Module 2 - Summary

Summary Notes Module 2

• Light propagates in the form of waves with a speed $v = \frac{c}{n}$ where $c = 3 \times 10^8$ m/s and n is the refractive index. Since $n \geq 1$ ($n = 1$ for air), the speed of light in any medium is $v \leq c$.

• The speed of light relates the wavelength λ and frequency f by $v = f\lambda$.

• An optical wave is described by its wavefunction $u(\mathbf{r}, t)$ where $\mathbf{r} = (x, y, z)$ represents the 3D position.

• Any optical wave satisfies the wave equation:

$$\nabla^2 u(\mathbf{r}, t) - \frac{1}{v^2} \frac{\partial^2 u(\mathbf{r}, t)}{\partial t^2} = 0 \tag{1}$$

where $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplacian operator in Cartesian coordinates.

• The intensity of a wave can be defined by the wavefunction as $I(\mathbf{r}) = \langle |u(\mathbf{r}, t)|^2 \rangle$, where $\langle \cdot \rangle$ means the average over a temporal interval.

• The power of a wave is calculated at the integrated intensity over an area normal to the light's propagation axis:

$$P = \int_{Area} I(\mathbf{r})\, dA \tag{2}$$

• The wavefunction for monochromatic waves is:

$$u(\mathbf{r}, t) = a(\mathbf{r}) \cos(2\pi f t + \varphi(\mathbf{r})) \tag{3}$$

where $a(\mathbf{r})$ is the amplitude and $\varphi(\mathbf{r})$ is the phase, f is frequency [Hz], and $\omega = 2\pi f$ is the angular frequency [rad/s], $T = \frac{1}{f}$ is the temporal period [s].

The intensity of monochromatic waves is $I(\mathbf{r}) = a^2(\mathbf{r})$, which is independent of time.

The complex wavefront of monochromatic waves is:

$$U(\mathbf{r}, t) = a(\mathbf{r}) e^{j\varphi(\mathbf{r})} e^{j2\pi f t} \tag{4}$$

where $U(\mathbf{r}) = a(\mathbf{r}) e^{j\varphi(\mathbf{r})}$ is the complex amplitude.

Whereas the complex wavefront $U(\mathbf{r}, t)$ satisfies the wave equation, the complex amplitude satisfies the Helmholtz equation:

$$\nabla^2 U(\mathbf{r}) + k^2 U(\mathbf{r}) = 0 \tag{5}$$

• The complex amplitude of a plane wave is:

$$U(\mathbf{r}) = a(\mathbf{r}) e^{-j2\pi \mathbf{k}\cdot\mathbf{r}} = a(\mathbf{r}) \exp(-j2\pi[k_x x + k_y y + k_z z]) \tag{6}$$

where $\mathbf{k} = (k_x, k_y, k_z)$ is the wave vector and $k^2 = k_x^2 + k_y^2 + k_z^2$.

The intensity of a plane wave is constant:

$$I(\mathbf{r}) = |U(\mathbf{r})|^2 = |U(\mathbf{r})| = |a(\mathbf{r}) e^{-j2\pi \mathbf{k}\cdot\mathbf{r}}|^2 = |a(\mathbf{r})|^2 \tag{7}$$

The complex amplitude of a plane wave traveling along the z-direction is:

$$U(\mathbf{r}, z) = a(\mathbf{r}) e^{-j2\pi \mathbf{k}\cdot\mathbf{r}} e^{-j2\pi k_z z} \tag{8}$$

The wavefunction of a plane monochromatic wave traveling along the z-direction is periodic in terms of time and z:

$$u(\mathbf{r}, t) = |a(\mathbf{r})| \cos(2\pi f t - k_z z + \arg(a(\mathbf{r}))) \tag{9}$$

or equivalently,

$$u(\mathbf{r}, t) = |a(\mathbf{r})| \cos\left(2\pi f \left(t - \frac{z}{c}\right) + \arg(a(\mathbf{r}))\right) \tag{10}$$

since $k=\frac{f}{c}$.

• As monochromatic waves propagate through media of different refractive indexes, their frequency remains the same, but their velocity, wavelength, and wavenumber change:

$$v=\frac{c}{n} \tag{11}$$

$$v=\lambda f \rightarrow \lambda=\frac{v}{f}=\frac{c}{nf}=\frac{\lambda_0}{n}, \quad \text{with} \quad \lambda_0=\frac{c}{f} \tag{12}$$

$$k=\frac{2\pi}{\lambda}=\frac{2\pi}{\lambda_0/n}=nk_0, \quad \text{where} \quad k_0=\frac{2\pi}{\lambda_0} \tag{13}$$

• The complex amplitude of a spherical wave $U(\mathbf{r})=\frac{a_0}{r}e^{-j\mathbf{k}\cdot\mathbf{r}}$ where a_0 is a constant, r is the radius of the spherical wave from the origin, and $k=\frac{2\pi}{\lambda}=\frac{2\pi f}{v}=\frac{w}{v}$.

In general, the origin of the spherical wave can be shifted: $U(\mathbf{r})=\frac{a_0}{|\mathbf{r}-\mathbf{r_0}|}e^{-j\mathbf{k}\cdot|\mathbf{r}-\mathbf{r_0}|}$.

The intensity of a spherical wave is inversely proportional to the square of the distance, $I(\mathbf{r})=|U(\mathbf{r})|^2=\frac{a_0^2}{r^2}$.

• The complex amplitude of a paraboloidal wave, $U(\mathbf{r})=\frac{a_0}{z}e^{-j\mathbf{k}\cdot z}\exp\left(-j\frac{k}{2z}(x^2+y^2)\right)$, can be understood as a plane wave $a_0e^{-j\mathbf{k}\cdot z}$, modulated by a factor $\frac{1}{z}\exp\left(-j\frac{k}{2z}(x^2+y^2)\right)$.

For larger z, the phase factor $\exp\left(-j\frac{k}{2z}(x^2+y^2)\right)\approx 0$, so the paraboloidal wave can be approximated as a plane wave.

The paraboloidal waves satisfy the paraxial Helmholtz equation, $\nabla_T^2 A(x,y,z)-j2k\frac{\partial A(x,y,z)}{\partial z}=0$, where $A(x,y,z)=a_0\exp\left(-j\frac{k}{2z}(x^2+y^2)\right)$.

A spherical wave that satisfies the Fresnel approximation becomes a paraboloidal wave. The Fresnel approximation is met for points (x,y) whose radius $a=\sqrt{x^2+y^2}$ is at least one order of magnitude smaller than $(8\lambda z^3)^{1/4}$, i.e., $a^4 \ll 8\lambda z^3$. If one defines the Fresnel number: $N_F=\frac{a^2}{\lambda z}$, and the maximum angle $\theta_m=\frac{a}{z}$, the Fresnel approximation is given by:

$$\frac{N_F\theta_m^2}{4}\ll 1. \tag{14}$$

• Interference:

Coherent: superposition of two or more waves whose wavefunctions are related to each other. For example, waves that come from the same point source. The resultant wavefunction is the sum of the individual ones.

Incoherent: superposition of two or more waves whose wavefunctions are not related. For example, the waves emitted from each point of a lamp are completely unrelated, therefore they do not interfere. Nonetheless, there is a superposition of their individual intensities.

In this module, interference relates to the coherent superposition, i.e., the sum of the wavefunctions so that $U=U_1+U_2$. Therefore, the resultant intensity of the sum of two or more coherent wavefunctions is not equal to the sum of their individual intensities, i.e., $I=|U|^2\neq I_1+I_2$.

Example of the interference of two coherent waves whose complex amplitude distributions are, respectively, $U_1=\sqrt{I_1}e^{j\varphi_1}$ and $U_2=\sqrt{I_2}e^{j\varphi_2}$. The resultant intensity is:

$$I=|U_1+U_2|^2=|U_1|^2+|U_2|^2+U_1U_2^*+U_2U_1^*. \tag{15}$$

$$I=I_1+I_2+\sqrt{I_1}e^{j\varphi_1}\sqrt{I_2}e^{-j\varphi_2}+\sqrt{I_1}e^{-j\varphi_1}\sqrt{I_2}e^{j\varphi_2}. \tag{16}$$

$$I=I_1+I_2+2\sqrt{I_1}\sqrt{I_2}\cos(\varphi_1-\varphi_2). \tag{17}$$

The term $2\sqrt{I_1}\sqrt{I_2}\cos(\varphi_1-\varphi_2)$ depends on the phase difference between both waves. If both waves have the same intensity, $I_0=I_1=I_2$, then:

$$I=2I_0[1+\cos(\varphi_1-\varphi_2)]. \tag{18}$$

Because the resultant intensity is $I=2I_0[1+\cos(\varphi_1-\varphi_2)]$, there are different scenarios:

- $I = 2I_0$ when $\cos(\varphi_1 - \varphi_2) = 0$. In other words, the phase difference between both waves is a multiple of $\frac{\pi}{2}$, i.e., $\varphi_1 - \varphi_2 = (m + \frac{1}{2})\pi$, where $m = 0, 1, 2, \ldots$
- $I_{\max} = 4I_0$ when $\cos(\varphi_1 - \varphi_2) = 1$. This is the condition for constructive interference (i.e., maximum value of the resultant intensity). In other words, the phase difference between both waves is a multiple of 2π, i.e., $\varphi_1 - \varphi_2 = 2\pi m$, where $m = 0, 1, 2, \ldots$
- $I_{\min} = 0$ when $\cos(\varphi_1 - \varphi_2) = -1$. This is the condition for destructive interference (i.e., minimum value of the resultant intensity). In other words, the phase difference between both waves is an odd multiple of π, i.e., $\varphi_1 - \varphi_2 = (2m + 1)\pi$, where $m = 0, 1, 2, \ldots$

The interference creates a fringe-like pattern whose visibility is:

$$V = \frac{I_{\max} + I_{\min}}{I_{\max} + I_{\min}}. \tag{19}$$

In a coherent interference, the intensity of the superposition between two coherent waves depends on the phase of the individual waves. The phase difference between the waves can be expressed as:

$$I = |U_1 + U_2|^2 = |U_1|^2 + |U_2|^2 + U_1 U_2^* + U_2 U_1^*. \tag{20}$$

Example of the interference of two tilted coherent waves with the same intensity:

$$U_1 = \sqrt{I_0} e^{-jkz}, \quad U_2 = \sqrt{I_0} e^{-jk(\cos\theta z + \sin\theta x)}. \tag{21}$$

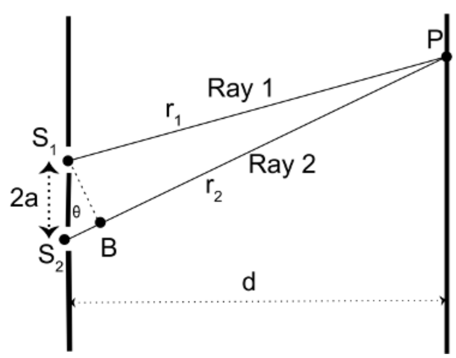
Thus, the intensity becomes:

$$I = 2I_0 + \sqrt{I_0}\sqrt{I_0} e^{-jkz} e^{jk(\cos\theta z + \sin\theta x)} + \sqrt{I_0}\sqrt{I_0} e^{jkz} e^{-jk(\cos\theta z + \sin\theta x)}. \tag{22}$$

$$I = 2I_0 \left[1 + \cos(k(1 - \cos\theta)z + k\sin\theta x)\right]. \tag{23}$$

The intensity changes axially and laterally. The axial and lateral periods are: $T_z = \frac{\lambda}{1-\cos\theta}, \quad T_x = \frac{\lambda}{\sin\theta}$.

• Young's Experiment



The optical path difference between the optical rays 1 and 2 is the distance S_2B.

$$r_1 = S_1P = \text{distance between the source } S_1 \text{ and the observation plane } P \tag{24}$$

$$r_2 = S_2P = S_2B + S_1P = \text{distance between the source } S_2 \text{ and the observation plane } P. \tag{25}$$

The interference pattern at the observation plane is:

$$I = 2I_0 \left[1 + \cos\left(\frac{2\pi}{\lambda}\frac{2a}{d}x\right)\right]. \tag{26}$$

Because the resultant intensity is: $I = 2I_0 \left[1 + \cos\left(\frac{2\pi}{\lambda}\frac{2a}{d}x\right)\right]$, we have the following conditions:

- $I_{\max} = 4I_0$ when $\cos\left(\frac{2\pi}{\lambda}\frac{2a}{d}x\right) = 1$.
- The maxima are located at $x_{\max} = m\frac{\lambda d}{2a}$.
- The separation between two consecutive maxima is $x_{\max,m+1} - x_{\max,m} = \frac{\lambda d}{2a}$.
- $I_{\min} = 0$ when $\cos\left(\frac{2\pi}{\lambda}\frac{2a}{d}x\right) = -1$.

- The minima are located at $x_{\min} = (m + \frac{1}{2})\frac{\lambda d}{2a}$.
- The separation between two consecutive minima is $x_{\min,m+1} - x_{\min,m} = \frac{\lambda d}{2a}$.