2(x-2) = 3$.

Answer

$x = 4$

Try It 9.4.6

Solve $\log_2 x + \log_2(x-6) = 4$,

Answer

$x = 8$

When there are logarithms on both sides, we condense each side into a single logarithm. Remember to use the Power Property as needed.

Example 9.4.7

Solve $\log_4(x+6) - \log_4(2x+5) = -\log_4 x$.

Solution

	$\log_4(x+6) - \log_4(2x+5) = -\log_4 x$
Use the Quotient Property on the left side and the PowerProperty on the right.	$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 x^{-1}$
Rewrite $x^{-1} = \frac{1}{x}$.	$\log_4\left(\frac{x+6}{2x+5}\right) = \log_4 \frac{1}{x}$
Use the One-to-One Property, if $\log_a M = \log_a N$, then $M = N$.	$\frac{x+6}{2x+5} = \frac{1}{x}$
Solve the rational equation.	$x(x+6) = 2x+5$
Distribute.	$x^2 + 6x = 2x + 5$
Write in standard form.	$x^2 + 4x - 5 = 0$
Factor.	$(x+5)(x-1) = 0$
Use the Zero-Product-Property.	$x+5=0 \quad \text{or} \quad x-1=0$
Solve each equation.	$\cancel{x=-5} \quad \text{or} \quad x=1$
Check.	We leave the check for you.

Try It 9.4.8

Solve $\log(x+2) - \log(4x+3) = -\log x$.

Answer

$x = 3$

Try It 9.4.9

Solve $\log(x-2)-\log(4x+16)=\log\left(\frac{1}{x}\right)$.

Answer

$x=8$

Example 9.4.10

Solve $5^x=11$. Find the exact answer and then approximate it to three decimal places.

Solution

	$5^x=11$
Since the exponential is isolated, take the logarithm of both sides.	$\log 5^x=\log 11$
Use the Power Property to get the x as a factor, not an exponent.	$x\log 5=\log 11$
Solve for x. Find the exact answer.	$x=\frac{\log 11}{\log 5}$
Approximate the answer.	$x\approx 1.490$ Since $5^1=5$ and $5^2=25$, does it makes sense that $5^{1.490}\approx 11$?

Try It 9.4.11

Solve $7^x=43$. Find the exact answer and then approximate it to three decimal places.

Answer

$x=\frac{\log 43}{\log 7}\approx 1.933$

Try It 9.4.12

Solve $8^x=98$. Find the exact answer and then approximate it to three decimal places.

Answer

$x=\frac{\log 98}{\log 8}\approx 2.205$

When we take the logarithm of both sides we will get the same result whether we use the common or the natural logarithm (try using the natural log in the last example. Did you get the same result?) When the exponential has base e, we use the natural logarithm.

Example 9.4.13

Solve $3e^{x+2}=24$. Find the exact answer and then approximate it to three decimal places.

Solution

	$3e^{x+2}=24$

	$3e^{x+2}=24$
Isolate the exponential by dividing both sides by 3.	$e^{x+2}=8$
Take the natural logarithm of both sides.	$\ln e^{x+2}=\ln 8$
Use the Power Property to get the x as a factor, not an exponent.	$(x+2)\ln e=\ln 8$
Use the property $\ln e=1$ to simplify.	$x+2=\ln 8$
Solve the equation. Find the exact answer.	$x=\ln 8-2$
Approximate the answer.	$x\approx 0.079$

Try It 9.4.14

Solve $2e^{x-2}=18$. Find the exact answer and then approximate it to three decimal places.

Answer

$x=\ln 9+2\approx 4.197$

Try It 9.4.15

Solve $5e^{2x}=25$. Find the exact answer and then approximate it to three decimal places.

Answer

$x=\frac{\ln 5}{2}\approx 0.805$

Use Exponential Models in Applications

In previous sections we were able to solve some applications that were modeled with exponential equations. Now that we have so many more options to solve these equations, we are able to solve more applications.

We will again use the Compound Interest Formulas and so we list them here for reference.

Compound Interest

For a principal, P, invested at an interest rate, r, for t years, the new balance, A is:

$$\begin{array}{ll} A=P\left(1+\frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ A=Pe^{rt} & \text{when compounded continuously.} \end{array}$$

Example 9.4.16

Jermael’s parents put $\$10,000$ in investments for his college expenses on his first birthday. They hope the investments will be worth $\$50,000$ when he turns 18. If the interest compounds continuously, approximately what rate of growth will they need to achieve their goal?

Solution

Identify the variables in the formula.	$A=\$50,000$ $P=\$10,000$ $r=?$ $t=17$ years $A=Pe^{rt}$

Identify the variables in the formula.	$A = \$50,000$ $P = \$10,000$ $r = ?$ $t = 17$ years $A = Pe^{rt}$
Substitute the values into the formula.	$50,000 = 10,000e^{r \cdot 17}$
Solve for r. Divide each side by 10, 000.	$5 = e^{17r}$
Take the natural log of each side.	$\ln 5 = \ln e^{17r}$
Use the Power Property.	$\ln 5 = 17r \ln e$
Simplify.	$\ln 5 = 17r$
Divide each side by 17.	$\frac{\ln 5}{17} = r$
Approximate the answer.	$r \approx 0.095$
Convert to a percentage.	$r \approx 9.5\%$
	They need the rate of growth to be approximately 9.5%.

Try It 9.4.17

Hector invests $\$10,000$ at age 21. He hopes the investments will be worth $\$150,000$ when he turns 50. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal?

Answer

$r \approx 9.3\%$

Try It 9.4.18

Rachel invests $\$15,000$ at age 25. She hopes the investments will be worth $\$90,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?

Answer

$r \approx 11.9\%$

We have seen that growth and decay are modeled by exponential functions. For growth and decay we use the formula $A = A_0 e^{kt}$. Exponential growth has a positive rate of growth or growth constant, k, and **exponential decay** has a negative rate of growth or decay constant, k.

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, k, for a certain time, t, the final amount, A, is:

$$A = A_0 e^{kt}$$

We can now solve applications that give us enough information to determine the rate of growth. We can then use that rate of growth to predict other situations.

Example 9.4.19

Researchers recorded that a certain bacteria population grew from 100 to 300 in 3 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Solution

This problem requires two main steps. First we must find the unknown rate, k. Then we use that value of k to help us find the unknown number of bacteria.

Identify the variables in the formula.	$\begin{aligned} A &= 300 \\ A_0 &= 100 \\ k &= ? \\ t &= 3 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$
Substitute the values in the formula.	$300 = 100e^{k\cdot 3}$
Solve for k. Divide each side by 100.	$3 = e^{3k}$
Take the natural log of each side.	$\ln 3 = \ln e^{3k}$
Use the Power Property.	$\ln 3 = 3k \ln e$
Simplify.	$\ln 3 = 3k$
Divide each side by 3.	$\frac{\ln 3}{3} = k$
Approximate the answer.	$k \approx 0.366$
We use this rate of growth to predict the number of bacteria there will be in 24 hours.	$\begin{aligned} A &= ? \\ A_0 &= 100 \\ k &= \frac{\ln 3}{3} \\ t &= 24 \text{ hours} \\ A &= A_0 e^{kt} \end{aligned}$
Substitute in the values.	$A = 100e^{\frac{\ln 3}{3}\cdot 24}$
Evaluate.	$A \approx 656,100$
	At this rate of growth, they can expect $656,100$ bacteria.

Try It 9.4.20

Researchers recorded that a certain bacteria population grew from 100 to 500 in 6 hours. At this rate of growth, how many bacteria will there be 24 hours from the start of the experiment?

Answer

There will be $62,500$bacteria.

Try It 9.4.21

Researchers recorded that a certain bacteria population declined from $700,000$ to $400,000$ in 5 hours after the administration of medication. At this rate of decay, how many bacteria will there be 24 hours from the start of the experiment?

Answer

There will be $5,870,061$bacteria.

Radioactive substances decay or decompose according to the exponential decay formula. The amount of time it takes for the substance to decay to half of its original amount is called the **half-life** of the substance.

Similar to the previous example, we can use the given information to determine the constant of decay, and then use that constant to answer other questions.

Example 9.4.22

The half-life of radium-226 is $1,590$ years. How much of a 100 mg sample will be left in 500 years?

Solution

This problem requires two main steps. First we must find the decay constant k. If we start with 100-mg, at the half-life there will be 50-mg remaining. We will use this information to find k. Then we use that value of k to help us find the amount of sample that will be left in 500 years.

Identify the variables in the formula.	$A = 50$ $A_0 = 100$ $k = ?$ $t = 1590$ years $A = A_0e^{kt}$
Substitute the values in the formula.	$50 = 100e^{k \cdot 1590}$
Solve for k. Divide each side by 100.	$0.5 = e^{1590k}$
Take the natural log of each side.	$\ln 0.5 = \ln e^{1590k}$
Use the Power Property.	$\ln 0.5 = 1590k \ln e$
Simplify.	$\ln 0.5 = 1590k$
Divide each side by 1590.	$\frac{\ln 0.5}{1590} = k$ exact answer
We use this rate of growth to predict the amount that will be left in 500 years.	$A = ?$ $A_0 = 100$ $k = \frac{\ln 0.5}{1590}$ $t = 500$ years $A = A_0e^{kt}$
Substitute in the values.	$A = 100e^{\frac{1n0.5}{1500} \cdot 500}$
Evaluate.	$A \approx 80.4\text{mg}$
	In 500 years there would be approximately 80.4 mg remaining.

Try It 9.4.23

The half-life of magnesium-27 is 9.45 minutes. How much of a 10-mg sample will be left in 6 minutes?

Answer

There will be 6.43mg left.

Try It 9.4.24

The half-life of radioactive iodine is 60 days. How much of a 50-mg sample will be left in 40 days?

Answer

There will be 31.5mg left.

Key Concepts

- **One-to-One Property of Logarithmic Equations:** For $M>0, N>0, a>0$, and $a \neq 1$ is any real number:

$$\text{if } \log_a M=\log_a N, \text{ then } M=N$$

- **Compound Interest:**
For a principal, P, invested at an interest rate, r, for t years, the new balance, A, is:

$$\begin{aligned} A &= P\left(1+\frac{r}{n}\right)^{nt} && \text{when compounded } n \text{ times a year.} \\ A &= Pe^{rt} && \text{when compounded continuously.} \end{aligned}$$

- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount, A, is $A=A_0e^{rt}$.

Practice Makes Perfect

Note that even-numbered answers are given in this section.

(Optional) Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

1. $\log_4 64=2\log_4 x$
2. $\log 49=2\log x$
3. $3\log_3 x=\log_3 27$
4. $3\log_6 x=\log_6 64$
5. $\log_5(4x-2)=\log_5 10$
6. $\log_3\left(x^2+3\right)=\log_3 4x$

Answer

2. $x=7$

4. $x=4$

6. $x=1, x=3$

Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

7. $3^x=89$
8. $2^x=74$
9. $5^x=110$
10. $4^x=112$
11. $e^x=16$
12. $e^x=8$
13. $\left(\frac{1}{2}\right)^x=6$

14. $\left(\frac{1}{3}\right)^x = 8$
15. $4e^{x+1} = 16$
16. $3e^{x+2} = 9$
17. $6e^{2x} = 24$
18. $2e^{3x} = 32$
19. $\frac{1}{4}e^x = 3$
20. $\frac{1}{3}e^x = 2$
21. $e^{x+1} + 2 = 16$
22. $e^{x-1} + 4 = 12$

Answer

8. $x = \frac{\log 74}{\log 2} \approx 6.209$

10. $x = \frac{\log 112}{\log 4} \approx 3.404$

12. $x = \ln 8 \approx 2.079$

14. $x = \frac{\log 8}{\log \frac{1}{3}} \approx -1.893$

16. $x = \ln 3 - 2 \approx -0.901$

18. $x = \frac{\ln 16}{3} \approx 0.924$

20. $x = \ln 6 \approx 1.792$

22. $x = \ln 8 + 1 \approx 3.079$

? Solve Exponential Equations Using Logarithms

In the following exercises, solve each equation.

23. $3^{3x+1} = 81$
24. $6^{4x-17} = 216$
25. $\frac{e^{x^2}}{e^{14}} = e^{5x}$
26. $\frac{e^{x^2}}{e^x} = e^{20}$
27. $\log_a 64 = 2$
28. $\log_a 81 = 4$
29. $\ln x = -8$
30. $\ln x = 9$
31. $\log_5(3x-8) = 2$
32. $\log_4(7x+15) = 3$
33. $\ln e^{5x} = 30$
34. $\ln e^{6x} = 18$
35. $3\log x = \log 125$
36. $7\log_3 x = \log_3 128$

Answer

24. $x = 5$

26. $x=-4, x=5$

28. $a=3$

30. $x=e^{9}$

32. $x=7$

34. $x=3$

36. $x=2$

Solve Exponential Equations Using Logarithms

In the following exercises, solve for x, giving an exact answer as well as an approximation to three decimal places.

37. $6^{x}=91$
38. $\left(\frac{1}{2}\right)^{x}=10$
39. $7e^{x-3}=35$
40. $8e^{x+5}=56$

Answer

38. $x=-\frac{\ln 10}{\ln 2} \approx 3.322$

40. $x=\ln 7-5 \approx -3.054$

Use Exponential Models in Applications

In the following exercises, solve.

41. Sung Lee invests $\$5,000$ at age 18. He hopes the investments will be worth $\$10,000$ when he turns 25. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?
42. Alice invests $\$15,000$ at age 30 from the signing bonus of her new job. She hopes the investments will be worth $\$30,000$ when she turns 40. If the interest compounds continuously, approximately what rate of growth will she need to achieve her goal?
43. Coralee invests $\$5,000$ in an account that compounds interest monthly and earns 7%. How long will it take for her money to double?
44. Simone invests $\$8,000$ in an account that compounds interest quarterly and earns 5%. How long will it take for his money to double?
45. Researchers recorded that a certain bacteria population declined from $100,000$ to 100 in 24 hours. At this rate of decay, how many bacteria will there be in 16 hours?
46. Researchers recorded that a certain bacteria population declined from $800,000$ to $500,000$ in 6 hours after the administration of medication. At this rate of decay, how many bacteria will there be in 24 hours?
47. A virus takes 6 days to double its original population $\left(A=2A_{0}\right)$. How long will it take to triple its population?
48. A bacteria doubles its original population in 24 hours $\left(A=2A_{0}\right)$. How big will its population be in 72 hours?
49. Carbon-14 is used for archeological carbon dating. Its half-life is $5,730$ years. How much of a 100-gram sample of Carbon-14 will be left in 1000 years?
50. Radioactive technetium-99m is often used in diagnostic medicine as it has a relatively short half-life but lasts long enough to get the needed testing done on the patient. If its half-life is 6 hours, how much of the radioactive material form a 0.5 ml injection will be in the body in 24 hours?

Answer

42. 6.9%

44. 13.9 years

46. $122,070$bacteria

48. 8 times as large as the original population

50. 0.03mL

Writing Exercises

51. Explain the method you would use to solve these equations: $3^{x+1}=81$, $3^{x+1}=75$. Does your method require logarithms for both equations? Why or why not?
52. What is the difference between the equation for exponential growth versus the equation for exponential decay?

Answer

52. Answers will vary.

Additional Exercises

53. Solve for x:

a) $\log_4 x+\log_4 x=3$

b) $\log_3 x+\log_3 (x+6)=3$

c) $\log x+\log (x-15)=2$

Answer

a) 8, b) 3, c) 20

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
solve logarithmic equations using the properties of logarithms.			
solve exponential equations using logarithms.			
use exponential models in applications.			

Figure 10.5.1

b. After looking at the checklist, do you think you are well-prepared for the next section? Why or why not?

9.5: Chapter 9 Review Exercises

Chapter Review Exercises

Evaluate and Graph Exponential Equations

Exercise $9.5.9$ Graph Exponential Equations

In the following exercises, graph each of the following equations.

1. $f(x)=4^x$
2. $f(x)=\left(\frac{1}{5}\right)^x$
3. $g(x)=(0.75)^x$
4. $g(x)=3^{x+2}$
5. $f(x)=(2.3)^x-3$
6. $f(x)=e^x+5$
7. $f(x)=-e^x$

Answer

1.

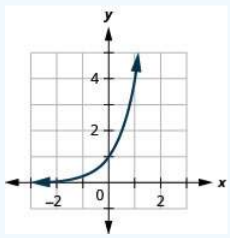


Figure 10.E.6

3.

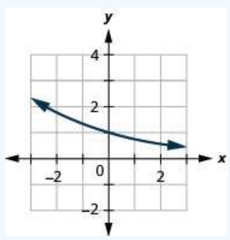


Figure 10.E.7

5.

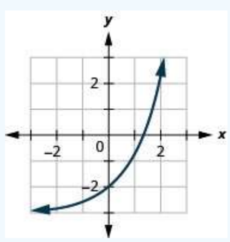


Figure 10.E.8

7.

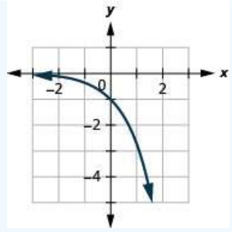


Figure 10.E.9

Exercise 9.5.10 Solve Exponential Equations

In the following exercises, solve each equation.

1. $3^{5x-6}=81$
2. $2^{x^2}=16$
3. $9^x=27$
4. $5^{x^2+2x}=\frac{1}{5}$
5. $e^{4x}\cdot e^7=e^{19}$
6. $\frac{e^{x^2}}{e^{15}}=e^{2x}$

Answer

2. $x=-2, x=2$

4. $x=-1$

6. $x=-3, x=5$

Exercise 9.5.11 Use Exponential Models in Applications

In the following exercises, solve.

1. Felix invested $\$12,000$ in a savings account. If the interest rate is 4% how much will be in the account in 12 years by each method of compounding?
   a. compound quarterly
   b. compound monthly
   c. compound continuously
2. Sayed deposits $\$20,000$ in an investment account. What will be the value of his investment in 30 years if the investment is earning 7% per year and is compounded continuously?
3. A researcher at the Center for Disease Control and Prevention is studying the growth of a bacteria. She starts her experiment with 150 of the bacteria that grows at a rate of 15% per hour. She will check on the bacteria every 24 hours. How many bacteria will he find in 24 hours?
4. In the last five years the population of the United States has grown at a rate of 0.7% per year to about $318,900,000$ If this rate continues, what will be the population in 5 more years?

Answer

2. $\$163,323.40$

4. $330,259,000$

Evaluate and Graph Logarithmic Equations

? Exercise 9.5.12 Convert Between Exponential and Logarithmic Form

In the following exercises, convert from exponential to logarithmic form.

1. $5^4 = 625$
2. $10^{-3} = \frac{1}{1{,}000}$
3. $63^{\frac{1}{5}} = \sqrt[5]{63}$
4. $e^y = 16$

Answer

2. $\log \frac{1}{1{,}000} = -3$

4. $\ln 16 = y$

? Exercise 9.5.13 Convert Between Exponential and Logarithmic Form

In the following exercises, convert each logarithmic equation to exponential form.

1. $7 = \log_2 128$
2. $5 = \log 100,000$
3. $4 = \ln x$

Answer

2. $100000 = 10^5$

? Exercise 9.5.14 Evaluate Logarithms

In the following exercises, solve for x.

1. $\log_x 125 = 3$
2. $\log_7 x = -2$
3. $\log_{\frac{1}{2}} \frac{1}{16} = x$

Answer

1. $x = 5$

3. $x = 4$

? Exercise 9.5.15 Evaluate Logarithms

In the following exercises, find the exact value of each logarithm without using a calculator.

1. $\log_2 32$
2. $\log_8 1$
3. $\log_3 \frac{1}{9}$

Answer

2. 0

? Exercise 9.5.16 Graph Logarithmic Equations

In the following exercises, graph each logarithmic function.

1. $y = \log_5 x$

2. $y = \log_{\frac{1}{4}} x$
3. $y = \log_{0.8} x$

Answer

1.

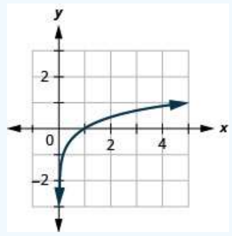


Figure 10.E.10

3.

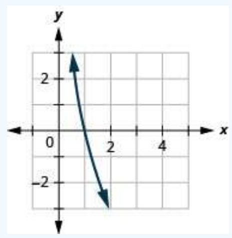


Figure 10.E.11

Exercise 9.5.17 Solve Logarithmic Equations

In the following exercises, solve each logarithmic equation.

1. $\log_a 36 = 5$
2. $\ln x = -3$
3. $\log_2(5x - 7) = 3$
4. $\ln e^{3x} = 24$
5. $\log\left(x^2 - 21\right) = 2$

Answer

2. $x = e^{-3}$

4. $x = 8$

Exercise 9.5.18 Use Logarithmic Models in Applications

What is the decibel level of a train whistle with intensity 10^{-3} watts per square inch?

Answer

90 dB

Use the Properties of Logarithms

Exercise 9.5.19 Use the Properties of Logarithms

In the following exercises, use the properties of logarithms to evaluate.

1. a. $\log_7 1$
   b. $\log_{12} 12$

2. a. $5^{\log_5 13}$
   b. $\log_3 3^{-9}$
3. a. $10^{\log \sqrt{5}}$
   b. $\log 10^{-3}$
4. a. $e^{\ln 8}$
   b. $\ln e^5$

Answer

2.

a. 13
b. -9

4.

a. 8
b. 5

? Exercise 9.5.20 Use the Properties of Logarithms

In the following exercises, use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

1. $\log_4(64xy)$
2. $\log 10,000m$

Answer

2. $4+\log m$

? Exercise 9.5.21 Use the Properties of Logarithms

In the following exercises, use the Quotient Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

1. $\log_7 \frac{49}{y}$
2. $\ln \frac{e^5}{2}$

Answer

2. $5-\ln 2$

? Exercise 9.5.22 Use the Properties of Logarithms

In the following exercises, use the Power Property of Logarithms to expand each logarithm. Simplify, if possible.

1. $\log x^{-9}$
2. $\log_4 \sqrt[7]{z}$

Answer

2. $\frac{1}{7}\log_4 z$

Exercise 9.5.23 Use the Properties of Logarithms

In the following exercises, use properties of logarithms to write each logarithm as a sum of logarithms. Simplify if possible.

1. $\log_3\left(\sqrt{4}x^7y^8\right)$
2. $\log_5\frac{8a^2b^6c}{d^3}$
3. $\ln\frac{\sqrt{3x^2-y^2}}{z^4}$
4. $\log_6\sqrt[3]{\frac{7x^2}{6y^3z^5}}$

Answer

2. $\log_5 8+2\log_5 a+6\log_5 b+\log_5 c-3\log_5 d$

4. $\frac{1}{3}(\log_6 7+2\log_6 x-1-3\log_6 y-5\log_6 z)$

Exercise 9.5.24 Use the Properties of Logarithms

In the following exercises, use the Properties of Logarithms to condense the logarithm. Simplify if possible.

1. $\log_2 56-\log_2 7$
2. $3\log_3 x+7\log_3 y$
3. $\log_5\left(x^2-16\right)-2\log_5(x+4)$
4. $\frac{1}{4}\log y-2\log(y-3)$

Answer

2. $\log_3 x^3y^7$

4. $\log\frac{\sqrt[4]{y}}{(y-3)^2}$

Exercise 9.5.25 Use the Change-of-Base Formula

In the following exercises, rounding to three decimal places, approximate each logarithm.

1. $\log_5 97$
2. $\log_{\sqrt{3}} 16$

Answer

2. 5.047

Solve Exponential and Logarithmic Equations

Exercise 9.5.26 Solve Logarithmic Equations Using the Properties of Logarithms

In the following exercises, solve for x.

1. $3\log_5 x=\log_5 216$
2. $\log_2 x+\log_2(x-2)=3$
3. $\log(x-1)-\log(3x+5)=-\log x$
4. $\log_4(x-2)+\log_4(x+5)=\log_4 8$
5. $\ln(3x-2)=\ln(x+4)+\ln 2$

Answer

2. $x=4$

4. $x=3$

Exercise $9.5.27$ Solve Exponential Equations Using Logarithms

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

1. $2^x = 101$
2. $e^x = 23$
3. $\left(\frac{1}{3}\right)^x = 7$
4. $7e^{x+3} = 28$
5. $e^{x-4} + 8 = 23$

Answer

1. $x = \frac{\log 101}{\log 2} \approx 6.658$

3. $x = \frac{\log 7}{\log \frac{1}{3}} \approx -1.771$

5. $x = \ln 15 + 4 \approx 6.708$

Exercise $9.5.28$ Use Exponential Models in Applications

1. Jerome invests $\$18,000$ at age 17. He hopes the investments will be worth $\$30,000$ when he turns 26. If the interest compounds continuously, approximately what rate of growth will he need to achieve his goal? Is that a reasonable expectation?
2. Elise invests $\$4500$ in an account that compounds interest monthly and earns 6%.How long will it take for her money to double?
3. Researchers recorded that a certain bacteria population grew from 100 to 300 in 8 hours. At this rate of growth, how many bacteria will there be in 24 hours?
4. Mouse populations can double in 8 months $(A = 2A_0)$. How long will it take for a mouse population to triple?
5. The half-life of radioactive iodine is 60 days. How much of a 50 mg sample will be left in 40 days?

Answer

2. 11.6years

4. 12.7months

Practice Test

6. Graph the equation $y = 2^{x-3}$.

7. Solve the equation $2^{2x-4} = 64$.

8. Solve the equation $\frac{e^{x^2}}{e^4} = e^{3x}$.

9. Megan invested $\$21,000$ in a savings account. If the interest rate is 5%, how much will be in the account in 8 years by each method of compounding?

a. compound quarterly
b. compound monthly
c. compound continuously

10. Convert the equation from exponential to logarithmic form: $10^{-2} = \frac{1}{100}$.

11. Convert the equation from logarithmic equation to exponential form: $3 = \log_7 343$.

12. Solve for x: $\log_5 x = -3$

13. Evaluate $\log_{11} 1$.

14. Evaluate $\log_4 \frac{1}{64}$.

15. Graph the equation $y = \log_3 x$.

16. Solve for x: $\log\left(x^2 - 39\right) = 1$

17. What is the decibel level of a small fan with intensity 10^{-8} watts per square inch?

18. Evaluate each.

a. $6^{\log_6 17}$
b. $\log_9 9^{-3}$

Answer

7. $x = 5$

9.

a. $\$31,250.74$
b. $\$31,302.29$
c. $\$31,328.32$

11. $343 = 7^3$

13. 0

15.

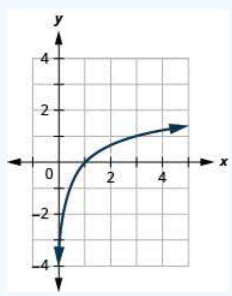


Figure 10.E.15

17. 40 dB

? Exercise 9.5.30

In the following exercises, use properties of logarithms to write each expression as a sum of logarithms, simplifying if possible.

1. $\log_5 25ab$
2. $\ln \frac{e^{12}}{8}$
3. $\log_2 \sqrt[4]{\frac{5x^3}{16y^2z^7}}$

Answer

1. $2 + \log_5 a + \log_5 b$

3. $\frac{1}{4}(\log_2 5 + 3\log_2 x - 4 - 2\log_2 y - 7\log_2 z)$

? Exercise 9.5.31

In the following exercises, use the Properties of Logarithms to condense the logarithm, simplifying if possible.

1. $5\log_4 x + 3\log_4 y$
2. $\frac{1}{6}\log x - 3\log(x + 5)$
3. Rounding to three decimal places, approximate $\log_4 73$.

4. Solve for x: $\log_7(x+2)+\log_7(x-3)=\log_7 24$

Answer

2. $\log \frac{\sqrt[6]{x}}{(x+5)^3}$

4. $x=6$

Exercise 9.5.32

In the following exercises, solve each exponential equation. Find the exact answer and then approximate it to three decimal places.

1. $\left(\frac{1}{5}\right)^x=9$
2. $5e^{x-4}=40$
3. Jacob invests $\$14,000$ in an account that compounds interest quarterly and earns 4%. How long will it take for his money to double?
4. Researchers recorded that a certain bacteria population grew from 500 to 700 in 5 hours. At this rate of growth, how many bacteria will there be in 20 hours?
5. A certain beetle population can double in 3 months $(A=2A_0)$. How long will it take for that beetle population to triple?

Answer

2. $x=\ln 8+4\approx 6.079$

4. $1,921$ bacteria

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