COLLEGE ALGEBRA AND TRIGONOMETRY-EXPRESSIONS, EQUATIONS, AND GRAPHS

Holly Carley and Ariane Masuda
New York City College of Technology

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College Algebra and Trigonometry-Expressions, Equations, and Graphs

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TABLE OF CONTENTS

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CHAPTER OVERVIEW

1: Expressions

SECTION OVERVIEW

1.1: Arithmetic

Featured articles

1.1.1: Integers

Learning Objectives

By the end of this section, you will be able to:

- Understand arithmetic with signed numbers
- Simplify numerical expressions with signed numbers

Be Prepared

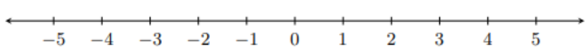
Before we get started, take this readiness quiz (see the appendix concerning arithmetic!).

1. Evaluate $2 \cdot 2$.

2. Evaluate $5 + 3$.

3. Evaluate $6 \div 2$.

We begin with a brief review of arithmetic with integers i.e. $\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$



Any number has a weight and a sign.

Definition 1.1.1.1

The **magnitude** (or **weight**) of a number is the distance it is from 0 on the number line.

Two numbers are **opposites** if, on the number line, they are on opposite sides of zero, but the same distance away from zero.

Example 1.1.1.2

The magnitude of -5 is 5 and the magnitude of 7 is 7 .

So, -5 is the opposite of 5, and 7 is the opposite of -7 and so on.

Try It 1.1.1.3

Find the opposite of

a. -3

b. 9

Answer

a. 3

b. -9

Addition

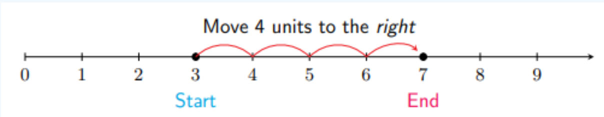
We can add two numbers with the help of a number line.

Example 1.1.1.4

Add $3 + 4$.

Solution

To add $3+4$, we start with 3 on the number line then move 4 units to the right. We land at 7 which is our answer.



So, $3+4=7$. Notice that the answer has the same sign as the signs of 3 and 4 (both positive) and its weight comes from adding the weights of 3 and 4.

You always move to the right when you add a positive number.

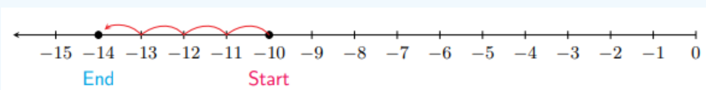
You always move to the left when you add a negative number (a debt).

✔ Example 1.1.1.5

Add $-10+(-4)$.

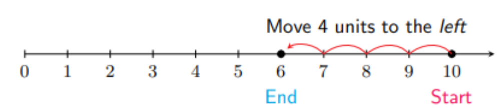
Solution

Adding two negative numbers, $-10+(-4)$ means you are adding a debt of $\$4$ to an already existing debt of $\$10$. So, we start at -10 on the number line and move 4 units to the left, to land at -14, which is the answer.



So, $-10+(-4)=-14$. Notice that the answer has the same sign as the signs of -10 and -4 (both negative) and its weight comes from adding the weights of -10 and -4

To add numbers of opposite signs, that is, a positive and a negative number, we can also use the number line. For example, to perform $10+(-4)$, we start at 10 on the number line and then move 4 units to the left. We land at 6, which is the answer. Think of $10+(-4)$ as having $\$10$ and adding a $\$4$ debt. Because we are adding a debt, we move to the left on the number line!



So, $10+(-4)=6$. Notice that the answer has the same sign as the sign of 10 (positive) because it is the number of larger weight, and its weight comes from finding the difference of the weights of 10 and -4

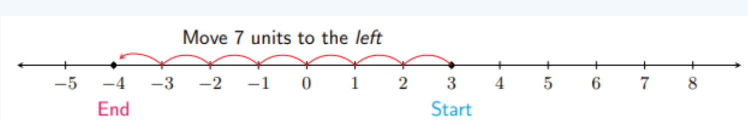
Notice that because we were adding two numbers of opposite signs, the answer ended up being the difference in weight (6) along with the sign of the number of larger weight (positive).

✔ Example 1.1.1.6

Add $3+(-7)$.

Solution

We start at 3 and move toward 7 units to the left, and we land at -4, which is our answer.



So, $3+(-7)=-4$. Notice that the answer has the same sign as the signs of -7 (negative) because it is the number of larger weight, and its weight comes from finding the difference of the weights of 3 and -7.

Definition 1.1.1.7

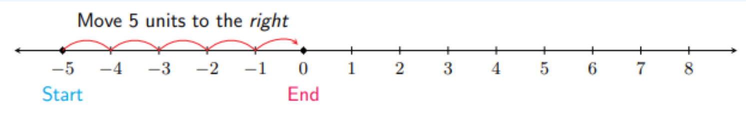
Two opposite numbers are called a **zero-pair**, because adding them always results in 0.

Example 1.1.1.8

Add $-5+5$.

Solution

We start at -5 on the number line and jump to the right 5 units to land finally at 0. So $-5+5=0$.



So, −5 and 5 are a zero-pair.

Adding Intergers

1. To **add two numbers of the same sign**, add their weights and place it after the sign.
2. To **add two numbers of opposite signs**, find the difference of their weights and place it after the sign of the number with the greater weight.

Example 1.1.1.9

Add:

a. $-8+19$

b. $-8+4$

c. $6+(-9)$

d. $7+(-2)$

e. $(-4)+(-7)$

f. $8+7$

Solution

a. $-8+19=11$

b. $-8+4=-4$

c. $6+(-9)=-3$

d. $7+(-2)=5$

e. $(-4)+(-7)=-11$

f. $8+7=15$

Note

While we can add in any order: $4+2=2+4$, it is sometimes convenient to add up all of the negative numbers and add up all of the positive numbers, and then add the results. There are also times when it is best to notice certain simplifications if the numbers are added in a different order.

For example

$$-5+4+5+(-8)=-5+(-8)+4+5(\text{ by reordering })$$

so,

$$-5+4+5+(-8)=-5+(-8)+4+5=-13+9=-4$$

We could have also simplified this by noting that (-5) and 5 are zero-pair, so we are left with $4+(-8)$ which is -4.

Example 1.1.1.10

Simplify $(-4)+(-5)+7+(-3)$.

Solution

$$(-4)+(-5)+7+(-3)=(-4)+(-5)+(-3)+7=(-12)+7=-5$$

We could have simplified this by noting that (-4) and (-3) make -7, and -7 and 7 are zero-pair, so the total is -5.

Try It 1.1.1.11

Add:

a. $9+7$

b. $5+(-7)$

c. $(-4)+(-3)$

d. $(-4)+2$

e. $(-4)+8$

f. $9+(-3)+7+(-9)+3+5+(-2)$

Answer

a. 16

b. -2

c. -7

d. -2

e. 4

f. 10

Subtraction (as Addition of the Opposite)

Once we know how to add numbers, we are set to subtract numbers because subtraction is nothing but addition of the opposite. That is, subtracting 8 − 3 (which reads: subtracting 3 from 8) is the same as 8 + (−3) (which reads: Adding −3 to 8.).

Example 1.1.1.12

Subtract $8-3$.

Solution

So, $8-3=8+(-3)$, and, we can use the rules of adding two numbers of opposite signs to find out that the answer is 5. We can also use the number line. We start at 8 and move 3 units to the left (adding -3 is adding a debt, so we move to the left). So $8-3=5$.

Example 1.1.1.13

Subtract $3-7$.

Solution

To calculate $3-7$ we first rewrite it as an addition problem. $3-7=3+(-7)$. We can either use the number line, or the rules of adding two numbers of opposite signs. And, $50, 3-7=3+(-7)=-4$

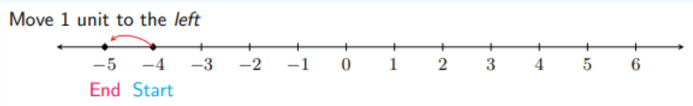
Example 1.1.1.14

Subtract $-4-1$.

Solution

To calculate $-4-1$, we first rewrite it as an addition problem. $-4-1=-4+(-1)$. We can either use the number line, or the rules of adding two numbers of the same signs. If we want to use the rules, both numbers are negative, so our answer will be negative, and, adding the weights of -4 and -1 is $5.50, -4-1=-4+(-1)=-5$

On the number line, we start at -4 and move 1 unit to the left, to land on -5 which is our answer.



Changing the subtractions to additions in this way is particularly useful when adding or subtracting several numbers (because we can add in any order).

Example 1.1.1.15

Simplify $-3-7+5+7+13-6-(-9)$.

Solution

$$\begin{aligned}
&-3-7+5+7+13-6-(-9)\\
=&-3+(-7)+5+7+13+(-6)+9\\
=&-3+(-7)+(-6)+5+7+13+9\\
=&-16+34\\
=&18
\end{aligned}$$

Remark 1.1.1.16

Warning: The symbol "-" is used in two different ways. When it is between two expressions, it means subtract (e.g., $3-4$). Otherwise, it means 'opposite' or 'negative' (e.g., $-3+4$). So in the expression $-4-(-3)$, the first and last "-" means opposite and the one in the middle one means subtract. The importance of understanding this can not be overestimated.

Try It 1.1.1.17

Simplify:

a. $-3-9$

b. $9-(-6)$

c. $4-9+8-(-3)+1$

d. $-2-(-4)+8+(-4)-(-7)$

Answer

a. -12

b. 15

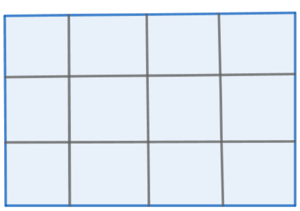
c. 1

d. 13

Multiplication and Division of Positive Numbers

Multiplication of positive integers is adding in the sense that $3\times 4=4+4+4$.

We can also view this multiplication as counting boxes in the image below, or calculating the area of the image below if each box is one square foot, say:



To multiply larger numbers, it is better to use the usual scheme of multiplication. For example:

Example 1.1.1.18

Multiply 152 by 34.

Solution

We will for convenience sake put the smaller number on the bottom (though it is not necessary). We have

$$\begin{array}{rrrr} & 1 & 5 & 2 \\ \times & & 3 & 4 \\ \hline & 6 & 0 & 8 \\ +\ 4 & 5 & 6 & 0 \\ \hline 5 & 1 & 6 & 8 \end{array}$$

And division is the opposite of multiplication in the sense that to compute $45 \div 9$ is to find a number so that when we multiply by 9 we get 45. We run through our multiplication tables (which are hopefully in our head) to discover that 5 does the trick: $5 \times 9 = 45$ so that $45 \div 9 = 5$. We will discuss division from a different point of view when we discuss fractions.

To divide larger numbers, we can use long division. For example,

Example 1.1.1.19

Divide 3571 by 11.

Solution

$$\begin{array}{r|l} & \quad 324 \\ \hline 11 & 3571 \\ & -33 \\ \hline & \quad 271 \\ & -22 \\ \hline & \quad 51 \\ & -44 \\ \hline & \quad 7 \quad = \text{remainder} \end{array}$$

Try It 1.1.1.20

a. Mulitply 36×102.

b. Divide $2345 \div 14$.

Answer

a. 3672

b. 167 with a remainder of 7

Multiplication involving negative numbers

Multiplication is a little tricky to understand without the notion of distribution (discussed later). We will begin with noting again what it means to multiply a number by a positive number. As with multiplication of positive integers, multiplication by a positive integer is really a short hand for a more lengthy addition problem:

$$4 \cdot (-7) = (-7) + (-7) + (-7) + (-7) = -28$$

Note that since $4 \cdot 7 = 28, 4 \cdot (-7) = -(4 \cdot 7)$. We can multiply positive numbers in any order: $4 \cdot 7 = 7 \cdot 4$. The same is true of positive and negative numbers:

$$(-7) \cdot 4 = 4 \cdot (-7) = -(4 \cdot 7) = -28$$

Example 1.1.1.21

Multitply:

a. $5 \cdot (-12)$

b. $(-3) \cdot (-2)$

Solution

a. $5 \cdot (-12) = -(5 \cdot 12) = -60$

b. $(-3) \cdot (-2) = -(3 \cdot (-2)) = -(-(3 \cdot 2)) = 6$

Note that the magnitude of the product of two numbers is the product of their magnitudes. The sign is positive if the signs are the same and negative if they are different.

Notation

Two quantities right next to each other, with no symbol between them (except for parentheses around either or both numbers), has an implicit multiplication. For example, $3(2) = 3 \times 2$.

Example 1.1.1.22

Multiply:

a. $(-5)(-8)$

b. $(-6) \cdot 7$

c. $4 \cdot 12$

d. $(-3)(-6) \cdot 4(-3)$

e. $(-3)(-5) \cdot 4(-2)$

Solution

a. $(-5)(-8) = 40$

b. $(-6) \cdot 7 = -42$

c. $4 \cdot 12 = 48$

d. $(-3)(-6) \cdot 4(-3) = 18 \cdot 4(-3) = 72(-3) = -216(\text{ multiplying from left to right })$

e. $(-3)(-5) \cdot 4(-2) = (-3) \cdot 4 \cdot (-5)(-2) = -12 \cdot 10 = -120(\quad$ since we can multiply in any order it is convenient to see that $-5 \cdot -2 = 10.\ )$

Try It 1.1.1.23

Multiply:

a. $(-3)(4)$

b. $(-2) \cdot (-3)$

c. $5 \cdot 11$

d. $(2)(-3) \cdot 4(-5)$

e. $(25)(-3) \cdot (-6)(-4)$

Answer

a. -12

b. 6

c. 55

d. 120

e. 1800

Division involving negative numbers

What does it mean to divide? $A \div B$ is the number you need to multiply B by to get A. Some will say 'B goes into A' $A \div B$ times. For example, $6 \div (-3)$ is -2 because $(-2)(-3) = 6.$.

Example 1.1.1.24

Divide:

a. $(-42) \div 7$

b. $81 \div (-9)$

c. $(-35) \div (-7)$

d. $14 \div 2$

e. $0 \div 5$

f. $-10 \div 0$

Solution

a. $(-42) \div 7 = -6$

b. $81 \div (-9) = -9$

c. $(-35) \div (-7) = 5$

d. $14 \div 2 = 7$

e. $0 \div 5 = 0$. Note When dividing 0 by any number, the answer is always 0.

f. $-10 \div 0 =$ undefined.

Note 1.1.1.25

Any number divided by 0 is undefined!

Multiplying and Dividing Integers

Consider two numbers at a time.

1. If the signs of the two numbers are the same, then the sign of the answer is positive.
2. If the signs of the two numbers are different, then the sign of the answer is negative.

Exeample 1.1.1.26

Divide:

a. $(-15) \div 0$

b. $-15 \div (-3)$

c. $(20) \div (-4)$

d. $0 \div 2$

e. $(-72) \div 3$

f. $10 \div 5$

Answer

a. Undefined

b. 5

c. -5

d. 0

e. -24

f. 2

Writing Exercises 1.1.1.27

1. Give an application of adding two positive numbers. Give an application of adding two negative numbers.
2. Give an application of multiplying two positive numbers.
3. Give an example of multiplying a positive number and a negative number.
4. Explain how addition and subtraction are related. Include an example.
5. Explain how multiplication and division are related. Include an example.

Exit Problem

Simplify:

1. $(-7)+5-(-8)$
2. $(-7)(5)(-8)$
3. $\frac{87}{4}$

Key Concepts

- Adding integers
- Subtracting integers
- Multiplying integers
- Dividing integers

1.1.2: Fractions

Learning Objectives

By the end of this section, you will be able to:

- Understand arithmetic of fractions
- Understand the meaning of fractions

Be Prepared

Before we get started, take this readiness quiz.

1. Evaluate $2 \cdot (-2)$.

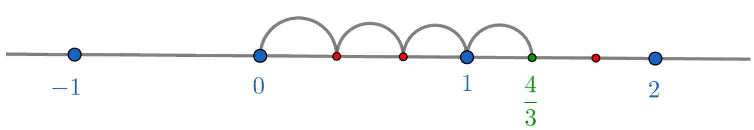
2. Evaluate $5 - (-3)$.

3. Evaluate $(-6) \div 2$.

In our previous section we identified integers: $0, \pm 1, \pm 2, \pm 3, \ldots$. To this set, now we will add all ratios of integers with non-zero denominators, like $\frac{2}{7}, \frac{-11}{17} \ldots$ We call this the set of **rational numbers**. Any rational number looks like $\frac{p}{q}$ where p and q are integers and q is not 0.. We have considered the division of integers in the previous section. We relate fractions to division in this way:

$$\frac{p}{q} = p \div q.$$

We understand the meaning of this when $p \div q$ is an integer. When it is not, we can still find the number on the numberline (at least in theory). We start with an example: $4/3$. The denominator is the number of equal pieces we divide up 1 into. And the numerator is the number of pieces. So, in this case we divide up 1 unit into 3 pieces (so we have 3 equal length pieces for each unit on the number line) and we count 4 of them.



If a pizza is your unit, then you can think of dividing pizzas each up into 3 equal pieces and then taking 4 pieces. Then you would have $\frac{4}{3}$ pizzas.

Just as we are able to perform arithmetic operations with integers we can also perform arithmetic operations with rational numbers (fractions). The two types of fractions we will encounter are called proper and improper:

- **Proper fractions** have value less than 1, for example $\frac{2}{5}$ and $\frac{1}{8}$. Observe that for these fractions the numerator is less than the denominator.
- **Improper fractions** have value greater than or equal to 1, for example $\frac{7}{6}$ and $\frac{3}{2}$. For these fractions the numerator is greater than the denominator.

Each fractional value can have many different, equivalent forms, for example $1 = \frac{2}{2} = \frac{-5}{-5} = \cdots$ In order to determine whether two fractions are equivalent we can use the fundamental principle of fractions.

Note

Equivalent fractions

For \{b,c\neq 0\),

$$\frac{a\cdot c}{b\cdot c}=\frac{a}{b}.$$

In this case, we say $\frac{a\cdot c}{b\cdot c}$ and $\frac{a}{b}$ are **equivalent**.

Reducing a fraction

For positive integers a and b, we say that $\frac{a}{b}$ is **reduced** if there is no equivalent fraction with smaller positive numerator/denominator. **Reducing a fraction** means to find the reduced equivalent fraction.

If a or b is negative, then we limit the sign to the numerator.

Since

$$\frac{2}{3}=\frac{2\cdot 4}{3\cdot 4}=\frac{8}{12},$$

$\frac{2}{3}$ and $\frac{8}{12}$ are equivalent. In addition, $\frac{2}{3}$ is reduced.

You should check these on your numberline so you see why this is the case. Check also with the idea of cutting up pizzas.

That is, as long as you multiply both numerator and denominator by the **same** number, the fraction value does not change, and you obtain equivalent fractions.

Example 1.1.2.1

Write a fraction that is equivalent to $\frac{3}{5}$.

Solution

Begin with our original fraction $\frac{3}{5}$ and apply the fundamental principle of fractions to get

$$\frac{3}{5}=\frac{3\cdot 2}{5\cdot 2}=\frac{6}{10}.$$

Example 1.1.2.2

Simplify the fraction $\frac{15}{35}$.

Solution

Begin with our original fraction and apply the fundamental principle of fractions in reverse to get

$$\frac{15}{35}=\frac{3\cdot 5}{7\cdot 5}=\frac{3}{7}.$$

Try It 1.1.2.3

a. Is $\frac{4}{10}$ equal to $\frac{8}{20}$? Explain.

b. Simplify $\frac{180}{140}$.

Answer

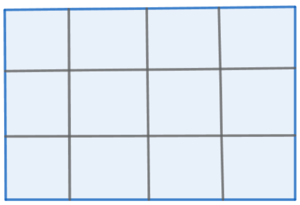
a. Yes.

b. $\frac{9}{7}$

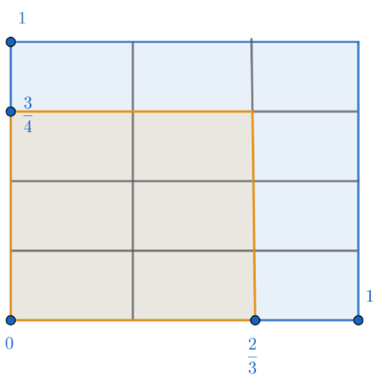
Multiplying Fractions

To understand multiplication of fractions, we can reconsider ways to understand multiplication of the numbers 3 and 4.

$3 \cdot 4 = 4 + 4 + 4$ and it is also the number of squares in the following picture:



We consider the same picture when we think of multiplying fractions. For an example, let's multiply $\frac{2}{3} \cdot \frac{3}{4}$. We make a square and divide up the units on horizontal side into 3 pieces and the units on the vertical side into 4 pieces. We then highlight the appropriate area.



There are a total of 12 pieces in the unit 1×1 square and there are 6 of those pieces highlighted. So the product is $\frac{6}{12} = \frac{1}{2}$.

So,

$$\frac{2}{3} \cdot \frac{3}{4} = \frac{6}{12}.$$

We see that the denominator comes from the product of the denominators and the numerator comes from the product of the numerators.

We multiply numerators together and denominators together:

$$\frac{a}{b} \cdot \frac{c}{d} = \frac{a \cdot c}{b \cdot d}$$

Example 1.1.2.4

Multiply and reduce $\frac{14}{3} \cdot \frac{9}{7}$.

Solution

The product of these two fractions is done as follows:

$$\frac{14 \cdot 9}{3 \cdot 7} = \frac{2 \cdot 7 \cdot 3 \cdot 3}{3 \cdot 7} = \frac{6}{1} = 6$$

Try It 1.1.2.5

Multiply and reduce:

a. $\frac{5}{14} \cdot \frac{7}{2}$

b. $\frac{20}{21} \cdot \frac{14}{15}$

Answer

a. $\frac{5}{4}$

b. $\frac{8}{9}$

Dividing Fractions

To divide fractions we recognize the relationship between fractions and division. Namely, $\frac{a}{b} = a \div b$. For example, $\frac{5}{2} = 5 \div 2$. So,

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b}}{\frac{c}{d}}$$

and we can multiply the numerator and denominator by $\frac{d}{c}$ to arrive at an equivalent fraction:

$$\frac{a}{b} \div \frac{c}{d} = \frac{\frac{a}{b} \cdot \frac{d}{c}}{\frac{c}{d} \cdot \frac{d}{c}} = \frac{a}{b} \cdot \frac{d}{c}!$$

The Reciprocal of a Fraction

The **reciprocal of a fraction** $\frac{p}{q}$ is the fraction formed by switching the numerator and denominator, namely $\frac{q}{p}$.

Example 1.1.2.6

Find the reciprocal of:

a. $\frac{3}{5}$

b. $\frac{-2}{7}$

c. $\frac{1}{8}$

d. 4

Solution

a. The reciprocal of $\frac{3}{5}$ is $\frac{5}{3}$.

b. The reciprocal of $\frac{-2}{7}$ is $\frac{7}{-2} = \frac{-7}{2} = -\frac{7}{2}$.

c. The reciprocal of $\frac{1}{8}$ is $\frac{8}{1} = 8$.

d. The reciprocal of $4 = \frac{4}{1}$ is $\frac{1}{4}$.

So to divide one fraction by another, we multiply the first fraction by the **reciprocal** of the second:

$$\frac{a}{b} \div \frac{c}{d} = \frac{a}{b} \cdot \frac{d}{c} = \frac{a \cdot d}{b \cdot c}$$

Example 1.1.2.7

Divide and reduce $\frac{8}{3} \div \frac{4}{5}$.

Solution

The quotient of these two fractions is found as follows:

$$\frac{8}{3} \div \frac{4}{5} = \frac{8}{3} \cdot \frac{5}{4} = \frac{8 \cdot 5}{3 \cdot 4} = \frac{2 \cdot 4 \cdot 5}{3 \cdot 4} = \frac{10}{3}$$

Try It 1.1.2.8

Divide and reduce

a. $\frac{5}{14} \div \frac{7}{2}$

b. $\frac{20}{21} \div \frac{14}{15}$

Answer

a. $\frac{5}{49}$

b. $\frac{50}{49}$

Adding and Subtracting Fractions (with same denominators)

The denominator simply indicates the size of the items we are counting. The numerator is the count. If the denominators are the same, we are counting things of the same size!

$$\frac{a}{b} + \frac{c}{b} = \frac{a+c}{b}$$

We can also think of $\frac{a}{b}$ as $a \cdot \frac{1}{b}$ etc. For example, $\frac{2}{6}$ is two (sixths) and $\frac{5}{6}$ is five (sixths) so

$$\frac{2}{6} + \frac{5}{6} \text{ is two (sixths) plus five (sixths)}$$

so, just like two apples plus five apples is seven apples, we have the result of seven (sixths) which is $\frac{7}{6}$. So, you can think if the denominator as indicating what object you are counting! This point of view might be helpful in working with and understanding fractions a bit more natural.

Similarly,

$$\frac{a}{b}-\frac{c}{b}=\frac{a-c}{b}$$

Example 1.1.2.9

a. Add $\frac{3}{5}+\frac{1}{5}$.

b. Subtract $\frac{3}{5}-\frac{1}{5}$.

Solution

a. $\frac{3}{5}+\frac{1}{5}=\frac{3+1}{5}=\frac{4}{5}$

b. $\frac{3}{5}-\frac{1}{5}=\frac{3-1}{5}=\frac{2}{5}$

Try It 1.1.2.10

Simplify:

a. $\frac{5}{14}+\frac{3}{14}$

b. $\frac{20}{21}-\frac{14}{21}$

Answer

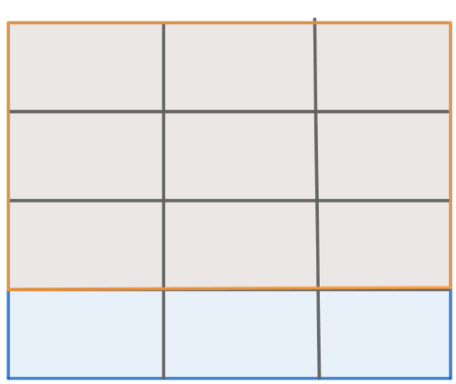
a. $\frac{8}{14}=\frac{4}{7}$

b. $\frac{6}{21}=\frac{2}{7}$

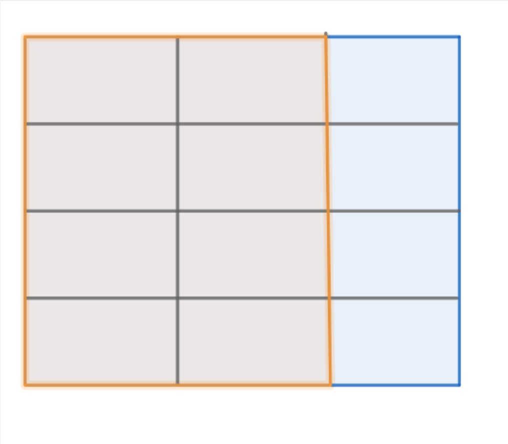
Adding or Subtracting Fractions (with unlike denominators)

Adding (or subtracting) fractions with unlike denominators requires us to first find a common denominator.

To understand the issues of adding fractions with unlike denominator, consider the problem of asking how much pizza Ariane and Holly have together if Ariane has $\frac{3}{4}$ pizza and Holly has $\frac{2}{3}$. Ariane,say, has three slices of a pizza cut into four pieces and Holly has two slices of a pizza cut into 3 pieces. While they have 3 slices together, the slices are not the same size. So it is tricky to see exactly how much pizza they have. To see how to do this we will assume the pizzas are square (it is easier to slice), so that both fractions are easily understood. We divide each of their pizzas into 12 pieces (3 times 4).



Ariane has $\frac{3}{4}=\frac{9}{12}$ slices.

	Holly has $\frac{2}{3} = \frac{8}{12}$ slices.
Since the slices are the same size we can not just count slices. They have 17 slices of pizzas which each have 12 equal slices. So they have together $\frac{17}{12}$ pizzas.	

One can always think of slicing pizzas this way, but sometimes we could manage with fewer slices. Consider what happens if Ariane has $\frac{1}{6}$ of a pizza and Holly has $\frac{3}{4}$. Can you do better than slicing each pizza into 24 slices each?

This leads us to the considering the smallest number of slices one needs to divide each pizza into in order that the amount of pizza that Ariane and Holly have can be some number of those slices.

The LCD or least common denominator is the smallest number that both denominators evenly divide. Once we rewrite each of our fractions so their denominator is the LCD, we may add or subtract fractions according to the above properties.

Finding the LCD

- Step 1: Make a list of (enough) multiples of each denominator.
- Step 2: Identify the lowest common multiple. If you can't see one, then your lists in Step 1. need to be expanded.

To be able to add or subtract fractions, we need to go one more step: Once you've identified the LCD, rewrite both fractions (by multiplying both numerator and denominator by the appropriate same number) to get the LCD as denominator.

Example 1.1.2.11

Find the LCD and then add and simplify $\frac{3}{12}+\frac{5}{8}$.

Solution

Let us first find the LCD by following our procedure.

List the denominators.	8 and 12
Make a list of (enough) multiples.	$8: 8, 16, 24, 32, \ldots$ $12: 12, 24, 36, 48, \ldots$
Identify the LCD.	24
Factor the LCD using 8 as one of the factors. Factor the LCD using 12 as one of the factors.	$24 = 8 \cdot 3$ $24 = 12 \cdot 2$
Rewrite each fraction using the LCD:	$\frac{3}{12} = \frac{3 \cdot 2}{12 \cdot 2} = \frac{6}{24}$ $\frac{5}{8} = \frac{5 \cdot 3}{8 \cdot 3} = \frac{15}{24}$

Now we are ready to add our fractions	$\frac{3}{12}+\frac{5}{8}$ $=\frac{6}{24}+\frac{15}{24}$ $=\frac{21}{24}$
Simplify.	$=\frac{3\cdot 7}{3\cdot 8}$ $=\frac{7}{8}$

Example 1.1.2.12

Find the LCD and then subtract and simplify $\frac{1}{9}-\frac{3}{5}$.

Solution

Let us first find the LCD by following our procedure.

List the denominators.	9 and 15
Make a list of (enough) their multiples.	$9: 9, 18, 27, 36, 45, 54, 63, \ldots$ $5: 5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 55, \ldots$
Identify the LCD.	45
Factor the LCD using 9 as one of the factors. Factor the LCD using 15 as one of the factors.	$45=9\cdot 5$ $45=15\cdot 3$
Rewrite each fraction using the LCD:	$\frac{1}{9}=\frac{1\cdot 5}{9\cdot 5}=\frac{5}{45}$ $\frac{3}{5}=\frac{3\cdot 9}{5\cdot 9}=\frac{27}{45}$
Now we are ready to subtract our fractions, but, first, we rewrite the subtraction as addition of the opposite:	$\frac{1}{9}-\frac{3}{5}$ $=\frac{1}{9}+\left(-\frac{3}{5}\right)$ $=\frac{5}{45}+\left(-\frac{27}{45}\right)$
Simplify.	$=\frac{5+(-27)}{45}$ $=\frac{-22}{45}$

Try It 1.1.2.13

Simplify

a. $\frac{5}{14}+\frac{3}{7}$

b. $\frac{2}{21}-\frac{4}{15}$

Answer

a. $\frac{11}{14}$

b. $\frac{-7}{15}$

Writing an Improper Fraction as a Mixed Number

By drawing a picture of pizzas, it is clear that, $\frac{4}{3}$ is the same as one whole pizza $\frac{3}{3}$ together with $\frac{1}{3}$, i.e., one entire pizza and one extra slice. We write $\frac{4}{3}=1\frac{1}{3}$. The number on the right is called a mixed number and means 'one whole plus one-third.' You might think of a general procedure before continuing with our discussion.

1. Divide the numerator by the denominator.
2. If there is a remainder, write it over the denominator.

Example 1.1.2.14

Write $\frac{42}{5}$ as a mixed number.

Solution

We begin by dividing the numerator 42 by the denominator 5 to get 8, with a remainder of 2. Our mixed number is $8\frac{2}{5}$.

Try It 1.1.2.15

Write $\frac{25}{3}$ as a mixed number.

Answer

$8\frac{1}{3}$

Writing a Mixed Number as an Improper Fraction

1. Multiply the whole number and the denominator then add the numerator. Use the result as your new numerator.
2. The denominator remains the same

Example 1.1.2.16

Write the mixed number $3\frac{5}{6}$ as an improper fraction.

Solution

1. Multiply the denominator by the whole number.
2. Add this result to the numerator.
3. Set this new numerator 23 over the denominator of 6.

We multiply the whole number 3 and the denominator 6 to get 18. Next, we add to this the numerator 5 to get 23. This is our new numerator and our improper fraction becomes $\frac{23}{6}$.

Try It 1.1.2.17

Write the $7\frac{3}{4}$ as an improper fraction.

Answer

$\frac{31}{4}$

Addition and Subtraction of Mixed Numbers

To add (or subtract) mixed numbers, we can convert the numbers into improper fractions, then add (or subtract) the fractions as we saw in this chapter.

Example 1.1.2.18

Subtract $7-2\frac{3}{8}$.

Solution

First we convert $2\frac{3}{8}=\frac{19}{8}$. Then, we rewrite the subtraction operation as addition of opposite:

$$7-\frac{19}{8}=7+\left(-\frac{19}{8}\right)=\frac{7}{1}+\left(-\frac{19}{8}\right)=\frac{56}{8}+\left(-\frac{19}{8}\right)=\frac{56+(-19)}{8}=\frac{37}{8}=4\frac{5}{8}$$

Also, we can keep mixed fractions and mixed fraction, and, add (or subtract) the integer parts together and the fraction parts together.

Example ✓ 1.1.2.19

Add $7\frac{3}{4}+3\frac{1}{5}$.

Solution

Here, we add $7+3=10$ and $\frac{3}{4}+\frac{1}{5}=\frac{15}{20}+\frac{4}{20}=\frac{19}{20}$.

And, our final answer is $10\frac{19}{20}$. Note that $\frac{19}{20}$ is a proper fraction, so, our work is done. But, if our answer ended up with an improper fraction, we would have had to make the conversion to write the answer in simplified form.

? Try It 1.1.2.20

Simplify $3\frac{3}{4}-2\frac{3}{5}$.

Answer

$$\frac{17}{20}$$

Multiplying and Dividing of Mixed Numbers

Be careful when multiplying mixed numbers. You must first convert them to improper fractions and use the rules for multiplying fractions to finish your problem.

Example ✓ 1.1.2.21

Multiply $2\frac{3}{5}$ and $3\frac{1}{2}$.

Solution

Begin by rewriting each mixed number as an improper fraction: $2\frac{3}{5}=\frac{13}{5}$ and $3\frac{1}{2}=\frac{7}{2}$. Now we proceed by multiplying the fractions

$$\frac{13}{5}\cdot\frac{7}{2}=\frac{13\cdot 7}{5\cdot 2}=\frac{91}{10}$$

We can now write the result (if we wish) as a mixed number: $9\frac{1}{10}$.

Example 1.1.2.22

Divide $\left(1\frac{4}{5}\right) \div \left(1\frac{1}{2}\right)$.

Solution

Begin by rewriting each mixed number as a improper fraction: $1\frac{4}{5} = \frac{9}{5}$ and $1\frac{1}{2} = \frac{3}{2}$. Now we proceed by dividing the fractions

$$\frac{9}{5} \div \frac{3}{2} = \frac{9}{5} \cdot \frac{2}{3} = \frac{9 \cdot 2}{5 \cdot 3} = \frac{3 \cdot 2}{5 \cdot 1} = \frac{6}{5} = 1\frac{1}{5}$$

Try It 1.1.2.23

Divide $7\frac{1}{2}$ by $2\frac{2}{3}$.

Answer

$\frac{45}{16}$

Writing Exercises 1.1.2.24

1. Give an application of fractions.
2. Explain why a common denominator is needed when adding or subtracting fractions.
3. Give an example of multiplication of two fractions which are not integers.
4. How might you go about converting a negative improper fraction into a mixed number? Give an example.
5. Give an example of adding two fractions with unlike denominators.
6. Give an example of how to divide two fractions which are not integers.

Exit Problem

Evaluate $\frac{3}{4} - 1\frac{5}{6}$.

Key Concepts

1. Fraction
2. Adding/subtracting fractions
3. Multiplying/dividing fractions
4. Mixed numbers
5. Improper fractions

1.1.3: Order of Operations and Introduction to Expressions

Learning Objectives

By the end of this section, you will be able to:

- Decide which operations to do first
- Simplify numerical expressions with multiple operations

Be Prepared

Before we get started, take this readiness quiz.

1. Evaluate $-5 \cdot 2$.

2. Evaluate $-5+3$.

3. Evaluate $-6 \div (-2)$.

What is the meaning of the expression '3 times 4 plus 5'. Some will answer 17, while others may answer 27. Why? To take the ambiguity out, we can write

$$(3 \cdot 4)+5=17$$

and

$$3 \cdot (4+5)=27,$$

where we must first evaluate the quantity in parentheses. Since it can be somewhat cumbersome to write a lot of parentheses, there is an important convention or agreement that if we just write $3 \cdot 4+5$ we mean $(3 \cdot 4)+5$. That is, in the absence of parentheses, we should multiply before we add. This is part of what is called **the order of operations**. This must be remembered.

The Order of Operations

When evaluating an expression, which only involves addition, subtraction, multiplication and division (no parentheses), we first perform, from left to right, all of the multiplications and divisions. Then, from left to right, the additions and subtractions. If there are parts of the expression enclosed by parentheses, what is within the parentheses must be evaluated first.

Remark 1.1.3.1

Subtraction can be turned into addition and then addition can be done in any order, not necessarily from left to right. This explains why addition and subtraction come together in the order of operations. For example, $5-2$ is also $5+(-2)$.

There is a similar statement for multiplication and division. For example, $8 \div 2$ is also $8 \cdot \frac{1}{2}$.

PE(MD)(AS) is a mnemonic device to remember the order of operations. This means that the order is: Parentheses, Exponents (this will be incorporated later), Multiplication and Division (taken together from left to right), and finally, Addition and Subtraction (taken together from left to right).

Let us try a few problems.

Example 1.1.3.2

Simplify:

a. $3+2(3+5)$

b. $3-2(-4+7)$

Solution

a. $3+2(3+5)=3+2(8)=3+16=19$

b. $3-2(-4+7)=3-2(3)=3-6=-3$

Try It 1.1.3.3

Simplify:

a. $-3-4-2(-2\cdot 6-5)$

b. $-(3-(-6))-(1-4\cdot(-5)+4)$

c. $-2(-14\div 7+7)$

Answer

a. $-3-4-2(-2\cdot 6-5)=-3-4-2(-12-5)=-3-4-2(-17)=-3-4-(-34)=-3-4+34=27$

b. $-(3-(-6))-(1-4\cdot(-5)+4)=-(3+6)-(1-(-20)+4)=-9-(1+20+4)=-9-25=-9+(-25)=-34$

c. $-2(-14\div 7+7)=-2(-2+7)=-2(5)=-10$

Try It 1.1.3.4

Simplify:

a. $-3(-2\cdot 7-(-5)(4)\div 2)$

b. $6\div 2\times 3$

c. $-2(3-1)2-(8-24)\div 4$

Answer

a. $-3(-2\cdot 7-(-5)(4)\div 2)=-3(-14-(-20)\div 2)=-3(-14-(-10))=-3(-4)=12$

b. $6\div 2\times 3=3\times 3=9$

c. $-2(3-1)2-(8-24)\div 4=-2(2)2-(-16)\div 4=-4\cdot 2-(-4)=-8+4=-4$

An expression is a combination of numbers, variables (letters that represent numbers), operations and parentheses that can be evaluated when 'appropriate' numbers are substituted in place of the variables following the order of operations which will be discussed. Which numbers are appropriate will depend on the particular example at hand. For example, $3\cdot(x+2)$ is an expression where 3 and 2 are numbers, x is a variable, $+$ and $\cdot$ are operations and the parentheses enclose a sub-expression, that is, an expression within an expression.

We concern ourselves with expressions in order to express a quantity that depends on another quantity which can vary. Keep the general idea in mind as we work through different ways of combining letters and numbers to form expressions in this unit. We will start with the simplest expressions that just involve multiplication of a number by a variable and addition of the same with numbers.

Expression An expression is a combination of numbers, variables (letters that represent numbers), operations and parentheses that can be evaluated when 'appropriate' numbers are substituted in place of the variables following the order of operations which will be discussed.

Variable letters that represent numbers

Given an expression, we can **evaluate** it by replacing every instance of a variable with a single number. Which numbers are being substituted for which variables should be made clear in the wording. The following examples involve only one variable so there is little chance of ambiguity.

Example 1.1.3.5

Evaluate the expression $-3x+2-2(x-2)$ at $x=-3$.

Solution

We replace every instance of x by -3 and evaluate the result using the order of operations.

We see that

$$-3(-3)+2-2((-3)-2)=9+2-2(-5)=9+2-(-10)=9+2+10=21.$$

The expression evaluated at -3 is 21.

Try It 1.1.3.6

Evaluate the expression $5x+1-3(-x+2)$ at $x=2$.

Answer

The expression evaluated at 2 is 11

Try It 1.1.3.7

Evaluate the expression $-3(2t-2)+5+2(7-t)$ at $t=-2$.

Answer

The expression evaluated at -2 is 41.

Writing Exercises 1.1.3.8

1. Explain the purpose of parentheses and give an example.
2. Explain why multiplication and division should be treated on the same level in the order of operations.
3. Give an example of an expression with one variable.
4. What reasons my we be interested in using variables?

Exit Problem

Evaluate $(2\cdot 3+5)\div 4-4(7-2)$.

Key Concepts

- The order of operations
- Expression
- Variable

1.1.4: Integer Exponents

Learning Objectives

By the end of this section, you will be able to:

- Understand the meaning of an integer exponent.
- Simplify numerical expressions with which involve integer exponents.

Be Prepared

Before we get started, take this readiness quiz.

1. Evaluate $-5 \cdot 2$.

2. Evaluate $-5+3$.

3. Evaluate $3 \cdot 3 \cdot 3 \cdot 3$.

We will begin with positive integer exponents where the meaning is straight forward. The positive integer exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the positive integer *exponent* m tells us how many times we use the *base* a as a factor.

$$a^m = \underbrace{a \cdot a \cdot \cdots \cdot a}_{m a' \mathrm{s}}$$

For example,

$$(-9)^5 = (-9)(-9)(-9)(-9)(-9).$$

Let's review the vocabulary for expressions with exponents.

Definition 1.1.4.1

$$a^m = \underbrace{a \cdot a \cdots a}_{m a' \mathrm{s}}$$

This is read a to the mth (**power**), or a to the **power** m.

In the expression a^m with positive integer m and $a \neq 0$, the **exponent** m tells us how many times we use the **base** a as a factor.

Example 1.1.4.2

a. Evaluate 2^3.

b. Evaluate -7^2.

c. Evaluate $(-1)^4$.

d. Write using exponents: $-2 \cdot 2 \cdot 2$.

e. Identify the base and the exponent: -4^3.

Solution

a. $2^3 = 2 \cdot 2 \cdot 2 = 8$

b. $-7^2 = -49$

c. $(-1)^4 = (-1)\cdot(-1)\cdot(-1)\cdot(-1) = 1$

d. $-2\cdot2\cdot2 = -2^3$

e. -4^3 has an exponent of 3 and the base is 4 since there are no parentheses that indicate including the "-".

Try It 1.1.4.3

a. Evaluate 3^4.

b. Evaluate -2^4.

c. Evaluate $(-2)^3$.

d. Write using exponents $-6\cdot6\cdot6\cdot6$.

e. Identify the base and the exponent: $-2\cdot5^7$.

Answer

a. 81

b. -16

c. -8

d. -6^4

e. The exponent is 7 and the base is 5 (since there are no parentheses that would include 2 or -2).

We will now investigate several properties of exponents. First we will look at an example that leads to the *Product Property for Positive Integer Exponents*.

	$7^2 7^3$
What does this mean?	$= \underbrace{7\cdot7}_{\text{2 factors}} \cdot \underbrace{7\cdot7\cdot7}_{\text{3 factors}}$ $= \underbrace{7\cdot7\cdot7\cdot7\cdot7}_{\text{5 factors}}$
	$= 7^5$

The base stayed the same and we added the exponents. Remember that the exponent counts the number of bases we multiply so, we are multiplying 2 sevens by 3 sevens which gives us a total of 5 sevens!

In general we have:

Product property for positive integer exponents

If a is a real number and m and n are positive integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Example 1.1.4.4

Simplify each expression:

a. $3^5 3^6$

b. $2^4\cdot2^{3\cdot4}$

c. $2 \cdot 5^7 \cdot 3 \cdot 5$

d. $11^4 11^5 11^2$

Solution

a.

	$3^5 3^6$
Use the Product Property, $a^m a^n = a^{m+n}$ or directly use the fact that the exponent counts.	$= 3^{5+6}$
Simplify.	$= 3^{11}$

b.

	$2^4 \cdot 2^{3 \cdot 4}$
Use the Product Property, $a^m a^n = a^{m+n}$ or or directly use the fact that the exponent counts..	$= 2^{4+3 \cdot 4}$
Simplify.	$= 2^{4 \cdot 4}$ $= 2^{16}$

c.

	$2 \cdot 5^7 \cdot 3 \cdot 5$
Rewrite, $a = a^1$.	$= 2 \cdot 5^7 \cdot 3 \cdot 5^1$
Use the Commutative Property and use the Product Property, $a^m a^n = a^{m+n}$ or or directly use the fact that the exponent counts..	$= 2 \cdot 3 \cdot 5^{7+1}$
Simplify.	$= 6 \cdot 5^8$

d.

	$11^4 11^5 11^2$
Add the exponents, since the bases are the same.	$= 11^{4+5+2}$
Simplify.	$= 11^{11}$

? Try It 1.1.4.5

Simplify each expression:

a. $7^9 7^8$

b. $4^{2 \cdot 3} \cdot 4^3$

c. $3 \cdot 9^5 \cdot 4 \cdot 9$

d. $3^6 3^4 3^8$

Answer

a. 7^{17}

b. 4^9

c. $12 \cdot 9^6$

d. 3^{18}

Try It 1.1.4.6

Simplify each expression:

a. $(-2)^{12}(-2)^4$

b. $10 \cdot 10^5$

c. $2 \cdot 4 \cdot 6 \cdot 4^7$

d. $6^5 6^9 6^5$

Answer

a. $(-2)^{16} = 2^{16}$

b. 10^6

c. $12 \cdot 4^8$

d. 6^{19}

Now let's look at an exponential expression that contains a power raised to a power. Let's see if we can discover a general property.

	$(5^2)^3$
What does this mean?	$= 5^2 5^2 5^2$
How many factors altogether?	$= \underbrace{5 \cdot 5}_{2\ factors} \cdot \underbrace{5 \cdot 5}_{2\text{ factors}} \cdot \underbrace{5 \cdot 5}_{2\text{ factors}}$ $= \underbrace{5 \cdot 5 \cdot 5 \cdot 5 \cdot 5 \cdot 5}_{6\text{ factors}}$
So we have	$= 5^6$

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(5^2)^3$ is $5^{2 \cdot 3}$ or x^6. We can also see that

In this example we multiplied the exponents.

We can check various examples to see that this leads us to the *Power Property for Positive Integer Exponents.*

Power Property for Integer Exponents

If a is a real number and m and n are positive integers, then

$$(a^m)^n = a^{mn}.$$

To raise a power to a power, multiply the exponents.

Example 1.1.4.13

Simplify each expression:

a. $(3^5)^9$

b. $(4^4)^7$

c. $(2^3)^6(2^5)^4$

Solution

a.

	$(3^5)^9$
Use the power property, $(a^m)^n = a^{mn}$ or use the fact that the exponent counts the number of bases being multiplied.	$3^{5 \cdot 9}$
Simplify.	3^{45}

b.

	$(4^4)^7$
Use the power property.	$= 4^{4 \cdot 7}$
Simplify.	$= 4^{28}$

c.

	$(2^3)^6(2^5)^4$
Use the power property.	$= 2^{18}2^{20}$
Add the exponents.	$= 2^{38}$

? Try It 1.1.4.14

Simplify each expression:

a. $((-3)^7)^5$

b. $(5^4)^3$

c. $(3^4)^5(3^7)^4$

Answer

a. $(-3)^{35} = -3^{35}$

b. 5^{12}

c. 3^{48}

? Try It 1.1.4.15

Simplify each expression:

a. $(9^6)^9$

b. $(3^7)^7$

c. $\left(\left(\frac{1}{2}\right)^4\right)^5 \left(\left(\frac{1}{2}\right)^3\right)^3$

Answer

a. 9^{54}

b. 3^{49}

c. $\left(\frac{1}{2}\right)^{29}$

We will now look at an expression containing a product that is raised to a power. Can we find this pattern?

	$(2 \cdot 5)^3$
What does this mean?	$= 2 \cdot 5 \cdot 2 \cdot 5 \cdot 2 \cdot 5$
We group the like factors together.	$= 2 \cdot 2 \cdot 2 \cdot 5 \cdot 5 \cdot 5$
How many factors of 2 and of 5?	$= 2^3 5^3$

Notice that each factor was raised to the power and $(2 \cdot 5)^3$ is $2^3 5^3$.

The exponent applies to each of the factors! We can say that the exponent distributes over multiplication. If we were to check various examples with exponents we would find the same pattern emerges. This leads to the *Product to a Power Property for Postive Integer Exponents.*

Product to a Power Property for Integer Exponents

If a and b are real numbers and m is a positive integer, then

$$(ab)^m = a^m b^m.$$

To raise a product to a power, raise each factor to that power.

Example 1.1.4.7

Simplify each expression using the Product to a Power Property:

a. $(-3 \cdot 2)^3$

b. $(6 \cdot 4^3)^2$

c. $(5 \cdot 4^3)^2$

Solution

a.

	$(-3 \cdot 2)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$= (-3)^3 2^3$
Simplify.	$= -27 \cdot 8 = -216$

b.

	$(6 \cdot 4^3)^2$
Use the Power of a Product Property, $(ab)^m = a^m b^m$.	$= 6^2 (4^3)^2$
Use the Power Property, $(a^m)^n = a^{mn}$.	$= 6^2 4^6$
Simplify.	$= 36 \cdot 4^6$

c.

	$(5 \cdot 4^3)^2$
Use the power of a product property, $(ab)^m = a^m b^m$.	$= 5^2 (4^3)^2$
Simplify.	$= 25 \cdot 4^6$

Try It 1.1.4.8

Simplify each expression using the Product to a Power Property:

a. $(2\cdot 3)^5$

b. $(2\cdot(-2)^3)^4$

c. $(8(-2)^4)^2$

Answer

a. $32\cdot 3^5$

b. $16(-2)^{12}=16\cdot 2^{12}=2^{16}$

c. $64(-2)^8=64\cdot 2^8=2^{14}$

Try It 1.1.4.9

Simplify each expression using the Product to a Power Property:

a. $(-3\cdot 2)^3$

b. $(-4\cdot 10^4)^2$

c. $(2\cdot 10^4)^3$

Answer

a. $-27\cdot 2^3=-27\cdot 8=-216$

b. $16\cdot 10^8$

c. $8\cdot 10^{12}$

Because division is multiplication by a recipricol, the exponent must distribute over division as well as multiplication. We give a specific example here:

$$\left(\frac{4}{3}\right)^3=\frac{4}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}=\frac{4^3}{3^3}.$$

So we have also the quotient property.

Quotient to a Power Property for Integer Exponents

If a and $b(\neq 0$ are real numbers and m is a positive integer, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}.$$

To raise a product to a power, raise each factor to that power.

Example 1.1.4.10

Rewrite $\left(\frac{3}{5}\right)^3$ using the Quotient to a Power property

Solution

$$\left(\frac{3}{5}\right)^3 = \frac{4^3}{5^3}.$$

Try It 1.1.4.11

Rewrite $\left(\frac{1}{2}\right)^5$ using the Quotient to a Power property

Answer

$\frac{1}{2^5}$.

Try It 1.1.4.12

Rewrite $\left(\frac{-2}{3}\right)^4$ using the Quotient to a Power property

Answer

$\frac{(-2)^4}{3^4}$. This is equal to $\frac{16}{81}$ but this isn't what is asked here.

Now we will extend the meaning to all integers.

Extending the Meaning of Exponent to Integers

Note that, while so far an exponent that is not a positive integer has no meaning, we see that blindly applying the above properties for such exponents leads to a couple definitions.

To give an idea of the argument we begin with a specific example. So far we do not have a meaning for 7^0 nor for 7^{-3} since the exponents in these examples are not counting numbers. We will assume the rules of exponents apply because we would like the meaning of integer exponents to be consistent with these rules.

We know

	$7^0 7^1$
Using the addition of exponents.	$= 7^{0+1}$
Simplify the exponent.	$= 7^1$
Conclude.	$7^0 7 = 7$

7^0 is the number that when you multiply it by 7 the result is 7.	$7^0 = 1.$

We could replace 7 with any non-zero number and have the same conclusion! So any non-zero number to the zeroth power is 1.

So, in general, **if** the Product Property is also to hold for the exponent zero we **must** define, for $a \neq 0$,

$$a^0 = 1.$$

Now, so far, 7^{-3} has no meaning. But if it did, and that meaning is consistent with the properties of exponents, then, for example

	$7^{-3} 7^3$
Blindly applying the product property.	$= 7^{-3+3}$
Simplify exponent.	$= 7^0$
Using our new definition: $x^0 = 1$.	$= 1$

Draw conclusion.	$7^{-3}7^{3} = 1$
Similarly applying the product property.	$= 7^{-3+3}$
Note the property of the reciprocal.	7^{-3} is the reciprocal of 7^{3}.
Simplify exponent.	$= 7^{0}$
Rewrite in symbols.	$7^{-3} = \frac{1}{7^{3}}$
Using our new definition: $x^0 = 1$.	$= 1$

Draw conclusion.	$7^{-3}7^{3} = 1$
Note the property of the reciprocal.	7^{-3} is the reciprocal of 7^{3}.
Rewrite in symbols.	$7^{-3} = \frac{1}{7^{3}}$

In the argument above, we can replace 7 with any non-zero number and 3 with any positive counting number and -3 with its opposite and arrive at a similar conclustion. So a negative exponent indicates a recipricol of the expression where the exponent is positive.

So, we **must** define

Definition 1.1.4.16

For a any non-zero real number

$$a^0 = 1$$

and for m any positive integer

$$a^{-m} = \frac{1}{a^m} \text{ or, equivalently, } \frac{1}{a^{-m}} = a^m.$$

Example 1.1.4.17

Simplify each expression:

a. 9^0

b. $(-2)^0$

c. $(-4 \cdot 5^2)^0$

d. -3^0

Solution

The definition says any non-zero number raised to the zero power is 1.

a. Use the definition of the zero exponent. $9^0 = 1$

b. Use the definition of the zero exponent. $(-2)^0 = 1$

c. Anything raised to the power zero is 1. Here the base is $-4 \cdot 5^2$, so $(-4 \cdot 5^2)^0 = 1$

d. Anything raised to the power zero is 1. Here the base is 3, so this is the opposite of 3^0, or, the opposite of 1. So, $-3^0 = -1$

Try It 1.1.4.18

Simplify each expression:

a. 11^0

b. $(-620)^0$

c. $(-12 \cdot 5^3 8^2)^0$

d. -7^0

Answer

a. 1

b. 1

c. 1

d. -1

Try It 1.1.4.18

Simplify each expression:

a. 23^0

b. π^0

c. $\left(2\left(\frac{-2}{7}\right)^5\right)^0$

d. -17^0

Answer

a. 1

b. 1

c. 1

d. -1

Example 1.1.4.19

Simplify each expression. Write your answer using positive exponents.

a. 2^{-5}

b. 10^{-3}

c. $\frac{1}{3^{-4}}$

d. $\frac{1}{3^{-2}}$

Solution

a.

	2^{-5}
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=\frac{1}{2^5}$ $=\frac{1}{32}$

b.

	10^{-3}
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=\frac{1}{10^3}$

10^{-3}	
Simplify.	$=\frac{1}{1000}$

c.

$\frac{1}{3^{-4}}$	
Use the definition of a negative exponent, $\frac{1}{a^{-n}}=a^n$.	$=3^4$ $=81$

d.

$\frac{1}{3^{-2}}$	
Use the definition of a negative exponent, $\frac{1}{a^{-n}}=a^n$.	$=3^2$
Simplify.	$=9$

Try It 1.1.4.20

Simplify each expression. Write your answer using positive exponents.

a. 2^{-3}

b. 10^{-7}

c. $\frac{1}{7^{-8}}$

d. $\frac{1}{4^{-3}}$

Answer

a. $\frac{1}{2^3}=\frac{1}{8}$

b. $\frac{1}{10,000,000}$

c. 7^8

d. 64

Try It 1.1.4.21

Simplify each expression. Write your answer using positive exponents.

a. 3^{-2}

b. 10^{-4}

c. $\frac{1}{(-2)2^{-7}}$

d. $\frac{1}{2^{-4}}$

Answer

a. $\frac{1}{9}$

b. $\frac{1}{10,000}$

c. $(-2)^7 = -128$

d. 16

Properties of Negative Exponents

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in *simplest form*. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

	$\left(\frac{3}{4}\right)^{-2}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$=\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$=\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$=\frac{16}{9}$
But we know that	$\frac{16}{9} = \left(\frac{4}{3}\right)^2$
This tells us that	$\left(\frac{3}{4}\right)^{-2} = \left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Integer Exponent Property*.

Quotient to a Negative Integer Exponent Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n.$$

Example 1.1.4.22

Simplify each expression. Write your answer using positive exponents.

$\left(\frac{5}{7}\right)^{-2}$

Solution

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(\frac{7}{5}\right)^{2}$
Simplify.	$=\frac{49}{25}$

? Try It 1.1.4.23

Simplify each expression. Write your answer using positive exponents.

$\left(\frac{2}{3}\right)^{-4}$

Answer

$\frac{81}{16}$

? Try It 1.1.4.24

Simplify each expression. Write your answer using positive exponents.

$\left(\frac{3}{5}\right)^{-3}$

Answer

$\frac{125}{27}$

We would like to verify that the properties of positive integer exponents can be extended to all integer exponents we will postpone this until a later section when we review these properties again in the chapter on polynomials where we will use of variables freely. Though we will postpone the demonstration, it turns out that all properties of exponents that are valid for all counting numbers are also valid for integers. And they are also valid for rational numbers, if such exponents are defined appropriately (we have no meaning at this time for rational exponents in general). We will treat this under the topic of radical expressions.

We provide for reference a table of properties of exponents.

Definition	**Description**
Definition of Zero Exponent	$a^0=1, a\neq 0$
Definition of Negative Exponents	$a^{-n}=\frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}}=a^n$

Property	**Description**
Product Property	$a^m\cdot a^n=a^{m+n}$

Property	Description
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Example 1.1.4.25

Simplify each expression by applying appropriate properties:

a. $(2.5 \times 10^8) \cdot (2 \times 10^4)$

b. $\left(\frac{2}{5}\right)^{-2} + 7^0 - \left(\frac{2}{3}\right)^2$

c. $\left(\frac{2 \cdot 5(-3)^2}{5^3(-3)^{-2}}\right)^2 \left(\frac{12 \cdot 5(-3)^3}{5^3(-3)^{-1}}\right)^{-1}$

Solution

a.

	$(2.5 \times 10^8) \cdot (2 \times 10^4)$
Rearrange product $(2.5 \cdot 2) cdot 10^8 \cdot 10^4$	$(2.5 \cdot 2) \cdot 10^8 \cdot 10^4$
Simplify.	$= 5 \cdot 10^8 10^4$
Use product property of exponents.	$= 5 \cdot 10^{12}$
Rewrite in original format.	$= 5 \times 10^{12}$

b.

	$\left(\frac{2}{5}\right)^{-2} + 7^0 - \left(\frac{2}{3}\right)^2$
Use the quotient to a negative exponent property and the definition of exponent 0.	$\left(\frac{5}{2}\right)^2 + 1 - \left(\frac{2}{3}\right)^2$
Use the meaning of the positive integer exponent.	$= \frac{5}{2}\frac{5}{2} + 1 - \frac{2}{3}\frac{2}{3}$
Multiply fractions.	$= \frac{25}{4} + 1 - \frac{4}{9}$
Add/Subtract fractions (find common denominator) and reduce if possible.	$= \frac{225 + 36 - 16}{36} = \frac{245}{36}$

c.

	$\left(\frac{2 \cdot 5(-3)^2}{5^3(-3)^{-2}}\right)^2 \left(\frac{12 \cdot 5(-3)^3}{5^3(-3)^{-1}}\right)^{-1}$

	$\left(\frac{2\cdot 5(-3)^2}{5^3(-3)^{-2}}\right)^2\left(\frac{12\cdot 5(-3)^3}{5^3(-3)^{-1}}\right)^{-1}$
Simplify inside the parentheses first: use the meaning of a negative exponent and the product property.	$=\left(\frac{2(-3)^2(-3)^2}{5^2}\right)^2\left(\frac{12(-3)^3(-3)^1}{5^2}\right)^{-1}$ $=\left(\frac{2(-3)^4}{5^2}\right)^2\left(\frac{12(-3)^4}{5^2}\right)^{-1}$
Use the quotient to a negative exponent property	$=\left(\frac{2(-3)^4}{5^2}\right)^2\left(\frac{5^2}{12(-3)^4}\right)^1$
Use meaning of the exponent or the quotient property and the Product to a Power Property, $(ab)^m = a^m b^m$.	$=\left(\frac{(2(-3)^4)^2}{(5^2)^2}\right)\cdot\left(\frac{5^2}{12(-3)^4}\right)$ $=\left(\frac{(2^2((-3)^4)^2}{(5^2)^2}\right)\cdot\left(\frac{5^2}{12(-3)^4}\right)$
Use the power property.	$=\left(\frac{(4(-3)^8}{(5^4}\right)\cdot\left(\frac{5^2}{12(-3)^4}\right)$
Multiply fractions.	$=\left(\frac{(4(-3)^8 5^2}{12(-3)^4 5^4}\right)$
Simplify.	$=\frac{(-3)^4}{35^2}$ $=\frac{3^3}{5^2}$ $=\frac{27}{25}$

❓ Try It 1.1.4.26

Simplify each expression by applying appropriate properties:

a. $(9\times 10^7)\div(3\times 10^{-4})$

b. $\left(\frac{3}{2}\right)^{-3} - 2^0 - \left(\frac{1}{2}\right)^{-2}$

c. $\left(\frac{3(-2)5^2}{(-2)^2(5)^{-3}}\right)^2$

Answer

a. $3\times 10^1 1$

b. $\frac{-227}{27}$

c. $\frac{95^{10}}{2^2} = \frac{9\cdot 5^1 0}{4}$ To find the actual value you might want to use a calculator.

❓ Try It 1.1.4.27

Simplify each expression by applying appropriate properties:

a. $\frac{8\times 10^{-3}}{2\times 10^{-7}}$

b. $\left(\frac{2}{3}\right)^{-1}+2^{-2}-\left(\frac{2}{3}\right)^{0}$

c. $\left(\frac{4\cdot 2^{3}3^{2}}{2^{2}3^{-1}}\right)^{2}\left(\frac{8\cdot 23^{-3}}{2^{2}\cdot 3}\right)^{-1}$

Answer

a. 4×10^{4}

b. $\frac{3}{4}$

c. $2\cdot 2^{3}3^{10}=16\cdot 3^{1}0$. To find the value you might want to use a calculator.

Writing Exercises 1.1.4.28

1. How is the negative exponent related to reciprocals? Give an example.
2. How are positive and negative exponents used in science to express large or small numbers?
3. What is the purpose in writing numbers this way?

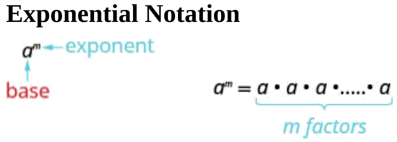
Exit Problem

a. Simplify $\left(\frac{3\cdot 5^{-3}}{(-2)^{-5}}\right)^{-3}$. Write your final answer with positive exponents only.

b. Simplify $\frac{36(-3)^{5}2^{10}}{70(-3)^{15}2^{5}}$. Write your final answer with positive exponents only.

Key Concepts

- **Exponential Notation**



  This is read a to the m^{th} power.
  In the expression a^{m}, the *exponent* m (when positive) tells us how many times we use the *base* a as a factor.
- **Zero Exponent (Definition)** If a is a non-zero number, then $a^{0}=1$.
- **Negative Exponent (Definition)** If n is an integer and $a\neq 0$, then $a^{-n}=\frac{1}{a^{n}}$ or, equivalently, $\frac{1}{a^{-n}}=a^{n}$.
- **Product Property for Exponents**
  If a is a real number and m and n are integers, then

$$a^{m}a^{n}=a^{m+n}$$

  To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
  If a is a real number, $a\neq 0$, and m and n are integers, then

$$\frac{a^{m}}{a^{n}}=a^{m-n},\quad m>n\quad\text{and}\quad\frac{a^{m}}{a^{n}}=\frac{1}{a^{n-m}},\quad n>m$$

- **Quotient to a Negative Exponent Property**
  If a and b are real numbers, $a\neq 0$, $b\neq 0$ and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n}=\left(\frac{b}{a}\right)^{n}$$

- **Power Property for Exponents**
  If a is a real number and m and n are integers, then
  $$(a^m)^n = a^{mn}$$
  To raise a power to a power, multiply the exponents.
- **Product to a Power Property for Exponents**
  If a and b are real numbers and m is a whole number, then
  $$(ab)^m = a^m b^m$$
  To raise a product to a power, raise each factor to that power.
- **Quotient to a Power Property for Exponents**
  If a and b are real numbers, $b \neq 0$, and m is an integer, then
  $$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$
  To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
  If a and b are real numbers, and m and n are integers, then
-

Property	Description
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

-

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

SECTION OVERVIEW

1.2: Polynomials

Topic hierarchy

1.2.1: Linear Expressions

Learning Objectives

By the end of this section, you will be able to:

- Recognize a linear expression
- Simplify sums and differences of linear expressions
- Evaluate a linear expression for any value of the variable

Be Prepared

Before we get started, take this readiness quiz.

1. Evaluate $5(-1)-4$.

2. Evaluate $3(4)-2(-3)$.

3. Evaluate $-3(-2)+2(-1)-5(2)-3(-3)$.

Linear Expressions with One Variable

Definition 1.2.1.1

An expression that can be written as

$$Ax+B$$

with A and B numbers, is called a **linear expression (with one variable)**, or more specifically, a **linear expression in** x. Ax and B are called **terms**. A is called the **coefficient** of x and B is called the **constant coefficient**.

Example 1.2.1.2

Decide if the following are linear expressions. If so, identify A and B such that $Ax+B$ is a linear expression.

a. $\frac{2}{3}x-4$

b. 3

c. $x\cdot x-2x$

Solution

a. $\frac{2}{3}x-4=\frac{2}{3}x+(-4)$ is linear with $A=\frac{2}{3}$ and $B=-4$.

b. $3=0\cdot x+3$ is linear with $A=0$ and $B=3$.

c. $x\cdot x-2x$ is not linear because it is not of the form $Ax+B$ where A and B are numbers.

Try It 1.2.1.3

Decide if the following are linear expressions. If so, identify A and B such that $Ax+B$ is a linear expression.

a. $-\frac{3}{4}x$

b. $4-2x$

c. $x\cdot x\cdot x-2x$

Answer

a. Yes, $A=-\frac{3}{4}$ and $B=0$.

b. Yes, $A=-2$ and $B=4$.

c. No.

We are interested in such expressions because they occur frequently in applications. The one we will concern ourselves with at this time is if x represents some variety of numbers. For example, suppose we are interested in constructing a fenced region that has as one side a preexisting wall and the length of the region is 2 less than the width. Then we don't know what we want the width to be, so we call it x (we could call it something else if we wanted) and then we can express the length of fencing we need as $3x+(-2)$. Now we can easily find the length of fence needed for various widths.

We will now treat this expression more abstractly (outside the context of the application above). To arrive at the expression $3x+(-2)$ we multiply a variable x (an unknown quantity which we call 'x') by the number 3 and then add another number -2. We can evaluate this expression for different values of x:

Values of x	Evaluating $3x+(-2)$ at x
$x=2$	$\begin{aligned}&3x+(-2)\\=&3(2)+(-2)\\=&6+(-2)\\=&4\end{aligned}$
$x=-1$	$\begin{aligned}&3x+(-2)\\=&3(-1)+(-2)\\=&-3+(-2)\\=&-5\end{aligned}$
$x=\frac{1}{2}$	$\begin{aligned}&3x+(-2)\\=&3\cdot\frac{1}{2}+(-2)\\=&\frac{3}{2}+(-2)\\=&\frac{3}{2}+\left(-\frac{4}{2}\right)\\=&-\frac{1}{2}\end{aligned}$

When we have a linear expression with a variable 'x', any value for 'x' is appropriate unless an application limits the meaningfulness of the expression. In the application above, only numbers greater than 2 make sense (why?).

We can add linear expressions and get another linear expression. To understand how this works you should treat x as a number:

$(3x-2)+(-2x+7)=3x+(-2x)+(-2)+7$ (because you can add in any order) means that you have 3 x's and you add -2 x's or, equivalently, you have 3 x's and you take away 2 x's, leaving you with one x, and you have -2 (units) and you add 7 (units) giving you 5. So the result is the sum $x+5$.

As it is with numbers, to subtract we add the 'opposite', so we therefore need to know how to find the opposite of a linear expression. The opposite of $A+Bx$ is $-A+(-B)x$ because when you add these together you get zero: $A+Bx+(-A)+(-B)x=A+(-A)+Bx+(-B)x=0$.

To demonstrate the process of subtraction, we consider the example: $(3x-2)-(-2x+7)$. The opposite of $-2x+7$ is $2x+(-7)=2x-7$ (you can check that adding these two expressions result in zero). So,

$$\begin{aligned}&(3x-2)-(-2x+7)\\=&3x-2+2x-7\\=&3x+2x-2-7\\=&5x-9.\end{aligned}$$

The last equality is because 3 x's plus 2 x's gives you 5 x's (for the same reason that having 3 dimes plus 2 dimes is the same as having 5 dimes). The process of arriving at the last equality is known as 'collecting like terms'.

This example also shows that not all linear expressions look like $Ax+B$ but those that don't can be written so that any evaluation for the original and $Ax+B$ for well chosen A and B is the same for any value of x. For example:

Values of x	Evaluating $(3x-2)-(-2x+7)$ at x	Evaluating $5x-9$ at x
$x=-2$	$(3x-2)-(-2x+7)$ $=3(-2)-2-(-2(-2)+7)$ $=-6-2-(4+7)$ $=-8-11$ $=-19$	$5x-9$ $=5(-2)-9$ $=-10-9$ $=-19$
$x=3$	$(3x-2)-(-2x+7)$ $=3(3)-2-(-2(3)+7)$ $=9-2-(-6+7)$ $=7-1$ $=6$	$5x-9$ $=5(3)-9$ $=15-9$ $=6$

In fact, since we didn't rely on x being any particular value, we can check ANY value of x and find that the two expressions give the same answer. We say that $(3x-2)-(-2x+7)$ is equivalent to $5x-9$. Since we can write $(3x-2)-(-2x+7)$ in the form $Ax+B$, with $A=5$ and $B=9$, we see that $(3x-2)-(-2x+7)$ is linear.

In general, if we can operate with a number in a certain way, we can do the same with a variable.

We can multiply a linear expression by a number and still have a linear expression.

For example, $2(2x+1)$ is $2x+1+2x+1=2(2x)+2(1)$. This is to say that we can 'distribute' the multiplication by 2 'over addition and subtraction', that is, we can multiply 2 by each term (understanding $2x$ and 1 as the *terms* of the linear expression $2x+1$).

So,

$$2(2x+1)=2(2x)+2(1)=4x+2$$

which is of the form $Ax+B$ where $A=4$ and $B=2$.

We noted above that $2(2x+1)$ is $2x+1+2x+1=2(2x)+2(1)$. The analagous statement would be true if that 2 were replaced by any number!

For example $7(2x+1)=7(2x)+7(1)$ and $\frac{8}{3}(2x+1)=\frac{8}{3}(2x)+\frac{8}{3}(1)$.

Example 1.2.1.4

Show $3(-3x+2)-2(x-1)+4x$ is a linear expression.

Solution

$$\begin{aligned}&3(-3x+2)-2(x-1)+4x\\=&-9x+2+(-2)x+(-2)(-1)+4x\\=&-9x+(-2)x+4x+2+2\\=&-11x+4\end{aligned}$$

Since we have written this in the form $Ax+B$ where A and B are numbers, we see this is a linear expression.

Try It 1.2.1.5

Show $-2(3x-7)-5(-2x+3)+4(-x-1)$ is a linear expression.

Answer

-5

Try It 1.2.1.6

Show $-\frac{2}{3}(3x-12)+\frac{1}{2}(-2x+3)+(-x-\frac{1}{3})$ is a linear expression.

Answer

$$-4x+\frac{19}{3}$$

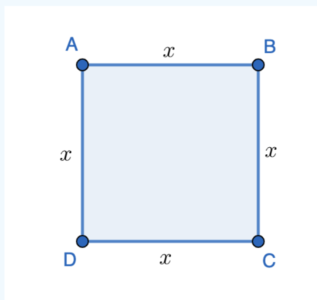
We now consider some applications.

Example 1.2.1.7

Write an expression that gives the perimeter of a square whose side is length x. Use the expression to find the perimeter of a square with side lengths 2 cm.

Solution

First we draw and label a picture which represents the situation:



The perimeter is the distance around the square which in this case is $x+x+x+x=4x$.

If $x=2$ cm. then the perimeter is $4 \cdot 2=8$ cm.

Try It 1.2.1.8

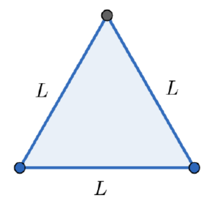
a. Give an expression for the perimeter of a equilateral triangle whose sides have length L. Use the formula to find the perimeter of a triangle whose sides are length 5 ft.

b. If a string has length L and we cut off 5 in, what is the remaining length in terms of L? Use the formula to find the length remaining if we cut 5 in off a string of length 7 in.

c. If we make a box of height x, with no top, by cutting square corners from a 30 in by 30 in piece or cardboard and folding them up what are the dimensions of the base of the box in terms of x? What is the biggest x can be?

Answer

a. $3L$; 15ft



b. $L-5$; $7-5=2\text{in}$.

c. $30-2x$; x must be smaller than 15in

Linear Expressions with Two Variables

Sometimes there is more than one unknown quantity or more than one quantity that has varying values. This leads us to consider the simplest type of expression with two variables, namely, a linear expression. The expression $x-4y+8$ is an example of a linear expression with two variables. A linear expression in two variables is formed by multiplying each variable by a number and then adding or subtracting the results.

Definition 1.2.1.9

An expression that can be written as

$$Ax+By+C$$

with A and B real numbers, not both zero, is called a **linear expression (with two variables)**, or more specifically, a **linear expression in x and y.**

Here is an example of a linear expression with two variables, x and y.

Example 1.2.1.10

Evaluate the linear expression $x-4y+8$ at

a. $x=-1$ and $y=5$

b. $x=2$ and $y=0$

Solution

This expression is in the form $Ax+By+C$ with $A=1$, $B=-4$, and $C=8$.

We can evaluate this expression for different values of x and y. For example:

Values of x and y	Evaluating $x-4y+8$ at x and y
$x=-1$ and $y=5$	$x-4y+8$ $=(-1-4(5)+8)$ $=-1-20+8$ $=-13$
$x=2$ and $y=0$	$x-4y+8$ $=(2-4(0)+8)$ $=2+8$ $=10$

Try It 1.2.1.11

Evaluate the linear expression $3x+2y-15$ at

a. $x=3$ and $y=0$

b. $x=-5$ and $y=-1$

c. $x=0$ and $y=2$

Answer

a. -6

b. -32

c. -11

We can add two linear expressions to get another linear expression. For example,

$$\begin{aligned} & x-4y+8+(-2x+y-6) \\ = & -x-3y+2, \end{aligned}$$

using the fact that we can add in any order and 1 x -2 x's is -1 x (if you have 1 dime and take away 2 dimes, you are down one dime), and -4 y's plus 1 y's is -3 y's (if you have have loaned 4 quarters and then got 1 quarters, you have loaned a total of 3 quarters).

The expression $x-4y+8+(-2x+y-6)$ is equivalent to $-x-3y+2$. The expression is of the form $Ax+By+C$, with $A=-1, B=-3$, and $C=2$, so $x-4y+8+(-2x+y-6)$ is a linear expression in two variables. We will check this equivalence for a couple of different values for x and y.

Values of x and y	Evaluating $x-4y+8+(-2x+y-6)$ at x and y	Evaluating $-x-3y+2$ at x and y
$x=-1$ and $y=2$	$\begin{aligned} & x-4y+8+(-2x+y-6) \\ = & (-1)-4(2)+8+(-2(-1)+2-6) \\ = & -1-8+8+2+2-6 \\ = & =-3 \end{aligned}$	$\begin{aligned} & -x-3y+2 \\ = & -(-1)-3(2)+2 \\ = & 1-6+2 \\ = & -3 \end{aligned}$
$x=1$ and $y=3$	$\begin{aligned} & x-4y+8+(-2x+y-6) \\ = & 1-4(3)+8+(-2(1)+3-6) \\ = & 1-12+8-2+3-6 \\ = & -8 \end{aligned}$	$\begin{aligned} & -x-3y+2 \\ = & -(1)-3(3)+2 \\ = & -1-9+2 \\ = & -8 \end{aligned}$

To subtract a linear expression $Ax+By+x$ in two variables, we must add the opposite $-Ax+(-B)y+-C=-Ax-By-C$.

For example,

$3x-2y-2-(2x-4y+2)=3x-2y-2+(-2)x+4y-2=3x-2x-2y+4y-2-2=x+2y-4.$

As in the linear expression with one variable, we can multiply a linear expression by a number and still have a linear expression.

For example, $2(2x-3y+1)$ is $2(2x)-2(3y)+2(1)$. This is to say that we can 'distribute' the multiplication by 2 'over addition and subtraction', that is, we can multiply 2 by each term. The same would be true if that 2 were replaced by any number!

So,

$$\begin{aligned} & 2(2x-3y+1) \\ = & 2(2x)-2(3y)+2(1) \\ = & 4x-6y+2 \end{aligned}$$

which is of the form $Ax+By+C$ where $A=4$, $B=-6$ and $C=2$.

Example 1.2.1.12

Simplify:

a. $3(x-3y+7)$

b. $3x-7y+2-(2x+y-1)$

c. $2a-4b+2(5b-2a-1)$

Solution

a.

	$3(x-3y+7)$
Distribute.	$=3x-(3)3y+3\cdot 7$
Multiply coefficients.	$=3x-9y+3$

b.

	$3x-7y+2-(2x+y-1)$
Distribute.	$=3x-7y+2-2x-y+1$
Collect like terms.	$=x-8y+3$

c.

	$2a-4b+2(5b-2a-1)$
Distribute.	$=2a-4b+2(5b)-2(2a)-2(1)$
Simplify.	$=2a-4b+10b-4a-2$
Collect like terms.	$=-2a+6b-2$

Try It 1.2.1.13

Simplify:

a. $-3(2x-y-2)$

b. $-(3x-7y+2)+(2x+y-1)$

c. $3(2a-4b)-2(5b-2a-1)$

Answer

a. $-6x+3y+6$

b. $-x+8y-3$

c. $10a-22b+2$

Example 1.2.1.14

Draw an appropriate picture and express the following quantities in terms of appropriate variables

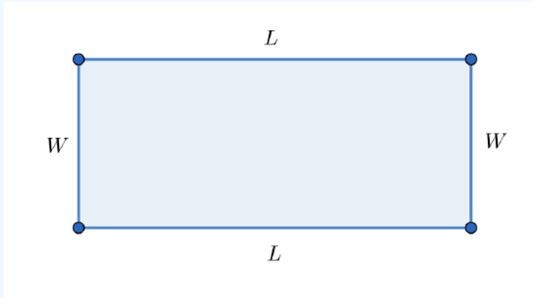
a. The perimeter of a rectangle of length L and width W.

b. The amount of fencing needed to enclose an area up against a wall whose length is L feet and width is W feet. Suppose that the side consisting of the wall is length L feet.

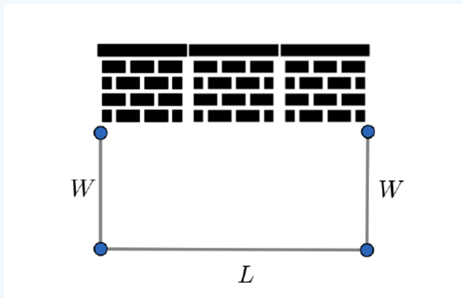
c. If we make a box of height 5 in, with no top, by cutting square corners from a L in by W in piece or cardboard and folding them up what are the dimensions of the base of the box? What is the biggest L can be?

Solution

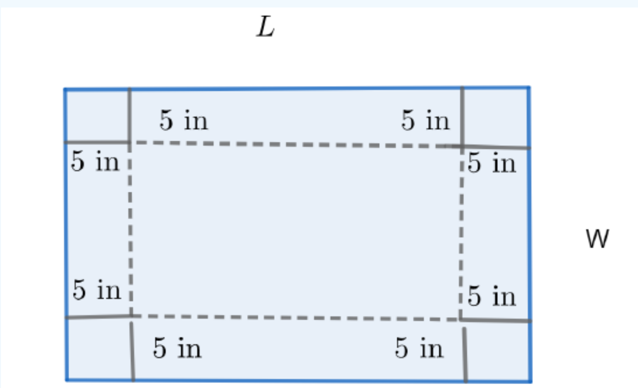
a. $L+W+L+W=2L+2W$



b. $W+L+W$



c. Draw a picture (see below). We are interested in the dimensions of the dotted figure. We see that it must be $L-10$ by $W-10$; L must be at least 10 but there is no upper bound to its length.

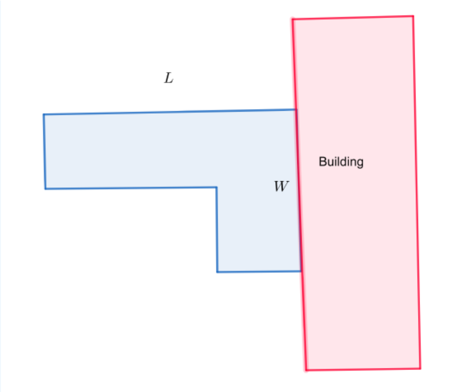


? Try It 1.2.1.15

Draw an appropriate picture and express the following quantities in terms of appropriate variables

a. The perimeter of an isosceles triangle with two sides of length x and one of length y.

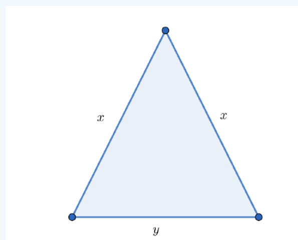
b. The amount of fencing needed to enclose an "L" shaped area below.



c. If we make a box of height x in, with no top, by cutting square corners from an L in by 10 in piece or cardboard and folding them up what are the dimensions of the base of the box? What is the biggest x can be if $L = 12$ in?

Answer

a. $2x + y$



b. $L + W + L$

c. $L - 2x$ by $10 - 2x$; if $L = 12$, the box is $12 - 2x$ by $10 - 2x$ so x must be less than 5 in.

Linear Expressions in more than Two Variables

This idea can be extended in a clear way to more than two variables leading to expressions that look like $2x - 3y + 3z - 6w + 2$. We won't discuss this any further because the concepts are similar to the above.

Additional Notes

Note that the linear expression of the standard forms given in the definitions are easy to evaluate. The process of writing an expression that is 'harder' to evaluate as an equivalent expression that is 'easier' to evaluate is called **simplification**. This concept of simplification is sometimes depends on the context or application involved.

Also, note that we have named our variables here x and y but it doesn't matter what you name them. You could use 'earth' and 'mars' or anything else that may be convenient. Usually the names are short (one letter if possible) because it saves a lot of writing.

Note that A, B, and C don't have to be integers. Important algebraic concepts that can be learned here in the process of simplification of linear expressions are

- What it means to collect like terms
- The 'opposite' of an expression is the one obtained by negating each term
- To subtract an expression, we add its opposite
- We can distribute multiplication over addition and subtraction

Key concepts

- **Variable**
- **Linear expressions in one and two variables**
- **Collecting like terms**
- **Opposite of a linear expression**

- **Adding and subtracting linear expressions**
- **Distribution of multiplication over addition and subtraction**
- **Simplification**

1.2.2: Evaluating, Adding and Subtracting Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Determine the degree of polynomials
- Add and subtract polynomials
- Evaluate a polynomial

Be Prepared

Before you get started, take this readiness quiz.

1. Subtract $(5n+8)-(2n-1)$.

2. Evaluate $4(-2)(5)-2(3)(3)$.

Positive Integer Exponents

Remember that a positive integer exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the positive integer *exponent*

m tells us how many times we use the *base* a as a factor.

$$a^m = \underbrace{a \cdot a \cdot \cdots \cdot a}_{ma's}$$

For example,

$$(-9)^5 = (-9)(-9)(-9)(-9)(-9).$$

Let's review the vocabulary for expressions with exponents.

Definition 1.2.2.1

$$a^m = \underbrace{a \cdot a \cdots a}_{ma's}$$

This is read a to the mth (**power**), or a to the **power** m.

In the expression a^m with positive integer m and $a \neq 0$, the **exponent** m tells us how many times we use the **base** a as a factor.

Example 1.2.2.2

a. Evaluate 2^3.

b. Evaluate -7^2.

c. Evaluate $(-1)^4$.

d. Write using exponents: $-2 \cdot 2 \cdot 2$.

e. Identify the base and the exponent: -4^3.

Solution

a. $2^3 = 2 \cdot 2 \cdot 2 = 8$

b. $-7^2 = -49$

c. $(-1)^4 = (-1) \cdot (-1) \cdot (-1) \cdot (-1) = 1$

d. $-2 \cdot 2 \cdot 2 = -2^3$

e. -4^3 has an exponent of 3 and the base is 4 since there are no parentheses that indicate including the "-".

Try It 1.2.2.3

a. Evaluate 3^4.

b. Evaluate -2^4.

c. Evaluate $(-2)^3$.

d. Write using exponents $-6 \cdot 6 \cdot 6 \cdot 6$.

e. Identify the base and the exponent: $-2 \cdot 5^7$.

Answer

a. 81

b. -16

c. -8

d. -6^4

e. The exponent is 7 and the base is 5 (since there are no parentheses that would include 2 or -2).

Polynomials

Just like we can add, subtract and multiply numbers, we can also do these things with variables (holding places for numbers). We will next define some words that help us highlight features of resulting expressions, and then, as we did in the case of a linear expression, we will see how to add and subtract.

Definition 1.2.2.4

A **monomial** is an expression formed by multiplying variables and numbers. A **polynomial** is a sum of monomials. We could also say that a **polynomial** is an expression formed by adding (or subtracting) and/or multiplying numbers and variables together.

A **term** of a polynomial is a monomial that is combined with other monomials using addition or subtraction.

- A polynomial with exactly two terms is called a **binomial**.
- A polynomial with exactly three terms is called a **trinomial**.

The **coefficient** of a term is the number multiplying the product of variables which are then combined via addition.

Here are some examples of polynomials.

Polynomial	$y+1$	$\mathbf{4a^2-7ab+2b^2}$	$4x^4+x^3+8x^2-9x+1$	0
Monomial	14	$8y^2$	$-9x^3y^5$	$-13a^3b^2c$
Binomial	$a+7ab+7b$	$4x^2-y^2$	y^2-16	$3p^3q-9p^2q$
Trinomial	$x^2-7x+12$	$\mathbf{9m^2+2mn-8n^2}$	$\mathbf{6k^4-k^3+8k}$	z^4+3z^2-1

Notice that every monomial, binomial, and trinomial is also a polynomial. They are just special members of the “family” of polynomials and so they have special names. We use the words *monomial*, *binomial*, and *trinomial* when referring to these special polynomials and just call all the rest *polynomials*.

The **degree of a polynomial** and the degree of its terms are determined by the exponents of the variable. A monomial that has no variable, just a constant, is a special case. The **degree of a constant** is 0.

Definition 1.2.2.5

- The **degree of a term** is the sum of the exponents of its variables.
- The **degree of a constant** is 0.
- The **degree of a polynomial** is the highest degree among all its terms.
- A polynomial of degree 1 is called a **linear** expression or polynomial.
- A polynomial of degree 2 is called a **quadratic** expression or polynomial.

Note that linear expressions from the last chapter are polynomials of degree 1 or 0 (in the case where there is no variable!). Also, linear expressions with one variable are either binomials or monomials. How many terms can a linear expression with two variables have?

Let's see how this works by looking at several polynomials. We'll take it step by step, starting with monomials, and then progressing to polynomials with more terms. Let's start by looking at a monomial. The monomial $8ab^2$ has two variables a and b. To find the degree we need to find the sum of the exponents. The variable a doesn't have an exponent written, but remember that means the exponent is 1. The exponent of b is 2. The sum of the exponents, $1+2$, is 3, so the degree is 3. The coefficient of the term is 8. Unless it is clear from the context, you should specify the term for which a number is the coefficient in some way. For example, 8 is the coefficient of the ab^2-term.

Monomial	$8ab^2$
Coefficient	8
Variables	a, b
Exponent of a	1
Exponent of b	2
Degree of the monomial	$1+2=3$

Here are some additional examples.

Monomial	14	$8ab^2$	$-9x^3y^5$	$-13a$
Term	14	$8ab^2$	$-9x^3y^5$	$-13a$
Coefficient of term	14	8	-9	-13
Degree of the monomial	0	3	8	1

Binomial	$h+7$	$7b^2-3b$	x^2y^2-25	$4n^3-8n^2$
Terms	h, 7	$7b^2$, $-3b$	x^2y^2, -25	$4n^3$, $-8n^2$
Coefficients or respective terms	1, 7	7, -3	1, -25	4, -8
Degree of respective terms	1,0	2,1	4,0	3,2
Degree of the binomial	1	2	4	3

Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Terms	x^2, $-12x$, 27	$9x^2$, $6ab$, b^2	$6m^4$, $-m^3n^2$, $8mn^5$	z^4, $3z^2$, -1
Coefficient of respective terms	1, -12, 27	9, 6, 1	6, -1, 8	1, 3, -1

Trinomial	$x^2-12x+27$	$9a^2+6ab+b^2$	$6m^4-m^3n^2+8mn^5$	z^4+3z^2-1
Degree of respective terms	2,1,0	2,2,2	4,5,6	4,2,0
Degree of the trinomial	2	2	6	4

Polynomial	$y-1$	$3y^2-2y-5$	$4x^4+x^3+8x^2-9x+1$
Terms	y, -1	$3y^2$, $-2y$, -5	$4x^4$, x^3, $8x^2$, $-9x$, 1
Coefficient of respective terms	1, -1	3, -2, -5	4, 1, 8, 9, 1
Degree of respective terms	1,0	2,1,0	4,3,2,1,0
Degree of the polynomial	1	2	4

Working with polynomials is easier (because its form is consistent) when you list the terms in descending order of degrees. When a polynomial is written this way, it is said to be in **standard form of a polynomial**. Get in the habit of writing the term with the highest degree first. This can be arranged because you can change the order in which terms are added (carry '-' signs along with the terms when rearranging).

For example,

$$\begin{aligned} & -5x^3-2x-x^5+3 \\ = & -5x^3+(-2x)+(-x^5)+3 \\ = & (-x^5)+(-5x^3)+(-2x)+3 \\ = & -x^5-5x^3-2x+3 \end{aligned}$$

so, the standard form of $-5x^3-2x-x^5+3$ is either $-x^5+(-5x^3)+(-2x)+3$ or, equivalently, $-x^5-5x^3-2x+3$.

Example 1.2.2.6

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $7y^2-5y+3$

b. $-2a^4b^2$

c. $3x^5-4x^3-6x^2+x-8$

d. $2y-8xy^3$

e. 15

Solution

	Polynomial	Number of terms	Type	Degree of each term	Degree of the polynomial
a.	$\boldsymbol{7y^2-5y+3}$	3	Trinomial	2, 1, 0	2
b.	$-2a^4b^2$	1	Monomial	6	6
c.	$3x^5-4x^3-6x^2+x-8$	5	Polynomial	5, 3, 2, 1, 0	5
d.	$\boldsymbol{2y-8xy^3}$	2	Binomial	1, 4	4
e.	15	1	Monomial	0	0

Try It 1.2.2.7

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. -5

b. $8y^3-7y^2-y-3$

c. $-3x^2y-5xy+9xy^3$

d. $81m^2-4n^2$

e. $-3x^6y^3z$

Answer a

It is a monomial of degree 0.

Answer b

It is a polynomial of degree 3.

Answer c

It is a trinomial of degree 4.

Answer d

It is a binomial of degree 2.

Answer e

It is a monomial of degree 10.

Try It 1.2.2.8

Determine whether each polynomial is a monomial, binomial, trinomial, or other polynomial. Then, find the degree of each polynomial.

a. $64k^3-8$

b. $9m^3+4m^2-2$

c. 56

d. $8a^4-7a^3b-6a^2b^2-4ab^3+7b^4$

e. $-p^4q^3$

Answer

a. It is a binomial of degree 3.

b. It is a trinomial of degree 3.

c. It is a monomial of degree 0.

d. It is a polynomial of degree 4.

e. It is a monomial of degree 7.

Evaluating Polynomials and the Order of Operations

When we have an expression that involves exponents, we should evaluate the exponents before multiplying, dividing, adding or subtracting. Also, parentheses should be evaluated before anything else unless you use the distribution property.

So, for example,

$3(-2)^3-2\cdot3^2-7$ $=3(-8)-2(9)-7$ $=-24-18-7$ $=-49$	$2-7(2-3)^3$ $=2-7(-1)^3$ $=2-7(-1)$ $=2+7$ $=9$ (Note here that you can not distribute the 7 because exponents come before multiplication in the order of operations.)	$2\left(\frac{3}{2}\right)^3-2$ $=2\cdot\frac{27}{4\cdot2}-2$ $=\frac{2\cdot27}{4\cdot2}-2$ $=\frac{27}{4}-2\cdot\frac{4}{4}$ $=\frac{27}{4}-\frac{8}{4}$ $=\frac{27-8}{4}$ $=\frac{19}{4}$ or $4\frac{3}{4}$.

Note that for the purpose of arithmetic improper fractions are preferable to mixed numbers, but once you want to estimate the value or find the quantity on the number line you may find mixed numbers more convenient. Occasionally you may find addition and subtraction can be simplified by a mixed approach.

Just as in the case of linear expressions (polynomials of degree 1) we can evaluate polynomials by substituting in values of the variables. For example,

Polynomial	Values of the variables	Evaluating the polynomial
$3x^3-2x^2-4$	$x=-2$	$3x^3-2x^2-4$ $=3(-2)^3-2(-2)^2-4$ $=3(-8)-2(4)-4$ $=-24-8-4$ $=-36$
$-a^2b+ab-4$	$a=2, b=-1$	$-a^2b+ab-4$ $=-(2)^2(-1)+(2)(-1)-4$ $=-4(-1)-2-4$ $=4-2-4$ $=-2$
b^2-4ac	$a=2,b=-2, c=1$	b^2-4ac $=(-2)^2-4(2)(1)$ $=4-8$ $=-4$

Example 1.2.2.9

Evaluate the trinomial $5x^2-8x+4$ at:

a. $x=4$

b. $x=-2$

c. $x=0$

Solution

a.

	$5x^2-8x+4$
Substitute 4 for x.	$5(4)^2-8(4)+4$
Simplify the exponents.	$=5(16)-8(4)+4$
Multiply.	$=80-32+4$
Simplify.	$=52$

b.

	$5x^2-8x+4$
Substitute -2 for x.	$5(-2)^2-8(-2)+4$
Simplify the exponents.	$=5(4)-8(-2)+4$
Multiply.	$=20+16+4$
Simplify.	$=40$

c.

	$5x^2-8x+4$
Substitute 0 for x.	$5(0)^2-8(0)+4$
Simplify the exponents.	$=5(0)-8(0)+4$
Multiply.	$=0+0+4$
Simplify.	$=4$

? Try It 1.2.2.10

Evaluate the trinomial $3x^2+2x-15$ at:

a. $x=3$

b. $x=-5$

c. $x=0$

Answer

a. 18

b. 50

c. -15

? Try It 1.2.2.11

Evaluate the trinomial $5x^2-x-4$ at:

a. $x=-2$

b. $x=-1$

c. $x=0$

Answer

a. 18

b. 2

c. -4

Adding and Subtracting Polynomials

Just like we could add and subtract linear expressions (polynomials of degree 1) we can add and subtract polynomials in general by combining like terms.

Two monomials are 'like' if one monomial is a constant times the other. For example,

$3x^2y^5$ and $-2y^5x^2$ are like because x^2y^5 is equivalent to y^5x^2 since multiplication of numbers and therefore variables can be done in any order (for example, $2^2 3^5 = 3^5 2^2$) and so $3x^2y^5 = \frac{3}{-2}(-2y^5x^2)$.

You can also think of like terms as terms that have the same variables to the same exponents so that the 'variable parts' are equivalent. Or, two terms are like if they are equivalent aside from their coefficients (numbers multiplying the variables).

We can add and subtract like terms. For example,

$$3x^2y^5 + (-2y^5x^2) = x^2y^5 \qquad \text{(3 dimes - 2 dimes is 1 dime)}$$

and

$$3x^2y^5 - (-2y^5x^2) = 5x^2y^5 \qquad \text{(3 dimes + 2 dimes is 5 dimes).}$$

If the monomials are like terms, we just combine them by adding or subtracting their coefficients.

Example 1.2.2.12

Add or subtract:

a. $25y^2 + 15y^2$

b. $16pq^3 - (-7pq^3)$

Solution

a.

	$25y^2 + 15y^2$
Identify like terms.	The like terms are $25y^2$ and $15y^2$.
Combine like terms.	$= 40y^2$

b.

	$16pq^3 - (-7pq^3)$
Rewrite without the parentheses.	$= 16pq^3 + 7pq^3$
Identify like terms.	The like terms are $16pq^3$ and $7pq^3$.
Combine like terms.	$= 23pq^3$

Try It 1.2.2.13

Add or subtract:

a. $12q^2 + 9q^2$

b. $8mn^3 - (-5mn^3)$

Answer

a. $21q^2$

b. $13mn^3$

Try It 1.2.2.14

Add or subtract:

a. $-15c^2 + 8c^2$

b. $-15y^2z^3 - (-5y^2z^3)$

Answer

a. $-7c^2$

b. $-10y^2z^3$

Remember that like terms must have the same variables with the same exponents.

Example 1.2.2.15

Simplify:

a. $a^2+7b^2-6a^2$

b. $u^2v+5u^2-3v^2$

Solution

a.

	$a^2+7b^2-6a^2$
Identify like terms.	The like terms are a^2 and $-6a^2$.
Combine like terms.	$=-5a^2+7b^2$

b.

	$u^2v+5u^2-3v^2$
Identify like terms.	none
Combine like terms.	There are no like terms to combine. In this case, the polynomial is unchanged. $u^2v+5u^2-3v^2$

Try It 1.2.2.16

Add:

a. $8y^2+3z^2-3y^2$

b. $m^2n^2-8m^2+4n^2$

Answer

a. $5y^2+3z^2$
b. $m^2n^2-8m^2+4n^2$

Try It 1.2.2.17

Add:

a. $3m^2+n^2-7m^2$

b. $pq^2-6p-5q^2$

Answer

a. $-4m^2+n^2$
b. $pq^2-6p-5q^2$

We can think of adding and subtracting polynomials as just adding and subtracting a series of monomials. Look for the like terms—those with the same variables and the same exponent. The Commutative Property allows us to rearrange the terms to put like terms together.

Example 1.2.2.18

Find the sum $(7y^2-2y+9)+(4y^2-8y-7)$.

Solution

	$(7y^2-2y+9)+(4y^2-8y-7)$
Rewrite without parentheses.	$=7y^2-2y+9+4y^2-8y-7$
Identify like terms.	The like terms are • $7y^2$ and $4y^2$ • $-2y$ and $-8y$ • 9 and -7
Rearrange to get the like terms together.	$=7y^2+4y^2-2y-8y+9-7$
Combine like terms.	$=11y^2-10y+2$

Try It 1.2.2.19

Find the sum $(7x^2-4x+5)+(x^2-7x+3)$.

Answer

$8x^2-11x+8$

Try It 1.2.2.20

Find the sum $(14y^2+6y-4)+(3y^2+8y+5)$.

Answer

$17y^2+14y+1$

Be careful with the signs as you distribute while subtracting the polynomials in the next example. As with linear expressions, we subtract a polynomial by adding its opposite which is obtained by negating each term.

Example 1.2.2.21

Find the difference $(9w^2-7w+5)-(2w^2-4)$.

Solution

	$(9w^2-7w+5)-(2w^2-4)$
Rewrite without parentheses.	$=9w^2-7w+5-2w^2+4$

	$(9w^2 - 7w + 5) - (2w^2 - 4)$
Rewrite without parentheses.	$= 9w^2 - 7w + 5 - 2w^2 + 4$
Distribute and identify like terms.	The like terms are: • $9w^2$ and $-2w^2$ • 5 and 4
Rearrange to get the like terms together.	$= 9w^2 - 2w^2 - 7w + 5 + 4$
Combine like terms.	$= 7w^2 - 7w + 9$

Distribute and identify like terms. The like terms are:
- $9w^2$ and $-2w^2$
- 5 and 4

Rearrange to get the like terms together. $= 9w^2 - 2w^2 - 7w + 5 + 4$

Combine like terms. $= 7w^2 - 7w + 9$

Try It 1.2.2.22

Find the difference $(8x^2 + 3x - 19) - (7x^2 - 14)$.

Answer

$x^2 + 3x - 5$

Try It 1.2.2.23

Find the difference $(9b^2 - 5b - 4) - (3b^2 - 5b - 7)$.

Answer

$6b^2 + 3$

Example 1.2.2.24

Subtract $p^2 + 10pq - 2q^2$ from $p^2 + q^2$.

Solution

	$(p^2 + q^2) - (p^2 + 10pq - 2q^2)$
Rewrite without parentheses.	$= p^2 + q^2 - p^2 - 10pq + 2q^2$
Identify like terms.	The like terms are: • p^2 and $-p^2$ • q^2 and $2q^2$
Rearrange to get like terms together.	$= p^2 - p^2 + q^2 + 2q^2 - 10pq$
Combine like terms.	$= -10pq + 3q^2$

Try It 1.2.2.25

Subtract $a^2 + 5ab - 6b^2$ from $a^2 + b^2$.

Answer

$-5ab + 7b^2$

Try It 1.2.2.26

Subtract $m^2 - 7mn - 3n^2$ from $m^2 + n^2$.

Answer

$7mn + 4n^2$

Example 1.2.2.27

Find the sum $(u^2-6uv+5v^2)+(3u^2+2uv)$.

Solution

	$(u^2-6uv+5v^2)+(3u^2+2uv)$
Distribute and identify like terms.	$=\underline{\underline{u^2}}\underline{-6uv}+5v^2+\underline{\underline{3u^2}}+\underline{2uv}$
Rearrange the terms to put like terms together.	$=\underline{\underline{u^2+3u^2}}\underline{-6uv+2uv}+5v^2$
Combine like terms.	$=4u^2-4uv+5v^2$

Try It 1.2.2.28

Find the sum $(3x^2-4xy+5y^2)+(2x^2-xy)$.

Answer

$5x^2-5xy+5y^2$

Try It 1.2.2.29

Find the sum $(2x^2-3xy-2y^2)+(5x^2-3xy)$.

Answer

$7x^2-6xy-2y^2$

When we add and subtract more than two polynomials, the process is the same.

Example 1.2.2.30

Simplify $(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$.

Solution

	$(a^3-a^2b)-(ab^2+b^3)+(a^2b+ab^2)$
Distribute.	$=a^3-a^2b-ab^2-b^3+a^2b+ab^2$
Identify like terms.	The like terms are: • $-a^2b$ and a^2b • $-ab^2$ and ab^2
Rearrange the terms to put like terms together.	$=a^3-a^2b+a^2b-ab^2+ab^2-b^3$
Combine like terms.	$=a^3-b^3$

Try It 1.2.2.31

Simplify $(x^3 - x^2y) - (xy^2 + y^3) + (x^2y + xy^2)$.

Answer

$x^3 + y^3$

Try It 1.2.2.32

Simplify $(p^3 - p^2q) + (pq^2 + q^3) - (p^2q + pq^2)$.

Answer

$p^3 - 2p^2q + q^3$

Applications

Polynomials similar to the one in the next example are used in many fields to model the height of an object at some time after it is projected into the air. The polynomial in the next example is used specifically to model the height of an object which is dropped from 250 ft.

Example 1.2.2.33

The polynomial $-16t^2 + 250$ gives the height of a ball t seconds after it is dropped from a 250-foot tall building. Find the height after $t = 2$ seconds. In this example the variable t is a place holder for any number you may be interested in rather than an 'unknown' number.

Solution

The interpretation of the expression when $t = 2$ is when evaluated it gives the height of the ball 2 seconds after it is dropped from a 250-foot tall building.

	$-16t^2 + 250$
To find the height at 2 seconds, we substitute 2 for t.	$-16(2)^2 + 250$
Simplify.	$= -16(4) + 250$
Simplify.	$= -64 + 250$
Simplify.	$= 186$
Answer the question.	After 2 seconds, the height of the ball is 186 feet.

Try It 1.2.2.34

The polynomial $-16t^2 + 150$ gives the height of a stone t seconds after it is dropped from a 150-foot tall cliff. Find the height after $t = 0$ seconds (the initial height of the object).

Answer

The height is 150 feet.

Try It 1.2.2.35

The polynomial $-16t^2 + 175$ gives the height of a ball t seconds after it is dropped from a 175-foot tall bridge. Find the height after $t = 3$ seconds.

Answer

The height is 31 feet.

Writing Exercises 1.2.2.36

1. Why are $-3x^2y^3$ and $2y^3x^2$ like terms and why can you add them to get $-x^2y^3$?
2. Why is $-x^2+x$ the opposite of x^2-x ?
3. What degree might the sum of a third degree and a 4th degree polynomial be?
4. What degree might the sum of two third degree polynomials be?

Exit Problem

Simplify $(-2x^2+3x-1)-(-3x^2-5x+2)$.

Key Concepts

- **Monomial**
  - A **monomial** is an algebraic expression with one term.
  - A monomial in one variable is a term of the form ax^m where a is a constant and m is a whole number.
- **Polynomials**
  - **Polynomial**—A monomial, or two or more algebraic terms combined by addition or subtraction is a polynomial.
  - **Monomial** —A polynomial with exactly one term is called a monomial.
  - **Binomial** — A polynomial with exactly two terms is called a binomial.
  - **Trinomial** —A polynomial with exactly three terms is called a trinomial.
- **Degree of a Polynomial**
  - The **degree of a term** is the sum of the exponents of its variables.
  - The **degree of a constant** is 0.
  - The **degree of a polynomial** is the highest degree of all its terms.
  - A linear expression is a polynomial of degree 0 or 1.
- **Term**
- **Coefficient**
- **Evaluating, Adding/Subtracting polynomials**

1.2.3: Multiplying Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Multiply monomials
- Multiply a polynomial by a monomial
- Multiply a binomial by a binomial
- Multiply a polynomial by a polynomial
- Multiply special products

Be Prepared

Before you get started, take this readiness quiz.

1. Distribute: $2(x+3)$.

2. Simplify:

a. 9^2

b. $(-9)^2$

c. -9^2

3. Evaluate $2x^2-5x+3$ for $x=-2$.

Multiplying Expressions Using a Property for Exponents

Remember that a positive integer exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the positive integer *exponent* m tells us how many times we use the *base* a as a factor.

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{m factors}}$$

For example,

$$(-9)^5 = \underbrace{(-9)\cdot(-9)\cdot(-9)\cdot(-9)\cdot(-9)}_{\text{5 factors}}$$

Let's review the vocabulary for expressions with exponents.

Definition 1.2.3.1

$$a^m = \underbrace{a \cdot a \cdot a \cdots a}_{m \text{ factors}}$$

This is read a to the m^{th} **power**.

In the expression a^m with positive integer m and $a \neq 0$, the **exponent** m tells us how many times we use the **base** a as a factor.

When we combine like terms by adding and subtracting, we need to have the same base with the same exponent. But when we multiply and divide, the exponents may be different, and sometimes the bases may be different, too.

First, we will look at an example that leads to the *Product Property for Positive Integer Exponents*.

	x^2x^3
What does this mean?	$=\underbrace{x\cdot x}_{2\text{ factors}}\cdot\underbrace{x\cdot x\cdot x}_{3\text{ factors}}$ $=\underbrace{x\cdot x\cdot x\cdot x\cdot x}_{5\text{ factors}}$
	$=x^5$

The base stayed the same and we added the exponents.

Product property for positive integer exponents

If a is a real number and m and n are positive integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Example 1.2.3.2

Simplify each expression:

a. y^5y^6

b. $2^4\cdot 2^{3\cdot 4}$

c. $2a^7\cdot 3a$

d. $d^4d^5d^2$

Solution

a.

	y^5y^6
Use the Product Property, $a^m a^n = a^{m+n}$.	$=y^{5+6}$
Simplify.	$=y^{11}$

b.

	$2^4\cdot 2^{3\cdot 4}$
Use the Product Property, $a^m a^n = a^{m+n}$.	$=2^{4+3\cdot 4}$
Simplify.	$=2^{4\cdot 4}$ $=2^{16}$

c.

	$2a^7\cdot 3a$
Rewrite, $a=a^1$.	$=2a^7\cdot 3a^1$
Use the Commutative Property and use the Product Property, $a^m a^n = a^{m+n}$.	$=2\cdot 3a^{7+1}$
Simplify.	$=6a^8$

d.

	$d^4d^5d^2$

	$d^4d^5d^2$
Add the exponents, since the bases are the same.	$= d^{4+5+2}$
Simplify.	$= d^{11}$

Try It 1.2.3.3

Simplify each expression:

a. b^9b^8

b. $4^{2\cdot3}\cdot4^3$

c. $3p^5\cdot4p$

d. $x^6x^4x^8$

Answer

a. b^{17}

b. 4^9

c. $12p^6$

d. x^{18}

Try It 1.2.3.4

Simplify each expression:

a. $x^{12}x^4$

b. $10\cdot10^5$

c. $2z\cdot6z^7$

d. $b^5b^9b^5$

Answer

a. x^{16}

b. 10^6

c. $12z^8$

d. b^{19}

Multiplying Monomials

We are ready to perform operations on polynomials. Since monomials are algebraic expressions, we can use the properties of exponents to multiply monomials.

Example 1.2.3.5

Multiply:

a. $(3x^2)(-4x^3)$

b. $\left(\frac{5}{6}x^3y\right)(12xy^2)$

Solution

a.

	$(3x^2)(-4x^3)$
Use the Commutative Property and use the Product Property, $a^m a^n = a^{m+n}$.	$= 3(-4)x^2x^3$
Simplify.	$= -12x^5$

b.

	$\left(\frac{5}{6}x^3y\right)(12xy^2)$
Use the Commutative Property and use the Product Property, $a^m a^n = a^{m+n}$.	$= \frac{5}{6} \cdot 12x^3x \cdot yy^2$
Simplify.	$= 10x^4y^3$

Try It 1.2.3.6

Multiply:

a. $(5y^7)(-7y^4)$

b. $(25a^4b^3)(15ab^3)$

Answer

a. $-35y^{11}$

b. $6a^5b^6$

Try It 1.2.3.7

Multiply:

a. $(-6b^4)(-9b^5)$

b. $(23r^5s)(12r^6s^7)$.

Answer

a. $54b^9$

b. $8r^{11}s^8$

Multiplying a Polynomial by a Monomial

Multiplying a polynomial by a monomial is really just applying the Distributive Property of multiplication over addition and subtraction.

Example 1.2.3.8

Multiply:

a. $-2y(4y^2+3y-5)$

b. $3x^3y(x^2-8xy+y^2)$

Solution

a.

	$-2y(4y^2+3y-5)$
Distribute.	$=-2y\cdot 4y^2+(-2y)3y+(-2y)(-5))$
Simplify.	$=-8y^3-6y^2+10y$

b.

	$3x^3y(x^2-8xy+y^2)$
Distribute.	$=3x^3yx^2-3x^3y\cdot 8xy+3x^3yy^2)$
Simplify.	$=3x^5y-24x^4y^2+3x^3y^3$

Try It 1.2.3.9

Multiply:

a. $-3y(5y^2+8y^{-7})$

b. $4x^2y^2(3x^2-5xy+3y^2)$

Answer

a. $-15y^3-24y^2+21y$

b. $12x^4y^2-20x^3y^3+12x^2y^4$

Try It 1.2.3.10

Multiply:

a. $4x^2(2x^2-3x+5)$

b. $-6a^3b(3a^2-2ab+6b^2)$

Answer

a. $-15y^3-24y^2+21y$

b. $-18a^5b+12a^4b^2-36a^3b^3$

Multiplying a Binomial by a Binomial

The multiplication of a binomial by a binomial can be performed by multiplying the each term of one binomial by the other binomial. This is seen by applying the Distributive Property twice.

$$\begin{aligned}(a+b)(c+d)\ &=a(c+d)+b(c+d)\\ &=ac+ad+bc+bd\end{aligned}$$

Example 1.2.3.11

Multiply:

a. $(y+5)(y+8)$

b. $(4y+3)(2y-5)$

Solution

a.

	$(y+5)(y+8)$
Distribute $y+8$.	$=y(y+8)+5(y+8)$
Distribute again.	$=y^2+8y+5y+40$
Combine like terms	$=y^2+13y+40$

Note that you could have distributed the $y+5$ instead on the first step and proceeded from there.

b.

	$(4y+3)(2y-5)$
Distribute $2y-5$.	$=4y(2y-5)+3(2y-5)$
Distribute again.	$=8y^2-20y+6y-15$
Combine like terms.	$=8y^2-14y-15$

Try It 1.2.3.12

Multiply:

a. $(x+8)(x+9)$

b. $(3c+4)(5c-2)$

Answer

a. $x^2+17x+72$

b. $15c^2+14c-8$

Try It 1.2.3.13

Multiply:

a. $(5x+9)(4x+3)$

b. $(5y+2)(6y-3)$

Answer

a. $20x^2+51x+27$

b. $30y^2-3y-6$

If you multiply binomials often enough you may notice a pattern. Notice that the first term in the result is the product of the *first* terms in each binomial. The second and third terms are the product of multiplying the two *outer* terms and then the two *inner* terms. And the last term results from multiplying the two *last* terms,

We abbreviate “First, Outer, Inner, Last” as FOIL. The letters stand for ‘First, Outer, Inner, Last’. We use this as another method of multiplying binomials. The word FOIL is easy to remember and ensures we find all **four** products.

Let’s multiply $(x+3)(x+7)$ using the Distributive Property (or FOIL).

Distribute Property	**FOIL**
$(x+3)(x+7)$	$(x+3)(x+7)$
$=x(x+7)+3(x+7)$	$=\underbrace{x^2}_{F}+\underbrace{7x}_{O}+\underbrace{3x}_{I}+\underbrace{21}_{L}$
$=x^2+7x+3x+21$	$=x^2+10x+21$

Distribute Property	FOIL
$= x^2 + 10x + 21$	

The following chart explains in details how to multiply binomials using FOIL. Note that FOIL does not work with other polynomials (why?)!

Use FOIL to multiply two binomials

		$(a+b)(c+d)$
	Identify first, outer, inner and last terms.	$= (\underbrace{\overbrace{a}^{\text{outer}}}_{\text{first}} + \underbrace{\overbrace{b}^{\text{inner}}}_{\text{last}})(\underbrace{\overbrace{c}^{\text{inner}}}_{\text{first}} + \underbrace{\overbrace{d}^{\text{outer}}}_{\text{last}})$
ou…	Multiply the first terms. (F)	$= \underbrace{\boldsymbol{ac}}_{\text{F}} + \underbrace{\_\_}_{\text{O}} + \underbrace{\_\_}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the outer terms. (O)	$= \underbrace{ac}_{\text{F}} + \underbrace{ad}_{\text{O}} + \underbrace{\_\_}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the inner terms. (I)	$= \underbrace{ac}_{\text{F}} + \underbrace{ad}_{\text{O}} + \underbrace{bc}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the last terms.(L)	$= \underbrace{ac}_{\text{F}} + \underbrace{ad}_{\text{O}} + \underbrace{bc}_{\text{I}} + \underbrace{bd}_{\text{L}}$
ou…	Combine like terms, when possible.	$= ac + ad + bc + bc$

When you multiply using FOIL, drawing the lines will help your brain focus on the pattern and make it easier to apply.

Now we will do an example where we use the FOIL pattern to multiply two binomials.

Example 1.2.3.14

Multiply:

a. $(y-7)(y+4)$

b. $(4x+3)(2x-5)$

Solution

a.

		$(y-7)(y+4)$
	Identify first, outer, inner and last terms.	$= (\underbrace{\overbrace{y}^{\text{outer}}}_{\text{first}} + \underbrace{\overbrace{(-7)}^{\text{inner}}}_{\text{last}})(\underbrace{\overbrace{y}^{\text{inner}}}_{\text{first}} + \underbrace{\overbrace{4}^{\text{outer}}}_{\text{last}})$
ou…	Multiply the first terms. (F)	$= \underbrace{\boldsymbol{y^2}}_{\text{F}} + \underbrace{\_\_}_{\text{O}} + \underbrace{\_\_}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the outer terms. (O)	$= \underbrace{y^2}_{\text{F}} + \underbrace{4y}_{\text{O}} + \underbrace{\_\_}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the inner terms. (I)	$= \underbrace{y^2}_{\text{F}} + \underbrace{4y}_{\text{O}} + \underbrace{(-7y)}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the last terms.(L)	$= \underbrace{y^2}_{\text{F}} + \underbrace{4y}_{\text{O}} + \underbrace{(-7y)}_{\text{I}} + \underbrace{(-28)}_{\text{L}}$

ou…	Combine like terms, when possible.	$= y^2 + 4y - 7y - 28$ $= y^2 - 3y - 28$

b.

		$(4x+3)(2x-5)$
	Identify first, outer, inner and last terms.	$= (\overbrace{\underbrace{\mathbf{4x}}_{\text{first}}}^{\text{outer}} + \overbrace{\underbrace{\mathbf{3}}_{\text{last}}}^{\text{inner}})(\overbrace{\underbrace{\mathbf{2x}}_{\text{first}}}^{\text{inner}} + \overbrace{\underbrace{(\mathbf{-5})}_{\text{last}}}^{\text{outer}})$
ou…	Multiply the first terms. (F)	$= \underbrace{\mathbf{8x^2}}_{\text{F}} + \underbrace{\_\_}_{\text{O}} + \underbrace{\_\_}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the outer terms. (O)	$= \underbrace{8x^2}_{\text{F}} + \underbrace{(-20x)}_{\text{O}} + \underbrace{\_\_}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the inner terms. (I)	$= \underbrace{8x^2}_{\text{F}} + \underbrace{(-20x)}_{\text{O}} + \underbrace{6x}_{\text{I}} + \underbrace{\_\_}_{\text{L}}$
ou…	Multiply the last terms.(L)	$= \underbrace{8x^2}_{\text{F}} + \underbrace{(-20x)}_{\text{O}} + \underbrace{6x}_{\text{I}} + \underbrace{(-15)}_{\text{L}}$
ou…	Combine like terms, when possible.	$= 8x^2 + (-20x) + 6x + (-15)$ $= 8x^2 - 14x - 15$

Try It 1.2.3.15

Multiply:

a. $(x-7)(x+5)$

b. $(3x+7)(5x-2)$

Answer

a. $x^2 - 2x - 35$
b. $15x^2 + 29x - 14$

Try It 1.2.3.16

Multiply:

a. $(b-3)(b+6)$

b. $(4y+5)(4y-10)$

Answer

a. $b^2 + 3b - 18$

b. $16y^2 - 20y - 50$

The final products in the last example were trinomials because we could combine the two middle terms. This is not always the case.

✔ Example 1.2.3.17

Multiply:

a. $(n^2+4)(n-1)$

b. $(3pq+5)(6pq-11)$

Solution

a.

		$(n^2+4)(n-1)$
	Identify first, outer, inner and last terms.	$=(\overbrace{\underbrace{\boldsymbol{n^2}}_{\text{first}}}^{\text{outer}}+\overbrace{\underbrace{\mathbf{4}}_{\text{last}}}^{\text{inner}})(\overbrace{\underbrace{\boldsymbol{n}}_{\text{first}}}^{\text{inner}}+\overbrace{\underbrace{\mathbf{(-1)}}_{\text{last}}}^{\text{outer}})$
ou…	Multiply the first terms. (F)	$=\underbrace{n^3}_{\text{F}}+\underbrace{\_\_\_}_{\text{O}}+\underbrace{\_\_\_}_{\text{I}}+\underbrace{\_\_\_}_{\text{L}}$
ou…	Multiply the outer terms. (O)	$=\underbrace{n^3}_{\text{F}}+\underbrace{(-n^2)}_{\text{O}}+\underbrace{\_\_\_}_{\text{I}}+\underbrace{\_\_\_}_{\text{L}}$
ou…	Multiply the inner terms. (I)	$=\underbrace{n^3}_{\text{F}}+\underbrace{(-n^2)}_{\text{O}}+\underbrace{4n}_{\text{I}}+\underbrace{\_\_\_}_{\text{L}}$
ou…	Multiply the last terms.(L)	$=\underbrace{n^3}_{\text{F}}+\underbrace{(-n^2)}_{\text{O}}+\underbrace{4n}_{\text{I}}+\underbrace{(-4)}_{\text{L}}$
ou…	Combine like terms, when possible.	$=n^3+(-n^2)+4n+(-4)$ $=n^3-n^2+4n-4$

b.

		$(3pq+5)(6pq-11)$
	Identify first, outer, inner and last terms.	$=(\overbrace{\underbrace{3pq}_{\text{first}}}^{\text{outer}}+\overbrace{\underbrace{5}_{\text{last}}}^{\text{inner}})(\overbrace{\underbrace{6pq}_{\text{first}}}^{\text{inner}}+\overbrace{\underbrace{(-11)}_{\text{last}}}^{\text{outer}})$
ou…	Multiply the first terms. (F)	$=\underbrace{15p^2q^2}_{\text{F}}+\underbrace{\_\_\_}_{\text{O}}+\underbrace{\_\_\_}_{\text{I}}+\underbrace{\_\_\_}_{\text{L}}$
ou…	Multiply the outer terms. (O)	$=\underbrace{15p^2q^2}_{\text{F}}+\underbrace{(-33pq)}_{\text{O}}+\underbrace{\_\_\_}_{\text{I}}+\underbrace{\_\_\_}_{\text{L}}$
ou…	Multiply the inner terms. (I)	$=\underbrace{15p^2q^2}_{\text{F}}+\underbrace{(-33pq)}_{\text{O}}+\underbrace{30pq}_{\text{I}}+\underbrace{\_\_\_}_{\text{L}}$
ou…	Multiply the last terms.(L)	$=\underbrace{15p^2q^2}_{\text{F}}+\underbrace{(-33pq)}_{\text{O}}+\underbrace{30pq}_{\text{I}}+\underbrace{(-55)}_{\text{L}}$
ou…	Combine like terms, when possible.	$=15p^2q^2+(-33pq)+30pq+(-55)$ $=15p^2q^2-3pq+4n-55$

? Try It 1.2.3.18

Multiply:

a. $(x^2+6)(x-8)$

b. $(2ab+5)(4ab-4)$

Answer

a. $x^3-8x^2+6x-48$

b. $8a^2b^2+12ab-20$

? Try It 1.2.3.19

Multiply:

a. $(y^2+7)(y-9)$

b. $(2xy+3)(4xy-5)$

Answer

a. $y^3-9y^2+7y-63$

b. $8x^2y^2+2xy-15$

The FOIL method is just one way to keep track of the four terms that appear in the product of two binomials, and it *only* works for binomials. Another way for obtaining the four term is using the Vertical format. It is very much like the procedure you use to multiply whole numbers. Look carefully at this example of multiplying two-digit numbers.

$$\begin{array}{r} 32 \\ \times 35 \\ \hline 160 \\ +96* \\ \hline 1120 \end{array}$$

Now we'll apply this same procedure to multiply two binomials.

✓ Example 1.2.3.20

Multiply $(3y-1)(2y-6)$ using the Vertical format.

Solution

It does not matter which binomial goes on the top.

	$(3y-1)(2y-6)$
Multiply $3y-1$ by -6. Multiply $3y-1$ by $2y$.	$-18y+6$ $+6y^2-2y$
Add like terms.	$=6y^2-20y+6$

? Try It 1.2.3.21

Multiply $(5m-7)(3m-6)$ using the Vertical format.

Answer

$15m^2-51m+42$

Try It 1.2.3.22

Multiply $(6b-5)(7b-3)$ using the Vertical format.

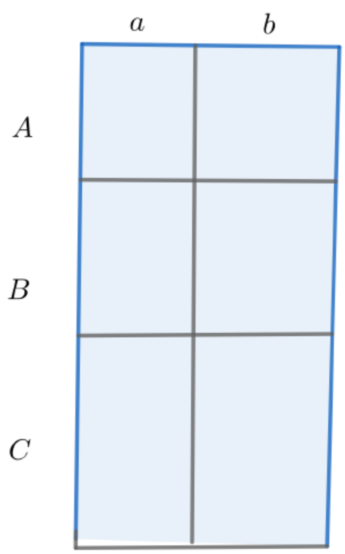
Answer

$42b^2-53b+15$

Multiplying a Polynomial by a Polynomial

We have multiplied monomials by monomials, monomials by polynomials, and binomials by binomials. Now we're ready to multiply a polynomial by a polynomial. Remember, FOIL will not work in this case, but we use the Distributive Property. There are various ways to keep track of the terms. The first is to just note one way to see the why we can distribute!

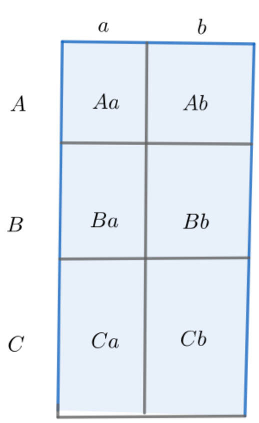
Consider finding the area of the marked figure below:



One way to calculate the area is to find the width $(a+b)$ and the length $(A+B+C)$ and then multiply to get:

$(a+b)(A+B+C)$

The other way is to find the area of each little rectangle (indicated here inside the corresponding rectangle)



and then add the results (in whichever order we like) to get, for example, that the area is also equal to

$Aa+Ab+Ba+Bb+Ca+Cb$

Notice that we have in this example, six terms, each of which is the result of the product of a length from the first grouping times a length of the second grouping.

Since the area is the same no matter how we compute it, we have, for example,

$(a+b)(A+B+C)=Aa+Ab+Ba+Bb+Ca+Cb.$

The picture only is valid for positive values since only such values represent length, but the equation is true for all values (and a picture can be drawn to specifically indicate negative quantities as length subtraction). The picture here then can generally be used for bookkeeping purposes no matter if the values of a, b, A, B, and C are all positive.

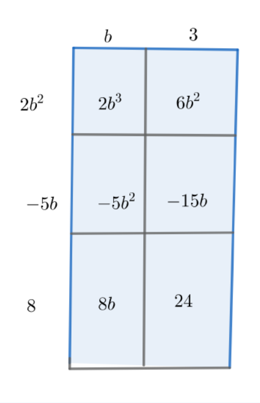
Example 1.2.3.23

Multiply $(b+3)(2b^2-5b+8)$.

Solution

We present three solutions.

First, we create the rectangle and labeling above in this case for the purpose of book-keeping (and ignore the fact that negative values can not be lengths):



To find $b+3)(2b^2-5b+8)=2b^3+6b^2-5b^2-15b+8b+24=2b^3+b^2-7b+24$

(the last equality we see by collecting like terms).

The second way is by using the Distributive Property of multiplication of monomials over addition/subtraction.

	$(b+3)(2b^2-5b+8)$
Distribute.	$=b(2b^2-5b+8)+3(2b^2-5b+8)$
Distribute.	$=2b^3-5b^2+8b+6b^2-15b+24$
Combine like terms.	$=2b^3+b^2-7b+24$

The third way is using the vertical format. It is easier to put the polynomial with fewer terms on the bottom because we get fewer partial products this way.

	$(b+3)(2b^2-5b+8)$
Multiply $(2b^2-5b+8)$ by 3. Multiply $(2b^2-5b+8)$ by b.	$=6b^2-15b+24$ $+2b^3-5b^2+8b$
Add like terms.	$=2b^3+b^2-7b+24$

Try It 1.2.3.24

Multiply $(y-3)(y^2-5y+2)$.

Answer

a. $y^3-8y2+17y-6$
b. $y^3-8y2+17y-6$

Try It 1.2.3.25

Multiply $(x+4)(2x^2-3x+5)$.

Answer

a. $2x^3+5x^2-7x+20$
b. $y^3-8y^2+17y-6$

Multiplying Special Products

Sometimes identifying patterns will make computations 'easier'. A good example of this is squaring binomials. While you can always get the product by writing the binomial twice and multiplying them, there is less work to do if you learn to use a pattern. Let's start by looking at three examples and look for a pattern.

Look at these results. Do you see any patterns?

$$(x-5)^2=x^2-10x+25, \quad (2y+3)^2=4y^2+12y+9, \quad (5z-2t)^2=25z^2-20zt+4t^2 \text{\nonumbe} \tag{1.2.3.1}$$

What about the number of terms? In each example we squared a binomial and the result was a trinomial.

$$(a+b)^2=\_\_\_+\_\_\_+\_\_\_$$

Now look at the *first term* in each result. Where did it come from?

The first term is the product of the first terms of each binomial. Since the binomials are identical, it is just the square of the first term!

$$(a+b)^2=a^2+\_\_\_+\_\_\_$$

To get the first term of the product, square the first term.

Where did the *last term* come from? Look at the examples and find the pattern.

The last term is the product of the last terms, which is the square of the last term.

$$(a+b)^2=\_\_\_+\_\_\_+b^2$$

To get the last term of the product, square the last term.

Finally, look at the *middle term*. Notice it came from adding the "outer" and the "inner" terms—which are both the same! So the middle term is double the product of the two terms of the binomial.

$$(a+b)^2=\_\_\_+2ab+\_\_\_$$

$$(a-b)^2=\_\_\_-2ab+\_\_\_$$

To get the middle term of the product, multiply the terms and double their product.

Putting it all together:

Binomial squares pattern

If a and b are real numbers,

$$\underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2},$$
$$\underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2(\text{product of terms})} + \underbrace{b^2}_{(\text{last term})^2}.$$

To square a binomial, square the first term, square the last term, double their product.

Example 1.2.3.26

Multiply:

a. $(x+5)^2$

b. $(2x-3y)^2$

Solution

a.

	$(x+5)^2$
Identify a and b.	$=(\underbrace{x}_{a}+\underbrace{5}_{b})^2 \qquad a=x,\ b=5$
Square the first term, a.	$=\underbrace{x^2}_{a^2}+\underbrace{\_\_\_}_{2ab}+\underbrace{\_\_\_}_{b^2}$
Double the product of a and b.	$=\underbrace{x^2}_{a^2}+\underbrace{2\cdot(x)\cdot(5)}_{2ab}+\underbrace{\_\_\_}_{b^2}$
Square the last term, b.	$=\underbrace{x^2}_{a^2}+\underbrace{2\cdot(x)\cdot(5)}_{2ab}+\underbrace{5^2}_{b^2}$
Simplify.	$=x^2+10x+25$

b.

	$(2x-3y)^2$
Identify a and b.	$=(\underbrace{2x}_{a}-\underbrace{3y}_{b})^2 \qquad a=2x,\ b=3y$
Square the first term, a.	$=\underbrace{(2x)^2}_{a^2}-\underbrace{\_\_\_}_{2ab}+\underbrace{\_\_\_}_{b^2}$
Double the product of a and b.	$=\underbrace{(2x)^2}_{a^2}-\underbrace{2\cdot(3x)\cdot(3y)}_{2ab}+\underbrace{\_\_\_}_{b^2}$
Square the last term, b.	$=\underbrace{x^2}_{a^2}-\underbrace{2\cdot(3x)\cdot(3y)}_{2ab}+\underbrace{(3y)^2}_{b^2}$
Simplify.	$=4x^2-12xy+9y^2$

Try It 1.2.3.27

Multiply:

a. $(x+9)^2$

b. $(2c-d)^2$

Answer

a. $x^2+18x+81$
b. $4c^2-4cd+d^2$

Try It 1.2.3.28

Multiply:

a. $(y+11)^2$

b. $(4x-5y)^2$

Answer

a. $y^2+22y+121$
b. $16x^2-40xy+25y^2$

We just saw a pattern for squaring binomials that we can use to make multiplying some binomials easier. Similarly, there is a pattern for another product of binomials. But before we get to it, we need to introduce some vocabulary.

A pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a **conjugate pair** and is of the form $(a-b)$, $(a+b)$.

Definition 1.2.3.29

A **conjugate pair** is two binomials of the form

$$(a-b),(a+b).$$

The pair of binomials each have the same first term and the same last term, but one binomial is a sum and the other is a difference.

There is a nice pattern for finding the product of conjugates. You could, of course, simply FOIL to get the product, but using the pattern makes your work easier. Let's look for the pattern by using FOIL to multiply some conjugate pairs.

$$(x-4)(x+4)=x^2+4x-4x-16=x^2-16$$

$$(2x+3)(2x-3)=4x^2-6x+6x-9=4x^2-9$$

$$(3z-2t)(3z+2t)=9z^2+6zt-6tz-4t^2=9z^2-4t^2$$

What do you observe about the products?

The product of the two binomials is also a binomial! Most of the products resulting from FOIL have been trinomials.

Each *first term* is the product of the first terms of the binomials, and since they are identical it is the square of the first term.

$$(a+b)(a-b)=a^2-\_\_\_$$

To get the first term, square the first term.

The *last term* came from multiplying the last terms, the square of the last term.

$$(a+b)(a-b)=a^2-b^2$$

To get the last term, square the last term.

Why is there no middle term? Notice the two middle terms you get from FOIL combine to 0 in every case, the result of one addition and one subtraction.

The product of conjugates is always of the form $a^2 - b^2$. This is called a **difference of squares**.

This leads to the pattern:

Product of conjugates pattern

For any a and b,

$$(a+b)(a-b) = a^2 - b^2$$

The product is called a **difference of squares**.

To multiply conjugates, square the first term, square the last term, write it as a difference of squares.

Example 1.2.3.30

Multiply using the product of conjugates pattern:

a. $(2x+5)(2x-5)$

b. $(5m-9n)(5m+9n)$

Solution

a.

	$(2x+5)(2x-5)$
Are the binomials conjugates? Identify a and b	The binomials $2x+5$ and $2x-5$ are conjugates. $a = 2x,\ b = 5$
It is the product of conjugates.	$= \underbrace{(2x+5)}_{a+b}\underbrace{(2x-5)}_{a-b}$
Square the first term, a.	$= \underbrace{(2x)^2}_{a^2} - \underbrace{\_\_\_}_{b^2}$
Square the last term, b.	$= \underbrace{(2x)^2}_{a^2} - \underbrace{(5)^2}_{b^2}$
Simplify. The product is a difference of squares.	$= 4x^2 - 25$

b.

	$(5m-9n)(5m+9n)$
Are the binomials conjugates? Identify a and b.	The binomials $5m-9n$ and $5m+9n$ are conjugates. $a = 5m,\ b = 9n$
It is the product of conjugates.	$= \underbrace{(5m-9n)}_{a-b}\underbrace{(5m+9n)}_{a+b}$
Square the first term, a.	$= \underbrace{(5m)^2}_{a^2} - \underbrace{\_\_\_}_{b^2}$
Square the last term, b.	$= \underbrace{(5m)^2}_{a^2} - \underbrace{(9n)^2}_{b^2}$

	$(5m - 9n)(5m + 9n)$
Simplify. The product is a difference of squares.	$= 25m^2 - 81n^2$

Try It 1.2.3.31

Multiply:

a. $(6x + 5)(6x - 5)$

b. $(4p - 7q)(4p + 7q)$

Answer

a. $36x^2 - 25$
b. $16p^2 - 49q^2$

Try It 1.2.3.32

Multiply:

a. $(2x + 7)(2x - 7)$

b. $(3x - y)(3x + y)$.

Answer

a. $4x^2 - 49$

b. $9x^2 - y^2$

We just developed special product patterns for Binomial Squares and for the Product of Conjugates. The products look similar, so it is important to recognize when it is appropriate to use each of these patterns and to notice how they differ. Look at the two patterns together and note their similarities and differences.

COMPARING THE SPECIAL PRODUCT PATTERNS

Binomial Squares	Product of Conjugates
$(a + b)^2 = a^2 + 2ab + b^2$	$(a - b)(a + b) = a^2 - b^2$
$(a - b)^2 = a^2 - 2ab + b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

Example 1.2.3.33

Choose the appropriate pattern and use it to find the product:

a. $(2x - 3)(2x + 3)$

b. $(5x - 8)^2$

c. $(6m + 7)^2$

d. $(5x - 6)(6x + 5)$

Solution

a. $(2x-3)(2x+3)$

These are conjugates. They have the same first numbers, and the same last numbers, and one binomial is a sum and the other is a difference. It fits the Product of Conjugates pattern.

	$\underbrace{(2x-3)}_{a-b}\underbrace{(2x+3)}_{a+b} \quad a=2x,\ b=3$
Use the pattern: $(a-b)(a+b)=a^2-b^2$.	$=\underbrace{(2x)^2}_{a^2}-\underbrace{(3)^2}_{b^2}$
Simplify.	$=4x^2-9$

b. $(8x-5)^2$

We are asked to square a binomial. It fits the binomial squares pattern.

	$(\underbrace{8x-5}_{a-b})^2 \quad a=8x,\ b=5$
Use the pattern: $(a-b)^2=a^2-2ab+b^2$.	$=\underbrace{(8x)^2}_{a^2}-\underbrace{2\cdot(8x)\cdot(5)}_{2ab}+\underbrace{(5)^2}_{b^2}$
Simplify.	$=64x^2+80x+25$

c. $(6m+7)^2$

Again, we will square a binomial so we use the binomial squares pattern.

	$(\underbrace{6m+7}_{a+b})^2 \quad a=6m,\ b=7$
Use the pattern: $(a+b)^2=a^2+2ab+b^2$.	$=\underbrace{(6m)^2}_{a^2}+\underbrace{2\cdot(6m)\cdot(7)}_{2ab}+\underbrace{(7)^2}_{b^2}$
Simplify.	$=36m^2+84m+49$

d. $(5x-6)(6x+5)$

This product does not fit the patterns, so we will use FOIL.

	$(5x-6)(6x+5)$
Use FOIL.	$=30x^2+25x-36x-30$
Simplify.	$=30x^2-11x-30$

? Try It 1.2.3.34

Choose the appropriate pattern and use it to find the product:

a. $(9b-2)(2b+9)$

b. $(9p-4)^2$

c. $(7y+1)^2$

d. $(4r-3)(4r+3)$

Answer

a. FOIL; $18b^2+77b-18$
b. Binomial Squares; $81p^2-72p+16$
c. Binomial Squares; $49y^2+14y+1$
d. Product of Conjugates; $16r^2-9$

Try It 1.2.3.35

Choose the appropriate pattern and use it to find the product:

a. $(6x+7)^2$

b. $(3x-4)(3x+4)$

c. $(2x-5)(5x-2)$

d. $(6n-1)^2$

Answer

a. Binomial Squares; $36x^2+84x+49$

b. Product of Conjugates; $9x^2-16$

c. FOIL; $10x^2-29x+10$

d. Binomial Squares; $36n^2-12n+1$

Writing Exercises 1.2.3.36

1. Explain how the 'FOIL' method is the same as a sequence of two distributions.
2. If you multiply two binomials, is the result a binomial? Give an example.
3. How many terms does a trinomial times a monomial have?
4. Is it possible that the product of a binomial and a trinomial has 7 terms? Explain and give an example.
5. What is the minimal amount of work you must do to determine the degree and the leading coefficient of $(5x^3+x^2-2x+1)(-2x^2-7x+2)$? Explain and give another example of this process.
6. Show using a picture that represents a possible situation that $(5-2)(6+2-3)=30+10-15-12-4+6$.
7. Which method do you prefer to use when multiplying two binomials: the rectangle, the distributive property for monomials (twice), or the FOIL method? Why? Which method do you prefer to use when multiplying a polynomial by a polynomial: the rectangle, the distributive property for monomials (twice), or the Vertical Method? Why?
8. Multiply the following:

   $(x+2)(x-2)$

   $(y+7)(y-7)$

   $(w+5)(w-5)$

   Explain the pattern that you see in your answers. Does that pattern persist for $(y+7)(y+7)$? Give an example.

Exit Problem

1. Multiply $3x^3(2x-3)(3x+1)$.
2. Multiply $3x(3x^3y)(2x^3y^4)$.

Key Concepts

- **How to use the FOIL method to multiply two binomials.**

$$(a+b)(c+d)=ac+ad+bc+bd \tag{1.2.3.2}$$

- **Multiplying Two Binomials:** To multiply binomials, use the:

  - Distributive Property
  - FOIL Method
- **Multiplying a Polynomial by a Polynomial:** To multiply a trinomial by a binomial, use the:
  - Distributive Property
  - Vertical setup
- **Binomial Squares Pattern**
  If a and b are real numbers, $(a+b)^2 = a^2 + 2ab + b^2$
- **Product of Conjugates Pattern**
  If a, b are real numbers

$$(a+b)(a-b) = a^2 - b^2 \tag{1.2.3.3}$$

  The product is called a difference of squares.
  To multiply conjugates, square the first term, square the last term, write it as a difference of squares.
- **Comparing the Special Product Patterns**

Binomial Squares	Product of Conjugates
$(a+b)^2 = a^2 + 2ab + b^2$	$(a-b)^2 = a^2 - 2ab + b^2$
$(a-b)(a+b) = a^2 - b^2$	
• Squaring a binomial	• Multiplying conjugates
• Product is a **trinomial**	• Product is a **binomial.**
• Inner and outer terms with FOIL are **the same.**	• Inner and outer terms with FOIL are **opposites.**
• Middle term is **double the product** of the terms	• There is **no** middle term.

1.2.4: Powers of Monomials and Binomials

Learning Objectives

By the end of this section, you will be able to:

- Expand a positive integer power of monomial and binomial expressions
- Identify coefficients of terms of a positive integer powers of a binomial expression

Be Prepared

Before we get started, take this readiness quiz.

1. Expand $(2x-3)^2$.

2. Simplify $(3 \cdot 5$.

3. Evaluate $(5x^3$ at $x=2$.

Powers of Monomials

Now let’s look at an exponential expression that contains a power raised to a power. Let's see if we can discover a general property.

	$(x^2)^3$
What does this mean?	$=x^2x^2x^2$
How many factors altogether?	$=\underbrace{x\cdot x}_{2\ factors}\cdot\underbrace{x\cdot x}_{2\text{ factors}}\cdot\underbrace{x\cdot x}_{2\text{ factors}}$ $=\underbrace{x\cdot x\cdot x\cdot x\cdot x\cdot x}_{6\text{ factors}}$
So we have	$=x^6$

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2\cdot 3}$ or x^6. We can also see that

In this example we multiplied the exponents.

We can check various examples to see that this leads us to the *Power Property for Positive Integer Exponents.*

Power Property for Integer Exponents

If a is a real number and m and n are positive integers, then

$$(a^m)^n=a^{mn}.$$

To raise a power to a power, multiply the exponents.

Example 1.2.4.1

Simplify each expression:

a. $(y^5)^9$

b. $(4^4)^7$

c. $(y^3)^6(y^5)^4$

Solution

a.

	$(y^5)^9$

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{mn}$.	$y^{5 \cdot 9}$
Simplify.	y^{45}

b.

	$(4^4)^7$
Use the power property.	$= 4^{4 \cdot 7}$
Simplify.	$= 4^{28}$

c.

	$(y^3)^6(y^5)^4$
Use the power property.	$= y^{18}y^{20}$
Add the exponents.	$= y^{38}$

Try It 1.2.4.2

Simplify each expression:

a. $(b^7)^5$

b. $(5^4)^3$

c. $(a^4)^5(a^7)^4$

Answer

a. b^{35}

b. 5^{12}

c. a^{48}

Try It 1.2.4.3

Simplify each expression:

a. $(z^6)^9$

b. $(3^7)^7$

c. $(q^4)^5(q^3)^3$

Answer

a. z^{54}

b. 3^{49}

c. q^{29}

We will now look at an expression containing a product that is raised to a power. Can we find this pattern?

	$(2x)^3$
What does this mean?	$= 2x \cdot 2x \cdot 2x$

	$(2x)^3$
We group the like factors together.	$= 2 \cdot 2 \cdot 2xxx$
How many factors of 2 and of x?	$= 2^3 x^3$

Notice that each factor was raised to the power and $(2x)^3$ is $2^3 x^3$.

The exponent applies to each of the factors! We can say that the exponent distributes over multiplication. If we were to check various examples with exponents we would find the same pattern emerges. This leads to the *Product to a Power Property for Postive Integer Exponents.*

Product to a Power Property for Integer Exponents

If a and b are real numbers and m is a positive integer, then

$$(ab)^m = a^m b^m.$$

To raise a product to a power, raise each factor to that power.

Example 1.2.4.4

Simplify each expression:

a. $(-3mn)^3$

b. $(6k^3)^2$

c. $(5x^3)^2$

Solution

a.

	$(-3mn)^3$
Use Power of a Product Property, $(ab)^m = a^m b^m$.	$= (-3)^3 m^3 n^3$
Simplify.	$= -27m^3 n^3$

b.

	$(6k^3)^2$
Use the Power of a Product Property, $(ab)^m = a^m b^m$.	$= 6^2 (k^3)^2$
Use the Power Property, $(a^m)^n = a^{mn}$.	$= 6^2 k^6$
Simplify.	$= 36k^6$

c.

	$(5x^3)^2$
Use the power of a product property, $(ab)^m = a^m b^m$.	$= 5^2 (x^3)^2$
Simplify.	$= 25x^6$

Try It 1.2.4.5

Simplify each expression:

a. $(2wx)^5$

b. $(2b^3)^4$

c. $(8a^4)^2$

Answer

a. $32w^5x^5$

b. $16b^{12}$

c. $64a^8$

Try It 1.2.4.6

Simplify each expression:

a. $(-3y)^3$

b. $(-4x^4)^2$

c. $(2c^4)^3$

Answer

a. $-27y^3$

b. $16x^8$

c. $8c^{12}$

The Binomial Theorem

In this section we consider powers of binomial expressions like

$$(x+y)^5, (x+3)^4, \text{ or } (2x-3)^{10}.$$

These are polynomials with degree equal to the exponent.

Let's consider

$$(x+y)^3 = (x+y)(x+y)(x+y).$$

To distribute, we take one term from each factor and multiply. We repeat for all possible choices then add the results.

So,

$$(x+y)^3 = (x+y)(x+y)(x+y) = xxx + xxy + xyx + yxx + xyy + yxy + yyx + yyy = x^3 + 3x^2y + 3xy^2 + y^3.$$

If we consider the coefficient of the x^2y term, we see it is the number of ways we can choose two x's and one y. Similarly, for the other coefficients.

It turns out these are well known numbers and we have

The Binomial Theorem 1.2.4.7

$$(x+y)^n = a_0x^n + a_1x^{n-1}y + \cdots + a_{n-1}xy^{n-1} + a_ny^n,$$

where the coefficients come from the nth row (counting from 0 of the Pascal's triangle):

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

1 7 21 35 35 21 7 1

See Pascal's triangle - Wikipedia -- The first paragraph. The image above is from the same.

To get from one row to the next you add the two numbers above (or 1 in the case of the first and last number). For example on the 4th row (counting from zero), above the 4 are 3 and 1, above the 6 is 3 and 3, and the ends have only 1 above.

For example,

$$\begin{aligned}(x+2)^4 &= x^4+4x^3\cdot 2+6x^2\cdot 2^2+4x\cdot 2^3+2^4\\ &= x^4+8x^3+24x^2+32x+16.\end{aligned}$$

Note that the numbers in Pascal's triangle are called binomial coefficients. The details are beyond the scope of the book but more information and many applications can be found in the Wikipedia article on the topic.

Example 1.2.4.8

Find the coefficient of x^3 in the expression $(2x-3)^6$.

Solution

Note that the exponent on x is 3. We find the 4th number in the 7th line of Pascal's triangle (this is because with Pascal's triangle the counting starts at 0). Thus, we see it is 20.

So the relevant term is $20(2x)^3(-3)^3$ (noting that the exponents add to 6). Therefore, the coefficient of x^3 is

$$20\cdot 2^3\cdot(-3)^3=-4320.$$

Try It 1.2.4.9

Find the coefficient of

a. x^4 in $(2-x)^7$

b. xy^2 in $(x-3y)^3$

c. x^4y^3 in $(2y-x^2)^5$

Answer

a. 280

b. 27

c. 80

Writing Exercises 1.2.4.10

1. Explain by writing out the full meaning why $(2^3)^4$.
2. Why is $(2^3)^4 = (2^4)^3$?
3. Verify the expansion of $(x+1)^4$ by evaluating at $x = 1$.

Exit Problem

1. Simplify $(3x^4)^3$ and check by evaluating this expression and your simplification at $x = 2$.
2. What is the coefficient of x^5y^2 in $2x - y)^7$

Key Concepts

- Power of a monomial
- Power of a binomial
- The Binomial Theorem
- Pascal's triangle

1.2.5: Dividing Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Reduce fractions
- Divide a monomial by a monomial
- Divide a polynomial by a monomial
- Divide polynomials using long division

Be Prepared

Before you get started, take this readiness quiz.

1. Find the greatest common factor (GCF) of 35 and 49.

2. Reduce $\frac{35}{49}$.

Reducing Fractions

Let's say we want to reduce the fraction $\frac{60}{84}$. We need to find the largest integer that goes into 60 and 84 at the same time. Note that 3 goes into 60 and 84 since

$$60 = 3 \cdot 20 \qquad \text{and} \qquad 84 = 3 \cdot 28.$$

So 3 is a common factor. However, 4 goes into 20 and 28 and so we can write

$$60 = 3 \cdot (4 \cdot 5) \qquad \text{and} \qquad 84 = 3 \cdot (4 \cdot 7)$$

or

$$60 = (3 \cdot 4) \cdot 5 \qquad \text{and} \qquad 84 = (3 \cdot 4) \cdot 7.$$

This means that $3 \cdot 4 = 12$ goes into 60 and 84 and we can write:

$$60 = 12 \cdot 5 \qquad \text{and} \qquad 84 = 12 \cdot 7.$$

So 12 is also a common factor of 60 and 84. Because 5 and 12 do not have any common factor other than 1, the greatest common factor of 60 and 84 is 12. Now we use the cancelation property:

$$\frac{a \cdot b}{a \cdot c} = \frac{b}{c}$$

to simplify $\frac{60}{84}$:

$$\begin{aligned}\frac{60}{84} &= \frac{12 \cdot 5}{12 \cdot 7} \\ &= \frac{5}{7}.\end{aligned}$$

The reduction is $\frac{5}{7}$. We want to keep the fraction format. The two fractions $\frac{60}{84}$ and $\frac{5}{7}$ are said to be **equivalent** as they represent the same number. If we had used the common factor 3 that we found in the beginning, we would have gotten

$$\begin{aligned}\frac{60}{84} &= \frac{3 \cdot 20}{3 \cdot 28} \\ &= \frac{20}{28}\end{aligned}$$

so that $\frac{60}{84}$, $\frac{5}{7}$ and $\frac{20}{28}$ are all equivalent, but $\frac{5}{7}$ is the one that cannot be simplified any further. This is the reduction we want.

Example 1.2.5.1

Simplify:

a. $\frac{36}{21}$

b. $\frac{72}{44}$

c. $\frac{8}{64}$

Solution

a. The greatest common factor of 36 and 21 is 3.

$$\begin{aligned}\frac{36}{21} &= \frac{3 \cdot 12}{3 \cdot 7} \\ &= \frac{12}{7}\end{aligned}$$

b. The greatest common factor of 72 and 44 is 4.

$$\begin{aligned}\frac{72}{44} &= \frac{4 \cdot 18}{4 \cdot 11} \\ &= \frac{18}{11}\end{aligned}$$

c. The greatest common factor of 8 and 64 is 8.

$$\begin{aligned}\frac{8}{64} &= \frac{8 \cdot 1}{8 \cdot 8} \\ &= \frac{1}{8}\end{aligned}$$

? Try It 1.2.5.2

Simplify:

a. $\frac{108}{16}$

b. $\frac{25}{100}$

c. $\frac{81}{3}$

Answer

a. $\frac{27}{4}$

b. $\frac{1}{4}$

c. 27

Dividing Monomials

Now we will look at some examples where dividing two monomials results in a monomial (which is not always the case!!).

Consider	$\frac{x^5}{x^2}$
What do they mean?	$=\frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$
Use the Equivalent Fractions Property.	$=\frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$
Simplify.	$=x^3$
Note.	$\frac{x^5}{x^2}=\frac{x^2x^3}{x^2}=x^3$ Reducing the fraction.

Example 1.2.5.3

Find the quotient $54a^2b^3 \div (-6ab^2)$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$54a^2b^3 \div (-6ab^2)$
Rewrite as a fraction.	$=\frac{54a^2b^3}{-6ab^2}$
Use fraction multiplication.	$=\frac{54}{-6}\cdot\frac{a^2}{a}\cdot\frac{b^3}{b^2}$
Reduce the fraction.	$=-9ab$

Try It 1.2.5.4

Find the quotient $-72a^7b^3 \div (8a^5b^2)$.

Answer

$-9a^2b$

Try It 1.2.5.5

Find the quotient $-63c^8d^3 \div (7c^2d)$.

Answer

$-9c^6d^2$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example 1.2.5.6

Find the quotient $\frac{14x^7y^{12}}{21x^3y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^3y^6}$
Use fraction multiplication.	$=\frac{14}{21}\cdot\frac{x^7}{x^3}\cdot\frac{y^{12}}{y^6}$
Reduce.	$=\frac{2}{3}x^4y^6$

Dividing a Polynomial by a Monomial

Do example with numbers here for distribution of division

Now that we know how to divide a monomial by a monomial, the next procedure is to divide a polynomial of two or more terms by a monomial. The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. So we'll start with an example to review fraction addition.

The sum $\frac{3}{7}+\frac{2}{7}=\frac{3+2}{7}$. So it is also true that $\frac{3+2}{7}=\frac{3}{7}+\frac{2}{7}$, so you can distribute the divion by 7. You may also recall that division by 7 is the same as multiplication by $\frac{1}{7}$ so, $\frac{3+2}{7}=\frac{1}{7}(3+2)=\frac{1}{7}\cdot 3+\frac{1}{7}\cdot 2=\frac{3}{7}+\frac{2}{7}$, or in words because you distribute multiplication over addition and subtraction, and division is multiplication is division by a reciprocal, you can also distribute division over addition and subtraction. Here is another example with a variable.

The sum $\frac{y}{5}+\frac{2}{5}$ simplifies to $\frac{y+2}{5}$. Now we will do this in reverse to split a single fraction into separate fractions. For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$. Or, thinking of this as distibution: $\frac{y+2}{5}=(y+2)\cdot\frac{1}{5}=\frac{y}{5}+\frac{2}{5}$.

This is the "reverse" of fraction addition and it states that if a, b, and c are numbers where $c\neq 0$, then $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. We will use this to divide polynomials by monomials.

Division of a polynomial by a monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial, or in other words, we distribute the division over addition and subtraction.

Example 1.2.5.7

Find the quotient $(18x^3y-36xy^2)\div(-3xy)$.

Solution

	$(18x^3y-36xy^2)\div(-3xy)$
Rewrite as a fraction.	$=\frac{18x^3y-36xy^2}{-3xy}$
Divide each term by the divisor (distribute the division). Be careful with the signs!	$=\frac{18x^3y}{-3xy}-\frac{36xy^2}{-3xy}$
Simplify.	$=-6x^2+12y$

Try It 1.2.5.8

Find the quotient $(32a^2b-16ab^2)\div(-8ab)$.

Answer

$-4a+2b$

Try It 1.2.5.9

Find the quotient $(-48a^8b^4-36a^6b^5)\div(-6a^3b^3)$.

Answer

$8a^5b+6a^3b^2$

We may, in certain situations, also divide a polynomial by a binomial as in the following example.

Example 1.2.5.10

Find the quotient $(6(x-2)(3x-2))\div(3(x-2))$.

Solution

	$(6(x-2)(3x-2))\div(3(x-2))$

Rewrite as a fraction.	$=\dfrac{6(x-2))(3x-2)}{3(x-2)}$
Identify the common factors.	$=\dfrac{3(x-2)2(3x-2)}{3(x-2)}$
Simplify.	$=2(3x-2)$ or $6x-4$

Try It 1.2.5.11

Find the quotient $((-4(2x-1))(2x+7))\div 2(2x+7)$.

Answer

$-2(2x-1)$

Try It 1.2.5.12

Find the quotient $((4(2x-1))(2x+7))\div -4(2x-1)$.

Answer

$-(2x+7)$

Dividing Polynomials Using Long Division

Divide a polynomial by a binomial, we follow a procedure very similar to long division of numbers. So let's look carefully the steps we take when we divide a 3-digit number, 875, by a 2-digit number, 25.

$$\begin{array}{r} 35 \\ 25\overline{)\,875} \\ \underline{-750} \\ 125 \\ \underline{-125} \\ 0 \end{array}$$

The quotient is 35 and the remainder is 0. We check division by multiplying the quotient by the divisor and then adding the remainder. If we did the division correctly, the result should equal the dividend.

$$35\cdot 25+0=875\checkmark$$

Now we will divide a trinomial by a binomial. As you read through the example, notice how similar the steps are to the numerical example above.

Example 1.2.5.13

Find the quotient and the remainder of $(x^2+9x+20)\div(x+5)$.

Solution

	$(x^2+9x+20)\div(x+5)$
Write it as a long division problem. Be sure the dividend is in standard form.	$x+5\overline{)\,x^2+9x+20}$
Divide x^2 by x. It may help to ask yourself, "What do I need to multiply x by to get x^2?" Put the answer, x, in the quotient over the x term.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \end{array}$
Multiply x by $x+5$. Change the sign of each term and put the answer under x^2+9x.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2-5x} \end{array}$

Add it to x^2+9x.	$\begin{array}{r} x \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2-5x} \\ 4x+20 \end{array}$
Divide $4x$ by x. It may help to ask yourself, “What do I need to multiply x by to get $4x$?” Put the answer, 4, in the quotient over the constant term.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2-5x} \\ 4x+20 \end{array}$
Multiply 4 by $x+5$. Change the sign of each term and put the answer under $4x+20$.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2-5x} \\ 4x+20 \\ \underline{-4x-20} \end{array}$
Add it to $4x+20$.	$\begin{array}{r} x+4 \\ x+5\overline{)\,x^2+9x+20} \\ \underline{-x^2+(-5x)} \\ 4x+20 \\ \underline{-4x-20} \\ 0 \end{array}$
Check.	Multiply the quotient by the divisor. $(x+4)(x+5)$ You should get the dividend. $x^2+9x+20\checkmark$
Conclude.	The quotient of $(x^2+9x+20)\div(x+5)$ is $x+4$, and the remainder is 0.

Try It 1.2.5.14

Find the quotient and the remainder of $(y^2+10y+21)\div(y+3)$.

Answer

The quotient is $y+7$. The remainder is 0.

Try It 1.2.5.15

Find the quotient and the remainder of $(m^2+9m+20)\div(m+4)$.

Answer

The quotient is $m+5$. The remainder is 0.

Look back at the dividends in previous examples. The terms were written in descending order of degrees, and there were no missing degrees. The dividend in this example will be x^4-x^2+5x-6 . It is missing an x^3 term. We will add in $0x^3$ as a placeholder.

Example 1.2.5.16

Find the quotient and the remainder of $(x^4-x^2+5x-6)\div(x+2)$.

Solution

Notice that there is no x^3 term in the dividend. We will add $0x^3$ as a placeholder.

$(x^4-x^2+5x-6)\div(x+2)$

	$(x^4 - x^2 + 5x - 6) \div (x+2)$
Write it as a long division problem. Be sure the dividend is in standard form with placeholders for missing terms.	$x+2\overline{\big)\ x^4+0x^3-x^2+5x-2}$
Divide x^4 by x. Put the answer, x^3, in the quotient over the x^3 term. Multiply x^3 by $x+2$. Change the sign of each term and put the answer under x^4+0x^3. Line up the like terms. Add them.	$\begin{array}{r} x^3 \qquad\qquad\quad \\ x+2\overline{\big)\ x^4+0x^3-x^2+5x-2} \\ \underline{-x^4-2x^3 \qquad\qquad\quad} \\ -2x^3-x^2+5x-2 \end{array}$
Divide $-2x^3$ by x. Put the answer, $-2x^2$, in the quotient over the x^2 term. Multiply $-2x^2$ times $x+1$. Change the sign of each term and put the answer under Line up the like terms. Add them.	$\begin{array}{r} x^3-2x^2 \qquad\quad \\ x+2\overline{\big)\ x^4+0x^3-x^2+5x-2} \\ \underline{-x^4-2x^3 \qquad\qquad\quad} \\ -2x^3-x^2+5x-2 \\ \underline{2x^3+4x^2 \qquad\quad} \\ 3x^2+5x-2 \end{array}$
Divide $3x^2$ by x. Put the answer, $3x$, in the quotient over the x term. Multiply $3x$ times $x+1$. Line up the like terms. Subtract and bring down the next term.	$\begin{array}{r} x^3-2x^2+3x \quad \\ x+2\overline{\big)\ x^4+0x^3-x^2+5x-2} \\ \underline{-x^4-2x^3 \qquad\qquad\quad} \\ -2x^3-x^2+5x-2 \\ \underline{2x^3+4x^2 \qquad\quad} \\ 3x^2+5x-2 \\ \underline{-3x^2-6x \quad\ } \\ -x-2 \end{array}$
Divide $-x$ by x. Put the answer, -1, in the quotient over the constant term. Multiply -1 times $x+1$. Line up the like terms. Change the signs, add.	$\begin{array}{r} x^3-2x^2+3x-1 \\ x+2\overline{\big)\ x^4+0x^3-x^2+5x-2} \\ \underline{-x^4-2x^3 \qquad\qquad\quad} \\ -2x^3-x^2+5x-2 \\ \underline{2x^3+4x^2 \qquad\quad} \\ 3x^2+5x-2 \\ \underline{-3x^2-6x \quad\ } \\ -x-2 \\ \underline{x+2} \\ 0 \end{array}$
Check.	Multiply $(x+2)(x^3-2x^2+3x-1-4x+2)$. The result should be x^4-x^2+5x-6.
Conclude.	The quotient of $(x^4-x^2+5x-6) \div (x+2)$ is x^3-2x^2+3x-1, and the remainder is 0.

? Try It 1.2.5.17

Find the quotient and the remainder of $(x^4-7x^2+7x+6) \div (x+3)$.

Answer

The quotient is $x^3-3x^2+2x+1+3x+3$. The remainder is 0.

? Try It 1.2.5.18

Find the quotient and the remainder of $(x^4-11x^2-7x-6) \div (x+3)$.

Answer

The quotient is $x^3-3x^2-2x-1-3x+3$. The remainder is 0.

In the next example, we will divide by $2a-3$. As we divide, we will have to consider the constants as well as the variables.

Example 1.2.5.19

Find the quotient and the remainder of $(8a^3+27)\div(2a+3)$.

Solution

This time we will show the division all in one step. We need to add two placeholders in order to divide.

$$\begin{array}{r} 4a^2-6a+9 \\ 2a+3\overline{\big)\ 8a^3+0a^2+0a+27} \\ \underline{-8a^3-12a^2\qquad\qquad} \\ -12a^2+0a+27 \\ \underline{12a^2+18a\quad} \\ 18a+27 \\ \underline{-18a-27} \\ 0 \end{array}$$

To check, multiply $(2a+3)(4a^2-6a+9)$. The result should be $8a^3+27$.

The quotient of $(8a^3+27)\div(2a+3)$ is $4a^2-6a+9$ and the remainder is 0.

Try It 1.2.5.20

Find the quotient and the remainder of $(x^3-64)\div(x-4)$.

Answer

The quotient is $x^2+4x+16$. The remainder is 0.

Try It 1.2.5.21

Find the quotient: $(125x^3-8)\div(5x-2)$.

Answer

The quotient is $25x^2+10x+4$. The remainder is 0.

When we divided 875 by 25, we had remainder zero. But sometimes division of numbers does leave a remainder different from zero. The same is true when we divide polynomials. In the next example, we'll have a division that leaves a remainder that is not zero. The degree of the remainder is always less than the degree of the divisor. To check, we need to verify that $(\text{quotient})(\text{divisor})+\text{remainder}=\text{dividend}$.

Example 1.2.5.22

Find the quotient and remainder when $2x^3+3x^2+x+8$ is divided by $x+2$.

Solution

	$(2x^3+3x^2+x+8)\div(x+2)$

Add it to $4x+20$.	$$\begin{array}{r} 2x^2-x+3 \\ x+2\overline{)\ 2x^3+3x^2+x+8} \\ \underline{-2x^3-4x^2} \\ -x^2+x+8 \\ \underline{x^2+2x} \\ 3x+8 \\ \underline{-3x-6} \\ 2 \end{array}$$
Check.	$$\begin{aligned} (\text{quotient})(\text{divisor})+\text{remainder} &= \text{dividend} \\ (2x^2-1x+3)(x+2)+2 &\stackrel{?}{=} 2x^3+3x^2+x+8 \\ 2x^3-x^2+3x+4x^2-2x+6+2 &\stackrel{?}{=} 2x^3+3x^2+x+8 \\ 2x^3+3x^2+x+8 &= 2x^3+3x^2+x+8\checkmark \end{aligned}$$
Conclude.	The quotient of $(2x^3+3x^2+x+8)\div(x+2)$ is $2x^2-x+3$, and the remainder is 2.

Try It 1.2.5.23

Find the quotient and remainder when $3x^3+10x^2+6x-2$ is divided by $x+2$.

Answer

The quotient is $3x^2+4x-2$. The remainder is 2.

Example 1.2.5.24

Find the quotient and remainder when $4x^3+5x^2-5x+3$ is divided by $x+2$.

Answer

The quotient is $4x^2-3x+1$. The remainder is 1.

Writing Exercises 1.2.5.25

1. Explain why you can distribute division over addition and subtraction.
2. Can you divide a polynomial by a monomial and not get a polynomial? Give an example.
3. Give another example of the type in the last exercise.
4. What is the first step to reducing a fraction (numerator and denominator whole numbers)? Explain.

Exit Problem

1. Divide $(27x^3y^7-18x^7y^3+9x^2y^3)\div(9x^2y^3)$.
2. Divide x^3-8x+3 by $x+5$ using long division.

Key Concepts

- **Dividend, divisor, quotient, remainder**
- **Division of a polynomial by a monomial**
- **Long division of polynomials**

1.2.6: The Greatest Common Factor and Factoring by Grouping

Learning Objectives

By the end of this section, you will be able to:

- Find the greatest common factor of two or more polynomials
- Factor the greatest common factor from a polynomial
- Factor by grouping

Be Prepared

Before you get started, take this readiness quiz.

1. Factor 56 into primes.

2. Find the least common multiple (LCM) of 18 and 24.

3. Multiply $-3a(7a+8b)$.

Introduction to Factoring

We saw in the last section that we may reduce fractions if the numerator and denominator have the same factor.

For example,

$$\frac{3\cdot 5}{2\cdot 3}=\frac{5}{2},$$
$$\frac{xy^2z}{yz}=xy$$

and

$$\frac{(x-2)(3x-4)}{2(3x-4)}=\frac{x-2}{2}=\frac{x}{2}-1.$$

Note that in the last example the numerator is actually of degree 2 since $(x-2)(3x-4)=3x^2-10x+8$ and we can also see then that

$$\frac{3x^2-10x+8}{6x-8}=\frac{(x-2)(3x-4)}{2(3x-4)}=\frac{x-2}{2}=\frac{x}{2}-1.$$

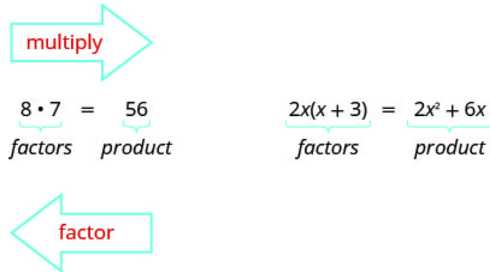
So, if we were given the expression $\frac{3x^2-10x+8}{6x-8}$ to simplify, the best way to go about it is to recognize the numerator and denominator are products of binomials! This is the same as it is with numbers. For example, to reduce $\frac{121}{77}$ you can recognize the factors of the numerator and denominator:

$$\frac{121}{77}=\frac{11\cdot 11}{7\cdot 11}=\frac{11}{7}.$$

Writing a polynomial as a product of other polynomials (of smaller degree) is the key to reducing fractions. We will spend some time practicing writing them in such a way (which is called **factoring**). Factoring will have other applications as we will see later. This section focuses writing a polynomial in an equivalent factored form.

Finding the Greatest Common Factor of a Polynomial

Earlier we multiplied factors together to get a *product*. Now, we will reverse this process; we will start with a product and then break it down into its factors. Splitting a product into factors is called *factoring*.



Above, we have recalled how to factor numbers to find the least common multiple (LCM) of two or more numbers. Our goal is to be able to factor polynomials. The first step is to find the greatest common factor of a given polynomial. The method we use is similar to what we can use to find the LCM.

Definition 1.2.6.1

1. If we write a polynomial P as a product of polynomials, we say that we have **factored** P.
2. A polynomial F is a **factor** of P if we can write P as $P = F \cdot G$ for some polynomial G.
3. The **greatest common factor (GCF) of two or more monomials** is a monomial F that satisfies the following conditions:
   - F is a factor of all the monomials, that is, F is a common factor, and
   - any other common factor of all the monomials is a factor of F.
4. The **greatest common factor (GCF) of a polynomial** is the GCF of its terms.

While we say *the* greatest common factor, there are actually two: one with a positive coefficient and one with a negative coefficient.

Factor as a noun and a verb

We use "factor" as both a noun and a verb:

Noun:	7 is a factor of 14
Verb:	factor 3 from $3a + 3$

We summarize the steps we use to find the greatest common factor.

Finding the Greatest Common Factor (GCF) of polynomials

1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
2. List all factors—matching common factors in a column. In each column, circle the common factors (this is suggested as a way of 'book-keeping').
3. Collect the factors (including repeats) that all polynomials share.
4. Multiply the factors.

The next example will show us the steps to find the greatest common factor of three monomials.

Example 1.2.6.2

Find the greatest common factor of $21x^3$, $9x^2$, and $15x$.

Solution

$21x^3, 9x^2, 15x$

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3, 9x^2, 15x$ $21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
Answer the question.	The GCF of $21x^3$, $9x^2$, and $15x$ is $3x$.

Factor each coefficient into primes and write the variables with exponents in expanded form. Circle the common factors in each column. Bring down the common factors.	$21x^3 = 3 \cdot 7 \cdot x \cdot x \cdot x$ $9x^2 = 3 \cdot 3 \cdot x \cdot x$ $15x = 3 \cdot 5 \cdot x$ $\text{GCF} = 3 \cdot x$
Multiply the factors.	$\text{GCF} = 3x$
Answer the question.	The GCF of $21x^3$, $9x^2$, and $15x$ is $3x$.

Try It 1.2.6.3

Find the greatest common factor of $25m^4$, $35m^3$, and $20m^2$.

Answer

The GCF is $5m^2$.

Try It 1.2.6.4

Find the greatest common factor of $14x^3$, $70x^2$, and $105x$.

Answer

The GCF is $7x$.

Factoring the Greatest Common Factor from a Polynomial

It is sometimes useful to represent a number as a product of factors, for example, 12 as $2 \cdot 6$ or $3 \cdot 4$. In algebra, it can also be useful to represent a polynomial in factored form. We will start with a product, such as $3x^2 + 15x$, and end with a product of its factors, $3x(x+5)$. To do this we apply the Distributive Property "in reverse."

We state the Distributive Property here just as you saw it in earlier chapters and "in reverse."

Distributive Property

If a, b, and c are real numbers, then

$$a(b+c) = ab + ac$$

and

$$ab + ac = a(b+c).$$

The form on the left is used to multiply. The form on the right is used to factor.

So how do you use the Distributive Property to (partially) factor a polynomial? You just find the GCF of all the terms and write the polynomial as a product!

Example 1.2.6.5

Factor $8m^3 - 12m^2n + 20mn^2$.

Solution

$8m^3 - 12m^2n + 20mn^2$

Find the GCF of all the termsof the polynomial.	Find the GCF of $8m^3$, $12m^2n$, and $20mn^2$.	$8m^3 = 2 \cdot 2 \cdot 2 \cdot m \cdot m \cdot m$ $12m^2n = 2 \cdot 2 \cdot 3 \cdot m \cdot m \cdot n$ $20mn^2 = 2 \cdot 2 \cdot 5 \cdot m \cdot n \cdot n$ $GCF = 2 \cdot 2 \cdot m$ $GCF = 4m$
Rewrite each term as a product using the GCF.	Rewrite $8m^3$, $12m^2n$, and $20mn^2$ as products of their GCF, $4m$.	$8m^3 - 12m^2n + 20mn^2$ $= 4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$
Use the "reverse" Distributive Property to factor the expression.		$= 4m(2m^2 - 3mn + 5n^2)$
Check by multiplying the factors.		$4m(2m^2 - 3mn + 5n^2)$ $= 4m \cdot 2m^2 - 4m \cdot 3mn + 4m \cdot 5n^2$ $= \mathbf{8m^3 - 12m^2n + 20n^2}$ ✓

? Try It 1.2.6.6

Factor $9xy^2 + 6x^2y^2 + 21y^3$.

Answer

$3y^2(3x + 2x^2 + 7y)$

? Try It 1.2.6.7

Factor $3p^3 - 6p^2q + 9pq^3$.

Answer

$3p(p^2 - 2pq + 3q^2)$

Factoring the Greatest Common Factor (GCF) from a polynomial

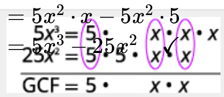
1. Find the GCF of all the terms of the polynomial.
2. Rewrite each term as a product using the GCF.
3. Use the “reverse” Distributive Property to factor the polynomial.
4. Check by multiplying the factors.

✓ Example 1.2.6.8

Factor out the GCF of $5x^3 - 25x^2$.

Solution

	$5x^3 - 25x^2$
Find the GCF of $5x^3$ and $25x^2$.	$5x^3 = 5 \cdot x \cdot x \cdot x$ $25x^2 = 5 \cdot 5 \cdot x \cdot x$ $GCF = 5 \cdot x \cdot x$ The GCF is $5x^2$.
Rewrite each term.	$5x^3 - 25x^2$ $= 5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$= 5x^2(x - 5)$

Check.	$5x^2(x-5)$ $= 5x^2 \cdot x - 5x^2 \cdot 5$ $= 5x^3 - 25x^2$ ✓
Find the GCF of $5x^3$ and $25x^2$.	 The GCF is $5x^2$.
Rewrite each term.	$5x^3 - 25x^2$ $= 5x^2 \cdot x - 5x^2 \cdot 5$
Factor the GCF.	$= 5x^2(x-5)$
Check.	$5x^2(x-5)$ $= 5x^2 \cdot x - 5x^2 \cdot 5$ $= 5x^3 - 25x^2$ ✓

Try It 1.2.6.9

Factor out the GCF of $2x^3+12x^2$.

Answer

$2x^2(x+6)$

Try It 1.2.6.10

Factor out the GCF of $6y^3-15y^2$.

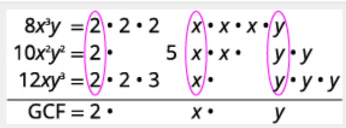
Answer

$3y^2(2y-5)$

Example 1.2.6.11

Factor out the GCF of $8x^3y-10x^2y^2+12xy^3$.

Solution

	$8x^3y-10x^2y^2+12xy^3$
The GCF of $8x^3y$, $-10x^2y^2$, and $12xy^3$ is $2xy$.	 The GCF is $2xy$.
Rewrite each term using the GCF, $2xy$.	$8x^3y-10x^2y^2+12xy^3$ $= 2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$
Factor the GCF.	$= 2xy(4x^2-5xy+6y^2)$
Check.	$2xy(4x^2-5xy+6y^2)$ $= 2xy \cdot 4x^2 - 2xy \cdot 5xy + 2xy \cdot 6y^2$ $= 8x^3y-10x^2y^2+12xy^3$ ✓

Try It 1.2.6.12

Factor out the GCF of $15x^3y-3x^2y^2+6xy^3$.

Answer

$3xy(5x^2-xy+2y^2)$

Try It 1.2.6.13

Factor out the GCF of $8a^3b+2a^2b^2-6ab^3$.

Answer

$2ab(4a^2+ab-3b^2)$

When the leading coefficient is negative, we factor the negative out as part of the GCF.

Example 1.2.6.14

Factor out the GCF of $-4a^3+36a^2-8a$.

Solution

The leading coefficient is negative, so the GCF will be negative.

	$-4a^3+36a^2-8a$
Rewrite each term using the GCF, $-4a$.	$=-4a\cdot a^2-4a\cdot(-9a)-4a\cdot 2$
Factor the GCF.	$=-4a(a^2-9a+2)$
Check.	$-4a(a^2-9a+2)$ $=-4a\cdot a^2-(-4a)\cdot 9a+(-4a)\cdot 2$ $=-4a^3+36a^2-8a \quad \checkmark$

Try It 1.2.6.15

Factor out the GCF of $-4b^3+16b^2-8b$.

Answer

$-4b(b^2-4b+2)$

Try It 1.2.6.16

Factor out the GCF of $-7a^3+21a^2-14a$.

Answer

$-7a(a^2-3a+2)$

In the next example, we extend the idea of factoring out the GCF to a binomial.

Example 1.2.6.17

Factor out the common factor of the two terms of $3y(y+7)-4(y+7)$.

Solution

	$3y(y+7)-4(y+7)$
The binomial $y+7$ is a common factor of the two terms.	$=3y(y+7)-4(y+7)$
Factor $(y+7)$.	$=(y+7)(3y-4)$

Check.	$3y(y+7)-4(y+7)$ $=3y\cdot y+3y\cdot 7-4y-4\cdot 7$ $=3y^2+21y-4y-28$ $=3y^2+17y-28$ $(y+7)(3y-4)$ $=y\cdot 3y+y(-4)+7\cdot 3y+7\cdot(-4)$ $=3y^2-4y+21y-28$ $=3y^2+17y-28\quad\checkmark$

Try It 1.2.6.18

Factor out the common factor of the two terms of $4m(m+3)-7(m+3)$.

Answer

$(m+3)(4m-7)$

Try It 1.2.6.19

Factor out the common factor of the two terms of $8n(n-4)+5(n-4)$.

Answer

$(n-4)(8n+5)$

Factoring by Grouping

Sometimes there is no common factor of all the terms of a polynomial. When there are four terms we separate the polynomial into two parts with two terms in each part. Then look for the **GCF** in each part. If the polynomial can be factored, you will find a common factor emerges from both parts. Not all polynomials can be factored.

Example 1.2.6.20

Factor $xy+3y+2x+6$ by grouping.

Solution

	$xy+3y+2x+6$	
Group terms with common factors.	Is there a greatest common factor of all four terms? No, so let's separate the first two terms from the second two.	$xy+3y+2x+6$ $=\underbrace{xy+3y}+\underbrace{2x+6}$
Factor out the common factor in each group.	Factor the GCF from the first two terms. Factor the GCF from the second two terms.	$=y(x+3)+\underbrace{2x+6}$ $=y(x+3)+2(x+3)$
Factor the common factor from the expression.	Notice that each terms has a common factor of $x+3$.	$=y(x+3)+2(x+3)$ $=(x+3)(y+2)$
Check.	Multiply $(x+3)(y+2)$. Is the product the original expression?	$(x+3)(y+2)$ $=xy+2x+3y+6$ $=xy+3y+2x+6\quad\checkmark$

Try It 1.2.6.21

Factor $xy + 8y + 3x + 24$ by grouping.

Answer

$(x+8)(y+3)$

Try It 1.2.6.22

Factor $ab + 7b + 8a + 56$ by grouping.

Answer

$(a+7)(b+8)$

Factoring by grouping

1. Group terms with common factors.
2. Factor out the common factor in each group.
3. Factor the common factor from the polynomial.
4. Check by multiplying the factors.

Example 1.2.6.23

Factor by grouping:

a. $x^2 + 3x - 2x - 6$

b. $6x^2 - 3x - 4x + 2$

Solution

a.

	$x^2 + 3x - 2x - 6$
There is no GCF in all four terms.	$= x^2 + 3x - 2x - 6$
Separate into two parts.	$= \underbrace{x^2 + 3x}\underbrace{-2x - 6}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$= x(x+3) - 2(x+3)$
Factor out the common factor, $x+3$.	$= (x+3)(x-2)$
Check.	$(x+3)(x-2)$ $= x^2 - 2x + 3x - 6$ $= x^2 + x - 6$

b.

	$6x^2 - 3x - 4x + 2$
There is no GCF in all four terms.	$= 6x^2 - 3x - 4x + 2$
Separate into two parts.	$= \underbrace{6x^2 - 3x}\underbrace{-4x + 2}$

There is no GCF in all four terms.	$6x^2-3x-4x+2$
Separate into two parts.	$\underbrace{6x^2-3x}\ \underbrace{-4x+2}$
Factor the GCF from both parts. Be careful with the signs when factoring the GCF from the last two terms.	$=3x(2x-1)-2(2x-1)$
Factor out the common factor, $2x-1$.	$=(2x-1)(3x-2)$
Check.	$(2x-1)(3x-2)$ $=6x^2-4x-3x+2$ $=6x^2-3x-4x+2$

? Try It 1.2.6.24

Factor by grouping:

a. $x^2+2x-5x-10$

b. $20x^2-16x-15x+12$

Answer

a. $(x-5)(x+2)$
b. $(5x-4)(4x-3)$

? Try It 1.2.6.25

Factor by grouping:

a. $y^2+4y-7y-28$

b. $42m^2-18m-35m+15$

Answer

a. $(y+4)(y-7)$
b. $(7m-3)(6m-5)$

? Writing Exercises 1.2.6.26

1. What does it mean to say a polynomial is in factored form?
2. What is a GCF?
3. Give your own example of factoring out the greatest common factor from a polynomial with two variables.
4. In your example above, factor out the negative of your GCF.
5. Explain how you know you have the *greatest* common factor.
6. How can you check your factoring process? Give an example.
7. When might you find it useful to factor a polynomial?
8. The greatest common factor of 3636 and 6060 is 1212. Explain what this means.
9. When should you look to factor by grouping? Is it always possible?

Exit Problem

a. Factor the GCF out $100x^3y^2-6xy$.

b. Factor $10x^2+5x-4x-2$ by grouping.

Key Concepts

- **Factor as a Noun and a Verb:** We use “factor” as both a noun and a verb.

$$\begin{array}{ll}\text{Noun:} & 7 \text{ is a factor of } 14\\ \text{Verb:} & \text{factor } 3 \text{ from } 3a+3\end{array}$$

- **How to find the greatest common factor (GCF) of two polynomials.**
  1. Factor each coefficient into primes. Write all variables with exponents in expanded form.
  2. List all factors—matching common factors in a column. In each column, circle the common factors.
  3. Bring down the common factors that all polynomials share.
  4. Multiply the factors.
- **Distributive Property:** If a, b and c are real numbers, then

$$a(b+c)=ab+ac \quad \text{and} \quad ab+ac=a(b+c)$$

  The form on the left is used to multiply. The form on the right is used to factor.
- **How to factor the greatest common factor from a polynomial.**
  1. Find the GCF of all the terms of the polynomial.
  2. Rewrite each term as a product using the GCF.
  3. Use the “reverse” Distributive Property to factor the polynomial.
  4. Check by multiplying the factors.
- **How to factor by grouping.**
  1. Group terms with common factors.
  2. Factor out the common factor in each group.
  3. Factor the common factor from the polynomial.
  4. Check by multiplying the factors.

1.2.7: Factoring Trinomials

Learning Objectives

By the end of this section, you will be able to:

- Factor trinomials of the form x^2+bx+c
- Factor trinomials of the form ax^2+bx+c using trial and error
- Factor trinomials of the form ax^2+bx+c using the 'ac' method
- Factor using substitution

Be Prepared

Before you get started, take this readiness quiz.

1. Find all the factors of 72.

2. Find the product $(3y+4)(2y+5)$.

3. Simplify $-9(6)$ and $-9(-6)$.

Factoring Trinomials of the Form x^2+bx+c

You have already learned how to multiply binomials using **FOIL**. Now you'll need to "undo" this multiplication. To factor the trinomial means to start with the product, and end with the factors.

To figure out how we would factor a **trinomial** of the form x^2+bx+c, such as x^2+5x+6 and factor it to $(x+2)(x+3)$, let's start with two general binomials of the form $(x+m)$ and $(x+n)$.

	$(x+m)(x+n)$
Foil to find the product.	$x^2+mx+nx+mn$
Factor the GCF from the middle terms.	$x^2+(m+n)x+mn$
Our trinomial is of the form x^2+bx+c.	$\underbrace{x^2+(m+n)x+mn}_{x^2+bx+c}$

This tells us that to factor a trinomial of the form x^2+bx+c, we need two factors $(x+m)$ and $(x+n)$ where the two numbers m and n multiply to c and add to b.

Example 1.2.7.1

Factor $x^2+11x+24$.

Solution

		$x^2+11x+24$	
1\)	Write the factors as two binomials with first terms x.	Write two sets of parentheses and put x as the first term.	$x^2+11x+24$ $=\underbrace{(x+\square)}_{(x+m)}\underbrace{(x+\square)}_{(x+n)}$
1\)	Identify b and c.	$\underbrace{x^2+11x+24}_{ax^2+bx+c}$	$b=11$ $c=24$

			$x^2+11x+24$
1\)	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn &= c \\ m+n &= b \end{cases}$	Find two numbers m and n that multiply to 24 and add to 11. $\begin{cases} mn &= 24 \\ m+n &= 11 \end{cases}$	Factors of 24: $1, 24$; $2, 12$; $3, 8$; $4, 6$. Sum of factors: $1+24=25$; $2+12=14$; $3+8=11^*$; $4+6=10$. $m=3$, $n=8$
1\)	Use m and n as the last terms of the factors.	Use 3 and 8 as the last terms of the binomials.	$\underbrace{(x+3)}_{(x+m)}\underbrace{(x+8)}_{(x+n)}$
1\)	Check by multiplying the factors.		$(x+3)(x+8)$ $=x^2+8x+3x+24$ $=x^2+11x+24$ ✓
1\)	Conclude.		The factorization is $(x+3)(x+8)$.

? Try It 1.2.7.2

Factor $q^2+10q+24$.

Answer

The factorization is $(q+4)(q+6)$.

? Try It 1.2.7.3

Factor $t^2+14t+24$.

Answer

The factorization is $(t+2)(t+12)$.

Let's summarize the steps we used to find the factors.

Factoring x^2+bx+c

1. Write the factors as two binomials with first terms x. $\quad x^2+bx+c$, $(x+\square)(x+\square)$
2. Find two numbers m and n that
   - multiply to c: $\quad mn=c$
   - add to b: $\quad m+n=b$
3. Use m and n as the last terms of the factors. $\quad (x+m)(x+n)$
4. Check by multiplying the factors.

In the first example, all terms in the trinomial were positive. What happens when there are negative terms? Well, it depends which term is negative. Let's look first at trinomials with only the middle term negative.

How do you get a *positive product* and a *negative sum*? We use two negative numbers.

Example 1.2.7.4

Factor $y^2 - 11y + 28$.

Solution

	$y^2 - 11y + 28$
Write the factors as two binomials with first terms y.	$y^2 - 11y + 28$ $= \underbrace{(y + \square)}_{(y+m)}\underbrace{(y + \square)}_{(y+n)}$
Identify b and c.	$\underbrace{y^2 - 11y + 28}_{ay^2+by+c}$ $b = -11$ $c = 28$
Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn &= c \\ m+n &= b \end{cases}$	Find m and n such that $\begin{cases} mn &= 28 \\ m+n &= -11 \end{cases}$ Factors of 28 / Sum of factors: $-1, -28$: $-1 + (-28) = -29$ $-2, -14$: $-2 + (-14) = -16$ $-4, -7$: $\mathbf{-4 + (-7) = -11^*}$ $m = -4$ $n = -7$
Use m and n as the last terms of the factors.	$\underbrace{(y-4)}_{(y+m)}\underbrace{(y-7)}_{(y+n)}$
Check.	$(y-4)(y-7)$ $= y^2 - 7y - 4y + 28$ $= y^2 - 11y + 28$ ✓
Conclude.	The factorization is $(y-4)(y-7)$.

Try It 1.2.7.5

Factor $u^2 - 9u + 18$.

Answer

The factorization is $(u-3)(u-6)$.

Try It 1.2.7.6

Factor $y^2 - 16y + 63$.

Answer

The factorization is $(y-7)(y-9)$.

Now, what if the last term in the trinomial is negative? Think about **FOIL**. The last term is the product of the last terms in the two binomials. A negative product results from multiplying two numbers with opposite signs. You have to be very careful to choose factors to make sure you get the correct sign for the middle term, too.

How do you get a *negative product* and a *positive sum*? We use one positive and one negative number.

When we factor trinomials, we must have the terms written in descending order—in order from highest degree to lowest degree.

Example 1.2.7.7

Factor $2x+x^2-48$.

Solution

		$2x+x^2-48$
	First we put the terms in decreasing degree order.	$x^2+2x-48$
he...	Write the factors as two binomials with first terms y.	$x^2+2x-48$ $=\underbrace{(x+\square)}_{(x+m)}\underbrace{(x+\square)}_{(x+n)}$
he...	Identify b and c.	$\underbrace{x^2+2x-48}_{ax^2+bx+c}$ $b=2$ $c=-48$
he...	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn &= c \\ m+n &= b \end{cases}$	Find m and n such that $\begin{cases} mn &= -48 \\ m+n &= 2 \end{cases}$ Factors of -48 / Sum of factors: $-1,48$: $-1+48=47$ $-2,24$: $-2+24=22$ $-3,16$: $-3+16=13$ $-4,12$: $-4+12=8$ $-6,8$: $-6+8=2^*$ $m=-6$ $n=8$
he...	Use m and n as the last terms of the factors.	$\underbrace{(x-6)}_{(x+m)}\underbrace{(x+8)}_{(x+n)}$
he...	Check.	$(x-6)(x+8)$ $=x^2+8x-6x-48$ $=x^2+2x-48$ ✓
he...	Conclude.	The factorization is $(x-6)(x+8)$.

Try It 1.2.7.8

Factor $9m+m^2+18$.

Answer

The factorization is $(m+3)(m+6)$.

Try It 1.2.7.9

Factor $-7n+12+n^2$.

Answer

The factorization is $(n-3)(n-4)$.

Sometimes you'll need to factor trinomials of the form $x^2+bxy+cy^2$ with two variables, such as $x^2+12xy+36y^2$. The first term, x^2, is the product of the first terms of the binomial factors, xx. The y^2 in the last term means that the second terms of the binomial factors must each contain y.

Example 1.2.7.10

Factor $r^2-8rs-9s^2$.

Solution

	$r^2-8rs-9s^2$
Write the factors as two binomials with r in the first term of each binomial and s in the second term.	$r^2-8rs-9s^2$ $=\underbrace{(r+\square s)}_{(r+ms)}\underbrace{(r+\square s)}_{(r+ns)}$
Identify b and c.	$\underbrace{r^2-8rs-9s^2}_{ar^2+brs+cs^2}$ $b=-8$ $c=-9$
Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn &= c \\ m+n &= b \end{cases}$	Find m and n such that $\begin{cases} mn &= -9 \\ m+n &= -8 \end{cases}$ Factors of -9 / Sum of factors: $1,-9$: $1+(-9)=-8^{*}$ $-1,9$: $-1+9=8$ $-3,3$: $-3+3=0$ $m=1$ $n=-9$
Use m and n as the last terms of the factors.	$\underbrace{(r+s)}_{(r+ms)}\underbrace{(r-9s)}_{(r+ns)}$
Check.	$(r+s)(r-9s)$ $=r^2-9rs+rs-9s^2$ $=r^2-8rs-9s^2$ ✓
Conclude.	The factorization is $(r+s)(r-9s)$.

Try It 1.2.7.11

Factor $a^2-11ab+10b^2$.

Answer

The factorization is $(a-b)(a-10b)$.

? Try It 1.2.7.12

Factor $m^2 - 13mn + 12n^2$.

Answer

The factorization is $(m-n)(m-12n)$.

Some trinomials are prime. The only way to be certain a trinomial is **prime** is to list all the possibilities and show that none of them work.

✔ Example 1.2.7.13

Factor $u^2 - 9uv - 12v^2$.

Solution

	$u^2 - 9uv - 12v^2$
.2) Write the factors as two binomials with u in the first term of each binomial and v in the second term.	$u^2 - 9uv - 12v^2$ $= \underbrace{(u + \square v)}_{(u+mv)} \underbrace{(u + \square v)}_{(u+nv)}$
.2) Identify b and c.	$\underbrace{\mathbf{u^2 - 9uv - 12v^2}}_{au^2+buv+cn^2}$ $b = -9$ $c = -12$
.2) Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn & = c \\ m+n & = b \end{cases}$	Find m and n such that $\begin{cases} mn & = -12 \\ m+n & = -9 \end{cases}$ (see factor table below) Note there are no factor pairs that give us -9 as a sum.
.2) Conclude.	The trinomial is prime.

Factors of -12	Sum of factors
$1, -12$	$1 + (-12) = -11$
$-1, 12$	$-1 + 12 = 11$
$2, -6$	$2 + (-6) = -4$
$-2, 6$	$-2 + 6 = 4$
$3, -4$	$3 + (-4) = -1$
$-3, 4$	$-3 + 4 = 1$

? Try It 1.2.7.14

Factor $x^2 - 7xy - 10y^2$.

Answer

The trinomial is prime.

Try It 1.2.7.15

Factor $p^2+15pq+20q^2$.

Answer

The trinomial is prime.

Let's summarize the method we just developed to factor trinomials of the form x^2+bx+c .

Strategy for factoring trinomials of the form x^2+bx+c

When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

x^2+bx+c $=(x+m)(x+n)$	
b positive	b negative
m, n positive	m, n negative
When c is positive, m and n have the same sign.	
For example, $x^2+5x+6=(x+2)(x+3)$.	For example, $x^2-6x+8=(x-4)(x-2)$.
When c is negative, m and n have the opposite sign.	
For example, $x^2+x-12=(x+4)(x-3)$.	For example, $x^2-2x-15=(x-5)(x+3)$.

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

Factoring Trinomials of the Form ax^2+bx+c Using Trial and Error

Our next step is to factor trinomials whose leading coefficient is not 1, trinomials of the form ax^2+bx+c .

Remember to always check for a **GCF** first! Sometimes, after you factor the GCF, the leading coefficient of the trinomial becomes 1 and you can factor it by the methods we've used so far. Let's do an example to see how this works.

Example 1.2.7.16

Factor $4x^3+16x^2-20x$.

Solution

	$4x^3+16x^2-20x$
What is the greatest common factor?	The GCF is $4x$.
Factor it.	$4x(x^2+4x-5)$
Is the GCF multiplied by a binomial, a trinomial, or a polynomial with more than three terms?	$=\underbrace{4x}_{\text{GCF}}\underbrace{(x^2+4x-5)}_{\text{trinomial}}$ It is a trinomial.
"Undo FOIL." Write the trinomial as factors of two binomials with first terms x.	$=\underbrace{4x}_{\text{GCF}}\underbrace{(x^2+4x-5)}_{(x+\square)(x+\square)}$
Identify b and c.	$=\underbrace{4x}_{\text{GCF}}\underbrace{(x^2+4x-5)}_{ax^2+bx+c}$ $b=4$ $c=-5$

\)	Find two numbers m and n that multiply to c and add to b. $\begin{cases} mn & = c \\ m+n & = b \end{cases}$	Find m and n such that $\begin{cases} mn & = -5 \\ m+n & = 4 \end{cases}$ Factors of -5: $-1, 5$; $1, -5$ — Sum of factors: $-1+5=4^{*}$; $1+(-5)=-4$ $m=-1$ $n=4$
\)	Use m and n as the last terms of the factors.	$\underbrace{4x}_{\text{GCF}}\underbrace{(x-1)(x+4)}_{(x+m)(x+n)}$
\)	Check.	$4x(x-1)(x+5)$ $=4x(x^2+5x-x-5$ $=4x(x^2+4x-5)$ $=4x^3+16x^2-20x$ ✓
\)	Conclude.	The factorization is $4x(x-1)(x+5)$.

? Try It 1.2.7.17

Factor $5x^3+15x^2-20x$.

Answer

The factorization is $5x(x-1)(x+4)$.

? Try It 1.2.7.18

Factor $6y^3+18y^2-60y$.

Answer

The factorization is $6y(y-2)(y+5)$.

What happens when the leading coefficient is not 1 and there is no GCF? There are several methods that can be used to factor these trinomials. First we will use the Trial and Error method.

Let's factor the trinomial $3x^2+5x+2$.

From our earlier work, we expect this will factor into two binomials.

$$3x^2+5x+2$$

$$(\quad)(\quad)$$

We know the first terms of the binomial factors will multiply to give us $3x^2$. The only factors of $3x^2$ are $1x$, $3x$. We can place them in the binomials.

$$(x+\quad)(3x+\quad)$$

Check: Does $1x\cdot 3x=3x^2$?

We know the last terms of the binomials will multiply to 2. Since this trinomial has all positive terms, we only need to consider positive factors. The only factors of 2 are 1 and 2. But we now have two cases to consider as it will make a difference if we write 1, 2 or 2, 1 (the opposites are also a possibility, but since the middle term is possitive, we can disregard this possibility).

$$(x+1)(3x+2) \text{ or } (x+2)(3x+1)$$

Which factors are correct? To decide that, we multiply the inner and outer terms.

$$3x^2+5x+2 \text{ or } 3x^2+7x+2.$$

Since the middle term of the trinomial is $5x$, the factors in the first case will work. Let's use FOIL to check.

$$(x+1)(3x+2)$$

$$3x^2+2x+3x+2$$

$$3x^2+5x+2\checkmark$$

Our result of the factoring is:

$$3x^2+5x+2$$

$$(x+1)(3x+2)$$

Example 1.2.7.19

Factor $3y^2+22y+7$ using trial and error.

Solution

Step 1. Write the trinomial in descending order.	The trinomial is already in descending order.	$3y^2+22y+7$
Step 2. Factor any GCF.	There is no GCF.	
Step 3. Find all the factor pairs of the first term.	The only of $3y^2$ are $1y, 3y$. Since there is only one pair, we can put them in the parentheses.	$3y^2+22y+7$ (1y, 3y) $3y^2+22y+7$ (1y, 3y) $(y\quad)(3y\quad)$
Step 4. Find all the factor pairs of the third term.	The only factors of 7 are 1, 7.	$3y^2+22y+7$ (1y, 3y; 1, 7) $(y\quad)(3y\quad)$
Step 5. Test all the possible combinations of the factors until the correct product is found.	$3y^2+22y+7$ (1y, 3y; 1, 7) $(y+1)(3y+7)$ $3y$ $7y$ $10y$ No! We need $22y$ $3y^2+22y+7$ (1y, 3y; 1, 7) $(y+7)(3y+1)$ $21y$ $+y$ $22y$	$3y^2+22y+7$ Possible factors: $(y+1)(3y+7)$ — Product: $3y^2+10y+7$ Possible factors: $(y+7)(3y+1)$ — Product: $3y^2+22y+7$
Step 6. Check by multiplying.		$(y+7)(3y+1)$ $3y^2+22y+7\checkmark$

Try It 1.2.7.20

Factor $2a^2+5a+3$ using trial and error.

Answer

The factorization is $(a+1)(2a+3)$.

Try It 1.2.7.21

Factor $4b^2+5b+1$ using trial and error.

Answer

The factorization is $(b+1)(4b+1)$.

Factor trinomials of the form ax^2+bx+c using trial and error

1. Write the trinomial in descending order of degrees as needed.
2. Factor any GCF.
3. Find all the factor pairs of the first term.
4. Find all the factor pairs of the third term.
5. Test all the possible combinations of the factors until the correct product is found.
6. Check by multiplying.

Remember, when the middle term is negative and the last term is positive, the signs in the binomials must both be negative.

Example 1.2.7.22

Factor $6b^2-13b+5$ using trial and error.

Solution

The trinomial is already in descending order.	$6b^2-13b+5$
Find the factors of the first term.	$6b^2-13b+5$ $1b\cdot 6b$ $2b\cdot 3b$
Find the factors of the last term. Consider the signs. Since the last term, 5, is positive its factors must both be positive or both be negative. The coefficient of the middle term is negative, so we use the negative factors.	$6b^2-13b+5$ $1b\cdot 6b$ $\quad -1,-5$ $2b\cdot 3b$

Consider all the combinations of factors.

$6b^2-13b+5$	
Possible factors	Product
$(b-1)(6b-5)$	$6b^2-11b+5$
$(b-5)(6b-1)$	$6b^2-31b+5$
$(2b-1)(3b-5)$	$6b^2-13b+5$*
$(2b-5)(3b-1)$	$6b^2-17b+5$

The correct factors are those whose product is the original trinomial.	$(2b-1)(3b-5)$
Check by multiplying:	

$$(2b-1)(3b-5)$$
$$6b^2-10b-3b+5$$
$$6b^2-13b+5\checkmark$$

? Try It 1.2.7.23

Factor $8x^2-13x+3$ using trial and error.

Answer

The factorization is $(2x-3)(4x-1)$.

? Try It 1.2.7.24

Factor $10y^2-37y+7$ using trial and error.

Answer

The factorization is $(2y-7)(5y-1)$.

When we factor an expression, we always look for a greatest common factor first. If the expression does not have a greatest common factor, there cannot be one in its factors either. This may help us eliminate some of the possible factor combinations.

✓ Example 1.2.7.25

Factor $18x^2-37xy+15y^2$ using trial and error.

Solution

The trinomial is already in descending order.	$18x^2-37xy+15y^2$
Find the factors of the first term.	$18x^2-37xy+15y^2$ $1x\cdot 18x$ $2x\cdot 9x$ $3x\cdot 6x$
Find the factors of the last term. Consider the signs. Since 15 is positive and the coefficient of the middle term is negative, we use the negative factors.	$18x^2-37xy+15y^2$ $1x\cdot 18x$ $-1, -15$ $2x\cdot 9x$ $-3, -5$ $3x\cdot 6x$

Consider all the combinations of factors.

$18x^2 - 37xy + 15y^2$	
Possible factors	**Product**
$(x - 1y)(18x - 15y)$	Not an option
$(x - 15y)(18x - 1y)$	$18x^2 - 271xy + 15y^2$
$(x - 3y)(18x - 5y)$	$18x^2 - 59xy + 15y^2$
$(x - 5y)(18x - 3y)$	Not an option
$(2x - 1y)(9x - 15y)$	Not an option
$(2x - 15y)(9x - 1y)$	$18x^2 - 137xy + 15y^2$
$(2x - 3y)(9x - 5y)$	$18x^2 - 37xy + 15y^2$*
$(2x - 5y)(9x - 3y)$	Not an option
$(3x - 1y)(6x - 15y)$	Not an option
$(3x - 15y)(6x - 1y)$	Not an option
$(3x - 3y)(6x - 5y)$	Not an option

If the trinomial has no common factors, then neither factor can contain a common factor. That means this combination is not an option.

The correct factors are those whose product is the original trinomial. $(2x - 3y)(9x - 5y)$

Check by multiplying:

$$(2x - 3y)(9x - 5y)$$
$$18x^2 - 10xy - 27xy + 15y^2$$
$$18x^2 - 37xy + 15y^2 \checkmark$$

? Try It 1.2.7.26

Factor $18x^2 - 3xy - 10y^2$ using trial and error.

Answer

The factorization is $(3x + 2y)(6x - 5y)$.

? Try It 1.2.7.27

Factor $30x^2 - 53xy - 21y^2$ using trial and error.

Answer

The factorization is $(3x + y)(10x - 21y)$.

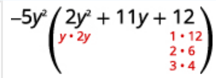
Don't forget to look for a GCF first and remember if the leading coefficient is negative, so is the GCF.

✓ Example 1.2.7.28

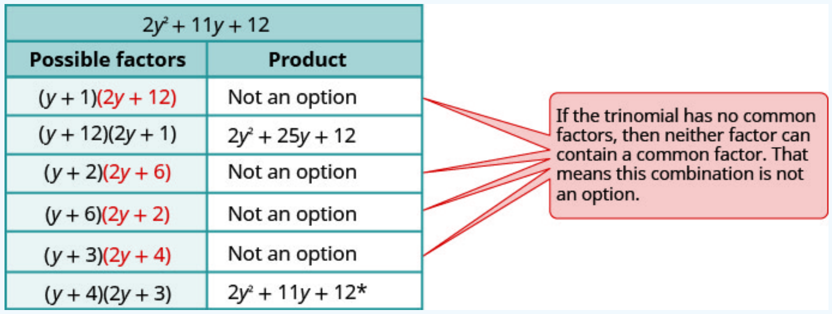
Factor $-10y^4 - 55y^3 - 60y^2$ using trial and error.

Solution

	$-10y^4 - 55y^3 - 60y^2$
Notice the greatest common factor, so factor it first.	$-5y^2(2y^2 + 11y + 12)$

Factor the trinomial.	

Consider all the combinations.



$2y^2+11y+12$	
Possible factors	Product
$(y+1)(2y+12)$	Not an option
$(y+12)(2y+1)$	$2y^2+25y+12$
$(y+2)(2y+6)$	Not an option
$(y+6)(2y+2)$	Not an option
$(y+3)(2y+4)$	Not an option
$(y+4)(2y+3)$	$2y^2+11y+12$*

The correct factors are those whose product is the original trinomial. Remember to include the factor -5^y2. $\quad -5y^2(y+4)(2y+3)$

Check by multiplying:

$$-5y^2(y+4)(2y+3)$$
$$-5y^2(2y^2+8y+3y+12)$$
$$-10y^4-55y^3-60y^2 \checkmark$$

Try It 1.2.7.29

Factor $15n^3-85n^2+100n$ using trial and error.

Answer

The factorization is $5n(n-4)(3n-5)$.

Try It 1.2.7.30

Factor $56q^3+320q^2-96q$ using trial and error.

Answer

The factorization is $8q(q+6)(7q-2)$.

Factoring Trinomials of the Form ax^2+bx+c Using the "ac" Method

Another way to factor trinomials of the form ax^2+bx+c is the "ac" method. (The "ac" method is sometimes called the grouping method.) The "ac" method is actually an extension of the methods you used in the last section to factor trinomials with leading coefficient one. This method is very structured (that is step-by-step), and it always works!

Example 1.2.7.31

Factor $6x^2+7x+2$ using the "ac" method.

Solution

Step 1. Factor any GCF.	Is there a greatest common factor? No!	$6x^2 + 7x + 2$
Step 2. Find the product ac.	$a \cdot c$ $6 \cdot 2$ 12	$ax^2 + bx + c$ $6x^2 + 7x + 2$
Step 3. Find two numbers m and n that: Multiply to ac. $m \cdot n = a \cdot c$ Add to b. $m + n = b$	Find two numbers that multiply to 12 and add to 7. Both factors must be positive. $3 \cdot 4 = 12 \quad 3 + 4 = 7$	
Step 4. Split the middle term using m and n. $ax^2 + bx + c$ bx $ax^2 + mx + nx + c$	Rewrite $7x$ as $3x + 4x$. It would also give the same result if we used $4x + 3x$. Notice that $6x^2 + 3x + 4x + 2$ is equal to $6x^2 + 7x + 2$. We just split the middle term to get a more useful form.	$6x^2 + 7x + 2$ $6x^2 + 3x + 4x + 2$
Step 5. Factor by grouping.		$3x(2x + 1) + 2(2x + 1)$ $(2x + 1)(3x + 2)$

Try It 1.2.7.32

Factor $6x^2 + 13x + 2$ using the "ac" method.

Answer

The factorization is $(x + 2)(6x + 1)$.

Try It 1.2.7.33

Factor $4y^2 + 8y + 3$ using the "ac" method.

Answer

The factorization is $(2y + 1)(2y + 3)$.

The "ac" method is summarized here.

Factor trinomials of the form $ax^2 + bx + c$ using the "ac" method

1. Factor any GCF.
2. Find the product ac.
3. Find two numbers m and n that:
   Multiply to ac $\quad mn = ac$
   Add to b $\quad m + n = b$
   $ax^2 + bx + c$
4. Split the middle term using m and n. $ax^2 + mx + nx + c$
5. Factor by grouping.
6. Check by multiplying the factors.

Don't forget to look for a common factor!

Example 1.2.7.34

Factor $10y^2 - 55y + 70$ using the "“ac” method.

Solution

	$10y^2 - 55y + 70$
Is there a greatest common factor?	Yes, the GCF is 5.
Factor it.	$5(\underbrace{2y^2 - 11y + 14}_{ax^2+bx+c}$
The trinomial inside the parentheses has a leading coefficient that is not 1. Find the product ac.	$ac = 28$
Find two numbers that multiply to ac and add to b.	$(-4)(-7) = 28$ $(-4) + (-7) = -11$
Split the middle term.	$5(2y^2 - 7y - 4y + 14)$ 
Factor the trinomial by grouping.	$5(y-2)(2y-7)$
Check by multiplying all three factors	$5(y-2)(2y-7)$ $= 5(2y^2 - 7y - 4y + 14)$ $= 5(2y^2 - 11y + 14)$ $= 410y^2 - 55y + 70$✓
Conclude.	The factorization is $5(y-2)(2y-7)$.

Try It 1.2.7.35

Factor $16x^2 - 32x + 12$ using the “ac” method.

Answer

The factorization is $4(2x-3)(2x-1)$.

Try It 1.2.7.36

Factor $18w^2 - 39w + 18$ using the “ac” method.

Answer

The factorization is $3(3w-2)(2w-3)$.

Factoring Using Substitution (optional)

Sometimes a trinomial does not appear to be in the $ax^2 + bx + c$ form. However, we can often make a thoughtful substitution that will allow us to make it fit the $ax^2 + bx + c$ form. This is called **factoring by substitution**. It is standard to use u for the substitution.

In the $ax^2 + bx + c$, the middle term has a variable, x, and its square, x^2, is the variable part of the first term. Look for this relationship as you try to find a substitution.

Example 1.2.7.37

Factor x^4-4x^2-5 by substitution (just using integers).

Solution

The variable part of the middle term is x^2 and its square, x^4, is the variable part of the first term. (We know $(x^2)^2=x^4$). If we let $u=x^2$, we can put our trinomial in the ax^2+bx+c form we need to factor it.

	x^4-4x^2-5
Rewrite the trinomial to prepare for the substitution.	$=(x^2)^2-4(x^2)-5$
Let $u=x^2$ and substitute.	$=u^2-4u-5$
Factor the trinomial.	$=(u+1)(u-5)$
Replace u with x^2.	$=(x^2+1)(x^2-5)$
Check:	$(x^2+1)(x^2-5)$ $x^4-5x^2+x^2-5$ $x^4-4x^2-5\checkmark$

Try It 1.2.7.38

Factor h^4+4h^2-12 by substitution (just using integers).

Answer

The factorization is $(h^2-2)(h^2+6)$.

Try It 1.2.7.39

Factor y^4-y^2-20 by substitution (just using integers).

Answer

The factorization is $(y^2+4)(y^2-5)$.

Sometimes the expression to be substituted is not a monomial.

Example 1.2.7.40

Factor $(x-2)^2+7(x-2)+12$ by substitution.

Solution

The binomial in the middle term, $(x-2)$ is squared in the first term. If we let $u=x-2$ and substitute, our trinomial will be in ax^2+bx+c form.

	$(x-2)^2+7(x-2)+12$
Rewrite the trinomial to prepare for the substitution.	$=(x-2)^2+7(x-2)+12$
Let $u=x-2$ and substitute.	$=u^2+7u+12$
Factor the trinomial.	$=(u+3)(u+4)$
Replace u with $x-2$.	$=((x-2)+3)((x-2)+4)$
Simplify inside the parentheses.	$=(x+1)(x+2)$

This could also be factored by first multiplying out the $(x-2)^2$ and the $7(x-2)$ and then combining like terms and then factoring. Most students prefer the substitution method.

Try It 1.2.7.41

Factor $(x-5)^2+6(x-5)+8$ by substitution.

Answer

The factorization is $(x-3)(x-1)$.

Try It 1.2.7.42

Factor $(y-4)^2+8(y-4)+15$ by substitution.

Answer

The factorization is $(y-1)(y+1)$.

Writing Exercises 1.2.7.43

1. Holly factored $(x^2-x-20)(x^2-x-20)$ as $(x+5)(x-4)(x+5)(x-4)$. Ariane factored it as $(x+4)(x-5)(x+4)(x-5)$. Lin factored it as $(x-5)(x-4)(x-5)(x-4)$. Who is correct? Explain why the other two are wrong.
2. How can you check whether you have factored correctly?
3. Create a multiple choice factoring question for a trinomial of degree 2 with leading coefficient different from 1 and provide the answer. (Make the incorrect choices so that they might capture a mistake that someone might make).
4. Is it possible to factor a trinomial of degree 2 with leading coefficient different from 1 into a monomial and a binomial?
5. Do you prefer the guess and check method or the 'ac'-method? Explain.
6. Explain with an example why the ac- method works.

Exit Problem

Factor $6x^2z^2-31xz^2+35z^2$.

Key Concepts

- **How to factor trinomials of the form x^2+bx+c.**
  1. Write the factors as two binomials with first terms x. x^2+bx+c $(x\quad)(x\quad)$
  2. Find two numbers m and n that
     multiply to $c,\ m\cdot n=c$
     add to $b,\ m+n=b$
  3. Use m and n as the last terms of the factors. $(x+m)(x+n)$
  4. Check by multiplying the factors.
- **Strategy for Factoring Trinomials of the Form** x^2+bx+c: When we factor a trinomial, we look at the signs of its terms first to determine the signs of the binomial factors.

  For trinomials of the form: $x^2+bx+c=(x+m)(x+n)$

  When c is positive, m and n must have the same sign (and this will be the sign of b).

  Examples: $x^2+5x+6=(x+2)(x+3)$, $x^2-6x+8=(x-4)(x-2)$

When c is negative, m and n have opposite signs. The larger of m and n will have the sign of b.

Examples: $x^2+x-12=(x+4)(x-3)$, $x^2-2x-15=(x-5)(x+3)$

Notice that, in the case when m and n have opposite signs, the sign of the one with the larger absolute value matches the sign of b.

- **How to factor trinomials of the form ax^2+bx+c using trial and error.**
  1. Write the trinomial in descending order of degrees as needed.
  2. Factor any GCF.
  3. Find all the factor pairs of the first term.
  4. Find all the factor pairs of the third term.
  5. Test all the possible combinations of the factors until the correct product is found.
  6. Check by multiplying.
- **How to factor trinomials of the form ax^2+bx+c using the "ac" method.**
  1. Factor any GCF.
  2. Find the product ac.
  3. Find two numbers m and n that:
     Multiply to ac. $\quad m \cdot n = a \cdot c$
     Add to b. $\quad m+n=b$
     ax^2+bx+c
  4. Split the middle term using m and n. $\quad ax^2+mx+nx+c$
  5. Factor by grouping.
  6. Check by multiplying the factors.

1.2.8: Factoring Special Products

Learning Objectives

By the end of this section, you will be able to:

- Factor perfect square trinomials
- Factor differences of squares
- Factor sums and differences of cubes

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $(3x^2)^3$.

2. Multiply $(m+4)^2$.

3. Multiply $(x-3)(x+3)$.

We have seen that some binomials and trinomials result from special products—squaring binomials and multiplying conjugates. If we recognize these kinds of polynomials, we can use the special products patterns to factor them much more quickly.

Factoring Differences of Squares

One special product we are familiar with is the Product of Conjugates pattern. We use this to multiply two binomials that were conjugates. Here's an example:

$$(2x-5)(2x+5)=4x^2-25.$$

A difference of squares factors to a product of conjugates (in this context, $a-b$ is said to be conjugate to $a+b$, and vice versa.)

Difference of Squares Pattern

If a and b are (or represent) real numbers,

$$a^2-b^2=(a+b)(a-b).$$

Remember, "difference" refers to subtraction. So, to use this pattern we must make sure we have a binomial in which two squares are being subtracted.

Example 1.2.8.1

Factor $64y^2-1$.

Solution

Note that $(8y)^2=64y^2$ and $1^2=1$, so that

$$\begin{aligned}&64y^2-1\\=&(8y)^2-(1)^2\\=&(8y+1)(8y-1).\end{aligned}$$

Try It 1.2.8.2

Factor $121m^2-1$.

Answer

$(11m-1)(11m+1)$

Try It 1.2.8.3

Factor $81y^2-1$.

Answer

$(9y-1)(9y+1)$

Factor Difference of Squares

Step 1.	Does the binomial fit the pattern?	a^2-b^2
	Is this a difference?	$\_\_\_\_-\_\_\_\_$
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2-(b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

It is important to remember that *sums of squares do not factor into a product of binomials*. There are no binomial factors that multiply together to get a sum of squares. After removing any GCF, the expression a^2+b^2 can not be factored any further!

The next example shows variables in both terms.

Example 1.2.8.4

Factor $144x^2-49y^2$.

Solution

	$144x^2-49y^2$
Is this a difference of squares? Yes.	$=(12x)^2-(7y)^2$
Factor as the product of conjugates.	$=(12x-7y)(12x+7y)$
Check by multiplying.	$=144x^2-49y^2\checkmark$
Conclude.	The factorization is $(12x-7y)(12x+7y)$

Try It 1.2.8.5

Factor $196m^2-25n^2$.

Answer

$(16m-5n)(16m+5n)$

Try It 1.2.8.6

Factor $121p^2-9q^2$.

Answer

$(11p-3q)(11p+3q)$

As always, we should look for a common factor first whenever we have an expression to factor. Sometimes a common factor may "disguise" the difference of squares and we won't recognize the perfect squares until we factor the GCF.

Also, to factor the binomial in the next example, we'll factor a difference of squares twice!

Example 1.2.8.7

Factor $48x^4y^2 - 243y^2$.

Solution

	$48x^4y^2 - 243y^2$
Is there a GCF? Yes, $3y^2$ - factor it out!	$= 3y^2(16x^4 - 81)$
Is the binomial a difference of squares? Yes.	$= 3y^2\left((4x^2)^2 - (9)^2\right)$
Factor as a product of conjugates.	$= 3y^2(4x^2 - 9)(4x^2 + 9)$
Notice the first binomial is also a difference of squares!	$= 3y^2((2x)^2 - (3)^2)(4x^2 + 9)$
Factor it as the product of conjugates.	$= 3y^2(2x - 3)(2x + 3)(4x^2 + 9)$
The last factor, the sum of squares, cannot be factored.	
Check by multiplying:	$3y^2(2x-3)(2x+3)(4x^2+9)$ $= 3y^2(4x^2-9)(4x^2+9)$ $= 3y^2(16x^4-81)$ $= 48x^4y^2 - 243y^2 \checkmark$
Conclude.	The factorization is $3y^2(2x - 3)(2x + 3)(4x^2 + 9)$.

Try It 1.2.8.8

Factor $2x^4y^2 - 32y^2$.

Answer

$2y^2(x-2)(x+2)(x^2+4)$

Try It 1.2.8.9

Factor $7a^4c^2 - 7b^4c^2$.

Answer

$7c^2(a-b)(a+b)(a^2+b^2)$

The next example has a polynomial with 4 terms. So far, when this occurred we grouped the terms in twos and factored from there. Here we will notice that the first three terms form a perfect square trinomial.

Example 1.2.8.10

Factor $x^2 - 6x + 9 - y^2$.

Solution

Notice that the first three terms form a perfect square trinomial.

	$x^2 - 6x + 9 - y^2$

Factor by grouping the first three terms. Use the perfect square trinomial pattern.	$(x-3)^2-y^2$
Is this a difference of squares? Yes.	
Yes—write them as squares.	(already in that form)
Factor as the product of conjugates.	$(x-3-y)(x-3+y)$
Conclude.	The factorization is $(x-y-3)(x+y-3)$.

Try It 1.2.8.11

Factor $x^2-10x+25-y^2$.

Answer

$(x-5-y)(x-5+y)$

Try It 1.2.8.12

Factor $x^2+6x+9-4y^2$.

Answer

$(x+3-2y)(x+3+2y)$

Factoring Perfect Square Trinomials (optional discussion)

Some trinomials are perfect squares. They result from multiplying a binomial times itself. We squared a binomial using the Binomial Squares pattern in a previous chapter.

$$\overset{(a\ +\ b)^2}{(3x+4)^2}$$

$$\overset{a^2\ +\ 2\cdot a\cdot b\ +\ b^2}{(3x)^2+2(3x\cdot 4)+4^2}$$

$$9x^2+24x+16$$

The trinomial $9x^2+24x+16$ is called a *perfect square trinomial*. It is the square of the binomial $3x+4$.

In this chapter, we will start with a perfect square trinomial and factor it into its as many factors as possible. We could factor this **trinomial** using the methods described in the last section, since it is of the form ax^2+bx+c . But if we recognize that the first and last terms are squares and the trinomial fits the perfect square trinomials pattern, we will save ourselves a lot of work. Here is the pattern—the reverse of the binomial squares pattern.

Perfect Square Trinomial Pattern

If a and b are (or represent) real numbers

$$a^2+2ab+b^2=(a+b)^2,$$

$$a^2-2ab+b^2=(a-b)^2.$$

To make use of this pattern, we have to recognize that a given trinomial fits it. Check first to see if the leading coefficient is a perfect square, a^2. Next check that the last term is a perfect square, b^2. Then check the middle term—is it the product, $2ab$? If everything checks, we can easily write the factors.

The sign of the middle term determines which pattern we will use. When the middle term is negative, we use the pattern $a^2-2ab+b^2$, which factors to $(a-b)^2$.

The steps are summarized here.

Factoring Perfect Square Trinomials

Step 1.	Does the trinomial fit the pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
	Check the middle term. Is it $2ab$?	$\searrow 2\cdot a\cdot b \swarrow$	$\searrow 2\cdot a\cdot b \swarrow$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

Remember the first step in factoring is to look for a greatest common factor. Perfect square trinomials may have a GCF in all three terms and it should be factored out first. And, sometimes, once the GCF has been factored, we will recognize a perfect square trinomial.

The above is for consideration, but we will not give any examples of this here.

? Writing Exercises 1.2.8.13

1. Why might it be useful to recognize the special products?
2. Explain why $(x+2)^2 \neq x^2+4$.
3. Give an example of the difference-of-squares pattern with three variables?
4. Use the difference of squares pattern to simplify and factor 49-16.
5. Is there a sum of squares formula?
6. Demonstrate why $a^2-b^2=(a-b)(a+b)$.

Exit Problem

Factor $25x^2-16y^6$.

Key Concepts

- **Difference of Squares Pattern:** If a, b are (or represent) real numbers,

$$a^2-b^2=(a+b)(a-b) \tag{1.2.8.1}$$

- **How to factor differences of squares**

Step 1.	Does the binomial fit the pattern?	a^2-b^2
	Is this a difference?	____ − ____
	Are the first and last terms perfect squares?	
Step 2.	Write them as squares.	$(a)^2-(b)^2$
Step 3.	Write the product of conjugates.	$(a-b)(a+b)$
Step 4.	Check by multiplying.	

- **Perfect Square Trinomials Pattern:** If a and b are (or represent) real numbers,

$$a^2+2ab+b^2=(a+b)^2$$
$$a^2-2ab+b^2=(a-b)^2$$

- **How to factor perfect square trinomials**

Step 1.	Does the trinomial fit the pattern?	$a^2+2ab+b^2$	$a^2-2ab+b^2$
	Are the first and last terms perfect squares?		
	Write them as squares.	$(a)^2 \quad (b)^2$	$(a)^2 \quad (b)^2$
	Check the middle term. Is it $2ab$?	$\searrow 2\cdot a\cdot b \swarrow$	$\searrow 2\cdot a\cdot b \swarrow$
Step 2.	Write the square of the binomial.	$(a+b)^2$	$(a-b)^2$
Step 3.	Check by multiplying.		

- **6.4: Factor Special Products** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/intermediate-algebra-2e.

1.2.9: General Strategy for Factoring Polynomials

Learning Objectives

By the end of this section, you will be able to:

- Recognize and use the appropriate method to factor a polynomial completely

Be Prepared

Before you get started, take this readiness quiz.

1. Find the GCF of $2x^4 - 14x^3 - 4x^2$.

2. Find the product $(3y+4)(3y-4)$.

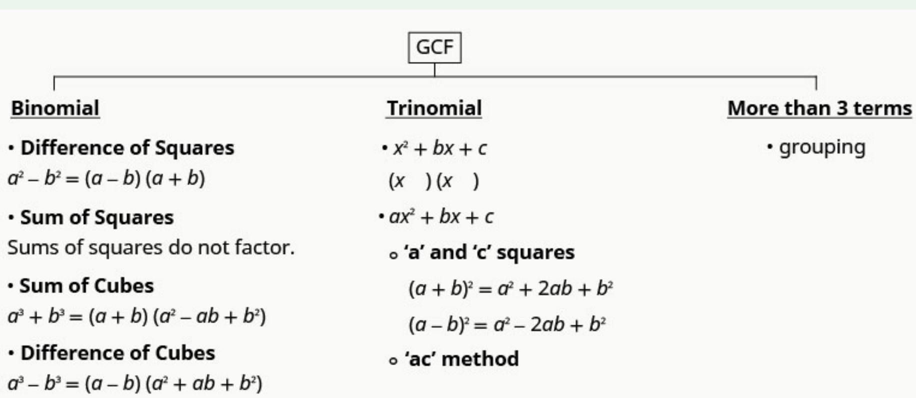
3. Find the coefficient of x in $(2x-7)(3x+3)$.

Recognizing and Using the Appropriate Method to Factor a Polynomial Completely

You have now become acquainted with all the methods of factoring that you will need in this course. The following chart summarizes all the factoring methods we have covered, and outlines a strategy you should use when factoring polynomials.

General Strategy for Factoring Polynomials

This chart shows the general strategies for factoring polynomials. It shows ways to find GCF of binomials, trinomials and polynomials with more than 3 terms. For binomials, we have difference of squares: a squared minus b squared equals a minus b, a plus b; sum of squares do not factor; sub of cubes: a cubed plus b cubed equals open parentheses a plus b close parentheses open parentheses a squared minus ab plus b squared close parentheses; difference of cubes: a cubed minus b cubed equals open parentheses a minus b close parentheses open parentheses a squared plus ab plus b squared close parentheses. For trinomials, we have x squared plus bx plus c where we put x as a term in each factor and we have a squared plus bx plus c. Here, if a and c are squares, we have a plus b whole squared equals a squared plus $2ab$ plus b squared and a minus b whole squared equals a squared minus $2ab$ plus b squared. If a and c are not squares, we use the "ac" method. For polynomials with more than 3 terms, we use grouping.



General Strategy for Factoring Polynomials

1. What is the greatest common factor?
   Factor it out if it is not 1.
2. Is the polynomial a binomial, trinomial, or are there more than three terms?
   If it is a binomial:
   - Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.

- Is it a difference?
  Of squares? Factor as the product of conjugates.
  Of cubes? Use the difference of cubes pattern.

If it is a trinomial:

- Is it of the form x^2+bx+c ? Undo FOIL.
- Is it of the form ax^2+bx+c ?
  If *a* and *c* are squares, check if it fits the trinomial square pattern.
  Use the trial and error or "*ac*" method.

If it has more than three terms:

- Use the grouping method.

3. Check.
Is it factored completely?
Do the factors multiply back to the original polynomial?

Remember, a polynomial is completely factored if, other than monomials, its factors are **prime**!

Example 1.2.9.1

Factor $7x^3-21x^2-70x$.

Solution

	$7x^3-21x^2-70x$
What is the GCF? $7x$. Factor out the GCF.	$=7x(x^2-3x-10)$
In the parentheses, is it a binomial, trinomial, or are there more terms Trinomial with leading coefficient 1.	$=7x(x+2)(x-5)$
Is the expression factored completely? Yes, neither binomial can be factored.	
Check your answer.	Multiply. $7x(x+2)(x-5)$ $=7x(x^2-5x+2x-10)$ $=7x^3-21x^2-70x$ ✓
Conclude.	The factorization of $7x^3-21x^2-70x$ is $7x(x+2)(x-5)$.

Try It 1.2.9.2

Factor $8y^3+16y^2-24y$.

Answer

$8y(y-1)(y+3)$

Try It 1.2.9.3

Factor $5y^3-15y^2-270y$.

Answer

$5y(y-9)(y+6)$

Be careful when you are asked to factor a binomial as there are several options!

✓ Example 1.2.9.4

Factor $24y^2-150$.

Solution

	$24y^2-150$
What is the GCF? 6. Factor out the GCF.	$=6(4y^2-25)$
In the parentheses, is it a binomial, trinomial or are there more than three terms? Binomial. Is it a difference? Of squares or cubes? Yes, squares.	$=6((2y)^2-(5)^2)$
Write as a product of conjugates.	$=6(2y-5)(2y+5)$
Is the expression factored completely? Yes, neither binomial can be factored.	
Check your answer.	Multiply. $6(2y-5)(2y+5)$ $=6(4y^2-25)$ $=24y^2-150$ ✓
Conclude.	The factorization of $24y^2-150$ is $6(2y-5)(2y+5)$.

Try It 1.2.9.5

Factor $16x^3-36x$.

Answer

$4x(2x-3)(2x+3)$

Try It 1.2.9.6

Factor $27y^2-48$.

Answer

$3(3y-4)(3y+4)$

The next example can be factored using several methods. Recognizing the trinomial squares pattern will make your work easier.

✓ Example 1.2.9.7

Factor $4a^2-12ab+9b^2$.

Solution

	$4a^2-12ab+9b^2$

What is the GCF? 1. Is it a binomial, trinomial, or are there more terms?Trinomial with $a \neq 1$ The first term is a perfect square. Is the last term a perfect square? Yes.	$= (2a)^2 - 12ab + (3b)^2$
Does it fit the pattern $a^2 - 2ab + b^2$? Yes	$= (2a)^2 - 2(2a)(3b) + (3b)^2$
Write it as a square.	$= (2a - 3b)^2$
Is the expression factored completely? Yes. The binomial cannot be factored.	
Check your answer.	Multiply. $(2a - 3b)^2$ $= (2a)^2 - 2(2a)(3b) + (3b)^2$ $= 4a^2 - 12ab + 9b^2 \quad \checkmark$
Conclude.	The factorization of $4a^2 - 12ab + 9b^2$ is $(2a - 3b)^2$.

Try It 1.2.9.8

Factor $4x^2 + 20xy + 25y^2$.

Answer

$(2x + 5y)^2$

Try It 1.2.9.9

Factor $9x^2 - 24xy + 16y^2$.

Answer

$(3x - 4y)^2$

Remember, sums of squares do not factor, but sums of cubes do!

Example 1.2.9.10

Factor $12x^3y^2 + 75xy^2$.

Solution

	$12x^3y^2 + 75xy^2$
What is the GCF? $3xy^2$. Factor out the GCF.	$= 3xy^2(4x^2 + 25)$
In the parentheses, is it a binomial, trinomial, or are there more than three terms? Binomial. Is it a sum? Of squares? Yes. Sums of squares are prime. Is the expression factored completely? Yes.	
Check.	Multiply. $3xy^2(4x^2 + 25)$ $= 12x^3y^2 + 75xy^2 \quad \checkmark$
Conclude.	The factorization of $12x^3y^2 + 75xy^2$ is $3xy^2(4x^2 + 25)$.

Try It 1.2.9.11

Factor $50x^3y+72xy$.

Answer

$2xy(25x^2+36)$

Try It 1.2.9.12

Factor $27xy^3+48xy$.

Answer

$3xy(9y^2+16)$

When using the sum or difference of cubes pattern, being careful with the signs.

Example 1.2.9.13

Factor $24x^3+81y^3$.

Solution

	$24x^3+81y^3$
What is the GCF? 3. Factor it out.	$=3(8x^3+27y^3)$
In the parentheses, is it a binomial, trinomial, of are there more than three terms? Binomial. Of squares or cubes? Sum of cubes.	\(=3((2x)^3+(3y)^3)
Write it using the sum of cubes pattern. $a^3+b^3=(a+b)(a^2-ab+b^2)$	$=3(2x+3y)((2x)^2-2x\cdot 3y+(3y)^2)$
Is the expression factored completely? Yes.	$=3(2x+3y)(4x^2-6xy+9y^2)$
Check by multiplying.	
Conclude.	The factorization of $24x^3+81y^3$ is $3(2x+3y)(4x^2-6xy+9y^2$.

Try It 1.2.9.14

Factor $250m^3+432n^3$.

Answer

$2(5m+6n)(25m^2-30mn+36n^2)$

Try It 1.2.9.15

Factor $2p^3+54q^3$.

Answer

$2(p+3q)(p^2-3pq+9q^2)$

Example 1.2.9.16

Factor $3x^5y - 48xy$.

Solution

	$3x^5y - 48xy$
Is there a GCF? Factor out $3xy$	$3xy(x^4 - 16)$
Is the binomial a sum or difference? Of squares or cubes? Write it as a difference of squares.	$3xy\left((x^2)^2 - (4)2\right)$
Factor it as a product of conjugates	$3xy(x^2 - 4)(x^2 + 4)$
The first binomial is again a difference of squares.	$3xy\left((x)^2 - (2)^2\right)(x^2 + 4)$
Factor it as a product of conjugates.	$3xy(x - 2)(x + 2)(x^2 + 4)$

Is the expression factored completely? Yes.
Check your answer.
Multiply.

$3xy(x - 2)(x + 2)(x^2 + 4)$

$3xy(x^2 - 4)(x^2 + 4)$

$3xy(x^4 - 16)$

$3x^5y - 48xy\checkmark$

Try It 1.2.9.17

Factor $4a^5b - 64ab$.

Answer

$4ab(a^2 + 4)(a - 2)(a + 2)$

Try It 1.2.9.18

Factor $7xy^5 - 7xy$.

Answer

$7xy(y^2 + 1)(y - 1)(y + 1)$

Example 1.2.9.19

Factor $4x^2 + 8bx - 4ax - 8ab$.

Solution

	$4x^2 + 8bx - 4ax - 8ab$
Is there a GCF? Factor out the GCF, 4.	$4(x^2 + 2bx - ax - 2ab)$
There are four terms. Use grouping.	$4[x(x + 2b) - a(x + 2b)]4(x + 2b)(x - a)$

Is the expression factored completely? Yes.
Check your answer.
Multiply.

$4(x + 2b)(x - a)$

$4(x^2 - ax + 2bx - 2ab)$

$4x^2 + 8bx - 4ax - 8ab\checkmark$

Try It 1.2.9.20

Factor $6x^2 - 12xc + 6bx - 12bc$.

Answer

$6(x+b)(x-2c)$

Try It 1.2.9.21

Factor $16x^2 + 24xy - 4x - 6y$.

Answer

$2(4x-1)(2x+3y)$

Taking out the complete GCF in the first step will always make your work easier.

Example 1.2.9.22

Factor $40x^2y + 44xy - 24y$.

Solution

	$40x^2y + 44xy - 24y$
Is there a GCF? Factor out the GCF, $4y$.	$4y(10x^2 + 11x - 6)$
Factor the trinomial with $a \neq 1$.	$4y(10x^2 + 11x - 6)$
	$4y(5x-2)(2x+3)$

Is the expression factored completely? Yes.
Check your answer.
Multiply.

$$4y(5x-2)(2x+3)$$
$$4y(10x^2 + 11x - 6)$$
$$40x^2y + 44xy - 24y \checkmark$$

Try It 1.2.9.23

Factor $4p^2q - 16pq + 12q$.

Answer

$4q(p-3)(p-1)$

Try It 1.2.9.24

Factor $6pq^2 - 9pq - 6p$.

Answer

$3p(2q+1)(q-2)$

When we have factored a polynomial with four terms, most often we separated it into two groups of two terms. Remember that we can also separate it into a trinomial and then one term.

Example 1.2.9.25

Factor $9x^2-12xy+4y^2-49$.

Solution

	$9x^2-12xy+4y^2-49$
What is the GCF? 1. With more than 3 terms, use grouping. Last 2 terms have GCF equals to 1. Try grouping first 3 terms. Factor the trinomial with $a\neq 1$. The first term is a perfect square. Is the last term of the trinomial a perfect square? Yes.	$=(3x)^2-12xy+(2y)^2-49$
Does the trinomial fit the pattern $a^2-2ab+b^2$? Yes.	$=(3x)^2-2(3x)(2y)+(2y)^2-49$
Write the trinomial as a square.	$=(3x-2y)^2-49$
Is this binomial a sum or difference of squares or cubes? Yes, a difference of squares. Write it as a difference of squares.	$=(3x-2y)^2-7^2$
Write it as a product of conjugates.	$=((3x-2y)-7)((3x-2y)+7)$ $=(3x-2y-7)(3x-2y+7)$
Is the expression factored completely? Yes. Check your answer.	Multiply. $(3x-2y-7)(3x-2y+7)$ $=9x^2-6xy-21x-6xy+4y^2+14y+21x-14y-49$ $=9x^2-12xy+4y^2-49$ ✓
Conclude.	The factorization of $9x^2-12xy+4y^2-49$ is $(3x-2y-7)(3x-2y+7)$.

Try It 1.2.9.26

Factor $4x^2-12xy+9y^2-25$.

Answer

$(2x-3y-5)(2x-3y+5)$

Try It 1.2.9.27

Factor $16x^2-24xy+9y^2-64$.

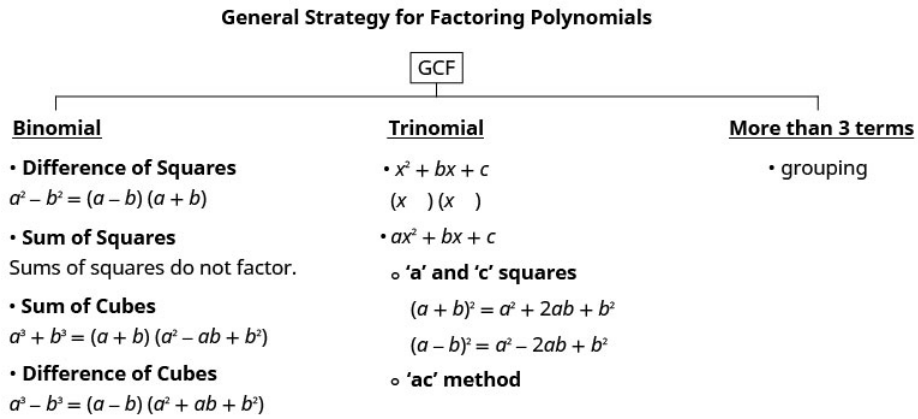
Answer

$(4x-3y-8)(4x-3y+8)$

Writing Exercises1.2.9.28

1. How do you check that you have factored correctly? Give an example.
2. How many binomial factors could a polynomial of degree 5 have? Explain and give an example of the maximum number.
3. Why might it be useful to factor?
4. Use factoring to carry out the division: $\dfrac{5x^3-5x^2+30x}{x^2+2x}$
5. If you know that $2x-5$ is a factor of $4x^2-4x-15$ help you to factor the latter? Explain.

Key Concepts



- **How to use a general strategy for factoring polynomials.**
  1. Is there a greatest common factor?
     Factor it out.
  2. Is the polynomial a binomial, trinomial, or are there more than three terms?
     If it is a binomial:
     Is it a sum?
     Of squares? Sums of squares do not factor.
     Of cubes? Use the sum of cubes pattern.
     Is it a difference?
     Of squares? Factor as the product of conjugates.
     Of cubes? Use the difference of cubes pattern.
     If it is a trinomial:
     Is it of the form x^2+bx+c ? Undo FOIL.
     Is it of the form ax^2+bx+c ?
     If \(a\) and c are squares, check if it fits the trinomial square pattern.
     Use the trial and error or “ac” method.
     If it has more than three terms:
     Use the grouping method.
  3. Check.
     Is it factored completely?
     Do the factors multiply back to the original polynomial?

SECTION OVERVIEW

1.3: Rational Expressions

1.3.1: Integer Exponents: a Review with Variables

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions using the properties for exponents
- Use the definition of a negative exponent

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $(-2)(-2)(-2)$.

2. Simplify $(2x^5)^3$.

3. Simplify $(2x^2)(4x^3)$.

A Review of Positive Integer Exponents

Remember that a positive integer exponent indicates repeated multiplication of the same quantity. For example, in the expression a^m, the positive integer *exponent* m tells us how many times we use the *base* a as a factor.

$$a^m = \underbrace{a \cdot a \cdot a \cdot \ldots \cdot a}_{\text{m factors}}$$

For example,

$$(-9)^5 = \underbrace{(-9) \cdot (-9) \cdot (-9) \cdot (-9) \cdot (-9)}_{\text{5 factors}}$$

Let's review the vocabulary for expressions with exponents.

Definition 1.3.1.1

$$a^m = \underbrace{a \cdot a \cdot a \cdots a}_{m \text{ factors}}$$

This is read a to the m^{th} **power**.

In the expression a^m with positive integer m and $a \neq 0$, the **exponent** m tells us how many times we use the **base** a as a factor.

Recall the following example which leads to the the *Product Property for Positive Integer Exponents*.

	x^2x^3
What does this mean?	$= \underbrace{x \cdot x}_{\text{2 factors}} \cdot \underbrace{x \cdot x \cdot x}_{\text{3 factors}}$ $= \underbrace{x \cdot x \cdot x \cdot x \cdot x}_{\text{5 factors}}$
	$= x^5$

The base stayed the same and we added the exponents.

Product Property for Positive Integer Exponents

If a is a real number and m and n are positive integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$=\frac{x\cdot x\cdot x\cdot x\cdot x}{x\cdot x}$		$=\frac{x\cdot x}{x\cdot x\cdot x}$
Use the Equivalent Fractions Property.	$=\frac{\cancel{x}\cdot\cancel{x}\cdot x\cdot x\cdot x}{\cancel{x}\cdot\cancel{x}}$		$=\frac{\cancel{x}\cdot\cancel{x}\cdot 1}{\cancel{x}\cdot\cancel{x}\cdot x}$
Simplify.	$=x^3$		$=\frac{1}{x}$
Note.	$x^3 = x^{5-2}$		$\frac{1}{x}=\frac{1}{x^1}=\frac{1}{x^{3-2}}$

Here we see how to use the initial exponents to arrive at the simplified expression. When the larger exponent was in the numerator, we were left with factors in the numerator. When the larger exponent was in the denominator, we were left with factors in the denominator--notice the numerator of 1. When all the factors in the numerator have been removed, remember this is really dividing the factors to one, and so we need a 1 in the numerator: $\frac{\cancel{x}}{\cancel{x}}=1$. This leads to the *Quotient Property for Positive Integer Exponents*.

Quotient Property for Positive Integer Exponents

If a is a real number, $a\neq 0$, and m and n are distinct positive integers, then

$$\frac{a^m}{a^n}=\begin{cases} a^{m-n} & \text{if } m>n \\ \frac{1}{a^{n-m}} & \text{if } n>m. \end{cases}$$

Example 1.3.1.2

Simplify each expression:

a. $\frac{x^9}{x^7}$

b. $\frac{3^{10}}{3^2}$

c. $\frac{b^8}{b^{12}}$

d. $\frac{7^3}{7^5}$

Solution

To simplify an expression with a quotient, we need to first compare the exponents in the numerator and denominator.

a.

	$\frac{x^9}{x^7}$
Since $9 > 7$, there are more factors of x in the numerator. Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, for $m > n$.	$= x^{9-7}$
Simplify.	$= x^2$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

b.

	$\frac{3^{10}}{3^2}$
Since $10 > 2$, there are more factors of 3 in the numerator. Use the Quotient Property, $\frac{a^m}{a^n} = a^{m-n}$, for $m > n$.	$= 3^{10-2}$
Simplify.	$= 3^8$

Notice that when the larger exponent is in the numerator, we are left with factors in the numerator.

c.

	$\frac{b^8}{b^{12}}$
Since $12 > 8$, there are more factors of b in the denominator. Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$, for $n > m$.	$= \frac{1}{b^{12-8}}$
Simplify.	$= \frac{1}{b^4}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

d.

	$\frac{7^3}{7^5}$
Since $5 > 3$, there are more factors of 3 in the denominator. Use the Quotient Property, $\frac{a^m}{a^n} = \frac{1}{a^{n-m}}$ for $n > m$.	$= \frac{1}{7^{5-3}}$
Simplify.	$= \frac{1}{7^2}$
Simplify.	$= \frac{1}{49}$

Notice that when the larger exponent is in the denominator, we are left with factors in the denominator.

Try It 1.3.1.3

Simplify each expression:

a. $\frac{x^{15}}{x^{10}}$

b. $\frac{6^{14}}{6^5}$

c. $\frac{x^{18}}{x^{22}}$

d. $\frac{12^{15}}{12^{30}}$

Answer

a. x^5

b. 6^9

c. $\frac{1}{x^4}$

d. $\frac{1}{12^{15}}$

Try It 1.3.1.4

Simplify each expression:

a. $\frac{y^{43}}{y^{37}}$

b. $\frac{10^{15}}{10^7}$

c. $\frac{m^7}{m^{15}}$

d. $\frac{9^8}{9^{19}}$

Answer

a. y^6

b. 10^8

c. $\frac{1}{m^8}$

d. $\frac{1}{9^{11}}$

Extending the Meaning of Exponent to Integers

Note that, while so far an exponent that is not a positive integer has no meaning, we see that blindly applying the above properties for such exponents leads to a couple definitions.

	1
For m a positive integer (so that x^m has meaning, $x \neq 0$).	$= \frac{x^m}{x^m}$
Blindly follow the quotient rule that we know to be true in another case.	$= x^{m-m}$
Reduce the fraction on the left.	$= x^0$
Conclude.	$x^0 = 1$

So **if** the Quotient Property is also to hold for the exponent zero we **must** define, for $x \neq 0$,

$$x^0 = 1.$$

Similarly, consider this expression where m is a positive integer (so that x^m has meaning).

	$x^{-m}x^m$

Blindly applying the product property.	$=x^{-m+m}$
Simplify exponent.	$=x^0$
Using our new definition: $x^0=1$.	$=1$
Draw conclusion.	$x^{-m}x^m=1$
Note the property of the reciprocal.	x^{-m} is the reciprocal of x^m.
Rewrite in symbols.	$x^{-m}=\frac{1}{x^m}$

So, **if** the product property of exponents holds also for (so far undefined) negative integers, we **must** define, for m a negative integer,

$$x^{-m}=\frac{1}{x^m}.$$

Also, since taking the reciprocal of both sides preserves the equality we also have, equivalently,

$$\frac{1}{x^{-m}}=x^m.$$

We could also write these two statements above simultaneously as

$$x^{-m}=\frac{1}{x^m}, m \text{ any integer.}$$

So, we **must** define

Definition 1.3.1.5

For a any non-zero real number

$$a^0=1$$

and for m any positive integer

$$a^{-m}=\frac{1}{a^m} \text{ or, equivalently, } \frac{1}{a^{-m}}=a^m.$$

In the above, the base can be anything (x can be anything) which we know to be different from zero, and in this text, we assume any variable that we raise to the zero power is not zero.

Example 1.3.1.6

Simplify each expression:

a. 9^0

b. n^0

c. $(-4a^2b)^0$

d. -3^0

Solution

The definition says any non-zero number raised to the zero power is 1.

a. Use the definition of the zero exponent. $9^0=1$

b. Use the definition of the zero exponent. $n^0 = 1$

To simplify the expression n raised to the zero power we just use the definition of the zero exponent. The result is 1.

c. Anything raised to the power zero is 1. Here the base is $-4a^2b$, so $(-4a^2b)^0 = 1$

d. Anything raised to the power zero is 1. Here the base is 3, so this is the opposite of 3^0, or, the opposite of 1. So, $-3^0 = -1$

Try It 1.3.1.7

Simplify each expression:

a. 11^0

b. q^0

c. $(-12p^3q^2)^0$

d. -7^0

Answer

- **a.** 1
- **b.** 1
- **c.** 1
- **d.** -1

Try It 1.3.1.8

Simplify each expression:

a. 23^0

b. r^0

c. $(2st^5)^0$

d. $-s^0$

Answer

- **a.** 1
- **b.** 1
- **c.** 1
- **d.** -1

Example 1.3.1.9

Simplify each expression. Write your answer using positive exponents.

a. x^{-5}

b. 10^{-3}

c. $\frac{1}{y^{-4}}$

d. $\frac{1}{3^{-2}}$

Solution

a.

	x^{-5}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{x^5}$

b.

	10^{-3}
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{10^3}$
Simplify.	$= \frac{1}{1000}$

c.

	$\frac{1}{y^{-4}}$
Use the definition of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	$= y^4$

d.

	$\frac{1}{3^{-2}}$
Use the definition of a negative exponent, $\frac{1}{a^{-n}} = a^n$.	$= 3^2$
Simplify.	$= 9$

? Try It 1.3.1.10

Simplify each expression. Write your answer using positive exponents.

a. z^{-3}

b. 10^{-7}

c. $\frac{1}{p^{-8}}$

d. $\frac{1}{4^{-3}}$

Answer

a. $\frac{1}{z^3}$

b. $\frac{1}{10,000,000}$

c. p^8

d. 64

Try It 1.3.1.11

Simplify each expression. Write your answer using positive exponents.

a. n^{-2}

b. 10^{-4}

c. $\frac{1}{q^{-7}}$

d. $\frac{1}{2^{-4}}$

Answer

a. $\frac{1}{n^2}$

b. $\frac{1}{10,000}$

c. q^7

d. 16

Properties of Negative Exponents

The negative exponent tells us we can rewrite the expression by taking the reciprocal of the base and then changing the sign of the exponent.

Any expression that has negative exponents is not considered to be in *simplest form*. We will use the definition of a negative exponent and other properties of exponents to write the expression with only positive exponents.

For example, if after simplifying an expression we end up with the expression x^{-3}, we will take one more step and write $\frac{1}{x^3}$. The answer is considered to be in simplest form when it has only positive exponents.

Suppose now we have a fraction raised to a negative exponent. Let's use our definition of negative exponents to lead us to a new property.

$\left(\frac{3}{4}\right)^{-2}$	
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=\frac{1}{\left(\frac{3}{4}\right)^2}$
Simplify the denominator.	$=\frac{1}{\frac{9}{16}}$
Simplify the complex fraction.	$=\frac{16}{9}$
But we know that	$\frac{16}{9}=\left(\frac{4}{3}\right)^2$
This tells us that	$\left(\frac{3}{4}\right)^{-2}=\left(\frac{4}{3}\right)^2$

To get from the original fraction raised to a negative exponent to the final result, we took the reciprocal of the base—the fraction—and changed the sign of the exponent.

This leads us to the *Quotient to a Negative Integer Exponent Property*.

Quotient to a Negative Integer Exponent Property

If a and b are real numbers, $a \neq 0$, $b \neq 0$, and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}.$$

Example 1.3.1.12

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{5}{7}\right)^{-2}$

b. $\left(-\frac{x}{y}\right)^{-3}$

Solution

a.

	$\left(\frac{5}{7}\right)^{-2}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(\frac{7}{5}\right)^{2}$
Simplify.	$=\frac{49}{25}$

b.

	$\left(-\frac{x}{y}\right)^{-3}$
Use the Quotient to a Negative Integer Exponent Property, $\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}$.	
Take the reciprocal of the fraction and change the sign of the exponent.	$=\left(-\frac{y}{x}\right)^{3}$
Simplify.	$=-\frac{y^3}{x^3}$

Try It 1.3.1.13

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{2}{3}\right)^{-4}$

b. $\left(-\frac{m}{n}\right)^{-2}$

Answer

a. $\frac{81}{16}$

b. $\frac{n^2}{m^2}$

? Try It 1.3.1.14

Simplify each expression. Write your answer using positive exponents.

a. $\left(\frac{3}{5}\right)^{-3}$

b. $\left(-\frac{a}{b}\right)^{-4}$

Answer

a. $\frac{125}{27}$

b. $\frac{b^4}{a^4}$

We would like to verify that the properties of positive integer exponents can be extended to all integer exponents. We will do some examples. Let m and n be non-negative integers, $m > n$. Let's simplify $x^m x^{-n}$.

	$x^m x^{-n}$
Use the definition: $x^{-n} = \frac{1}{x^n}$.	$= x^m \cdot \frac{1}{x^n}$
Multiply fractions.	$= \frac{x^m}{x^n}$
Use the Quotient Property of exponents.	x^{m-n} or $x^{m+(-n)}$
Conclude.	$x^m x^{-n} = x^{m-n} = x^{m+(-n)}$

So, the product property holds in this case.

And if $m < n$,

	$x^m x^{-n}$
Use the definition: $x^{-n} = \frac{1}{x^n}$.	$= x^m \cdot \frac{1}{x^n}$
Multiply fractions.	$= \frac{x^m}{x^n}$
Use the Quotient Property of exponents.	$= \frac{1}{x^{n-m}}$
Use second variation of the definition of negative exponents.	$= x^{-(n-m)}$
Simplify.	x^{m-n} or $x^{m+(-n)}$
Conclude.	$x^m x^{-n} = x^{m-n} = x^{m+(-n)}$

So the Product Property for Positive Integer Exponents holds in this case, too. We can in a similar way check other combinations to see that the Product Property for Positive Integer Exponents holds for all integers. The Product Property for Integer Exponents follows directly

$$\frac{x^m}{x^n} = x^m \cdot \frac{1}{x^n} = x^m \cdot x^{-n} = x^{m-n}$$

and

$$\frac{x^m}{x^n} = \frac{1}{x^{-m}} \cdot \frac{1}{x^n} = \frac{1}{x^{-m} \cdot x^n} = \frac{1}{x^{-m+n}} \quad \text{or} \quad \frac{1}{x^{n-m}}.$$

So, both the Product Property and the Quotient Property hold for all integer exponents.

Product Property for Integer Exponents

If a is a real number and m and n are integers, then

$$a^m a^n = a^{m+n}.$$

To multiply with like bases, add the exponents.

Quotient Property for Integer Exponents

If a is a real number, $a \neq 0$, and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}$$

and

$$\frac{a^m}{a^n} = \frac{1}{a^{n-m}}.$$

To divide with like bases, subtract the exponents as above.

We will now use the product property with expressions that have negative exponents. We can choose to use the meaning of negative exponents instead of the quotient property above, so we will focus here on the product property.

Example 1.3.1.15

Simplify each expression:

a. $z^{-5}z^{-3}$

b. $(m^4n^{-3})(m^{-5}n^{-2})$

c. $(2x^{-6}y^8)(-5x^5y^{-3})$

Solution

a.

	$z^{-5}z^{-3}$
Add the exponents, since the bases are the same.	$= z^{-5-3}$
Simplify.	$= z^{-8}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{z^8}$

b.

	$(m^4n^{-3})(m^{-5}n^{-2})$
Use the Commutative Property to get like bases together.	$= m^4m^{-5}n^{-2}n^{-3}$
Add the exponents for each base.	$= m^{-1}n^{-5}$
Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{1}{m^1} \cdot \frac{1}{n^5}$

	$(m^4n^{-3})(m^{-5}n^{-2})$
Simplify.	$=\frac{1}{mn^5}$

c.

	$(2x^{-6}y^8)(-5x^5y^{-3})$
Rewrite with the like bases together.	$=2(-5)(x^{-6}x^5)(y^8y^{-3})$
Multiply the coefficients and add the exponents of each variable.	$=-10x^{-1}y^5$
Use the definition of a negative exponent, $a^{-n}=\frac{1}{a^n}$.	$=-10\frac{1}{x}y^5$
Simplify.	$=-\frac{10y^5}{x}$

? Try It 1.3.1.16

Simplify each expression:

a. $z^{-4}z^{-5}$

b. $(p^6q^{-2})(p^{-9}q^{-1})$

c. $(3u^{-5}v^7)(-4u^4v^{-2})$

Answer

a. $\frac{1}{z^9}$

b. $\frac{1}{p^3q^3}$

c. $-\frac{12v^5}{u}$

? Try It 1.3.1.17

Simplify each expression:

a. $c^{-8}c^{-7}$

b. $(r^5s^{-3})(r^{-7}s^{-5})$

c. $(-6c^{-6}d^4)(-5c^{-2}d^{-1})$

Answer

a. $\frac{1}{c^{15}}$

b. $\frac{1}{r^2s^8}$

c. $\frac{30d^3}{c^8}$

Now let’s look at an exponential expression that contains a power raised to a power. Let's see if we can discover a general property.

$(x^2)^3$

	$(x^2)^3$
What does this mean?	$= x^2 x^2 x^2$
How many factors altogether?	$= \underbrace{x \cdot x}_{2\ factors} \cdot \underbrace{x \cdot x}_{2\text{ factors}} \cdot \underbrace{x \cdot x}_{2\text{ factors}}$ $= \underbrace{x \cdot x \cdot x \cdot x \cdot x \cdot x}_{6\text{ factors}}$
So we have	$= x^6$

Notice the 6 is the *product* of the exponents, 2 and 3. We see that $(x^2)^3$ is $x^{2 \cdot 3}$ or x^6. We can also see that

$$(x^{-2})^3 = \left(\frac{1}{x^2}\right)^3 = \frac{1}{x^2} \cdot \frac{1}{x^2} \cdot \frac{1}{x^2} = \frac{1}{x^6} = x^{-6}$$

so that $(x^{-2})^3 = x^{-6}$. In these examples we multiplied the exponents.

We can check various combinations of signs of the exponents which leads us to the *Power Property for Integer Exponents.*

Power Property for Integer Exponents

If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}.$$

To raise a power to a power, multiply the exponents.

Example 1.3.1.18

Simplify each expression:

a. $(y^5)^9$

b. $(4^{-4})^7$

c. $(y^3)^6(y^5)^4$

Solution

a.

	$(y^5)^9$
Use the power property, $(a^m)^n = a^{mn}$.	$y^{5 \cdot 9}$
Simplify.	y^{45}

b.

	$(4^{-4})^7$
Use the power property.	$= 4^{-4 \cdot 7}$
Simplify.	$= 4^{-28}$

c.

	$(y^3)^6(y^5)^4$
Use the power property.	$= y^{18} y^{20}$
Add the exponents.	$= y^{38}$

Try It 1.3.1.19

Simplify each expression:

a. $(b^7)^5$

b. $(5^4)^{-3}$

c. $(a^4)^5(a^7)^4$

Answer

a. b^{35}

b. 5^{-12}

c. a^{48}

Try It 1.3.1.20

Simplify each expression:

a. $(z^6)^9$

b. $(3^{-7})^7$

c. $(q^4)^5(q^3)^3$

Answer

a. z^{54}

b. 3^{-49}

c. q^{29}

We will now look at an expression containing a product that is raised to a power. Can we find this pattern?

	$(2x)^3$
What does this mean?	$= 2x \cdot 2x \cdot 2x$
We group the like factors together.	$= 2 \cdot 2 \cdot 2xxx$
How many factors of 2 and of x?	$= 2^3x^3$

Notice that each factor was raised to the power and $(2x)^3$ is 2^3x^3.

The exponent applies to each of the factors! We can say that the exponent distributes over multiplication. If we were to check various examples with exponents which are negative or zero, then we would find the same pattern emerges. This leads to the *Product to a Power Property for Integer Exponents.*

Product to a Power Property for Integer Exponents

If a and b are real numbers and m is an integer, then

$$(ab)^m = a^mb^m.$$

To raise a product to a power, raise each factor to that power.

> **Example 1.3.1.21**
>
> Simplify each expression:
>
> **a.** $(-3mn)^3$
>
> **b.** $(6k^3)^{-2}$
>
> **c.** $(5x^{-3})^2$
>
> ### Solution
>
> **a.**
>
> | | |
> |---|---|
> | | $(-3mn)^3$ |
> | Use Power of a Product Property, $(ab)^m = a^m b^m$. | $= (-3)^3 m^3 n^3$ |
> | Simplify. | $= -27m^3n^3$ |
>
> **b.**
>
> | | |
> |---|---|
> | | $(6k^3)^{-2}$ |
> | Use the Power of a Product Property, $(ab)^m = a^m b^m$. | $= 6^{-2}(k^3)^{-2}$ |
> | Use the Power Property, $(a^m)^n = a^{mn}$. | $= 6^{-2}k^{-6}$ |
> | Use the definition of a negative exponent, $a^{-n} = \frac{1}{a^n}$. | $= \frac{1}{6^2} \cdot \frac{1}{k^6}$ |
> | Simplify. | $= \frac{1}{36k^6}$ |
>
> **c.**
>
> | | |
> |---|---|
> | | $(5x^{-3})^2$ |
> | Use the power of a product property, $(ab)^m = a^m b^m$. | $= 5^2(x^{-3})^2$ |
> | Simplify. | $= 25x^{-6}$ |
> | Rewrite x^{-6} using $a^{-n} = \frac{1}{a^n}$. | $= 25\frac{1}{x^6}$ |
> | Simplify. | $= \frac{25}{x^6}$ |

> **Try It 1.3.1.22**
>
> Simplify each expression:
>
> **a.** $(2wx)^5$
>
> **b.** $(2b^3)^{-4}$
>
> **c.** $(8a^{-4})^2$

Answer

a. $32w^5x^5$

b. $\frac{1}{16b^{12}}$

c. $\frac{64}{a^8}$

Try It 1.3.1.23

Simplify each expression:

a. $(-3y)^3$

b. $(-4x^4)^{-2}$

c. $(2c^{-4})^3$

Answer

a. $-27y^3$

b. $\frac{1}{16x^8}$

c. $8c^{12}$

Now we will look at an example that will lead us to the quotient to a power property.

	$\left(\frac{x}{y}\right)^3$
This means	$=\frac{x}{y}\cdot\frac{x}{y}\cdot\frac{x}{y}$
Multiply the fractions.	$=\frac{xxx}{yyy}$
Write with exponents.	$=\frac{x^3}{y^3}$

Notice that the exponent applies to both the numerator and the denominator.

We see that $\left(\frac{x}{y}\right)^3$ is $\frac{x^3}{y^3}$.

This leads to the *Quotient to a Power Property for Integer Exponents*. We can say that the exponent distributes over division as well as multiplication.

Quotient to a Power Property for Integer Exponents

If a and b are real numbers, $b\neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}.$$

To raise a fraction to a power, raise the numerator and denominator to that power.

Example 1.3.1.24

Simplify each expression:

a. $\left(\frac{b}{3}\right)^4$

b. $\left(\frac{k}{j}\right)^{-3}$

c. $\left(\frac{2xy^2}{z}\right)^3$

d. $\left(\frac{4p^{-3}}{q^2}\right)^2$

Solution

a.

	$\left(\frac{b}{3}\right)^4$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$= \frac{b^4}{3^4}$
Simplify.	$= \frac{b^4}{81}$

b.

	$\left(\frac{k}{j}\right)^{-3}$
Raise the numerator and denominator to the power.	$= \frac{k^{-3}}{j^{-3}}$
Use the definition of negative exponent, $a^{-n} = \frac{1}{a^n}$ and $\frac{1}{a^{-n}} = a^n$.	$= \frac{1}{k^3} \cdot \frac{j^3}{1}$
Multiply.	$= \frac{j^3}{k^3}$

c.

	$\left(\frac{2xy^2}{z}\right)^3$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$= \frac{(2xy^2)^3}{z^3}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$, and the Power Property, $(a^m)^n = a^{mn}$.	$= \frac{8x^3y^6}{z^3}$

d.

	$\left(\frac{4p^{-3}}{q^2}\right)^2$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$.	$= \frac{(4p^{-3})^2}{(q^2)^2}$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$= \frac{16p^{-6}}{q^4}$
Use the definition of negative exponent, $a^{-n} = \frac{1}{a^n}$.	$= \frac{16}{q^4} \cdot \frac{1}{p^6}$
Simplify.	$= \frac{16}{p^6q^4}$

Try It 1.3.1.25

Simplify each expression:

a. $\left(\frac{p}{10}\right)^4$

b. $\left(\frac{m}{n}\right)^{-7}$

c. $\left(\frac{3ab^3}{c^2}\right)^4$

d. $\left(\frac{3x^{-2}}{y^3}\right)^3$

Answer

a. $\frac{p^4}{10000}$

b. $\frac{n^7}{m^7}$

c. $\frac{81a^4b^{12}}{c^8}$

d. $\frac{27}{x^6y^9}$

Try It 1.3.1.26

Simplify each expression:

a. $\left(\frac{-2}{q}\right)^3$

b. $\left(\frac{w}{x}\right)^{-4}$

c. $\left(\frac{xy^3}{3z^2}\right)^2$

d. $\left(\frac{2m^{-2}}{n^{-2}}\right)^3$

Answer

a. $\frac{-8}{q^3}$

b. $\frac{x^4}{w^4}$

c. $\frac{x^2y^6}{9z^4}$

d. $\frac{8n^6}{m^6}$

We now have several properties for exponents. Let's summarize them and then we'll do some more examples that use more than one of the properties. We note that there are many ways to simplify the expressions, but the final simplification should be equivalent.

Summary of Integer Exponent Definitions and Properties

If a and b are real numbers, and m and n are integers, then

Definition	Description
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

Example 1.3.1.27

Simplify each expression by applying several properties:

a. $(3x^2y)^4(2xy^2)^3$

b. $\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$

c. $\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$

Solution

a.

	$(3x^2y)^4(2xy^2)^3$
Use the Product to a Power Property, $(ab)^m = a^m b^m$.	$= (3^4x^8y^4)(2^3x^3y^6)$
Simplify.	$= (81x^8y^4)(8x^3y^6)$
Use the Commutative Property.	$= 81 \cdot 8x^8x^3y^4y^6$
Multiply the constants and add the exponents.	$= 648x^{11}y^{10}$

b.

	$\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$
Use the Power Property, $(a^m)^n = a^{m \cdot n}$.	$= \frac{x^{12}x^{-10}}{x^{30}}$

	$\frac{(x^3)^4(x^{-2})^5}{(x^6)^5}$
Add the exponents in the numerator.	$=\frac{x^2}{x^{30}}$
Use the Quotient Property, $\frac{a^m}{a^n}=\frac{1}{a^{n-m}}$	$=\frac{1}{x^{28}}$

c.

	$\left(\frac{2xy^2}{x^3y^{-2}}\right)^2\left(\frac{12xy^3}{x^3y^{-1}}\right)^{-1}$
Simplify inside the parentheses first.	$=\left(\frac{2y^4}{x^2}\right)^2\left(\frac{12y^4}{x^2}\right)^{-1}$
Use the Quotient to a Power Property, $\left(\frac{a}{b}\right)^m=\frac{a^m}{b^m}$.	$=\frac{(2y^4)^2}{(x^2)^2}\frac{(12y^4)^{-1}}{(x^2)^{-1}}$
Use the Product to a Power Property, $(ab)^m=a^mb^m$.	$=\frac{4y^8}{x^4}\cdot\frac{12^{-1}y^{-4}}{x^{-2}}$
Simplify.	$=\frac{4y^4}{12x^2}$
Simplify.	$=\frac{y^4}{3x^2}$

Try It 1.3.1.28

Simplify each expression:

a. $(c^4d^2)^5(3cd^5)^4$

b. $\frac{(a^{-2})^3(a^2)^4}{(a^4)^5}$

c. $\left(\frac{3xy^2}{x^2y^{-3}}\right)^2$

Answer

a. $81c^{24}d^{30}$

b. $\frac{1}{a^{18}}$

c. $\frac{9y^{10}}{x^2}$

Try It 1.3.1.29

Simplify each expression:

a. $(a^3b^2)^6(4ab^3)^4$

b. $\frac{(p^{-3})^4(p^5)^3}{(p^7)^6}$

c. $\left(\frac{4x^3y^2}{x^2y^{-1}}\right)^2\left(\frac{8xy^{-3}}{x^2y}\right)^{-1}$

Answer

a. $256a^{22}b^{24}$

b. $\frac{1}{p^{39}}$

c. $2x^3y^{10}$

Writing Exercises 1.3.1.30

1. Give an example of distributing division over subtraction.
2. Give an example to show that you can not distribute division over multiplication.
3. How is the negative exponent related to reciprocals? Give an example.
4. How are positive and negative exponents used in science to express large or small numbers?
5. What is the purpose in writing numbers this way?

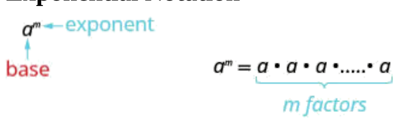
Exit Problem 1.3.1.31

a. Simplify $\left(\frac{3a^{-3}}{b^{-5}}\right)^{-3}$. Write your final answer with positive exponents only.

b. Simplify $\frac{36x^5y^{10}}{70x^{15}y^5}$. Write your final answer with positive exponents only.

Key Concepts

- **Exponential Notation**



This is read a to the m^{th} power.
In the expression a^m, the *exponent* m (when positive) tells us how many times we use the *base* a as a factor.
- **Zero Exponent (Definition)** If a is a non-zero number, then $a^0 = 1$.
- **Negative Exponent (Definition)** If n is an integer and $a \neq 0$, then $a^{-n} = \frac{1}{a^n}$ or, equivalently, $\frac{1}{a^{-n}} = a^n$.
- **Product Property for Exponents**
If a is a real number and m and n are integers, then

$$a^m a^n = a^{m+n}$$

To multiply with like bases, add the exponents.
- **Quotient Property for Exponents**
If a is a real number, $a \neq 0$, and m and n are integers, then

$$\frac{a^m}{a^n} = a^{m-n}, \quad m > n \quad \text{and} \quad \frac{a^m}{a^n} = \frac{1}{a^{n-m}}, \quad n > m$$

- **Quotient to a Negative Exponent Property**
If a and b are real numbers, $a \neq 0$, $b \neq 0$ and n is an integer, then

$$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$$

- **Power Property for Exponents**
If a is a real number and m and n are integers, then

$$(a^m)^n = a^{mn}$$

To raise a power to a power, multiply the exponents.

- **Product to a Power Property for Exponents**
  If a and b are real numbers and m is a whole number, then

$$(ab)^m = a^m b^m$$

  To raise a product to a power, raise each factor to that power.
- **Quotient to a Power Property for Exponents**
  If a and b are real numbers, $b \neq 0$, and m is an integer, then

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}$$

  To raise a fraction to a power, raise the numerator and denominator to that power.
- **Summary of Exponent Properties**
  If a and b are real numbers, and m and n are integers, then
-

Property	Description
Definition of Zero Exponent	$a^0 = 1, a \neq 0$
Definition of Negative Exponents	$a^{-n} = \frac{1}{a^n}$, or equivalently, $\frac{1}{a^{-n}} = a^n$

-

Property	Description
Product Property	$a^m \cdot a^n = a^{m+n}$
Power Property	$(a^m)^n = a^{m \cdot n}$
Product to a Power	$(ab)^n = a^n b^n$
Quotient Property	$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
Quotient to a Power Property	$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
Quotient to a Negative Exponent	$\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^n$

1.3.2: Simplifying, Multiplying and Dividing Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Determine the values for which a rational expression is undefined
- Simplify rational expressions
- Multiply rational expressions
- Divide rational expressions

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\frac{90y}{15y^2}$.
2. Multiply $\frac{14}{15} \cdot \frac{6}{35}$.
3. Divide $\frac{12}{10} \div \frac{8}{25}$.

Above are examples of some properties of fractions and their operations. Recall that these fractions where the numerators and denominators are integers are called rational numbers. In this chapter, we will work with fractions whose numerators and denominators are polynomials. We call this kind of expression a **rational expression**.

Definition 1.3.2.1

A **rational expression** is an expression of the form $\frac{p}{q}$, where p and q are polynomials and $q \neq 0$.

Here are some examples of rational expressions:

$$-\frac{24}{56} \qquad \frac{5x}{12y} \qquad \frac{4x+1}{x^2-9} \qquad \frac{4x^2+3x-1}{2x-8}$$

Notice that the first rational expression listed above, $-\frac{24}{56}$, is just a fraction. Since a constant is a polynomial with degree zero, the ratio of two constants is a rational expression, provided the denominator is not zero.

We will do the same operations with rational expressions that we did with fractions. We will simplify, add, subtract, multiply, divide and use them in applications.

Simplify Rational Expressions

Here we recall for example, $\frac{1}{2} = \frac{6}{12}$, which can be understood by imagining dividing a pizza into 2 pieces and eating 1, and imagining dividing the pizza up into 12 pieces and eating 6 of them! In general, $\frac{a}{b} = \frac{ca}{cb}$ when c and b are not zero. The process of replacing $\frac{ca}{cb}$ by the equivalent expression $\frac{a}{b}$ is called reducing the fraction. It is commonly also called 'canceling' the common factor c.

A fraction is considered simplified if there are no common factors, other than 1 and -1, in its numerator and denominator. Similarly, a **simplified rational expression** has no common factors, other than 1 and -1, in its numerator and denominator.

✎ Simplified Rational Expression

A rational expression is considered **simplified** if there are no common factors other than 1 and -1 in its numerator and denominator. There should be no more "$-$"s than necessary.

For example,

$$\frac{x+2}{x+3} \text{ is simplified because there are no common factors of } x+2 \text{ and } x+3.$$
$$\frac{2x}{3x} \text{ is not simplified because } x \text{ is a common factor of } 2x \text{ and } 3x.$$

We use the Equivalent Fractions Property to simplify numerical fractions. We restate it here as we will also use it to simplify rational expressions.

✎ Equivalent Fractions Property (Reducing Fractions)

If a, b, and c are numbers where $b \neq 0, c \neq 0$,

$$\text{then } \frac{a}{b} = \frac{a \cdot c}{b \cdot c} \text{ and } \frac{a \cdot c}{b \cdot c} = \frac{a}{b}$$

Notice that in the Equivalent Fractions Property, the values that would make the denominators zero are specifically disallowed. We see $b \neq 0$, $c \neq 0$ clearly stated.

To simplify rational expressions, we first write the numerator and denominator in factored form. Then we remove the common factors using the Equivalent Fractions Property.

Be very careful as you remove common factors. Factors are multiplied to make a product. You can remove a factor from a product. You cannot remove a term from a sum.

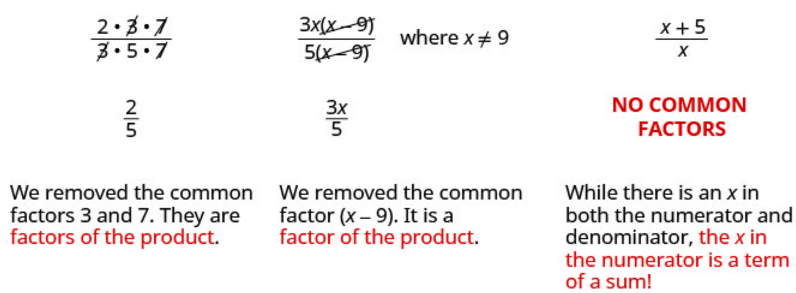


Figure 1.3.2.1.

Removing the x's from $\dfrac{x+5}{x}$ would be like cancelling the 2's in the fraction $\dfrac{2+5}{2}$!

Simplifying Rational Expressions: Monomials over Monomials

Now we will look at an exponent property for division. As before, we'll try to discover a property by looking at some examples.

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
What do they mean?	$= \frac{x \cdot x \cdot x \cdot x \cdot x}{x \cdot x}$		$= \frac{x \cdot x}{x \cdot x \cdot x}$
Use the Equivalent Fractions Property.	$= \frac{\cancel{x} \cdot \cancel{x} \cdot x \cdot x \cdot x}{\cancel{x} \cdot \cancel{x}}$		$= \frac{\cancel{x} \cdot \cancel{x} \cdot 1}{\cancel{x} \cdot \cancel{x} \cdot x}$
Simplify.	$= x^3$		$= \frac{1}{x}$ (This is not a polynomial!)

Consider	$\frac{x^5}{x^2}$	and	$\frac{x^2}{x^3}$
Note.	$\frac{x^5}{x^2} = \frac{x^2x^3}{x^2} = x^3$ Reducing the fraction.		$\frac{x^2}{x^3} = \frac{x^2}{x^2x} = \frac{1}{x}$ Reducing the fraction.

Note that the result when you divide monomials is not necessarily a polynomial!

Example 1.3.2.2

Simplify $\frac{54a^2b^3}{-6ab^5}$.

Solution

When we divide monomials with more than one variable, we write one fraction for each variable.

	$\frac{54a^2b^3}{-6ab^5}$
Rewrite as a fraction.	$= \frac{54a^2b^3}{-6ab^5}$
Use fraction multiplication.	$= \frac{54}{-6} \cdot \frac{a^2}{a} \cdot \frac{b^3}{b^5}$
Simplify and use the Quotient Property.	$= -9a \cdot \frac{1}{b^2}$
Multiply.	$= -\frac{9a}{b^2}$

Try It 1.3.2.3

Simplify $-\frac{72a^7b^3}{(8a^{12}b^4}$.

Answer

$-\frac{9}{a^5b}$

Try It 1.3.2.4

Simplify $-\frac{63c^8d^3}{7c^{12}d^2}$.

Answer

$\frac{-9d}{c^4}$

Once you become familiar with the process and have practiced it step by step several times, you may be able to simplify a fraction in one step.

Example 1.3.2.5

Simplify $\frac{14x^7y^{12}}{21x^{11}y^6}$.

Solution

Be very careful to simplify $\frac{14}{21}$ by dividing out a common factor, and to simplify the variables by subtracting their exponents.

	$\frac{14x^7y^{12}}{21x^{11}y^6}$
Simplify and use the Quotient Property.	$=\frac{2y^6}{3x^4}$

Try It 1.3.2.6

Simplify $\frac{28x^5y^{14}}{49x^9y^{12}}$.

Answer

$\frac{4y^2}{7x^4}$

Try It 1.3.2.7

Simplify $\frac{30m^5n^{11}}{48m^{10}n^{14}}$.

Answer

$\frac{5}{8m^5n^3}$

Simplifying Rational Expressions: Polynomials over Monomials

Now that we know how to divide a monomial by a monomial, the next procedure is to divide a polynomial of two or more terms by a monomial. The method we'll use to divide a polynomial by a monomial is based on the properties of fraction addition. It is clear that $\frac{2+3}{7}=\frac{2}{7}+\frac{3}{7}$ because the left side is 5 sevenths and the right side is 2 sevenths plus 3 sevenths. You may also see this as $\frac{1}{7}(2+3)=\frac{1}{7}2+\frac{1}{7}3=\frac{2}{7}+\frac{3}{7}$. So in other words, we distribute division over addition (and subtraction) just as we did multiplication. Since variables are place-holders for numbers, it will be the same if instead of numbers we have variables.

For example, the sum $\frac{y}{5}+\frac{2}{5}$ simplifies to $\frac{y+2}{5}$. Now we will do this in reverse to split a single fraction into separate fractions. For example, $\frac{y+2}{5}$ can be written $\frac{y}{5}+\frac{2}{5}$.

This is the "reverse" of fraction addition and it states that if a, b, and c are numbers where $c\neq 0$, then $\frac{a+b}{c}=\frac{a}{c}+\frac{b}{c}$. We will use this to divide polynomials by monomials.

Division of a polynomial by a monomial

To divide a polynomial by a monomial, divide each term of the polynomial by the monomial.

Example 1.3.2.8

Simplify $\frac{18x^3y-36xy^2}{-3xy}$.

Solution

	$\frac{18x^3y-36xy^2}{-3xy}$
Rewrite as a fraction.	$=\frac{18x^3y-36xy^2}{-3xy}$
Divide each term by the divisor. Be careful with the signs!	$=\frac{18x^3y}{-3xy}-\frac{36xy^2}{-3xy}$
Simplify.	$=-6x^2+12y$

Try It 1.3.2.9

Simplify $\frac{32a^2b-16ab^2}{-8ab}$.

Answer

$-4a+2b$

Try It 1.3.2.10

Simplify $\frac{-48a^8b^4-36a^6b^5}{-6a^3b^3}$.

Answer

$8a^5b+6a^3b^2$

Simplifying Rational Expressions: Polynomials over Polynomials

Example 1.3.2.11

Simplify $\frac{x^2+5x+6}{x^2+8x+12}$.

Solution

	$\frac{x^2+5x+6}{x^2+8x+12}$
Factor the numerator and denominator completely.	$=\frac{(x+2)(x+3)}{(x+2)(x+6)}$

Simplify by dividing out common factors.	$=\frac{\cancel{(x+2)}(x+3)}{\cancel{(x+2)}(x+6)}$ $=\frac{x+3}{x+6}$ for the values of x where \ $\frac{x^2+5x+6}{x^2+8x+12}=\frac{(x+2)(x+3)}{(x+2)(x+6)}$ and $\frac{x+3}{x+6}$ are both defined.

? Try It 1.3.2.12

Simplify $\frac{x^2-x-2}{x^2-3x+2}$.

Answer

$\frac{x+1}{x-1}$ for values of x where $\frac{x^2-x-2}{x^2-3x+2}$ is defined.

? Try It 1.3.2.13

Simplify $\frac{x^2-3x-10}{x^2+x-2}$.

Answer

$\frac{x-5}{x-1}$ where $\frac{x^2-3x-10}{x^2+x-2}$ is defined.

Note that in the example it is noted that the equality is valid only where $\frac{x^2+5x+6}{x^2+8x+12}=\frac{(x+2)(x+3)}{(x+2)(x+6)}$ and $\frac{x+3}{x+6}$ are defined. The latter is defined everywhere because one can always ad and multiply numbers. The problem is what when $\frac{x^2+5x+6}{x^2+8x+12}=\frac{(x+2)(x+3)}{(x+2)(x+6)}$ is not defined. For example, if $x=-2$ the denominator is 0 (check by evaluating) and so the quotient is undefined whereas $\frac{x+3}{x+6}=\frac{-2+3}{-2+6}=\frac{1}{4}$ which is defined so that for $x=-2$, $\frac{x^2+5x+6}{x^2+8x+12}\neq\frac{x+3}{x+6}$. Maybe you can spot another value that doesn't give equality. If the two expressions are defined then they are equal, which is most values of x. In the context of an application, it will be necessary to go back and make sure the equality holds for the value of interest. We will see this in action in the next unit.

We now summarize the steps you should follow to simplify rational expressions.

Simplify a Rational Expression

1. Factor the numerator and denominator completely.
2. Simplify by noting common factors.

Usually, we leave the simplified rational expression in factored form. This way, it is easy to check that we have removed *all* the common factors.

We'll use the methods we have learned to factor the polynomials in the numerators and denominators in the following examples.

Every time we write a rational expression, we should make a statement disallowing values that would make a denominator zero. However, to let us focus on the work at hand, we will omit writing it in the examples.

Example 1.3.2.14

Simplify $\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$.

Solution

	$\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}$
Factor the numerator and denominator, first factoring out the GCF.	$=\dfrac{3(a^2-4ab+4b^2)}{6(a^2-4b^2)}$ $=\dfrac{3(a-2b)(a-2b)}{6(a+2b)(a-2b)}$
Remove the common factors of $a-2b$ and 3.	$=\dfrac{\cancel{3}(a-2b)\cancel{(a-2b)}}{\cancel{3}\cdot 2(a+2b)\cancel{(a-2b)}}$ $=\dfrac{a-2b}{2(a+2b)}$
Conclude.	$\dfrac{3a^2-12ab+12b^2}{6a^2-24b^2}==\dfrac{a-2b}{2(a+2b)}$ where both sides are defined.

Try It 1.3.2.15

Simplify $\dfrac{2x^2-12xy+18y^2}{3x^2-27y^2}$.

Answer

$\dfrac{2(x-3y)}{3(x+3y)}$

Try It 1.3.2.16

Simplify $\dfrac{5x^2-30xy+25y^2}{2x^2-50y^2}$.

Answer

$\dfrac{5(x-y)}{2(x+5y)}$

Now we will see how to simplify a rational expression whose numerator and denominator have opposite factors. We previously introduced opposite notation: the opposite of a is $-a$ and $-a=-1\cdot a$.

The numerical fraction, say $\dfrac{7}{-7}$ simplifies to -1. We also recognize that the numerator and denominator are opposites.

The fraction $\dfrac{a}{-a}$, whose numerator and denominator are opposites also simplifies to -1.

$$\begin{array}{ll} \text{Let's look at the expression } b-a. & b-a \\ \text{Rewrite.} & -a+b \\ \text{Factor out } -1. & -1(a-b) \end{array}$$

This tells us that $b-a$ is the opposite of $a-b$.

In general, we could write the opposite of $a-b$ as $b-a$. So the rational expression $\dfrac{a-b}{b-a}$ simplifies to -1.

Opposites in a Rational Expression

The **opposite** of $a-b$ is $b-a$.

$$\frac{a-b}{b-a}=-1 \quad a \neq b$$

An expression and its opposite divide to -1.

We will use this property to simplify rational expressions that contain opposites in their numerators and denominators. Be careful not to treat $a+b$ and $b+a$ as opposites. Recall that in addition, order doesn't matter so $a+b=b+a$. So if $a \neq -b$, then $\frac{a+b}{b+a}=1$.

Example 1.3.2.17

Simplify $\frac{x^2-4x-32}{64-x^2}$.

Solution

	$\frac{x^2-4x-32}{64-x^2}$
Factor the numerator and the denominator.	$=\frac{(x-8)(x+4)}{(8-x)(8+x)}$
Recognize the factors that are opposites.	$=(-1)\frac{\cancel{(x-8)}(x+4)}{\cancel{(8-x)}(8+x)}$
Simplify.	$=-\frac{x+4}{x+8}$
Conclude.	$\frac{x^2-4x-32}{64-x^2}=-\frac{x+4}{x+8}$ where both sides are defined.

Try It 1.3.2.18

Simplify $\frac{x^2-4x-52}{5-x^2}$

Answer

$-\frac{x+1}{x+5}$

Try It 1.3.2.19

Simplify $\frac{x^2+x-2}{1-x^2}$.

Answer

$-\frac{x+2}{x+1}$

Multiplying Rational Expressions

To multiply rational expressions, we do just what we did with numerical fractions. We multiply the numerators and multiply the denominators. Then, if there are any common factors, we remove them to simplify the result.

Multiplication of Rational Expressions

If p, q, r, and s are polynomials where $q \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \cdot \frac{r}{s} = \frac{pr}{qs}.$$

To multiply rational expressions, multiply the numerators and multiply the denominators.

Remember, throughout this chapter, we will assume that all numerical values that would make the denominator be zero are excluded. We will not write the restrictions for each rational expression, but keep in mind that the denominator can never be zero. So in this next example, $x \neq 0$, $x \neq 3$, and $x \neq 4$.

Example 1.3.2.20

Simplify $\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$.

Solution

Step 1. Factor each numerator and denominator completely.	Factor x^2-9 and $x^2-7x+12$.	$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2}$ $= \frac{2x}{(x-3)(x-4)} \cdot \frac{(x-3)(x+3)}{6x^2}$
Step 2. Multiply the numerators and denominators.	Multiply the numerators and denominators. It is helpful to write the monomials first.	$= \frac{2x(x-3)(x+3)}{6x^2(x-3)(x-4)}$
Step 3. Reduce the fraction.	Recognize factors common to the numerator and denominator and reduce the fraction. "Divide out" the common factors (divide numerator and denominator by those common factors). Leave the result in factored form.	$\frac{\cancel{2}\cancel{x}\cancel{(x-3)}(x+3)}{\cancel{2}\cdot 3\cdot \cancel{x}\cdot x\cancel{(x-3)}(x-4)}$ $= \frac{x+3}{3x(x-4)}$
Conclude.		$\frac{2x}{x^2-7x+12} \cdot \frac{x^2-9}{6x^2} = \frac{x+3}{3x(x-4)}$ where both sides are defined.

Note that we will stop writing the qualification that the expressions of interest are defined. It will be tacitly understood for the rest of this chapter.

Try It 1.3.2.21

Simplify $\frac{5x}{x^2+5x+6} \cdot \frac{x^2-4}{10x}$.

Answer

$$\frac{x-2}{2(x+3)}$$

Try It 1.3.2.22

Simplify $\frac{9x^2}{x^2+11x+30}] \cdot \frac{x^2-36}{3x^2}$.

Answer

$$\frac{3(x-6)}{x+5}$$

Multiplying Rational Expressions

1. Factor each numerator and denominator completely.
2. Multiply the numerators and denominators.
3. Simplify by dividing out the common factors.

Example 1.3.2.23

Multiply $\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$.

Solution

$$\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$$

Factor the numerators and denominators and then multiply.

$$\frac{(3a+1)(a-3)(a+5)(a+5)}{(a-5)(a+5)(3a+1)(a-5)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(3a+1)}(a-3)\cancel{(a+5)}(a+5)}{(a-5)\cancel{(a+5)}\cancel{(3a+1)}(a-5)},$$

Simplify.

$$\frac{(a-3)(a+5)}{(a-5)(a-5)}$$

Rewrite $(a-5)(a-5)$ using an exponent.

$$\frac{(a-3)(a+5)}{(a-5)^2}$$

where $\frac{3a^2-8a-3}{a^2-25}\cdot\frac{a^2+10a+25}{3a^2-14a-5}$ is defined (then $\frac{(a-3)(a+5)}{(a-5)^2}$ is also defined).

Try It 1.3.2.24

Simplify $\frac{2x^2+5x-12}{x^2-16}\cdot\frac{x^2-8x+16}{2x^2-13x+15}$.

Answer

$$\frac{x-4}{x-5}$$

Try It 1.3.2.25

Simplify $\frac{4b^2+7b-2}{1-b^2}\cdot\frac{b^2-2b+1}{4b^2+15b-4}$.

Answer

$$-\frac{(b+2)(b-1)}{(1+b)(b+4)}$$

Dividing Rational Expressions

Just like we did for numerical fractions, to divide rational expressions, we multiply the first fraction by the reciprocal of the second.

Division of Rational Expressions

If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}.$$

To divide rational expressions, multiply the first fraction by the reciprocal of the second.

Once we rewrite the division as multiplication of the first expression by the reciprocal of the second, we then factor everything and look for common factors.

Example 1.3.2.26

Divide $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$. Use the fact that $p^3+q^3=(p+q)(p^2-pq+q^2)$.

Solution

Step 1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.	"Flip" the second fraction and change the division sign to multiplication.	$\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \div \dfrac{p^2-q^2}{6}$ $\dfrac{p^3+q^3}{2p^2+2pq+2q^2} \cdot \dfrac{6}{p^2-q^2}$
Step 2. Factor the numerators and denominators completely.	Factor the numerators and denominators.	$\dfrac{(p+q)(p^2-pq+q^2)}{2(p^2+pq+q^2)} \cdot \dfrac{2 \cdot 3}{(p-q)(p+q)}$

Step 3 is to multiply the numerators and denominators. The result is the quantity p plus q times the quantity p squared minus p q plus q squared times 2 times 3 all divided by the 2 times the quantity p squared plus p q plus q squared times the quantity p minus q times the quantity p plus q.

Step 4. Simplify by dividing out common factors.	Divide out the common factors.	$\dfrac{\cancel{(p+q)}(p^2-pq+q^2)\cancel{2} \cdot 3}{\cancel{2}(p^2+pq+q^2)(p-q)\cancel{(p+q)}}$ $\dfrac{3(p^2-pq+q^2)}{(p-q)(p^2+pq+q^2)}$

So, \(\dfrac{p^3+q^3}{2p^2+2pq+2q^2}÷\dfrac{p^2−q^2}{6}=\dfrac

Try It 1.3.2.27

Simplify $\dfrac{x^3-8}{3x^2-6x+12} \div \dfrac{x^2-4}{6}$. Use the fact that $x^3-8=(x-2)(x^2+2x+4)$.

Answer

$$\frac{2(x^2+2x+4)}{(x+2)(x^2-2x+4)}$$

Try It 1.3.2.28

Simplify $\dfrac{2z^2}{z^2-1} \div \dfrac{z^3-z^2+z}{z^3+1}$. Use $z^3+1=(z+1)(z^2-z+1)$ and don't forget to factor out greatest common factors.

Answer

$\frac{2z}{z-1}$

Division of Rational Expressions

1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
2. Factor the numerators and denominators completely.
3. Multiply the numerators and denominators together.
4. Simplify by dividing out common factors.

A complex fraction is a fraction that contains a fraction in the numerator, the denominator or both. Recall that a fraction bar means division. A complex fraction is another way of writing division of two fractions.

Example 1.3.2.29

Divide $\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$.

Solution

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-7x+3}{x^2-5x+6}}$$

Rewrite with a division sign.

$$\frac{6x^2-7x+2}{4x-8} \div \frac{2x^2-7x+3}{x^2-5x+6}$$

Rewrite as product of first times reciprocal of second.

$$\frac{6x^2-7x+2}{4x-8} \cdot \frac{x^2-5x+6}{2x^2-7x+3}$$

Factor the numerators and the denominators, and then multiply.

$$\frac{(2x-1)(3x-2)(x-2)(x-3)}{4(x-2)(2x-1)(x-3)}$$

Simplify by dividing out common factors.

$$\frac{\cancel{(2x-1)}(3x-2)\cancel{(x-2)}\cancel{(x-3)}}{4\cancel{(x-2)}\cancel{(2x-1)}\cancel{(x-3)}}$$

Simplify.

$$\frac{3x-2}{4}$$

Try It 1.3.2.30

Simplify $\frac{\frac{3x^2+7x+2}{4x+24}}{\frac{3x^2-14x-5}{x^2+x-30}}$.

Answer

$\frac{x+2}{4}$

Try It 1.3.2.31

Simplify $\dfrac{\dfrac{y^2-36}{2y^2+11y-6}}{\dfrac{2y^2-2y-60}{8y-4}}$.

Answer

$\dfrac{2}{y+5}$

If we have more than two rational expressions to work with, we still follow the same procedure. The first step will be to rewrite any division as multiplication by the reciprocal. Then, we factor and multiply.

Example 1.3.2.32

Perform the indicated operations: $\dfrac{3x-6}{4x-4}\cdot\dfrac{x^2+2x-3}{x^2-3x-10}\div\dfrac{2x+12}{8x+16}$.

Solution

	$\dfrac{3x-6}{4x-4}\cdot\dfrac{x^2+2x-3}{x^2-3x-10}\div\dfrac{2x+12}{8x+16}$
Rewrite the division as multiplication by the reciprocal.	$\dfrac{3x-6}{4x-4}\cdot\dfrac{x^2+2x-3}{x^2-3x-10}\cdot\dfrac{8x+16}{2x+12}$
Factor the numerators and the denominators.	$\dfrac{3(x-2)}{4(x-1)}\cdot\dfrac{(x+3)(x-1)}{(x+2)(x-5)}\cdot\dfrac{8(x+2)}{2(x+6)}$
Multiply the fractions. Bringing the constants to the front will help when removing common factors.	
Simplify by dividing out common factors.	
Simplify.	$\dfrac{3(x-2)(x+3)}{(x-5)(x+6)}$

Try It 1.3.2.33

Perform the indicated operations: $\dfrac{4m+4}{3m-15}\cdot\dfrac{m^2-3m-10}{m^2-4m-32}\div\dfrac{12m-36}{6m-48}$.

Answer

$\dfrac{2(m+1)(m+2)}{3(m+4)(m-3)}$

Try It 1.3.2.34

Perform the indicated operations: $\dfrac{2n^2+10n}{n-1}\div\dfrac{n^2+10n+24}{n^2+8n-9}\cdot\dfrac{n+4}{8n^2+12n}$.

Answer

$\dfrac{(n+5)(n+9)}{2(n+6)(2n+3)}$

Writing Exercises 1.3.2.35

1. What is a rational expression?
2. What is the first goal when aiming at simplifying rational expressions? Why is this the first goal?
3. Give an example of dividing a rational expression where the reduced result is a quotient of linear expressions.
4. a. Multiply $\frac{7}{4}\cdot\frac{9}{10}$ and explain all your steps.
   b. Multiply $\frac{n}{n-3}\cdot\frac{9}{n+3}$ and explain all your steps.
   c. Evaluate your answer to part b. when $n=7$. Did you get the same answer you got in part a.? Why or why not
5. a. Divide $\frac{24}{5}\div 6$ and explain all your steps.
   b. Divide $\frac{x^2-1}{x}\div(x+1)$ and explain all your steps.
   c. Evaluate your answer to part b. when $x=5$. Did you get the same answer you got in part a.? Why or why not?

Exit Problem 1.3.2.36

Simplify:

a. $\frac{5p^2}{p^2-5p-36}\cdot\frac{p^2-16}{10p}$

b. $\frac{2y^2-10y}{y^2}+\frac{10y+2}{5}\cdot\frac{y+5}{6y}$

Key Concepts

- **Determine the values for which a rational expression is undefined.**
  1. Set the denominator equal to zero.
  2. Solve the equation.
- **Equivalent Fractions Property**
  If a, b, and c are numbers where $b\neq 0$, $c\neq 0$, then

  $$\frac{a}{b}=\frac{a\cdot c}{b\cdot c}\text{ and }\frac{a\cdot c}{b\cdot c}=\frac{a}{b}.$$
- **How to simplify a rational expression.**
  1. Factor the numerator and denominator completely.
  2. Simplify by dividing out common factors. (The resulting equality holds for values of the variable where the initial expression is defined.)
- **Opposites in a Rational Expression**
  The opposite of $a-b$ is $b-a$.

  $$\frac{a-b}{b-a}=-1 \qquad a\neq b$$

  An expression and its opposite divide to -1.
- **Multiplication of Rational Expressions**
  If p, q, r, and s are polynomials where $q\neq 0$, $s\neq 0$, then

  $$\frac{p}{q}\cdot\frac{r}{s}=\frac{pr}{qs}$$
- **How to multiply rational expressions.**
  1. Factor each numerator and denominator completely.
  2. Multiply the numerators and denominators.
  3. Simplify by dividing out common factors.

- **Division of Rational Expressions**
  If p, q, r, and s are polynomials where $q \neq 0$, $r \neq 0$, $s \neq 0$, then

$$\frac{p}{q} \div \frac{r}{s} = \frac{p}{q} \cdot \frac{s}{r}$$

- **How to divide rational expressions.**
  1. Rewrite the division as the product of the first rational expression and the reciprocal of the second.
  2. Factor the numerators and denominators completely.
  3. Multiply the numerators and denominators together.
  4. Simplify by dividing out common factors.

1.3.3: Adding and Subtracting Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract rational expressions with a common denominator
- Add and subtract rational expressions whose denominators are opposites
- Find the least common denominator of rational expressions
- Add and subtract rational expressions with unlike denominators
- Add and subtract rational functions

Be Prepared

Before you get started, take this readiness quiz.

1. Add $\frac{7}{10}+\frac{8}{15}$.
2. Subtract $\frac{3x}{4}-\frac{8}{9}$.
3. Subtract $6(2x+1)-4(x-5)$.

Add and Subtract Rational Expressions with a Common Denominator

What is the first step we take when we add numerical fractions? We check if they have a common denominator. If they do, we add the numerators and place the sum over the common denominator. If they do not have a common denominator, we find one before we add.

It is the same with rational expressions. To add rational expressions, they must have a common denominator. When the denominators are the same, we add the numerators and place the sum over the common denominator.

Rational Expression Addition and Subtraction

If p, q, and r are polynomials where $r\neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}.$$

To add or subtract rational expressions with a common denominator, add or subtract the numerators and place the result over the common denominator.

We always simplify rational expressions. Be sure to factor, if possible, after subtracting the numerators so you can identify any common factors.

Remember, too, we do not allow values that would make the denominator zero. What value of x should be excluded in the next example?

Example 1.3.3.1

Add: $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$.

Solution

Since the denominator is $x+4$, we must exclude the value $x=-4$.

$$\frac{11x+28}{x+4}+\frac{x^2}{x+4},\ x\neq -4$$

The fractions have a common denominator, so add the numerators and place the sum over the common denominator.

$$\frac{11x+28+x^2}{x+4}$$

Write the degrees in descending order.

$$\frac{x^2+11x+28}{x+4}$$

Factor the numerator.

$$\frac{(x+4)(x+7)}{x+4}$$

Simplify by removing common factors.

$$\frac{\cancel{(x+4)}(x+7)}{\cancel{x+4}}$$

Simplify.

$$x+7$$

This simplification is valid for values of x where $\frac{11x+28}{x+4}+\frac{x^2}{x+4}$ is defined.

As before, we will tacitly understand that the simplifications are valid for all values of the variables for which the original expression is defined.

? Try It 1.3.3.2

Simplify $\frac{9x+14}{x+7}+\frac{x^2}{x+7}$.

Answer

$x+2$

? Try It 1.3.3.3

Simplify $\frac{x^2+8x}{x+5}+\frac{15}{x+5}$.

Answer

$x+3$

To subtract rational expressions, they must also have a common denominator. When the denominators are the same, we subtract the numerators and place the difference over the common denominator. Be careful of the signs when subtracting a binomial or trinomial.

? Example 1.3.3.4

Subtract: $\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$.

Solution

	$\frac{5x^2-7x+3}{x^2-3x+18}-\frac{4x^2+x-9}{x^2-3x+18}$
Subtract the numerators and place the difference over the common denominator.	$\frac{5x^2-7x+3-(4x^2+x-9)}{x^2-3x+18}$
Distribute the sign in the numerator.	$\frac{5x^2-7x+3-4x^2-x+9}{x^2-3x-18}$
Combine like terms.	$\frac{x^2-8x+12}{x^2-3x-18}$
Factor the numerator and the denominator.	$\frac{(x-2)(x-6)}{(x+3)(x-6)}$
Simplify by removing common factors.	$\frac{(x-2)\,\cancel{(x-6)}}{(x+3)\,\cancel{(x-6)}}$
	$(x-2)(x+3)$

? Try It 1.3.3.5

Subtract: $\frac{4x^2-11x+8}{x^2-3x+2}-\frac{3x^2+x-3}{x^2-3x+2}$.

Answer

$\frac{x-11}{x-2}$

? Try It 1.3.3.6

Subtract: $\frac{6x^2-x+20}{x^2-81}-\frac{5x^2+11x-7}{x^2-81}$.

Answer

$\frac{x-3}{x+9}$

Add and Subtract Rational Expressions Whose Denominators are Opposites

When the denominators of two rational expressions are opposites, it is easy to get a common denominator. We just have to multiply one of the fractions by $\frac{-1}{-1}$.

Let's see how this works when simplifying $\frac{2}{x-1}-\frac{3}{1-x}$.

	$\frac{2}{x-1}-\frac{3}{1-x}$
Multiply the second fraction by $\frac{-1}{-1}$.	$\frac{2}{x-1}-\frac{-3}{-(1-x)}$

The denominators are the same.	$\frac{2}{x-1}-\frac{-3}{x-1}$
Simplify.	$\frac{2+3}{x-1}=\frac{5}{x-1}$

Be careful with the signs as we work with the opposites when the fractions are being subtracted.

Example 1.3.3.7

Subtract: $\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$.

Solution

	$\frac{m^2-6m}{m^2-1}-\frac{3m+2}{1-m^2}$
The denominators are opposites, so multiply the second fraction by $\frac{-1}{-1}$.	$\frac{m^2-6m}{m^2-1}-\frac{-(3m+2)}{-(1-m^2)}$
Simplify the second fraction.	$\frac{m^2-6m}{m^2-1}-\frac{-3m-2)}{m^2-1)}$
The denominators are the same. Subtract the numerators.	$\frac{m^2-6m-(-3m-2)}{m^2-1}$
Distribute.	$\frac{m^2-6m+3m+2}{m^2-1}$
Combine like terms.	$\frac{m^2-3m+2}{m^2-1}$
Factor the numerator and denominator.	$\frac{(m-1)(m-2)}{(m+1)(m-1)}$
Simplify by dividing out common factors.	$\frac{(m-2)}{(m+1)}$
Simplify.	$dfracm-2m+1$

Try It 1.3.3.8

Subtract: $\frac{y^2-5y}{y^2-4}-\frac{6y-6}{4-y^2}$.

Answer

$\frac{y+3}{y+2}$

Try It 1.3.3.9

Subtract: $\frac{2n^2+8n-1}{n^2-1}-\frac{n^2-7n-1}{1-n^2}$.

Answer

$\frac{3n-2}{n-1}$

Find the Least Common Denominator of Rational Expressions

When we add or subtract rational expressions with unlike denominators, we will need to get common denominators. If we review the procedure we used with numerical fractions, we will know what to do with rational expressions.

Let's look at this example: $\frac{7}{12}+\frac{5}{18}$. Since the denominators are not the same, the first step was to find the least common denominator (LCD).

To find the LCD of the fractions, we factored 12 and 18 into primes: $12=2\cdot 2\cdot 3$ and $18=2\cdot 3\cdot 3$. We recognize each of these factorizations contain a 2 and a 3. But we need two 2s to accomodate 12 and two 3s to accomodate 18 and there are no other factors present. So, the LCD is $2^2\cdot 3^2=36$.

We could arrange the factors of 12 and 18 in a table as well as follows:

factors of 12	2	2	3	
factors of 18	2		3	3
factors of the LCD	2	2	3	3

Here we have aligned common factors and then 'brought down the common factor so that the last column consists of the factors of the LCD.

When we add numerical fractions, once we found the LCD, we rewrote each fraction as an equivalent fraction with the LCD by multiplying the numerator and denominator by the same number (multiplying the fraction by one doesn't change its value!). We are now ready to add.

$$\frac{7}{12}+\frac{5}{18}=\frac{7\cdot 3}{12\cdot 3}+\frac{5\cdot 2}{18\cdot 2}=\frac{21+10}{36}=\frac{31}{36}. \tag{1.3.3.1}$$

We do the same thing for rational expressions. However, we leave the LCD in factored form.

Find the Least Common Denominator (LCD) of Rational Expressions

1. Factor each denominator completely.
2. List the factors of each denominator. Match factors vertically when possible.
3. Bring down the columns by including all factors, but do not include common factors twice.
4. Write the LCD as the product of the factors.

Remember, we always exclude values that would make the denominator zero. What values of x should we exclude in this next example?

Example 1.3.3.10

a. Find the LCD for the expressions $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$ and b. rewrite them as equivalent rational expressions with the lowest common denominator.

Solution

a.

Find the LCD for $\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$.	

Factor each denominator completely, lining up common factors. Bring down the columns.	<table><tr><td>factors of x^2-2x-3</td><td>$x-3$</td><td>$x+1$</td><td></td></tr><tr><td>factors of x^2+4x+3</td><td></td><td>$x+1$</td><td>$x+3$</td></tr><tr><td>factors of the LCD</td><td>$x-3$</td><td>$x+1$</td><td>$x+3$</td></tr></table>
Write the LCD as the product of the factors.	$(x-3)(x+1)(x+3)$

b.

	$\frac{8}{x^2-2x-3}, \frac{3x}{x^2+4x+3}$
Factor each denominator.	$\frac{8}{(x-3)(x+1)}, \frac{3x}{(x+1)(x+3)}$
Multiply each denominator by the 'missing' LCD factor and multiply each numerator by the same factor.	$\frac{8(x+3)}{(x-3)(x+1)(x+3)}, \frac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$\frac{8x+24}{(x-3)(x+1)(x+3)}, \frac{3x^2-9x)}{(x+1)(x+3)(x-3)}$

? Try It 1.3.3.11

a. Find the LCD for the expressions $\frac{2}{x^2-x-12}, \frac{1}{x^2-16}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x-4)(x+3)(x+4)$
b. $\frac{2x+8}{(x-4)(x+3)(x+4)}$, $\frac{x+3}{(x-4)(x+3)(x+4)}$

? Try It 1.3.3.12

a. Find the LCD for the expressions $\frac{3x}{x^2-3x+10}, \frac{5}{x^2+3x+2}$ b. rewrite them as equivalent rational expressions with the lowest common denominator.

Answer

a. $(x+2)(x-5)(x+1)$
b. $\frac{3x^2+3x}{(x+2)(x-5)(x+1)}$, $\frac{5x-25}{(x+2)(x-5)(x+1)}$

Add and Subtract Rational Expressions with Unlike Denominators

Now we have all the steps we need to add or subtract rational expressions with unlike denominators.

Example 1.3.3.13

Add: $\frac{3}{x-3}+\frac{2}{x-2}$.

Solution

$$\frac{3}{x-3}+\frac{2}{x-2}=\frac{3(x-2)}{(x-3)(x-2)}+\frac{2(x-3)}{(x-2)(x-3)}=\frac{3x-6)}{(x-3)(x-2)}+\frac{2x-6)}{(x-2)(x-3)}$$

Now that we have common denominators, we can add the numerators (the number of 'parts' which are now of equal sizes):

$$\frac{3}{x-3}+\frac{2}{x-2}=\frac{3x-6+2x-6}{(x-2)(x-3)}=\frac{5x}{(x-2)(x-3)} \tag{1.3.3.2}$$

Try It 1.3.3.14

Add: $\frac{2}{x-2}+\frac{5}{x+3}$.

Answer

$$\frac{7x-4}{(x-2)(x+3)}$$

Try It 1.3.3.15

Add: $\frac{4}{m+3}+\frac{3}{m+4}$.

Answer

$$\frac{7m+25}{(m+3)(m+4)}$$

The steps used to add rational expressions are summarized here.

Add or Subtract Rational Expressions

1. Determine if the expressions have a common denominator.
   - **Yes** – go to step 2.
   - **No** – Rewrite each rational expression with the LCD.
     - Find the LCD.
     - Rewrite each rational expression as an equivalent rational expression with the LCD.
2. Add or subtract the rational expressions.
3. Simplify, if possible.

Avoid the temptation to simplify too soon. In the example above, we must leave the first rational expression as $\frac{3x-6}{(x-3)(x-2)}$ to be able to add it to $\frac{2x-6}{(x-2)(x-3)}$. Simplify *only* after combining the numerators.

Example 1.3.3.16

Add: $\frac{8}{x^2-2x-3}+\frac{3x}{x^2+4x+3}$.

Solution

	$\dfrac{8}{x^2-2x-3}+\dfrac{3x}{x^2+4x+3}$
Do the expressions have a common denominator?	No.
Rewrite each expression with the LCD.	Find the LCD. $\begin{array}{r} x^2-2x-3=(x+1)(x-3) \\ \underline{x^2+4x+3=(x+1)\quad(x+3)} \\ LCD=(x+1)(x-3)(x+3) \end{array}$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\dfrac{8}{(x+1)(x-3)}+\dfrac{3x}{(x+1)(x+3)}=\dfrac{8(x+3)}{(x+1)(x-3)(x+3)}+\dfrac{3x(x-3)}{(x+1)(x+3)(x-3)}$
Simplify the numerators.	$=\dfrac{8x+24}{(x+1)(x-3)(x+3)}+\dfrac{3x^2-9x}{(x+1)(x+3)(x-3)}$
Add the rational expressions.	$=\dfrac{8x+24+3x^2-9x}{(x+1)(x-3)(x+3)}$
Simplify the numerator.	$=\dfrac{3x^2-x+24}{(x+1)(x-3)(x+3)}$
	The numerator is prime, so there are no common factors.

? Try It 1.3.3.17

Add: $\dfrac{1}{m^2-m-2}+\dfrac{5m}{m^2+3m+2}$.

Answer

$$\dfrac{5m^2-9m+2}{(m+1)(m-2)(m+2)}$$

? Try It 1.3.3.18

Add: $\dfrac{2n}{n^2-3n-10}+\dfrac{6}{n^2+5n+6}$.

Answer

$$\dfrac{2n^2+12n-30}{(n+2)(n-5)(n+3)}$$

The process we use to subtract rational expressions with different denominators is the same as for addition. We just have to be very careful of the signs when subtracting the numerators.

? Example 1.3.3.19

Subtract: $\dfrac{8y}{y^2-16}-\dfrac{4}{y-4}$.

Solution

	$\dfrac{8y}{y^2-16}-\dfrac{4}{y-4}$
Do the expressions have a common denominator?	No.

Rewrite each expression with the LCD.	Find the LCD. $\begin{aligned} y^2-16&=(y-4)(y+4)\\ y-4&=y-4\\ \hline LCD&=(y-4)(y+4)\end{aligned}$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{8y}{y^2-16}-\frac{4}{y-4}$ $=\frac{8y}{(y+4)(y-4)}-\frac{4(y+4)}{(y-4)(y+4)}$
Simplify the numerators.	$=\frac{8y}{(y+4)(y-4)}-\frac{4y+16)}{(y-4)(y+4)}$
Subtract the rational expressions.	$=\frac{8y-(4y+16)}{(y+4)(y-4)}$
Simplify the numerator.	$=\frac{8y-4y-16)}{(y+4)(y-4)}=\frac{4y-16}{(y+4)(y-4)}$
Factor the numerator to look for common factors.	$=\frac{4(y-4)}{(y+4)(y-4)}$
Remove common factors	$=\frac{4}{y+4}$, for values of y that make sense in the original expression.
Simplify. This one is already simplified.	$=\frac{4}{y+4}$

Try It 1.3.3.20

Subtract: $\frac{2x}{x^2-4}-\frac{1}{x+2}$.

Answer

$\frac{1}{x-2}$

Try It 1.3.3.21

Subtract: $\frac{3}{z+3}-\frac{6z}{z^2-9}$.

Answer

$\frac{-3}{z-3}$

There are lots of negative signs in the next example. Be extra careful.

Example 1.3.3.22

Subtract: $\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$.

Solution

	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}$
Factor the denominator.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{n+3}{2-n}$
Since $n-2$ and $2-n$ are opposites, we will multiply the second rational expression by $\frac{-1}{-1}$.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{-(n+3)}{-(2-n)}$

Simplify. Remember, $a-(-b)=a+b$.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{-n-3)}{-2+n)}$
Do the rational expressions have a common denominator? No.	
Find the LCD.	Find the LCD. $\begin{aligned} n^2+n-6 &= (n-2)(n+3) \\ n-2 &= (n-2) \\ LCD &= \quad (n-2)(n+3) \end{aligned}$
Rewrite each rational expression as an equivalent rational expression with the LCD.	$\frac{-3n-9}{n^2+n-6}-\frac{n+3}{2-n}=\frac{-3n-9}{(n+3)(n-2)}-\frac{(-n-3)(n+3)}{-2+n)(n+3)}$
Simplify the numerators.	$=\frac{-3n-9}{(n+3)(n-2)}-\frac{-n^2-6n-9}{n-2)(n+3)}$
Add the rational expressions.	$=\frac{-3n-9-(-n^2-6n-9)}{(n+3)(n-2)}$
Simplify the numerator.	$=\frac{-3n-9+n^2+6n+9)}{(n+3)(n-2)}=\frac{n^2+3n)}{(n+3)(n-2)}$
Factor the numerator to look for common factors.	$=\frac{n(n+3)}{(n+3)(n-2)}$
Simplify.	$=\frac{n}{n-2}$ for values of n which make sense in the original expression (all values except -3 and 2).
You can verify this simplification by choosing a few values for n to make sure you get the same result upon simplifying. This isn't mathematically necessary, but a good check for errors.	

Try It 1.3.3.23

Subtract: $\frac{3x-1}{x^2-5x-6}-\frac{2}{6-x}$.

Answer

$$\frac{5x+1}{(x-6)(x+1)}$$

Try It 1.3.3.24

Subtract: $\frac{-2y-2}{y^2+2y-8}-\frac{y-1}{2-y}$.

Answer

$$\frac{y+3}{y+4}$$

Things can get very messy when both fractions must be multiplied by a binomial to get the common denominator.

Example 1.3.3.25

Subtract: $\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$.

Solution

	$\frac{4}{a^2+6a+5}-\frac{3}{a^2+7a+10}$

Factor the denominators.	$=\frac{4}{(a+5)(a+1)}-\frac{3}{(a+5)(a+2)}$
Do the rational expressions have a common denominator? No.	
Find the LCD. $\begin{array}{l} a^2+6a+5=(a+1)(a+5) \\ a^2+7a+10=(a+5)(a+2) \\ \hline LCD=(a+1)(a+5)(a+2) \end{array}$	
Rewrite each rational expression as an equivalent rational expression with the LCD.	$=\frac{4(a+2)}{(a+5)(a+1)(a+2)}-\frac{3(a+1)}{(a+5)(a+2)(a+1)}$
Simplify the numerators.	$=\frac{4a+8}{(a+5)(a+1)(a+2)}-\frac{3a+3}{(a+5)(a+2)(a+1)}$
Subtract the rational expressions.	$=\frac{4a+8-(3a+3)}{(a+5)(a+1)(a+2)}$
Simplify the numerator.	$=\frac{4a+8-3a-3}{(a+5)(a+1)(a+2)}=\frac{a+5}{(a+5)(a+1)(a+2)}$
Look for common factors and simplify.	$=\frac{1}{(a+1)(a+2)}$ for values that make sense in the original expression (all values except $-5, -1$, and -2).

Try It 1.3.3.26

Subtract: $\frac{3}{b^2-4b-5}-\frac{2}{b^2-6b+5}$.

Answer

$$\frac{1}{(b+1)(b-1)}$$

Try It 1.3.3.27

Subtract: $\frac{4}{x^2-4}-\frac{3}{x^2-x-2}$.

Answer

$$\frac{1}{(x+2)(x+1)}$$

Writing Exercises 1.3.3.28

1. What is the LCD?
2. When is the LCD used? Why?
3. Why can't $\frac{x+5}{x-5}$ be simplified?
4. When I add the two fractions $\frac{8y}{y^2-16}-\frac{4}{y-4}$ to get $\frac{4}{y+4}$, are the two rational expressions the same for all values of y? Explain.
5. What is the first goal when adding fractions?
6. When adding fractions, why might you prefer a least common denominator rather than another common denominator? Give an example.

Exit Question 1.3.3.29

Simplify the expression $\dfrac{x-1}{x^2+x-6}-\dfrac{5}{3x-6}$.

Key Concepts

- **Rational Expression Addition and Subtraction**
  If p, q, and r are polynomials where $r \neq 0$, then

$$\frac{p}{r}+\frac{q}{r}=\frac{p+q}{r} \quad \text{and} \quad \frac{p}{r}-\frac{q}{r}=\frac{p-q}{r}$$

- **How to find the least common denominator of rational expressions.**
  1. Factor each expression completely.
  2. List the factors of each expression. Match factors vertically when possible.
  3. Bring down the columns.
  4. Write the LCD as the product of the factors.
- **How to add or subtract rational expressions.**
  1. Determine if the expressions have a common denominator.
     - Yes – go to step 2.
     - No – Rewrite each rational expression with the LCD.
       - Find the LCD.
       - Rewrite each rational expression as an equivalent rational expression with the LCD.
  2. Add or subtract the rational expressions.
  3. Simplify, if possible.

1.3.4: Complex Rational Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify a complex rational expression by writing it as division
- Simplify a complex rational expression by using the LCD

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify: $\dfrac{\frac{3}{5}}{\frac{9}{10}}$.
2. Simplify: $\dfrac{1-\frac{1}{3}}{4^2+4\cdot 5}$.

Simplify a Complex Rational Expression by Writing it as Division

Complex fractions are fractions in which the numerator or denominator contains a fraction. We previously simplified complex fractions like these:

$$\frac{\frac{3}{4}}{\frac{5}{8}} \quad \text{and} \quad \frac{\frac{x}{2}}{\frac{xy}{6}}.$$

In this section, we will simplify complex rational expressions, which are rational expressions with rational expressions in the numerator or denominator.

Definition 1.3.4.1

A **complex rational expression** is a rational expression in which the numerator and/or the denominator contains a rational expression.

Here are some complex rational expressions:

$$\frac{\frac{4}{y-3}}{\frac{8}{y^2-9}}, \quad \frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}} \quad \text{and} \quad \frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Remember, we always exclude values that would make any denominator zero.

We will use two methods to simplify complex rational expressions.

We have already seen this complex rational expression earlier in this chapter:

$$\frac{\frac{6x^2-7x+2}{4x-8}}{\frac{2x^2-8x+3}{x^2-5x+6}}.$$

We noted that fraction bars tell us to divide, and rewrote it as the division problem:

$$\left(\frac{6x^2-7x+2}{4x-8}\right) \div \left(\frac{2x^2-8x+3}{x^2-5x+6}\right).$$

Then, we multiplied the first rational expression by the reciprocal of the second, just like we do when we divide two fractions.

This is one method to simplify complex rational expressions. We make sure the complex rational expression is of the form where one fraction is over one fraction. We then write it as if we were dividing two fractions.

Example 1.3.4.2

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}.$$

Solution

$\frac{\frac{6}{x-4}}{\frac{3}{x^2-16}}$	
Rewrite the complex fraction as division.	$=\frac{6}{x-4} \div \frac{3}{x^2-16}$
Rewrite as the product of first times the reciprocal of the second.	$=\frac{6}{x-4} \cdot \frac{x^2-16}{3}$
Factor.	$=\frac{3 \cdot 2}{x-4} \cdot \frac{(x-4)(x+4)}{3}$
Multiply.	$=\frac{3 \cdot 2(x-4)(x+4)}{3(x-4)}$
Remove common factors.	$=\frac{\cancel{3} \cdot 2\,\cancel{(x-4)}(x+4)}{\cancel{3}\,\cancel{(x-4)}}$
Simplify.	$=2(x+4)$

Are there any value(s) of x that should not be allowed? The original complex rational expression had denominators of $x-4$ and x^2-16. This expression would be undefined if $x=4$ or $x=-4$.

Try It 1.3.4.3

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{2}{x^2-1}}{\frac{3}{x+1}}.$$

Answer

$$\frac{2}{3(x-1)}$$

Try It 1.3.4.4

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x^2-7x+12}}{\frac{2}{x-4}}.$$

Answer

$\frac{1}{2(x-3)}$

Fraction bars act as grouping symbols. So to follow the Order of Operations, we simplify the numerator and denominator as much as possible before we can do the division.

Example 1.3.4.5

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}.$$

Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
Find the LCD and add the fractions in the numerator. Find the LCD and subtract the fractions in the denominator.	$=\frac{\frac{1\cdot 2}{3\cdot 2}+\frac{1}{6}}{\frac{1\cdot 3}{2\cdot 3}-\frac{1\cdot 2}{3\cdot 2}}$
Simplify the numerator and denominator.	$=\frac{\frac{2}{6}+\frac{1}{6}}{\frac{3}{6}-\frac{2}{6}}$ $=\frac{\frac{3}{6}}{\frac{1}{6}}$
Rewrite the complex rational expression as a division problem.	$=\frac{3}{6}\div\frac{1}{6}$
Multiply the first by the reciprocal of the second.	$=\frac{3}{6}\cdot\frac{6}{1}$
Simplify.	$=3$

Try It 1.3.4.6

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{2}+\frac{2}{3}}{\frac{5}{6}+\frac{1}{12}}$$

Answer

$\frac{14}{11}$

Try It 1.3.4.7

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{3}{4}-\frac{1}{3}}{\frac{1}{8}+\frac{5}{6}}$$

Answer

$$\frac{10}{23}$$

We follow the same procedure when the complex rational expression contains variables.

Example 1.3.4.8

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}.$$

Solution

$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$		
Simplify the numerator.	We will simplify the sum in the numerator and the difference in the denominator.	$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
	Find the LCD in the numerator and denominator.	$=\frac{\frac{1\cdot y}{x\cdot y}+\frac{1\cdot x}{y\cdot x}}{\frac{x\cdot x}{y\cdot x}-\frac{y\cdot y}{x\cdot y}}$
	Simplify.	$=\frac{\frac{y}{xy}+\frac{x}{xy}}{\frac{x^2}{xy}-\frac{y^2}{xy}}$
	Add the fractions in the numerator and subtract the fractions in the denominator. We now have just one rational expression in the numerator and one in the denominator.	$=\frac{\frac{y+x}{xy}}{\frac{x^2-y^2}{xy}}$
Rewrite the complex rational expression as a division problem.	We write the numerator divided by the denominator.	$=\left(\frac{y+x}{xy}\right)\div\left(\frac{x^2-y^2}{xy}\right)$
Divide the expressions.	Multiply the first by the reciprocal of the second.	$=\left(\frac{y+x}{xy}\right)\cdot\left(\frac{xy}{x^2-y^2}\right)$
	Factor any expressions if possible.	$=\frac{xy(y+x)}{xy(x-y)(x+y)}$
	Remove common factors.	$=\frac{\cancel{xy}\,\cancel{(y+x)}}{\cancel{xy}(x-y)\,\cancel{(x+y)}}$
	Simplify.	$=\frac{1}{x-y}$

Try It 1.3.4.9

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{1}{x}-\frac{1}{y}}.$$

Answer

$$\frac{y+x}{y-x}$$

Try It 1.3.4.10

Simplify the complex rational expression by writing it as division:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{1}{a^2}-\frac{1}{b^2}}.$$

Answer

$$\frac{ab}{b-a}$$

We summarize the steps here.

How to Simplify a Complex Rational Expression by Writing It as Division

1. Rewrite the complex rational expression as a division problem.
2. Divide the expressions.

Example 1.3.4.11

Simplify the complex rational expression by writing it as division:

$$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}.$$

Solution

	$\frac{n-\frac{4n}{n+5}}{\frac{1}{n+5}+\frac{1}{n-5}}$
Simplify the numerator and denominator. Find common denominators for the numerator and denominator.	$=\frac{\frac{n(n+5)}{1(n+5)}-\frac{4n}{n+5}}{\frac{1(n-5)}{(n+5)(n-5)}+\frac{1(n+5)}{(n-5)(n+5)}}$
Simplify the numerators.	$=\frac{\frac{n^2+5n}{n+5}-\frac{4n}{n+5}}{\frac{n-5}{(n+5)(n-5)}+\frac{n+5}{(n-5)(n+5)}}$

	$\dfrac{n-\dfrac{4n}{n+5}}{\dfrac{1}{n+5}+\dfrac{1}{n-5}}$
Subtract the rational expressions in the numerator and add in the denominator.	$=\dfrac{\dfrac{n^2+5n-4n}{n+5}}{\dfrac{n-5+n+5}{(n+5)(n-5)}}$
Simplify. (We now have one rational expression over one rational expression.)	$=\dfrac{\dfrac{n^2+n}{n+5}}{\dfrac{2n}{(n+5)(n-5)}}$
Rewrite as fraction division.	$=\dfrac{n^2+n}{n+5}\div\dfrac{2n}{(n+5)(n-5)}$
Multiply the first times the reciprocal of the second.	$=\dfrac{n^2+n}{n+5}\cdot\dfrac{(n+5)(n-5)}{2n}$
Factor any expressions if possible.	$=\dfrac{n(n+1)(n+5)(n-5)}{(n+5)2n}$
Remove common factors.	$=\dfrac{\cancel{n}(n+1)\,\cancel{(n+5)}(n-5)}{\cancel{(n+5)}2\,\cancel{n}}$
Simplify.	$=\dfrac{(n+1)(n-5)}{2}$

Try It 1.3.4.12

Simplify the complex rational expression by writing it as division:

$$\frac{b-\dfrac{3b}{b+5}}{\dfrac{2}{b+5}+\dfrac{1}{b-5}}.$$

Answer

$$\frac{b(b+2)(b-5)}{3b-5}$$

Try It 1.3.4.13

Simplify the complex rational expression by writing it as division:

$$\frac{1-\dfrac{3}{c+4}}{\dfrac{1}{c+4}+\dfrac{c}{3}}.$$

Answer

$$\frac{3}{c+3}$$

Simplify a Complex Rational Expression by Using the LCD

We “cleared” the fractions by multiplying by the LCD when we solved equations with fractions. We can use that strategy here to simplify complex rational expressions. We will multiply the numerator and denominator by the LCD of all the rational expressions.

Let's look at the complex rational expression we simplified one way in Example 7.4.2. We will simplify it here by multiplying the numerator and denominator by the LCD. When we multiply by $\frac{\text{LCD}}{\text{LCD}}$ we are multiplying by 1, so the value stays the same.

Example 1.3.4.14

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}.$$

Solution

	$\frac{\frac{1}{3}+\frac{1}{6}}{\frac{1}{2}-\frac{1}{3}}$
The LCD of all the fractions in the whole expression is 6. Clear the fractions by multiplying the numerator and denominator by that LCD.	$=\frac{6\cdot\left(\frac{1}{3}+\frac{1}{6}\right)}{6\cdot\left(\frac{1}{2}-\frac{1}{3}\right)}$
Distribute.	$=\frac{6\cdot\frac{1}{3}+6\cdot\frac{1}{6}}{6\cdot\frac{1}{2}-6\cdot\frac{1}{3}}$
Simplify.	$=\frac{2+1}{3-2}$ $=\frac{3}{1}$ $=3$

Try It 1.3.4.15

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{2}+\frac{1}{5}}{\frac{1}{10}+\frac{1}{5}}.$$

Answer

$\frac{7}{3}$

Try It 1.3.4.16

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{4}+\frac{3}{8}}{\frac{1}{2}-\frac{5}{16}}.$$

Answer

$\frac{10}{3}$

We will use the same example as in Example 7.4.3. Decide which method works better for you.

Example 1.3.4.17

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}.$$

Solution

		$\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Find the LCD of all fractions in the is complex rational expression.	The LCD of all the fractions is xy.	$=\frac{\frac{1}{x}+\frac{1}{y}}{\frac{x}{y}-\frac{y}{x}}$
Multiply the numerator and denominator by the LCD.	Multiply both the numerator and denominator by xy.	$=\frac{xy\cdot\left(\frac{1}{x}+\frac{1}{y}\right)}{xy\cdot\left(\frac{x}{y}-\frac{y}{x}\right)}$
Simplify the expression.	Distribute.	$=\frac{xy\cdot\frac{1}{x}+xy\cdot\frac{1}{y}}{xy\cdot\frac{x}{y}-xy\cdot\frac{y}{x}}$ $=\frac{y+x}{x^2-y^2}$
	Simplify.	$=\frac{(y+x)}{(x-y)(x+y)}$
	Remove common factors.	$=\frac{1}{x-y}$

Try It 1.3.4.18

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{a}+\frac{1}{b}}{\frac{a}{b}+\frac{b}{a}}.$$

Answer

$$\frac{b+a}{a^2+b^2}$$

Try It 1.3.4.19

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{1}{x^2}-\frac{1}{y^2}}{\frac{1}{x}+\frac{1}{y}}.$$

Answer

$$\frac{y-x}{xy}$$

How to Simplify a Complex Rational Expression by Using the LCD

1. Multiply the numerator and denominator by the LCD of all rational expressions.
2. Simplify the expression.

Be sure to start by factoring all the denominators so you can find the LCD.

Example 1.3.4.20

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}.$$

Solution

	$\frac{\frac{2}{x+6}}{\frac{4}{x-6}-\frac{4}{x^2-36}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is: $x^2-36=(x+6)(x-6)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}-\frac{4}{(x+6)(x-6)}\right)}$
Distribute in the denominator.	$=\frac{(x+6)(x-6)\frac{2}{x+6}}{(x+6)(x-6)\left(\frac{4}{x-6}\right)-(x+6)(x-6)\left(\frac{4}{(x+6)(x-6)}\right)}$
Simplify.	$=\frac{\cancel{(x+6)}(x-6)\frac{2}{\cancel{x+6}}}{(x+6)\cancel{(x-6)}\left(\frac{4}{\cancel{x-6}}\right)-\cancel{(x+6)(x-6)}\left(\frac{4}{\cancel{(x+6)(x-6)}}\right)}$
Simplify.	$=\frac{2(x-6)}{4(x+6)-4}$
To simplify the denominator, distribute and combine like terms.	$=\frac{2(x-6)}{4x+20}$
Factor the denominator.	$=\frac{2(x-6)}{4(x+5)}$
Remove common factors.	$=\frac{\cancel{2}(x-6)}{\cancel{2}\cdot 2(x+5)}$
Simplify.	$=\frac{x-6}{2(x+5)}$ Notice that there are no more factors common to the numerator and denominator.

Try It 1.3.4.21

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x+2}}{\frac{5}{x-2}-\frac{3}{x^2-4}}.$$

Answer

$\frac{3(x-2)}{5x+7}$

Try It 1.3.4.22

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{2}{x-7}-\frac{1}{x+7}}{\frac{6}{x+7}-\frac{1}{x^2-49}}.$$

Answer

$\frac{x+21}{6x-43}$

Be sure to factor the denominators first. Proceed carefully as the math can get messy!

Example 1.3.4.23

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}.$$

Solution

	$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$
Find the LCD of all fractions in the complex rational expression.	The LCD is $(m-3)(m-4)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(m-3)(m-4)\frac{4}{(m-3)(m-4)}}{(m-3)(m-4)\left(\frac{3}{m-3}-\frac{2}{m-4}\right)}$
Simplify.	$=\frac{\cancel{(m-3)(m-4)}\frac{4}{\cancel{(m-3)(m-4)}}}{\cancel{(m-3)}(m-4)\left(\frac{3}{\cancel{m-3}}\right)-(m-3)\cancel{(m-4)}\left(\frac{2}{\cancel{m-4}}\right)}$
Simplify.	$=\frac{4}{3(m-4)-2(m-3)}$
Distribute.	$=\frac{4}{3m-12-2m+6}$

	$$\frac{\frac{4}{m^2-7m+12}}{\frac{3}{m-3}-\frac{2}{m-4}}$$
Combine like terms.	$=\frac{4}{m-6}$

Try It 1.3.4.24

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{3}{x^2+7x+10}}{\frac{4}{x+2}+\frac{1}{x+5}}.$$

Answer

$$\frac{3}{5x+22}$$

Try It 1.3.4.25

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{4y}{y+5}+\frac{2}{y+6}}{\frac{3y}{y^2+11y+30}}.$$

Answer

$$\frac{2\left(2y^2+13y+5\right)}{3y}$$

Example 1.3.4.26

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}.$$

Solution

	$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$
Find the LCD of all fractions in the complex rational expression.	The LCD is $(y+1)(y-1)$.
Multiply the numerator and denominator by the LCD.	$=\frac{(y+1)(y-1)\frac{y}{y+1}}{(y+1)(y-1)\left(1+\frac{1}{y-1}\right)}$

	$$\frac{\frac{y}{y+1}}{1+\frac{1}{y-1}}$$
Distribute in the denominator and simplify.	$$=\frac{\cancel{(y+1)}(y-1)\frac{y}{\cancel{y+1}}}{(y+1)(y-1)(1)+(y+1)\,\cancel{(y-1)}\left(\frac{1}{\cancel{y-1}}\right)}$$
Simplify.	$$=\frac{(y-1)y}{(y+1)(y-1)+(y+1)}$$
Simplify the denominator and leave the numerator factored.	$$=\frac{y(y-1)}{y^2-1+y+1}$$ $$=\frac{y(y-1)}{y^2+y}$$
Factor the denominator and remove factors common with the numerator.	$$=\frac{\cancel{y}(y-1)}{\cancel{y}(y+1)}$$
Simplify.	$$=\frac{y-1}{y+1}$$

Try It 1.3.4.27

Simplify the complex rational expression by using the LCD:

$$\frac{\frac{x}{x+3}}{1+\frac{1}{x+3}}.$$

Answer

$$\frac{x}{x+4}$$

Try It 1.3.4.28

Simplify the complex rational expression by using the LCD:

$$\frac{1+\frac{1}{x-1}}{\frac{3}{x+1}}.$$

Answer

$$\frac{x(x+1)}{3(x-1)}$$

Writing Exercise 1.3.4.29

1. What is the LCD (least common denominator)?
2. What is the purpose in multiplying numerator and denominator by the LCD in Method II?
3. What happens if you multiply by a common denominator that is not the least common denominator?

Exit Problem 1.3.4.30

Simplify the expression $\dfrac{\dfrac{3}{y^2}+\dfrac{4}{y}+1}{\dfrac{3}{y^2}+\dfrac{1}{y}}$.

Key Concepts

- **Complex rational expression**
- **How to simplify a complex rational expression by writing it as division.**
  1. Simplify the numerator and denominator.
  2. Rewrite the complex rational expression as a division problem.
  3. Divide the expressions.
- **How to simplify a complex rational expression by using the LCD.**
  1. Find the LCD of all fractions in the complex rational expression.
  2. Multiply the numerator and denominator by the LCD.
  3. Simplify the expression.

SECTION OVERVIEW

1.4: Radical Expressions

Topic hierarchy

Thumbnail: The mathematical expression "The (principal) square root of x". (GPL, David Vignoni (original icon); Flamurai (SVG convertion); bayo (color)).

1.4.1: Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with roots
- Estimate and approximate roots
- Simplify variable expressions with roots

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify

a. $(-9)^2$

b. -9^2

2. Round 3.846 to the nearest hundredth.

3. Simplify

a. $x^3 \cdot x^3$

b. $y^2 \cdot y^2$

In this section we deal with radical expressions of index 2 called square roots.

Simplify Expressions with Roots

In Foundations, we briefly looked at square roots. Remember that when a real number n is multiplied by itself, we write n^2 and read it 'n squared'. This number is called the **square** of n, and n is called the **square root**. For example,

13^2 is read "13 squared"

169 is called the square of 13, since $13^2 = 169$

13 is called a square root of 169

Definition 1.4.1.1

Square

If $n^2 = m$, then m is the **square** of n.

Square Root

If $n^2 = m$, then n is a **square root** of m.

In words,

a square root of m is a number whose square is m.

Notice $(-13)^2 = 169$ also, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169.

So, every positive number has two square roots—one positive and one negative. What if we only wanted the positive square root of a positive number? We use a *radical sign*, and write, $\sqrt{m}$, which denotes the positive square root of m. The non-negative square root is also called the **principal square root**. This is the square root approximated by using the root symbol of your calculator!

We also use the radical sign for the square root of zero. Because $0^2 = 0$, $\sqrt{0} = 0$. Notice that zero has only one square root.

Definition 1.4.1.2

$\sqrt{m}$ is read "**the square root of m.**"

If $n^2 = m$, then $n = \sqrt{m}$, for $n \geq 0$.

$$\text{radical sign} \longrightarrow \sqrt{m} \longleftarrow \text{radicand}$$

We know that every positive number has two square roots and the radical sign indicates the positive one. We write $\sqrt{169} = 13$. If we want to find the negative square root of a number, we place a negative in front of the radical sign. For example, $-\sqrt{169} = -13$.

Example 1.4.1.3

Simplify:

a. $\sqrt{144}$

b. $-\sqrt{289}$

Solution

a.

$$\sqrt{144}$$

Since $12^2 = 144$, and $12 \geq 0$

$$12$$

b.

$$-\sqrt{289}$$

Since $17^2 = 289$, $17 \geq 0$, and the negative is in front of the radical sign.

$$-17$$

Try It 1.4.1.4

Simplify:

a. $-\sqrt{64}$

b. $\sqrt{225}$

Answer

a. -8

b. 15

Try It 1.4.1.5

Simplify:

a. $\sqrt{100}$

b. $-\sqrt{121}$

Answer

a. 10

b. -11

Can we simplify $\sqrt{-49}$? Is there a number whose square is -49?

$$(\_\_\_)^2 = -49$$

Any positive number squared is positive. Any negative number squared is positive. There is no real number equal to $\sqrt{-49}$. The square root of a negative number is not a real number.

Example 1.4.1.6

Simplify:

a. $\sqrt{-196}$

b. $-\sqrt{64}$

Solution

a.

$$\sqrt{-196}$$

There is no real number whose square is -196.

$$\sqrt{-196} \text{ is not a real number.}$$

b.

$$-\sqrt{64}$$

The negative is in front of the radical.

$$-8$$

Try It 1.4.1.7

Simplify:

a. $\sqrt{-169}$

b. $-\sqrt{81}$

Answer

a. not a real number

b. -9

Try It 1.4.1.8

Simplify:

a. $-\sqrt{49}$

b. $\sqrt{-121}$

Answer

a. -7

b. not a real number

Properties of $\sqrt{a}$

When

- $a \geq 0$, then $\sqrt{a}$ is a real number.
- $a < 0$, then $\sqrt{a}$ is not a real number.

Simplify Variable Expressions with Square Roots

Note, for example,

$$\sqrt{4^2} = \sqrt{16} = 4$$

but,

$$\sqrt{(-4)^2} = \sqrt{16} = 4,$$

So that the result is positive.

How can we make sure the square root of -5 squared is 5? We can use the absolute value. $|-5| = 5$:

$$\sqrt{a^2} = |a|. \tag{1.4.1.1}$$

This guarantees the principal root is positive.

Note that the 'root button' and the 'square button' are the same on most calculators and if a is positive, applying the square button and then the root button (or vice versa) will result in the return of a.

Summary

We have

$$\sqrt{a^2} = |a|$$

Example 1.4.1.9

Simplify $\sqrt{x^2}$.

Solution

We use the absolute value to be sure to get the positive root.

$$\sqrt{x^2} = |x|$$

Try It 1.4.1.10

Simplify $\sqrt{b^2}$.

Answer

$|b|$

What about square roots of higher powers of variables? The power property of exponents says $(a^m)^n = a^{m \cdot n}$. So if we square a^m, the exponent will become $2m$.

$$(a^m)^2 = a^{2m}$$

Looking now at the square root.

$$\sqrt{a^{2m}} = \sqrt{(a^m)^2}$$

Since 2 is even, $\sqrt[2]{x^2} = |x|$. So

$$\sqrt{a^{2m}} = |a^m|\,. \tag{1.4.1.2}$$

We apply this concept in the next example.

Example 1.4.1.11

Simplify:

a. $\sqrt{x^6}$

b. $\sqrt{y^{16}}$

Solution

a.

$$\sqrt{x^6}$$

Since $\left(x^3\right)^2 = x^6$, this is equal to

$$\sqrt{\left(x^3\right)^2}.$$

Since $\sqrt{a^2} = |a|$, this is equal to

$$\left|x^3\right|$$

b.

$$\sqrt{y^{16}}$$

Since $\left(y^8\right)^2 = y^{16}$, this is equal to

$$\sqrt{\left(y^8\right)^2}.$$

Since $\sqrt{a^2} = |a|$, this is equal to

$$y^8$$

In this case the absolute value sign is not needed as y^8 is positive.

Try It 1.4.1.12

Simplify:

a. $\sqrt{y^{18}}$

b. $\sqrt{z^{12}}$

Answer

a. $|y^9|$

b. z^6

Try It 1.4.1.13

Simplify:

a. $\sqrt{m^4}$

b. $\sqrt{b^{10}}$

Answer

a. m^2

b. $\left|b^{5}\right|$

Note that if the variables are positive, then the exponent gets halved which is the same action you would do to simplify $()^{\frac{1}{2}}$ if rational exponents followed the same rules as integer exponents. For example, if x is non-negative,

$\sqrt{x^{6}}=(x^{6})^{\frac{1}{2}}=x^{6\cdot\dfrac{1}{2}}=x^{3}.$

We will treat this in greater detail a little later in this chapter.

In the next example, we now have a coefficient in front of the variable. The concept $\sqrt{a^{2m}}=|a^{m}|$ works in much the same way.

$$\sqrt{16r^{22}}=4\left|r^{11}\right| \text{ because } \left(4r^{11}\right)^{2}=16r^{22}.$$

But notice $\sqrt{25u^{8}}=5u^{4}$ and no absolute value sign is needed as u^{4} is always non-negative.

Example 1.4.1.14

Simplify:

a. $\sqrt{16n^{2}}$

b. $-\sqrt{81c^{2}}$

Solution

a.

$$\sqrt{16n^{2}}$$

Since $(4n)^{2}=16n^{2}$, this is equal to

$$\sqrt{(4n)^{2}}.$$

Since $\sqrt{a^{2}}=|a|$, this is equal to

$$4|n|.$$

b.

$$-\sqrt{81c^{2}}$$

Since $(9c)^{2}=81c^{2}$, this is equal to

$$-\sqrt{(9c)^{2}}.$$

Since $\sqrt{a^{2}}=|a|$, this is then equal to

$$-9|c|.$$

Try It 1.4.1.15

Simplify:

a. $\sqrt{64x^{2}}$

b. $-\sqrt{100p^{2}}$

Answer

a. $8|x|$

b. $-10|p|$

Try It 1.4.1.16

Simplify:

a. $\sqrt{169y^2}$

b. $-\sqrt{121y^2}$

Answer

a. $13|y|$

b. $-11|y|$

The next examples have two variables.

Example 1.4.1.17

Simplify:

a. $\sqrt{36x^2y^2}$

b. $\sqrt{121a^6b^8}$

Solution

a.

$$\sqrt{36x^2y^2}$$

Since $(6xy)^2 = 36x^2y^2$

$$\sqrt{(6xy)^2}$$

Take the square root.

$$6|xy|$$

b.

$$\sqrt{121a^6b^8}$$

Since $\left(11a^3b^4\right)^2 = 121a^6b^8$

$$\sqrt{\left(11a^3b^4\right)^2}$$

Take the square root.

$$11\left|a^3\right|b^4$$

Try It 1.4.1.18

Simplify:

a. $\sqrt{100a^2b^2}$

b. $\sqrt{144p^{12}q^{20}}$

Answer

a. $10|ab|$

b. $12p^6q^{10}$

? Try It 1.4.1.19

Simplify:

a. $\sqrt{225m^2n^2}$

b. $\sqrt{169x^{10}y^{14}}$

Answer

a. $15|mn|$

b. $13\left|x^5y^7\right|$

? Writing Exercises 1.4.1.20

1. What is a square root?
2. Explain why $\sqrt{9}=3$.
3. What happens if we square a square root?
4. What is the index? Radicand?

Exit Problem 1.4.1.21

Simplify $\sqrt{81a^4b^6}$.

Key Concepts

- **Square Root Notation**
  - $\sqrt{m}$ is read 'the square root of m'
  - If $n^2=m$, then $n=\sqrt{m}$, for $n\geq 0$.

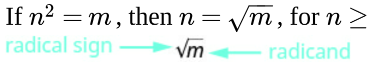


Figure 8.1.1

  - The square root of m, $\sqrt{m}$, is a positive number whose square is m.
- **Properties of $\sqrt{a}$**
  - $a\geq 0$, then $\sqrt{a}$ is a real number
  - $a<0$, then $\sqrt{a}$ is not a real number
- **Simplifying Odd and Even Roots**
  - $\sqrt{a^2}=|a|$. We must use the absolute value signs when we take a square root of an expression with a variable in the radical.

1.4.2: Simplifying Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Use the Product Property to simplify radical expressions
- Use the Quotient Property to simplify radical expressions

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\frac{x^9}{x^4}$.
2. Simplify $\frac{y^3}{y^{11}}$.
3. Simplify $\left(n^2\right)^6$.

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator. A radical expression, $\sqrt[n]{a}$, is considered simplified if it has no factors of m^n. So, to simplify a radical expression, we look for any factors in the radicand that are powers of the index.

Use the Product Property to Simplify Radical Expressions

We will simplify radical expressions in a way similar to how we simplified fractions. A fraction is simplified if there are no common factors in the numerator and denominator. To simplify a fraction, we look for any common factors in the numerator and denominator.

A **radical expression**, $\sqrt{a}$, is considered simplified if it has no factors of the form m^2. So, to simplify a radical expression, we look for any factors in the radicand that are squares.

Definition 1.4.2.1

For non-negative integers a and m,

$$\sqrt{\boldsymbol{a}}\textbf{ is considered simplified} \text{ if } a \text{ has no factors of the form } m^2.$$

For example, $\sqrt{5}$ is considered simplified because there are no perfect square factors in 5. But $\sqrt{12}$ is not simplified because 12 has a perfect square factor of 4.

To simplify radical expressions, we will also use some properties of roots. The properties we will use to simplify radical expressions are similar to the properties of exponents. We know that

$$(ab)^n = a^n b^n. \tag{1.4.2.1}$$

The corresponding of **Product Property of Roots** says that

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}. \tag{1.4.2.2}$$

To see why this is true we note that

$$(\sqrt{ab})^2 = ab \tag{1.4.2.3}$$

since $\sqrt{ab}$ is the non-negative quantity you square to get ab by definition.

Also,

$$(\sqrt{a} \cdot \sqrt{b})^2 = (\sqrt{a})^2 \cdot (\sqrt{b})^2 = ab, \tag{1.4.2.4}$$

where the first equality follows from the product property of exponents and the second by the definition of the square root (as above).

So, the left and the right hand sides, being both non-negative, are square roots of ab, and therefore are equal.

As mentioned in the last section, due to this property, it is written that $\sqrt{a} = a^{\frac{1}{2}}$ and the properties of exponents can be shown to be extended to the exponents obtained this way.

Fact 1.4.2.2

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, and $n \geq 2$ is an integer, then

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b}$$

Note that you can also read the equality: $\sqrt{a} \cdot \sqrt{b} = \sqrt{ab}$.

We use the Product Property of Roots to remove all perfect square factors from a square root.

Example 1.4.2.3

Simplify $\sqrt{98}$.

Solution

Find the largest factor in the radicand that is a perfect power of the index.	We see that 49 is the largest factor of 98 that has a power of 2.	$\sqrt{98}$
Rewrite the radicand as a product of two factors, using that factor.	In other words 49 is the largest perfect square factor of 98. $98 = 49 \cdot 2$ Always write the perfect square factor first.	$\sqrt{49 \cdot 2}$
Use the product rule to rewrite the radical as the product of two radicals.		$\sqrt{49} \cdot \sqrt{2}$
Simplify the root of the perfect power.		$7\sqrt{2}$

Try It 1.4.2.4

Simplify $\sqrt{48}$.

Answer

$4\sqrt{3}$

Try It 1.4.2.5

Simplify $\sqrt{45}$.

Answer

$3\sqrt{5}$

Notice in the previous example that the simplified form of $\sqrt{98}$ is $7\sqrt{2}$, which is the product of an integer and a square root. We always write the integer in front of the square root.

Be careful to write your integer so that it is not confused with the index (which we will discuss later). The expression $7\sqrt{2}$ is very different from $\sqrt[7]{2}$.

Simplify a Radical Expression Using the Product Property

1. Find the largest factor in the radicand that is a perfect power of the index. Rewrite the radicand as a product of two factors, using that factor.
2. Use the product rule to rewrite the radical as the product of two radicals.
3. Simplify the root of the perfect power.

We will apply this method in the next example. It may be helpful to have a table of perfect squares.

Example 1.4.2.6

Simplify $\sqrt{500}$.

Solution

$$\sqrt{500}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{100 \cdot 5}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{100} \cdot \sqrt{5}$$

Simplify.

$$10\sqrt{5}$$

Try It 1.4.2.7

Simplify $\sqrt{288}$.

Answer

$12\sqrt{2}$

Try It 1.4.2.8

Simplify $\sqrt{432}$.

Answer

$12\sqrt{3}$

The next example is much like the previous examples, but with variables. Don't forget to use the absolute value signs when taking an even root of an expression with a variable in the radical.

Example 1.4.2.9

Simplify $\sqrt{x^3}$.

Solution

$$\sqrt{x^3}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{x^2 \cdot x}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{x^2} \cdot \sqrt{x}$$

Simplify.

$$|x|\sqrt{x}$$

? Try It 1.4.2.10

Simplify $\sqrt{b^5}$.

Answer

$b^2\sqrt{b}$

? Try It 1.4.2.11

Simplify $\sqrt{p^9}$.

Answer

$p^4\sqrt{p}$

We follow the same procedure when there is a coefficient in the radicand. In the next example, both the constant and the variable have perfect square factors.

? Example 1.4.2.12

Simplify $\sqrt{72n^7}$.

Solution

$$\sqrt{72n^7}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{36n^6 \cdot 2n}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{36n^6} \cdot \sqrt{2n}$$

Simplify.

$$6\left|n^3\right|\sqrt{2n}$$

? Try It 1.4.2.13

Simplify $\sqrt{32y^5}$.

Answer

$4y^2\sqrt{2y}$

? Try It 1.4.2.14

Simplify $\sqrt{75a^9}$.

Answer

$5a^4\sqrt{3a}$

In the next example, we continue to use the same methods even though there are more than one variable under the radical.

? Example 1.4.2.15

Simplify $\sqrt{63u^3v^5}$.

Answer

$$\sqrt{63u^3v^5}$$

Rewrite the radicand as a product using the largest perfect square factor.

$$\sqrt{9u^2v^4 \cdot 7uv}$$

Rewrite the radical as the product of two radicals.

$$\sqrt{9u^2v^4} \cdot \sqrt{7uv}$$

Rewrite the first radicand as $\left(3uv^2\right)^2$.

$$\sqrt{\left(3uv^2\right)^2} \cdot \sqrt{7uv}$$

Simplify.

$$3|u|v^2\sqrt{7uv}$$

? Try It 1.4.2.16

Simplify $\sqrt{98a^7b^5}$.

Answer

$7\left|a^3\right|b^2\sqrt{2ab}$

? Try It 1.4.2.17

Simplify $\sqrt{180m^9n^{11}}$.

Answer

$6m^4\left|n^5\right|\sqrt{5mn}$

Use the Quotient Property to Simplify Radical Expressions

Whenever you have to simplify a radical expression, the first step you should take is to determine whether the radicand is a perfect square. If not, check the numerator and denominator for any common factors, and remove them. You may find a fraction in which both the numerator and the denominator are perfect powers of the index.

? Example 1.4.2.18

Simplify $\sqrt{\frac{45}{80}}$.

Solution

$$\sqrt{\frac{45}{80}}$$

Simplify inside the radical first. Rewrite showing the common factors of the numerator and denominator.

$$\sqrt{\frac{5 \cdot 9}{5 \cdot 16}}$$

Simplify the fraction by removing common factors.

$$\sqrt{\frac{9}{16}}$$

Simplify. Note $\left(\frac{3}{4}\right)^2 = \frac{9}{16}$.

$$\frac{3}{4}$$

Try It 1.4.2.19

Simplify $\sqrt{\frac{75}{48}}$.

Answer

$\frac{5}{4}$

Try It 1.4.2.20

Simplify $\sqrt{\frac{98}{162}}$.

Answer

$\frac{7}{9}$

In the last example, our first step was to simplify the fraction under the radical by removing common factors. In the next example we will use the **Quotient Property** to simplify under the radical. We divide the like bases by subtracting their exponents,

$$\frac{a^m}{a^n} = a^{m-n}, \quad a \neq 0$$

Example 1.4.2.21

Simplify $\sqrt{\frac{m^6}{m^4}}$.

Solution

$$\sqrt{\frac{m^6}{m^4}}$$

Simplify the fraction inside the radical first. Divide the like bases by subtracting the exponents.

$$\sqrt{m^2}$$

Simplify.

$$|m|$$

Try It 1.4.2.22

Simplify $\sqrt{\frac{a^8}{a^6}}$.

Answer

$|a|$

Try It 1.4.2.23

Simplify $\sqrt{\frac{x^{14}}{x^{10}}}$.

Answer

x^2

Remember the **Quotient to a Power Property**? It said we could raise a fraction to a power by raising the numerator and denominator to the power separately.

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Quotient Property of Radical Expressions

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then,

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}.$$

Example 1.4.2.24

Simplify $\sqrt{\frac{27m^3}{196}}$.

Solution

Simplify the fraction in the radicand, if possible.

$\frac{27m^3}{196}$ cannot be simplified.

$$\sqrt{\frac{27m^3}{196}}$$

Use the Quotient Property to rewrite the radical as the quotient of two radicals.

We rewrite $\sqrt{\frac{27m^3}{196}}$ as the quotient of $\sqrt{27m^3}$ and $\sqrt{196}$.

$$\frac{\sqrt{27m^3}}{\sqrt{196}}$$

Simplify the radicals in the numerator and the denominator.

$9m^2$ and 196 are perfect squares.

$$\frac{\sqrt{9m^2} \cdot \sqrt{3m}}{\sqrt{196}}$$

$$\frac{3m\sqrt{3m}}{14}$$

Try It 1.4.2.25

Simplify $\sqrt{\frac{24p^3}{49}}$.

Answer

$$\frac{2|p|\sqrt{6p}}{7}$$

Try It 1.4.2.26

Simplify $\sqrt{\frac{48x^5}{100}}$.

Answer

$$\frac{2x^2\sqrt{3x}}{5}$$

Simplify a Square Root Using the Quotient Property

1. Simplify the fraction in the radicand, if possible.
2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
3. Simplify the radicals in the numerator and the denominator.

Example 1.4.2.27

Simplify $\sqrt{\frac{45x^5}{y^4}}$.

Solution

$$\sqrt{\frac{45x^5}{y^4}}$$

We cannot simplify the fraction in the radicand. Rewrite using the Quotient Property.

$$\frac{\sqrt{45x^5}}{\sqrt{y^4}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9x^4}\cdot\sqrt{5x}}{y^2}$$

Simplify.

$$\frac{3x^2\sqrt{5x}}{y^2}$$

Try It 1.4.2.28

Simplify $\sqrt{\frac{80m^3}{n^6}}$.

Answer

$$\frac{4|m|\sqrt{5m}}{|n^3|}$$

Try It 1.4.2.29

Simplify $\sqrt{\dfrac{54u^7}{v^8}}$.

Answer

$\dfrac{3u^3\sqrt{6u}}{v^4}$

Be sure to simplify the fraction in the radicand first, if possible.

Example 1.4.2.30

Simplify $\sqrt{\dfrac{18p^5q^7}{32pq^2}}$.

Solution

$$\sqrt{\frac{18p^5q^7}{32pq^2}}$$

Simplify the fraction in the radicand, if possible.

$$\sqrt{\frac{9p^4q^5}{16}}$$

Rewrite using the Quotient Property.

$$\frac{\sqrt{9p^4q^5}}{\sqrt{16}}$$

Simplify the radicals in the numerator and the denominator.

$$\frac{\sqrt{9p^4q^4}\cdot\sqrt{q}}{4}$$

Simplify.

$$\frac{3p^2q^2\sqrt{q}}{4}$$

Try It 1.4.2.31

Simplify $\sqrt{\dfrac{50x^5y^3}{72x^4y}}$.

Answer

$\dfrac{5|y|\sqrt{x}}{6}$

Try It 1.4.2.32

Simplify $\sqrt{\dfrac{48m^7n^2}{100m^5n^8}}$.

Answer

$$\frac{2|m|\sqrt{3}}{5\left|n^3\right|}$$

In the next example, there is nothing to simplify in the denominators. Since the index on the radicals is the same, we can use the **Quotient Property** again, to combine them into one radical. We will then look to see if we can simplify the expression.

Example 1.4.2.33

Simplify $\frac{\sqrt{48a^7}}{\sqrt{3a}}$.

Solution

$$\frac{\sqrt{48a^7}}{\sqrt{3a}}$$

The denominator cannot be simplified, so use the Quotient Property to write as one radical.

$$\sqrt{\frac{48a^7}{3a}}$$

Simplify the fraction under the radical.

$$\sqrt{16a^6}$$

Simplify.

$$4\left|a^3\right|$$

Try It 1.4.2.34

Simplify $\frac{\sqrt{98z^5}}{\sqrt{2z}}$.

Answer

$7z^2$

Try It 1.4.2.35

Simplify $\frac{\sqrt{128m^9}}{\sqrt{2m}}$.

Answer

$8m^4$

Writing Exercises 1.4.2.36

1. What is the goal in simplifying a radical expression?
2. How can you recognize if a radical expression is simplified?

Exit Problem 1.4.2.37

Simplify $-4x^3y\sqrt{32x^6y^9}$.

Key Concepts

- **Simplified Radical Expression**
  - For real numbers a, m and $n \geq 2$
    $\sqrt{a}$ is considered simplified if a has no factors of m^2
- **Product Property of n^{th} Roots**
  - For any real numbers, $\sqrt{a}$ and $\sqrt{b}$, and for any integer $n \geq 2$
    $\sqrt{ab} = \sqrt{a}\sqrt{b}$ and $\sqrt{a}\sqrt{b} = \sqrt{ab}$
- **How to simplify a radical expression using the Product Property**
  1. Find the largest factor in the radicand that is a perfect power of the index.
     Rewrite the radicand as a product of two factors, using that factor.
  2. Use the product rule to rewrite the radical as the product of two radicals.
  3. Simplify the root of the perfect power.
- **Quotient Property of Radical Expressions**
  - If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, $b \neq 0$, and for any integer $n \geq 2$ then, $\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}}$.
- **How to simplify a radical expression using the Quotient Property.**
  1. Simplify the fraction in the radicand, if possible.
  2. Use the Quotient Property to rewrite the radical as the quotient of two radicals.
  3. Simplify the radicals in the numerator and the denominator.

1.4.3: Rational Exponents

Learning Objectives

By the end of this section, you will be able to:

- Simplify expressions with $a^{\frac{1}{n}}$
- Simplify expressions with $a^{\frac{m}{n}}$
- Use the properties of exponents to simplify expressions with rational exponents

Be Prepared

Before you get started, take this readiness quiz.

1. Add $\dfrac{7}{15}+\dfrac{5}{12}$.
2. Simplify $(4x^2y^5)^3$.
3. Simplify 5^{-3}.

The n^{th} root

In the previous sections we worked with the simplest radical expressions, i.e. square roots. This can be generalized as follows.

Definition 1.4.3.1

The $\boldsymbol{n}^{\mathbf{th}}$ **root** of a is b if $b^n = a$. If n is even, take b to be positive and we write $\sqrt[n]{a}=b$. We call n the **index** and a the **radicand**.

Example 1.4.3.2

Evaluate:

a. $\sqrt{4}10000$

b. $\sqrt[3]{-27}$

c. $\sqrt[5]{-32x^{10}y^5z^{20}}$

Solution

a. $10^4 = 10000$, so $\sqrt[4]{10000} = 10$

b. $(-3)^3 = -27$, so $\sqrt[3]{-27} = -3$

c. $(-2x^2yz^4)^5 = -32x^{10}y^5z^{20}$, so $\sqrt[5]{-32x^{10}y^5z^{20}} = -2x^2yz^4$

Try It 1.4.3.3

Evaluate:

a. $\sqrt[3]{64}$

b. $\sqrt[5]{-243}$

Answer

a. 4

b. -3

Try It 1.4.3.4

Evaluate:

a. $\sqrt[4]{16x^8y^{12}z^4}$

b. $\sqrt[5]{-100000p^{-25}}$

Answer

a. $2x^2y^3z$

b. $-\frac{10}{p^5}$

Simplify Expressions with $a^{\frac{1}{n}}$

Rational exponents are another way of writing expressions with radicals. When we use rational exponents, we can apply the properties of exponents to simplify expressions.

The Power Property for Exponents says that $(a^m)^n = a^{mn}$ when m and n are integers. Let's assume we are now not limited to integers.

Suppose we want to find a number p such that $(8^p)^3 = 8$. We will use the Power Property of Exponents to find the value of p.

$$(8^p)^3 = 8$$

Multiply the exponents on the left.

$$8^{3p} = 8$$

Write the exponent 1 on the right.

$$8^{3p} = 8^1$$

Since the bases are the same, the exponents must be equal.

$$3p = 1$$

Solve for p.

$$p = \frac{1}{3}$$

So $\left(8^{\frac{1}{3}}\right)^3 = 8$. But we know also $(\sqrt[3]{8})^3 = 8$. Then it must be that $8^{\frac{1}{3}} = \sqrt[3]{8}$.

This same logic can be used for any positive integer exponent n to show that $a^{\frac{1}{n}} = \sqrt[n]{a}$.

Definition 1.4.3.5

If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then

$$a^{\frac{1}{n}} = \sqrt[n]{a}.$$

The denominator of the rational exponent is the index of the radical.

There will be times when working with expressions will be easier if you use rational exponents and times when it will be easier if you use radicals. In the first few examples, you'll practice converting expressions between these two notations.

Example 1.4.3.6

Write as a radical expression:

a. $x^{\frac{1}{2}}$

b. $y^{\frac{1}{3}}$

c. $z^{\frac{1}{4}}$

Solution

We want to write each expression in the form $\sqrt[n]{a}$.

a.

	$x^{\frac{1}{2}}$
The denominator of the rational exponent is 2, so the index of the radical is 2. We do not show the index when it is 2.	$\sqrt{x}$

b.

	$y^{\frac{1}{3}}$
The denominator of the exponent is 3, so the index is 3.	$\sqrt[3]{y}$

c.

	$z^{\frac{1}{4}}$
The denominator of the exponent is 4, so the index is 4.	$\sqrt[4]{z}$

Try It 1.4.3.7

Write as a radical expression:

a. $t^{\frac{1}{2}}$

b. $m^{\frac{1}{3}}$

c. $r^{\frac{1}{4}}$

Answer

a. $\sqrt{t}$

b. $\sqrt[3]{m}$

c. $\sqrt[4]{r}$

? Try It 1.4.3.8

Write as a radical expression:

a. $b^{\frac{1}{6}}$

b. $z^{\frac{1}{5}}$

c. $p^{\frac{1}{4}}$

Answer

a. $\sqrt[6]{b}$

b. $\sqrt[5]{z}$

c. $\sqrt[4]{p}$

In the next example, we will write each radical using a rational exponent. It is important to use parentheses around the entire expression in the radicand since the entire expression is raised to the rational power.

? Example 1.4.3.9

Write with a rational exponent:

$$\sqrt{5y} \tag{1.4.3.1}$$

Solution

We want to write each radical in the form $a^{\frac{1}{n}}$

$$\sqrt{5y}$$

No index is shown, so it is 2.

The denominator of the exponent will be 2.

Put parentheses around the entire expression $5y$.

$$(5y)^{\frac{1}{2}}$$

? Try It 1.4.3.10

Write with a rational exponent:

$$\sqrt{10m} \tag{1.4.3.2}$$

Answer

$(10m)^{\frac{1}{2}}$

In the next example, you may find it easier to simplify the expressions if you rewrite them as radicals first.

Example 1.4.3.11

Simplify:

a. $25^{\frac{1}{2}}$

b. $64^{\frac{1}{3}}$

c. $256^{\frac{1}{4}}$

Solution

a.

	$25^{\frac{1}{2}}$
Rewrite as a square root.	$\sqrt{25}$
Simplify.	5

b.

	$64^{\frac{1}{3}}$
Rewrite as a cube root.	$\sqrt[3]{64}$
Recognize 64 is a perfect cube.	$\sqrt[3]{4^3}$
Simplify.	4

c.

	$256^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{256}$
Recognize 256 is a perfect fourth power.	$\sqrt[4]{4^4}$
Simplify.	4

Try It 1.4.3.12

Simplify:

a. $36^{\frac{1}{2}}$

b. $8^{\frac{1}{3}}$

c. $16^{\frac{1}{4}}$

Answer

a. 6

b. 2

c. 2

Try It 1.4.3.13

Simplify:

a. $100^{\frac{1}{2}}$

b. $27^{\frac{1}{3}}$

c. $81^{\frac{1}{4}}$

Answer

a. 10

b. 3

c. 3

Be careful of the placement of the negative signs in the next example. We will need to use the definition $a^{-n} = \frac{1}{a^n}$ in one case.

Example 1.4.3.14

Simplify:

a. $(-16)^{\frac{1}{4}}$

b. $-16^{\frac{1}{4}}$

c. $(16)^{-\frac{1}{4}}$

Solution

a.

	$(-16)^{\frac{1}{4}}$
Rewrite as a fourth root.	$\sqrt[4]{-16}$
Simplify.	$\sqrt[4]{(-2)^4}$

No real solution

b.

	$-16^{\frac{1}{4}}$
The exponent only applies to the 16. Rewrite as a fourth root.	$-\sqrt[4]{16}$
Rewrite 16 as 2^4.	$-\sqrt[4]{2^4}$
Simplify.	-2

c.

	$(16)^{-\frac{1}{4}}$
Rewrite using the definition $a^{-n} = \frac{1}{a^n}$.	$\frac{1}{(16)^{\frac{1}{4}}}$
Rewrite as a fourth root.	$\frac{1}{\sqrt[4]{16}}$

Rewrite 16 as 2^4.	$\frac{1}{\sqrt[4]{2^4}}$
Simplify.	$\frac{1}{2}$

Try It 1.4.3.15

Simplify:

a. $(-64)^{-\frac{1}{2}}$

b. $-64^{\frac{1}{2}}$

c. $(64)^{-\frac{1}{2}}$

Answer

a. No real solution

b. -8

c. $\frac{1}{8}$

Try It 1.4.3.16

Simplify:

a. $(-256)^{\frac{1}{4}}$

b. $-256^{\frac{1}{4}}$

c. $(256)^{-\frac{1}{4}}$

Answer

a. No real solution

b. -4

c. $\frac{1}{4}$

Simplify Expressions with $a^{\frac{m}{n}}$

We can look at $a^{\frac{m}{n}}$ in two ways. Remember the Power Property tells us to multiply the exponents and so $\left(a^{\frac{1}{n}}\right)^m$ and $(a^m)^{\frac{1}{n}}$ both equal $a^{\frac{m}{n}}$. If we write these expressions in radical form, we get

$$a^{\frac{m}{n}}=\left(a^{\frac{1}{n}}\right)^m=(\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}}=(a^m)^{\frac{1}{n}}=\sqrt[n]{a^m}$$

This leads us to the following definition.

Definition 1.4.3.17

For any positive integers m and n,

$$a^{\frac{m}{n}}=(\sqrt[n]{a})^m \quad \text{and} \quad a^{\frac{m}{n}}=\sqrt[n]{a^m}.$$

Which form do we use to simplify an expression? We usually take the root first—that way we keep the numbers in the radicand smaller, before raising it to the power indicated.

Example 1.4.3.18

Write with a rational exponent:

a. $\sqrt{y^3}$

b. $(\sqrt[3]{2x})^4$

c. $\sqrt{\left(\frac{3a}{4b}\right)^3}$

Solution

We want to use $a^{\frac{m}{n}} = \sqrt[n]{a^m}$ to write each radical in the form $a^{\frac{m}{n}}$

a.

	$\sqrt{y^3}$
When the index is missing, it is implicitly 2.	$\sqrt[2]{y^3}$
$m=3, n=2$ in $\sqrt[n]{a^m} = a^{\frac{m}{n}}$	$y^{\frac{3}{2}}$

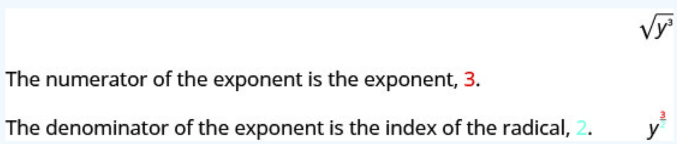


Figure 8.3.1

b.

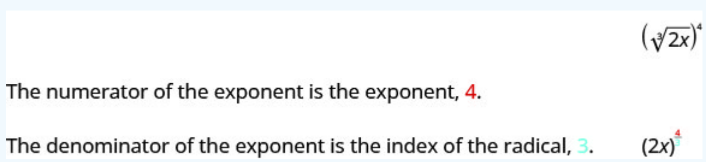


Figure 8.3.2

c.

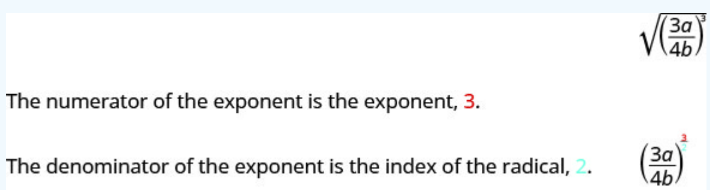


Figure 8.3.3

Try It 1.4.3.19

Write with a rational exponent:

a. $\sqrt{x^5}$

b. $(\sqrt[4]{3y})^3$

c. $\sqrt{\left(\frac{2m}{3n}\right)^5}$

Answer

a. $x^{\frac{5}{2}}$

b. $(3y)^{\frac{3}{4}}$

c. $\left(\frac{2m}{3n}\right)^{\frac{5}{2}}$

? Try It 1.4.3.20

Write with a rational exponent:

a. $\sqrt[5]{a^2}$

b. $\left(\sqrt[3]{5ab}\right)^5$

c. $\sqrt{\left(\frac{7xy}{z}\right)^3}$

Answer

a. $a^{\frac{2}{5}}$

b. $(5ab)^{\frac{5}{3}}$

c. $\left(\frac{7xy}{z}\right)^{\frac{3}{2}}$

Remember that $a^{-n} = \frac{1}{a^n}$. The negative sign in the exponent does not change the sign of the expression.

? Example 1.4.3.21

Simplify:

a. $125^{\frac{2}{3}}$

b. $16^{-\frac{3}{2}}$

c. $32^{-\frac{2}{5}}$

Solution

We will rewrite the expression as a radical first using the defintion, $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$. This form lets us take the root first and so we keep the numbers in the radicand smaller than if we used the other form.

a.

	$125^{\frac{2}{3}}$
The power of the radical is the numerator of the exponent, 2. The index of the radical is the denominator of the exponent, 3.	$\left(\sqrt[3]{125}\right)^2$
Simplify.	$(5)^2$
	25

b.

	$16^{-\frac{3}{2}}$

We will rewrite each expression first using $a^{-n}=\frac{1}{a^n}$ and then change to radical form.	$\frac{1}{16^{\frac{3}{2}}}$
Change to radical form. The power of the radical is the numerator of the exponent, 3. The index is the denominator of the exponent, 2.	$\frac{1}{\left(\sqrt{16}\right)^3}$
Simplify.	$\frac{1}{4^3}$ $\frac{1}{64}$

c.

	$32^{-\frac{2}{5}}$
Rewrite using $a^{-n}=\frac{1}{a^n}$.	$\frac{1}{32^{\frac{2}{5}}}$
Change to radical form.	$\frac{1}{(\sqrt[5]{32})^2}$
Rewrite the radicand as a power.	$\frac{1}{\left(\sqrt[5]{2^5}\right)^2}$
Simplify.	$\frac{1}{2^2}$ $\frac{1}{4}$

? Try It 1.4.3.22

Simplify:

a. $27^{\frac{2}{3}}$

b. $81^{-\frac{3}{2}}$

c. $16^{-\frac{3}{4}}$

Answer

a. 9

b. $\frac{1}{729}$

c. $\frac{1}{8}$

? Try It 1.4.3.23

Simplify:

a. $4^{\frac{3}{2}}$

b. $27^{-\frac{2}{3}}$

c. $625^{-\frac{3}{4}}$

Answer

a. 8

b. $\frac{1}{9}$

c. $\frac{1}{125}$

Example 1.4.3.24

Simplify:

a. $-25^{\frac{3}{2}}$

b. $-25^{-\frac{3}{2}}$

c. $(-25)^{\frac{3}{2}}$

Solution

a.

	$-25^{\frac{3}{2}}$
Rewrite in radical form.	$-(\sqrt{25})^3$
Simplify the radical.	$-(5)^3$
Simplify.	-125

b.

	$-25^{-\frac{3}{2}}$
Rewrite using $a^{-n}=\frac{1}{a^n}$.	$-\left(\frac{1}{25^{\frac{3}{2}}}\right)$
Rewrite in radical form.	$-\left(\frac{1}{(\sqrt{25})^3}\right)$
Simplify the radical.	$-\left(\frac{1}{5^3}\right)$
Simplify.	$-\frac{1}{125}$

c.

	$(-25)^{\frac{3}{2}}$
Rewrite in radical form.	$(\sqrt{-25})^3$ There is no real number whose square root is -25. Not a real number.

Try It 1.4.3.25

Simplify:

a. $-16^{\frac{3}{2}}$

b. $-16^{-\frac{3}{2}}$

c. $(-16)^{-\frac{3}{2}}$

Answer

a. -64

b. $-\frac{1}{64}$

c. Not a real number

Try It 1.4.3.26

Simplify:

a. $-81^{\frac{3}{2}}$

b. $-81^{-\frac{3}{2}}$

c. $(-81)^{-\frac{3}{2}}$

Answer

a. -729

b. $-\frac{1}{729}$

c. Not a real number

Use the Properties of Exponents to Simplify Expressions with Rational Exponents

The same properties of exponents that we have already used for integers also apply to rational exponents. We will not show this here, but doing some examples will convince you. For example,

$$16^{\frac{1}{2}} \cdot 16^{\frac{3}{4}} = 4 \cdot 8 = 32$$

and

$$16^{\frac{1}{2}+\frac{3}{4}} = 16^{\frac{5}{4}} = 32.$$

So $16^{\frac{1}{2}} \cdot 16^{\frac{3}{4}} = 16^{\frac{1}{2}+\frac{3}{4}}$.

We will list the Properties of Exponents below to have them for reference as we simplify expressions.

Properties of Exponents

If a and b are real numbers and m and n are rational numbers, then

Product Property

$$a^m \cdot a^n = a^{m+n}$$

Power Property

$$(a^m)^n = a^{mn}$$

Product to a Power

$$(ab)^m = a^m b^m$$

Quotient Property

$$\frac{a^m}{a^n} = a^{m-n}, a \neq 0$$

Zero Exponent Definition

$$a^0 = 1, a \neq 0$$

Quotient to a Power Property

$$\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$$

Negative Exponent Property

$$a^{-n} = \frac{1}{a^n}, a \neq 0$$

We will apply these properties in the next example.

Example 1.4.3.27

Simplify:

a. $x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$

b. $\left(z^9\right)^{\frac{2}{3}}$

c. $\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

Solution

a.

The Product Property tells us that when we multiple the same base, we add the exponents.	$x^{\frac{1}{2}} \cdot x^{\frac{5}{6}}$
The bases are the same, so we add the exponents.	$x^{\frac{1}{2}+\frac{5}{6}}$
Add the fractions.	$x^{\frac{8}{6}}$
Simplify the exponent.	$x^{\frac{4}{3}}$

b.

The Power Property tells us that when we raise a power to a power, we multiple the exponents.	$\left(z^9\right)^{\frac{2}{3}}$
To raise a power to a power, we multiple the exponents.	$z^{9 \cdot \frac{2}{3}}$
Simplify.	z^6

c.

The Quotient Property tells us that when we divide with the same base, we subtract the exponents.	$\frac{x^{\frac{1}{3}}}{x^{\frac{5}{3}}}$

To divide with the same base, we subtract the exponents.	$\frac{1}{x^{\frac{5}{3}-\frac{1}{3}}}$
Simplify.	$\frac{1}{x^{\frac{4}{3}}}$

Try It 1.4.3.28

Simplify:

a. $x^{\frac{1}{6}} \cdot x^{\frac{4}{3}}$

b. $\left(x^6\right)^{\frac{4}{3}}$

c. $\frac{x^{\frac{2}{3}}}{x^{\frac{5}{3}}}$

Answer

a. $x^{\frac{3}{2}}$

b. x^8

c. $\frac{1}{x}$

Try It 1.4.3.29

Simplify:

a. $y^{\frac{3}{4}} \cdot y^{\frac{5}{8}}$

b. $\left(m^9\right)^{\frac{2}{9}}$

c. $\frac{d^{\frac{1}{5}}}{d^{\frac{6}{5}}}$

Answer

a. $y^{\frac{11}{8}}$

b. m^2

c. $\frac{1}{d}$

Writing Exercises 1.4.3.30

1. How is the square root related to rational exponents?
2. What about a cube root?
3. Show two different algebraic methods to simplify $4^{\frac{3}{2}}$. Explain all your steps.
4. Explain why the expression $(-16)^{\frac{3}{2}}$ cannot be evaluated.
5. Explain why $x^{\frac{1}{2}} = \sqrt{x}$.
6. Give and example of the rules $(ab)^n = a^n b^n$ and $a^n a^m = a^{n+m}$ with rational exponents.

Exit Problem 1.4.3.31

Simplify $(-27b^9)^{1/3}$.

Key Concepts

- **Rational Exponent $a^{\frac{1}{n}}$**
  - If $\sqrt[n]{a}$ is a real number and $n \geq 2$, then $a^{\frac{1}{n}} = \sqrt[n]{a}$.
- **Rational Exponent $a^{\frac{m}{n}}$**
  - For any positive integers m and n,
    $a^{\frac{m}{n}} = (\sqrt[n]{a})^m$ and $a^{\frac{m}{n}} = \sqrt[n]{a^m}$
- **Properties of Exponents**
  - If a, b are real numbers and m, n are rational numbers, then
    - **Product Property** $a^m \cdot a^n = a^{m+n}$
    - **Power Property** $(a^m)^n = a^{mn}$
    - **Product to a Power** $(ab)^m = a^m b^m$
    - **Quotient Property** $\frac{a^m}{a^n} = a^{m-n}, a \neq 0$
    - **Zero Exponent Definition** $a^0 = 1, a \neq 0$
    - **Quotient to a Power Property** $\left(\frac{a}{b}\right)^m = \frac{a^m}{b^m}, b \neq 0$
    - **Negative Exponent Property** $a^{-n} = \frac{1}{a^n}, a \neq 0$

1.4.4: Adding, Subtracting and Multiplying Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Add and subtract radical expressions
- Multiply radical expressions
- Use polynomial multiplication to multiply radical expressions

Be Prepared

Before you get started, take this readiness quiz.

1. Add $3x^2+9x-5-(x^2-2x+3)$.
2. Simplify $(2+a)(4-a)$.
3. Simplify $(9-5y)^2$.

Add and Subtract Radical Expressions

Adding radical expressions with the same index and the same radicand is just like adding like terms. We call radicals with the same index and the same radicand **like radicals** to remind us they work the same as like terms.

Definition 1.4.4.1

Like radicals are radical expressions with the same index and the same radicand.

We add and subtract like radicals in the same way we add and subtract like terms. We know that $3x+8x$ is $11x$. Similarly we add $3\sqrt{x}+8\sqrt{x}$ and the result is $11\sqrt{x}$.

Let's think about adding like terms with variables as we do the next few examples. When we have like radicals, we just add or subtract the coefficients. When the radicals are not like, we cannot combine the terms.

Example 1.4.4.2

Simplify $2\sqrt{2}-7\sqrt{2}$.

One should think of $\sqrt{2}$ as an object and then count: 2 objects -7 objects is -5 objects.

Solution

	$2\sqrt{2}-7\sqrt{2}$
Since the radicals are like, we subtract the coefficients.	$=-5\sqrt{2}$

Try It 1.4.4.3

Simplify $8\sqrt{2}-9\sqrt{2}$.

Answer

$-\sqrt{2}$

Try It 1.4.4.4

Simplify $5\sqrt{3}-9\sqrt{3}$.

Answer

$-4\sqrt{3}$

For radicals to be like, they must have the same index and radicand. When the radicands contain more than one variable, as long as all the variables and their exponents are identical, the radicands are the same.

Example 1.4.4.5

Simplify $2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$.

Solution

	$2\sqrt{5n}-6\sqrt{5n}+4\sqrt{5n}$
Since the radicals are like, we combine them.	$=0\sqrt{5n}$
Simplify.	$=0$

Try It 1.4.4.6

Simplify $\sqrt{7x}-7\sqrt{7x}+4\sqrt{7x}$.

Answer

$-2\sqrt{7x}$

Try It 1.4.4.7

Simplify $4\sqrt{3y}-7\sqrt{3y}+2\sqrt{3y}$.

Answer

$-\sqrt{3y}$

Remember that we always simplify radicals by removing the largest factor from the radicand that is a power of the index. Once each radical is simplified, we can then decide if they are like radicals.

Example 1.4.4.8

Simplify $\sqrt{20}+3\sqrt{5}$.

Solution

	$\sqrt{20}+3\sqrt{5}$
Simplify the radicals, when possible.	$=\sqrt{4}\cdot\sqrt{5}+3\sqrt{5}$
Simplify.	$=2\sqrt{5}+3\sqrt{5}$
Combine the like radicals.	$=5\sqrt{5}$

Try It 1.4.4.9

Simplify $\sqrt{18}+6\sqrt{2}$.

Answer

$9\sqrt{2}$

Try It 1.4.4.10

Simplify $\sqrt{27}+4\sqrt{3}$.

Answer

$7\sqrt{3}$

In the next example, we will remove both constant and variable factors from the radicals. Now that we have practiced taking both the even and odd roots of variables, it is common practice at this point for us to assume all variables are greater than or equal to zero so that absolute values are not needed. We will use this assumption thoughout the rest of this chapter.

Example 1.4.4.11

Simplify $9\sqrt{50m^2}-6\sqrt{48m^2}$.

Solution

	$9\sqrt{50m^2}-6\sqrt{48m^2}$
Simplify the radicals.	$=9\sqrt{25m^2}\cdot\sqrt{2}-6\sqrt{16m^2}\cdot\sqrt{3}$
Simplify,	$=9\cdot 5m\cdot\sqrt{2}-6\cdot 4m\cdot\sqrt{3}$
The radicals are not like and so cannot be combined.	$=45m\sqrt{2}-24m\sqrt{3}$

Try It 1.4.4.12

Simplify $\sqrt{32m^7}-\sqrt{50m^7}$.

Answer

$-m^3\sqrt{2m}$

Try It 1.4.4.13

Simplify $\sqrt{27p^3}-\sqrt{48p^3}$.

Answer

$-p\sqrt{3p}$

Multiply Radical Expressions

We have used the **Product Property of Roots** to simplify square roots by removing the perfect square factors. We can use the Product Property of Roots ‘in reverse’ to multiply square roots. Remember, we assume all variables are greater than or equal to zero.

We will rewrite the Product Property of Roots so we see both ways together.

Product Property of Roots

For any real numbers, $\sqrt{a}$ and $\sqrt{b}$, we have

$$\sqrt{ab} = \sqrt{a} \cdot \sqrt{b} \quad \text{and} \quad \sqrt{a} \cdot \sqrt{b} = \sqrt{ab}.$$

When we multiply two radicals, they must have the same index. Once we multiply the radicals, we then look for factors that are a power of the index and simplify the radical whenever possible.

Multiplying radicals with coefficients is much like multiplying variables with coefficients. To multiply $4x \cdot 3y$ we multiply the coefficients together and then the variables. The result is $12xy$. Keep this in mind as you do these examples.

Example 1.4.4.14

Simplify $(6\sqrt{2})(3\sqrt{10})$.

Solution

	$(6\sqrt{2})(3\sqrt{10})$
Multiply using the Product Property.	$= 18\sqrt{20}$
Simplify the radical.	$= 18\sqrt{4} \cdot \sqrt{5}$
Simplify.	$= 18 \cdot 2 \cdot \sqrt{5}$
Simplify.	$= 36\sqrt{5}$

Try It 1.4.4.15

Simplify $(3\sqrt{2})(2\sqrt{30})$.

Answer

$12\sqrt{15}$

Try It 1.4.4.16

Simplify $(3\sqrt{3})(3\sqrt{6})$.

Answer

$27\sqrt{2}$

We follow the same procedures when there are variables in the radicands.

Example 1.4.4.17

Simplify $\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$.

Solution

	$\left(10\sqrt{6p^3}\right)\left(4\sqrt{3p}\right)$
Multiply.	$= 40\sqrt{18p^4}$
Simplify the radical.	$= 40\sqrt{9p^4} \cdot \sqrt{2}$
Simplify.	$= 40 \cdot 3p^2 \cdot \sqrt{2}$

Simplify.	$= 120p^2\sqrt{2}$

Try It 1.4.4.18

Simplify $\left(6\sqrt{6x^2}\right)\left(8\sqrt{30x^4}\right)$.

Answer

$36x^3\sqrt{5}$

Try It 1.4.4.19

Simplify $\left(2\sqrt{6y^4}\right)\left(12\sqrt{30y}\right)$.

Answer

$144y^2\sqrt{5y}$

Use Polynomial Multiplication to Multiply Radical Expressions

In the next a few examples, we will use the **Distributive Property** to multiply expressions with radicals. First we will distribute and then simplify the radicals when possible.

Example 1.4.4.20

Simplify $\sqrt{6}(\sqrt{2}+\sqrt{18})$.

Solution

	$\sqrt{6}(\sqrt{2}+\sqrt{18})$
Multiply.	$=\sqrt{12}+\sqrt{108}$
Simplify.	$=\sqrt{4}\cdot\sqrt{3}+\sqrt{36}\cdot\sqrt{3}$
Simplify.	$=2\sqrt{3}+6\sqrt{3}$
Combine like radicals.	$=8\sqrt{3}$

Try It 1.4.4.21

Simplify $\sqrt{6}(1+3\sqrt{6})$.

Answer

$18+\sqrt{6}$

Try It 1.4.4.22

Simplify $\sqrt{8}(2-5\sqrt{8})$.

Answer

$-40+4\sqrt{2}$

When we worked with polynomials, we multiplied binomials by binomials. Remember, this gave us four products before we combined any like terms. To be sure to get all four products, we organized our work—usually by the FOIL method.

Example 1.4.4.23

Simplify $(3-2\sqrt{7})(4-2\sqrt{7})$.

Solution

	$(3-2\sqrt{7})(4-2\sqrt{7})$
Multiply.	$=12-6\sqrt{7}-8\sqrt{7}+4(\sqrt{7})^2$
Simplify.	$=12-6\sqrt{7}-8\sqrt{7}+4\cdot 7$ $=12-6\sqrt{7}-8\sqrt{7}+28$
Combine like terms.	$=40-14\sqrt{7}$

Try It 1.4.4.24

Simplify $(6-3\sqrt{7})(3+4\sqrt{7})$.

Answer

$-66+15\sqrt{7}$

Try It 1.4.4.25

Simplify $(2-3\sqrt{11})(4-\sqrt{11})$.

Answer

$41-14\sqrt{11}$

Example 1.4.4.26

Simplify $(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$.

Solution

	$(3\sqrt{2}-\sqrt{5})(\sqrt{2}+4\sqrt{5})$
Multiply.	$=3(\sqrt{2})^2+12\sqrt{2}\sqrt{5}-\sqrt{5}\sqrt{2}-4(\sqrt{5})^2$
Simplify.	$=6+12\sqrt{10}-\sqrt{10}-20$
Combine like terms.	$=-14+11\sqrt{10}$

Try It 1.4.4.27

Simplify $(5\sqrt{3}-\sqrt{7})(\sqrt{3}+2\sqrt{7})$.

Answer

$1+9\sqrt{21}$

Try It 1.4.4.28

Simplify $(\sqrt{6}-3\sqrt{8})(2\sqrt{6}+\sqrt{8})$.

Answer

$-12-20\sqrt{3}$

Recognizing some special products made our work easier when we multiplied binomials earlier. This is true when we multiply radicals, too. The special product formulas we used are shown here.

Special Products

Binomial Squares

Product of Conjugates

$(a+b)(a-b)=a^2-b^2$

We will use the special product formulas in the next few examples. We will start with the **Product of Binomial Squares Pattern**.

Example 1.4.4.29

Simplify:

a. $(2+\sqrt{3})^2$

b. $(4-2\sqrt{5})^2$

Solution

a.

	$\underbrace{(2+\sqrt{3})^2}_{(a+b)^2}$
Multiply using the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$, or FOIL $(a+b)(a+b)$.	$=\underbrace{2^2+2\cdot 2\sqrt{3}+(\sqrt{3})^2}_{a^2+2ab+b^2}$
Simplify.	$=4+4\sqrt{3}+3$
Combine like terms.	$=7+4\sqrt{3}$

b.

	$\underbrace{(4-2\sqrt{5})^2}_{(a-b)^2}$
Multiple, using the Product of Binomial Squares Pattern, $(a-b)^2=a^2-2ab+b^2$, or FOIL $(a-b)(a-b)$.	$=\underbrace{4^2+2\cdot 4\cdot 2\sqrt{5}+(2\sqrt{5})^2}_{a^2-2ab+b^2}$
Simplify.	$=16-16\sqrt{5}+4\cdot 5$ $=16-16\sqrt{5}+20$
Combine like terms.	$=36-16\sqrt{5}$

Try It 1.4.4.30

Simplify:

a. $(10+\sqrt{2})^2$

b. $(1+3\sqrt{6})^2$

Answer

a. $102+20\sqrt{2}$

b. $55+6\sqrt{6}$

Try It 1.4.4.31

Simplify:

a. $(6-\sqrt{5})^2$

b. $(9-2\sqrt{10})^2$

Answer

a. $41-12\sqrt{5}$

b. $121-36\sqrt{10}$

In the next example, we will use the Product of Conjugates Pattern. Notice that the final product has no radical.

Example 1.4.4.32

Simplify $(5-2\sqrt{3})(5+2\sqrt{3})$.

Solution

	$\underbrace{(5-2\sqrt{3})(5+2\sqrt{3})}_{(a-b)(a+b)}$
Multiply using the Product of Conjugates Pattern.	$=\underbrace{5^2-(2\sqrt{3})^2}_{a^2-b^2}$
Simplify.	$=25-4\cdot 3$
Simplify.	$=13$

Try It 1.4.4.33

Simplify $(3-2\sqrt{5})(3+2\sqrt{5})$.

Answer

-11

Try It 1.4.4.34

Simplify $(4+5\sqrt{7})(4-5\sqrt{7})$.

Answer

-159

Writing Exercises 1.4.4.35

1. Explain how the product property of roots is related to a property of exponents.
2. How can you see that $2\sqrt{7}-4\sqrt{7}=-2\sqrt{7}$ by using a variable and what you know about simplifying polynomials?
3. Why do often need to simplify radical expressions before adding them? Give an example.
4. What are like radicals? What is their role in adding and subtracting radicals?
5. Based on the definition of $\sqrt{x}$, why is $(\sqrt{x})^2=x$?
6. If we write the $\sqrt{\ }$ using rational exponents, what property of exponents is this equality related to?

Exit Problem 1.4.4.36

(a) Simplify $3\sqrt{80}-5\sqrt{45}$.

(b) Expand $\left(2\sqrt{a}-\sqrt{ab}\right)^2$.

Key Concepts

- **Product Property of Roots**
  - For any real numbers, $\sqrt{a}$ and $\sqrt{b}$, and $\sqrt{ab}=\sqrt{a}\cdot\sqrt{b}$ and $\sqrt{a}\cdot\sqrt{b}=\sqrt{ab}$, or in the language of exponents: $(ab)^{\frac{1}{2}}=a^{\frac{1}{2}}b^{\frac{1}{2}}$.
- **Special Products**

1.4.5: Dividing Radical Expressions

Learning Objectives

By the end of this section, you will be able to:

- Divide radical expressions
- Rationalize a one term denominator
- Rationalize a two term denominator

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\frac{30}{48}$.
2. Simplify $x^2 \cdot x^4$.
3. Multiply $(7+3x)(7-3x)$.

Divide Radical Expressions

We have used the **Quotient Property of Radical Expressions** to simplify roots of fractions. We will need to use this property 'in reverse' to simplify a fraction with radicals. We give the Quotient Property of Radical Expressions again for easy reference. Remember, we assume all variables are greater than or equal to zero so that no absolute value bars re needed.

Quotient Property of Radical Expressions

If $\sqrt{a}$ and $\sqrt{b}$ are real numbers with $b \neq 0$, then

$$\sqrt{\frac{a}{b}} = \frac{\sqrt{a}}{\sqrt{b}},$$

or, in the language of exponents:

$$\left(\frac{a}{b}\right)^{\frac{1}{2}} = \frac{a^{\frac{1}{2}}}{b^{\frac{1}{2}}}.$$

We will use the Quotient Property of Radical Expressions when the fraction we start with is the quotient of two radicals, and neither radicand is a perfect power of the index. When we write the fraction in a single radical, we may find common factors in the numerator and denominator.

Example 1.4.5.1

Simplify $\frac{\sqrt{72x^3}}{\sqrt{162x}}$.

Solution

	$\frac{\sqrt{72x^3}}{\sqrt{162x}}$
Rewrite using the quotient property,	$=\sqrt{\frac{72x^3}{162x}}$
Remove common factors.	$=\sqrt{\frac{\cancel{18}\cdot 4\cdot x^2\cdot \cancel{x}}{\cancel{18}\cdot 9\cdot \cancel{x}}}$
Simplify.	$=\sqrt{\frac{4x^2}{9}}$

Simplify the radical.	$=\dfrac{2x}{3}$

Try It 1.4.5.2

Simplify $\dfrac{\sqrt{50s^3}}{\sqrt{128s}}$.

Answer

$\dfrac{5s}{8}$

Try It 1.4.5.3

Simplify $\dfrac{\sqrt{75q^5}}{\sqrt{108q}}$.

Answer

$\dfrac{5q^2}{6}$

Example 1.4.5.4

Simplify $\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$.

Solution

	$\dfrac{\sqrt{147ab^8}}{\sqrt{3a^3b^4}}$
Rewrite using the quotient property.	$=\sqrt{\dfrac{147ab^8}{3a^3b^4}}$
Remove common factors in the fraction.	$=\sqrt{\dfrac{49b^4}{a^2}}$
Simplify the radical.	$=\dfrac{7b^2}{a}$

Try It 1.4.5.5

Simplify $\dfrac{\sqrt{162x^{10}y^2}}{\sqrt{2x^6y^6}}$.

Answer

$\dfrac{9x^2}{y^2}$

Try It 1.4.5.6

Simplify $\dfrac{\sqrt{300m^3n^7}}{\sqrt{3m^5n}}$.

Answer

$\frac{10n^3}{m}$

Example 1.4.5.7

Simplify $\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$.

Solution

	$\frac{\sqrt{54x^5y^3}}{\sqrt{3x^2y}}$
Rewrite using the quotient property.	$=\sqrt{\frac{54x^5y^3}{3x^2y}}$
Remove common factors in the fraction.	$=\sqrt{18x^3y^2}$
Rewrite the radicand as a product using the largest perfect square factor.	$=\sqrt{9x^2y^2\cdot 2x}$
Rewrite the radical as the product of two radicals.	$=\sqrt{9x^2y^2}\cdot\sqrt{2x}$
Simplify.	$=3xy\sqrt{2x}$

Try It 1.4.5.8

Simplify $\frac{\sqrt{64x^4y^5}}{\sqrt{2xy^3}}$.

Answer

$4xy\sqrt{2x}$

Try It 1.4.5.9

Simplify $\frac{\sqrt{96a^5b^4}}{\sqrt{2a^3b}}$.

Answer

$4ab\sqrt{3b}$

Rationalize a One-Term Denominator

Before the calculator became a tool of everyday life, approximating the value of a fraction with a radical in the denominator was a very cumbersome process! For this reason, a process called **rationalizing the denominator** was developed. A fraction with a radical in the denominator is converted to an equivalent fraction whose denominator is an integer. Square roots of numbers that are not perfect squares are irrational numbers. When we rationalize the denominator, we write an equivalent fraction with a rational number in the denominator. This process is still used today, and is useful in other areas of mathematics too. In particular, when such expressions will be added or further simplified it is handy to have them in a form where there are no radicals in the denominator.

Definition 1.4.5.10

Rationalizing the denominator is the process of converting a fraction with a radical in the denominator to an equivalent fraction whose denominator is an integer.

Even though we have calculators available nearly everywhere, a fraction with a radical in the denominator should still be rationalized. It is not considered simplified if the denominator contains a radical.

Similarly, a **radical expression** is not considered simplified if the radicand contains a fraction.

Simplified Radical Expressions

Simplified Radical Expressions

A radical expression is considered **simplified** if there are

- no factors in the radicand have perfect powers of the index
- no fractions in the radicand
- no radicals in the denominator of a fraction

To rationalize a denominator with a square root, we use the property that $(\sqrt{a})^2 = a$. If we square an irrational square root, we get a rational number.

We will use this property to rationalize the denominator in the next example.

Example 1.4.5.11

Simplify:

a. $\frac{4}{\sqrt{3}}$

b. $\sqrt{\frac{3}{20}}$

c. $\frac{3}{\sqrt{6x}}$

Solution

To rationalize a denominator with one term, we can multiply a square root by itself. To keep the fraction equivalent, we multiply both the numerator and denominator by the same factor.

a.

	$\frac{4}{\sqrt{3}}$
Multiply both the numerator and denominator by $\sqrt{3}$.	$= \frac{4 \cdot \sqrt{3}}{\sqrt{3} \cdot \sqrt{3}}$
Simplify.	$= \frac{4\sqrt{3}}{3}$

b. We always simplify the radical in the denominator first, before we rationalize it. This way the numbers stay smaller and easier to work with.

	$\sqrt{\frac{3}{20}}$
The fraction is not a perfect square, so rewrite using the Quotient Property.	$= \frac{\sqrt{3}}{\sqrt{20}}$

Simplify the denominator.	$=\dfrac{\sqrt{3}}{2\sqrt{5}}$
Multiply the numerator and denominator by $\sqrt{5}$.	$=\dfrac{\sqrt{3}\cdot\sqrt{5}}{2\sqrt{5}\cdot\sqrt{5}}$
Simplify.	$=\dfrac{\sqrt{15}}{2\cdot 5}$
Simplify.	$=\dfrac{\sqrt{15}}{10}$

c.

	$\dfrac{3}{\sqrt{6x}}$
Multiply the numerator and denominator by $\sqrt{6x}$.	$=\dfrac{3\cdot\sqrt{6x}}{\sqrt{6x}\cdot\sqrt{6x}}$
Simplify.	$=\dfrac{3\sqrt{6x}}{6x}$
Simplify.	$=\dfrac{\sqrt{6x}}{2x}$

Try It 1.4.5.12

Simplify:

a. $\dfrac{5}{\sqrt{3}}$

b. $\sqrt{\dfrac{3}{32}}$

c. $\dfrac{2}{\sqrt{2x}}$

Answer

a. $\dfrac{5\sqrt{3}}{3}$

b. $\dfrac{\sqrt{6}}{8}$

c. $\dfrac{\sqrt{2x}}{x}$

Try It 1.4.5.13

Simplify:

a. $\dfrac{6}{\sqrt{5}}$

b. $\sqrt{\dfrac{7}{18}}$

c. $\dfrac{5}{\sqrt{5x}}$

Answer

a. $\frac{6\sqrt{5}}{5}$

b. $\frac{\sqrt{14}}{6}$

c. $\frac{\sqrt{5x}}{x}$

When we rationalized a square root, we multiplied the numerator and denominator by a square root that would give us a perfect square under the radical in the denominator. When we took the square root, the denominator no longer had a radical.

Rationalize a Two-Term Denominator

When the denominator of a fraction is a sum or difference with square roots, we use the **Product of Conjugates Pattern** to **rationalize the denominator**.

$$\begin{array}{ll} (a-b)(a+b) & (2-\sqrt{5})(2+\sqrt{5}) \\ =a^2-b^2 & =2^2-(\sqrt{5})^2 \\ & =4-5 \\ & =-1 \end{array}$$

When we multiple a binomial that includes a square root by its conjugate, the product has no square roots.

Example 1.4.5.14

Simplify $\frac{5}{2-\sqrt{3}}$.

Solution

	$\frac{5}{2-\sqrt{3}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$=\frac{5\,(2+\sqrt{3})}{(2-\sqrt{3})\,(2+\sqrt{3})}$
Multiply the conjugates in the denominator.	$=\frac{5\,(2+\sqrt{3})}{2^2-(\sqrt{3})^2}$
Simplify the denominator.	$=\frac{5\,(2+\sqrt{3})}{4-3}$
Simplify the denominator.	$=\frac{5\,(2+\sqrt{3})}{1}$
Simplify.	$=5\,(2+\sqrt{3})$

Try It 1.4.5.15

Simplify $\frac{3}{1-\sqrt{5}}$.

Answer

$-\frac{3(1+\sqrt{5})}{4}$

Try It 1.4.5.16

Simplify $\frac{2}{4-\sqrt{6}}$.

Answer

$$\frac{4+\sqrt{6}}{5}$$

Notice we did not distribute the 5 in the answer of the last example. By leaving the result factored we can see if there are any factors that may be common to both the numerator and denominator.

Example 1.4.5.17

Simplify $\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$.

Solution

	$\frac{\sqrt{3}}{\sqrt{u}-\sqrt{6}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$=\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u}-\sqrt{6})(\sqrt{u}+\sqrt{6})}$
Multiply the conjugates in the denominator.	$=\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{(\sqrt{u})^2-(\sqrt{6})^2}$
Simplify the denominator.	$=\frac{\sqrt{3}(\sqrt{u}+\sqrt{6})}{u-6}$

Try It 1.4.5.18

Simplify $\frac{\sqrt{5}}{\sqrt{x}+\sqrt{2}}$.

Answer

$$\frac{\sqrt{5}(\sqrt{x}-\sqrt{2})}{x-2}$$

Try It 1.4.5.19

Simplify $\frac{\sqrt{10}}{\sqrt{y}-\sqrt{3}}$.

Answer

$$\frac{\sqrt{10}(\sqrt{y}+\sqrt{3})}{y-3}$$

Be careful of the signs when multiplying. The numerator and denominator look very similar when you multiply by the conjugate.

Example 1.4.5.20

Simplify $\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$.

Solution

	$\dfrac{\sqrt{x}+\sqrt{7}}{\sqrt{x}-\sqrt{7}}$
Multiply the numerator and denominator by the conjugate of the denominator.	$=\dfrac{(\sqrt{x}+\sqrt{7})\,(\sqrt{x}+\sqrt{7})}{(\sqrt{x}-\sqrt{7})\,(\sqrt{x}+\sqrt{7})}$
Multiply the conjugates in the denominator.	$=\dfrac{(\sqrt{x}+\sqrt{7})\,(\sqrt{x}+\sqrt{7})}{(\sqrt{x})^2-(\sqrt{7})^2}$
Simplify the denominator.	$=\dfrac{(\sqrt{x}+\sqrt{7})^2}{x-7}$

We do not square the numerator. Leaving it in factored form, we can see there are no common factors to remove from the numerator and denominator.

Try It 1.4.5.21

Simplify $\dfrac{\sqrt{p}+\sqrt{2}}{\sqrt{p}-\sqrt{2}}$.

Answer

$$\frac{(\sqrt{p}+\sqrt{2})^2}{p-2}$$

Try It 1.4.5.22

Simplify $\dfrac{\sqrt{q}-\sqrt{10}}{\sqrt{q}+\sqrt{10}}$.

Answer

$$\frac{(\sqrt{q}-\sqrt{10})^2}{q-10}$$

Writing Exercises 1.4.5.23

1. Explain why the process of rationalizing the denominator described above works.
2. What is one reason you may want to rationalize the denominator?
3. Why does the ‘conjugate’ play a role in accomplishing this?

Exit Problem 1.4.5.24

Simplify:

a. $\dfrac{3-3\sqrt{3a}}{4\sqrt{8a}}$

b. $\dfrac{\sqrt{5}+3}{4-\sqrt{5}}$

Key Concepts

- **Quotient Property of Radical Expressions**
  - If $\sqrt{a}$ and $\sqrt{b}$ are real numbers, $b \neq 0$, and then, $\sqrt{\dfrac{a}{b}} = \dfrac{\sqrt{a}}{\sqrt{b}}$.
- **Simplified Radical Expressions**
  - A radical expression is considered simplified if there are:
    - no factors in the radicand that have perfect powers of the index
    - no fractions in the radicand
    - no radicals in the denominator of a fraction

1.4.6: Complex Numbers

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the square root of a negative number
- Add and subtract complex numbers
- Multiply complex numbers
- Divide complex numbers
- Simplify powers of i

Be Prepared

Before you get started, take this readiness quiz.

1. Given the numbers $-4, -\sqrt{7}, 0.\overline{5}, \frac{7}{3}, 3, \sqrt{81}$, list the
   a. rational numbers
   b. irrational numbers
   c. real numbers
2. Multiply: $(x-3)(2x+5)$.
3. Rationalize the denominator: $\frac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$

Evaluate the Square Root of a Negative Number

Whenever we have a situation where we have a square root of a negative number we say there is no real number that equals that square root. For example, to simplify $\sqrt{-1}$, we are looking for a real number x so that $x^2=-1$. Since all real numbers squared are positive numbers, there is no real number that equals – 1 when squared.

Mathematicians have often expanded their numbers systems as needed. They added 0 to the counting numbers to get the whole numbers. When they needed negative balances, they added negative numbers to get the integers. When they needed the idea of parts of a whole they added fractions and got the rational numbers. Adding the irrational numbers allowed numbers like $\sqrt{5}$. All of these together gave us the real numbers and so far in your study of mathematics, that has been sufficient.

But now we will expand the real numbers to include the square roots of negative numbers. We start by defining the **imaginary unit** i as a number whose square is – 1.

Definition1.4.6.1

The **imaginary unit** i is a number whose square is -1, i.e., so that $i^2=-1$.

The treatment of radical expressions is almost identical to the treatment of numbers involving i since it is just a replacement of $a^2=b$ with $i^2=-1$ so where there was a replacement of $\sqrt{b}^2$ with b, here we will replace i^2 with -1. Look for the similarities as you move through this section!

We will use the imaginary unit to simplify the square roots of negative numbers.

Square Root of a Negative Number

If b is a positive real number, then

$$\sqrt{-b}=\sqrt{b}i$$

We will use this definition in the next example. Be careful that it is clear that the i is not under the radical. Sometimes you will see this written as $\sqrt{-b}=i\sqrt{b}$ to emphasize the i is not under the radical. But the $\sqrt{-b}=\sqrt{b}i$ is considered standard form.

Example 1.4.6.2

Write each expression in terms of i and simplify is possible:

a. $\sqrt{-25}$

b. $\sqrt{-7}$

c. $\sqrt{-12}$

Solution

a.

	$\sqrt{-25}$
Use the definition of the square root of negative numbers. Be careful that it is clear that i is not under the radical sign.	$=\sqrt{25}i$
Simplify.	$=5i$

b.

	$\sqrt{-7}$
Use the definition of the square root of negative numbers. Be careful that it is clear that i is not under the radical sign.	$=\sqrt{7}i$

c.

	$\sqrt{-12}$
Use the definition of the square root of negative numbers. Be careful that it is clear that i is not under the radical sign.	$=\sqrt{12}i$
Simplify $\sqrt{12}$.	$=2\sqrt{3}i$

Try It 1.4.6.3

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-81}$

b. $\sqrt{-5}$

c. $\sqrt{-18}$

Answer

a. $9i$

b. $\sqrt{5}i$

c. $3\sqrt{2}i$

Try It 1.4.6.4

Write each expression in terms of i and simplify if possible:

a. $\sqrt{-36}$

b. $\sqrt{-3}$

c. $\sqrt{-27}$

Answer

a. $6i$

b. $\sqrt{3}i$

c. $3\sqrt{3}i$

Now that we are familiar with the imaginary number i, we can expand the real numbers to include imaginary numbers. The **complex number system** includes the real numbers and the imaginary numbers. A **complex number** is of the form $a+bi$, where a, b are real numbers. We call a the real part and b the imaginary part.

Definition 1.4.6.5

A **complex number** is of the form $a+bi$, where a and b are real numbers.

The number a is called the **real part** of $a+bi$.

The number (\b\) is called the **imaginary part** of $a+bi$.

A complex number is in standard form when written as $a+bi$, where a and b are real numbers.

If $b=0$, then $a+bi$ becomes $a+0\cdot i=a$, and is a real number.

If $b\neq 0$, then $a+bi$ is an imaginary number.

If $a=0$, then $a+bi$ becomes $0+bi=bi$, and is called a pure imaginary number.

We summarize this here.

Table 8.8.1

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0\mathrm{R}$	$0+bi$ bi	Pure imaginary numbe4

The standard form of a complex number is $a+bi$, so this explains why the preferred form is $\sqrt{-b}=\sqrt{b}i$ when $b>0$.

The diagram helps us visualize the complex number system. It is made up of both the real numbers and the imaginary numbers.

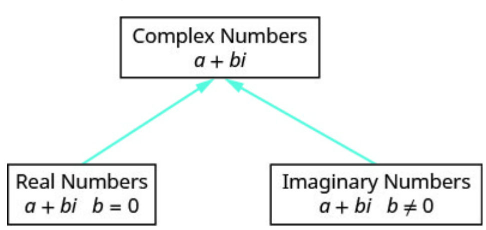


Figure 8.8.2

Add or Subtract Complex Numbers

We are now ready to perform the operations of addition, subtraction, multiplication and division on the complex numbers—just as we did with the real numbers.

Adding and subtracting complex numbers is much like adding or subtracting like terms. We add or subtract the real parts and then add or subtract the imaginary parts. Our final result should be in standard form.

Example 1.4.6.6

Add $\sqrt{-12}+\sqrt{-27}$.

Solution

	$\sqrt{-12}+\sqrt{-27}$
Use the definition of the square root of negative numbers.	$=\sqrt{12}i+\sqrt{27}i$
Simplify the square roots.	$=2\sqrt{3}i+3\sqrt{3}i$
Add.	$=5\sqrt{3}i$

Try It 1.4.6.7

Add $\sqrt{-8}+\sqrt{-32}$.

Answer

$6\sqrt{2}i$

Try It 1.4.6.8

Add $\sqrt{-27}+\sqrt{-48}$

Answer

$7\sqrt{3}i$

Remember to add both the real parts and the imaginary parts in this next example.

Example 1.4.6.9

Simplify:

a. $(4-3i)+(5+6i)$

b. $(2-5i)-(5-2i)$

Solution:

a.

	$(4-3i)+(5+6i)$
Use the Associative Property to put the real parts and the imaginary parts together.	$=(4+5)+(-3+6)i$
Simplify.	$=9+3i$

b.

	$(2-5i)-(5-2i)$

	$(2-5i)-(5-2i)$
Distribute.	$=2-5i)+(-5+2i)$
Use the Associative Property to put the real parts and the imaginary parts together.	$=(2-5)+(-5+2)i$
Simplify.	$=-3-3i$

Try It 1.4.6.10

Simplify:

a. $(2+7i)+(4-2i)$

b. $(8-4i)-(2-i)$

Answer

a. $6+5i$

b. $6-3i$

Try It 1.4.6.11

Simplify:

a. $(3-2i)+(-5-4i)$

b. $(4+3i)-(2-6i)$

Answer

a. $-2-6i$

b. $2+9i$

Multiply Complex Numbers

Multiplying complex numbers is also much like multiplying expressions with coefficients and variables. There is only one special case we need to consider. We will look at that after we practice in the next two examples.

Example 1.4.6.12

Multiply $2i(7-5i)$.

Solution:

	$2i(7-5i)$
Distribute.	$=14i-10i^2)$
Simplify i^2.	$=14i-10(-1)$ $=14i+10$
Write in standard form.	$=10+14i$

Try It 1.4.6.13

Multiply $4i(5-3i)$.

Answer

$12+20i$

Try It 1.4.6.14

Multiply $-3i(2+4i)$.

Answer

$12-6i$

In the next example, we multiply the binomials using the **Distributive Property** or **FOIL**.

Example 1.4.6.15

Multiply $(3+2i)(4-3i)$.

Solution

	$(3+2i)(4-3i)$
Use FOIL.	$=12-9i+8i-6i^2$
Simplify i^2 and combine like terms.	$=12-i-6(-1)$
Multiply.	$=12-i+6$
Combine the real parts.	$=18-i$

Try It 1.4.6.16

Multiply $(5-3i)(-1-2i)$.

Answer

$-11-7i$

Try It 1.4.6.17

Multiply $(-4-3i)(2+i)$.

Answer

$-5-10i$

In the next example, we could use FOIL or the **Product of Binomial Squares Pattern**.

Example 1.4.6.18

Multiply $(3+2i)^2$.

Solution:

	$\underbrace{(3+2i)^2}_{(a+b)^2}$
Use the Product of Binomial Squares Pattern, $(a+b)^2=a^2+2ab+b^2$.	$=\underbrace{3^2+2\cdot 3\cdot 2i+(2i)^2}_{a^2+2ab+b^2}$
Simplify.	$=9+12i+4i^2$

$\underbrace{(3+2i)^2}_{(a+b)^2}$	
Simplify i^2.	$=9+12i+4(-1)$
Simplify.	$=9+12i-4$
Write in standard form.	$=5+12i$

? Try It 1.4.6.19

Multiply using the Binomial Squares pattern: $(-2-5i)^2$.

Answer

$-21-20i$

? Try It 1.4.6.20

Multiply using the Binomial Squares pattern: $(-5+4i)^2$.

Answer

$9-40i$

Since the square root of a negative number is not a real number, we cannot use the Product Property for Radicals. In order to multiply square roots of negative numbers we should first write them as complex numbers, using $\sqrt{-b}=\sqrt{b}i$.This is one place students tend to make errors, so be careful when you see multiplying with a negative square root.

✓ Example 1.4.6.21

Multiply $\sqrt{-36}\cdot\sqrt{-4}$.

Solution:

$\sqrt{-36}\cdot\sqrt{-4}$	
To multiply square roots of negative numbers, we first write them as complex numbers. Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.	$=\sqrt{-36}\cdot\sqrt{-4}$
Simplify.	$=6i\cdot 2i$
Multiply.	$=12i^2$
Simplify i^2 and multiply.	$=-12$

? Try It 1.4.6.22

Multiply $\sqrt{-49}\cdot\sqrt{-4}$.

Answer

-14

Try It 1.4.6.23

Multiply $\sqrt{-36} \cdot \sqrt{-81}$.

Answer

-54

In the next example, each binomial has a square root of a negative number. Before multiplying, each square root of a negative number must be written as a complex number.

Example 1.4.6.24

Multiply $(3-\sqrt{-12})(5+\sqrt{-27})$.

Solution:

	$(3-\sqrt{-12})(5+\sqrt{-27})$
To multiply square roots of negative numbers, we first write them as complex numbers. Write as complex numbers using $\sqrt{-b}=\sqrt{b}i$.	$=(3-\sqrt{12}i)(5+\sqrt{27}i)$
Use FOIL.	$=15+3\sqrt{27}i-5\sqrt{12}i-\sqrt{12}\sqrt{27}i^2$
Simplify the radical terms.	$=15+3\cdot 3\sqrt{3}i-5\cdot 2\sqrt{3}i-2\sqrt{3}\cdot 3\sqrt{3}i^2$ $=15+9\sqrt{3}i-10\sqrt{3}i-18i^2$
Combine like terms and simplify i^2.	$=15-\sqrt{3}i-18(-1)$
Multiply and combine like terms.	$=15-\sqrt{3}i+18$ $=33-\sqrt{3}i$

Try It 1.4.6.25

Multiply $(4-\sqrt{-12})(3-\sqrt{-48})$.

Answer

$-12-22\sqrt{3}i$

Try It 1.4.6.26

Multiply $(-2+\sqrt{-8})(3-\sqrt{-18})$.

Answer

$6+12\sqrt{2}i$

We first looked at conjugate pairs when we studied polynomials. We said that a pair of binomials that each have the same first term and the same last term, but one is a sum and one is a difference is called a *conjugate pair* and is of the form $(a-b),(a+b)$.

A **complex conjugate pair** is very similar. For a complex number of the form $a+bi$, its conjugate is $a-bi$. Notice they have the same first term and the same last term, but one is a sum and one is a difference.

Definition 1.4.6.27

A **complex conjugate pair** is of the form $a+bi, a+bi$.

We will multiply a complex conjugate pair in the next example.

Example 1.4.6.28

Multiply $(3-2i)(3+2i)$.

Solution:

	$(3-2i)(3+2i)$
Use FOIL.	$=9+6i-6i-4i^2$
Combine like terms and simplify i^2.	$=9-4(-1)$
Multiply and combine like terms.	$=-13$

Try It 1.4.6.29

Multiply $(4-3i)\cdot(4+3i)$.

Answer

25

Try It 1.4.6.30

Multiply $(-2+5i)\cdot(-2-5i)$.

Answer

29

From our study of polynomials, we know the product of conjugates is always of the form $(a-b)(a+b)=a^2-b^2$ .The result is called a **difference of squares**. We can multiply a complex conjugate pair using this pattern.

The last example we used FOIL. Now we will use the **Product of Conjugates Pattern**.

$$\overset{a\ -\ b}{(3-2i)}\overset{a\ +\ b}{(3+2i)}$$

$$\overset{a^2\ -\ b^2}{(3)^2-(2i)^2}$$

$$9-4i^2$$

$$9-4(-1)$$

$$13$$

Figure 8.8.8

Notice this is the same result we found in Example 8.8.9.

When we multiply complex conjugates, the product of the last terms will always have an i^2 which simplifies to -1.

$$(a-bi)(a+bi)$$
$$a^2-(bi)^2$$
$$a^2-b^2i^2$$
$$a^2-b^2(-1)$$
$$a^2+b^2$$

This leads us to the Product of Complex Conjugates Pattern: $(a-bi)(a+bi)=a^2+b^2$

Product of Complex Conjugates

If a and b are real numbers, then

$$\backslash(a-bi)(+bi)=a^2=b^2. \tag{1.4.6.1}$$

Example 1.4.6.31

Multiply using the Product of Complex Conjugates Pattern: $(8-2i)(8+2i)$.

Solution

Table 8.8.3

	$\underbrace{(8-2i)(8+2i)}_{(a=bi)(a+bi)}$
Use the Product of Complex Conjugates Pattern, $(a-bi)(a+bi)=a^2+b^2$.	$=\underbrace{8^2+2^2}_{a^2+b^2}$
Simplify the squares.	$=64+4$
Add.	$=68$

Try It 1.4.6.32

Multiply using the Product of Complex Conjugates Pattern: $(3-10i)(3+10i)$.

Answer

109

Try It 1.4.6.33

Multiply using the Product of Complex Conjugates Pattern: $(-5+4i)(-5-4i)$.

Answer

41

Divide Complex Numbers

Dividing complex numbers is much like rationalizing a denominator. We want our result to be in standard form with no imaginary numbers in the denominator.

Example 1.4.6.34

Divide $\frac{4+3i}{3-4i}$.

Solution

Table 8.8.4

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 2: Multiply the numerator and denominator by the complex conjugate of the denominator.	The complex conjugate of $3-4i$ is $3+4i$.	$\frac{(4+3i)(3+4i)}{(3-4i)(3+4i)}$

Step 1: Write both the numerator and denominator in standard form.	They are both in standard form.	$\frac{4+3i}{3-4i}$
Step 3: Simplify and write the result in standard form.	Use the pattern $(a-bi)(a+bi)=a^2+b^2$ in the denominator. Combine like terms. Simplify. Write the result in standard form.	$\frac{12+16i+9i+12i^2}{9+16}$ $\frac{12+25i-12}{25}$ $\frac{25i}{25}$ i

b...

Try It 1.4.6.35

Divide $\frac{2+5i}{5-2i}$.

Answer

i

Try It 1.4.6.36

Divide $\frac{1+6i}{6-i}$.

Answer

i

We summarize the steps here.

How to Divide Complex Numbers

1. Write both the numerator and denominator in standard form.
2. Multiply both the numerator and denominator by the complex conjugate of the denominator.
3. Simplify and write the result in standard form.

Example 1.4.6.37

Divide, writing the answers in standard form: $\frac{-3}{5+2i}$.

Solution

	$\frac{-3}{5+2i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$=\frac{-3(5-2i)}{(5+2i)(5-2i)}$
Multiply in the numerator and use the Product of Complex Conjugates Pattern in the denominator.	$=\frac{-15+6i}{5^2+2^2}$
Simplify.	$=\frac{-15+6i}{29}$
Write in standard form.	$=-\frac{15}{29}+\frac{6}{29}i$

? Ty It 1.4.6.38

Divide, writing the answer in standard form: $\frac{4}{1-4i}$.

Answer

$\frac{4}{17}+\frac{16}{17}i$

? Try It 1.4.6.39

Divide, writing the answer in standard form: $\frac{-2}{-1+2i}$.

Answer

$\frac{2}{5}+\frac{4}{5}i$

Be careful as you find the conjugate of the denominator.

✓ Example 1.4.6.40

Divide $\frac{5+3i}{4i}$.

Solution

	$\frac{5+3i}{4i}$
Write the denominator in standard form.	$=\frac{5+3i}{0+4i}$
Multiply the numerator and denominator by the complex conjugate of the denominator.	$=\frac{(5+3i)(0-4i)}{(0+4i)(0-4i)}$ $=\frac{-20i-12i^2}{16}$
Simplify the i^2.	$=\frac{-20i-12(-1)}{-16(-1)}$
Multiply.	$=\frac{-20i+12}{16}$
Rewrite in standard form.	$=\frac{12}{16}-\frac{20}{16}i$
Simplify the fractions.	$=\frac{3}{4}-\frac{5}{4}i$

? Try It 1.4.6.41

Divide $\frac{3+3i}{2i}$.

Answer

$\frac{3}{2}-\frac{3}{2}i$

Try It 1.4.6.42

Divide $\dfrac{2+4i}{5i}$.

Answer

$\dfrac{4}{5}-\dfrac{2}{5}i$

Simplify Powers of i (Optional)

The powers of i make an interesting pattern that will help us simplify higher powers of i. Let's evaluate the powers of i to see the pattern.

$$\begin{array}{cccc} i^1 & i^2 & i^3 & i^4 \\ i & -1 & i^2\cdot i & i^2\cdot i^2 \\ & & -1\cdot i & (-1)(-1) \\ & & -i & 1 \end{array}$$

$$\begin{array}{cccc} i^5 & i^6 & i^7 & i^8 \\ i^4\cdot i & i^4\cdot i^2 & i^4\cdot i^3 & i^4\cdot i^4 \\ 1\cdot i & 1\cdot i^2 & 1\cdot i^3 & 1\cdot 1 \\ i & i^2 & i^3 & 1 \\ & -1 & -i & \end{array}$$

We summarize this now.

$$\begin{array}{ll} i^1=i & i^5=i \\ i^2=-1 & i^6=-1 \\ i^3=-i & i^7=-i \\ i^4=1 & i^8=1 \end{array}$$

If we continued, the pattern would keep repeating in blocks of four. We can use this pattern to help us simplify powers of i. Since $i^4=1$, we rewrite each power, i^n, as a product using i^4 to a power and another power of i.

We rewrite it in the form $i^n=\left(i^4\right)^q\cdot i^r$, where the exponent, q, is the quotient of n divided by 4 and the exponent, r, is the remainder from this division. For example, to simplify i^{57}, we divide 57 by 4 and we get 14 with a remainder of 1. In other words, $57=4\cdot 14+1$. So we write $i^{57}=\left(1^4\right)^{14}\cdot i^1$ and then simplify from there.

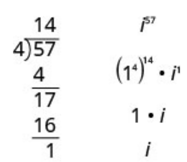


Figure 8.8.13

Example 1.4.6.43

Simplify i^{86}.

Solution

	i^{86}
Divide 86 by 4 and rewrite i^{86} in the $i^n=\left(i^4\right)^q\cdot i^r$ form.	$\left(1^4\right)^{21}\cdot i^2$
Simplify.	$(1)^{21}\cdot(-1)$
Simplify.	-1

? Try It 1.4.6.44

Simplify: i^{75}.

Answer

-1

? Try It 1.4.6.45

Simplify i^{92}.

Answer

1

? Writing Exercises 1.4.6.46

1. What is an imaginary number?
2. What does it mean for a complex number to be written in standard form?
3. What is the complex conjugate?
4. How is the arithmetic with complex numbers similar to arithmetic with square roots?
5. Why is there a similarity?
6. Mulitply $(2+3\sqrt{7})(-5-2\sqrt{7})$ and multiply $(2+3i)(-5-2i)$. What are the similarities and differences in your computation?
7. Add $(2+3\sqrt{7})+(-5-2\sqrt{7})$ and add $(2+3i)+(-5-2i)$. What are the similarities and differences in your computation?
8. Divide $\frac{2+3\sqrt{7}}{-5-2\sqrt{7}}$ and divide $\frac{2+3i}{-5-2i}$. What are the similarities and differences in your computation?

Exit Problem 1.4.6.47

Simplify:

a. $\dfrac{3+2i}{4+6i}$

b. i^{83}

Key Concepts

- **Square Root of a Negative Number**
  - If b is a positive real number, then \(\sqrt{-b}=\sqrt{b} i\

	$a+bi$	
$b=0$	$a+0\cdot i$ a	Real number
$b\neq 0$	$a+bi$	Imaginary number
$a=0$	$0+bi$ bi	Pure imaginary number

- - A complex number is in **standard form** when written as $a + bi$, where a, b are real numbers.

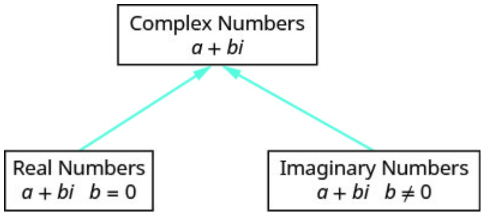


Figure 8.8.2

- **Product of Complex Conjugates**
  - If a, b are real numbers, then
    $(a - bi)(a + bi) = a^2 + b^2$
- **How to Divide Complex Numbers**
  1. Write both the numerator and denominator in standard form.
  2. Multiply the numerator and denominator by the complex conjugate of the denominator.
  3. Simplify and write the result in standard form.

complex conjugate pair

A complex conjugate pair is of the form $a + bi, a - bi$.

complex number

A complex number is of the form $a + bi$, where a and b are real numbers. We call a the real part and b the imaginary part.

complex number system

The complex number system is made up of both the real numbers and the imaginary numbers.

imaginary unit

The imaginary unit i is the number whose square is -1. $i^2 = -1$ or $i = \sqrt{-1}$.

standard form

A complex number is in standard form when written as $a + bi$, where a, b are real numbers.

CHAPTER OVERVIEW

2: Equations with One Variable

2.1: Linear Equations

Learning Objectives

By the end of this section, you will be able to:

- Understand what it means to solve a linear equation.
- Solve linear equations with one unknown, i.e., one-variable.

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $3(2+x)$.
2. Simplify $\frac{1}{3}+\frac{9}{4}\cdot\frac{5}{7}$.
3. Is the expression $9x^2-12x+4$ linear?

Introduction to Equations

In this unit we will use these expressions in applications!In the previous unit we used expressions to represent quantities when it involved a quantity that is either unknown or could be any of a number of values. For example, we may want to represent the total amount of money in a bin that had 5 dollars and a certain number of quarters. How could we do this? We first did some examples where we chose a specific number of quarters:

If we had 3 quarters then we have $\$5.75$. If the bin contains 7 quarters then the bin has $\$5+1.75=\$6.75$. Using this as an example, we can try to discover what we do if we don't know how many quarters we are dealing with. We let the variable q represent the number of quarters. Then we can translate the sum made by having q quarters: $\frac{q}{4}$. We can test this on our 7 quarters: $\frac{7}{4}=1.75$. To then find the total in the bin we add the 5 dollars. So the total amount in the bin is $5+q/4$.

We can test this on 3 quarters:

$$5+\frac{3}{4}=5.75 \tag{2.1.1}$$

and on 7 quarters:

$$5+\frac{7}{4}=5+1.75=6.75. \tag{2.1.2}$$

But in this unit, we recognize this expression represents the amount of money in the bin no matter what q is and we use this fact to solve problems. For example, suppose we know that the money in the bin is equal to $\$8.25$ and we want to know how many quarters there are. Then we represent the amount of money in the bin in two ways which then must be equal (since they represent the same quantity):

$$8.25=5+\frac{q}{4}.$$

This is an assertion about q. Sometimes we have used equal signs during a simplification of an expression. Such equal signs implicitly mean that equation holds for all values of the variables except those that leave an expression undefined. In the case of this equation and others in this unit, these equalities say something about the variable, and for 'most' values of the variable the equation is false. We will seek the values which make the equality true, .i.e., we seek to solve these equations.

We can imagine ourselves looking at this bin and putting aside the $\$5$. We then know that what is left is both 3.25 and $\frac{q}{4}$, so that the contribution of the quarters is $\$3.25$ which we can figure out using our experience with counting money that there are 13 quarters.

The expression $5+\frac{q}{4}$ has two basic interpretations: If there are several such bins then we could use the expression $5+\frac{q}{4}$ and substitute the number of quarters, one at a time, in for q to find the total amount of money in each bin; or maybe, for example we have only one bin but someone will tell us later how many quarters are in the bin and we will use this expression, substituting what they tell us for q, to find out how much money is in that bin. The latter interpretation is what was used above. We know there were a certain but currently unknown number of quarters in the bin. Combining this with the information that the total amount of money in the bin was $\$8.25$ we learned what q must be.

The above is just a simple example to illustrate the basic notions we will be using in this section. We will do many examples with the expressions we used in the last unit. Before continuing, make sure you understand this most simple example.

Linear Equations and Their Solutions

Just as linear expressions are the simplest expressions involving only multiplication and addition of numbers, here we consider equations involving these expressions.

We begin by returning to the example in the introduction and considering the 'solution.'

You may be familiar with how to solve this equation, but the basics of solving equations lie in this one example.

Recall that in the example we had a bin with 5 dollars and a certain number of quarters. We arrived at an expression equal to the amount of money in the bin once the number of quarters (which we will call 'q') is known and substituted. The expression is

$$5+\frac{q}{4}.$$

The question raised in the introduction was to suppose we know something about the amount of money in the bin, namely, that it contains $\$8.25$ and we want to find the number of quarters. We wrote the amount of money in two ways (which then must be equal!). On the one hand we know the bin contains $\$8.25$ and on the other hand we know that if it has q quarters, it has $5+\frac{q}{4}.$ dollars. So,

$$8.25=5+\frac{q}{4}.$$

We figured out how many quarters there are (13) using our heads and experience. Here, we would like to proceed differently so that it sets the stage for more complicated problems.

The equality $8.25=5+\frac{q}{4}.$ means that if we substitute the number of quarters in for the variable q, we have a true statement. We already saw that there are 13 quarters and we see that

$$8.25=5+\frac{q}{4}.$$

	$5+q/4$
	$=5+13/4$
	$=5+3.25$
	$=8.25$
So,	$8.25=5+13/4,$ or $8.25=5+q/4,$ when $q=13.$

We say that $8.25=5+\frac{q}{4}$ is an equation with one variable and $q=13$ is a solution. This equation is in fact a linear equation since the expressions on both sides are linear expressions.

This is a verification of the solution. If we were unable to figure this out by using our experience, how could we find the solution?

Notice that we are interested in finding q. If we knew what q was we could evaluate q by first dividing by 4 and then adding the result to 5. Here we must use the order of operations. So to undo these actions, we must undo them in the opposite order. And to maintain the truth of the equation, we must do the same thing to both sides.

	$8.25=5+\frac{q}{4}.$

The last thing we do to evaluate the right side is add 5. This is the first thing we undo! How do we undo addition? Subtraction! So let's subtract 5 on both sides of the equation.	$8.25-5=5+\frac{q}{4}-5$, or, simplifying, $3.25=\frac{q}{4}$.
Next, q is being divided by 4 and we must undo this division to 'get at' q. So, what is the opposite of the action of division? Multiplication! So, lets multiply both sides (to maintain the truth of the equality) of the equation by 4.	$3.25\cdot 4=\frac{q}{4}\cdot 4$ so, $(3+1/4)\cdot 4=\frac{q}{4}\cdot\frac{4}{1}$ and so $12+1=\frac{4q}{4}$ or, $q=13$.

Note that if you do the arithmetic correctly you should see the operation you are undoing disappearing as you wished!

We have had extensive practice with simplifications in the first unit, so from now on we will just indicate when we have simplified.

We introduce some general vocabulary:

Definition

An **equation** is a mathematical sentence of the form (an expression)= (another expression).

A **solution** to an equation in a variable (lets call it x) is a number that we can substitute in for x that makes the equation true.

Two equations are **equivalent** if they have the same solutions.

Finding the solutions of an equation is called **solving** the equation.

Definition

A **linear equation with one variable** x is an equation that is equivalent to an equation $Ax+B=0$, where $A\neq 0$.

To **solve** a linear equation with one variable means to find the number that when substituted makes the equation true. If a is a solution to the equation with the variable x, then we may also say $x=a$, is a solution (it is a simplification of the original assertion).

Example 2.1.4

Why is the equation $5=3x-2$ linear? Solve the equation.

Solution

The equation is linear because both sides can be written in the form $ax+b$ and so are linear expressions. We note that for most values of x the equation is not satisfied.

Is $x=1$ a solution?	Does $5=3(1)-2$? That is, does $5=1$? No. So $x=1$ is not a solution.
Is $x=-2$ a solution?	Does $5=3(-2)-2$? That is, does $5=-8$? No. So $x=1$ is not a solution.

What is the solution then? Or maybe there is more than one?

The assertion about x is that if we multiply by 3 and then subtract 2 from the result, we should get 5. We need to undo this in the reverse order. And, to preserve the equality, whatever we do to one side of the equal sign we must do to the other. We need to write an equivalent equation of the form $x=a$, that is, to see to it that x is on one side of an equation and a number is on the other side.

	$5=3x-2$
The opposite of subtracting 2 is adding 2. So we now add 2 to both sides.	$5+2=3x-2+2$, or $7=3x$
The x is being multiplied by 3 and to undo multiplication, we must divide!	$7/3=3x/3$or $x=7/3$
We may check to see that our conclusion about x is true.	$3(7/3)-2=7-2=5$ So the equation is indeed true!

We see that by undoing the actions on x, we were led to the conclusion that x must be 7/3. So there is only one solution!

? Try It 2.1.5

Why is the equation $2x+5=3$ linear? Solve the equation.

Answer

Answers vary. $x=-1$

? Try It 2.1.6

Why is the equation $2m+5=-m+3$ linear? Solve the equation. Note that you can do anything to both sides of the equation to get your desired outcome!

Answer

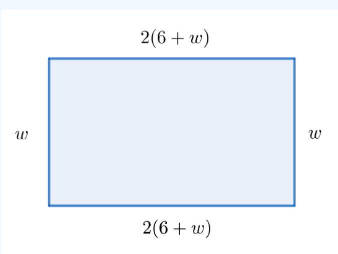
Answers vary. $m=-2/3$

✓ Example 2.1.7

The perimeter of an enclosed rectangle is 60 feet. The length is twice six more than the width. What are the dimensions of the enclosure?

Solution

Let's start with some example to understand the relation between the width and the length. If the width is 4 feet, for example, and the length is twice (six more than the width), then the length is 2 times 10 feet or, 20 feet. Since this leads to a perimeter of 20+4+20+4=48 feet (which isn't the correct perimeter, but has the correct width/length relation). We see here also that our answer should have a width larger than 4 feet since the perimeter of 48 feet is less than the required 60 feet. If the width is 10 feet, then the length is twice (six more than the width), or twice 16 feet, or, 32 feet. The perimeter in this example will be greater than 64 so the width here is too long. But we can use our examples to lead us to write down an expression for the length if w stands for the unknown width. The length is twice (six more than the width), or twice (6+w). Now we can't actually add since we don't know what w is, but we can still multiply by 2 to get that the length is $2(6+w)$. We indicate this on a picture. This picture is only to help us with relationships and will very likely not have the final length or width correct!



So, we know then that the perimeter for the width w we are looking for is

$w+2(6+w)+w+2(6+w)=2w+4(6+w)$.

We know that the perimeter is also 60 feet, so 60 must be equal to the expression above for the w we seek:

$60 = 2w + 4(6 + w)$.

We can use our skills of simplifying to rewrite the right side to get the equivalent equation

	$60 = 2w + 4(6 + w)$
	$60 = 2w + 24 + 4w$
	$60 = 6w + 24$

Notice that the right hand side says to multiply w by 6 and then add 24. To undo these actions we must undo them in the opposite order!

Now, the two sides of this equation are equal for the w we are looking for, so the equality is preserved if we subtract 24 from both sides.

This gives us:

$60 - 24 = 6w + 24 - 24$, or, simplifying both sides, $36 = 6w$.

We can then divide both sides by 6 and since 36 is the same as $6w$ for the w we are looking for, dividing both sides by 6 will result in an equality:

$36/6 = 6w/6$, or, $w = 6$ feet.

We recall that the length is $2(6 + w) = 2(6 + 6) = 24$ feet.

So the rectangular enclosure has a width of 6 feet and a length of 24 feet.

(We can check that the length is twice six more than the width and that the perimeter is 60 feet).

Try It 2.1.8

The perimeter of an enclosed isosceles triangle is 40 feet. The length of the shortest side is 7 less than half of the two longest (equal) sides. What are the length of the sides of the enclosure? Give some examples of any expression you develop and write your solution in detail.

Answer

Answers vary. $94/5$ ft, $94/5$ ft, $12/5$ ft.

Writing Exercises 2.1.9

1. How do you recognize a linear equation?
2. Give an example of a linear equation.
3. Do all linear equations have a solution?
4. Explain the relationship between the order of operations and the process of solving $3x - 4 = 7$.

Exit Problem

A rectangular garden has a length the is 10 feet longer than its width. The perimeter is 42 feet. What are the dimensions of the garden?

Key Concepts

Linear equation

Solution to a linear equation

Solving a linear equation

SECTION OVERVIEW

2.2: Quadratic Equations

Thumbnail: Plot of the quadratic function. (Public Domain; N.Mori).

2.2.1: Solving Quadratic Equations Using the Zero-Product Property

Learning Objectives

By the end of this section, you will be able to:

- Understand what it means to solve a quadratic equation.
- Solve quadratic equations with one unknown, i.e., one-variable by using the zero-product property.

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $3(2+x)(3x-2)$.
2. Solve: $5y-3=0$.
3. Factor completely: n^3-9n^2-22n.
4. Simplify $1/3+9/4 \cdot 5/7(3-2/5)$.
5. Why is the following expression quadratic? $9x^2-12x+4$.

Quadratic Equations

Just like a linear equation in one variable is an equation that is equivalent to (a linear expression$=0$), a quadratic equation is an equation in one variable, is an equation that is equivalent to (a $\text{quadratic expression}\;=0$), and the equation isn't equivalent to a linear equation. For example, suppose that the area of a rectangular region with the length 3 feet more than twice its width (in feet) is equal to a square region of the twice the width of the other. We don't know what the width is (let's call it w) but we can write down this assertion about the width.

The area of a square region with width w is	$(2w)^2$ or, $4x^2$
If the length of the rectangular region is 3 more than twice its width, then its length is	$2w+3$
So the area of the rectangular region is length times width, which in terms of the unknown w is	$(2w+3)w$, or $2w^2+3w$
The assertion that these two areas are the same can be written	$4w^2=2w^2+3w$.

Now, this looks like a quadratic equation, at least on the surface, but we need to make sure it isn't a linear equation in disguise so we subtract $4w^2$ from both sides of the equation to get

$0=-2w^2+3w$.

We will not show this in detail, but an equation of the form 0 =(something quadratic) is not a linear equation.

This is the assertion about the width in the problem! Whatever w is, it satisfies the equation

$0=-2w^2+3w$.

For example, $w \neq 1$ since if it were, $-2w^2+3w=-2(1)^2+3(1)=1$ which is not 0 as asserted.

Our question is then, how do you find the width (so that if you substitute that width in for w the assertion is true!

In general we will be interested in finding the solutions to quadratic equations.

Definition

A **solution** to an equation in a variable (lets call it x) is a number that we can substitute in for x that makes the equation true.

Two equations are **equivalent** if they have the same solutions.

A **quadratic equation with one variable**, say, x, has polynomials on both sides of the equation and can be written in **standard form**: $Ax^2+Bx+C=0$ where A is not zero, i.e., there is an equivalent equation of the form $Ax^2 + Bx + C = 0$.

It is important to understand the difference between expressions and equations. We give here examples to demonstrate the two concepts.

Expressions vs Equations

A quadratic expression and a quadratic equation are not the same types of objects. Here are some examples.

Expressions	Equation
$2x+1, 0$	$2x+1=0$ (linear equation)
$x^2, x-2$	$x^2=x-2$ (quadratic equation)
$3x^2+5x-1, 0$	$3x^2+5x-1=0$ (quadratic equation)

Note that $x^2+5x=x(x+5)$ is not a polynomial equation that we aim to solve. This is an identity of polynomials that was developed in the process of factoring out the GCF.

It does not make sense to "solve a polynomial." We can only solve equations. For example, we cannot solve $2x+1$ as there is no statement to assess. Whereas $2x+1=0$ is either true or false for a particular value of x. Below we check whether or not some values of x are solutions to the equation $2x+1=0$.

x	True or False	Solution or Not a Solution
5	$2\cdot 5+1=0$ is false	Not a Solution
0	$2\cdot 0+1=0$ is false	Not a Solution
$-\frac{1}{2}$	$2\cdot\left(-\frac{1}{2}\right)+1=0$ is true	Solution

To solve quadratic equations we need methods different from the ones we used in solving linear equations. We will look at one method here and then several others in a later chapter.

Example 2.2.1.1

Is $x^3-2x+1=x^3+2x^2+x-7$ a quadratic equation? If so, is $x=2$ a solution?

Solution

The left and the right sides of the equality are not quadratic, so the first impulse may be to say that this is not a quadratic equation. But let's be patient and see if we can put it in the required form:

	$x^3-2x+1=x^3+2x^2+x-7$
We do not change the solutions to the equation if we subtract the same quantity on both sides of the equality. Let's subtract x^3-2x+1, which is the same as adding $-x^3+2x-1$.	$x^3-2x+1-x^3+2x-1=x^3+2x^2+x-7-x^3+2x-1$
And upon simplifying (which also doesn't change the solutions) we see	$0=2x^2+3x-8$, or, $2x^2+3x-8=0$
This is in the correct form $Ax^2+Bx+C=0$,	$A=2, B=3$, and $C=-8$.

We see that A is not zero, so this is indeed a quadratic equation.

To see whether 2 is a solution, we can substitute it into any of the equivalent equations to check it's truth. We'll check to see if 2 is a solution to $2x^2+3x-8=0$.

The left hand side is $2(2)^2+3(2)-8=6$ which is not equal to the right hand side (0). So 2 is not a solution to the equation!

We could've checked the original equation instead and arrived at the same conclusion.

Finding solutions to a quadratic equation is, in general, a little more difficult than finding solution to a linear equation. But we know that we can sometimes write a quadratic equation as a product of two linear factors (or two linear expressions and a number). As we will see this can be vary handy!!

The zero-product property and solutions to quadratic equations

We know that $a \cdot 0=0$ and $0 \cdot a=0$ no matter what a is. Suppose though that $a \cdot b=0$. What does this tell us about a and b?

	$a \cdot b=0$
Either $a=0$ or it is not. If $a \neq 0$, then we can divide both sides of the equality by a.	so, $\frac{a \cdot b}{a}=\frac{0}{a}$, or, simplifying, $b=0$!
So, either $a=0$ or $b=0$, or both $a=0$ and $b=0$.	

This is called the zero product property:

The zero product property

If $ab=0$ then $a=0$ or $b=0$ or both.

This property will allow us to sometimes solve quadratic equations by reducing the problem to something we already know how to do, namely, solve linear equations.

We have seen that it is sometimes the case that we can write a quadratic expression as a product of two linear expressions. If we know this product is zero, then we can use the zero-product property!

We start with an example,

Example 2.2.1.1

Lets solve $(2x-3) \cdot(3x+5)=0$.

Solution

If the equation is true, then this asserts something about the unknown which we call x. This equation is a product of numbers (that we don't know) and that product is zero. So we can use the zero-product property to conclude that

either $(2x-3)=0$ or $(3x+5)=0$ (or both).

In other words, if $(2x-3) \cdot(3x+5)=0$ is true about x, then either $(2x-3)=0$ is true or $3x+5)=0$ is true (or both).

We have experience solving these linear equation from the last section!

If $(2x-3)=0$ then $2x-3+3=0+3$, or, $2x=3$. x is still being multiplied by 2 so we must divide both sides of the equal sign by 2

and we see

$2x/2=3/2$ or, simplifying, $x=3/2$.

Similarly, if $3x+5)=0$ then $x=-5/3$.

To conclude, if $(2x-3) \cdot(3x+5)=0$ then it must be that either $x=3/2$ or $x=-5/3$.

So, we have solved the equation by finding the values that when substituted in for x yield a true statement, namely, $3/2$ and $-5/3$.

In the context of an application, we may have more information about x that will help us pin down our unknown to one single answer.

Try It 2.2.1.4

Solve $(3m-2)(2m+1)=0$.

Answer

$m=\frac{2}{3}$, $m=-\frac{1}{2}$

Try It 2.2.1.5

Solve $(4p+3)(4p-3)=0$.

Answer

$p=-\frac{3}{4}$, $p=\frac{3}{4}$

Solve Quadratic Equations by Factoring

The Zero Product Property works very nicely to solve quadratic equations. The quadratic equation must be factored, with zero isolated on one side. So be sure to start with the quadratic equation in **standard form**, $ax^2+bx+c=0$. Then factor the expression on the left.

Example 2.2.1.6

Solve $2y^2=13y+45$.

Solution

Step 1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.	Write the equation in standard form.	$2y^2=13y+45$ $2y^2-13y-45=0$
Step 2. Factor the quadratic expression.	Factor $2y^2-13y+45$ $(2y+5)(y-9)$	$(2y+5)(y-9)=0$
Step 3. Use the Zero Product Property.	Set each factor equal to zero. We have two linear equations.	$2y+5=0$ $y-9=0$

Step 4. Solve the linear equations.		$y=-\frac{5}{2}$ $y=9$
Step 5. Check. Substitute each solution separately into the original equation.	Substitute each solution separately into the original equation.	$y=-\frac{5}{2}$ $2y^2=13y+45$ $2\left(-\frac{5}{2}\right)^2 \stackrel{?}{=} 13\left(-\frac{5}{2}\right)+45$ $2\left(\frac{25}{4}\right) \stackrel{?}{=} \left(-\frac{65}{2}\right)+\frac{90}{2}$ $\frac{25}{2}=\frac{25}{2}\checkmark$ $y=9$ $2y^2=13y+45$ $2(9)^2 \stackrel{?}{=} 13(9)+45$ $2(81) \stackrel{?}{=} 117+45$ $162=162\checkmark$

? Try It 2.2.1.7

Solve $3c^2=10c-8$.

Answer

$$c=2,\ c=\frac{4}{3}$$

? Try It 2.2.1.8

Solve $2d^2-5d=3$.

Answer

$$d=3,\ d=-\frac{1}{2}$$

Solve a Quadratic Equation by Factoring

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
2. Factor the quadratic expression.
3. Use the Zero Product Property.
4. Solve the linear equations.
5. Optional: Check answer by substituting each solution separately into the original equation to see if a mistake has been made.

Before we factor, we must make sure the **quadratic equation** is in **standard form**.

Solving quadratic equations by factoring will make use of all the factoring techniques we have learned in this chapter! Do we recognize the special product pattern in the next example?

? Example 2.2.1.9

Solve $169q^2=49$.

Solution

	$169q^2 = 49$
Write the quadratic equation in standard form.	$169q^2 - 49 = 0$
Factor. It is a difference of squares.	$(13q-7)(13q+7) = 0$
Use the Zero Product Property to set each factor to 0.	$13q - 7 = 0 \quad 13q + 7 = 0$
Solve each equation.	$13q = 7 \quad 13q = -7$ $q = \frac{7}{13} \quad q = -\frac{7}{13}$
Check the answers.	The check is left as an exercise.

Try It 2.2.1.10

Solve $25p^2 = 49$.

Answer

$p = \frac{7}{5}, p = -\frac{7}{5}$

Try It 2.2.1.11

Solve $36x^2 = 121$.

Answer

$x = \frac{11}{6}, x = -\frac{11}{6}$

In the next example, the left side of the equation is factored, but the right side is not zero. In order to use the **Zero Product Property**, one side of the equation must be zero. We'll multiply the factors and then write the equation in standard form.

Example 2.2.1.12

Solve $(3x-8)(x-1) = 3x$.

Solution

	$(3x-8)(x-1) = 3x$
Multiply the binomials.	$3x^2 - 11x + 8 = 3x$
Write the quadratic equation in standard form.	$3x^2 - 14x + 8 = 0$
Factor the trinomial.	$(3x-2)(x-4) = 0$
Use the Zero Product Property to set each factor to 0.	$3x - 2 = 0 \quad x - 4 = 0$
Solve each equation.	$3x = 2 \quad x = 4$ $x = \frac{2}{3}$
Check the answers.	The check is left as an exercise.

Try It 2.2.1.13

Solve $(2m+1)(m+3) = 12m$.

Answer

$m = 1, m = \frac{3}{2}$

Try It 2.2.1.14

Solve $(k+1)(k-1)=8$.

Answer

$k=3, k=-3$

In the next example, when we factor the quadratic equation we will get three factors. However the first factor is a constant. We know that factor cannot equal 0.

Example 2.2.1.15

Solve $3x^2=12x+63$.

Solution

	$3x^2=12x+63$
Write the quadratic equation in standard form.	$3x^2-12x-63=0$
Factor the greatest common factor first.	$3(x^2-4x-21)=0$
Factor the trinomial.	$3(x-7)(x+3)=0$
Use the Zero Product Property to set each factor to 0.	$3\neq 0 \quad x-7=0 \quad x+3=0$
Solve each equation.	$3\neq 0 \quad x=7 \quad x=-3$
Check the answers.	The check is left as an exercise.

Try It 2.2.1.16

Solve $18a^2-30=-33a$.

Answer

$a=-\frac{5}{2}, a=\frac{2}{3}$

Try It 2.2.1.17

Solve $123b=-6-60b^2$.

Answer

$b=-2, b=-\frac{1}{20}$

The **Zero Product Property** also applies to the product of three or more factors. If the product is zero, at least one of the factors must be zero. We can solve some equations of degree greater than two by using the Zero Product Property, just like we solved quadratic equations.

Example 2.2.1.18

Solve $9m^3+100m=60m^2$.

Solution

	$9m^3+100m=60m^2$
Bring all the terms to one side so that the other side is zero.	$9m^3-60m^2+100m=0$
Factor the greatest common factor first.	$m(9m^2-60m+100)=0$
Factor the trinomial.	$m(3m-10)^2=0$
Use the Zero Product Property to set each factor to 0.	
Solve each equation.	$m=0 \quad 3m-10=0$ $m=0 \quad m=\frac{10}{3}$
Check your answers.	The check is left to you.

Try It 2.2.1.19

Solve $8x^3=24x^2-18x$.

Answer

$x=0, x=\frac{3}{2}$

Try It 2.2.1.20

Solve $16y^2=32y^3+2y$.

Answer

$y=0, y=14$

We will now continue with our application:

Example 2.2.1.1

Suppose that the area of a rectangular region with the length 3 feet more than twice its width (in feet) is equal to a square region twice the width of the other. Find the dimensions of the two regions.

Solution

We saw in the introduction that if we call the width w, that the length is then $2w+3$ and the assertion is

$(2w)^2=(2w+3)w$

which is equivalent to

$-2w^2+3w=0.$

So, let's factor this so that it looks more like our previous example (which we now know how to go about solving).

We have experience factoring from Unit 1, but we will remind you to factor out the GCF which is w.

An equivalent equation is therefore

$w(-2w+3)=0.$

Again, this is an assertion about the width, i.e., the width w that we are looking for is a solution to this equation. Using the zero-product property we see that

if $w(-2w+3)=0$ then $w=0$ or $-2w+3=0$

or, equivalently,

if $w(-2w+3)=0$ then $w=0$ or $w=3/2$.

Since w is a width, we see that it doesn't make sense for w to be zero. So, w must be 3/2.

The length of the rectangle is then $2(3/2)+3=6$ feet. So the rectangle is 6 ft by 3/2 ft and the square is 3 ft by 3 ft. (we can see that the areas of both regions is 9 sq ft).

Example 2.2.1.21

The product of two consecutive odd integers is 323. Find the integers.

Solution

Step 1. Read the problem.

Step 2. Identify what we are looking for. — We are looking for two consecutive integers.

Step 3. Name what we are looking for. — Let $n=$ the first integer. $n+2=$ next consecutive odd integer

Step 4. Translate into an equation. Restate the problem in a sentence. — The product of the two consecutive odd integers is 323. $n(n+2)=323$

Step 5. Solve the equation. $n^2+2n=323$

Bring all the terms to one side. — $n^2+2n-323=0$

Factor the trinomial. — $(n-17)(n+19)=0$

Use the Zero Product Property. — $n-17=0$ $\quad n+19=0$

Solve the equations. — $n=17$ $\quad n=-19$

There are two values for n that are solutions to this problem. So there are two sets of consecutive odd integers that will work.

If the first integer is $n=17$ then the next odd integer is

$$n+2$$
$$17+2$$
$$19$$
$$17,19$$

If the first integer is $n=-19$ then the next odd integer is

$$n+2$$
$$-19+2$$
$$-17$$
$$-17,-19$$

Step 6. Check the answer.

The results are consecutive odd integers $17,\ 19$ and $-19,\ -17$.

$17\cdot 19=323\checkmark$ $\qquad -19(-17)=323\checkmark$

Both pairs of consecutive integers are solutions.

Step 7. Answer the question — The consecutive integers are $17, 19$ and $-19, -17$.

Try It 2.2.1.22

The product of two consecutive odd integers is 255. Find the integers.

Answer

$-15, -17$ and $15, 17$

Try It 2.2.1.23

The product of two consecutive odd integers is 483 Find the integers.

Answer

$-23, -21$ and $21, 23$

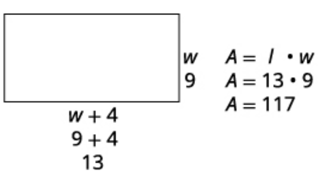
Were you surprised by the pair of negative integers that is one of the solutions to the previous example? The product of the two positive integers and the product of the two negative integers both give positive results.

In some applications, negative solutions will result from the algebra, but will not be realistic for the situation.

Example 2.2.1.24

A rectangular bedroom has an area 117 square feet. The length of the bedroom is four feet more than the width. Find the length and width of the bedroom.

Solution

Step 1. Read the problem. In problems involving geometric figures, a sketch can help you visualize the situation.	
Step 2. Identify what you are looking for.	We are looking for the length and width.
Step 3. Name what you are looking for.	Let $w =$ the width of the bedroom .
The length is four feet more than the width.	$w+4=$ the length of the garden
Step 4. Translate into an equation.	
Restate the important information in a sentence.	The area of the bedroom is 117 square feet.
Use the formula for the area of a rectangle.	$A=l\cdot w$
Substitute in the variables.	$117=(w+4)w$
Step 5. Solve the equation Distribute first.	$117=w^2+4w$
Get zero on one side.	$117=w^2+4w$
Factor the trinomial.	$0=w^2+4w-117$
Use the Zero Product Property.	$0=(w^2+13)(w-9)$
Solve each equation.	$0=w+13 \quad 0=w-9$
Since w is the width of the bedroom, it does not make sense for it to be negative. We eliminate that value for w.	~~$w=-13$~~ $\quad w=9$
	$w=9$ The width is 9 feet.
Find the value of the length.	$w+4$ $9+4$ 13 The length is 13 feet.
Step 6. Check the answer. Does the answer make sense?  Yes, this makes sense.	
Step 7. Answer the question.	The width of the bedroom is 9 feet and the length is 13 feet.

Try It 2.2.1.25

A rectangular sign has area 30 square feet. The length of the sign is one foot more than the width. Find the length and width of the sign.

Answer

The width is 5 feet and length is 6 feet.

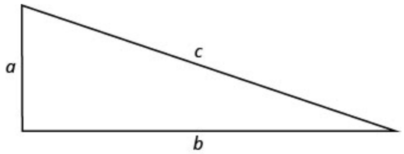
Try It 2.2.1.26

A rectangular patio has area 180 square feet. The width of the patio is three feet less than the length. Find the length and width of the patio.

Answer

The length of the patio is 12 feet and the width 15 feet.

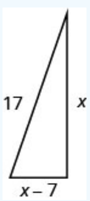
In the next example, we will use the Pythagorean Theorem $(a^2+b^2=c^2)$. This formula gives the relation between the legs and the hypotenuse of a right triangle.



We will use this formula to in the next example.

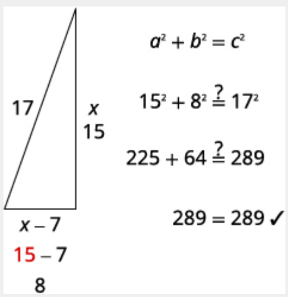
Example 2.2.1.27

A boat's sail is in the shape of a right triangle as shown. The hypotenuse will be 17 feet long. The length of one side will be 7 feet less than the length of the other side. Find the lengths of the sides of the sail.



Solution

Step 1. Read the problem	
Step 2. Identify what you are looking for.	We are looking for the lengths of the sides of the sail.
Step 3. Name what you are looking for. One side is 7 less than the other.	Let $x=$ length of a side of the sail . $x-7=$ length of other side
Step 4. Translate into an equation. Since this is a right triangle we can use the Pythagorean Theorem.	$a^2+b^2=c^2$
Substitute in the variables.	$x^2+(x-7)^2=17^2$
Step 5. Solve the equation Simplify.	$x^2+x^2-14x+49=289$
	$2x^2-14x+49=289$
It is a quadratic equation, so get zero on one side.	$2x^2-14x-240=0$

Factor the greatest common factor.	$2(x^2 - 7x - 120) = 0$
Factor the trinomial.	$2(x - 15)(x + 8) = 0$
Use the Zero Product Property.	$2 \neq 0 \quad x - 15 = 0 \quad x + 8 = 0$
Solve.	$2 \neq 0 \quad x = 15 \quad x = -8$
Since x is a side of the triangle, $x = -8$ does not make sense.	$2 \neq 0 \quad x = 15 \quad$ ~~$x = -8$~~
Find the length of the other side.	
If the length of one side is then the length of the other side is	$x = 15$ $x - 7$ $15 - 7$ 8 is the length of the other side.
Step 6. Check the answer in the problem Do these numbers make sense?  	
Step 7. Answer the question	The sides of the sail are 8, 15 and 17 feet.

? Try It 2.2.1.28

Justine wants to put a deck in the corner of her backyard in the shape of a right triangle. The length of one side of the deck is 7 feet more than the other side. The hypotenuse is 13. Find the lengths of the two sides of the deck.

Answer

5 feet and 12 feet

? Try It 2.2.1.29

A meditation garden is in the shape of a right triangle, with one leg 7 feet. The length of the hypotenuse is one more than the length of the other leg. Find the lengths of the hypotenuse and the other leg.

Answer

24 feet and 25 feet

✓ Writing Exercises 2.2.1.30

1. What is the Zero Product Property?
2. Is there an analogue of the Zero Product Property if we replace 0 with 1? Give an example supporting your answer.
3. What is the goal of solving a quadratic equation?
4. Must we check our answers even if we know we didn't make a mistake?

Exit Problem 2.2.1.31

Solve the equation $x(x+1)=-x+35$ by using the Zero Product Property.

Key Concepts

- **Polynomial Equation:** A polynomial equation is an equation that contains a polynomial expression. The degree of the polynomial equation is the degree of the polynomial.
- **Quadratic Equation:** An equation of the form $ax^2+bx+c=0$, where a, b, c, are real numbers and $a \neq 0$, is called a quadratic equation.
- **Zero Product Property:** If $a \cdot b=0$, then either $a=0$ or $b=0$ or both.
- **How to use the Zero Product Property**
  1. Set each factor equal to zero.
  2. Solve the linear equations.
  3. Check.
- **How to solve a quadratic equation by factoring.**
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$.
  2. Factor the quadratic expression.
  3. Use the Zero Product Property.
  4. Solve the linear equations.
  5. Check. Substitute each solution separately into the original equation.
- **How to use a problem solving strategy to solve word problems.**
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.
  4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
  5. **Solve** the equation using appropriate algebra techniques.
  6. **Check** the answer in the problem and make sure it makes sense.
  7. **Answer** the question with a complete sentence.

2.2.2: Solving Quadratic Equations Using the Square Root Property

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations of the form $ax^2 = k$ using the Square Root Property
- Solve quadratic equations of the form $a(x-h)^2 = k$ using the Square Root Property

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\sqrt{128}$.
2. Simplify $\sqrt{\frac{32}{5}}$.
3. Factor $9x^2 - 12x + 4$.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$. Quadratic equations differ from linear equations by including a quadratic term with the variable raised to the second power of the form ax^2. We use different methods to solve quadratic equations than linear equations, because just adding, subtracting, multiplying, and dividing terms will not isolate the variable.

We have seen that some quadratic equations can be solved by factoring. In this chapter, we will learn three other methods to use in case a quadratic equation cannot be factored.

Solve Quadratic Equations of the Form $ax^2 = k$ using the Square Root Property

We have already solved some quadratic equations by factoring. Let's review how we used factoring to solve the quadratic equation $x^2 = 9$.

	$x^2 = 9$
Put the equation in standard form.	$x^2 - 9 = 0$
Factor the difference of squares.	$(x-3)(x+3) = 0$
Use the Zero Produce Property.	$x - 3 = 0 \quad \text{or} \quad x - 3 = 0$
Solve each equation.	$x = 3 \quad \text{or} \quad x = -3$

We can easily use factoring to find the solutions of similar equations, like $x^2 = 16$ and $x^2 = 25$, because 16 and 25 are perfect squares. In each case, we would get two solutions, $x = 4, x = -4$ and $x = 5, x = -5$

But what happens when we have an equation like $x^2 = 7$? Since 7 is not a perfect square, we cannot solve the equation by factoring.

Previously we learned that since 169 is the square of 13, we can also say that 13 is a *square root* of 169. Also, $(-13)^2 = 169$, so -13 is also a square root of 169. Therefore, both 13 and -13 are square roots of 169. So, every positive number has two square roots—one positive and one negative. We earlier defined the square root of a number in this way:

If $n^2 = m$, then n is a square root of m.

Since these equations are all of the form $x^2 = k$, the square root definition tells us the solutions are the two square roots of k. This leads to the **Square Root Property**.

Square Root Property

If $x^2 = k$, then

$$x = \sqrt{k} \quad \text{or} \quad x = -\sqrt{k} \quad .$$

These two solutions are often written

$$x=\pm\sqrt{k}.$$

Notice that the Square Root Property gives two solutions to an equation of the form $x^2=k$, the principal square root of k and its opposite. We could also write the solution as $x=\pm\sqrt{k}$. We read this as x equals positive or negative the square root of k.

Now we will solve the equation $x^2=9$ again, this time using the Square Root Property.

	$x^2=9$
Use the Square Root Property.	$x=\pm\sqrt{9}$
Simplify.	$x=\pm 3$
Rewrite to show two solutions.	$x=3$ or $x=-3$

What happens when the constant is not a perfect square? Let's use the Square Root Property to solve the equation $x^2=7$.

	$x^2=7$
Use the Square Root Property.	$x=\pm\sqrt{7}$
Rewrite to show two solutions.	$x=\sqrt{7}$ or $x=-\sqrt{7}$

We cannot simplify $\sqrt{7}$, so we leave the answer as a radical.

Example 2.2.2.1

Solve $x^2-50=0$.

Solution

		$x^2-50=0$
Isolate the quadratic term and make its coefficient one.	Add 50 to both sides to get x^2 by itself.	$x^2=50$
Use the Square Root Property.	Remember to write the $\pm$ symbol or list the solutions.	$x=\pm\sqrt{50}$
Simplify the radical.		$x=\pm\sqrt{25}\cdot\sqrt{2}$ $x=\pm 5\sqrt{2}$
Rewrite to show two solutions.		$x=5\sqrt{2}$ or $x=-5\sqrt{2}$
Check the solutions in order to detect errors.		

Try It 2.2.2.2

Solve $x^2-48=0$.

Answer

$x=4\sqrt{3}$ or $x=-4\sqrt{3}$

Try It 2.2.2.3

Solve $y^2 - 27 = 0$.

Answer

$$y = 3\sqrt{3} \quad \text{or} \quad y = -3\sqrt{3}$$

The steps to take to use the **Square Root Property** to solve a quadratic equation are listed here.

Solve a Quadratic Equation Using the Square Root Property

1. Isolate the quadratic term and make its coefficient one.
2. Use Square Root Property.
3. Simplify the radical.
4. Check the solutions in order to detect errors.

In order to use the Square Root Property, the coefficient of the variable term must equal one. In the next example, we must divide both sides of the equation by the coefficient 3 before using the Square Root Property.

Example 2.2.2.4

Solve $3z^2 = 108$.

Solution

	$3z^2 = 108$
The quadratic term is isolated. Divide by 3 to make its coefficient 1.	$\frac{3z^2}{3} = \frac{108}{3}$
Simplify.	$z^2 = 36$
Use the Square Root Property.	$z = \pm\sqrt{36}$
Simplify the radical.	$z = \pm 6$
Rewrite to show two solutions.	$z = 6 \quad \text{or} \quad z = -6$
Check the solutions.	$3z^2 = 108$, $3(6)^2 \stackrel{?}{=} 108$, $3(36) \stackrel{?}{=} 108$, $108 = 108$ ✓ ; $3z^2 = 108$, $3(-6)^2 \stackrel{?}{=} 108$, $3(36) \stackrel{?}{=} 108$, $108 = 108$ ✓

Try It 2.2.2.5

Solve $2x^2 = 98$.

Answer

$$x = 7 \quad \text{or} \quad x = -7$$

Try It 2.2.2.6

Solve $5m^2 = 80$.

Answer

$m=4 \quad \text{or} \quad m=-4$

The Square Root Property states 'If $x^2=k$,' What will happen if $k<0$? This will be the case in the next example.

✓ Example 2.2.2.7

Solve $x^2+72=0$.

Solution

	$x^2+72=0$
Isolate the quadratic term.	$x^2=-72$
Use the Square Root Property.	$x=\pm\sqrt{-72}$
Simplify using complex numbers.	$x=\pm\sqrt{72}i$
Simplify the radical.	$x=\pm 6\sqrt{2}i$
Rewrite to show two solutions	$x=6\sqrt{2}i \quad \text{or} \quad x=-6\sqrt{2}i$
Check the solutions.	$x^2+72=0$; $(6\sqrt{2}i)^2+72\stackrel{?}{=}0$; $6^2(\sqrt{2})^2i^2+72\stackrel{?}{=}0$; $36\cdot 2\cdot(-1)+72\stackrel{?}{=}0$; $0=0$ ✓ $x^2+72=0$; $(6\sqrt{2}i)^2+72\stackrel{?}{=}0$; $(-6)^2(\sqrt{2})^2i^2+72\stackrel{?}{=}0$; $36\cdot 2\cdot(-1)+72\stackrel{?}{=}0$; $0=0$ ✓

? Try It 2.2.2.8

Solve $c^2+12=0$.

Answer

$c=2\sqrt{3}i \quad \text{or} \quad c=-2\sqrt{3}i$

? Try It 2.2.2.9

Solve $q^2+24=0$.

Answer

$c=2\sqrt{6}i \quad \text{or} \quad c=-2\sqrt{6}i$

Our method also works when fractions occur in the equation, we solve as any equation with fractions. In the next example, we first isolate the quadratic term, and then make the coefficient equal to one.

✓ Example 2.2.2.10

Solve $\frac{2}{3}u^2+5=17$.

Solution

	$\frac{2}{3}u^2+5=17$
Isolate the quadratic term.	$\frac{2}{3}u^2=12$
Multiply by $\frac{3}{2}$ to make the coefficient 1.	$\frac{3}{2}\cdot\frac{2}{3}u^2=\frac{3}{2}\cdot 12$
Simplify.	$u^2=18$
Use the Square Root Property.	$u=\pm\sqrt{18}$
Simplify the radical.	$u=\pm\sqrt{9\cdot 2}$
Simplify.	$u=\pm 3\sqrt{2}$
Rewrite to show two solutions.	$u=3\sqrt{2}$ or $u=-3\sqrt{2}$
Check.	$\frac{2}{3}u^2+5=17$; $\frac{2}{3}(3\sqrt{2})^2+5\stackrel{?}{=}17$; $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$; $12+5\stackrel{?}{=}17$; $17=17$ ✓ $\frac{2}{3}u^2+5=17$; $\frac{2}{3}(-3\sqrt{2})^2+5\stackrel{?}{=}17$; $\frac{2}{3}\cdot 18+5\stackrel{?}{=}17$; $12+5\stackrel{?}{=}17$; $17=17$ ✓

Try It 2.2.2.11

Solve $\frac{1}{2}x^2+4=24$.

Answer

$x=2\sqrt{10}$ or $x=-2\sqrt{10}$

Try It 2.2.2.12

Solve $\frac{3}{4}y^2-3=18$.

Answer

$y=2\sqrt{7}$ or $y=-2\sqrt{7}$

The solutions to some equations may have fractions inside the radicals. When this happens, we must **rationalize the denominator**.

Example 2.2.2.13

Solve $2x^2-8=41$.

Solution

	$2x^2-8=41$
Isolate the quadratic term.	$2x^2=49$
Divide by 2 to make the coefficient 1.	$\frac{2x^2}{2}=\frac{49}{2}$
Simplify.	$x^2=\frac{49}{2}$

Use the Square Root Property.	$x=\pm\sqrt{\frac{49}{2}}$
Rewrite the radical as a fraction of square roots.	$x=\pm\frac{\sqrt{49}}{\sqrt{2}}$
Rationalize the denominator.	$x=\pm\frac{\sqrt{49}\cdot\sqrt{2}}{\sqrt{2}\cdot\sqrt{2}}$
Simplify.	$x=\pm\frac{7\sqrt{2}}{2}$
Rewrite to show two solutions.	$x=\frac{7\sqrt{2}}{2}$ or $x=-\frac{7\sqrt{2}}{2}$
Check: We leave the check for you.	

Try It 2.2.2.14

Solve $5r^2-2=34$.

Answer

$$r=\frac{6\sqrt{5}}{5} \quad \text{or} \quad r=-\frac{6\sqrt{5}}{5}$$

Try It 2.2.2.15

Solve $3t^2+6=70$.

Answer

$$t=\frac{8\sqrt{3}}{3} \quad \text{or} \quad t=-\frac{8\sqrt{3}}{3}$$

Solve Quadratic Equation of the Form $a(x-h)^2=k$ Using the Square Root Property

We can use the **Square Root Property** to solve an equation of the form $a(x-h)^2=k$ as well. Notice that the quadratic term, x, in the original form $ax^2=k$ is replaced with $(x-h)$.

$$ax^2=k \qquad a(x-h)^2=k$$

The first step, like before, is to isolate the term that has the variable squared. In this case, a binomial is being squared. Once the binomial is isolated, by dividing each side by the coefficient of a, then the Square Root Property can be used on $(x-h)^2$.

Example 2.2.2.16

Solve $4(y-7)^2=48$.

Solution

	$4(y-7)^2=48$
Divide both sides by the coefficient 4.	$(y-7)^2=12$
Use the Square Root Property on the binomial.	$y-7=\pm\sqrt{12}$
Simplify the radical.	$y-7=\pm2\sqrt{3}$
Solve for y.	$y=7\pm2\sqrt{3}$

Rewrite to show two solutions.	$y=7+2\sqrt{3}$ or $y=7-2\sqrt{3}$
Check.	$4(y-7)^2=48$ $\quad$ $4(y-7)^2=48$ $4(7+2\sqrt{3}-7)^2\stackrel{?}{=}48$ $\quad$ $4(7-2\sqrt{3}-7)^2\stackrel{?}{=}48$ $4(2\sqrt{3})^2\stackrel{?}{=}48$ $\quad$ $4(-2\sqrt{3})^2\stackrel{?}{=}48$ $4(12)\stackrel{?}{=}48$ $\quad$ $4(12)\stackrel{?}{=}48$ $48=48$ ✓ $\quad$ $48=48$ ✓

? Try It 2.2.2.17

Solve $3(a-3)^2=54$.

Answer

$a=3+3\sqrt{2}$ or $a=3-3\sqrt{2}$

? Try It 2.2.2.18

Solve $2(b+2)^2=80$.

Answer

$b=-2+2\sqrt{10}$ or $b=-2-2\sqrt{10}$

Remember when we take the square root of a fraction, we can take the square root of the numerator and denominator separately.

✓ Example 2.2.2.19

Solve $\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$.

Solution

	$\left(x-\frac{1}{3}\right)^2=\frac{5}{9}$
Rewrite the radical as a fraction of square roots.	$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{\sqrt{9}}$
Simplify the radical.	$x-\frac{1}{3}=\pm\frac{\sqrt{5}}{3}$
Solve for x.	$x=\frac{1}{3}\pm\frac{\sqrt{5}}{3}$
Rewrite to show two solutions.	$x=\frac{1}{3}+\frac{\sqrt{5}}{3}$ or $x=\frac{1}{3}-\frac{\sqrt{5}}{3}$
Check.	We leave the check for you.

Try It 2.2.2.20

Solve $\left(x-\frac{1}{2}\right)^2=\frac{5}{4}$.

Answer

$$x=\frac{1}{2}+\frac{\sqrt{5}}{2} \quad \text{or} \quad x=\frac{1}{2}-\frac{\sqrt{5}}{2}$$

Try It 2.2.2.21

Solve $\left(y+\frac{3}{4}\right)^2=\frac{7}{16}$.

Answer

$$y=-\frac{3}{4}+\frac{\sqrt{7}}{4} \quad \text{or} \quad y=-\frac{3}{4}-\frac{\sqrt{7}}{4}$$

We will start the solution to the next example by isolating the binomial term.

Example 2.2.2.22

Solve $2(x-2)^2+3=57$.

Solution

	$2(x-2)^2+3=57$
Subtract 3 from both sides to isolate the binomial term.	$2(x-2)^2=54$
Divide both sides by 2.	$(x-2)^2=27$
Use the Square Root Property.	$x-2=\pm\sqrt{27}$
Simplify the radical.	$x-2=\pm 3\sqrt{3}$
Solve for x.	$x=2\pm 3\sqrt{3}$
Rewrite to show two solutions.	$x=2+3\sqrt{3}$ or $x=2-3\sqrt{3}$
Check.	We leave the check for you.

Try It 2.2.2.23

Solve $5(a-5)^2+4=104$.

Answer

$$a=5+2\sqrt{5} \quad \text{or} \quad a=5-2\sqrt{5}$$

Try It 2.2.2.24

Solve $3(b+3)^2-8=88$.

Answer

$$b=-3+4\sqrt{2} \quad \text{or} \quad b=-3-4\sqrt{2}$$

Sometimes the solutions are complex numbers.

Example 2.2.2.25

Solve $(2x-3)^2=-12$.

Solution

	$(2x-3)^2=-12$
Use the Square Root Property.	$2x-3=\pm\sqrt{-12}$
Simplify the radical.	$2x-3=\pm 2\sqrt{3}i$
Add 3 to both sides.	$2x=3\pm 2\sqrt{3}i$
Divide both sides by 2.	$x=\frac{3\pm 2\sqrt{3}i}{2}$
Rewrite in standard form.	$x=\frac{3}{2}\pm\frac{2\sqrt{3}i}{2}$
Simplify.	$x=\frac{3}{2}\pm\sqrt{3}i$
Rewrite to show two solutions.	$x=\frac{3}{2}+\sqrt{3}i$ or $x=\frac{3}{2}-\sqrt{3}i$
Check.	We leave the check for you.

Try It 2.2.2.26

Solve $(3r+4)^2=-8$.

Answer

$$r=-\frac{4}{3}+\frac{2\sqrt{2}i}{3} \quad \text{or} \quad r=-\frac{4}{3}-\frac{2\sqrt{2}i}{3}$$

Try It 2.2.2.27

Solve $(2t-8)^2=-10$.

Answer

$$t=4+\frac{\sqrt{10}i}{2} \quad \text{or} \quad t=4-\frac{\sqrt{10}i}{2}$$

The left sides of the equations in the next two examples do not seem to be of the form $a(x-h)^2$. But they are perfect square trinomials, so we will factor to put them in the form we need.

Example 2.2.2.28

Solve $4n^2+4n+1=16$.

Solution

We notice the left side of the equation is a perfect square trinomial. We will factor it first.

	$4n^2+4n+1=16$
Factor the perfect square trinomial.	$(2n+1)^2=16$
Use the Square Root Property.	$2n+1=\pm\sqrt{16}$
Simplify the radical.	$2n+1=\pm 4$
Solve for n.	$2n=-1\pm 4$
Divide each side by 2.	$\frac{2n}{2}=\frac{-1\pm 4}{2}$
Simplify.	$n=\frac{-1\pm 4}{2}$
Rewrite to show two solutions.	$n=\frac{-1+4}{2}$ or $n=\frac{-1-4}{2}$
Simplify each equation.	$n=\frac{3}{2}$ or $n=-\frac{5}{2}$
Check.	$4n^2+4n+1=16$ $4\left(\frac{3}{2}\right)^2+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{9}{4}\right)+4\left(\frac{3}{2}\right)+1\stackrel{?}{=}16$ $9+6+1\stackrel{?}{=}16$ $16=16\checkmark$ $4n^2+4n+1=16$ $4\left(-\frac{5}{2}\right)^2+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $4\left(\frac{25}{4}\right)+4\left(-\frac{5}{2}\right)+1\stackrel{?}{=}16$ $25-10+1\stackrel{?}{=}16$ $16=16\checkmark$

? Try It 2.2.2.30

Solve $9m^2-12m+4=25$.

Answer

$$m=\frac{7}{3} \quad \text{or} \quad m=-1$$

? Try It 2.2.2.31

Solve $16n^2+40n+25=4$.

Answer

$$n=-\frac{3}{4} \quad \text{or} \quad n=-\frac{7}{4}$$

✓ Writing Exercises 2.2.2.31

1. In your own words, explain the Square Root Property.
2. In your own words, explain how to use the Square Root Property to solve the quadratic equation $(x+2)^2=16$.

Exit Problem 2.2.2.32

Solve $(5x-2)^2-3=4$ by using the Square Root Property.

Key Concepts

- Square Root Property
  - If $x^2 = k$, then $x = \sqrt{k}$ or $x = -\sqrt{k}$or $x = \pm\sqrt{k}$

  How to solve a quadratic equation using the square root property.

  1. Isolate the quadratic term and make its coefficient one.
  2. Use Square Root Property.
  3. Simplify the radical.
  4. Check the solutions.

2.2.3: Solving Quadratic Equations by Completing the Square

Learning Objectives

By the end of this section, you will be able to:

- Complete the square of a binomial expression
- Solve quadratic equations of the form $x^2+bx+c=0$ by completing the square
- Solve quadratic equations of the form $ax^2+bx+c=0$ by completing the square

Be Prepared

Before you get started, take this readiness quiz.

1. Expand $(x+9)^2$.
2. Factor $y^2-14y+49$.
3. Factor $5n^2+40n+80$.

So far we have solved quadratic equations by factoring and using the Square Root Property. In this section, we will solve quadratic equations by a process called **completing the square**, which is important for our work on conics later.

Exercise 2.2.3.1

Add exercises text here.

Answer

Add texts here. Do not delete this text first.

Example 2.2.3.1

Add example text here.

Solution

Add example text here.

Exercise 2.2.3.1

Add exercises text here.

Answer

Add texts here. Do not delete this text first.

Complete the Square of a Binomial Expression

In the last section, we were able to use the Square Root Property to solve the equation $(y-7)^2=12$ because the left side was a perfect square.

$$\begin{aligned}(y-7)^2 &= 12\\ y-7 &= \pm\sqrt{12}\\ y-7 &= \pm 2\sqrt{3}\\ y &= 7\pm 2\sqrt{3}\end{aligned}$$

We also solved an equation in which the left side was a perfect square trinomial, but we had to rewrite it the form $(x-k)^2$ in order to use the Square Root Property.

$$\begin{aligned} x^2-10x+25 &= 18 \\ (x-5)^2 &= 18 \\ x-5 &= \pm 3\sqrt{2} \\ x &= 5 \pm 3\sqrt{2} \end{aligned}$$

What happens if the variable is not part of a perfect square? Can we use algebra to make a perfect square?

Let's look at two examples to help us recognize the patterns.

$$\begin{aligned} &(x+9)^2 \\ &=(x+9)(x+9) \\ &=x^2+9x+9x+81 \\ &=x^2+18x+81 \end{aligned} \qquad \begin{aligned} &(y-7)^2 \\ &=(y-7)(y-7) \\ &=y^2-7y-7y+49 \\ &=y^2-14y+49 \end{aligned}$$

We restate the patterns here for reference.

Binomial Squares Pattern

If a and b are real numbers,

$$(a+b)^2 = a^2+2ab+b^2,$$

and

$$(a-b)^2 = a^2-2ab+b^2.$$

We can use this pattern to "make" a perfect square.

We will start with the expression x^2+6x. Since there is a plus sign between the two terms, we will use the $(a+b)^2$ pattern, $a^2+2ab+b^2=(a+b)^2$.

$$\underbrace{x^2+6x+\ldots}_{a^2+2ab+b^2}$$

We ultimately need to find the last term of this trinomial that will make it a perfect square trinomial. To do that we will need to find b. But first we start with determining a. Notice that the first term of x^2+6x is a square, x^2. This tells us that $a=x$.

$$\underbrace{x^2+2xb+b^2}_{a^2+2ab+b^2}$$

What number, b, when multiplied with $2x$ gives $6x$? It would have to be 3, which is $\frac{1}{2}(6)$. So $b=3$.

$$\underbrace{x^2+2\cdot 3x+\ldots}_{a^2+2ab+b^2}$$

Now to complete the perfect square trinomial, we will find the last term by squaring b, which is $3^2=9$.

$$\underbrace{x^2+6x+9}_{a^2+2ab+b^2}$$

We can now factor.

$$\underbrace{(x+3)^2}_{(a+b)^2}$$

So we found that adding 9 to x^2+6x 'completes the square', and we write it as $(x+3)^2$.

Complete a Square of x^2+bx

1. Identify b, the coefficient of x.
2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
3. Add the $\left(\frac{1}{2}b\right)^2$ to x^2+bx.
4. Factor the perfect square trinomial, writing it as a binomial squared.

Example 2.2.3.1

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. x^2-26x

b. y^2-9y

c. $n^2+\frac{1}{2}n$

Solution

a.

	$\underbrace{x^2-26x}_{x^2+bx}$
The coefficient of x is -26.	$b=-26$
Find $\left(\frac{1}{2}b\right)^2$.	$\left(\frac{1}{2}\cdot(-26)\right)^2$ $=(-13)^2$ $=169$
Add 169 to the binomial to complete the square.	$x^2-26x+169$
Factor the perfect square trinomial, writing it as a binomial squared.	$=(x-13)^2$

b.

	$\underbrace{y^2-9y}_{x^2+bx}$
The coefficient of y is -9.	$b=-9$
Find $\left(\frac{1}{2}b\right)^2$.	$\left(\frac{1}{2}\cdot(-9)\right)^2$ $=\left(-\frac{9}{2}\right)^2$ $=\frac{81}{4}$
Add $\frac{81}{4}$ to the binomial to complete the square.	$y^2-9y+\frac{81}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$=\left(y-\frac{9}{2}\right)^2$

c.

	$\underbrace{n^2+\frac{1}{2}n}_{x^2+bx}$

	$\underbrace{n^2+\frac{1}{2}n}_{x^2+bx}$
The coefficient of n is $\frac{1}{2}$.	$b=\frac{1}{2}$
Find $\left(\frac{1}{2}b\right)^2$.	$\left(\frac{1}{2}\cdot\frac{1}{2}\right)^2$ $=\left(\frac{1}{4}\right)^2$ $=\frac{1}{16}$
Add $\frac{1}{16}$ to the binomial to complete the square.	$n^2+\frac{1}{2}n+\frac{1}{16}$
Rewrite as a binomial square.	$=\left(n+\frac{1}{4}\right)^2$

Try It 2.2.3.2

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. a^2-20a

b. m^2-5m

c. $p^2+\frac{1}{4}p$

Answer

a. $(a-10)^2$

b. $\left(m-\frac{5}{2}\right)^2$

c. $\left(p+\frac{1}{8}\right)^2$

Try It 2.2.3.3

Complete the square to make a perfect square trinomial. Then write the result as a binomial squared.

a. b^2-4b

b. n^2+13n

c. $q^2-\frac{2}{3}q$

Answer

a. $(b-2)^2$

b. $\left(n+\frac{13}{2}\right)^2$

c. $\left(q-\frac{1}{3}\right)^2$

Solve Quadratic Equations of the Form $x^2+bx+c=0$ by Completing the Square

In solving equations, we must always do the same thing to both sides of the equation. This is true, of course, when we solve a **quadratic equation** by **completing the square** too. When we add a term to one side of the equation to make a perfect square trinomial, we must also add the same term to the other side of the equation.

For example, if we start with the equation $x^2+6x=40$, and we want to complete the square on the left, we will add 9 to both sides of the equation.

	$x^2+6x=40$
	$x^2+6x+\cdots=40+\cdots$
Add 9 to both sides to complete the square.	$x^2+6x+9=40+9$
Rewrite it as a binomial square.	$(x+3)^2=49$

Now the equation is in the form to solve using the **Square Root Property**! Completing the square is a way to transform an equation into the form we need to be able to use the Square Root Property.

Example 2.2.3.4

Solve by completing the square: $x^2+8x=48$.

Solution

		$x^2+8x=48$
Isolate the variable terms on one side and the constant terms on the other.	This equation has all the variables on the left.	$x^2+bx \quad c$ $x^2+8x=48$
Find $\left(\frac{1}{2}\cdot b\right)^2$, the number to complete the square. Add it to both sides of the equation.	$b=8$ Take half of 8 and square it. $4^2=16$ Add 16 to BOTH sides of the equation.	$x^2+8x+\underline{\quad\left(\frac{1}{2}\cdot 8\right)^2\quad}=48$ $x^2+8x+16=48+16$
Factor the perfect square trinomial as a binomial square.	$x^2+8x+16=(x+4)^2$ Add the terms on the right.	$(x+4)^2=64$
Use the Square Root Property.		$x+4=\pm\sqrt{64}$
Simplify the radical.		$x+4=\pm 8$
Solve the two resulting equations.		$x+4=8 \quad x+4=-8$ $x=4 \quad x=-12$
Check the solutions.	Put each answer in the original equation to check. Substitute $x=4$ and $x=-12$.	$x^2+8x=48$ $x=4: \quad (4)^2+8(4)\stackrel{?}{=}48$ $16+32\stackrel{?}{=}48$ $48\stackrel{?}{=}48$ True $x^2+8x=48$ $x=-12: \quad (-12)^2+8(-12)\stackrel{?}{=}48$ $144-96\stackrel{?}{=}48$ $48\stackrel{?}{=}48$ True
		The solutions are $x=4$ or $x=-12$.

Try It 2.2.3.5

Solve by completing the square: $x^2+4x=5$.

Answer

$x=-5 \quad \text{or} \quad x=1$

Try It 2.2.3.6

Solve by completing the square: $y^2-10y=-9$.

Answer

$y=1 \quad \text{or} \quad y=9$

The steps to solve a quadratic equation by completing the square are listed here.

Solve a Quadratic Equation of the Form $x^2+bx+c=0$ by Completing the Square

1. Isolate the variable terms on one side and the constant terms on the other.
2. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
3. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
4. Use the Square Root Property.
5. Simplify the radical and then solve the two resulting equations.
6. Check the solutions.

When we solve an equation by completing the square, the answers will not always be integers.

Example 2.2.3.7

Solve by completing the square: $x^2+4x=-21$.

Solution

	$x^2+4x=-21$
The variable terms are on the left side.	$x^2+bx \quad c$ $x^2+4x=-21$
Take half of 4 and square it. $\left(\frac{1}{2}(4)\right)^2=4$	$x^2+4x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot 4\right)^2}=-21$
Add 4 to both sides.	$x^2+4x+4=-21+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+2)^2=-17$
Use the Square Root Property.	$x+2=\pm\sqrt{-17}$
Simplifying using complex numbers.	$x+2=\pm\sqrt{17}i$
Subtract 2 from each side.	$x=-2\pm\sqrt{17}i$
Rewrite to show two solutions.	$x=-2+\sqrt{17}i \quad \text{or} \quad x=-2-\sqrt{17}i$
Check.	We leave the check to you.
	The solutions are $x=-2+\sqrt{17}i \quad \text{or} \quad x=-2-\sqrt{17}i$.

Try It 2.2.3.8

Solve by completing the square: $y^2 - 10y = -35$.

Answer

$y = 5 + \sqrt{10}i \quad$ or $\quad y = 5 - \sqrt{15}i$

Try It 2.2.3.9

Solve by completing the square: $z^2 + 8z = -19$.

Answer

$z = -4 + \sqrt{3}i \quad$ or $\quad z = -4 - \sqrt{3}i$

In the previous example, our solutions were complex numbers. In the next example, the solutions will be irrational numbers.

Example 2.2.3.10

Solve by completing the square: $y^2 - 18y = -6$.

Solution

	$y^2 - 18y = -6$
The variable terms are on the left side.	$x^2 + bx \quad c$ $y^2 - 18y = -6$
Take half of -18 and square it. $\left(\frac{1}{2}(-18)\right)^2 = 81$	$y^2 - 18y + \underbrace{\ldots}_{\left(\frac{1}{2}\cdot(-18)\right)^2} = -6$
Add 81 to both sides.	$y^2 - 18y + 81 = -6 + 81$
Factor the perfect square trinomial, writing it as a binomial squared.	$(y-9)^2 = 75$
Use the Square Root Property.	$y - 9 = \pm\sqrt{75}$
Simplify the radical.	$y - 9 = \pm 5\sqrt{3}$
Solve for y.	$y = 9 \pm 5\sqrt{3}$
Rewrite to show two solutions.	$y = 9 + 5\sqrt{3} \quad$ or $\quad y = 9 - 5\sqrt{3}$
Check.	$y^2 - 18y = -6$ $(9 + 5\sqrt{3})^2 - 18(9 + 5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6 \checkmark$ $y^2 - 18y = -6$ $(9 - 5\sqrt{3})^2 - 18(9 - 5\sqrt{3}) \stackrel{?}{=} -6$ $81 + 90\sqrt{3} + 75 - 162 + 90\sqrt{3} \stackrel{?}{=} -6$ $-6 = -6 \checkmark$

Another way to check this would be to use a calculator. Evaluate $y^2 - 18y$ for both solutions. The answer should be -6.

Try It 2.2.3.11

Solve by completing the square: $x^2 - 16x = -16$.

Answer

$x = 8 + 4\sqrt{3} \quad$ or $\quad x = 8 - 4\sqrt{3}$

Try It 2.2.3.12

Solve by completing the square: $y^2 + 8y = 11$.

Answer

$y = -4 + 3\sqrt{3}$ or $y = -4 - 3\sqrt{3}$

We will start the next example by isolating the variable terms on the left side of the equation.

Example 2.2.3.13

Solve by completing the square: $x^2 + 10x + 4 = 15$.

Solution

	$x^2 + 10x + 4 = 15$
Isolate the variable terms on the left side. Subtract 4 to get the constant terms on the right side.	$x^2 + 10x = 11$
Take half of 10 and square it. $\left(\frac{1}{2}(10)\right)^2 = 25$	$x^2 - 10x + \underbrace{\ldots}_{\left(\frac{1}{2}\cdot(10)\right)^2} = 11$
Add 25 to both sides.	$x^2 + 10x + 25 = 11 + 25$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+5)^2 = 36$
Use the Square Root Property.	$x + 5 = \pm\sqrt{36}$
Simplify the radical.	$x + 5 = \pm 6$
Solve for x.	$x = -5 \pm 6$
Rewrite to show two solutions.	$x = -5 + 6$ or $x = -5 - 6$
Solve the equations.	$x = 1$ or $x = -11$
Check.	$x^2 + 10x + 4 = 15$ $(1)^2 + 10(1) + 4 \stackrel{?}{=} 15$ $1 + 10 + 4 \stackrel{?}{=} 15$ $15 = 15$ ✓ $x^2 + 10x + 4 = 15$ $(-11)^2 + 10(-11) + 4 \stackrel{?}{=} 15$ $121 + 110 + 4 \stackrel{?}{=} 15$ $15 = 15$ ✓
	The solutions are $x = 1$ or $x = -11$.

Try It 2.2.3.14

Solve by completing the square: $a^2 + 4a + 9 = 30$.

Answer

$a = -7$ or $a = 3$

Try It 2.2.3.15

Solve by completing the square: $b^2 + 8b - 4 = 16$.

Answer

$b = -10$ or $b = 2$

To solve the next equation, we must first collect all the variable terms on the left side of the equation. Then we proceed as we did in the previous examples.

Example 2.2.3.16

Solve by completing the square: $n^2 = 3n + 11$.

Answer

	$n^2 = 3n + 11$
Subtract $3n$ to get the variable terms on the left side.	$n^2 - 3n = 11$
Take half of -3 and square it.	$\left(\frac{1}{2}(-3)\right)^2 = \frac{9}{4}$
	$n^2 - 18y + \underbrace{\ldots}_{\left(\frac{1}{2}\cdot(-18)\right)^2} = -6$
Add $\frac{9}{4}$ to both sides.	$n^2 - 3n + \frac{9}{4} = 11 + \frac{9}{4}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(n - \frac{3}{2}\right)^2 = \frac{44}{4} + \frac{9}{4}$
Add the fractions on the right side.	$\left(n - \frac{3}{2}\right)^2 = \frac{53}{4}$
Use the Square Root Property.	$n - \frac{3}{2} = \pm\sqrt{\frac{53}{4}}$
Simplify the radical.	$n - \frac{3}{2} = \pm\frac{\sqrt{53}}{2}$
Solve for n.	$n = \frac{3}{2} \pm \frac{\sqrt{53}}{2}$
Rewrite to show two solutions.	$n = \frac{3}{2} + \frac{\sqrt{53}}{2} \quad \text{or} \quad n = \frac{3}{2} - \frac{\sqrt{53}}{2}$
Check.	We leave the check for you!

Try It 2.2.3.17

Solve by completing the square: $p^2 = 5p + 9$.

Answer

$$p = \frac{5}{2} + \frac{\sqrt{61}}{2} \quad \text{or} \quad p = \frac{5}{2} - \frac{\sqrt{61}}{2}$$

Try It 2.2.3.18

Solve by completing the square: $q^2 = 7q - 3$.

Answer

$$q = \frac{7}{2} + \frac{\sqrt{37}}{2} \quad \text{or} \quad q = \frac{7}{2} - \frac{\sqrt{37}}{2}$$

Notice that the left side of the next equation is in factored form. But the right side is not zero. So, we cannot use the **Zero Product Property** since it says "If $a \cdot b = 0$, then $a = 0$ or $b = 0$." Instead, we multiply the factors and then put the equation into standard form to solve by completing the square.

Example 2.2.3.19

Solve by completing the square: $(x-3)(x+5)=9$.

Solution

	$(x-3)(x+5)=9$
We multiple the binomials on the left.	$x^2+2x-15=9$
Add 15 to isolate the constant terms on the right.	$x^2+2x=24$
	$x^2+2x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(2)\right)^2}=24$
Take half of 2 and square it.	$\left(\frac{1}{2}\cdot(2)\right)^2=1$
Add 1 to both sides.	$x^2+2x+1=24+1$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x+1)^2=25$
Use the Square Root Property.	$x+1=\pm\sqrt{25}$
Solve for x.	$x=-1\pm 5$
Rewrite to show two solutions.	$x=-1+5 \quad \text{or} \quad x=-1-6$
Simplify.	$x=4 \quad \text{or} \quad x=-6$
Check.	We leave the check for you!

Try It 2.2.3.20

Solve by completing the square: $(c-2)(c+8)=11$.

Answer

$c=-9 \quad \text{or} \quad c=3$

Try It 2.2.3.21

Solve by completing the square: $(d-7)(d+3)=56$.

Answer

$d=11 \quad \text{or} \quad =-7$

Solve Quadratic Equations of the Form $ax^2+bx+c=0$ by Completing the Square

The process of **completing the square** works best when the coefficient of x^2 is 1, so the left side of the equation is of the form x^2+bx+c. If the x^2 term has a coefficient other than 1, we take some preliminary steps to make the coefficient equal to 1.

Sometimes the coefficient can be factored from all three terms of the trinomial. This will be our strategy in the next example.

Example 2.2.3.22

Solve by completing the square: $3x^2-12x-15=0$.

Solution

To complete the square, we need the coefficient of x^2 to be one. If we factor out the coefficient of x^2 as a common factor, we can continue with solving the equation by completing the square.

	$3x^2-12x-15=0$
Factor out the greatest common factor.	$3(x^2-4x-5)=0$
Divide both sides by 3 to isolate the trinomial with coefficient 1.	$\frac{3(x^2-4x-5)}{3}=\frac{0}{3}$
Simplify.	$x^2-4x-5=0$
Add 5 to get the constant terms on the right side.	$x^2-4x=5$
Take half of 4 and square it.	$\left(\frac{1}{2}(-4)\right)^2=4$
	$x^2-4x+\underbrace{\ldots}_{\left(\frac{1}{2}\cdot(4)\right)^2}=5$
Add 4 to both sides.	$x^2-4x+4=5+4$
Factor the perfect square trinomial, writing it as a binomial squared.	$(x-2)^2=9$
Use the Square Root Property.	$x-2=\pm\sqrt{9}\backslash$
Solve for x.	$x-2=\pm 3$
Rewrite to show two solutions.	$x=2+3 \quad \text{or} \quad x=2-3$
Simplify.	$x=5 \quad \text{or} \quad x=-1$
Check.	$x=5$: $3x^2-12x-15=0$; $3(5)^2-12(5)-15\overset{?}{=}0$; $75-60-15\overset{?}{=}0$; $0=0$ ✓ $x=-1$: $3x^2-12x-15=0$; $3(-1)^2-12(-1)-15\overset{?}{=}0$; $3+12-15\overset{?}{=}0$; $0=0$ ✓

Try It 2.2.3.23

Solve by completing the square: $2m^2+16m+14=0$.

Answer

$m=-7 \quad \text{or} \quad m=-1$

Try It 2.2.3.24

Solve by completing the square: $4n^2-24n-56=8$.

Answer

$n=-2 \quad \text{or} \quad n=8$

To complete the square, the coefficient of the x^2 must be 1. When the **leading coefficient** is not a factor of all the terms, we will divide both sides of the equation by the leading coefficient! This will give us a fraction for the second coefficient. We have already seen how to complete the square with fractions in this section.

Example 2.2.3.25

Solve by completing the square: $2x^2-3x=20$.

Solution

To complete the square we need the coefficient of x^2 to be one. We will divide both sides of the equation by the coefficient of x^2. Then we can continue with solving the equation by completing the square.

	$2x^2 - 3x = 20$
Divide both sides by 2 to get the coefficient of x^2 to be 1.	$\frac{2x^2 - 3x}{2} = \frac{20}{2}$
Simplify.	$x^2 - \frac{3}{2}x = 10$
Take half of $-\frac{3}{2}$ and square it.	$\left(\frac{1}{2}\left(-\frac{3}{2}\right)\right)^2 = \frac{9}{16}$
	$x^2 - \frac{3}{2}x + \backslash(x^2 - 4x + \underbrace{\ldots}_{\left(\frac{1}{2}\cdot(4)\right)^2} = 5$ $x^2 - \frac{3}{2}x + \underline{\qquad\qquad}_{\left(\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\right)^2} = 10$
Add $\frac{9}{16}$ to both sides.	$x^2 - \frac{3}{2}x + \frac{9}{16} = 10 + \frac{9}{16}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x - \frac{3}{4}\right)^2 = \frac{169}{16} + \frac{9}{16}$
Add the fractions on the right side.	$\left(x - \frac{3}{4}\right)^2 = \frac{160}{16} + \frac{9}{16}$
Use the Square Root Property.	$x - \frac{3}{4} = \pm\sqrt{\frac{169}{16}}$
Simplify the radical.	$x - \frac{3}{4} = \pm\frac{13}{4}$
Solve for x.	$x = \frac{3}{4} \pm \frac{13}{4}$
Rewrite to show two solutions.	$x = \frac{3}{4} + \frac{13}{4} \quad \text{or} \quad x = \frac{3}{4} \pm \frac{13}{4}$
Simplify.	$x = 4 \quad \text{or} \quad x = -\frac{5}{2}$
Check.	We leave the check for you!

Try It 2.2.3.26

Solve by completing the square: $3r^2 - 2r = 21$.

Answer

$$r = -\frac{7}{3} \quad \text{or} \quad r = 3$$

Try It 2.2.3.27

Solve by completing the square: $4t^2 + 2t = 20$.

Answer

$$t = -\frac{5}{2} \quad \text{or} \quad t = 2$$

Now that we have seen that the coefficient of x^2 must be 1 for us to complete the square, we update our procedure for solving a **quadratic equation** by completing the square to include equations of the form $ax^2 + bx + c = 0$.

Solve a Quadratic Equation of the Form $ax^2+bx+c=0$ by Completing the Square

1. Divide by aa to make the coefficient of x^2 term 1.
2. Isolate the variable terms on one side and the constant terms on the other.
3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right
5. Use the Square Root Property.
6. Simplify the radical and then solve the two resulting equations.
7. Check the solutions.

Example 2.2.3.28

Solve by completing the square: $3x^2+2x=4$.

Solution

Again, our first step will be to make the coefficient of x^2 one. By dividing both sides of the equation by the coefficient of x^2, we can then continue with solving the equation by completing the square.

	$3x^2+2x=4$
Divide both sides by 3 to make the coefficient of x^2 equal 1.	$\frac{3x^2+2x}{3}=\frac{4}{3}$
Simplify.	$x^2+\frac{2}{3}x=\frac{4}{3}$
Take half of $\frac{2}{3}$ and square it.	
$\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2=\frac{1}{9}$	
	$x^2+\frac{2}{3}x+\underset{\left(\frac{1}{2}\cdot\frac{2}{3}\right)^2}{\underline{\qquad}}=\frac{4}{3}$
Add $\frac{1}{9}$ to both sides.	$x^2+\frac{2}{3}x+\frac{1}{9}=\frac{4}{3}+\frac{1}{9}$
Factor the perfect square trinomial, writing it as a binomial squared.	$\left(x+\frac{1}{3}\right)^2=\frac{12}{9}+\frac{1}{9}$
Use the Square Root Property.	$x+\frac{1}{3}=\pm\sqrt{\frac{13}{9}}$
Simplify the radical.	$x+\frac{1}{3}=\pm\frac{\sqrt{13}}{3}$
Solve for x.	$x=-\frac{1}{3}\pm\frac{\sqrt{13}}{3}$
Rewrite to show two solutions.	$x=-\frac{1}{3}+\frac{\sqrt{13}}{3},\quad x=-\frac{1}{3}-\frac{\sqrt{13}}{3}$
Check.	We leave the check for you!

Try It 2.2.3.29

Solve by completing the square: $4x^2+3x=2$.

Answer

$$x=-\frac{3}{8}+\frac{\sqrt{41}}{8} \quad \text{or} \quad x=-\frac{3}{8}-\frac{\sqrt{41}}{8}$$

Try It 2.2.3.30

Solve by completing the square: $3y^2 - 10y = -5$.

Answer

$$y = \frac{5}{3} + \frac{\sqrt{10}}{3} \quad \text{or} \quad y = \frac{5}{3} - \frac{\sqrt{10}}{3}$$

Exit Problem 2.2.3.31

Solve $x^2 + x = 5$ by completing the square and applying the Square Root Property.

Key Concepts

- Binomial Squares Pattern
  If a and b are real numbers,

$$(a+b)^2 = a^2 + 2ab + b^2 \qquad \underbrace{(a+b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} + \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

$$(a-b)^2 = a^2 - 2ab + b^2 \qquad \underbrace{(a-b)^2}_{(\text{binomial})^2} = \underbrace{a^2}_{(\text{first term})^2} - \underbrace{2ab}_{2\cdot(\text{product of terms})} + \underbrace{b^2}_{(\text{second term})^2}$$

- How to Complete a Square
  1. Identify b, the coefficient of x.
  2. Find $\left(\frac{1}{2}b\right)^2$, the number to complete the square.
  3. Add the $\left(\frac{1}{2}b\right)^2$ to $x^2 + bx$
  4. Rewrite the trinomial as a binomial square
- How to solve a quadratic equation of the form $ax^2 + bx + c = 0$ by completing the square.
  1. Divide by a to make the coefficient of x^2 term 1.
  2. Isolate the variable terms on one side and the constant terms on the other.
  3. Find $\left(\frac{1}{2}\cdot b\right)^2$, the number needed to complete the square. Add it to both sides of the equation.
  4. Factor the perfect square trinomial, writing it as a binomial squared on the left and simplify by adding the terms on the right.
  5. Use the Square Root Property.
  6. Simplify the radical and then solve the two resulting equations.
  7. Check the solutions.

2.2.4: Solving Quadratic Equations Using the Quadratic Formula

Learning Objectives

By the end of this section, you will be able to:

- Solve quadratic equations using the Quadratic Formula
- Use the discriminant to predict the number and type of solutions of a quadratic equation
- Identify the most appropriate method to use to solve a quadratic equation

Be Prepared

Before you get started, take this readiness quiz.

1. Evaluate b^2-4ab when $a=3$ and $b=-2$.
2. Simplify $\sqrt{108}$.
3. Simplify $\sqrt{50}$.

Solve Quadratic Equations Using the Quadratic Formula

When we solved quadratic equations in the last section by completing the square, we took the same steps every time. By the end of the Try It set, you may have been wondering ‘isn’t there an easier way to do this?’ The answer is ‘yes’. Mathematicians look for patterns when they do things over and over in order to make their work easier. In this section we will derive and use a formula to find the solution of a quadratic equation.

We have already seen how to solve a formula for a specific variable ‘in general’, so that we would do the algebraic steps only once, and then use the new formula to find the value of the specific variable. Now we will go through the steps of completing the square using the general form of a quadratic equation to solve a quadratic equation for x.

We start with the standard form of a quadratic equation and solve it for x by completing the square.

	$ax^2+bx+c=0, \quad a\neq 0$
Isolate the variable terms on one side.	$ax^2+bx \quad =-c$
Make the coefficient of x^2 equal to 1, by dividing by a.	$\frac{ax^2}{a}+\frac{b}{a}x \quad =-\frac{c}{a}$
Simplify.	$x^2+\frac{b}{a}x \quad =-\frac{c}{a}$
To complete the square, find $\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2$ and add it to both sides of the equation.	
$\left(\frac{1}{2}\cdot\frac{b}{a}\right)^2=\frac{b^2}{4a^2}$	$x^2+\frac{b}{a}x+\frac{b^2}{4a^2}=-\frac{c}{a}+\frac{b^2}{4a^2}$
The left side is a perfect square, factor it.	$\left(x+\frac{b}{2a}\right)^2=-\frac{c}{a}+\frac{b^2}{4a^2}$
Find the common denominator of the right side and write equivalent fractions with the common denominator.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c\cdot 4a}{a\cdot 4a}$
Simplify.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{4ac}{4a^2}$
Combine to one fraction.	$\left(x+\frac{b}{2a}\right)^2=\frac{b^2-4ac}{4a^2}$
Use the square root property.	$x+\frac{b}{2a}=\pm\sqrt{\frac{b^2-4ac}{4a^2}}$
Simplify the radical.	$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}$

	$ax^2+bx+c=0, \quad a\neq 0$
Add $-\frac{b}{2a}$ to both sides of the equation.	$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}$
Combine the terms on the right side.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$

The final equation is called the "Quadratic Formula." We have just solved all quadratic equations!

Definition 2.2.4.1

The solutions to a **quadratic equation** of the form $ax^2+bx+c=0$, where $a\neq 0$ are given by the **Quadratic Formula**:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

To use the **Quadratic Formula**, we substitute the values of a, b, and c from the standard form into the expression on the right side of the formula. Then we simplify the expression. The result is the pair of solutions to the quadratic equation.

Notice the Quadratic Formula is an equation. Make sure you use both sides of the equation. This equation requires an understanding of the meaning of a, b, and c.

Example 2.2.4.2

Solve by using the Quadratic Formula: $2x^2+9x-5=0$.

Solution

Step 1: Write the quadratic equation in standard form. Identify the a,b,c values.	This equation is in standard form.	
Step 2: Write the quadratic formula. Then substitute in the values of a,b,c .	Substitute in $a=2, b=9, c=-5$	$x=\frac{-9\pm\sqrt{9^2-4\cdot 2\cdot(-5)}}{2\cdot 2}$
Step 3: Simplify the fraction, and solve for x.		
Step 4: Check the solutions to detect errors.	Put each answer in the original equation to check. Substitute $x=\frac{1}{2}$ and $x=-5$.	$2x^2+9x-5=0$ $2\left(\frac{1}{2}\right)^2+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+0\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $2\cdot\frac{1}{4}+9\cdot\frac{1}{2}-5\stackrel{?}{=}0$ $\frac{1}{2}+\frac{9}{2}-5\stackrel{?}{=}0$ $\frac{10}{2}-5\stackrel{?}{=}0$ $5-5\stackrel{?}{=}0$ $0=0$ $2x^2+9x-5=0$ $2(-5)^2+9(-5)-5\stackrel{?}{=}0$ $2\cdot 25-45-5\stackrel{?}{=}0$ $50-45-5\stackrel{?}{=}0$ $0=0$

Note:

Step 4 is not strictly necessary since, provided that we didn't make a mistake, the Quadratic Formula tells us what the solutions are!

Try It 2.2.4.3

Solve by using the Quadratic Formula: $3y^2-5y+2=0$.

Answer

$$y=1, y=\frac{2}{3}$$

Try It 2.2.4.4

Solve by using the Quadratic Formula: $4z^2+2z-6=0$.

Answer

$$z=1, z=-\frac{3}{2}$$

Solve a Quadratic Equation Using the Quadratic Formula

1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, and c.
2. Write the Quadratic Formula. Then substitute in the values of a, b, and c.
3. Simplify.
4. Check the solutions to detect errors. This step is only to guard against mistakes!

If you say the formula as you write it in each problem, you'll have it memorized in no time! And remember, the Quadratic Formula is an EQUATION. Be sure you start with "$x=$".

Example 2.2.4.5

Solve by using the Quadratic Formula: $x^2-6x=-5$.

Solution:

	$x^2-6x=-5$
Write the equation in standard form by adding 5 to each side.	$x^2-6x+5=0$
This equation is now in standard form.	$ax^2+bx+c=0$ $x^2-6x+5=0$
Identify the values of a, b, c.	$a=1$, $b=-6$, $c=5$
Write the Quadratic Formula.	
Then substitute in the values of a,b,c .	
Simplify.	$x=\frac{6\pm\sqrt{36-20}}{2}$ $x=\frac{6\pm\sqrt{16}}{2}$ $x=\frac{6\pm 4}{2}$
Rewrite to show two solutions.	$x=\frac{6+4}{2}, \quad x=\frac{6-4}{2}$
Simplify.	$x=\frac{10}{2}, \quad x=\frac{2}{2}$
	$x=5, \quad x=1$
Check:	$(5)^2-6(5)=-5$ $(1)^2-6(1)=-5$ Both are true statements and so if $x^2-6x=-5$ then either $x=5$ pr $x=1$.

Try It 2.2.4.6

Solve by using the Quadratic Formula: $a^2 - 2a = 15$.

Answer

$a = -3, a = 5$

Try It 2.2.4.7

Solve by using the Quadratic Formula: $b^2 + 24 = -10b$.

Answer

$b = -6, b = -4$

When we solved quadratic equations by using the Square Root Property, we sometimes got answers that had radicals. That can happen, too, when using the **Quadratic Formula**. If we get a **radical** as a solution, the final answer should have the radical in its simplified form.

Example 2.2.4.8

Solve by using the Quadratic Formula: $2x^2 + 10x + 11 = 0$.

Solution:

	$2x^2 + 10x + 11 = 0$
This equation is in standard form.	$\underbrace{2x^2 + 10x + 11 = 0}_{ax^2+bx+c=0}$
Identify the values of a, b and c.	$a = 2, b = 10, c = 11$
Write the Quadratic Formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(10) \pm \sqrt{(10)^2 - 4 \cdot 2 \cdot 11}}{2 \cdot 2}$
Simplify.	$x = \frac{-10 \pm \sqrt{100 - 88}}{4}$
	$x = \frac{-10 \pm \sqrt{12}}{4}$
Simplify the radical.	$x = \frac{-10 \pm 2\sqrt{3}}{4}$
Factor out the common factor in the numerator.	$x = \frac{2(-5 \pm \sqrt{3})}{4}$
Remove the common factors.	$x = \frac{-5 \pm \sqrt{3}}{2}$
Rewrite to show two solutions.	$x = \frac{-5 + \sqrt{3}}{2}, \quad x = \frac{-5 - \sqrt{3}}{2}$
Check: We leave the check for you!	

Try It 2.2.4.9

Solve by using the Quadratic Formula: $3m^2 + 12m + 7 = 0$.

Answer

$$m=\frac{-6+\sqrt{15}}{3}, m=\frac{-6-\sqrt{15}}{3}$$

Try It 2.2.4.10

Solve by using the Quadratic Formula: $5n^2+4n-4=0$.

Answer

$$n=\frac{-2+2\sqrt{6}}{5}, n=\frac{-2-2\sqrt{6}}{5}$$

When we substitute $a, b,$ and c into the Quadratic Formula and the **radicand** is negative, the quadratic equation will have imaginary or complex solutions. We will see this in the next example.

Example 2.2.4.11

Solve by using the Quadratic Formula: $3p^2+2p+9=0$.

Solution:

	$3p^2+2p+9=0$
This equation is in standard form.	$\underbrace{3p^2+2p+9=0}_{ax^2+bx+c=0}$
Identify the values of a, b, c .	$a=3, b=2, c=9$
Write the Quadratic Formula.	$p=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a, b, c .	$p=\frac{-(2)\pm\sqrt{(2)^2-4\cdot 3\cdot 9}}{2\cdot 3}$
Simplify.	$p=\frac{-2\pm\sqrt{4-108}}{6}$ $p=\frac{-2\pm\sqrt{-104}}{6}$
Simplify the radical using complex numbers.	$p=\frac{-2\pm\sqrt{104}i}{6}$
Simplify the radical.	$p=\frac{-2\pm 2\sqrt{26}i}{6}$
Factor the common factor in the numerator.	$p=\frac{2(-1\pm\sqrt{26}i)}{6}$
Remove the common factors.	$p=\frac{-1\pm\sqrt{26}i}{3}$
Rewrite in standard $a+bi$ form.	$p=\frac{-1}{3}\pm\frac{\sqrt{26}i}{3}$
Write as two solutions.	$p=\frac{-1}{3}+\frac{\sqrt{26}i}{3}$ $p=\frac{-1}{3}-\frac{\sqrt{26}i}{3}$

Try It 2.2.4.12

Solve by using the Quadratic Formula: $4a^2-2a+8=0$.

Answer

$$a=\frac{1}{4}+\frac{\sqrt{31}}{4}i, \quad a=\frac{1}{4}-\frac{\sqrt{31}}{4}i$$

Try It 2.2.4.13

Solve by using the Quadratic Formula: $5b^2+2b+4=0$.

Answer

$$b=-\frac{1}{5}+\frac{\sqrt{19}}{5}i, \quad b=-\frac{1}{5}-\frac{\sqrt{19}}{5}i$$

Remember, to use the Quadratic Formula, the equation must be written in standard form, $ax^2+bx+c=0$. Sometimes, we will need to do some algebra to get the equation into standard form before we can use the Quadratic Formula.

Example 2.2.4.14

Solve by using the Quadratic Formula: $x(x+6)+4=0$.

Solution:

Our first step is to get the equation in standard form.

	$x(x+6)+4=0$
Distribute to get the equation in standard form.	$x^2+6x+4=0$
This equation is now in standard form.	$\underbrace{x^2+6x+4=0}_{ax^2+bx+c=0}$
Identify the values of a,b,c .	$a=1, b=6, c=4$
Write the Quadratic Formula.	$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of a,b,c .	$x=\frac{-(6)\pm\sqrt{(6)^2-4(1)(4)}}{2\cdot 1}$
Simplify.	$x=\frac{-6\pm\sqrt{36-16}}{2}$ $x=\frac{-6\pm\sqrt{20}}{2}$
Simplify the radical.	$x=\frac{-6\pm 2\sqrt{5}}{2}$
Factor the common factor in the numerator.	$x=\frac{2(-3\pm\sqrt{5})}{2}$
Remove the common factors.	$x=-3\pm\sqrt{5}$
Write as two solutions.	$x=-3+\sqrt{5}$ $x=-3-\sqrt{5}$
Check: We leave the check for you!	

Try It 2.2.4.15

Solve by using the Quadratic Formula: $x(x+2)-5=0$.

Answer

$$x=-1+\sqrt{6}, x=-1-\sqrt{6}$$

Try It 2.2.4.16

Solve by using the Quadratic Formula: $3y(y-2)-3=0$.

Answer

$y=1+\sqrt{2}, y=1-\sqrt{2}$

When we solved linear equations, if an equation had too many fractions we cleared the fractions by multiplying both sides of the equation by the LCD. This gave us an equivalent equation—without fractions— to solve. We can use the same strategy with quadratic equations.

Example 2.2.4.17

Solve by using the Quadratic Formula: $\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$.

Solution:

Our first step is to clear the fractions.

	$\frac{1}{2}u^2+\frac{2}{3}u=\frac{1}{3}$
Multiply both sides by the LCD, 6, to clear the fractions.	$\frac{6}{2}u^2+\frac{6\cdot 2}{3}u=\frac{6}{3}$
Reduce fractions and simplify..	$3u^2+4u=2$
Subtract 2 to get the equation in standard form.	$3u^2+4u-2=0$
Identify the values of a,b, and c.	$(a=3, b=4, c=-2$
Write the Quadratic Formula.	$u=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$
Then substitute in the values of $a,b,$ and c.	$u=\frac{-4\pm\sqrt{(4)^2-4(3)(-2)}}{2(3)}$
Simplify.	$u=\frac{-4\pm\sqrt{4(4+6)}}{6}$
Simplify the radical.	$u=\frac{-4\pm 2\sqrt{10}}{6}$
Factor the common factor in the numerator.	$u=\frac{2(-2\pm\sqrt{10})}{2\cdot 3}$
Remove the common factors.	$u=\frac{-2\pm\sqrt{10}}{3}$
Rewrite to show two solutions.	$u=\frac{-2+\sqrt{10}}{3}$ or $u=\frac{-2-\sqrt{10}}{3}$
Check: We leave the check for you!	

Try It 2.2.4.18

Solve by using the Quadratic Formula: $\frac{1}{4}c^2-\frac{1}{3}c=\frac{1}{12}$.

Answer

$c=\frac{2+\sqrt{7}}{3}, \quad c=\frac{2-\sqrt{7}}{3}$

Try It 2.2.4.19

Solve by using the Quadratic Formula: $\frac{1}{9}d^2 - \frac{1}{2}d = -\frac{1}{3}$.

Answer

$d = \frac{9 + \sqrt{33}}{4}, d = \frac{9 - \sqrt{33}}{4}$

Think about the equation $(x-3)^2 = 0$. We know from the **Zero Product Property** that this equation has only one solution, $x = 3$.

We will see in the next example how using the **Quadratic Formula** to solve an equation whose standard form is a perfect square **trinomial** equal to 0 gives just one solution. Notice that once the radicand is simplified it becomes 0, which leads to only one solution.

Example 2.2.4.20

Solve by using the Quadratic Formula: $4x^2 - 20x = -25$.

Solution:

	$4x^2 - 20x = -25$
Add 25 to get the equation in standard form.	$4x^2 - 20x + 25 = 0$
Identify the values of a, b, and c.	$(a = 4, b = -20, c = 25)$
Write the quadratic formula.	$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, and c.	$x = \frac{-(-20) \pm \sqrt{(-20)^2 - 4(4)(25)}}{2 \cdot (4)}$
Simplify.	$x = \frac{20 \pm \sqrt{0}}{8}$
Simplify the fraction.	$x = \frac{5}{2}$
Check: We leave the check for you!	

Did you recognize that $4x^2 - 20x + 25$ is a perfect square trinomial. It is equivalent to $(2x-5)^2$? If you solve $4x^2 - 20x + 25 = 0$ by factoring and then using the Square Root Property, do you get the same result?

Try It 2.2.4.21

Solve by using the Quadratic Formula: $r^2 + 10r + 25 = 0$.

Answer

$r = -5$

Try It 2.2.4.22

Solve by using the Quadratic Formula: $25t^2 - 40t = -16$.

Answer

$t = \frac{4}{5}$

Use the Discriminant to Predict the Number and Type of Solutions of a Quadratic Equation

When we solved the quadratic equations in the previous examples, sometimes we got two real solutions, one real solution, and sometimes two complex solutions. Is there a way to predict the number and type of solutions to a quadratic equation without actually solving the equation?

Yes, the expression under the radical of the Quadratic Formula makes it easy for us to determine the number and type of solutions. This expression is called the **discriminant**.

Definition 2.2.4.23

If $ax^2 + bx + c = 0$, the quantity

$$b^2 - 4ac$$

is called the **discriminant**. It is the radicand in the quadratic formula.

Let's look at the discriminant of the equations in some of the examples and the number and type of solutions to those quadratic equations.

Quadratic Equation (in standard form)	Discriminant $b^2 - 4ac$	Value of the Discriminant	Number and Type of Solutions
$2x^2 + 9x - 5 = 0$	$9^2 - 4 \cdot 2(-5)$ 121	$+$	2 real
	0	0	1 real
$3p^2 + 2p + 9 = 0$	-104	$-$	2 complex

When the value under the radical in the Quadratic Formula, the discriminant, is positive, the equation has two real solutions. When the value under the radical in the Quadratic Formula, the discriminant, is zero, the equation has one real solution. When the value under the radical in the Quadratic Formula, the discriminant, is negative, the equation has two complex solutions.

Figure 9.3.86

Using the Discriminant $b^2 - 4ac$, to Determine the Number and Type of Solutions of a Quadratic Equation

For a quadratic equation of the form $ax^2 + bx + c = 0$, $a \neq 0$,

- If $b^2 - 4ac > 0$, the equation has 2 real solutions.
- if $b^2 - 4ac = 0$, the equation has 1 real solution.
- if $b^2 - 4ac < 0$, the equation has 2 complex solutions.

Example 2.2.4.24

Determine the number of solutions to each quadratic equation.

a. $3x^2 + 7x - 9 = 0$

b. $5n^2 + n + 4 = 0$

c. $9y^2 - 6y + 1 = 0$

Solution:

To determine the number of solutions of each quadratic equation, we will look at its discriminant.

a.

$$3x^2 + 7x - 9 = 0$$

The equation is in standard form, identify a, b, and c.

$$a = 3, \quad b = 7, \quad c = -9$$

Write the discriminant.

Substitute in the values of a, b, and c.

Simplify.

$$49+108$$
$$157$$

Since the discriminant is positive, there are 2 real solutions to the equation.

b.

$$5n^2+n+4=0$$

The equation is in standard form, identify a, b, and c.

$$a=5, \quad b=1, \quad c=4$$

Write the discriminant.

Substitute in the values of a, b, and c.

Simplify.

$$1-80$$
$$-79$$

Since the discriminant is negative, there are 2 complex solutions to the equation.

c.

The equation is in standard form, identify a, b, and c.

$a=9, \quad b=-6, \quad c=1$

Write the discriminant.

Substitute in the values of a, b, and c.

Simplify.

$$36-36$$
$$0$$

Since the discriminant is 0, there is 1 real solution to the equation.

Try It 2.2.4.25

Determine the number and type of solutions to each quadratic equation.

a. $8m^2-3m+6=0$

b. $5z^2+6z-2=0$

c. $9w^2+24w+16=0$

Answer

a. 2 complex solutions

b. 2 real solutions

c. 1 real solution

Try It 2.2.4.26

Determine the number and type of solutions to each quadratic equation.

a. $b^2 + 7b - 13 = 0$

b. $5a^2 - 6a + 10 = 0$

c. $4r^2 - 20r + 25 = 0$

Answer

a. 2 real solutions

b. 2 complex solutions

c. 1 real solution

Identify the Most Appropriate Method to Use to Solve a Quadratic Equation

We summarize the four methods that we have used to solve quadratic equations below.

Methods for Solving Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

Given that we have four methods to use to solve a quadratic equation, how do you decide which one to use? Factoring is often the quickest method and so we try it first. If the equation is $ax^2 = k$ or $a(x - h)^2 = k$ we use the Square Root Property. For any other equation, it is probably best to use the Quadratic Formula. Remember, you can solve any quadratic equation by using the Quadratic Formula, but that is not always the easiest method.

What about the method of Completing the Square? Most people find that method cumbersome and prefer not to use it. We needed to include it in the list of methods because we completed the square in general to derive the Quadratic Formula. You will also use the process of Completing the Square in other areas of algebra.

Identify the Most Appropriate Method to Solve a Quadratic Equation

1. Try **Factoring** first. If the quadratic factors easily, this method is very quick.
2. Try the **Square Root Property** next. If the equation fits the form $ax^2 = k$ or $a(x - h)^2 = k$, it can easily be solved by using the Square Root Property.
3. Use the **Quadratic Formula**. Any other quadratic equation is best solved by using the Quadratic Formula.

The next example uses this strategy to decide how to solve each quadratic equation.

Example 2.2.4.27

Identify the most appropriate method to use to solve each quadratic equation.

a. $5z^2 = 17$
b. $4x^2 - 12x + 9 = 0$
c. $8u^2 + 6u = 11$

Solution:

a.

$$5z^2 = 17$$

Since the equation is in the $ax^2 = k$, the most appropriate method is to use the Square Root Property.

b.

We recognize that the left side of the equation is a perfect square trinomial, and so factoring will be the most appropriate method.

c.

$8u^2 + 6u = 11$

Put the equation in standard form.

$$8u^2 + 6u - 11 = 0$$

While our first thought may be to try factoring, thinking about all the possibilities for trial and error method leads us to choose the Quadratic Formula as the most appropriate method.

Try It 2.2.4.28

Identify the most appropriate method to use to solve each quadratic equation.

a. $x^2 + 6x + 8 = 0$
b. $(n-3)^2 = 16$
c. $5p^2 - 6p = 9$

Answer

a. Factoring
b. Square Root Property
c. Quadratic Formula

Try It 2.2.4.29

Identify the most appropriate method to use to solve each quadratic equation.

a. $8a^2 + 3a - 9 = 0$
b. $4b^2 + 4b + 1 = 0$
c. $5c^2 = 125$

Answer

a. Quadratic Formula
b. Factoring or Square Root Property
c. Square Root Property

? Writing Exercises 2.2.4.30

1. How is the quadratic formula related to the process of completing the square?
2. Why would we ever need or want to factor if we have the quadratic formula?
3. Suppose we use the quadratic formula to solve a quadratic equation, what does what is under the radical symbol tell us about the solution(s) ?

Exit Problem 2.2.4.31

Solve $3^2 - 4 = 2x$ by using the quadratic formula.

Key Concepts

- Quadratic Formula
  - The solutions to a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$ are given by the formula:

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$

- How to solve a quadratic equation using the Quadratic Formula.
  1. Write the quadratic equation in standard form, $ax^2+bx+c=0$. Identify the values of a, b, c.
  2. Write the Quadratic Formula. Then substitute in the values of a, b, c.
  3. Simplify.
  4. Check the solutions to detect errors.
- Using the Discriminant, b^2-4ac, to Determine the Number and Type of Solutions of a Quadratic Equation
- For a quadratic equation of the form $ax^2+bx+c=0, a\neq 0$,
  - If $b^2-4ac>0$, the equation has 2 real solutions.
  - If $b^2-4ac=0$, the equation has 1 real solution.
  - If $b^2-4ac<0$, the equation has 2 complex solutions.

2.2.5: Applications of Quadratic Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve applications modeled by quadratic equations

Be Prepared

Before you get started, take this readiness quiz.

1. The sum of two consecutive odd numbers is -100. Find the numbers.
2. Find the length of the hypotenuse of a right triangle with legs 5 inches and 12 inches.

Solve Applications Modeled by Quadratic Equations

We solved some applications that are modeled by quadratic equations earlier, when the only method we had to solve them was factoring. Now that we have more methods to solve quadratic equations, we will take another look at applications.

Let's first summarize the methods we now have to solve quadratic equations.

Methods to Solve Quadratic Equations

1. Factoring
2. Square Root Property
3. Completing the Square
4. Quadratic Formula

As we solve each equation, we choose the method that is most convenient for us to work the problem. As a reminder, we will copy our usual Problem-Solving Strategy here so we can follow the steps.

Use a Problem-Solving Strategy

1. **Read** the problem. Make sure all the words and ideas are understood.
2. **Identify** what we are looking for.
3. **Name** what we are looking for. Choose a variable to represent that quantity.
4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebraic equation.
5. **Solve** the equation using algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

We have solved a number of applications that involved consecutive even and odd integers, by modeling the situation with linear equations. Remember, we noticed each even integer is 2 more than the number preceding it. If we call the first one n, then the next one is $n+2$. The next one would be $n+2+2$ or $n+4$. This is also true when we use odd integers. One set of even integers and one set of odd integers are shown below.

$$\begin{array}{rl} & \text{Consecutive even integers} \\ & 64, 66, 68 \\ n & 1^{\text{st}} \text{ even integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive even integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive even integer} \end{array}$$

$$\begin{array}{ll} & \text{Consecutive odd integers} \\ & 77, 79, 81 \\ n & 1^{\text{st}} \text{ odd integer} \\ n+2 & 2^{\text{nd}} \text{ consecutive odd integer} \\ n+4 & 3^{\text{rd}} \text{ consecutive odd integer} \end{array}$$

Some applications of odd or even consecutive integers are modeled by quadratic equations. The notation above will be helpful as we name the variables.

Example 2.2.5.1

The product of two consecutive odd integers is 195. Find the integers.

Solution

Read the problem.	
Identify what we are looking for.	We are looking for two consecutive odd integers.
Name what we are looking for.	Let $n =$ the first odd integer. Then (n+2=\) the next odd integer.
Translate into an equation. State the problem in one sentence.	"The product of two consecutive odd integers is 195." The product of the first odd integer and the second odd integer is 195.
Translate into an equation.	$n(n+2)=195$
Solve the equation. Distribute.	$n^2+2n=195$
Write the equation in standard form.	$n^2+2n-195=0$
Factor.	$(n+15)(n-13)=0$
Use the Zero Product Property.	$n+15=0 \quad \text{or} \quad n-13=0$
Solve each equation.	$n=-15 \quad \text{or} \quad n=13$
There are two values of n that are solutions. This will give us two pairs of consecutive odd integers for our solution.	First odd integer: $n=-15$ Next odd integer $n+2$ $=-15+2$ $=-13$ First odd integer: $n=13$ Next odd integer $n+2$ $=13+2$ $=15$
Check the answer. Do these pairs work? Are they consecutive odd integers?	• $-15, -13$ are consecutive odd integers. Is their product 195? Yes: $-13(-15)=195$ • $13, 15$ are consecutive odd integers. Is their product 195? Yes: $13 \cdot 15=195$
Answer the question.	Two consecutive odd integers whose product is 195 are $-15, -13$ and $13, 15$.

Try It 2.2.5.2

The product of two consecutive odd integers is 99. Find the integers.

Answer

The two consecutive odd integers whose product is 99 are $9, 11$, and $-9, -11$.

Try It 2.2.5.3

The product of two consecutive even integers is 168. Find the integers.

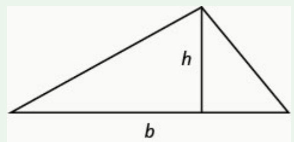
Answer

The two consecutive even integers whose product is 128 are $12, 14$ and $-12, -14$.

We will use the formula for the area of a triangle to solve the next example.

Area of a Triangle

For a triangle with base, b, and height, h, the area, A, is given by the formula $A = \frac{1}{2}bh$.

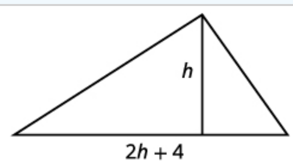


Recall that when we solve geometric applications, it is helpful to draw the figure.

Example 2.2.5.4

An architect is designing the entryway of a restaurant. She wants to put a triangular window above the doorway. Due to energy restrictions, the window can only have an area of 120 square feet and the architect wants the base to be 4 feet more than twice the height. Find the base and height of the window.

Solution

Read the problem. Draw a picture.	
Identify what we are looking for.	We are looking for the base and height.
Name what we are looking for.	Let h = the height of the triangle. Then $2h + 4$ = the base of the triangle.
Translate into an equation. We know the area. Write the formula for the area of a triangle.	$A = \frac{1}{2}bh$
Solve the equation. Substitute in the values.	$120 = \frac{1}{2}(2h + 4)h$
Distribute.	$120 = h^2 + 2h$
This is a quadratic equation, rewrite it in standard form.	$h^2 + 2h - 120 = 0$
Factor.	$(h - 10)(h + 12) = 0$
Use the Zero Product Property.	$h - 10 = 0 \quad h + 12 = 0$
Simplify.	$h = 10, \quad h = \cancel{-12}$
	Since h is the height of a window, a value of $h = -12$ does not make sense. The height of the triangle $h = 10$. The base of the triangle $2h + 4$. $2 \cdot 10 + 4$ 24

Check the answer. Does a triangle with height 10 and base 24 have area 120? Yes.	
Answer the question.	The height of the triangular window is 10 feet and the base is 24 feet.

? Try It 2.2.5.5

Find the base and height of a triangle whose base is four inches more than six times its height and has an area of 456 square inches.

Answer

The height of the triangle is 12 inches and the base is 76 inches.

? Try It 2.2.5.6

If a triangle that has an area of 110 square feet has a base that is two feet less than twice the height, what is the length of its base and height?

Answer

The height of the triangle is 11 feet and the base is 20 feet.

In the two preceding examples, the number in the radical in the **Quadratic Formula** was a perfect square and so the solutions were rational numbers. If we get an irrational number as a solution to an application problem, we will use a calculator to get an approximate value.

We will use the formula for the area of a rectangle to solve the next example.

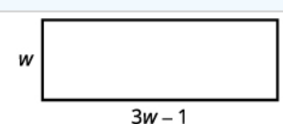
Area of a Rectangle

For a rectangle with length, L, and width, W, the area, A, is given by the formula $A = LW$.

? Example 2.2.5.7

Mike wants to put 150 square feet of artificial turf in his front yard. This is the maximum area of artificial turf allowed by his homeowners association. He wants to have a rectangular area of turf with length one foot less than 3 times the width. Find the length and width. Round to the nearest tenth of a foot.

Solution

Read the problem. Draw a picture.	
Identify what we are looking for.	We are looking for the length and width.
Name what we are looking for.	Let $W =$ the width of the rectangle. Then $3W - 1 =$ the length of the rectangle

Translate into an equation. We know the area. Write the formula for the area of a rectangle.	$A = L \cdot W$
Solve the equation. Substitute in the values.	$150 = (3W - 1)W$
Distribute.	$150 = 3W^2 - W$
This is a quadratic equation; rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\underbrace{3W^2 - W - 150 = 0}_{ax^2+bx+c=0}$
Identify the a, b, c values.	$a = 3$ $b = -1$ $c = -150$
Write the Quadratic Formula.	$W = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$W = \frac{-(-1) \pm \sqrt{(-1)^2 - 4 \cdot 3 \cdot (-150)}}{2 \cdot 3}$
Simplify.	$W = \frac{1 \pm \sqrt{1 + 1800}}{6}$ $W = \frac{1 \pm \sqrt{1801}}{6}$
Rewrite to show two solutions.	$W = \frac{1 + \sqrt{1801}}{6}, \quad \frac{1 - \sqrt{1801}}{6}$
Approximate the answers using a calculator. We eliminate the negative solution for the width.	$w \approx 7.2$, ~~$w \approx -6.9$~~ Width $w \approx 7.2$ Length $\approx 3w - 1$ $\approx 3(7.2) - 1$ ≈ 20.6
Check the answer. Make sure that the answers make sense. Since the answers are approximate, the area will not come out exactly to 150.	
Answer the question.	The width of the rectangle is approximately 7.2 feet and the length is approximately 20.6 feet.

? Try It 2.2.5.8

The length of a 200 square foot rectangular vegetable garden is four feet less than twice the width. Find the length and width of the garden, to the nearest tenth of a foot.

Answer

The length of the garden is approximately 18 feet and the width 11 feet.

? Try It 2.2.5.9

A rectangular tablecloth has an area of 80 square feet. The width is 5 feet shorter than the length.What are the length and width of the tablecloth to the nearest tenth of a foot?

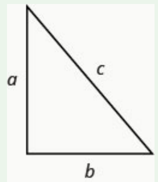
Answer

The length of the tablecloth is approximately 11.8 feet and the width 6.8 feet.

The **Pythagorean Theorem** gives the relation between the legs and hypotenuse of a right triangle. We will use the Pythagorean Theorem to solve the next example.

Pythagorean Theorem

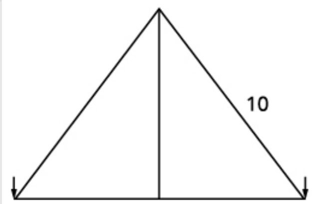
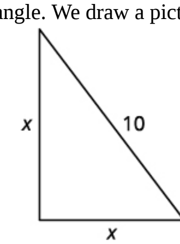
In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2 + b^2 = c^2$.



Example 2.2.5.10

Rene is setting up a holiday light display. He wants to make a 'tree' in the shape of two right triangles, as shown below, and has two 10-foot strings of lights to use for the sides. He will attach the lights to the top of a pole and to two stakes on the ground. He wants the height of the pole to be the same as the distance from the base of the pole to each stake. How tall should the pole be?

Solution

Read the problem. Draw a picture.	
Identify what we are looking for.	We are looking for the height of the pole.
Name what we are looking for.	The distance from the base of the pole to either stake is the same as the height of the pole. Let $x =$ the height of the pole. Then $x =$ the distance from pole to stake Each side is a right triangle. We draw a picture of one of them. 
Translate into an equation. We can use the Pythagorean Theorem to solve for x. Write the Pythagorean Theorem.	$a^2 + b^2 = c^2$
Solve the equation. Substitute.	$x^2 + x^2 = 10^2$
Simplify.	$2x^2 = 100$
Divide by 2 to isolate the variable.	$\frac{2x^2}{2} = \frac{100}{2}$
Simplify.	$x^2 = 50$
Use the Square Root Property.	$x = \pm\sqrt{50}$
Simplify the radical.	$x = \pm 5\sqrt{2}$
Rewrite to show two solutions.	$x = 5\sqrt{2}$ or $x = \cancel{-5\sqrt{2}}$

	If we approximate $x = 5\sqrt{2}$ to the nearest tenth with a calculator, we find $x \approx 7.1$.
Check the answer. Check on your own in the Pythagorean Theorem.	
Answer the question.	The pole should be about 7.1 feet tall.

Try It 2.2.5.11

The sun casts a shadow from a flag pole. The height of the flag pole is three times the length of its shadow. The distance between the end of the shadow and the top of the flag pole is 20 feet. Find the length of the shadow and the length of the flag pole. Round to the nearest tenth.

Answer

The length of the flag pole's shadow is approximately 6.3 feet and the height of the flag pole is 18.9 feet.

Try It 2.2.5.12

The distance between opposite corners of a rectangular field is four more than the width of the field. The length of the field is twice its width. Find the distance between the opposite corners. Round to the nearest tenth.

Answer

The distance between the opposite corners is approximately 7.2 feet.

OPTIONAL APPLICATIONS

The height of a projectile shot upward from the ground is modeled by a quadratic equation. The initial velocity, v_0, propels the object up until gravity causes the object to fall back down.

Projectile Motion

The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula

$$h = -16t^2 + v_0 t\ .$$

We can use this formula to find how many seconds it will take for a firework to reach a specific height.

Example 2.2.5.13

A firework is shot upwards with initial velocity 130 feet per second. How many seconds will it take to reach a height of 260 feet? Round to the nearest tenth of a second.

Solution

Read the problem.	
Identify what we are looking for.	We are looking for the number of seconds, which is time.
Name what we are looking for.	Let $t =$ the number of seconds.
Translate into an equation. Use the formula.	$h = -16t^2 + v_0 t$\)
Solve the equation. We know the velocity v_0 is 130 feet per second. The height is 260 feet. Substitute the values.	$260 = -16t^2 + 130t$
This is a quadratic equation, rewrite it in standard form. Solve the equation using the Quadratic Formula.	$\underbrace{16t^2 - 130t + 260 = 0}_{ax^2+bx+c=0}$

Identify the values of a, b, c.	$a = 16$ $b = -130$ $c = 260$
Write the Quadratic Formula.	$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
Then substitute in the values of a, b, c.	$t = \frac{-(-130) \pm \sqrt{(-130)^2 - 4 \cdot 16 \cdot 260}}{2 \cdot 16}$
Simplify.	$t = \frac{130 \pm \sqrt{16{,}900 - 16{,}640}}{32}$ $t = \frac{130 \pm \sqrt{260}}{32}$
Rewrite to show two solutions.	$t = \frac{130 + \sqrt{260}}{32}, \quad t = \frac{130 - \sqrt{260}}{32}$
Approximate the answer with a calculator.	$t \approx 4.6$ seconds $t \approx 3.6$ seconds
Check the answer. The check is left to you.	
Answer the question.	The firework will go up and then fall back down. As the firework goes up, it will reach 260 feet after approximately 3.6 seconds. It will also pass that height on the way down at 4.6 seconds.

Try It 2.2.5.14

An arrow is shot from the ground into the air at an initial speed of 108 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the arrow will be 180 feet from the ground. Round the nearest tenth.

Answer

The arrow will reach 180 feet on its way up after 3 seconds and again on its way down after approximately 3.8 seconds.

Try It 2.2.5.15

A man throws a ball into the air with a velocity of 96 ft/s. Use the formula $h = -16t^2 + v_0 t$ to determine when the height of the ball will be 48 feet. Round to the nearest tenth.

Answer

The ball will reach 48 feet on its way up after approximately .6 second and again on its way down after approximately 5.4 seconds.

We have solved uniform motion problems using the formula $D = rt$ in previous chapters. We used a table like the one below to organize the information and lead us to the equation.

	Rate •	Time =	Distance

Figure 9.5.29

The formula $D = rt$ assumes we know r and t and use them to find D. If we know D and r and need to find t, we would solve the equation for t and get the formula $t = \frac{D}{r}$.

Some uniform motion problems are also modeled by quadratic equations.

Example 2.2.5.16

Professor Smith just returned from a conference that was $2,000$ miles east of his home. His total time in the airplane for the round trip was 9 hours. If the plane was flying at a rate of 450 miles per hour, what was the speed of the jet stream?

Solution

This is a uniform motion situation. A diagram will help us visualize the situation.

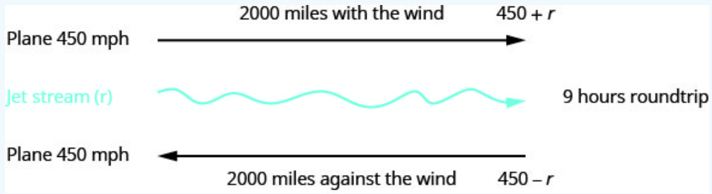


Figure 9.5.30

We fill in the chart to organize the information.

We are looking for the speed of the jet stream. Let $r =$ the speed of the jet stream.

When the plane flies with the wind, the wind increases its speed and so the rate is $450 + r$.

When the plane flies against the wind, the wind decreases its speed and the rate is $450 - r$.

Write in the rates.
Write in the distances.
Since $D = r \cdot t$, we solve for t and get $t = \frac{D}{r}$.
We divide the distance by the rate in each row, and place the expression in the time column.

Type	Rate	·	Time	=	Distance
Headwind	$450 - r$		$\frac{2000}{450 - r}$		2000
Tailwind	$450 + r$		$\frac{2000}{450 + r}$		2000
			9		

We know the times add to 9 and so we write our equation.	$\frac{2000}{450 - r} + \frac{2000}{450 + r} = 9$
We multiply both sides by the LCD.	$(450 - r)(450 + r)\left(\frac{2000}{450 - r} + \frac{2000}{450 + r}\right) = 9(450 - r)(450 + r)$
Simplify.	$2000(450 + r) + 2000(450 - r) = 9(450 - r)(450 + r)$
Factor the $2,000$.	$2000(450 + r + 450 - r) = 9\left(450^2 - r^2\right)$
Solve.	$2000(900) = 9\left(450^2 - r^2\right)$
Divide by 9.	$2000(100) = 450^2 - r^2$
Simplify.	$200000 = 202500 - r^2$ $-2500 = -r^2$ \ $50 = r$ The speed of the jet stream is 50 mph.
Check:	Is 50 mph a reasonable speed for the jet stream? Yes. If the plane is traveling 450 mph and the wind is 50 mph, Tailwind $450 + 50 = 500\text{mph} \quad \frac{2000}{500} = 4\text{hours}$ Headwind $450 - 50 = 400\text{mph} \quad \frac{2000}{400} = 5\text{hours}$ The times add to 9 hours, so it checks.
Answer the question.	The speed of the jet stream was 50 mph.

Try It 2.2.5.17

MaryAnne just returned from a visit with her grandchildren back east. The trip was 2400 miles from her home and her total time in the airplane for the round trip was 10 hours. If the plane was flying at a rate of 500 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 100 mph.

Try It 2.2.5.18

Gerry just returned from a cross country trip. The trip was 3000 miles from his home and his total time in the airplane for the round trip was 11 hours. If the plane was flying at a rate of 550 miles per hour, what was the speed of the jet stream?

Answer

The speed of the jet stream was 50 mph.

Work applications can also be modeled by quadratic equations. We will set them up using the same methods we used when we solved them with rational equations.We'll use a similar scenario now.

Example 2.2.5.19

The weekly gossip magazine has a big story about the presidential election and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 12 hours more than Press #2 to do the job and when both presses are running they can print the job in 8 hours. How long does it take for each press to print the job alone?

Solution

This is a work problem. A chart will help us organize the information.

We are looking for how many hours it would take each press separately to complete the job.

Let $x =$ the number of hours for Press #2 to complete the job. Enter the hours per job for Press #1, Press #2, and when they work together.	(see chart below)
The part completed by Press #1 plus the part completed by Press #2 equals the amount completed together. Translate to an equation.	Work completed by Press #1 + Press #2 = Together $\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$
Solve.	$\frac{1}{x+12} + \frac{1}{x} = \frac{1}{8}$
Multiply by the LCD, $8x(x+12)$.	$8x(x+12)\left(\frac{1}{x+12} + \frac{1}{x}\right) = \left(\frac{1}{8}\right)8x(x+12)$
Simplify.	$8x + 8(x+12) = x(x+12)$ $8x + 8x + 96 = x^2 + 12x$ Figure 9.5.37 $0 = x^2 - 4x - 96$ Figure 9.5.38

	Number of hours needed to complete the job.	Part of job completed/hour
Press #1	$x + 12$	$\frac{1}{x+12}$
Press #2	x	$\frac{1}{x}$
Together	8	$\frac{1}{8}$

Solve.	$0 = (x - 12)(x + 8)$ $x - 12 = 0, x + 8 = 0$ Figure 9.5.40 $x = 12, \cancel{x = -8}$ hours Figure 9.5.41
Since the idea of negative hours does not make sense, we use the values $x = 12$.	$12 + 12$ 24 hours 12 12 hours Figure 9.5.43
Write our sentence answer.	Press #1 would take 24 hours and Press #2 would take 12 hours to do the job alone.

Try It 2.2.5.20

The weekly news magazine has a big story naming the Person of the Year and the editor wants the magazine to be printed as soon as possible. She has asked the printer to run an extra printing press to get the printing done more quickly. Press #1 takes 6 hours more than Press #2 to do the job and when both presses are running they can print the job in 4 hours. How long does it take for each press to print the job alone?

Answer

Press #1 would take 12 hours, and Press #2 would take 6 hours to do the job alone.

Try It 2.2.5.21

Erlinda is having a party and wants to fill her hot tub. If she only uses the red hose it takes 3 hours more than if she only uses the green hose. If she uses both hoses together, the hot tub fills in 2 hours. How long does it take for each hose to fill the hot tub?

Answer

The red hose take 6 hours and the green hose take 3 hours alone.

Writing Exercises 2.2.5.22

1. When solving a word problem, why might it be important to draw a picture representing the situation?
2. What aspects of the drawing are important?

Exit Problem 2.2.5.23

The length of a rectangular photograph is 7in. more than the width. If the area is 78in^2, what are the dimensions?

Key Concepts

- Methods to Solve Quadratic Equations
  - Factoring
  - Square Root Property
  - Completing the Square
  - Quadratic Formula
- How to use a Problem-Solving Strategy.
  1. **Read** the problem. Make sure all the words and ideas are understood.
  2. **Identify** what we are looking for.
  3. **Name** what we are looking for. Choose a variable to represent that quantity.

4. **Translate** into an equation. It may be helpful to restate the problem in one sentence with all the important information. Then, translate the English sentence into an algebra equation.
5. **Solve** the equation using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

- Area of a Triangle
  - For a triangle with base, b, and height, h, the area, A, is given by the formula $A=\frac{1}{2}bh$.

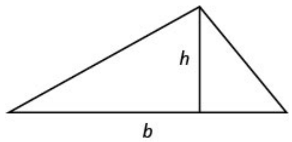


Figure 9.5.1

- Area of a Rectangle
  - For a rectangle with length,L, and width, W, the area, A, is given by the formula $A=LW$.

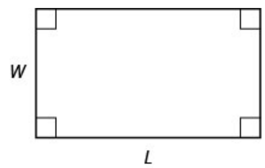


Figure 9.5.3

- Pythagorean Theorem
  - In any right triangle, where a and b are the lengths of the legs, and c is the length of the hypotenuse, $a^2+b^2=c^2$.

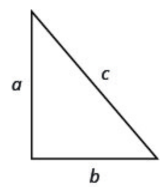


Figure 9.5.16

- Projectile motion
  - The height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds is given by the formula $h=-16t^2+v_0t$.

2.3: Polynomial Equations

Learning Objectives

By the end of this section, you will be able to:

- Understand what it means to solve a polynomial equation
- Solve certain polynomial equations with one variable.

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $(x-2)(3x-7)=0$.
2. Divide $(x^3-6x^2+11x-6)$ by $(x-1)$.
3. Solve $2x^2-5x-1=0$.

Polynomial equations with one variable have features that are similar to the quadratic equations, the kind of polynomial equations we treated in the previous section.

For example, consider the polynomial equation

$$0=2(x-1)(2x-3)(x+7).$$

Note that the right hand side of the equation is a polynomial of degree 3 with leading coefficient 4 which we can see by imagining distributing the product. We see that if we replace x with 1 we arrive at the true statement $0=0$ so that 1 is a solution. Also, when we replace x with $\frac{3}{2}$, the factor $(2x-3)$ has the value 0 so that the right hand side is also 0 and we have again the true statement $0=0$. Similarly, if we replace x with -7 the factor $(x+7)$ becomes 0 and again we arrive at $0=0$.

So, 1, $\frac{3}{2}$, and -7 are solutions. For other values of x, none of the factors are zero, so the product on the right hand side is not zero. It follows that these are the only solutions.

Definition

A number c is a root of a polynomial of a single variable if when we substitute c for that variable the result is equal to 0.

So, in the above example, the roots of $2(x-1)(2x-3)(x+7)$ are $1, \frac{3}{2}$ and -7. Note that, then, $1, \frac{3}{2}$ and -7 are solutions to the equation $2(x-1)(2x-3)(x+7)=0$.

We can also easily write down a polynomial with prescribed roots. For example, suppose we want to write down a polynomial of degree 3 with roots 3, -2, and 4 and leading coefficient 7. We simply try to arrange the situation that we have above. We attempt to find factors so that one of the factors is zero when we substitute 3, another is zero when we substitute -2 and yet another is zero when we substitute 4. We see that $(x-3)(x+2)(x-4)$ is a polynomial with the appropriate zeros. But the leading coefficient is 1. To arrange the leading coefficient to be 7, we need only multiply (so that $(x-3)$, $(x+2)$ and $(x-4)$ are still factors) by 7. So,

$$7(x-3)(x+2)(x-4) \tag{2.3.1}$$

is a polynomial of degree 7 with leading coefficient 7 and roots $4, -2$, and 4.

More generally, we have the following

Theorem 2.3.1

If c is a root of a polynomial, then $(x-c)$ is a factor of that polynomial, i.e., the polynomial is equivalent to $(x-c)(\text{some polynomial})$.

Now suppose we want to solve

$$0 = x^3 + 2x^2 - 11x - 12. \tag{2.3.2}$$

The right hand side is not quadratic and in general when the degree is 3 or larger the equation can be very difficult to solve.

However, in this case, we can observe (via trial and error or with the help of a calculator) that

$$0 = (-1)^3 + 2(-1)^2 - 11(-1) - 12. \tag{2.3.3}$$

It follows that $(x+1)$ is a factor! What is the other factor? We can use long division to find

$$x^3 + 2x^2 - 11x - 12 = (x+1)(x^2 + x - 12). \tag{2.3.4}$$

This is a product of a linear expression and a quadratic expression (which can then be factored!!).

We see then that

$$x^3 + 2x^2 - 11x - 12 = (x+1)(x+4)(x-3). \tag{2.3.5}$$

So,

$$0 = x^3 + 2x^2 - 11x - 12 \tag{2.3.6}$$

is equivalent to

$$0 = (x+1)(x+4)(x-3). \tag{2.3.7}$$

So we can see the solutions are -1, -4 and 3.

Example 2.3.9

Solve $2x^3 + x^2 - 5x + 2 = 0$.

Solution

To begin solving this equation, we note that by guess and checking some simple values for x , perhaps with the help of a calculator,

$2(1)^3 + (1)^2 - 5(1) + 2 = 0$

so, 1 is a root of $2x^3 + x^2 - 5x + 2$! It follows that $(x-1)$ is a factor, i.e.,

$2x^3 + x^2 - 5x + 2 = (x-1)(\text{some polynomial})$.

We can proceed by guessing and checking or by dividing $2x^3 + x^2 - 5x + 2$ by $(x-1)$. By guessing at coefficients we can easily determine that that mystery polynomial has a leading term of $2x^2$ and a constant term of -2. So, this mystery polynomial looks like $2x^2 + bx - 2$. But, the coefficient of x when we multiply $(x-1)(2x^2 + bx - 2)$ is $-b-2$ but, being equal to $2x^3 + x^2 - 5x + 2$ must also be -5 so b must be 3 and

$$2x^3 + x^2 - 5x + 2 = (x-1)(2x^2 - 3x - 2). \tag{2.3.8}$$

This should be checked by multiplying the right hand side.

We can now hope to factor the quadratic factor by guessing and checking, the AC method, or by finding roots (and therefore corresponding factors) by using the quadratic formula as we will see in a later section. Here we see

$$2x^3 + x^2 - 5x + 2 = (x-1)(2x^2 - 3x - 2) = (x-1)(2x+1)(x-2) \tag{2.3.9}$$

so the equation

$$2x^3 + x^2 - 5x + 2 = 0 \tag{2.3.10}$$

is equivalent to

$$(x-1)(2x+1)(x-2) = 0. \tag{2.3.11}$$

We can use the zero-product-property then to find that the solutions are $1, -\frac{1}{2}$ and 2.

We may check that substituting any of the three values lead to a true statement in the original equation.

Try It 2.3.10

Solve $y^3 - 6y^2 - y + 30 = 0$.

Answer

$y = -2, 3$ or 5.

Try It 2.3.11

Solve $2x^3 - 5x^2 = x - 6$.

Answer

$x = -1, \frac{3}{2},$ or 2

Writing Exercises 2.3.12

1. Explain the relationships between roots and factors of a polynomial.
2. What can you say about the degree of a polynomial which has roots $1, 0$ and -2?
3. If the height at time t is a polynomial with variable t, what do the roots represent?

Exit Problem

1. Give a polynomial of degree 4 with roots $2, 3, -1$ and 0.
2. Find all solutions to the polynomial equation $x^3 + 3x^2 - 5x = 6$.

Key Concepts

roots of a polynomial

Factors of a polynomial

2.4: Rational Equations

Learning Objectives

- Solve rational equations
- Use rational functions
- Solve a rational equation for a specific variable

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $\frac{1}{6}x+\frac{1}{2}=\frac{1}{3}$.

2. Solve $n^2-5n-36=0$.

3. Solve the formula $5x+2y=10$ for y.

After defining the terms 'expression' and 'equation' earlier, we have used them throughout this book. We have simplified many kinds of expressions and solved many kinds of equations. We have simplified many rational expressions so far in this chapter. Now we will solve **rational equations**.

Definition 2.4.1

A **rational equation** is an equation that contains a rational expression.

Let's recall the difference between rational expressions and rational equations. The equation contains an equal sign.

Rational Expression	Rational Equation
$\frac{1}{8}x+\frac{1}{2}$	$\frac{1}{8}x+\frac{1}{2}=\frac{1}{4}$
$\frac{y+6}{y^2-36}$	$\frac{y+6}{y^2-36}=y+1$
$\frac{1}{n-3}+\frac{1}{n+4}$	$\frac{1}{n-3}+\frac{1}{n+4}=\frac{15}{n^2+n-12}$

As we explained in the introduction to this unit, in the previous unit we have used equal signs during a simplification of an expression. These equal signs implicitly mean that equation holds for all values of the variables except those that leave an expression undefined. In the case of the equations dealt with here, these equalities say something about the variable, and for 'most' values of the variable the equation is false. We will seek the values which make the equality true, i.e., we seek to solve these equations.

Solve Rational Equations

We have already solved linear equations that contained fractions. We found the LCD of all the fractions in the equation and then multiplied both sides of the equation by the LCD to "clear" the fractions.

We will use the same strategy to solve rational equations. As is often the case when facing a new kind of problem, we aim to somehow reformulate it in a way that is more familiar. So, we will aim at rewriting the equation as a polynomial equation. We will multiply both sides of the equation by the LCD. Then, we will have an equation that does not contain rational expressions and thus is much easier for us to solve since we already know how to solve linear and quadratic equations. But because the original equation may have a variable in a denominator, we must be careful that we don't end up with a solution that would make a denominator equal to zero.

So before we begin solving a rational equation, we examine it first to find the values that would make any denominators zero. That way, when we solve a rational equation we will know if there are any algebraic solutions we must discard. Alternatively, we can check to make sure your possible solutions make sense in our equation.

A solution to an an equation that is equivalent (except for a few values) to a rational equation for which the rational expressions are undefined is called an **extraneous solution to a rational equation**.

Definition 2.4.2

An **extraneous solution to a rational equation** is a solution to an equation which is that is equivalent to the original except for a certain finite number of values that would cause any of the expressions in the original equation to be undefined.

We note any possible extraneous solutions, c, by writing $x\neq c$ next to the equation.

✓ Example 2.4.3

Solve $\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$.

Solution

	$\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$
Note any value of the variable that would make any denominator zero.	If $x=0$, then $\frac{1}{x}$ is undefined. So we'll write $x \neq 0$ next to the equation. $\frac{1}{x}+\frac{1}{3}=\frac{5}{6}, x \neq 0$
Find the least common denominator of all denominators in the equation.	Find the LCD of $\frac{1}{x}$, $\frac{1}{3}$, and $\frac{5}{6}$. The LCD is $6x$.
Clear the fractions by multiplying both sides of the equation by the LCD.	Multiply both sides of the equation by the LCD, $6x$. $6x \cdot\left(\frac{1}{x}+\frac{1}{3}\right)=6x \cdot\left(\frac{5}{6}\right)$ Use the Distributive Property. $6x \cdot \frac{1}{x}+6x \cdot \frac{1}{3}=6x \cdot\left(\frac{5}{6}\right)$ Simplify - and notice, no more fractions! $6+2x=5x$
Solve the resulting equation.	Simplify. $6=3x$ $2=x$
Check. If any values found in Step 1 are extraneous solutions, discard them. Check any remaining solutions in the original equation. (This is not strictly necessary, but is helpful in case an error was made).	Our solution is not an extraneous solution. $\frac{1}{x}+\frac{1}{3}=\frac{5}{6}$ We substitute $x=2$ into the original equation. $\frac{1}{2}+\frac{1}{3} \stackrel{?}{=} \frac{5}{6}$ $\frac{3}{6}+\frac{2}{6} \stackrel{?}{=} \frac{5}{6}$ $\frac{5}{6}=\frac{5}{6}\checkmark$
Conclude.	The solution is $x=2$.

? Try It 2.4.4

Solve $\frac{1}{y}+\frac{2}{3}=\frac{1}{5}$.

Answer

$y=-\frac{15}{7}$

? Try It 2.4.5

Solve $\frac{2}{3}+\frac{1}{5}=\frac{1}{x}$.

Answer

$x=\frac{15}{13}$

The steps of this method are shown.

How to solve equations with rational expressions.

- Step 1. Note any value of the variable that would make any denominator zero.
- Step 2. Find the least common denominator of all denominators in the equation.
- Step 3. Clear the fractions by multiplying both sides of the equation by the LCD.
- Step 4. Solve the resulting equation.
- Step 5. Check:
  - If any values found in Step 1 are algebraic solutions, discard them.
  - Check any remaining solutions in the original equation.

We always start by noting the values that would cause any denominators to be zero.

Example 2.4.6

Solve $1-\frac{5}{y}=-\frac{6}{y^2}$.

Solution

	$1-\frac{5}{y}=-\frac{6}{y^2}$
Note any value of the variable that would make any denominator zero.	$1-\frac{5}{y}=-\frac{6}{y^2}, y\neq 0$
Find the least common denominator of all denominators in the equation.	The LCD is y^2.
Clear the fractions by multiplying both sides of the equation by the LCD.	$y^2\left(1-\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Distribute.	$y^2\cdot 1-y^2\left(\frac{5}{y}\right)=y^2\left(-\frac{6}{y^2}\right)$
Multiply.	$y^2-5y=-6$
Solve the resulting equation. First write the quadratic equation in standard form.	$y^2-5y+6=0$
Factor.	$(y-2)(y-3)=0$
Use the Zero Product Property.	$y-2=0$ or $y-3=0$
Solve.	$y=2$ or $y=3$
Check. There are no extraneous solutions. Check $y=2$ and $y=3$ in the original equation.	$1-\frac{5}{y}=-\frac{6}{y^2}$ $1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{2^2}$ $1-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$ $\frac{2}{2}-\frac{5}{2}\stackrel{?}{=}-\frac{6}{4}$ $-\frac{3}{2}\stackrel{?}{=}-\frac{6}{4}$ $-\frac{3}{2}=-\frac{3}{2}\checkmark$ $1-\frac{5}{y}=-\frac{6}{y^2}$ $1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{3^2}$ $1-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$ $\frac{3}{3}-\frac{5}{3}\stackrel{?}{=}-\frac{6}{9}$ $-\frac{2}{3}\stackrel{?}{=}-\frac{6}{9}$ $-\frac{2}{3}=-\frac{2}{3}\checkmark$
Conclude.	The solutions are $y=2$ and $y=3$.

Try It 2.4.7

Solve $1-\frac{2}{x}=\frac{15}{x^2}$.

Answer

$x=-3, x=5$

Try It 2.4.8

Solve $1-\frac{4}{y}=\frac{12}{y^2}$.

Answer

$y=-2, y=6$

In the next example, the last denominators is a difference of squares. Remember to factor it first to find the LCD.

Example 2.4.9

Solve $\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$.

Solution

	$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$

Note any value of the variable that would make any denominator zero.	$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{(x+2)(x-2)}, x \neq -2, x \neq 2$
Find the least common denominator of all denominators in the equation.	The LCD is $(x+2)(x-2)$.
Clear the fractions by multiplying both sides of the equation by the LCD.	$(x+2)(x-2)\left(\frac{2}{x+2}+\frac{4}{x-2}\right)=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Distribute.	$(x+2)(x-2)\frac{2}{x+2}+(x+2)(x-2)\frac{4}{x-2}=(x+2)(x-2)\left(\frac{x-1}{x^2-4}\right)$
Remove common factors.	$\cancel{(x+2)}(x-2)\frac{2}{\cancel{x+2}}+(x+2)\cancel{(x-2)}\frac{4}{\cancel{x-2}}=\cancel{(x+2)(x-2)}\left(\frac{x-1}{\cancel{x^2-4}}\right)$
Simplify.	$2(x-2)+4(x+2)=x-1$
Distribute.	$2x-4+4x+8=x-1$
Solve.	$6x+4=x-1$ $5x=-5$ $x=-1$
Check: We have no extraneous solutions. Check $x=-1$ in the original equation.	$\frac{2}{x+2}+\frac{4}{x-2}=\frac{x-1}{x^2-4}$ $\frac{2}{(-1)+2}+\frac{4}{(-1)-2}\stackrel{?}{=}\frac{(-1)-1}{(-1)^2-4}$ $\frac{2}{1}+\frac{4}{-3}\stackrel{?}{=}\frac{-2}{-3}$ $\frac{6}{3}-\frac{4}{3}\stackrel{?}{=}\frac{2}{3}$ $\frac{2}{3}=\frac{2}{3}\checkmark$
Conclude.	The solution is $x=-1$.

Try It 2.4.10

Solve $\frac{2}{x+1}+\frac{1}{x-1}=\frac{1}{x^2-1}$.

Answer

$x=\frac{2}{3}$

Try It 2.4.11

Solve $\frac{5}{y+3}+\frac{2}{y-3}=\frac{5}{y^2-9}$.

Answer

$y=2$

In the next example, the first denominator is a trinomial. Remember to factor it first to find the LCD.

Example 2.4.12

Solve $\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$.

Solution

	$\frac{m+11}{m^2-5m+4}=\frac{5}{m-4}-\frac{3}{m-1}$
Note any value of the variable that would make any denominator zero. Use the factored form of the quadratic denominator.	$\frac{m+11}{(m-4)(m-1)}=\frac{5}{m-4}-\frac{3}{m-1}, m \neq 4, m \neq 1$
Find the least common denominator of all denominators in the equation.	The LCD is $(m-4)(m-1)$.
Clear the fractions by multiplying both sides of the equation by the LCD.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\left(\frac{5}{m-4}-\frac{3}{m-1}\right)$

Distribute.	$(m-4)(m-1)\left(\frac{m+11}{(m-4)(m-1)}\right)=(m-4)(m-1)\frac{5}{m-4}-(m-4)(m-1)\frac{3}{m-}$
Remove common factors.	$\cancel{(m-4)}\cancel{(m-1)}\left(\frac{m+11}{\cancel{(m-4)}\cancel{(m-1)}}\right)=\cancel{(m-4)}(m-1)\frac{5}{\cancel{m-4}}-(m-4)\,(m-$
Simplify.	$m+11=5(m-1)-3(m-4)$
Solve the resulting equation.	$m+11=5m-5-3m+12$ $4=m$
Check.	We see that substituting 4 for m in the original equation would make a denominator equal to zero. The only solution to the nearly equivalent linear equation is an extraneous solution.
Conclude.	There is no solution to the original equation.

Try It 2.4.13

Solve $\frac{x+13}{x^2-7x+10}=\frac{6}{x-5}-\frac{4}{x-2}$.

Answer

There is no solution.

Try It 2.4.14

Solve $\frac{y-6}{y^2+3y-4}=\frac{2}{y+4}+\frac{7}{y-1}$.

Answer

There is no solution.

The equation we solved in the previous example had only one algebraic solution, but it was an extraneous solution. That left us with no solution to the equation. In the next example we get two algebraic solutions. Here one or both could be extraneous solutions.

Example 2.4.15

Solve $\frac{y}{y+6}=\frac{72}{y^2-36}+4$.

Solution

	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{y}{y+6}=\frac{72}{(y-6)(y+6)}+4, y\neq 6, y\neq -6$
Find the least common denominator.	The LCD is $(y-6)(y+6)$.
Clear the fractions.	$(y-6)(y+6)\left(\frac{y}{y+6}\right)=(y-6)(y+6)\left(\frac{72}{(y-6)(y+6)}+4\right)$ Note that the above equation has exactly the same solutions as the original except for, perhaps, $y=-6$ and $y=6$.
Simplify.	$(y-6)\cdot y=72+(y-6)(y+6)\cdot 4$
Simplify.	$y(y-6)=72+4\left(y^2-36\right)$
Solve the resulting equation.	$y^2-6y=72+4y^2-144$ $0=3y^2+6y-72$ $0=3\left(y^2+2y-24\right)$ $0=3(y+6)(y-4)$ $y=-6, y=4$

Check. $y=-6$ is an extraneous solution. Check $y=4$ in the original equation (not strictly necessary).	$\frac{y}{y+6}=\frac{72}{y^2-36}+4$ $\frac{4}{4+6}\overset{?}{=}\frac{72}{4^2-36}+4$ $\frac{4}{10}\overset{?}{=}\frac{72}{-20}+4$ $\frac{4}{10}\overset{?}{=}-\frac{36}{10}+\frac{40}{10}$ $\frac{4}{10}=\frac{4}{10}\checkmark$
Conclude.	The solution is $y=4$.

? Try It 2.4.16

Solve $\frac{x}{x+4}=\frac{32}{x^2-16}+5$.

Answer

$x=3$

? Try It 2.4.17

Solve $\frac{y}{y+8}=\frac{128}{y^2-64}+9$.

Answer

$y=7$

In some cases, all the algebraic solutions are extraneous.

✓ Example 2.4.18

Solve $\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$.

Solution

	$\frac{x}{2x-2}-\frac{2}{3x+3}=\frac{5x^2-2x+9}{12x^2-12}$
We will start by factoring all denominators, to make it easier to identify extraneous solutions and the LCD.	$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}$
Note any value of the variable that would make any denominator zero.	$\frac{x}{2(x-1)}-\frac{2}{3(x+1)}=\frac{5x^2-2x+9}{12(x-1)(x+1)}, x\neq 1, x\neq -1$
Find the least common denominator.	The LCD is $12(x-1)(x+1)$.
Clear the fractions.	$12(x-1)(x+1)\left(\frac{x}{2(x-1)}-\frac{2}{3(x+1)}\right)=12(x-1)(x+1)\left(\frac{5x^2-2x+9}{12(x-1)(x+1)}\right)$
Simplify.	$6(x+1)\cdot x-4(x-1)\cdot 2=5x^2-2x+9$
Simplify.	$6x(x+1)-4\cdot 2(x-1)=5x^2-2x+9$
Solve the resulting equation.	$6x^2+6x-8x+8=5x^2-2x+9$ $x^2-1=0$ $(x-1)(x+1)=0$ $x=1$ or $x=-1$
Check.	$x=1$ and $x=-1$ are extraneous solutions.
Conclude.	The equation has no solution.

Try It 2.4.19

Solve $\frac{y}{5y-10}-\frac{5}{3y+6}=\frac{2y^2-19y+54}{15y^2-60}$.

Answer

There is no solution.

Try It 2.4.20

Solve $\frac{z}{2z+8}-\frac{3}{4z-8}=\frac{3z^2-16z-16}{8z^2+2z-64}$.

Answer

There is no solution.

Example 2.4.21

Solve $\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$.

Solution

	$\frac{4}{3x^2-10x+3}+\frac{3}{3x^2+2x-1}=\frac{2}{x^2-2x-3}$
Factor all the denominators, so we can note any value of the variable that would make any denominator zero.	$\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}=\frac{2}{(x-3)(x+1)}, x\neq -1, x\neq \frac{1}{3}, x\neq 3$
Find the least common denominator.	The LCD is $(3x-1)(x+1)(x-3)$.
Clear the fractions.	$(3x-1)(x+1)(x-3)\left(\frac{4}{(3x-1)(x-3)}+\frac{3}{(3x-1)(x+1)}\right)$ $=(3x-1)(x+1)(x-3)\left(\frac{2}{(x-3)(x+1)}\right)$
Simplify.	$4(x+1)+3(x-3)=2(3x-1)$
Distribute.	$4x+4+3x-9=6x-2$
Simplify.	$7x-5=6x-2$ $x=3$ But this is an extraneous solution.
Conclude.	There is no solution to this equation.

Try It 2.4.22

Solve $\frac{15}{x^2+x-6}-\frac{3}{x-2}=\frac{2}{x+3}$.

Answer

There is no solution.

Try It 2.4.23

Solve $\frac{5}{x^2+2x-3}-\frac{3}{x^2+x-2}=\frac{1}{x^2+5x+6}$.

Answer

There is no solution.

Solve a Rational Equation for a Specific Variable (Optional)

When we solved linear equations, we learned how to solve a formula for a specific variable. Many formulas used in business, science, economics, and other fields use rational equations to model the relation between two or more variables. We will now see how to solve a rational equation for a specific variable.

When we developed the point-slope formula from our slope formula, we cleared the fractions by multiplying by the LCD.

$$\begin{aligned} m &= \frac{y-y_1}{x-x_1} \\ m\left(x-x_1\right) &= \left(\frac{y-y_1}{x-x_1}\right)\left(x-x_1\right) \quad \text{Multiply both sides of the equation by } x-x_1. \\ m\left(x-x_1\right) &= y-y_1 \quad \text{Simplify.} \\ y-y_1 &= m\left(x-x_1\right) \quad \text{Rewrite the equation with the y terms on the left.} \end{aligned}$$

In the next example, we will use the same technique with the formula for slope that we used to get the point-slope form of an equation of a line through the point $(2,3)$. We will add one more step to solve for y.

Example 2.4.24

Solve $m=\frac{y-2}{x-3}$ for y.

Solution

	$m=\frac{y-2}{x-3}$
Note any value of the variable that would make any denominator zero.	$m=\frac{y-2}{x-3}, x\neq 3$
Clear the fractions by multiplying both sides of the equation by the LCD, $x-3$.	$(x-3)m=(x-3)\left(\frac{y-2}{x-3}\right)$
Simplify.	$xm-3m=y-2$
Isolate the term with y.	$xm-3m+2=y$
Conclude.	$y=xm-3m+2$

Try It 2.4.25

Solve $m=\frac{y-5}{x-4}$ for y.

Answer

$y=mx-4m+5$

Try It 2.4.26

Solve $m=\frac{y-1}{x+5}$ for y.

Answer

$y=mx+5m+1$

Remember to multiply both sides by the LCD in the next example.

Example 2.4.27

Solve $\frac{1}{c}+\frac{1}{m}=1$ for c.

Solution

	$\frac{1}{c}+\frac{1}{m}=1$ for c
Note any value of the variable that would make any denominator zero.	$\frac{1}{c}+\frac{1}{m}=1, c\neq 0, m\neq 0$
Clear the fractions by multiplying both sides of the equations by the LCD, cm.	$cm\left(\frac{1}{c}+\frac{1}{m}\right)=cm(1)$
Distribute.	$cm\left(\frac{1}{c}\right)+cm\frac{1}{m}=cm(1)$
Simplify.	$m+c=cm$
Collect the terms with c to the right.	$m=cm-c$
Factor the expression on the right.	$m=c(m-1)$
To isolate c, divide both sides by $m-1$.	$\frac{m}{m-1}=\frac{c(m-1)}{m-1}$

Simplify by removing common factors.	$\frac{m}{m-1}=c$
Conclude.	$c=\frac{m}{m-1}$

Notice that even though we excluded $c=0$ and $m=0$ from the original equation, we must also now state that $m\neq 1$.

Try It 2.4.28

Solve $\frac{1}{a}+\frac{1}{b}=c$ for a.

Answer

$$a=\frac{b}{cb-1}$$

Try It 2.4.29

Solve $\frac{2}{x}+\frac{1}{3}=\frac{1}{y}$ for y.

Answer

$$y=\frac{3x}{x+6}$$

Writing Exercises 2.4.30

1. What is the difference between an expression and an equation?
2. What does it mean to solve an equation?
3. Is it necessary to check your answer if you know you have not made a mistake? Explain.

Exit Problem

Solve $\frac{y}{y+3}+\frac{3}{y-3}=\frac{18}{y^2-9}$.

Key Concepts

Rational equation

Solultion to a rational equation

Extraneous solution to a rational equation

Solving a rational equation

2.5: Radical Equations

Learning Objectives

By the end of this section, you will be able to:

- Solve radical equations
- Solve radical equations with two radicals
- Use radicals in applications

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $(y-3)^2$.
2. Solve $2x-5=0$.
3. Solve $n^2-6n+8=0$.

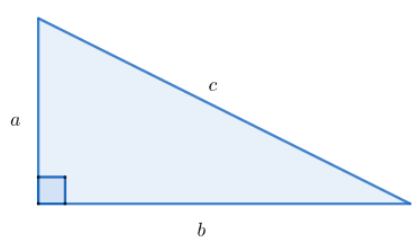
Radical Equations

In this section we will solve equations that have a variable in the radicand of a radical expression. An equation of this type is called a **radical equation**.

Definition 2.5.1

An equation in which a variable is in the radicand of a radical expression is called a **radical equation**.

As always, these equations arise as assertions about an unknown quantity. It becomes the task then to solve these equations. Radicals sometimes arise in geometric problems which we will talk more about in the next unit. We can talk about those coming from the Pythagoras' Theorem, or the resulting distance formula.



$$a^2+b^2=c^2$$

$$c=\sqrt{a^2+b^2}$$

Suppose for example you know that a boat is 10 meters further from an observer at a light house than the it is from the shore and the observer is 80 meters from the point on the shore that is closest to boat. Suppose that the triangle that is formed by the observer, the boat, and the point on the shore closest to the boat is a right triangle where the hypotenuse is formed by the segment connecting the observer to the boat. How close is the boat to the shore? There are ways to arrive at an equation without radicals, but you may also do the following:

First call the distance from the boat to the shore d. Then we can determine the distance from the observer to the boat is given by

$\sqrt{80^2+d^2}$,

but we know that this is also $d+10$ since "boat is 10 meters further from an observer at a light house than the it is from the shore". So for the d we seek, we see

$d+10=\sqrt{80^2+d^2}$.

The distance we seek satisfies this equation! So, we must now solve this equation for d.

Notice that the difficulty with this equation is that there is a square root on the right side of the equation. The order of operations says that if we were to evaluate the right side, we would evaluate the square root last. So we try to undo that square root first! What is the opposite of the square root? The square!! So we should square both sides of the equation. Since the two sides of the equation

$d+10=\sqrt{80^2+d^2}$

are equal (that is why it's an equation), the squares will also be equal.

So we learn that d must also satisfy

$(d+10)^2=80^2+d^2.$

These two equations may not be equivalent since to undo this step we would need to take a square root of both sides and every positive number has two square roots! How would we know which to use? For example, if $4=sqrta$ then $a=16$, BUT if $a^2=16$ then either $a=\sqrt{16}=4$ or $a=-\sqrt{16}=-4$.

Now this is an equation which we know how to solve (and which if we had gone about it differently could have arrived at initially),

	$(d+10)^2=80^2+d^2.$
Distribute on the left side to arrive at	$d^2+20d+100=80^2+d^2$
Subtract d^2 on both sides (we see that these should cancel!!)	$d^2-d^2+20d+100=80^2+d^2-d^2$ or, equivalently, $20d+100=80^2$
Notice that this is a linear equation and our variable appears on the left side only. So the last step if we evaluated the left side for a particular d is adding 100, so to undo this we subtract 100 on both sides of the equation to get	$20d+100-100=80^2-100$ or, equivalently, $20d=6300$
And then undoing the multiplication of d by 20, we divide both sides of the equation by 20 to get	$20d/20=6300/20$ or, equivalently, $d=315$

So, the distance we seek is 315 meters. We must check our answer since

while d satisfies $(d+10)^2=80^2+d^2$,

it must satisfy either

$d+10=\sqrt{80^2+d^2}$ or $d+10=-\sqrt{80^2+d^2}$

but we want to make sure it is

$d+10=\sqrt{80^2+d^2}$ that it satisfies.

Let's check it!

The left side is $315+10=316$

and the right side is $\sqrt{80^2+315^2}=325.$

So, indeed we found the distance we seek and conclude that the boat is 325 meters from the shore.

Solving Equations with Square Roots

As usual, when solving these equations, what we do to one side of an equation we must do to the other side as well. Once we isolate the square root, our strategy will be to square both sides of the equation. This will eliminate the radical.

Solving radical equations with square roots by squaring both sides may introduce an algebraic solution that would not be a solution to the original radical equation. Again, we call this an extraneous solution. As explained in the context above, the issue is that $x^2=a$ (where a is some fixed number) is not equivalent to $x=\sqrt{a}$, but rather to $x=\sqrt{a}$ or $x=-\sqrt{a}$. So is we find a solution to $x^2=a$ it may be a solution to $x=-\sqrt{a}$ and not to $x=\sqrt{a}$ as desired. We therefore must check our answer to see if it is a solution to the equation we wanted to solve originally.

In the next example, we will see how to solve a radical equation. Our strategy is based on squaring a square root. This will eliminate the radical.

$$\text{For } a \geq 0, \left(\sqrt{a}\right)^2 = a.$$

This is saying that if we want to undo a square root we need to square (notice where the square and the square-root button are on your calculator)!

Keep the order of operations in mind as you solve equations. We will demonstrate this concept in the next example.

Example 2.5.2

Solve $\sqrt{5n-4}-9=0$.

Solution

Consider the equation.

	$\sqrt{5n-4}-9=0$
Isolate the radical on one side of the equation.	$\sqrt{5n-4}-9+9=0+9$
Simplify.	$\sqrt{5n-4}=9$
Square both sides of the equation. Remember that $\left(\sqrt{a}\right)^2=a$.	$\left(\sqrt{5n-4}\right)^2=(9)^2$ $5n-4=81$
Solve the new equation.	$5n-4+4=81+4$ $5n=85$ $n=\frac{85}{5}$ $n=17$
Check the answer in the original equation.	$\sqrt{5n-4}-9=0$ $\sqrt{5\cdot 17-4}-9\overset{?}{=}0$ $\sqrt{85-4}-9\overset{?}{=}0$ $\sqrt{81}-9\overset{?}{=}0$ $9-9\overset{?}{=}0$ True
	The solution is $n=17$.

Try It 2.5.3

Solve $\sqrt{3m+2}-5=0$.

Answer

$m=\frac{23}{3}$

Try It 2.5.4

Solve $\sqrt{10z+1}-2=0$.

Answer

$z=\frac{3}{10}$

Solve a Radical Equation With One Square Root

1. Isolate the radical on one side of the equation.
2. Square both sides of the equation.
3. Solve the new equation.
4. Check the answer in the original equation.

When we use a radical sign, it indicates the principal or positive root. If an equation has a square root equal to a negative number, that equation will have no solution.

Example 2.5.5

Solve $\sqrt{9k-2}+1=0$.

Solution

	$\sqrt{9k-2}+1=0$
Isolate the radical on one side of the equation.	$\sqrt{9k-2}+1-1=0-1$
Simplify.	$\sqrt{9k-2}=-1$

Because the square root is equal to a negative number, the equation has no solution.

Try It 2.5.6

Solve $\sqrt{2r-3}+5=0$.

Answer

no solution

Try It 2.5.7

Solve $\sqrt{7s-3}+2=0$.

Answer

no solution

If one side of an equation with a square root is a binomial, we use the Product of Binomial Squares Pattern when we square it.

Binomial Squares

$$(a+b)^2=a^2+2ab+b^2$$
$$(a-b)^2=a^2-2ab+b^2$$

Don't forget the middle term!

Example 2.5.8

Solve $\sqrt{p-1}+1=p$.

Solution

	$\sqrt{p-1}+1=p$

To isolate the radical, subtract 1 from both sides.	$\sqrt{p-1}+1-1=p-1$
Simplify.	$\sqrt{p-1}=p-1$
Square both sides of the equation.	$\left(\sqrt{p-1}\right)^2=(p-1)^2$
Simplify, using the Product of Binomial Squares Pattern on the right, then solve the new equation.	$p-1=p^2-2p+1$
It is a quadratic equation, so get zero on one side.	$0=p^2-3p+2$
Factor the right side.	$0=(p-1)(p-2)$ $(p-1)(p-2)=0$
Use the Zero Product Property.	$p-1=0 \quad \text{or} \quad p-2=0$
Solve each equation.	$p=1 \quad \text{or} \quad p=2$
Check the answers.	$p=1 \quad \sqrt{p-1}+1=p$ $\sqrt{1-1}+1\stackrel{?}{=}1$ $0+1\stackrel{?}{=}1$ $1\stackrel{?}{=}1$ True $p=2 \quad \sqrt{p-1}+1=p$ $\sqrt{2-1}+1\stackrel{?}{=}2$ $\sqrt{1}+1\stackrel{?}{=}2$ $2\stackrel{?}{=}2$ True
	The solutions are $p=1$ or $p=2$.

? Try It 2.5.9

Solve $\sqrt{x-2}+2=x$.

Answer

$x=2$ or $x=3$

? Try It 2.5.10

Solve $\sqrt{y-5}+5=y$.

Answer

$y=5$ or $y=6$

Sometimes the solution of a radical equation results in two algebraic solutions, but one of them may be an **extraneous solution**!

? Example 2.5.11

Solve $\sqrt{r+4}-r+2=0$.

Solution

	$\sqrt{r+4}-r+2=0$
Isolate the radical.	$\sqrt{r+4}=r-2$
Square both sides of the equation.	$(\sqrt{r+4})^2=(r-2)^2$
Simplify and then solve the equation.	
If it is a quadratic equation, so get zero on one side.	

Factor the right side.	$0 = r(r-5)$
Use the Zero Product Property.	$0 = r$ or $0 = r-5$
Solve the equation.	$r = 0 \quad r = 5$
Check your answer.	$r = 0 \quad \sqrt{r+4} - r + 2 = 0$ $\sqrt{0+4} - 0 + 2 \stackrel{?}{=} 0$ $\sqrt{4} + 2 \stackrel{?}{=} 0$ $4 \stackrel{?}{=} 0$ False $r = 5 \quad \sqrt{r+4} - r + 2 = 0$ $\sqrt{5+4} - 5 + 2 \stackrel{?}{=} 0$ $\sqrt{9} - 3 \stackrel{?}{=} 0$ $0 \stackrel{?}{=} 0$ True
	$r = 0$ is an extraneous solution. The solution is $r = 5$.

Try It 2.5.12

Solve $\sqrt{m+9} - m + 3 = 0$.

Answer

$m = 7$

Try It 2.5.13

Solve $\sqrt{n+1} - n + 1 = 0$.

Answer

$n = 3$

When there is a coefficient in front of the radical, we must raise it to the power of the index, too.

Example 2.5.14

Solve $3\sqrt{3x-5} - 8 = 4$.

Solution

	$3\sqrt{3x-5} - 8 = 4$
Isolate the radical term.	$3\sqrt{3x-5} = 12$
Isolate the radical by dividing both sides by 3.	$\sqrt{3x-5} = 4$
Square both sides of the equation.	$(\sqrt{3x-5})^2 = (4)^2$
Simplify, then solve the new equation.	$3x - 5 = 16$
	$3x = 21$
Solve the equation.	$x = 7$

Check the answer.	$x=7 \quad 3\sqrt{3x-5}-8 = 4$ $3\sqrt{3(7)-5}-8 \stackrel{?}{=} 4$ $3\sqrt{21-5}-8 \stackrel{?}{=} 4$ $3\sqrt{16}-8 \stackrel{?}{=} 4$ $3(4)-8 \stackrel{?}{=} 4$ $4 \stackrel{?}{=} 4$ True
	The solution is $x=7$.

Try It 2.5.15

Solve $2\sqrt{4a+4}-16=16$.

Answer

$a=63$

Try It 2.5.16

Solve $3\sqrt{2b+3}-25=50$

Answer

$b=311$

Solve Radical Equations with Two Radicals

If the radical equation has two radicals, we start out by isolating one of them. It often works out easiest to isolate the more complicated radical first.

In the next example, when one radical is isolated, the second radical is also isolated.

Example 2.5.17

Solve $\sqrt{4x-3}=\sqrt{3x+2}$.

Solution

The radical terms are isolated.	$\sqrt{4x-3}=\sqrt{3x+2}$
Square both sides of the equation.	$(\sqrt{4x-3})^2=(\sqrt{3x+2})^2$
Simplify, then solve the new equation.	$4x-3=3x+2$ $4x-3-3x+3=3x+2-3x+3$ $x=5$
Check the answer.	$x=5 \quad \sqrt{4x-3} = \sqrt{3x+2}$ $\sqrt{4\cdot 5-3} \stackrel{?}{=} \sqrt{3\cdot 5+2}$ $\sqrt{17} \stackrel{?}{=} \sqrt{17}$ True
	The solution is $x=5$.

Try It 2.5.18

Solve $\sqrt{5x-4}=\sqrt{2x+5}$.

Answer

$x=3$

> **Try It 2.5.19**
>
> Solve $\sqrt{7x+1}=\sqrt{2x-5}$.
>
> **Answer**
>
> There is no real solution.

Sometimes after raising both sides of an equation to a power, we still have a variable inside a radical. When that happens, we repeat Step 1 and Step 2 of our procedure. We isolate the radical and square both sides of the equation again.

> **Example 2.5.20**
>
> Solve $\sqrt{m}+1=\sqrt{m+9}$.
>
> **Solution**

Isolate one of the radical terms on one side of the equation.	The radical on the right is isolated.	$\sqrt{m}+1=\sqrt{m+9}$
Raise both sides of the equation to the power of the index.	We square both sides. Simplify--be very careful as you multiply!	$(\sqrt{m}+1)^2=(\sqrt{m+9})^2$
Are there any more radicals? If yes, repeat Step 1 and Step 2 again. If no, solve the new equation.	There is still a radical in the equation. So we must repeat the previous steps. Isolate the radical term. Here, we can easily isolate the radical by dividing both sides by 2. Square both sides.	$m+2\sqrt{m}+1=m+9$ $2\sqrt{m}=8$ $\sqrt{m}=4$ $(\sqrt{m})^2=(4)^2$ $m=16$
Check the answer in the original equation.		$\sqrt{m}+1=\sqrt{m+9}$ $\sqrt{16}+1\stackrel{?}{=}\sqrt{16+9}$ $4+1\stackrel{?}{=}5$ $5=5$ The solution is $m=16$.

> **Try It 2.5.21**
>
> Solve $3-\sqrt{x}=\sqrt{x-3}$.
>
> **Answer**
>
> $x=4$

> **Try It 2.5.22**
>
> Solve $\sqrt{x}+2=\sqrt{x+16}$.
>
> **Answer**
>
> $x=9$

We summarize the steps here. We have adjusted our previous steps to include more than one radical in the equation This procedure will now work for any radical equations.

Solve a Radical Equation

1. Isolate one of the radical terms on one side of the equation.
2. Raise both sides of the equation to the power of the index.
3. Are there any more radicals?
   If yes, repeat Step 1 and Step 2 again.
   If no, solve the new equation.
4. Check the answer in the original equation.

Be careful as you square binomials in the next example. Remember the pattern in $(a+b)^2=a^2+2ab+b^2$ or $(a-b)^2=a^2-2ab+b^2$.

Example 2.5.23

Solve $\sqrt{q-2}+3=\sqrt{4q+1}$.

Solution

Try It 2.5.24

Solve $\sqrt{x-1}+2=\sqrt{2x+6}$.

Answer

$x=5$

Try It 2.5.25

Solve $\sqrt{x}+2=\sqrt{3x+4}$.

Answer

$x=0x=4$

Use Radicals in Applications

As you progress through your college courses, you'll encounter formulas that include radicals in many disciplines. We will modify our Problem Solving Strategy for Geometry Applications slightly to give us a plan for solving applications with formulas from any discipline.

Use a Problem Solving Strategy for Applications with Formulas

1. **Read** the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
2. **Identify** what we are looking for.
3. **Name** what we are looking for by choosing a variable to represent it.
4. **Translate** into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
5. **Solve the equation** using good algebra techniques.
6. **Check** the answer in the problem and make sure it makes sense.
7. **Answer** the question with a complete sentence.

One application of radicals has to do with the effect of **gravity** on falling objects. The formula allows us to determine how long it will take a fallen object to hit the gound.

Falling Objects

On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is

$$\frac{\sqrt{h}}{4}.$$

For example, if an object is dropped from a height of 64 feet, we can find the time it takes to reach the ground by substituting $h = 64$ into the formula.

$h = 64$ and so the time we seek satisfies	$t = \frac{\sqrt{64}}{4}$
Take the square root of 64.	$t = \frac{8}{4}$
Simplify the fraction.	$t = 2$

It would take 2 seconds for an object dropped from a height of 64 feet to reach the ground.

Example 2.5.26

Marissa dropped her sunglasses from a bridge 400 feet above a river. The time it takes for an object to hit the ground is $\frac{\sqrt{h}}{4}$ seconds. Find how many seconds it took for the sunglasses to reach the river.

Solution

Read the problem.	
Identify what we are looking.	The time it takes for the sunglasses to reach the river.
Name what we are looking.	Let t be the time we are looking for.
Translate into an equation by equating two different expressions for the same quantity.	$t = \frac{\sqrt{400}}{4}$
Solve the equation.	$t = \frac{20}{4}$
	$t = 5$
Check the answer in the problem and make sure it makes sense. Here, it will solve the original problem, but we must see that it makes sense in the particular application.	$5 \stackrel{?}{=} \frac{\sqrt{400}}{4}$ $5 \stackrel{?}{=} \frac{20}{4}$ $5 = 5$ ✓
Does 5 seconds seem like a reasonable length of time?	Yes.
Answer the equation.	It will take 5 seconds for the sunglasses to reach the river.

Try It 2.5.27

A helicopter dropped a rescue package from a height of $1,296$ feet. The time it takes for it to hit the ground is $\frac{\sqrt{h}}{4}$ seconds. Find how many seconds it took for the package to reach the ground.

Answer

9 seconds

Try It 2.5.28

A window washer dropped a squeegee from a platform 196 feet above the sidewalk. The time it takes for it to hit the ground is $\frac{\sqrt{h}}{4}$ seconds. Find how many seconds it took for the squeegee to reach the sidewalk.

Answer

3.5 seconds

Police officers investigating car accidents measure the length of the skid marks on the pavement. Then they use square roots to determine the **speed**, in miles per hour, a car was going before applying the brakes.

Skid Marks and Speed of a Car

If the length of the skid marks (on a certain surface) is d feet, then the speed of the car before the brakes were applied is

$$\sqrt{24d} \text{ miles per hour.}$$

Example 2.5.29

After a car accident, the skid marks for one car measured 190 feet. The speed of the car before the brakes were applied is $\sqrt{24d}$ feet/second. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Solution

Read the problem.	
Identify what we are looking for.	The speed of a car.
Name what we are looking for.	Let $s =$ the speed.
Translate into an equation by writing the appropriate formula. Substitute in the given information.	$s=\sqrt{24d}$, and $d=190$ $s=\sqrt{24(190)}$
Solve the equation.	$s=\sqrt{4,560}$
	$s=67.52777...$
Round to 1 decimal place.	$s\approx 67.5$
	$67.5 \overset{?}{\approx} \sqrt{24(190)}$ $67.5 \overset{?}{\approx} \sqrt{4560}$ $67.5 \approx 67.5277... \checkmark$
	The speed of the car before the brakes were applied was 67.5 miles per hour.

Try It 2.5.30

An accident investigator measured the skid marks of the car. The length of the skid marks was 76 feet. The speed of the car before the brakes were applied is $\sqrt{24d}$ feet/second. Find the speed of the car before the brakes were applied. Round your answer to the nearest tenth.

Answer

42.7 miles/hour

Try It 2.5.31

The skid marks of a vehicle involved in an accident were 122 feet long. The speed of the car before the brakes were applied is $\sqrt{24d}$ feet/second. Find the speed of the vehicle before the brakes were applied. Round your answer to the nearest tenth.

Answer

54.1miles/hour

Writing Exercises 2.5.32

1. When we are trying to solve a radical equation, what are we trying to accomplish?
2. Why do we isolate the term with the radical?
3. Is it necessary to check our answers even if we know we did not make a mistake? Give an example to support your answer.

Exit Problem 2.5.33

Solve $2\sqrt{p-3}-3=-p$.

Key Concepts

- **Binomial Squares**
  $(a+b)^2=a^2+2ab+b^2$
  $(a-b)^2=a^2-2ab+b^2$
- **Solve a Radical Equation**
  1. Isolate one of the radical terms on one side of the equation.
  2. Raise both sides of the equation to the power of the index.
  3. Are there any more radicals?
     If yes, repeat Step 1 and Step 2 again.
     If no, solve the new equation.
  4. Check the answer in the original equation.
- **Problem Solving Strategy for Applications with Formulas**
  1. Read the problem and make sure all the words and ideas are understood. When appropriate, draw a figure and label it with the given information.
  2. Identify what we are looking for.
  3. Name what we are looking for by choosing a variable to represent it.
  4. Translate into an equation by writing the appropriate formula or model for the situation. Substitute in the given information.
  5. Solve the equation using good algebra techniques.
  6. Check the answer in the problem and make sure it makes sense.
  7. Answer the question with a complete sentence.
- **Falling Objects**
  - On Earth, if an object is dropped from a height of h feet, the time in seconds it will take to reach the ground is found by using the formula $t=\frac{\sqrt{h}}{4}$.
- **Skid Marks and Speed of a Car**
  - If the length of the skid marks is d feet, then the speed of the car before the brakes were applied (on a certain surface) is $\sqrt{24d}$ miles per hour.

Glossary

radical equation

An equation in which a variable is in the radicand of a radical expression is called a radical equation.

CHAPTER OVERVIEW

3: Graphs and Equations with Two Variables

In the previous unit we were able to solve a number of types of equations for the unknown. Sometimes there were no solutions, sometimes there was one solution and sometimes there were two. In all of these cases we were able to write down all of the solutions in a list.

In this unit we will be looking at equations with two variables. In this case, finding a solution involves specifying a value of each variable so that the equality after substituting in the values for the variables, respectively, is true. So a single solution involves specifying more than one number. Often it is the case that there are many more solutions, and in fact infinitely many, so that writing them in a list would be impossible! We therefore will devise here a way of illustrating the solutions.

Topic hierarchy

SECTION OVERVIEW

3.1: Linear Equations with Two Variables

3.1.1: Graphing Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Plot points in a rectangular coordinate system
- Graph a linear equation by plotting points
- Graph vertical and horizontal lines
- Find the x- and y-intercepts
- Graph a line using the intercepts

Be Prepared

Before we get started, take this readiness quiz.

1. Evaluate $5x-4$ when $x=-1$.
2. Evaluate $3x-2y$ when $x=4$ and $y=-3$.
3. Solve for y: $8-3y=20$.

In the previous unit we treated equations with one variable which we hoped to solve. Sometimes there were no solutions, and sometimes there were one or two. In this unit we will treat equations with two variables (often x and y but sometimes named other things for convenience). We will deal with equations that are familiar in the sense that if we replace either variable with a number, we are left with an equation that we have discussed previously.

A simple example of an equation with two variables is $3x+2y=6$. For particular values of x and y, this statement may be either true or false. For example if $x=1$ and $y=2$ this equation says $3\cdot1+2\cdot2=6$ which is false, but if $x=2$ and $y=0$ then the equation says $3\cdot2+2\cdot0=6$ which is true. In this case, we say that $x=2$ and $y=0$ is a solution. It turns out that unlike the previous unit, our equations have solutions that require two numbers to specify, and there are often infinitely many solutions. We will examine in the context of the first section how to represent the solutions to such an equation.

Linear Expressions and Linear Equations with One Variable

The expressions in the Be Prepared section are examples of what we call *linear expressions*.

Definition 3.1.1.1

1. An expression that can be written as

$$Ax+B$$

with A and B real numbers, $A\neq0$, is called a **linear expression (with one variable)**, or more specifically, a **linear expression in x**.

2. An equation that can be written as

$$Ax+B=0$$

with A and B real numbers, $A\neq0$, is called a **linear equation (with one variable)**, or more specifically, a **linear equation in x**.

3. A **solution** to a linear equation with one variable, say x, is a number, say a, that when substituted in for that variable yields a true statement. In this case we say that $x=a$ is a solution.

Be Prepared (3) can be written as $0=3y+12$ by adding $3y$ and subtracting 8 from both sides. So $8-3y=20$ is a linear equation in y. Notice that substituting $y=1$, for example, into the equation gives

$$\begin{aligned} 8-3(1) &= 20 \quad \text{or} \\ 5 &= 20, \end{aligned}$$

which is not true. However, substituting $y=-4$ into the equation gives

$$\begin{aligned} 8-3(-4) &= 20 \quad \text{or} \\ 20 &= 20, \end{aligned}$$

which is true. So $y=-4$ is a solution to $8-3y=20$, and $y=1$ is not. In solving Be Prepared 3, we found that $y=-4$ is the only solution.

Introduction to Linear Expressions and Linear Equations with Two Variables

In this section we will be looking at equations that have two variables, x and y, and more than one solution. We want to represent the solutions in a picture. Consider the equation $2x-3y=6$. The expression $2x-3y$ is an example of a linear expression with two variables. We can evaluate the expressions on both sides of the equal sign for any particular choice of x and y. For example, we can choose $x=3$ and $y=0$ and substitute them into the equation to get

$$\begin{aligned} 2(3)-3(0) &= 6 \quad \text{or} \\ 6 &= 6, \end{aligned}$$

which is a true statement. We can also choose $x=-3$ and $y=-4$ and substitute them into the equation to get

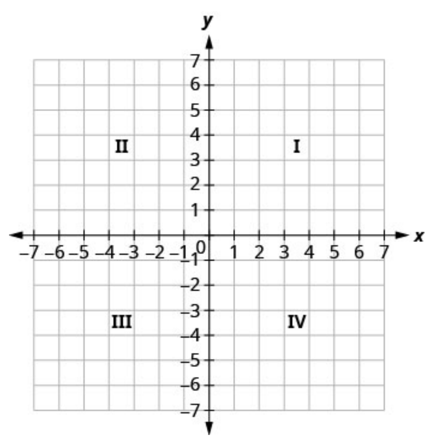
$$\begin{aligned} 2(-3)-3(-4) &= 6 \quad \text{or} \\ 6 &= 6, \end{aligned}$$

which is also true. We say that $x=3$ and $y=0$ is one solution, and $x=-3$ and $y=-4$ is another solution. We will actually see that equations like $2x-3y=6$ have infinitely many solutions. Next we introduce what is needed to make a picture of the solutions.

Plotting Points on a Rectangular Coordinate System

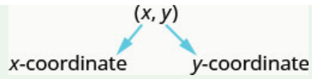
Just like maps use a grid system to identify locations, a grid system, or a rectangular coordinate system, is used in algebra to represent ordered pairs of numbers, and ultimately, to show a relationship between two variables. The rectangular coordinate system is also called the xy*-plane* or the "*coordinate plane*."

The rectangular coordinate system is formed by two intersecting number lines, one horizontal and one vertical. The horizontal number line is called the x*-axis*. The vertical number line is called the y*-axis* (note that in the context of an application these may take on different names). These axes divide a plane into four regions, called *quadrants*. The quadrants are identified by Roman numerals, beginning on the upper right and proceeding counterclockwise.



Definition 3.1.1.2

An **ordered pair**, (x,y), gives the coordinates of a point in a rectangular coordinate system. The first number is the x-**coordinate**. The second number is the y-**coordinate**.



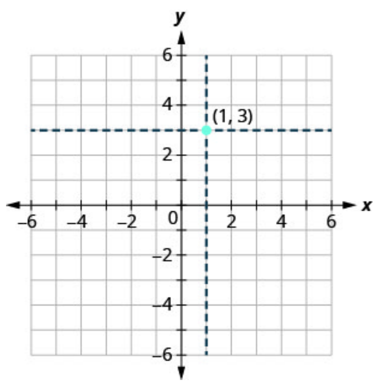
The phrase “ordered pair” means that the order is important. For example, $(2, 5)$ and $(5, 2)$ are different points.

What is the ordered pair of the point where the axes cross? At that point both coordinates are zero, so its ordered pair is $(0, 0)$. The point $(0, 0)$ has a special name.

Definition 3.1.1.3

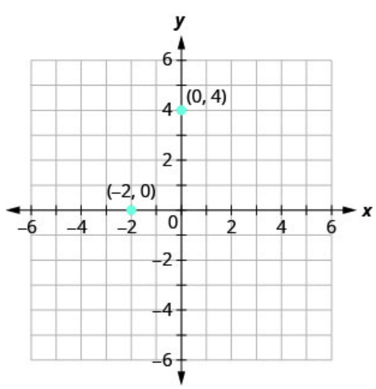
The point $(0, 0)$ is called the **origin**. It is the point where the x-axis and y-axis intersect.

We use the coordinates to locate a point on the xy-plane. Let’s plot the point $(1, 3)$ as an example. First, locate the x-coordinate 1 on the x-axis and lightly sketch a vertical line through $x = 1$. Then, locate the y-coordinate 3 on the y-axis and sketch a horizontal line through $y = 3$. Now, find the point where these two lines meet -- that is the point with coordinates $(1, 3)$.



Notice that the vertical line through $x = 1$ and the horizontal line through $y = 3$ are not part of the graph. We just used them to help us locate the point $(1, 3)$.

When one of the coordinates is zero, the point lies on one of the axes. The graph below shows that the point $(0, 4)$ is on the y-axis and the point $(-2, 0)$ is on the x-axis.



Points on the x- or y-axis

1. Points with a y-coordinate equal to 0 are on the x-axis, and have the form $(p, 0)$, where p is some real number.
2. Points with an x-coordinate equal to 0 are on the y-axis, and have the form $(0, q)$, where q is some real number.

Example 3.1.1.4

Plot each point in the rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-5, 4)$

b. $(-3, -4)$

c. $(2, -3)$

d. $(0, -1)$

e. $\left(3, \frac{5}{2}\right)$

f. $(-2, 3)$

Solution

The first number of the coordinate pair is the x-coordinate, and the second number is the y-coordinate. To plot each point, sketch a vertical line through the x-coordinate and a horizontal line through the y-coordinate. Their intersection is the point.

a. Since the x-coordinate is -5, the point is to the left of the y-axis. Also, since the y-coordinate is 4, the point is above the x-axis. The point $(-5, 4)$ is in Quadrant II.

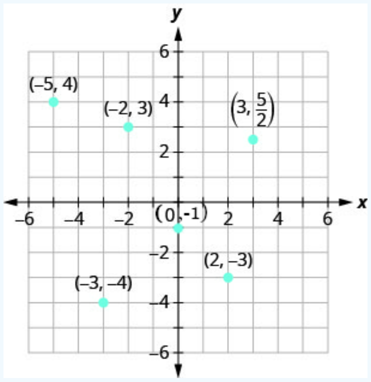
b. Since the x-coordinate is -3, the point is to the left of the y-axis. Also, since the y-coordinate is -4, the point is below the x-axis. The point $(-3, -4)$ is in Quadrant III.

c. Since the x-coordinate is 2, the point is to the right of the y-axis. Since the y-coordinate is -3, the point is below the x-axis. The point $(2, -3)$ is in Quadrant IV.

d. Since the x-coordinate is 0, the point whose coordinates are $(0, -1)$ is on the y-axis.

e. Since the x-coordinate is 3, the point is to the right of the y-axis. Since the y-coordinate is $\frac{5}{2}$, the point is above the x-axis. (It may be helpful to write $\frac{5}{2}$ as a mixed number, $2\frac{1}{2}$, or decimal, 2.5, so that we know $\frac{5}{2}$ is between 2 and 3.) The point $\left(3, \frac{5}{2}\right)$ is in Quadrant I.

f. Since the x-coordinate is -2, the point is to the left of the y-axis. Since the y-coordinate is 3, the point is above the x-axis. The point $(-2, 3)$ is in Quadrant II.



? Try It 3.1.1.5

Plot each point in a rectangular coordinate system and identify the quadrant in which the point is located:

a. $(-2, 1)$

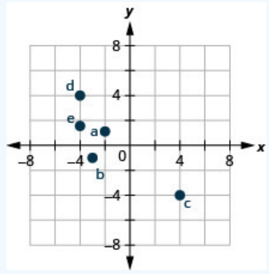
b. $(-3, -1)$

c. $(4, -4)$

d. $(-4, 4)$

e. $\left(-4, \dfrac{3}{2}\right)$

Answer



The points $(-2,1)$, $(-4,4)$, and $\left(-4, \dfrac{3}{2}\right)$ are in Quadrant II.

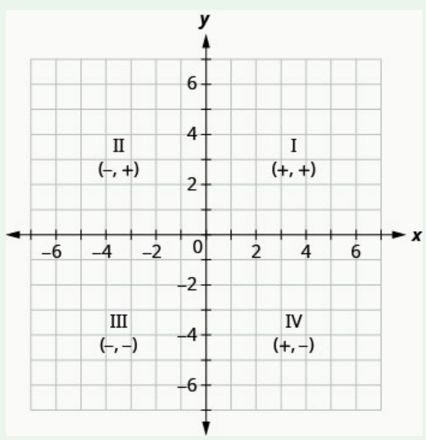
The point $(-3,-1)$ is in Quadrant III.

The point $(4,-4)$ is in Quadrant IV.

The signs of the x-coordinate and y-coordinate affect the location of the points. We may have noticed some patterns as we graphed the points in the previous example. We can summarize sign patterns of the quadrants in this way:

Quadrants

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$



Linear Expressions and Linear Equations with Two Variables

Up to now, all the equations we have solved were equations with just one variable. In almost every case, when we solved the equation we got exactly one solution. But equations can have more than one variable. Equations with two variables may be of the form $Ax+By=C$. An equation of this form is called a *linear equation with two variables*.

Definition 3.1.1.6

1. An expression that can be written as

$$Ax+By$$

with A and B real numbers, not both zero, is called a **linear expression (with two variables)**, or more specifically, a **linear expression in x and y.**

2. An equation that can be written as

$$Ax + By = C$$

with A and B real numbers, not both zero, is called a **linear equation (with two variables)**, or more specifically, a **linear equation in x and y.**

Here is an example of a linear equation with two variables, x and y.

$$x + 4y = 8$$

This equation is in the form $Ax + By = C$ with $A = 1$, $B = 4$, and $C = 8$. The equation

$$y = -3x + 5$$

is also a linear equation with two variables, x and y, but it does not appear to be in the form $Ax + By = C$. We can rewrite it in $Ax + By = C$ form in the following way.

	$y = -3x + 5$
Add $3x$ to both sides.	$y + 3x = -3x + 5 + 3x$
Simplify.	$y + 3x = 5$
Use the Commutative Property to put it in $Ax + By = C$ form.	$3x + y = 5$

By rewriting $y = -3x + 5$ as $3x + y = 5$, we can easily see that it is a linear equation with two variables because it is of the form $Ax + By = C$ with $A = 3$, $B = 1$, and $C = 5$. When an equation is in the form $Ax + By = C$, we say it is in *standard form of a linear equation*.

Definition 3.1.1.7

A linear equation with two variables, x and y, is in **standard form** when it is written as $Ax + By = C$.

Most people prefer to have A, B, and C be integers and $A \geq 0$ when writing a linear equation in standard form, although it is not strictly necessary.

Linear equations with two variables have infinitely many solutions. For example, if $A \neq 0$, for every number that is substituted for x there is a corresponding y-value. This pair of values is a *solution* to the linear equation and is represented by the ordered pair (x, y). When we substitute these values of x and y into the equation, the result is a true statement, because the value on the left side is equal to the value on the right side.

Definition 3.1.1.8

An ordered pair (p, q) is a **solution of the linear equation** $Ax + By = C$, if the equation is a true statement when the x- and y-coordinates of the ordered pair, p and q, respectively, are substituted into the equation. We can also say in this case that $(x, y) = (p, q)$ is a solution, or $x = p$ and $y = q$ is a solution.

We can represent the solutions as points in the rectangular coordinate system. The points will line up perfectly in a straight line. We use a straight-edge to draw this line, and put arrows on the ends of each side of the line to indicate that the line continues in both directions.

A graph is a visual representation of all the solutions of the equation. It is an example of the saying, “A picture is worth a thousand words.” The line (with the arrows) shows us *all* the solutions to that equation. Every point on the line corresponds a solution of the equation. And, every solution of this equation corresponds to a point on this line. This line is called the graph of the equation. Points *not* on the line do not correspond to solutions! We may say, as is common, then that the points on the line *are* the solutions.

Definition 3.1.1.9

The **graph of the linear equation** $Ax + By = C$ is the collection of all solutions (x, y).

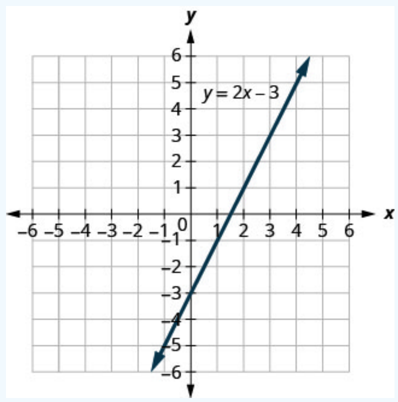
We can represent the graph on the coordinate plane. The **representation** is a straight line so that

- every solution of the equation is a point on this line, and
- every point on the line is a solution of the equation.

As universally accepted, a representation is also called a **graph of the linear equation**.

Example 3.1.1.10

The graph of $y = 2x - 3$ is shown below.



For each ordered pair,

$$A(0, -3), \quad B(3, 3), \quad C(2, -3), \quad \text{and} \quad D(-1, -5),$$

decide:

a. is the ordered pair a solution to the equation?

b. is the point on the line?

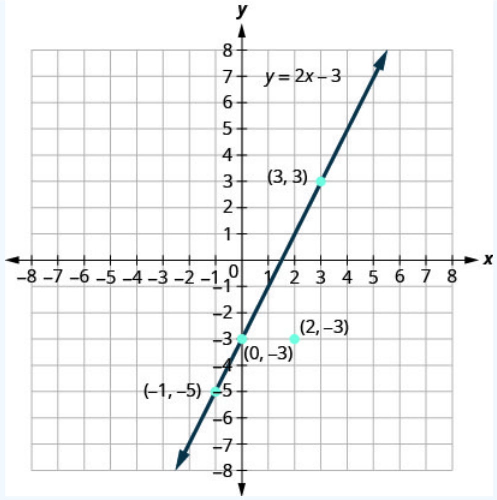
Solution

Substitute the x- and y-values into the equation to check if the ordered pair is a solution to the equation.

a.

Point	$A(0,-3)$	$B(3,3)$	$C(2,-3)$	$D(-1,-5)$
Write the equation of the line.	$y = 3x - 3$	$y = 2x - 3$	$y = 2x - 3$	$y = 2x - 3$
Substitute the x- and y-values.	$-3 \stackrel{?}{=} 2 \cdot 0 - 3$	$3 \stackrel{?}{=} 2 \cdot 3 - 3$	$-3 \stackrel{?}{=} 2 \cdot 2 - 3$	$-5 \stackrel{?}{=} 2 \cdot (-1) - 3$
Simplify.	$-3 \stackrel{?}{=} -3$	$3 \stackrel{?}{=} 3$	$-3 \stackrel{?}{=} 1$	$-5 \stackrel{?}{=} 5$
True or false?	True	True	False	True
Answer the question.	$(0,-3)$ is a solution	$(3,3)$ is a solution	$(2,-3)$ is not a solution	$(-1,-5)$ is a solution

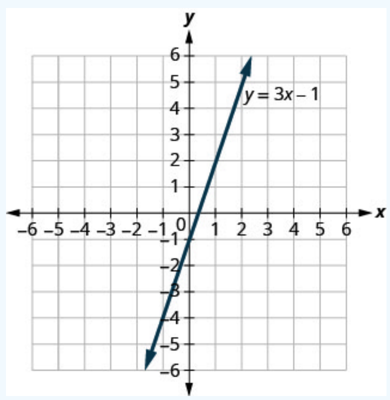
b. Plot the points $(0, -3)$, $(3, 3)$, $(2, -3)$, and $(-1, -5)$.



The points $(0, 3)$, $(3, -3)$, and $(-1, -5)$ are on the line $y = 2x - 3$, and the point $(2, -3)$ is not on the line. The points that are solutions to $y = 2x - 3$ are on the line, but the points that are not solutions are not on the line.

? Try It 3.1.1.11

The graph of $y = 3x - 1$ is shown below.



For each ordered pair,

$$A(0, -1) \qquad \text{and} \qquad B(2, 5),$$

decide:

a. is the ordered pair a solution to the equation?
b. is the point on the line?

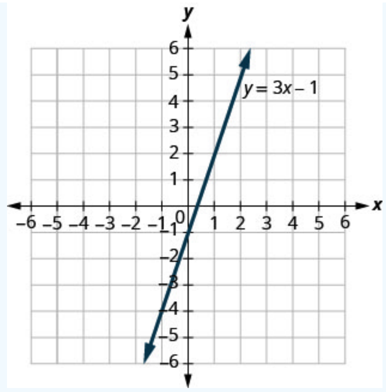
Answer

a. Both pairs are solutions.

b. Both pairs are on the line.

? Try It 3.1.1.12

The graph of $y = 3x - 1$ is shown below.



For each ordered pair,

$$A(3,-1) \qquad \text{and} \qquad B(-1,-4),$$

decide:

a. Is the ordered pair a solution to the equation?
b. Is the point on the line?

Answer

a. A is a solution.

B is not a solution.

b. A is on the line.

B is not on the line.

Graph a Linear Equation by Plotting Points

There are several methods that can be used to graph a linear equation. The first method we will use is called *plotting points*. We find three points whose coordinates are solutions to the equation and then plot them in a rectangular coordinate system. By connecting these points in a line, we have the graph of the linear equation. While two points are enough to determine a line, using three points helps us detect errors.

Example 3.1.1.13

Graph the equation $y = 2x + 1$ by plotting points.

Solution

Step 1. Find three points whose coordinates are solutions to the equation.

You can choose any values for *x* or *y*.

In this case, since *y* is isolated on the left side of the equation, it is easier to choose values for *x*.

$y = 2x + 1$

$x = 0$

$y = 2x + 1$

$y = 2 \cdot 0 + 1$

$y = 0 + 1$

$y = 1$

$x = 1$

$y = 2x + 1$

$y = 2 \cdot 1 + 1$

$y = 2 + 1$

$y = 3$

$x = -2$

$y = 2x + 1$

$y = 2(-2) + 1$

$y = -4 + 1$

$y = -3$

Organize the solutions in a table.

Put the three solutions in a table.

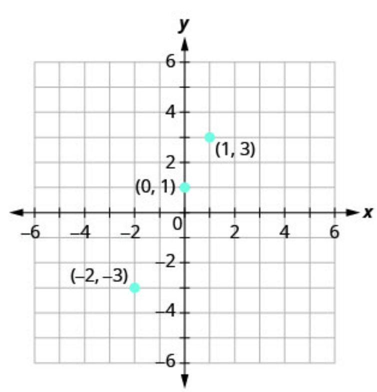
$y = 2x + 1$		
x	y	(x, y)
0	1	(0, 1)
1	3	(1, 3)
–2	–3	(–2, –3)

Step 2. Plot the points in a rectangular coordinate system.

Check that the points line up. If they do not, carefully check your work!

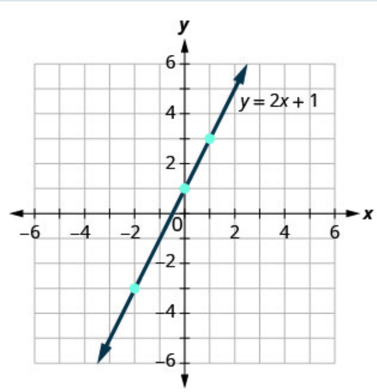
Plot:
(0, 1), (1, 3), (–2, –3).

Do the points line up?
Yes, the points line up.



Step 3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

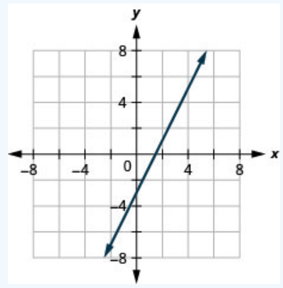
This line is the graph of $y = 2x + 1$.



Try It 3.1.1.14

Graph the equation $y=2x-3$ by plotting points.

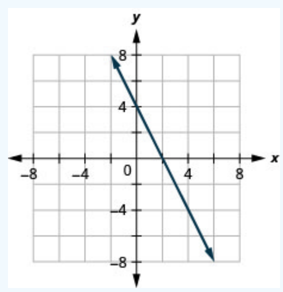
Answer



Try It 3.1.1.15

Graph the equation $y=-2x+4$ by plotting points.

Answer



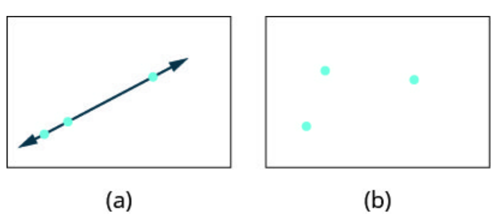
The steps to take when graphing a linear equation by plotting points are summarized here.

Graph a linear equation by plotting points

1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.

It is true that it only takes two points to determine a line, but it is a good habit to use three points. If we only plot two points and one of them is incorrect, we can still draw a line but it will not represent the solutions to the equation. It will be the wrong line.

If we use three points, and one is incorrect, the points will not line up. This tells us something is wrong and we need to check our work. Look at the difference between these illustrations.



When an equation includes a fraction as the coefficient of x, we can still substitute any numbers for x. But the arithmetic is easier if we make “good” choices for the values of x. This way we will avoid fractional answers, which are hard to plot precisely.

Example 3.1.1.16

Graph the equation $y = \frac{1}{2}x + 3$.

Solution

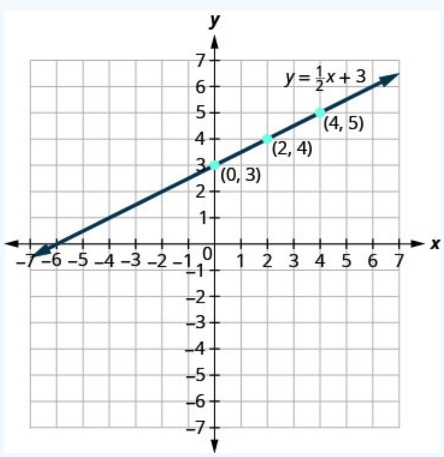
Find three points that are solutions to the equation. Since this equation has the fraction $\frac{1}{2}$ as a coefficient of x, we will choose values of x carefully. We will use zero as one choice and multiples of 2 for the other choices. Why are multiples of two a good choice for values of x? By choosing multiples of 2 the multiplication by $\frac{1}{2}$ simplifies to a whole number.

	Choose a value for x that is a multiple of 2.	$x = 0$	$x = 2$	$x = 4$
value f…	Write the equation of the line.	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$	$y = \frac{1}{2}x + 3$
value f…	Substitute the x-value into the equation.	$y = \frac{1}{2}(0) + 3$	$y = \frac{1}{2}(2) + 3$	$y = \frac{1}{2}(4) + 3$
value f…	Simplify.	$y = 0 + 3$	$y = 1 + 3$	$y = 2 + 3$
value f…	Find y.	$y = 3$	$y = 4$	$y = 5$

We organize the three solutions in a table.

$y = \frac{1}{2}x + 3$		
x	y	(x, y)
0	3	(0,3)
2	4	(2,4)
4	5	(4,5)

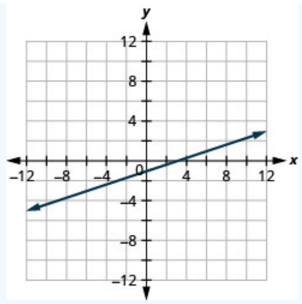
Plot the points, check that they line up, and draw the line.



Try It 3.1.1.17

Graph the equation $y = \frac{1}{3}x - 1$.

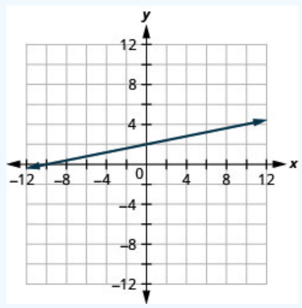
Answer



? Try It 3.1.1.18

Graph the equation $y=\frac{1}{4}x+2$.

Answer



Graph Vertical and Horizontal Lines

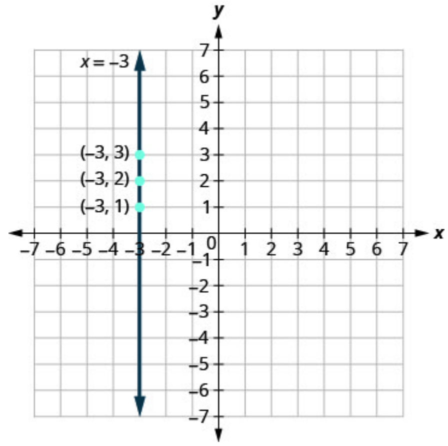
Some linear equations have only one variable. They may have just x and no y, or just y without an x. This changes how we make a table of values to get the points to plot.

Let’s consider the equation $x=-3$. This equation has only one variable, x. The equation says that the x-coordinate of any solution is equal to -3, so its value does not depend on the y-coordinate. No matter what is the value of the y-coordinate, the value of the x-coordinate is always -3.

So to make a table of values, write -3 in for all the x-coordinates. Then choose any values for the y-coordinate. Since the x-coordinate does not depend on the y-coordinate, we can choose any numbers we like. But to fit the points on our coordinate graph, we’ll use 1, 2, and 3 for the y-coordinates.

$x=-3$		
x	y	(x,y)
-3	1	$(-3,1)$
-3	2	$(-3,2)$
-3	3	$(-3,3)$

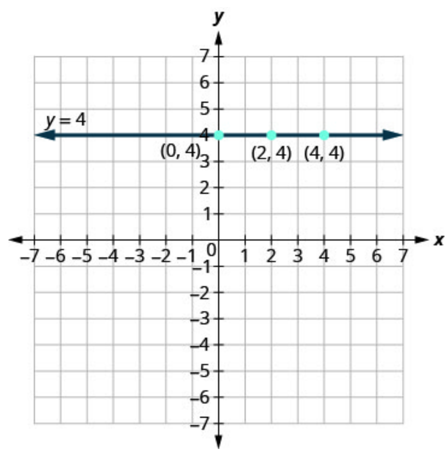
Plot the points from the table and connect them with a straight line. Notice that we have graphed a **vertical line**.



What if the equation has y but no x? Let's graph the equation $y=4$. This time the y-coordinate of any solution is 4, so for this equation, the y-coordinate of the solution does not depend on the x-coordinate. Fill in 4 for all the y-coordinates in the table and then choose any values for x-coordinates. We will use 0, 2, and 4 for the x-coordinates.

$y=4$		
x	y	(x,y)
0	4	(0,4)
2	4	(2,4)
4	4	(4,4)

In this figure, we have graphed a **horizontal line** passing through the y-axis at 4.



Definition 3.1.1.19

1. A **vertical line** is the graph of an equation (with two variables x and y) of the form $x=a$.

The line passes through the x-axis at $(a,0)$.

2. A **horizontal line** is the graph of an equation (with two variables x and y) of the form $y=b$.

The line passes through the y-axis at $(0,b)$.

Example 3.1.1.20

Graph:

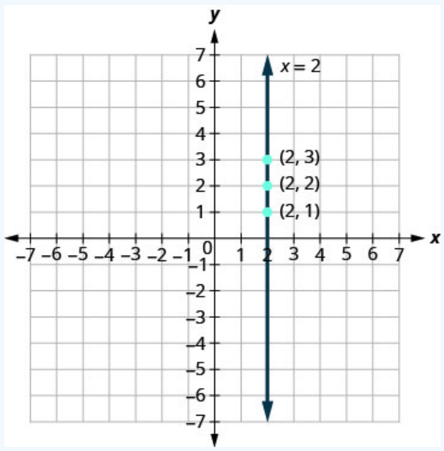
a. $x=2$

b. $y=-1$

Solution

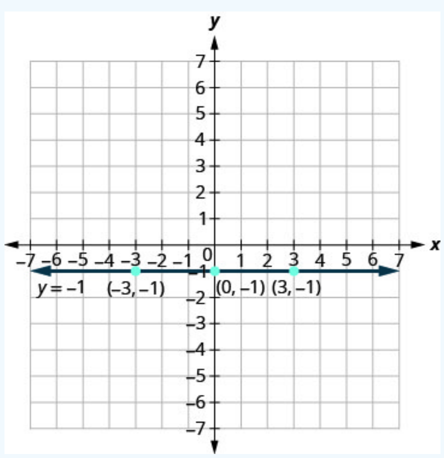
a. The equation has only one variable, x, and x is always equal to 2. We create a table where the x-coordinate is always 2 and then put in any values for the y-coordinate. The graph is a vertical line passing through the x-axis at 2.

$x=2$		
x	y	(x,y)
2	1	$(2,1)$
2	2	$(2,2)$
2	3	$(2,3)$



b. Similarly, the equation $y=-1$ has only one variable, y. The value of the y-coordinate of any solution is -1. All the ordered pairs in the next table have the same y-coordinate. The graph is a horizontal line passing through the y-axis at -1.

$y=-1$		
x	y	(x,y)
0	-1	$(0,-1)$
3	-1	$(3,-1)$
-3	-1	$(-3,-1)$



Try It 3.1.1.21

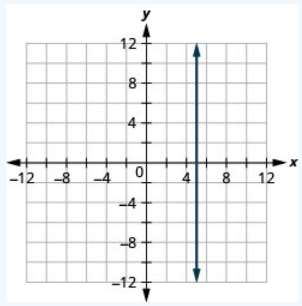
Graph:

a. $x=5$

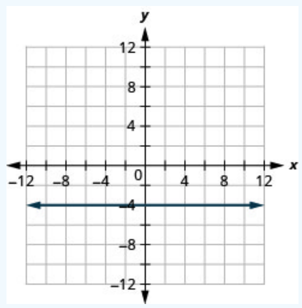
b. $y=-4$

Answer

a.



b.



Try It 3.1.1.22

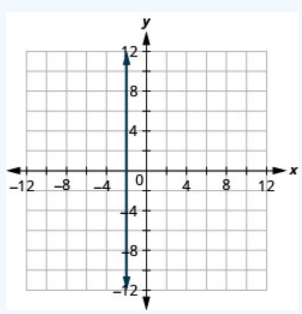
Graph:

a. $x=-2$

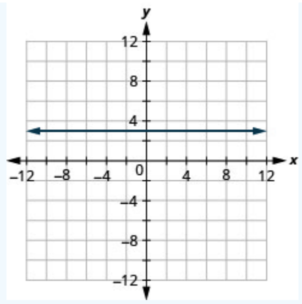
b. $y=3$

Answer

a.



b.

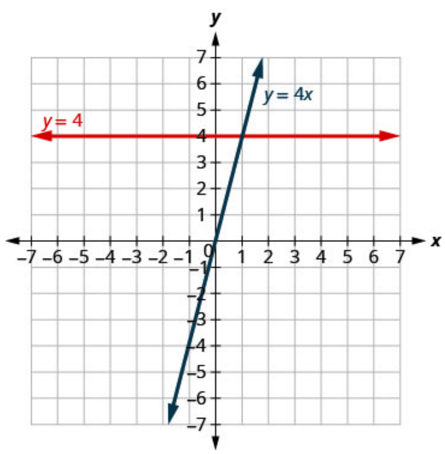


What is the difference between the equations $y=4x$ and $y=4$?

The equation $y=4x$ has both x and y. The value of the y-coordinate of a solution depends on the value of the x-coordinate, so the y-coordinate changes according to the value of the x-coordinate. The equation $y=4$ has only one variable. The value of y-coordinate of any solution is 4, it does not depend on the value of the x-coordinate.

y = 4x		
x	y	(x, y)
0	0	(0, 0)
1	4	(1, 4)
2	8	(2, 8)

y = 4		
x	y	(x, y)
0	4	(0, 4)
1	4	(1, 4)
2	4	(2, 4)



Notice, in the graph, the equation $y=4x$ gives a slanted line, while $y=4$ gives a horizontal line.

? Example 3.1.1.23

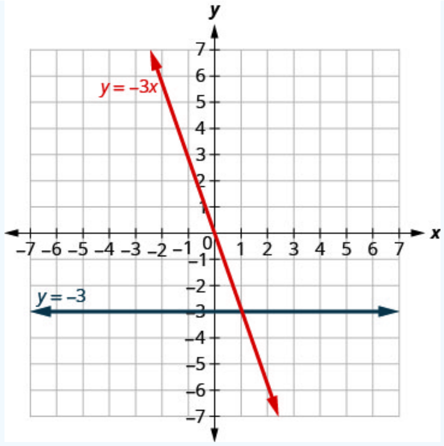
Graph $y=-3x$ and $y=-3$ in the same rectangular coordinate system.

Solution

We notice that the first equation has the variable x, while the second does not. We make a table of points for each equation and then graph the lines. The two graphs are shown.

y = –3x		
x	y	(x, y)
0	0	(0, 0)
1	–3	(1, –3)
2	–6	(2, –6)

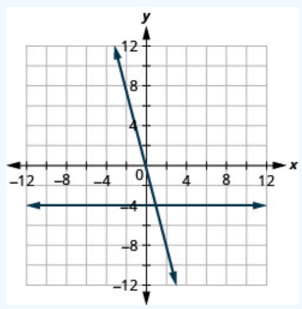
y = –3		
x	y	(x, y)
0	–3	(0, –3)
1	–3	(1, –3)
2	–3	(2, –3)



? Try It 3.1.1.24

Graph $y=-4x$ and $y=-4$ in the same rectangular coordinate system.

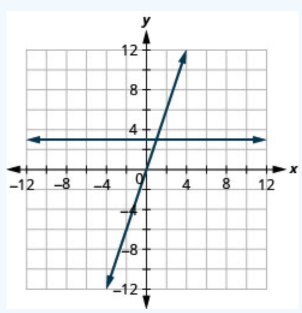
Answer



? Try It 3.1.1.25

Graph $y=3$ and $y=3x$ in the same rectangular coordinate system.

Answer



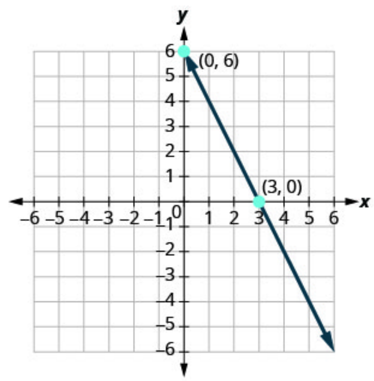
Find x- and y-intercepts

Every linear equation can be represented by a line. We have seen that when graphing a line by plotting points, we can use any three solutions to graph. This means that two people graphing the line might use different sets of three points.

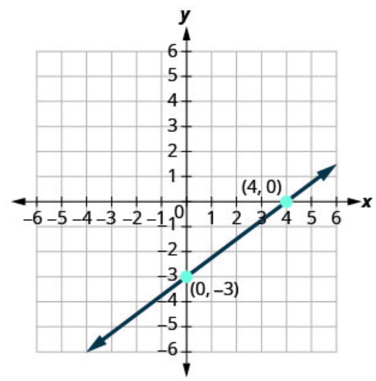
At first glance, their two lines might not appear to be the same, since they would have different points labeled. But if all the work was done correctly, the lines should be exactly the same. One way to recognize that they are indeed the same line is to look at where the line intersects the x-axis and the y-axis. These points are called the *intercepts of a line*.

> **Definition 3.1.1.26**
>
> The points where a graph crosses the x-axis and the y-axis are called the **intercepts of the graph**.

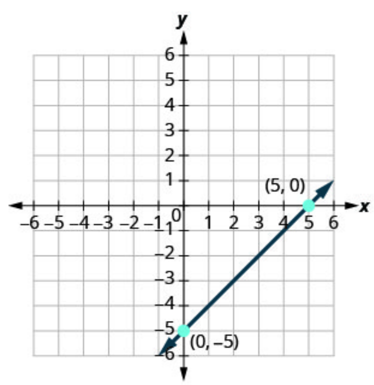
Let's look at the graphs of the lines.



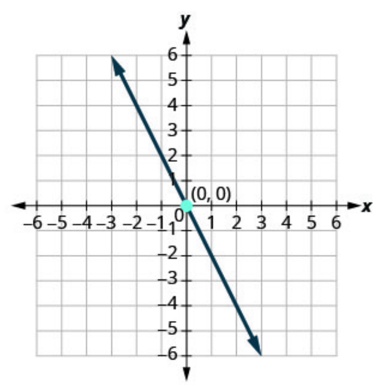
(a) $2x + y = 6$



(b) $3x - 4y = 12$



(c) $x - y = 5$



(d) $y = -2x$

First, notice where each of these lines crosses the x-axis. Now, let's look at the points where these lines cross the y-axis.

Figure	The line crosses the x-axis at	Ordered pair for this point	The line crosses the y-axis at	Ordered pair for this point
Figure (a)	3	$(3,0)$	6	$(0,6)$
Figure (b)	4	$(4,0)$	-3	$(0,-3)$
Figure (c)	5	$(5,0)$	-5	$(0,-5)$
Figure (d)	0	$(0,0)$	0	$(0,0)$
General Figure	a	$(a,0)$	b	$(0,b)$

Is there a pattern?

For each line, the y-coordinate of the point where the line crosses the x-axis is zero. The point where the line crosses the x-axis has the form $(a, 0)$ and is called the *x-intercept* of the line. The x-intercept occurs when y is zero.

In each line, the x-coordinate of the point where the line crosses the y-axis is zero. The point where the line crosses the y-axis has the form $(0, b)$ and is called the *y-intercept* of the line. The y-intercept occurs when x is zero.

Definition 3.1.1.27

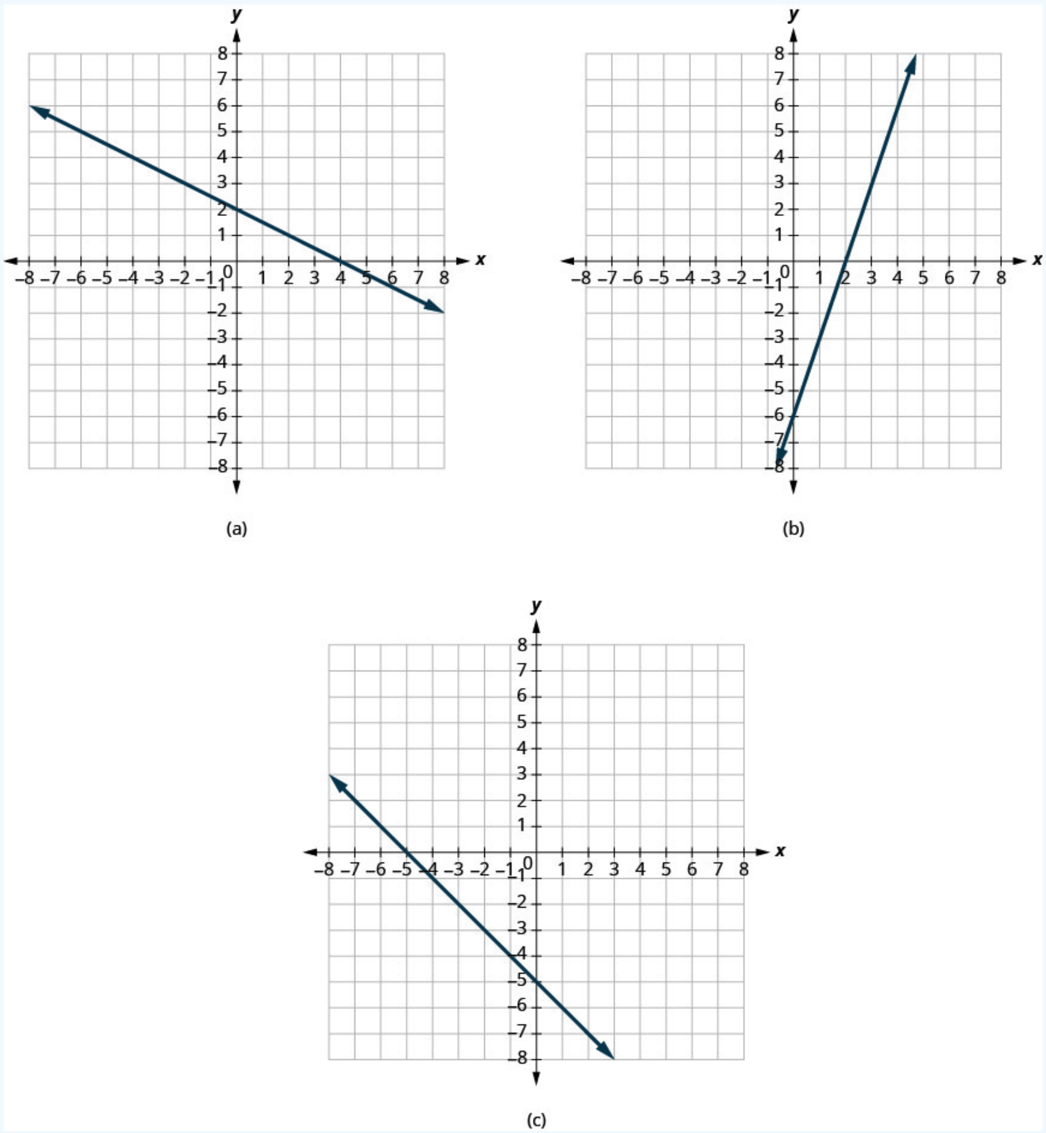
1. The x**-intercept** of a line is the point $(a, 0)$ where the line crosses the x-axis.
2. The y**-intercept** of a line is the point $(0, b)$ where the line crosses the y-axis.

- The x-intercept occurs when y is zero.
- The y-intercept occurs when x is zero.

x	y
a	0
0	b

Example 3.1.1.28

Find the x- and y-intercepts on each graph shown.



Solution

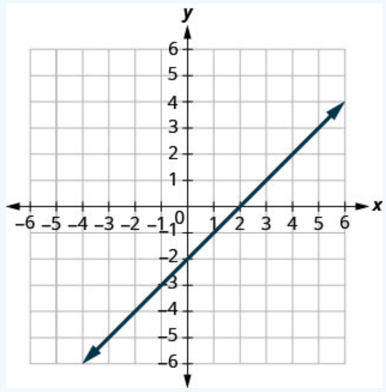
a. The graph crosses the x-axis at the point $(4, 0)$. The x-intercept is $(4, 0)$.
The graph crosses the y-axis at the point $(0, 2)$. The y-intercept is $(0, 2)$.

b. The graph crosses the x-axis at the point $(2, 0)$. The x-intercept is $(2, 0)$.
The graph crosses the y-axis at the point $(0, -6)$. The y-intercept is $(0, -6)$.

c. The graph crosses the x-axis at the point $(-5, 0)$. The x-intercept is $(-5, 0)$.
The graph crosses the y-axis at the point $(0, -5)$. The y-intercept is $(0, -5)$.

? Try It 3.1.1.29

Find the x- and y-intercepts on the graph.

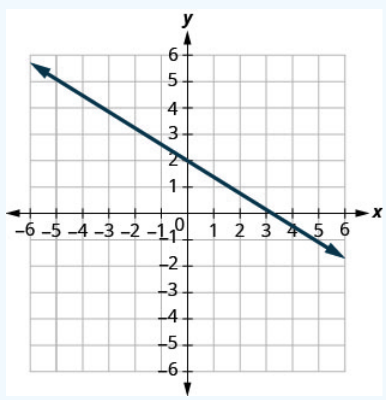


Answer

The x-intercept is $(2, 0)$.
The y-intercept is $(0, -2)$.

? Try It 3.1.1.30

Find the x- and y-intercepts on the graph.



Answer

The x-intercept is $(3, 0)$.
The y-intercept is $(0, 2)$.

Recognizing that the x-intercept occurs when y is zero and that the y-intercept occurs when x is zero gives us a method to find the intercepts of a line from its equation. To find the x-intercept, let $y = 0$ and solve for x. To find the y-intercept, let $x = 0$ and solve for y.

Intercepts from the equation of a line

To find:

- the x-intercept of the line, let $y=0$ and solve for x.
- the y-intercept of the line, let $x=0$ and solve for y.

Example 3.1.1.31

Find the intercepts of the graph of $2x+y=8$.

Solution

We will let $y=0$ to find the x-intercept, and let $x=0$ to find the y-intercept. We will fill in a table, which reminds us of what we need to find.

$2x+y=8$		
x	y	
	0	x-intercept
0		y-intercept

Let's find the x-intercept first.

	$2x+y=8$	
	To find the x-intercept, let $y=0$.	
To fin…	Let $y=0$.	$2x+0=8$
To fin…	Simplify.	$2x=8$
To fin…		$x=4$
To fin…	Write the x-intercept as a point (x,y).	$(4,0)$

Now, the y-intercept.

	$2x+y=8$	
	To find the y-intercept, let $x=0$.	
To fin…	Let $x=0$.	$2\cdot 0+y=8$
To fin…	Simplify.	$y=8$
To fin…	Write the y-intercept as a point (x,y).	$(0,8)$

The intercepts are the points $(4,0)$ and $(0,8)$ as shown in the table.

$2x+y=8$		
x	y	(x,y)
4	0	$(4,0)$
0	8	$(0,8)$

The x-intercept is $(4,0)$.

The y-intercept is $(0,8)$.

Try It 3.1.1.32

Find the intercepts of the graph of $3x+y=12$.

Answer

The x-intercept is $(4,0)$.
The y-intercept is $(0,12)$.

Try It 3.1.1.33

Find the intercepts of the graph of $x+4y=8$.

Answer

The x-intercept is $(8,0)$.
The y-intercept is $(0,2)$.

Graph a Line Using the Intercepts

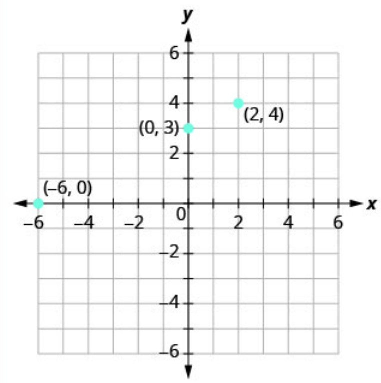
To graph a linear equation by plotting points, we need to find three points whose coordinates are solutions to the equation. We can use the x- and y- intercepts as two of our three points. Find the intercepts, and then find a third point to ensure accuracy. Make sure the points line up—then draw the line. This method is often the quickest way to graph a line.

Example 3.1.1.34

Graph $-x+2y=6$ using the intercepts.

Solution

Step 1. Find the x- and y-intercepts of the line. Let $y=0$ and solve for x.	Find the x-intercept.	Let $y=0$ $-x+2y=6$ $-x+2(0)=6$ $-x=6$ $x=-6$ The x-intercept is $(-6, 0)$.
Let $x=0$ and solve for y.	Find the y-intercept.	Let $x=0$. $-x+2y=6$ $-0+2y=6$ $2y=6$ $y=3$ The y-intercept is $(0, 3)$.
Step 2. Find another solution to the equation.	We'll use $x=2$.	Let $x=2$. $-x+2y=6$ $-2+2y=6$ $2y=8$ $y=4$ A third point is $(2, 4)$.

<table>
<tr>
<td>Step 3. Plot the three points. Check that the points line up.</td>
<td>
<table>
<tr><th>x</th><th>y</th><th>(x, y)</th></tr>
<tr><td>–6</td><td>0</td><td>(–6, 0)</td></tr>
<tr><td>0</td><td>3</td><td>(0, 3)</td></tr>
<tr><td>2</td><td>4</td><td>(2, 4)</td></tr>
</table>
</td>
<td>
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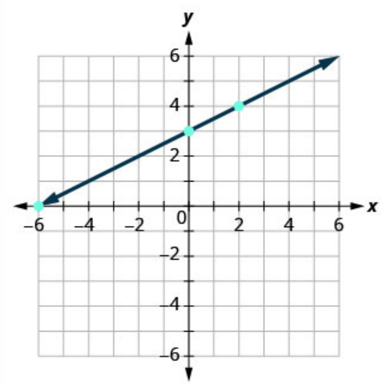
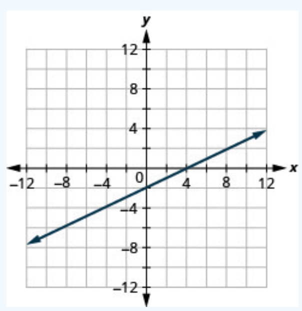
</tr>
<tr>
<td>Step 4. Draw the line.</td>
<td>See the graph.</td>
<td>
</td>
</tr>
</table>

? Try It 3.1.1.35

Graph $x - 2y = 4$ using the intercepts.

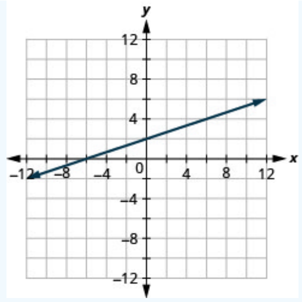
Answer



? Try It 3.1.1.36

Graph $-x + 3y = 6$ using the intercepts.

Answer



When the line passes through the origin, the x-intercept and the y-intercept are the same point.

Example 3.1.1.37

Graph $y=5x$ using the intercepts.

Solution

$y=5x$	
x-intercept	y-intercept
Let $y=0$.	Let $x=0$.
$0=5x$	$y=5\cdot 0$
$0=x$	$y=0$
$(0,0)$	$(0,0)$

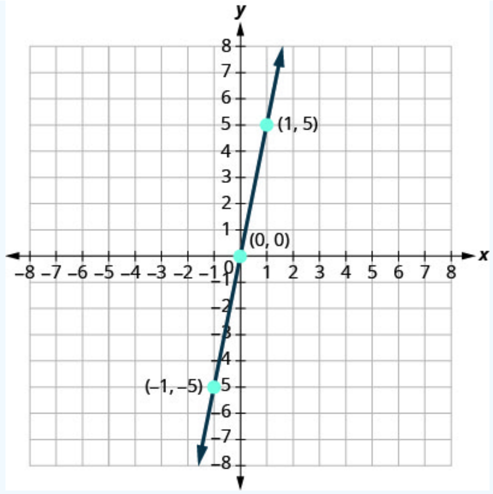
This line has only one intercept. It is the point $(0,0)$.
To check accuracy, we need to plot three points. Since the x- and y-intercepts are the same point, we need *two* more points to graph the line.

$y=5x$	
Let $x=1$.	Let $x=-1$.
$y=5\cdot 1$	$y=5\cdot(-1)$
$y=5$	$y=-5$

The resulting three points are summarized in the table.

$y=5x$		
x	y	(x,y)
0	0	$(0,0)$
1	5	$(1,5)$
-1	-5	$(-1,-5)$

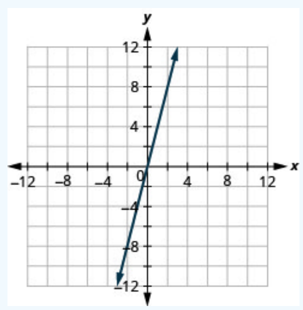
Plot the three points, check that they line up, and draw the line.



Try It 3.1.1.38

Graph $y=4x$ using the intercepts.

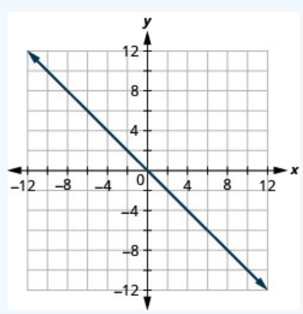
Answer



Try It 3.1.1.39

Graph $y=-x$ using the intercepts.

Answer



Writing Exercises 3.1.1.40

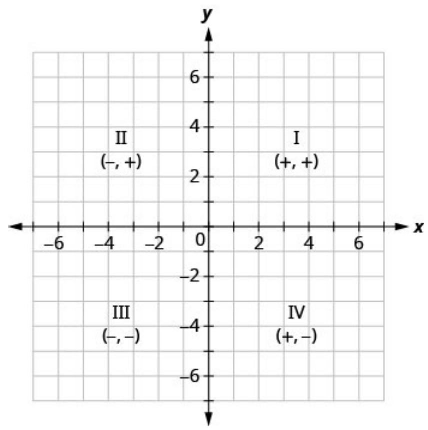
1. Explain how you would choose three x-values to make a table to graph the line $y=\frac{1}{5}x-2$.
2. What is the difference between the equations of a vertical and a horizontal line?

3. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $4x+y=-4$? Why?
4. Do you prefer to use the method of plotting points or the method using the intercepts to graph the equation $y=\frac{2}{3}x-2$? Why?

Key Concepts

- **Points on the Axes**
  - Points with a y-coordinate equal to 0 are on the x-axis, and have coordinates $(a, 0)$.
  - Points with an x-coordinate equal to 0 are on the y-axis, and have coordinates $(0, b)$.
- **Quadrant**

Quadrant I	Quadrant II	Quadrant III	Quadrant IV
(x,y)	(x,y)	(x,y)	(x,y)
$(+,+)$	$(-,+)$	$(-,-)$	$(+,-)$



- **Graph of a Linear Equation:** The graph of a linear equation $Ax+By=C$ is a straight line.
  Every point on the line is a solution of the equation.
  Every solution of this equation is a point on this line.
- **How to graph a linear equation by plotting points.**
  1. Find three points whose coordinates are solutions to the equation. Organize them in a table.
  2. Plot the points in a rectangular coordinate system. Check that the points line up. If they do not, carefully check your work.
  3. Draw the line through the three points. Extend the line to fill the grid and put arrows on both ends of the line.
- **x-intercept and y-intercept of a Line**
  - The x-intercept is the point $(a, 0)$ where the line crosses the x-axis.
  - The y-intercept is the point $(0, b)$ where the line crosses the y-axis.

  - The x-intercept occurs when y is zero.
  - The y-intercept occurs when x is zero.

x	y
a	0
0	b

- **Find the x- and y-intercepts from the Equation of a Line**
  - Use the equation of the line. To find:
    the x-intercept of the line, let $y=0$ and solve for x.
    the y-intercept of the line, let $x=0$ and solve for y.
- **How to graph a linear equation using the intercepts.**
  1. Find the x- and y-intercepts of the line.
     Let $y=0$ and solve for x.

Let $x = 0$ and solve for y.
2. Find a third solution to the equation.
3. Plot the three points and check that they line up.
4. Draw the line

3.1.2: Slope of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find the slope of a line
- Graph a line given a point and the slope
- Graph a line using its slope and intercept
- Choose the most convenient method to graph a line
- Graph and interpret applications of slope-intercept
- Use slopes to identify parallel and perpendicular lines

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\frac{1-4}{8-2}$.
2. Divide $\frac{0}{4}$ and $\frac{4}{0}$.
3. Simplify $\frac{15}{-3}$, $\frac{-15}{3}$, and $\frac{-15}{-3}$.

Find the Slope of a Line

When we graph linear equations, we may notice that some lines tilt up and some lines tilt down as they go from left to right. Some lines are very steep and some lines are flatter.

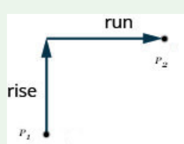
In mathematics, the measure of the steepness of a line is called the *slope* of the line.

The concept of slope has many applications in the real world. In construction, the pitch of a roof, the slant of the plumbing pipes, and the steepness of the stairs are all applications of slope.

We can assign a numerical value to the inclination of a line by finding the ratio of the rise and run. This is the slope.

Defintion 3.1.2.1

Given two points P_1 and P_2 on a line, the **rise** is the vertical "distance" and the **run** is the horizontal "distance" traveled and moving from P_1 to P_2, as shown in this illustration.



The "distance" is positive when we are moving up or to the right, and negative when we are moving down or to the left.

The **slope of a line** is $m = \frac{\text{rise}}{\text{run}}$.

To find the slope of a line, we locate any two points on the line, preferably whose coordinates are integers. Then we sketch a right triangle where the two points are vertices and one side is horizontal and one side is vertical.

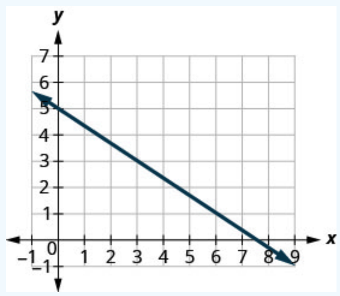
To find the slope of the line, we measure the "distance" along the vertical and horizontal sides of the triangle, that is, the rise and the run, respectively. The rise and the run can be positive, negative or zero.

Find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$

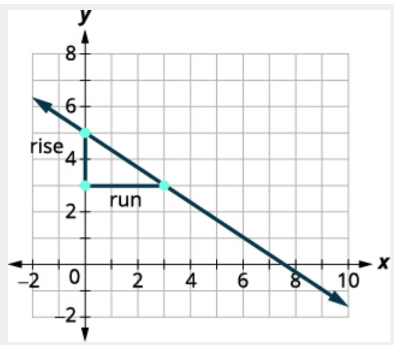
1. Locate two points on the line whose coordinates are integers.
2. Starting with one point, sketch a right triangle, going from the first point to the second point.
3. Determine the rise and the run on the legs of the triangle. This can be done by counting jumps vertically and horizontally when moving from the first point to the second.
4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.

Example 3.1.2.2

Find the slope of the line shown.

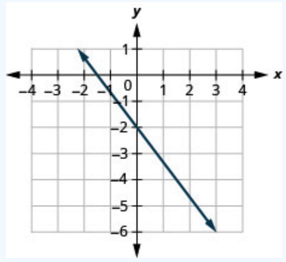


Solution

	Find the slope of a line from its graph.	
lope o…	Locate two points on the graph whose coordinates are integers.	$P_1 = (0,5)$ and $P_2 = (3,3)$
lope o…	Starting at $(0,5)$, sketch a right triangle to $(3,3)$ as shown in this graph.	
lope o…	Determine the rise. Since it goes down, it is negative.	The rise is -2.
lope o…	Determine the run.	The run is 3.
lope o…	Write the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
lope o…	Substitute the values of the rise and run.	$m = \frac{-2}{3}$
lope o…	Simplify.	$m = -\frac{2}{3}$
lope o…	Answer the question.	The slope of the line is $-\frac{2}{3}$.
lope o…	Interpret this as a rate of change.	As x increases by 3 units, y decreases by 2 units.

Try It 3.1.2.3

Find the slope of the line shown.

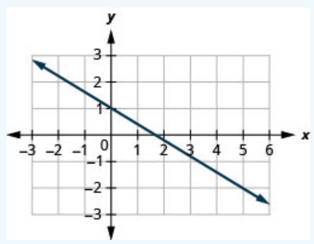


Answer

The slope of the line is $-\frac{4}{3}$.

Try It 3.1.2.4

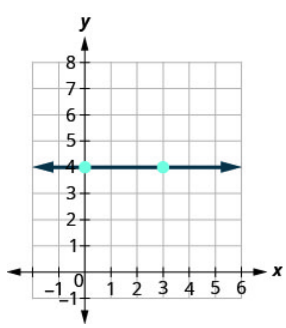
Find the slope of the line shown.



Answer

The slope of the line is $-\frac{3}{5}$.

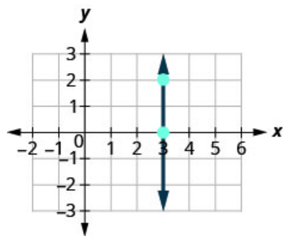
How do we find the slope of horizontal and vertical lines? To find the slope of the horizontal line, $y=4$, we could graph the line, find two points on it, and determine the rise and the run. Let's see what happens when we do this, as shown in the graph below.



Find the slope of a line from its graph.	
Locate two points on the graph whose coordinates are integers.	$P_1=(0,4)$ and $P_2=(3,4)$
Determine the rise.	The rise is 0.
Determine the run.	The run is 3.
Write the slope formula.	$m=\frac{\text{rise}}{\text{run}}$

…	Substitute the values of the rise and run.	$m=\frac{0}{3}$
…	Simplify.	$m=0$
…	Answer the question.	The slope of the horizontal line $y=4$ is 0.

Let's also consider a vertical line, the line $x=3$, as shown in the graph.



	Find the slope of a line from its graph.	
…	Locate two points on the graph whose coordinates are integers.	$P_1=(3,0)$ and $P_2=(3,2)$
…	Determine the rise.	The rise is 2.
…	Determine the run.	The run is 0.
…	Write the slope formula.	$m=\frac{\text{rise}}{\text{run}}$
…	Substitute the values of the rise and run.	$m=\frac{2}{0}$
…	Simplify.	m is undefined
…	Answer the question.	The slope of the vertical line $x=3$ is undefined.

The slope is undefined since division by zero is undefined. So we say that the slope of the vertical line $x=3$ is undefined.

All horizontal lines have slope 0. When the y-coordinates are the same, the rise is 0.

The slope of any vertical line is undefined. When the x-coordinates of a line are all the same, the run is 0.

Slope of a horizontal and vertical line

The **slope of a horizontal line**, $y=b$, is 0.

The **slope of a vertical line**, $x=a$, is undefined.

Example 3.1.2.5

Find the slope of each line:

a. $x=8$

b. $y=-5$

Solution

a. $x=8$
This is a vertical line. Its slope is undefined.

b. $y=-5$
This is a horizontal line. It has slope 0.

Try It 3.1.2.6

Find the slope of the line $x = -4$.

Answer

The slope of the line is undefined.

Try It 3.1.2.7

Find the slope of the line $y = 7$.

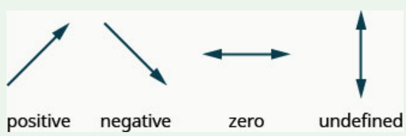
Answer

The slope of the line is 0.

How does the sign of the slope manifest itself in the line?

- When both the rise and the run are positive, the slope is positive. In this case, the line is "going up" from left to right.
- When both the rise and the run are negative, the slope is also positive. In this case, the line is "going up" from left to right.
- If the rise is positive and the run is negative, the slope is negative. In this case, the line is "going down" from left to right.
- If the rise is negative and the run is positive, the slope is negative. In this case, the line is "going down" from left to right.
- If the rise is zero, the slope is zero. In this case, the line is horizontal.
- If the run is zero, the slope is undefined. In this case, the line is vertical.

Quick guide to the slope of lines



Sometimes we’ll need to find the slope of a line between two points when we don’t have a graph to help determine the rise and the run. We could plot the points on grid paper, then count out the jumps to determine the rise and the run, but as we’ll see, there is a way to find the slope without graphing. Before we get to it, we need to introduce some algebraic notation.

We have seen that an ordered pair (x, y) gives the coordinates of a point. But when we work with slopes, we use two points. How can the same symbol (x, y) be used to represent two different points? Mathematicians use subscripts to distinguish the points.

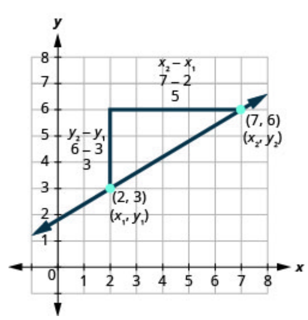
(x_1, y_1) read “x sub 1, y sub 1.”

(x_2, y_2) read “x sub 2, y sub 2.”

We will use (x_1, y_1) to identify the first point and (x_2, y_2) to identify the second point.

If we had more than two points, we could use (x_3, y_3), (x_4, y_4), and so on.

Let’s see how the rise and run relate to the coordinates of the two points by taking another look at the slope of the line between the points $(2, 3)$ and $(7, 6)$, as shown in this graph.



Find the slope of the line given two points.	
Since we have two points, we will use subscript notation.	$(x_1, y_1) = (2,3)$ and $(x_2, y_2) = (7,6)$
Write the slope formula.	$m = \frac{\text{rise}}{\text{run}}$
Determine the rise and the run by counting jumps on the grid.	On the graph, we counted jumps and found a rise of 3 and a run of 5.
Substitute the values of the rise and the run.	$m = \frac{3}{5}$ Notice that the rise of 3 can be found by subtracting the y-coordinates, 6 and 3, and the run of 5 can be found by subtracting the x-coordinates 7 and 2.
We rewrite the rise and run by putting in the coordinates.	$m = \frac{6-3}{7-2}$ But 6 is y_2 the y-coordinate of the second point and 3 is y_1, the y-coordinate of the first point.
We can rewrite the slope using subscript notation.	$m = \frac{y_2 - y_1}{7-2}$ Also 7 is the x-coordinate of the second point and 2 is the x-coordinate of the first point.
So again we rewrite the slope using subscript notation.	$m = \frac{y_2 - y_1}{x_2 - x_1}$
Conclusion.	We've shown that $m = \frac{y_2 - y_1}{x_2 - x_1}$ is $m = \frac{\text{rise}}{\text{run}}$. We can use this formula to find the slope of a line when we have two points on the line.

Definition 3.1.2.8

The **slope of the line** between two points $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

The slope is

$$y \text{ of the second point minus } y \text{ of the first point}$$

$$\text{over}$$

$$x \text{ of the second point minus } x \text{ of the first point.}$$

Example 3.1.2.9

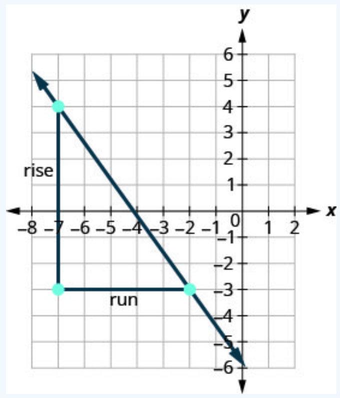
Use the slope formula to find the slope of the line through the points $(-2, -3)$ and $(-7, 4)$.

Solution

Find the slope of the line given two points.	
Write the two points.	$P_1 = (-2, -3)$ and $P_2 = (-7, 4)$
Identify x_1 and y_1.	$P_1 = \underbrace{(-2, -3)}_{(x_1, y_1)}$ $x_1 = -2, y_1 = -3$
Identify x_2 and y_2.	$P_2 = \underbrace{(-7, 4)}_{(x_2, y_2)}$ $x_2 = -7, y_2 = 4$
Write the slope formula.	$m = \frac{y_2 - y_1}{x_2 - x_1}$

	Find the slope of the line given two points.	
	Write the two points.	$P_1 = (-2,-3)$ and $P_2 = (-7,4)$
lope o…	Substitute the values.	$m = \dfrac{4-(-3)}{-7-(-2)}$
lope o…	Simplify.	$m = \dfrac{7}{-5}$
lope o…	Simplify.	$m = -\dfrac{7}{5}$
lope o…	Answer the question.	The slope of the line is $m = -\frac{7}{5}$.

Let's verify this slope on the graph shown.



$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{7}{-5} \\ &= -\frac{7}{5} \end{aligned}$$

? Try It 3.1.2.10

Use the slope formula to find the slope of the line through the points $(-3,4)$ and $(2,-1)$.

Answer

The slope of the line is -1.

? Try It 3.1.2.11

Use the slope formula to find the slope of the line through the points $(-2,6)$ and $(-3,-4)$.

Answer

The slope of the line is 10.

Graph a Line Given a Point and the Slope

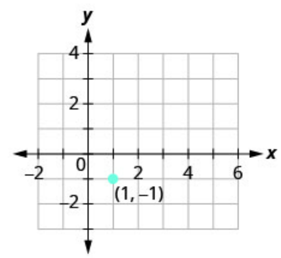
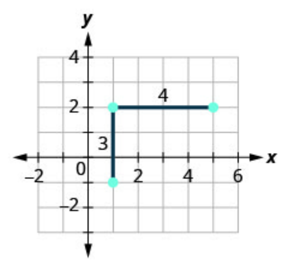
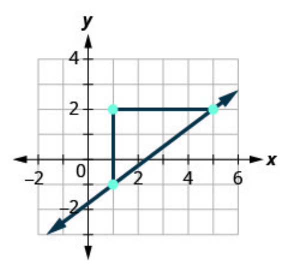
Up to now, we have graphed lines by plotting points, by using intercepts, and by recognizing horizontal and vertical lines.

We can also graph a line when we know one point and the slope of the line. We will start by plotting the point and then use the definition of slope to draw the graph of the line.

Example 3.1.2.12

Graph the line passing through the point $(1, -1)$ whose slope is $m = \frac{3}{4}$.

Solution

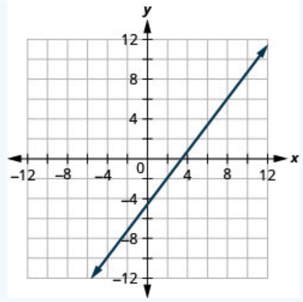
Step 1. Plot the given point.	Plot (1, –1).	
Step 2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.	Identify the rise and the run.	$m = \frac{3}{4}$ $\frac{\text{rise}}{\text{run}} = \frac{3}{4}$ $\text{rise} = 3$ $\text{run} = 4$
Step 3. Starting at the given point, count out the rise and run to mark the second point.	Start at (1, –1) and count the rise and the run. Up 3 units, right 4 units.	
Step 4. Connect the points with a line.	Connect the two points with a line.	

We can check our work by finding a third point. Since the slope is $m = \frac{3}{4}$, it can also be written as $m = \frac{-3}{-4}$ (negative divided by negative is positive!). Go back to $(1, -1)$ and count out the rise, -3, and the run, -4.

Try It 3.1.2.13

Graph the line passing through the point $(2, -2)$ with the slope $m = \frac{4}{3}$.

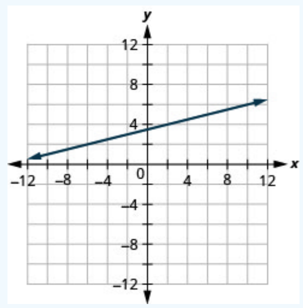
Answer



Try It 3.1.2.14

Graph the line passing through the point $(-2, 3)$ with the slope $m = \frac{1}{4}$.

Answer



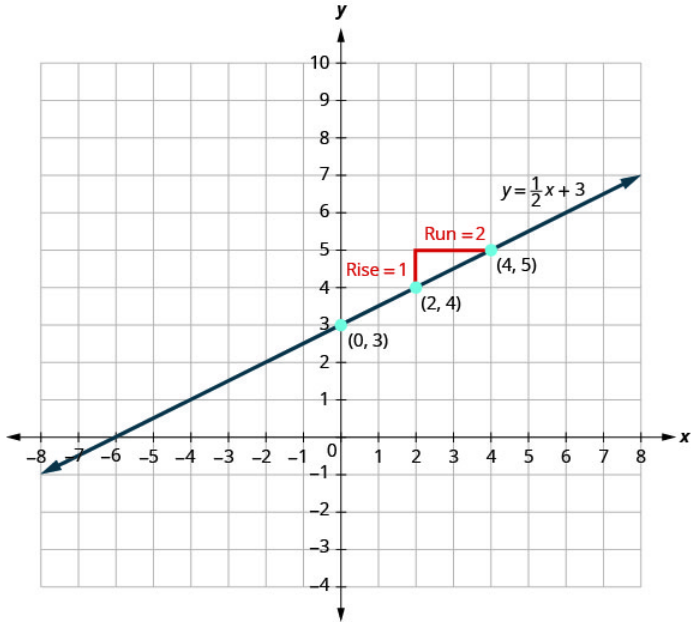
Graph a line given a point and the slope

1. Plot the given point.
2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
3. Starting at the given point, count jumps for the rise and run to mark the second point.
4. Draw a line passing through the points.

Graph a Line Using its Slope and Intercept

We have graphed linear equations by plotting points, using intercepts, recognizing horizontal and vertical lines, and using one point and the slope of the line. Once we see how an equation in slope-intercept form and its graph are related, we will have one more method we can use to graph lines.

Let's look at the graph of the equation $y = \frac{1}{2}x + 3$ and find its slope and y-intercept.



The red lines in the graph show us the rise is 1 and the run is 2. Substituting into the slope formula:

$$\begin{aligned} m &= \frac{\text{rise}}{\text{run}} \\ &= \frac{1}{2} \end{aligned}$$

The y-intercept is $(0, 3)$.

Look at the equation of this line.

$$y = \frac{1}{2}x + 3$$

Look at the slope and y-intercept.

$$\begin{aligned} \text{slope: } m &= \frac{1}{2} \\ y\text{-intercept: } &(0, 3) \end{aligned}$$

When a linear equation is solved for y, the coefficient of the x term is the slope m and the constant term is the y-coordinate of the y-intercept, say b. We say that the equation $y = \frac{1}{2}x + 3$ is in slope-intercept form.

$$y = \underbrace{\frac{1}{2}}_{m} x + \underbrace{3}_{b}$$

$$y = mx + b$$

Slope-intercept form of an equation of a line

The **slope-intercept form** of an equation of a line with slope m and y-intercept, $(0, b)$, is $y = mx + b$.

Let's practice finding the values of the slope and y-intercept from the equation of a line.

Example 3.1.2.15

Identify the slope and y-intercept of the line from the equation:

a. $y = -\frac{4}{7}x - 2$

b. $x+3y=9$

Solution

a. We compare our equation to the slope-intercept form of the equation.

	$y=-\frac{4}{7}x-2$
Write the slope-intercept form of the equation of the line.	$y=mx+b$
Write the equation of the line. Note that it is in slope-intercept form.	$y=-\frac{4}{7}x-2$
Identify the slope.	The slope is $m=-\frac{4}{7}$.
Identify the y-intercept.	The y-intercept is $(0,-2)$.

b. When an equation of a line is not given in slope-intercept form, our first step will be to solve the equation for y.

	$x+3y=9$
Solve for y.	$x+3y=9$
Subtract x from each side.	$3y=-x+9$
Divide both sides by 3.	$\frac{3y}{3}=\frac{-x+9}{3}$
Simplify.	$y=-\frac{1}{3}x+3$
Write the slope-intercept form of the equation of the line.	$y=mx+b$
Write the equation of the line in slope-intercept form.	$y=-\frac{1}{3}x+3$
Identify the slope.	The slope is $m=-\frac{1}{3}$.
Identify the y-intercept.	The y-intercept is $(0,3)$.

? Try It 3.1.2.16

Identify the slope and y-intercept from the equation of the line.

a. $y=\frac{2}{5}x-1$

b. $x+4y=8$

Answer

a. The slope is $m=\frac{2}{5}$, and the y-intercept is $(0,-1)$.

b. The slope is $m=-\frac{1}{4}$, and the y-intercept is $(0,2)$.

? Try It 3.1.2.17

Identify the slope and y-intercept from the equation of the line.

a. $y=-\frac{4}{3}x+1$

b. $3x+2y=12$

Answer

a. The slope is $m = -\frac{4}{3}$, and the y-intercept is $(0, 1)$.

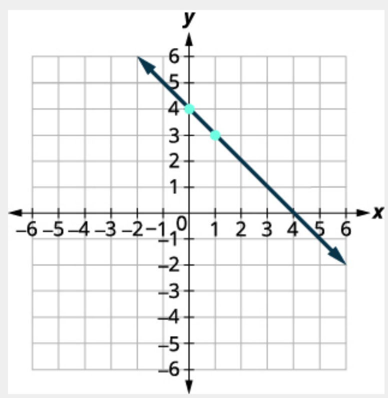
b. The slope is $m = -\frac{3}{2}$, and the y-intercept is $(0, 6)$.

We have graphed a line using the slope and a point. Now that we know how to find the slope and y-intercept of a line from its equation, we can use the y-intercept as the point, and then count jumps determined by the slope from there to find a second point.

? Example 3.1.2.18

Graph the line of the equation $y = -x + 4$ using its slope and y-intercept.

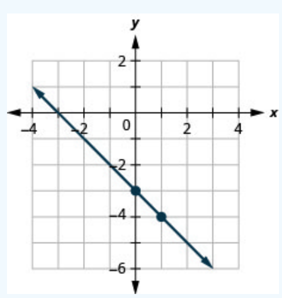
Solution

	$y = -x + 4$
The equation is in slope–intercept form, $y = mx + b$.	$y = -x + 4$
Identify the slope and y-intercept.	The slope is $m = -1$. The y-intercept is $(0, 4)$.
Plot the y-intercept.	See the graph.
Rewrite the slope in the fraction form.	$m = \frac{-1}{1}$
Identify the rise and the run.	rise = -1 run =1
Count jumps using the rise and run to mark the second point. Draw the line as shown in the graph.	

? Try It 3.1.2.19

Graph the line of the equation $y = -x - 3$ using its slope and y-intercept.

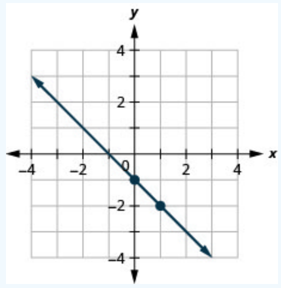
Answer



Try It 3.1.2.20

Graph the line of the equation $y=-x-1$ using its slope and y-intercept.

Answer



Now that we have graphed lines by using the slope and y-intercept, let's summarize all the methods we have used to graph lines.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y (table with three empty rows)	$y = mx + b$	x | y: 0 | ; | 0 ; (third row empty)	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

Choose the Most Convenient Method to Graph a Line

Now that we have seen several methods we can use to graph lines, how do we know which method to use for a given equation?

While we could plot points, use the slope-intercept form, or find the intercepts for *any* equation, if we recognize the most convenient way to graph a certain type of equation, our work will be easier.

Generally, plotting points is not the most efficient way to graph a line. Let's look for some patterns to help determine the most convenient method to graph a line.

Here are five equations we graphed in this section, and the method we used to graph each of them.

	Equation	**Method**
#1	$x=2$	Vertical line
#2	$y=-1$	Horizontal line
#3	$-x+2y=6$	Intercepts
#4	$4x-3y=12$	Intercepts
#5	$y=-x+4$	Slope-intercept

Equations #1 and #2 each have just one variable. Remember, in equations of this form the value of that one variable is the same for every solution; it does not depend on the value of the other variable. Equations of this form have graphs that are vertical or horizontal lines.

In Equations #3 and #4, both x and y are on the same side of the equation. These two equations are of the form $Ax+By=C$. We substituted $y=0$ to find the x-intercept and $x=0$ to find the y-intercept, and then found a third point by choosing another value for x or y.

Equation #5 is written in slope-intercept form. After identifying the slope and y-intercept from the equation we used them to graph the line.

This leads to the following strategy.

Strategy for choosing the most convenient method to graph a line

Consider the form of the equation.

- If it only has one variable, it is a vertical or horizontal line.
  - $x=a$ is a vertical line passing through the x-axis at a.
  - $y=b$ is a horizontal line passing through the y-axis at b.
- If y is isolated on one side of the equation, in the form $y=mx+b$, graph by using the slope and y-intercept.
  - Identify the slope and y-intercept and then graph.
- If the equation is of the form $Ax+By=C$, find the intercepts.
  - Find the x- and y-intercepts, a third point, and then graph.

Example 3.1.2.21

Determine the most convenient method to graph each line:

a. $y=5$

b. $4x-5y=20$

c. $x=-3$

d. $y=-\frac{5}{9}x+8$

Solution

a. $y=5$
This equation has only one variable, y. Its graph is a horizontal line crossing the y-axis at 5.

b. $4x-5y=20$
This equation is of the form $Ax+By=C$. The easiest way to graph it will be to find the intercepts and one more point.

c. $x=-3$
There is only one variable, x. The graph is a vertical line crossing the x-axis at -3.

d. $y=-\frac{5}{9}x+8$
Since this equation is in $y=mx+b$ form, it will be easiest to graph this line by using the slope and y-intercepts.

Try It 3.1.2.22

Determine the most convenient method to graph each line:

a. $3x+2y=12$

b. $y=4$

c. $y=\frac{1}{5}x-4$

d. $x=-7$

Answer

a. intercepts

b. horizontal line

c. slope-intercept

d. vertical line

Try It 3.1.2.23

Determine the most convenient method to graph each line:

a. $x=6$

b. $y=-\frac{3}{4}x+1$

c. $y=-8$

d. $4x-3y=-1$

Answer

a. vertical line

b. slope-intercept

c. horizontal line

d. intercepts

Graph and Interpret Applications of Slope-Intercept

Many real-world applications are modeled by linear equations. We will take a look at a few applications here so we can see how equations written in slope-intercept form relate to real-world situations.

Usually, when a linear equation models uses real-world data, different letters are used for the variables, instead of using only x and y. The variable names remind us of what quantities are being measured.

Also, we often will need to extend the axes in our rectangular coordinate system to bigger positive and negative numbers to accommodate the data in the application.

Example 3.1.2.24

The equation $F=\frac{9}{5}C+32$ is used to convert temperatures, C, on the Celsius scale to temperatures, F, on the Fahrenheit scale.

a. Find the Fahrenheit temperature for a Celsius temperature of 0.

b. Find the Fahrenheit temperature for a Celsius temperature of 20.

c. Interpret the slope and F-intercept of the equation.

d. Graph the equation.

Solution

a.

	Find the Fahrenheit temperature for a Celsius temperature of 0.	
ahren…	Write the conversion equation.	$F=\frac{9}{5}C+32$
ahren…	Find F when $C=0$.	$F=\frac{9}{5}(0)+32$
ahren…	Simplify.	$F=32$
ahren…	Answer the question.	The Fahrenheit temperature for a Celsius temperature of 0 is 20.

b.

	Find the Fahrenheit temperature for a Celsius temperature of 20.	
ren…	Write the conversion equation.	$F=\frac{9}{5}C+32$
ren…	Find F when $C=20$.	$F=\frac{9}{5}(20)+32$
ren…	Simplify.	$F=36+32$
ren…	Simplify.	$F=68$
ren…	Answer the question.	The Fahrenheit temperature for a Celsius temperature of 20 is 68.

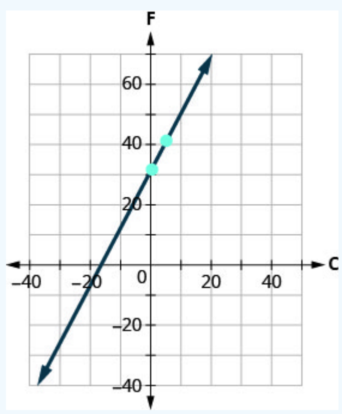
c.
Interpret the slope and F-intercept of the equation.
Even though this equation uses F and C, it is still in slope-intercept form.

$$y=mx+b$$
$$F=mC+b$$
$$F=\frac{9}{5}C+32$$

The slope, $\frac{9}{5}$, means that the temperature Fahrenheit (F) increases 9 degrees when the temperature Celsius (F) increases 5 degrees.

The F-intercept means that when the temperature is 0° on the Celsius scale, it is 32° on the Fahrenheit scale.

d. Graph the equation. We will need to use a larger scale than our usual. Start at the F-intercept $(0,32)$, and then count out the rise of 9 and the run of 5 to get a second point as shown in the graph.



? Try It 3.1.2.25

The equation $h=2s+50$ is used to estimate a woman's height in inches, h, based on her shoe size, s.

a. Estimate the height of a child who wears women's shoe size 0.

b. Estimate the height of a woman with shoe size 8.

c. Interpret the slope and h-intercept of the equation.

d. Graph the equation.

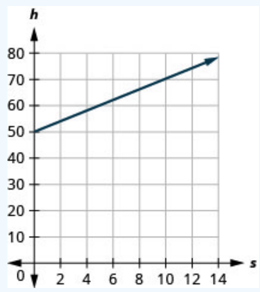
Answer

a. 50 inches

b. 66 inches

c. The slope, 2, means that the height, h, increases by 2 inches when the shoe size, s, increases by 1. The h-intercept means that when the shoe size is 0, the height is 50 inches.

d.



Try It 3.1.2.26

The equation $T=\frac{1}{4}n+40$ is used to estimate the temperature in degrees Fahrenheit, T, based on the number of cricket chirps, n, in one minute.

a. Estimate the temperature when there are no chirps.

b. Estimate the temperature when the number of chirps in one minute is 100.

c. Interpret the slope and T-intercept of the equation.

d. Graph the equation.

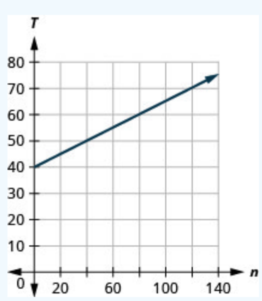
Answer

a. 40 degrees

b. 65 degrees

c. The slope, $\frac{1}{4}$, means that the temperature Fahrenheit (T) increases 1 degree when the number of chirps, n, increases by 4. The T-intercept means that when the number of chirps is 0, the temperature is 40°.

d.



The cost of running some types business have two components—a *fixed cost* and a *variable cost*. The fixed cost is always the same regardless of how many units are produced. This is the cost of rent, insurance, equipment, advertising, and other items that must be paid regularly. The variable cost depends on the number of units produced. It is for the material and labor needed to produce each item.

Example 3.1.2.27

Sam drives a delivery van. The equation $C = 0.5m + 60$ models the relation between his weekly cost, C, in dollars and the number of miles, n, that he drives.

a. Find Sam's cost for a week when he drives 0 miles.

b. Find the cost for a week when he drives 250 miles.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

Solution

a.

	Find Sam's cost for a week when he drives 0 miles.	
’s cost …	Write the equation that relates Sam's cost (C) per week when he drives n miles.	$C = 0.5n + 60$
’s cost …	Find C when $n = 0$.	$C = 0.5(0) + 60$
’s cost …	Simplify.	$C = 60$
’s cost …	Answer the question.	Sam's costs are \$60 when he drives 0 miles.

b.

	Find Sam's cost for a week when he drives 250 miles.	
’s cost …	Write the equation that relates Sam's cost (C) per week when he drives n miles.	$C = 0.5n + 60$
’s cost …	Find C when $n = 250$.	$C = 0.5(250) + 60$
’s cost …	Simplify.	$C = 185$
’s cost …	Answer the question.	Sam's costs are \$185 when he drives 250 miles.

c. Interpret the slope and C-intercept of the equation.

$$y = mx + b$$
$$C = 0.5n + 60$$

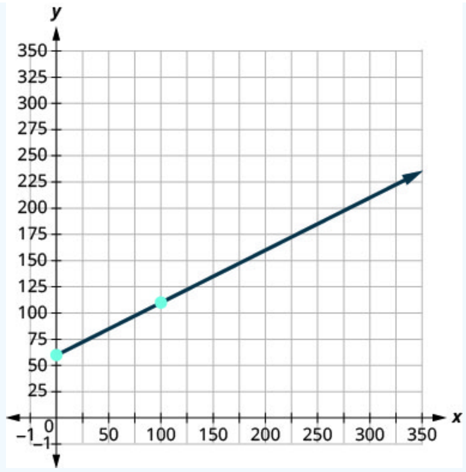
The slope, 0.5, means that the weekly cost, C, increases by \$0.50 when the number of miles driven, n, increases by 1. The C-intercept means that when the number of miles driven is 0, the weekly cost is \$60.

d. Graph the equation. We will need to use a larger scale than our usual. Start at the C-intercept $(0, 60)$.

To count out the slope $m = 0.5$, we rewrite it as an equivalent fraction that will make our graphing easier.

	$m = 0.5$
Rewrite as a fraction.	$m = \frac{0.5}{1}$
Multiply numerator and denominator by 100.	$m = \frac{0.5(100)}{1(100)}$
Simplify.	$m = \frac{50}{100}$

So to graph the next point go up 50 from the intercept of 60 and then to the right 100. The second point will be $(100, 110)$



Try It 3.1.2.28

Stella has a home business selling gourmet pizzas. The equation $C=4p+25$ models the relation between her weekly cost, C, in dollars and the number of pizzas, p, that she sells.

a. Find Stella's cost for a week when she sells no pizzas.

b. Find the cost for a week when she sells 15 pizzas.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

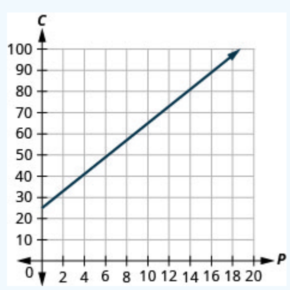
Answer

a. $25

b. $85

c. The slope, 4, means that the weekly cost, C, increases by $4 when the number of pizzas sold, p, increases by 1. The C-intercept means that when the number of pizzas sold is 0, the weekly cost is $25.

d.



Try It 3.1.2.29

Loreen has a calligraphy business. The equation $C=1.8n+35$ models the relation between her weekly cost, C, in dollars and the number of wedding invitations, n, that she writes.

a. Find Loreen's cost for a week when she writes no invitations.

b. Find the cost for a week when she writes 75 invitations.

c. Interpret the slope and C-intercept of the equation.

d. Graph the equation.

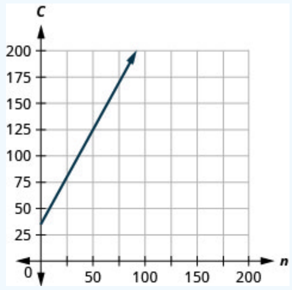
Answer

a. $\$35$

b. $\$170$

c. The slope, 1.8, means that the weekly cost, C, increases by $\$1.80$ when the number of invitations, n, increases by 1. The C-intercept means that when the number of invitations is 0, the weekly cost is $\$35$.

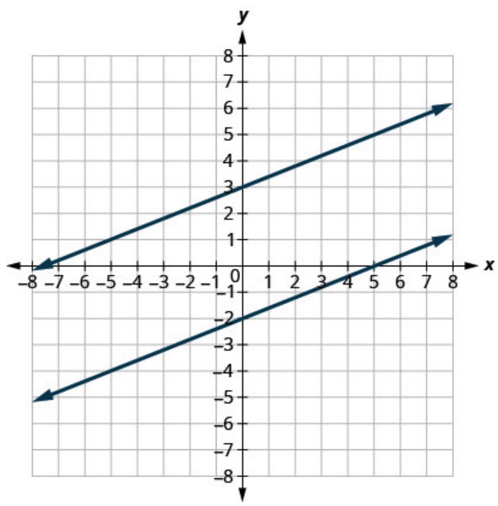
d.



Use Slopes to Identify Parallel and Perpendicular Lines

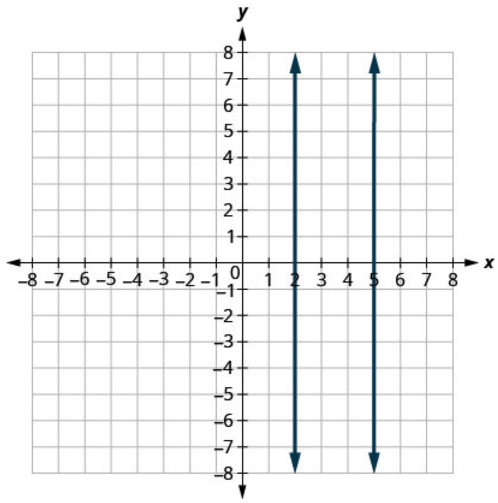
Two lines that don't intersect are called parallel. Parallel lines have the same steepness and never intersect.

We say this more formally in terms of the rectangular coordinate system. Two lines that have the same slope and different y-intercepts are called parallel lines.



Verify that both lines have the same slope, $m = \frac{2}{5}$, and different y-intercepts.

What about vertical lines? The slope of a vertical line is undefined but we see that vertical lines that have different x-intercepts are parallel, like the lines shown in this graph.



Definition 3.1.2.30

Parallel lines are lines in the same plane that do not intersect.

Parallel lines

- Parallel lines have the same slope and different y-intercepts.
- If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
- Parallel vertical lines have different x-intercepts.

Since non-vertical parallel lines have the same slope and different y-intercepts, we can now just look at the slope-intercept form of the equations of lines and decide if the lines are parallel.

Example 3.1.2.31

Use slopes and y-intercepts to determine if the lines are parallel:

a. $3x - 2y = 6$ and $y = \frac{3}{2}x + 1$

b. $y = 2x - 3$ and $-6x + 3y = -9$

Solution

a.

Verify that the lines are parallel.		
	$3x - 2y = 6$	$y = \frac{3}{2}x + 1$
Is the equation in slope-intercept form, $y = mx + b$?	No	Yes
Solve the first equation for y and write it in slope-intercept form.	$-2y = -3x + 6$ $\frac{-2y}{-2} = \frac{-3x + 6}{-2}$ $y = \frac{3}{2}x - 3$	
Now both equations are in slope-intercept form.	$y = \frac{3}{2}x - 3$	$y = \frac{3}{2}x + 1$
Identify the slope and y-intercept of both lines.	The slope is $m = \frac{3}{2}$. The y-intercept is $(0, -3)$.	The slope is $m = \frac{3}{2}$. The y-intercept is $(0, 1)$.

	Verify that the lines are parallel.	
Conclusion.	The lines have the same slope and different y-intercepts and so they are parallel. You may want to graph the lines to confirm whether they are parallel.	

b.

Verify that the lines are parallel.		
	$y=2x-3$	$-6x+3y=-9$
Is the equation in slope-intercept form, $y=mx+b$?	Yes	No
Solve the second equation for y and write it in slope-intercept form.		$3y=6x-9$ $\frac{3y}{3}=\frac{6x-9}{3}$ $y=2x-3$
Now both equations are in slope-intercept form.	$y=2x-3$	$y=2x-3$
Identify the slope and y-intercept of both lines.	The slope is $m=2$. The y-intercept is $(0,-3)$.	The slope is $m=2$. The y-intercept is $(0,-3)$.
Conclusion.	The lines have the same slope, but they also have the same y-intercepts. Their equations represent the same line and we say the lines are coincident. They are not parallel; they are the same line.	

? Try It 3.1.2.32

Use slopes and y-intercepts to determine if the lines are parallel:

a. $2x+5y=5$ and $y=-\frac{2}{5}x-4$

b. $y=-\frac{1}{2}x-1$ and $x+2y=-2$

Answer

a. The lines are parallel.

b. The lines are not parallel as the equations represent the same line.

? Try It 3.1.2.33

Use slopes and y-intercepts to determine if the lines are parallel:

a. $4x-3y=6$ and $y=\frac{4}{3}x-1$

b. $y=\frac{3}{4}x-3$ and $3x-4y=12$

Answer

a. The lines are parallel.

b. The lines are not parallel as the equations represent the same line.

Example 3.1.2.34

Use slopes and y-intercepts to determine if the lines are parallel:

a. $y=-4$ and $y=3$

b. $x=-2$ and $x=-5$

Solution

a. $y=-4$ and $y=3$

We recognize right away from the equations that these are horizontal lines, and so we know their slopes are both 0. Since the horizontal lines cross the y-axis at $y=-4$ and at $y=3$, we know the y-intercepts are $(0,-4)$ and $(0,3)$. The lines have the same slope and different y-intercepts and so they are parallel.

b. $x=-2$ and $x=-5$

We recognize right away from the equations that these are vertical lines, and so we know their slopes are undefined. Since the vertical lines cross the x-axis at $x=-2$ and $x=-5$, we know the y-intercepts are $(-2,0)$ and $(-5,0)$. The lines are vertical and have different x-intercepts and so they are parallel.

Try It 3.1.2.35

Use slopes and y-intercepts to determine if the lines are parallel:

a. $y=8$ and $y=-6$

b. $x=1$ and $x=-5$

Answer

a. The lines are parallel.

b. The lines are parallel.

Try It 3.1.2.36

Use slopes and y-intercepts to determine if the lines are parallel:

a. $y=1$ and $y=-5$

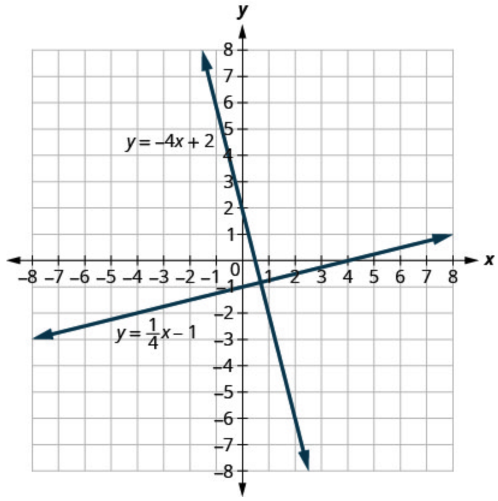
b. $x=8$ and $x=-6$

Answer

a. The lines are parallel.

b. The lines are parallel.

Let’s look at the lines whose equations are $y=\frac{1}{4}x-1$ and $y=-4x+2$.



These lines lie in the same plane and intersect in right angles. We call these lines *perpendicular*.

If we look at the slope of the first line, $m_1 = \frac{1}{4}$, and the slope of the second line, $m_2 = -4$, we can see that they are *negative reciprocals* of each other. If we multiply them, their product is -1.

$$\begin{aligned} & m_1 m_2 \\ = & \frac{1}{4}(-4) \\ = & -1 \end{aligned}$$

This is always true for **perpendicular lines** and leads us to the following definition.

Definition 3.1.2.37

Perpendicular lines are lines in the same plane that intersect at a right angle.

Perpendicular lines

- If m_1 and m_2 are the slopes of two perpendicular lines, then their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$. In other words, the product of their slopes is -1, that is, $m_1 m_2 = -1$.
- A vertical line and a horizontal line are always perpendicular to each other.

We were able to look at the slope-intercept form of linear equations and determine whether or not the lines were parallel. We can do the same thing for perpendicular lines.

We find the slope-intercept form of the equation, and then see if the slopes are opposite reciprocals. If the product of the slopes is -1, the lines are perpendicular.

Example 3.1.2.38

Use slopes to determine if the lines are perpendicular:

a. $y = -5x - 4$ and $x - 5y = 5$

b. $7x + 2y = 3$ and $2x + 7y = 5$

Solution

a.

Verify that the lines are perpendicular.		
	$y=-5x-4$	$x-5y=5$
Is the equation in slope-intercept form, $y=mx+b$?	Yes	No
Solve the second equation for y and write it in slope-intercept form.		$-5y=-x+5$ $y=-\frac{1}{5}x-1$
Now both equations are in slope-intercept form.	$y=-5x-4$	$y=-\frac{1}{5}x-1$
Identify the slope of each line.	$m_1=-5$	$m_2=\frac{1}{5}$
Conclusion.	The slopes are negative reciprocals of each other, so the lines are perpendicular. We check by multiplying the slopes. Since $$m_1m_2=-5\cdot\frac{1}{5}=-1,$$ it checks.	

b.

Verify that the lines are perpendicular.		
	$7x+2y=3$	$2x+7y=5$
Is the equation in slope-intercept form, $y=mx+b$?	No	No
Solve both equations for y and write it in slope-intercept form.	$2y=-7x+3$ $y=-\frac{7}{2}x+\frac{3}{2}$	$7y=-2x+5$ $y=-\frac{2}{7}x+\frac{5}{7}$
Now both equations are in slope-intercept form.	$y=-\frac{7}{2}x+\frac{3}{2}$	$y=-\frac{1}{5}x-1$
Identify the slope of each line.	$m_1=-\frac{7}{2}$	$m_2=-\frac{2}{7}$
Conclusion.	The slopes are reciprocals of each other, but they have the same sign. Since they are not negative reciprocals, the lines are not perpendicular.	

? Try It 3.1.2.39

Use slopes to determine if the lines are perpendicular:

a. $y=-3x+2$ and $x-3y=4$

b. $5x+4y=1$ and $4x+5y=3$

Answer

a. The lines are perpendicular.

b. The lines are not perpendicular.

? Try It 3.1.2.40

Use slopes to determine if the lines are perpendicular:

a. $y=2x-5$ and $x+2y=-6$

b. $2x-9y=3$ and $9x-2y=1$

Answer

a. The lines are perpendicular.

b. The lines are not perpendicular.

? Writing Exercises 3.1.2.41

1. How does the graph of a line with slope $m = 12$ differ from the graph of a line with slope $m = 2$?
2. Why is the slope of a vertical line "undefined"?
3. Explain how you can graph a line given a point and its slope.
4. Explain in your own words how to decide which method to use to graph a line.

Key Concepts

- **Slope of a Line**
  - The slope of a line is $m = \frac{\text{rise}}{\text{run}}$.
  - The rise measures the vertical change and the run measures the horizontal change when moving from one point on the line to another on the line.
- **How to find the slope of a line from its graph using $m = \frac{\text{rise}}{\text{run}}$.**
  1. Locate two points on the line whose coordinates are integers.
  2. Starting with one point, sketch a right triangle, going from the first point to the second point.
  3. Count vertical and horizontal jumps needed when moving along the legs of the triangle to find the rise and the run.
  4. Take the ratio of rise to run to find the slope: $m = \frac{\text{rise}}{\text{run}}$.
- **Slope of a line between two points.**

The slope of the line between two points (x_1, y_1) and (x_2, y_2) is

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

- **How to graph a line given a point and the slope.**
  1. Plot the given point.
  2. Use the slope formula $m = \frac{\text{rise}}{\text{run}}$ to identify the rise and the run.
  3. Starting at the given point, count jumps corresponding to the rise and run to mark the second point.
  4. Draw the line passing through the points.
- **Slope Intercept Form of an Equation of a Line**
  - The slope–intercept form of an equation of a line with slope m and y-intercept, $(0, b)$ is $y = mx + b$.

Methods to Graph Lines			
Point Plotting	**Slope–Intercept**	**Intercepts**	**Recognize Vertical and Horizontal Lines**
x | y (blank table)	$y = mx + b$	x | y; 0 | ; | 0	
Find three points. Plot the points, make sure they line up, then draw the line.	Find the slope and y-intercept. Start at the y-intercept, then count the slope to get a second point.	Find the intercepts and a third point. Plot the points, make sure they line up, then draw the line.	The equation has only one variable. $x = a$ vertical $y = b$ horizontal

- **Parallel Lines**
  - Parallel lines are lines in the same plane that do not intersect.
  - Parallel lines have the same slope and different y-intercepts.
  - If m_1 and m_2 are the slopes of two parallel lines then $m_1 = m_2$.
  - Parallel vertical lines have different x-intercepts.
- **Perpendicular Lines**
  - Perpendicular lines are lines in the same plane that intersect at a right angle.
  - If m_1 and m_2 are the slopes of two perpendicular lines, then their slopes are negative reciprocals of each other, $m_1 = -\frac{1}{m_2}$. Equivalently, the product of their slopes is -1, that is, $m_1 m_2 = -1$.
  - A vertical line and a horizontal line are always perpendicular to each other.

3.1.3: Finding the Equation of a Line

Learning Objectives

By the end of this section, you will be able to:

- Find an equation of the line given the slope and y-intercept
- Find an equation of the line given the slope and a point
- Find an equation of the line given two points
- Find an equation of a line parallel to a given line
- Find an equation of a line perpendicular to a given line

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $\frac{2}{5}(x+15)$.
2. Simplify $-3(x-(-2))$.
3. Solve for y: $y-3=-2(x+1)$.

How do online companies know that "we may also like" a particular item based on something we just ordered? How can economists know how a rise in the minimum wage will affect the unemployment rate? How do medical researchers create drugs to target cancer cells? How can traffic engineers predict the effect on our commuting time of an increase or decrease in gas prices? It's all mathematics.

The physical sciences, social sciences, and the business world are full of situations that can be modeled with linear equations relating two variables. To create a mathematical model of a linear relation between two variables, we must be able to find an equation of a line. In this section, we will look at several ways to write an equation of a line. The specific method we use will be determined by what information we are given.

Find an Equation of a Line Given the Slope and y-intercept

If we have an equation of a line which is in slope-intercept form, $y=mx+b$, we can easily determine the line's slope and y-intercept. Now we will do the reverse - we will start with the slope and the y-intercept and find an equation of the line. A line has only one equation which is in slope-intercept form. So, if an equation is requested to be in slope-intercept form, we will write "the equation" instead of "an equation".

Example 3.1.3.1

Find the slope-intercept form of an equation of a line with slope -9 and y-intercept $(0,-4)$.

Solution

Since we are given the slope and y-intercept of the line, we can substitute the needed values into the slope-intercept form, $y=mx+b$.

Identify the slope m.	$m=-9$
Identify the y-intercept $(0,b)$ and b.	The y-intercept is $(0,-4)$, so $b=-4$.
Substitute the values into $y=mx+b$.	$y=mx+b$ $y=-9x+(-4)$
Simplify.	$y=-9x-4$
Answer the question in slope-intercept form.	The equation of the line is $y=-9x-4$.

Try It 3.1.3.2

Find the slope-intercept form of an equation of a line with slope $\frac{2}{5}$ and y-intercept $(0, 4)$.

Answer

The equation of the line is $y = \frac{2}{5}x + 4$.

Try It 3.1.3.3

Find the slope-intercept form of an equation of a line with slope -1 and y-intercept $(0, -3)$.

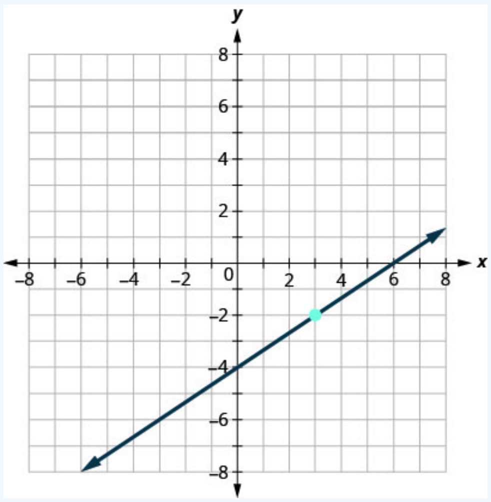
Answer

The equation of the line is $y = -x - 3$.

Sometimes, the slope and intercept need to be determined from the graph.

Example 3.1.3.4

Find the slope-intercept form of an equation of the line shown in the graph.



Solution

We need to find the slope and y-intercept of the line from the graph so that we can substitute the needed values into the slope-intercept form, $y = mx + b$.

To find the slope, we choose two points on the graph.

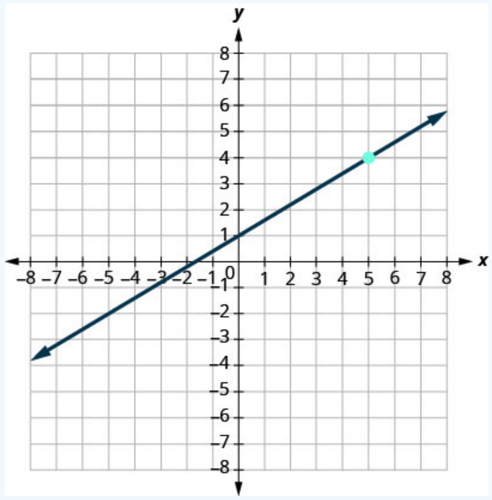
The y-intercept is $(0, -4)$ and the graph passes through $(3, -2)$.

Pick two points on the line.	$(0, -4)$ and $(3, -2)$
Determine the rise and run.	The rise is 2, and the run is 3.
Substitute the rise and run into the slope formula, $m = \frac{\text{rise}}{\text{run}}$.	$m = \frac{\text{rise}}{\text{run}}$ $m = \frac{2}{3}$
Find the y-intercept.	The y-intercept is $(0, -4)$.
Substitute the values into $y = mx + b$.	$y = mx + b$ $y = \frac{2}{3}x + (-4)$

Simplify.	$y=\frac{2}{3}x-4$
Answer the question.	The equation of the line is $y=\frac{2}{3}x-4$.

Try It 3.1.3.5

Find the slope-intercept form of an equation of the line shown in the graph.

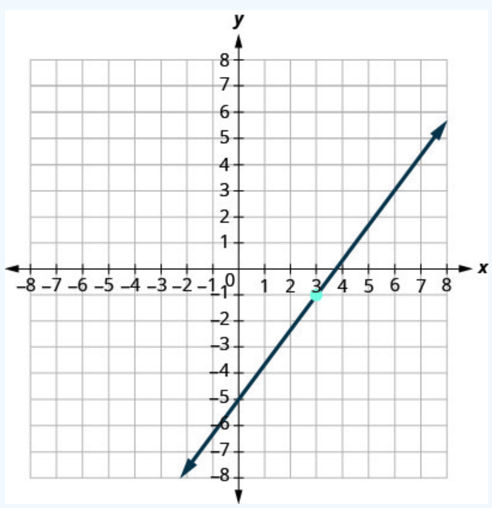


Answer

The equation of the line is $y=\frac{3}{5}x+1$.

Try It 3.1.3.6

Find the slope-intercept form of an equation of the line shown in the graph.



Answer

The equation of the line is $y=\frac{4}{3}x-5$.

Finding an Equation of the Line Given the Slope and a Point

Finding an equation of a line using the slope-intercept form of the equation works well when we are given the slope and y-intercept or when we read them off a graph. But what happens when we have another point instead of the y-intercept?

We are going to use the slope formula to derive another form of an equation of the line. By 'another form' we mean that it is an equation that has the line as it's graph though the equation has a different structure, or looks different, from what we have seen before.

Suppose we have a line that has slope m and that contains some specific point (x_1, y_1) and some other point, which we will just call (x, y). We can write the slope of this line and then change it to a different form. We will write our answer in slope-intercept form, or of the form $x = a$, since this gives a unique answer.

Write the slope formula based on two points, (x_1, y_1) and (x, y).	$m = \frac{y - y_1}{x - x_1}$
Multiply both sides of the equation by $x - x_1$.	$m(x - x_1) = \left(\frac{y - y_1}{x - x_1}\right) \cdot (x - x_1)$
Simplify.	$m(x - x_1) = y - y_1$
Rewrite the equation with the y terms on the left.	$y - y_1 = m(x - x_1)$ This format is called the **point-slope form** of an equation of a line.

Point-slope form of an equation of a line

The **point-slope form** of an equation of a line with slope m and containing the point (x_1, y_1) is

$$y - y_1 = m(x - x_1).$$

We can use the point-slope form of an equation to find an equation of a line when we know the slope and at least one point.

Finding an Equation of a Line Given a Point and the Slope

In the example below we will find an equation for a line with given attributes. In order to have a unique answer, we specify that the answer should be the equation which is in slope-intercept form, or in the form $x = a$.

Example 3.1.3.7

Find an equation of a line with slope $m = -\frac{1}{3}$ that contains the point $(6, -4)$. Write the equation in slope-intercept form.

Solution

In order to use the point-slope form, we need the slope and one point.

Identify the slope.	$m = -\frac{1}{3}$
Identify the point, (x_1, y_1).	$\underbrace{(6, -4)}_{(x_1, y_1)}$
Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$ $y - (-4) = -\frac{1}{3}(x - 6)$
Simplify.	$y + 4 = -\frac{1}{3}x + 2$
Write the equation in slope-intercept form, $y = mx + b$.	$y = -\frac{1}{3}x - 2$
Answer the question in slope-intercept form, or in the form $x = a$.	The equation of the line is $y = -\frac{1}{3}x - 2$.

Try It 3.1.3.8

Find an equation of a line with slope $m=-\frac{2}{5}$ that contains the point $(10,-5)$.

Answer

The equation of the line in slope-intercept form is $y=-\frac{2}{5}x-1$.

Try It 3.1.3.9

Find the equation of a line with slope $m=-\frac{3}{4}$ that contains the point $(4,-7)$.

Answer

The equation of the line in slope-intercept form is $y=-\frac{3}{4}x-4$.

We list the steps for easy reference.

To find an equation of a line given the slope and a point

1. Identify the slope.
2. Identify the point.
3. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
4. Write the equation in slope-intercept form or in the form $x=a$.

Example 3.1.3.10

Find an equation of a horizontal line that contains the point $(-2,-6)$. Write the equation in slope-intercept form.

Solution

Every horizontal line has slope 0. We can substitute the slope and a point into the point-slope form, $y-y_1=m(x-x_1)$. We need the slope and the point.

Identify the slope.	$m=0$
Identify the point.	$\underbrace{(-2,-6)}_{(x_1,y_1)}$
Substitute the values into $y-y_1=m(x-x_1)$.	$y-y_1=m(x-x_1)$ $y-(-6)=0(x-(-2))$
Simplify.	$y+6=0$
Solve for y.	$y=-6$
Write in slope-intercept form.	It is in y-form, but it could be written as $y=0x-6$.
Answer the question.	The equation of the horizontal line is $y=-6$ or $y=0x-6$.

Did we end up with the form of a horizontal line, $y=a$?

Try It 3.1.3.11

Find an equation of a horizontal line that contains the point $(-3,8)$. Write the equation in slope-intercept form.

Answer

The equation is $y=8$.

Try It 3.1.3.12

Find an equation of a horizontal line that contains the point $(-1, 4)$. Write the equation in slope-intercept form.

Answer

The equation is $y = 4$.

Finding an Equation of the Line Given Two Points

When real-world data is collected, a linear model can be created from two data points. In the next example we will see how to find an equation of a line when just two points are given.

So far, we have two options for finding an equation of a line: by using the slope-intercept form or by using the point-slope form. When we start with two points, it makes more sense to use the point-slope form. But then we need the slope. Can we find the slope with just two points? Yes. Then, once we have the slope, we can use it and one of the given points to find an equation. While our equation is an equation for the given line, we will then rewrite our equation in slope-intercept form, or in the form $x = a$ so that our answer is unique.

Example 3.1.3.13

Find an equation of the line that contains the points $(-3, -1)$ and $(2, -2)$. Write the equation in slope-intercept form.

Solution

In order to find the equation of the line using the point-slope form, we need the slope and the point.

Write the given points.	$(-3, -1)$ and $(2, -2)$
Choose one point and identify x_1 and y_1.	$\underbrace{(-3,-1)}_{(x_1,y_1)}$ $x_1 = -3$ $y_1 = -1$
Pick the other point and identify x_2 and y_2.	$\underbrace{(2,-2)}_{(x_2,y_2)}$ $x_2 = 2$ $y_2 = -2$
Find the slope, using $m = \frac{y_2 - y_1}{x_2 - x_1}$.	$m = \frac{y_2 - y_1}{x_2 - x_1}$ $m = \frac{-2-(-1)}{2-(-3)}$ $m = \frac{-1}{5}$ $m = -\frac{1}{5}$ The slope is $m = -\frac{1}{5}$.
Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	$y - y_1 = m(x - x_1)$ $y - (-1) = -\frac{1}{5}(x - (-3))$
Simplify to write the equation in slope-intercept form, $y = mx + b$.	$y + 1 = -\frac{1}{5}(x + 3)$ $y + 1 = -\frac{1}{5}x - \frac{3}{5}$ $y = -\frac{1}{5}x - \frac{8}{5}$
Answer the question.	The equation of the line is $y = -\frac{1}{5}x - \frac{8}{5}$.

Try It 3.1.3.14

Find the equation of a line that contains the points $(-2,-4)$ and $(1,-3)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=\frac{1}{3}x-\frac{10}{3}$.

Try It 3.1.3.15

Find the equation of a line that contains the points $(-4,-3)$ and $(1,-5)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=-\frac{2}{5}x-\frac{23}{5}$.

The steps are summarized here.

To find an equation of a line given two points

1. Find the slope using the given points: $m=\frac{y_2-y_1}{x_2-x_1}$.
2. Choose one point.
3. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
4. Write the equation in slope-intercept form.

Example 3.1.3.16

Find an equation of a line that contains the points $(-3,5)$ and $(-3,4)$. Write the equation in slope-intercept form.

Solution

In order to find the equation of the line using the point-slope form, we need the slope and the point.

Write the given points.	$(-3,5)$ and $(-3,4)$
Choose one point, name it (x_1,y_1), and identify x_1 and y_1.	$\underbrace{(-3,5)}_{(x_1,y_1)}$ $x_1=-3$ $y_1=5$
Pick the other point, name it (x_2,y_2), and identify x_2 and y_2.	$\underbrace{(-3,4)}_{(x_2,y_2)}$ $x_2=-3$ $y_2=4$
Find the slope, using $m=\frac{y_2-y_1}{x_2-x_1}$.	$m=\frac{y_2-y_1}{x_2-x_1}$ $m=\frac{4-5}{-3-(-3)}$ $m=\frac{-1}{0}$ The slope is undefined.
Conclusion.	This tells us it is a vertical line. Both of our points have an x-coordinate of -2. So our equation of the line is $x=-2$. Since there is no y, we cannot write it in slope-intercept form.

We may want to sketch a graph using the two given points. Does the graph agree with our conclusion that this is a vertical line?

? Try It 3.1.3.17

Find the equation of a line that contains the points $(5,1)$ and $(5,-4)$.

Answer

The equation of the line is $x=5$.

? Try It 3.1.3.18

Find the equation of a line that contains the points $(-4,4)$ and $(-4,3)$.

Answer

The equation of the line is $x=-4$.

We have seen that we can use the slope-intercept form or the point-slope form to find an equation of a line. Which form we use will depend on the information we are given.

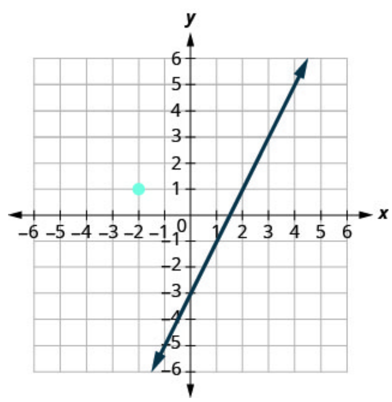
To write an equation of a line		
If given:	**Use:**	**Form:**
Slope and y-intercept	slope-intercept	$y=mx+b$
Slope and a point	point-slope	$y-y_1=m(x-x_1)$
Two points	point-slope	$y-y_1=m(x-x_1)$

Finding an Equation of a Line Parallel to a Given Line

Suppose we need to find an equation of a line that passes through a specific point and is parallel to a given line. We can use the fact that parallel lines have the same slope. So we will have a point and the slope - just what we need to use the point-slope equation.

First, let’s look at this graphically.

This graph shows $y=2x-3$. We want to graph a line parallel to this line and passing through the point $(-2,1)$.

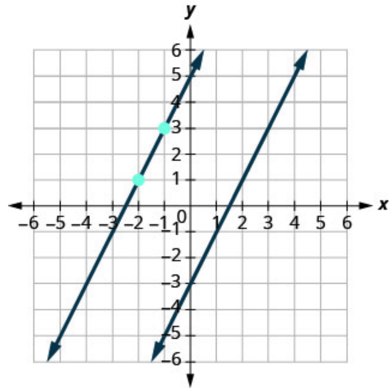


We know that parallel lines have the same slope. So the second line will have the same slope as $y=2x-3$. That slope is $m_{||}=2$. We’ll use the notation $m_{||}$ to represent the slope of a line parallel to a line with slope m. (Notice that the subscript $||$ looks like two parallel lines.)

The second line will pass through $(-2,1)$ and have $m=2$.

To graph the line, we start at $(-2,1)$ and count out the rise and run.

With $m=2$ (or $m=\frac{2}{1}$), we count out the rise 2 and the run 1. We draw the line, as shown in the graph.



Do the lines appear parallel? Does the second line pass through $(-2, 1)$?

We were asked to graph the line, now let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. Here we know one point and can find the slope. So we will use the point-slope form.

Example 3.1.3.19

Find an equation of a line parallel to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

$y = 2x - 3$		
Find the slope of the given line.	The line is in slope-intercept form, $y = 2x - 3$.	$m = 2$
Find the slope of the parallel line.	Parallel lines have the same slopes.	$m_{\parallel} = 2$
Identify the point.	The given point is $(-2, 1)$.	$\underbrace{(-2,1)}_{(x_1,y_1)}$
Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = 2(x - (-2))$ $y - 1 = 2(x + 2)$ $y - 1 = 2x + 4$
Write the equation in slope-intercept form.		$y = 2x + 5$

Look at graph with the parallel lines shown previously. Does this equation make sense? What is the y-intercept of the line? What is the slope?

Try It 3.1.3.20

Find an equation of a line parallel to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y = 3x - 10$.

Try It 3.1.3.21

Find an equation of a line parallel to the line $y = \frac{1}{2}x - 3$ that contains the point $(6, 4)$. Write the equation in slope-intercept form.

Write the equation in slope-intercept form.

Answer

The equation of the line is $y=\frac{1}{2}x+1$.

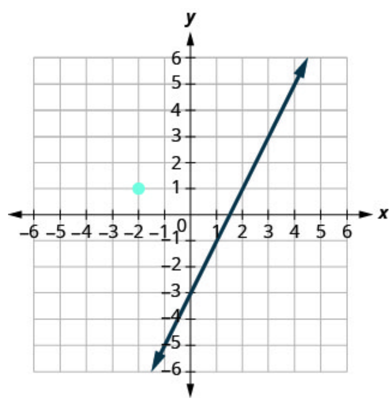
Find an equation of a line parallel to a given line

1. Find the slope of the given line.
2. Find the slope of the parallel line.
3. Identify the point.
4. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
5. Write the equation in slope-intercept form.

Finding an Equation of a Line Perpendicular to a Given Line

Now, let’s consider perpendicular lines. Suppose we need to find the line passing through a specific point and which is perpendicular to a given line. We can use the fact that perpendicular lines have slopes that are negative reciprocals. We will again use the point-slope equation, like we did with parallel lines.

This graph shows $y=2x-3$. Now, we want to graph a line perpendicular to this line and passing through $(-2,1)$.



We know that perpendicular lines have slopes that are negative reciprocals.

We will use the notation $m_{\perp}$ to represent the slope of a line perpendicular to a line with slope m. (Notice that the subscript $\perp$ looks like the right angles made by two perpendicular lines.)

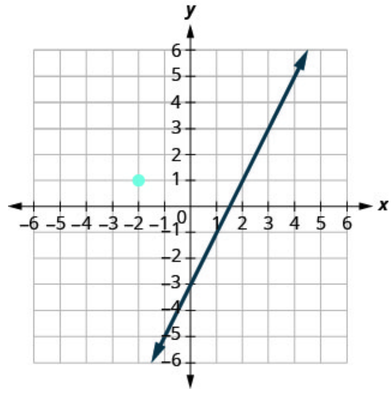
$$y=2x-3 \qquad \text{perpendicular line}$$

$$m=2 \qquad m_{\perp}=-\frac{1}{2}$$

We now know the perpendicular line will pass through $(-2,1)$ with $m_{\perp}=-\frac{1}{2}$.

To graph the line, we will start at $(-2,1)$ and count out the rise -1 and the run 2. Then we draw the line.

This figure has a graph of two perpendicular straight lines on the x y-coordinate plane. The x and y-axes run from negative 8 to 8. The first line goes through the points (0, negative 3), (1, negative 1), and (2, 1). The points (negative 2, 1) and (0, 0) are plotted. A right triangle is drawn connecting the points (negative 2, 1), (negative 2, 0), and (0, 0). The second line goes through the points (negative 2, 1) and (0, 0).



Do the lines appear perpendicular? Does the second line pass through $(-2, 1)$?

We were asked to graph the line, now, let's see how to do this algebraically.

We can use either the slope-intercept form or the point-slope form to find an equation of a line. In this Example we know one point, and can find the slope, so we will use the point-slope form.

Example 3.1.3.22

Find an equation of a line perpendicular to $y = 2x - 3$ that contains the point $(-2, 1)$. Write the equation in slope-intercept form.

Solution

$y = 2x - 3$		
Find the slope of the given line.	The line is in slope-intercept form, $y = 2x - 3$.	$m = 2$
Find the slope of the perpendicular line.	The slopes of perpendicular lines are negative.	$m_{\perp} = -\frac{1}{2}$
Identify the point.	The given point is $(-2,1)$.	$\underbrace{(-2,1)}_{(x_1,y_1)}$
Substitute the values into the point-slope form, $y - y_1 = m(x - x_1)$.	Simplify.	$y - y_1 = m(x - x_1)$ $y - 1 = -\frac{1}{2}(x - (-2))$ $y - 1 = -\frac{1}{2}(x + 2)$ $y - 1 = -\frac{1}{2}x - 1$
Write the equation in slope-intercept form.		$y = -\frac{1}{2}x$
Answer the question.	The equation of the line is $y = -\frac{1}{2}x$.	

Try It 3.1.3.23

Find an equation of a line perpendicular to the line $y = 3x + 1$ that contains the point $(4, 2)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y = -\frac{1}{3}x + \frac{10}{3}$.

Try It 3.1.3.24

Find an equation of a line perpendicular to the line $y=\frac{1}{2}x-3$ that contains the point $(6,4)$. Write the equation in slope-intercept form.

Answer

The equation of the line is $y=-2x+16$.

Find an equation of a line perpendicular to a given line

1. Find the slope of the given line.
2. Find the slope of the perpendicular line.
3. Identify the point.
4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
5. Write the equation in slope-intercept form.

Example 3.1.3.25

Find an equation of a line perpendicular to $x=5$ that contains the point $(3,-2)$. Write the equation in slope-intercept form, or in the form $x=a$.

Solution

Again, since we know one point, the point-slope option seems more promising than the slope-intercept option. We need the slope to use this form, and we know the new line will be perpendicular to $x=5$. This line is vertical, so its perpendicular will be horizontal. This tells us the $m_{\perp}=0$.

Identify the point.	$(3,-2)$
Identify the slope of the perpendicular line.	$m_{\perp}=0$
Substitute the values into $y-y_1=m(x-x_1)$.	$y-(-2)=0(x-3)$
Simplify.	$y+2=0$
Write the equation in slope-intercept form.	$y=-2$
Answer the question.	The equation of the line is $y=-2$.

Sketch the graph of both lines. On the graph, do the lines appear to be perpendicular?

Try It 3.1.3.26

Find an equation of a line that is perpendicular to the line $x=4$ that contains the point $(4,-5)$. Write the equation in slope-intercept form, or in the form $x=a$.

Answer

The equation of the line is $y=-5$.

Try It 3.1.3.27

Find an equation of a line that is perpendicular to the line $x=2$ that contains the point $(2,-1)$. Write the equation in slope-intercept form, or in the form $x=a$.

Answer

The equation of the line is $y=-1$.

In Example 1.3.25, we used the point-slope form to find the equation. We could have looked at this in a different way.

We want to find a line that is perpendicular to $x=5$ that contains the point $(3,-2)$. This graph shows us the line $x=5$ and the point $(3,-2)$.

This figure has a graph of a straight vertical line and a point on the x y-coordinate plane. The x and y-axes run from negative 8 to 8. The line goes through the points (5, 0), (5, 1), and (5, 2). The point (3, negative 2) is plotted. The line does not go through the point (3, negative 2).

We know every line perpendicular to a vertical line is horizontal, so we will sketch the horizontal line through $(3,-2)$.

This figure has a graph of a straight vertical line and a straight horizontal line on the x y-coordinate plane. The x and y-axes run from negative 8 to 8. The vertical line goes through the points (5, 0), (5, 1), and (5, 2). The horizontal line goes through the points (negative 2, negative 2), (0, negative 2), (3, negative 2), and (6, negative 2).

Do the lines appear perpendicular?

If we look at a few points on this horizontal line, we notice they all have y-coordinates of -2. So, the equation of the line perpendicular to the vertical line $x=5$ is $y=-2$.

Example 3.1.3.28

Find an equation of a line that is perpendicular to $y=-3$ that contains the point $(-3,5)$. Write the equation in slope-intercept form, or in the form $x=a$.

Solution

The line $y=-3$ is a horizontal line. Any line perpendicular to it must be vertical, in the form $x=a$. Since the perpendicular line is vertical and passes through $(-3,5)$, every point on it has an x-coordinate of -3. The equation of the perpendicular line is $x=-3$.

We may want to sketch the lines. Do they appear perpendicular?

Try It 3.1.3.29

Find an equation of a line that is perpendicular to the line $y=1$ that contains the point $(-5,1)$. Write the equation in slope-intercept form, or in the form $x=a$.

Answer

The equation of the line is $x=-5$.

Try It 3.1.3.30

Find an equation of a line that is perpendicular to the line $y=-5$ that contains the point $(-4,-5)$. Write the equation in slope-intercept form, or in the form $x=a$.

Answer

The equation of the line is $x=-4$.

Writing Exercises 3.1.3.31

1. Explain the notion of the slope and give reasoning behind the slope formula.
2. How do you write a given linear equation in slope-intercept form?
3. If two lines are perpendicular/parallel, what is the relationship between their slopes?
4. How do you graph the solutions to a linear equation in two variables if it is given in point-slope form?
5. How does the information you can see from the graph inform your production of an equation whose solution is that graph?
6. If a point $(2,3)$ is not on the graph of an equation, what can you say about the equation?

Exit Problem 3.1.3.32

Write an equation for a line perpendicular to $5x-2y=6$ passing through $(-1,2)$.

Key Concepts

- **How to find an equation of a line given the slope and a point.**
  1. Identify the slope.
  2. Identify the point.
  3. Substitute the values into the point-slope form,$y-y_1=m(x-x_1)$.
  4. Write the equation in slope-intercept form.
- **How to find an equation of a line given two points.**
  1. Find the slope using the given points. $m=\frac{y_2-y_1}{x_2-x_1}$
  2. Choose one point.
  3. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
  4. Write the equation in slope-intercept form.

To Write an Equation of a Line		
If given:	**Use:**	**Form:**
Slope and y-intercept	**slope-intercept**	$y=mx+b$
Slope and a point	**point-slope**	$y-y_1=m(x-x_1)$
Two points	**point-slope**	$y-y_1=m(x-x_1)$

- **How to find an equation of a line parallel to a given line.**
  1. Find the slope of the given line.
  2. Find the slope of the parallel line.
  3. Identify the point.
  4. Substitute the values into the point-slope form: $y-y_1=m(x-x_1)$.
  5. Write the equation in slope-intercept form
- **How to find an equation of a line perpendicular to a given line.**
  1. Find the slope of the given line.
  2. Find the slope of the perpendicular line.
  3. Identify the point.
  4. Substitute the values into the point-slope form, $y-y_1=m(x-x_1)$.
  5. Write the equation in slope-intercept form.

SECTION OVERVIEW

3.2: Quadratic Equations: Conics

Topic hierarchy

3.2.1: Geometric Description and Solutions of Two Particular Equations: the Circle and the Parabola

Learning Objectives

By the end of this section, you will be able to:

- Graph a circle with the origin at its center
- Write the equation of a circle with the center at its origin
- Graph a particular parabola
- Write the equation of a particular parabola
- Find the distance between two points
- Find the midpoint of a line segment

Be Prepared

Before you get started, take this readiness quiz.

1. Check to see if (2,-3) is a solution to $x^2+y^2=7$.
2. Find the solution of $y=x^2$ of the form $(-2,\cdot)$.
3. Plot the point $(2,-3)$ on the coordinate plane.

Introduction to Quadratic Equations with Two Variables

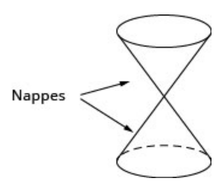
A quadratic equation with two variables x and y is an equation that is equivalent to

$$Ax^2+By^2+Cx+Dy+Exy+F=0,$$

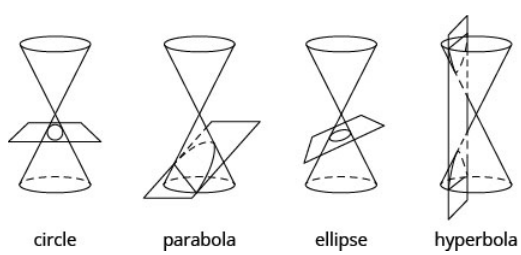
where at least one of A or B is not zero.

In general, the graph of the solution to this type of equation is called a **conic section** (a circle, parabola, ellipse, hyperbola, line, two intersecting lines, or a point).

The conics are curves that result from a plane intersecting a double cone—two cones placed point-to-point. Each half of a double cone is called a nappe.



There are four conics—the **circle**, **parabola**, **ellipse**, and **hyperbola** and the degenerate ones also mentioned above. The next figure shows how the plane intersecting the double cone results in each curve.



Each of the curves has many applications that affect your daily life, from your cell phone to acoustics and navigation systems.

We will discuss the solutions to this type of equation in certain cases. We'll treat with the case where $A=E=0$ or $B=E=0$ (i.e., where the equation is linear in one of the variables) and the case where $A=B$ and $E=0$ in which case, the graph is a circle.

The other cases could be treated in a similar way.

We start our discussion of conics with two particular important examples (and one family of examples).

A Particular Circle: Its geometric description and equation

We may be familiar with drawing a circle using a compass, or a pencil and a string. What is the figure that you draw this way: fix one end of a string on a paper. The place where this is fixed is called the center, and attach a pencil to the other end of the string. Then keeping the string taut, draw the curve that results from moving the pencil everywhere possible under the constraint of the string.

Let's suppose the length of the string is 1 unit. Then the distance between the center and any point on the drawn curve is 1 unit! Also, every point 1 unit away from the center must be part of the drawn curve. If we place our figure on a coordinate plane with the center at $(0,0)$ we have:

To find an equation which has this circle as its solution we label an arbitrary point on the circle (x,y). We now form a right triangle as seen below.

Notice that in the picture the base of the triangle has length equal to the absolute value of the x-coordinate of the arbitrary point. Similarly, the height of the triangle is equal to the absolute value of the y-coordinate of the arbitrary point. Confirm by choosing this arbitrary point in different quadrants that this is true.

Now, Pythagoras' Theorem gives us that

$$x^2+y^2=1$$

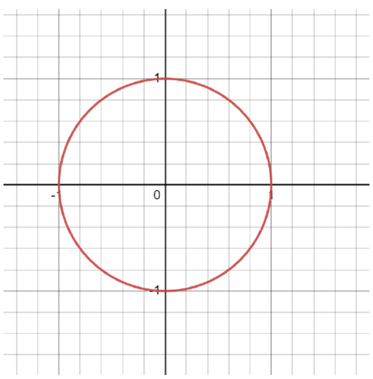
which is the equation we are looking for! Every point on the circle satisfies this equation and every pair of numbers (a,b) satisfying equation are the coordinates of a point distance 1 unit away from the center!

Notice that if the length of the string were r, then Pythagoras' Theorem would tell us

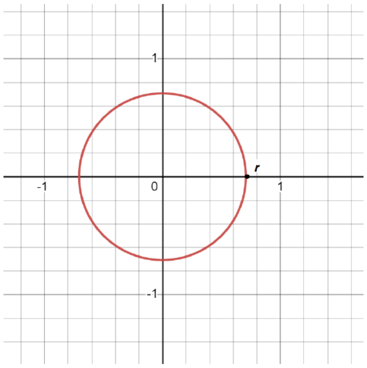
$$x^2+y^2=r^2.$$

For such a circle, the center is $(0,0)$ and r is called the radius.

So, the graph, following the geometric description, of the equation $x^2+y^2=1$ is



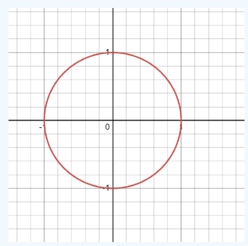
More generally, the graph, following the geometric description, of the equation $x^2+y^2=r^2$ is



? Try It 3.2.1.1

Graph the circle centered at $(0,0)$ with radius 1.

Answer



? Try It 3.2.1.2

Graph the circle centered at $(0,0)$ with radius $7/2$.

Answer

Hint: Find four guiding points: above, below, left and right of the center (these are the points for which the grid is especially helpful!).

Check your answer on Desmos.

A Particular Parabola: A Geometric Description and Equation

Consider a line (called the directrix) and a point (called the focus) not on that line. The set of points that are the same distance to the focus as they are to the line (the length of the shortest line segment connecting the point to the line): see below.

The collection of points is called a parabola. Note that the point $(0,0)$ satisfies that condition: the distance from $(0,0)$ to the focus is $\frac{1}{4}$ as is the distance from $(0,0)$ to the directrix.

Let's consider a particular case and place it on a coordinate plane so that we can derive an equation for the collection of points satisfying the property above.

We will take (for later convenience) the focus to be $\left(0, \frac{1}{4}\right)$ and the directrix to be the horizontal line with equation $y=-\frac{1}{4}$.

Then, as we did with the circle, we label an arbitrary point on the parabola with coordinates (x,y). We see from the picture that the y coordinate of the point is the distance to the x-axis and we have an additional distance 1 unit to the line! So the distance from the point to the line is $y+\frac{1}{4}$. If the arbitrary point were in the second quadrant this would still be the case. It will be more convenient (but equivalent) to think of the parabola as the points whose distance squared is the same to the focus as it is to the directrix. The square of the distance to the line is then

$$\left(y+\frac{1}{4}\right)^2=y^2+\frac{1}{2}y+\frac{1}{16}.$$

To find the distance from (x,y) to the focus $\left(0,\frac{1}{4}\right)$ we look at the right triangle below and apply Pythagoras' Theorem.

We see the length of the base of the triangle is the absolute value of the x-coordinate of the point, i.e., $|x|$ (check to see this is true no matter where the point is), and the height of the triangle is $\left|y-\frac{1}{4}\right|$. So, using Pythagoras' Theorem, we see that the square of the distance we are looking for is

$$x^2+\left|y-\frac{1}{4}\right|^2$$

and because $\left|y-\frac{1}{4}\right|^2=\left(y-\frac{1}{4}\right)^2$ the square of the distance from the point (x,y) to the focus is

$$x^2+y^2-\frac{1}{2}y+\frac{1}{16}.$$

Returning to the description of the parabola as the points which are the same distance to the focus as they are to the directrix, or, equivalently, so that the square of the distance from the point to the focus is the same as the square of the distance from the point to the directrix, we see

$$x^2+y^2-\frac{1}{2}y+\frac{1}{16}=y^2+\frac{1}{2}y+\frac{1}{16}$$

or, by adding $-y^2+\frac{1}{2}y-\frac{1}{16}$ to both sides of the equation,

$$x^2=y$$

or,

$$y=x^2.$$

Every point on the parabola satisfies $y=x^2$ and every solution (a,b) to this equation has the property that it is the same distance to the focus as it is to the directrix.

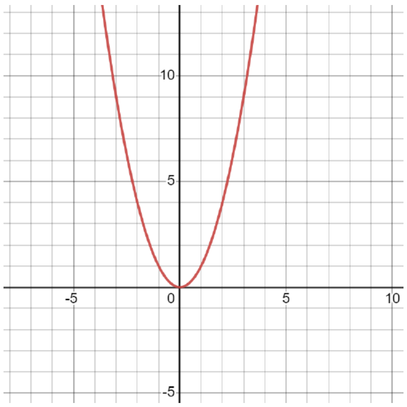
We can find some solutions to this particular equation by completing the table

x	y
-3	
-2	
-1	
0	
1	
2	
3	

by finding the values for y that create solutions:

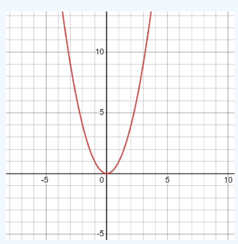
x	y
-3	9
-2	4
-1	1
0	0
1	1
2	4
3	9

This helps us get an idea of what the graph looks like:



Try It 3.2.1.3

Write the equation whose solutions are graphed here and verify by finding three solutions to your equation which are on the graph.



Answer

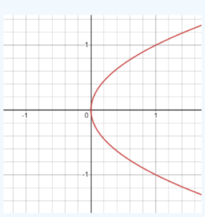
The equation is $y = x^2$. The three examples will vary.

Try It 3.2.1.4

Using the ideas presented in this subsection, graph the equation $x = y^2$.

Answer

Check your answer on Desmos.



The Distance Formula

Here we mention, as we are discussing geometry in this section, the formula for the distance d between to points and the midpoint of a line segment.

Consider two points on the coordinate plane (x_1, y_1) and (x_2, y_2). As we did above, we form the triangle shown below

The length of the base is $|x_2 - x_1|$ and the height is $|y_2 - y_1|$. We could do some examples with numbers to get a feel for these expressions. Putting these points in various quadrants will convince you of the need for the absolute values. Now, as before, we use Pythagoras' Theorem to obtain

$$|x_2 - x_1|^2 + |y_2 - y_1|^2 = d^2,$$

or, equivalently,

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 = d^2,$$

so that the distance d between (x_1, y_1) and (x_2, y_2) is

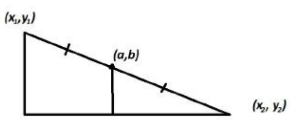
$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,.$$

This is called the distance formula.

Similar Triangles and the Midpoint Formula

Two triangles are similar if they are the same shape (have the same angles). And if two triangles are similar, corresponding ratios of sides are equal.

The midpoint of the line segment connecting (x_1, y_1) and (x_2, y_2) is the point on the line segment which is the same distance from each of the endpoints.Look at the midpoint (label it (a, b)) of the line segment connecting (x_1, y_1) and (x_2, y_2) and form the triangle below. Here we don't draw the coordinate plane that this rests in so as to not clutter the picture.



We see two triangles and since they are both right triangles and share an angle (since the sum of the measure of the interior angles of a triangle is 180 degrees, two angles are enough to determine the third angle), they are similar. Further, because (a, b) is the midpoint, the length of the hypotenuse of the small triangle is half the length of the hypotenuse of the larger triangle. So, the same relation is true of the legs! It follows that $2(a - x_1) = x_2 - x1$, or, solving for a, $a = \frac{x_1 + x_2}{2}$. Similarly, $b = \frac{y_1 + y_2}{2}$. By drawing the points in different relationships to each other, we can convince ourselves that these equations are true no matter where the points are located. So we have what is known as the midpoint formula. The midpoint of the line segment connecting (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right).$$

Example 3.2.1.5

Find the midpoint of the line segment connecting $(-2, 1)$ and $(3, 5)$. Verify your answer by using the distance formula.

Solution

We find the midpoint by averaging the x and y components of the points (that is what the midpoint formula says):

The midpoint is $\left(\frac{-2+3}{2}, \frac{1+5}{2}\right) = \left(\frac{1}{2}, 3\right)$.

To verify that this is the midpoint, we will check that the distance between each end point and the proposed midpoint are the same and that this is half the length of the segment.

The distance between $(-2,1)$ and $\left(\frac{1}{2},3\right)$ is $\sqrt{\left(-2-\frac{1}{2}\right)^2+(1-3)^2} = \sqrt{\frac{25}{4}+4} = \frac{\sqrt{41}}{2}$

and

The distance between $(3,5)$ and $\left(\frac{1}{2},3\right)$ is $\sqrt{\left(3-\frac{1}{2}\right)^2+(5-3)^2} = \sqrt{\frac{25}{4}+4} = \frac{\sqrt{41}}{2}$.

The distance between $(-2,1)$ and $(3,5)$ is $\sqrt{(-2-3)^2+(1-5)^2} = \sqrt{25+16} = \sqrt{41}$

Because the midpoint is the point that is the same distance from the two endpoints and this distance is half the total distance, we have verified that $\left(\frac{1}{2},3\right)$ is the midpoint of the line segment connecting $(-2,1)$ and $(3,5)$.

? Try It 3.2.1.6

Find the midpoint of the line segment connecting $(2,-1)$ and $(-3,5)$. Verify your answer by using the distance formula.

Answer

The midpoint is $\left(-\frac{1}{2},2\right)$. The exposition of the verification will vary. The length of the line segment is $\sqrt{61}$.

? Try It 3.2.1.7

Find the midpoint of the line segment connecting $(3,-1)$ and $(-3,-5)$. Verify your answer by using the distance formula.

Answer

The midpoint is $(0,-3)$. The exposition of the verification will vary. The length of the line segment is $\sqrt{52}$.

A question for discussion is: Why isn't it enough to check any two of the three conditions we checked in the above example.

Applications

✓ Example 3.2.1.8

Find the center to a circle whose diameter has endpoints $(2,1)$ and $(-3,4)$.

Solution

The center of the circle is at the midpoint of this segment. So, to find the center we average the coordinates to find it is at $(-1/2, 5/2)$.

? Try It 3.2.1.9

Find the center to a circle whose diameter has endpoints $(-2,1)$ and $(3,5)$.

Answer

The center is at

$(1/2, 3)$.

Example 3.2.1.10

Use the distance formula to find the length of the hypotenuse of the triangle with vertices $(1,3)$, $(5,0)$, and $(1,0)$.

Solution

We can see that the first point is above $(1,0)$ and the second point is to the right of $(1,0)$. The segments with $(1,0)$ as an enpoint form a right angle. Therefore the hypotenuse has endpoints $(1,3)$ and $(5,0)$. So, to find the distance we look at the vertical component of the distance and the horizontal component of the distance: 3 and -4 to see that

$d^2 = 3^2 + (-4)^2 = 25$ so that $d = 5$.

Alternatively, we can substitute $(1,3)$ for (x_1, y_1) and $(5,0)$ for (x_2, y_2) to see

$$d = \sqrt{(5-1)^2 + (0-3)^2} = 5.$$

Try It 3.2.1.11

Use the distance formula to find the length of the hypotenuse of the triangle with vertices $(4,-1)$, $(2,1)$, and $(2,-1)$.

Answer

The hypotenuse has length $2\sqrt{2}$.

Writing Exercises 3.2.1.12

1. Explain reasoning behind the midpoint formula.
2. Can you describe what a circle is using the words: center, radius and distance?
3. How is the distance formula related to the Pythagorean Theorem?
4. How can you recognize solutions to a quadratic equation in two variables as forming a line, a parabola, or a circle?
5. What is the value of finding 4 points on the circle when graphing by hand?
6. What is a perpendicular bisector?
7. What information about a line is convenient to deal with if we want to write it's equation in slope-intercept form?
8. What about point-slope form?

Exit Problem 3.2.1.13

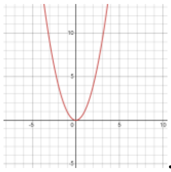
Identify the center and the radius of the circle given by the equation $x^2 - 4x + y^2 + 6y - 23 = 0$.

Graph the circle and label four points on it.

Key Concepts

- The circle with center $(0,0)$ and radius r has the equation $x^2 + y^2 = r^2$
- A particular parabola has the equation $y = x^2$ and its graph is

$$(3.2.1.1)$$

.

- The distance d between (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}\,. \tag{3.2.1.2}$$

- The midpoint of the line segment connecting (x_1, y_1) and $(x_2, y_2) = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$, or in words the coordinates of the midpoint is the average of the coordinates.

3.2.2: Graphs of Certain Quadratic Equations: Part I

Learning Objectives

By the end of this section, you will be able to:

- Understand what it means to be a solution of a quadratic equation with two variables.
- Represent solutions of certain quadratic equations with two variables.

Be Prepared

Before you get started, take this readiness quiz.

1. Find a solution of $x-3y=7$ of the form $(2,\cdot)$.
2. Find two solutions of $x-2y=4$ and put them on a graph. Represent all solutions on the graph and determine if $(2,4)$ is a solution by using that graph.
3. Graph the solutions to $(y-3)=2(x+1)$ in the simplest way you can think of.

x	y
2	3

We learned in the last section that the graph of $x^2+y^2=r^2$, where r is some positive number, looks like

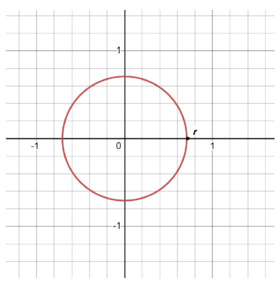
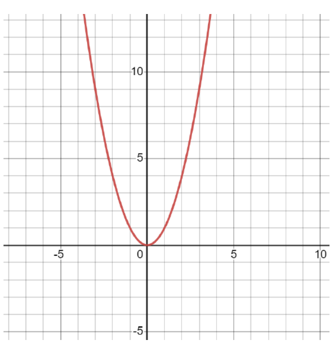


Figure 3.2.2.1: Copy and Paste Caption here. (Copyright; author via source)

and the graph of $y=x^2$ looks like



Switching x and y

If we switch the roles of x and y in the equation $x^2+y^2=r^2$ we obtain $y^2+x^2=r^2$ which has exactly the same solutions.

If we switch the roles of x and y in the equation $y=x^2$ we obtain $x=y^2$. Here, note that the table of values we had will just have the numbers reversed:

x	y
4	-2
1	-1
0	0
1	1
4	2
9	3

So, the graph of the equation $x=y^2$ is basically switching the axes. You may also think of it as reflecting the graph with respect to the diagonal $y=x$. Note that the same was also true with the circle, but the graph turned out to be the same! This feature is a type of symmetry. We will discuss other symmetries in the next section.

? Try It 3.2.2.1

Graph $y^2-x^3=2$ using desmos and try to sketch the graph of $x^2-y^3=2$. Check your answer on Desmos.

Answer

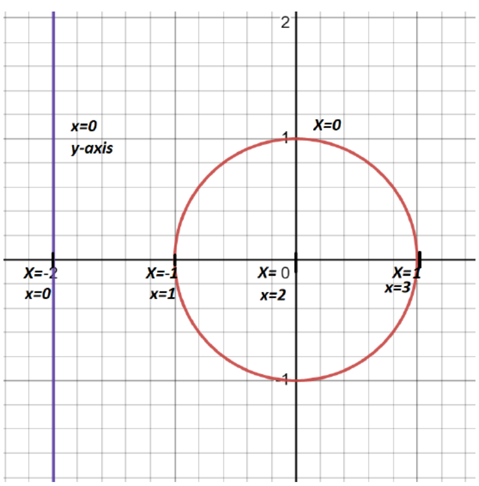
This is to be answered with the help of Desmos.

Shifts

Let's investigate the equation $(x-2)^2+y^2=1$. Note that if we call $x-2$, X, (or, $X=x+2$) then the equation looks familiar:

$$X^2+y^2=1,$$

and we know what the graph looks like, except that the horizontal axis is now the 'X'-axis. Note that $x=0$, which is the equation to the usual y-axis, is equivalent to $X+2=0$ or equivalently, $X=-2$. (Alternatively, we can see that the axis $X=0$ is equivalent to $x=2$). The markings are still one unit apart since for example $X=1$ is equivalent to $x=3$, $X=-1$ is equivalent to $x=1$, and so forth. So, on the usual coordinate plane with x and y -coordinates labeled, the circle has center $(2,0)$ and the radius is still 1.



The same argument holds for any number replacing the '2'. So, the graph of $(x-h)^2+y^2=r^2$ is a circle of radius $(h,0)$ with radius r.

Also, we can argued the same way when we consider the equation: $x^2+(y-4)^2=1$. That is we can replace $y-4$ with Y (or, $y=Y+4$ and see that the equation becomes

$$x^2+Y^2=1$$

and so on the $x-Y$ coordinate plane we see that this is a circle centered at the origin with radius 1. But the $x-$axis on the $x-y$ plane has the equation $y=0$ which is equivalent to $Y+4=0$ or $Y=-4$. The markings are length one unit apart (whether you are using y or Y). So, we see that on the original plane, we see that the center of the circle is $(0,4)$ and the radius is still 1. In general, using the same argument, the graph of $x^2+(y-k)^2=r^2$ is a circle with center $(0,k)$ and radius r.

Putting these so-called shifts together we see that replacing the x with $x-h$ and the y with $y-k$ in the original equation $x^2+y^2=r^2$ has the effect of shifting the original graph h units along the x-axis and shifting it k units along the y-axis so that the center of the circle described by

$(x-h)^2+(y-k)^2=r^2$

is at (h,k) and the radius is r. Note that if we substitute (h,k) in for (x,y) we get 0 on the left side. This is a useful way to determine or check the values of h and k.

Now, consider the basic equation for the parabola $y=x^2$. In exactly the same way (though tracking the vertex instead of the center), we can explore the graph of $y-k=(x-h)^2$.

Consider the equation $y=(x-2)^2$). Letting $X=x-2$ the equation becomes $y=X^2$ which we know how to graph on the $X-y$ coordinate plane. The y-axis in the $x-y$ plane is given by the equation $x=0$ or $X=-2$. (Or, equivalently, $X=0$ is equivalent to $x-2=0$ or $x=2$. So the x-coordinate of the vertex is 2. The markings are the same distance apart no matter if you are considering the $x-y$- plane or the $X-y$ plane. So, the parabola has vertex $(2,0)$. In general, if we replace x by $x-h$, the graph can be obtained by shifting the graph of $y=x^2$ along the x-axis by h units.

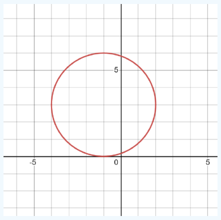
Similarly, the graph of $y-k=x^2$ is obtained by shifting the graph of $y=x^2$ along the y-axis by k units.

Putting these together, we see that the graph of $y-k=(x-h)^2$ is obtained by shifting the graph of $y=x^2$ along the x-axis by h units and along the y-axis by k units. Note that if we substitute (h,k) in for (x,y) we get the equation $0=0$. This can be a nice way of verifying or finding the values of h and k.

Generally, if you know what the graph of an equation in x and y looks like you can graph the equation obtained by replacing x with $x-h$ and y with $y-k$ by shifting the graph of the original. Pinpointing particular features (here it was the center of the circle, and the vertex of the parabola), may help you form your new graph.

Example 3.2.2.2

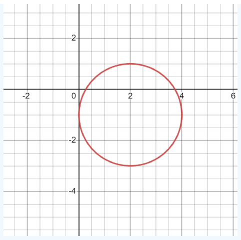
1. Graph $(x-2)^2+(y+1)^2=4$
2. Write the equation of the following graph



Solution

1. We know this is a circle due to the fact that it is of the form $(x-h)^2+(y-k)^2=r^2$. Now we set about graphing it. We see that $(2,-1)$ makes each of the terms on the left side 0 so it is the center of the circle. We start our sketch by finding this point on the coordinate plane. Since $r^2=4$, the radius is 2 (the radius is always non-negative). So, to guide our graph, we move up two from the center and place a point there. This point must be on our circle. Then, from the center we move two units to the right and place a point there since that point must also be on our circle. Similarly, we move down two units from the center and also left to units from the center to find a total of four points that are on our circle.

We know this is a circle so we connect the 4 points in as close to a circle figure as we can. The end result should look like:



2. We see that this is a circle but is shifted so that its center is not $(0,0)$. We see that its center is in fact $(-1,3)$. We can verify this by checking the distance from this point to the right most point on the circle which we find to be 3, by checking the distance between this point and the highest point which is also 3, by checking the distance from this point to the left most point on the circle which is again 3, and by checking the distance from this point to the lowest point on the circle which is again 3. (We in fact only need three of these equations since in the general form of the circle, there are only 3 parameters: h, k, and r.) We also, discovered that the radius is 3! The general form of the circle is:

$$(x-h)^2+(y-k)^2=r^2.$$

We determine h and k by noting that when we substitute $(-1,3)$ for (x,y) we should get zero on the left side, so $h=-1$ and $k=3$. Also, since $r=3$, the right hand side is $3^2=9$. So the equation for this circle is:

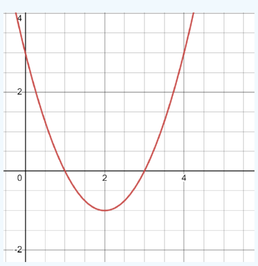
$$(x-(-1))^2+(y-3)^2=9$$

or, equivalently,

$$(x+1)^2+(y-3)^2=9.$$

Try It 3.2.2.3

1. Graph $y-3=(x+1)^2$
2. Write the equation of the following graph



Answer

1. The first can be checked on Desmos.
2. Here we note that this has the shape of the parabola we discussed in the last section. Its vertex is not at $(0,0)$ but it is instead at $(2,-1)$. The standard form of a parabola with this shape (opening up and the same shape as the one in the last section) is

$$y-h=(x-k)^2.$$

To determine h and k we note that when we substitute $(2,-1)$ in for (x,y) we should get 0 on both sides of the equation. This forces $h=2$ and $k=-1$. We therefore have the equation

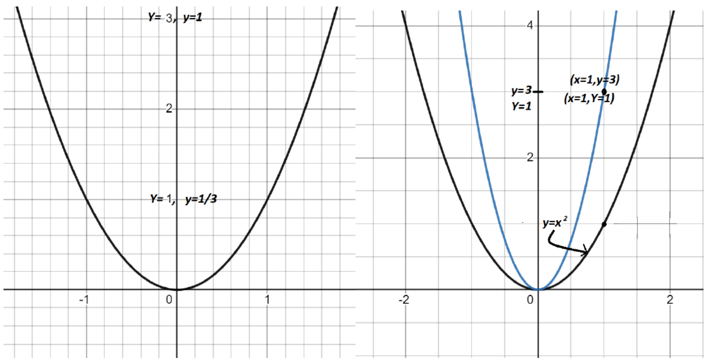
$$y-2=(x-(-1))^2$$

or, equivalently,

$$y-2=(x+1)^2.$$

Scaling

Recall the graph for the basic parabola $y=x^2$. What does the graph of $3y=x^2$ look like? If we let $Y=3y$ then we can write this equation as $Y=x^2$ and we know that on the $x-Y$ -plane, this looks like a parabola (the vertex is where $x=0$ and $Y=0$). Unlike before, the $x-y$ axes is the same as the $x-Y$ axes. However, when $y=1$, $Y=3$ so the $y-$units occur every 3 of the Y-units. So when we shrink the $xY-$plane so that the $x-y$-units are the same as on the original, we see that the graph has been squeezed by a factor of 3 in the y direction. Similarly, compared to the standard parabola $y=x^2$, $\frac{1}{3}y=x^2$ appears to be stretched by a factor of 3! Note that when we plug in $(1,3)$ for (x,y) we get the equation $1=1$. You can think of 1 as the $x-$stretch factor and 3as the $x-$stretch factor. This can be handy to identify the stretching along the axis. The idea is that $(1,1)$ is a point on the standard graph and the question we can ask here is, what is the corresponding point on the graph of $\frac{1}{3}y=x^2$.



In general, if you know what the graph of an equation in x and y, the graph of that equation with every occurrence of y replaced with ay $(a>0)$, can be obtained by stretching or shrinking the graph along the y-axis.

Even more, if you know what the graph of an equation in x and y, the graph of that equation with every occurrence of x replaced with ax can be obtained by stretching or shrinking the graph along the x-axis. More generally, the graph of that equation with every occurrence of x replaced with ax and every occurrence of y replaced with by will stretch/shrink the graph along the x-axis by a factor of a and along the y-axis by a factor of b.

Note that if we consider the equation $(2x)^2+(2y)^2=r^2$, we can rewrite the equation as $X^2+Y^2=r^2$, where $X=2x$ and $Y=2y$ so that when $x=1$ is the same as $X=2$ so, the graph of the circle on the XY-plane is centered at $(0,0)$ with radius r, but since the distance between two adjacent marks on the X-axis is only $\frac{1}{2}$ on the $x-$axis, or equivalently, when $X=r, x=\frac{r}{2}$. So rhe radius is actually $r/2$. We can see this another way:

$$(2x)^2+(2y)^2=r^2 \implies 4x^2+4y^2=r^2 \implies x^2+y^2=\frac{r^2}{4} \implies x^2+y^2=\left(\frac{r}{2}\right)^2$$

so we see that the radius is $\frac{r}{2}$.

Finally consider $y=-x^2$. We can see by finding solutions the pattern presented looks like the graph of $y=x^2$ which is reflected across the x-axis.

Alternatively, we can rewrite the equation as $-y=-x^2$ and in a similar way as before, let $Y=-y$. The graph of $Y=x^2$ is the graph of a parabola. But since y and Y are opposites, 1 on the Y-axis is -1 on the $y-$axis. So the Y axis is numbered in a way that is opposite to the numbering on the y-axis. So, to align them requires us to reflect with respect to the x-axis.

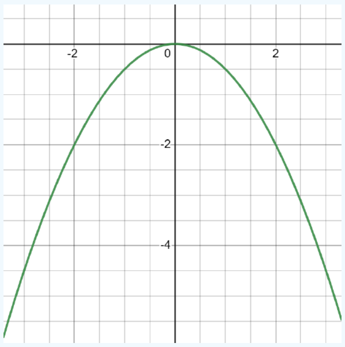
If we do the same with the x, we will see that the graph is identical! There is s symmetry!

In general if we know the graph of an equation of x and y and we replace y by $-y$ then we can graph the resulting equation by "flipping the y-axis" or, reflecting the graph about the x-axis. Also, if we replace x by $-x$ then we can graph the resulting equation by "flipping the x-axis" or, reflecting the graph about the y-axis.

Notice again, that nothing changes with the circle if you replace x by $-x$ or if you replace y with $-y$! This is another example of symmetry.

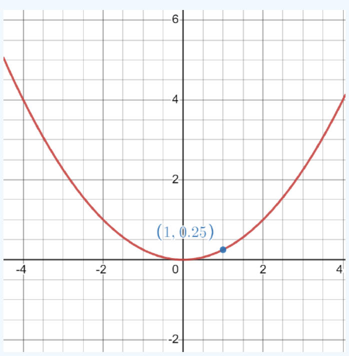
Example 3.2.2.4

1. Graph $4y = x^2$.
2. Identify the equation of



Solution

1. We see that this equation has a parabola as its graph because there is one quadratic term and one linear term. The vertex is at $(0,0)$ because there is no shifting of the x or y. Here we note that to arrive at the equation $1 = 1$ we need to substitute $(1, 1/4)$. So the point $(1, 1/4)$ is on the graph of this parabola. This will help us determine the scaling. The next point to plot will be $\left(2, 4 \cdot \frac{1}{4}\right)$ and then the points symmetricly located on the other side of the vertex.



2. The parabola here has $ay = x^2$. Notice that $(1, -1/2)$ is on the graph and has an x-coordinate of 1. When we substitute $(1, -1/2)$ in for (x, y) we should get $1 = 1$. This leads us to notice that $a \cdot (-\frac{1}{2}) = 1$ so that $a = -2$. The equation of this parabola is therefore

$$-2y = x^2.$$

Try It 3.2.2.5

1. Graph $-\frac{1}{2}y = x^2$.
2. Identify the equation of

Answer

1. Check your graph on Desmos.
2.

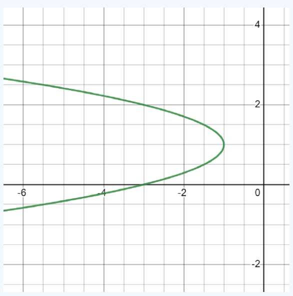
> **Try It 3.2.2.6**
>
> 1. Graph $y = x^3$ on desmos and try to graph $4y = x^3$. Find at least two particular solutions and check your answer on Desmos.
> 2. Sketch the graph of $-\frac{1}{3}y = x^3$ (indicating at least two particular solutions) and check your answer on Desmos.
> 3. Challenge: Sketch the graph of $y = (2x)^2$ (indicating at least two particular solutions) and check your answer on desmos.
>
> **Answer**
>
> Graphs should be checked on Desmos. Answers will vary.

Combined Examples

Example 3.2.2.7

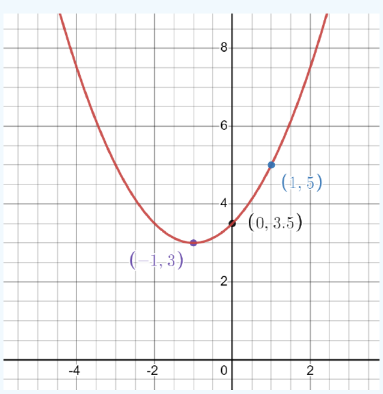
1. Graph $2(y-3) = (x+1)^2$.

2. Find an equation whose solution has the following graph.



Solution

1. We note that this is a parabola. The vertex (h, k) can be found by substituting in (h, k) for (x, y) for which we should get $0 = 0$. This leads to the equation $k - 3 = 0$ and $h + 1 = 0$ so that the vertex is $(-1, 3)$. Now, there is some scaling on the y-axis. So, we will ask ourselves which values of (x, y) result in $(1, 1)$. We find $2(y-3) = 1$ gives us $y = 7/2$ and $x + 1 = 1$ gives us $x = 0$. So a point on our parabola is $(0, 7/2)$. Notice that as we move from the vertex to the point $(0, 7/2)$, the y-coordinate moved up by $1/2$, which is our scaling factor. Our next point moving to the right of the vertex is $\left(1, 3 + 4 \cdot \frac{1}{2}\right) =$ \((1, 5). Using the symmetry of the parabola to find points on the other side of the vertex, we find the graph is:



2. This parabola is of the form $a(x-h) = (y-k)^2$. The vertex to this parabola is $(-1, 1)$ so when we substitute this for (x, y) we should arrive at the equation $(0, 0)$. This leads us to $h = -1$ and $k = 1$ so that our equation is $a(x-(-1)) = (y-1)^2$ or $a(x+1) = (y-1)^2$. Now to find our point that corresponds to $(1, 1)$ in the standard parabola.

We look for points whose y-coordinate is 1 away from the y-coordinate of the vertex: $1+1=2$. We see that $(-3,2)$ is a point on the parabola and has a y-coordinate of 2. When we plug this point into the equation we should get the equation $1=1$ which will allow us to determine the value of a:

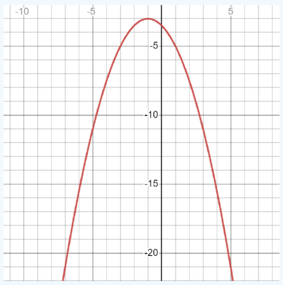
We find that $a(-3+1)=1$ so that $a=-1/2$.

The parabola therefore has the equation $-\frac{1}{2}(x+1)=(y-1)^2$, which can be verified with Desmos.

Example 3.2.2.8

1. Graph $\frac{1}{2}(x-1)=(y+2)^2$. Label at least two points.

2. Find an equation whose solution has the following graph.



Answer

1. Check the graph on Desmos. Add texts here.
2. $-2(y+3)=(x+1)^2$

An interesting point to consider is an equation where there is scaling of both axes in the case of a circle: a(x-h)^2+a(y-k)^2=r^2. What is the effect? Is this still a circle?

Other Quadratic Equations

We have only dealt with certain types of equations which lead to circles and parabolas.

We could ask:

What is the graph of $(ax)^2+y^2=r^2$? We stretch/shrink along the x-axis and the result is called an ellipse!

What about $xy=1$? This one is a new one, called a hyperbola.

There are also all that we've discussed rotated! To discuss all of these is beyond the scope of this book.

More Graphing

We now look at certain quadratic equations in x and y and try to write them in a way that we can use the previous section to graph.

We have already learned a process by which you can turn an expression that looks like ax^2+bx into one that looks like $a(x+B)^2$. It was called completing the square. This is the key to graphing the equations in this section.

Example 3.2.2.9

1. Graph $x^2+y^2-4y-2x-2=0$.

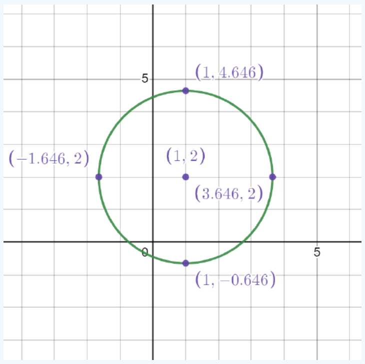
2. Graph $x^2-5x+3y=0$.

Solution

1. We see this equation is quadratic in x and y so we anticipate that this is a circle and proceed to put it in the form $(x-h)^2+(y-k)^2=r^2$. To do this we recall the process of completing the square:

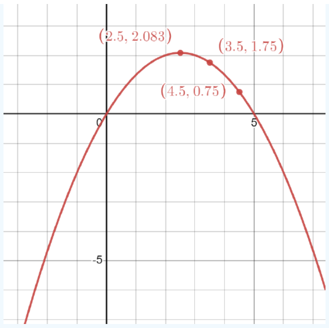
	$x^2+y^2-4y-2x-2=0$
Note that $x^2-2x=(x-1)^2-1$ and $y^2-4y=(y-2)^2-4$.	$(x-1)^2-1+(y-2)^2-4-2=0$
Combine constants on the right hand side.	$(x-1)^2+(y-2)^2=7$
Identify the center and the radius.	The center is $(1,2)$ and the radius is $\sqrt{7}$
Locate four particular points on the circle to aid in the sketch.	$(1+\sqrt{7},2)$ $(1-\sqrt{7},2)$ $(1,2+\sqrt{7})$ $(1,2-\sqrt{7})$

Graph the points and form a circle:



2. Note that this is quadratic in x and linear in y. We therefore anticipate that this is a parabola and aim to put it in the form $a(y-k)=(x-h)^2$.

	$x^2-5x+3y=0$
$x^2-5x=(x-\frac{5}{2})^2-\frac{25}{4}$	$-3y=(x-\frac{5}{2})^2-\frac{25}{4}$
The constant on the right side needs to be brought to join the y-term.	$-3y+\frac{25}{4}=(x-\frac{5}{2})^2$
Finally, let's factor out the -3 to bring it into form.	$-3(y-\frac{25}{12})=(x-\frac{5}{2})^2$
Identify the vertex (the values that can be substituted for x and y that will result in $0=0$).	The vertex is $(\frac{5}{2},\frac{25}{12})$.
Identify the point corresponding to the point $(1,1)$ of the basic parabola.	$1=-3\left(y-\frac{25}{12}\right) \implies y=\frac{25}{12}-\frac{1}{3}=\frac{7}{4}$ $1=x-\frac{5}{2} \implies x=\frac{7}{2}$ So, $\left(\frac{7}{2},\frac{7}{4}\right)$ is on the graph.
Identify one more point by noting that in moving from the vertex to $\left(\frac{7}{4},\frac{7}{4}\right)$, the $y-$ coordinate moved down by $\frac{1}{3}$.	Another point on our graph is $\left(\frac{9}{2},\frac{25}{12}-4\cdot\frac{1}{3}\right)$, or, more simply, $\left(\frac{9}{2},\frac{9}{12}\right)$.
Plot these three points and the symmetric points on the other side of the vertex and complete the sketch.	



Try It 3.2.2.10

1. Graph $x^2+y^2+6x+2y-1=0$.

2. Graph $y^2-5x+3y=0$.

Answer

Expositions of solutions will vary. Final graphs can be checked on Desmos. We will give the form after completing the square here.

1. The graph is a circle. $(x+3)^2+(y+1)^2=11$ so the center is $(-3,-1)$ and its radius is $\sqrt{11}$.

2. The graph is a parabola. $(y+3/2)^2-9/4=5x$ which leads to $5(x+9/20)=(y+3/2)^2$ with vertex $(-9/20,-3/2)$. The point corresponding to the $(1,1)$ of the basic parabola is: $\left(1/5-9/20,-\frac{1}{2}\right)=\left(-5/20,-\frac{1}{2}\right)$. The difference of the y-coordinates between this point and the vertex is $1/5$. So, another point on the parabola is: $\left(-9/20+4\cdot\frac{1}{5},1/2\right)=(7/20,1/2)$. Look at the graph and these points plotted on Desmos.

Try It 3.2.2.11

1. Graph: $x^2-x+y-3=0$

2. Graph: $x(x+3)+y(y+4)=0$

Answer

Expositions of solutions will vary. Final graphs can be checked on Desmos. We will give the form after completing the square here.Add texts here.

1. The graph is a parabola. $(x-1/2)^2-1/4+y-3=0$ which gives $-(y-13/4)=(x-1/2)^2$. The vertex is $(1/2,13/4)$

2. Expanding this gives $x^2+3x+y^2+4y=0$. The graph is a circle. $(x+3/2)^2-9/4+(y+2)^2-4=0$

Applications

Example 3.2.2.12

A ball is thrown up from a height of 4 feet and the height h of its trajectory has the equation $h=-16t^2+32t+4$ where t is measured in seconds. When is the ball at its highest and how high does it go? When does it land?

Solution

A very very rough sketch of the solutions to the equation tells us that the ball goes up then comes down (the sketch here really starts at $(t=0,h=4)$ and is a downward opening parabola (as is indicated by the negative coefficient of t^2 or by the context of the problem). If $(t=a,h=b)$ is on the graph that means that the ball is at height b feet when $t=a$ seconds. The solutions (the graphed points) are exactly the time-height pairs that occur in this situation. So, the largest occurring height is the h-coordinate of the vertex.

We rewrite the equation in order to identify its vertex:

$h = -16(t^2 - 2t) + 4$

so, completing the square of the expression in parentheses gives

$h = -16((t^2 - 2t + 1) - 1) + 4$, or $h = -16((t-1)^2 - 1) + 4$.

Distributing the -16 gives

$h = -16(t-1)^2 + 20$.

So, $\frac{-1}{16}(h-20) = (t-1)^2$.

We see that the vertex of this parabola is when $t = 1$ and $h = 20$. The parabola opens downward due to the negative factor $\frac{-1}{16}$. So when $t = 1$ sec the height reaches its peak at 20 feet.

When will the ball land? The question can be rephrased: At what time t will $h = 0$. This means we should solve the equation: $\frac{-1}{16}(0-20) = (t-1)^2$ for t. We simplify the equation to see $\frac{5}{4} = (t-1)^2$. It follows that

$\pm\frac{\sqrt{5}}{2} = t - 1$ and adding 1 to both sides gives us that

$t = 1 + \pm\frac{\sqrt{5}}{2}$. Since $\frac{\sqrt{5}}{2} > 1$ $\frac{5}{4} > 1^2$, the solution to the equation with a minus sign gives a negative time which isn't what we are looking for here. So the ball lands at $1 + \frac{\sqrt{5}}{2}$ seconds.

? Try It 3.2.2.13

A ball is launched into the air from a cliff with a height of 20 feet and the height h of its trajectory has the equation $h = -16t^2 + 48t + 20$ where t is measured in seconds. When is the ball at its highest and how high does it go? When does it land?

Answer

Height is 56 feet when the watch hits 1.5 seconds. It lands at $\frac{3}{2} + \sqrt{\frac{7}{2}}$.

The focus of a parabola $a(y-k) = (x-k)^2$ is $(h, k + \frac{a}{4})$. If a light source very far away is hitting the 'inside' of the parabola then the light rays reflect off the parabola and all intersect at that focus. See https://en.wikipedia.org/wiki/Parabolic_reflector#/media/File:Parabola_with_focus_and_arbitrary_line.svg and, in general the article on the topic of parabolic reflectors in wikipedia.

✓ Example 3.2.2.14

If you wanted to make a parabola (this is really only a 2-dimensional version of a 3 dimensional problem, but they are related) so that its focus is one foot away (where a soup needed to be heated), what equation would your parabola have?

Solution

The vertex of the parabola is at (h, k) and the focus is at $(h, k + \frac{a}{4})$. The distance is $\frac{a}{4}$ which should be 1. It follows that $a = 4$. If I position my parabola that I am making so that it opens up (as indicated by the equation) then the equation is $4(y-k) = (x-k)^2$. If my vertex is at $(0, 0)$ then the equation is $4y = x^2$. Maybe I can use my 3-D printer to make it.

Try It 3.2.2.15

If you wanted to make a parabola (this is really only a 2-dimensional version of a 3 dimensional problem, but they are related) so that its focus is 4 foot away (where a soup needed to be heated), what equation would your parabola have?

If you had a parabola with equation $8(y-3)=(x+7)^2$ where would you need to put your soup to heat it?

Answer

$16y=x^2$. The soup should be placed at $(-7,5)$.

Writing Exercises 3.2.2.16

1. Change the title to exclude "functions". How is the graph of a quadratic equation in two variables related to the equation itself?
2. In other words what is the graph of an equation in general?
3. Which elements of the graph can we easily see from the equation when it is in standard form?
4. Which elements of the graph can we easily see when we write it in factored form: $y=(x-a)(x-b)$?
5. Which elements of the graph can we easily see when we complete the square and write it in the form: $y-k=m(x-h)^2$ or $\frac{1}{m}(y-k)=(x-h)^2$.
6. In this last form, what is the relationship between m and the shape of the graph? What does the graph of $x-k=(y-h)^2$ look like?

Exit Problem

1. Let $y=-x^2-6x-7$.

a. Write the equation in the form $y=a(x-h)^2+k$ by completing the square.

b. Identify the vertex and the axis of symmetry.

c. Find the x- and y-intercepts.

d. Graph the parabola and the axis of symmetry on the graph paper. Label the coordinates of the vertex and write the equation of the axis of symmetry.

2. Use the vertex formula to find the coordinates of the vertex of the parabola given by $y=-2x^2+x+1$.

Key Concepts

- The equation $(x-h)^2+(y-k)^2=r^2$ can be obtained by shifting the circle centered at $(0,0)$ with radius r so that its center is (h,k). Note that this is exactly where the left hand side evaluates to 0.
- The equation $a(y-k)=(x-h)^2$ is a parabola opening up or down which is shifted from the standard parabola whose equation is $y=x^2$ so that its vertex is at (h,k) (the point that gives $0=0$ when substituted into the equation). The scale of the y-axis is adjusted according to a. The point $(1,1)$ of the standard parabola is played by what gives $1=1$ when substituted: $(h+1,k+1/a)$.
- The equation $a(x-k)=(y-h)^2$ has a graph that is a reflection about the line $y=x$ (x and y coordinates are exchanged) of the graph of $a(y-k)=(x-h)^2$.

3.2.3: Graphs of Certain Quadratic Equations: Part II

Learning Objectives

By the end of this section, you will be able to:

- Graph certain quadratic equations with two variables using the idea of intercepts.
- Use the intercepts of certain quadratic equations with two variables to find other features of the solutions.

Be Prepared

Before you get started, take this readiness quiz.

1. Solve $(x-2)(3x-7)=0$.
2. Find the midpoint of $(3,0)$ and $(7,0)$.
3. Find a solution of $y=-5x-1$ of the form $(-2,\cdot)$.

Parabolas are somewhat easier to recognize. So we will devote some time here to an alternative strategy.

This involves finding intercepts and making note of the symmetries that we have already recognized. This strategy doesn't always work, but nonetheless can be helpful.

Consider the parabola $y=4x^2-x-6$. We could follow the ideas from the last section. But, we could also take an approach that used the fact that we have an idea of what the parabola looks like already. It opens upward.

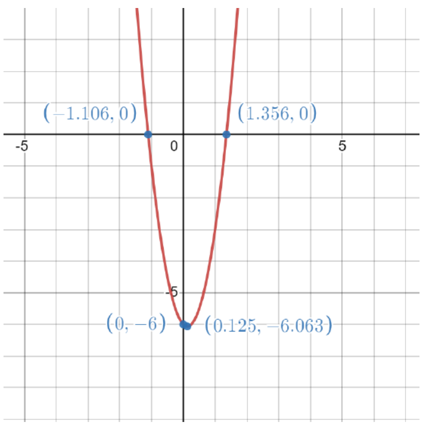
Let's start by finding some special points:

The y-intercept (where the graph crosses the y-axis) has the form $(0,b)$. So, we can find b be substituting this into the equation to find $b=-6$.

The x-intercept(s) (where the graph crosses the x-axis) has the form $(a,0)$. So we need to solve $0=4a^2-a-6$, which we will use the quadratic formula for. We arrive at two intercepts: $\left(\frac{1}{8}+\frac{\sqrt{97}}{8},0\right)$ and $\left(\frac{1}{8}-\frac{\sqrt{97}}{8},0\right)$.

Note that the formula gives us two solutions: $\frac{1}{8}+$ something and $\frac{1}{8}-$ the same something. The average of these two is $\frac{1}{8}$ which is the x-coordinate of the vertex!! To find the y-coordinate we need to find a solution of the equation of the form $\left(\frac{1}{8},b\right)$ which turns out to give $\left(\frac{1}{8},\frac{-97}{16}\right)$ as the vertex.

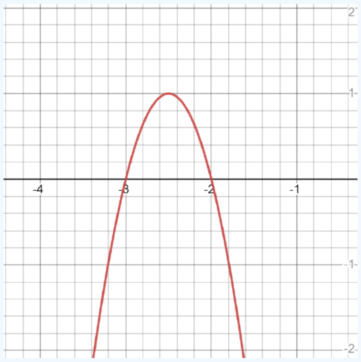
Using symmetry of the parabola we can form the sketch which we provide with Desmos:



This idea is especially nice if you can put it in a particular form where the intercepts are easy to find.

✓ Example 3.2.3.1

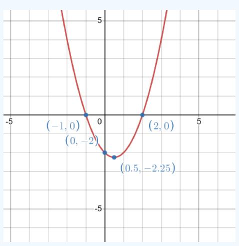
1. Sketch the graph of $y=(x-2)(x+1)$.
2. Write an equation for the following graph



Solution

1. Here the y-intercept is $(0,-2)$ and the x-intercept(s), being of the form $(a,0)$, are $(2,0)$ and $(-1,0)$.

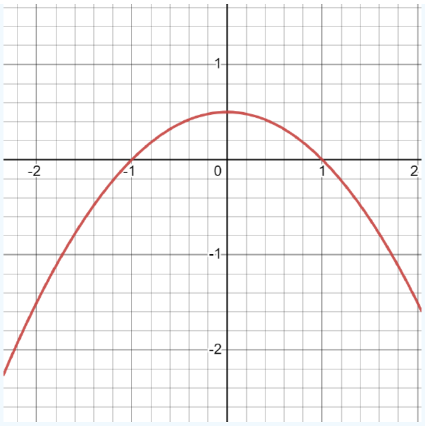
The x-coordinate of the vertex is midway between the x-intercepts which is $\dfrac{2+-1}{2}=\dfrac{1}{2}$. So the vertex is at $\left(\dfrac{1}{2},-\dfrac{9}{4}\right)$. We graph this information and complete the sketch.



2. The parabola opens downward and has x-intercepts $(-2,0)$ and $(-3,0)$. It follows that the parabola has the form $ay=(x+2)(x+3)$. We can find a by noting that $\left(-\dfrac{5}{2},1\right)$ is on the graph and should therefore satisfy the equation. This gives us that $a=-\dfrac{1}{4}$. So our equation is $-\dfrac{y}{4}=(x+2)(x+3)$. This can be checked on Desmos.

? Exercise 3.2.3.1

1. Sketch the graph of $x=(y+2)(y-1)$.
2. Write an equation for the following graph



Answer

1. Check your answer on Desmos.
2. $-2y=(x+1)(x-1)$.

These ideas are helpful, but sometimes not sufficient. For example maybe there are no x-intercepts. In which case, the vertex is still 'in between' as if there were. Or, in the case of an upward opening parabola, for example, the vertex coincides with the x-intercepts, and therefore also with the vertex. You may in these and similar cases plot some points to help.

Key Concepts

When sketching a parabola, depending on its form, you may want to do the following to guide your sketch:

- Find the intercepts
- Locate the vertex (note the symmetry of the parabola)

SECTION OVERVIEW

3.3: Systems of Equations

3.3.1: Systems of Linear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Determine whether an ordered pair is a solution of a system of equations
- Solve a system of linear equations by graphing
- Solve a system of equations by substitution
- Solve a system of equations by elimination
- Choose the most convenient method to solve a system of linear equations

Be Prepared

Before you get started, take this readiness quiz.

1. For the equation $y=\frac{2}{3}x-4$,

a. Is $(6,0)$ a solution?

b. Is $(-3,-2)$ a solution?

2. Find the slope and y-intercept of the line $3x-y=12$.
3. Find the x- and y-intercepts of the line $2x-3y=12$.

Determine Whether an Ordered Pair is a Solution of a System of Equations

We have learned about solutions to linear equations with two variables. Now we will work with two or more linear equations grouped together, which is known as a *system of linear equations*.

Definition 3.3.1.1

When two or more linear equations are grouped together, they form a **system of linear equations**.

In this section, we will focus our work on systems of two linear equations with two unknowns. We will solve larger systems of equations later in this chapter.

An example of a system of two linear equations is shown below. We use a brace to show the two equations are grouped together to form a system of equations.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

A linear equation in two variables, such as $2x+y=7$, has an infinite number of solutions. Its graph is a line. Remember, every point on the line is a solution to the equation and every solution to the equation is a point on the line.

To solve a system of two linear equations, we want to find the values of the variables that are solutions to *both* equations. In other words, we are looking for the ordered pairs (p,q) that when substituted in for (x,y) (p for x and q for y) make both equations true. These are called the *solutions of a system of equations*.

Definition 3.3.1.2

The **solutions of a system of equations** are the values of the variables that make *all* the equations true. A solution of a system of two linear equations is represented by an ordered pair (p,q). We say (x,y)=(p,q) is a solution to the system if when when we substitute p for x and q for y, both resulting equations are true.

To determine if an ordered pair is a solution to a system of two equations, we substitute the values of the variables into each equation. If the ordered pair makes both equations true, it is a solution to the system.

Example 3.3.1.3

Determine whether the ordered pair is a solution to the system $\begin{cases} x-y=-1 \\ 2x-y=-5 \end{cases}$.

a. $(-2,-1)$

b. $(-4,-3)$

Solution

a.

	$x-y=-1$	$2x-y=-5$
We substitute -2 for x and -1 for y in both equations.	$x-y=-1$ $-2-(-1)\stackrel{?}{=}-1$ $-1\stackrel{?}{=}-1$ True	$2x-y=-5$ $2(-2)-(-1)\stackrel{?}{=}-5$ $-3\stackrel{?}{=}-5$ False
Is $(-2,1)$ a solution to the equation?	Yes	No
Answer the question.	Since substituting in $(-2,-1)$ does not make both equations true, $(-2,-1)$ is not a solution.	

b.

	$x-y=-1$	$2x-y=-5$
We substitute -4 for x and -3 for y in both equations.	$x-y=-1$ $-4-(-3)\stackrel{?}{=}-1$ $-1\stackrel{?}{=}-1$ True	$2x-y=-5$ $2(-4)-(-3)\stackrel{?}{=}-5$ $-5\stackrel{?}{=}-5$ True
Is $(-4,-3)$ a solution to the equation?	Yes	Yes
Answer the question.	Since substituting in $(-4,-3)$ does make both equations true, $(-4,-3)$ is not a solution.	

Try It 3.3.1.4

Determine whether the ordered pair is a solution to the system $\begin{cases} 3x+y=0 \\ x+2y=-5 \end{cases}$.

a. $(1,-3)$

b. $(0,0)$

Answer

a. It is a solution.

b. It is not a solution.

Try It 3.3.1.5

Determine whether the ordered pair is a solution to the system $\begin{cases} x-3y=-8 \\ -3x-y=4 \end{cases}$.

a. $(2,-2)$

b. $(-2,2)$

Answer

a. It is not a solution.

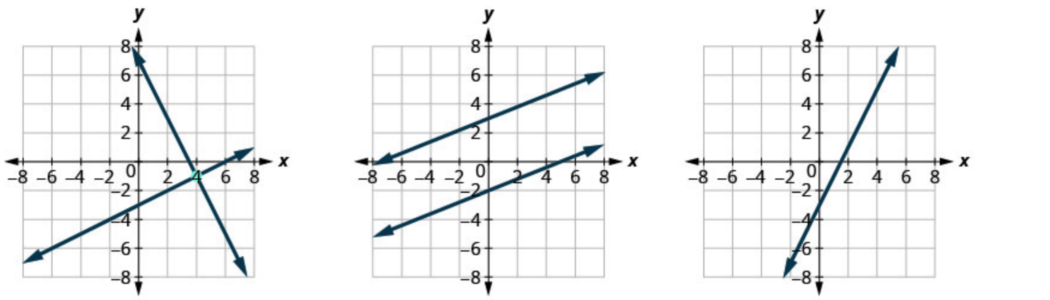
b. It is a solution.

Solve a System of Linear Equations by Graphing

In this section, we will use three methods to solve a system of linear equations. The first method we will use is graphing.

The graph of a linear equation is a line. With the representation of ordered pairs of numbers on the coordinate plane, we will call these ordered pairs points and imagine them on the coordinate plane. So, we will say, for example, that each point on the line *is* a solution to the equation instead of that each point on the line *represents* a solution to the equation. For a system of two equations, we will graph two lines. Then we can see all the points that are solutions to each equation. And, by finding what the lines have in common, we will find the solution to the system.

When we solve a system of two linear equations represented by a graph of two lines in the same plane. There are three possible cases, as shown.



The lines intersect. Intersecting lines have one point in common. There is one solution to this system.

The lines are parallel. Parallel lines have no points in common. There is no solution to this system.

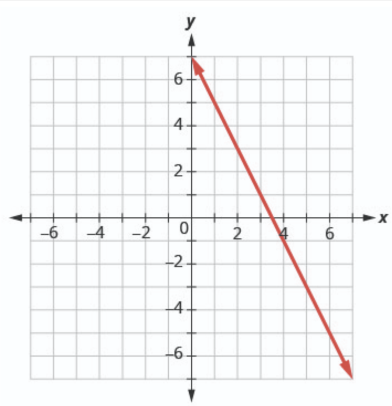
Both equations give the same line. Because we have just one line, there are infinitely many solutions.

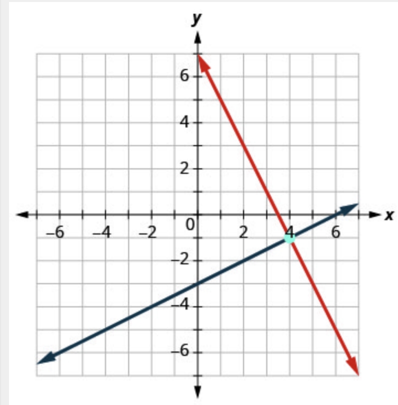
Each time we demonstrate a new method, we will use it on the same system of linear equations. At the end of the section we will decide which method was the most convenient way to solve this system.

Example 3.3.1.6

Solve the system $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ by graphing.

Solution

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Graph the first equation.	To graph the first line, write the equation in slope-intercept form. $2x+y=7$ $y=-2x+7$ The slope is $m=-2$. The y-intercept is $(0,7)$, so $b=7$.	

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Graph the second equation on the same rectangular coordinate system.	To graph the second line, $x-2y=6$ use intercepts. The x-intercept is $(6,0)$. The y-intercept is $(0,-3)$.	
Determine whether the lines intersect, are parallel, or are the same line.	Look at the graph of the lines.	The lines intersect.
Identify the solution to the system. • If the lines intersect, identify the point of intersection. Check to make sure it is a solution to both equations. This is the solution to the system. • If the lines are parallel, the system has no solution. • If the lines are the same, the system has an infinite number of solutions.	Since the lines intersect, find the point of intersection. Check the point in both equations.	The lines intersect at $(4,-1)$. $2x+y=7$ / $2(4)+(-1)\stackrel{?}{=}7$ / $7\stackrel{?}{=}7$ True $x-2y=6$ / $4-2(-1)\stackrel{?}{=}6$ / $6\stackrel{?}{=}6$ True Substituting $(4,-1)$ in for (x,y) makes both equations true.
Answer the question.		The solution is $(4,-1)$.

? Try It 3.3.1.7

Solve the system $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$ by graphing.

Answer

The solution is $(3,2)$.

? Try It 3.3.1.8

Solve the system $\begin{cases} -x+y=1 \\ 3x+2y=12 \end{cases}$ by graphing.

Answer

The solution is $(2,3)$.

The steps to use to solve a system of linear equations by graphing are shown here.

Solve a system of linear equations by graphing

1. Graph the first equation.
2. Graph the second equation on the same rectangular coordinate system.
3. Determine whether the lines intersect, are parallel, or are the same line.
4. Identify the solution to the system.

- If the lines intersect, identify the point of intersection. This is the solution to the system.
- If the lines are parallel, the system has no solution.
- If the lines are the same, the system has an infinite number of solutions.

5. Check the solution in both equations.

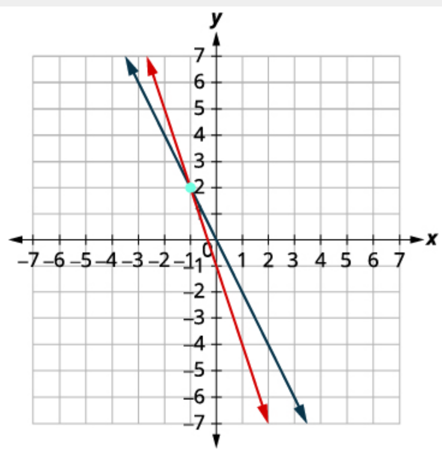
In the next example, we will first rewrite the equations into slope-intercept form as this will make it easy for us to quickly graph the lines.

Example 3.3.1.9

Solve the system $\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$ by graphing.

Solution

We will solve both of these equations for y so that we can easily graph them using their slopes and y-intercepts.

	$\begin{cases} 3x+y=-1 \\ 2x+y=0 \end{cases}$
Solve the first equation for y.	$3x+y=-1$ $y=-3x-1$
Find the slope and y-intercept.	The slope is $m=-3$. The y-intercept is $(0,-1)$.
Solve the second equation for y.	$2x+y=0$ $y=-2x$
Find the slope and y-intercept.	The slope is $m=-2$. The y-intercept is $(0,0)$.
Graph the lines.	
Determine the point of intersection.	The lines intersect at $(-1,2)$.
Check the solution in both equations.	$3x+y=-1$ $2x+y=0$ $3(-1)+2 \stackrel{?}{=} -1$ $2(-1)+2 \stackrel{?}{=} 0$ $-1 \stackrel{?}{=} -1$ $0 \stackrel{?}{=} 0$ True True Substituting $(-1,2)$ in for (x,y) makes both equations true.
Answer the question.	The solution is $(-1,2)$.

Try It 3.3.1.10

Solve the system $\begin{cases} -x+y=1 \\ 2x+y=10 \end{cases}$ by graphing.

Answer

The solution is $(3, 4)$.

Try It 3.3.1.11

Solve the system $\begin{cases} 2x+y=6 \\ x+y=1 \end{cases}$ by graphing.

Answer

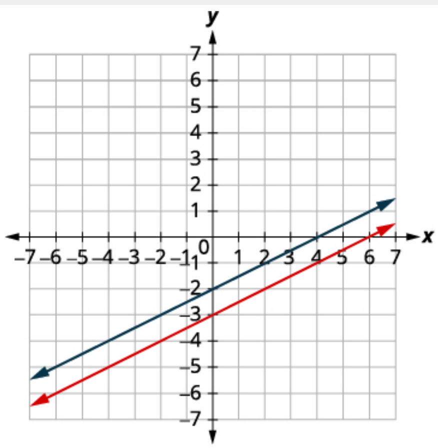
The solution is $(5, -4)$.

In all the systems of linear equations so far, the lines intersected and the solution was one point. In the next two examples, we'll look at a system of equations that has no solution and at a system of equations that has an infinite number of solutions.

Example 3.3.1.12

Solve the system $\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$ by graphing.

Solution

	$\begin{cases} y=\frac{1}{2}x-3 \\ x-2y=4 \end{cases}$
To graph the first equation, we will use its slope and y-intercept.	$y=\frac{1}{2}x-3$ The slope is $m=\frac{1}{2}$. The y-intercept is $(0, -3)$.
To graph the second equation, we will use the intercepts.	$x-2y=4$
Find the x- and y-intercepts of the second equation.	(see table below) The x-intercept is $(4, 0)$. The y-intercept is $(0, -2)$.
Graph the lines.	

x	y	(x, y)
0	-2	$(0, -2)$
4	0	$(4, 0)$

Determine the points of intersection.	The lines are parallel. Since no point is on both lines, there is no ordered pair that makes both equations true upon substitution.
Answer the question.	There is no solution to the system.

Try It 3.3.1.13

Solve the system $\begin{cases} y=-\frac{1}{4}x+2 \\ x+4y=4 \end{cases}$ by graphing.

Answer

The system has no solution.

Try It 3.3.1.14

Solve the system $\begin{cases} y=3x-1 \\ 6x-2y=6 \end{cases}$ by graphing.

Answer

The system has no solution.

Sometimes the equations in a system represent the same line. Since every point on the line makes both equations true, there are infinitely many ordered pairs that, when substituted, make both equations true. Therefore there are infinitely many solutions to the system.

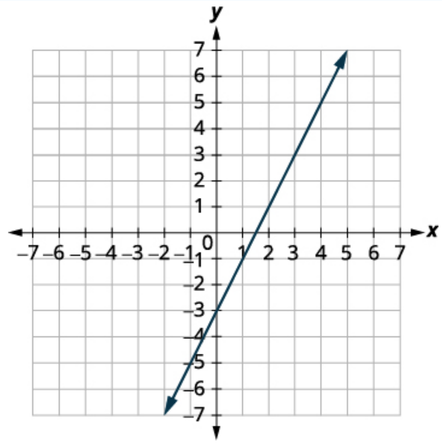
Example 3.3.1.15

Solve the system $\begin{cases} y=2x-3 \\ -6x+3y=9 \end{cases}$ by graphing.

Solution

	$\begin{cases} y=2x-3 \\ -6x+3y=9 \end{cases}$
Find the slope and y-intercept of the first equation.	$y=2x-3$ The slope is $m=2$. The y-intercept is $(0,-3)$.
To graph the second equation, we will use the intercepts.	$-6x+3y=9$
Find the x- and y-intercepts of the second equation.	see table below The x-intercept is $\left(\frac{3}{2},0\right)$. The y-intercept is $(0,-3)$.

x	y	(x,y)
0	-3	$(0,-3)$
$\frac{3}{2}$	0	$\left(\frac{3}{2},0\right)$

Graph the lines.	
Determine the points of intersection.	The lines are the same! Any point on the line is a point of intersection. Since every point on the line, when substituted, makes both equations true, there are infinitely many ordered pairs that make both equations true.
Answer the question.	There are infinitely many solutions to the system.

If we write the second equation in slope-intercept form, we may recognize that the equations have the same slope and same y-intercept.

Try It 3.3.1.16

Solve the system $\begin{cases} y = -3x - 6 \\ 6x + 2y = -12 \end{cases}$ by graphing.

Answer

The system has infinitely many solutions.

Try It 3.3.1.17

Solve the system $\begin{cases} y = \frac{1}{2}x - 4 \\ 2x - 4y = 16 \end{cases}$ by graphing.

Answer

The system has infinitely many solutions.

When we graphed the second line in the last example, we drew it right over the first line. We say the two lines are *coincident*. Coincident lines have the same slope and same y-intercept.

Definition 3.3.1.18

Coincident lines have the same slope and same y-intercept.

The systems of equations in Example and Example each had two intersecting lines. Each system had one solution.

In Example, the equations gave coincident lines, and so the system had infinitely many solutions.

The systems in those three examples had at least one solution. A system of equations that has at least one solution is called a *consistent* system.

A system with parallel lines, like Example, has no solution. We call a system of equations like this *inconsistent.* It has no solution.

Definition 3.3.1.19

A **consistent system of equations** is a system of equations with at least one solution.

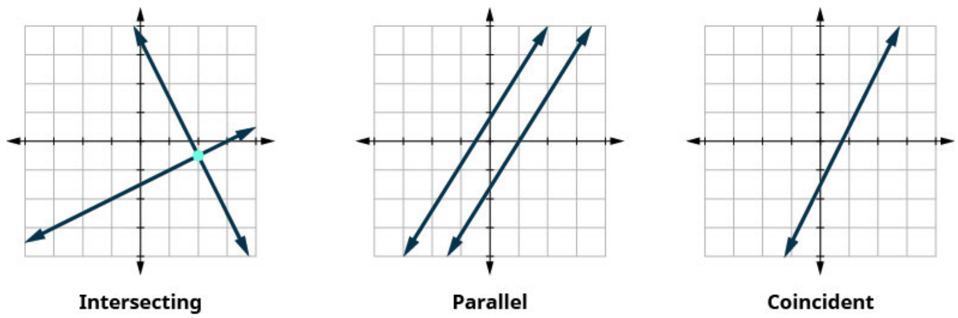
An **inconsistent system of equations** is a system of equations with no solution.

We also categorize the equations in a system of equations by calling the equations *independent* or *dependent*. If two equations are dependent, all the solutions of one equation are also solutions of the other equation. When we graph two dependent equations, we get coincident lines. Otherwise, they each have their own different set of solutions and we say the two equations are *independent*. Intersecting lines and parallel lines are independent.

Definition 3.3.1.20

Two equations are **dependent** if they have the same set of solutions. Two equations are **independent** if their solutions differ.

Let's sum this up by looking at the graphs of the three types of systems. See below.



Lines	Intersecting	Parallel	Coincident
Number of solutions	One point	No solution	Infinitely many
Consistent/Inconsistent	Consistent	Inconsistent	Consistent
Dependent/Independent	Independent	Independent	Dependent

Solving systems of linear equations by graphing is a good way to visualize the types of solutions that may result. However, there are many cases where solving a system by graphing is inconvenient or imprecise. If the graphs extend beyond the small grid with x and y both between -10 and 10, graphing the lines may be cumbersome. And if the solutions to the system are not integers, it can be hard to read their values precisely from a graph.

Solve a System of Equations by Substitution

We will now solve systems of linear equations by the *substitution method*.

We will use the same system we used first for graphing.

$$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$$

We will first solve one of the equations for either x or y. We can choose either equation and solve for either variable - but we'll try to make a choice that will keep the work easy.

Then we substitute that expression into the other equation. The result is an equation with just one variable—and we know how to solve those!

After we find the value of one variable, we will substitute that value into one of the original equations and solve for the other variable. Finally, we check our solution and make sure it is a solution to both equations.

Example 3.3.1.21

Solve the system $\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$ by substitution.

Solution

	$\begin{cases} 2x+y=7 \\ x-2y=6 \end{cases}$	
Solve one of the equations for either variable.	We will solve the first equation for y.	$2x+y=7$ $y=7-2x$
Substitute the expression from the previous step into the other equation.	We replace y in the second equation with the expression $7-2x$.	$x-2y=6$ $x-2(7-2x)=6$
Solve the resulting equation.	Now we have an equation with just one variable. We know how to solve this.	$x-14+4x=6$ $5x=20$ $x=4$
Substitute the solution just found into one of the original equations to find the other variable.	We will use the first equation and replace x with 4.	$2x+y=7$ $2\cdot4+y=7$ $8+y=7$ $y=-1$
Write the solution as an ordered pair.	Write $x=4$ and $y=-1$ as an ordered pair.	$(4,-1)$
Check that the ordered pair is a solution to **both** original equations.	Substitute $x=4$ and $y=-1$ into both equations and make sure they are both true.	$2x+y=7$ $2(4)+(-1)\stackrel{?}{=}7$ $7\stackrel{?}{=}7$ True $x-2y=6$ $4-2(-1)\stackrel{?}{=}6$ $6\stackrel{?}{=}6$ True Substituting $(4,-1)$ in for (x,y) makes both equations true.
Answer the question.		$(4,-1)$ is a solution to the system.

Try It 3.3.1.22

Solve the system $\begin{cases} -2x+y=-11 \\ x+3y=9 \end{cases}$ by substitution.

Answer

The solution is $(6,1)$.

Try It 3.3.1.23

Solve the system $\begin{cases} 2x+y=-1 \\ 4x+3y=3 \end{cases}$ by substitution.

Answer

The solution is $(-3,5)$.

Solve a system of equations by substitution

1. Solve one of the equations for either variable.
2. Substitute the expression from Step 1 into the other equation.
3. Solve the resulting equation.
4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
5. Write the solution as an ordered pair.

6. Check that the ordered pair is a solution to **both** original equations. This step is added here in order to check for errors.

Be very careful with the signs in the next example.

Example 3.3.1.24

Solve the system $\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$ by substitution.

Solution

We need to solve one equation for one variable. We will solve the first equation for y.

	$\begin{cases} 4x+2y=4 \\ 6x-y=8 \end{cases}$
Solve the first equation for y.	$4x+2y=4$ $2y=-4x+4$ $y=-2x+2$
Replace the y with $-2x+2$ in the second equation.	$6x-y=8$ $6x-(-2x+2)=8$
Solve the equation for x.	$6x+2x-2=8$ $8x=10$ $x=\frac{5}{4}$
Substitute $x=\frac{5}{4}$ into $4x+2y=4$ to find y.	$4x+2y=4$ $4\cdot\frac{5}{4}+2y=4$ $5+2y=4$ $2y=-1$ $y=-\frac{1}{2}$
Write the solution as an ordered pair.	The ordered pair is $\left(\frac{5}{4},-\frac{1}{2}\right)$.
Check the ordered pair in both equations.	$4x+2y=4$ $4\left(\frac{5}{4}\right)+2\left(-\frac{1}{2}\right)\stackrel{?}{=}4$ $5-1\stackrel{?}{=}4$ $4\stackrel{?}{=}4$ True $6x-y=8$ $6\left(\frac{5}{4}\right)-\left(-\frac{1}{2}\right)\stackrel{?}{=}8$ $\frac{15}{4}+\frac{1}{2}\stackrel{?}{=}8$ $\frac{16}{2}\stackrel{?}{=}8$ $4\stackrel{?}{=}4$ True Substituting $\left(\frac{5}{4},-\frac{1}{2}\right)$ in for (x,y) makes both equations true.
Answer the question.	$\left(\frac{5}{4},-\frac{1}{2}\right)$ is a solution to the system.

Try It 3.3.1.25

Solve the system $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$ by substitution.

Answer

The solution is $\left(2,\frac{3}{2}\right)$.

Try It 3.3.1.26

Solve the system $\begin{cases} 4x - y = 0 \\ 2x - 3y = 5 \end{cases}$ by substitution.

Answer

The solution is $\left(-\frac{1}{2}, -2\right)$.

Solve a System of Equations by Elimination

We have solved systems of linear equations by graphing and by substitution. Graphing works well when the variable coefficients are small and the solution has integer values. Substitution works well when we can easily solve one equation for one of the variables and not have too many fractions in the resulting expression.

The third method of solving systems of linear equations is called the *Elimination Method*. When we solved a system by substitution, we started with two equations with two variables and reduced it to one equation with one variable. This is what we will do with the elimination method, too, but we will get there a different way.

The Elimination Method is based on the Addition Property of Equality. The Addition Property of Equality says that when we add the same quantity to both sides of an equation, we still have equality. We will extend the Addition Property of Equality to say that when we add equal quantities to both sides of an equation, the results are equal. In other words, for any expressions a, b, c, and d,

$$\begin{array}{lrcl} \text{if} & a & = & b \\ \text{and} & c & = & d \\ \text{then} & a+c & = & b+d. \end{array}$$

To solve a system of equations by elimination, we start with both equations in standard form. Then we decide which variable will be easiest to eliminate. How do we decide? We want to have the coefficients of one variable be opposites, so that we can add the equations together and eliminate that variable.

Notice how that works when we add these two equations together:

$$\begin{cases} 3x + y = 5 \\ \underline{2x - y = 0} \end{cases}$$
$$5x = 5$$

The y's add to zero and we have one equation with one variable.

Let's try another one:

$$\begin{cases} x + 4y = 2 \\ 2x + 5y = -2 \end{cases}$$

This time we don't see a variable that can be immediately eliminated if we add the equations.

But if we multiply the first equation by -2, we will make the coefficients of x opposites. We must multiply every term on both sides of the equation by -2.

$$\begin{cases} (-2)(x + 4y) = (-2)2 \\ 2x + 5y = -2 \end{cases}$$

Then rewrite the system of equations.

$$\begin{cases} -2x - 8y = -4 \\ 2x + 5y = -2 \end{cases}$$

Now we see that the coefficients of the x terms are opposites, so x will be eliminated when we add these two equations.

$$\begin{cases} -2x - 8y = -4 \\ \underline{\;\;2x + 5y = -2} \end{cases}$$
$$-3y = -6$$

Once we get an equation with just one variable, we solve it. Then we substitute that value into one of the original equations to solve for the remaining variable. And, as always, we check our answer to make sure it is a solution to both of the original equations.

Now we will see how to use elimination to solve the same system of equations we solved by graphing and by substitution.

Example 3.3.1.27

Solve the system $\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$ by elimination.

Solution

	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$	
Write both equations in standard form. • If any coefficients are fractions, clear them.	Both equations are in standard form, $Ax + By = C$. There are no fractions	$\begin{cases} 2x + y = 7 \\ x - 2y = 6 \end{cases}$
Make the coefficients of one variable opposites. • Decide which variable you will eliminate. • Multiply one of both equations so that the coefficients of that variable are opposites.	We can eliminate the y's by multiplying the first equation by 2. Multiply both sides of $2x + y = 7$ by 2.	$\begin{cases} 2(2x + y) = 2(7) \\ x - 2y = 6 \end{cases}$
Add the equations resulting from the previous step to eliminate one variable.	We add the x's, y's, and constants.	$\begin{cases} 4x + 2y = 14 \\ \underline{x - 2y = 6} \\ 5x = 20 \end{cases}$
Solve for the remaining variable.	Solve for x.	$x = 4$
Substitute the solution from the previous step into one of the original equations. Then solve for the other variable.	Substitute $x = 4$ into the second equation, $x - 2y = 6$. Then solve for y.	$x - 2y = 6$ $4 - 2y = 6$ $-2y = 2$ $y = -1$
Write the solution as an ordered pair.	Write it as (x, y).	$(4, -1)$
Check that the ordered pair is a solution to **both** original equations.	Substitute $x = 4$, $y = -1$ into $2x + y = 7$ and $x - 2y = 6$. Do they make both equations true?	$2x + y = 7$ $2(4) + (-1) \stackrel{?}{=} 7$ $7 \stackrel{?}{=} 7$ True $x - 2y = 6$ $4 - 2(-1) \stackrel{?}{=} 6$ $6 \stackrel{?}{=} 6$ True Substituting $(x, y) = (4, -1)$ makes both equations true.
Answer the question.		The solution is $(4, -1)$

Try It 3.3.1.28

Solve the system $\begin{cases} 3x + y = 5 \\ 2x - 3y = 7 \end{cases}$ by elimination.

Answer

The solution is $(2, -1)$.

Try It 3.3.1.29

Solve the system $\begin{cases} 4x+y=-5 \\ -2x-2y=-2 \end{cases}$ by elimination.

Answer

The solution is $(-2,3)$.

The steps are listed here for easy reference.

Solve a system of equations by elimination

1. Write both equations in standard form. If any coefficients are fractions, clear them.
2. Make the coefficients of one variable opposites.
   - Decide which variable you will eliminate.
   - Multiply one or both equations by appropriate numbers so that the coefficients of that variable are opposites.
3. Add the equations resulting from Step 2 to eliminate one variable.
4. Solve for the remaining variable.
5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
6. Write the solution as an ordered pair.
7. Check that the ordered pair is a solution to **both** original equations. This step is included for the purpose of detecting errors.

Now we will do an example where we need to multiply both equations by constants in order to make the coefficients of one variable opposites.

Example 3.3.1.30

Solve the system $\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$ by elimination.

Solution

In this example, we cannot multiply just one equation by any constant to get opposite coefficients. So we will strategically multiply both equations by different constants to get the opposites.

	$\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$
Both equations are in standard form. To get opposite coefficients of y, we will multiply the first equation by 2 and the second equation by 3.	$\begin{cases} 2(4x-3y)=2(9) \\ 3(7x+2y)=3(-6) \end{cases}$
Simplify.	$\begin{cases} 8x-6y=18 \\ 21x+6y=-18 \end{cases}$
Add the two equations to eliminate y.	$29x=0$
Solve for x.	$x=0$
Substitute $x=0$ into one of the original equations.	$7x+2y=-6$ $7(0)+2y=-6$
Simplify.	$2y=-6$
Solve for y.	$y=-3$

	$\begin{cases} 4x-3y=9 \\ 7x+2y=-6 \end{cases}$
Check that the ordered pair is a solution to **both** original equations.	$4x-3y=9$, $4(0)-3(-3)\stackrel{?}{=}9$, $9\stackrel{?}{=}9$ True; $7x+2y=-6$, $7(0)+2(-3)\stackrel{?}{=}-6$, $-6\stackrel{?}{=}-6$ True. Substituting $(x,y)=(0,-3)$ makes both equations true.
Answer the question.	The solution is $(0,-3)$.

Try It 3.3.1.31

Solve the system $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$ by elimination.

Answer

The solution is $(1,3)$.

Try It 3.3.1.32

Solve the system $\begin{cases} 7x+8y=4 \\ 3x-5y=27 \end{cases}$ by elimination.

Answer

The solution is $(4,-3)$.

When the system of equations contains fractions, we will first clear the fractions by multiplying each equation by the LCD of all the fractions in the equation.

Example 3.3.1.33

Solve the system $\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$ by elimination.

Solution

In this example, both equations have fractions. Our first step will be to multiply each equation by the LCD of all the fractions in the equation to clear the fractions.

	$\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$
To clear the fractions, multiply each equation by its LCD.	$\begin{cases} 2\left(x+\frac{1}{2}y\right)=2(6) \\ 6\left(\frac{3}{2}x+\frac{2}{3}y\right)=6\left(\frac{17}{2}\right) \end{cases}$
Simplify	$\begin{cases} 2x+y=12 \\ 9x+4y=51 \end{cases}$
Now we are ready to eliminate one of the variables. Notice that both equations are in standard form. We can eliminate y by multiplying the top equation by -4.	$\begin{cases} -4(2x+y)=-4(12) \\ 9x+4y=51 \end{cases}$

	$\begin{cases} x+\frac{1}{2}y=6 \\ \frac{3}{2}x+\frac{2}{3}y=\frac{17}{2} \end{cases}$
Simplify.	$\begin{cases} -8x-4y=-48 \\ 9x+4y=51 \end{cases}$
Add.	$x=3$
Substitute $x=3$ into one of the original equations.	$x+\frac{1}{2}y=6$ $3+\frac{1}{2}y=6$
Simplify.	$\frac{1}{2}y=3$
Solve for y.	$y=6$
Write the solution as an ordered pair.	The ordered pair is $(3,6)$.
Check that the ordered pair is a solution to both original equations.	$x+\frac{1}{2}y=6$ $3+\frac{1}{2}(6)\stackrel{?}{=}6$ $3+3\stackrel{?}{=}6$ $6\stackrel{?}{=}6$ True $\frac{3}{2}x+\frac{2}{3}y\stackrel{?}{=}\frac{17}{2}$ $\frac{3}{2}(3)+\frac{2}{3}(6)\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+4\stackrel{?}{=}\frac{17}{2}$ $\frac{9}{2}+\frac{8}{2}\stackrel{?}{=}\frac{17}{2}$ $\frac{17}{2}\stackrel{?}{=}\frac{17}{2}$ True Substituting $(x,y)=(3,6)$ makes both equations true.
Answer the question.	The solution is $(3,6)$.

Try It 3.3.1.34

Solve the system $\begin{cases} \frac{1}{3}x-\frac{1}{2}y=1 \\ \frac{3}{4}x-y=\frac{5}{2} \end{cases}$ by elimination.

Answer

The solution is $(6,2)$.

Try It 3.3.1.35

Solve the system $\begin{cases} x+\frac{3}{5}y=-\frac{1}{5} \\ -\frac{1}{2}x-\frac{2}{3}y=\frac{5}{6} \end{cases}$ by elimination.

Answer

The solution is $(1,-2)$.

When we solved the system by graphing, we saw that not all systems of linear equations have a single ordered pair as a solution. When the two equations were really the same line, there were infinitely many solutions. We called that a consistent system. When the two equations described parallel lines, there was no solution. We called that an inconsistent system.

The same is true using substitution or elimination. If the equation at the end of substitution or elimination is a true statement, we have a consistent but dependent system and the system of equations has infinitely many solutions. If the equation at the end of substitution or elimination is a false statement, we have an inconsistent system and the system of equations has no solution.

? Example 3.3.1.36

Solve the system $\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$ by elimination.

Solution

	$\begin{cases} 3x+4y=12 \\ y=3-\frac{3}{4}x \end{cases}$
Write the second equation in standard form.	$\begin{cases} 3x+4y=12 \\ \frac{3}{4}x+y=3 \end{cases}$
Clear the fractions by multiplying the second equation by 4.	$\begin{cases} 3x+4y=12 \\ 4(\frac{3}{4}x+y)=4(3) \end{cases}$
Simplify.	$\begin{cases} 3x+4y=12 \\ 3x+4y=12 \end{cases}$
To eliminate a variable, we multiply the second equation by -1 and simplify.	$\begin{cases} 3x+4y=12 \\ -3x-4y=-12 \end{cases}$
Add the equations.	$0=0$
Conclusion.	This is a true statement. The equations are consistent but dependent. Their graphs would be the same line.
Answer the question.	The system has infinitely many solutions.

After we cleared the fractions in the second equation, did you notice that the two equations were the same? That means we have coincident lines.

? Try It 3.3.1.37

Solve the system $\begin{cases} 5x-3y=15 \\ 5y=-5+\frac{5}{3}x \end{cases}$ by elimination.

Answer

The system has infinitely many solutions.

? Try It 3.3.1.38

Solve the system $\begin{cases} x+2y=6 \\ y=-\frac{1}{2}x+3 \end{cases}$ by elimination.

Answer

The system has infinitely many solutions.

Choose the Most Convenient Method to Solve a System of Linear Equations

When we solve a system of linear equations in an application, we will not be told which method to use. We will need to make that decision ourselves. So we will want to choose the method that is easiest to do and minimizes our chance of making mistakes.

Choose the Most Convenient Method to Solve a System of Linear Equations		
Graphing	**Substitution**	**Elimination**
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations are in standard form.

Example 3.3.1.40

For each system of linear equations, decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a. $\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$

b. $\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$

Solution

a.

$$\begin{cases} 3x+8y=40 \\ 7x-4y=-32 \end{cases}$$

Since both equations are in standard form, using elimination will be most convenient.

b.

$$\begin{cases} 5x+6y=12 \\ y=\frac{2}{3}x-1 \end{cases}$$

Since one equation is already solved for y, using substitution will be most convenient (especially since we see that the fraction will be reduced to an integer in the first step of simplification).

Try It 3.3.1.41

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a. $\begin{cases} 4x-5y=-32 \\ 3x+2y=-1 \end{cases}$

b. $\begin{cases} x=2y-1 \\ 3x-5y=-7 \end{cases}$

Answer

a. Since both equations are in standard form, using elimination will be most convenient.

b. Since one equation is already solved for x, using substitution will be most convenient.

Try It 3.3.1.42

For each system of linear equations decide whether it would be more convenient to solve it by substitution or elimination. Explain your answer.

a. $\begin{cases} y=2x-1 \\ 3x-4y=-6 \end{cases}$

b. $\begin{cases} 6x-2y=12 \\ 3x+7y=-13 \end{cases}$

Answer

a. Since one equation is already solved for y, using substitution will be most convenient.

b. Since both equations are in standard form, using elimination will be most convenient.

Writing Exercises 3.3.1.43

1. What does it mean to be a solution to a system of equations?
2. What does it mean to solve a system of equations?
3. How do you check if $(2,3)$ is a solution to your system?
4. What is the purpose of the elimination method?
5. Explain the graphical approach to solving a 2-D system of equations, and the reasoning behind it. What are this method's advantages and disadvantages?
6. Explain each algebraic approach and describe any advantages or disadvantages.

Exit Problem 3.3.1.44

Solve $\begin{cases} 5x+3y=2 \\ 3x-2y=-14 \end{cases}$

Key Concepts

- **How to solve a system of linear equations by graphing.**
  1. Graph the first equation.
  2. Graph the second equation on the same rectangular coordinate system.
  3. Determine whether the lines intersect, are parallel, or are the same line.
  4. Identify the solution to the system.
     If the lines intersect, identify the point of intersection. This is the solution to the system.
     If the lines are parallel, the system has no solution.
     If the lines are the same, the system has an infinite number of solutions.
  5. Check the solution in both equations. This step is included to make sure there was no error and that the intersection point was correctly identified.
- **How to solve a system of equations by substitution.**
  1. Solve one of the equations for either variable.
  2. Substitute the expression from Step 1 into the other equation.
  3. Solve the resulting equation.
  4. Substitute the solution in Step 3 into either of the original equations to find the other variable.
  5. Write the solution as an ordered pair.
  6. In order to detect errors, check that the ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Write both equations in standard form. If any coefficients are fractions, clear them.
  2. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations by appropriate numbers so that the coefficients of that variable are opposites.
  3. Add the equations resulting from Step 2 to eliminate one variable.
  4. Solve for the remaining variable.
  5. Substitute the solution from Step 4 into one of the original equations. Then solve for the other variable.
  6. Write the solution as an ordered pair.

7. In order to detect errors, check that the ordered pair is a solution to **both** original equations.

Choose the Most Convenient Method to Solve a System of Linear Equations

Graphing	Substitution	Elimination
Use when you need a picture of the situation.	Use when one equation is already solved or can be easily solved for one variable.	Use when the equations a rein standard form.

3.3.2: Systems of Nonlinear Equations with Two Variables

Learning Objectives

By the end of this section, you will be able to:

- Solve a system of nonlinear equations using graphing
- Solve a system of nonlinear equations using substitution
- Solve a system of nonlinear equations using elimination
- Use a system of nonlinear equations to solve applications

Be Prepared

Before you get started, take this readiness quiz.

1. Solve the system by graphing: $\begin{cases} x-3y=-3 \\ x+y=5 \end{cases}$
2. Solve the system by substitution: $\begin{cases} x-4y=-4 \\ -3x+4y=0 \end{cases}$
3. Solve the system by elimination: $\begin{cases} 3x-4y=-9 \\ 5x+3y=14 \end{cases}$

Solve a System of Nonlinear Equations using Graphing

We learned how to solve systems of linear equations with two variables by graphing, substitution and elimination. We will be using these same methods as we look at nonlinear systems of equations with two equations and two variables. A **system of nonlinear equations** is a system where at least one of the equations is not linear.

For example each of the following systems is a **system of nonlinear equations**.

$$\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$$

$$\begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases}$$

$$\begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$$

Definition 3.3.2.1

A **system of nonlinear equations** is a system where at least one of the equations is not linear.

Just as with systems of linear equations, a solution of a nonlinear system is an ordered pair that makes both equations true. In a nonlinear system, there may be more than one solution. We will see this as we solve a system of nonlinear equations by graphing.

When we solved systems of linear equations, the solution of the system was the point of intersection of the two lines. With systems of nonlinear equations, the graphs may be circles, parabolas or hyperbolas and there may be several points of intersection, and so several solutions. Once you identify the graphs, visualize the different ways the graphs could intersect and so how many solutions there might be.

To solve systems of nonlinear equations by graphing, we use basically the same steps as with systems of linear equations modified slightly for nonlinear equations. The steps are listed below for reference.

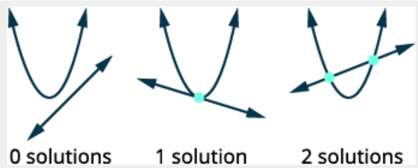
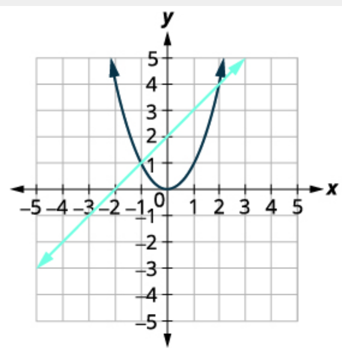
Solve a System of Nonlinear Equations by Graphing

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Graph the first equation.
3. Graph the second equation on the same rectangular coordinate system.
4. Determine whether the graphs intersect.
5. Identify the points of intersection.
6. Check that each ordered pair is a solution to both original equations.

Example 3.3.2.2

Solve the system by graphing: $\begin{cases} x-y=-2 \\ y=x^2 \end{cases}$

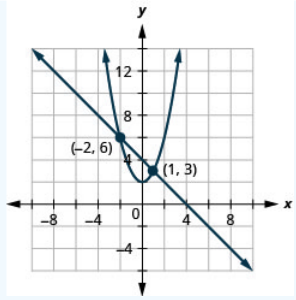
Solution

Identify each graph.	$\begin{cases} x-y=-2 & \text{line} \\ y=x^2 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
Graph the line, $x-y=-2$. Slope-intercept form $y=x+2$. Graph the parabola, $y=x^2$.	
Identify the points of intersection.	The points of intersection appear to be $(2,3)$ and $(-1,1)$.
Check to make sure each solution makes both equations true.	$(2,4)$ $(-1,1)$
	The solutions are $(2,4)$ and $(-1,1)$.

Try It 3.3.2.3

Solve the system by graphing: $\begin{cases} x+y=4 \\ y=x^2+2 \end{cases}$.

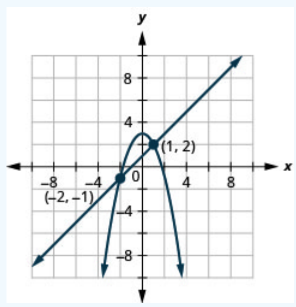
Answer



Try It 3.3.2.4

Solve the system by graphing: $\begin{cases} x-y=-1 \\ y=-x^2+3 \end{cases}$

Answer



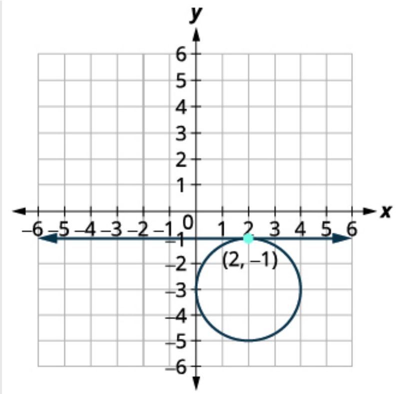
To identify the graph of each equation, keep in mind the characteristics of the x^2 and y^2 terms of each conic.

Example 3.3.2.5

Solve the system by graphing: $\begin{cases} y=-1 \\ (x-2)^2+(y+3)^2=4 \end{cases}$.

Solution

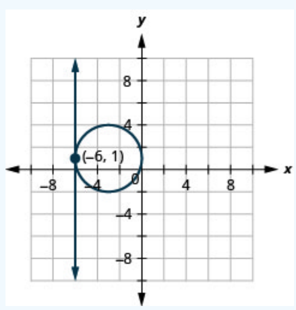
Identify each graph.	$\begin{cases} y=-1 & \text{line} \\ (x-2)^2+(y+3)^2=4 & \text{circle} \end{cases}$
Sketch the possible options for the intersection of a circle and a line.	

Graph the circle, $(x-2)^2+(y+3)^2=4$ Center: $(2,-3)$ radius: 2 Graph the line, $y=-1$. It is a horizontal line.	
Identify the points of intersection.	The point of intersection appears to be $(2,-1)$.
Check to make sure the solution makes both equations true.	$(2,-1)$ $(x-2)^2+(y+3)^2=4 \quad y=-1$ $(2-2)^2+(-1+3)^2\stackrel{?}{=}4 \quad -1=-1$ $(0)^2+(2)^2\stackrel{?}{=}4$ $4=4$
	The solution is $(2,-1)$

Try It 3.3.2.6

Solve the system by graphing: $\begin{cases} x=-6 \\ (x+3)^2+(y-1)^2=9 \end{cases}$

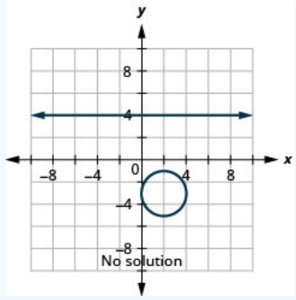
Answer



Try It 3.3.2.7

Solve the system by graphing: $\begin{cases} y=4 \\ (x-2)^2+(y+3)^2=4 \end{cases}$

Answer



We may encounter systems of equations where we don't know what the graph looks like. In what follows some of the equations may be unfamiliar to us. We may graph these on Desmos, to get an idea of how many solutions to expect, but the methods presented will not depend on graphing the equations.

Solve a System of Nonlinear Equations Using Substitution

The graphing method works well when the points of intersection are integers and so easy to read off the graph. But more often it is difficult to read the coordinates of the points of intersection. The substitution method is an algebraic method that will work well in many situations. It works especially well when it is easy to solve one of the equations for one of the variables.

The substitution method is very similar to the substitution method that we used for systems of linear equations. The steps are listed below for reference.

Solve a System of Nonlinear Equations by Substitution

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Solve one of the equations for either variable.
3. Substitute the expression from Step 2 into the other equation.
4. Solve the resulting equation.
5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
6. Write each solution as an ordered pair.
7. Check that each ordered pair is a solution to **both** original equations.

Example 3.3.2.8

Solve the system by using substitution: $\begin{cases} 9x^2+y^2=9 \\ y=3x-3 \end{cases}$

Solution

Identify each graph. In this example, the first equation is one we haven't graphed by hand but you may graph this on Desmos. It is called an ellipse.	$\begin{cases} 9x^2+y^2=9 & \text{ellipse} \\ y=3x-3 & \text{line} \end{cases}$
Sketch the possible options for intersection of an ellipse and a line.	
The equation $y=3x-3$ is solved for y.	$y=3x-3$
Substitute $3x-3$ for y in the first equation.	$9x^2+y^2=9$ $9x^2+(3x-3)^2=9$
Solve the equation for x.	$9x^2+9x^2-18x+9=9$ $18x^2-18x=0$ $18x(x-1)=0$ $x=0$ or $x=1$

Substitute $x=0$ and $x=1$ into $y=3x-3$ to find y-.	$y=3x-3 \quad y=3x-3$ $y=3\cdot 0-3 \quad y=3\cdot 1-3$ $y=-3 \quad y=0$
	The ordered pairs are $(0,-3)$ and $(1,0)$.
Check **both** ordered pairs in **both** equations.	$(0,-3)$ $9x^2+y^2=9 \quad y=3x-3$ $9\cdot 0^2+(-3)^2\stackrel{?}{=}9 \quad -3\stackrel{?}{=}3\cdot 0-3$ $0+9\stackrel{?}{=}9 \quad -3\stackrel{?}{=}0-3$ $9=9 \quad -3=-3$ $(1,0)$ $9x^2+y^2=9 \quad y=3x-3$ $9\cdot 1^2+(0)^2\stackrel{?}{=}9 \quad 0\stackrel{?}{=}3\cdot 1-3$ $9+0\stackrel{?}{=}9 \quad 0\stackrel{?}{=}3-3$ $9=9 \quad 0=0$
	The solutions are $(0,-3),(1,0)$.

Try It 3.3.2.9

Solve the system by using substitution: $\begin{cases} x^2+9y^2=9 \\ y=\frac{1}{3}x-3 \end{cases}$

Answer

No solution

Try It 3.3.2.10

Solve the system by using substitution: $\begin{cases} 4x^2+y^2=4 \\ y=x+2 \end{cases}$

Answer

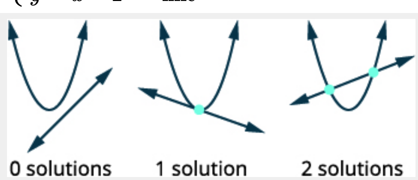
$\left(-\frac{4}{5},\frac{6}{5}\right),(0,2)$

So far, each system of nonlinear equations has had at least one solution. The next example will show another option.

Example 3.3.2.11

Solve the system by using substitution: $\begin{cases} x^2-y=0 \\ y=x-2 \end{cases}$

Solution

Identify each graph.	$\begin{cases} x^2-y=0 & \text{parabola} \\ y=x-2 & \text{line} \end{cases}$
Sketch the possible options for intersection of a parabola and a line.	
The equation $y=x-2$ is solved for y.	$y=x-2$
	$x^2-y=0$

Substitute $x-2$ for y in the first equation.	$x^2-(x-2)=0$
Solve the equation for x.	$x^2-x+2=0$
This doesn't factor easily, so we can check the discriminant.	
	The discriminant is negative, so there is no real solution. The system has no solution.

Try It 3.3.2.12

Solve the system by using substitution: $\begin{cases} x^2-y=0 \\ y=2x-3 \end{cases}$

Answer

No solution

Try It 3.3.2.13

Solve the system by using substitution: $\begin{cases} y^2-x=0 \\ y=3x-2 \end{cases}$

Answer

$\left(\frac{4}{9},-\frac{2}{3}\right),(1,1)$

Solve a System of Nonlinear Equations Using Elimination

When we studied systems of linear equations, we used the method of elimination to solve the system. We can also use elimination to solve systems of nonlinear equations. It works well when the equations have both variables squared. When using elimination, we try to make the coefficients of one variable to be opposites, so when we add the equations together, that variable is eliminated.

The elimination method is very similar to the elimination method that we used for systems of linear equations. The steps are listed for reference.

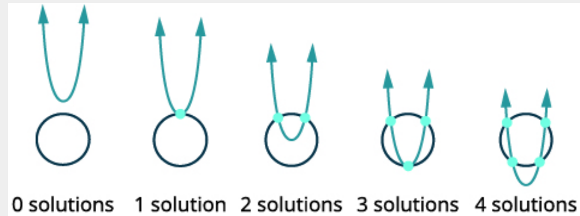
Solve a System of Equations by Elimination

1. Identify the graph of each equation. Sketch the possible options for intersection.
2. Write both equations in standard form.
3. Make the coefficients of one variable opposites.
   Decide which variable you will eliminate.
   Multiply one or both equations so that the coefficients of that variable are opposites.
4. Add the equations resulting from Step 3 to eliminate one variable.
5. Solve for the remaining variable.
6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
7. Write each solution as an ordered pair.
8. Check that each ordered pair is a solution to **both** original equations.

Example 3.3.2.14

Solve the system by elimination: $\begin{cases} x^2+y^2=4 \\ x^2-y=4 \end{cases}$

Solution

Identify each graph.	$\begin{cases} x^2+y^2=4 & \text{circle} \\ x^2-y=4 & \text{parabola} \end{cases}$
Sketch the possible options for intersection of a circle and a parabola.	
Both equations are in standard form.	$\begin{cases} x^2+y^2=4 \\ x^2-y\ =4 \end{cases}$
To get opposite coefficients of x^2, we will multiply the second equation by -1.	$\begin{cases} x^2+y^2=4 \\ -1(x^2-y)\ =-1(4) \end{cases}$
Simplify.	$\begin{cases} x^2+y^2=4 \\ -x^2+y\ =4 \end{cases}$
Add the two equation to eliminate x^2/	$\begin{cases} x^2+y^2=4 \\ -x^2+y\ =4 \end{cases}$ $y^2+y\ =0$
Solve for y.	$y(y+1)=0$
	$y=0 \quad y+1=0$ $y=-1$
Substitute $y=0$ and $y=-1$ into one of the original equations. Then solve for x.	$y=0 \quad y=-1$
	$x^2-y\ =4 \qquad x^2-y\ =4$ $x^2-0\ =4 \qquad x^2-(-1)=4$ $x^2=4 \qquad x^2=3$ $x=\pm2 \qquad x=\pm\sqrt{3}$
Write each solution as an ordered pair.	The ordered pairs are $(-2,0)(2,0)$. $(\sqrt{3},-1)(-\sqrt{3},-1)$
Check that each ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,0),(2,0),(\sqrt{3},-1)$, and $(-\sqrt{3},-1)$.

Try It 3.3.2.15

Solve the system by elimination: $\begin{cases} x^2+y^2=9 \\ x^2-y=9 \end{cases}$

Answer

$(-3,0),(3,0),(-2\sqrt{2},-1),(2\sqrt{2},-1)$

Try It 3.3.2.16

Solve the system by elimination: $\begin{cases} x^2+y^2=1 \\ -x+y^2=1 \end{cases}$

Answer

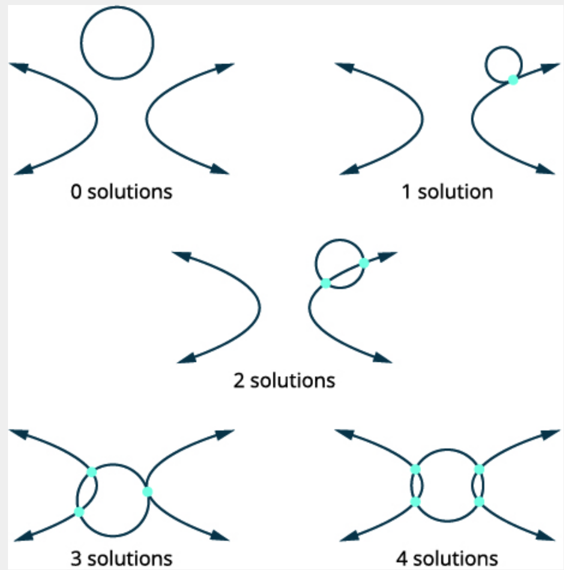
$(-1,0),(0,1),(0,-1)$

There are also four options when we consider a circle and a hyperbola.

Example 3.3.2.17

Solve the system by elimination: $\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$

Solution

Identify each graph. The second equation is one we have not graphed (and we don't need to). To get an idea of the number of solutions we need only that it is a hyperbola.	$\begin{cases} x^2+y^2=7 & \text{circle} \\ x^2-y^2=1 & \text{hyperbola} \end{cases}$
Sketch the possible options for intersection of a circle and hyperbola.	
Both equations are in standard form.	$\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$
The coefficients of y^2 are opposite, so we will add the equations.	$\begin{cases} x^2+y^2=7 \\ x^2-y^2=1 \end{cases}$ $2x^2=8$
Simplify.	$x^2=4$ $x=\pm 2$ $x=2 \quad x=-2$
Substitue $x=2$ and $x=-2$ into one of the original equations. Then solve for y.	$x^2+y^2=7 \qquad x^2+y^2=7$ $2^2+y^2=7 \qquad (-2)^2+y^2=7$ $4+y^2=7 \qquad 4+y^2=7$ $y^2=3 \qquad y^2=3$ $y=\pm\sqrt{3} \qquad y=\pm\sqrt{3}$
Write each solution as an ordered pair.	The ordered pairs are $(-2,\sqrt{3}),(-2,-\sqrt{3})$, $(2,\sqrt{3})$, and $(2,-\sqrt{3})$.
Check that the ordered pair is a solution to **both** original equations.	
We will leave the checks for each of the four solutions to you.	The solutions are $(-2,\sqrt{3}),(-2,-\sqrt{3}),(2,\sqrt{3})$, and $(2,-\sqrt{3})$.

Try It 3.3.2.18

Solve the system by elimination: $\begin{cases} x^2+y^2=25 \\ y^2-x^2=7 \end{cases}$

Answer

$(-3,-4),(-3,4),(3,-4),(3,4)$

Try It 3.3.2.19

Solve the system by elimination: $\begin{cases} x^2+y^2=4 \\ x^2-y^2=4 \end{cases}$

Answer

$(-2,0),(2,0)$

Use a System of Nonlinear Equations to Solve Applications

Systems of nonlinear equations can be used to model and solve many applications. We will look at an everyday geometric situation as our example.

Example 3.3.2.20

The difference of the squares of two numbers is 15. The sum of the numbers is 5. Find the numbers.

Solution

Identify what we are looking for.	Two different numbers.
Define the variables.	x=first number y=second number
Translate the information into a system of equations.	
First sentence.	The difference of the squares of two numbers is 15.
	$x^2-y^2=15$
Second sentence.	The sum of the numbers is 5.
	$x+y=5$
Solve the system by substitution.	$x^2-y^2=15$ $x+y=5$
Solve the second equation for x.	$x=5-y$
Substitute x into the first equation.	$x^2-y^2=15$
	$(5-y)^2-y^2=15$
Expand and simplify.	$(25-10y+y^2)-y^2=15$
	$25-10y+y^2-y^2=15$ $25-10y=15$
Solve for y.	$-10y=-10$
	$y=1$
Substitute back into the second equation.	$x+y=5$
	$x+1=5$ $x=4$
	The numbers are 1 and 4.

Try It 3.3.2.21

The difference of the squares of two numbers is -20. The sum of the numbers is 10. Find the numbers.

Answer

4 and 6

Try It 3.3.2.22

The difference of the squares of two numbers is 35. The sum of the numbers is -1. Find the numbers.

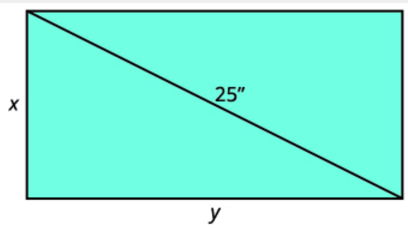
Answer

-18 and 17

Example 3.3.2.23

Myra purchased a small 25" TV for her kitchen. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 300 square inches. What are the length and width of the TV screen?

Solution

Identify what we are looking for.	The length and width of the rectangle.
Define the variables.	Let x= width of the rectangle y=length of the rectangle
Draw a diagram to help visualize the situation.	
	Area is 300 square inches.
Translate the information into a system of equations.	The diagonal of the right triangle is 25 inches.
	$x^2+y^2=25^2$ $x^2+y^2=625$
	The area of the rectangle is 300 square inches.
	$xy=300$ $x^2+y^2=625$
Solve the system using substitution.	$xy=300$
Solve the second equation for x.	$x=\dfrac{300}{y}$
Substitute x into the first equation.	$x^2+y^2=625$ $\left(\dfrac{300}{y}\right)+y^2=625$
Simplify.	$\dfrac{90000}{y^2}+y^2=625$
Multiply by y^2 to clear the fractions.	$90000+y^4=625y^2$
Put in standard form.	$y^4-625y^2+90000=0$
Solve by factoring.	$(y^2-225)(y^2-400)=0$ $y^2-225=0 \quad \text{or} \quad y^2-400=0$

	$y^2 - 225 = 0 \quad y^2 - 400 = 0$
	$y^2 = 225 \quad y^2 = 400$ $y = \pm 15 \quad y = \pm 20$
Since y is a side of the rectangle, we discard the negative values.	$y = 15 \quad y = 20$
Substitute back into the second equation.	$x \cdot y = 300 \quad x \cdot y = 300$
	$x \cdot 15 = 300 \quad x \cdot 20 = 300$ $x = 20 \quad x = 15$
	If the length is 15 inches, the width is 20 inches.
	If the length is 20 inches, the width is 15 inches.

Try It 3.3.2.24

Edgar purchased a small 20” TV for his garage. The size of a TV is measured on the diagonal of the screen. The screen also has an area of 192 square inches. What are the length and width of the TV screen?

Answer

If the length is 12 inches, the width is 16 inches. If the length is 16 inches, the width is 12 inches.

Try It 3.3.2.25

The Harper family purchased a small microwave for their family room. The diagonal of the door measures 15 inches. The door also has an area of 108 square inches. What are the length and width of the microwave door?

Answer

If the length is 12 inches, the width is 9 inches. If the length is 9 inches, the width is 12 inches.

? Writing Exercises 3.3.2.26

1. What is the difference between a linear and a nonlinear equation?
2. What does it mean to solve a system of equations?
3. What is the graphical description of solving a system of nonlinear equations?
4. How can this be helpful in solving the system?
5. Must we check our answers?
6. How is this similar or different from solving a system of two linear equations?

Exit Problem 3.3.2.27

Solve the system of equations.

$$\begin{cases} 3x^2 + 4y^2 &= 16 \\ 2x^2 - 3y^2 &= 5 \end{cases}$$

Key Concepts

- **How to solve a system of nonlinear equations by graphing.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Graph the first equation.
  3. Graph the second equation on the same rectangular coordinate system.
  4. Determine whether the graphs intersect.
  5. Identify the points of intersection.
  6. Check that each ordered pair is a solution to both original equations.

- **How to solve a system of nonlinear equations by substitution.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Solve one of the equations for either variable.
  3. Substitute the expression from Step 2 into the other equation.
  4. Solve the resulting equation.
  5. Substitute each solution in Step 4 into one of the original equations to find the other variable.
  6. Write each solution as an ordered pair.
  7. Check that each ordered pair is a solution to **both** original equations.
- **How to solve a system of equations by elimination.**
  1. Identify the graph of each equation. Sketch the possible options for intersection.
  2. Write both equations in standard form.
  3. Make the coefficients of one variable opposites.
     Decide which variable you will eliminate.
     Multiply one or both equations so that the coefficients of that variable are opposites.
  4. Add the equations resulting from Step 3 to eliminate one variable.
  5. Solve for the remaining variable.
  6. Substitute each solution from Step 5 into one of the original equations. Then solve for the other variable.
  7. Write each solution as an ordered pair.
  8. Check that each ordered pair is a solution to **both** original equations.

CHAPTER OVERVIEW

4: Introduction to Trigonometry and Transcendental Expressions

SECTION OVERVIEW

4.1: Trigonometric Expressions

Topic hierarchy

4.1.1: Angles and Triangles

Learning Objectives

By the end of this section, you will be able to:

- Communicate about triangles.
- Know what it means for two triangles to be similar.
- Know what a right triangle is and what relationship its sides have.
- Determine the measure of a missing interior angle of a triangle if the other two are known.

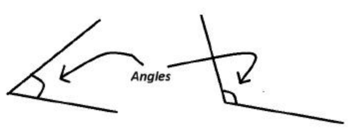
Be Prepared

Before you get started, take this readiness quiz.

1. Solve $x+40=18$.
2. Reduce $\dfrac{90}{2}$.

In elementary geometry, angles are always considered to be positive and not larger than 360°. You also learned that the sum of the angles in a triangle equals 180°, and that an isosceles triangle is a triangle with two sides of equal length. Recall that in a right triangle one of the angles is a right angle. Thus, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles).

Angles are what is formed when two rays, lines or line segments come together. Here are two examples:



In the above the angle on the right seems 'bigger' than the angle on the left. In order to quantify this observation, we first clarify some vocabulary.

Properly defining an angle first requires that we define a *ray*. A ray consists of one point on a line and all points extending in one direction from that point. The first point is called the endpoint of the ray. We can refer to a specific ray by stating its *endpoint* and any other point on it. The ray in Figure 4.1.1.1can be named as ray EF, or in symbol form $\overrightarrow{EF}$.

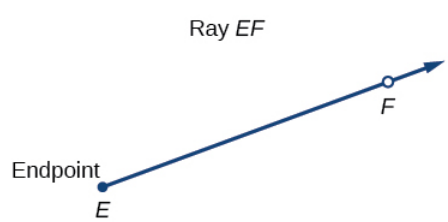


Figure 4.1.1.1

An *angle* is the union of two rays having a common endpoint. The endpoint is called the **vertex** of the angle, and the two rays are the sides of the angle. The angle in Figure 4.1.1.2is formed from $\overrightarrow{ED}$ and $\overrightarrow{EF}$. Angles can be named using a point on each ray and the vertex, such as angle DEF, or in symbol form $\angle DEF$ or, if no confusion will arise, it can be labeled by the vertex alone $(\angle E)$ or some other letter that may be indicated on the picture.

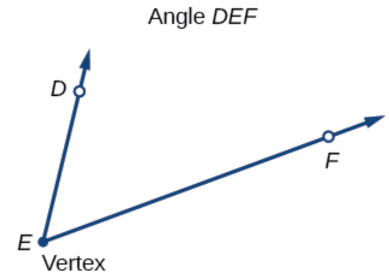


Figure 4.1.1.2

We can think of angle creation is a dynamic process. We start with two rays lying on top of one another. We leave one fixed in place, and rotate the other. The fixed ray is the **initial side,** and the rotated ray is the **terminal side**. In order to identify the different sides, we indicate the rotation with a small arc and arrow close to the vertex as in Figure 4.1.1.3.

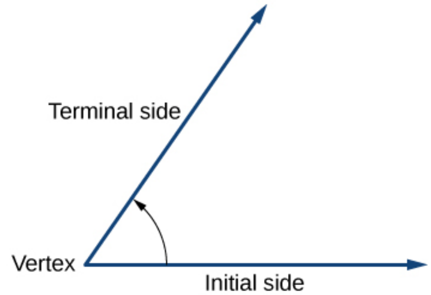
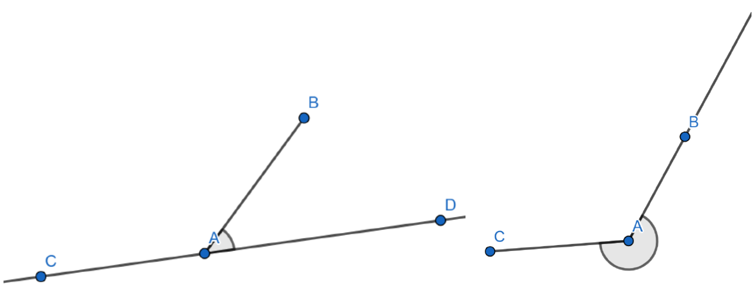


Figure 4.1.1.3

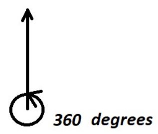
Here are some examples with the angles indicated by an arc.

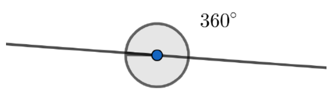


To quantify the comparison at the beginning of this section, we must be able to measure them.

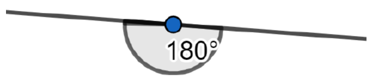
The **measure of an angle** is the amount of rotation from the initial side to the terminal side. Probably the most familiar unit of angle measurement is the degree. To begin we agree that the measure of an angle formed when the two segments are identical forms a 360 degree angle.

So, one **degree** is $\frac{1}{360}$ of a circular rotation. An angle measured in degrees should always include the unit "degrees" after the number, or include the degree symbol $^{\text{o}}$. For example, 90 degrees $= 90^{\text{o}}$.

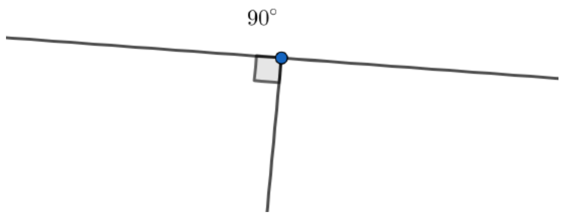




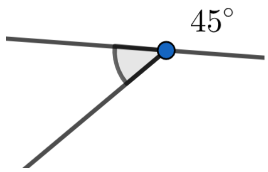
Half of that angle is then 180°. This is called a straight angle:



Half of a straight angle has a measure of 90° and is called a right angle:

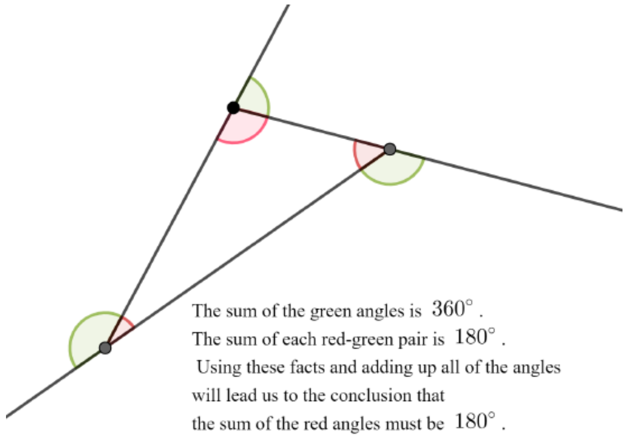


Half of a right angle has a measure of 45°:

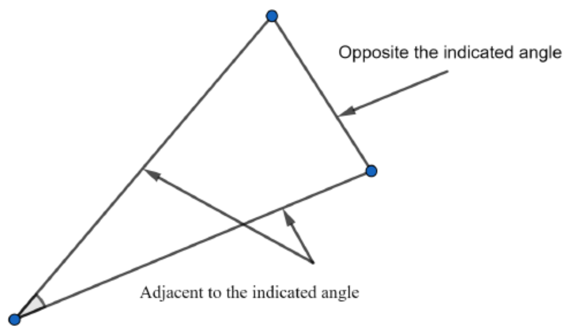


And so on. There is an instrument, called a protractor that helps us to measure angles that are drawn on paper.

When three line segments are drawn in a way that encloses an area we have what is called a triangle. There are six angles formed by the sides of a triangle, three are outside and three are inside the triangle. We could, if we had an accurate picture of a triangle, measure these angles with a protractor. If we measured the interior angles and added them up, we would discover a remarkable fact that the sum is 180˚. We should at this point, draw some triangles and measure them to check this out. We show a picture giving an idea of why this is true here:

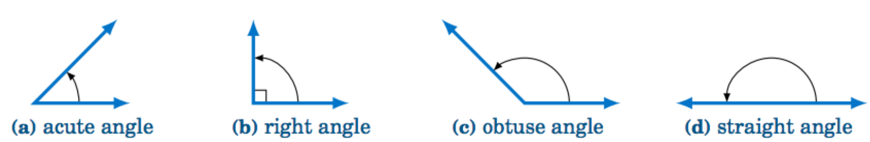


For any triangle, there are 3 side lengths and 3 (interior) angle measurements. We say that an angle is adjacent to a side (or the side is adjacent to the angle) if the side is one of the two forming the angle. We say that an angle is opposite a side (or the side is opposite the angle) if the side is not adjacent to the angle.



The sides and angles are sometimes labeled with variables representing their measure. Often (but not always) the the variable representing the measure of an angle is a greek letter (usually θ or ψ).

An angle is **acute** if it is between 0° and 90° and is **obtuse** if it is between 90° and 180°.

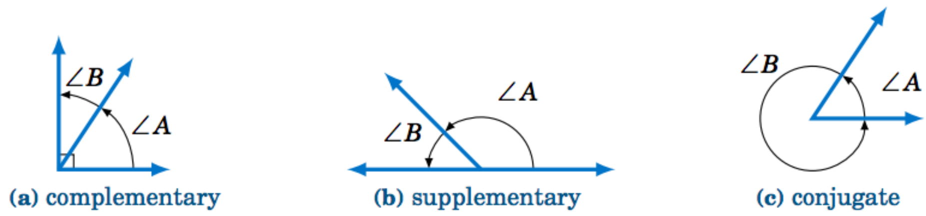


In elementary geometry, angles are always considered to be positive and not larger than 360°. For now we will only consider such angles. The following definitions are also sometimes used when discussing angles.

It is often the case that the name of the angle is used as the value. For example, if we call the angle $\angle A$ and the measure of angle $\angle A$ is 45 degrees (notated 45°) then we may also say $\angle A = 45^\circ$.

a. Two acute angles are **complementary** if their sum equals 90°. In other words, if $0^\circ \leq \angle A, \angle B \leq 90^\circ$ then $\angle A$ and $\angle B$ are complementary if $\angle A + \angle B = 90^\circ$.
b. Two angles between 0° and 180° are **supplementary** if their sum equals 180°. In other words, if $0^\circ \leq \angle A, \angle B \leq 180^\circ$ then $\angle A$ and $\angle B$ are supplementary if $\angle A + \angle B = 180^\circ$.

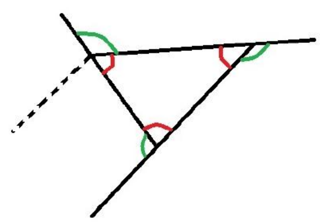
c. Two angles between 0° and 360° are **conjugate** if their sum equals 360°. In other words, if $0^\circ \leq \angle A, \angle B \leq 360^\circ$ then $\angle A$ and $\angle B$ are conjugate if $\angle A + \angle B = 360^\circ$.



Sometimes the measure of angles are named by the vertex or they may be given variable names like x, y, or z. Often they are given names from the greek alphabet, most commonly $\theta, \phi, \psi, \alpha, \beta$, or γ (theta, phi, alpha, beta, or gamma, respectively).

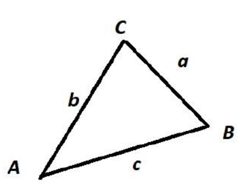
Triangles

When three line segments are drawn in a way that encloses an area we have what is called a triangle. There are six angles formed by the sides of a triangle, three are outside and three are inside the triangle. We could, if we had an accurate picture of a triangle, measure these angles with a protractor. If we measured the interior angles and added them up, we would discover a remarkable fact that the sum is 180^o. We should at this point, draw some triangles and measure them to check this out. We show a picture giving an idea of why this is true here:



The sum of the green angles (outside) is 360° (noting corresponding angles with the rays parallel to the dotted ray) and the sum of each of the red green pairs is 180°. So, the sum of all of the red green pairs is $3(180)^\circ$ which is the same as the sum of the red angles plus the sum of the green angles (360°). It follows that the sum of the red angles is $(3(180) - 360)^\circ = 180^\circ$.

For any triangle, there are 3 side lengths (here, a, b, and c) and 3 angle measurements (here A, B, and C):



Recall that we will sometimes identify the measure of an angle with the angle itself. We will do the same with sides. For example, we may refer to side b when b is the length of the side to which we refer. So, in the diagram above, for example we may refer to the angle A as well as assign a value to A which is the measure of that angle and we may refer to the side a as well as the length of that side as a.

We say that an angle is adjacent to a side (or the side is adjacent to the angle) if the side is one of the two forming the angle.

For, example,

- side b is adjacent to angle A and adjacent to angle C
- angle B is adjacent to side a and adjacent to side c

We say that an angle is opposite a side (or the side is opposite the angle) if the side is not adjacent to the angle.

In the above picture, we can say, for example,

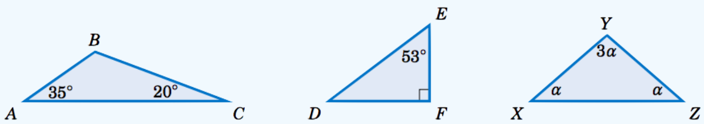
- side a is opposite angle A
- angle B is opposite side b
- side c is opposite angle C.

Recall the following definitions from elementary geometry:

An **isosceles triangle** is a triangle with two sides of equal length and a **right triangle** is a triangle where one of the angles is a right angle. Because the sum of the interior angles of a triangle equals 180°, in a right triangle one of the angles is 90° and the other two angles are acute angles whose sum is 90° (i.e. the other two angles are complementary angles). We can use our knowledge of solving equations to discover measures of angles if enough information is known.

Example 4.1.1.4

For each triangle below, determine the unknown angle(s):



Note: We will sometimes refer to the angles of a triangle by their vertex points. For example, in the first triangle above we will simply refer to the angle $\angle BAC$ as angle A.

Solution:

For triangle $\triangle ABC$, $A = 35^\circ$ and $C = 20^\circ$, and we know that $A + B + C = 180^\circ$, so

$$35^\circ + B + 20^\circ = 180^\circ \Rightarrow B = 180^\circ - 35^\circ - 20^\circ \Rightarrow \boxed{B = 125^\circ}.$$

For the right triangle $\triangle DEF$, $E = 53^\circ$ and $F = 90^\circ$, and we know that the two acute angles D and E are complementary, so

$$D + E = 90^\circ \Rightarrow D = 90^\circ - 53^\circ \Rightarrow \boxed{D = 37^\circ}.$$

For triangle $\triangle XYZ$, the angles are in terms of an unknown number α, but we do know that $X + Y + Z = 180^\circ$, which we can use to solve for α and then use that to solve for $X, Y,$ and Z:

$$\alpha + 3\alpha + \alpha = 180^\circ \Rightarrow 5\alpha = 180^\circ \Rightarrow \alpha = 36^\circ \Rightarrow \boxed{X = 36^\circ,\ Y = 3 \times 36^\circ = 108^\circ,\ Z = 36^\circ}$$

Try It 4.1.1.5

If $\triangle ABC$ has $A = 25^\circ$ and $B = 15^\circ$, then what is the measure of angle C?

Answer

$C = 140^\circ$

? Try It 4.1.1.6

If $\triangle ABC$ has $B = 125^\circ$ and $C = 10^\circ$, then what is the measure of angle A?

Answer

Add texts here. $A = 65^\circ$.

✔ Example 4.1.1.7 *Thales' Theorem*

Thales' Theorem states that if A, B, and C are (distinct) points on a circle such that the line segment $\overline{AB}$ is a diameter of the circle, then the angle $\angle ACB$ is a right angle (see Figure 1.1.3(a)). In other words, the triangle $\triangle ABC$ is a right triangle.

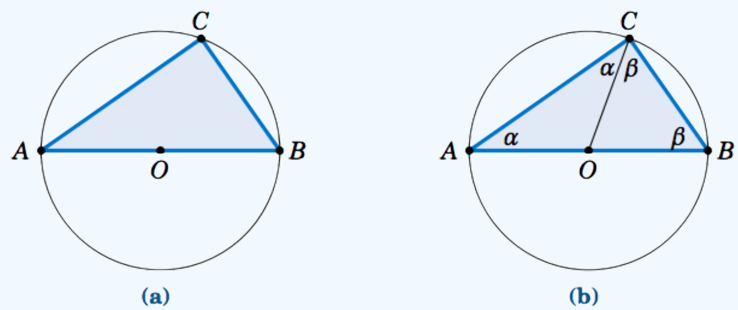


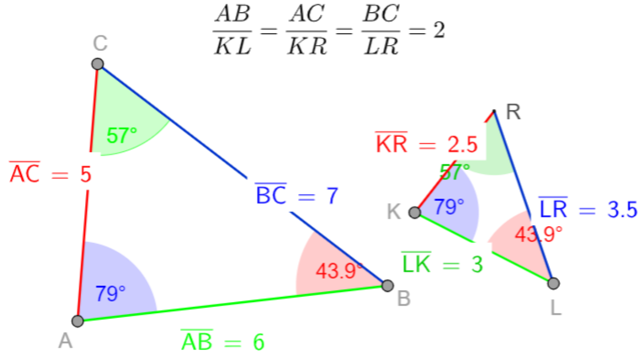
Figure 1.1.4 Thales' Theorem: $\angle ACB = 90^\circ$

To prove this, let O be the center of the circle and draw the line segment $\overline{OC}$, as in Figure 1.1.3(b). Let $\alpha = \angle BAC$ and $\beta = \angle ABC$. Since $\overline{AB}$ is a diameter of the circle, $\overline{OA}$ and $\overline{OC}$ have the same length (namely, the circle's radius). This means that $\triangle OAC$ is an isosceles triangle, and so $\angle OCA = \angle OAC = \alpha$. Likewise, $\triangle OBC$ is an isosceles triangle and $\angle OCB = \angle OBC = \beta$. So we see that $\angle ACB = \alpha + \beta$. And since the angles of $\triangle ABC$ must add up to 180°, we see that $180^\circ = \alpha + (\alpha + \beta) + \beta = 2(\alpha + \beta)$, so $\alpha + \beta = 90^\circ$. Thus, $\angle ACB = 90^\circ$.

Similar Triangles

Similar triangles are triangles that have the same shape, i.e., the same angles. Two angles of two similar triangles (one angle from each triangle) are called corresponding if they have the same measure, and sides opposite corresponding angles are called corresponding sides. Ratios of corresponding sides is the same for each pair as seen in the figure below. In the figure, the sides are labeled (order is unimportant) by listing the names of the two points that define the side and to avoid clutter in the ratios, the bar indicating the segment is dropped.

$$\frac{AB}{KL} = \frac{AC}{KR} = \frac{BC}{LR} = 2$$



See https://www.geogebra.org/m/z9NHEdrz for the interactive version of the figure above. In this instance, notice that the side opposite the 57° angle is 6 on the larger and 3 on the smaller. The larger length is twice the smaller length. The same is true of the other lengths. This is confirmed by the ratio calculation the triangles.

Looking at the first equality above we also see that $\frac{\overline{KL}}{\overline{KR}} = \frac{\overline{AB}}{\overline{AC}}$ since $\frac{2.5}{3} = \frac{2.5 \cdot 2}{3 \cdot 2} = \frac{5}{6}$ (the factor of increase of lengths cancels in the fraction).

So, the ratios of lengths remain the same if the size is changed.

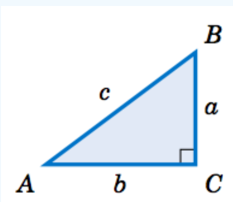
Right Triangles

While we have used what is known as the Pythagorean Theorem, or Pythagoras' Theorem, in this section we provide a proof. We recall that a right triangle is one with a 90° angle. We can conclude that the sum of the measure of the other two angles is 90°.

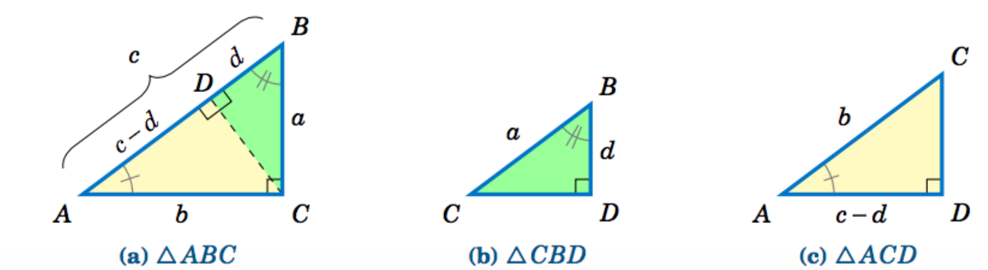
Like the angles, the sides of a right triangle are also related so that knowing the length of two sides leads to discovery of the length of the third side. Unlike the angle case where we encountered linear equations, here we will encounter a quadratic equation by using the **Pythagorean Theorem**:

Theorem 4.1.1.8 Pythagorean Theorem

The square of the length of the hypotenuse of a right triangle is equal to the sum of the squares of the lengths of its legs. So if the triangle is labeled in the following way, we have $c^2 = a^2 + b^2$:



Here is one way to see this. Create a fourth segment so that we have three triangles and label them as shown.



Similar triangles $\triangle ABC, \triangle CBD, \triangle ACD$

Two triangles are **similar** if their corresponding angles are equal, and that similarity implies that corresponding sides are proportional. Thus, since $\triangle ABC$ is similar to $\triangle CBD$, by proportionality of corresponding sides we see that

$$\overline{AB} \text{ is to } \overline{CB} \text{ (hypotenuses) as } \overline{BC} \text{ is to } \overline{BD} \text{ (vertical legs)} \quad \Rightarrow \quad \frac{c}{a} = \frac{a}{d} \quad \Rightarrow \quad cd = a^2 .$$

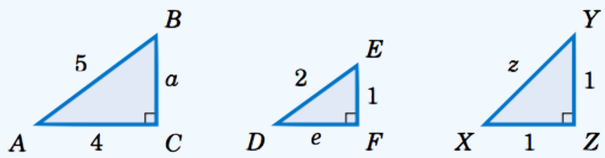
Since $\triangle ABC$ is similar to $\triangle ACD$, comparing horizontal legs and hypotenuses gives

$$\frac{b}{c-d} = \frac{c}{b} \quad \Rightarrow \quad b^2 = c^2 - cd = c^2 - a^2 \quad \Rightarrow \quad a^2 + b^2 = c^2 . \textbf{ QED}$$

Note: The symbols $\perp$ and $\sim$ denote perpendicularity and similarity, respectively. For example, in the above proof we had $\overline{CD} \perp \overline{AB}$ and $\triangle ABC \sim \triangle CBD \sim \triangle ACD$.

Example 4.1.1.9

For each right triangle below, determine the length of the unknown side:



Solution:

For triangle $\triangle ABC$, the Pythagorean Theorem says that

$$a^2 + 4^2 = 5^2 \quad \Rightarrow \quad a^2 = 25 - 16 = 9 \quad \Rightarrow \quad \boxed{a = 3}.$$

For triangle $\triangle DEF$, the Pythagorean Theorem says that

$$e^2 + 1^2 = 2^2 \quad \Rightarrow \quad e^2 = 4 - 1 = 3 \quad \Rightarrow \quad \boxed{e = \sqrt{3}}.$$

For triangle $\triangle XYZ$, the Pythagorean Theorem says that

$$1^2 + 1^2 = z^2 \quad \Rightarrow \quad z^2 = 2 \quad \Rightarrow \quad \boxed{z = \sqrt{2}}.$$

? Try It 4.1.1.10

If $\triangle ABC$ has $C = 90^\circ$, $b = 10$ in and $c = 15$ in, then what is the length of a?

Answer

$a = 5\sqrt{5}$ in

? Try It 4.1.1.11

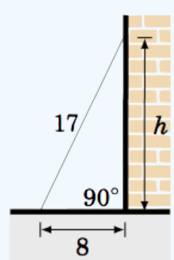
If $\triangle ABC$ has $a = 12,\ b = 10$, and $c = 4$ (all measured in the same unit), then is $\triangle ABC$ a right triangle?

Answer

No, it can not be a right triangle.

✓ Example 4.1.1.12

A 17 ft ladder leaning against a wall has its foot 8 ft from the base of the wall. At what height is the top of the ladder touching the wall?



Solution

Let h be the height at which the ladder touches the wall. We can assume that the ground makes a right angle with the wall, as in the picture on the right. Then we see that the ladder, ground, and wall form a right triangle with a hypotenuse of length 17 ft (the length of the ladder) and legs with lengths 8 ft and h ft. So by the Pythagorean Theorem, we have

$$h^2 + 8^2 = 17^2 \quad \Rightarrow \quad h^2 = 289 - 64 = 225 \quad \Rightarrow \quad \boxed{h = 15 \text{ ft}}.$$

Try It 4.1.1.13

You and your friend are talking at a street corner and upon parting, you walk north at 2 mph and your friend walks east at 3 mph. You have a walkie talkie system that you are using that has a range of 5 miles. Approximately, how long will you be able to continue talking? Explain reasoning clearly.

Answer

Answers vary. About 83 minutes.

Try It 4.1.1.14

How long of a ladder do you need if you need to reach a light attached to a wall 10 feet from the ground if for stability's sake you need the base of the ladder to be 2.5 feet from the base of the wall? Be sure to draw a picture that represents the situation.

Answer

The ladder should be $\dfrac{5\sqrt{17}}{2}$ feet (about 10.3 feet) . Here we have assumed the ladder meets the wall 10 feet above the ground.

Written Exercises 4.1.1.15

1. How can you recognize if two triangles are similar?
2. Do you know of examples in your life where similar triangles (or similar objects) appear?
3. Construct a triangle which is not a right triangle and show that the Pythagorean Theorem is not true.
4. Give an example of an acute angle and its complement. Draw a picture showing the relationship.

Exit Problem

1. If two of the three interior angles of a triangle are 40 degrees, what is the third angle? Draw a picture of the situation. Are all triangles that you could draw similar to one another?

2. If the legs of a right triangle measure 7 in and 9 in, how long is the hypotenuse? Draw a picture of this situation.

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- A right angle is $\frac{1}{4}$ of a complete revolution.
- A triangle is a made by connecting 3 points (vertices) pairwise by line segments.
- A right triangle is a triangle with one of the interior angles being a right angle.
- Interior angles of a triangle add up to 180°.
- Two triangles are similar if they are the same shape but not necessarily the same size.
- Two triangles which are similar have equal corresponding ratios of sides.
- Pythagoras' Theorem

4.1.2: Right Triangles and Trigonometric Ratios

Learning Objectives

By the end of this section, you will be able to:

- Understand what it means for two right triangles to be similar to each other.
- Be able to produce two special triangles.
- Be able to use ratios to determine missing side lengths of a triangle (with and without (if possible) a calculator).
- Be able to find missing angles of a right triangle if the lengths of the sides are known.

Be Prepared

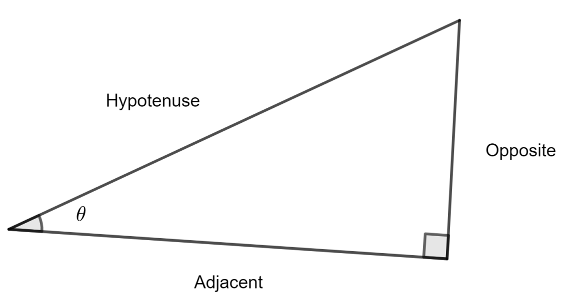
Before you get started, take this readiness quiz.

1. Solve $\frac{2}{x}=\frac{3}{5}$.
2. Solve $\frac{x}{4}=\frac{4}{5}$.
3. Draw a picture of a right triangle and identify its hypotenuse.

Similar Triangles and Trigonometric Ratios

A right triangle is a triangle with one of the angles being 90^{o}. This is a special angle and the side opposite is called the hypotenuse. There are two other angles which are not distinguished but we may name them or assign a variable to represent their measure.

So if we label one of the two right angles with its measure θ then of the two undistinguished sides (not the hypotenuse) there is one that is opposite and one that is adjacent.



Two right triangles are similar if one (and therefore both) of the non-right interior angles are equal! So, for example, all right triangles with an interior angle of 40^{o} are similar to each other. Recall that if two triangles are similar, the corresponding ratios are equal. We consider such ratios (determined only by the shape/interior angles) for a right triangle.

There are 6 ratios that we can consider (three are reciprocals of the other three). We would like to name these and to do this we will label an angle θ representing
its measure. Then the three sides now have names: hypotenuse, opposite, and adjacent. The six ratios are opposite/hypotenuse, adjacent/hypotenuse, and opposite/adjacent together with their reciprocals.

The names though are with reference to one of the angles. We then call these ratios:

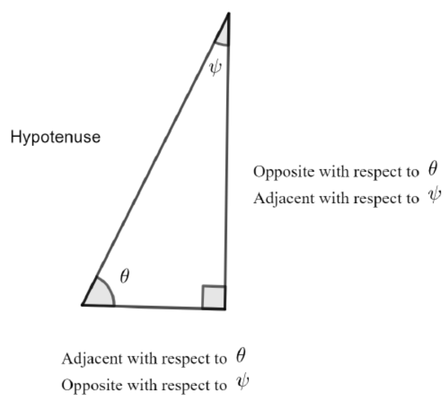
- $\sin\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}$ (read 'sine of theta')

- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ (read 'cosine of theta')
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ (read 'tangent of theta')

and their corresponding reciprocals:

- $\csc\theta = \dfrac{1}{\sin\theta}$ (read 'cosecant of theta')
- $\sec\theta = \dfrac{1}{\cos\theta}$ (read 'secant of theta')
- $\cot\theta = \dfrac{1}{\tan\theta}$ (read 'cotangent of theta')

Notice that if we had used the other non-right angle as our reference angle instead our ratios would be formed the same way but adjacent and opposite are
switched and so the names would change accordingly (putting a co- or removing a co-):



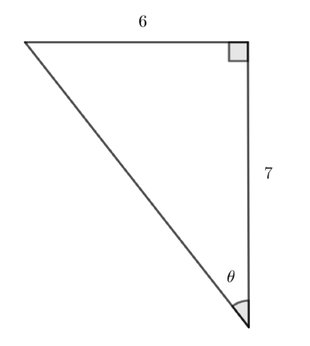
$\sin\theta = \cos\psi, \cos\theta = \sin\psi, \tan\theta = \cot\psi, \ldots$ etc

So, for instance, when seeing the expression $\sin\theta$ we understand the reference angle is the one with measure θ and the ratio is the length of the side opposite that angle divided by the length of the hypotenuse. The word 'sin' makes no sense by itself because it lacks a reference angle. Sometimes parentheses are present and sometimes we may write $\sin\theta$ as $\sin\theta$ and should not be confused with multiplication.

In all of the examples and figures, we should not treat the figures as anything but representing relationships. So, for example, you should not think that measuring with any instrument will lead to a solution. We have also not written units.

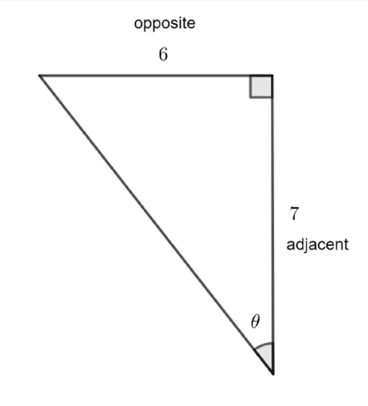
Example 4.1.2.1

Given the figure below, relate θ to the given numbers using one of the three ratios $(\sin\theta, \cos\theta, \tan\theta)$. Then find the other 5 ratios with reference angle θ.



Solution

We first identify the where the measured sides are as it relates to the marked angle.



Now we look at the ratios:

- $\sin\theta = \dfrac{\text{opposite}}{\text{hypotenuse}}$ (read 'sine of theta')
- $\cos\theta = \dfrac{\text{adjacent}}{\text{hypotenuse}}$ (read 'cosine of theta')
- $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$ (read 'tangent of theta')

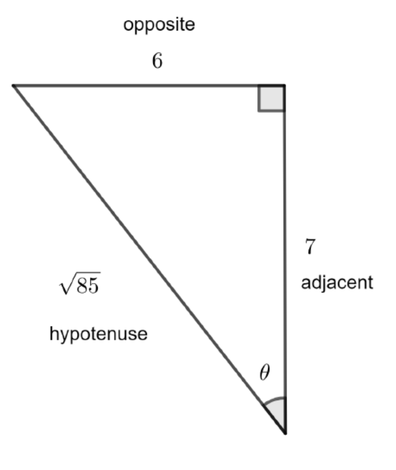
and see that the one that we could use is $\tan\theta = \dfrac{\text{opposite}}{\text{adjacent}}$. So,

$$\tan\theta = \frac{6}{7}.$$

To find the other five ratios we need the length of the hypotenuse. Since this is a right triangle, Pythagoras tells us that the square of the length of the hypotenuse is the sum of the squares of the length of the legs. If we call the length of the hypotenuse c, then, in this case,

$$c^2 = 6^2 + 7^2 = 85,$$

so that $c = \sqrt{85}$ since the length of the hypotenuse can not be negative.

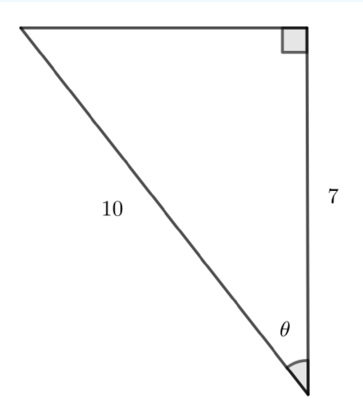


Forming the other ratios gives us

- $\sin\theta = \dfrac{6}{\sqrt{85}}$
- $\cos\theta = \dfrac{7}{\sqrt{85}}$
- $\csc\theta = \dfrac{1}{\sin(\theta)} = \dfrac{\sqrt{85}}{6}$
- $\sec\theta = \dfrac{1}{\cos(\theta)} = \dfrac{\sqrt{85}}{7}$
- $\cot\theta = \dfrac{1}{\tan(\theta)} = \dfrac{7}{6}$

? Try It 4.1.2.2

Given the figure below, relate θ to the given numbers using one of the three ratios $(\sin\theta, \cos\theta, \tan\theta)$. Then find the other 5 ratios with reference angle θ.



Answer

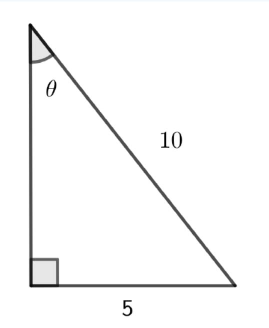
$\cos\theta = \dfrac{7}{10}$. The unknown leg length is $\sqrt{51}$.

- $\sin\theta = \dfrac{\sqrt{51}}{10}$
- $\cos\theta = \dfrac{7}{10}$

- $\tan\theta = \frac{\sqrt{51}}{7}$
- $\csc\theta = \frac{1}{\sin(\theta)} = \frac{10}{\sqrt{51}}$
- $\sec\theta = \frac{1}{\cos(\theta)} = \frac{10}{7}$
- $\cot\theta = \frac{1}{\tan(\theta)} = \frac{7}{\sqrt{51}}$

? Try It 4.1.2.3

Given the figure below, relate θ to the given numbers using one of the three ratios $(\sin\theta, \cos\theta, \tan\theta)$. Then find the other 5 ratios with reference angle θ.



Answer

$\sin\theta = \frac{5}{10} = \frac{1}{2}$ and the hypotenuse is $\sqrt{75} = 5\sqrt{3}$.

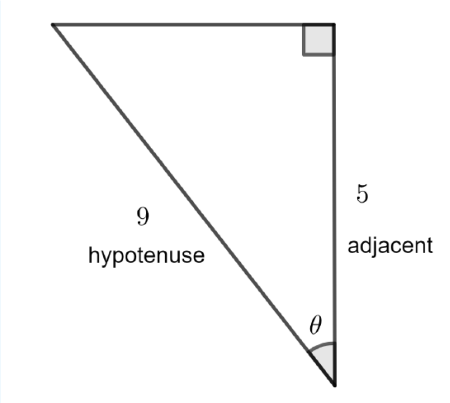
- $\sin\theta = \frac{1}{2}$
- $\cos\theta = \frac{\sqrt{3}}{2}$
- $\tan\theta = \frac{\sqrt{1}}{\sqrt{3}}$
- $\csc\theta = \frac{1}{\sin\theta} = 2$
- $\sec\theta = \frac{1}{\cos\theta} = \frac{2}{\sqrt{3}}$
- $\cot\theta = \frac{1}{\tan\theta} = \sqrt{3}$

✓ Example 4.1.2.4

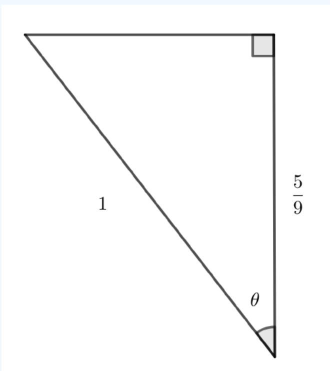
Suppose you know that for a particular angle of a right triangle, θ, that $\cos\theta = \frac{5}{9}$. Draw a picture that shows the relationship between θ, 5, and 9. Find the other ratios.

Solution

We know that the ratio $\cos\theta$ is $\frac{\text{adjacent}}{\text{hypotenuse}}$ (relative to the angle with measure θ). Since $\frac{5}{9}$ is a fraction, we will simply use the numerator as the length of the adjacent leg, and the 9 as the length of the hypotenuse:



There are many pictures you could draw, but most are a little 'unnatural'. For example, we can view $\frac{5}{9} = \frac{\frac{5}{9}}{1}$ to give



To find the other ratios we will need the length of the opposite, which we will call a. When we write down an equation for a it may be easier to use the first of the two triangles, though the ratios will be the same.

So, using the first triangle we see that a satisfies

$$9^2 = a^2 + 5^2.$$

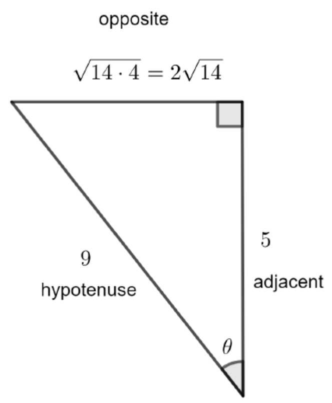
subtracting 5^2 gives us

$$a^2 = 9^2 - 5^2 = (9+5)(9-5) = 14 \cdot 4$$

Here we anticipated needing to simplify a square root and took advantage of the difference-of-squares form. We hope, though perhaps in vain, that there will be some simplifications in the ratios.

So,

$$a = 2\sqrt{14}.$$



Now we form the remaining ratios:

- $\sin\theta=\dfrac{2\sqrt{14}}{9}$
- $\tan\theta=\dfrac{2\sqrt{14}}{5}$
- $\csc\theta=\dfrac{1}{\sin\theta}=\dfrac{9}{2\sqrt{14}}$
- $\sec\theta=\dfrac{1}{\cos\theta}=\dfrac{9}{5}$
- $\cot\theta=\dfrac{1}{\tan\theta}=\dfrac{5}{2\sqrt{14}}$

? Try It 4.1.2.5

Suppose you know that for a particular angle of a right triangle, θ, that $\tan\theta=3$. Draw a picture that shows the relationship between θ and 3. Find the other ratios.

Answer

Triangles vary.

- $\sin\theta=\dfrac{3}{\sqrt{10}}$
- $\cos\theta=\dfrac{1}{\sqrt{10}}$
- $\tan\theta=3$
- $\csc\theta=\dfrac{1}{\sin\theta}=\dfrac{\sqrt{10}}{3}$
- $\sec\theta=\dfrac{1}{\cos\theta}=\sqrt{10}$
- $\cot\theta=\dfrac{1}{\tan\theta}=\dfrac{1}{3}$

? Try It 4.1.2.6

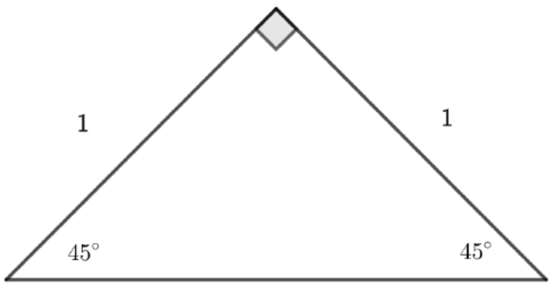
Suppose you know that for a particular angle of a right triangle, θ, that $\sin\theta=\dfrac{4}{5}$. Draw a picture that shows the relationship between θ, 4, and 5. Find the other ratios.

Answer

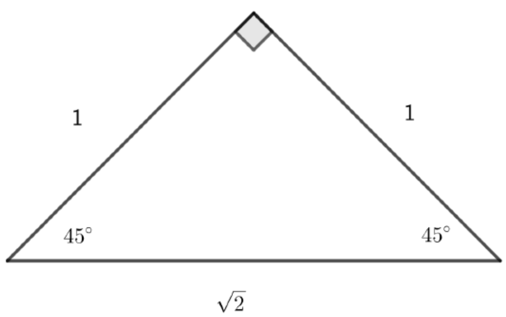
Answers vary.

- $\sin\theta = \frac{4}{5}$
- $\cos\theta = \frac{3}{5}$
- $\tan\theta = \frac{4}{3}$
- $\csc\theta = \frac{1}{\sin\theta} = \frac{5}{4}$
- $\sec\theta = \frac{1}{\cos\theta} = \frac{5}{3}$
- $\cot\theta = \frac{1}{\tan\theta} = \frac{3}{4}$

There are two special triangles that will be important in what follows where the ratios are known exactly (in terms of square roots). First, consider an isosceles right triangle:

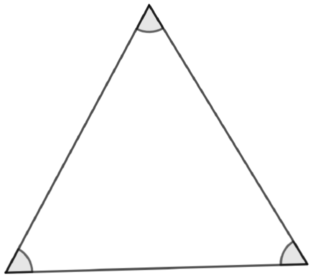


We can use the Pythagorean Theorem to find that the hypotenuse is $\sqrt{2}$.

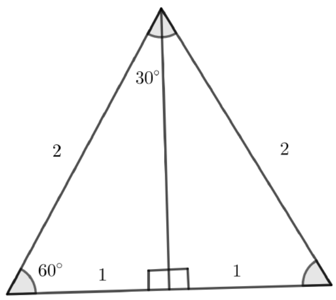


So, for example, $\cos 45^{o} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$.

For the second special triangle, consider an equilateral triangle.



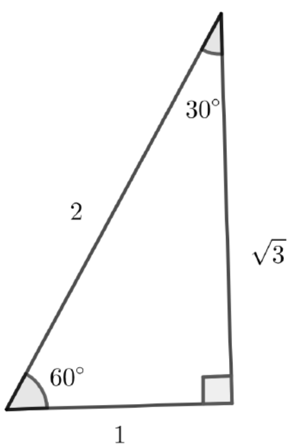
In anticipation of simpler arithmetic, we will make all of the sides have length 2 units. Now we form a line that bisects the upper angle thereby forming two smaller triangles. The bottom inner angle must by a right angle. The resulting triangles are the same shape since their angles are the same and they are the same size since the hypotenuse is 2 for each. So the line bisects the base as well leaving us with:



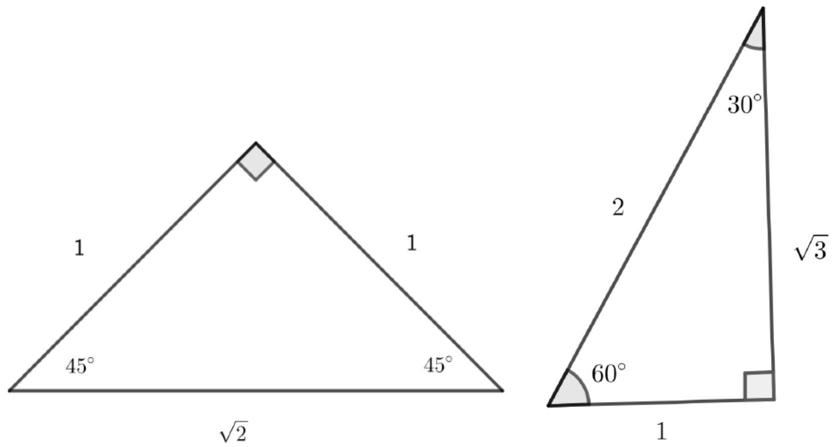
If we focus on the left (extracting it from the picture), we can use the Pythagorean Theorem to find the length of the missing leg (which we can call a:

$$2^2 = a^2 + 1^2,$$

which leads us to discover that $a = \sqrt{3}$ and so we have the following picture:



You will need to be able to recall (either by memory or re-derivation) these two special triangles often called "the 45-45-90" triangle and "the 30-60-90" triangle. These are just two particular representatives.



For the 30-60-90 triangle, it may be helpful to note that opposite the smallest angle is the shortest side, and opposite the largest angle is the longest side.

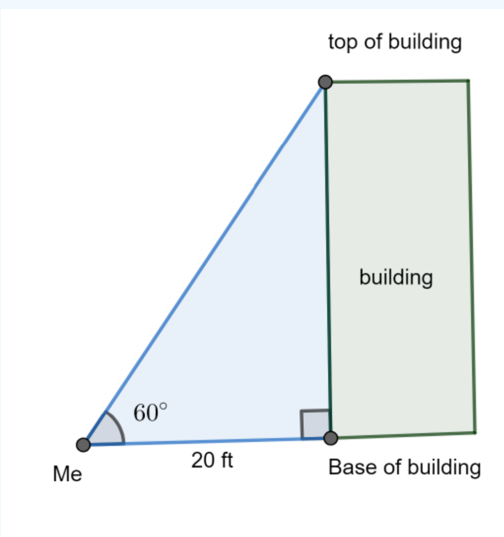
It follows that, for example, $\tan 60^o = \sqrt{3}$ etc. It is possibly better to recall the triangle rather than all of ratios.

Example 4.1.2.7

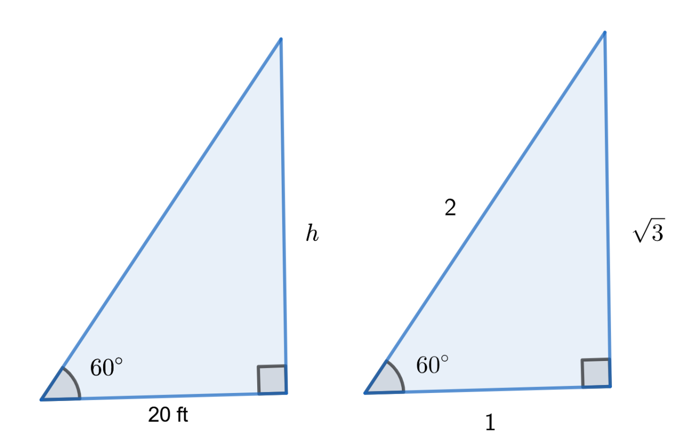
Suppose that I am standing at the tip of a shadow of a building and look at a 60^o angle to the top of a building. The shadow is measured to be 20 feet. Neglecting my height, approximately how tall is the building?

Solution

We first draw a picture and label the information we have:



We want to find the height of the building, so we will call this h and extract just the triangle from the picture. We note that the triangle is a special triangle and we draw by its side the one we know (not to scale) noting that opposite the 30^o angle is 1.



There are two approaches:

The first approach is to use that corresponding ratios are equal:

$$\frac{h}{20} = \frac{\sqrt{3}}{1}$$

and then solve for h to get $h = 20\sqrt{3}$. We conclude that the height of the building is approximately 34 feet (because we are neglecting my height that this is only an approximation).

The second approach is to use the ratios:

From the left triangle we see that

$$\frac{h}{20} = \tan 60^{\mathrm{o}}$$

and the right triangle helps us see that the ratio $\tan 60^{\mathrm{o}}$ is $\sqrt{3}$. We again find that

$$\frac{h}{20} = \sqrt{3}$$

and proceed as above.

You may find that you prefer one over the other in certain situations.

Note that the angle in the previous example is called the angle of elevation.

? Try It 4.1.2.8

Suppose we are looking out a window that is 50 ft from the ground and we look down at a 45^{o} angle (this is called the angle of depression) to spot a fire on the ground. Approximately how far away from us in is the fire? Draw a picture of the situation. Why is this only an approximation? Is it a good one?

Answer

Answers vary. The fire is approximately $50\sqrt{2}$ ft from the building.

? Try It 4.1.2.9

You are standing in a large field. A drone takes off from your feet at a 30^{o} angle and runs out of power after traveling 100 meters. Approximately how far will you have to walk to retrieve the drone? Draw a picture of the situation. Explain why this is an approximation and discuss its accuracy.

Answer

Explanation varies. You will have to walk approximately $50\sqrt{3}$ meters, or 86.6m.

> **Writing Exercise 4.1.2.10**
>
> 1. Why must the ratios sine and cosine be less than or equal to one?
> 2. Does the tangent ratio have to be less than or equal to one?
> 3. In the examples in this section, why doesn't the particular triangle we draw matter as long as it demonstrates the given ratio?
> 4. In Example 4.1.2.7, What might the picture look like if I don't neglect my height? Is this a good approximation? We can explore this later.
> 5. What is the relationship between $\sin 40^{\circ}$ and $\cos 50^{\circ}$? Explain and draw a picture supporting your explanation.

> **Exit Problem**
>
> 1. If two right triangles have an interior angle of 30^{o} give the 6 trigonometric ratios with respect to the 30^{o} angle.
>
> 2. If you are standing a distance of 25 feet from a structure and the angle of elevation to the top of the structure is 45^{o}, how tall is the structure? How far would you have to throw a rock to reach the top? Draw a picture of the situation and support your answer.

Key Concepts

- The sides of a right triangle with respect to an angle θ which is not a right angle are called hypotenuse, opposite, and adjacent according to whether the sides is opposite the right angle, opposite the specified angle, or adjacent (but not the hypotenuse) to the specified angle.
- The ratios of the sides of a right triangle are called $\sin\theta=\dfrac{\text{opposite}}{\text{hypotenuse}}$, $\cos\theta=\dfrac{\text{adjacent}}{\text{hypotenuse}}$, and $\tan\theta=\dfrac{\text{opposite}}{\text{adjacent}}$.
- There are two families of special triangles: 30-60-90 and 45-45-90 whose ratios are known exactly.

4.1.3: Angles on the Coordinate Plane

Learning Objectives

- Draw angles in standard position.
- Convert between degrees and radians.
- Find coterminal angles.
- Find the length of a circular arc.
- Use linear and angular speed to describe motion on a circular path.

Be Prepared

Before you get started, take this readiness quiz.

1. Find an integer n so that $900+n360$ is between 0 an 360.
2. Draw an angle measuring 300°
3. Reduce $\frac{270}{180}$.

An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle. An angle is in standard position if its vertex is at the origin and its initial side lies along the positive x-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.

Angles that are larger than 90° occur in applications and we would like to extend our ideas to deal with these angles. For example, we would like to be able to relate lengths and angles of triangles that are not right triangles and may have an obtuse interior angle. Here we will extend our trigonometric ratios to angles that are not interior angles of a right triangle.

Drawing Angles in Standard Position

To formalize our work, we will begin by drawing angles on an xy-coordinate plane. Angles can occur in any position on the coordinate plane, but for the purpose of comparison, the convention is to illustrate them in the same position whenever possible. An angle is in **standard position** if its vertex is located at the origin, and its initial side extends along the positive x-axis. See Figure $4.1.3.1$.

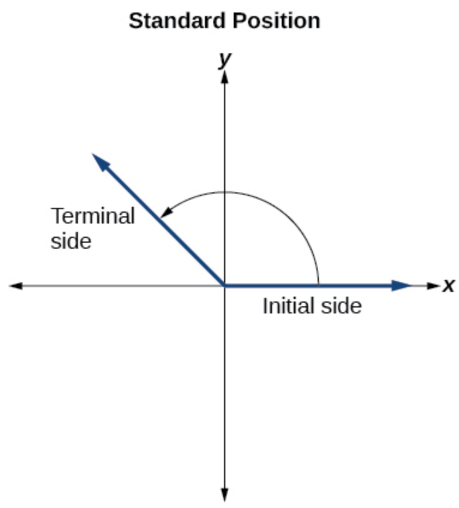


Figure 4.1.3.1

If the angle is measured in a counterclockwise direction from the initial side to the terminal side, the angle is said to be a **positive angle**. If the angle is measured in a clockwise direction, the angle is said to be a **negative angle**.

Drawing an angle in standard position always starts the same way—draw the initial side along the positive x-axis. To place the terminal side of the angle, we must calculate the fraction of a full rotation the angle represents. We do that by dividing the angle measure in degrees by 360°. For example, to draw a 90° angle, we calculate that $\frac{90^\circ}{360^\circ} = \frac{1}{4}$. So, the terminal side will be one-fourth of the way around the circle, moving counterclockwise from the positive x-axis. To draw a 360° angle, we calculate that $\frac{360^\circ}{360^\circ} = 1$. So the terminal side will be 1 complete rotation around the circle, moving counterclockwise from the positive x-axis. In this case, the initial side and the terminal side overlap. See Figure 4.1.3.2.

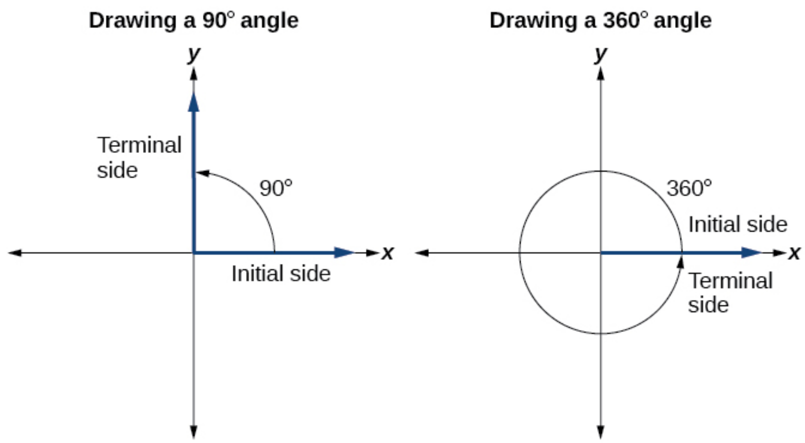


Figure 4.1.3.2

Given an angle measure in degrees, draw the angle in standard position

1. Express the angle measure as a fraction of 360°.
2. Reduce the fraction to simplest form.
3. Draw an angle that contains that same fraction of the circle, beginning on the positive x-axis and moving counterclockwise for positive angles and clockwise for negative angles.

Example 4.1.3.1

1. Sketch an angle of 30° in standard position.

2. Sketch an angle of −135° in standard position.

Solution

1. Divide the angle measure by 360°.

$$\frac{30^\circ}{360^\circ} = \frac{1}{12}$$

To rewrite the fraction in a more familiar fraction, we can recognize that

$$\frac{1}{12} = \frac{1}{3}\left(\frac{1}{4}\right)$$

One-twelfth equals one-third of a quarter, so by dividing a quarter rotation into thirds, we can sketch a line at 30° as in Figure 4.1.3.3.

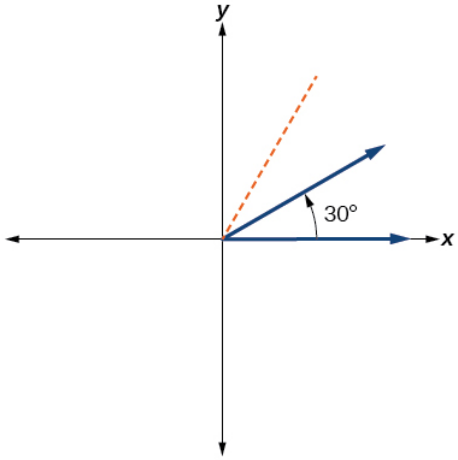


***Figure* 4.1.3.3**

2. Divide the angle measure by 360°.

$$\frac{-135^\circ}{360^\circ} = -\frac{3}{8}$$

In this case, we can recognize that

$$-\frac{3}{8} = -\frac{3}{2}\left(\frac{1}{4}\right)$$

Negative three-eighths is one and one-half times a quarter, so we place a line by moving clockwise one full quarter and one-half of another quarter, as in Figure 4.1.3.4.

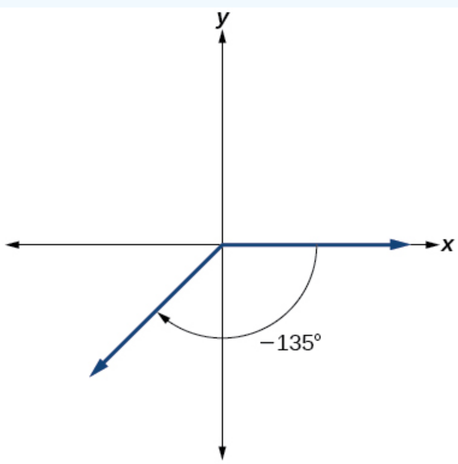


***Figure* 4.1.3.4**

? Try It 4.1.3.5

Show an angle of 240° on a circle in standard position.

Answer

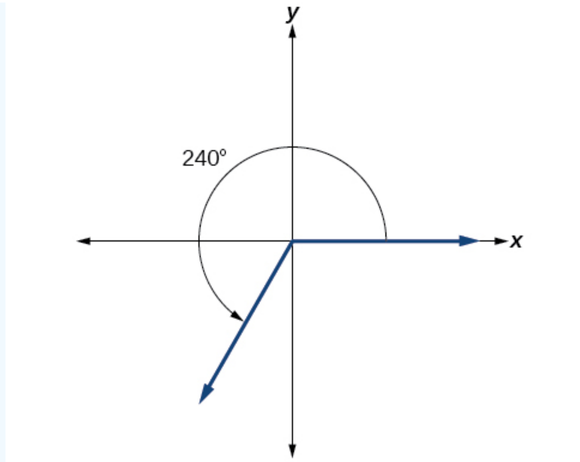


Figure 4.1.3.1: Copy and Paste Caption here. (Copyright; author via source)

Converting Between Degrees and Radians

Dividing a circle into 360 parts is an arbitrary choice, although it creates the familiar degree measurement. We may choose other ways to divide a circle. To find another unit, think of the process of drawing a circle. Imagine that you stop before the circle is completed. The portion that you drew is referred to as an arc. An **arc** may be a portion of a full circle, a full circle, or more than a full circle, represented by more than one full rotation. The length of the arc around an entire circle is called the **circumference** of that circle.

The circumference of a circle is $C = 2\pi r$. If we divide both sides of this equation by r, we create the ratio of the circumference to the radius, which is always 2π regardless of the length of the radius. So the circumference of any circle is $2\pi \approx 6.28$ times the length of the radius. That means that if we took a string as long as the radius and used it to measure consecutive lengths around the circumference, there would be room for six full string-lengths and a little more than a quarter of a seventh, as shown in Figure 4.1.3.6.

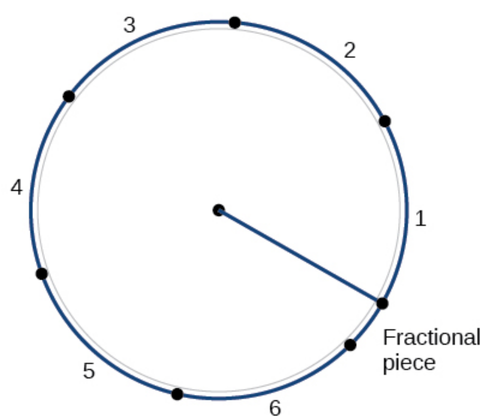


Figure 4.1.3.6

This brings us to our new angle measure. One **radian** is the measure of a central angle of a circle that intercepts an arc equal in length to the radius of that circle. A central angle is an angle formed at the center of a circle by two radii. Because the total circumference equals 2π times the radius, a full circular rotation is 2π radians. So

$$2\pi \text{ radians} = 360^o \tag{4.1.3.1}$$

$$\pi \text{ radians} = \frac{360^o}{2} = 180^o \tag{4.1.3.2}$$

$$1 \text{ radian} = \frac{180^o}{\pi} \approx 57.3^o \tag{4.1.3.3}$$

See Figure 4.1.3.7. Note that when an angle is described without a specific unit, it refers to radian measure. For example, an angle measure of 3 indicates 3 radians. In fact, radian measure is dimensionless, since it is the quotient of a length (circumference) divided by a length (radius) and the length units cancel out.

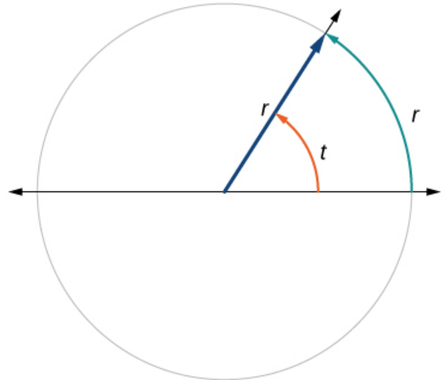


Figure 4.1.3.7: The angle t sweeps out a measure of one radian. Note that the length of the intercepted arc is the same as the length of the radius of the circle.

Relating Arc Lengths to Radius

An **arc length** s is the length of the curve along the arc. Just as the full circumference of a circle always has a constant ratio to the radius, the arc length produced by any given angle also has a constant relation to the radius, regardless of the length of the radius.

This ratio, called the **radian measure**, is the same regardless of the radius of the circle—it depends only on the angle. This property allows us to define a measure of any angle as the ratio of the arc length s s to the radius r. See Figure 4.1.3.8.

$$s = r\theta \tag{4.1.3.4}$$

$$\theta = \frac{s}{r} \tag{4.1.3.5}$$

If $s = r$, then $\theta = \frac{r}{r} = 1$ radian.

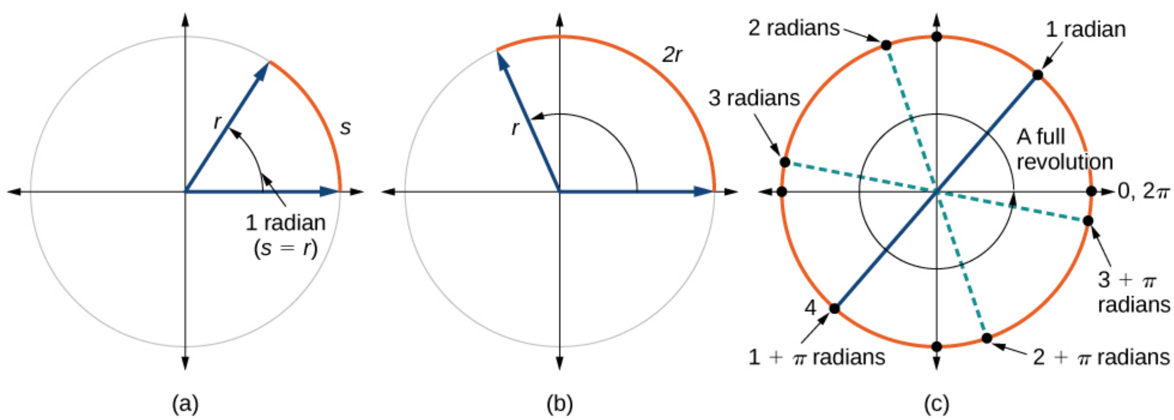


Figure 4.1.3.8: (a) In an angle of 1 radian, the arc length s equals the radius r. (b) An angle of 2 radians has an arc length $s = 2r$. (c) A full revolution is 2π or about 6.28 radians.

To elaborate on this idea, consider two circles, one with radius 2 and the other with radius 3. Recall the circumference of a circle is $C = 2\pi r$, where r is the radius. The smaller circle then has circumference $2\pi(2) = 4\pi$ and the larger has circumference $2\pi(3) = 6\pi$. Now we draw a 45° angle on the two circles, as in Figure 4.1.3.9.

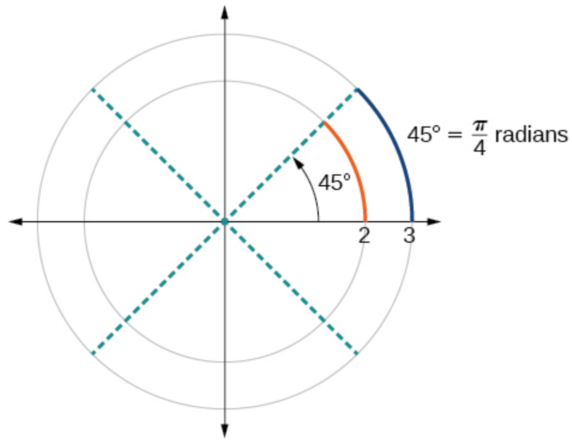


Figure 4.1.3.9: A 45° angle contains one-eighth of the circumference of a circle, regardless of the radius.

Notice what happens if we find the ratio of the arc length divided by the radius of the circle.

$$\text{Smaller circle:} \quad \frac{\frac{1}{2}\pi}{2} = \frac{1}{4}\pi \tag{4.1.3.6}$$

$$\text{Larger circle:} \quad \frac{\frac{3}{4}\pi}{3} = \frac{1}{4}\pi \tag{4.1.3.7}$$

Since both ratios are $\frac{1}{4}\pi$, the angle measures of both circles are the same, even though the arc length and radius differ.

RADIANS

One **radian** is the measure of the central angle of a circle such that the length of the arc between the initial side and the terminal side is equal to the radius of the circle. A full revolution (360°) equals 2π radians. A half revolution (180°) is equivalent to π radians.

The **radian measure** of an angle is the ratio of the length of the arc subtended by the angle to the radius of the circle. In other words, if s is the length of an arc of a circle, and r is the radius of the circle, then the central angle containing that arc measures $\frac{s}{r}$ radians. In a circle of radius 1, the radian measure corresponds to the length of the arc.

A measure of 1 radian looks to be about 60°. Is that correct?

Yes. It is approximately 57.3°. Because 2π radians equals 360°, 1 radian equals $\frac{360^\circ}{2\pi} \approx 57.3^\circ$.

Using Radians

Because **radian** measure is the ratio of two lengths, it is a unitless measure. For example, in Figure 4.1.3.9, suppose the radius were 2 inches and the distance along the arc were also 2 inches. When we calculate the radian measure of the angle, the "inches" cancel, and we have a result without units. Therefore, it is not necessary to write the label "radians" after a radian measure, and if we see an angle that is not labeled with "degrees" or the degree symbol, we can assume that it is a radian measure.

Considering the most basic case, the **unit circle** (a circle with radius 1), we know that 1 rotation equals 360 degrees, 360°. We can also track one rotation around a circle by finding the circumference, $C = 2\pi r$, and for the unit circle $C = 2\pi$. These two different ways to rotate around a circle give us a way to convert from degrees to radians.

$$\begin{aligned} 1 \text{ rotation} &= 360^\circ &&= 2\pi \text{ radians} \\ \frac{1}{2} \text{ rotation} &= 180^\circ &&= \pi \text{ radians} \\ \frac{1}{4} \text{ rotation} &= 90^\circ &&= \frac{\pi}{2} \text{ radians} \end{aligned}$$

Identifying Special Angles Measured in Radians

In addition to knowing the measurements in degrees and radians of a quarter revolution, a half revolution, and a full revolution, there are other frequently encountered angles in one revolution of a circle with which we should be familiar. It is common to encounter multiples of 30, 45, 60, and 90 degrees. These values are shown in Figure 4.1.3.10.

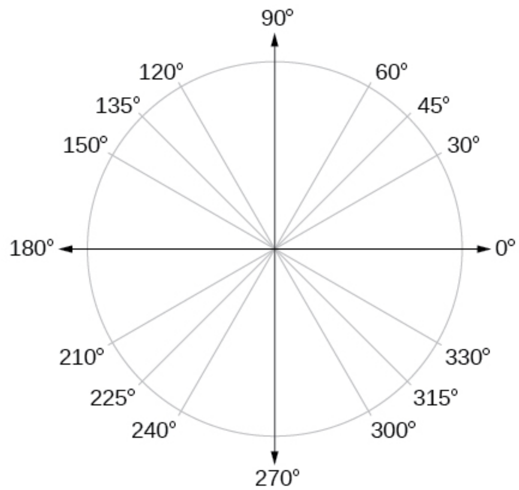


Figure 4.1.3.10: Commonly encountered angles measured in degrees

Now, we can list the corresponding radian values for the common measures of a circle corresponding to those listed in Figure 4.1.3.10, which are shown in Figure 4.1.3.11. Be sure you can verify each of these measures.

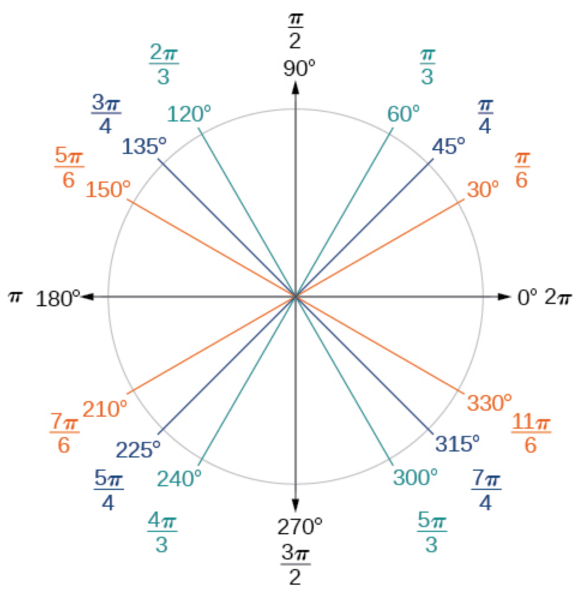


Figure 4.1.3.11: Commonly encountered angles measured in radians

Example 4.1.3.12

Find the radian measure of one-third of a full rotation.

Solution

For any circle, the arc length along such a rotation would be one-third of the circumference. We know that

$$1 \text{ rotation} = 2\pi r$$

So,

$$\begin{aligned} s &= \frac{1}{3}(2\pi r) \\ &= \frac{2\pi r}{3} \end{aligned}$$

The radian measure would be the arc length divided by the radius.

$$\begin{aligned} \text{radian measure} &= \frac{\frac{2\pi r}{3}}{r} \\ &= \frac{2\pi r}{3r} \\ &= \frac{2\pi}{3} \end{aligned}$$

Try It 4.1.3.13

Find the radian measure of three-fourths of a full rotation.

Solution

$\frac{3\pi}{2}$

Converting between Radians and Degrees

Because degrees and radians both measure angles, we need to be able to convert between them. We can easily do so using a proportion. We will use θ to represent the measure in degrees of an angle and x the measurement in radians of the same angle.

$$\frac{\theta}{180} = \frac{x}{\pi}. \tag{4.1.3.8}$$

This proportion shows that the measure of angle θ in degrees divided by 180 equals the measure of angle x in radians divided by π. Or, phrased another way, degrees is to 180 as radians is to π.

$$\frac{\text{Degrees}}{180} = \frac{\text{Radians}}{\pi} \tag{4.1.3.9}$$

CONVERTING BETWEEN RADIANS AND DEGREES

To convert between degrees and radians, use the proportion

$$\frac{\theta}{180} = \frac{x}{\pi} \tag{4.1.3.10}$$

Example 4.1.3.14: Converting Radians to Degrees

Convert each radian measure to degrees.

1. $\frac{\pi}{6}$
2. 3

Solution

Because we are given radians and we want degrees, we should set up a proportion and solve it.

1. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{x}{\pi}$$

$$\frac{\theta}{180} = \frac{\frac{\pi}{6}}{\pi}$$

$$\theta = \frac{180}{6}$$

$$\theta = 30^{o}$$

2. We use the proportion, substituting the given information.

$$\frac{\theta}{180} = \frac{x}{\pi}$$

$$\frac{\theta}{180} = \frac{3}{\pi}$$

$$\theta = \frac{3(180)}{\pi}$$

$$\theta \approx 172^{\circ}$$

Try It 4.1.3.15

Convert $-\frac{3\pi}{4}$ radians to degrees.

Solution

−135°

Example 4.1.3.16

Convert 15 degrees to radians.

Solution

In this example, we start with degrees and want radians, so we again set up a proportion and solve it, but we substitute the given information into a different part of the proportion.

$$\frac{\theta}{180} = \frac{x}{\pi}$$

$$\frac{15}{180} = \frac{x}{\pi}$$

$$\frac{15\pi}{180} = x$$

$$\frac{\pi}{12} = x$$

Analysis

Another way to think about this problem is by remembering that $30^{o} = \frac{\pi}{6}$. Because $15^{\circ} = \frac{1}{2}(30^{\circ})$, ,we can find that $\frac{1}{2}\left(\frac{\pi}{6}\right)$ is $\frac{\pi}{12}$.

Try It 4.1.3.17

Convert 126° to radians.

Solution

$\frac{7\pi}{10}$

Finding Coterminal Angles

Converting between degrees and radians can make working with angles easier in some applications. For other applications, we may need another type of conversion. Negative angles and angles greater than a full revolution are more awkward to work with than those in the range of 0^o to 360^o, or 0 to 2π. It would be convenient to replace those out-of-range angles with a corresponding angle within the range of a single revolution.

It is possible for more than one angle to have the same terminal side. Look at Figure 4.1.3.18. The angle of 140^o is a **positive angle**, measured counterclockwise. The angle of $–220^\text{o}$ is a **negative angle**, measured clockwise. But both angles have the same terminal side. If two angles in standard position have the same terminal side, they are **coterminal** angles. Every angle greater than 360^o or less than 0^o is coterminal with an angle between 0^o and 360^o, and it is often more convenient to find the coterminal angle within the range of 0^o to 360^o than to work with an angle that is outside that range.

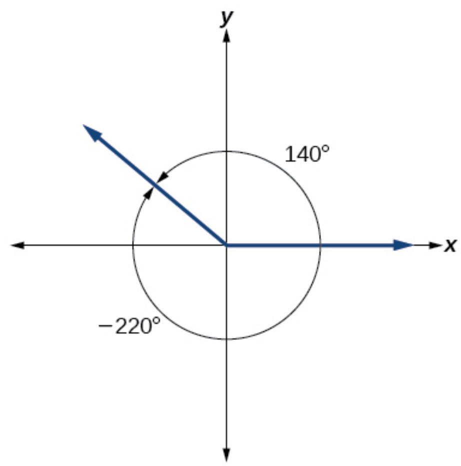


Figure 4.1.3.18: An angle of 140^o and an angle of $–220^\text{o}$ are coterminal angles.

Any angle has infinitely many **coterminal angles** because each time we add 360^o to that angle—or subtract 360^o from it—the resulting value has a terminal side in the same location. For example, 100^o and 460^o are coterminal for this reason, as is -260^o. Recognizing that any angle has infinitely many coterminal angles explains the repetitive shape in the graphs of trigonometric functions.

An angle's reference angle is the measure of the smallest, positive, acute angle t formed by the terminal side of the angle t and the horizontal axis. Thus positive reference angles have terminal sides that lie in the first quadrant and can be used as models for angles in other quadrants. See Figure 4.1.3.19 for examples of reference angles for angles in different quadrants.

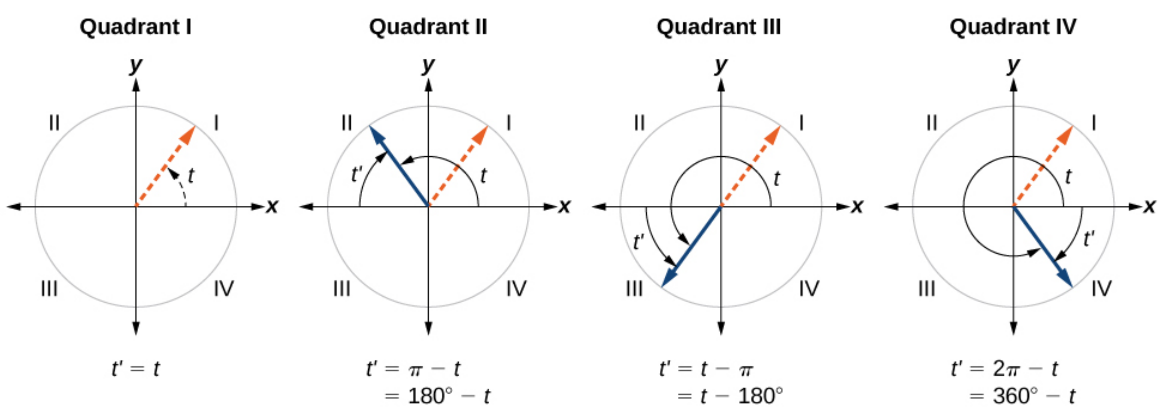


Figure 4.1.3.19

> **COTERMINAL AND REFERENCE ANGLES**
>
> - *Coterminal angles* are two angles in standard position that have the same terminal side.
> - An angle's *reference angle* is the size of the smallest acute angle, t', formed by the terminal side of the angle t and the horizontal axis.

Given an angle greater than 360^\text{o}, find a coterminal angle between 0^\text{o} and 360^\text{o}

1. Subtract 360^o from the given angle.
2. If the result is still greater than 360^o, subtract 360^o again till the result is between 0^\text{o} and 360^\text{o}.
3. The resulting angle is coterminal with the original angle.

Example 4.1.3.20

Find the least positive angle θ that is coterminal with an angle measuring 800^o, where $0^o \leq \theta < 360^o$.

Solution

An angle with measure 800^o is coterminal with an angle with measure $800^o - 360^o = 440^o$, but 440^o is still greater than 360^o, so we subtract 360^o again to find another coterminal angle: $440^o - 360^o = 80^o$.

The angle $\theta = 80^o$ is coterminal with 800^\text{o}. To put it another way, 800^o equals 80^o plus two full rotations, as shown in Figure 4.1.3.21.

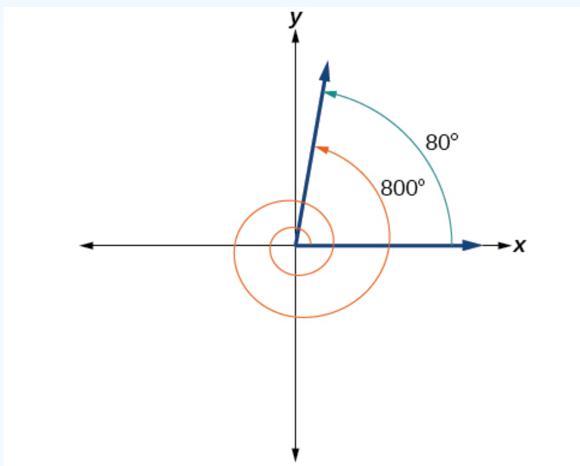


Figure 4.1.3.21

Try It 4.1.3.22

Find an angle α that is coterminal with an angle measuring 870^o, where $0^o \leq \alpha < 360^o$.

Solution

$\alpha = 150^o$

Given an angle with measure less than 0^\text{o}, find a coterminal angle having a measure between 0^\text{o} and 360^\text{o}.

1. Add 360^o to the given angle.
2. If the result is still less than 0^o, add 360^o again until the result is between 0^o and 360^o.
3. The resulting angle is coterminal with the original angle.

Example 4.1.3.23

Show the angle with measure -45^o on a circle and find a positive coterminal angle α such that $0^o \leq \alpha < 360^o$.

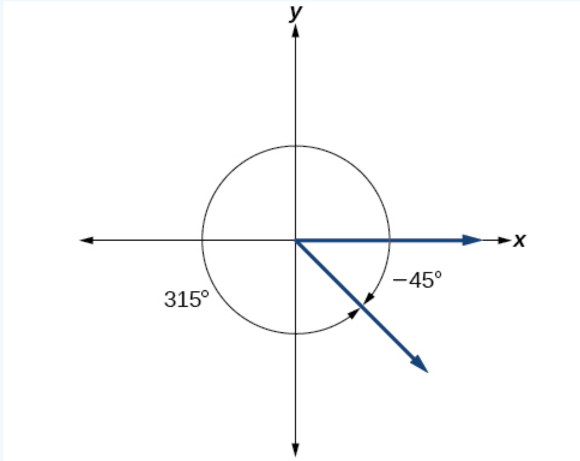
Solution

Since 45^o is half of 90^o, we can start at the positive horizontal axis and measure clockwise half of a 90^o angle.

Because we can find coterminal angles by adding or subtracting a full rotation of 360^o, we can find a positive coterminal angle here by adding 360^o:

$$-45^o + 360^o = 315^o \tag{4.1.3.11}$$

We can then show the angle on a circle, as



Try It 4.1.3.24

Find an angle β that is coterminal with an angle measuring -300° such that $0^{\circ} \leq \beta < 360^{\circ}$.

Solution

$\beta = 60^{\circ}$

Finding Coterminal Angles Measured in Radians

We can find **coterminal** angles measured in radians in much the same way as we have found them using degrees. In both cases, we find coterminal angles by adding or subtracting one or more full rotations.

Given an angle greater than 2π, find a coterminal angle between 0 and 2π.

1. Subtract 2π from the given angle.
2. If the result is still greater than 2π, subtract 2π again until the result is between 0 and 2π.
3. The resulting angle is coterminal with the original angle.

Example 4.1.3.25

Find an angle β that is coterminal with $\frac{19\pi}{4}$, where $0 \leq \beta < 2\pi$.

Solution

When working in degrees, we found coterminal angles by adding or subtracting 360 degrees, a full rotation. Likewise, in radians, we can find coterminal angles by adding or subtracting full rotations of 2π radians:

$$\frac{19\pi}{4} - 2\pi = \frac{19\pi}{4} - \frac{8\pi}{4} \tag{4.1.3.12}$$

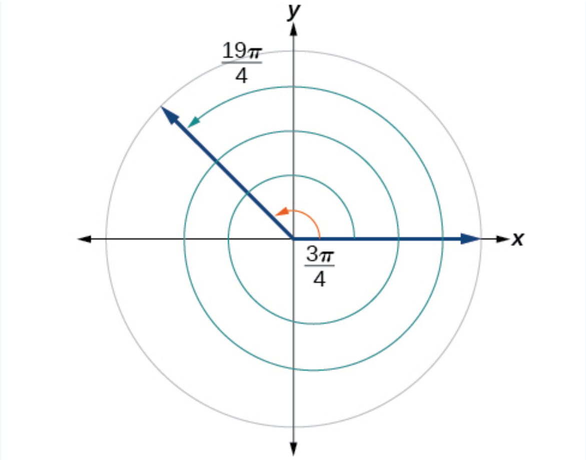
$$= \frac{11\pi}{4} \tag{4.1.3.13}$$

The angle $\frac{11\pi}{4}$ is coterminal, but not less than 2π, so we subtract another rotation:

$$\frac{11\pi}{4} - 2\pi = \frac{11\pi}{4} - \frac{8\pi}{4} \tag{4.1.3.14}$$

$$= \frac{3\pi}{4} \tag{4.1.3.15}$$

The angle $\frac{3\pi}{4}$ is coterminal with $\frac{19\pi}{4}$, as shown



Try It 4.1.3.26

Find an angle of measure θ that is coterminal with an angle of measure $-\frac{17\pi}{6}$ where $0 \leq \theta < 2\pi$.

Solution

$\frac{7\pi}{6}$

Determining the Length of an Arc (optional)

Recall that the **radian measure** θ of an angle was defined as the ratio of the **arc length** s of a circular arc to the radius r of the circle, $\theta = \frac{s}{r}$. From this relationship, we can find arc length along a circle, given an angle.

ARC LENGTH ON A CIRCLE

In a circle of radius r, the length of an arc s subtended by an angle with measure θ in radians, shown in Figure 4.1.3.27, is

$$s = r\theta \tag{4.1.3.16}$$

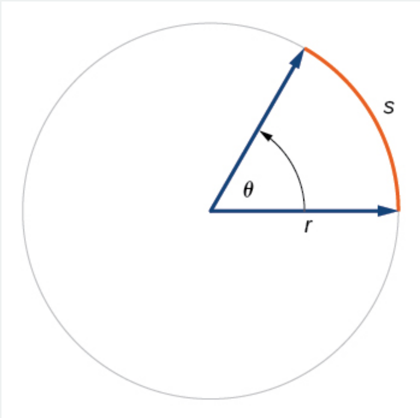


Figure 4.1.3.27

Given a circle of radius r, calculate the length s of the arc subtended by a given angle of measure θ.

1. If necessary, convert θ to radians.
2. Multiply the radius r by the radian measure of θ: $s = r\theta$.

Example 4.1.3.28: Finding the Length of an Arc

Assume the orbit of Mercury around the sun is a perfect circle. Mercury is approximately 36 million miles from the sun.

1. In one Earth day, Mercury completes 0.0114 of its total revolution. How many miles does it travel in one day?
2. Use your answer from part (a) to determine the radian measure for Mercury's movement in one Earth day.

Solution

1. Let's begin by finding the circumference of Mercury's orbit.

$$C = 2\pi r \tag{4.1.3.17}$$
$$= 2\pi(\text{ 36 million miles}) \tag{4.1.3.18}$$
$$\approx 226 \text{ million miles} \tag{4.1.3.19}$$

Since Mercury completes 0.0114 of its total revolution in one Earth day, we can now find the distance traveled:

$$(0.0114)226 \text{ million miles} = 2.58 \text{ million miles} \tag{4.1.3.20}$$

2. Now, we convert to radians:

$$\text{radian} = \frac{\text{arc length}}{\text{radius}} \tag{4.1.3.21}$$
$$= \frac{2.58 \text{ million miles}}{36 \text{ million miles}} \tag{4.1.3.22}$$
$$= 0.0717 \tag{4.1.3.23}$$

Try It 4.1.3.29

Find the arc length along a circle of radius 10 units subtended by an angle of 215°.

Solution

$$\frac{215\pi}{18} = 37.525 \text{ units} \tag{4.1.3.24}$$

Finding the Area of a Sector of a Circle (optional)

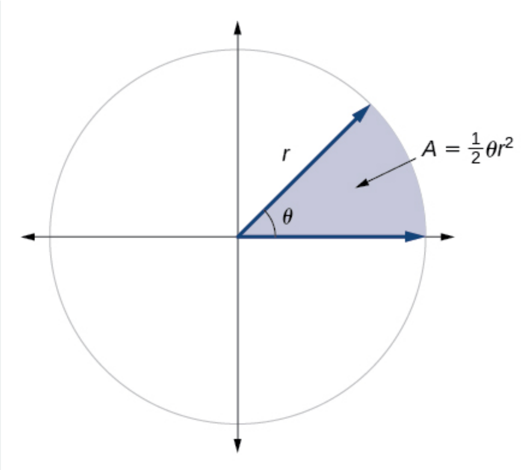
In addition to arc length, we can also use angles to find the area of a **sector of a circle**. A sector is a region of a circle bounded by two radii and the intercepted arc, like a slice of pizza or pie. Recall that the area of a circle with radius r can be found using the formula $A = \pi r^2$. If the two radii form an angle of θ, measured in radians, then $\frac{\theta}{2\pi}$ is the ratio of the angle measure to the measure of a full rotation and is also, therefore, the ratio of the area of the sector to the area of the circle. Thus, the **area of a sector** is the fraction $\frac{\theta}{2\pi}$ multiplied by the entire area. (Always remember that this formula only applies if θ is in radians.)

$$\text{Area of sector} = \left(\frac{\theta}{2\pi}\right)\pi r^2 \tag{4.1.3.25}$$
$$= \frac{\theta\pi r^2}{2\pi} \tag{4.1.3.26}$$
$$= \frac{1}{2}\theta r^2 \tag{4.1.3.27}$$

AREA OF A SECTOR

The area of a **sector of a circle** with radius r subtended by an angle θ, measured in radians, is

$$A = \frac{1}{2}\theta r^2 \tag{4.1.3.28}$$



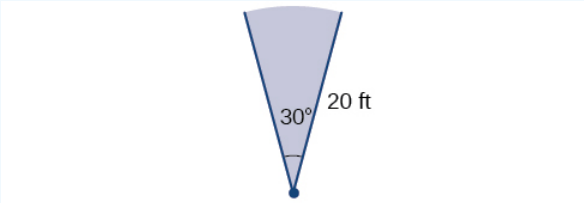
The area of the sector equals half the square of the radius times the central angle measured in radians.

Given a circle of radius r, find the area of a sector defined by a given angle θ.

1. If necessary, convert θ to radians.
2. Multiply half the radian measure of θ by the square of the radius r : $A = \frac{1}{2}\theta r^2$.

Example 4.1.3.30

An automatic lawn sprinkler sprays a distance of 20 feet while rotating 30 degrees, as shown in the figure below. What is the area of the sector of grass the sprinkler waters?



The sprinkler sprays 20 ft within an arc of 30°.

Solution

First, we need to convert the angle measure into radians. Because 30 degrees is one of our special angles, we already know the equivalent radian measure, but we can also convert:

$$30 \text{ degrees} = 30 \cdot \frac{\pi}{180} \tag{4.1.3.29}$$

$$= \frac{\pi}{6} \text{ radians} \tag{4.1.3.30}$$

The area of the sector is then

$$\text{Area} = \frac{1}{2}(\frac{\pi}{6})(20)^2 \tag{4.1.3.31}$$

$$\approx 104.72 \tag{4.1.3.32}$$

So the area is about 104.72 ft^2.

Try It 4.1.3.31

In central pivot irrigation, a large irrigation pipe on wheels rotates around a center point. A farmer has a central pivot system with a radius of 400 meters. If water restrictions only allow her to water 150 thousand square meters a day, what angle should she set the system to cover? Write the answer in radian measure to two decimal places.

Solution

1.88

Use Linear and Angular Speed to Describe Motion on a Circular Path (optional)

In addition to finding the area of a sector, we can use angles to describe the speed of a moving object. An object traveling in a circular path has two types of speed. **Linear speed** is speed along a straight path and can be determined by the distance it moves along (its **displacement**) in a given time interval. For instance, if a wheel with radius 5 inches rotates once a second, a point on the edge of the wheel moves a distance equal to the circumference, or 10\pi inches, every second. So the linear speed of the point is 10π in./s. The equation for linear speed is as follows where v is linear speed, s is displacement, and t is time.

$$v=\frac{s}{t} \tag{4.1.3.33}$$

Angular speed results from circular motion and can be determined by the angle through which a point rotates in a given time interval. In other words, angular speed is angular rotation per unit time. So, for instance, if a gear makes a full rotation every 4 seconds, we can calculate its angular speed as $\frac{360 \text{ degrees}}{4 \text{ seconds}}=90$ degrees per second. Angular speed can be given in radians per second, rotations per minute, or degrees per hour for example. The equation for angular speed is as follows, where ω (read as omega) is angular speed, θ is the angle traversed, and t is time.

$$\omega=\frac{\theta}{t} \tag{4.1.3.34}$$

Combining the definition of angular speed with the arc length equation, $s=r\theta$, we can find a relationship between angular and linear speeds. The angular speed equation can be solved for θ, giving $\theta=\omega t$. Substituting this into the arc length equation gives:

$$s=r\theta \tag{4.1.3.35}$$

$$=r\omega t \tag{4.1.3.36}$$

Substituting this into the linear speed equation gives:

$$v=\frac{s}{t} \qquad =\frac{r\omega t}{t}=r\omega \tag{4.1.3.37}$$

ANGULAR AND LINEAR SPEED

As a point moves along a circle of radius r, its **angular speed**, ω, is the angular rotation θ per unit time, t.

$$\omega=\frac{\theta}{t} \tag{4.1.3.38}$$

The **linear speed.** v, of the point can be found as the distance traveled, arc length s, per unit time, t.

$$v=\frac{s}{t} \tag{4.1.3.39}$$

When the angular speed is measured in radians per unit time, linear speed and angular speed are related by the equation

$$v=r\omega \tag{4.1.3.40}$$

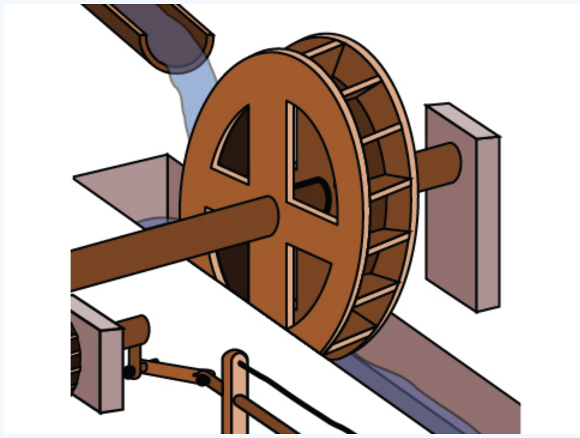
This equation states that the angular speed in radians, ω, representing the amount of rotation occurring in a unit of time, can be multiplied by the radius r to calculate the total arc length traveled in a unit of time, which is the definition of linear speed.

Given the amount of angle rotation and the time elapsed, calculate the angular speed

1. If necessary, convert the angle measure to radians.
2. Divide the angle in radians by the number of time units elapsed: $\omega=\frac{\theta}{t}$.
3. The resulting speed will be in radians per time unit.

Example 4.1.3.32: Finding Angular Speed

A water wheel, shown in the figure, completes 1 rotation every 5 seconds. Find the angular speed in radians per second.



Solution

The wheel completes 1 rotation, or passes through an angle of 2π radians in 5 seconds, so the angular speed would be $\omega=\frac{2\pi}{5}\approx 1.257$ radians per second.

Try It 4.1.3.33

An old vinyl record is played on a turntable rotating clockwise at a rate of 45 rotations per minute. Find the angular speed in radians per second.

Solution

$-\frac{3\pi}{2}$ rad/s

Given the radius of a circle, an angle of rotation, and a length of elapsed time, determine the linear speed

1. Convert the total rotation to radians if necessary.
2. Divide the total rotation in radians by the elapsed time to find the angular speed: apply $\omega=\frac{\theta}{t}$.
3. Multiply the angular speed by the length of the radius to find the linear speed, expressed in terms of the length unit used for the radius and the time unit used for the elapsed time: apply $v=r\omega$.

Example 4.1.3.34: Finding a Linear Speed

A bicycle has wheels 28 inches in diameter. A tachometer determines the wheels are rotating at 180 RPM (revolutions per minute). Find the speed the bicycle is traveling down the road.

Solution

Here, we have an angular speed and need to find the corresponding linear speed, since the linear speed of the outside of the tires is the speed at which the bicycle travels down the road.

We begin by converting from rotations per minute to radians per minute. It can be helpful to utilize the units to make this conversion:

$$180\frac{\text{rotations}}{\text{minute}}\cdot\frac{2\pi\text{ radians}}{\text{rotation}}=360\pi\frac{\text{radians}}{\text{minute}} \tag{4.1.3.41}$$

Using the formula from above along with the radius of the wheels, we can find the linear speed:

$$v = (14 \text{ inches})(360\pi \frac{\text{radians}}{\text{minute}}) \tag{4.1.3.42}$$

$$= 5040\pi \frac{\text{inches}}{\text{minute}} \tag{4.1.3.43}$$

Remember that radians are a unitless measure, so it is not necessary to include them.unitless measure, so it is not necessary to include them.

Finally, we may wish to convert this linear speed into a more familiar measurement, like miles per hour.

$$5040\pi \frac{\cancel{\text{inches}}}{\cancel{\text{minute}}} \cdot \frac{1 \ \cancel{\text{feet}}}{12 \ \cancel{\text{inches}}} \cdot \frac{1 \ \cancel{\text{mile}}}{5280 \ \cancel{\text{feet}}} \cdot \frac{60 \ \cancel{\text{minutes}}}{1 \text{ hour}} \approx 14.99 \text{ miles per hour (mph)} \tag{4.1.3.44}$$

Try It 4.1.3.35

A satellite is rotating around Earth at 0.25 radians per hour at an altitude of 242 km above Earth. If the radius of Earth is 6378 kilometers, find the linear speed of the satellite in kilometers per hour.

Solution

1655 kilometers per hour

Written Exercises 4.1.3.36

1. Give an example of how to convert radians to degrees.
2. Describe how to find an angle x, $[0, 2\pi)$, which is coterminal with $\frac{19\pi}{2}$ in two different ways. Give examples.
3. How can you find the angle 300° in radians without using the conversion ratio (thinking in radians directly)?

Exit Question

1. Convert 80° to radian measure.
2. Find an angle θ, $0 \leq \theta < 360^\circ$, which is co-terminal with -960°.

Key Concepts

- An angle is formed from the union of two rays, by keeping the initial side fixed and rotating the terminal side. The amount of rotation determines the measure of the angle.
- An angle is in standard position if its vertex is at the origin and its initial side lies along the positive *x*-axis. A positive angle is measured counterclockwise from the initial side and a negative angle is measured clockwise.
- To draw an angle in standard position, draw the initial side along the positive *x*-axis and then place the terminal side according to the fraction of a full rotation the angle represents.
- In addition to degrees, the measure of an angle can be described in radians.
- To convert between degrees and radians, use the proportion $\frac{\theta}{180} = \frac{x}{\pi}$.
- Two angles that have the same terminal side are called coterminal angles.
- We can find coterminal angles by adding or subtracting 360° or 2π.
- Coterminal angles can be found using radians just as they are for degrees.
- The length of a circular arc is a fraction of the circumference of the entire circle.
- The area of sector is a fraction of the area of the entire circle.
- An object moving in a circular path has both linear and angular speed.
- The angular speed of an object traveling in a circular path is the measure of the angle through which it turns in a unit of time.
- The linear speed of an object traveling along a circular path is the distance it travels in a unit of time.

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4.1.4: The Unit Circle

Learning Objectives

- Find function values for the sine and cosine of 30^{o} or $\frac{\pi}{6}$, 45^{o} or $\frac{\pi}{4}$, and 60^{o} or $\frac{\pi}{3}$.
- Find reference angles.
- Use reference angles to evaluate trigonometric functions.

Be Prepared

Before you get started, take this readiness quiz.

1. Draw a unit circle: $x^2+y^2=1$.
2. Determine how many points on the unit circle have $-\frac{1}{3}$ as their x-coordinate. Indicate these on the graph.
3. Determin how many points on the unit circle have $-\frac{1}{3}$ as their y-coordinate. Indicate these on the graph.

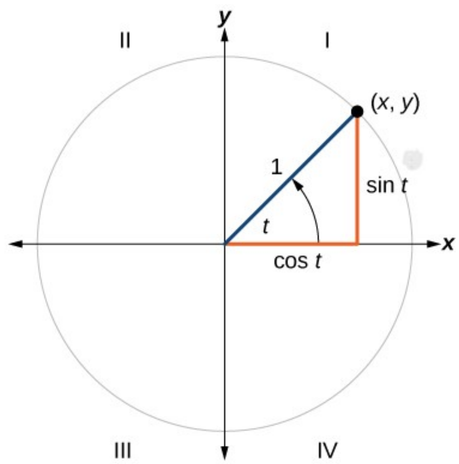
In this section, we will examine this type of revolving motion around a circle. To do so, we need to define the type of circle first, and then place that circle on a coordinate system. Then we can discuss circular motion in terms of the coordinate pairs.

Defining Sine, Cosine, and Tangent ratios for any angle

To define our trigonometric ratios, we begin by drawing a unit circle (a circle of radius 1 centered at the origin $(0,0)$).

Recall that the *x*- and *y*-axes divide the coordinate plane into four quarters called quadrants. We label these quadrants to mimic the direction a positive angle would sweep. The four quadrants are labeled I, II, III, and IV.

For an acute angle t, we can label the intersection of the terminal side and the unit circle as by its coordinates, (x,y). Since the circle has radius 1, using our trigonometric ratios we see that the triangle in orange below has base $\cos t$ and height $\sin t$. The coordinates x and y therefore can be related to the angle t: $x=\cos t$ and $y=\sin t$. Also, $\tan t=\frac{\sin t}{\cos t}$.



Unit circle with the angle is t radians or degrees

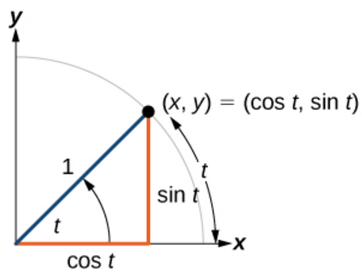
UNIT CIRCLE

A unit circle has a center at $(0,0)$ and radius 1. Form the angle with measure t with initial side coincident with the x-axis.

Let (x,y) be point where the terminal side of the angle and unit circle meet. Then $(x,y)=(\cos t,\sin t)$. Further, $\tan t=\frac{\sin t}{\cos t}$.

Defining Sine, Cosine, and Tangent for any Angle

Now that we have our unit circle labeled, we can learn how the (x, y) coordinates relate to the angle t. For an acute angle, the **sine ratio** $\sin t$ is the y-coordinate of the point where the corresponding terminal side of the angle intersects the unit circle and the **cosine ratio** $\cos t$ is the x-coordinate of the point where the corresponding terminal side of the angle intersects the unit circle. We extend this definition to all angles. So, we define the **sine ratio** $\sin t$ to be the y-coordinate of the point where the corresponding terminal side of the angle intersects the unit circle and the **cosine ratio** $\cos t$ to be the x-coordinate of the point where the corresponding terminal side of the angle intersects the unit circle. The **tangent ratio** $\tan t$ is $\frac{\sin t}{\cos t}$.



Here, if t is measure in radians then t is also the length of the indicated arc.

We may enclose the angle in parentheses or not depending on how clear the expression is: $\sin t$ is the same as $\sin(t)$ and $\cos t$ is the same as $\cos(t)$. When in doubt, use the extra parentheses when entering calculations into a calculator or computer.

SINE, COSINE, and TANGENT RATIOS

If t is an angle measurement and a point (x, y) is both on the unit circle and the terminal side of the angle in standard position, then

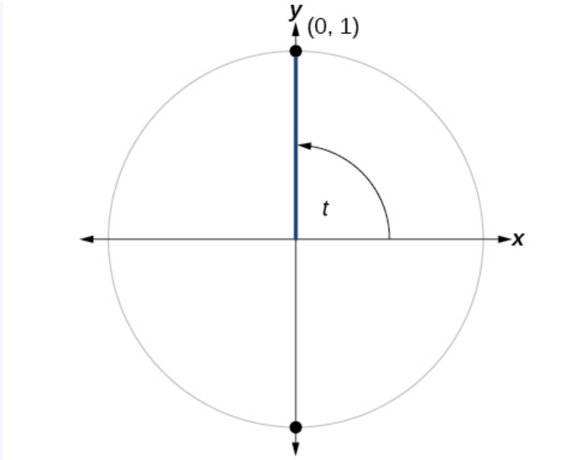
$$\cos t = x \tag{4.1.4.1}$$
$$\sin t = y \tag{4.1.4.2}$$
$$\tan t = \frac{\sin t}{\cos t} \tag{4.1.4.3}$$

Example 4.1.4.1

Find $\cos(90^o)$ and $\sin(90^o)$.

Solution

Moving 90^o counterclockwise around the unit circle from the positive x-axis brings us to the top of the circle, where the (x, y) coordinates are $(0, 1)$, as shown here:



Using our definitions of cosine and sine,

$$x = \cos t = \cos(90^{\text{o}}) = 0$$
$$y = \sin t = \sin(90^{\text{o}}) = 1$$

The cosine of 90^{o} is 0; the sine of 90^{o} is 1.

Try It 4.1.4.2

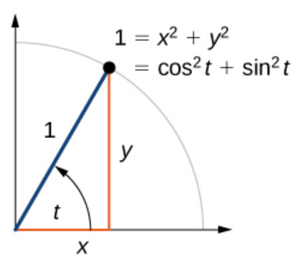
Find cosine and sine of the angle π.

Answer

$\cos(\pi) = -1, \ \sin(\pi) = 0$

The Pythagorean Identity

Now that we can define sine and cosine, we will learn how they relate to each other and the unit circle. Recall that the equation for the unit circle is $x^2 + y^2 = 1$. Because $x = \cos\, t$ and $y = \sin\, t$, we can substitute for x and y to get $\cos^2 t + \sin^2 t = 1$. This equation, $\cos^2 t + \sin^2 t = 1$, is known as the **Pythagorean Identity**. See below:



We can use the Pythagorean Identity to find the cosine of an angle if we know the sine, or vice versa. However, because the equation yields two solutions, we need additional knowledge of the angle to choose the solution with the correct sign. If we know the quadrant where the angle is, we can easily choose the correct solution.

PYTHAGOREAN IDENTITY

The **Pythagorean Identity** states that, for any angle t,

$$\cos^2 t + \sin^2 t = 1 \tag{4.1.4.4}$$

How To: Given the sine of some angle t and its quadrant location, find the cosine of t

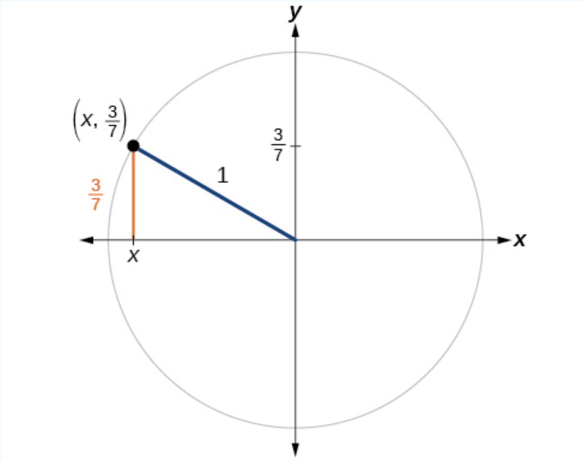
1. Substitute the known value of $\sin(t)$ into the Pythagorean Identity.
2. Solve for $\cos(t)$.
3. Choose the solution with the appropriate sign for the x-values in the quadrant where t is located.

Example 4.1.4.3

If $\sin(t) = \frac{3}{7}$ and t is in the second quadrant, find $\cos(t)$.

Solution

If we drop a vertical line from the point on the unit circle corresponding to t, we create a right triangle, from which we can see that the Pythagorean Identity is simply one case of the Pythagorean Theorem.



Substituting the known value for sine into the Pythagorean Identity,

$$\begin{aligned} \cos^2(t) + \sin^2(t) &= 1 \\ \cos^2(t) + \frac{9}{49} &= 1 \\ \cos^2(t) &= \frac{40}{49} \\ \cos(t) &= \pm\sqrt{\frac{40}{49}} = \pm\frac{\sqrt{40}}{7} = \pm\frac{2\sqrt{10}}{7} \end{aligned}$$

Because the angle is in the second quadrant, we know the x-value is a negative real number, so the cosine is also negative. So

$$\cos(t) = -\frac{2\sqrt{10}}{7}$$

Try It 4.1.4.4

If $\cos(t) = \frac{24}{25}$ and t is in the fourth quadrant, find $\sin(t)$.

Answer

$$\sin(t) = -\frac{7}{25}$$

Finding Sines, Cosines and Tangent Ratios of Special Angles

When possible, we would like to be precise about evaluation of the sine and cosine ratios. We give an example here.

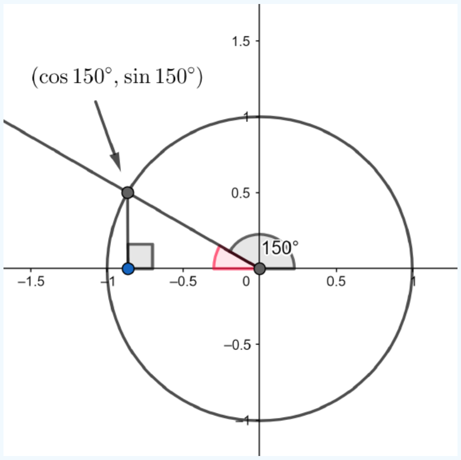
Example 4.1.4.5

a) Find the value of $\sin 150^{\mathrm{o}}$

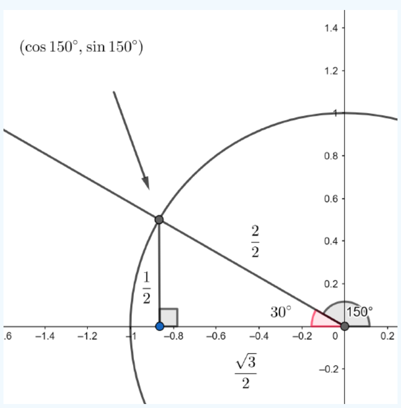
b) Find the value of $\cos \dfrac{7\pi}{4}$.

Solution

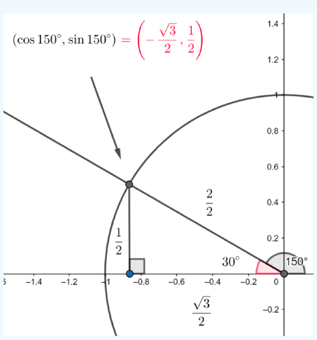
a) We first draw a picture and identify a triangle (while there are two natural possibilities, we will use the triangle shown below (often called "the reference triangle)).



Then determine the reference angle indicated in red. To do this we note that the straight angle (black angle plus the red angle is 180^{o}. So the red angle measures 30^{o}. Then we use our knowledge of the 30-60-90 triangle (choosing the one with hypotenuse 1) and note the lengths on the diagram.

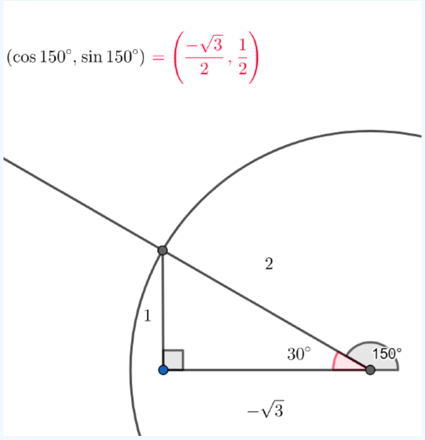


Finally, making note of the position of the triangle, we see that the x-coordinate must be negative and the y-coordinate must be positive, we find

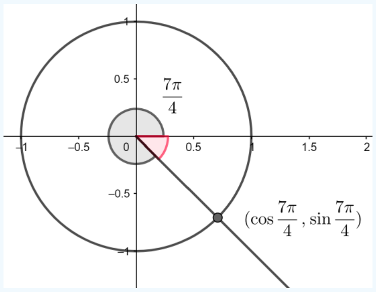


So we conclude, in particular, that $\sin 150^{\circ} = \frac{1}{2}$.

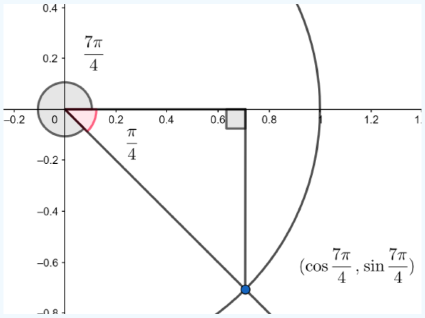
Note that we could also use the standard triangle (using the circle of radius 2) and use signed lengths and form the sine ratio as before:



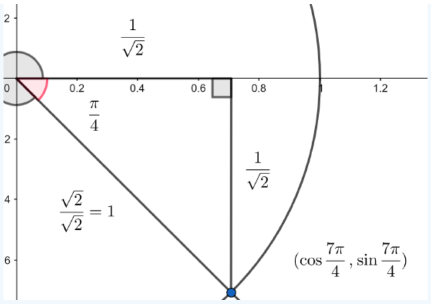
b) We first draw the angle together with the unit circle and label the intersection:



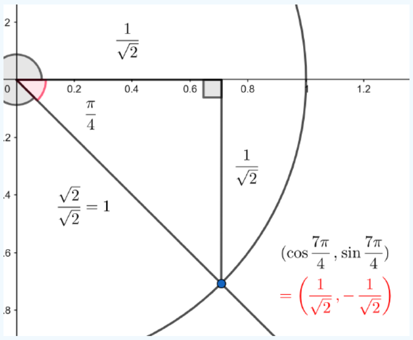
We determine the value of the red angle (noting that the black angle plus the red angle measures 2π. We construct a right triangle that will help us determine the coordinates of the labeled intersection (here we will use, again, what is often called 'the reference triangle').



Now, we recall the 45-45-90 triangle from memory (or rederive it) and scale it to have the hypotenuse have length 1.

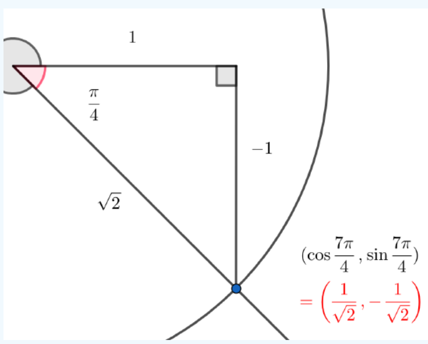


Now, we note from the location of the point that the x-coordinate must be positive and the y-coordinate must be negative. We determine the coordinates of the labeled intersection.



It follows that $\cos\dfrac{7\pi}{4} = \dfrac{1}{\sqrt{2}} = \dfrac{\sqrt{2}}{2}$.

As in part a) we could also use the standard triangle (using the circle of radius 2) and use signed lengths and form the sine ratio as before:



To find a reference angle, draw a sketch noting in which quadrant your terminal side is and use your familiarity with right angles and straight angles to figure out the reference angle. Try these exercises using the above example as a model.

? Try It 4.1.4.6

Find the value of

a) $\sin\dfrac{-5\pi}{3}$

b) $\cos 225^{o}$

c) $\tan 225^{o}$ (Here use the extension that $\tan t = \dfrac{\sin t}{\cos t}$.

Answer

a) $\dfrac{\sqrt{3}}{2}$

b) $\dfrac{-1}{\sqrt{2}}$

c) 1

Using a Calculator to Find Sine and Cosine

To find the cosine and sine of angles other than the **special angles**, we turn to a computer or calculator. **Be aware**: Most calculators can be set into "degree" or "radian" mode, which tells the calculator the units for the input value. When we evaluate $\cos(30^o)$ on our calculator, it will evaluate it as the cosine of 30 degrees if the calculator is in degree mode, or the cosine of 30 radians if the calculator is in radian mode.

Example 4.1.4.7: Using a Graphing Calculator to Find Sine and Cosine

a) Evaluate $\sin 150^o$ using a calculator or computer.

b) Evaluate cos $\dfrac{7\pi}{4}$

Solution

a) This depends on your calculator. But we know the answer is $\dfrac{1}{2}$. Make sure the mode on the calculator is set to 'degree'.

b) This depends on your calculator. But we know the answer is $\dfrac{1}{\sqrt{2}}$ so if you square your answer you should get $\dfrac{1}{2}$ and your answer should be positive. Make sure the mode on the calculator is set to 'radian'.

Try It 4.1.4.8

Evaluate $\sin(-50)$.

Answer

Approximately -7.66

Writing Exercises 4.1.4.9

1. Explain how $\cos(300o)$ can be evaluated using the unit circle.
2. Give four angles θ, α, β and γ for which $\cos(300^o) = \cos(\theta) = \cos(\alpha) = \cos(\beta) = \cos(\gamma)$
3. Explain what a reference triangle is and how you can create it. Is there another triangle that would give you the same benefit?
4. If you know $\sin(x) = \dfrac{4}{5}$, how can you find $\cos(x)$ and $\tan(x)$ without first finding x. How might you use similar triangles/figures to make use of a circle of radius 5 instead of the unit circle?

Exit Problem

Evaluate $\cos(-210^o)$ and $\sin(-210^o)$. Support your answer with an appropriate picture of the unit circle. Then evaluate $\tan(210^o)$.

Key Concepts

- Finding the function values for the sine and cosine begins with drawing a unit circle, which is centered at the origin and has a radius of 1 unit.
- Using the unit circle, the sine of an angle t equals the y-value of the endpoint on the unit circle of an arc of length t whereas the cosine of an angle t equals the x-value of the endpoint.
- The sine and cosine values are most directly determined when the corresponding point on the unit circle falls on an axis.
- When the sine or cosine is known, we can use the Pythagorean Identity to find the other. The Pythagorean Identity is also useful for determining the sines and cosines of special angles.
- Calculators and graphing software are helpful for finding sines and cosines if the proper procedure for entering information is known.
- The domain of the sine and cosine functions is all real numbers.
- The range of both the sine and cosine functions is $[-1, 1]$.
- The sine and cosine of an angle have the same absolute value as the sine and cosine of its reference angle.
- The signs of the sine and cosine are determined from the \(*x*\)- and y-values in the quadrant of the original angle.
- An angle's reference angle is the size angle, t, formed by the terminal side of the angle t and the horizontal axis.
- Reference angles can be used to find the sine and cosine of the original angle.
- Reference angles can also be used to find the coordinates of a point on a circle.

4.2: Trigonometric Equations

Learning Objectives

By the end of this section, you will be able to:

- Find two specific solutions to simple trigonometric equations.
- Draw a picture of the unit circle illustrating the solutions to certain simple trigonometric equations.

Be Prepared

Before you get started, take this readiness quiz.

1. Draw a picture of a triangle where one vertex is the origin, one is on the unit circle in the 2nd quadrant and one is on the x-axis.
2. Draw a picture of a triangle where on vertex is the origin, one is on the unit circle and in the 4th quadrant and one is on th x-axis.
3. If in the picture above, the coordinates of the vertex which is on the unit circle has coordinates $(3,-4)$, find the lengths of the sides of triangle.

An equation involving trigonometric functions is called a *trigonometric equation.* For example, an equation like

$$\cos A=0.75,$$

which we encountered in while solving right triangles, is a trigonometric equation. There we were concerned only with finding a single solution (say, between 0° and 90°). In this section we will be concerned with finding the most general solution to such equations.

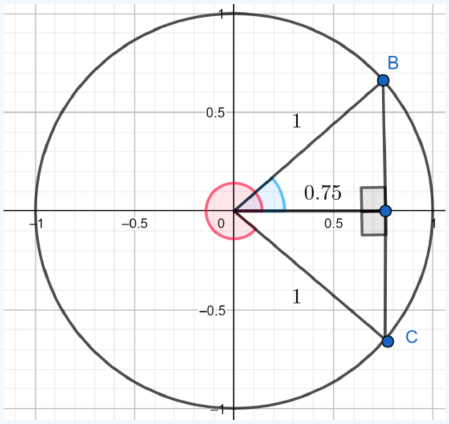
To see what that means, we first look at the example:

Example 4.2.1

Solve $\cos A=0.75$ where $0^\circ \leq A<360^\circ$.

Solution

Using the $\boxed{\cos^{-1}}$ calculator button in degree mode, we get $A=41.41^\circ$. However, we know the $\cos A$ is the x-coordinate of the point on the unit circle intersecting with the terminal side of the angle A. This x-coordinate is given as positive, viz., .75, so we draw two possibilities (the drawing needn't be precise but only indicate relationships accurately):



The points B and C are the points on the unit circle with x-coordinate 0.75. To these coorespond two angles, one in blue and one in red. Calling the blue angle A for the moment, we find that A is an angle of a right triangle and $\cos A = .75$ so that $A = 41.41^\circ$ as found above by using the calculator. The angle of the other right triangle (in the 4th quadrant) is congruent (same size and shape) of the triangle in the first quadrant and so the angle at the origin is also 38.68°. It follows that, since the sum of the red angle and 41.41° is 360°, the red angle is $(360 - 41.41)^\circ = 319.59^\circ$.

These are the solutions of the equation that are in $[0, 360^\circ)$. There are many more since we should also look at all possible co-terminal angles. For example $360 + 41.41$ degrees. There are an infinite number of solutions if we do not consider the restriction $0^\circ \leq A < 360^\circ$.

Try It 4.2.2

Solve $\cos\theta = -\dfrac{1}{2}$ if $0^\circ \leq \theta < 360^\circ$.

Answer

$\theta = 120^\circ$ or $\theta = 240^\circ$

Try It 4.2.3

Solve $\tan x = -\dfrac{1}{3}$ if $0 \leq \theta < 2\pi$ (radians). Round your answer to the nearest hundredth.

Answer

$x = 2.82$ or $x = 5.96$ radians.

Try the following examples before looking at the detailed solutions.

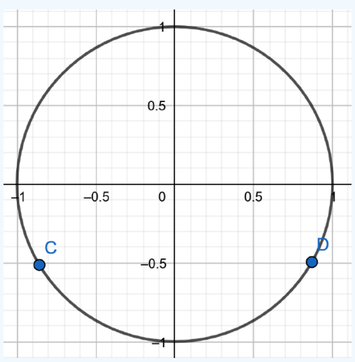
Example 4.2.4

Solve the equation $2\sin\theta + 1 = 0$.

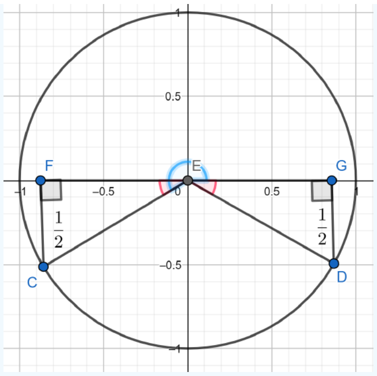
Solution:

Isolating $\sin\theta$ gives $\sin\theta = -\dfrac{1}{2}$.

Recall that the sin ratio is y-coordinate on the unit circle. This is negative (negative $\dfrac{1}{2}$). So, we can sketch the possibilities by marking the point on the unit circle with y-coordinate equal to $-\dfrac{1}{2}$



and then to construct two congruent triangles (called reference triangles).



The two triangles are congruent and the red angles are the same.

Noting that the triangle is a similar to the 30-60-90 triangle we see that the red angles are $\frac{\pi}{6}$. We could also use the $\boxed{\sin^{-1}}$ calculator button in degree mode which gives us $\theta = -30^\circ$ only the angle in quadrant IV. To find the blue angle, we have to see how it relates to the red angle. The blue angle minus the red angle is a straight angle. So our blue angle is $180 + 30 = 210^\circ$ or in radians $\pi + \frac{\pi}{6} = \frac{7\pi}{6}$. So two solutions are (and all others are coterminal)

$$\theta = -30^\circ \text{ or } \theta = 210^\circ. \tag{4.2.1}$$

If we want find the solutions in $[0, 360^\circ]$ we will need to add 360° to the -30° to get 330° so,

If $2\sin\theta + 1 = 0$ and θ is in $[0, 360^\circ)$ then

$$\theta = 330^\circ \text{ or } \theta = 210^\circ. \tag{4.2.2}$$

In radians, the solution is:

If $2\sin\theta + 1 = 0$ and θ is in $[0, 2\pi)$ then

$$\boxed{\theta = \frac{11\pi}{6} \quad or \quad \frac{7\pi}{6}} \tag{4.2.3}$$

All other solutions are co-terminal to these.

Example 4.2.5

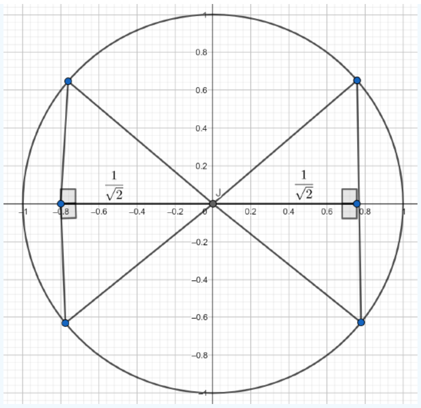
Solve the equation $2\cos^2\theta - 1 = 0$, where θ is in $[0, 2\pi)$.

Solution:

Isolating $\cos^2\theta$ gives us

$$\cos^2\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \pm\frac{1}{\sqrt{2}} \tag{4.2.4}$$

Noting that this means the x-coordinate of points on the unit circle associated with θ are $\pm\frac{1}{\sqrt{2}}$.



Each of the triangles are special triangles and the angles at the origin are all $\frac{\pi}{4}$. So we have that

if $2\cos^2\theta - 1 = 0$, where θ is in $[0, 2\pi)$,

then

$$\theta = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \text{ or } \frac{7\pi}{4}. \tag{4.2.5}$$

Example 4.2.6

Solve the equation $2\sec\theta = 1$.

Solution:

Isolating $\sec\theta$ gives us

$$\sec\theta = \frac{1}{2} \quad \Rightarrow \quad \cos\theta = \frac{1}{\sec\theta} = 2,$$

which is impossible. Thus, there is $\boxed{\text{no solution}}$.

Example 4.2.7

Solve the equation $\cos\theta = \tan\theta$.

Solution:

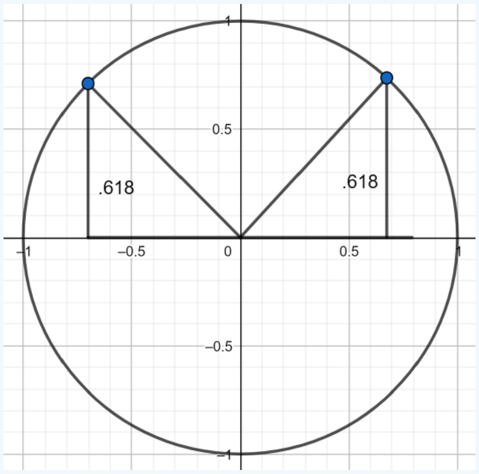
The idea here is to use identities to put everything in terms of a single trigonometric function:

$$\begin{aligned} \cos\theta &= \tan\theta \\ \cos\theta &= \frac{\sin\theta}{\cos\theta} \\ \cos^2\theta &= \sin\theta \\ 1 - \sin^2\theta &= \sin\theta \\ 0 &= \sin^2\theta + \sin\theta - 1 \end{aligned}$$

The last equation looks more complicated than the original equation, but notice that it is actually a *quadratic* equation: making the substitution $x = \sin\theta$, we have

$$x^2 + x - 1 = 0 \quad \Rightarrow \quad x = \frac{-1 \pm \sqrt{1 - (4)(-1)}}{2(1)} = \frac{-1 \pm \sqrt{5}}{2} = -1.618,\ 0.618$$

by the quadratic formula from elementary algebra. But -1.618 can not be a y-coordinate of a point on the unit circle, so it is impossible that $\sin\theta = x = -1.618$. Thus, we must have $\sin\theta = x = 0.618$. Hence, there are two possible solutions corresponding to the points on the unit circle with y-coordinate 0.618:



So, if $\cos\theta = \tan\theta$,

then

$\theta = 0.666$ radians or its reflection $\pi - .666 = 2.476$ radians (or, more accurately, $\pi - \sin^{-1} .618 = 2.475$.

Challenge examples

Try to solve the equations before looking at the solution.

Example 4.2.8

Solve the equation $\sin\theta = \tan\theta$.

Solution:

Trying the same method as in the previous example, we get

$$\begin{aligned} \sin\theta &= \tan\theta \\ \sin\theta &= \frac{\sin\theta}{\cos\theta} \\ \sin\theta\cos\theta &= \sin\theta \\ \sin\theta\cos\theta - \sin\theta &= 0 \\ \sin\theta(\cos\theta - 1) &= 0 \\ \Rightarrow \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = 1 \\ \Rightarrow \quad \theta = 0, \pi \quad \text{or} \quad \theta = 0 \\ \Rightarrow \quad \theta = 0, \pi, \end{aligned}$$

plus multiples of 2π. So since the above angles are multiples of π, and every multiple of 2π is a multiple of π, we can combine the two answers into one for the general solution:

$$\boxed{\theta = \pi k} \qquad \text{for } k = 0, \pm1, \pm2, \ldots$$

Example 4.2.9

Solve the equation $\cos(3\theta) = \frac{1}{2}$.

Solution:

The idea here is to solve for 3θ first, using the most general solution, and then divide that solution by 3. So since $\cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$, there are two possible solutions for 3θ: $3\theta = \frac{\pi}{3}$ in QI and its reflection $-3\theta = -\frac{\pi}{3}$ around the x-axis in QIV. Adding multiples of 2π to these gives us:

$$3\theta = \pm\frac{\pi}{3} + 2\pi k \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

So dividing everything by 3 we get the general solution for θ:

$$\boxed{\theta = \pm\frac{\pi}{9} + \frac{2\pi}{3}k} \qquad \text{for } k = 0, \pm 1, \pm 2, \ldots$$

Challenge example using advanced trigonometric identities

Solve the equation $2\sin\theta - 3\cos\theta = 1$. Use the fact (not treated here) that for any angles A and B,

$$\sin(A - B) = \sin(A)\cos(B) - \sin(B)\cos(A). \tag{4.2.6}$$

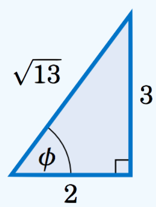


Figure 6.1.1: Copy and Paste Caption here. (Copyright; author via source)

Solution

We will use the technique which we discussed in Chapter 5 for finding the amplitude of a combination of sine and cosine functions. Take the coefficients 2 and 3 of $\sin\theta$ and $-\cos\theta$, respectively, in the above equation and make them the legs of a right triangle, as in Figure 6.1.1. Let ϕ be the angle shown in the right triangle. The leg with length $3 > 0$ means that the angle ϕ is above the x-axis, and the leg with length $2 > 0$ means that ϕ is to the right of the y-axis. Hence, ϕ must be in QI. The hypotenuse has length $\sqrt{13}$ by the Pythagorean Theorem, and hence $\cos\phi = \frac{2}{\sqrt{13}}$ and $\sin\theta = \frac{3}{\sqrt{13}}$. We can use this to transform the equation to solve as follows:

$$\begin{aligned}
2\sin\theta - 3\cos\theta &= 1 \\
\sqrt{13}\left(\frac{2}{\sqrt{13}}\sin\theta - \frac{3}{\sqrt{13}}\cos\theta\right) &= 1 \\
\sqrt{13}(\cos\phi\sin\theta - \sin\phi\cos\theta) &= 1 \\
\sqrt{13}\sin(\theta - \phi) &= 1 \quad \text{(by the sine subtraction formula)} \\
\sin(\theta - \phi) &= \frac{1}{\sqrt{13}} \\
\Rightarrow \quad \theta - \phi = 0.281 \quad &\text{or} \quad \theta - \phi = \pi - 0.281 = 2.861 \\
\Rightarrow \quad \theta = \phi + 0.281 \quad &\text{or} \quad \theta = \phi + 2.861
\end{aligned}$$

Now, since $\cos\phi = \frac{2}{\sqrt{13}}$ and ϕ is in QI, the most general solution for ϕ is $\phi = 0.983 + 2\pi k$ for $k = 0, \pm 1, \pm 2, \ldots$. So since we needed to add multiples of 2π to the solutions 0.281 and 2.861 anyway, the most general solution for θ is:

$$\begin{aligned} \theta &= 0.983 + 0.281 + 2\pi k \quad \text{and} \quad 0.983 + 2.861 + 2\pi k \\ \Rightarrow & \boxed{\theta = 1.264 + 2\pi k \quad \text{and} \quad 3.844 + 2\pi k} \quad \text{for } k = 0, \pm 1, \pm 2, \ldots \end{aligned}$$

Note: In Example 6.8 if the equation had been $2\sin\theta + 3\cos\theta = 1$ then we still would have used a right triangle with legs of lengths 2 and 3, but we would have used the sine addition formula instead of the subtraction formula.

? Written Exercises 4.2.10

1. If you know one solution to a trigonometric equaiton $\cos(x) = a$ for a given a, how can you find another which is not coterminal?
2. How many solutions are there to the equation $\sin(x) = -\frac{1}{2}$? Do not solve this here.
3. If we replaced $-\frac{1}{2}$ by any other number in the previous question, would we get the same answer? Explain.
4. Solve $\cos(x) = -2$ and explain.

Exit Problem

1. Solve $\cos(x) = -\frac{1}{\sqrt{2}}$, where $0 \leq x < 2\pi$.
2. Solve $\sin(\theta) = -\frac{1}{3}$, where $0 \leq \theta < 360^\circ$.

Key Concepts

- Unit circle and the trigonometric ratios
- Solutions to equations in one variable

SECTION OVERVIEW

4.3: Exponential and Logarithmic Expressions

4.3.1: Evaluating Exponential Expressions

Learning Objectives

By the end of this section, you will be able to:

- Graph basic exponential equations
- Solve exponential equations
- Use exponential models in applications

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\frac{x^3}{x^2}$.

2. Evaluate:

a. 2^0

b. $\left(\frac{1}{3}\right)^0$

3. Evaluate:

a. 2^{-1}

b. $\left(\frac{1}{3}\right)^{-1}$

Basic Exponential Expressions

The expressions we have studied so far do not give us a model for many naturally occurring phenomena. From the growth of populations and the spread of viruses to radioactive decay and compounding interest, the models are very different from what we have studied so far. These models involve exponential expressions.

An **exponential expression** is an expression of the form a^x where $a>0$ and $a\neq 1$.

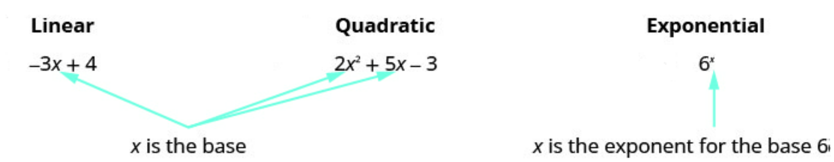
Definition 4.3.1.1

An **exponential expression**, where $a>0$ and $a\neq 1$, is an expression of the form

$$a^x,$$

or an expression containing expressions of that form.

Notice that in this expression, the variable is the exponent. In our expressions so far, the variables were the base.



Our definition says $a\neq 1$. If we let $a=1$, then a^x becomes 1^x. But we know $1^x=1$ for all real numbers.

Our definition also says $a>0$. If we let a base be negative, say -4, then $(-4)^x$ is not a real number when $x=\frac{1}{2}$.

In fact, $(-4)^x$ would not be a real number any time x is a fraction with an even denominator. So our definition requires $a>0$.

We will assume that the base in an exponential expression is positive.

We notice $a^0 = 1$ for any a.

Also, $a^1 = a$ and $a^{-1} = \frac{1}{a}$ for any a.

The expression a^x, $a > 1$ makes sense for any value of x (we know that it is true for x rational and will accept this as fact for other values of x).

It is clear that $a^x > 0$ for rational values of x and is in fact true for all values of x.

Also, $\left(\frac{a}{b}\right)^{-x} = \left(\frac{b}{a}\right)^x$, so for example, $(2)^{-x} = \left(\frac{1}{2}\right)^x$.

Note 4.3.1.2

Natural Base e

In applications, it happens that different bases are convenient for writing relevant expressions. There is one number which is written "e" . We will not delve into the details of this number except to say that this number is irrational and

$$e \approx 2.718281827.$$

Most calculators will include this particular base on a button often labeled e^x.

Notice that for each x, $2^x < e^x < 3^x$ since $2 < e < 3$. So you may get a rough idea of the values for particular values of x without using a calculator!

Note

One-to-One Property of Exponential Equations

For $a > 0$ and $a \neq 1$,

$$\text{if } a^x = a^y \text{, then } x = y.$$

Using Exponential Models in Applications

Exponential equations model many situations. If you own a bank account, you have experienced the use of an exponential equation. There are two formulas that are used to determine the balance in the account when interest is earned. If a principal, P, is invested at an interest rate, r, for t years, the new balance, A, will depend on how often the interest is compounded, i.e., how often the interest is calculated and then added to the new balance. If the interest is compounded n times a year we use the formula $A = P\left(1 + \frac{r}{n}\right)^{nt}$. If the interest is compounded continuously, we use the formula $A = Pe^{rt}$. These are the formulas for **compound interest**.

Definitions and Formulae 4.3.1.3

Compound Interest is interest that accumulates on the interest earned.

The **principal** is the amount invested.

Interest is compounded when the interest is calculated and then added to the new balance.

For a **principal**, P, invested at an interest **rate**, r, for t years, the new balance is:

$$\begin{array}{ll} P\left(1+\frac{r}{n}\right)^{nt} & \text{when compounded } n \text{ times a year.} \\ Pe^{rt} & \text{when continuously.} \end{array}$$

To see why the first formula works, work out some examples by writing down how the compounding works in detail. As you work with the Interest formulas, it is often helpful to identify the values of the variables first and then substitute them into the formula.

Example 4.3.1.4

A total of $\$10,000$ was invested in a college fund for a new grandchild. If the interest rate is 5%, how much will be in the account in 18 years by each method of compounding?

a. compound quarterly

b. compound monthly

c. compound continuously

Solution:

a.

Identify the values of each variable in the formulas. Remember to express the percent as a decimal.	$P = \$10,000$ $r = 0.05$ $t = 18$ years
For quarterly compounding, $n = 4$. There are 4 quarters in a year.	$P\left(1+\frac{r}{n}\right)^{nt}$
Substitute the values in the formula.	$10,000\left(1+\frac{0.05}{4}\right)^{4\cdot 18}$
Compute the amount. Be careful to consider the order of operations as you enter the expression into your calculator.	The amount in the account will be $\$24,459.20$

b.

For monthly compounding, $n = 12$.There are 12 months in a year.	$P\left(1+\frac{r}{n}\right)^{nt}$
Substitute the values in the formula.	$10,000\left(1+\frac{0.05}{12}\right)^{12\cdot 18}$
Compute the amount.	The amount in the account will be $\$24,550.08$

c.

For compounding continuously,	Pe^{rt}
Substitute the values in the formula.	$10,000e^{0.05\cdot 18}$
Compute the amount.	The amount in the account will be \(\$ 24,596.03\

Try It 4.3.1.5

Angela invested $\$15,000$ in a savings account. If the interest rate is 4%, how much will be in the account in 10 years by each method of compounding?

a. compound quarterly

b. compound monthly

c. compound continuously

Answer

a. $\$22,332.96$

b. $\$22,362.49$

c. $\$22,377.37$

Try It 4.3.1.6

Allan invested $\$10,000$ in a mutual fund. If the interest rate is 5%, how much will be in the account in 15 years by each method of compounding?

a. compound quarterly

b. compound monthly

c. compound continuously

Answer

a. $\$21,071.81$

b. $\$21,137.04$

c. $\$21,170.00$

Other topics that are modeled by exponential equations involve growth and decay. Both also use the expression Pe^{rt} we used for the growth of money. For growth and decay, generally we use A_0 (Amount at time 0), as the original amount instead of calling it P, the principal. We see that **exponential growth** has a positive rate r of growth and **exponential decay** has a negative rate r of growth.

Definition 4.3.1.7

Exponential Growth and Decay

For an original amount, A_0, that grows or decays at a rate, r, for a certain time, t, the final amount is:

$$A_0e^{rt}$$

Exponential growth is typically seen in the growth of populations of humans or animals or bacteria. Our next example looks at the growth of a virus.

Example 4.3.1.8

Chris is a researcher at the Center for Disease Control and Prevention and he is trying to understand the behavior of a new and dangerous virus. He starts his experiment with 100 of the virus that grows at a rate of 25% per hour. He will check on the virus in 24 hours. How many viruses will he find?

Solution:

Identify the values of each variable in the formulas. Be sure to put the percent in decimal form. Be sure the units match--the rate is per hour and the time is in hours.	$A_0 = 100$ $r = 0.25/\text{ hour}$ $t = 24\text{ hours}$
Substitute the values in the expression: A_0e^{rt}.	$100e^{0.25\cdot 24}$
Compute the amount.	$40,342.88$
Round to the nearest whole virus.	$40,343$
Conclude.	The researcher will find $40,343$ viruses.

Try It 4.3.1.9

Another researcher at the Center for Disease Control and Prevention, Lisa, is studying the growth of a bacteria. She starts his experiment with 50 of the bacteria that grows at a rate of 15% per hour. He will check on the bacteria every 8 hours. How many bacteria will he find in 8 hours?

Answer

She will find 166 bacteria.

Try It 4.3.1.10

Maria, a biologist is observing the growth pattern of a virus. She starts with 100 of the virus that grows at a rate of 10% per hour. She will check on the virus in 24 hours. How many viruses will she find?

Answer

She will find $1,102$ viruses.

Writing Exercises 4.3.1.11

1. Why are we dividing by n in the interest formula?
2. Why is 1 being added to r/n in the formula?

Exit Question

How much do you have to invest today at 2% compounded monthly to obtain $\$50,000$ in return in 3 years? What if it is compounded continuously? Which one is better?

Key Concepts

- **Compound Interest:** For a principal, P, invested at an interest rate, r, for t years, the new balance is

  $P\left(1+\frac{r}{n}\right)^{nt}$ when compounded n times a year.

  Pe^{rt} when compounded continuously.
- **Exponential Growth and Decay:** For an original amount, A_0 that grows or decays at a rate, r, for a certain time t, the final amount is A_0e^{rt}.

Glossary

exponential expression

An exponential expression, where $a>0$ and $a\neq 1$, is an expression involving an expression of the form a^x.

natural base

The number e is an irrational number that appears in many applications. $e\approx 2.718281827\ldots$

4.3.2: Evaluating Logarithmic Expressions

Learning Objectives

By the end of this section, you will be able to:

- Convert between exponential and logarithmic form
- Understand what a logarithm is.
- Estimate the value of logarithms and evaluate certain logarithms exactly without a calculator.
- Evaluate logarithmic expressions
- Graph basic logarithmic equations
- Solve logarithmic equations
- Use logarithmic models in applications

Be Prepared

Before you get started, take this readiness quiz.

1. Simplify $\left(\sqrt{81}\right)^2$.

2. Evaluate 2^3.

3. Evaluate 3^{-2}.

4. Write $\sqrt[3]{7}$ using exponents.

It works well to ‘undo’ an operation with another operation. Subtracting ‘undoes’ addition, multiplication ‘undoes’ division, taking the square root ‘undoes’ squaring.

Subtraction 'undoes' addition:	$x+3-3=x$ $x-3+3=x$
Division 'undoes' multiplication:	$\frac{1}{3}\cdot(3x)=x$ or $\frac{3x}{3}=x$ $3\cdot\left(\frac{x}{3}\right)=x$
square root 'undoes' squaring (same button on calculator!)	$(\sqrt{x})^2=x$ $\sqrt{x^2}=x$ for $x\geq 0$.
What undoes the exponential e^x? What about 10^x? and more generally, a^x?	The calculator likely carries the answer or at least part of it!

In the previous section we learned about exponential expressions and how to evaluate them. In this section we will learn about a related type of expression, namely, the logarithmic expression.

Suppose we want to solve a simple exponential equation $2^x=16$. Because we know how to evaluate it, we can use trial and error with a calculator to find an approximation to the value of x. What we are looking for is the value of the exponent we need to raise the base 2 to to arrive at 16. This exponent (which depends on the base and the result) is called $\log_2(16)$.

In general,

Definition: Logarithm

Assume $b>0$. Then $\log_b(a)$ ('the logarithm with base b of a') is the exponent we need on the base b to arrive at a value of a, or in other words, $\log_b(a)$ is the solution to

$b^x=a$.

You might wonder why we assume the base of the logarithm should be positive. What are the issues when this isn't the case?

Example 4.3.2.1

Evaluate $\log_5 25$.

Solution

We are looking here for the exponent we need to raise 5 to in order to arrive at 25. But we know $5^2 = 25$. It follows that $\log_5 25 = 2$.

Try It \PageIndex{1}\)

Evaluate

1. $\log_3 27$
2. $\log_{10}\left(\frac{1}{100}\right)$
3. $\log_{16} 4$

Answer

1. 3
2. -2
3. $\frac{1}{2}$

Now, we will also note that

$$\log_b b^c = c \tag{4.3.2.1}$$

since we need to raise the base b to the exponent c in order to arrive at b^c.

Example 4.3.2.1

Evaluate $\log_5(25\sqrt{5})$.

Solution

We are looking here for the exponent we need to raise 5 to in order to arrive at $25\sqrt{5}$. But we know $25\sqrt{5} = 5^2 \cdot 5^{\frac{1}{2}} = 5^{2+\frac{1}{2}} = 5^{\frac{5}{2}}$. It follows that $\log_5(25\sqrt{5}) = \frac{5}{2}$

Try It \PageIndex{2}\)

Evaluate:

1. $\log_3 27\sqrt[4]{3}$
2. $\log_{10}\left(\frac{\sqrt[3]{100}}{100}\right)$
3. $\log_{16}\left(\frac{4}{\sqrt[3]{16}}\right)$

Answer

1. $\frac{13}{3}$
2. $-\frac{4}{3}$
3. $\frac{1}{6}$

Note that just solving equations $x^2 = 15$ involves the use of the square root, solving an equation $2^x = 15$ involves the logarithm. Most calculators reflect this analogy by the placement of the buttons. There are typically two easily accessible logarithm buttons: 'log' which is shorthand for $\log_{10}$ and 'ln' which is shorthand for $\log_e$. Others logarithms will require additional understanding or a fancy calculator to approximate. Here we will give some examples using only the trial and error method for evaluation.

Example 4.3.2.3

Approximate $\log_5 7$.

Solution

This is equivalent to finding an approximate solution to $5^x = 7$.

We note that $5^1 = 5$ and $5^2 = 25$, so the exponent we seek is between these to exponents since $5 < 7 < 25$. In fact 7 is closer to 5 so we expect that our sought-for exponent is closer to 1. We use our calculator to evaluate $5^{1.1}$, $5^{1.2}$ and $5^{1.3}$. we find that our desired exponent is between 1.2 and 1.3 and probably closer to 1.2. Evaluating $5^{1.21} \approx 7.01$ is enough to know the exponent we seek is between 1.2 and 1.21, probably closer to the latter. Evaluating $5^{1.209} \approx 6.9992$ is enough to conclude that our exponent is between 1.209 and 1.21 and, in particular, rounded to the nearest hundredth,

$\log_5(7) \approx 1.21$.

Try It \PageIndex{1}\)

Approximate $\log_{\frac{1}{2}}(7)$ to the nearest hundredth

Answer

-2.81

Definition 4.3.2.1

The solution to the equation $x = a^y$ is written

$\log_a x$ and is called the **logarithm** of x with **base** a, where $a > 0, x > 0$, and $a \neq 1$ if

$\log_a(a^x) = x$ and $a^{\log_a x} = x$,

or,

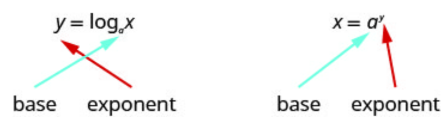
$y = \log_a x$ is equivalent to $x = a^y$.

So the chart becomes:

Subtraction 'undoes' addition:	$x+3-3=x$ $x-3+3=x$
Division 'undoes' multiplication:	$\frac{1}{3}\cdot(3x)=x$ or $\frac{3x}{3}=x$ $3\cdot\left(\frac{x}{3}\right)=x$
square root 'undoes' squaring (same button on calculator!)	$(\sqrt{x})^2=x$ $\sqrt{x^2}=x$ for $x\geq 0$.
logarithm 'undoes' exponential $\ln x$ 'undoes' e^x? $\log x$ 'undoes' 10^x? and more generally $\log_a x$ 'undoes' a^x?	**Notation:** $\ln e^x=x$ and $e^{\ln x}=x$ $\log 10^x=x$ and $10^{\log x}=x$ $\log_a a^x=x$ and $a^{\log_a x}=x$

Converting Between Exponential and Logarithmic Form (DELETE???)

Since the equations $y=\log_a x$ and $x=a^y$ are equivalent, we can go back and forth between them. This will often be the method to solve some exponential and logarithmic equations. To help with converting back and forth let's take a close look at the equations. See the figure below. Notice the positions of the exponent and base.



If we realize the logarithm is the exponent it makes the conversion easier. You may want to repeat, "base to the exponent give us the number."

Example 4.3.2.2

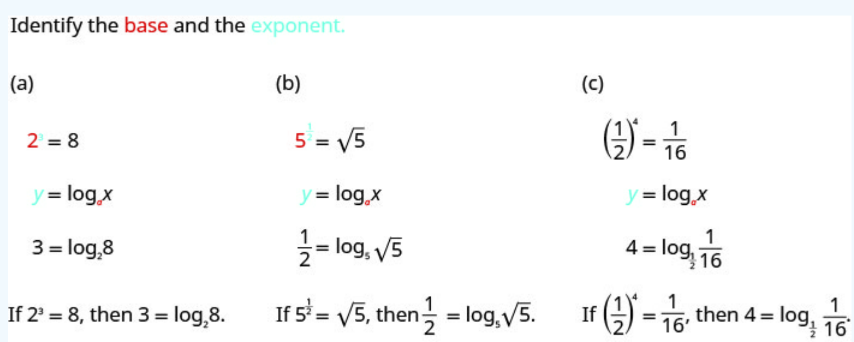
Convert to logarithmic form:

a. $2^3=8$

b. $5^{1/2}=\sqrt{5}$

c. $\left(\frac{1}{2}\right)^x=\frac{1}{16}$

Solution:



Try It 4.3.2.3

Convert to logarithmic form:

a. $3^2=9$

b. $7^{1/2} = \sqrt{7}$

c. $\left(\frac{1}{3}\right)^x = \frac{1}{27}$

Answer

a. $\log_3(9) = 2$

b. $\log_7(\sqrt{7}) = \frac{1}{2}$

c. $\log_{1/3}\left(\frac{1}{27}\right) = x$

Try It 4.3.2.4

Convert to logarithmic form:

a. $4^3 = 64$

b. $4^{1/3} = \sqrt[3]{4}$

c. $\left(\frac{1}{2}\right)^x = \frac{1}{32}$

Answer

a. $\log_4(64) = 3$

b. $\log_4(\sqrt[3]{4}) = \frac{1}{3}$

c. $\log_{1/2}\left(\frac{1}{32}\right) = x$

In the next example we do the reverse—convert logarithmic form to exponential form.

Example 4.3.2.5

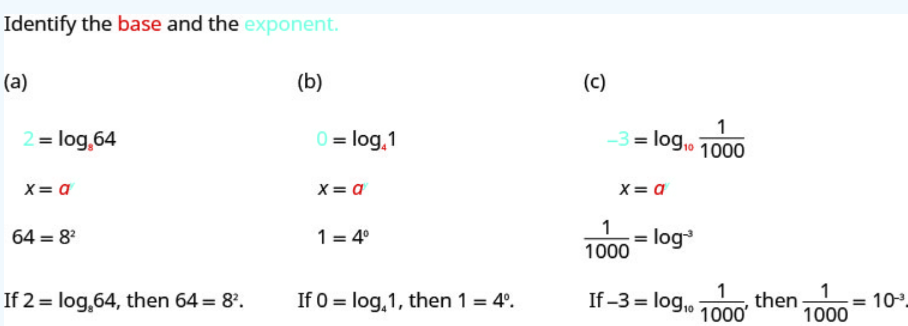
Convert to exponential form:

a. $2 = \log_8(64)$

b. $0 = \log_4(1)$

c. $-3 = \log_{10}\left(\frac{1}{1000}\right)$

Solution:



Try It 4.3.2.6

Convert to exponential form:

a. $3 = \log_4(64)$

b. $0 = \log_x(1)$

c. $-2 = \log_{10}\left(\frac{1}{100}\right)$

Answer

a. $64 = 4^3$

b. $1 = x^0$

c. $\frac{1}{100} = 10^{-2}$

Try It 4.3.2.7

Convert to exponential form:

a. $3 = \log_3(27)$

b. $0 = \log_x(1)$

c. $-1 = \log_{10}\left(\frac{1}{10}\right)$

Answer

a. $27 = 3^3$

b. $1 = x^0$

c. $\frac{1}{10} = 10^{-1}$

Evaluating Logarithmic Expressions

We can solve and evaluate logarithmic equations by using the technique of converting the equation to its equivalent exponential equation.

Example 4.3.2.8

Evaluate:

a. $\log_6(36)$

b. $\log_4(64)$

c. $\log_{1/2}\left(\frac{1}{8}\right)$

Solution:

a.

$\log_6 36$

This is the power you need to raise 6 to in order to get 36. Since $6^2 = 36$, this is 2!

So, $\log_6 36 = 2$.

b.

$\log_4 64$ is the power you need to raise 4 to to get 64.

Since $4^3 = 64$, we see the power is 3.

Therefore, $\log_4 64 = 3$.

c.

$\log_{1/2}\left(\frac{1}{8}\right)$ is the power we need to raise $\frac{1}{2}$ to to get $\frac{1}{8}$.

But

$$\left(\frac{1}{2}\right)^3 = \left(\frac{1}{8}\right)$$

Therefore $\log_{1/2}\left(\frac{1}{8}\right) = 3$.

? Try It 4.3.2.9

Evaluate:

a. $\log_8(64)$

b. $\log_5(125)$

c. $\log_{1/2}\left(\frac{1}{4}\right)$

Answer

a. 2

b. 3

c. 2

? Try It 4.3.2.10

Evaluate:

a. $\log_9(81)$

b. $\log_3(243)$

c. $\log_{1/3}\left(\frac{1}{27}\right)$

Answer

a. 2

b. 5

c. 3

When see an expression such as $\log_3 27$, we can find its exact value two ways. By inspection we realize it means "3 to what power will be 27"? Since $3^3 = 27$, we know $\log_3 27 = 3$. An alternate way is to set the expression equal to x and then convert it into an exponential equation.

Example 4.3.2.11

Find the exact value of each logarithm without using a calculator:

a. $\log_5(25)$

b. $\log_9(3)$

c. $\log_2\left(\frac{1}{16}\right)$

Solution:

a.

	$\log_5(25)$
5 to what power will be 25?	2
Conclude	$\log_5(25)=2$

Or

	$\log_5(25)$
Set the expression equal to x.	$\log_5(25)=x$
Change to exponential form.	$5^x=25$
Rewrite 25 as 5^2.	$5^x=5^2$
With the same base the exponents must be equal.	$x=2$
Conclude.	Therefore $\log_5(25)=2$.

b.

	$\log_9(3)$
Set the expression equal to x.	$\log_9(3)=x$
Change to exponential form.	$9^x=3$
Rewrite 9 as 3^2.	$\left(3^2\right)^x=3^1$
Simplify the exponents.	$3^{2x}=3^1$
With the same base the exponents must be equal.	$2x=1$
Solve the equation.	$x=\frac{1}{2}$
Conclude.	Therefore $\log_9(3)=\frac{1}{2}$.

c.

	$\log_2\left(\frac{1}{16}\right)$
Set the expression equal to x.	$\log_2\left(\frac{1}{16}\right)=x$

Change to exponential form.	$2^x = \frac{1}{16}$
Rewrite 16 as 2^4.	$2^x = \frac{1}{2^4}$
With the same base the exponents must be equal.	$2^x = 2^{-4}$ $x = -4$
Conclude.	Therefore $\log_2\left(\frac{1}{16}\right) = -4$.

? Try It 4.3.2.12

Find the exact value of each logarithm without using a calculator:

a. $\log_{12}(144)$

b. $\log_4(2)$

c. $\log_2\left(\frac{1}{32}\right)$

Answer

a. 2

b. $\frac{1}{2}$

c. -5

? Try It 4.3.2.13

Find the exact value of each logarithm without using a calculator:

a. $\log_9(81)$

b. $\log_8(2)$

c. $\log_3\left(\frac{1}{9}\right)$

Answer

a. 2

b. $\frac{1}{3}$

c. -2

It will be important for you to use your calculator to evaluate both common and natural logarithms.

Look for the log and ln keys on your calculator.

In general we have expressions $\ln x$, $\log x$, and $\log_a x$ for any value of x as can be approximated using your calculator (for now in the first two cases but also in general which we won't go into here).

We can therefore also have logarithmic expressions:

Definition 4.3.2.14

A **logarithmic expression** is an expression containing any of $\ln x$, $\log x$, and $\log_a x$ with x being replaced by any combinations of variables and numbers.

Using Logarithmic Models in Applications

There are many applications that are modeled by logarithmic expressions. We will first look at the logarithmic equation that gives the decibel (dB) level of sound. Decibels range from 0, which is barely audible to 160, which can rupture an eardrum. The 10^{-12} in the formula represents the intensity of sound that is barely audible.

Definition 4.3.2.15

Decibel Level of Sound

The loudness level measured in decibels, of a sound of intensity, I, measured in watts per square inch is

$$10\log\left(\frac{I}{10^{-12}}\right)$$

Example 4.3.2.16

Extended exposure to noise that measures 85 dB can cause permanent damage to the inner ear which will result in hearing loss. What is the decibel level of music coming through ear phones with intensity 10^{-2} watts per square inch?

Solution:

	$10\log\left(\frac{I}{10^{-12}}\right)$
Substitute in the intensity level, I.	$10\log\left(\frac{10^{-2}}{10^{-12}}\right)$
Simplify.	$10\log(10^{10})$
Since $\log(10^{10}) = 10$.	$10 \cdot 10$
Multiply.	100
	The decibel level of music coming through earphones is 100 dB.

Try It 4.3.2.17

What is the decibel level of one of the new quiet dishwashers with intensity 10^{-7} watts per square inch?

Answer

The quiet dishwashers have a decibel level of 50 dB.

Try It 4.3.2.18

What is the decibel level heavy city traffic with intensity 10^{-3} watts per square inch?

Answer

The decibel level of heavy traffic is 90 dB.

The magnitude R of an earthquake is measured by a logarithmic scale called the Richter scale. The model is $\log I$, where I is the intensity of the shock wave. This model provides a way to measure **earthquake intensity**.

Definition 4.3.2.19: Earthquake Intensity

The magnitude of an earthquake is measured by $\log I$, where I is the intensity of its shock wave.

Example 4.3.2.20

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. Over 80% of the city was destroyed by the resulting fires. In 2014, Los Angeles experienced a moderate earthquake that measured 5.1 on the Richter scale and caused $\$108$ million dollars of damage. Compare the intensities of the two earthquakes.

Solution:

To compare the intensities, we first need to convert the magnitudes to intensities using the log formula. Then we will set up a ratio to compare the intensities.

The logarithm of the intensity, I, of the 1906 earthquake is 7.8,	$7.8 = \log I.$
This means that 7.8 is the exponent of 10 needed to arrive at the Intensity:	$I = 10^{7.8}$
The magnitude of the intensity, J (we already used I for something else!), of the 2014 earthquake is	$5.1 = \log J$
This means that 5.1 is the exponent with base 10 needed to arrive at the intensity J:	$J = 10^{5.1}$
Form a ratio of the intensities.	$\frac{\text{Intensity for 1906}}{\text{Intensity for 2014}}$
Substitute in the values.	$\frac{10^{7.8}}{10^{5.1}}$
Divide by subtracting the exponents.	$10^{2.7}$
Evaluate.	501
	The intensity of the 1906 earthquake was about 501 times the intensity of the 2014 earthquake.

Try It 4.3.2.21

In 1906, San Francisco experienced an intense earthquake with a magnitude of 7.8 on the Richter scale. In 1989, the Loma Prieta earthquake also affected the San Francisco area, and measured 6.9 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the 1906 earthquake was about 8 times the intensity of the 1989 earthquake.

Try It 4.3.2.22

In 2014, Chile experienced an intense earthquake with a magnitude of 8.2 on the Richter scale. In 2014, Los Angeles also experienced an earthquake which measured 5.1 on the Richter scale. Compare the intensities of the two earthquakes.

Answer

The intensity of the earthquake in Chile was about $1,259$ times the intensity of the earthquake in Los Angeles.

Writing Exercises 4.3.2.23

1. How is approximating the solution to $10^x = 2$ similar to solving $x^2 = 5$ on a calculator?

2. What is $\log_b(a)$?

3. In the definition of $\log_b(a)$, why don't we allow b to be 1?

Exit Problem

Evaluate $\log_{1/2}(25)$.

Key Concepts

- **Evaluation of logarithms**
- **(Optional) Basic shape of the graph of** $y = \log_a x$ for various a values.
- **Decibel Level of Sound:** The loudness level measured in decibels, of a sound of intensity, I, measured in watts per square inch is $10 \log\left(\frac{I}{10^{-12}}\right)$.
- **Earthquake Intensity:** The magnitude of an earthquake is measured by $\log I$, where I is the intensity of its shock wave.

Glossary

common logarithm

$\log_{10} x = \log x$ is the common logarithm, where $x > 0$, which is the exponent with base 10 needed to evaluate to x.

Equivalently,

$$y = \log x \text{ is equivalent to } x = 10^y$$

logarithm

For $a > 0, a \neq 1$ and $x > 0$, the logarithm of x with base a, $\log_a x$ is the exponent with base a needed to evaluate to x. Or equivalently, $y = \log_a x$ is equivalent to $x = a^y$

natural logarithm

$\log_e x = \ln x$ is the natural logarithm, where $x > 0$. It is the exponent needed with base e to evaluate to x, or, equivalently,

$$y = \ln x \text{ is equivalent to } x = e^y$$

4.3.3: Properties of Logarithms

Learning Objectives

By the end of this section, you will be able to:

- Use the properties of logarithms
- Use the Change of Base Formula

Be Prepared

Before you get started, take this readiness quiz.

1. Evaluate

a. b^0

b. b^1

2. Write $\sqrt[3]{x^2y}$ with a rational exponent.

3. Round $2.5646415$to three decimal places.

Use the Properties of Logarithms

Now that we have learned about exponential and logarithmic expressions, we can introduce some of the properties of logarithms. These will be very helpful as we continue to solve both exponential and logarithmic equations.

The first two properties derive from the definition of logarithms. Since $b^0=1$, we can convert this to logarithmic form and get $\log_b(1)=0$. Also, since $b^1=b$, we get $\log_b(b)=1$.

Properties of Logarithms

$$\log_b(1)=0$$

$$\log_b(b)=1$$

In the next example we could evaluate the logarithm by converting to exponential form, as we have done previously, but recognizing and then applying the properties saves time.

Example 4.3.3.1

Evaluate using the properties of logarithms:

a. $\log_8(1)$

b. $\log_6(6)$

Solution

a.

	$\log_8(1)$
Use the property, $\log_b(1)=0$.	$=0$

b.

	$\log_6(6)$
Use the property, $\log_b(b)=1$.	$=1$

Try It 4.3.3.2

Evaluate using the properties of logarithms:

a. $\log_{13}(1)$

b. $\log_9(9)$

Answer

a. 0

b. 1

Try It 4.3.3.3

Evaluate using the properties of logarithms:

a. $\log_5(1)$

b. $\log_7(7)$

Answer

a. 0

b. 1

The next two properties can also be verified by converting them from exponential form to logarithmic form, or the reverse.

The exponential equation $b^{\log_b(x)} = x$ converts to the logarithmic equation $\log_b(x) = \log_b(x)$, which is a true statement for positive values for x only.

The logarithmic equation $\log_b(b^x) = x$ converts to the exponential equation $b^x = b^x$, which is also a true statement (for all values of x).

These two properties are called inverse properties because, when we have the same base, raising to a power "undoes" the log and taking the log "undoes" raising to a power.

Inverse Properties of Logarithms

For $b > 0$ and $b \neq 1$,

$$b^{\log_b(x)} = x \text{ for } x > 0 \text{ and } \log_b(b^x) = x.$$

In the next example, we apply the inverse properties of logarithms.

Example 4.3.3.4

Evaluate using the properties of logarithms:

a. $4^{\log_4(9)}$

b. $\log_3(3^5)$

Solution

a.

	$4^{\log_4(9)}$
Use the property, $b^{\log_b(x)} = x$.	$= 9$

b.

	$\log_3(3^5)$
Use the property, $\log_b(b^x) = x$.	$= 5$

Try It 4.3.3.5

Evaluate using the properties of logarithms:

a. $5^{\log_5(15)}$

b. $\log_7(7^4)$

Answer

a. 15

b. 4

Try It 4.3.3.6

Evaluate using the properties of logarithms:

a. $2^{\log_2(8)}$

b. $\log_2(2^{15})$

Answer

a. 8

b. 15

There are three more properties of logarithms that will be useful in our work. We know exponential expressions and logarithmic expressions are very interrelated. Our definition of logarithm shows us that a logarithm is the exponent of the equivalent exponential. The properties of exponents have related properties for exponents.

In the Product Property of Exponents, $b^m b^n = b^{m+n}$, we see that to multiply the same base, we add the exponents. The **Product Property of Logarithms**, $\log_b(MN) = \log_b(M) + \log_b(N)$, tells us that to take the log of a product, we add the log of the factors (here we use $M = b^m$ and $N = b^n$ and note that $\log_b(b^{m+n}) = m + n = \log_b(b^m) + \log_b(b^n)$.

Product Property of Logarithms

If $M > 0$, $N > 0$, $b > 0$ and $b \neq 1$, then

$$\log_b(MN) = \log_b(M) + \log_b(N).$$

The logarithm of a product is the sum of the logarithms.

We use this property to write the log of a product as a sum of the logs of each factor.

Example 4.3.3.7

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible:

a. $\log_3(7x)$

b. $\log_4(64xy)$

Solution

a.

$\log_3(7x)$	
Use the Product Property, $\log_b(MN) = \log_b(M) + \log_b(N)$.	$= \log_3(7) + \log_3(x)$

b.

$\log_4(64xy)$	
Use the Product Property, $\log_b(MN) = \log_b(M) + \log_b(N)$.	$= \log_4(64) + \log_4(x) + \log_4(y)$
Simplify by evaluating $\log_4(64)$.	$= 3 + \log_4(x) + \log_4(y)$

Try It 4.3.3.8

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a. $\log_3(3x)$

b. $\log_2(8xy)$

Answer

a. $1 + \log_3(x)$

b. $3 + \log_2(x) + \log_2(y)$

Try It 4.3.3.9

Use the Product Property of Logarithms to write each logarithm as a sum of logarithms. Simplify, if possible.

a. $\log_9(9x)$

b. $\log_3(27xy)$

Answer

a. $1 + \log_9(x)$

b. $3 + \log_3(x) + \log_3(y)$

Similarly, in the Quotient Property of Exponents, $\frac{b^m}{b^n} = b^{m-n}$, we see that to divide the same base, we subtract the exponents. The **Quotient Property of Logarithms**, $\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N)$ tells us to take the log of a quotient, we subtract the log of the numerator and denominator.

Quotient Property of Logarithms

If $M > 0$, $N > 0$, $b > 0$ and $b \neq 1$, then

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

The logarithm of a quotient is the difference of the logarithms.

Note that $\log_b(M) - \log_b(N) \neq \log_b(M - N)!$.

We use this property to write the log of a quotient as a difference of the logs of each factor.

Example 4.3.3.10

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_5\left(\frac{5}{7}\right)$

b. $\log\left(\frac{x}{100}\right)$

Solution

a.

$\log_5\left(\frac{5}{7}\right)$	
Use the Quotient Property, $\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$.	$=\log_5(5)-\log_5(7)$
Simplify.	$=1-\log_5(7)$

b.

$\log\left(\frac{x}{100}\right)$	
Use the Quotient Property, Use the Quotient Property, $\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$.	$=\log(x)-\log(100)$
Simplify.	$=\log(x)-2$

Try It 4.3.3.11

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_4\left(\frac{3}{4}\right)$

b. $\log\left(\frac{x}{1000}\right)$

Answer

a. $\log_4(3)-1$

b. $\log(x)-3$

Try It 4.3.3.12

Use the Quotient Property of Logarithms to write each logarithm as a difference of logarithms. Simplify, if possible.

a. $\log_2\left(\frac{5}{4}\right)$

b. $\log\left(\frac{10}{y}\right)$

Answer

a. $\log_2(5) - 2$

b. $1 - \log(y)$

The third property of logarithms is related to the Power Property of Exponents, $(a^m)^n = a^{m \cdot n}$, we see that to raise a power to a power, we multiply the exponents. The **Power Property of Logarithms**, $\log_b M^p = p \log_b M$ tells us to take the log of a number raised to a power, we multiply the power times the log of the number.

Power Property of Logarithms

If $M > 0$, $b > 0$, $b \neq 1$ and p is any real number then,

$$\log_b(M^p) = p \log_b(M).$$

The log of a number raised to a power as the product product of the power times the log of the number.

We use this property to write the log of a number raised to a power as the product of the power times the log of the number. We essentially take the exponent and throw it in front of the logarithm.

Example 4.3.3.13

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_5(4^3)$

b. $\log(x^{10})$

Solution

a.

	$\log_5(4^3)$
Use the Power Property, $\log_b(M^p) = p\log_b(M)$.	$= 3\log_5(4)$

b.

	$\log(x^{10})$
Use the Power Property, $\log_b(M^p) = p\log_b(M)$.	$= 10\log(x)$

Try It 4.3.3.14

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_7(5^4)$

b. $\log(x^{100})$

Answer

a. $4\log_7(5)$

b. $100\log(x)$

Try It 4.3.3.15

Use the Power Property of Logarithms to write each logarithm as a product of logarithms. Simplify, if possible.

a. $\log_2\left(3^7\right)$

b. $\log\left(x^{20}\right)$

Answer

a. $7\log_2(3)$

b. $20\log(x)$

We summarize the Properties of Logarithms here for easy reference. While the natural logarithms are a special case of these properties, it is often helpful to also show the natural logarithm version of each property.

Properties of Logarithms

If $M>0$, $b>0$, $b\neq 1$ and p is any real number, then

Property	**Base b**	**Base e**
	$\log_b(1)=0$	$\ln(1)=0$
	$\log_b(b)=1$	$\ln(e)=1$
Inverse Properties	$b^{\log_b(x)}=x$ $\log_b(b^x)=x$	$e^{\ln(x)}=x$ $\ln(e^x)=x$
Product Property of Logarithms	$\log_b(MN)=\log_b(M)+\log_b(N)$	$\ln(MN)=\ln(M)+\ln(N)$
Quotient Property of Logarithms	$\log_b\left(\frac{M}{N}\right)=\log_b(M)-\log_b(N)$	$\ln\left(\frac{M}{N}\right)=\ln(M)-\ln(N)$
Power Property of Logarithms	$\log_b(M^p)=p\log_b(M)$	$\ln(M^p)=p\ln(M)$

Now that we have the properties we can use them to “expand” a logarithmic expression. This means to write the logarithm as a sum or difference and without any powers.

We generally apply the Product and Quotient Properties before we apply the Power Property.

Example 4.3.3.16

Use the Properties of Logarithms to expand the logarithm $\log_4\left(2x^3y^2\right)$. Simplify, if possible.

Solution

	$\log_4\left(2x^3y^2\right)$
Use the Product Property, $\log_b(MN)=\log_b(M)+\log_b(N)$.	$=\log_4(2)+\log_4\left(x^3\right)+\log_4\left(y^2\right)$
Use the Power Property, $\log_b(M^p)=p\log_b(M)$, on the last two terms.	$=\log_4\left(4^{1/2}\right)+3\log_4(x)+2\log_4(y)$ $=\frac{1}{2}\log_4(4)+3\log_4(x)+2\log_4(y)$
Simplify.	$=\frac{1}{2}+\frac{3}{2}\log_4(x)+\log_4(y)$

? Try It 4.3.3.17

Use the Properties of Logarithms to expand the logarithm $\log_2\left(5x^4y^2\right)$. Simplify, if possible.

Answer

$$\log_2(5)+4\log_2(x)+2\log_2(y)$$

? Try It 4.3.3.18

Use the Properties of Logarithms to expand the logarithm $\log_3\left(7x^5y^3\right)$. Simplify, if possible.

Answer

$$\log_3(7)+5\log_3(x)+3\log_3(y)$$

When we have a radical in the logarithmic expression, it is helpful to first write its radicand as a rational exponent.

✓ Example 4.3.3.19

Use the Properties of Logarithms to expand the logarithm $\log_2\left(\sqrt[4]{\frac{x^3}{3y^2z}}\right)$. Simplify, if possible.

Solution

	$\log_2\left(\sqrt[4]{\frac{x^3}{3y^2z}}\right)$
Rewrite the radical with a rational exponent.	$=\log_2\left(\frac{x^3}{3y^2z}\right)^{1/4}$
Use the Power Property, $\log_b(M^p)=p\log_b(M)$.	$=\frac{1}{4}\log_2\left(\frac{x^3}{3y^2z}\right)$
Use the Quotient Property, $\log_b(MN)=\log_b(M)-\log_b(N)$.	$=\frac{1}{4}\left(\log_2\left(x^3\right)-\log_2\left(3y^2z\right)\right)$
Use the Product Property, $\log_b(MN)=\log_b(M)+\log_b(N)$, in the second term.	$=\frac{1}{4}\left(\log_2\left(x^3\right)-\left(\log_2(3)+\log_2\left(y^2\right)+\log_2(z)\right)\right)$
Use the Power Property, $\log_b(M^p)=p\log_b(M)$, inside the parentheses.	$=\frac{1}{4}\left(3\log_2(x)-\left(\log_2(3)+2\log_2(y)+\log_2(z)\right)\right)$
Simplify by distributing.	$=\frac{3}{4}\log_2(x)-\frac{1}{4}\log_2(3)-\frac{1}{2}\log_2(y)-\frac{1}{4}\log_2(z)$

? Try It 4.3.3.20

Use the Properties of Logarithms to expand the logarithm $\log_4\left(\sqrt[5]{\frac{x^4}{2y^3z^2}}\right)$. Simplify, if possible.

Answer

$$\frac{4}{5}\log_4(x)-\frac{1}{10}-\frac{3}{5}\log_4(y)-\frac{2}{5}\log_4(z)$$

Try It 4.3.3.21

Use the Properties of Logarithms to expand the logarithm $\log_3\left(\sqrt[3]{\frac{x^2}{5yz}}\right)$. Simplify, if possible.

Answer

$$\frac{2}{3}\log_3(x)-\frac{1}{3}\log_3(5)-\frac{1}{3}\log_3(y)-\frac{1}{3}\log_3(z)$$

The opposite of expanding a logarithm is to condense a sum or difference of logarithms that have the same base into a single logarithm. We again use the properties of logarithms to help us, but in reverse.

To condense logarithmic expressions with the same base into one logarithm, we start by using the Power Property to get the coefficients of the log terms to be one and then the Product and Quotient Properties as needed.

Example 4.3.3.22

Use the Properties of Logarithms to condense the logarithm $\log_4(3)+\log_4(x)-\log_4(y)$. Simplify, if possible.

Solution

	$\log_4(3)+\log_4(x)-\log_4(y)$
The log expressions all have the same base, 4.	
The first two terms are added, so we use the Product Property, $\log_b(M)+\log_b(N)=\log_b(MN)$.	$=\log_4(3x)-\log_4(y)$
Since the logs are subtracted, we use the Quotient Property, $\log_b(M)-\log_b(N)=\log_b\left(\frac{M}{N}\right)$.	$=\log_4\left(\frac{3x}{y}\right)$

Try It 4.3.3.23

Use the Properties of Logarithms to condense the logarithm $\log_2(5)+\log_2(x)-\log_2(y)$. Simplify, if possible.

Answer

$$\log_2\left(\frac{5x}{y}\right)$$

Try It 4.3.3.24

Use the Properties of Logarithms to condense the logarithm $\log_3(6)-\log_3(x)-\log_3(y)$. Simplify, if possible.

Answer

$$\log_3\left(\frac{6}{xy}\right)$$

Example 4.3.3.25

Use the Properties of Logarithms to condense the logarithm $2\log_3(x)+4\log_3(x+1)$. Simplify, if possible.

Solution

	$2\log_3(x)+4\log_3(x+1)$
The log expressions have the same base, 3.	$=2\log_3(x)+4\log_3(x+1)$
Use the Power Property, $\log_b(M)+\log_b(N)=\log_b(MN)$.	$=\log_3(x^2)+\log_3((x+1)^4)$
The terms are added, so we use the Product Property, $\log_b(M)+\log_b(N)=\log_b(MN)$.	$=\log_3(x^2(x+1)^4)$

Try It 4.3.3.26

Use the Properties of Logarithms to condense the logarithm $3\log_2(x)+2\log_2(x-1)$. Simplify, if possible.

Answer

$\log_2(x^3(x-1)^2)$

Try It 4.3.3.27

Use the Properties of Logarithms to condense the logarithm $2\log(x)+2\log(x+1)$. Simplify, if possible.

Answer

$\log(x^2(x+1)^2)$

Use the Change-of-Base Formula

To evaluate a logarithm with any other base, we can use the **Change-of-Base Formula**. We will show how this is derived.

	$\log_a(M)$
Suppose we want to evaluate $\log_a(M)$. Set it to be equal to y.	$y=\log_a(M)$
Rewrite the expression in exponential form.	$a^y=M$
Take the $\log_b$ of each side.	$\log_b(a^y)=\log_b(M)$
Use the Power Property.	$y\log_b(a)=\log_b(M)$
Solve for y.	$y=\frac{\log_b(M)}{\log_b(a)}$
Substitute}$y=\log_a(M)$ and conclude.	$\log_a(M)=\frac{\log_b(M)}{\log_b(a)}$

The Change-of-Base Formula introduces a new base b. This can be any base b we want where $b>0, b\neq 1$. Because our calculators have keys for logarithms base 10 and base e, we will rewrite the Change-of-Base Formula with the new base as 10 or e.

Change-of-Base Formula

For any logarithmic bases a, b and $M>0$,

$$\log_a(M)=\frac{\log_b(M)}{\log_b(a)} \qquad \log_a(M)=\frac{\log(M)}{\log(a)} \qquad \log_a(M)=\frac{\ln(M)}{\ln(a)}$$

$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

When we use a calculator to find the logarithm value, we usually round to three decimal places. This gives us an approximate value and so we use the approximately equal symbol $(\approx)$.

Example 4.3.3.28

Rounding to three decimal places, approximate $\log_4(35)$.

Solution

	$\log_4(35)$
Use the Change-of-Base Formula, $\log_a(M) = \frac{\log_b(M)}{\log_b(a)}$.	
Identify a and M. Choose 10 for b.	$= \frac{\log(35)}{\log(4)}$
Enter the expression $\frac{\log(35)}{\log(4)}$ in the calculator using the log button for base 10. Round to three decimal places.	≈ 2.565

Try It 4.3.3.29

Rounding to three decimal places, approximate $\log_3(42)$.

Answer

3.402

Try It 4.3.3.30

Rounding to three decimal places, approximate $\log_5(46)$.

Answer

2.379

Writing Exercises 4.3.3.31

1. In $3^x = 4$, does the base of the logarithm used when solve that equation matter?
2. How do the properties of logarithms correspond to properties of exponents?

Exit Problem

1. Assuming that $x, y, z > 0$, combine to an expression with one logarithm only.

$$\frac{2}{3}\log_5(x) - 5\log_5(y) - 3\log_5(z)$$

2. Assuming that $x, y > 0$, expand

$$\log_6\left(\frac{\sqrt[5]{x^2}}{36y^3}\right).$$

Key Concepts

- $\log_a 1 = 0 \quad \log_a a = 1$
- **Inverse Properties of Logarithms**
  - For $a > 0, x > 0$ and $a \neq 1$

$$a^{\log_a x} = x \quad \log_a a^x = x$$

- **Product Property of Logarithms**
  - If $M>0, N>0, a>0$ and $a \neq 1$, then,

$$\log_a M \cdot N = \log_a M + \log_a N$$

  The logarithm of a product is the sum of the logarithms.
- **Quotient Property of Logarithms**
  - If $M>0, N>0, \mathrm{a}>0$ and $a \neq 1$, then,

$$\log_a \frac{M}{N} = \log_a M - \log_a N$$

  The logarithm of a quotient is the difference of the logarithms.
- **Power Property of Logarithms**
  - If $M>0, a>0, a \neq 1$ and p is any real number then,

$$\log_a M^p = p \log_a M$$

  The log of a number raised to a power is the product of the power times the log of the number.
- **Properties of Logarithms Summary**
  If $M>0, a>0, a \neq 1$ and p is any real number then,

Table 10.4.1

Property	Base a	Base e
	$\log_a 1 = 0$	$\ln 1 = 0$
	$\log_a a = 1$	$\ln e = 1$
Inverse Properties	$a^{\log_a x} = x$ $\log_a a^x = x$	$e^{\ln x} = x$ $\ln e^x = x$
Product Property of Logarithms	$\log_a (M \cdot N) = \log_a M + \log_a N$	$\ln(M \cdot N) = \ln M + \ln N$
Quotient Property of Logarithms	$\log_a \frac{M}{N} = \log_a M - \log_a N$	$\ln \frac{M}{N} = \ln M - \ln N$
Power Property of Logarithms	$\log_a M^p = p \log_a M$	$\ln M^p = p \ln M$

- **Change-of-Base Formula**
  For any logarithmic bases a and b, and $M>0$,

$$\log_a M = \frac{\log_b M}{\log_b a} \qquad \log_a M = \frac{\log M}{\log a} \qquad \log_a M = \frac{\ln M}{\ln a}$$
$$\text{new base } b \qquad \text{new base } 10 \qquad \text{new base } e$$

CHAPTER OVERVIEW

5: Appendix

SECTION OVERVIEW

5.1: Appendix A- An Alternative Treatment of Conics

In this chapter, you will learn about some conic sections, including circles and parabolas. Then you will use what you learn to investigate systems of nonlinear equations.

Topic hierarchy

5.1.1: A.1- Graphing Quadratic Equations Using Properties and Applications

Learning Objectives

By the end of this section, you will be able to:

- Recognize the graph of a quadratic equation in 2-variables
- Find the axis of symmetry and vertex of a parabola
- Find the intercepts of a parabola
- Graph quadratic equations using properties
- Solve maximum and minimum applications

Be Prepared

Before you get started, take this readiness quiz.

1. Graph the equation $y=x^2$ by plotting points.
2. Solve: $2x^2+3x-2=0$.
3. Evaluate $-\frac{b}{2a}$ when $a=3$ and $b=-6$.

Recognizing the Graph of a Quadratic Equation (in 2 variables)

Previously we very briefly looked at the equation $y=x^2$. It was one of the first non-linear equations we looked at. Now we will graph equations of the form $y=ax^2+bx+c$ if $a\neq 0$. We call the kind of expression on the right hand side a quadratic expression.

I THINK THE BELOW ISN'T CORRECT.. ISN'T A QUADRATIC EQUATION IN TWO VARIABLES POTENTIALLY QUADRATIC IN X AND Y--I propose deleting this?

Definition 5.1.1.1

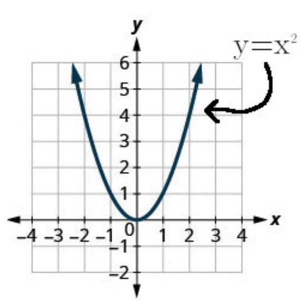
Let A, B, and C be real numbers with $A\neq 0$. We call

$$Ax^2+Bx+C\ ,$$

a **quadratic expression** (in x). We will call an equation $y=Ax^2+Bx+C$ a **quadratic equation in (in two variables)** , or more specifically, an **equation quadratic in x.**

Recall that the graph of an equation is a picture of the solutions we get by using an identification of solutions of the equation with points on a coordinate plane. This is ideal in numerous ways, for example, our drawing instruments are not precise or our space is limited (which it always is!). In what follows (as we did with lines before) we will be happy to capture certain features of the graph.

We graph can the quadratic equation $y=x^2$ by plotting points. That is, we can recognize that $(-3,9)$ is a solution to the equation since $9=(-3)^2$. Similarly, $(-2,4)$ is a solution because $4=(-2)^2$. Continuing down the chart below, we can substitute the value a for x to find a value b for y so that when we substitute (a,b) in for (x,y) we have a solution. When we do this, we see that for each value we substitute in for x there is only one value for y that will do the trick. We proceed to find these solutions on the Cartesian plane:



x	$y=x^2$	(x,y)
–3	9	(–3, 9)
–2	4	(–2, 4)
–1	1	(–1, 1)
0	0	(0, 0)
1	1	(1, 1)
2	4	(2, 4)
3	9	(3, 9)

We call this figure a **parabola**. Parabolas are figures that can be squeezed vertically and horizontally, flipped or rotated, and translated so as to lay directly on the one above. We will look at a few special types. Let's practice graphing a parabola by plotting a few points.

Example 5.1.1.2

Graph $y = x^2 - 1$.

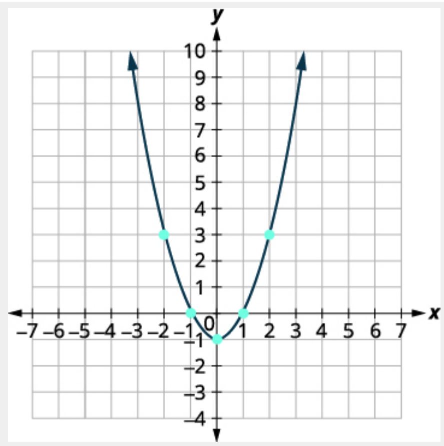
Solution

We will graph the equation by plotting points.

Choose integer values for x, substitute them into the equation and simplify to find y. Record the values of the ordered pairs in the chart.

$y = x^2 - 1$	
x	y
0	–1
1	0
–1	0
2	3
–2	3

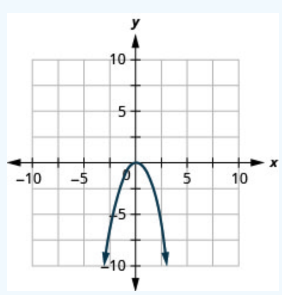
Plot the points, and then connect them with a smooth curve. The result will be the graph of the equation $y = x^2 - 1$.



Try It 5.1.1.3

Graph $y = -x^2$.

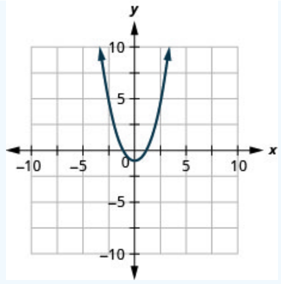
Answer



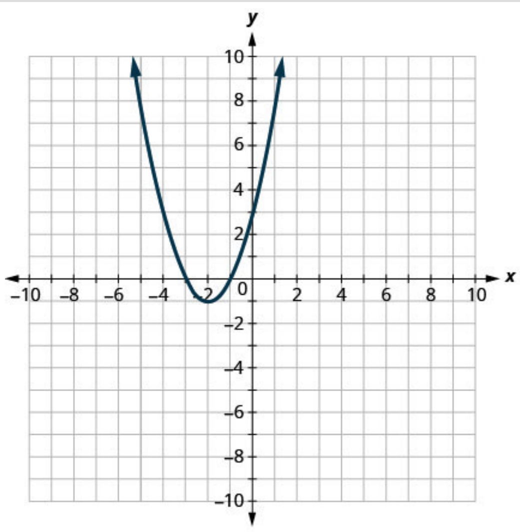
Try It 5.1.1.4

Graph $y = x^2 + 1$.

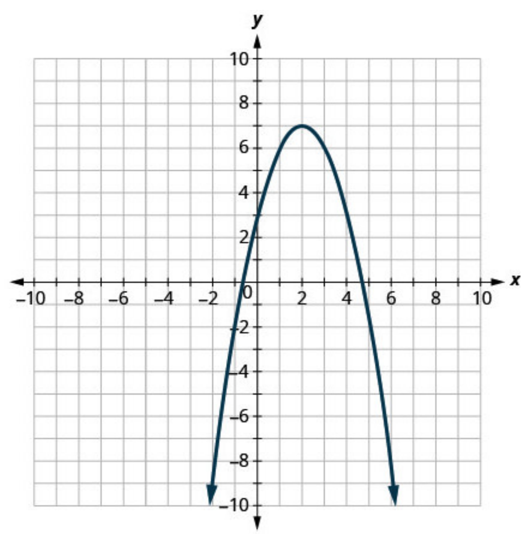
Answer



All graphs of quadratic equations of the form $y = ax^2 + bx + c$ are parabolas that open upward or downward.



$y = ax^2 + bx + c$
$y = x^2 + 4x + 3$
$a > 0$
opens upward



$y = ax^2 + bx + c$
$y = -x^2 + 4x + 3$
$a < 0$
opens downward

Notice that the only difference in the two functions is the negative sign before the quadratic term (x^2 in the equation of the graph in *Figure 9.6.6*). When the quadratic term, is positive, the parabola opens upward, and when the quadratic term is negative, the parabola opens downward.

Definition 5.1.1.5

Parabola Orientation

For the graph of the quadratic equation $y = ax^2 + bx + c$, if

- $a > 0$, the parabola opens upward 
- $a < 0$, the parabola opens downward 

Example 5.1.1.6

Determine whether each parabola described by the following opens upward or downward:

a. $y = -3x^2 + 2x - 4$

b. $y = 6x^2 + 7x - 9$

Solution

a. Find the value of a.

Comparing $y = ax^2 + bx + c$ to $y = -3x^2 + 2x - 4$, we see that $a = -3$, which is negative. The associated parabola opens downward.

b. Find the value of a.

Comparing $y = ax^2 + bx + c$ to $y = 6x^2 + 7x - 9$, we see that $a = 6$, which is positive. The associated parabola opens upward.

Try It 5.1.1.7

Determine whether each parabola described by the following opens upward or downward:

a. $y = 2x^2 + 5x - 2$

b. $y = -3x^2 - 4x + 7$

Answer

a. up

b. down

Try It 5.1.1.8

Determine whether each parabola described by the following opens upward or downward:

a. $y = -2x^2 - 2x - 3$

b. $y = 5x^2 - 2x - 1$

Answer

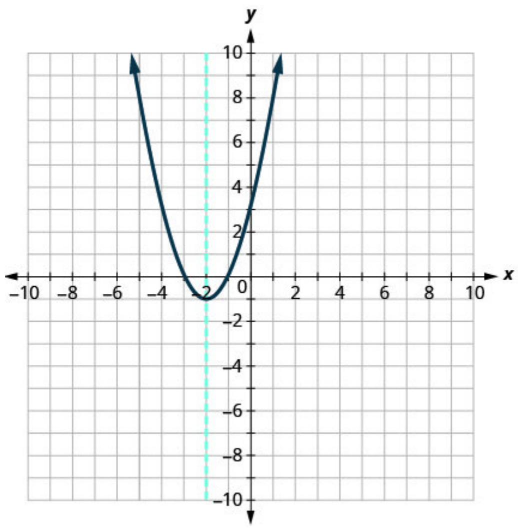
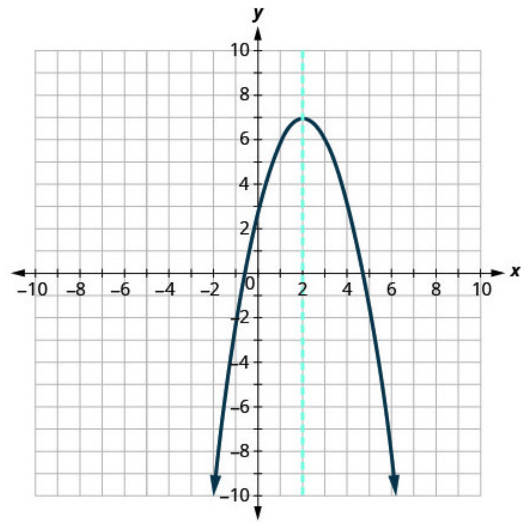
a. down

b. up

Find the Axis of Symmetry and Vertex of a Parabola

Look again at *Figure 9.6.10*. Do you see that we could fold each parabola in half and then one side would lie on top of the other? The 'fold line' is a line of symmetry. We call it the **axis of symmetry** of the parabola.

We show the same two graphs again with the axis of symmetry.

	
$y=x^2+4x+3$	$y=-x^2+4x+3$

The equation of the axis of symmetry can be derived by using the Quadratic Formula. We will omit the derivation here and proceed directly to using the result. The equation of the axis of symmetry of the graph of $y=ax^2+bx+c$ is $x=-\frac{b}{2a}$.

So to find the equation of symmetry of each of the parabolas we graphed above, we will substitute into the formula $x=-\frac{b}{2a}$.

$y=ax^2+bx+c$	$y=ax^2+bx+c$
$y=x^2+4x+3$	$y=-x^2+4x+3$
$x=-\frac{b}{2a}$	$x=-\frac{b}{2a}$
$x=-\frac{4}{2\cdot 1}$	$x=-\frac{4}{2(-1)}$
$x=-2$	$x=2$

Notice that these are the equations of the dashed blue lines on the graphs.

The point on the parabola that is the lowest (parabola opens up), or the highest (parabola opens down), lies on the axis of symmetry. This point is called the **vertex** of the parabola.

We can easily find the coordinates of the vertex, because we know it is on the axis of symmetry. This means its x-coordinate is $-\frac{b}{2a}$. To find the y-coordinate of the vertex we substitute the value of the x-coordinate into the quadratic equation.

$y=x^2+4x+3$	$y=-x^2+4x+3$
axis of symmetry is $x=-2$	axis of symmetry is $x=2$
vertex is (-2, __)	vertex is (2, __)
$y=x^2+4x+3$	$y=-x^2+4x+3$
$y=(-2)^2+4(-2)+3$	$y=-(2)^2+4(2)+3$
$y=-1$	$y=7$
vertex is $(-2,-1)$	vertex is $(2,7)$

Axis of Symmetry and Vertex of a Parabola

The graph of the equation $y = ax^2 + bx + c$ is a parabola where:

- the axis of symmetry is the vertical line $x = -\frac{b}{2a}$.
- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$
- the y-coordinate of the vertex is found by substituting $x = -\frac{b}{2a}$ into the quadratic equation.

Example 5.1.1.9

For the graph of $y = 3x^2 - 6x + 2$ find:

a. the axis of symmetry

b. the vertex

Solution

a.

	$a = 3, b = -6, c = 2$
The axis of symmetry is the vertical line $x = -\frac{b}{2a}$.	
Substitute the values a, b into the equation.	$x = -\frac{-6}{2 \cdot 3}$
Simplify.	$x = 1$
	The axis of symmetry is the line $x = 1$.

b.

The vertex is a point on the line of symmetry, so its x-coordinate will be $x = 1$. Find the y-coordinate so that $(1, y$ is a solution.	$y = 3(1)^2 - 6(1) + 2$
Simplify.	$y = 3 \cdot 1 - 6 + 2 = -1$
The result is the y-coordinate.	$y = -1$
	$(1, -1)$ is the solution of the equation on the axis of symmetry. The vertex is $(1, -1)$.

Try It 5.1.1.10

For the graph of $y = 2x^2 - 8x + 1$ find:

a. the axis of symmetry

b. the vertex

Answer

a. $x = 2$

b. $(2, -7)$

Try It 5.1.1.11

For the graph of $y = 2x^2 - 4x - 3$ find:

a. the axis of symmetry

b. the vertex

Answer

a. $x=1$

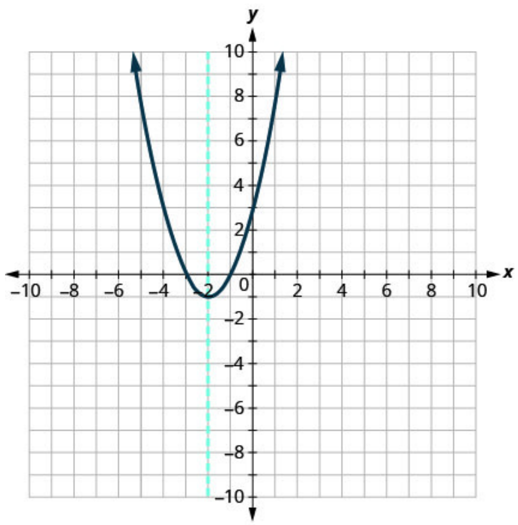
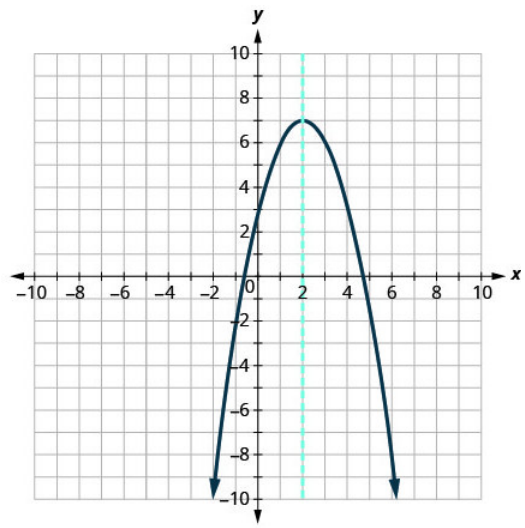
b. $(1,-5)$

Find the Intercepts of a Parabola

When we graphed linear equations, we often used the x- and y-intercepts to help us graph the lines. Finding the coordinates of the intercepts will help us to graph parabolas, too.

Remember, at the y-intercept the value of x is zero. So to find the y-intercept, we substitute $x=0$ into the equation.

Let's find the y-intercepts of the two parabolas shown in *Figure 9.6.20*.

	
$y=x^2+4x+3$ Look for a solution of the form $(0,b)$. $b=0^2+4\cdot 0+3$ $b=3$ The y-intercept is $(0,3)$.	$y=x^2+4x+3$ Look for a solution of the form $(0,b)$. $b=-0^2+4\cdot 0+3$ $b=3$ The y-intercept is $(0,3)$.

An x-intercept results when the value of $f(x)$ is zero. To find an x-intercept, we let $f(x)=0$. In other words, we will need to solve the equation $0=ax^2+bx+c$ for x.

$$y=ax^2+bx+c$$
$$0=ax^2+bx+c$$

Solving quadratic equations like this (one variable) is exactly what we have done earlier in this chapter!

We can now find the x-intercepts of the two parabolas we looked at. First we will find the x-intercepts of the parabola whose equation is $y=x^2+4x+3$.

	$y=x^2+4x+3$
Let $y=0$.	$0=x^2+4x+3$
Factor.	$0=(x+1)(x+3)$
Use the Zero Product Property.	$x+1=0 \quad x+3=0$
Solve.	
	The x-intercepts are $(-1,0)$ and $(-3,0)$.

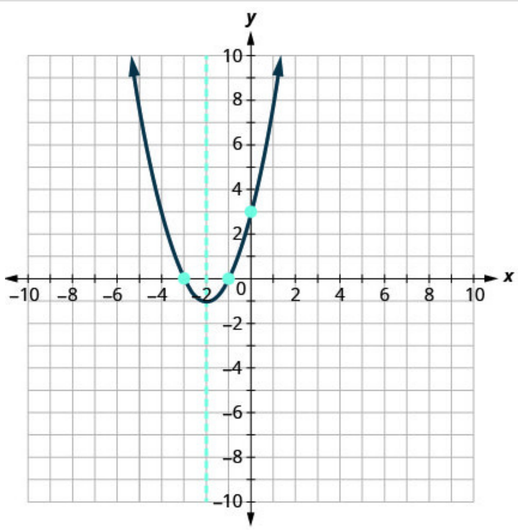
Now we will find the x-intercepts of the parabola whose equation is $y=-x^2+4x+3$.

	$y=-x^2+4x+3$
Let $y=0$.	$0=-x^2+4x+3$
This quadratic does not factor, so we use the Quadratic Formula.	$a=-1, b=4, c=3$
Simplify.	
	$x=\frac{-2(2\pm\sqrt{7})}{-2}$
	$x=2\pm\sqrt{7}$
	The x-intercepts are $(2+\sqrt{7},0)$ and $(2-\sqrt{7},0)$.

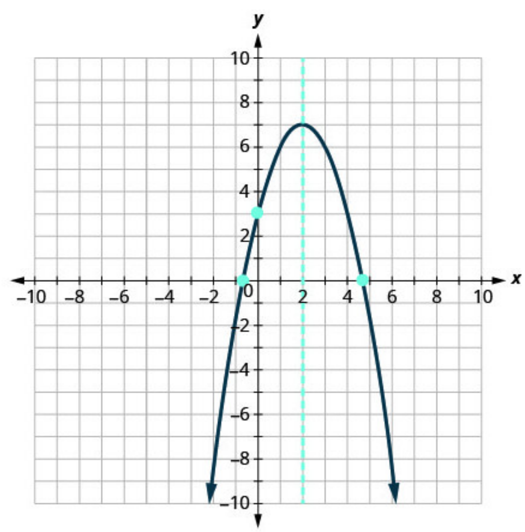
We will use the decimal approximations of the x-intercepts, so that we can locate these points on the graph,

$$(2+\sqrt{7},0)\approx(4.6,0)\quad(2-\sqrt{7},0)\approx(-0.6,0)$$

Do these results agree with our graphs? See *Figure 9.6.34*



$y=x^2+4x+3$
y-intercept is $(0,3)$
x-intercepts are $(-1,0)$ and $(-3,0)$



$y=-x^2+4x+3$
y-intercept is $(0,3)$
x-intercepts are $(2+\sqrt{7},0)\approx(4.6,0)$ and $(2-\sqrt{7},0)\approx(-0.6,0)$

Find the Intercepts of a Parabola

To find the intercepts of a parabola whose equation is $y=ax^2+bx+c$:

y-intercept

Let $x=0$ and solve for y.

x-intercepts

Let $y=0$ and solve for x

Example 5.1.1.12

Find the intercepts of the parabola whose equation is $y=x^2-2x-8$.

Solution

To find the y-intercept, let $x=0$ and solve for y.	
	$y=-8$
	When $x=0$, then $y=-8$. The y-intercept is the point $(0,-8)$.
To find the x-intercept, let $y=0$ and solve for x.	
Solve by factoring.	$0=(x-4)(x+2)$
	$4=x \quad -2=x$
	When $y=0$, then $x=4$ or $x=-2$. The x-intercepts are the points $(4,0)$ and $(-2,0)$.

Try It 5.1.1.13

Find the intercepts of the parabola whose equation is $y=x^2+2x-8$.

Answer

y -intercept: $(0,-8)$

x -intercepts: $(-4,0),(2,0)$

Try It 5.1.1.14

Find the intercepts of the parabola whose equation is $y=x^2-4x-12$.

Answer

y -intercept: $(0,-12)$

x -intercepts: $(-2,0),(6,0)$

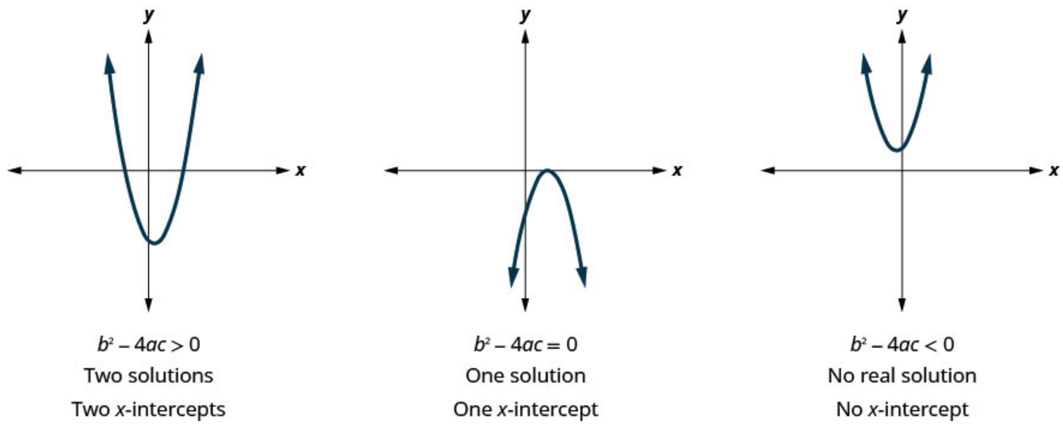
In this chapter, we have been solving quadratic equations of the form $ax^2+bx+c=0$. We solved for x and the results were the solutions to the equation.

We are now looking at quadratic equations of the form $y=ax^2+bx+c$. The graphs of these equations are parabolas. The x-intercepts of the parabolas occur where $y=0$.

The solutions of the quadratic equation resulting from replacing y with 0 are the x values of the x-intercepts.

Earlier, we saw that quadratic equations have $2,1$, or 0 solutions. The graphs below show examples of parabolas for these three cases. Since the solutions of the functions give the x-intercepts of the graphs, the number of x-intercepts is the same as the number of solutions.

Previously, we used the **discriminant** to determine the number of solutions of a quadratic function of the form $ax^2+bx+c=0$. Now we can use the discriminant to tell us how many x-intercepts there are on the graph.



Before you to find the values of the x-intercepts, you may want to evaluate the discriminant so you know how many solutions to expect.

Example 5.1.1.15

Find the intercepts of the parabola for the equation $y = 5x^2 + x + 4$.

Solution

		$y = 5x^2 + x + 4.$	
To find the y-intercept, let $x = 0$ and solve for y.		$y = 50^2 + 0 + 4.$	
		$y = 4$	
		When $x = 0$, then $y = 4$. The y-intercept is the point $(0, 4)$.	
To find the x-intercept, let $f(x) = 0$ and solve for x.		$0 = 5x^2 + x + 4.$	
Find the value of the discriminant to predict the number of solutions which is also the number of x-intercepts.		$a = 5, b = 1, c = 4$ $b^2 - 4ac$ $= 1^2 - 4 \cdot 5 \cdot 4 = -79$	
		Since the value of the discriminant is negative, there is no real solution to the equation. There are no x-intercepts.	

Try It 5.1.1.16

Find the intercepts of the parabola whose equation is $y = 3x^2 + 4x + 4$.

Answer

y-intercept: $(0, 4)$

no x-intercept

Try It 5.1.1.17

Find the intercepts of the parabola whose equation is $y = x^2 - 4x - 5$.

Answer

y-intercept: $(0, -5)$

x-intercepts: $(-1, 0), (5, 0)$

Graphing Quadratic Equations Using Properties

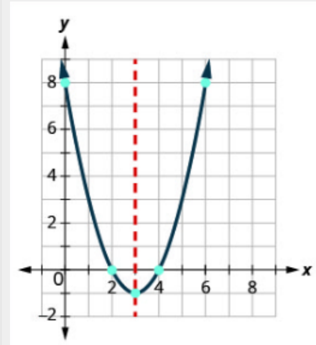
Now we have all the pieces we need in order to graph a quadratic equation. We just need to put them together. In the next example we will see how to do this.

Example 5.1.1.18

Graph $y = x^2 - 6x + 8$ by using its properties.

Solution

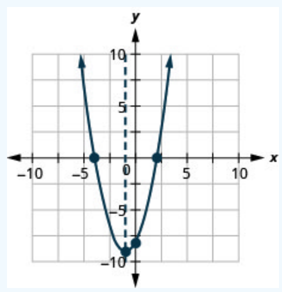
Determine whether the parabola opens upward or downward.	Look a in the equation $y = x^2 - 6x + 8$ Since a is positive, the parabola opens upward.	$y = x^2 - 6x + 8$ $a = 1, b = -6, c = 8$ **The parabola opens upward.**
Find the axis of symmetry.	$y = x^2 - 6x + 8$ The axis of symmetry is the line $x = -\frac{b}{2a}$.	Axis of Symmetry $x = -\frac{b}{2a}$ $x = -\frac{(-6)}{2\cdot 1}$ $x = 3$ **The axis of symmetry is the line $x = 3$.**
Find the vertex.	The vertex is on the axis of symmetry. Substitute $x = 3$ into the function.	Vertex $y = x^2 - 6x + 8$ the y-coordinate of the vertex $= (3)^2 - 6(3) + 8$ $= -1$ **The vertex is** $(3, -1)$**.**
Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.	We find solutions of the form $(0, y)$. We use the axis of symmetry to find a point symmetric to the y-intercept. The y-intercept is 3 units left of the axis of symmetry, $x = 3$. A point 3 units to the right of the axis of symmetry has $x = 6$.	y-intercept The y-coordinate of the y-intercept $= (0)^2 - 6(0) + 8$ $= 8$ **The y-intercept is** $(0, 8)$**.** The point symmetric to the y-intercept: **The point is** $(6, 8)$**.**
Find the x-intercepts. Find additional points if needed.	We look for solutions of the form $(x, 0)$. We can solve this quadratic equation by factoring.	x-intercepts We substitute 0 for y in the equation: $0 = x^2 - 6x + 8$ $0 = (x - 2)(x - 4)$ $x = 2 or x = 4$ **The x-intercepts are** $(2, 0)$ **and** $(4, 0)$**.**

Graph the parabola.	We graph the vertex, intercepts, and the point symmetric to the y-intercept. We connect these 5 points to sketch the parabola.	

Try It 5.1.1.19

Graph $y=x^2+2x-8$ by using its properties.

Answer



Try It 5.1.1.20

Graph $y=x^2-8x+12$ by using its properties.

Answer

This figure shows an upward-opening parabola graphed on the x y-coordinate plane. The x-axis of the plane runs from negative 10 to 10. The y-axis of the plane runs from negative 10 to 15. The axis of symmetry, x equals 4, is graphed as a dashed line. The parabola has a vertex at (4, negative 4). The y-intercept of the parabola is the point (0, 12). The x-intercepts of the parabola are the points (2, 0) and (6, 0).

We list the steps to take in order to graph a quadratic function here.

To Graph a Parabola Using Properties

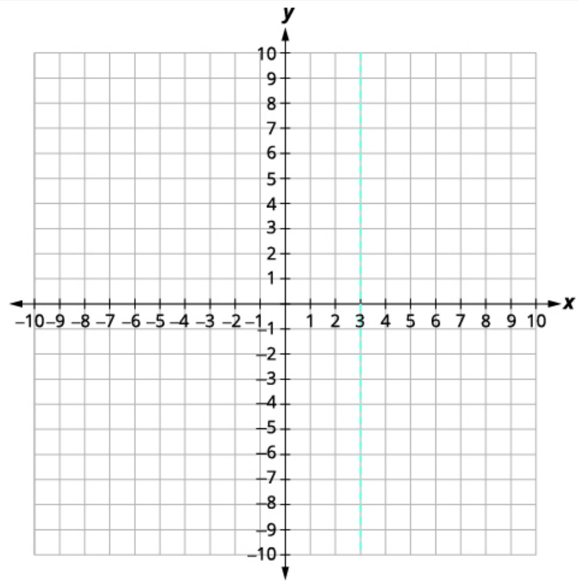
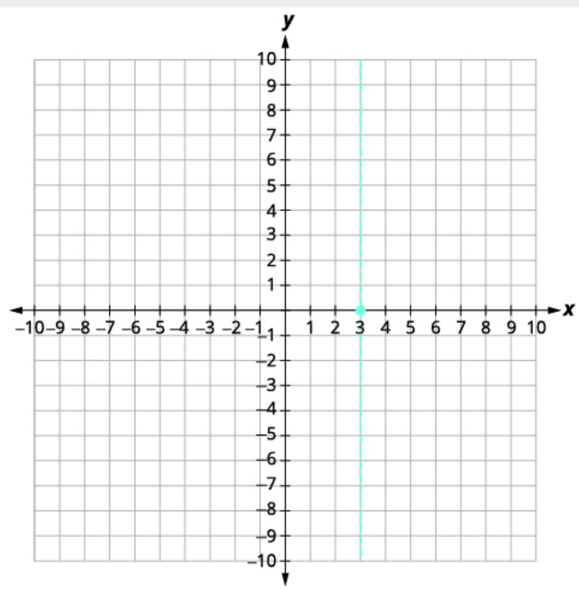
1. Determine whether the parabola opens upward or downward.
2. Find the equation of the axis of symmetry.
3. Find the vertex.
4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
5. Find the x-intercepts. Find additional points if needed.
6. Graph the parabola.

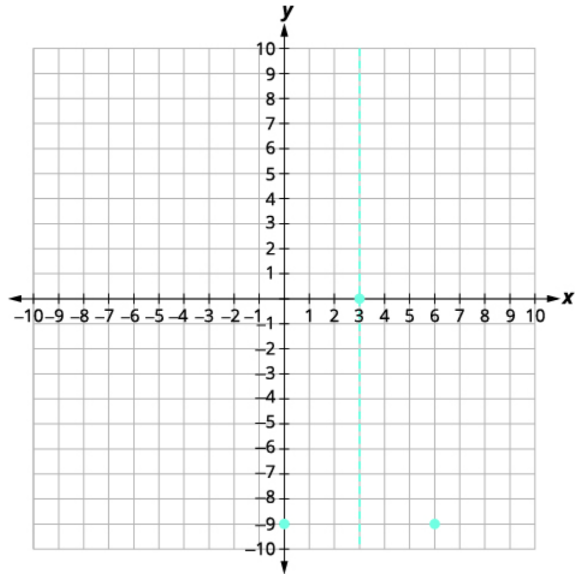
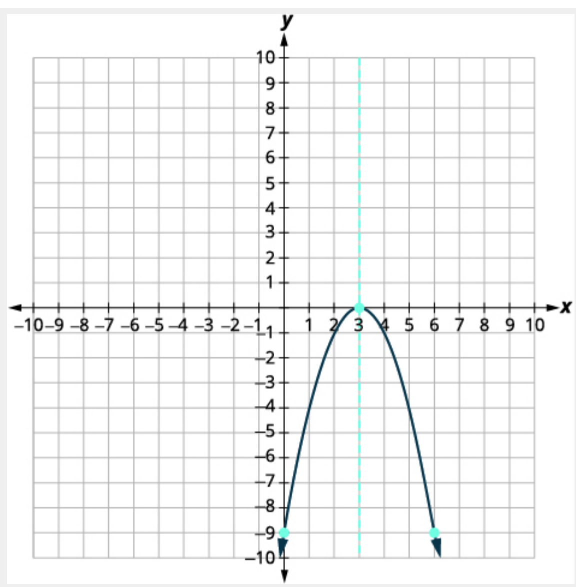
We were able to find the x-intercepts in the last example by factoring. We find the x-intercepts in the next example by factoring, too.

Example 5.1.1.21

Graph $y=x^2+6x-9$ by using its properties.

Solution

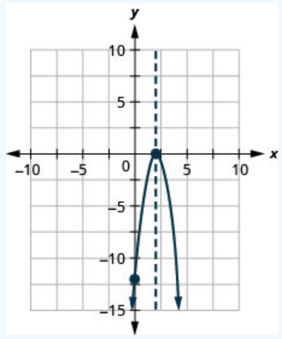
	$y = x^2 + 6x - 9$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x = -\frac{b}{2a}$.	$x = -\frac{b}{2a}$
	$x = -\frac{6}{2(-1)}$
	$x = 3$
	The axis of symmetry is $x = 3$. The vertex is on the line $x = 3$.
	
Find a solution of the form $(3, y)$.	$y = -x^2 + 6x - 9$
	$y = -3^2 + 6 \cdot 3 - 9$
	$y = -9 + 18 - 9$
	$y = 0$
	The vertex is $(3, 0)$.
	

	$y = x^2 + 6x - 9$
The y-intercept occurs when $x = 0$. Find a solution of the form $(0, y)$.	$y = -x^2 + 6x - 9$
xSubstitute $x = 0$.	$y = -0^2 + 6 \cdot 0 - 9$
Simplify.	$y = -9$
The point $(0, -9)$ is three units to the left of the line of symmetry. The point three units to the right of the line of symmetry is $(6, -9)$.	
	Point symmetric to the y-intercept is $(6, -9)$
The x-intercept occurs when $y = 0$. So we look for a solution of the form $x, 0)$	$y = -x^2 + 6x - 9$
Find $y = 0$.	$0 = -x^2 + 6x - 9$
Factor the GCF.	$0 = -(x^2 - 6x + 9)$
Factor the trinomial.	$0 = -(x - 3)^2$
Solve for x.	$x = 3$
Connect the points to graph the parabola.	

? Try It 5.1.1.22

Graph $y=3x^2+12x-12$ by using its properties.

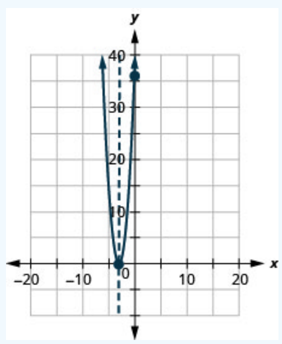
Answer



? Try It 5.1.1.23

Graph $y=4x^2+24x+36$ by using its properties.

Answer



For the graph of $y=-x^2+6x-9$, the vertex and the x-intercept were the same point. Remember how the discriminant determines the number of solutions of a quadratic equation? The discriminant of the equation $0=-x^2+6x-9$ is 0, so there is only one solution. That means there is only one x-intercept, and it is the vertex of the parabola.

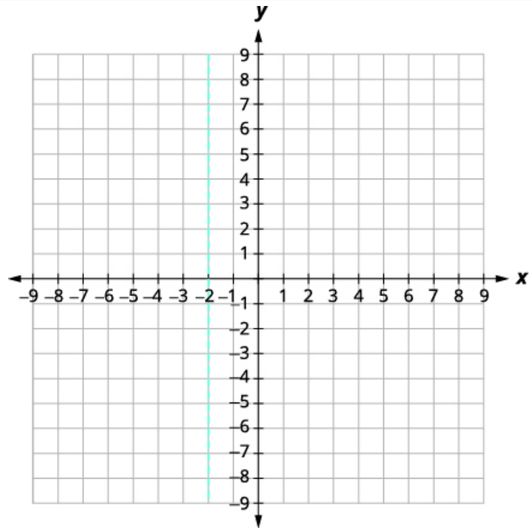
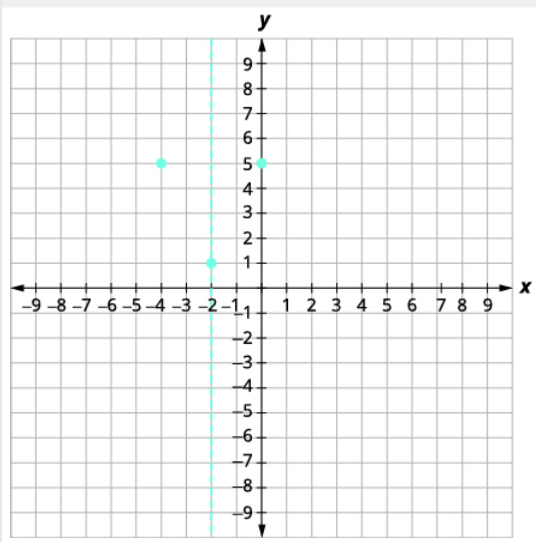
How many x-intercepts would you expect to see on the graph of $f(x)=x^2+4x+5$?

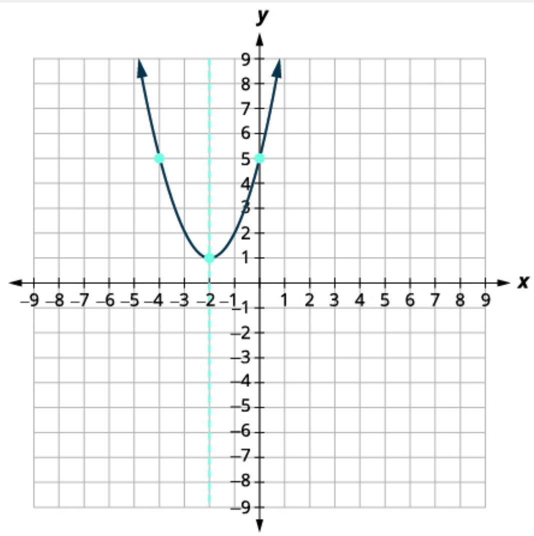
? Example 5.1.1.24

Graph $y=x^2+4x+5$ by using its properties.

Solution

	$y=x^2+4x+5$
Since a is -1, the parabola opens downward.	
	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-2$
	The equation of the axis of symmetry is $x=-2$.

	$y=x^2+4x+5$
	
The vertex is on the line $x=-2$.	
Find y when $x=-2$.	$y=x^2+4x+5$
	$y=(-2)^2+4(-2)+5$
	$y=4-8+5$
	$y=1$
	The vertex is $(-2,1)$.
The y-intercept occurs when $x=0$.	$y=x^2+4x+5$
Find the solution of the form $(0,y)$.	$y=0^2+4(0)+5$
Simplify.	$y=5$
	The y-intercept is $(0,5)$.
The point $(-4,5)$ is two units to the left of the line of symmetry. The point to units to the right of the line of symmetry is \((0,5)\.	
	Point symmetric to the y-intercept is $(-4,5)$.
The x-intercept occurs when $y=0$.	
Find the solution of the form $(x,0)$.	

	$y=x^2+4x+5$
Test the discriminant.	$a=1, b=4, c=5$
	$4^2-4 \cdot 1 \cdot 5$
	$16-20$
	-4
Since the value of the discriminant is negative, there is no real solution and so no x-intercept.	
Connect the points to graph the parabola. You may want to choose two more points for greater accuracy. For example, you could try to find a solution of the form $(-1, y)$ which would lead to a solution $(-1, 2$.	

Try It 5.1.1.25

Graph $y=x^2-2x+3$ by using its properties.

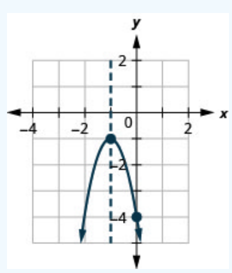
Answer

This figure shows an upward-opening parabola graphed on the x y-coordinate plane. The x-axis of the plane runs from negative 2 to 4. The y-axis of the plane runs from negative 1 to 5. The parabola has a vertex at (1, 2). The y-intercept (0, 3) is plotted as is the line of symmetry, x equals 1.

Try It 5.1.1.26

Graph $y=-3x^2-6x-4$ by using its properties.

Answer



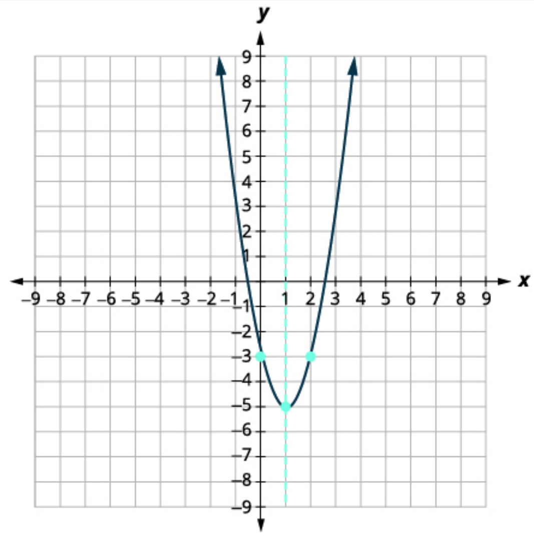
Finding the y-intercept of the graph of an equation that looks like $y=ax^2+bx+c$ is easy, isn't it? Finding the x-intercepts of such a graph can be more challenging. Sometimes we need to use the **Quadratic Formula**!

❓ Example 5.1.1.27

Graph $y=2x^2-4x-3$ by using its properties.

Solution

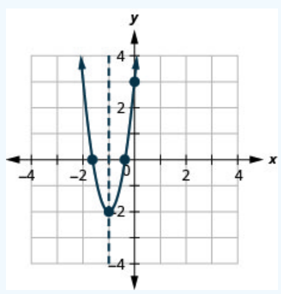
	$y=2x^2-4x-3$
Since a is 2, the parabola opens upward.	
To find the equation of the axis of symmetry, use $x=-\frac{b}{2a}$.	$x=-\frac{b}{2a}$
	$x=-\frac{-4}{2\cdot 2}$
	$x=1$
	The equation of the axis of symmetry is $x=1$.
The vertex is on the line $x=1$.	
Find a solution of the form $(1,y)$.	$y=21^2-4\cdot 1-3$
	$y=2-4-3$
	\(y=-5)
	The vertex is $(1,-5)$.
The y-intercept occurs when $x=0$.	
Find a solution of the form $(0,y)$.	$y=20^2-4\cdot 0-3$
Simplify.	$y=-3$
	The y-intercept is $(0,-3)$.
The point $(0,-3)$ is one unit to the left of the line of symmetry.	Point symmetric to the y-intercept is $(2,-3)$
The point one unit to the right of the line of symmetry is $(2,3)$.	
The x-intercept occurs when $y=0$.	
Find a solution of the form $(x,0)$.	
Use the Quadratic Formula.	
Substitute in the values of a, b and c.	$x=\frac{-(-4)\pm\sqrt{(-4)^2-4(2)(3)}}{2(2)}$
Simplify.	
Simplify inside the radical.	$x=\frac{4\pm\sqrt{40}}{4}$
Simplify the radical.	$x=\frac{4\pm 2\sqrt{10}}{4}$
Factor the GCF.	$x=\frac{2(2\pm\sqrt{10})}{4}$
Remove common factors.	$x=\frac{2\pm\sqrt{10}}{2}$
Write as two equations.	$x=\frac{2+\sqrt{10}}{2}, \quad x=\frac{2-\sqrt{10}}{2}$
Approximate the values.	$x\approx 2.5, \quad x\approx -0.6$
	The approximate values of the x-intercepts are $(2.5,0)$ and $(-0.6,0)$.

	$y=2x^2-4x-3$
Graph the parabola using the points found.	

? Try It 5.1.1.28

Graph $y=5x^2+10x+3$ by using its properties.

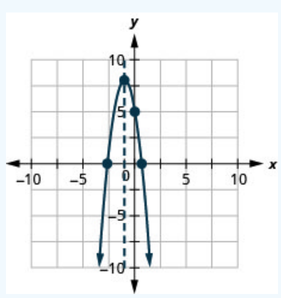
Answer



? Try It 5.1.1.29

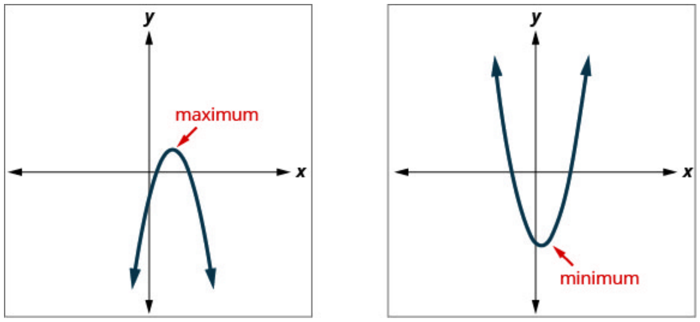
Graph $y=-3x^2-6x+5$ by using its properties.

Answer



Solve Maximum and Minimum Applications

Knowing that the **vertex** of a parabola is the lowest or highest point of the parabola gives us an easy way to determine the minimum or maximum value of a quadratic expression. Given a quadratic expression, with variable x, say, we consider the equation in two variables \y\) equals the given expression. The *y*-coordinate of the vertex is the **minimum** value of that expression if the graph of the equation is a parabola that opens upward. It is the **maximum** value of the expression if the graph of the equation is a parabola that opens downward.



Optional: Minimum or Maximum Values of a Quadratic Expression

Minimum and Maximum Values

The y-coordinate of the vertex of the graph of a quadratic equation is the

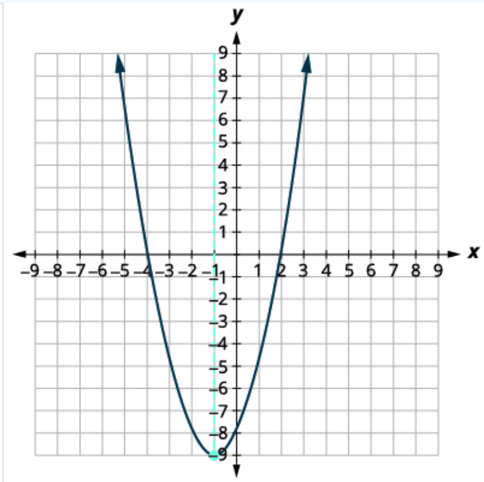
- *minimum* value of the quadratic expression if the parabola opens *upward*.
- *maximum* value of the quadratic expression if the parabola opens *downward*.

Example 5.1.1.30

Find the minimum or maximum value of the quadratic expression x^2+2x-8 .

Solution

Consider the equation	$y=x^2+2x-8$
Since a is positive, the parabola opens upward. The quadratic equation has a minimum.	
Find the equation of the axis of symmetry.	$x=-\frac{b}{2a}$
	$x=-\frac{2}{2\times1}$
	$x=-1$
	The equation of the axis of symmetry is $x=-1$.
The vertex is on the line $x=-1$.	$y=x^2+2x-8$
Find the y-coordinate of the vertex.	$y=(-1)^2+2(-1)-8$
	$y=1-2-8$
	$y=-9$
	The vertex is $(-1,-9)$.
Since the parabola has a minimum, the y-coordinate of the vertex is the minimum y-value of the quadratic equation. The minimum value of the quadratic is -9 and it occurs when $x=-1$.	



Show the graph to verify the result.

The minimum value of the expression is -9 and it occurs when $x=-1$.

Try It 5.1.1.31

Find the maximum or minimum value of the quadratic expression $x^2-8x+12$.

Answer

The minimum value of the quadratic expression is -4 and it occurs when $x=4$.

Try It 5.1.1.32

Find the maximum or minimum value of the quadratic expression $-4x^2+16x-11$.

Answer

The maximum value of the quadratic expression is 5 and it occurs when $x=2$.

We have used the formula

$$h=-16t^2+v_0t$$

to calculate the height in feet, h, of an object shot upwards into the air with initial velocity, v_0, after t seconds (assuming the initial shot occurs at height 0). The solutions to this equation in the context of the application are precisely the pairs (t,h) so that h is the height of the object at time t.

This formula is a quadratic equation in t, so its graph is a parabola. By solving for the coordinates of the vertex (t,h), we can find how long it will take the object to reach its maximum height. Then we can calculate the maximum height.

Example 5.1.1.33

The quadratic expression $-16t^2+176t+4$ models the height of a volleyball hit straight upwards with velocity 176 feet per second from a height of 4 feet.

a. How many seconds will it take the volleyball to reach its maximum height?

b. Find the maximum height of the volleyball.

Solution

Let us consider the equation $h=-16t^2+176t+4$ (here we have used h for 'height' instead of y). So again, the solutions of the equation are exactly the ordered pairs (t,h) so that h is the height at time t.

Since a is negative, the parabola opens downward. The quadratic function has a maximum.

a. Find the equation of the axis of symmetry.

$t=-\frac{b}{2a}$

$t=-\frac{176}{2(-16)}$

$t=5.5$

The equation of the axis of symmetry is $t=5.5$.

The vertex is on the line $t=5.5$.

The maximum occurs when $t=5.5$ seconds.

b. Find the height at time $t=5.5$. This is the same as searching for a solution of the form $(h,5.5)$.

Use a calculator to simplify.

$h=488$

The vertex is $(5.5,488)$.

Since the parabola has a maximum, the h-coordinate of the vertex is the maximum value of the quadratic expression.

The maximum value of the quadratic expression is then 488 feet and it occurs when $t=5.5$ seconds.

After 5.5 seconds, the volleyball will reach its maximum height of 488 feet.

? Exercise 5.1.1.34

Solve, rounding answers to the nearest tenth.

The quadratic function $h=-16t^2+128t+32$ is used to find the height of a stone thrown upward from a height of 32 feet at a rate of 128 ft/sec. How long will it take for the stone to reach its maximum height? What is the maximum height?

Answer

It will take 4 seconds for the stone to reach its maximum height of 288 feet.

? Exercise 5.1.1.35

A path of a toy rocket thrown upward from the ground at a rate of 208 ft/sec is modeled by the quadratic function of $h=-16t^2+208t$. When will the rocket reach its maximum height? What will be the maximum height?

Answer

It will take 6.5 seconds for the rocket to reach its maximum height of 676 feet.

Key Concepts

- Parabola Orientation
  - For the graph of the quadratic equation $y=ax^2+bx+c$, if
    - $a>0$, the parabola opens upward.
    - $a<0$, the parabola opens downward.
- Axis of Symmetry and Vertex of a Parabola The graph of the equation $y=ax^2+bx+c$ is a parabola where:
  - the axis of symmetry is the vertical line $x=-\frac{b}{2a}$.

- the vertex is a point on the axis of symmetry, so its x-coordinate is $-\frac{b}{2a}$.
- the y-coordinate of the vertex is found by substituting $x=-\frac{b}{2a}$ into the quadratic equation.

- Find the Intercepts of a Parabola
  - To find the intercepts of a parabola whose equation is $y=ax^2+bx+c$:
    - **y-intercept**
      - Let $x=0$ and solve for y.
    - **x-intercepts**
      - Let \(y=0\) and solve for x.
- How to graph a quadratic equation using properties.
  1. Determine whether the parabola opens upward or downward.
  2. Find the equation of the axis of symmetry.
  3. Find the vertex.
  4. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  5. Find the x-intercepts. Find additional points if needed.
  6. Graph the parabola.
- Minimum or Maximum Values of a Quadratic Expression
  - The y-coordinate of the vertex of the graph of a quadratic equation formed by setting the expression (in x) equal to y is the
  - *minimum* value of the quadratic expression if the parabola opens *upward*.
  - *maximum* value of the quadratic expression if the parabola opens *downward*.

Glossary

quadratic expression (in x)

An expression of the form ax^2+bx+c , where a, b, and c are real numbers and $a\neq 0$, is called a quadratic expression (in x).

quadratic equation (in x)

An equation which can be written in the form $y=ax^2+bx+c$, where a, b, and c are real numbers and $a\neq 0$ is a quadratic equation (in x)

5.1.1.1: A.1.1- Introduction to Quadratic Equations with Two Variables

A quadratic equation with two variables x and y is an equation that is equivalent to

$$Ax^2 + By^2 + Cx + Dy + Exy + F = 0, \tag{5.1.1.1.1}$$

where at least one of A or B is not zero.

In general, the graphs of the solution to this type of equation is called a conic section (a circle, parabola, ellipse, hyperbola, line, two intersecting lines, or a point).

We will discuss the solutions to this type of equation in certain cases. We'll begin with the case that $A = E = 0$ or $B = E = 0$. The solutions to these equations are called parabolas.

We will then discuss the case where $A = B$ and $E = 0$ in which case, the graph is a circle.

The other cases could be treated in a similar way.

5.1.2: A.2- Graphing Quadratic Equations Using Transformations

Learning Objectives

By the end of this section, you will be able to:

- Graph quadratic expressions of the form $y=x^2+k$
- Graph quadratic expressions of the form $y=(x-h)^2$
- Graph quadratic expressions of the form $y=ax^2$
- Graph quadratic expressions using transformations
- Find a quadratic expressions from its graph

Be Prepared

Before you get started, take this readiness quiz.

1. Graph the expressions $y=x^2$ by plotting points.
2. Factor completely: $y^2-14y+49$.
3. Factor completely: $2x^2-16x+32$.

Graphing Quadratic Expressions of the Form $y=x^2+k$

In the last section, we learned how to graph quadratic expressions using their properties. Another method involves starting with the basic graph of $y=x^2$ and 'moving' it according to information given in the equation. We call this graphing quadratic equations using transformations.

In the first example, we will graph the quadratic equation $y=x^2$ by plotting points. Then we will see what effect adding a constant, k, to the equation will have on the graph of the new equation $y=x^2+k$.

Example 5.1.2.1

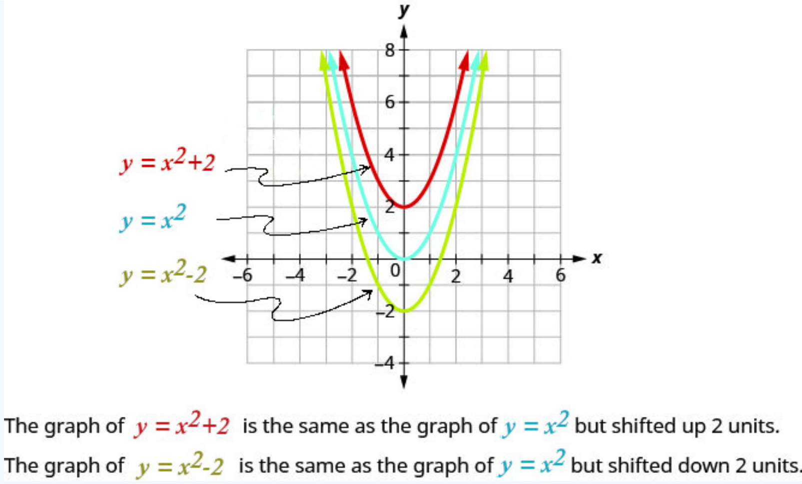
Graph $y=x^2$, $y=x^2+2$, and $y=x^2-2$ on the same rectangular coordinate system. Describe what effect adding a constant to the right side of the equation has on the basic parabola.

Solution:

Plotting points will help us see the effect of the constants on the basic $y=x^2$ graph. We fill in the chart for all three equations.

x	$y=x^2$	(x,y)	$y=x^2+2$	(x,y)	$y=x^2-2$	(x,y)
–3	9	(–3, 9)	9 + 2	(–3, 11)	9 – 2	(–3, 7)
–2	4	(–2, 4)	4 + 2	(–2, 6)	4 – 2	(–2, 2)
–1	1	(–1, 1)	1 + 2	(–1, 3)	1 – 2	(–1, 1)
0	0	(0, 0)	0 + 2	(0, 2)	0 – 2	(0, –2)
1	1	(1, 1)	1 + 2	(1, 3)	1 – 2	(1, –1)
2	4	(2, 4)	4 + 2	(2, 6)	4 – 2	(2, 2)
3	9	(3, 9)	9 + 2	(3, 11)	9 – 2	(3, 7)

The y-coordinates of the solutions to $y=x^2+2$ are two more than the y-coordinates of the solutions to $y=x^2$ values. Also, the y-coordinates of the solutions to $y=x^2-2$ values are two less than the y-coordinates of the solutions to $y=x^2$ values. Now we will graph all three equations on the same rectangular coordinate system.



The graph of $y=x^2+2$ is the same as the graph of $y=x^2$ but shifted up 2 units.

The graph of $y=x^2-2$ is the same as the graph of $y=x^2$ but shifted down 2 units.

? Exercise 5.1.2.1

a. Graph $y=x^2, y=x^2+1,$ and $y=x^2-1$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the equation has on the basic parabola.

Answer

a.

This figure shows 3 upward-opening parabolas on the x y-coordinate plane. The middle graph is of y equals x squared has a vertex of (0, 0). Other points on the curve are located at (negative 1, 1) and (1, 1). The top curve has been moved up 1 unit, and the bottom has been moved down 1 unit.

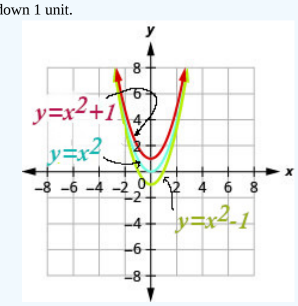


Figure 9.7.3

b. The graph of $y=x^2+1$ is the same as the graph of $y=x^2$ but shifted up 1 unit. The graph of $y=x^2-1$ is the same as the graph of $y=x^2$ but shifted down 1 unit.

? Exercise 5.1.2.2

a. Graph $y=x^2, y=x^2+6,$ and $y=x^2-6$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the equation has on the basic parabola.

Answer

a.

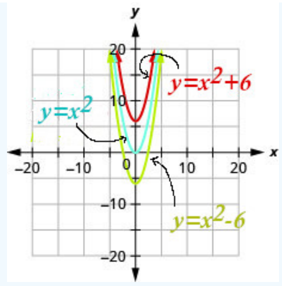


Figure 9.7.4

b. The graph of $y=x^2+6$ is the same as the graph of $y=x^2$ but shifted up 6 units. The graph of $y=x^2-6$ is the same as the graph of $y=x^2$ but shifted down 6 units.

The last example shows us that to graph a quadratic equation of the form $y=x^2+k$, we take the basic parabola graph of $y=x^2$ and vertically shift it up $(k>0)$ or shift it down $(k<0)$.

This transformation is called a vertical shift.

Graphing a Quadratic equation of the Form $y=x^2+k$ Using a Vertical Shift

The graph of $y=x^2+k$ shifts the graph of $y=x^2$ vertically k units.

- If $k>0$, shift the parabola vertically up k units.
- If $k<0$, shift the parabola vertically down $|k|$ units.

Now that we have seen the effect of the constant, k, it is easy to graph equations of the form $y=x^2+k$. We just start with the basic parabola of $y=x^2$ and then shift it up or down.

It may be helpful to practice sketching $y=x^2$ quickly. We know the values and can sketch the graph from there.

This figure shows an upward-opening parabola on the x y-coordinate plane, with vertex (0, 0). Other points on the curve are located at (negative 4, 16), (negative 3, 9), (negative 2, 4), (negative 1, 1), (1, 1), (2, 4), (3, 9), and (4, 16).

Figure 9.7.5

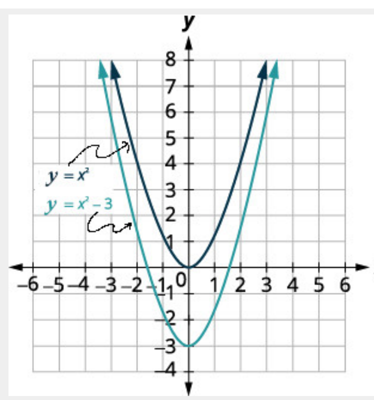
Once we know this parabola, it will be easy to apply the transformations. The next example will require a vertical shift.

Example 5.1.2.2

Graph $y=x^2-3$ using a vertical shift.

Solution:

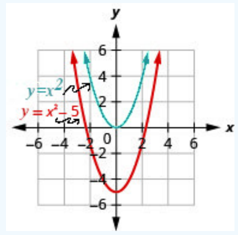
Table 9.7.1

We first draw the graph of $y=x^2$ on the grid.	This figure shows an upward-opening parabola on the x y-coordinate plane with a vertex of (0, 0) with other points on the curve located at (negative 1, 1) and (1, 1). It is the graph of y equals x squared.
Determine k.	$y=x^2+k$ $y=x^2-3$
Shift the graph $y=x^2$ down 3.	

Exercise 5.1.2.3

Graph $y = x^2 - 5$ using a vertical shift.

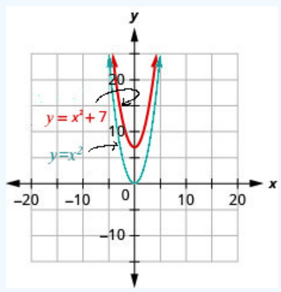
Answer



Exercise 5.1.2.4

Graph $y = x^2 + 7$ using a vertical shift.

Answer



Graphing Quadratic equations of the Form $y = (x - h)^2$

In the first example, we graphed the quadratic equation $y = x^2$ by plotting points and then saw the effect of adding a constant k to the right side of the equation (or, equivalently, subtracting k from y) had on the resulting graph of the new equation $y = x^2 + k$.

We will now explore the effect of subtracting a constant, h, from x has on the resulting graph of the new equation $y = (x - h)^2$.

Example 5.1.2.3

Graph $y = x^2, y = (x-1)^2,$ and $y = (x+1)^2$ on the same rectangular coordinate system. Describe what effect adding a constant to the equation has on the basic parabola.

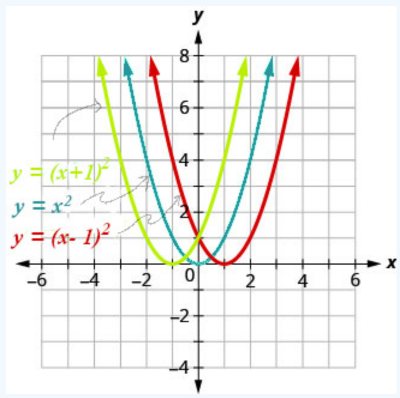
Solution:

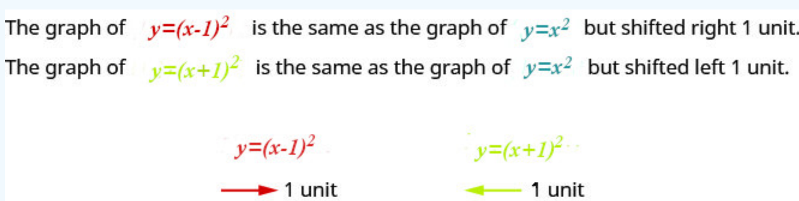
Plotting points will help us see the effect of the constants on the basic $y = x^2$ graph. We fill in the chart for all three equations.

x	$y=x^2$	(x,y)	$y=(x-1)^2$	(x,y)	$y=(x+1)^2$	(x,y)
–3	9	(–3, 9)	16	(–3, 16)	4	(–3, 4)
–2	4	(–2, 4)	9	(–2, 9)	1	(–2, 1)
–1	1	(–1, 1)	4	(–1, 4)	0	(–1, 0)
0	0	(0, 0)	1	(0, 1)	1	(0, 1)
1	1	(1, 1)	0	(1, 0)	4	(1, 4)
2	4	(2, 4)	1	(2, 1)	9	(2, 9)
3	9	(3, 9)	4	(3, 4)	16	(3, 16)

Figure 9.7.12

The y-coordinates share the common numbers $0, 1, 4, 9$, and 16, but are shifted, in that they correspond to different x-coordinates in the different cases.



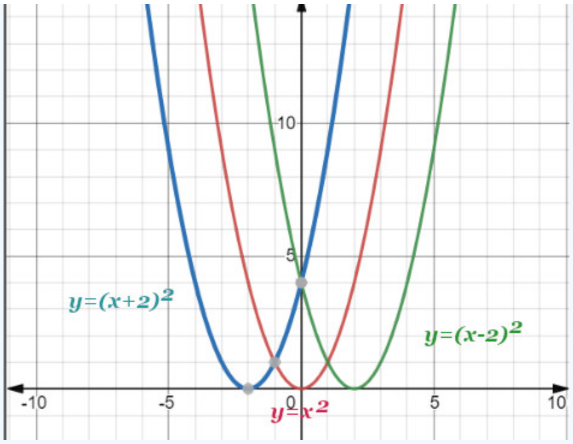


Exercise 5.1.2.5

a. Graph $y=x^2, y=(x+2)^2,$ and $y=(x-2)^2$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the x variable has on the basic parabola.

Answer

a.



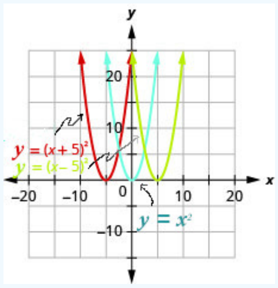
b. The graph of $y=(x+2)^2$ is the same as the graph of $y=x^2$ but shifted left 2 units. The graph of $y=(x-2)^2$ is the same as the graph of $y=x^2$ but shifted right 2 units.

Exercise 5.1.2.6

a. Graph $y=x^2, y=x^2+5,$ and $y=x^2-5$ on the same rectangular coordinate system.
b. Describe what effect adding a constant to the equation has on the basic parabola.

Answer

a.



b. The graph of $y=(x+5)^2$ is the same as the graph of $y=x^2$ but shifted left 5 units. The graph of $y=(x-5)^2$ is the same as the graph of $y=x^2$ but shifted right 5 units.

The last example shows us that to graph a quadratic equation of the form $y=(x-h)^2$, we take the basic parabola graph of $y=x^2$ and shift it left $(h>0)$ or shift it right $(h<0)$.

This transformation is called a horizontal shift.

Graphing a Quadratic Equation of the Form $y=(x-h)^2$ Using a Horizontal Shift

The graph of $y=(x-h)^2$ shifts the graph of $y=x^2$ horizontally h units.

- If $h>0$, shift the parabola horizontally left h units.
- If $h<0$, shift the parabola horizontally right $|h|$ units.

Now that we have seen the effect of the constant, h, it is easy to graph equations of the form $y=(x-h)^2$. We just start with the basic parabola of $y=x^2$ and then shift it left or right.

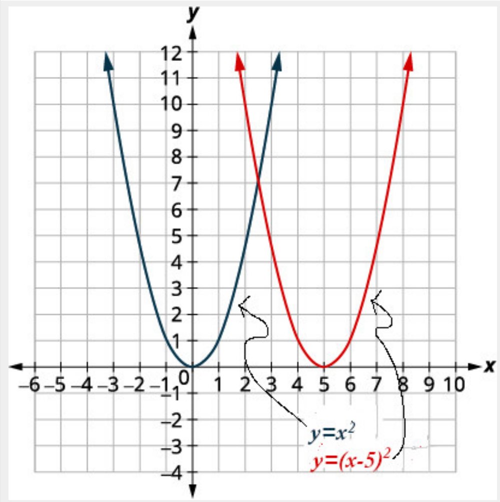
The next example will require a horizontal shift.

Example 5.1.2.4

Graph $y=(x-5)^2$ using a horizontal shift.

Solution:

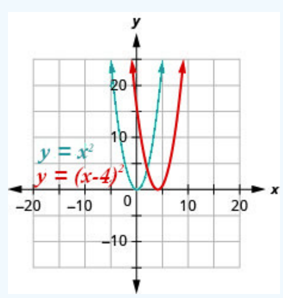
Table 9.7.2

We first draw the graph of $y=x^2$ on the grid.	
Determine h.	$y=(x-h)^2$ $y=(x-5)^2$
	$h=5$
Shift the graph $y=x^2$ to the right 5 units.	

Exercise 5.1.2.7

Graph $y=(x-4)^2$ using a horizontal shift.

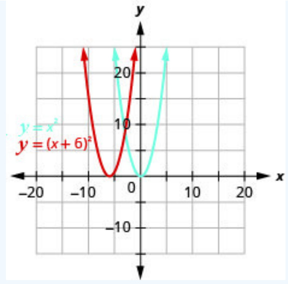
Answer



Exercise 5.1.2.8

Graph $y=(x+6)^2$ using a horizontal shift.

Answer



(Note that you may also view the consideration of the equation $y=x^2+k$ as $y-k=x^2$ and think of this as a shift along the y-axis. Viewing it this way allows for a similar approach to the transformation of graphs of equations formed by adding (or subtracting) constants to (or from) variables.)

Now that we know the effect of the constants h and k, we will graph a quadratic equation of the form $y=(x-h)^2+k$ by first drawing the basic parabola and then making a horizontal shift followed by a vertical shift. We could do the vertical shift followed by the horizontal shift, but most students prefer the horizontal shift followed by the vertical.

Example 5.1.2.5

Graph $y=(x+1)^2-2$ using transformations.

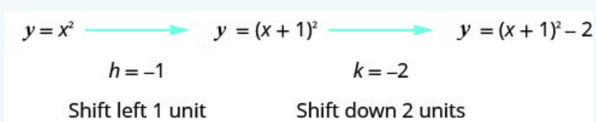
Solution:

This equation will involve two transformations and we need a plan.

Let's first identify the constants h, k.

$$y=(x+1)^2-2$$
$$y=(x-h)^2+k$$
$$y=(x-(-1))^2-2$$
$$h=-1, k=-2$$

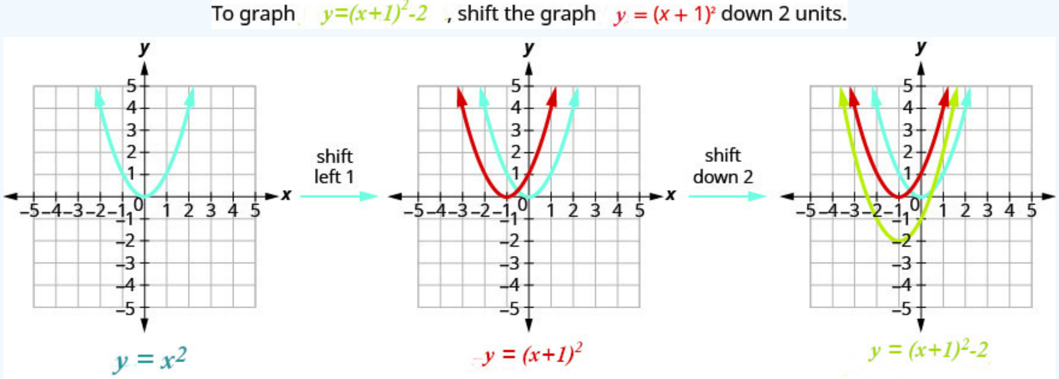
The h constant gives us a horizontal shift and the k gives us a vertical shift.



We first draw the graph of $y=x^2$ on the grid.

To graph $y=(x+1)^2$, shift the graph $y=x^2$ to the left 1 unit.

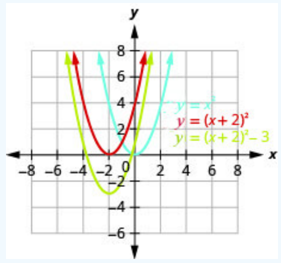
To graph $y=(x+1)^2-2$, shift the graph $y=(x+1)^2$ down 2 units.



Exercise 5.1.2.9

Graph $y=(x+2)^2-3$ using transformations.

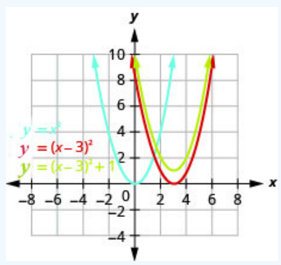
Answer



Exercise 5.1.2.10

Graph $y=(x-3)^2+1$ using transformations.

Answer



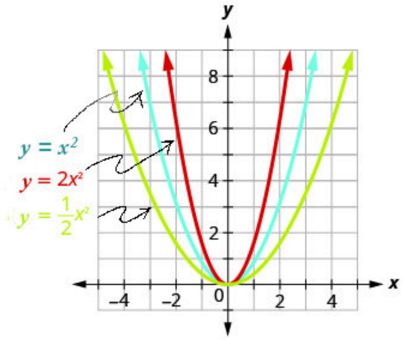
Graphing Quadratic Equations of the Form $y=ax^2$

So far we graphed the quadratic equation $y=x^2$ and then saw the effect of including a constant h or k in the equation had on the resulting graph of the new equation. We will now explore the effect of the coefficient a on the resulting graph of the new equation $y=ax^2$.

Let's look at the quadratic functions $y=x^2$, $y=2x^2$ and $y=\frac{1}{2}x^2$.

x	$y=x^2$	(x,y)	$y=2x^2$	(x,y)	$y=\frac{1}{2}x^2$	(x,y)
–2	4	(–2, 4)	2 • 4	(–2, 8)	$\frac{1}{2}\cdot 4$	(–2, 2)
–1	1	(–1, 1)	2 • 1	(–1, 2)	$\frac{1}{2}\cdot 1$	$\left(-1,\frac{1}{2}\right)$
0	0	(0, 0)	2 • 0	(0, 0)	$\frac{1}{2}\cdot 0$	(0, 0)
1	1	(1, 1)	2 • 1	(1, 2)	$\frac{1}{2}\cdot 1$	$\left(1,\frac{1}{2}\right)$
2	4	(2, 4)	2 • 4	(2, 8)	$\frac{1}{2}\cdot 4$	(2, 2)

If we graph these equations, we can see the effect of the constant a, assuming $a>0$.



The graph of the equation $y = 2x^2$ is "skinnier" than the graph of $y = x^2$.

The graph of the equation $y = \frac{1}{2}x^2$ is "wider" than the graph of $y = x^2$.

To graph a equation with constant a it is easiest to choose a few points on $y = x^2$ and multiply the y-coordinates by a.

Graphing a Quadratic equation of the Form $y = ax^2$

The coefficient a in the equation $y = ax^2$ affects the graph of $y = x^2$ by stretching or compressing it.

- If $0 < |a| < 1$, the graph of $y = ax^2$ will be “wider” than the graph of $y = x^2$.
- If $|a| > 1$, the graph of $y = ax^2$ will be “skinnier” than the graph of $y = x^2$.

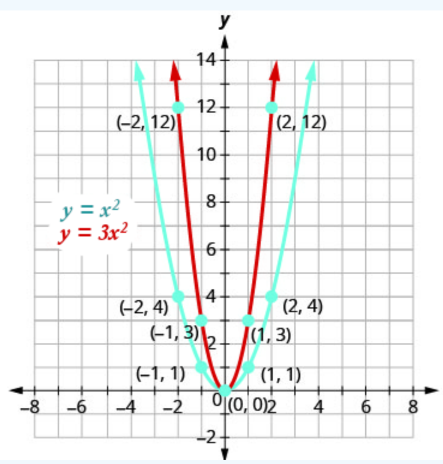
Example 5.1.2.6

Graph $y = 3x^2$.

Solution:

We will graph the equations $y = x^2$ and $y = 3x^2$ on the same grid. We will choose a few points on $y = x^2$ and then multiply the y-values by 3 to get the points for $y = 3x^2$.

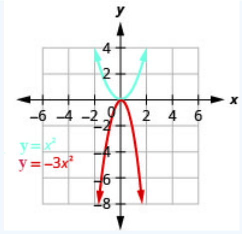
	$y = x^2$	$y = 3x^2$	
x	(x,y)	(x,y)	
–2	(–2, 4)	(–2, 12)	3 • 4 = 12
–1	(–1, 1)	(–1, 3)	3 • 1 = 3
0	(0, 0)	(0, 0)	3 • 0 = 0
1	(1, 1)	(1, 3)	3 • 1 = 3
2	(2, 4)	(2, 12)	3 • 4 = 12



Exercise 5.1.2.11

Graph $y = -3x^2$.

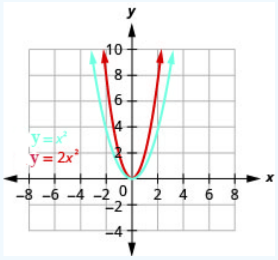
Answer



Exercise 5.1.2.12

Graph $y=2x^2$.

Answer



Graphing Quadratic Equations of the Form $y=ax^2+bx+c$ Using Transformations

We have learned how the constants a, h, and k in the equations, $y=x^2+k, y=(x-h)^2$, and $y=ax^2$ affect their graphs. We can now put this together and graph quadratic equations $y=ax^2+bx+c$ by first putting them into the form $y=a(x-h)^2+k$ by completing the square. This form is sometimes known as the vertex form or standard form.

We must be careful to both add and subtract the number to the expression to complete the square. We cannot add the number to 'both sides' (both sides of what?) as we did when we completed the square with quadratic equations. Note that we could also add and subtract a number from the same side in the case of the quadratic equation as well.

Quadratic Equation	Quadratic Expression
$x^2+8x+6=0$	x^2+8x+6
$x^2+8x=-6$	x^2+8x+6
$x^2+8x+16=-6+16$ -- add 16 to both sides	$x^2+8x+16-16+6$ --add and subtract 16 from the expression
$(x+4)^2=10$	$(x+4)^2-10$

When we complete the square in a equation with a coefficient of x^2 that is not one, we have to factor that coefficient from just the x-terms. We do not factor it from the constant term. It is often helpful to move the constant term a bit to the right to make it easier to focus only on the x-terms.

Once we get the constant we want to complete the square, we must remember to multiply it by that coefficient before we then subtract it.

Example 5.1.2.7

Rewrite $y=-3x^2-6x-1$ in the $y=a(x-h)^2+k$ form by completing the square.

Solution:

	$y=-3x^2-6x-1$
Separate the x terms from the constant.	$y=-3x^2-6x \quad -1$

Factor the coefficient of $x^2, -3$.	$y=-3(x^2+2x) \quad -1$
Prepare to complete the square.	$y=-3(x^2+2x \qquad) \quad -1$
Take half of 2 and then square it to complete the square $(\frac{1}{2}\cdot 2)^2=1$	
The constant 1 completes the square in the parentheses, but the parentheses is multiplied by -3. So we are really adding -3. We must then add 3 to not change the value of the equation.	$y=-3((x^2+2x+1-1))-1$ $y=-3(x^2+2x+1)+3-1$
Rewrite the trinomial as a square and subtract the constants.	$y=-3(x+1)^2+2$
The equation is now in the $y=a(x-h)^2+k$ form.	$y=a(x-h)^2+k$ $y=-3(x+1)^2+2$

? Exercise 5.1.2.13

Rewrite $y=-4x^2-8x+1$ in the $y=a(x-h)^2+k$ form by completing the square.

Answer

$y=-4(x+1)^2+5$

? Exercise 5.1.2.14

Rewrite $y=2x^2-8x+3$ in the $y=a(x-h)^2+k$ form by completing the square.

Answer

$y=2(x-2)^2-5$

Once we put the equation into the $y=(x-h)^2+k$ form, we can then use the transformations as we did in the last few problems. The next example will show us how to do this.

✓ Example 5.1.2.8

Graph $y=x^2+6x+5$ by using transformations.

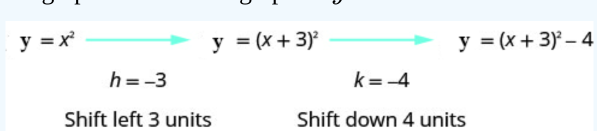
Solution:

Step 1: Rewrite the equation in $y=a(x-h)^2+k$ vertex form by completing the square.

	$y=x^2+6x+5$
Separate the x terms from the constant.	$y=x^2+6x+5$
Take half of 6 and then square it to complete the square. $(\frac{1}{2}\cdot 6)^2=9$	
We both add 9 and subtract 9 to not change the value of the equation.	$y=x^2+6x+9-9+5$
Rewrite the trinomial as a square and subtract the constants.	$y=(x+3)^2-4$
The equation is now in the $y=(x-h)^2+k$ form.	$y=(x-h)^2+k$ $y=(x+3)^2-4$

Step 2: Graph the equation using transformations.

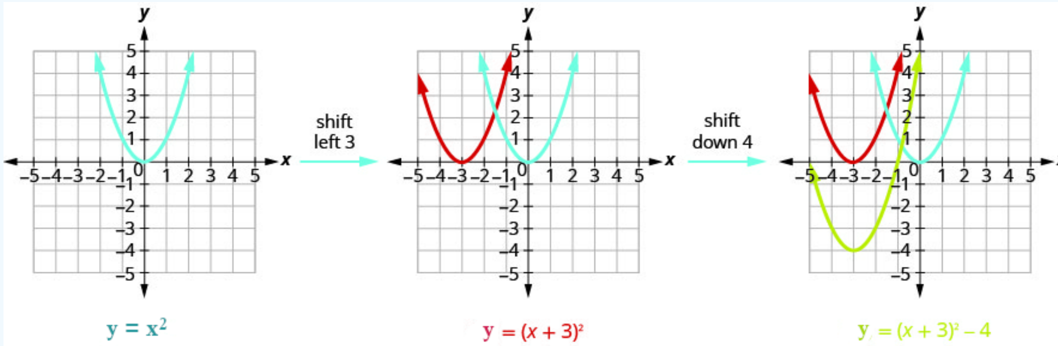
Looking at the h, k values, we see the graph will take the graph of $y=x^2$ and shift it to the left 3 units and down 4 units.



We first draw the graph of $y=x^2$ on the grid.

To graph $y=(x+3)^2$, shift the graph $y=x^2$ to the left 3 units.

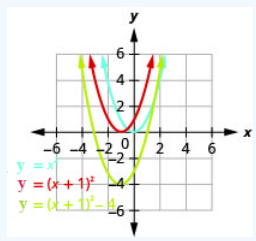
To graph $y=(x+3)^2-4$, shift the graph $y=(x+3)^2$ down 4 units.



Exercise 5.1.2.15

Graph $y=x^2+2x-3$ by using transformations.

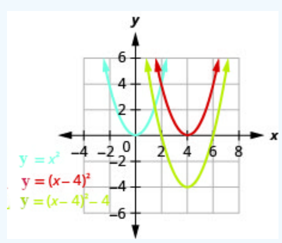
Answer



Exercise 5.1.2.16

Graph $y=x^2-8x+12$ by using transformations.

Answer



We list the steps to take a graph a quadratic equation using transformations here.

Graphing a Quadratic Equation of the Form $y=ax^2+bx+c$ Using Transformations

1. Rewrite the equation in $y=a(x-h)^2+k$ form by completing the square.
2. Graph the equation using transformations.

Example 5.1.2.9

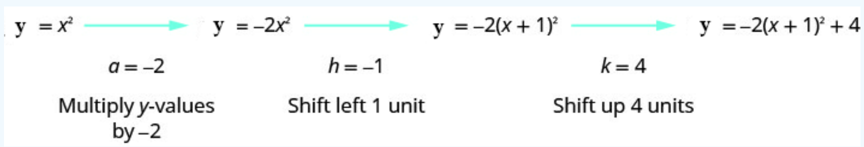
Graph $y=-2x^2-4x+2$ by using transformations.

Solution:

Step 1: Rewrite the equation in $y=a(x-h)^2+k$ vertex form by completing the square.

	$y=-2x^2-4x+2$
Separate the x terms from the constant.	$y=-2x^2-4x \quad +2$
We need the coefficient of x^2 to be one. We factor -2 from the x-terms.	$y=-2(x^2+2x \quad)+2$
Take half of 2 and then square it to complete the square. $(\frac{1}{2}\cdot 2)^2=1$	
We add 1 to complete the square in the parentheses, but the parentheses is multiplied by -2. So we are really adding -2. To not change the value of the equation we add 2.	$y=-2((x^2+2x+1)-1)+2$ $y=-2(x^2+2x+1)+2+2$
Rewrite the trinomial as a square ad subtract the constants.	$y=-2(x+1)^2+4$
The equation is now in the $y=a(x-h)^2+k$ form.	$y=a(x-h)^2+k$ $y=-2(x+1)^2+4$

Step 2: Graph the equation using transformations.

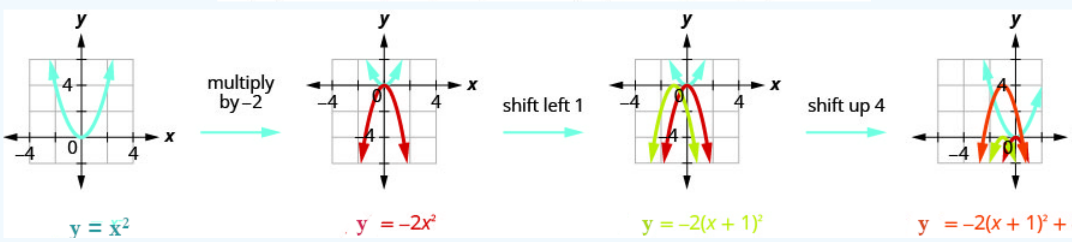


We first draw the graph of $y=x^2$ on the grid.

To graph $y=-2x^2$, multiply the y-values in parabola of $y=x^2$ by -2.

To graph $y=-2(x+1)^2$, shift the graph $y=-2x^2$ to the left 1 unit.

To graph $y=-2(x+1)^2+4$, shift the graph $y=(x+1)^2$ up 4 units.



? Exercise 5.1.2.17

Graph $y=-3x^2+12x-4$ by using transformations.

Answer

This figure shows a downward-opening parabola on the x y-coordinate plane with a vertex of (2,8) and other points of (1,5) and (3,5).

? Exercise 5.1.2.18

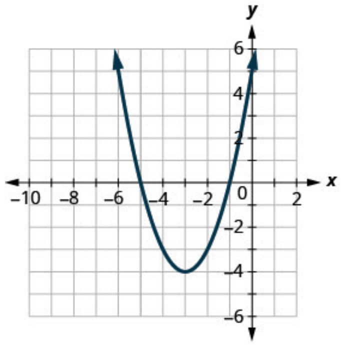
Graph $y=-2x^2+12x-9$ by using transformations.

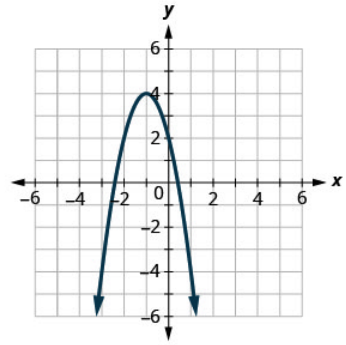
Answer

This figure shows a downward-opening parabola on the x y-coordinate plane with a vertex of (3, 9) and other points of (1, 1) and (5, 1).

Now that we have completed the square to put a quadratic equation into $y=a(x-h)^2+k$ form, we can also use this technique to graph the equation using its properties as in the previous section.

If we look back at the last few examples, we see that the vertex is related to the constants h and k.





$y = (x - h)^2 + k$
$y = (x+3)^2 - 4$
$h = -3 \quad k = -4$
Vertex $= (-3, -4)$

$y = a(x - h)^2 + k$
$y = -2(x+1)^2 + 4$
$h = -1 \quad k = 4$
Vertex $= (-1, 4)$

In each case, the vertex is (h, k). Also the **axis of symmetry** is the line $x = h$.

We rewrite our steps for graphing a quadratic equation using properties for when the equation is in $y = a(x-h)^2 + k$ form.

Graphing a Quadratic Equation of the Form $y = a(x-h)^2 + k$ Using Properties

1. Rewrite the equation $y = a(x-h)^2 + k$ form.
2. Determine whether the parabola opens upward, $a > 0$, or downward, $a < 0$.
3. Find the axis of symmetry, $x = h$.
4. Find the vertex, (h, k).
5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
6. Find the x-intercepts.
7. Graph the parabola.

Example 5.1.2.10

a. Rewrite $y = 2x^2 + 4x + 5$ in $y = a(x-h)^2 + k$ form
b. Graph the equation using properties

Solution:

Rewrite the equation in $y = a(x-h)^2 + k$ form by completing the square.	$y = 2x^2 + 4x + 5$
	$y = 2\left(x^2 + 2x\right) + 5$
	$y = 2\left(x^2 + 2x + 1\right) + 5 - 2$
	$y = 2(x+1)^2 + 3$
Identify the constants a, h, k.	
Since $a = 2$, the parabola opens upward.	
The axis of symmetry is $x = h$.	The axis of symmetry is $x = -1$.
The vertex is (h, k).	The vertex is $(-1, 3)$.
Find the y-intercept by finding $f(0)$.	$f(0) = 2 \cdot 0^2 + 4 \cdot 0 + 5$
	$f(0) = 5$
	y-intercept $(0, 5)$
Find the point symmetric to $(0, 5)$ across the axis of symmetry.	$(-2, 5)$
Find the x-intercepts.	The discriminant is negative, so there are no x-intercepts. Graph the parabola.

? Exercise 5.1.2.19

a. Rewrite $y=3x^2-6x+5$ in $y=a(x-h)^2+k$ form
b. Graph the equation using properties

Answer

a. $y=3(x-1)^2+2$
b.

Figure 9.7.66

? Exercise 5.1.2.20

a. Rewrite $y=-2x^2+8x-7$ in $y=a(x-h)^2+k$ form
b. Graph the equation using properties

Answer

a. $y=-2(x-2)^2+1$
b.

The graph shown is a downward facing parabola with vertex (2, 1) and x-intercepts (1, 0) and (3, 0). The axis of symmetry is shown, x equals 2.

Figure 9.7.67

Challenge section:

Find a Quadratic Equation from its Graph

So far we have started with a equation and then found its graph.

Now we are going to reverse the process. Starting with the graph, we will find the equation.

✓ Example 5.1.2.11

Determine the quadratic equation whose graph is shown.

Figure 9.7.68

Solution:

Since it is quadratic, we start with the $y=a(x-h)^2+k$ form.

The vertex, (h,k), is $(-2,-1)$ so $h=-2$ and $k=-1$.

$$y=a(x-(-2))^2-1$$

To find a, we use the y-intercept, $(0,7)$.

So $f(0)=7$.

$$7=a(0+2)^2-1$$

Solve for a.

$$\begin{aligned}7&=4a-1\\8&=4a\\2&=a\end{aligned}$$

Write the equation.

$$y=a(x-h)^2+k$$

Substitute in $h=-2, k=-1$ and $a=2$.

$$y=2(x+2)^2-1$$

> **?** Exercise 5.1.2.21
>
> Write the quadratic equation in $y=a(x-h)^2+k$ form whose graph is shown.
>
> The graph shown is an upward facing parabola with vertex (3, negative 4) and y-intercept (0, 5).
>
> Figure 9.7.69
>
> **Answer**
>
> $y=(x-3)^2-4$

> **?** Exercise 5.1.2.22
>
> Determine the quadratic equation whose graph is shown.
>
> The graph shown is an upward facing parabola with vertex (negative 3, negative 1) and y-intercept (0, 8).
>
> Figure 9.7.70
>
> **Answer**
>
> $y=(x+3)^2-1$

Key Concepts

- Graph a Quadratic equation of the form $y=x^2+k$ Using a Vertical Shift
  - The graph of $y=x^2+k$ shifts the graph of $y=x^2$ vertically k units.
    - If $k>0$, shift the parabola vertically up k units.
    - If $k<0$, shift the parabola vertically down $|k|$ units.
- Graph a Quadratic equation of the form $y=(x-h)^2$ Using a Horizontal Shift
  - The graph of $y=(x-h)^2$ shifts the graph of $y=x^2$ horizontally h units.
    - If $h>0$, shift the parabola horizontally left h units.
    - If $h<0$, shift the parabola horizontally right $|h|$ units.
- Graph of a Quadratic equation of the form $y=ax^2$
  - The coefficient a in the equation $y=ax^2$ affects the graph of $y=x^2$ by stretching or compressing it.
    If $0<|a|<1$, then the graph of $y=ax^2$ will be “wider” than the graph of $y=x^2$.
    If $|a|>1$, then the graph of $y=ax^2$ will be “skinnier” than the graph of $y=x^2$.
- How to graph a quadratic equation using transformations
  1. Rewrite the equation in $y=a(x-h)^2+k$ form by completing the square.
  2. Graph the equation using transformations.
- Graph a quadratic equation in the vertex form $y=a(x-h)^2+k$ using properties
  1. Rewrite the equation in $y=a(x-h)^2+k$ form.
  2. Determine whether the parabola opens upward, $a>0$, or downward, $a<0$.
  3. Find the axis of symmetry, $x=h$.
  4. Find the vertex, (h,k).
  5. Find the y-intercept. Find the point symmetric to the y-intercept across the axis of symmetry.
  6. Find the x-intercepts, if possible.
  7. Graph the parabola.

5.1.3: A.3- Distance and Midpoint Formulas and Circles

Learning Objectives

By the end of this section, you will be able to:

- Use the Distance Formula
- Use the Midpoint Formula
- Write the equation of a circle in standard form
- Graph a circle

Be Prepared

Before you get started, take this readiness quiz.

1. Find the length of the hypotenuse of a right triangle whose legs are 12 and 16 inches.
2. Factor: $x^2 - 18x + 81$.
3. Solve by completing the square: $x^2 - 12x - 12 = 0$.

Here we will discuss the next conic section: the circle. We need some preliminary discussions.

The Distance Formula

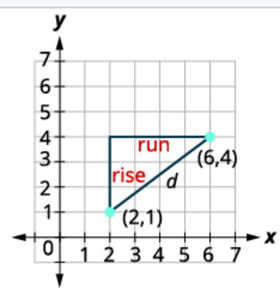
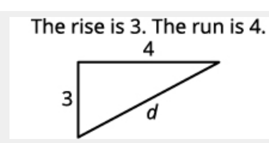
We have used the Pythagorean Theorem to find the lengths of the sides of a right triangle. Here we will use this theorem again to find distances on the rectangular coordinate system. By finding distance on the rectangular coordinate system, we can make a connection between the geometry of a conic and algebra—which opens up a world of opportunities for application.

Our first step is to develop a formula to find distances between points on the rectangular coordinate system. We will plot the points and create a right triangle much as we did when we found slope of a line. We then take it one step further and use the Pythagorean Theorem to find the length of the hypotenuse of the triangle—which is the distance between the points.

Example 5.1.3.1

Use the rectangular coordinate system to find the distance between the points $(6, 4)$ and $(2, 1)$.

Solution

Plot the two points. Connect the two points with a line. Draw a right triangle as if you were going to find slope.	
Find the length of each leg.	
Use the Pythagorean Theorem to find d, the distance between the two points.	$a^2 + b^2 = c^2$, where c is the length of the hypotenuse (opposite the right angle) and a and b are the lengths of the other two sides.
Substitute in the values.	$3^2 + 4^2 = d^2$

Simplify.	$9+16=d^2$
	$25=d^2$
Use the Square Root Property.	$d=5$ or, ~~$d=-5$~~
Since distance, d is positive, we can eliminate $d=-5$.	The distance between the points $(6,4)$ and $(2,1)$ is 5.

? Try It 5.1.3.2

Use the rectangular coordinate system to find the distance between the points $(6,1)$ and $(2,-2)$.

Answer

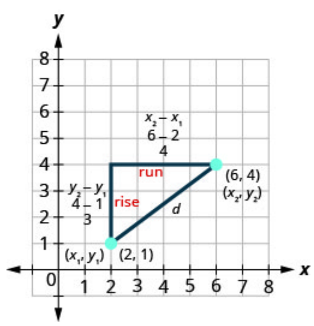
$d=5$

? Try It 5.1.3.3

Use the rectangular coordinate system to find the distance between the points $(5,3)$ and $(-3,-3)$.

Answer

$d=10$



The method we used in the last example leads us to the formula to find the distance between the two points (x_1,y_1) and (x_2,y_2).

When we found the length of the horizontal leg we subtracted $6-2$ which is x_2-x_1.

When we found the length of the vertical leg we subtracted $4-1$ which is y_2-y_1.

If the triangle had been in a different position, we may have subtracted x_1-x_2 or y_1-y_2. The expressions x_2-x_1 and x_1-x_2 vary only in the sign of the resulting number. To get the positive value-since distance is positive- we can use absolute value. So to generalize we will say $|x_2-x_1|$ and $|y_2-y_1|$.

In the Pythagorean Theorem, we substitute the general expressions $|x_2-x_1|$ and $|y_2-y_1|$ rather than the numbers.

	$a^2+b^2=c^2$
Substitute in the values.	$(\lvert x_2-x_1\rvert)^2+(\lvert y_2-y_1\rvert)^2=d^2$
Squaring the expressions makes them positive so eliminate the absolute value.	$(x_2-x_1)^2+(y_2-y_1)^2=d^2$
Use the Square Root Property.	$d=\pm\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Distance is positive, so eliminate the negative solution.	$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$

This is the Distance Formula we use to find the distance d between the two points (x_1,y_1) and (x_2,y_2).

The Distance Formula

The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 5.1.3.4

Use the Distance Formula to find the distance between the points $(-5, -3)$ and $(7, 2)$.

Solution

	$(-5,-3)$ and $(7,2)$
Write the Distance Formula.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Label the points.	$\overset{(-5,-3)}{(x_1,y_1)}$ and $\overset{(7,2)}{(x_2,y_2)}$
Substitute.	$d = \sqrt{(7-(-5))^2 + (2-(-3))^2}$
Simplify.	$d = \sqrt{12^2 + 5^2}$ $d = \sqrt{144 + 25}$ $d = \sqrt{169}$ $d = 13$

Try It 5.1.3.5

Use the Distance Formula to find the distance between the points $(-4, -5)$ and $(5, 7)$.

Answer

$d = 15$

Try It 5.1.3.6

Use the Distance Formula to find the distance between the points $(-2, -5)$ and $(-14, -10)$.

Answer

$d = 13$

Example 5.1.3.7

Use the Distance Formula to find the distance between the points $(10, -4)$ and $(-1, 5)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Solution

Write the Distance Formula.	$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$
Label the points,	$\overset{(10,-4)}{(x_1,y_1)}$ and $\overset{(-1,5)}{(x_2,y_2)}$
and substitute.	$d = \sqrt{(-1-10)^2 + (5-(-4))^2}$
Simplify.	$d = \sqrt{(-11)^2 + 9^2}$ $d = \sqrt{121 + 81}$ $d = \sqrt{202}$
Since 202 is not a perfect square, we can leave the answer in exact form or if we need to, we can find a decimal approximation.	$d = \sqrt{202}$, $d \approx 14.2$

? Try It 5.1.3.8

Use the Distance Formula to find the distance between the points $(-4,-5)$ and $(3,4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Answer

$d=\sqrt{130}, d\approx 11.4$

? Try It 5.1.3.9

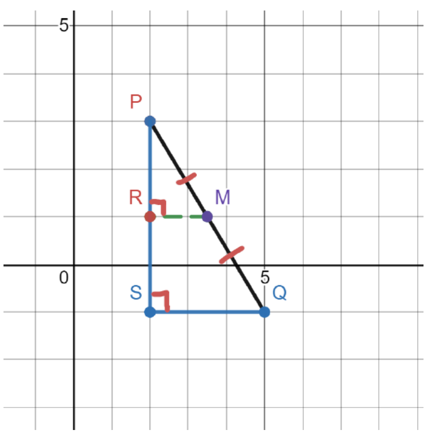
Use the Distance Formula to find the distance between the points $(-2,-5)$ and $(-3,-4)$. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

Answer

$d=\sqrt{2}, d\approx 1.4$

Use the Midpoint Formula

It is often useful to be able to find the midpoint of a segment, i.e., the point that divides a segment in half. For example, if you have the endpoints of the diameter of a circle, you may want to find the center of the circle which is the midpoint of the diameter. To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints. To see why this is the case, consider two points, P and Q, together with the midpoint M of the segment PQ and the triangle formed below:



The triangle $\triangle PQS$ is similar to the triangle $\triangle PMR$ because $\angle PRM$ and $\angle PSQ$ both right triangles and the two triangles share the angle P. Since M is the midpoint the length of PQ, $|PQ|$, is twice the length of PM, $|PM|$, i.e.,

$$|PM|=\frac{1}{2}\cdot|PQ|. \tag{5.1.3.1}$$

When two triangles are similar, the ratios of corresponding sides are equal. So,

$$\frac{|PR|}{|PS|}=\frac{|PM|}{|PQ|}=\frac{|PQ|}{2|PQ|}=\frac{1}{2}. \tag{5.1.3.2}$$

So, we see that

$$|PR|=\frac{1}{2}\cdot|PS|. \tag{5.1.3.3}$$

We see then that R is the midpoint of PS, and since the segment is vertical we can calculate the y-coordinate of the midpoint R by calculating the average of the y coordinates of P and S. But the y-coordinates of $P, M, andQ$ are the y-coordinates of P, R, and

S, respectively, so we can calculate the y-coordinate of M by calculating the average of the y coordinates of P and Q. A similar story unfolds for the x coordinates.

The Midpoint Formula

The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

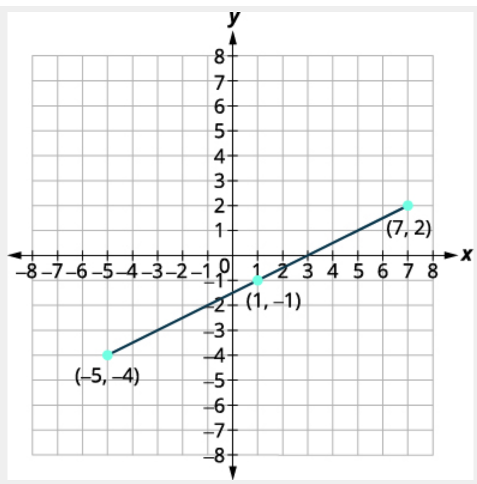
$$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$$

To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.

Example 5.1.3.10

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-5, -4)$ and $(7, 2)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

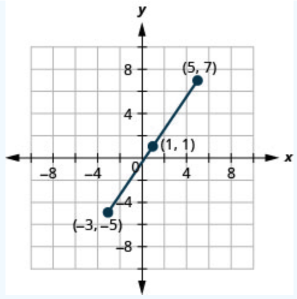
Solution:

Write the Midpoint Formula.	$\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$
Label the points,	$\overset{(-5,-4)}{(x_1,y_1)}$ and $\overset{(7,2)}{(x_2,y_2)}$
and substitute.	$\left(\frac{-5+7}{2}, \frac{-4+2}{2}\right)$
Simplify.	$\left(\frac{2}{2}, \frac{-2}{2}\right)$
	$(1,-1)$ The midpoint of the segment is the point $(1,-1)$.
Plot the endpoints and midpoint.	

Try It 5.1.3.11

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-3, -5)$ and $(5, 7)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

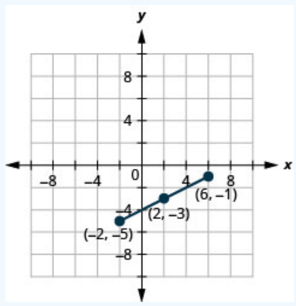
Answer



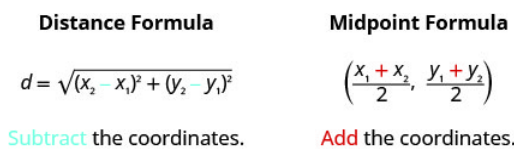
? Try It 5.1.3.12

Use the Midpoint Formula to find the midpoint of the line segments whose endpoints are $(-2, -5)$ and $(6, -1)$. Plot the endpoints and the midpoint on a rectangular coordinate system.

Answer



Both the Distance Formula and the Midpoint Formula depend on two points, (x_1, y_1) and (x_2, y_2). It is easy to confuse which formula requires addition and which subtraction of the coordinates. If we remember where the formulas come from, is may be easier to remember the formulas.



Finding the Perpendicular Bisector of a Line Segment

Definition 5.1.3.13

The **Perpendicular Bisector** of a line segment PQ is the line that is perpendicular to PQ and passes through though the midpoint of PQ.

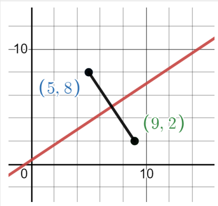
Example 5.1.3.14

Find the perpendicular bisector of the line segment whose endpoints are $5, 8$ and $9, 2$.

Solution

A line is determined by a point on the line (in this case the midpoint of the given segment) and the slope.

The perpendicular bisector passes through the midpoint of the given line segment. So we must calculate the midpoint by averaging the x-coordinates, and averaging the y-coordinates.	$M = \left(\frac{5+9}{2}, \frac{8+2}{2}\right)$
Simplify.	$M = (7,5)$

We now need to find the slope of the line. To do this, we first find the slope of the given segment.	$m_{\text{segment}} = \frac{2-8}{9-5} = \frac{-6}{4} = -\frac{3}{2}$
The slope of a line perpendicular to this segment has a slope that is the negative reciprocal of the slope of this segment. We calculate the slope of the desired line.	$m_{\text{p.b.}} = -\frac{1}{m_{\text{segment}}} = -\frac{1}{-\frac{3}{2}} = \frac{2}{3}$
Use the slope obtained together with the midpoint to find an equation of the perpendicular bisector.	$M = (7,5), \quad m = \frac{2}{3}$ $y - 5 = \frac{2}{3}(x-7)$
If desired, write in standard form (or any desired form).	$3(y-5) = 2(x-7)$ $3y - 15 = 2x - 14$ $2x - 3y = -1$
We could check our answer by graphing which will help to detect major errors.	 The line appears to be perpendicular to the line segment and also pass through the midpoint of the line segment.

? Try It 5.1.3.15

Find the perpendicular bisector of the line segment connecting the points $(-3, 5)$ and $(5, -7)$.

Answer

$$y + 1 = \frac{2}{3}(x-1)$$

or

$$2x - 3y = 5$$

? Try It 5.1.3.16

Find the perpendicular bisector of the line segment connecting the points $\left(-2, \frac{7}{2}\right)$ and $\left(-5, -\frac{5}{2}\right)$.

Answer

$$y - \frac{1}{2} = -\frac{1}{2}\left(x + \frac{7}{2}\right)$$

Write the Equation of a Circle in Standard Form

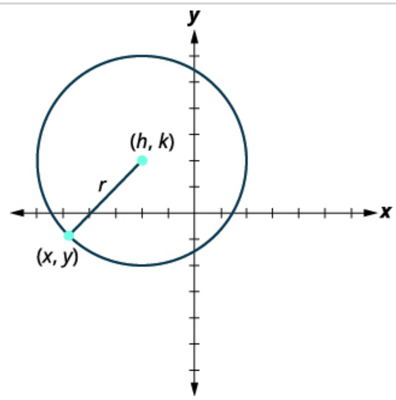
As we mentioned, our goal is to connect the geometry of a conic with algebra. By using the coordinate plane, we are able to do this easily.

circle

We define a **circle** as all points in a plane that are a fixed distance from a given point in the plane. The given point is called the *center,* (h,k), and the fixed distance is called the *radius*, r, of the circle.

Definition 5.1.3.17

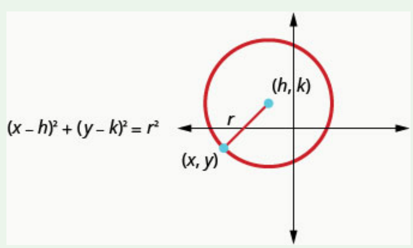
A **circle** is all points in a plane that are a fixed distance from a given point in the plane. The given point is called the **center**, (h,k), and the fixed distance is called the **radius**, r, of the circle.

We look at a circle in the rectangular coordinate system. The radius is the distance from the center, (h,k),to a point on the circle, (x,y).	
To derive the equation of a circle, we can use the distance formula with the points (h,k), (x,y) and the distance, r.	$d=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Substitute the values.	$r=\sqrt{(x-h)^2+(y-k)^2}$
Square both sides.	$r^2=(x-h)^2+(y-k)^2$

This is the standard form of the equation of a circle with center, (h,k), and radius, r.

Definition 5.1.3.18

The **standard form** of the equation of a circle with center, (h,k), and radius, r, is



Example 5.1.3.19

Write the standard form of the equation of the circle with radius 3 and center $(0,0)$.

Solution:

Use the standard form of the equation of a circle.	$(x-h)^2+(y-k)^2=r^2$
Identify the center and radius.	$\overset{(0,0)}{(h,k)}$ and $\overset{3}{r}$
Substitute in the values $r=3, h=0$, and $k=0$.	$(x-0)^2+(y-0)^2=3^2$

Simplify.	$x^2 + y^2 = 9$

Try It 5.1.3.20

Write the standard form of the equation of the circle with a radius of 6 and center $(0,0)$.

Answer

$x^2 + y^2 = 36$

Try It 5.1.3.21

Write the standard form of the equation of the circle with a radius of 8 and center $(0,0)$.

Answer

$x^2 + y^2 = 64$

In the last example, the center was $(0,0)$. Notice what happened to the equation. Whenever the center is $(0,0)$, the standard form becomes $x^2 + y^2 = r^2$.

Example 5.1.3.22

Write the standard form of the equation of the circle with radius 2 and center $(-1,3)$.

Solution:

Use the standard form of the equation of a circle.	$(x-h)^2 + (y-k)^2 = r^2$
Identify the center and radius.	$\overset{(-1,3)}{(h,k)}$ and $\overset{2}{r}$
Substitute in the values.	$(x-(-1))^2 + (y-3)^2 = 2^2$
Simplify.	$(x+1)^2 + (y-3)^2 = 4$

Try It 5.1.3.23

Write the standard form of the equation of the circle with a radius of 7 and center $(2,-4)$.

Answer

$(x-2)^2 + (y+4)^2 = 49$

Try It 5.1.3.24

Write the standard form of the equation of the circle with a radius of 9 and center $(-3,-5)$.

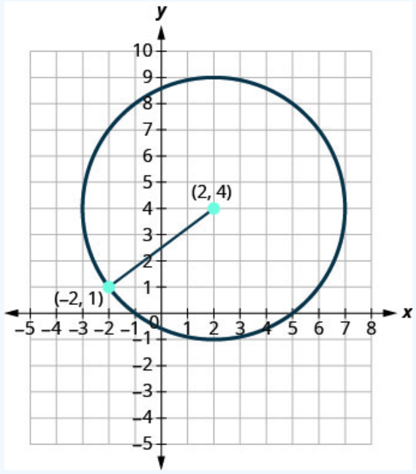
Answer

$(x+3)^2 + (y+5)^2 = 81$

In the next example, the radius is not given. To calculate the radius, we use the Distance Formula with the two given points.

Example 5.1.3.25

Write the standard form of the equation of the circle with center $(2, 4)$ that also contains the point $(-2, 1)$.



Solution:

The radius is the distance from the center to any point on the circle so we can use the distance formula to calculate it. We will use the center $(2,4)$ and point $(-2,1)$ Use the Distance Formula to find the radius.	$r=\sqrt{(x_2-x_1)^2+(y_2-y_1)^2}$
Label the points,	$\overset{(2,4)}{(x_1,y_1)}$ and $\overset{(-2,1)}{(x_2,y_2)}$
and substitute the values to find r.	$r=\sqrt{(-2-2)^2+(1-4)^2}$
Simplify.	$r=\sqrt{(-4)^2+(-3)^2}$ $r=\sqrt{16+9}$ $r=\sqrt{25}$ $r=5$
Now that we know the radius, $r=5$, and the center, $(2,4)$, we can use the standard form of the equation of a circle to find the equation.	$(x-h)^2+(y-k)^2=r^2$
Substitute in the values.	$(x-2)^2+(y-4)^2=5^2$
Simplify.	$(x-2)^2+(y-4)^2=25$

Try It 5.1.3.26

Write the standard form of the equation of the circle with center $(2, 1)$ that also contains the point $(-2, -2)$.

Answer

$(x-2)^2+(y-1)^2=25$

Try It 5.1.3.27

Write the standard form of the equation of the circle with center $(7, 1)$ that also contains the point $(-1, -5)$.

Answer

$(x-7)^2+(y-1)^2=100$

Graph a Circle

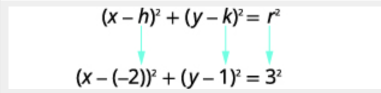
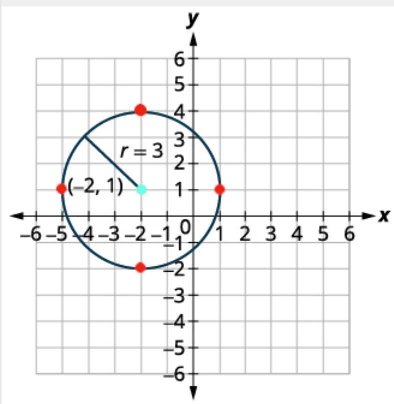
Any equation of the form $(x-h)^2+(y-k)^2=r^2$ is the standard form of the equation of a **circle** with center, (h,k), and radius, r. We can then graph the circle on a rectangular coordinate system.

Note that the standard form calls for subtraction from x and y. In the next example, the equation has $x+2$, so we need to rewrite the addition as subtraction of a negative.

Example 5.1.3.28

Find the center and radius, then graph the circle: $(x+2)^2+(y-1)^2=9$.

Solution:

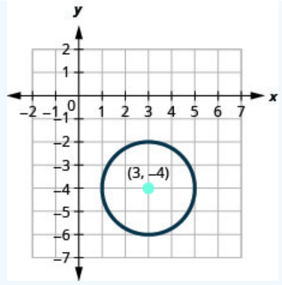
	$(x+2)^2+(y-1)^2=9$
Use the standard form of the equation of a circle. Identify the center, (h,k) and radius, r.	 $(x-h)^2+(y-k)^2=r^2$ $(x-(-2))^2+(y-1)^2=3^2$
	Center: $(-2,1)$ radius: 3
Use four points (N,S,E, and W of center) to guide the sketch by adding and subtracting the radius to the y-coordinate and the x-coordinate in turn.	Center: $(-2,1)$ $(-2,1+3)=(-2,4)$ $(-2,1-3)=(-2,-2)$ $(-2+3,1)=(1,1)$ $(-2-3,1)=(-5,1)$
Graph the circle.	

Try It 5.1.3.29

a. Find the center and radius, then
b. Find the 4 points N, S, E, and W of the center. Graph the circle: $(x-3)^2+(y+4)^2=4$.

Answer

a. The circle is centered at $(3,-4)$ with a radius of 2.
b. $(3,-2),(3,-6),(5,-4),(1,-4)$

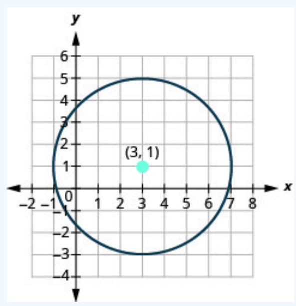


? Try It 5.1.3.30

a. Find the center and radius, then
b. Find the four points N, S, E, and W of center. Graph the circle: $(x-3)^2+(y-1)^2=16$.

Answer

a. The circle is centered at $(3,1)$ with a radius of 4.
b. $(3,5),(3,-3),(7,1),(-1,1)$



If we expand the equation from Example 11.24, we see a different form:

	$(x+2)^2+(y-1)^2=9$
Square the binomials.	$x^2+4x+4+y^2-2y+1=9$
Arrange the terms in descending degree order, and get zero on the right	$x^2+y^2+4x-2y-4=0.$

This form of the equation is called the general form of the equation of the **circle**.

Definition 5.1.3.31

The **general form of the equation of a circle** is

$$x^2+y^2+ax+by+c=0$$

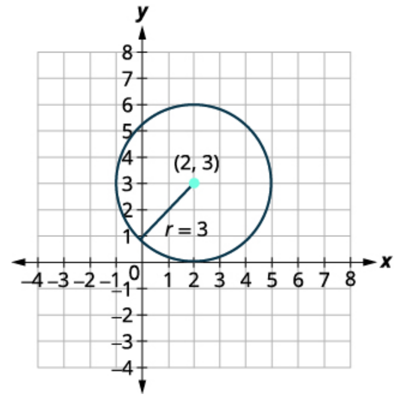
If we are given an equation in general form, we can change it to standard form by completing the squares in both x and y. Then we can graph the circle using its center and radius.

Example 5.1.3.32

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2-4x-6y+4=0$

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

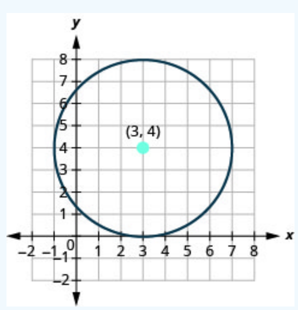
	$x^2+y^2-4x-6y+4=0$
Group the x-terms and y-terms. Collect the constants on the right side.	$x^2-4x+y^2-6y=-4$
Complete the squares.	$x^2-4x+4-4+y^2-6y+9-9=-4$ $x^2-4x+4+y^2-6y+9=-4+4+9$
Rewrite as binomial squares.	$(x-2)^2+(y-3)^2=9$
Identify the center and radius.	Center: $(2,3)$ radius: 3
Identify four points on the circle to guide the sketch.	$(2,6),(2,0),(5,3),(-1,3)$
Graph the circle.	 Figure 11.1.31

? Try It 5.1.3.33

a. Find the center and radius, then
b. Identify four points to guide the sketch of the graph. Graph the circle: $x^2+y^2-6x-8y+9=0$.

Answer

a. The circle is centered at $(3,4)$ with a radius of 4.
b. $(3,8)$, $(3,0)$, $(7,4)$, $(-1,4)$



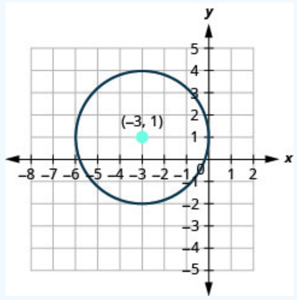
? Try It 5.1.3.34

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2+6x-2y+1=0$

Answer

a. The circle is centered at $(-3,1)$ with a radius of 3.

b. $(-3,4),(-3,-2),(0,1),(-6,1)$



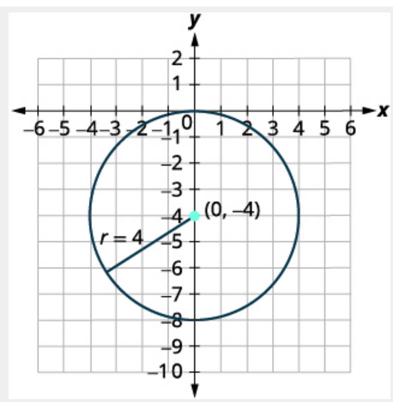
In the next example, there is a y-term and a y^2-term. But notice that there is no x-term, only an x^2-term. We have seen this before and know that it means h is 0. We will need to complete the square for the y terms, but not for the x terms.

Example 5.1.3.35

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2+8y=0$

Solution:

We need to rewrite this general form into standard form in order to find the center and radius.

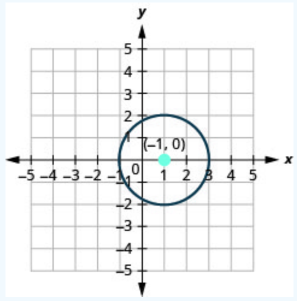
	$x^2+y^2+8y=0$
Group the x-terms and y-terms.	$x^2+y^2+8y=0$
There are no constants to collect on the right side.	
Complete the square for y^2+8y.	$x^2+y^2+8y+16-16=0$ $x^2+y^2+8y+16=16$
Rewrite as binomial squares.	$(x-0)^2+(y+4)^2=16$
Identify the center and radius.	Center: $(0,-4)$ radius: 4
Identify four points to guide our sketch.	$(0,0),(0,-16),(4,0),(-4,0)$
Graph the circle.	

Try It 5.1.3.36

a. Find the center and radius, then
b. Graph the circle: $x^2+y^2-2x-3=0$.

Answer

a. The circle is centered at $(-1, 0)$ with a radius of 2.
b. $(-1, 2), (-1, -2), (1, 0), (-3, 0)$

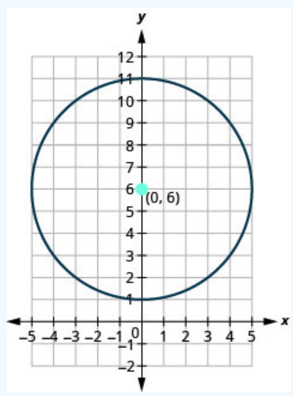


? Try It 5.1.3.37

a. Find the center and radius, then
b. Graph the circle: $x^2 + y^2 - 12y + 11 = 0$.

Answer

a. The circle is centered at $(0, 6)$ with a radius of 5.
b. $(0, 11), (0, 1), (5, 6), (-5, 6)$



Key Concepts

- **Distance Formula:** The distance d between the two points (x_1, y_1) and (x_2, y_2) is

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

- **Midpoint Formula:** The midpoint of the line segment whose endpoints are the two points (x_1, y_1) and (x_2, y_2) is

$$\left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}\right)$$

  To find the midpoint of a line segment, we find the average of the x-coordinates and the average of the y-coordinates of the endpoints.
- **Circle:** A circle is all points in a plane that are a fixed distance from a fixed point in the plane. The given point is called the *center,* (h, k), and the fixed distance is called the *radius,* r, of the circle.
- **Standard Form of the Equation a Circle:** The standard form of the equation of a circle with center, (h, k), and radius, r, is

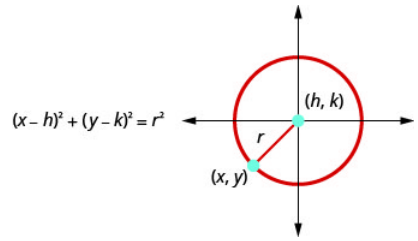


Figure 11.1.41

- **General Form of the Equation of a Circle:** The general form of the equation of a circle is

$$x^2 + y^2 + ax + by + c = 0$$

Glossary

circle

A circle is all points in a plane that are a fixed distance from a fixed point in the plane.

Practice Makes Perfect

Use the Distance Formula

In the following exercises, find the distance between the points. Write the answer in exact form and then find the decimal approximation, rounded to the nearest tenth if needed.

1. $(2, 0)$ and $(5, 4)$
2. $(-4, -3)$ and $(2, 5)$
3. $(-4, -3)$ and $(8, 2)$
4. $(-7, -3)$ and $(8, 5)$
5. $(-1, 4)$ and $(2, 0)$
6. $(-1, 3)$ and $(5, -5)$
7. $(1, -4)$ and $(6, 8)$
8. $(-8, -2)$ and $(7, 6)$
9. $(-3, -5)$ and $(0, 1)$
10. $(-1, -2)$ and $(-3, 4)$
11. $(3, -1)$ and $(1, 7)$
12. $(-4, -5)$ and $(7, 4)$

Answer

1. $d = 5$

3. 13

5. 5

7. 13

9. $d = 3\sqrt{5}, d \approx 6.7$

11. $d = \sqrt{68}, d \approx 8.2$

Use the Midpoint Formula

In the following exercises,

a. find the midpoint of the line segments whose endpoints are given and
b. plot the endpoints and the midpoint on a rectangular coordinate system.

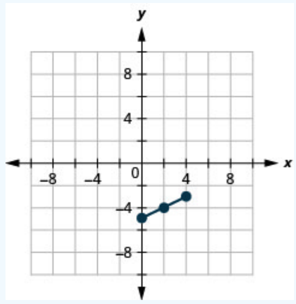
13. $(0, -5)$ and $(4, -3)$
14. $(-2, -6)$ and $(6, -2)$

15. $(3,-1)$ and $(4,-2)$
16. $(-3,-3)$ and $(6,-1)$

Answer

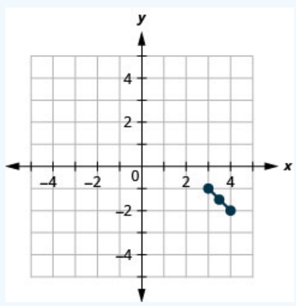
13.

a. Midpoint: $(2,-4)$
b.



15.

a. Midpoint: $\left(3\frac{1}{2},-1\frac{1}{2}\right)$
b.



? Write an equation for the perpendicular bisector of the line segment with the given endpoints.

17. $(-9,9),(25,-25)$

18. $(0.02,-3.5),(1.06,-11.7)$

? Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center $(0,0)$.

19. Radius: 7
20. Radius: 9
21. Radius: $\sqrt{2}$
22. Radius: $\sqrt{5}$

Answer

19. $x^2+y^2=49$

21. $x^2+y^2=2$

? Write the Equation of a Circle in Standard Form

In the following exercises, write the standard form of the equation of the circle with the given radius and center

23. Radius: 1, center: $(3, 5)$
24. Radius: 10, center: $(-2, 6)$
25. Radius: 2.5, center: $(1.5, -3.5)$
26. Radius: 1.5, center: $(-5.5, -6.5)$

Answer

23. $(x-3)^2+(y-5)^2=1$

25. $(x-1.5)^2+(y+3.5)^2=6.25$

? Write the Equation of a Circle in Standard Form

For the following exercises, write the standard form of the equation of the circle with the given center with point on the circle.

27. Center $(3, -2)$ with point $(3, 6)$
28. Center $(6, -6)$ with point $(2, -3)$
29. Center $(4, 4)$ with point $(2, 2)$
30. Center $(-5, 6)$ with point $(-2, 3)$

Answer

27. $(x-3)^2+(y+2)^2=64$

29. $(x-4)^2+(y-4)^2=8$

? Graph a Circle

In the following exercises,

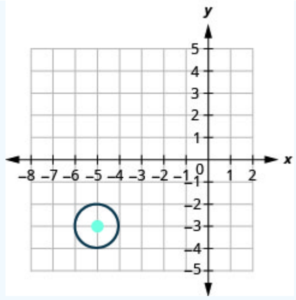
a. find the center and radius, then
b. graph each circle.

31. $(x+5)^2+(y+3)^2=1$
32. $(x-2)^2+(y-3)^2=9$
33. $(x-4)^2+(y+2)^2=16$
34. $(x+2)^2+(y-5)^2=4$
35. $x^2+(y+2)^2=25$
36. $(x-1)^2+y^2=36$
37. $(x-1.5)^2+(y+2.5)^2=0.25$
38. $(x-1)^2+(y-3)^2=\frac{9}{4}$
39. $x^2+y^2=64$
40. $x^2+y^2=49$

Answer

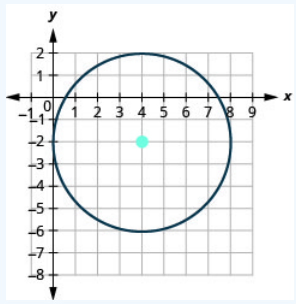
31.

a. The circle is centered at $(-5, -3)$ with a radius of 1.
b.



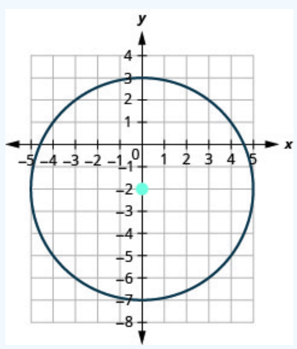
33.

a. The circle is centered at $(4, -2)$ with a radius of 4.
b.



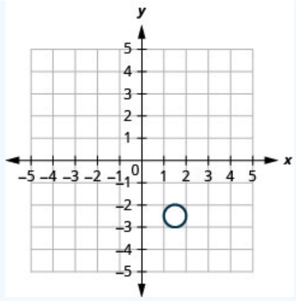
35.

a. The circle is centered at $(0, -2)$ with a radius of 5.
b.



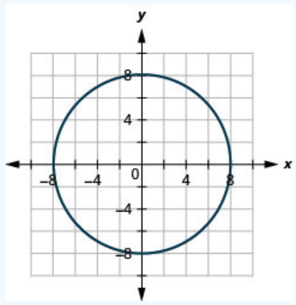
37.

a. The circle is centered at $(1.5, 2.5)$with a radius of 0.5.
b.



39.

a. The circle is centered at $(0,0)$ with a radius of 8.
b.



? Graph a Circle

In the following exercises,

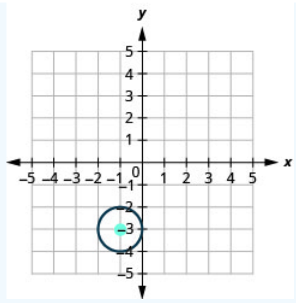
a. identify the center and radius and
b. graph.

41. $x^2+y^2+2x+6y+9=0$
42. $x^2+y^2-6x-8y=0$
43. $x^2+y^2-4x+10y-7=0$
44. $x^2+y^2+12x-14y+21=0$
45. $x^2+y^2+6y+5=0$
46. $x^2+y^2-10y=0$
47. $x^2+y^2+4x=0$
48. $x^2+y^2-14x+13=0$

Answer

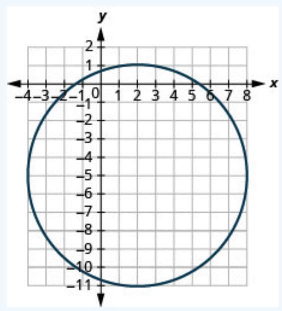
41.

a. Center: $(-1,-3)$, radius: 1
b.



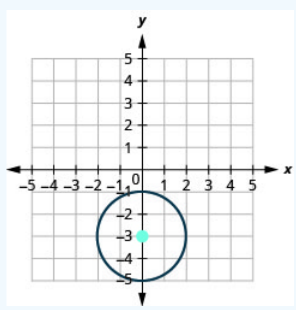
43.

a. Center: $(2, -5)$, radius: 6
b.



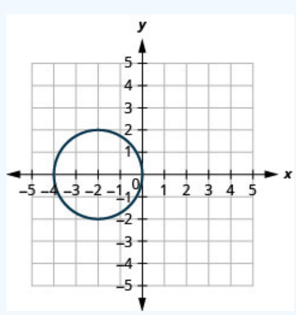
45.

a. Center: $(0, -3)$, radius: 2
b.



47.

a. Center: $(-2, 0)$, radius: -2
b.

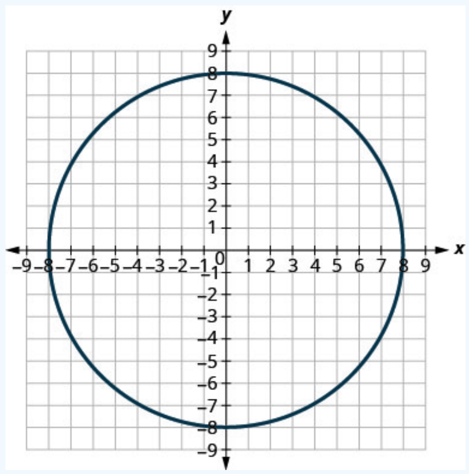


? Mixed Practice

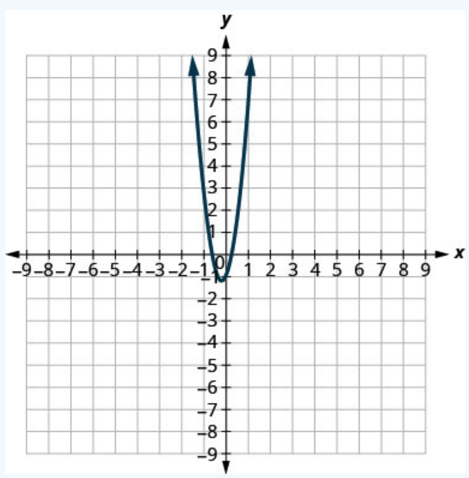
In the following exercises, match each graph to one of the following equations:

a. $x^2+y^2=64$
b. $x^2+y^2=49$
c. $(x+5)^2+(y+2)^2=4$
d. $(x-2)^2+(y-3)^2=9$
e. $y=-x^2+8x-15$
f. $y=6x^2+2x-1$

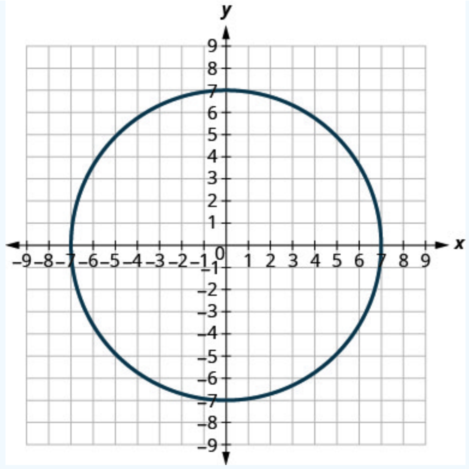
49.



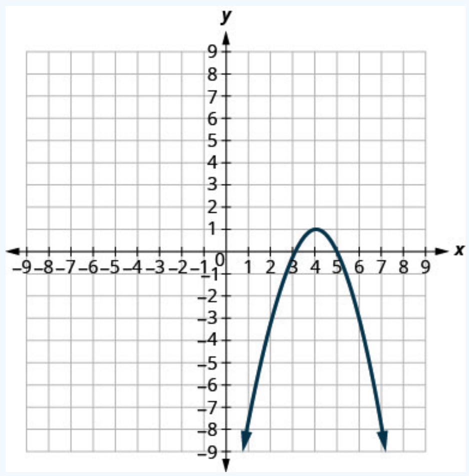
50.



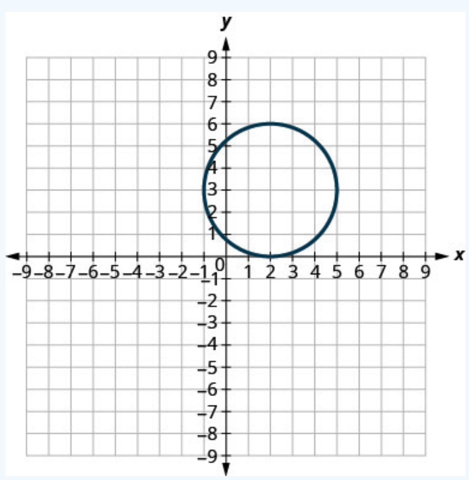
51.



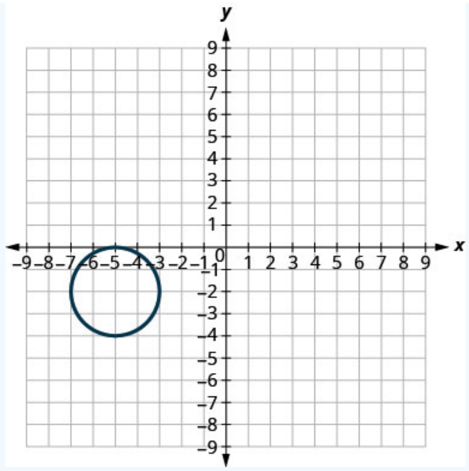
52.



53.



54.



Answer

49. a

51. b

53. d

Writing Exercises

53. Explain the relationship between the distance formula and the equation of a circle.
54. Is a circle a function? Explain why or why not.
55. In your own words, state the definition of a circle.
56. In your own words, explain the steps you would take to change the general form of the equation of a circle to the standard form.

Answer

53. Answers will vary.

55. Answers will vary.

Self Check

a. After completing the exercises, use this checklist to evaluate your mastery of the objectives of this section.

I can...	Confidently	With some help	No-I don't get it!
use the distance formula.			
use the midpoint formula.			
write the equation of a circle in standard form.			
graph a circle.			

Figure 11.1.53

b. If most of your checks were:

…confidently. Congratulations! You have achieved the objectives in this section. Reflect on the study skills you used so that you can continue to use them. What did you do to become confident of your ability to do these things? Be specific.

…with some help. This must be addressed quickly because topics you do not master become potholes in your road to success. In math every topic builds upon previous work. It is important to make sure you have a strong foundation before you move on. Who can you ask for help? Your fellow classmates and instructor are good resources. Is there a place on campus where math tutors are available? Can your study skills be improved?

…no - I don't get it! This is a warning sign and you must not ignore it. You should get help right away or you will quickly be overwhelmed. See your instructor as soon as you can to discuss your situation. Together you can come up with a plan to get you the help you need.

5.2: Appendix B- Decimal Numbers

Consider the number 23.7456. Each digit occupies a 'place". The 2 is in the tens place, the three in the ones place, the 7 in the tenths place, the 4 in the hundredths place, the 5 in the thousandths place, and the 6 in the ten thousandths place. Why? Because:

$$23.7456 = 2 \cdot 10 + 3 \cdot 1 + 7 \cdot \frac{1}{10} + 4 \cdot \frac{1}{100} + 5 \cdot \frac{1}{1000} + 6 \cdot \frac{1}{10,000}$$

Rounding

Rounding is associated with cutting or truncating a number, and that the rounding compensates for the lost tail of the number. For example, to round a given number to the nearest tenth, we look one digit to the right of the tenths place (the hundredths place) and if it is greater than or equal to 5, we add one to the tenths place and remove all the digits to the right, otherwise we leave the tenths place as it is and remove all the digits to the right.

Example 5.2.1

a. 234.45 rounded to the nearest ten is 230.
b. 45.6 rounded to the nearest ones (whole number) is 46.
c. 34.555 rounded to the nearest tenth is 34.6.
d. 34.54 rounded to the nearest tenth is 34.5.
e. 34.95 rounded to the nearest tenth is 35.0.
f. 34.554 rounded to the nearest hundredth is 34.55.
g. 56.7874778 rounded to the nearest ten-thousandth is 56.7875

Adding and Subtracting Decimal Numbers

To add decimals, we line up the decimal points, and wherever there is a missing digit, we fill it in with a zero. For example, add 45.23 and 2.3:

$$\begin{array}{ccccc} 4 & 5 & . & 2 & 3 \\ + & 2 & . & 3 & 0 \\ \hline 4 & 7 & . & 5 & 3 \end{array}$$

Subtracting is similar. To subtract 45.23 from 2.3 we first note that the answer should be negative and proceed to subtract 2.3 from 45.23:

$$\begin{array}{ccccc} 4 & 5 & . & 2 & 3 \\ - & 2 & . & 3 & 0 \\ \hline 4 & 2 & . & 9 & 3 \end{array}$$

So, the answer of $2.3 - 45.23 = -42.93$

Example 3.2

a. $2.4 + 32.032 = 34.432$
b. $3.44 + 12.035 = 15.475$
c. $34.3 - 0.05 = 34.25$
d. $6.3 - 9.72 = -3.42$

Multiplying and Dividing Decimal Numbers

Multiplying and Dividing Decimal Numbers by 10, 100, 1000, . . .

We first look at the special multiplication of decimals by 10, 100, 1000, . . .

$$\begin{aligned}12.415\times 10 &= 124.15\\ 12.415\times 100 &= 1241.5\\ 12.415\times 1000 &= 12415\end{aligned}$$

When we multiply by 10 we move the decimal point to the right one place (because 10 has one decimal place). Multiplying by 100 moves the decimal point two places (because 100 has two decimal places), etc.

$12.415 \div 10 = 1.2415$
$12.415 \div 100 = 0.12415$
$12.415 \div 1000 = 0.012415$

When we divide by 10 we move the decimal point to the left one place (because 10 has one decimal place). Dividing by 100 moves the decimal point to the left two places (because 100 has two decimal places), etc.

10^n notation

$$\begin{aligned}10 &= 10^1\\ 100 &= 10\times 10 = 10^2\\ 1000 &= 10\times 10\times 10 = 10^3\end{aligned}$$

Notice that the exponent of 10 in 10^n notation reflects the number of zeros! So, $10000 = 10^4$ $(4$ zeros, exponent is $4)$ and $100,000 = 10^5, \ldots$

Multiplying by 10^n

Multiplying a decimal number by 10^n moves the decimal place n spots to the right. For example:

$$\begin{aligned}5.435\times 10 &= 54.35\\ 5.435\times 100 &= 543.5\\ 5.435\times 10000 &= 54350\end{aligned}$$

Multiplying Decimal Numbers

To multiply two decimal numbers, we multiply as if there is no decimal point, then place a decimal point as described in the next example.

Example 3.3

Multiply 5.4 by 1.21.

Solution

$$\begin{array}{rrrr} & 1 & 2 & 1\\ \times & & 5 & 4\\ \hline & 4 & 8 & 4\\ 6 & 0 & 5 & 0\\ \hline 6 & 5 & 3 & 4\end{array}$$

Now, to write out the answer, we notice that there are two digits after the decimal point in the first number 1.21, and one digit after the decimal point in the second number 5.4. The product then should have 3 digits after the decimal point. So, $5.4 \times 1.21 = 6.534$

Example 3.4

Multiply 3.72 by 13.

Solution

$$\begin{array}{rrrrr} & & 3 & 7 & 2 \\ \times & & & 1 & 3 \\ \hline & 1 & 1 & 1 & 6 \\ & 3 & 7 & 2 & 0 \\ \hline & 4 & 8 & 3 & 6 \end{array}$$

Now, to write out the answer, we notice that there are two digits after the decimal point in 3.72 while 13 has no decimal part. The product then should have 2 digits after the decimal point: $3.72 \times 13 = 48.36$

Dividing Decimal Numbers

Dividing a decimal number is a lot like dividing a whole number, except you use the position of the decimal point in the dividend to determine the decimal places in the result.

Example 3.5

a. $6.5 \div 2$
b. $55.318 \div 3.4$

Solution

a) Divide as usual:

If the divisor does not go into the dividend evenly, add zeros to the right of the last digit in the dividend and continue until either the remainder is 0, or a repeating pattern appears. Place the position of the decimal point in your answer directly above the decimal point in the dividend.

$$\begin{array}{r|l} & 3.25 \\ \hline 2 & 6.5 \\ & -6 \\ \hline & 0\,5 \\ & -4 \\ \hline & 10 \\ & -10 \\ \hline & 0 \end{array}$$

b)

If the divisor in not a whole number, move the decimal point in the divisor all the way to the right (to make it a whole number). Then move the decimal point in the dividend the same number of places.

In this example, move the decimal point one place to the right for the divisor from 3.4 to 34. Therefore, also move the decimal point one place to the right for the dividend, from 55.318 to 553.18 .

$$\begin{array}{r|l} & 16.27 \\ \hline 34 & 553.18 \\ & -34 \\ \hline & 213 \\ & -204 \\ \hline & 9\,1 \\ & -6\,8 \\ \hline & 2\,38 \\ & -2\,38 \\ \hline & 0 \end{array}$$

Converting Decimals to Fractions

To convert a decimal to a fraction is as simple as recognizing the place of the right most digit.

Example: Note that in the number 2.45, the right most digit 5 is in the hundredths place so $2.45 = \dfrac{245}{100} = \dfrac{49}{20}$ or $2\dfrac{9}{20}$

Example 3.6

Here are a few more examples:

a. $1.2=\frac{12}{10}=\frac{6}{5}$ or $1\frac{1}{5}$

b. $0.0033=\frac{33}{10,000}$

c. $0.103=\frac{103}{1000}$

Converting Fractions to Decimals

To convert a fraction to a decimal you simply perform long division.

Example 3.7

Convert the given fraction to a decimal:

a. $\frac{4}{5}=4\div 5=0.8$

b. $3\frac{4}{5}=3+4\div 5=3.8$

c. $\frac{13}{2}=6\frac{1}{2}=6+1\div 2=6.5$

d. (round to the nearest tenth) $\frac{3}{7}=3\div 7=0.42857\cdots\approx 0.4$

Converting Decimals to Percents and Percents to Decimals

"Percent" comes from Latin and means per hundred. We use the sign \% for Por example, if you know that 25% of the students speak Spanish fluently, it means that 25 of every 100 students speak fluent Spanish. Presented as fraction, it would be $\frac{25}{100}$ and as a decimal 0.25.

Example 3.8

Convert the given percent to fraction then to a decimal:

a. 17% is $\frac{17}{100}=0.17$

b. 31% is $\frac{31}{100}=0.31$

c. 23.44% is $\frac{23.44}{100}=0.2344$

Example 3.9

Convert the given decimal to a fraction then to percent:

a. $0.55=\frac{55}{100}$ which is 55%

b. $8.09=\frac{809}{100}$ which is 809%

c. $98.08=\frac{9808}{100}$ which is 9808%

d. $0.5=\frac{50}{100}$ which is 50%

Exit Problem

Divide: $782.56 \div 3.2$

Index

Glossary

Sample Word 1 | Sample Definition 1

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- College Algebra and Trigonometry: Expressions, Equations and Graphs - *Undeclared*
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    - InfoPage - *Undeclared*
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  - 1: Expressions - *Undeclared*
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    - 1.2: Polynomials - *Undeclared*
      - 1.2.1: Linear Expressions - *Undeclared*
      - 1.2.2: Evaluating, Adding and Subtracting Polynomials - *Undeclared*
      - 1.2.3: Multiplying Polynomials - *CC BY 4.0*
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    - 1.3: Rational Expressions - *CC BY 4.0*
      - 1.3.1: Integer Exponents: a Review with Variables - *CC BY 4.0*
      - 1.3.2: Simplifying, Multiplying and Dividing Rational Expressions - *CC BY 4.0*
      - 1.3.3: Adding and Subtracting Rational Expressions - *CC BY 4.0*
      - 1.3.4: Complex Rational Expressions - *CC BY 4.0*
    - 1.4: Radical Expressions - *CC BY 4.0*
      - 1.4.1: Radical Expressions - *CC BY 4.0*
      - 1.4.2: Simplifying Radical Expressions - *CC BY 4.0*
      - 1.4.3: Rational Exponents - *CC BY 4.0*
      - 1.4.4: Adding, Subtracting and Multiplying Radical Expressions - *CC BY 4.0*
      - 1.4.5: Dividing Radical Expressions - *CC BY 4.0*
      - 1.4.6: Complex Numbers - *Undeclared*
  - 2: Equations with One Variable - *Undeclared*
    - 2.1: Linear Equations - *Undeclared*
    - 2.2: Quadratic Equations - *CC BY 4.0*
      - 2.2.1: Solving Quadratic Equations Using the Zero-Product Property - *Undeclared*
      - 2.2.2: Solving Quadratic Equations Using the Square Root Property - *CC BY 4.0*
      - 2.2.3: Solving Quadratic Equations by Completing the Square - *CC BY 4.0*
      - 2.2.4: Solving Quadratic Equations Using the Quadratic Formula - *CC BY 4.0*
      - 2.2.5: Applications of Quadratic Equations - *CC BY 4.0*
    - 2.3: Polynomial Equations - *Undeclared*
    - 2.4: Rational Equations - *CC BY 4.0*
    - 2.5: Radical Equations - *Undeclared*
  - 3: Graphs and Equations with Two Variables - *Undeclared*
    - 3.1: Linear Equations with Two Variables - *CC BY 4.0*
      - 3.1.1: Graphing Linear Equations with Two Variables - *CC BY 4.0*
      - 3.1.2: Slope of a Line - *CC BY 4.0*
      - 3.1.3: Finding the Equation of a Line - *CC BY 4.0*