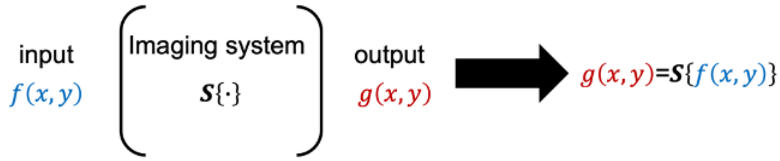
Module 4 - Summary

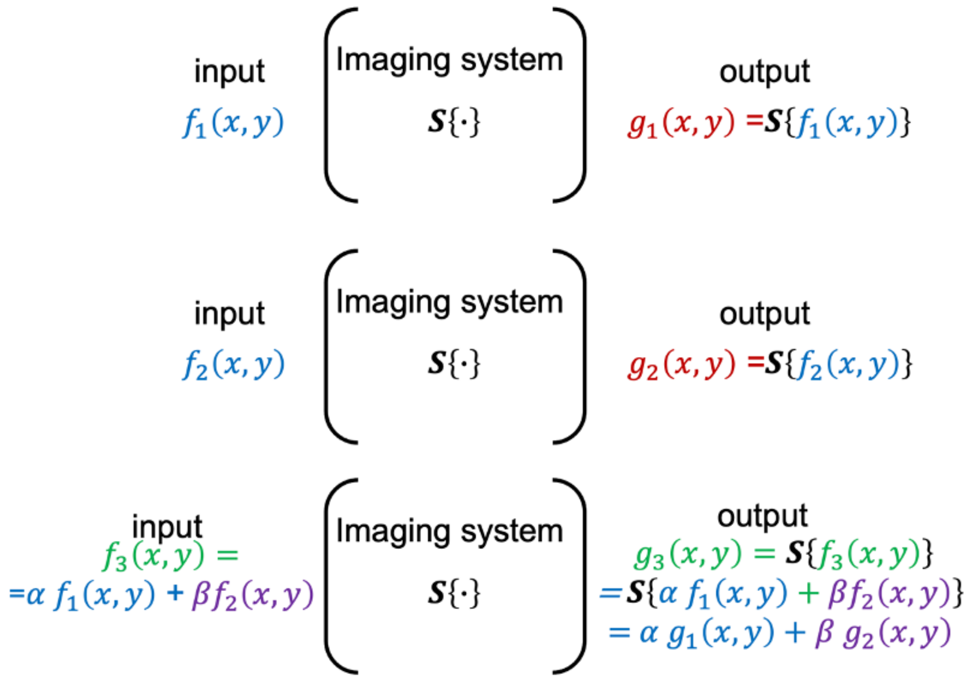
Summary Notes Module 4

• Any amplitude distribution can be expressed as a sum of plane waves, i.e., $\exp[j2\pi(ux+vy)]$. Compared to the mathematical expression of a plane wave in Module 2, $\exp[j(k_x x+k_y y)]$, then $k_x = 2\pi u = \frac{2\pi}{\lambda}\sin\theta_x$ and $k_y = 2\pi v = \frac{2\pi}{\lambda}\sin\theta_y$. This means that the angle of the plane waves along the x- and y-direction can be estimated by $\theta_x = \sin^{-1}(\lambda u)$ and $\theta_y = \sin^{-1}(\lambda v)$.

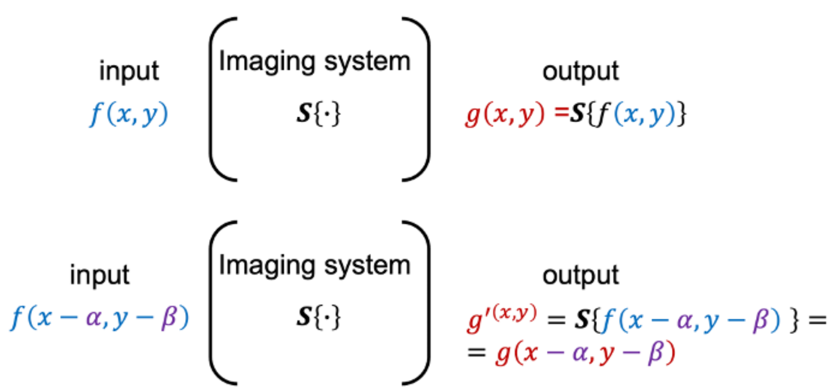
• An imaging system must be a linear shift-invariant system. Consider that an imaging system is represented by a $\mathbf{S}$ operator.



A system is linear if the output is equal to the sum of the individual outputs.



A system is shift-invariant if the output of a displaced input signal is the same function but displaced by the same amount.



Only when the optical system is linear space-invariant (LSI), the output signal, $g(x,y)$, can be expressed as the two-dimensional (2D) convolution ($\otimes_2$) between the input signal, $f(x,y)$, and the impulse response of the system, $h(x,y)$:

$$g(x,y) = f(x,y) \otimes_2 h(x,y). \tag{1}$$

In the Fourier domain, the spectrum of the output signal, $G(u,v)$, is the product between the spectrum of the input signal, $F(u,v)$, and the optical transfer function, $H(u,v)$:

$$G(u,v) = F(u,v)H(u,v). \tag{2}$$

This means that the transfer function $H(u,v)$ is the 2D Fourier transform of the impulse response, $h(x,y)$:

$$H(u,v) = \mathrm{FT}[h(x,y)]. \tag{3}$$

• Free-space propagator under the Fresnel approximation

$$h(x,y) = \frac{e^{jkz}}{j\lambda z} \exp\left[j\frac{k}{2z}(x^2+y^2)\right], \tag{4}$$

and

$$H(u,v) = \frac{e^{jkz}}{j\lambda z} \exp\left[-j\pi\lambda z(u^2+v^2)\right]. \tag{5}$$

Condition of the Fresnel region:

$$N_F \frac{\theta_m^2}{4} \ll 1, \tag{6}$$

where $N_F = \frac{a^2}{\lambda z}$ is the Fresnel number and $\theta_m = \frac{a}{z}$ is the maximum aperture angle.

• Free-space propagator under the Fraunhofer approximation ($z \gg$)

$$g(x,y) = f(x,y) \otimes_2 h(x,y) = F\left(\frac{x}{\lambda z}, \frac{y}{\lambda z}\right) = F\left(\frac{x}{M}, \frac{y}{M}\right), \tag{7}$$

where M is the magnification factor.

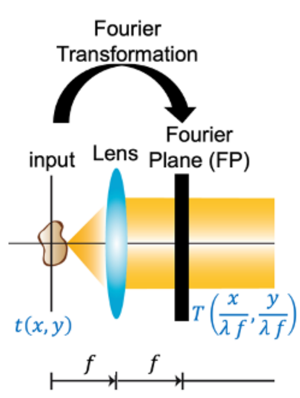
For $z \gg$, the amplitude distribution (also known as the diffraction pattern) of an object transmittance $f(x,y)$ is a scaled version of its Fourier Transform under Fraunhofer approximation.

Condition of the Fraunhofer region:

$$z \gg \frac{b_{\text{input}}^2}{\lambda} \quad \text{and} \quad z \gg \frac{a_{\text{output}}^2}{\lambda}, \tag{8}$$

where b_{input} is the maximum lateral extent in the input plane and a_{output} is the maximum lateral extent in the output plane.

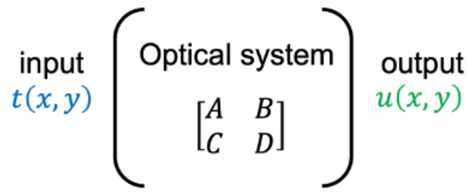
• Converging and diverging (i.e., optical) lenses make Fourier transforms.



Consider a converging lens. If an object with amplitude transmittance $t(x, y)$ is placed at the front focal plane of a lens (i.e., the distance between the object/input and lens planes is equal to the lens' focal length, f), a scaled replica of the Fourier transform of the object transmittance is found at its back focal plane:

$$T\left(\frac{x}{\lambda f}, \frac{y}{\lambda f}\right). \tag{9}$$

• Calculation of the complex amplitude distribution of an arbitrary optical system with an input transmittance $t(x, y)$. The arbitrary optical system is represented by its ABCD matrix.



If $B = 0$, the complex amplitude distribution at the output plane is the product of a quadratic phase wavefront and a scaled replica of the object transmittance:

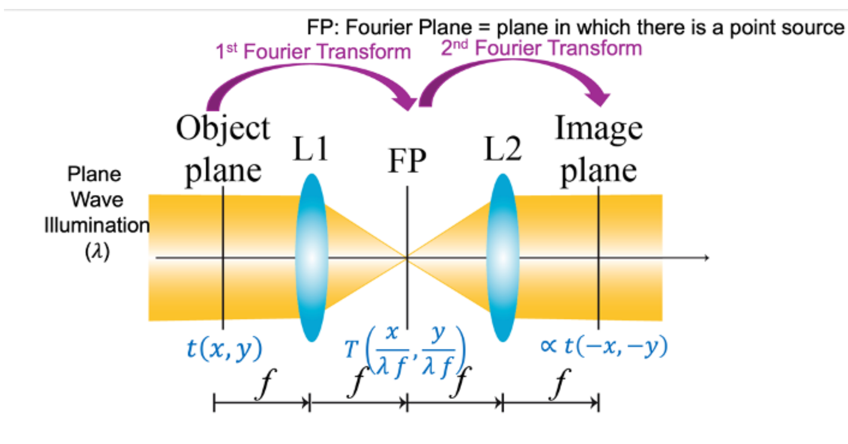
$$u(x, y) = e^{jkL_0/A} \exp\left[j\frac{kC}{2A}(x^2 + y^2)\right] t\left(\frac{x}{A}, \frac{y}{A}\right), \tag{10}$$

where L_0 is the geometrical distance between the input and output planes and $k = \frac{2\pi}{\lambda}$ is the wave number.

Else, if $B \neq 0$, the complex amplitude distribution at the output plane becomes:

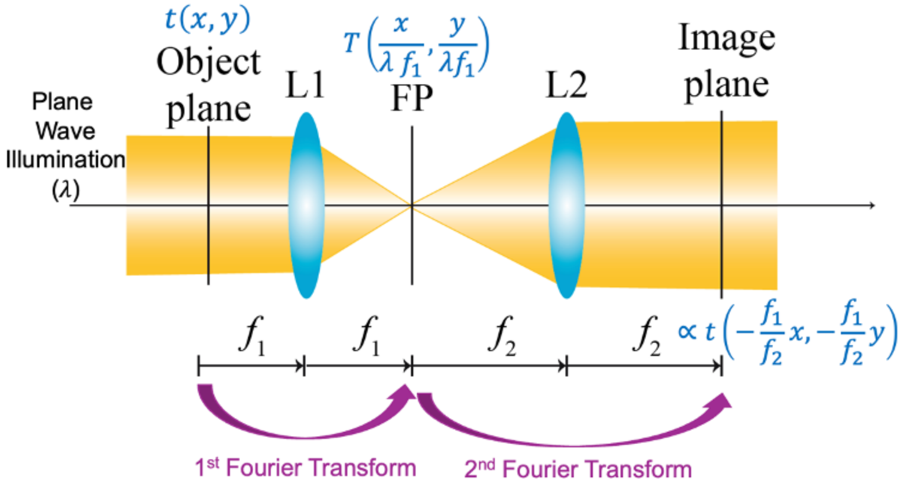
$$u(x, y) = \frac{e^{jkL_0}}{j\lambda B} \exp\left[j\frac{kD}{2B}(x^2 + y^2)\right] \int_{-\infty}^{\infty} t(x_0, y_0) \exp\left[j\frac{kA}{2B}(x_0^2 + y_0^2)\right] \exp\left[-j\frac{2\pi}{\lambda B}(xx_0 + yy_0)\right] dx_0\, dy_0. \tag{11}$$

• 4f imaging system if both lenses have the same focal length

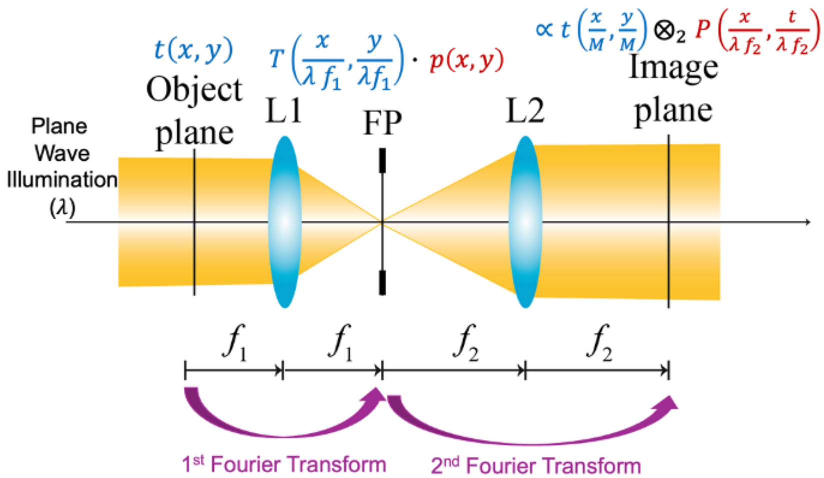


Hint – The Fourier Transform of a plane wave is a Delta function (i.e., point source)

• 4f imaging system if both lenses have different focal lengths



• Spatial filtering in a 4f imaging system by inserting a pupil with transmittance $p(x, y)$ at the Fourier plane of a 4f system. The pupil filters out some object frequencies.



Thus, the complex amplitude at the image plane (i.e., back focal plane of the L_2 lens) is:

$$u(x,y)=\frac{1}{M^2}t\left(\frac{x}{M},\frac{y}{M}\right)\otimes_2 P\left(\frac{x}{\lambda f_2},\frac{y}{\lambda f_2}\right)=\frac{1}{M^2}t\left(\frac{x}{M},\frac{y}{M}\right)\otimes_2 h(x,y), \tag{12}$$

where $M=-\frac{f_2}{f_1}$ is the lateral magnification, $\otimes_2$ is the 2D convolution operator, and $P(u,v)$ is the 2D Fourier transform of the pupil transmittance. Based on the equation, the impulse response of the 4f imaging system is a scaled replica of the pupil's Fourier transform, $h(x,y)=P\left(\frac{x}{\lambda f_2},\frac{y}{\lambda f_2}\right)$. Therefore, the transfer function is a scaled replica of the pupil's transmittance:

$$H(u,v)=\mathrm{FT}[h(x,y)]=p\left(-\frac{\lambda f_2 u}{f_1},-\frac{\lambda f_2 v}{f_1}\right). \tag{13}$$