13.3: Intensity in Single-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Calculate the intensity relative to the central maximum of the single-slit diffraction peaks
- Calculate the intensity relative to the central maximum of an arbitrary point on the screen

To calculate the intensity of the diffraction pattern, we follow the phasor method used for calculations with ac circuits in Alternating-Current Circuits. If we consider that there are N Huygens sources across the slit shown previously, with each source separated by a distance a/N from its adjacent neighbors, the path difference between waves from adjacent sources reaching the arbitrary point P on the screen is $(a/N)\ \sin\theta$. This distance is equivalent to a phase difference of $(2\pi a/\lambda N)\ \sin\,\theta$. The phasor diagram for the waves arriving at the point whose angular position is θ is shown in Figure $13.3.1$. The amplitude of the phasor for each Huygens wavelet is ΔE_0, the amplitude of the resultant phasor is E, and the phase difference between the wavelets from the first and the last sources is

$$\phi = \left(\frac{2\pi}{\lambda}\right)\ a\ \sin\theta.$$

With $N \to \infty$, the phasor diagram approaches a circular arc of length $N\Delta E_0$ and radius r. Since the length of the arc is $N\Delta E_0$ for any ϕ, the radius r of the arc must decrease as ϕ increases (or equivalently, as the phasors form tighter spirals).

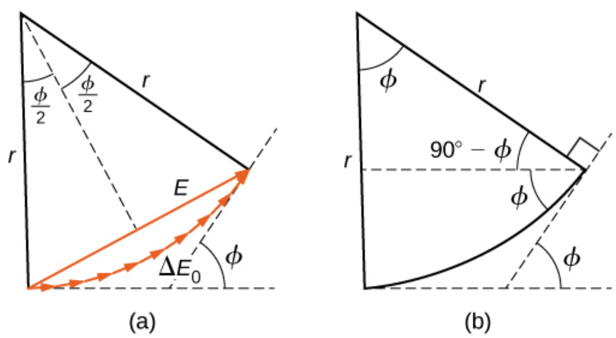


Figure 13.3.1: (a) Phasor diagram corresponding to the angular position θθ in the single-slit diffraction pattern. The phase difference between the wavelets from the first and last sources is $\phi = (2\pi/\lambda)a\,sin\,\theta$. (b) The geometry of the phasor diagram.

The phasor diagram for ϕ = 0 (the center of the diffraction pattern) is shown in Figure $13.3.1a$ using N=30. In this case, the phasors are laid end to end in a straight line of length $N\Delta E_0$, the radius **r** goes to infinity, and the resultant has its maximum value $E = N\Delta E_0$. The intensity of the light can be obtained using the relation $I = \frac{1}{2}c\epsilon_0 E^2$ from Electromagnetic Waves. The intensity of the maximum is then

$$I_0 = \frac{1}{2}c\epsilon_0(N\Delta E_0)^2 = \frac{1}{2\mu_0 c}(N\Delta E_0)^2,$$

where $\epsilon_0 = 1/\mu_0 c^2$. The phasor diagrams for the first two zeros of the diffraction pattern are shown in Figure $13.3.1b$ and Figure $13.3.1d$. In both cases, the phasors add to zero, after rotating through $\phi = 2\pi$ rad for m = 1 and 4π rad for m = 2.

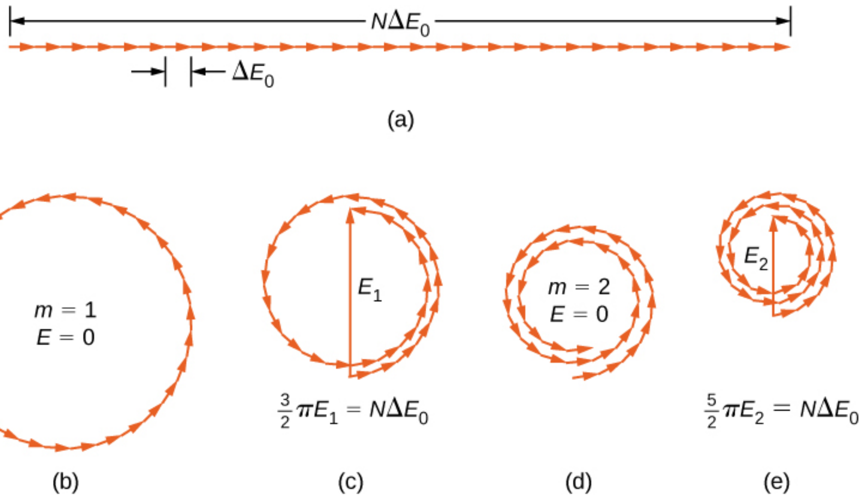


Figure 13.3.2: Phasor diagrams (with 30 phasors) for various points on the single-slit diffraction pattern. Multiple rotations around a given circle have been separated slightly so that the phasors can be seen. (a) Central maximum, (b) first minimum, (c) first maximum beyond central maximum, (d) second minimum, and (e) second maximum beyond central maximum.

The next two maxima beyond the central maxima are represented by the phasor diagrams of parts (c) and (e). In part (c), the phasors have rotated through $\phi = 3\pi$ rad and have formed a resultant phasor of magnitude E_1. The length of the arc formed by the phasors is $N\Delta E_0$. Since this corresponds to 1.5 rotations around a circle of diameter E_1, we have

$$\frac{3}{2}\pi E_1 = N\Delta E_0,$$

so

$$E_1 = \frac{2N\Delta E_0}{3\pi}$$

and

$$I_1 = \frac{1}{2\mu_0 c}E_1^2 = \frac{4(N\Delta E_0)^2}{(9\pi^2)(2\mu_0 c)} = 0.045 I_0,$$

where

$$I_0 = \frac{(N\Delta E_0)^2}{2\mu_0 c}.$$

In part (e), the phasors have rotated through $\phi = 5\pi$ rad, corresponding to 2.5 rotations around a circle of diameter E_2 and arc length $N\Delta E_0$. This results in $I_2 = 0.016 I_0$. The proof is left as an exercise for the student (Exercise 4.119).

These two maxima actually correspond to values of ϕ slightly less than 3π rad and 5π rad. Since the total length of the arc of the phasor diagram is always $N\Delta E_0$, the radius of the arc decreases as ϕ increases. As a result, E_1 and E_2 turn out to be slightly larger for arcs that have not quite curled through 3π rad and 5π rad, respectively. The exact values of ϕ for the maxima are investigated in Exercise 4.120. In solving that problem, you will find that they are less than, but very close to, $\phi = 3\pi,\ 5\pi,\ 7\pi, \ldots$ rad.

To calculate the intensity at an arbitrary point P on the screen, we return to the phasor diagram of Figure 13.3.1. Since the arc subtends an angle ϕ at the center of the circle,

$$N\Delta E_0 = r\phi \tag{13.3.1}$$

and

$$\sin\left(\frac{\phi}{2}\right) = \frac{E}{2r}. \tag{13.3.2}$$

where E is the amplitude of the resultant field. Solving the Equation 13.3.2 for E and then substituting r from Equation 13.3.1, we find

$$\begin{aligned} E &= 2r \sin \frac{\phi}{2} \\ &= 2\frac{N\Delta E_0}{\phi} \sin \frac{\phi}{2}. \end{aligned}$$

Now defining

$$\beta = \frac{\phi}{2} = \frac{\pi a \sin \theta}{\lambda} \tag{13.3.3}$$

we obtain

$$E = N\Delta E_0 \frac{\sin \beta}{\beta} \tag{13.3.4}$$

Equation 13.3.4 relates the amplitude of the resultant field at any point in the diffraction pattern to the amplitude $N\Delta E_0$ at the central maximum. The intensity is proportional to the square of the amplitude, so

$$I = I_0 \left(\frac{\sin \beta}{\beta} \right)^2 \tag{13.3.5}$$

where $I_0 = (N\delta E_0)^2/2\mu_0 c$ is the intensity at the center of the pattern.

For the central maximum, ϕ = 0, β is also zero and we see from l'Hôpital's rule that $\lim_{\beta\to 0}(sin\,\beta/\beta) = 1$, so that $lim_{\phi\to 0} I = I_0$. For the next maximum, $\phi = 3\pi$ rad, we have $\beta = 3\pi/2$ rad and when substituted into Equation 13.3.5, it yields

$$I_1 = I_0 \left(\frac{\sin 3\pi/2}{3\pi/2} \right)^2 = 0.045 I_0,$$

in agreement with what we found earlier in this section using the diameters and circumferences of phasor diagrams. Substituting $\phi = 5\pi$ rad into Equation 13.3.5 yields a similar result for I_2.

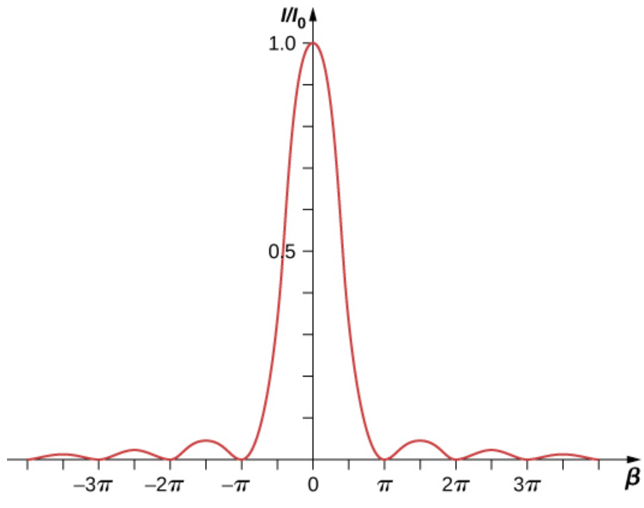
A plot of Equation 13.3.5 is shown in Figure $13.3.3$ and directly below it is a photograph of an actual diffraction pattern. Notice that the central peak is much brighter than the others, and that the zeros of the pattern are located at those points where $sin\,\beta = 0$, which occurs when $\beta = m\pi$ rad. This corresponds to

$$\frac{\pi a \sin\theta}{\lambda} = m\pi,$$

or

$$a \sin \theta = m\lambda,$$

which we derived for the destructive interference ro a single slit previously.



(b)

Figure 13.3.3: (a) The calculated intensity distribution of a single-slit diffraction pattern. (b) The actual diffraction pattern.

Example 13.3.1: Intensity in Single-Slit Diffraction

Light of wavelength 550 nm passes through a slit of width 2.00 μm and produces a diffraction pattern similar to that shown in Figure $13.3.3a$.

a. Find the locations of the first two minima in terms of the angle from the central maximum.
b. Determine the intensity relative to the central maximum at a point halfway between these two minima.

Strategy

The minima are given by Equation 4.2.1, $a\,sin\,\theta = m\lambda$. The first two minima are for m = 1 and m = 2. Equation 13.3.5 and Equation 13.3.3 can be used to determine the intensity once the angle has been worked out.

Solution

1. Solving Equation 4.2.1 for θ gives us $\theta_m = \sin^{-1}(m\lambda/a)$, so that

$$\theta_1 = \sin^{-1}\left(\frac{(+1)(550\times10^{-9}m)}{2.00\times10^{-6}m}\right) = +16.0^\circ$$

and

$$\theta_2 = \sin^{-1}\left(\frac{(+2)(550\times10^{-9}m)}{2.00\times10^{-6}m}\right) = +33.4^\circ.$$

2. The halfway point between θ_1 and θ_2 is

$$\theta = (\theta_1+\theta_2)/2 = (16.0^\circ+33.4^\circ)/2 = 24.7^\circ.$$

Equation 13.3.3 gives

$$\beta = \frac{\pi a\,sin\,\theta}{\lambda} = \frac{\pi(2.00\times10^{-6}m)\,\sin(24.7^\circ)}{(550\times10^{-9}m)} = 1.52\pi\;or\;4.77\;rad.$$

From Equation 13.3.5, we can calculate

$$\frac{I}{I_0} = \left(\frac{\sin\beta}{\beta}\right)^2 = \left(\frac{\sin(4.77)}{4.77}\right)^2 = \left(\frac{-0.9985}{4.77}\right)^2 = 0.044.$$

Significance

This position, halfway between two minima, is very close to the location of the maximum, expected near $\beta = 3\pi/2$, or 1.5π.

Exercise 13.3.1

For the experiment in Example 13.3.1, at what angle from the center is the third maximum and what is its intensity relative to the central maximum?

Answer

74.3^o, $0.0083I_0$

If the slit width a is varied, the intensity distribution changes, as illustrated in Figure 13.3.4. The central peak is distributed over the region from $sin\,\theta = -\lambda/a$ to $sin\,\theta = +\lambda/a$. For small θ, this corresponds to an angular width $\Delta\theta \approx 2\lambda/a$. Hence, an increase in the slit width results in a decrease in the **width of the central peak**. For a slit with a ≫ λ, the central peak is very sharp, whereas if a ≈ λ, it becomes quite broad.

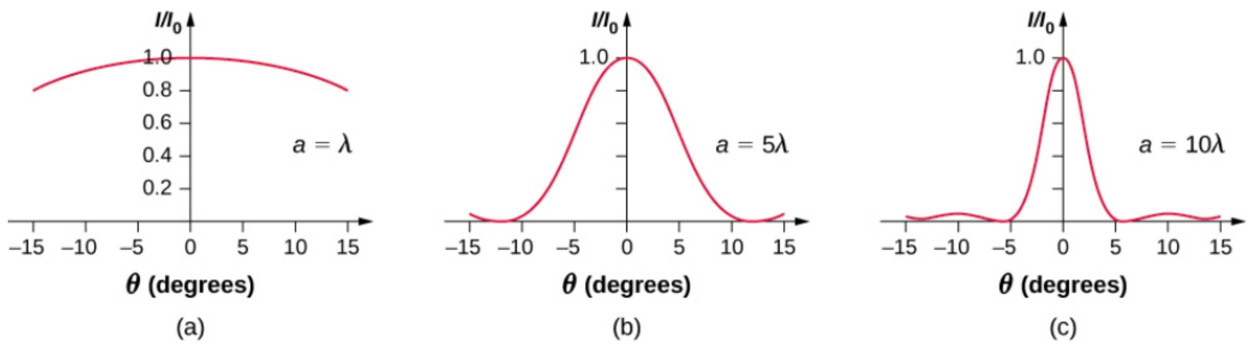


Figure 13.3.4: Single-slit diffraction patterns for various slit widths. As the slit width **a** increases from **a**=λ to 5λ and then to 10λ, the width of the central peak decreases as the angles for the first minima decrease as predicted by Equation 4.2.1.

Diffraction Simulation

A diffraction experiment in optics can require a lot of preparation but this simulation by Andrew Duffy offers not only a quick set up but also the ability to change the slit width instantly. Run the simulation and select “Single slit.” You can adjust the slit width and see the effect on the diffraction pattern on a screen and as a graph.