PH246 Calculus Physics II (2025)

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This text was compiled on 04/11/2025

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Detailed Licensing

Licensing

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CHAPTER OVERVIEW

1: Electric Charges and Fields

In this chapter, we begin the study of the electric force, which acts on all objects with a property called charge. The electric force is much stronger than gravity (in most systems where both appear), but it can be a force of attraction or a force of repulsion, which leads to very different effects on objects. The electric force helps keep atoms together, so it is of fundamental importance in matter. But it also governs most everyday interactions we deal with, from chemical interactions to biological processes.

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1.1: Prelude to Electric Charges and Fields

Back when we were studying Newton's laws, we identified several physical phenomena as forces. We did so based on the effect they had on a physical object: Specifically, they caused the object to accelerate. Later, when we studied impulse and momentum, we expanded this idea to identify a force as any physical phenomenon that changed the momentum of an object. In either case, the result is the same: We recognize a force by the effect that it has on an object.

Figure 1.1.1: Electric charges exist all around us. They can cause objects to be repelled from each other or to be attracted to each other. (credit: modification of work by Sean McGrath)

In Gravitation, we examined the force of gravity, which acts on all objects with mass. In this chapter, we begin the study of the electric force, which acts on all objects with a property called charge. The electric force is much stronger than gravity (in most systems where both appear), but it can be a force of attraction or a force of repulsion, which leads to very different effects on objects. The electric force helps keep atoms together, so it is of fundamental importance in matter. But it also governs most everyday interactions we deal with, from chemical interactions to biological processes.

1.2: Electric Charge

Learning Objectives

By the end of this section, you will be able to:

- Describe the concept of electric charge
- Explain qualitatively the force electric charge creates

You are certainly familiar with electronic devices that you activate with the click of a switch, from computers to cell phones to television. And you have certainly seen electricity in a flash of lightning during a heavy thunderstorm. But you have also most likely experienced electrical effects in other ways, maybe without realizing that an electric force was involved. Let’s take a look at some of these activities and see what we can learn from them about electric charges and forces.

Discoveries

You have probably experienced the phenomenon of **static electricity**: When you first take clothes out of a dryer, many (not all) of them tend to stick together; for some fabrics, they can be very difficult to separate. Another example occurs if you take a woolen sweater off quickly—you can feel (and hear) the static electricity pulling on your clothes, and perhaps even your hair. If you comb your hair on a dry day and then put the comb close to a thin stream of water coming out of a faucet, you will find that the water stream bends toward (is attracted to) the comb (Figure $1.2.1$).

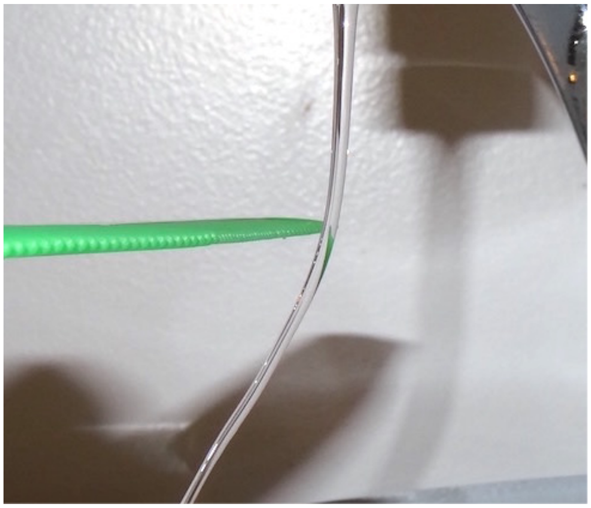


Figure $1.2.1$: An electrically charged comb attracts a stream of water from a distance. Note that the water is not touching the comb. (credit: Jane Whitney)

Suppose you bring the comb close to some small strips of paper; the strips of paper are attracted to the comb and even cling to it (Figure $1.2.2$). In the kitchen, quickly pull a length of plastic cling wrap off the roll; it will tend to cling to most any nonmetallic material (such as plastic, glass, or food). If you rub a balloon on a wall for a few seconds, it will stick to the wall. Probably the most annoying effect of static electricity is getting shocked by a doorknob (or a friend) after shuffling your feet on some types of carpeting.

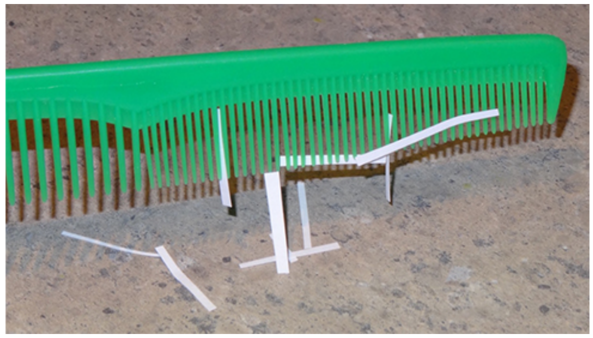


Figure 1.2.2: After being used to comb hair, this comb attracts small strips of paper from a distance, without physical contact. Investigation of this behavior helped lead to the concept of the electric force.

Many of these phenomena have been known for centuries. The ancient Greek philosopher Thales of Miletus (624–546 BCE) recorded that when amber (a hard, translucent, fossilized resin from extinct trees) was vigorously rubbed with a piece of fur, a force was created that caused the fur and the amber to be attracted to each other (Figure 1.2.3). Additionally, he found that the rubbed amber would not only attract the fur, and the fur attract the amber, but they both could affect other (nonmetallic) objects, even if not in contact with those objects (Figure 1.2.4).

Figure 1.2.3: Borneo amber is mined in Sabah, Malaysia, from shale-sandstone-mudstone veins. When a piece of amber is rubbed with a piece of fur, the amber gains more electrons, giving it a net negative charge. At the same time, the fur, having lost electrons, becomes positively charged. (credit: “Sebakoamber”/Wikimedia Commons)

The English physicist William Gilbert (1544–1603) also studied this attractive force, using various substances. He worked with amber, and, in addition, he experimented with rock crystal and various precious and semi-precious gemstones. He also experimented with several metals. He found that the metals never exhibited this force, whereas the minerals did. Moreover, although an electrified amber rod would attract a piece of fur, it would repel another electrified amber rod; similarly, two electrified pieces of fur would repel each other.

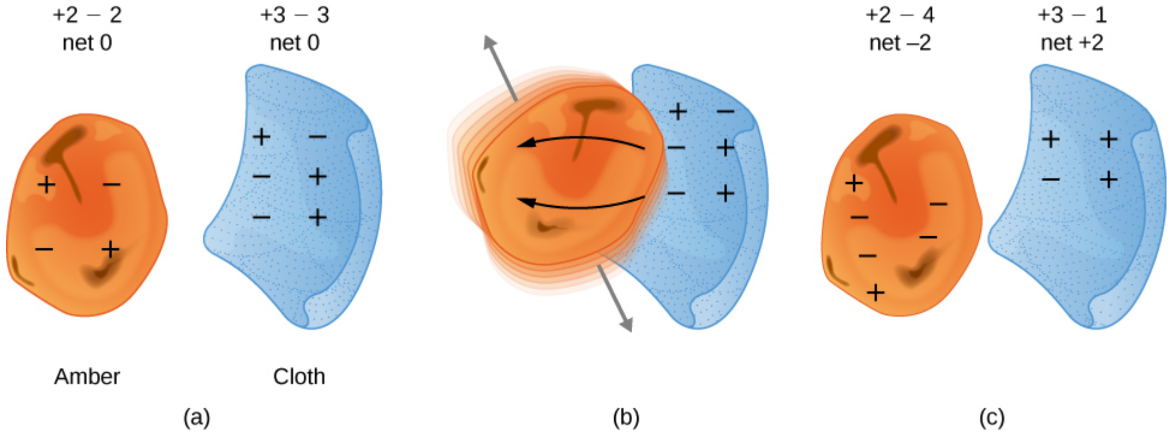


Figure 1.2.4: When materials are rubbed together, charges can be separated, particularly if one material has a greater affinity for electrons than another. (a) Both the amber and cloth are originally neutral, with equal positive and negative charges. Only a tiny fraction of the charges are involved, and only a few of them are shown here. (b) When rubbed together, some negative charge is transferred to the amber, leaving the cloth with a net positive charge. (c) When separated, the amber and cloth now have net charges, but the absolute value of the net positive and negative charges will be equal.

This suggested there were two types of an electric property; this property eventually came to be called **electric charge**. The difference between the two types of electric charge is in the directions of the electric forces that each type of charge causes: These forces are repulsive when the same type of charge exists on two interacting objects and attractive when the charges are of opposite types. The SI unit of electric charge is the **coulomb** (C), after the French physicist Charles Augustine de Coulomb (1736–1806).

The most peculiar aspect of this new force is that it does not require physical contact between the two objects in order to cause an acceleration. This is an example of a so-called “long-range” force. (Or, as James Clerk Maxwell later phrased it, “action at a distance.”) With the exception of gravity, all other forces we have discussed so far act only when the two interacting objects actually touch.

The American physicist and statesman Benjamin Franklin found that he could concentrate charge in a “Leyden jar,” which was essentially a glass jar with two sheets of metal foil, one inside and one outside, with the glass between them (Figure 1.2.4). This created a large electric force between the two foil sheets.

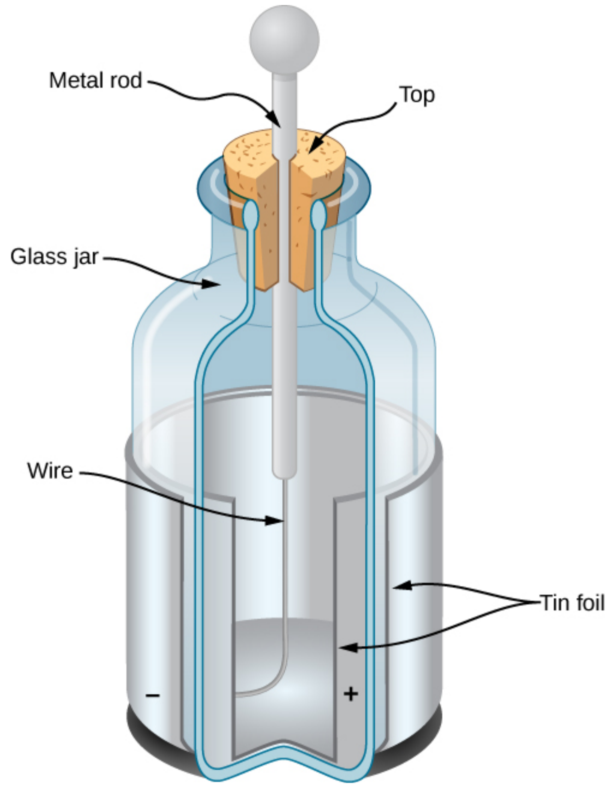


Figure 1.2.4: A Leyden jar (an early version of what is now called a capacitor) allowed experimenters to store large amounts of electric charge. Benjamin Franklin used such a jar to demonstrate that lightning behaved exactly like the electricity he got from the equipment in his laboratory.

Franklin pointed out that the observed behavior could be explained by supposing that one of the two types of charge remained motionless, while the other type of charge flowed from one piece of foil to the other. He further suggested that an excess of what he called this "electrical fluid" be called "positive electricity" and the deficiency of it be called "negative electricity." His suggestion, with some minor modifications, is the model we use today. (With the experiments that he was able to do, this was a pure guess; he had no way of actually determining the sign of the moving charge. Unfortunately, he guessed wrong; we now know that the charges that flow are the ones Franklin labeled negative, and the positive charges remain largely motionless. Fortunately, as we'll see, it makes no practical or theoretical difference which choice we make, as long as we stay consistent with our choice.)

Let's list the specific observations that we have of this **electric force**:

- The force acts without physical contact between the two objects.
- The force can be either attractive or repulsive: If two interacting objects carry the same sign of charge, the force is repulsive; if the charges are of opposite sign, the force is attractive. These interactions are referred to as **electrostatic repulsion** and **electrostatic attraction**, respectively.
- Not all objects are affected by this force.
- The magnitude of the force decreases (rapidly) with increasing separation distance between the objects.

To be more precise, we find experimentally that the magnitude of the force decreases as the square of the distance between the two interacting objects increases. Thus, for example, when the distance between two interacting objects is doubled, the force between them decreases to one fourth what it was in the original system. We can also observe that the surroundings of the charged objects affect the magnitude of the force. However, we will explore this issue in a later chapter.

Properties of Electric Charge

In addition to the existence of two types of charge, several other properties of charge have been discovered.

- **Charge is quantized.** This means that electric charge comes in discrete amounts, and there is a smallest possible amount of charge that an object can have. In the SI system, this smallest amount is $e \equiv 1.602 \times 10^{-19}\ \mathrm{C}$. No free particle can have less charge than this, and, therefore, the charge on any object—the charge on all objects—must be an integer multiple of this amount. All macroscopic, charged objects have charge because electrons have either been added or taken away from them, resulting in a net charge.

- **The magnitude of the charge is independent of the type.** Phrased another way, the smallest possible positive charge (to four significant figures) is $+1.602 \times 10^{-19}$ C, and the smallest possible negative charge is -1.602×10^{-19}; these values are exactly equal. This is simply how the laws of physics in our universe turned out.
- **Charge is conserved.** Charge can neither be created nor destroyed; it can only be transferred from place to place, from one object to another. Frequently, we speak of two charges "canceling"; this is verbal shorthand. It means that if two objects that have equal and opposite charges are physically close to each other, then the (oppositely directed) forces they apply on some other charged object cancel, for a net force of zero. It is important that you understand that the charges on the objects by no means disappear, however. The net charge of the universe is constant.
- **Charge is conserved in closed systems.** In principle, if a negative charge disappeared from your lab bench and reappeared on the Moon, conservation of charge would still hold. However, this never happens. If the total charge you have in your local system on your lab bench is changing, there will be a measurable flow of charge into or out of the system. Again, charges can and do move around, and their effects can and do cancel, but the net charge in your local environment (if closed) is conserved. The last two items are both referred to as the **law of conservation of charge**.

The Source of Charges: The Structure of the Atom

Once it became clear that all matter was composed of particles that came to be called atoms, it also quickly became clear that the constituents of the atom included both positively charged particles and negatively charged particles. The next question was, what are the physical properties of those electrically charged particles?

The negatively charged particle was the first one to be discovered. In 1897, the English physicist J. J. Thomson was studying what was then known as *cathode rays*. Some years before, the English physicist William Crookes had shown that these "rays" were negatively charged, but his experiments were unable to tell any more than that. (The fact that they carried a negative electric charge was strong evidence that these were not rays at all, but particles.) Thomson prepared a pure beam of these particles and sent them through crossed electric and magnetic fields, and adjusted the various field strengths until the net deflection of the beam was zero. With this experiment, he was able to determine the charge-to-mass ratio of the particle. This ratio showed that the mass of the particle was much smaller than that of any other previously known particle—1837 times smaller, in fact. Eventually, this particle came to be called the **electron**.

Since the atom as a whole is electrically neutral, the next question was to determine how the positive and negative charges are distributed within the atom. Thomson himself imagined that his electrons were embedded within a sort of positively charged paste, smeared out throughout the volume of the atom. However, in 1908, the New Zealand physicist Ernest Rutherford showed that the positive charges of the atom existed within a tiny core—called a nucleus—that took up only a very tiny fraction of the overall volume of the atom, but held over 99% of the mass (see Linear Momentum and Collisions.) In addition, he showed that the negatively charged electrons perpetually orbited about this nucleus, forming a sort of electrically charged cloud that surrounds the nucleus (Figure $1.2.5$). Rutherford concluded that the nucleus was constructed of small, massive particles that he named **protons**.

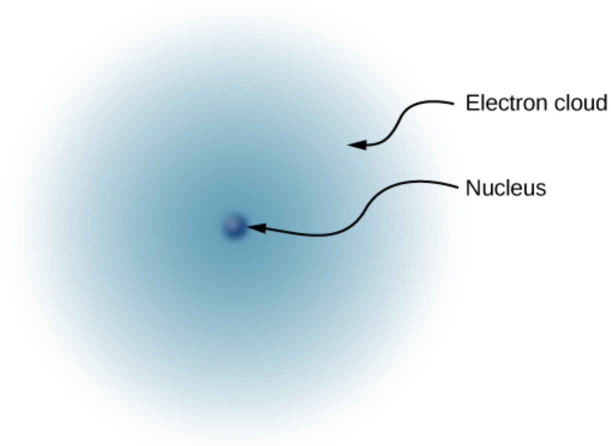


Figure $1.2.5$: This simplified model of a hydrogen atom shows a positively charged nucleus (consisting, in the case of hydrogen, of a single proton), surrounded by an electron “cloud.” The charge of the electron cloud is equal (and opposite in sign) to the charge of the nucleus, but the electron does not have a definite location in space; hence, its representation here is as a cloud. Normal macroscopic amounts of matter contain immense numbers of atoms and molecules, and, hence, even greater numbers of individual negative and positive charges.

Since it was known that different atoms have different masses, and that ordinarily atoms are electrically neutral, it was natural to suppose that different atoms have different numbers of protons in their nucleus, with an equal number of negatively charged electrons orbiting about the positively charged nucleus, thus making the atoms overall electrically neutral. However, it was soon discovered that although the lightest atom, hydrogen, did indeed have a single proton as its nucleus, the next heaviest atom—helium—has twice the number of protons (two), but *four* times the mass of hydrogen.

This mystery was resolved in 1932 by the English physicist James Chadwick, with the discovery of the **neutron**. The neutron is, essentially, an electrically neutral twin of the proton, with no electric charge, but (nearly) identical mass to the proton. The helium nucleus therefore has two neutrons along with its two protons. (Later experiments were to show that although the neutron is electrically neutral overall, it does have an internal charge *structure*. Furthermore, although the masses of the neutron and the proton are *nearly* equal, they aren’t exactly equal: The neutron’s mass is very slightly larger than the mass of the proton. That slight mass excess turned out to be of great importance. That, however, is a story that will have to wait until our study of modern physics in Nuclear Physics.)

Thus, in 1932, the picture of the atom was of a small, massive nucleus constructed of a combination of protons and neutrons, surrounded by a collection of electrons whose combined motion formed a sort of negatively charged “cloud” around the nucleus (Figure $1.2.6$). In an electrically neutral atom, the total negative charge of the collection of electrons is equal to the total positive charge in the nucleus. The very low-mass electrons can be more or less easily removed or added to an atom, changing the net charge on the atom (though without changing its type). An atom that has had the charge altered in this way is called an **ion**. Positive ions have had electrons removed, whereas negative ions have had excess electrons added. We also use this term to describe molecules that are not electrically neutral.

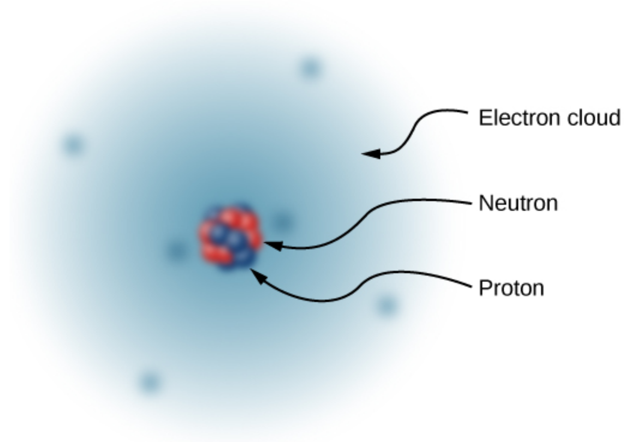


Figure 1.2.6: The nucleus of a carbon atom is composed of six protons and six neutrons. As in hydrogen, the surrounding six electrons do not have definite locations and so can be considered to be a sort of cloud surrounding the nucleus.

The story of the atom does not stop there, however. In the latter part of the twentieth century, many more subatomic particles were discovered in the nucleus of the atom: pions, neutrinos, and quarks, among others. With the exception of the photon, none of these particles are directly relevant to the study of electromagnetism, so we defer further discussion of them until the chapter on particle physics (Particle Physics and Cosmology).

A Note on Terminology

As noted previously, electric charge is a property that an object can have. This is similar to how an object can have a property that we call mass, a property that we call density, a property that we call temperature, and so on. Technically, we should always say something like, "Suppose we have a particle that carries a charge of μC." However, it is very common to say instead, "Suppose we have a μC charge." Similarly, we often say something like, "Six charges are located at the vertices of a regular hexagon." A charge is not a particle; rather, it is a *property* of a particle. Nevertheless, this terminology is extremely common (and is frequently used in this book, as it is everywhere else). So, keep in the back of your mind what we really mean when we refer to a "charge."

1.3: Conductors, Insulators, and Charging by Induction

Learning Objectives

By the end of this section, you will be able to:

- Explain what a conductor is
- Explain what an insulator is
- List the differences and similarities between conductors and insulators
- Describe the process of charging by induction

In the preceding section, we said that scientists were able to create electric charge only on nonmetallic materials and never on metals. To understand why this is the case, you have to understand more about the nature and structure of atoms. In this section, we discuss how and why electric charges do—or do not—move through materials (Figure $1.3.1$). A more complete description is given in a later chapter.

Figure $1.3.1$: This power adapter uses metal wires and connectors to conduct electricity from the wall socket to a laptop computer. The conducting wires allow electrons to move freely through the cables, which are shielded by rubber and plastic. These materials act as insulators that don't allow electric charge to escape outward. (credit: modification of work by "Evan-Amos"/Wikimedia Commons)

Conductors and Insulators

As discussed in the previous section, electrons surround the tiny nucleus in the form of a (comparatively) vast cloud of negative charge. However, this cloud does have a definite structure to it. Let's consider an atom of the most commonly used conductor, copper.

For reasons that will become clear in Atomic Structure, there is an outermost electron that is only loosely bound to the atom's nucleus. It can be easily dislodged; it then moves to a neighboring atom. In a large mass of copper atoms (such as a copper wire or a sheet of copper), these vast numbers of outermost electrons (one per atom) wander from atom to atom, and are the electrons that do the moving when electricity flows. These wandering, or "free," electrons are called **conduction electrons**, and copper is therefore an excellent **conductor** (of electric charge). All conducting elements have a similar arrangement of their electrons, with one or two conduction electrons. This includes most metals.

Insulators, in contrast, are made from materials that lack conduction electrons; charge flows only with great difficulty, if at all. Even if excess charge is added to an insulating material, it cannot move, remaining indefinitely in place. This is why insulating materials exhibit the electrical attraction and repulsion forces described earlier, whereas conductors do not; any excess charge placed on a conductor would instantly flow away (due to mutual repulsion from existing charges), leaving no excess charge around to create forces. Charge cannot flow along or through an **insulator**, so its electric forces remain for long periods of time. (Charge will dissipate from an insulator, given enough time.) As it happens, amber, fur, and most semi-precious gems are insulators, as are materials like wood, glass, and plastic.

Charging by Induction

Let's examine in more detail what happens in a conductor when an electrically charged object is brought close to it. As mentioned, the conduction electrons in the conductor are able to move with nearly complete freedom. As a result, when a charged insulator (such as a positively charged glass rod) is brought close to the conductor, the (total) charge on the insulator exerts an electric force on the conduction electrons. Since the rod is positively charged, the conduction electrons (which themselves are negatively charged) are attracted, flowing toward the insulator to the near side of the conductor (Figure $1.3.2$).

Now, the conductor is still overall electrically neutral; the conduction electrons have changed position, but they are still in the conducting material. However, the conductor now has a charge *distribution*; the near end (the portion of the conductor closest to the insulator) now has more negative charge than positive charge, and the reverse is true of the end farthest from the insulator. The relocation of negative charges to the near side of the conductor results in an overall positive charge in the part of the conductor farthest from the insulator. We have thus created an electric charge distribution where one did not exist before. This process is referred to as *inducing polarization*—in this case, polarizing the conductor. The resulting separation of positive and negative charge is called **polarization**, and a material, or even a molecule, that exhibits polarization is said to be polarized. A similar situation occurs with a negatively charged insulator, but the resulting polarization is in the opposite direction.

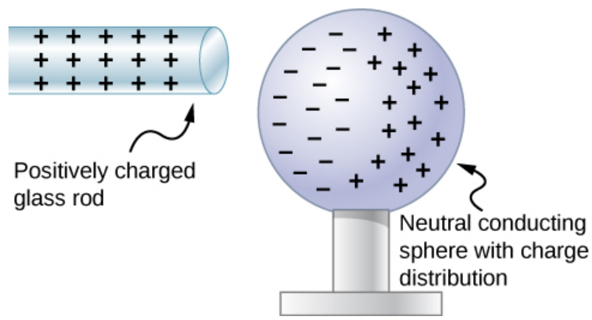


Figure $1.3.2$: Induced polarization. A positively charged glass rod is brought near the left side of the conducting sphere, attracting negative charge and leaving the other side of the sphere positively charged. Although the sphere is overall still electrically neutral, it now has a charge distribution, so it can exert an electric force on other nearby charges. Furthermore, the distribution is such that it will be attracted to the glass rod.

The result is the formation of what is called an electric **dipole**, from a Latin phrase meaning "two ends." The presence of electric charges on the insulator—and the electric forces they apply to the conduction electrons—creates, or "induces," the dipole in the conductor.

Neutral objects can be attracted to any charged object. The pieces of straw attracted to polished amber are neutral, for example. If you run a plastic comb through your hair, the charged comb can pick up neutral pieces of paper. Figure $1.3.3$ shows how the polarization of atoms and molecules in neutral objects results in their attraction to a charged object.

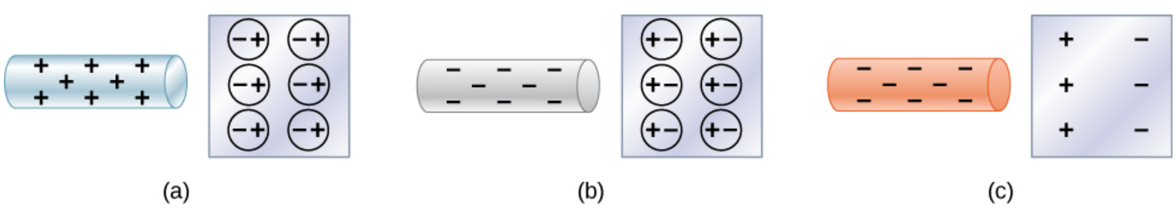


Figure $1.3.3$: Both positive and negative objects attract a neutral object by polarizing its molecules. (a) A positive object brought near a neutral insulator polarizes its molecules. There is a slight shift in the distribution of the electrons orbiting the molecule, with unlike charges being brought nearer and like charges moved away. Since the electrostatic force decreases with distance, there is a net attraction. (b) A negative object produces the opposite polarization, but again attracts the neutral object. (c) The same effect occurs for a conductor; since the unlike charges are closer, there is a net attraction.

When a charged rod is brought near a neutral substance, an insulator in this case, the distribution of charge in atoms and molecules is shifted slightly. Opposite charge is attracted nearer the external charged rod, while like charge is repelled. Since the electrostatic force decreases with distance, the repulsion of like charges is weaker than the attraction of unlike charges, and so there is a net attraction. Thus, a positively charged glass rod attracts neutral pieces of paper, as will a negatively charged rubber rod. Some molecules, like water, are polar molecules. Polar molecules have a natural or inherent separation of charge, although they are neutral overall. Polar molecules are particularly affected by other charged objects and show greater polarization effects than molecules with naturally uniform charge distributions.

When the two ends of a dipole can be separated, this method of **charging by induction** may be used to create charged objects without transferring charge. In Figure $1.3.4$, we see two neutral metal spheres in contact with one another but insulated from the rest of the world. A positively charged rod is brought near one of them, attracting negative charge to that side, leaving the other sphere positively charged.

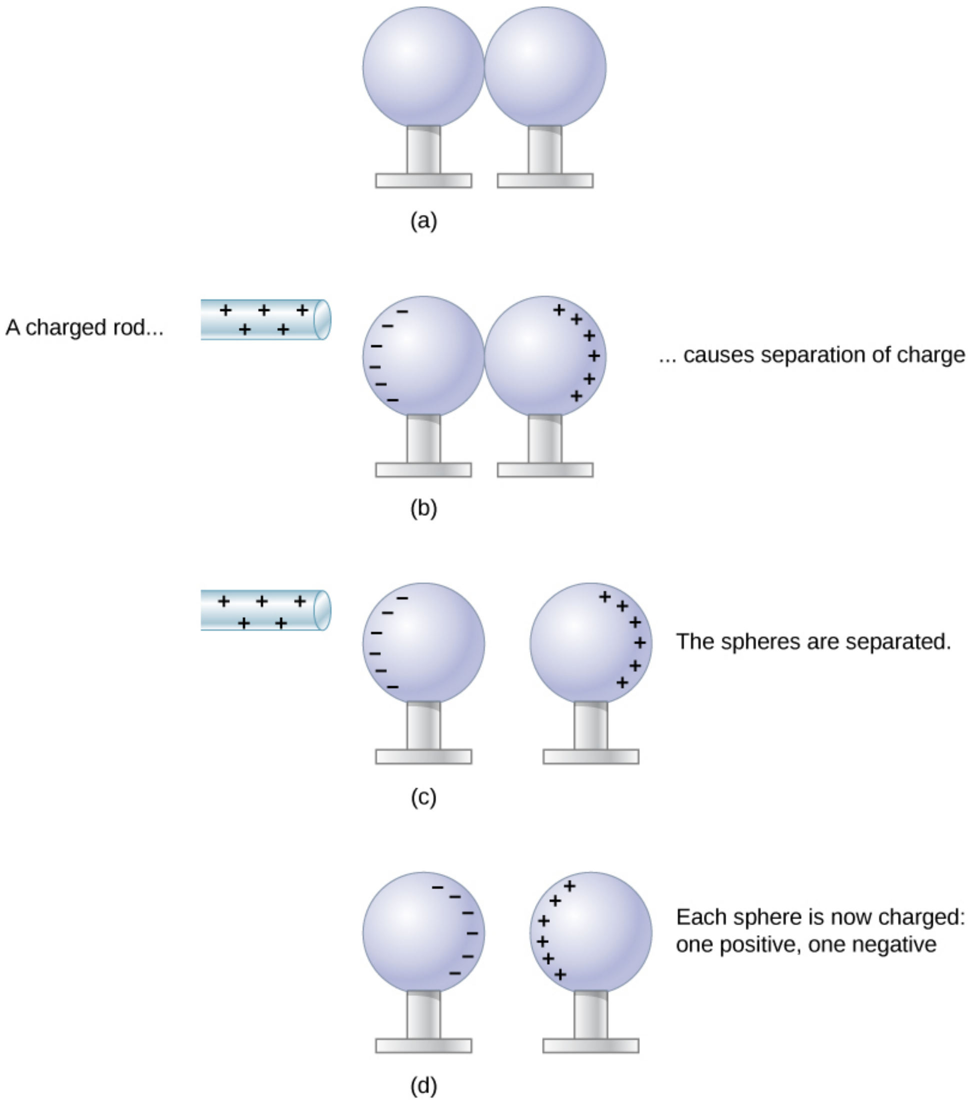


Figure $1.3.4$: Charging by induction. (a) Two uncharged or neutral metal spheres are in contact with each other but insulated from the rest of the world. (b) A positively charged glass rod is brought near the sphere on the left, attracting negative charge and leaving the other sphere positively charged. (c) The spheres are separated before the rod is removed, thus separating negative and positive charges. (d) The spheres retain net charges after the inducing rod is removed—without ever having been touched by a charged object.

Another method of charging by induction is shown in Figure $1.3.5$. The neutral metal sphere is polarized when a charged rod is brought near it. The sphere is then grounded, meaning that a conducting wire is run from the sphere to the ground. Since Earth is large and most of the ground is a good conductor, it can supply or accept excess charge easily. In this case, electrons are attracted to the sphere through a wire called the ground wire, because it supplies a conducting path to the ground. The ground connection is broken before the charged rod is removed, leaving the sphere with an excess charge opposite to that of the rod. Again, an opposite charge is achieved when charging by induction, and the charged rod loses none of its excess charge.

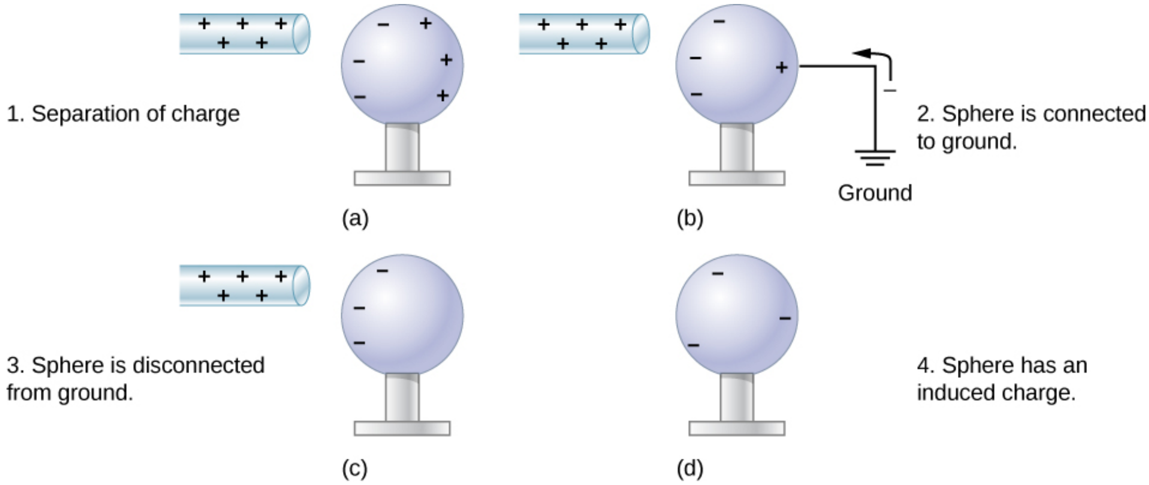


Figure 1.3.5: Charging by induction using a ground connection. (a) A positively charged rod is brought near a neutral metal sphere, polarizing it. (b) The sphere is grounded, allowing electrons to be attracted from Earth's ample supply. (c) The ground connection is broken. (d) The positive rod is removed, leaving the sphere with an induced negative charge.

1.4: Coulomb's Law

Learning Objectives

By the end of this section, you will be able to:

- Describe the electric force, both qualitatively and quantitatively
- Calculate the force that charges exert on each other
- Determine the direction of the electric force for different source charges
- Correctly describe and apply the superposition principle for multiple source charges

Experiments with electric charges have shown that if two objects each have electric charge, then they exert an electric force on each other. The magnitude of the force is linearly proportional to the net charge on each object and inversely proportional to the square of the distance between them. (Interestingly, the force does not depend on the mass of the objects.) The direction of the force vector is along the imaginary line joining the two objects and is dictated by the signs of the charges involved.

Let

- $q_1, q_2 =$ the net electric charge of the two objects;
- $\vec{r}_{12} =$ the vector displacement from q_1 to q_2.

The electric force $\vec{F}$ on one of the charges is proportional to the magnitude of its own charge and the magnitude of the other charge, and is inversely proportional to the square of the distance between them:

$$F \propto \frac{q_1 q_2}{r_{12}^2}.$$

This proportionality becomes an equality with the introduction of a proportionality constant. For reasons that will become clear in a later chapter, the proportionality constant that we use is actually a collection of constants. (We discuss this constant shortly.)

Coulomb's Law

The magnitude of the electric force (or **Coulomb force**) between two electrically charged particles is equal to

$$|\mathbf{F}_{12}| = \frac{1}{4\pi\varepsilon_0}\frac{|q_1 q_2|}{r_{12}^2} \tag{1.4.1}$$

The unit vector r has a magnitude of 1 and points along the axis as the charges. If the charges have the same sign, the force is in the same direction as r showing a repelling force. If the charges have different signs, the force is in the opposite direction of r showing an attracting force. (Figure 1.4.1).

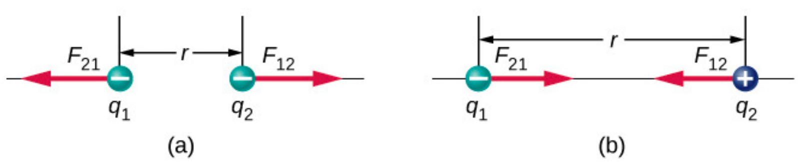


Figure 1.4.1: The electrostatic force $\vec{F}$ between point charges q_1 and q_2 separated by a distance r is given by Coulomb's law. Note that Newton's third law (every force exerted creates an equal and opposite force) applies as usual—the force on q_1 is equal in magnitude and opposite in direction to the force it exerts on q_2. (a) Like charges; (b) unlike charges.

It is important to note that the electric force is not constant; it is a function of the separation distance between the two charges. If either the test charge or the source charge (or both) move, then $\vec{r}$ changes, and therefore so does the force. An immediate consequence of this is that direct application of Newton's laws with this force can be mathematically difficult, depending on the specific problem at hand. It can (usually) be done, but we almost always look for easier methods of calculating whatever physical quantity we are interested in. (Conservation of energy is the most common choice.)

Finally, the new constant ϵ_0 in Coulomb's law is called the *permittivity of free space*, or (better) the **permittivity of vacuum**. It has a very important physical meaning that we will discuss in a later chapter; for now, it is simply an empirical proportionality constant. Its numerical value (to three significant figures) turns out to be

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2}.$$

These units are required to give the force in Coulomb's law the correct units of newtons. Note that in Coulomb's law, the permittivity of vacuum is only part of the proportionality constant. For convenience, we often define a Coulomb's constant:

$$k_e = \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}.$$

Example 1.4.1: The Force on the Electron in Hydrogen

A hydrogen atom consists of a single proton and a single electron. The proton has a charge of $+e$ and the electron has $-e$. In the "ground state" of the atom, the electron orbits the proton at most probable distance of 5.29×10^{-11}m (Figure $1.4.2$). Calculate the electric force on the electron due to the proton.

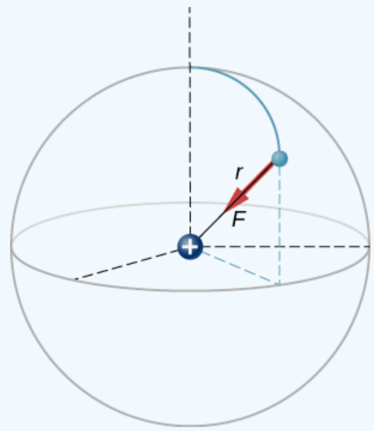


Figure $1.4.2$: A schematic depiction of a hydrogen atom, showing the force on the electron. This depiction is only to enable us to calculate the force; the hydrogen atom does not really look like this.

Strategy

For the purposes of this example, we are treating the electron and proton as two point particles, each with an electric charge, and we are told the distance between them; we are asked to calculate the force on the electron. We thus use Coulomb's law (Equation 1.4.1).

Solution

Our two charges are,

$$\begin{aligned} q_1 &= +e \\ &= +1.602 \times 10^{-19} \text{ C} \\ q_2 &= -e \\ &= -1.602 \times 10^{-19} \text{ C} \end{aligned}$$

and the distance between them

$$r = 5.29 \times 10^{-11} \text{ m}.$$

The magnitude of the force on the electron (Equation 1.4.1) is

$$\begin{aligned} F &= \frac{1}{4\pi\epsilon_0}\frac{|q_1 q_2|}{r_{12}^2} \\ &= \frac{1}{4\pi\left(8.85\times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N}\cdot\mathrm{m}^2}\right)}\frac{(1.602\times 10^{-19}\ \mathrm{C})^2}{(5.29\times 10^{-11}\ \mathrm{m})^2} \\ &= 8.25\times 10^{-8}\ \mathrm{N}. \end{aligned}$$

As for the direction, since the charges on the two particles are opposite, the force is attractive; the force on the electron points radially directly toward the proton, everywhere in the electron's orbit. The force is thus expressed as

$$\vec{F} = (8.25\times 10^{-8}\ \mathrm{N})\hat{r}.$$

Significance

This is a three-dimensional system, so the electron (and therefore the force on it) can be anywhere in an imaginary spherical shell around the proton. In this "classical" model of the hydrogen atom, the electrostatic force on the electron points in the inward centripetal direction, thus maintaining the electron's orbit. But note that the quantum mechanical model of hydrogen (discussed in Quantum Mechanics) is utterly different.

? Exercise 1.4.1

What would be different if the electron also had a positive charge?

Answer

The force would point outward.

Multiple Source Charges

The analysis that we have done for two particles can be extended to an arbitrary number of particles; we simply repeat the analysis, two charges at a time. Specifically, we ask the question: Given N charges (which we refer to as source charge), what is the net electric force that they exert on some other point charge (which we call the test charge)? Note that we use these terms because we can think of the test charge being used to test the strength of the force provided by the source charges.

Like all forces that we have seen up to now, the net electric force on our test charge is simply the vector sum of each individual electric force exerted on it by each of the individual test charges. Thus, we can calculate the net force on the test charge Q by calculating the force on it from each source charge, taken one at a time, and then adding all those forces together (as vectors). This ability to simply add up individual forces in this way is referred to as the **principle of superposition**, and is one of the more important features of the electric force. In mathematical form, this becomes

$$\vec{F}(r) = \frac{1}{4\pi\epsilon_0} Q \sum_{i=1}^{N} \frac{q_i}{r_i^2}\hat{r}_i^2. \tag{1.4.2}$$

In this expression, Q represents the charge of the particle that is experiencing the electric force $\vec{F}$, and is located at $\vec{r}$ from the origin; the q_i's are the N source charges, and the vectors $\vec{r}_i = r_i \hat{r}_i$ are the displacements from the position of the ith charge to the position of Q. Each of the N unit vectors points directly from its associated source charge toward the test charge. All of this is depicted in Figure 1.4.3. Please note that there is no physical difference between Q and q_i; the difference in labels is merely to allow clear discussion, with Q being the charge we are determining the force on.

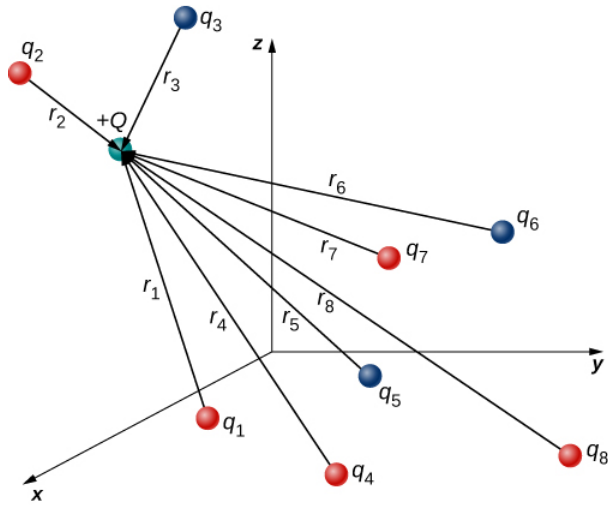


Figure 1.4.3: The eight source charges each apply a force on the single test charge Q. Each force can be calculated independently of the other seven forces. This is the essence of the superposition principle.

(Note that the force vector $\vec{F}_i$ does not necessarily point in the same direction as the unit vector $\hat{r}_i$; it may point in the opposite direction, $-\hat{r}_i$. The signs of the source charge and test charge determine the direction of the force on the test charge.)

There is a complication, however. Just as the source charges each exert a force on the test charge, so too (by Newton's third law) does the test charge exert an equal and opposite force on each of the source charges. As a consequence, each source charge would change position. However, by Equation 1.4.1, the force on the test charge is a function of position; thus, as the positions of the source charges change, the net force on the test charge necessarily changes, which changes the force, which again changes the positions. Thus, the entire mathematical analysis quickly becomes intractable. Later, we will learn techniques for handling this situation, but for now, we make the simplifying assumption that the source charges are fixed in place somehow, so that their positions are constant in time. (The test charge is allowed to move.) With this restriction in place, the analysis of charges is known as **electrostatics**, where "statics" refers to the constant (that is, static) positions of the source charges and the force is referred to as an **electrostatic force**.

✓ Example 1.4.2: The Net Force from Two Source Charges

Three different, small charged objects are placed as shown in Figure 1.4.4. The charges q_1 and q_3 are fixed in place; q_2 is free to move. Given $q_1 = 2e$, $q_2 = -3e$, and $q_3 = -5e$, and that $d = 2.0 \times 10^{-7}\ \text{m}$, what is the net force on the middle charge q_2?

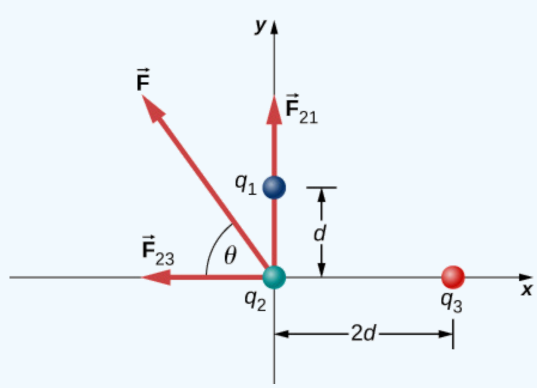


Figure 1.4.4: Source charges q_1 and q_3 each apply a force on q_2.

Strategy

We use Coulomb's law again. The way the question is phrased indicates that q_2 is our test charge, so that q_1 and q_3 are source charges. The principle of superposition says that the force on q_2 from each of the other charges is unaffected by the presence of the other charge. Therefore, we write down the force on q_2 from each and add them together as vectors.

Solution

We have two source charges q_1 and q_3 a test charge q_2, distances r_{21} and r_{23} and we are asked to find a force. This calls for Coulomb's law and superposition of forces. There are two forces:

$$\vec{F} = \vec{F}_{21} + \vec{F}_{23} = \frac{1}{4\pi\epsilon_0}\left[\frac{q_2 q_1}{r_{21}^2}\hat{j} + \left(-\frac{q_2 q_3}{r_{23}^2}\hat{i}\right)\right].$$

We cannot add these forces directly because they don't point in the same direction: $\vec{F}_{12}$ points only in the $-x$-direction, while $\vec{F}_{13}$ points only in the $+y$-direction. The net force is obtained from applying the Pythagorean theorem to its x- and y-components:

$$F = \sqrt{F_x^2 + F_y^2}$$

and

$$\begin{aligned} F_x &= -F_{23} = -\frac{1}{4\pi\epsilon_0}\frac{q_2 q_3}{r_{23}^2} \\ &= -\left(8.99 \times 10^9 \frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}\right)\frac{(4.806 \times 10^{-19}\,\mathrm{C})(8.01 \times 10^{-19}\,\mathrm{C})}{(4.00 \times 10^{-7}\,\mathrm{m})^2} \\ &= -2.16 \times 10^{-14}\,\mathrm{N} \end{aligned}$$

and

$$\begin{aligned} F_y &= F_{21} = \frac{1}{4\pi\epsilon_0}\frac{q_2 q_1}{r_{21}} \\ &= \left(9.99 \times 10^9 \frac{\mathrm{N \cdot m^2}}{\mathrm{C^2}}\right)\frac{(4.806 \times 10^{-19}\,\mathrm{C})(3.204 \times 10^{-19}\,\mathrm{C})}{(2.00 \times 10^{-7}\,\mathrm{m})^2} \\ &= 3.46 \times 10^{-14}\,\mathrm{N}. \end{aligned}$$

We find that

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= 4.08 \times 10^{-14}\,\mathrm{N} \end{aligned}$$

at an angle of

$$\begin{aligned} \phi &= \tan^{-1}\left(\frac{F_y}{F_x}\right) \\ &= \tan^{-1}\left(\frac{3.46 \times 10^{-14}\,\mathrm{N}}{-2.16 \times 10^{-14}\,\mathrm{N}}\right) \\ &= -58^\circ, \end{aligned}$$

that is, 58° above the $-x$-axis, as shown in the diagram.

Significance

Notice that when we substituted the numerical values of the charges, we did not include the negative sign of either q_1 or q_3. Recall that negative signs on vector quantities indicate a reversal of direction of the vector in question. But for electric forces, the direction of the force is determined by the types (signs) of both interacting charges; we determine the force directions by considering whether the signs of the two charges are the same or are opposite. If you also include negative signs from negative charges when you substitute numbers, you run the risk of mathematically reversing the direction of the force you are calculating. Thus, the safest thing to do is to calculate just the magnitude of the force, using the absolute values of the charges, and determine the directions physically.

It's also worth noting that the only new concept in this example is how to calculate the electric forces; everything else (getting the net force from its components, breaking the forces into their components, finding the direction of the net force) is the same as force problems you have done earlier.

Exercise 1.4.2

What would be different in Example $1.4.2$ if q_1 were negative rather than positive?

Answer

The net force would point 58° below the $-x$-axis.

1.5: Electric Field

> **LEARNING OBJECTIVES**
>
> By the end of this section, you will be able to:
>
> - Explain the purpose of the electric field concept
> - Describe the properties of the electric field
> - Calculate the field of a collection of source charges of either sign

As we showed in the preceding section, the net electric force on a test charge is the vector sum of all the electric forces acting on it, from all of the various source charges, located at their various positions. But what if we use a different test charge, one with a different magnitude, or sign, or both? Or suppose we have a dozen different test charges we wish to try at the same location? We would have to calculate the sum of the forces from scratch. Fortunately, it is possible to define a quantity, called the **electric field**, which is independent of the test charge. It only depends on the configuration of the source charges, and once found, allows us to calculate the force on any test charge.

Defining a Field

Suppose we have N source charges $q_1, q_2, q_3, \dots q_N$ located at positions $\vec{r}_1, \vec{r}_2, \vec{r}_3, \dots \vec{r}_N$, applying N electrostatic forces on a test charge Q. The net force on Q is

$$\vec{F} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \dots + \vec{F}_N \tag{1.5.1}$$

$$= \frac{1}{4\pi\epsilon_0}\left(\frac{Qq_1}{r_1^2}\hat{r}_1 + \frac{Qq_2}{r_2^2}\hat{r}_2 + \frac{Qq_3}{r_3^2}\hat{r}_3 + \dots + \frac{Qq_N}{r_1^2}\hat{r}_N\right) \tag{1.5.2}$$

$$= Q\left[\frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \frac{q_3}{r_3^2}\hat{r}_3 + \dots + \frac{q_N}{r_1^2}\hat{r}_N\right)\right] \tag{1.5.3}$$

We can rewrite this as

$$\vec{F} = Q\vec{E} \tag{1.5.4}$$

where

$$\vec{E} \equiv \frac{1}{4\pi\epsilon_0}\left(\frac{q_1}{r_1^2}\hat{r}_1 + \frac{q_2}{r_2^2}\hat{r}_2 + \frac{q_3}{r_3^2}\hat{r}_3 + \dots + \frac{q_N}{r_1^2}\hat{r}_N\right) \tag{1.5.5}$$

or, more compactly,

$$\vec{E}(P) \equiv \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i. \tag{1.5.6}$$

This expression is called the electric field at position $P = P(x, y, z)$ of the N source charges. Here, P is the location of the point in space where you are calculating the field and is relative to the positions $\vec{r}_i$ of the source charges (Figure $1.5.1$). Note that we have to impose a coordinate system to solve actual problems.

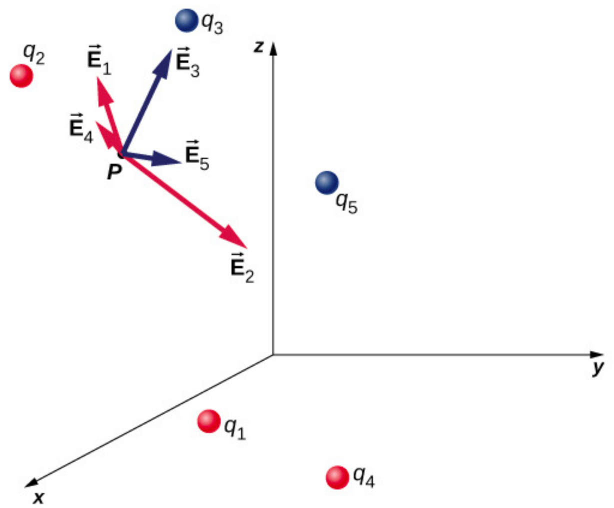


Figure 1.5.1: Each of these five source charges creates its own electric field at every point in space; shown here are the field vectors at an arbitrary point P. Like the electric force, the net electric field obeys the superposition principle.

Notice that the calculation of the electric field makes no reference to the test charge. Thus, the physically useful approach is to calculate the electric field and then use it to calculate the force on some test charge later, if needed. Different test charges experience different forces Equation 1.5.4, but it is the same electric field Equation 1.5.6. That being said, recall that there is no fundamental difference between a test charge and a source charge; these are merely convenient labels for the system of interest. Any charge produces an electric field; however, just as Earth's orbit is not affected by Earth's own gravity, a charge is not subject to a force due to the electric field it generates. Charges are only subject to forces from the electric fields of other charges.

In this respect, the electric field $\vec{E}$ of a point charge is similar to the gravitational field $\vec{g}$ of Earth; once we have calculated the gravitational field at some point in space, we can use it any time we want to calculate the resulting force on any mass we choose to place at that point. In fact, this is exactly what we do when we say the gravitational field of Earth (near Earth's surface) has a value of $9.81\,\mathrm{m/s^2}$ and then we calculate the resulting force (i.e., weight) on different masses. Also, the general expression for calculating $\vec{g}$ at arbitrary distances from the center of Earth (i.e., not just near Earth's surface) is very similar to the expression for E:

$$\vec{g} = G\frac{M}{r^2}\hat{r}$$

where G is a proportionality constant, playing the same role for $\vec{g}$ as $\dfrac{1}{4\pi\epsilon_0}$ does for $\vec{E}$. The value of $\vec{g}$ is calculated once and is then used in an endless number of problems.

To push the analogy further, notice the units of the electric field: From $F = QE$, the units of **E** are newtons per coulomb, N/C, that is, the electric field applies a force on each unit charge. Now notice the units of g: From $w = mg$ the units of g are newtons per kilogram, N/kg, that is, the gravitational field applies a force on each unit mass. We could say that the gravitational field of Earth, near Earth's surface, has a value of 9.81 N/kg.

The Meaning of "Field"

Recall from your studies of gravity that the word "field" in this context has a precise meaning. A field, in physics, is a physical quantity whose value depends on (is a function of) position, relative to the source of the field. In the case of the electric field, Equation 1.5.6 shows that the value of $\vec{E}$ (both the magnitude and the direction) depends on where in space the point P is located, measured from the locations $\vec{r}_i$ of the source charges q_i.

In addition, since the electric field is a vector quantity, the electric field is referred to as a *vector field*. (The gravitational field is also a vector field.) In contrast, a field that has only a magnitude at every point is a *scalar field*. The temperature in a room is an example of a scalar field. It is a field because the temperature, in general, is different at different locations in the room, and it is a scalar field because temperature is a scalar quantity.

Also, as you did with the gravitational field of an object with mass, you should picture the electric field of a charge-bearing object (the source charge) as a continuous, immaterial substance that surrounds the source charge, filling all of space—in principle, to $\pm\infty$ in all directions. The field exists at every physical point in space. To put it another way, the electric charge on an object alters the space around the charged object in such a way that all other electrically charged objects in space experience an electric force as a result of being in that field. The electric field, then, is the mechanism by which the electric properties of the source charge are transmitted to and through the rest of the universe. (Again, the range of the electric force is infinite.)

We will see in subsequent chapters that the speed at which electrical phenomena travel is the same as the speed of light. There is a deep connection between the electric field and light.

Superposition

Yet another experimental fact about the field is that it obeys the superposition principle. In this context, that means that we can (in principle) calculate the total electric field of many source charges by calculating the electric field of only q_1 at position **P**, then calculate the field of q_2 at **P**, while—and this is the crucial idea—ignoring the field of, and indeed even the existence of, q_1. We can repeat this process, calculating the field of each individual source charge, independently of the existence of any of the other charges. The total electric field, then, is the vector sum of all these fields. That, in essence, is what Equation 1.5.6 says.

In the next section, we describe how to determine the shape of an electric field of a source charge distribution and how to sketch it.

The Direction of the Field

Equation 1.5.6 enables us to determine the magnitude of the electric field, but we need the direction also. We use the convention that the direction of any electric field vector is the same as the direction of the electric force vector that the field would apply to a positive test charge placed in that field. Such a charge would be repelled by positive source charges (the force on it would point away from the positive source charge) but attracted to negative charges (the force points toward the negative source).

Direction of the Electric Field

By convention, all electric fields $\vec{E}$ point away from positive source charges and point toward negative source charges.

Phet Simulation: Electric Field of Dreams

Download the Electric Field of Dreams PhET simulation and add charges to the and see how they react to the electric field. Turn on a background electric field and adjust the direction and magnitude.

Example 1.5.1A: The E-field of an Atom

In an ionized helium atom, the most probable distance between the nucleus and the electron is $r = 26.5 \times 10^{-12}\ \mathrm{m}$. What is the electric field due to the nucleus at the location of the electron?

Strategy

Note that although the electron is mentioned, it is not used in any calculation. The problem asks for an electric field, not a force; hence, there is only one charge involved, and the problem specifically asks for the field due to the nucleus. Thus, the electron is a red herring; only its distance matters. Also, since the distance between the two protons in the nucleus is much,

much smaller than the distance of the electron from the nucleus, we can treat the two protons as a single charge $+2e$ (Figure $1.5.2$).

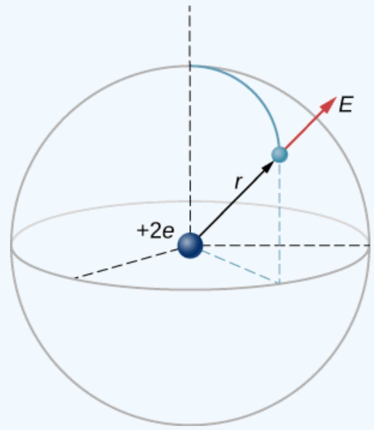


Figure $1.5.2$: A schematic representation of a helium atom. Again, helium physically looks nothing like this, but this sort of diagram is helpful for calculating the electric field of the nucleus.

Solution

The electric field is calculated by

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\frac{q_i}{r_i^2}\hat{r}_i.$$

Since there is only one source charge (the nucleus), this expression simplifies to

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}.$$

Here, $q = 2e = 2(1.6 \times 10^{-19}\ \text{C})$ (since there are two protons) and r is given; substituting gives

$$\begin{aligned}\vec{E} &= \frac{1}{4\pi\left(8.85 \times 10^{-12}\frac{\text{C}^2}{\text{N}\cdot\text{m}^2}\right)}\frac{2(1.6 \times 10^{-19}\ \text{C})}{26.5 \times 10^{-12}\ \text{m})^2}\hat{r} \\ &= 4.1 \times 10^{12}\frac{\text{N}}{\text{C}}\hat{r}.\end{aligned}$$

The direction of $\vec{E}$ is radially away from the nucleus in all directions. Why? Because a positive test charge placed in this field would accelerate radially away from the nucleus (since it is also positively charged), and again, the convention is that the direction of the electric field vector is defined in terms of the direction of the force it would apply to positive test charges.

Example $1.5.1B$: The E-Field above Two Equal Charges

a. Find the electric field (magnitude and direction) a distance z above the midpoint between two equal charges $+q$ that are a distance d apart (Figure $1.5.3$). Check that your result is consistent with what you'd expect when $z \gg d$.
b. The same as part (a), only this time make the right-hand charge $-q$ instead of $+q$.

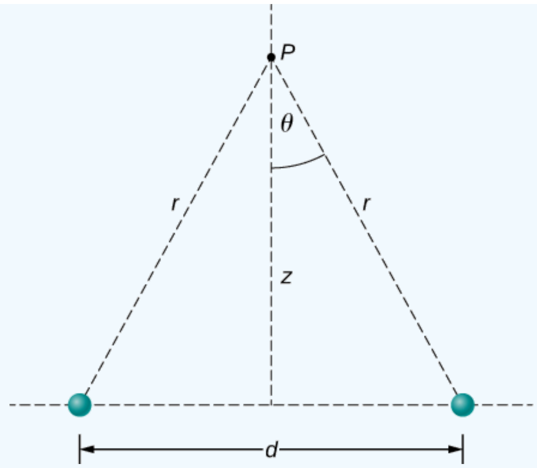


Figure 1.5.3: Finding the field of two identical source charges at the point P. Due to the symmetry, the net field at P is entirely vertical. (Notice that this is **not** true away from the midline between the charges.)

Strategy

We add the two fields as vectors, per Equation 1.5.6. Notice that the system (and therefore the field) is symmetrical about the vertical axis; as a result, the horizontal components of the field vectors cancel. This simplifies the math. Also, we take care to express our final answer in terms of only quantities that are given in the original statement of the problem: q, z, d, and constants (π, ϵ_0).

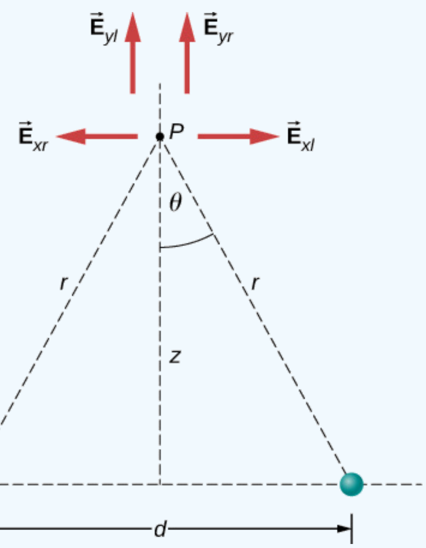


Figure 1.5.4. Note that the horizontal components of the electric fields from the two charges cancel each other out, while the vertical components add together.

Solution

a. By symmetry, the horizontal (x)-components of $\vec{E}$ cancel (Figure 1.5.4);

$$\begin{aligned} E_x &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\sin\theta - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\sin\theta \\ &= 0. \end{aligned}$$

The vertical (z)-component is given by

$$\begin{aligned} E_z &= \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta + \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\theta \\ &= \frac{1}{4\pi\epsilon_0}\frac{2q}{r^2}\cos\theta. \end{aligned}$$

Since none of the other components survive, this is the entire electric field, and it points in the $\hat{k}$ direction. Notice that this calculation uses the principle of **superposition**; we calculate the fields of the two charges independently and then add them together.

What we want to do now is replace the quantities in this expression that we don't know (such as r), or can't easily measure (such as $\cos\theta$ with quantities that we do know, or can measure. In this case, by geometry,

$$r^2 = z^2 + \left(\frac{d}{2}\right)^2$$

and

$$\cos\,\theta = \frac{z}{R} = \frac{z}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}.$$

Thus, substituting,

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2q}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^2}\frac{z}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{1/2}}\hat{k}.$$

Simplifying, the desired answer is

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2qz}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}\hat{k}. \tag{1.5.7}$$

b. If the source charges are equal and opposite, the vertical components cancel because

$$E_z = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,cos\,\theta - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\cos\,\theta = 0$$

and we get, for the horizontal component of $\vec{E}$.

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{qd}{\left[z^2 + \left(\frac{d}{2}\right)^2\right]^{3/2}}\hat{i}. \tag{1.5.8}$$

Significance

It is a very common and very useful technique in physics to check whether your answer is reasonable by evaluating it at extreme cases. In this example, we should evaluate the field expressions for the cases $d = 0$, $z \gg d$, and $z \to \infty$, and confirm that the resulting expressions match our physical expectations. Let's do so:

Let's start with Equation 1.5.7, the field of two identical charges. From far away (i.e., $z >> d$), the two source charges should "merge" and we should then "see" the field of just one charge, of size $2q$. So, let $z \gg d$; then we can neglect d^2 in Equation 1.5.7 to obtain

$$\begin{aligned}\lim_{d\to 0}\vec{E} &= \frac{1}{4\pi\epsilon_0}\frac{2qz}{[z^2]^{3/2}}\hat{k}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2qz}{z^3}\hat{k}\\ &= \frac{1}{4\pi\epsilon_0}\frac{2q}{z^2}\hat{k},\end{aligned}$$

which is the correct expression for a field at a distance z away from a charge $2q$.

Next, we consider the field of equal and opposite charges, Equation 1.5.8. It can be shown (via a Taylor expansion) that for $d \ll z \ll \infty$, , this becomes

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{qd}{z^3}\hat{\imath},$$

which is the field of a dipole, a system that we will study in more detail later. (Note that the units of $\vec{E}$ are still correct in this expression, since the units of d in the numerator cancel the unit of the "extra" z in the denominator.) If z is *very* large $(z \rightarrow \infty)$, then $E \rightarrow 0$, as it should; the two charges "merge" and so cancel out.

Exercise 1.5.1

What is the electric field due to a single point particle?

Answer

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r}$$

Phe Simulation: Electric Field Hockey

Try this simulation of electric field hockey to get the charge in the goal by placing other charges on the field.

1.6: Electric Field Lines

Learning Objectives

By the end of this section, you will be able to:

- Explain the purpose of an electric field diagram
- Describe the relationship between a vector diagram and a field line diagram
- Explain the rules for creating a field diagram and why these rules make physical sense
- Sketch the field of an arbitrary source charge

Now that we have some experience calculating electric fields, let's try to gain some insight into the geometry of electric fields. As mentioned earlier, our model is that the charge on an object (the source charge) alters space in the region around it in such a way that when another charged object (the test charge) is placed in that region of space, that test charge experiences an electric force. The concept of electric **field lines**, and of electric field line diagrams, enables us to visualize the way in which the space is altered, allowing us to visualize the field. The purpose of this section is to enable you to create sketches of this geometry, so we will list the specific steps and rules involved in creating an accurate and useful sketch of an electric field.

It is important to remember that electric fields are three-dimensional. Although in this book we include some pseudo-three-dimensional images, several of the diagrams that you'll see (both here, and in subsequent chapters) will be two-dimensional projections, or cross-sections. Always keep in mind that in fact, you're looking at a three-dimensional phenomenon.

Our starting point is the physical fact that the electric field of the source charge causes a test charge in that field to experience a force. By definition, electric field vectors point in the same direction as the electric force that a (hypothetical) positive test charge would experience, if placed in the field (Figure $1.6.1$).

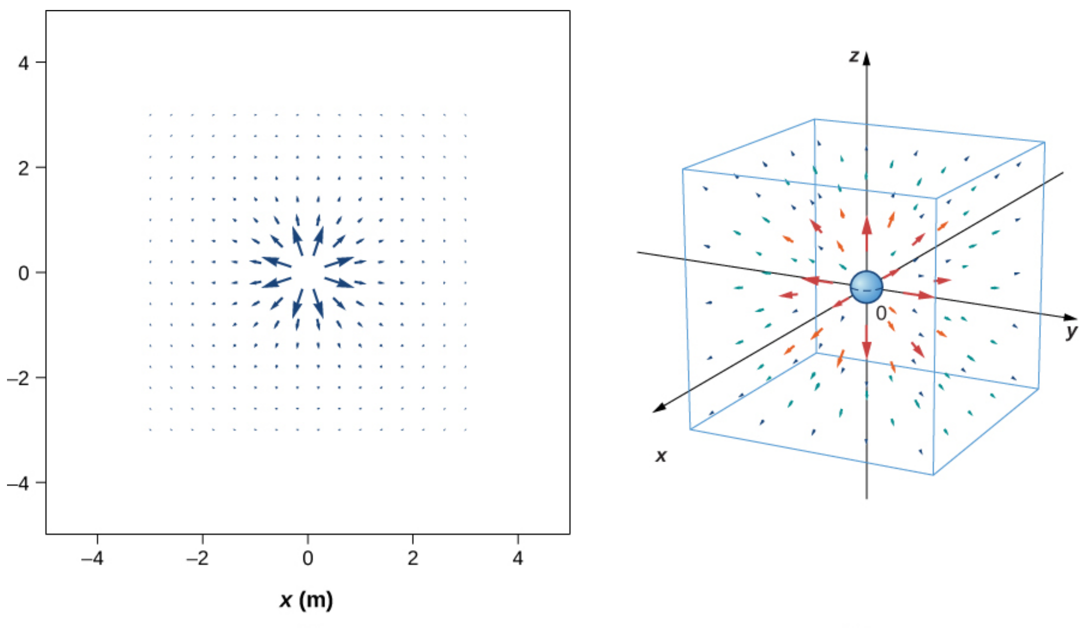


Figure $1.6.1$: The electric field of a positive point charge. A large number of field vectors are shown. Like all vector arrows, the length of each vector is proportional to the magnitude of the field at each point. (a) Field in two dimensions; (b) field in three dimensions.

We've plotted many field vectors in the figure, which are distributed uniformly around the source charge. Since the electric field is a vector, the arrows that we draw correspond at every point in space to both the magnitude and the direction of the field at that point. As always, the length of the arrow that we draw corresponds to the magnitude of the field vector at that point. For a point source charge, the length decreases by the square of the distance from the source charge. In addition, the direction of the field vector is radially away from the source charge, because the direction of the electric field is defined by the direction of the force that a positive test charge would experience in that field. (Again, keep in mind that the actual field is three-dimensional; there are also field lines pointing out of and into the page.)

This diagram is correct, but it becomes less useful as the source charge distribution becomes more complicated. For example, consider the vector field diagram of a dipole (Figure $1.6.2$).

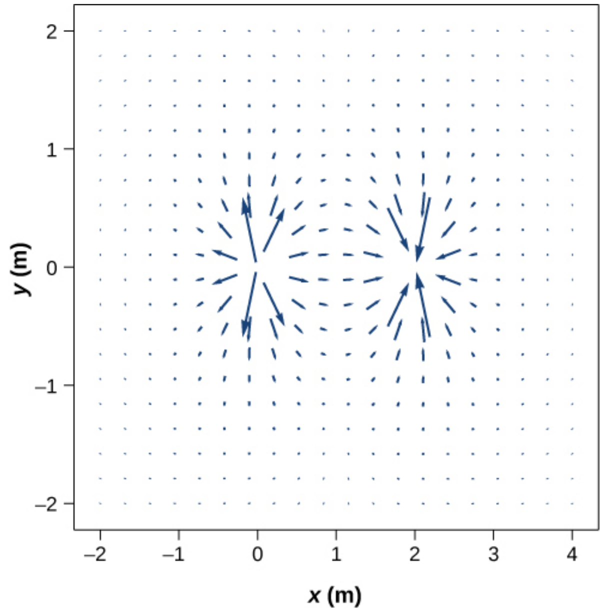


Figure $1.6.2$: The vector field of a dipole. Even with just two identical charges, the vector field diagram becomes difficult to understand.

There is a more useful way to present the same information. Rather than drawing a large number of increasingly smaller vector arrows, we instead connect all of them together, forming continuous lines and curves, as shown in Figure $1.6.3$.

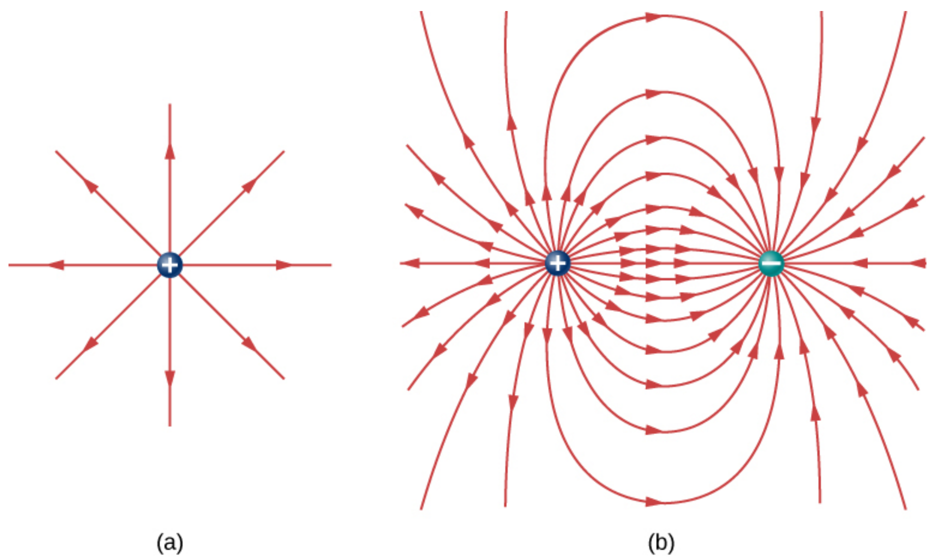


Figure $1.6.3$: (a) The electric field line diagram of a positive point charge. (b) The field line diagram of a dipole. In both diagrams, the magnitude of the field is indicated by the field line density. The field **vectors** (not shown here) are everywhere tangent to the field lines.

Although it may not be obvious at first glance, these field diagrams convey the same information about the electric field as do the vector diagrams. First, the direction of the field at every point is simply the direction of the field vector at that same point. In other words, at any point in space, the field vector at each point is tangent to the field line at that same point. The arrowhead placed on a field line indicates its direction.

As for the magnitude of the field, that is indicated by the **field line density**—that is, the number of field lines per unit area passing through a small cross-sectional area perpendicular to the electric field. This field line density is drawn to be proportional to the magnitude of the field at that cross-section. As a result, if the field lines are close together (that is, the field line density is greater), this indicates that the magnitude of the field is large at that point. If the field lines are far apart at the cross-section, this indicates the magnitude of the field is small. Figure 1.6.4 shows the idea.

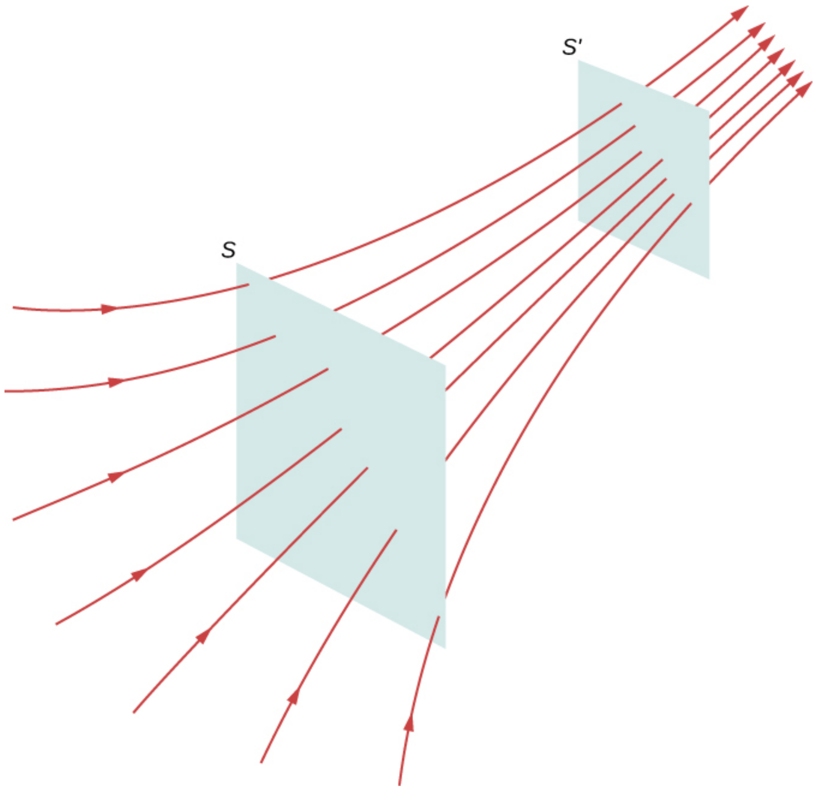


Figure 1.6.4: Electric field lines passing through imaginary areas. Since the number of lines passing through each area is the same, but the areas themselves are different, the field line density is different. This indicates different magnitudes of the electric field at these points.

In Figure 1.6.4, the same number of field lines passes through both surfaces S and S', but the surface S is larger than surface S'. Therefore, the density of field lines (number of lines per unit area) is larger at the location of S', indicating that the electric field is stronger at the location of S' than at S. The rules for creating an electric field diagram are as follows.

Problem-Solving Strategy: Drawing Electric Field Lines

1. Electric field lines either originate on positive charges or come in from infinity, and either terminate on negative charges or extend out to infinity.
2. The number of field lines originating or terminating at a charge is proportional to the magnitude of that charge. A charge of $2q$ will have twice as many lines as a charge of q.
3. At every point in space, the field vector at that point is tangent to the field line at that same point.
4. The field line density at any point in space is proportional to (and therefore is representative of) the magnitude of the field at that point in space.
5. Field lines can never cross. Since a field line represents the direction of the field at a given point, if two field lines crossed at some point, that would imply that the electric field was pointing in two different directions at a single point. This in turn would suggest that the (net) force on a test charge placed at that point would point in two different directions. Since this is obviously impossible, it follows that field lines must never cross.

Always keep in mind that field lines serve only as a convenient way to visualize the electric field; they are not physical entities. Although the direction and relative intensity of the electric field can be deduced from a set of field lines, the lines can also be misleading. For example, the field lines drawn to represent the electric field in a region must, by necessity, be discrete. However, the actual electric field in that region exists at every point in space.

Field lines for three groups of discrete charges are shown in Figure 1.6.5. Since the charges in parts (a) and (b) have the same magnitude, the same number of field lines are shown starting from or terminating on each charge. In (c), however, we draw three times as many field lines leaving the $+3q$ charge as entering the $-q$. The field lines that do not terminate at $-q$ emanate outward from the charge configuration, to infinity.

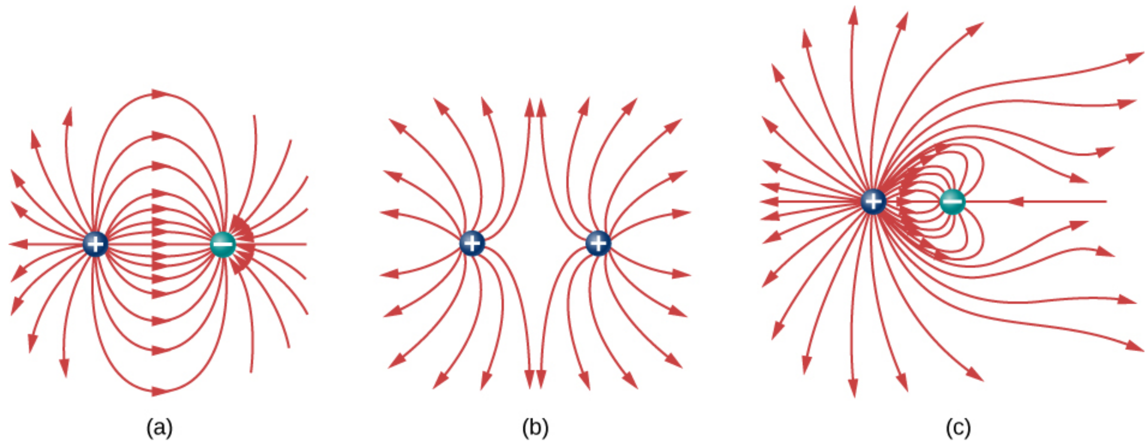


Figure 1.6.5: Three typical electric field diagrams. (a) A dipole. (b) Two identical charges. (c) Two charges with opposite signs and different magnitudes. Can you tell from the diagram which charge has the larger magnitude?

The ability to construct an accurate electric field diagram is an important, useful skill; it makes it much easier to estimate, predict, and therefore calculate the electric field of a source charge. The best way to develop this skill is with software that allows you to place source charges and then will draw the net field upon request. We strongly urge you to search the Internet for a program. Once you've found one you like, run several simulations to get the essential ideas of field diagram construction. Then practice drawing field diagrams, and checking your predictions with the computer-drawn diagrams.

PhET: Charges and Fields

Arrange positive and negative charges in space and view the resulting electric field and electrostatic potential. Plot equipotential lines and discover their relationship to the electric field. Create models of dipoles, capacitors, and more!

1.6.1: Conductors and the Electric Field

An ideal conductor is chock full of charged particles that are perfectly free to move around within the conductor. Like all macroscopic samples of material, an ideal conductor consists of a huge amount of positive charge, and, when neutral, the same amount of negative charge. When not neutral, there is a tiny fractional imbalance one way or the other. In an ideal conductor, some appreciable fraction of the charge is completely free to move around within the conducting material. The ideal (perfect) conductor is well approximated by some materials familiar to you, in particular, metals. In some materials, it is positive charge that is free to move about, in some, it is negative, and in others, it is both. For our purposes, the observable effects of positive charge moving in one direction are so close to being indistinguishable from negative charge moving in the opposite direction that, we will typically treat the charge carriers as being positive without concern for what the actual charge carriers are.

Here, we make one point about conductors by means of an analogy. The analogy involves a lake full of fish. Let the lake represent the conductor and the fish the charge carriers. The fish are free to move around anywhere within the lake, but, and this is the point, they can't, under ordinary circumstances, escape the lake. They can go to every boundary of the body of water, you might even see some on the surface, but, they cannot leave the water. This is similar to the charge carriers in a conductor surrounded by vacuum or an insulating medium such as air. The charges can go everywhere in and on the conductor, but, they cannot leave the conductor.

The facts we have presented on the nature of charge, electric fields, and conductors allow one to draw some definite conclusions about the electric field and unbalanced charge within the material of, and at or on the surface of, an ideal conductor. Please try to reason out the answers to the following questions:

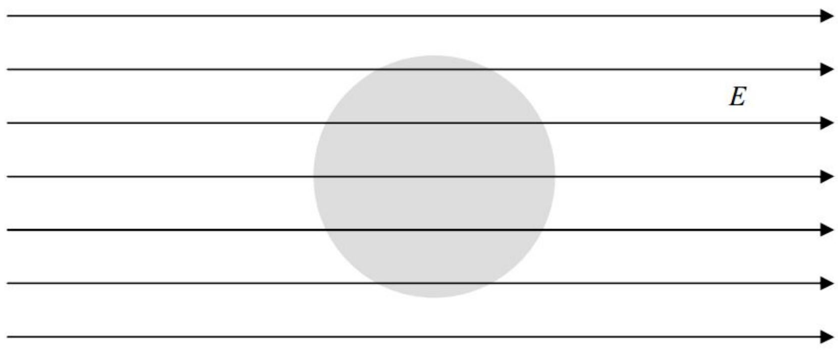
1. Suppose you put a neutral ideal conducting solid sphere in a region of space in which there is, initially, a uniform electric field. Describe (as specifically as possible) the electric field inside the conductor and the electric field at the surface of the conductor. Describe the distribution of charge in and on the conductor.
2. Repeat question 1 for the case of a non-uniform field.
3. Suppose you put some charge on an initially-neutral, solid, perfectly-conducting sphere (where the sphere is not in a pre-existing electric field). Describe the electric field inside the conductor, at the surface of the conductor, and outside the conductor as a result of the unbalanced charge. Describe the distribution of the charge in and on the conductor.
4. Repeat questions 1-3 for the case of a hollow perfectly-conducting spherical shell (with the interior being vacuum).
5. How would your answers to questions 1-4 change if the conductor had some shape other than spherical?

Here we provide the answers (preceded in each case, with the corresponding question).

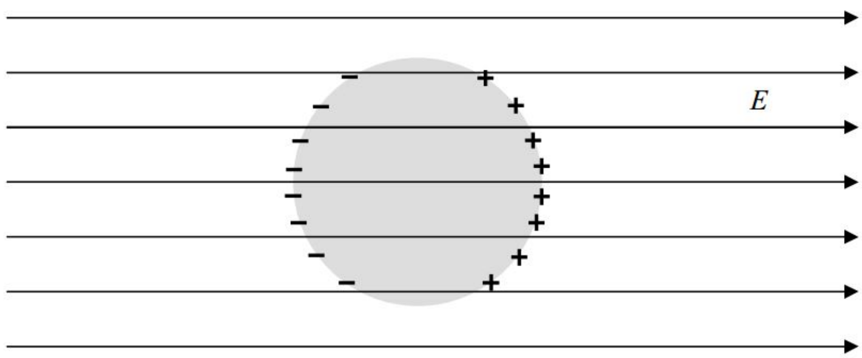
> 1) Suppose you put a neutral ideal conducting solid sphere in a region of space in which there is, initially, a uniform electric field. Describe (as specifically as possible) the electric field inside the conductor and the electric field at the surface of the conductor. Describe the distribution of charge in and on the conductor.

Answer: We start with a uniform electric field.

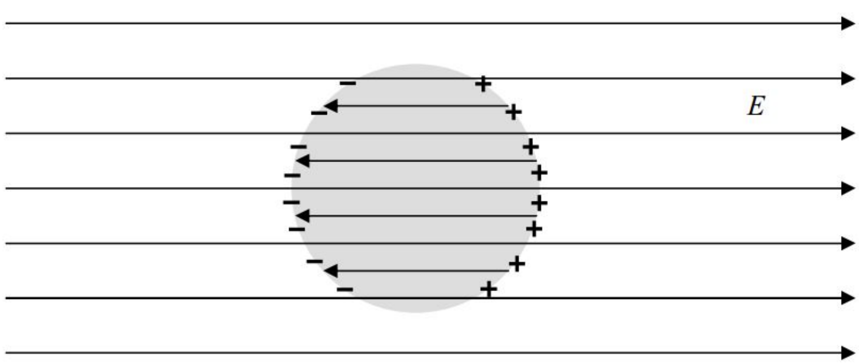
We put a solid, ideal conductor in it. The electric field permeates everything, including the conductor.



The charged particles in the conductor respond to the force exerted on them by the electric field. (The force causes acceleration, the acceleration of particles that are initially at rest causes them to acquire some velocity. In short, they move.) All this occurs in less than a microsecond. The net effect is a redistribution of the charged particles.

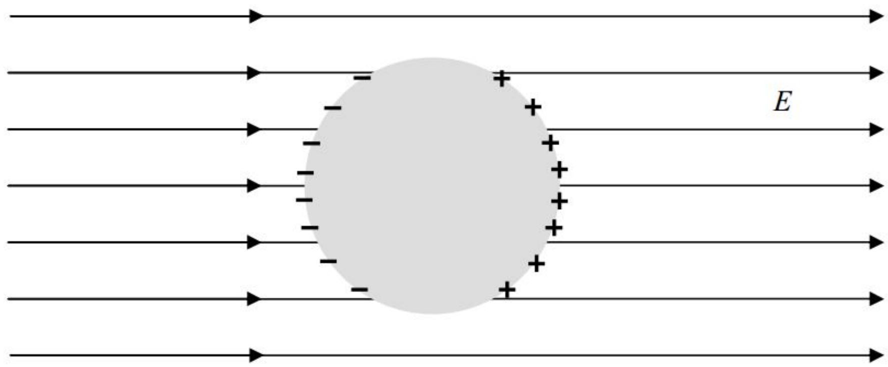


Now, get this! The charged particles create their own electric field.

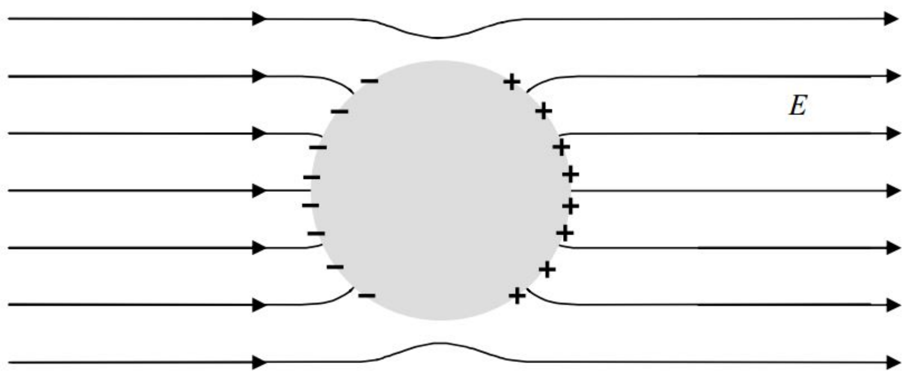


The total electric field at any point in the conductor is the vector sum of the original electric field and the electric field due to the redistributed charged particles. Since they are oppositely directed, the two contributions to the electric field inside the conductor tend to cancel each other. Now comes the profound part of the argument: the two contributions to the electric field at any point in the conductor exactly cancel. We know they have to completely cancel because, if they didn't, the free-to-move-charge in the conductor would move as a result of the force exerted on it by the electric field. And the force on the charge is always in a direction that causes the charge to be redistributed to positions in which it will create its own electric field that tends to cancel the electric field that caused the charge to move. The point is that the charge will not stop responding to the electric field until the net electric field at every point in the conductor is zero.

So far, in answer to the question, we have: The electric field is zero at all points inside the conductor, and, while the total charge is still zero, the charge has been redistributed as in the following diagram:

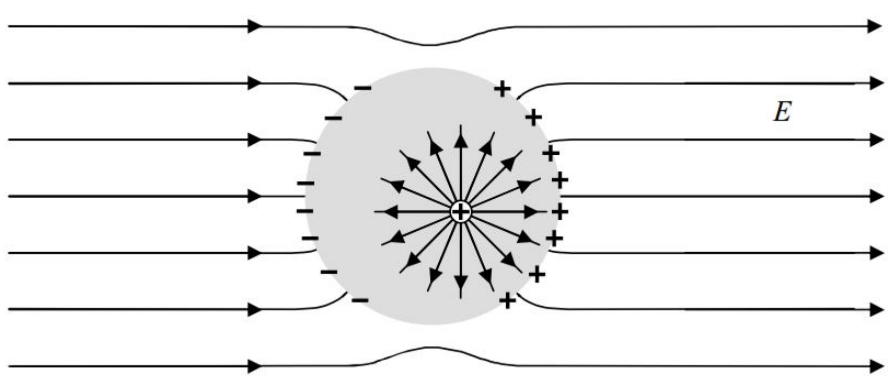


Recall that we were also called upon to describe the electric field at the surface of the conductor. Note that the charge on the surface of the sphere will not only contribute to the electric field inside the conductor, it will also contribute to the electric field outside. The net effect of all the contributions to the electric field in the near vicinity of the sphere is to cause the electric field to be normal to (perpendicular to) the surface of the sphere at all points where it meets the sphere.



How is it that we are able to assert this without doing any calculations? Here's the argument: If the electric field at the surface had a component parallel to the surface, then the charged particles on the surface of the conductor would experience a force directed along the surface. Since those particles are free to move anywhere in the conductor, they would be redistributed. In their new positions, they would make their own contribution to the electric field in the surface and their contribution would cancel the electric field that caused the charge redistribution.

About the charge distribution: The object started out neutral and no charge has left or entered the conductor from the outside world so it is still neutral. But we do see a separation of the two different kinds of charge. Something that we have depicted but not discussed is the assertion that all the charge resides on the surface. (In the picture above, there is positive charge on the right surface of the sphere and an equal amount of negative charge on the left side.) How do we know that all charge must be on the surface? Assume that there was a positive point charge at some location within the conductor:

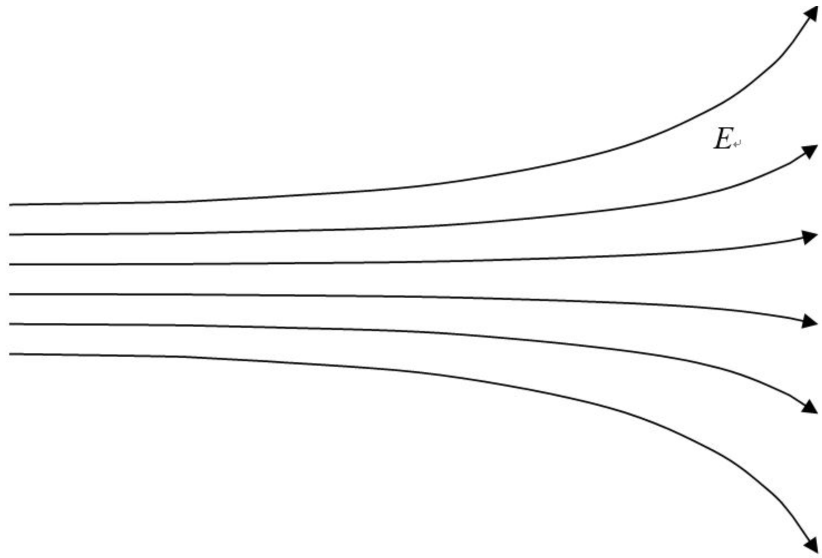


The electric field of that point charge would cause the free-to-move charge in the conductor to move, and it would keep moving as long as there was an electric field. So where would the charge move in order to cancel out the electric field of the positive point charge. You can try any arrangement of charge that you want to, around that positive point charge, but, if it is stipulated that there be a net positive charge at that location, there is no way to cancel out the electric field of that positive charge. So the situation doesn't even occur. If it did happen, the particle would repel the conductor's free-to-move-positive charge away from the stipulated positive charge, so that (excluding the stipulated positive charge under consideration) the conductor would have a net negative charge at that location, an amount of negative charge exactly equal to the originally stipulated positive charge. Taking the positive charge into account as well, the point, after the redistribution of charge, would be neutral. The point of our argument is that, under static conditions, there can be no net charge inside the material of a perfect conductor. Even if you assume there to be some, it would soon be neutralized by the nearly instantaneous charge redistribution that it would cause.

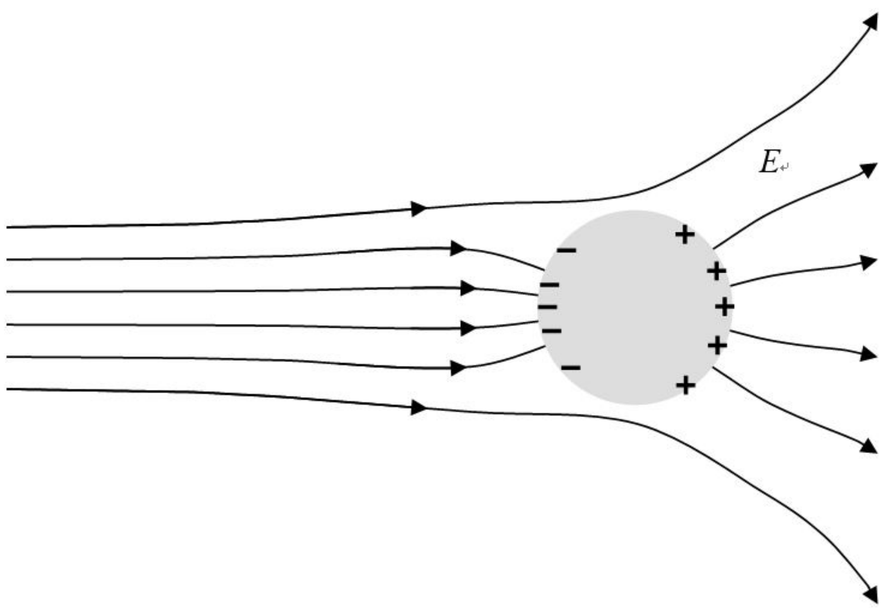
Next question:

> 2) Repeat question 1 for the case of a non-uniform field. (Question 1 asked for a description of the charge distribution that develops on a solid neutral conducting sphere when you place it in a uniform electric field.)

Answer: Here is a depiction of an example of a non-uniform field:



If we put a solid, perfectly-conducting sphere in it we get:

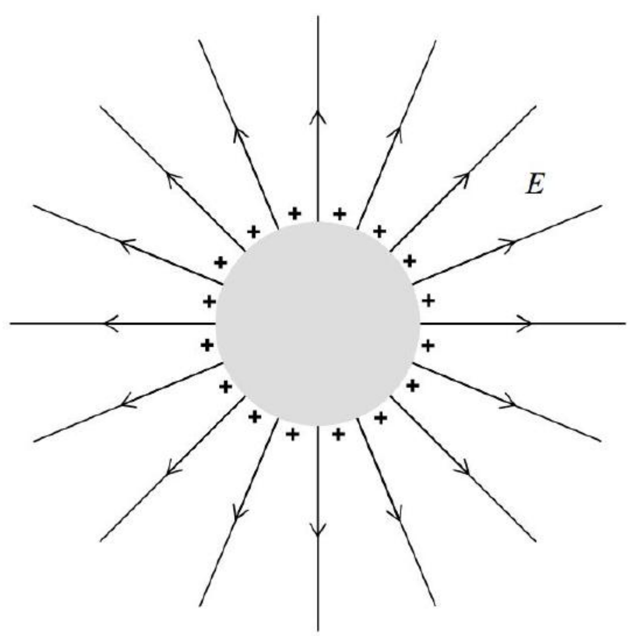


The same arguments lead to the same conclusions. When, after less than a microsecond, the new static conditions are achieved: There can be no electric field inside the conductor or else the free-to-move-charges in it would still be moving around within the volume of the conductor. There can be no unbalanced charge within the volume of the conductor or else there would be an electric field inside the conductor. Hence, any locally unbalanced charge (overall, the initially neutral sphere remains neutral) must be on the surface. The electric field has to be normal to the surface of the sphere or else the free-to-move-charge at the surface would still be moving around on the surface. The only thing that is different in this case, as compared to the initially-uniform electric field case, is the way the charge is distributed on the surface. We see that the negative charge is more bunched up than the positive charge in the case at hand. In the initially-uniform electric field case, the positive charge distribution was the mirror image of the negative charge distribution.

Next Question:

> 3) Suppose you put some charge on an initially-neutral, solid, perfectly-conducting sphere (where the sphere is not in a pre-existing electric field). Describe the electric field inside the conductor, at the surface of the conductor, and outside the conductor as a result of the unbalanced charge. Describe the distribution of the charge in and on the conductor.

Again, we assume that we have waited long enough (less than a microsecond) for static conditions to have been achieved. There can be no charge within the bulk of the conductor or else there would be an electric field in the conductor and there can't be an electric field in the conductor or else the conductor's free-to-move charge would move and static conditions would not be prevailing. So, all the unbalanced charge must be on the surface. It can't be bunched up more at any location on the surface than it is at any other location on the surface or else the charge on the edge of the bunch would be repelled by the bunch and it would move, again in violation of our stipulation that we have waited until charge stopped moving. So, the charge must be distributed uniformly over the surface of the sphere. Inside the sphere there is no electric field. Where the outside electric field meets the surface of the sphere, the electric field must be normal to the surface of the sphere. Otherwise, the electric field at the surface would have a vector component parallel to the surface which would cause charge to move along the surface, again in violation of our static conditions stipulations. Now, electric field lines that are perpendicular to the surface of a sphere lie on lines that pass through the center of the sphere. Hence, outside the sphere, the electric field lines form the same pattern as the pattern that would be formed by a point charge at the location of the center of the sphere (with the sphere gone). Furthermore, if you go so far away from the sphere that the sphere "looks like" a point, the electric field will be the same as that due to a point charge at the location of the center of the sphere. Given that outside the sphere, it has the same pattern as the field due to a point charge at the center of the sphere, the only way it can match up with the point charge field at a great distance from the sphere, is if it is identical to the point charge field everywhere that it exists. So, outside the sphere, the electric field is indistinguishable from the electric field due to the same amount of charge that you put on the sphere, all concentrated at the location of the center of the sphere (with the sphere gone).



Next question:

> 4) Repeat questions 1-3 for the case of a hollow perfectly-conducting spherical shell (with the interior being vacuum).

In all three cases we have considered so far, the interior of the sphere has played no role. It is initially neutral and it is neutral after the sphere is placed in a pre-existing electric field or some charge is placed on it. Nothing would change if we removed all that neutral material making up the bulk of the conductor, leaving nothing but a hollow shell of a sphere. Hence all the results that we found for the solid sphere apply to the hollow sphere. In particular, the electric field at all points inside an empty hollow perfectly-conducting spherical shell is, under all conditions, zero.

Last question:

> 5) How would your answers to questions 1-4 change if the conductor had some shape other than spherical?

For a solid perfect conductor, the electric field and the charge everywhere inside would have to be zero for the same reasons discussed above. Furthermore, the electric field would have to be normal to the surface for the same reasons as before. Again, it would not make any difference if we hollow out the conductor by removing a bunch of neutral material. The only things that would be different for a non-spherical conductor are the way the charge would be distributed on the surface, and, the outside electric field. In particular, if you put some charge on a perfectlyconducting object that is not a sphere, the electric field in the vicinity of the object will not be the same as the electric field due to a point charge at the center of the object (although the difference would be negligible at great enough distances from the object).

1.7: Calculating Electric Fields of Charge Distributions

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Explain what a continuous source charge distribution is and how it is related to the concept of quantization of charge
- Describe line charges, surface charges, and volume charges
- Calculate the field of a continuous source charge distribution of either sign

The charge distributions we have seen so far have been discrete: made up of individual point particles. This is in contrast with a **continuous charge distribution**, which has at least one nonzero dimension. If a charge distribution is continuous rather than discrete, we can generalize the definition of the electric field. We simply divide the charge into infinitesimal pieces and treat each piece as a point charge.

Note that because charge is quantized, there is no such thing as a "truly" continuous charge distribution. However, in most practical cases, the total charge creating the field involves such a huge number of discrete charges that we can safely ignore the discrete nature of the charge and consider it to be continuous. This is exactly the kind of approximation we make when we deal with a bucket of water as a continuous fluid, rather than a collection of H_2O molecules.

Our first step is to define a charge density for a charge distribution along a line, across a surface, or within a volume, as shown in Figure 1.7.1.

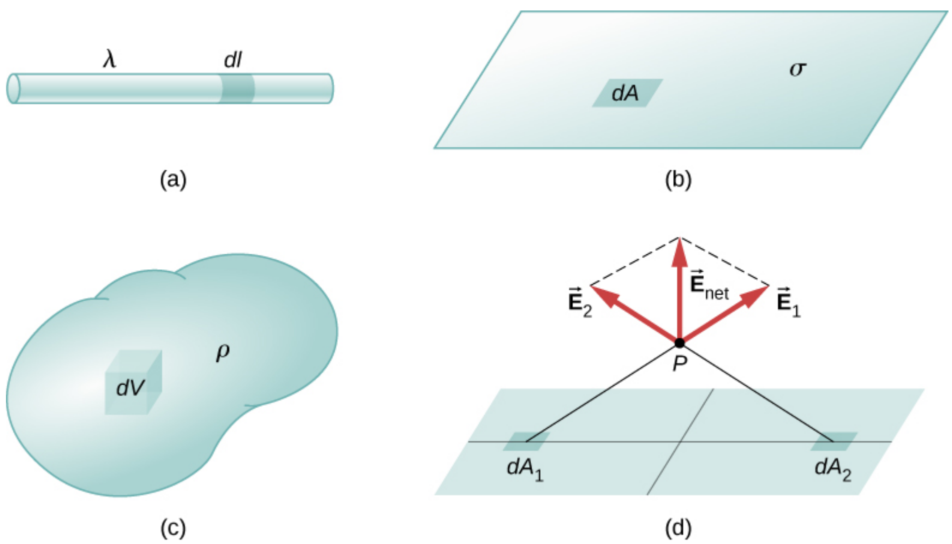


Figure 1.7.1: The configuration of charge differential elements for a (a) line charge, (b) sheet of charge, and (c) a volume of charge. Also note that (d) some of the components of the total electric field cancel out, with the remainder resulting in a net electric field.

Definitions: Charge Densities

Definitions of charge density:

- **linear charge density:** $\lambda \equiv$ charge per unit length (Figure 1.7.1a); units are coulombs per meter $(\mathrm{C/m})$
- **surface charge density:** $\sigma \equiv$ charge per unit area (Figure 1.7.1b); units are coulombs per square meter $(\mathrm{C/m^2})$
- **volume charge density:** $\rho \equiv$ charge per unit volume (Figure 1.7.1c); units are coulombs per square meter $(\mathrm{C/m^3})$

For a line charge, a surface charge, and a volume charge, the summation in the definition of an Electric field discussed previously becomes an integral and q_i is replaced by $dq = \lambda\, dl$, $\sigma\, dA$, or $\rho\, dV$, respectively:

$$\vec{E}(P)=\underbrace{\frac{1}{4\pi\epsilon_0}\sum_{i=1}^{N}\left(\frac{q_i}{r^2}\right)\hat{r}}_{\text{Point charges}} \tag{1.7.1}$$

$$\vec{E}(P)=\underbrace{\frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda\,dl}{r^2}\right)\hat{r}}_{\text{Line charge}} \tag{1.7.2}$$

$$\vec{E}(P)=\underbrace{\frac{1}{4\pi\epsilon_0}\int_{\text{surface}}\left(\frac{\sigma\,dA}{r^2}\right)\hat{r}}_{\text{Surface charge}} \tag{1.7.3}$$

$$\vec{E}(P)=\underbrace{\frac{1}{4\pi\epsilon_0}\int_{\text{volume}}\left(\frac{\rho\,dV}{r^2}\right)\hat{r}}_{\text{Volume charge}} \tag{1.7.4}$$

The integrals in Equations 1.7.1-1.7.4 are generalizations of the expression for the field of a point charge. They implicitly include and assume the principle of superposition. The "trick" to using them is almost always in coming up with correct expressions for dl, dA, or dV, as the case may be, expressed in terms of **r**, and also expressing the charge density function appropriately. It may be constant; it might be dependent on location.

Note carefully the meaning of r in these equations: It is the distance from the charge element (q_i, $\lambda\,dl$, $\sigma\,dA$, $\rho\,dV$) to the location of interest, $P(x,y,z)$ (the point in space where you want to determine the field). However, don't confuse this with the meaning of $\hat{r}$; we are using it and the vector notation $\vec{E}$ to write three integrals at once. That is, Equation 1.7.2 is actually

$$E_x(P)=\frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda\,dl}{r^2}\right)_x, \tag{1.7.5}$$

$$E_y(P)=\frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda\,dl}{r^2}\right)_y, \tag{1.7.6}$$

$$E_z(P)=\frac{1}{4\pi\epsilon_0}\int_{\text{line}}\left(\frac{\lambda\,dl}{r^2}\right)_z \tag{1.7.7}$$

Example 1.7.1: Electric Field of a Line Segment

Find the electric field a distance z above the midpoint of a straight line segment of length L that carries a uniform line charge density λ.

Strategy

Since this is a continuous charge distribution, we conceptually break the wire segment into differential pieces of length dl, each of which carries a differential amount of charge

$$dq=\lambda\,dl.$$

Then, we calculate the differential field created by two symmetrically placed pieces of the wire, using the symmetry of the setup to simplify the calculation (Figure 1.7.2). Finally, we integrate this differential field expression over the length of the wire (half of it, actually, as we explain below) to obtain the complete electric field expression.

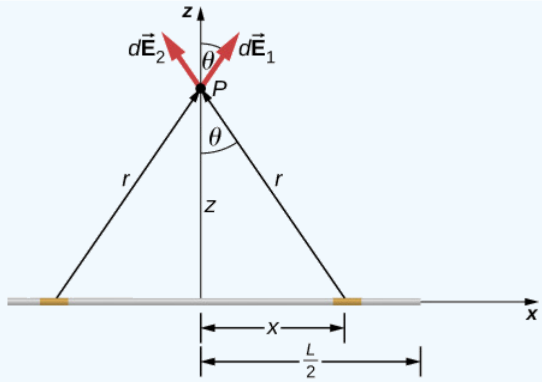


Figure 1.7.2: A uniformly charged segment of wire. The electric field at point P can be found by applying the superposition principle to symmetrically placed charge elements and integrating.

Solution

Before we jump into it, what do we expect the field to "look like" from far away? Since it is a finite line segment, from far away, it should look like a point charge. We will check the expression we get to see if it meets this expectation.

The electric field for a line charge is given by the general expression

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}} \frac{\lambda dl}{r^2}\hat{r}.$$

The symmetry of the situation (our choice of the two identical differential pieces of charge) implies the horizontal (x)-components of the field cancel, so that the net field points in the z-direction. Let's check this formally.

The total field $\vec{E}(P)$ is the vector sum of the fields from each of the two charge elements (call them $\vec{E}_1$ and $\vec{E}_2$, for now):

$$\begin{aligned}\vec{E}(P) &= \vec{E}_1 + \vec{E}_2 \\ &= E_{1x}\hat{i} + E_{1z}\hat{k} + E_{2x}(-\hat{i}) + E_{2z}\hat{k}.\end{aligned}$$

Because the two charge elements are identical and are the same distance away from the point P where we want to calculate the field, $E_{1x} = E_{2x}$, so those components cancel. This leaves

$$\begin{aligned}\vec{E}(P) &= E_{1z}\hat{k} + E_{2z}\hat{k} \\ &= E_1\,\cos\theta\,\hat{k} + E_2\,\cos\theta\,\hat{k}.\end{aligned}$$

These components are also equal, so we have

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\int\frac{\lambda dl}{r^2}\cos\theta\,\hat{k} + \frac{1}{4\pi\epsilon_0}\int\frac{\lambda dl}{r^2}\cos\theta\,\hat{k} \\ &= \frac{1}{4\pi\epsilon_0}\int_0^{L/2}\frac{2\lambda dx}{r^2}\cos\theta\,\hat{k}\end{aligned}$$

where our differential line element dl is dx, in this example, since we are integrating along a line of charge that lies on the x-axis. (The limits of integration are 0 to $\frac{L}{2}$, not $-\frac{L}{2}$ to $+\frac{L}{2}$, because we have constructed the net field from two differential pieces of charge dq. If we integrated along the entire length, we would pick up an erroneous factor of 2.)

In principle, this is complete. However, to actually calculate this integral, we need to eliminate all the variables that are not given. In this case, both r and θ change as we integrate outward to the end of the line charge, so those are the variables to get rid of. We can do that the same way we did for the two point charges: by noticing that

$$r = (z^2 + x^2)^{1/2}$$

and

$$\cos\theta = \frac{z}{r} = \frac{z}{(z^2+x^2)^{1/2}}.$$

Substituting, we obtain

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\int_0^{L/2}\frac{2\lambda dx}{(z^2+x^2)}\frac{z}{(z^2+x^2)^{1/2}}\hat{k} \\ &= \frac{1}{4\pi\epsilon_0}\int_0^{L/2}\frac{2\lambda z}{(z^2+x^2)^{3/2}}dx\,\hat{k} \\ &= \frac{2\lambda z}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]_0^{L/2}\hat{k}.\end{aligned}$$

which simplifies to

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z\sqrt{z^2+\frac{L^2}{4}}}\hat{k}. \tag{1.7.8}$$

Significance

Notice, once again, the use of symmetry to simplify the problem. This is a very common strategy for calculating electric fields. The fields of nonsymmetrical charge distributions have to be handled with multiple integrals and may need to be calculated numerically by a computer.

? Exercise 1.7.1

How would the strategy used above change to calculate the electric field at a point a distance z above one end of the finite line segment?

Answer

We will no longer be able to take advantage of symmetry. Instead, we will need to calculate each of the two components of the electric field with their own integral.

✓ Example 1.7.2: Electric Field of an Infinite Line of Charge

Find the electric field a distance z above the midpoint of an infinite line of charge that carries a uniform line charge density λ.

Strategy

This is exactly like the preceding example, except the limits of integration will be $-\infty$ to $+\infty$.

Solution

Again, the horizontal components cancel out, so we wind up with

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda dx}{r^2}\cos\,\theta\hat{k}$$

where our differential line element dl is dx, in this example, since we are integrating along a line of charge that lies on the x-axis. Again,

$$\begin{aligned}\cos\theta &= \frac{z}{r} \\ &= \frac{z}{(z^2+x^2)^{1/2}}.\end{aligned}$$

Substituting, we obtain

$$\begin{aligned}\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda dx}{(z^2+x^2)}\frac{z}{(z^2+x^2)^{1/2}}\hat{k} \\ &= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty}\frac{\lambda z}{(z^2+x^2)^{3/2}}dx\,\hat{k} \\ &= \frac{1}{4\pi\epsilon_0}\left[\frac{x}{z^2\sqrt{z^2+x^2}}\right]_{-\infty}^{\infty}\hat{k}\end{aligned}$$

which simplifies to

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{k}.$$

Significance

Our strategy for working with continuous charge distributions also gives useful results for charges with infinite dimension.

In the case of a finite line of charge, note that for $z \gg L$, z^2 dominates the L in the denominator, so that Equation 1.7.8 simplifies to

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0}\frac{\lambda L}{z^2}\hat{k}.$$

If you recall that $\lambda L = q$ the total charge on the wire, we have retrieved the expression for the field of a point charge, as expected.

In the limit $L \to \infty$ on the other hand, we get the field of an **infinite straight wire**, which is a straight wire whose length is much, much greater than either of its other dimensions, and also much, much greater than the distance at which the field is to be calculated:

$$\vec{E}(z) = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{z}\hat{k}. \tag{1.7.9}$$

An interesting artifact of this infinite limit is that we have lost the usual $1/r^2$ dependence that we are used to. This will become even more intriguing in the case of an infinite plane.

Example 1.7.3A: Electric Field due to a Ring of Charge

A ring has a uniform charge density λ, with units of coulomb per unit meter of arc. Find the electric field at a point on the axis passing through the center of the ring.

Strategy

We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle and use polar coordinates shown in Figure 1.7.3.

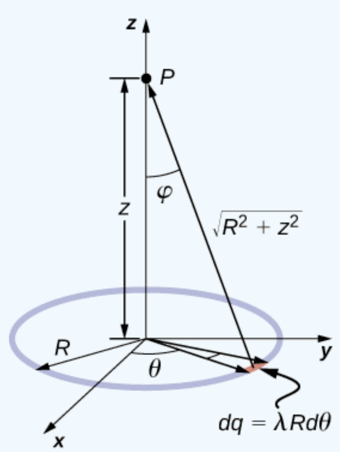


Figure 1.7.3: The system and variable for calculating the electric field due to a ring of charge.

Solution

The electric field for a line charge is given by the general expression

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{line}} \frac{\lambda dl}{r^2}\hat{r}.$$

A general element of the arc between θ and $\theta + d\theta$ is of length $R\,d\theta$ and therefore contains a charge equal to $\lambda R\,d\theta$. The element is at a distance of $r = \sqrt{z^2 + R^2}$ from P, the angle is $\cos\phi = \frac{z}{\sqrt{z^2+R^2}}$ and therefore the electric field is

$$\begin{aligned}
\vec{E}(P) &= \frac{1}{4\pi\epsilon_0}\int_{\text{line}} \frac{\lambda dl}{r^2}\hat{r} = \frac{1}{4\pi\epsilon_0}\int_0^{2\pi} \frac{\lambda R d\theta}{z^2+R^2}\frac{z}{\sqrt{z^2+R^2}}\hat{z} \\
&= \frac{1}{4\pi\epsilon_0}\frac{\lambda R z}{(z^2+R^2)^{3/2}}\hat{z}\int_0^{2\pi} d\theta \\
&= \frac{1}{4\pi\epsilon_0}\frac{2\pi\lambda R z}{(z^2+R^2)^{3/2}}\hat{z} \\
&= \frac{1}{4\pi\epsilon_0}\frac{q_{tot} z}{(z^2+R^2)^{3/2}}\hat{z}.
\end{aligned}$$

Significance

As usual, symmetry simplified this problem, in this particular case resulting in a trivial integral. Also, when we take the limit of $z \gg R$, we find that

$$\vec{E} \approx \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{z^2}\hat{z},$$

as we expect.

Example 1.7.3B: The Field of a Disk

Find the electric field of a circular thin disk of radius R and uniform charge density at a distance z above the center of the disk (Figure 1.7.4)

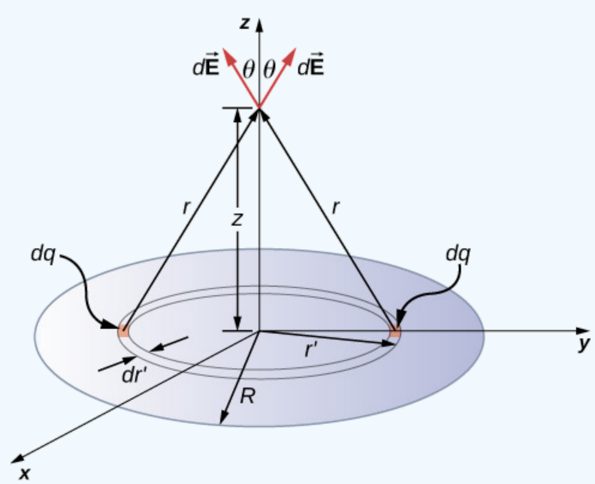


Figure 1.7.4: A uniformly charged disk. As in the line charge example, the field above the center of this disk can be calculated by taking advantage of the symmetry of the charge distribution.

Strategy

The electric field for a surface charge is given by

$$\vec{E}(P) = \frac{1}{4\pi\epsilon_0}\int_{\text{surface}} \frac{\sigma\, dA}{r^2}\hat{r}.$$

To solve surface charge problems, we break the surface into symmetrical differential "stripes" that match the shape of the surface; here, we'll use rings, as shown in the figure. Again, by symmetry, the horizontal components cancel and the field is entirely in the vertical $(\hat{k})$ direction. The vertical component of the electric field is extracted by multiplying by θ, so

$$\vec{E}(P)=\frac{1}{4\pi\epsilon_0}\int_{surface}\frac{\sigma\, dA}{r^2}\cos\theta\,\hat{k}.$$

As before, we need to rewrite the unknown factors in the integrand in terms of the given quantities. In this case,

$$dA=2\pi r'\,dr'$$

$$r^2=r'^2+z^2$$

$$\cos\theta=\frac{z}{(r'^2+z^2)^{1/2}}.$$

(Please take note of the two different "r's" here; r is the distance from the differential ring of charge to the point P where we wish to determine the field, whereas r' is the distance from the center of the disk to the differential ring of charge.) Also, we already performed the polar angle integral in writing down dA.

Solution

Substituting all this in, we get

$$\begin{aligned}\vec{E}(P)&=\vec{E}(z)\\ &=\frac{1}{4\pi\epsilon_0}\int_0^R\frac{\sigma(2\pi r'\,dr')z}{(r'^2+z^2)^{3/2}}\hat{k}\\ &=\frac{1}{4\pi\epsilon_0}(2\pi\sigma z)\left(\frac{1}{z}-\frac{1}{\sqrt{R^2+z^2}}\right)\hat{k}\end{aligned}$$

or, more simply,

$$\vec{E}(z)=\frac{1}{4\pi\epsilon_0}\left(2\pi\sigma-\frac{2\pi\sigma z}{\sqrt{R^2+z^2}}\right)\hat{k}. \tag{1.7.10}$$

Significance

Again, it can be shown (via a Taylor expansion) that when $z\gg R$, this reduces to

$$\vec{E}(z)\approx\frac{1}{4\pi\epsilon_0}\frac{\sigma\pi R^2}{z^2}\hat{k},$$

which is the expression for a point charge $Q=\sigma\pi R^2$.

? Exercise 1.7.3

How would the above limit change with a uniformly charged rectangle instead of a disk?

Answer

The point charge would be $Q=\sigma ab$ where a and b are the sides of the rectangle but otherwise identical.

As $R\rightarrow\infty$, Equation 1.7.10 reduces to the field of an infinite plane, which is a flat sheet whose area is much, much greater than its thickness, and also much, much greater than the distance at which the field is to be calculated:

$$\vec{E}=\lim_{R\rightarrow\infty}\frac{1}{4\pi\epsilon_0}\left(2\pi\sigma-\frac{2\pi\sigma z}{\sqrt{R^2+z^2}}\right)\hat{k} \tag{1.7.11}$$

$$=\frac{\sigma}{2\epsilon_0}\hat{k}. \tag{1.7.12}$$

Note that this field is constant. This surprising result is, again, an artifact of our limit, although one that we will make use of repeatedly in the future. To understand why this happens, imagine being placed above an infinite plane of constant charge. Does the plane look any different if you vary your altitude? No—you still see the plane going off to infinity, no matter how far you are from it. It is important to note that Equation 1.7.12 is because we are above the plane. If we were below, the field would point in the $-\hat{k}$ direction.

✔ Example 1.7.4: The Field of Two Infinite Planes

Find the electric field everywhere resulting from two infinite planes with equal but opposite charge densities (Figure 1.7.5).

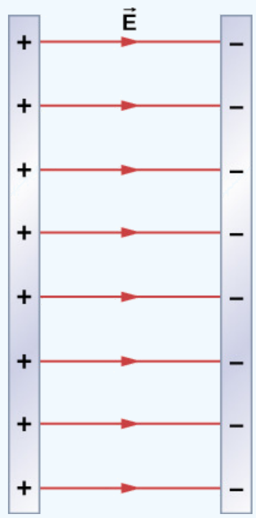


Figure 1.7.5: Two charged infinite planes. Note the direction of the electric field.

Strategy

We already know the electric field resulting from a single infinite plane, so we may use the principle of superposition to find the field from two.

Solution

The electric field points away from the positively charged plane and toward the negatively charged plane. Since the σ are equal and opposite, this means that in the region outside of the two planes, the electric fields cancel each other out to zero. However, in the region between the planes, the electric fields add, and we get

$$\vec{E} = \frac{\sigma}{\epsilon_0}\hat{i}$$

for the electric field. The $\hat{i}$ is because in the figure, the field is pointing in the $+x$-direction.

Significance

Systems that may be approximated as two infinite planes of this sort provide a useful means of creating uniform electric fields.

? Exercise 1.7.1

What would the electric field look like in a system with two parallel positively charged planes with equal charge densities?

Answer

The electric field would be zero in between, and have magnitude $\frac{\sigma}{\epsilon_0}$ everywhere else.

1.7.1: The Electric Field Due to a Continuous Distribution of Charge on a Line

> *Every integral must include a differential (such as dx, dt, dq, etc.). An integral is an infinite sum of terms. The differential is necessary to make each term infinitesimal (vanishingly small).* $\int f(x) dx$ *is okay,* $\int g(y) dy$ *is okay, and* $\int h(t) dt$ *is okay, but never write* $\int f(x)$*, never write* $\int g(y)$ *and never write* $\int h(t)$*.*

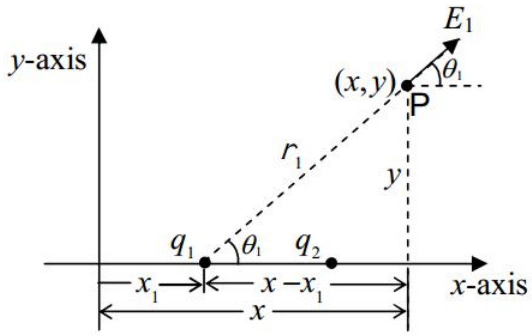
Here we revisit Coulomb's Law for the Electric Field. Recall that Coulomb's Law for the Electric Field gives an expression for the electric field, at an empty point in space, due to a charged particle. You have had practice at finding the electric field at an empty point in space due to a single charged particle and due to several charged particles. In the latter case, you simply calculated the contribution to the electric field at the one empty point in space due to each charged particle, and then added the individual contributions. You were careful to keep in mind that each contribution to the electric field at the empty point in space was an electric field vector, a vector rather than a scalar, hence the individual contributions had to be added like vectors.

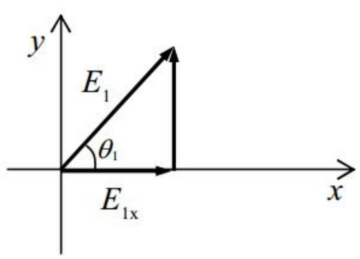
A Review Problem for the Electric Field due to a Discrete Distribution of Charge

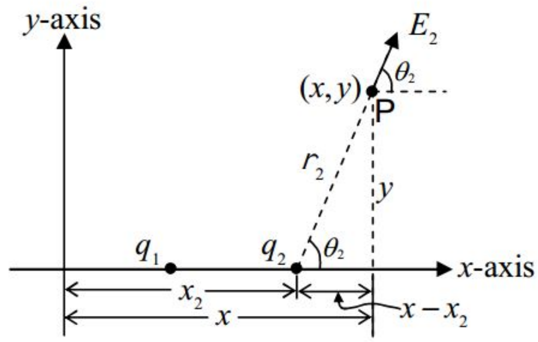
Let's kick this chapter off by doing a review problem. The following example is one of the sort that you learned how to do when you first encountered Coulomb's Law for the Electric Field. You are given a discrete distribution of source charges and asked to find the electric field (in the case at hand, just the x component of the electric field) at an empty point in space.

The example is presented on the next page. Here, a word about one piece of notation used in the solution. The symbol P is used to identify a point in space so that the writer can refer to that point, unambiguously, as "point P." The symbol P in this context does not stand for a variable or a constant. It is just an identification tag. It has no value. It cannot be assigned a value. It does not represent a distance. It just labels a point.

There are two charged particles on the x-axis of a Cartesian coordinate system, q_1 at $x = x_1$ and q_2 at $x = x_2$ where $x_2 > x_1$. Find the x component of the electric field, due to this pair of particles, valid for all points on the x-y plane for which $x > x_2$.







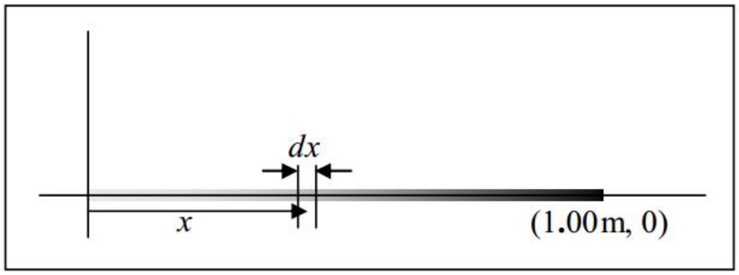
Linear Charge Density

Okay, enough review, now lets consider the case in which we have a continuous distribution of charge along some line segment. In practice, we could be talking about a charged piece of string or thread, a charged thin rod, or even a charged piece of wire. First we need to discuss how one even specifies such a situation. We do so by stating what the linear charge density, the charge per-length, λ is. For now we'll consider the meaning of λ for a few different situations (before we get to the heart of the matter, finding the electric field due to the linear charge distribution). Suppose for instance we have a one-meter string extending from the origin to $x = 1.00m$ along the x axis, and that the linear charge density on that string is given by:

$$\lambda = 1.56 \frac{\mu}{m^2} x.$$

(Just under the equation, we have depicted the linear charge density graphically by drawing a line whose darkness represents the charge density.)

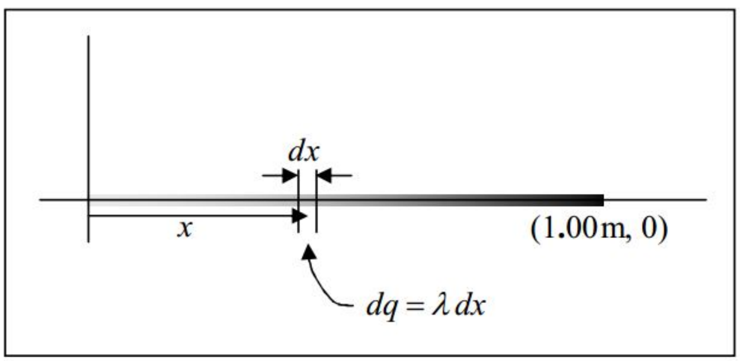
Note that if the value of x is expressed in meters, λ will have units of $\frac{\mu C}{m}$, units of charge-per-length, as it must. Further note that for small values of x, λ is small, and for larger values of x, λ is larger. That means that the charge is more densely packed near the far (relative to the origin) end of the string. To further familiarize ourselves with what λ is, let's calculate the total amount of charge on the string segment. What we'll do is to get an expression for the amount of charge on any infinitesimal length dx of the string, and add up all such amounts of charge for all of the infinitesimal lengths making up the string segment.



The infinitesimal amount of charge dq on the infinitesimal length dx of the string is just the charge per length λ times the length dx of the infinitesimal string segment.

$$dq = \lambda dx$$

Note that you can't take the amount of charge on a finite length (such as $15 cm$) of the string to be λ times the length of the segment because λ varies over the length of the segment. In the case of an infinitesimal segment, every part of it is within an infinitesimal distance of the position specified by one and the same value of x. The linear charge density doesn't vary on an infinitesimal segment because x doesn't—the segment is simply too short.



To get the total charge we just have to add up all the dq's. Each dq is specified by its corresponding value of x. To cover all the dq's we have to take into account all the values of x from 0 to 1.00 m. Because each dq is the charge on an infinitesimal length of the line of charge, the sum is going to have an infinite number of terms. An infinite sum of infinitesimal pieces is an integral. When we integrate

$$dq = \lambda dx$$

we get, on the left, the sum of all the infinitesimal pieces of charge making up the whole. By definition, the sum of all the infinitesimal amounts of charge is just the total charge Q (which by the way, is what we are solving for); we don't need the tools of integral calculus to deal with the left side of the equation. Integrating both sides of the equation yields:

$$Q = \int_0^{1.00m} \lambda dx$$

Using the given expression $\lambda = 1.56 \frac{\mu}{m^2} x$ we obtain

$$Q = \int_0^{1.00m} 2.56 \frac{\mu C}{m^2} x dx = 2.56 \frac{\mu C}{m^2} \int_0^{1.00m} x dx = 2.56 \frac{\mu C}{m^2} \frac{x^2}{2} \Big|_0^{1.00m} = 2.56 \frac{\mu C}{m^2} \left[\frac{(1.00m)^2}{2} - \frac{(0)^2}{2} \right] = 1.28 \mu C$$

A few more examples of distributions of charge follow:

For instance, consider charge distributed along the x axis, from $x=0$ to $x=L$ for the case in which the charge density is given by

$$\lambda=\lambda_{MAX}\sin(\pi\text{rad}x/L$$

where λ_{MAX} is a constant having units of charge-per-length, rad stands for the units radians, x is the position variable, and L is the length of the charge distribution. Such a charge distribution has a maximum charge density equal to λ_{MAX} occurring in the middle of the line segment.

Another example would be a case in which charge is distributed on a line segment of length L extending along the y axis from $y=a$ to $y=a+L$ with a being a constant and the charge density given by

$$\lambda=\frac{38\mu C\cdot m}{y^2}$$

In this case the charge on the line is more densely packed in the region closer to the origin. (The smaller y is, the bigger the value of λ, the charge-per-length.)

The simplest case is the one in which the charge is spread out uniformly over the line on which there is charge. In the case of a uniform linear charge distribution, the charge density is the same everywhere on the line of charge. In such a case, the linear charge density λ is simply a constant. Furthermore, in such a simple case, and only in such a simple case, the charge density λ is just the total amount of charge Q divided by the length L of the line along which that charge is uniformly distributed. For instance, suppose you are told that an amount of charge $Q=2.45C$ is uniformly distributed along a thin rod of length $L=0.840m$. Then λ is given by:

$$\lambda=\frac{Q}{L}$$

$$\lambda=\frac{2.45C}{0.840m}$$

$$\lambda=2.92\frac{C}{m}$$

The Electric Field Due to a Continuous Distribution of Charge along a Line

Okay, now we are ready to get down to the nitty-gritty. We are given a continuous distribution of charge along a straight line segment and asked to find the electric field at an empty point in space in the vicinity of the charge distribution. We will consider the case in which both the charge distribution and the empty point in space lie in the x-y plane. The values of the coordinates of the empty point in space are not necessarily specified. We can call them x and y. In solving the problem for a single point in space with unspecified coordinates (x,y), our final answer will have the symbols x and y in it, and our result will actually give the answer for an infinite set of points on the x-y plane.

The plan for solving such a problem is to find the electric field, due to an infinitesimal segment of the charge, at the one empty point in space. We do that for every infinitesimal segment of the charge, and then add up the results to get the total electric field.

Now once we chop up the charge distribution (in our mind, for calculational purposes) into infinitesimal (vanishingly small) pieces, we are going to wind up with an infinite number of pieces and hence an infinite sum when we go to add up the contributions to the electric field at the one single empty point in space due to all the infinitesimal segments of the linear charge distribution. That is to say, the result is going to be an integral.

An important consideration that we must address is the fact that the electric field, due to each element of charge, at the one empty point in space, is a vector. Hence, what we are talking about is an infinite sum of infinitesimal vectors. In general, the vectors being added are all in different directions from each other. (Can you think of a case so special that the infinite set of infinitesimal electric field vectors are all in the same direction as each other? Note that we are considering the general case, not such a special case.) We know better than to simply add the magnitudes of the vectors, infinite sum or not. Vectors that are not all in the same direction as each other, add like vectors, not like numbers. The thing is, however, the x components of all the infinitesimal electric field vectors at the one empty point in space do add like numbers. Likewise for the y components. Thus, if, for each infinitesimal element of the charge distribution, we find, not just the electric field at the empty point in space, but the x component of that electric field, then we can add up all the x components of the electric field at the empty point in space to get the x component of the electric field, due to the entire charge distribution, at the one empty point in space. The sum is still an infinite sum, but this time it is an infinite sum of scalars rather than vectors, and we have the tools for handling that. Of course, if we are asked for the total electric field, we have to repeat the entire procedure to get the y component of the electric field and then combine the two components of the electric field to get the total.

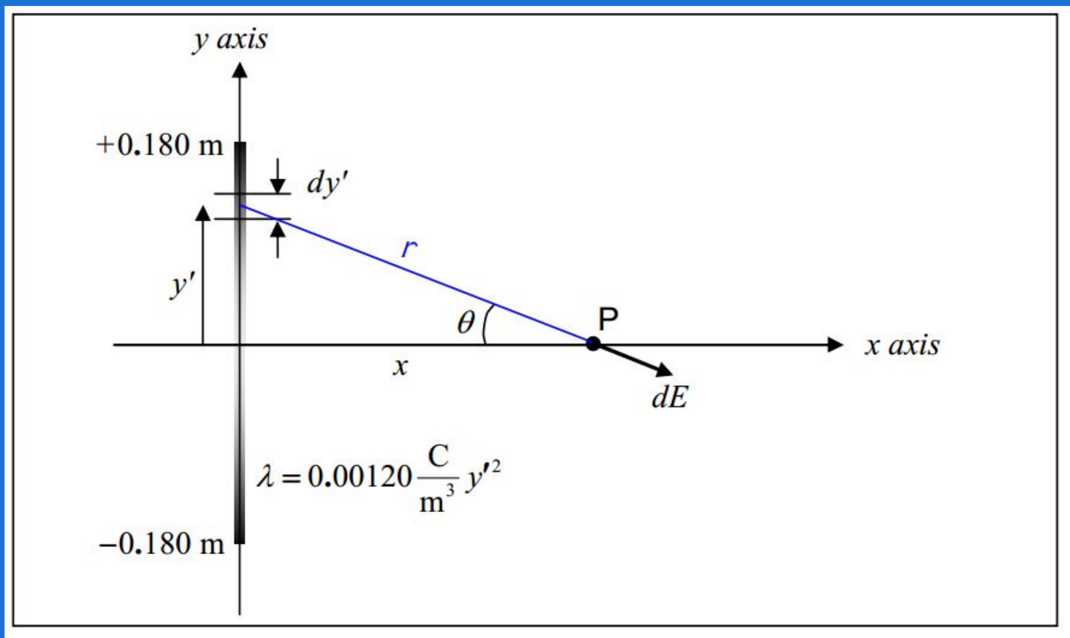
The easy way to do the last step is to use $\hat{\imath}, \hat{\jmath}, \hat{k}$ notation. That is, once we have E_x and E_y, we can simply write:

$$\vec{E} = E_x \hat{\imath} + E_y \hat{\jmath}$$

Find the electric field valid for any point on the positive x axis due a $36.0cm$ long line of charge, lying on the y axis and centered on the origin, for which the charge density is given by

$$\lambda = 0.00120\frac{C}{m^2}y^2$$

As usual, we'll start our solution with a diagram:



Note that we use (and strongly recommend that you use) primed quantities (x', y') to specify a point on the charge distribution and unprimed quantities (x, y) to specify the empty point in space at which we wish to know the electric field. Thus, in the diagram, the infinitesimal segment of the charge distribution is at $(0, y')$ and point P, the point at which we are finding the electric field, is at $(x, 0)$. Also, our expression for the given linear charge density $\lambda = 0.00120\frac{C}{m^2}y^2$ expressed in terms of y' rather than y is:

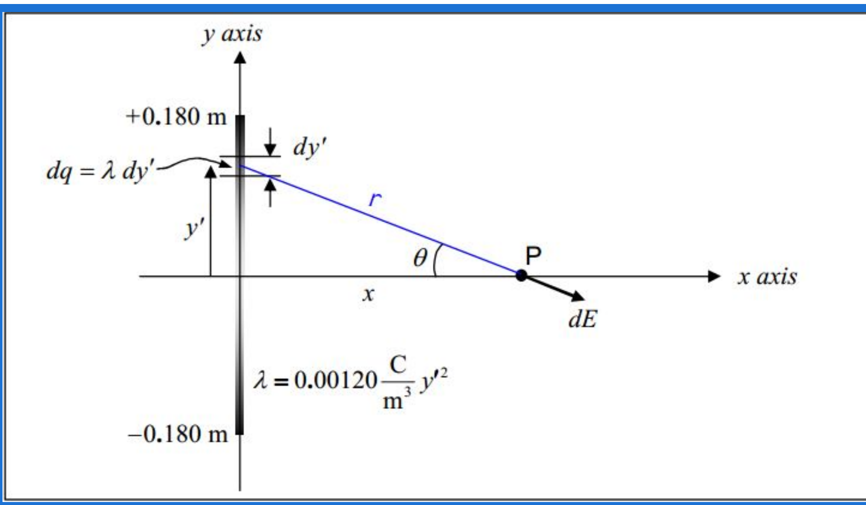
$$\lambda = 0.00120\frac{C}{m^2}y'^2$$

The plan here is to use Coulomb's Law for the Electric Field to get the magnitude of the infinitesimal electric field vector $\overrightarrow{dE}$ at point P due to the infinitesimal amount of charge dq in the infinitesimal segment of length dy'.

$$dE = \frac{kdq}{r^2}$$

The amount of charge dq in the infinitesimal segment dy' of the linear charge distribution is given by

$$dq = \lambda dy'$$



From the diagram, it clear that we can use the Pythagorean theorem to express the distance r that point P is from the infinitesimal amount of charge dq under consideration as:

$$r=\sqrt{x^2+y'^2}$$

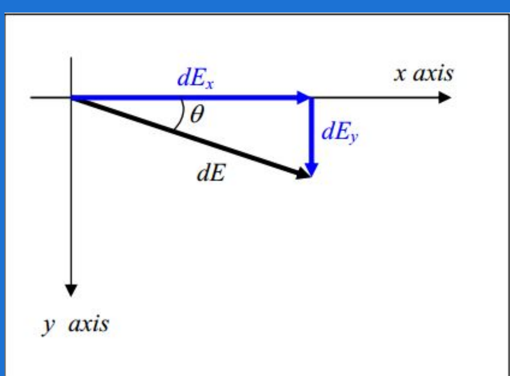
Substituting this and $dq=\lambda dy'$ into our equation for dE ($dE=\frac{kdq}{r^2}$) we obtain

$$dE=\frac{k\lambda dy'}{x^2+y'^2}$$

Recall that our plan is to find E_x, then E_y and then put them together using $\vec{E}=E_x\hat{\imath}+E_y\hat{\jmath}$. So for now, let's get an expression for E_x.

Based on the vector component diagram at right we have

$$dE_x=dE\cos\theta$$



The θ appearing in the diagram at right is the same θ that appears in the diagram above. Based on the plane geometry evident in that diagram (above), we have:

$$\cos\theta=\frac{x}{r}=\frac{x}{\sqrt{x^2+y'^2}}$$

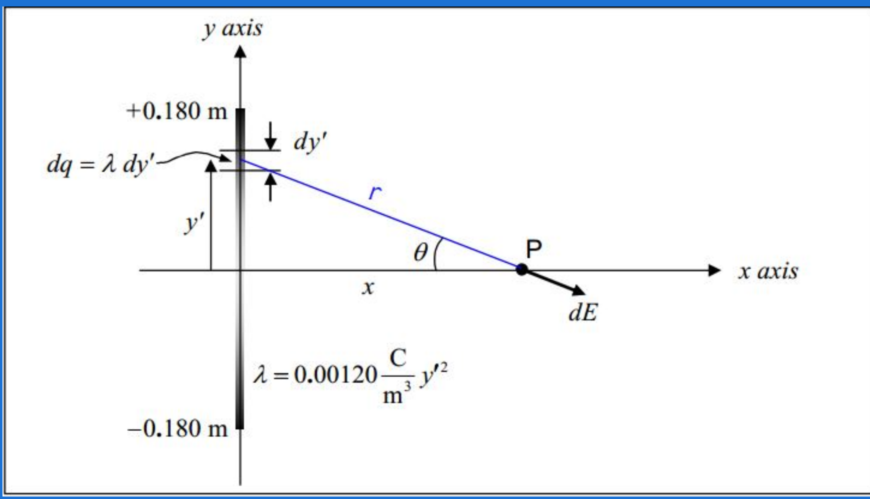
Substituting both this expression for $\cos\theta (\cos\theta=\frac{x}{\sqrt{x^2+y'^2}})$ and the expression we derived for dE above $(dE=\frac{k\lambda dy'}{(x^2+y'^2)^2})$ into the expression $dE_x=dE\cos\theta$ from the vector component diagram yields:

$(dE = \frac{k\lambda dy'}{(x^2+y'^2)^{\frac{3}{2}}})$

Also, let's go ahead and replace λ with the given expression $\lambda = 0.00120\frac{C}{m^3}y'^2$:

$$dE_x = \left(0.00120\frac{C}{m^3}\right)\frac{ky'^2 x dy'}{(x^2+y'^2)^{\frac{3}{2}}}$$

Now we have an expression for dE_x that includes only one quantity, namely y', that depends on which bit of the charge distribution is under consideration. Furthermore, although in the diagram



it appears that we picked out a particular infinitesimal line segment dy', in fact, the value of y' needed to establish its position is not specified. That is, we have an equation for dE_x that is good for any infinitesimal segment dy' of the given linear charge distribution. To identify a particular dy' we just have to specify the value of y'. Thus to sum up all the dE_x's we just have to add, to a running total, the dE_x for each of the possible values of y'. Thus we need to integrate the expression for dE_x for all the values of y' from $-0.180m$ to $+0.180m$.

$$\int dE_x = \int_{-0.180m}^{+0.180m}\left(0.00120\frac{C}{m^3}\right)\frac{ky'^2 x dy'}{(x^2+y'^2)^{\frac{3}{2}}}$$

Copying that equation here:

$$\int dE_x = \int_{-0.180m}^{+0.180m}\left(0.00120\frac{C}{m^3}\right)\frac{ky'^2 x dy'}{(x^2+y'^2)^{\frac{3}{2}}}$$

we note that on the left is the infinite sum of all the contributions to the x component of the electric field due to all the infinitesimal elements of the line of charge. We don't need any special mathematics techniques to evaluate that. The sum of all the parts is the whole. That is, on the left, we have E_x.

The right side, we can evaluate. First, let's factor out the constants:

$$E_x = \left(0.00120\frac{C}{m^3}\right)kx\int_{-0.180m}^{+0.180m}\frac{y'^2 dy'}{(x^2+y'^2)^{\frac{3}{2}}}$$

The integral is given on your formula sheet. Carrying out the integration yields:

$$E_x=\left(0.00120\frac{C}{m^3}\right)kx\left[\frac{y'}{\sqrt{x^2+y'^2}}+ln(y'+\sqrt{x^2+y'^2})\right]_{-0.180m}^{+0.180m}$$

$$E_x=\left(.00120\frac{C}{m^3}\right)kx\cdot\left\{\left(\left[\frac{+.018m}{\sqrt{x^2+(+.180m)^2}}+ln\left(+.018m+\sqrt{x^2+(+.018m)^2}\right)\right]-\right.\right.$$

$$\left.\left.\left[\frac{-.180m}{\sqrt{x^2+(-.180)^2}}+ln(-.180m+\sqrt{x^2+(-.180m)^2})\right]\right\}$$

$$E_x=\left(.00120\frac{C}{m^3}\right)kx\cdot\left[\frac{.360m}{\sqrt{x^2+(.180m)^2}}+\ln\frac{\sqrt{x^2+(.180m)^2}+.180m}{\sqrt{x^2+(.180m)^2}-.180m}\right]$$

Substituting the value of the Coulomb constant k from the formula sheet we obtain

$$E_x=\left(.00120\frac{C}{m^3}\right)8.99\times10^9\frac{N\cdot m^2}{C^2}x\cdot\left[\frac{.360m}{\sqrt{x^2+(.180m)^2}}+\ln\frac{\sqrt{x^2+(.180m)^2}+.180m}{\sqrt{x^2+(.180m)^2}-.180m}\right]$$

Finally we have

$$E_x=1.08\times10^7\frac{N}{C\cdot m}\cdot\left[\frac{.360m}{\sqrt{x^2+(.180m)^2}}+\ln\frac{\sqrt{x^2+(.180m)^2}+.180m}{\sqrt{x^2+(.180m)^2}-.180m}\right]$$

It is interesting to note that while the position variable x (which specifies the location of the empty point in space at which the electric field is being calculated) is a constant for purposes of integration (the location of point P does not change as we include the contribution to the electric field at point P of each of the infinitesimal segments making up the charge distribution), an actual value x was never specified. Thus our final result

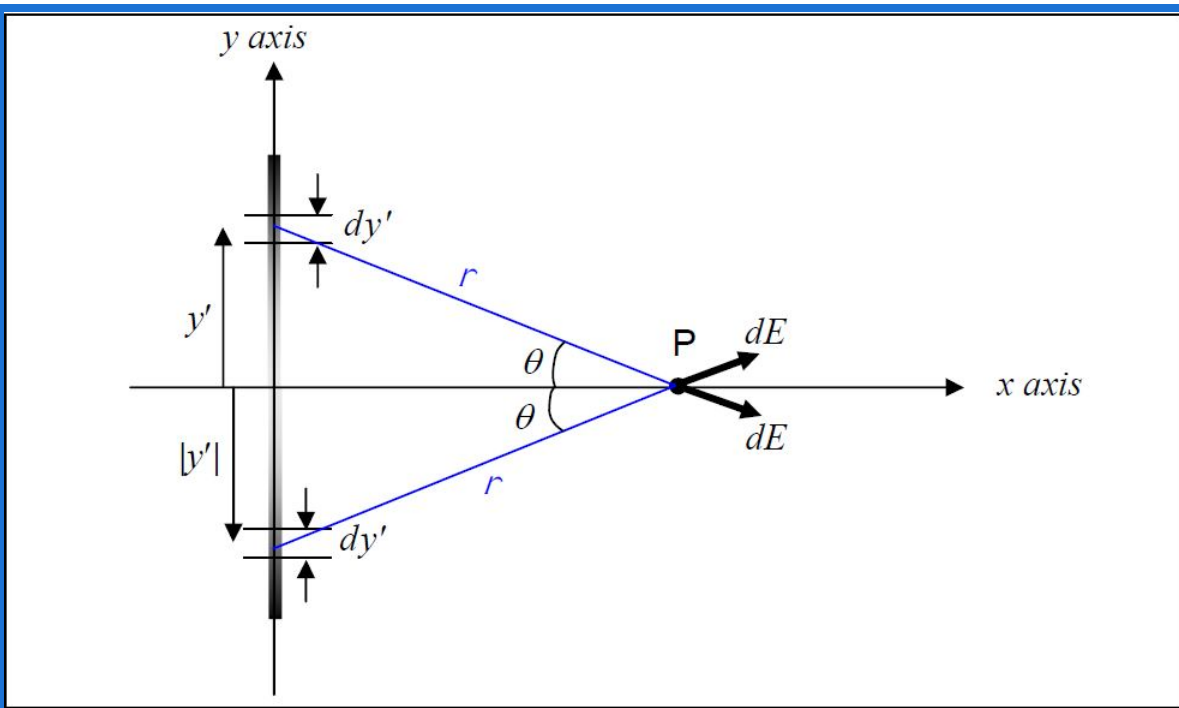
$$E_x=1.08\times10^7\frac{N}{C\cdot m}x\cdot\left[\frac{.360m}{\sqrt{x^2+(.180m)^2}}+\ln\frac{\sqrt{x^2+(.180m)^2}+.180m}{\sqrt{x^2+(.180m)^2}-.180m}\right]$$

for E_x is a function of the position variable x.

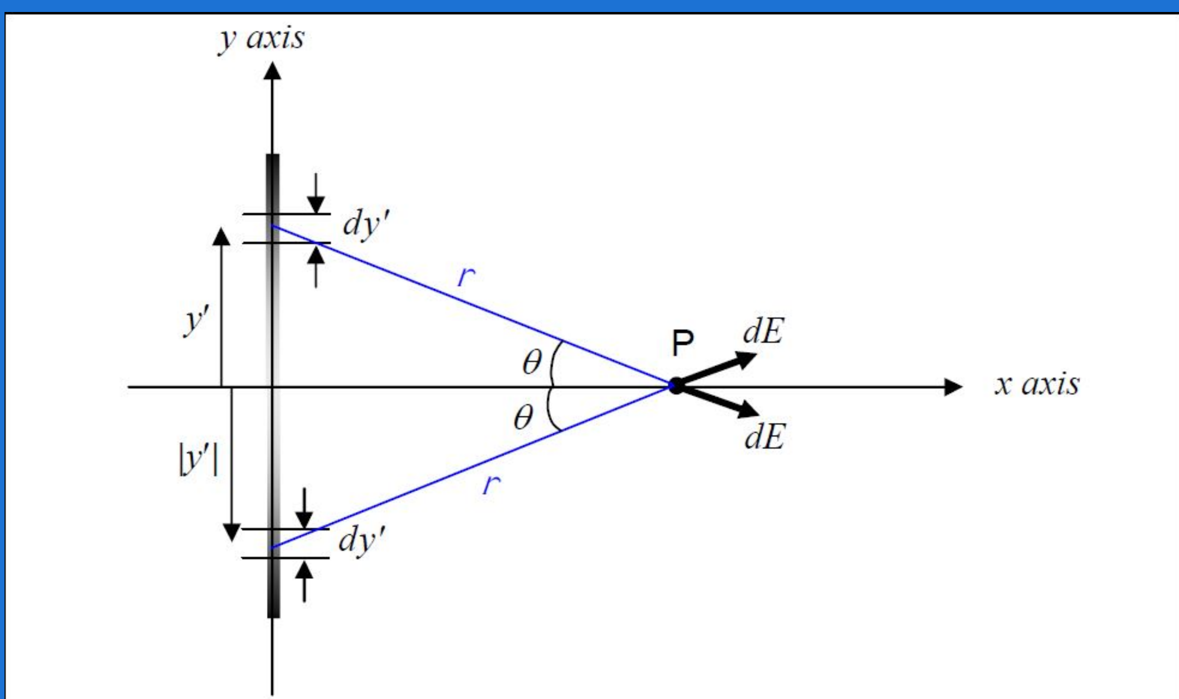
Getting the y-component of the electric field can be done with a lot less work than it took to get E_x if we take advantage of the symmetry of the charge distribution with respect to the x axis. Recall that the charge density λ, for the case at hand, is given by:

$$\lambda=0.00120\frac{C}{m^3}y'^2$$

Because λ is proportional to y'^2, the value of λ is the same at the negative of a specified y' value as it is at the y' value itself. More specifically, the amount of charge in each of the two samesize infinitesimal elements dy' of the charge distribution depicted in the following diagram:



is one and the same value because one element is the same distance below the x axis as the other is above it. This position circumstance also makes the distance r that each element is from point P the same as that of the other, and, it makes the two angles (each of which is labeled θ in the diagram) have one and the same value. Thus the two $\overrightarrow{dE}$ vectors have one and the same magnitude. As a result of the latter two facts (same angle, same magnitude of $\overrightarrow{dE}$), the y components of the two $\overrightarrow{dE}$ vectors cancel each other out. As can be seen in the diagram under consideration:



one is in the $+y$ direction and the other in the $-y$ direction. The y components are “equal and opposite.”) In fact, for each and every charge distribution element dy' that is above the x axis and is thus creating a downward contribution to the y component of the electric field at point P, there is an element dy' that is the same distance below the x axis that is creating an upward contribution to the y component of the electric field at point P, canceling the y component of the former. Thus the net sum of all the electric field y components (since they cancel pair-wise) is zero. That is to say that due to the symmetry of the charge distribution with respect to the x axis, $E_y = 0$. Thus,

$$\vec{E} = E_x \hat{i}$$

Using the expression for E_x that we found above, we have, for our final answer:

$$\vec{E} = 1.08 \times 10^7 \frac{N}{C \cdot m} x \left[\frac{.360m}{\sqrt{x^2 + (.180m)^2}} + \ln \frac{\sqrt{x^2 + (.180m)^2} + .180m}{\sqrt{x^2 + (.180m)^2} - .180m} \right] \hat{i}$$

1.8: Electric Charges and Fields (Exercises)

Conceptual Questions

5.2 Electric Charge

1. There are very large numbers of charged particles in most objects. Why, then, don't most objects exhibit static electricity?

2. Why do most objects tend to contain nearly equal numbers of positive and negative charges?

3. A positively charged rod attracts a small piece of cork.

(a) Can we conclude that the cork is negatively charged?

(b) The rod repels another small piece of cork. Can we conclude that this piece is positively charged?

4. Two bodies attract each other electrically. Do they both have to be charged? Answer the same question if the bodies repel one another.

5. How would you determine whether the charge on a particular rod is positive or negative?

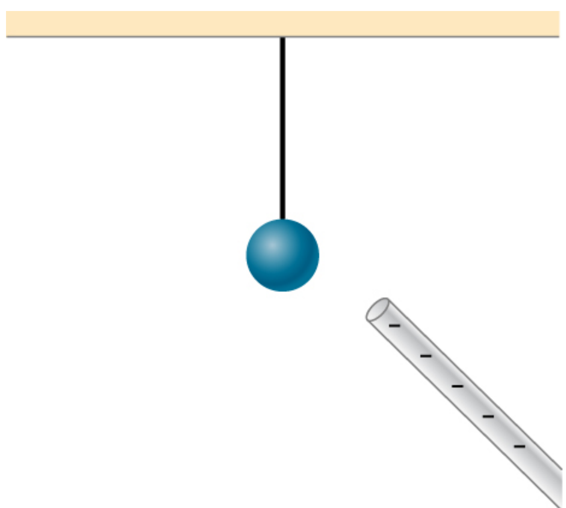
5.3 Conductors, Insulators, and Charging by Induction

6. An eccentric inventor attempts to levitate a cork ball by wrapping it with foil and placing a large negative charge on the ball and then putting a large positive charge on the ceiling of his workshop. Instead, while attempting to place a large negative charge on the ball, the foil flies off. Explain.

7. When a glass rod is rubbed with silk, it becomes positive and the silk becomes negative—yet both attract dust. Does the dust have a third type of charge that is attracted to both positive and negative? Explain.

8. Why does a car always attract dust right after it is polished? (Note that car wax and car tires are insulators.)

9. Does the uncharged conductor shown below experience a net electric force?



10. While walking on a rug, a person frequently becomes charged because of the rubbing between his shoes and the rug. This charge then causes a spark and a slight shock when the person gets close to a metal object. Why are these shocks so much more common on a dry day?

11. Compare charging by conduction to charging by induction.

12. Small pieces of tissue are attracted to a charged comb. Soon after sticking to the comb, the pieces of tissue are repelled from it. Explain.

13. Trucks that carry gasoline often have chains dangling from their undercarriages and brushing the ground. Why?

14. Why do electrostatic experiments work so poorly in humid weather?

15. Why do some clothes cling together after being removed from the clothes dryer? Does this happen if they're still damp?

16. Can induction be used to produce charge on an insulator?

17. Suppose someone tells you that rubbing quartz with cotton cloth produces a third kind of charge on the quartz. Describe what you might do to test this claim.

18. A handheld copper rod does not acquire a charge when you rub it with a cloth. Explain why.

19. Suppose you place a charge **q** near a large metal plate.

(a) If **q** is attracted to the plate, is the plate necessarily charged?

(b) If **q** is repelled by the plate, is the plate necessarily charged?

5.4 Coulomb's Law

20. Would defining the charge on an electron to be positive have any effect on Coulomb's law?

21. An atomic nucleus contains positively charged protons and uncharged neutrons. Since nuclei do stay together, what must we conclude about the forces between these nuclear particles?

22. Is the force between two fixed charges influenced by the presence of other charges?

5.5 Electric Field

23. When measuring an electric field, could we use a negative rather than a positive test charge?

24. During fair weather, the electric field due to the net charge on Earth points downward. Is Earth charged positively or negatively?

25. If the electric field at a point on the line between two charges is zero, what do you know about the charges?

26. Two charges lie along the **x**-axis. Is it true that the net electric field always vanishes at some point (other than infinity) along the **x**-axis?

5.6 Calculating Electric Fields of Charge Distributions

27. Give a plausible argument as to why the electric field outside an infinite charged sheet is constant.

28. Compare the electric fields of an infinite sheet of charge, an infinite, charged conducting plate, and infinite, oppositely charged parallel plates.

29. Describe the electric fields of an infinite charged plate and of two infinite, charged parallel plates in terms of the electric field of an infinite sheet of charge.

30. A negative charge is placed at the center of a ring of uniform positive charge. What is the motion (if any) of the charge? What if the charge were placed at a point on the axis of the ring other than the center?

5.7 Electric Field Lines

31. If a point charge is released from rest in a uniform electric field, will it follow a field line? Will it do so if the electric field is not uniform?

32. Under what conditions, if any, will the trajectory of a charged particle not follow a field line?

33. How would you experimentally distinguish an electric field from a gravitational field?

34. A representation of an electric field shows 10 field lines perpendicular to a square plate. How many field lines should pass perpendicularly through the plate to depict a field with twice the magnitude?

35. What is the ratio of the number of electric field lines leaving a charge 10q and a charge q?

5.8 Electric Dipoles

36. What are the stable orientation(s) for a dipole in an external electric field? What happens if the dipole is slightly perturbed from these orientations?

Problems

5.2 Electric Charge

37. Common static electricity involves charges ranging from nanocoulombs to microcoulombs.

(a) How many electrons are needed to form a charge of −2.00 nC?

(b) How many electrons must be removed from a neutral object to leave a net charge of **0.500μC**?

38. If 1.80×10^{20} electrons move through a pocket calculator during a full day's operation, how many coulombs of charge moved through it?

39. To start a car engine, the car battery moves 3.75×10^{21} electrons through the starter motor. How many coulombs of charge were moved?

40. A certain lightning bolt moves 40.0 C of charge. How many fundamental units of charge is this?

41. A 2.5-g copper penny is given a charge of $-2.0 \times 10^{-9}C$.

(a) How many excess electrons are on the penny?

(b) By what percent do the excess electrons change the mass of the penny?

42. A 2.5-g copper penny is given a charge of $4.0 \times 10^{-9}C$.

(a) How many electrons are removed from the penny?

(b) If no more than one electron is removed from an atom, what percent of the atoms are ionized by this charging process?

5.3 Conductors, Insulators, and Charging by Induction

43. Suppose a speck of dust in an electrostatic precipitator has 1.0000×10^{12} protons in it and has a net charge of −5.00 nC (a very large charge for a small speck). How many electrons does it have?

44. An amoeba has 1.00×10^{16} protons and a net charge of 0.300 pC.

(a) How many fewer electrons are there than protons?

(b) If you paired them up, what fraction of the protons would have no electrons?

45. A 50.0-g ball of copper has a net charge of **2.00μC**. What fraction of the copper's electrons has been removed? (Each copper atom has 29 protons, and copper has an atomic mass of 63.5.)

46. What net charge would you place on a 100-g piece of sulfur if you put an extra electron on 1 in 10^{12} of its atoms? (Sulfur has an atomic mass of 32.1 u.)

47. How many coulombs of positive charge are there in 4.00 kg of plutonium, given its atomic mass is 244 and that each plutonium atom has 94 protons?

5.4 Coulomb's Law

48. Two point particles with charges **+3μC** and **+5μC** are held in place by 3-N forces on each charge in appropriate directions. (a) Draw a free-body diagram for each particle. (b) Find the distance between the charges.

49. Two charges **+3μC** and **+12μC** are fixed 1 m apart, with the second one to the right. Find the magnitude and direction of the net force on a −2-nC charge when placed at the following locations:

(a) halfway between the two

(b) half a meter to the left of the **+3μC** charge

(c) half a meter above the **+12μC** charge in a direction perpendicular to the line joining the two fixed charges

50. In a salt crystal, the distance between adjacent sodium and chloride ions is $2.82 \times 10^{-10}m$. What is the force of attraction between the two singly charged ions?

51. Protons in an atomic nucleus are typically $10^{-15}m$ apart. What is the electric force of repulsion between nuclear protons?

52. Suppose Earth and the Moon each carried a net negative charge **−Q**. Approximate both bodies as point masses and point charges.

(a) What value of **Q** is required to balance the gravitational attraction between Earth and the Moon?

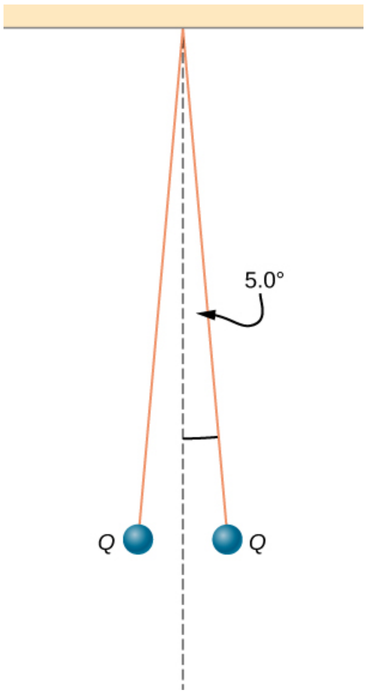
(b) Does the distance between Earth and the Moon affect your answer? Explain.

(c) How many electrons would be needed to produce this charge?

53. Point charges $q_1 = 50\mu C$ and $q_2 = -25\mu C$ are placed 1.0 m apart. What is the force on a third charge $q_3 = 20\mu C$ placed midway between q_1 and q_2?

54. Where must q_3 of the preceding problem be placed so that the net force on it is zero?

55. Two small balls, each of mass 5.0 g, are attached to silk threads 50 cm long, which are in turn tied to the same point on the ceiling, as shown below. When the balls are given the same charge **Q**, the threads hang at **5.0°** to the vertical, as shown below. What is the magnitude of **Q**? What are the signs of the two charges?

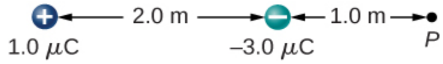


56. Point charges $Q_1 = 2.0\mu C$ and $Q_2 = 4.0\mu C$ are located at $\vec{r_1} = (4.0\hat{i} - 2.0\hat{j} + 5.0\hat{k})m$ and $\vec{r_2} = (8.0\hat{i} + 5.0\hat{j} - 9.0\hat{k})m$. What is the force of Q_2 on Q_1?

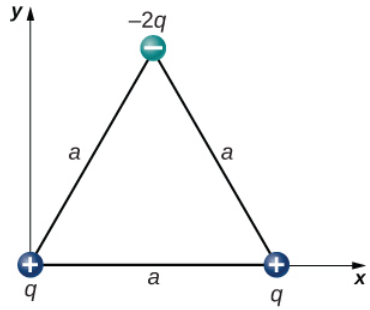
57. The net excess charge on two small spheres (small enough to be treated as point charges) is **Q**. Show that the force of repulsion between the spheres is greatest when each sphere has an excess charge **Q/2**. Assume that the distance between the spheres is so large compared with their radii that the spheres can be treated as point charges.

58. Two small, identical conducting spheres repel each other with a force of 0.050 N when they are 0.25 m apart. After a conducting wire is connected between the spheres and then removed, they repel each other with a force of 0.060 N. What is the original charge on each sphere?

59. A charge **q=2.0μC** is placed at the point **P** shown below. What is the force on **q**?

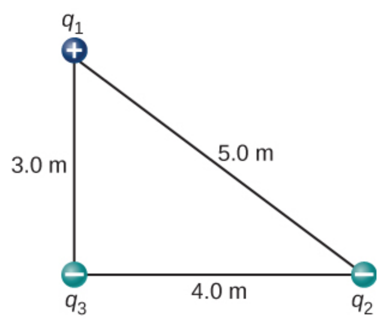


60. What is the net electric force on the charge located at the lower right-hand corner of the triangle shown here?

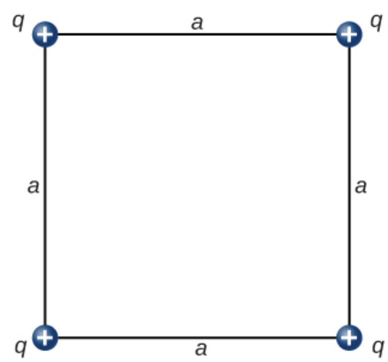


61. Two fixed particles, each of charge $5.0 \times 10^{-6} C$, are 24 cm apart. What force do they exert on a third particle of charge $-2.5 \times 10^{-6} C$ that is 13 cm from each of them?

62. The charges $q_1 = 2.0 \times 10^{-7} C, q_2 = -4.0 \times 10^{-7} C$, and $q_3 = -1.0 \times 10^{-7} C$ are placed at the corners of the triangle shown below. What is the force on q_1?



63. What is the force on the charge q at the lower-right-hand corner of the square shown here?



64. Point charges $q_1 = 10 \mu C$ and $q_2 = -30 \mu C$ are fixed at $r_1 = (3.0\hat{i} - 4.0\hat{j})m$ and $r_2 = (9.0\hat{i} + 6.0\hat{j})m$. What is the force of q_2 on q_1?

5.5 Electric Field

65. A particle of charge $2.0 \times 10^{-8} C$ experiences an upward force of magnitude $4.0 \times 10^{-6} N$ when it is placed in a particular point in an electric field.

(a) What is the electric field at that point?

(b) If a charge $q = -1.0 \times 10^{-8} C$ is placed there, what is the force on it?

66. On a typical clear day, the atmospheric electric field points downward and has a magnitude of approximately 100 N/C. Compare the gravitational and electric forces on a small dust particle of mass $2.0 \times 10^{-15} g$ that carries a single electron charge. What is the acceleration (both magnitude and direction) of the dust particle?

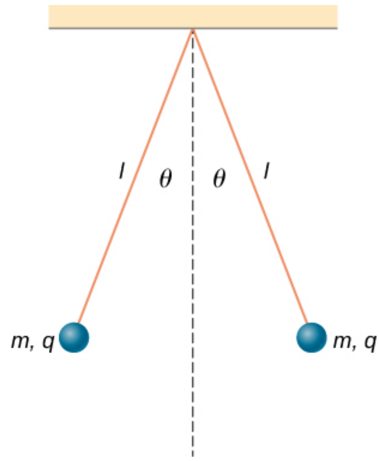
67. Consider an electron that is 10−10m10−10m from an alpha particle (q=3.2×10−19C).(q=3.2×10−19C).

(a) What is the electric field due to the alpha particle at the location of the electron?

(b) What is the electric field due to the electron at the location of the alpha particle?

(c) What is the electric force on the alpha particle? On the electron?

68. Each the balls shown below carries a charge q and has a mass m. The length of each thread is l, and at equilibrium, the balls are separated by an angle 2θ. How does θ vary with **q** and **l**? Show that θ satisfies \(\displaystyle sin(θ)^2tan(θ)=\frac{q^2}{16πε0gl^2m\).



69. What is the electric field at a point where the force on a $-2.0 \times 10^{-6} - C$ charge is $(4.0\hat{i} - 6.0\hat{j}) \times 10^{-6} N$?

70. A proton is suspended in the air by an electric field at the surface of Earth. What is the strength of this electric field?

71. The electric field in a particular thundercloud is $2.0 \times 10^5 N/C$. What is the acceleration of an electron in this field?

72. A small piece of cork whose mass is 2.0 g is given a charge of $5.0 \times 10^{-7} C$. What electric field is needed to place the cork in equilibrium under the combined electric and gravitational forces?

73. If the electric field is **100N/C** at a distance of 50 cm from a point charge **q**, what is the value of **q**?

74. What is the electric field of a proton at the first Bohr orbit for hydrogen $(r = 5.29 \times 10^{-11} m)$? What is the force on the electron in that orbit?

75. (a) What is the electric field of an oxygen nucleus at a point that is $10^{-10} m$ from the nucleus?

(b) What is the force this electric field exerts on a second oxygen nucleus placed at that point?

76. Two point charges, $q_1 = 2.0 \times 10^{-7} C$ and $q_2 = -6.0 \times 10^{-8} C$, are held 25.0 cm apart.

(a) What is the electric field at a point 5.0 cm from the negative charge and along the line between the two charges?

(b)What is the force on an electron placed at that point?

77. Point charges $q_1 = 50\mu C$ and $q_2 = -25\mu C$ are placed 1.0 m apart.

(a) What is the electric field at a point midway between them?

(b) What is the force on a charge $q_3 = 20\mu C$ situated there?

78. Can you arrange the two point charges $q_1 = -2.0 \times 10^{-6} C$ and $q_2 = 4.0 \times 10^{-6} C$ along the **x**-axis so that $E = 0$ at the origin?

79. Point charges $q_1 = q_2 = 4.0 \times 10^{-6} C$ are fixed on the **x**-axis at $x = -3.0m$ and $x = 3.0m$. What charge **q** must be placed at the origin so that the electric field vanishes at **x=0,y=3.0m**?

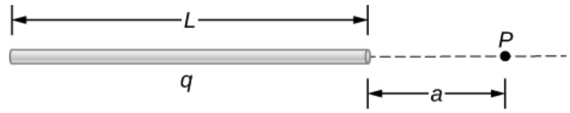
5.6 Calculating Electric Fields of Charge Distributions

80. A thin conducting plate 1.0 m on the side is given a charge of $-2.0 \times 10^{-6} C$. An electron is placed 1.0 cm above the center of the plate. What is the acceleration of the electron?

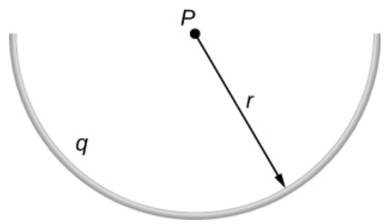
81. Calculate the magnitude and direction of the electric field 2.0 m from a long wire that is charged uniformly at $\lambda = 4.0 \times 10^{-6} C/m$.

82. Two thin conducting plates, each 25.0 cm on a side, are situated parallel to one another and 5.0 mm apart. If 10^{11} electrons are moved from one plate to the other, what is the electric field between the plates?

83. The charge per unit length on the thin rod shown below is λ. What is the electric field at the point **P**? (Hint: Solve this problem by first considering the electric field $d\vec{E}$ at **P** due to a small segment **dx** of the rod, which contains charge $dq = \lambda dx$. Then find the net field by integrating $d\vec{E}$ over the length of the rod.)



84. The charge per unit length on the thin semicircular wire shown below is **λ**. What is the electric field at the point **P**?



85. Two thin parallel conducting plates are placed 2.0 cm apart. Each plate is 2.0 cm on a side; one plate carries a net charge of **8.0μC,** and the other plate carries a net charge of **−8.0μC**. What is the charge density on the inside surface of each plate? What is the electric field between the plates?

86. A thin conducting plate 2.0 m on a side is given a total charge of **−10.0μC**.

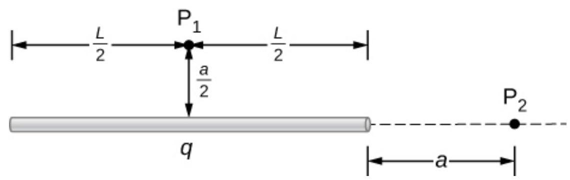
(a) What is the electric field **1.0cm** above the plate?

(b) What is the force on an electron at this point?

(c) Repeat these calculations for a point 2.0 cm above the plate.

(d) When the electron moves from 1.0 to 2,0 cm above the plate, how much work is done on it by the electric field?

87. A total charge **q** is distributed uniformly along a thin, straight rod of length **L** (see below). What is the electric field at P_1? At P_2?



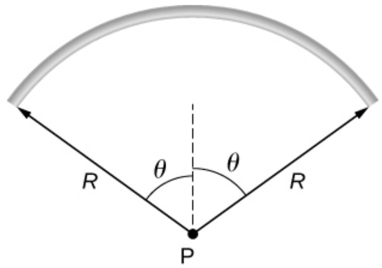
88. Charge is distributed along the entire **x**-axis with uniform density **λ**. How much work does the electric field of this charge distribution do on an electron that moves along the **y**-axis from $y = a$ to $y = b$?

89. Charge is distributed along the entire **x**-axis with uniform density λ_x and along the entire **y**-axis with uniform density λ_y. Calculate the resulting electric field at

(a) $\vec{r} = a\hat{i} + b\hat{j}$ and

(b) $\vec{r} = c\hat{k}$.

90. A rod bent into the arc of a circle subtends an angle 2θ at the center **P** of the circle (see below). If the rod is charged uniformly with a total charge **Q**, what is the electric field at **P**?.

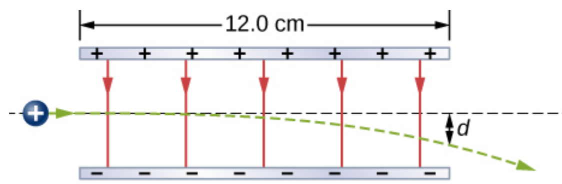


91. A proton moves in the electric field $\vec{E} = 200\hat{i}\,N/C$. (a) What are the force on and the acceleration of the proton? (b) Do the same calculation for an electron moving in this field.

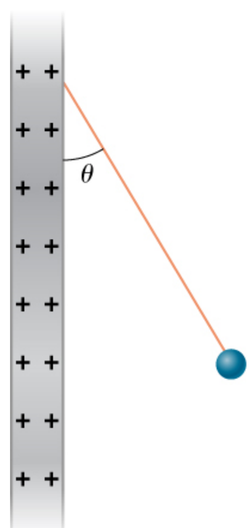
92. An electron and a proton, each starting from rest, are accelerated by the same uniform electric field of 200 N/C. Determine the distance and time for each particle to acquire a kinetic energy of $3.2 \times 10^{-16} J$.

93. A spherical water droplet of radius **25μm** carries an excess 250 electrons. What vertical electric field is needed to balance the gravitational force on the droplet at the surface of the earth?

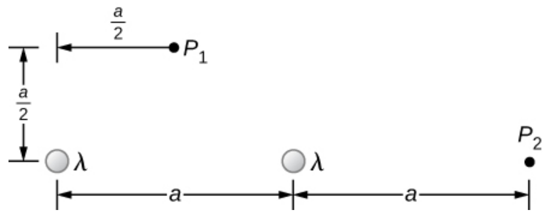
94. A proton enters the uniform electric field produced by the two charged plates shown below. The magnitude of the electric field is $4.0 \times 10^5 N/C$, and the speed of the proton when it enters is $1.5 \times 10^7 m/s$. What distance **d** has the proton been deflected downward when it leaves the plates?



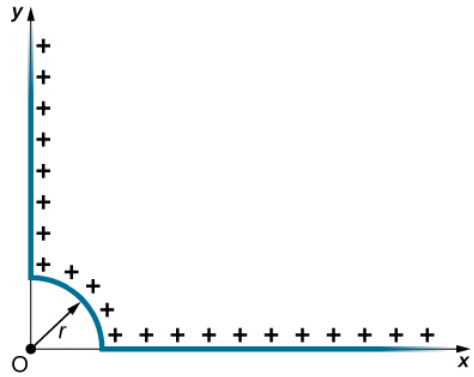
95. Shown below is a small sphere of mass 0.25 g that carries a charge of $9.0 \times 10^{-10} C$. The sphere is attached to one end of a very thin silk string 5.0 cm long. The other end of the string is attached to a large vertical conducting plate that has a charge density of $30 \times 10^{-6} C/m^2$. What is the angle that the string makes with the vertical?



96. Two infinite rods, each carrying a uniform charge density λ, are parallel to one another and perpendicular to the plane of the page. (See below.) What is the electrical field at P_1? At P_2?



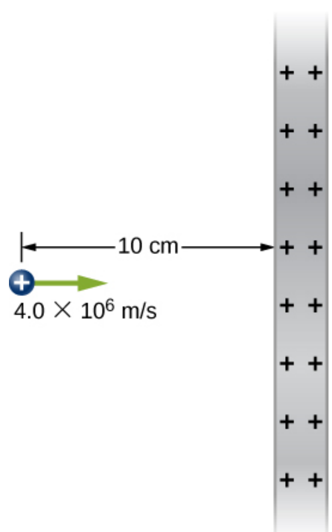
97. Positive charge is distributed with a uniform density λ along the positive **x**-axis from r to ∞, along the positive **y**-axis from r to ∞, and along a **90°** arc of a circle of radius r, as shown below. What is the electric field at **O**?



98. From a distance of 10 cm, a proton is projected with a speed of $v = 4.0 \times 10^6 m/s$ directly at a large, positively charged plate whose charge density is $\sigma = 2.0 \times 10^{-5} C/m^2$.. (See below.)

(a) Does the proton reach the plate?

(b) If not, how far from the plate does it turn around?



99. A particle of mass m and charge $-q$ moves along a straight line away from a fixed particle of charge **Q**. When the distance between the two particles is r_0, $-q$ is moving with a speed v_0.

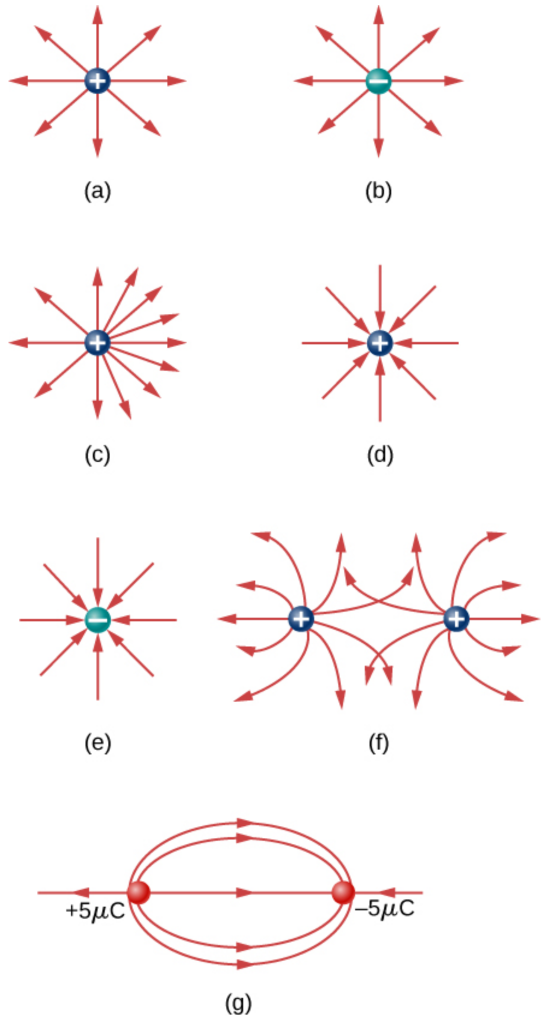
(a) Use the work-energy theorem to calculate the maximum separation of the charges.

(b) What do you have to assume about v_0 to make this calculation?

(c) What is the minimum value of v_0 such that $-q$ escapes from **Q**?

5.7 Electric Field Lines

100. Which of the following electric field lines are incorrect for point charges? Explain why.



101. In this exercise, you will practice drawing electric field lines. Make sure you represent both the magnitude and direction of the electric field adequately. Note that the number of lines into or out of charges is proportional to the charges.

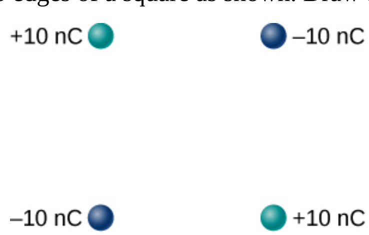
(a) Draw the electric field lines map for two charges **+20μC** and **−20μC** situated 5 cm from each other.

(b) Draw the electric field lines map for two charges **+20μC** and **+20μC** situated 5 cm from each other.

(c) Draw the electric field lines map for two charges **+20μC** and **−30μC** situated 5 cm from each other.

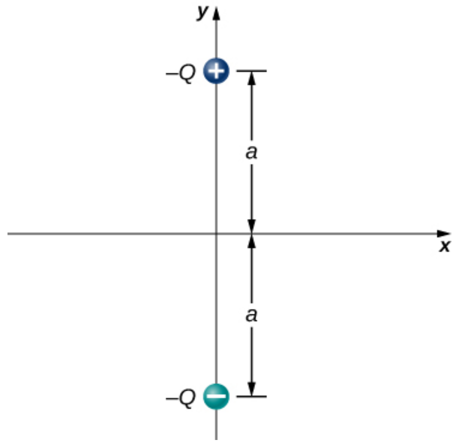
102. Draw the electric field for a system of three particles of charges **+1μC, +2μC** and **−3μC** fixed at the corners of an equilateral triangle of side 2 cm.

103. Two charges of equal magnitude but opposite sign make up an electric dipole. A quadrupole consists of two electric dipoles that are placed anti-parallel at two edges of a square as shown. Draw the electric field of the charge distribution.



104. Suppose the electric field of an isolated point charge decreased with distance as $1/r^{2+\delta}$ rather than as $1/r^2$. Show that it is then impossible to draw continuous field lines so that their number per unit area is proportional to **E**.

105. Consider the equal and opposite charges shown below. (a) Show that at all points on the **x**-axis for which $|x| \gg a, E \approx Qa/2\pi\varepsilon_0 x^3$. (b) Show that at all points on the **y**-axis for which $|y| \gg a, E \approx Qa/\pi\varepsilon_0 y^3$.



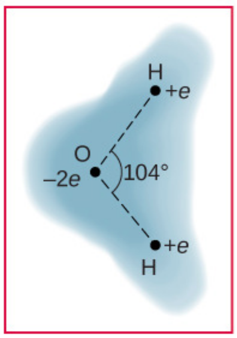
106. (a) What is the dipole moment of the configuration shown above? If **Q=4.0μC**,

(b) what is the torque on this dipole with an electric field of $4.0 \times 10^5 N/C\hat{i}$?

(c) What is the torque on this dipole with an electric field of $-4.0 \times 10^5 N/C\hat{i}$?

(d) What is the torque on this dipole with an electric field of $\pm 4.0 \times 10^5 N/C\hat{j}$?

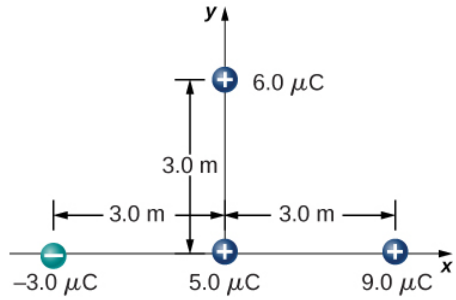
107. A water molecule consists of two hydrogen atoms bonded with one oxygen atom. The bond angle between the two hydrogen atoms is **104°** (see below). Calculate the net dipole moment of a hypothetical water molecule where the charge at the oxygen molecule is −2e and at each hydrogen atom is +e. The net dipole moment of the molecule is the vector sum of the individual dipole moment between the two O-Hs. The separation O-H is 0.9578 angstroms.



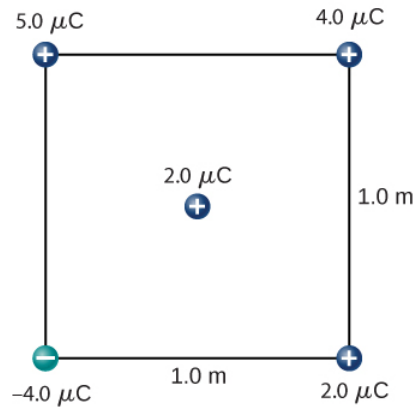
Additional Problems

108. Point charges $q_1 = 2.0\mu C$ and $q_1 = 4.0\mu C$ are located at $r_1 = (4.0\hat{i} - 2.0\hat{j} + 2.0\hat{k})m$ and $r_2 = (8.0\hat{i} + 5.0\hat{j} - 9.0\hat{k})m$. What is the force of q_2 on q_1?

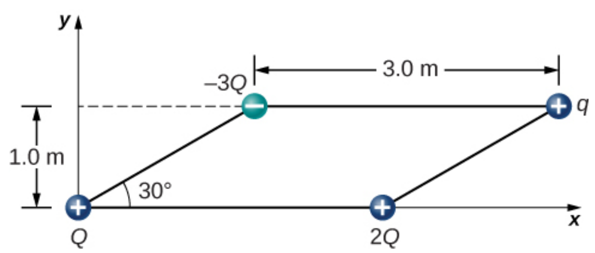
109. What is the force on the **5.0-μC** charges shown below?



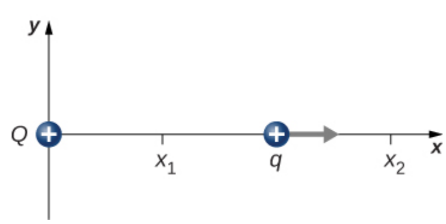
110. What is the force on the **2.0-μC** charge placed at the center of the square shown below?



111. Four charged particles are positioned at the corners of a parallelogram as shown below. If $q=5.0\mu C$ and $Q=8.0\mu C$, what is the net force on **q**?

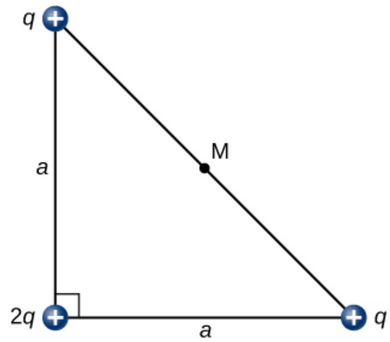


112. A charge **Q** is fixed at the origin and a second charge **q** moves along the **x**-axis, as shown below. How much work is done on **q** by the electric force when **q** moves from x_1 to x_2?

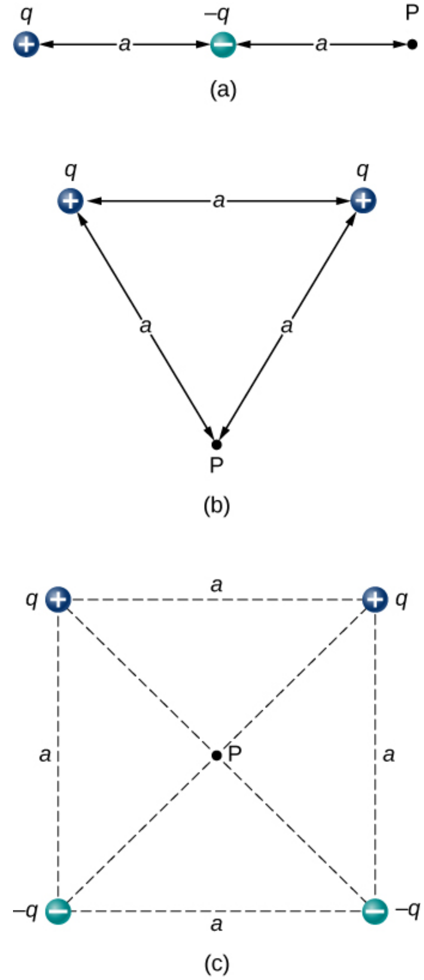


113. A charge **q=−2.0μC** is released from rest when it is 2.0 m from a fixed charge $Q=6.0\mu C$. What is the kinetic energy of **q** when it is 1.0 m from **Q**?

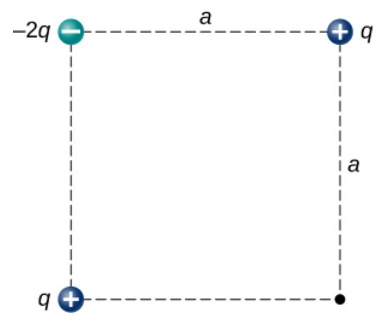
114. What is the electric field at the midpoint M of the hypotenuse of the triangle shown below?



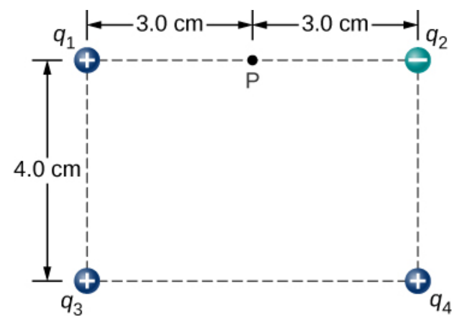
115. Find the electric field at P for the charge configurations shown below.



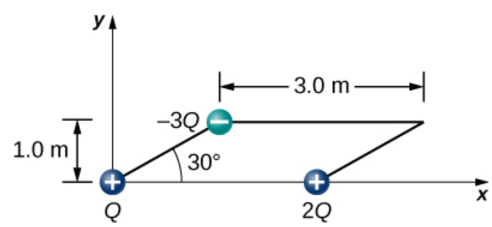
116. (a) What is the electric field at the lower-right-hand corner of the square shown below? (b) What is the force on a charge **q** placed at that point?



117. Point charges are placed at the four corners of a rectangle as shown below: $q_1 = 2.0 \times 10^{-6} C, q_2 = -2.0 \times 10^{-6} C, q_3 = 4.0 \times 10^{-6} C,$ and $q_4 = 1.0 \times 10^{-6} C$. What is the electric field at **P**?



118. Three charges are positioned at the corners of a parallelogram as shown below. (a) If $Q = 8.0 \mu C$, what is the electric field at the unoccupied corner? (b) What is the force on a **5.0-μC** charge placed at this corner?



119. A positive charge **q** is released from rest at the origin of a rectangular coordinate system and moves under the influence of the electric field $\vec{E} = E_0(1 + x/a)\hat{i}$. What is the kinetic energy of **q** when it passes through $x = 3a$?

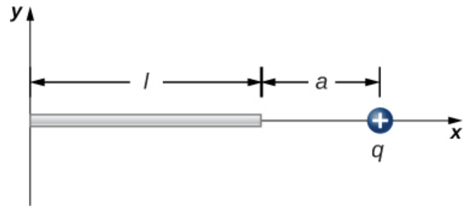
120. A particle of charge $-q$ and mass **m** is placed at the center of a uniformaly charged ring of total charge **Q** and radius **R**. The particle is displaced a small distance along the axis perpendicular to the plane of the ring and released. Assuming that the particle is constrained to move along the axis, show that the particle oscillates in simple harmonic motion with a frequency $f = \frac{1}{2\pi}\sqrt{\frac{qQ}{4\pi\varepsilon_0 m R^3}}$.

121. Charge is distributed uniformly along the entire **y**-axis with a density y_λ and along the positive **x**-axis from $x = a$ to $x = b$ with a density λ_x. What is the force between the two distributions?

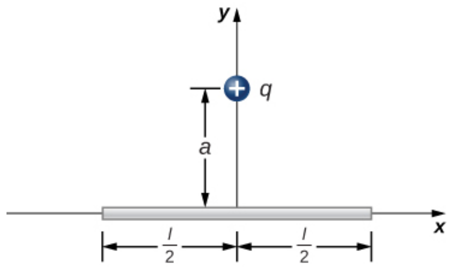
122. The circular arc shown below carries a charge per unit length $\lambda = \lambda_0 cos\theta$, where θ is measured from the **x**-axis. What is the electric field at the origin?

123. Calculate the electric field due to a uniformly charged rod of length **L**, aligned with the **x**-axis with one end at the origin; at a point **P** on the **z**-axis.

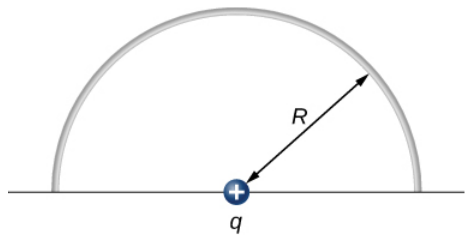
124. The charge per unit length on the thin rod shown below is λ. What is the electric force on the point charge **q**? Solve this problem by first considering the electric force $d\vec{F}$ on **q** due to a small segment dx of the rod, which contains charge λdx. Then, find the net force by integrating $d\vec{F}$ over the length of the rod.



125. The charge per unit length on the thin rod shown here is λ. What is the electric force on the point charge **q**? (See the preceding problem.)



126. The charge per unit length on the thin semicircular wire shown below is λ. What is the electric force on the point charge **q**? (See the preceding problems.)



2: Gauss's Law

2.1: Prelude to Gauss's Law

Flux is a general and broadly applicable concept in physics. However, in this chapter, we concentrate on the flux of the electric field. This allows us to introduce Gauss's law, which is particularly useful for finding the electric fields of charge distributions exhibiting spatial symmetry. The main topics discussed here are

1. **Electric flux.** We define electric flux for both open and closed surfaces.
2. **Gauss's law.** We derive Gauss's law for an arbitrary charge distribution and examine the role of electric flux in Gauss's law.
3. **Calculating electric fields with Gauss's law.** The main focus of this chapter is to explain how to use Gauss's law to find the electric fields of spatially symmetrical charge distributions. We discuss the importance of choosing a Gaussian surface and provide examples involving the applications of Gauss's law.
4. **Electric fields in conductors.** Gauss's law provides useful insight into the absence of electric fields in conducting materials.

Figure $2.1.1$: This chapter introduces the concept of flux, which relates a physical quantity and the area through which it is flowing. Although we introduce this concept with the electric field, the concept may be used for many other quantities, such as fluid flow. (credit: modification of work by "Alessandro"/Flickr)

So far, we have found that the electrostatic field begins and ends at point charges and that the field of a point charge varies inversely with the square of the distance from that charge. These characteristics of the electrostatic field lead to an important mathematical relationship known as Gauss's law. This law is named in honor of the extraordinary German mathematician and scientist Karl Friedrich **Gauss** (Figure $2.1.2$). Gauss's law gives us an elegantly simple way of finding the electric field, and, as you will see, it can be much easier to use than the integration method described in the previous chapter. However, there is a catch—Gauss's law has a limitation in that, while always true, it can be readily applied only for charge distributions with certain symmetries.

Figure $2.1.2$: Karl Friedrich Gauss (1777–1855) was a legendary mathematician of the nineteenth century. Although his major contributions were to the field of mathematics, he also did important work in physics and astronomy.

2.2: Electric Flux

Learning Objectives

By the end of this section, you will be able to:

- Define the concept of flux
- Describe electric flux
- Calculate electric flux for a given situation

The concept of **flux** describes how much of something goes through a given area. More formally, it is the dot product of a vector field (in this chapter, the electric field) with an area. You may conceptualize the flux of an electric field as a measure of the number of electric field lines passing through an area (Figure $2.2.1$). The larger the area, the more field lines go through it and, hence, the greater the flux; similarly, the stronger the electric field is (represented by a greater density of lines), the greater the flux. On the other hand, if the area rotated so that the plane is aligned with the field lines, none will pass through and there will be no flux.

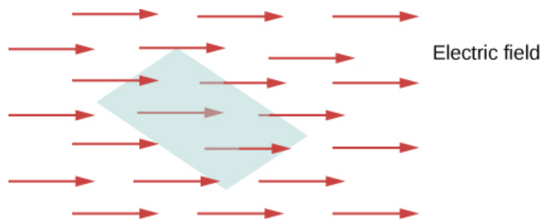


Figure $2.2.1$: The flux of an electric field through the shaded area captures information about the "number" of electric field lines passing through the area. The numerical value of the electric flux depends on the magnitudes of the electric field and the area, as well as the relative orientation of the area with respect to the direction of the electric field.

A macroscopic analogy that might help you imagine this is to put a hula hoop in a flowing river. As you change the angle of the hoop relative to the direction of the current, more or less of the flow will go through the hoop. Similarly, the amount of flow through the hoop depends on the strength of the current and the size of the hoop. Again, flux is a general concept; we can also use it to describe the amount of sunlight hitting a solar panel or the amount of energy a telescope receives from a distant star, for example.

To quantify this idea, Figure $2.2.1a$ shows a planar surface S_1 of area A_1 that is perpendicular to the uniform electric field $\vec{E} = E\hat{y}$. If N field lines pass through S_1, then we know from the definition of electric field lines (Electric Charges and Fields) that $N/A \propto E$, or $N \propto EA_1$.

The quantity EA_1 is the **electric flux** through S_1. We represent the electric flux through an open surface like S_1 by the symbol Φ. Electric flux is a scalar quantity and has an SI unit of newton-meters squared per coulomb ($\mathrm{N \cdot m^2/C}$). Notice that $N \propto EA_1$ may also be written as $N \propto \Phi$, demonstrating that *electric flux is a measure of the number of field lines crossing a surface.*

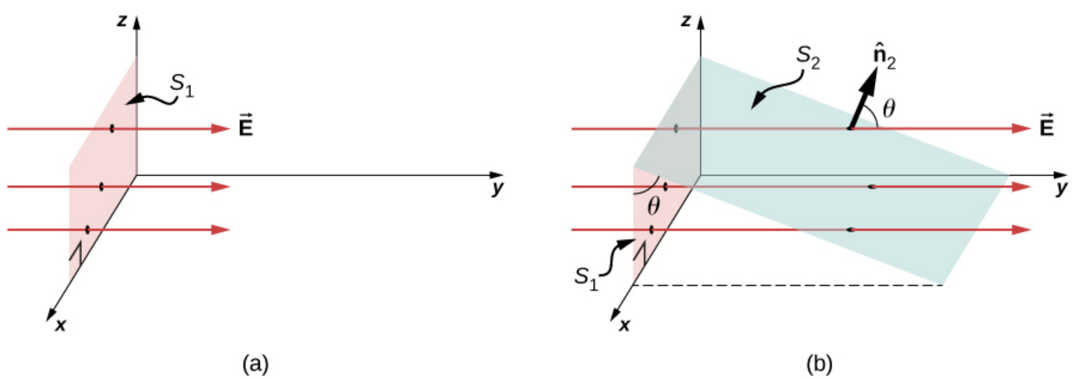


Figure 2.2.2: (a) A planar surface S_1 of area A_1 is perpendicular to the electric field $E\hat{j}$. N field lines cross surface S_1. (b) A surface S_2 of area A_2 whose projection onto the xz-plane is S_1. The same number of field lines cross each surface.

Now consider a planar surface that is not perpendicular to the field. How would we represent the electric flux? Figure $2.2.2b$ shows a surface S_2 of area A_2 that is inclined at an angle θ to the xz-plane and whose projection in that plane is S_1 (area A_1). The areas are related by $A_2 \, cos\, \theta = A_1$. Because the same number of field lines crosses both S_1 and S_2, the fluxes through both surfaces must be the same. The flux through S_2 is therefore $\Phi = EA_1 = EA_2 \cos\theta$. Designating $\hat{n}_2$ as a unit vector normal to S_2 (see Figure $2.2.2b$), we obtain

$$\Phi = \vec{E} \cdot \hat{n}_2 A_2.$$

> **Note**
>
> Check out this video to observe what happens to the flux as the area changes in size and angle, or the electric field changes in strength.

Area Vector

For discussing the flux of a vector field, it is helpful to introduce an area vector $\vec{A}$. This allows us to write the last equation in a more compact form. What should the magnitude of the area vector be? What should the direction of the area vector be? What are the implications of how you answer the previous question?

The **area vector** of a flat surface of area $\vec{A}$ has the following magnitude and direction:

- Magnitude is equal to area (A)
- Direction is along the normal to the surface $(\hat{n})$; that is, perpendicular to the surface.

Since the normal to a flat surface can point in either direction from the surface, the direction of the area vector of an open surface needs to be chosen, as shown in Figure $2.2.3$.

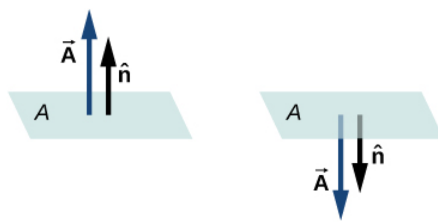


Figure $2.2.3$: The direction of the area vector of an open surface needs to be chosen; it could be either of the two cases displayed here. The area vector of a part of a closed surface is defined to point from the inside of the closed space to the outside. This rule gives a unique direction.

Since $\hat{n}$ is a unit normal to a surface, it has two possible directions at every point on that surface (Figure $2.2.1a$). For an open surface, we can use either direction, as long as we are consistent over the entire surface. $2.2.1c$of the figure shows several cases.

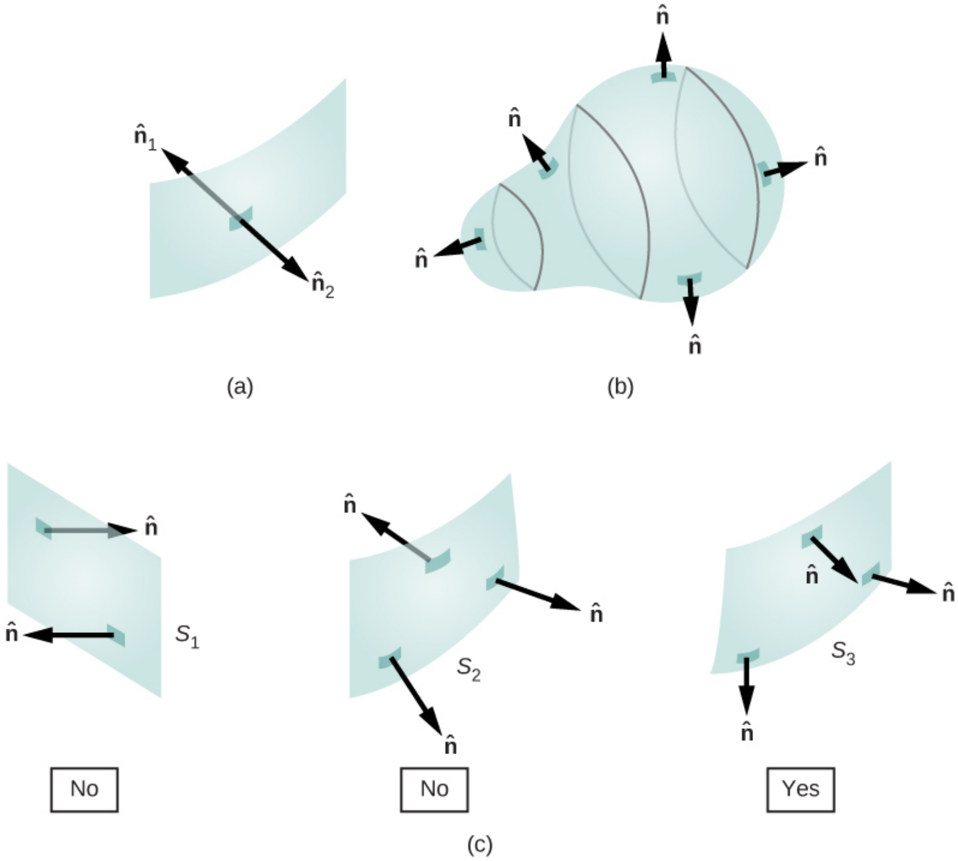


Figure 2.2.4: (a) Two potential normal vectors arise at every point on a surface. (b) The outward normal is used to calculate the flux through a closed surface. (c) Only S_3 has been given a consistent set of normal vectors that allows us to define the flux through the surface.

However, if a surface is closed, then the surface encloses a volume. In that case, the direction of the normal vector at any point on the surface points from the inside to the outside. On a *closed surface* such as that of Figure $2.2.1b$, $\hat{n}$ is chosen to be the *outward normal* at every point, to be consistent with the sign convention for electric charge.

Electric Flux

Now that we have defined the area vector of a surface, we can define the electric flux of a uniform electric field through a flat area as the scalar product of the electric field and the area vector:

$$\Phi = \vec{E} \cdot \vec{A} \;\; (\text{uniform } \vec{E}, \text{flat surface}).$$

Figure $2.2.5$ shows the electric field of an oppositely charged, parallel-plate system and an imaginary box between the plates. The electric field between the plates is uniform and points from the positive plate toward the negative plate. A calculation of the flux of this field through various faces of the box shows that the net flux through the box is zero. Why does the flux cancel out here?

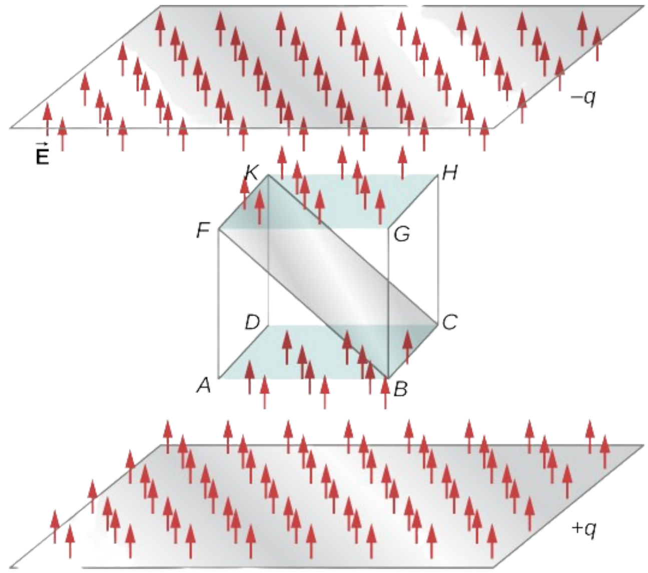


Figure 2.2.5: Electric flux through a cube, placed between two charged plates. Electric flux through the bottom face ($ABCD$) is negative, because $\vec{E}$ is in the opposite direction to the normal to the surface. The electric flux through the top face ($FGHK$) is positive, because the electric field and the normal are in the same direction. The electric flux through the other faces is zero, since the electric field is perpendicular to the normal vectors of those faces. The net electric flux through the cube is the sum of fluxes through the six faces. Here, the net flux through the cube is equal to zero. The magnitude of the flux through rectangle $BCKF$ is equal to the magnitudes of the flux through both the top and bottom faces.

The reason is that the sources of the electric field are outside the box. Therefore, if any electric field line enters the volume of the box, it must also exit somewhere on the surface because there is no charge inside for the lines to land on. Therefore, quite generally, electric flux through a closed surface is zero if there are no sources of electric field, whether positive or negative charges, inside the enclosed volume. In general, when field lines leave (or "flow out of") a closed surface, Φ is positive; when they enter (or "flow into") the surface, Φ is negative.

Any smooth, non-flat surface can be replaced by a collection of tiny, approximately flat surfaces, as shown in Figure $2.2.6$. If we divide a surface S into small patches, then we notice that, as the patches become smaller, they can be approximated by flat surfaces. This is similar to the way we treat the surface of Earth as locally flat, even though we know that globally, it is approximately spherical.

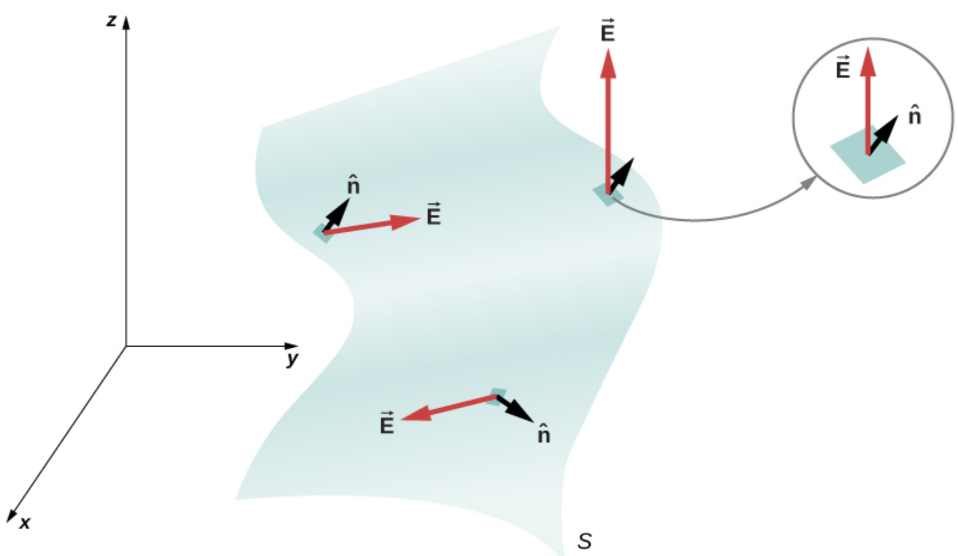


Figure 2.2.6: A surface is divided into patches to find the flux.

To keep track of the patches, we can number them from 1 through N. Now, we define the area vector for each patch as the area of the patch pointed in the direction of the normal. Let us denote the area vector for the ith patch by $\delta\vec{A}_i$. (We have used the symbol δ

to remind us that the area is of an arbitrarily small patch.) With sufficiently small patches, we may approximate the electric field over any given patch as uniform. Let us denote the average electric field at the location of the ith patch by $\vec{E}_i$.

$$\vec{E}_i = \text{average electric field over the } i\text{th patch.}$$

Therefore, we can write the electric flux Φ through the area of the ith patch as

$$\Phi_i = \vec{E}_i \cdot \delta\vec{A}_i \ \ (i\text{th patch}).$$

The flux through each of the individual patches can be constructed in this manner and then added to give us an estimate of the net flux through the entire surface S, which we denote simply as Φ.

$$\Phi = \sum_{i=1}^{N} \Phi_i = \sum_{i=1}^{N} \vec{E}_i \cdot \delta\vec{A}_i \ \ (N \text{ patch estimate}).$$

This estimate of the flux gets better as we decrease the size of the patches. However, when you use smaller patches, you need more of them to cover the same surface. In the limit of infinitesimally small patches, they may be considered to have area dA and unit normal $\hat{n}$. Since the elements are infinitesimal, they may be assumed to be planar, and $\vec{E}_i$ may be taken as constant over any element. Then the flux $d\Phi$ through an area dA is given by $d\Phi = \vec{E} \cdot \hat{n} dA$. It is positive when the angle between $\vec{E}_i$ and $\hat{n}$ is less than 90° and negative when the angle is greater than 90^o. The net flux is the sum of the infinitesimal flux elements over the entire surface. With infinitesimally small patches, you need infinitely many patches, and the limit of the sum becomes a surface integral. With $\int_S$ representing the integral over S,

$$\Phi = \int_S \vec{E} \cdot \hat{n}\, dA = \int_S \vec{E} \cdot d\vec{A} \ \ (\text{open surface}).$$

In practical terms, surface integrals are computed by taking the antiderivatives of both dimensions defining the area, with the edges of the surface in question being the bounds of the integral.

To distinguish between the flux through an open surface like that of Figure $2.2.2$ and the flux through a closed surface (one that completely bounds some volume), we represent flux through a closed surface by

$$\Phi = \oint_S \vec{E} \cdot \hat{n}\, dA = \oint_S \vec{E} \cdot d\vec{A} \ \ (\text{closed surface})$$

where the circle through the integral symbol simply means that the surface is closed, and we are integrating over the entire thing. If you only integrate over a portion of a closed surface, that means you are treating a subset of it as an open surface.

✔ Example 2.2.1: Flux of a Uniform Electric Field

A constant electric field of magnitude E_0 points in the direction of the positive z-axis (Figure $2.2.7$). What is the electric flux through a rectangle with sides a and b in the (a) xy-plane and in the (b) xz-plane?

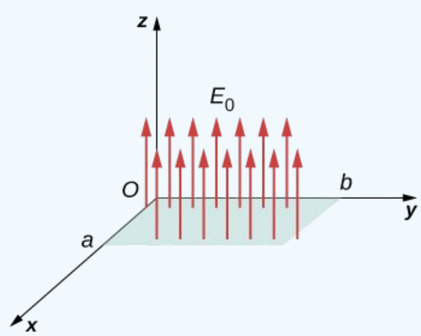


Figure $2.2.7$: Calculating the flux of E_0 through a rectangular surface.

Strategy

Apply the definition of flux: $\Phi = \vec{E} \cdot \vec{A} \ (\text{uniform } \vec{E})$, where the definition of dot product is crucial.

Solution

1. In this case, $\Phi = \vec{E}_0 \cdot \vec{A} = E_0 A = E_0 ab$.
2. Here, the direction of the area vector is either along the positive y-axis or toward the negative y-axis. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

Significance

The relative directions of the electric field and area can cause the flux through the area to be zero.

✓ Flux of a Uniform Electric Field through a Closed Surface

A constant electric field of magnitude E_0 points in the direction of the positive z-axis (Figure $2.2.8$). What is the net electric flux through a cube?

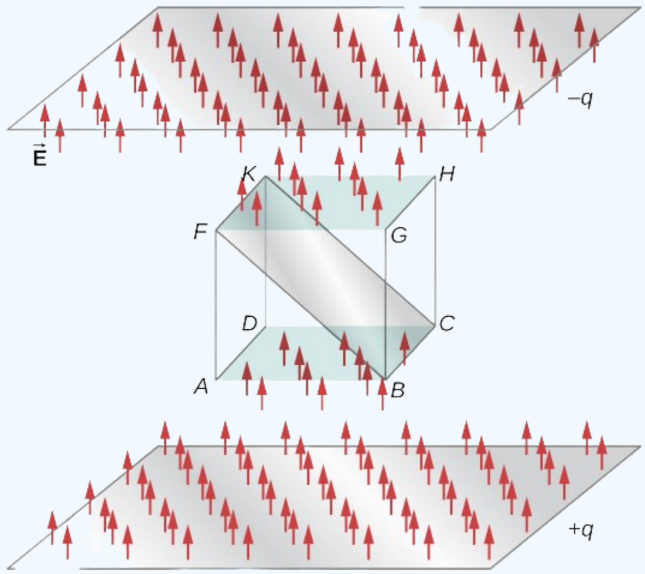


Figure $2.2.8$: Calculating the flux of E_0 through a closed cubic surface.

Strategy

Apply the definition of flux: $\Phi = \vec{E} \cdot \vec{A}\ (\text{uniform } \vec{E})$, noting that a closed surface eliminates the ambiguity in the direction of the area vector.

Solution

Through the top face of the cube $\Phi = \vec{E}_0 \cdot \vec{A} = E_0 A$.

Through the bottom face of the cube, $\Phi = \vec{E}_0 \cdot \vec{A} = -E_0 A$, because the area vector here points downward.

Along the other four sides, the direction of the area vector is perpendicular to the direction of the electric field. Therefore, the scalar product of the electric field with the area vector is zero, giving zero flux.

The net flux is $\Phi_{net} = E_0 A - E_0 A + 0 + 0 + 0 + 0 = 0$.

Significance

The net flux of a uniform electric field through a closed surface is zero.

✓ Example 2.2.3: Electric Flux through a Plane, Integral Method

A uniform electric field $\vec{E}$ of magnitude 10 N/C is directed parallel to the yz-plane at 30° above the xy-plane, as shown in Figure $2.2.9$. What is the electric flux through the plane surface of area $6.0\ \mathrm{m}^2$ located in the xz-plane? Assume that $\hat{n}$ points in the positive y-direction.

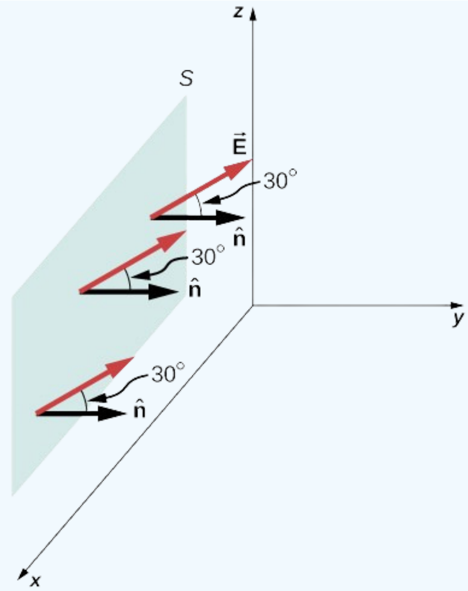


Figure 2.2.9: The electric field produces a net electric flux through the surface S.

Strategy

Apply $\Phi = \int_S \vec{E} \cdot \hat{n} dA$, where the direction and magnitude of the electric field are constant.

Solution

The angle between the uniform electric field $\vec{E}$ and the unit normal $\hat{n}$ to the planar surface is $30°$. Since both the direction and magnitude are constant, E comes outside the integral. All that is left is a surface integral over dA, which is A. Therefore, using the open-surface equation, we find that the electric flux through the surface is

$$\Phi = \int_S \vec{E} \cdot \hat{n}\, dA = EA\, \cos\theta$$

$$= (10\,\mathrm{N/C})(6.0\,\mathrm{m}^2)(\cos 30°) = 52\,\mathrm{N} \cdot \mathrm{m}^2/\mathrm{C}.$$

Significance

Again, the relative directions of the field and the area matter, and the general equation with the integral will simplify to the simple dot product of area and electric field.

? Exercise 2.2.1

What angle should there be between the electric field and the surface shown in Figure $2.2.9$ in the previous example so that no electric flux passes through the surface?

Answer

Place it so that its unit normal is perpendicular to $\vec{E}$.

✓ Example 2.2.4 : Inhomogeneous Electric Field

What is the total flux of the electric field $\vec{E} = cy^2\hat{k}$ through the rectangular surface shown in Figure $2.2.10$?

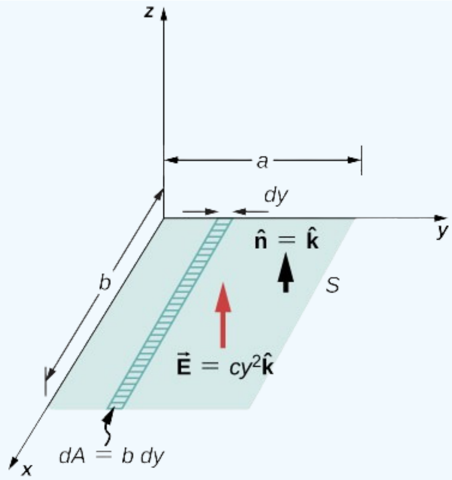


Figure 2.2.10: Since the electric field is not constant over the surface, an integration is necessary to determine the flux.

Strategy

Apply $\Phi = \int_S \vec{E} \cdot \hat{n}\, dA$. We assume that the unit normal $\hat{n}$ to the given surface points in the positive z-direction, so $\hat{n} = \hat{k}$. Since the electric field is not uniform over the surface, it is necessary to divide the surface into infinitesimal strips along which $\vec{E}$ is essentially constant. As shown in Figure $2.2.10$, these strips are parallel to the x-axis, and each strip has an area $dA = b\, dy$.

Solution

From the open surface integral, we find that the net flux through the rectangular surface is

$$\begin{aligned}\Phi &= \int_S \vec{E} \cdot \hat{n} dA = \int_0^a (cy^2\hat{k}) \cdot \hat{k}(b\, dy) \\ &= cb\int_0^a y^2 dy = \frac{1}{3}a^3bc.\end{aligned}$$

Significance

For a non-constant electric field, the integral method is required.

Exercise 2.2.2

If the electric field in Example $2.2.4$ is $\vec{E} = mx\hat{k}$. what is the flux through the rectangular area?

Answer

$mab^2/2$

2.3: Explaining Gauss's Law

Learning Objectives

By the end of this section, you will be able to:

- State Gauss's law
- Explain the conditions under which Gauss's law may be used
- Apply Gauss's law in appropriate systems

We can now determine the electric flux through an arbitrary closed surface due to an arbitrary charge distribution. We found that if a closed surface does not have any charge inside where an electric field line can terminate, then any electric field line entering the surface at one point must necessarily exit at some other point of the surface. Therefore, if a closed surface does not have any charges inside the enclosed volume, then the electric flux through the surface is zero. Now, what happens to the electric flux if there are some charges inside the enclosed volume? Gauss's law gives a quantitative answer to this question.

To get a feel for what to expect, let's calculate the electric flux through a spherical surface around a positive point charge q, since we already know the electric field in such a situation. Recall that when we place the point charge at the origin of a coordinate system, the electric field at a point P that is at a distance r from the charge at the origin is given by

$$\vec{E}_p = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{r},$$

where $\hat{r}$ is the radial vector from the charge at the origin to the point P. We can use this electric field to find the flux through the spherical surface of radius r, as shown in Figure $2.3.1$.

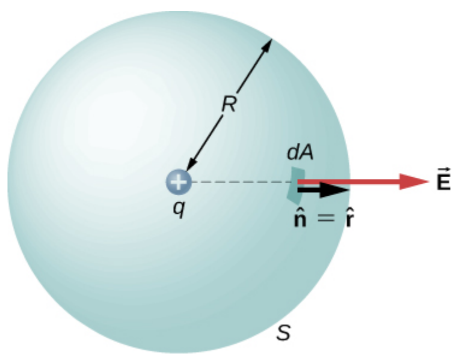


Figure $2.3.1$: A closed spherical surface surrounding a point charge q.

Then we apply $\Phi = \int_S \vec{E}\cdot\hat{n}dA$ to this system and substitute known values. On the sphere, $\hat{n}$ and $r = R$ so for an infinitesimal area dA,

$$\begin{aligned} d\Phi &= \vec{E}\cdot\hat{n}dA \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\hat{r}\cdot\hat{r}dA \\ &= \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}dA. \end{aligned}$$

We now find the net flux by integrating this flux over the surface of the sphere:

$$\Phi = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\oint_S dA = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}(4\pi R^2) = \frac{q}{\epsilon_0}.$$

where the total surface area of the spherical surface is $4\pi R^2$. This gives the flux through the closed spherical surface at radius r as

$$\Phi = \frac{q}{\epsilon_0}.$$

A remarkable fact about this equation is that the flux is independent of the size of the spherical surface. This can be directly attributed to the fact that the electric field of a point charge decreases as $1/r^2$ with distance, which just cancels the r^2 rate of increase of the surface area.

Electric Field Lines Picture

An alternative way to see why the flux through a closed spherical surface is independent of the radius of the surface is to look at the electric field lines. Note that every field line from q that pierces the surface at radius R_1 also pierces the surface at R_2 (Figure $2.3.2$).

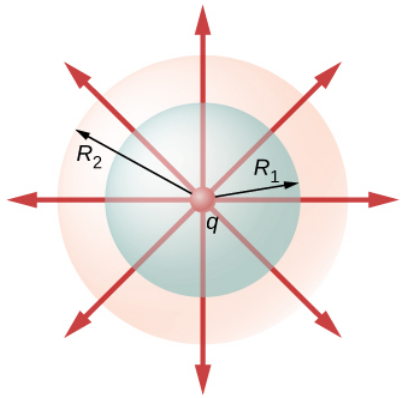


Figure $2.3.2$: Flux through spherical surfaces of radii R_1 and R_2 enclosing a charge q are equal, independent of the size of the surface, since all E-field lines that pierce one surface from the inside to outside direction also pierce the other surface in the same direction.

Therefore, the net number of electric field lines passing through the two surfaces from the inside to outside direction is equal. This net number of electric field lines, which is obtained by subtracting the number of lines in the direction from outside to inside from the number of lines in the direction from inside to outside gives a visual measure of the electric flux through the surfaces.

You can see that if no charges are included within a closed surface, then the electric flux through it must be zero. A typical field line enters the surface at dA_1 and leaves at dA_2. Every line that enters the surface must also leave that surface. Hence the net "flow" of the field lines into or out of the surface is zero (Figure $2.3.3a$). The same thing happens if charges of equal and opposite sign are included inside the closed surface, so that the total charge included is zero (Figure $2.3.3b$). A surface that includes the same amount of charge has the same number of field lines crossing it, regardless of the shape or size of the surface, as long as the surface encloses the same amount of charge (Figure $2.3.3c$).

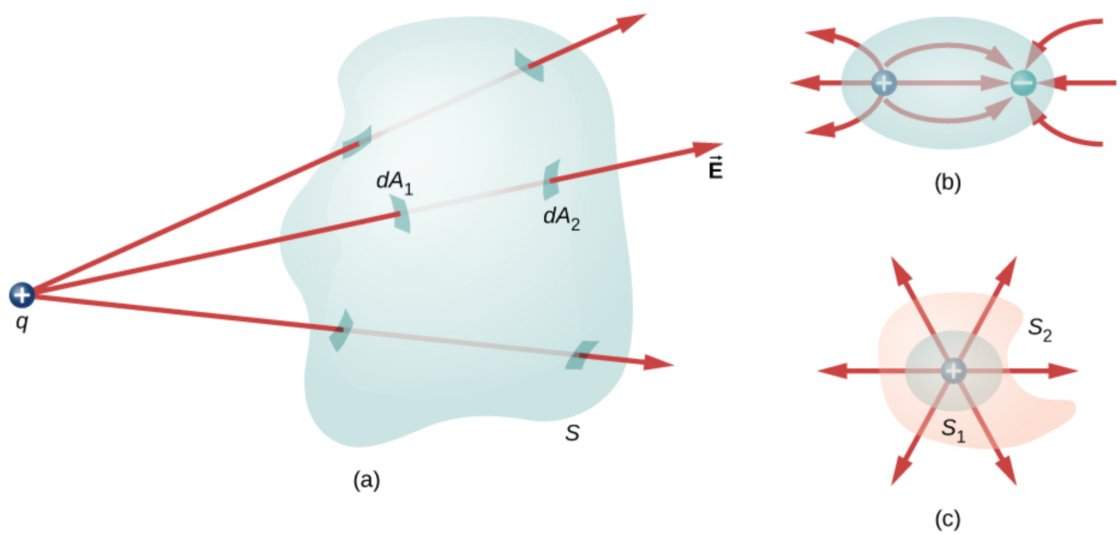


Figure $2.3.3$: Understanding the flux in terms of field lines. (a) The electric flux through a closed surface due to a charge outside that surface is zero. (b) Charges are enclosed, but because the net charge included is zero, the net flux through the closed surface is also zero. (c) The shape and size of the surfaces that enclose a charge does not matter because all surfaces enclosing the same charge have the same flux.

Statement of Gauss's Law

Gauss's law generalizes this result to the case of any number of charges and any location of the charges in the space inside the closed surface. According to Gauss's law, the flux of the electric field $\vec{E}$ through any closed surface, also called a **Gaussian surface**, is equal to the net charge enclosed (q_{enc}) divided by the permittivity of free space (ϵ_0):

$$\Phi_{\text{Closed Surface}} = \frac{q_{enc}}{\epsilon_0}.$$

This equation holds for *charges of either sign*, because we define the area vector of a closed surface to point outward. If the enclosed charge is negative (Figure $2.3.4b$), then the flux through either S or S' is negative.

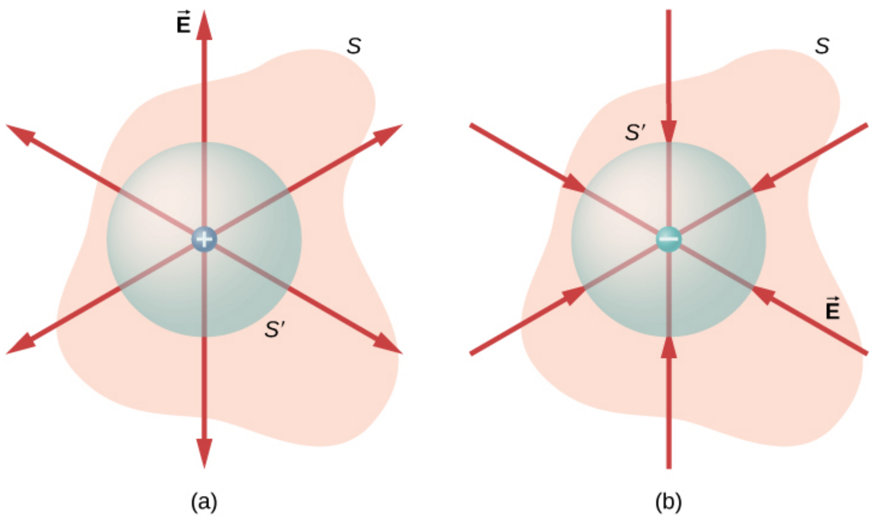


Figure $2.3.4$: The electric flux through any closed surface surrounding a point charge q is given by Gauss's law. (a) Enclosed charge is positive. (b) Enclosed charge is negative.

The Gaussian surface does not need to correspond to a real, physical object; indeed, it rarely will. It is a mathematical construct that may be of any shape, provided that it is closed. However, since our goal is to integrate the flux over it, we tend to choose shapes that are highly symmetrical.

If the charges are discrete point charges, then we just add them. If the charge is described by a continuous distribution, then we need to integrate appropriately to find the total charge that resides inside the enclosed volume. For example, the flux through the Gaussian surface S of Figure $2.3.5$ is

$$\Phi = (q_1 + q_2 + q_5)/\epsilon_0.$$

Note that q_{enc} is simply the sum of the point charges. If the charge distribution were continuous, we would need to integrate appropriately to compute the total charge within the Gaussian surface.

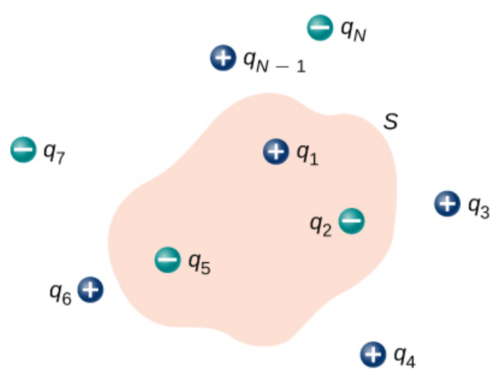


Figure $2.3.5$: The flux through the Gaussian surface shown, due to the charge distribution, is $\Phi = (q_1 + q_2 + q_5)/\epsilon_0$.

Recall that the principle of superposition holds for the electric field. Therefore, the total electric field at any point, including those on the chosen Gaussian surface, is the sum of all the electric fields present at this point. This allows us to write Gauss's law in terms of the total electric field.

Gauss's Law

The flux Φ of the electric field $\vec{E}$ through any closed surface **S** (a Gaussian surface) is equal to the net charge enclosed (q_{enc}) divided by the permittivity of free space (ϵ_0):

$$\Phi = \oint_S \vec{E} \cdot \hat{n} dA = \frac{q_{enc}}{\epsilon_0}.$$

To use Gauss's law effectively, you must have a clear understanding of what each term in the equation represents. The field $\vec{E}$ is the **total electric field** at every point on the Gaussian surface. This total field includes contributions from charges both inside and outside the Gaussian surface. However, q_{enc} is just the charge **inside** the Gaussian surface. Finally, the Gaussian surface is any closed surface in space. That surface can coincide with the actual surface of a conductor, or it can be an imaginary geometric surface. The only requirement imposed on a Gaussian surface is that it be closed (Figure $2.3.5$).

Figure $2.3.6$: A **Klein bottle** partially filled with a liquid. Could the Klein bottle be used as a Gaussian surface?

Example $2.3.1$: Electric Flux through Gaussian Surfaces

Calculate the electric flux through each Gaussian surface shown in Figure $2.3.7$.

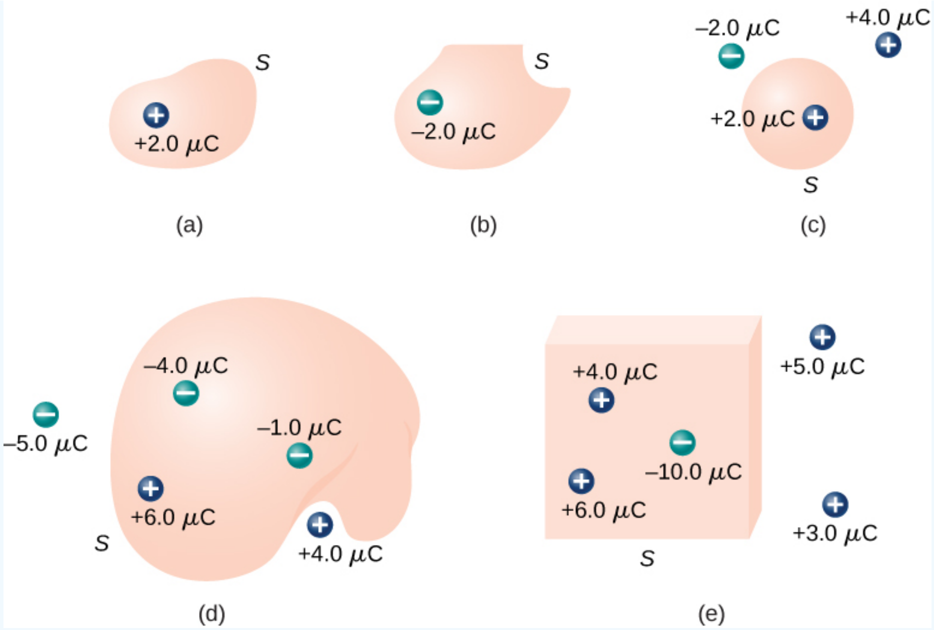


Figure 2.3.7: Various Gaussian surfaces and charges.

Strategy

From Gauss's law, the flux through each surface is given by q_{enc}/ϵ_0, where q_{enc} is the charge enclosed by that surface.

Solution

For the surfaces and charges shown, we find

a. $\Phi = \frac{2.0\ \mu\mathrm{C}}{\epsilon_0} = 2.3 \times 10^5\ \mathrm{N \cdot m^2/C}$.

b. $\Phi = \frac{-2.0\ \mu\mathrm{C}}{\epsilon_0} = -2.3 \times 10^5\ \mathrm{N \cdot m^2/C}$.

c. $\Phi = \frac{2.0\ \mu\mathrm{C}}{\epsilon_0} = 2.3 \times 10^5\ \mathrm{N \cdot m^2/C}$.

d. $\Phi = \frac{-4.0\ \mu\mathrm{C} + 6.0\ \mu\mathrm{C} - 1.0\ \mu\mathrm{C}}{\epsilon_0} = 1.1 \times 10^5\ \mathrm{N \cdot m^2/C}$.

e. $\Phi = \frac{4.0\ \mu\mathrm{C} + 6.0\ \mu\mathrm{C} - 10.0\ \mu\mathrm{C}}{\epsilon_0} = 0$.

Significance

In the special case of a closed surface, the flux calculations become a sum of charges. In the next section, this will allow us to work with more complex systems.

? Exercise 2.3.1

Calculate the electric flux through the closed cubical surface for each charge distribution shown in Figure 2.3.8.

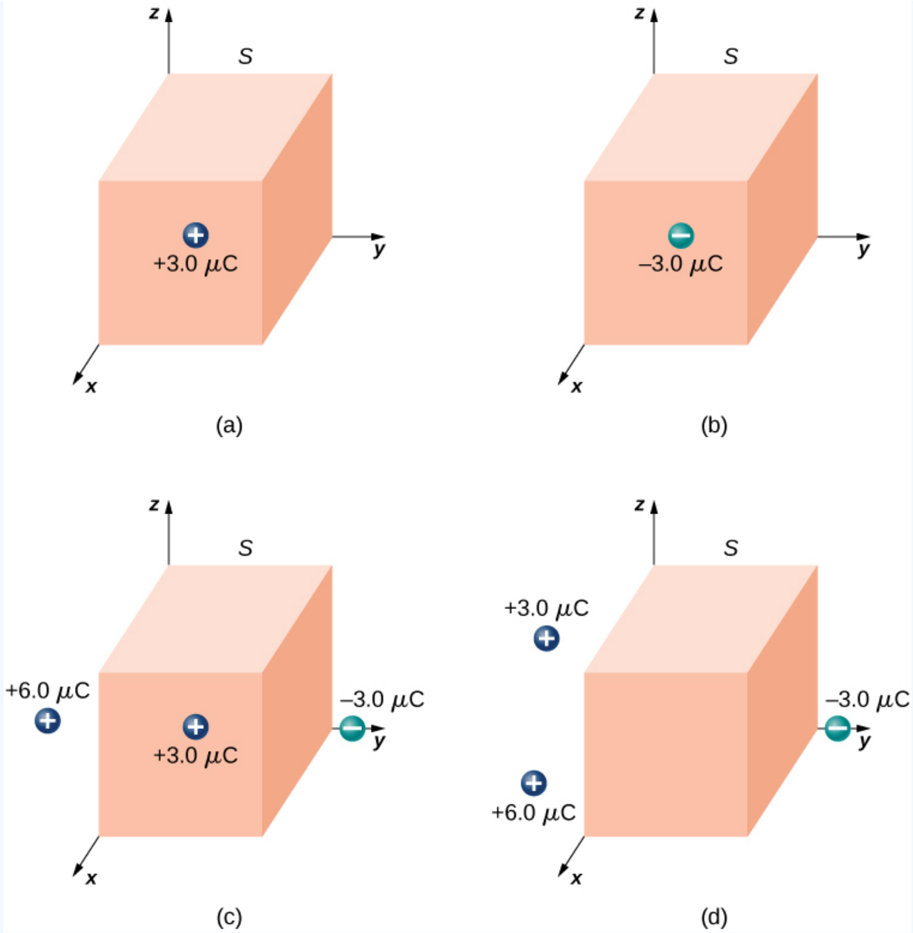


Figure 2.3.8: A cubical Gaussian surface with various charge distributions.

Answer a

$3.4 \times 10^5 \, \mathrm{N \cdot m^2/C}$

Answer b

$-3.4 \times 10^5 \, \mathrm{N \cdot m^2/C}$

Answer c

$3.4 \times 10^5 \, \mathrm{N \cdot m^2/C}$

Answer d

0

Use this simulation to adjust the magnitude of the charge and the radius of the Gaussian surface around it. See how this affects the total flux and the magnitude of the electric field at the Gaussian surface.

2.4: Applying Gauss's Law

Learning Objectives

By the end of this section, you will be able to:

- Explain what spherical, cylindrical, and planar symmetry are
- Recognize whether or not a given system possesses one of these symmetries
- Apply Gauss's law to determine the electric field of a system with one of these symmetries

Gauss's law is very helpful in determining expressions for the electric field, even though the law is not directly about the electric field; it is about the electric flux. It turns out that in situations that have certain symmetries (spherical, cylindrical, or planar) in the charge distribution, we can deduce the electric field based on knowledge of the electric flux. In these systems, we can find a Gaussian surface S over which the electric field has constant magnitude. Furthermore, if $\vec{E}$ is parallel to $\hat{n}$ everywhere on the surface, then $\vec{E}\cdot\hat{n}=E$. (If $\vec{E}$ and $\hat{n}$ are antiparallel everywhere on the surface, $\vec{E}\cdot\hat{n}=-E$.) Gauss's law then simplifies to

$$\Phi=\oint_S \vec{E}\cdot\hat{n}dA=E\oint_S dA=EA=\frac{q_{enc}}{\epsilon_0},$$

where A is the area of the surface. Note that these symmetries lead to the transformation of the flux integral into a product of the magnitude of the electric field and an appropriate area. When you use this flux in the expression for Gauss's law, you obtain an algebraic equation that you can solve for the magnitude of the electric field, which looks like

$$E\approx\frac{q_{enc}}{\epsilon_0\,\text{area}}.$$

The direction of the electric field at point P is obtained from the symmetry of the charge distribution and the type of charge in the distribution. Therefore, Gauss's law can be used to determine $\vec{E}$. Here is a summary of the steps we will follow:

Problem-Solving Strategy: Gauss's Law

1. *Identify the spatial symmetry of the charge distribution.* This is an important first step that allows us to choose the appropriate Gaussian surface. As examples, an isolated point charge has spherical symmetry, and an infinite line of charge has cylindrical symmetry.
2. *Choose a Gaussian surface with the same symmetry as the charge distribution and identify its consequences.* With this choice, $\vec{E}\cdot\hat{n}$ is easily determined over the Gaussian surface.
3. *Evaluate the integral* $\oint_S \vec{E}\cdot\hat{n}\,dA$ *over the Gaussian surface, that is, calculate the flux through the surface.* The symmetry of the Gaussian surface allows us to factor $\vec{E}\cdot\hat{n}$ outside the integral.
4. *Determine the amount of charge enclosed by the Gaussian surface.* This is an evaluation of the right-hand side of the equation representing Gauss's law. It is often necessary to perform an integration to obtain the net enclosed charge.
5. *Evaluate the electric field of the charge distribution.* The field may now be found using the results of steps 3 and 4.

Basically, there are only three types of symmetry that allow Gauss's law to be used to deduce the electric field. They are

- A charge distribution with spherical symmetry
- A charge distribution with cylindrical symmetry
- A charge distribution with planar symmetry

To exploit the symmetry, we perform the calculations in appropriate coordinate systems and use the right kind of Gaussian surface for that symmetry, applying the remaining four steps.

Charge Distribution with Spherical Symmetry

A charge distribution has **spherical symmetry** if the density of charge depends only on the distance from a point in space and not on the direction. In other words, if you rotate the system, it doesn't look different. For instance, if a sphere of radius R is uniformly charged with charge density ρ_0 then the distribution has spherical symmetry (Figure $2.4.1a$). On the other hand, if a sphere of radius R is charged so that the top half of the sphere has uniform charge density ρ_1 and the bottom half has a uniform charge density $\rho_2 \neq \rho_1$ then the sphere does not have spherical symmetry because the charge density depends on the direction (Figure $2.4.1b$). Thus, it is not the shape of the object but rather the shape of the charge distribution that determines whether or not a system has spherical symmetry.

Figure $2.4.1c$ shows a sphere with four different shells, each with its own uniform charge density. Although this is a situation where charge density in the full sphere is not uniform, the charge density function depends only on the distance from the center and not on the direction. Therefore, this charge distribution does have spherical symmetry.

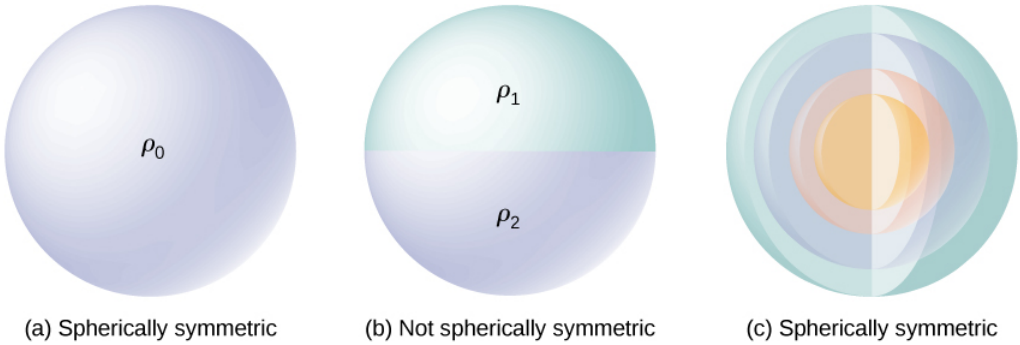


Figure $2.4.1$: Illustrations of spherically symmetrical and nonsymmetrical systems. Different shadings indicate different charge densities. Charges on spherically shaped objects do not necessarily mean the charges are distributed with spherical symmetry. The spherical symmetry occurs only when the charge density does not depend on the direction. In (a), charges are distributed uniformly in a sphere. In (b), the upper half of the sphere has a different charge density from the lower half; therefore, (b) does not have spherical symmetry. In (c), the charges are in spherical shells of different charge densities, which means that charge density is only a function of the radial distance from the center; therefore, the system has spherical symmetry.

One good way to determine whether or not your problem has spherical symmetry is to look at the charge density function in spherical coordinates, $\rho(r, \theta, \phi)$. If the charge density is only a function of r, that is $\rho = \rho(r)$, then you have spherical symmetry. If the density depends on θ or ϕ, you could change it by rotation; hence, you would not have spherical symmetry.

Consequences of symmetry

In all spherically symmetrical cases, the electric field at any point must be radially directed, because the charge and, hence, the field must be invariant under rotation. Therefore, using spherical coordinates with their origins at the center of the spherical charge distribution, we can write down the expected form of the electric field at a point P located at a distance r from the center:

$$\text{Spherical symmetry}: \; \vec{E}_P = E_P(r)\hat{r},$$

where $\hat{r}$ is the unit vector pointed in the direction from the origin to the field point P. The radial component E_p of the electric field can be positive or negative. When $E_p > 0$, the electric field at P points away from the origin, and when $E_p < 0$, the electric field at P points toward the origin.

Gaussian surface and flux calculations

We can now use this form of the electric field to obtain the flux of the electric field through the Gaussian surface. For spherical symmetry, the Gaussian surface is a closed spherical surface that has the same center as the center of the charge distribution. Thus, the direction of the area vector of an area element on the Gaussian surface at any point is parallel to the direction of the electric field at that point, since they are both radially directed outward (Figure $2.4.2$).

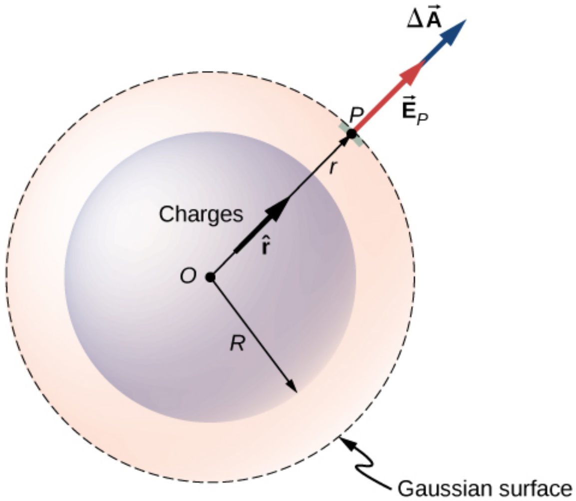


Figure $2.4.2$: The electric field at any point of the spherical Gaussian surface for a spherically symmetrical charge distribution is parallel to the area element vector at that point, giving flux as the product of the magnitude of electric field and the value of the area. Note that the radius R of the charge distribution and the radius r of the Gaussian surface are different quantities.

The magnitude of the electric field $\vec{E}$ must be the same everywhere on a spherical Gaussian surface concentric with the distribution. For a spherical surface of radius r:

$$\Phi = \oint_S \vec{E}_p \cdot \hat{n} dA = E_p \oint_S dA = E_p \, 4\pi r^2.$$

Using Gauss's law

According to Gauss's law, the flux through a closed surface is equal to the total charge enclosed within the closed surface divided by the permittivity of vacuum ϵ_0. Let q_{enc} be the total charge enclosed inside the distancer$S$ from the origin, which is the space inside the Gaussian spherical surface of radius r. This gives the following relation for Gauss's law:

$$4\pi r^2 E = \frac{q_{enc}}{\epsilon_0}.$$

Hence, the electric field at point P that is a distance r from the center of a spherically symmetrical charge distribution has the following magnitude and direction:

$$\text{Magnitude: } E(r) = \frac{1}{4\pi\epsilon_0} \frac{q_{enc}}{r^2}$$

Direction: radial from O to P or from P to O.

The direction of the field at point P depends on whether the charge in the sphere is positive or negative. For a net positive charge enclosed within the Gaussian surface, the direction is from O to P, and for a net negative charge, the direction is from P to O. This is all we need for a point charge, and you will notice that the result above is identical to that for a point charge. However, Gauss's law becomes truly useful in cases where the charge occupies a finite volume.

Computing Enclosed Charge

The more interesting case is when a spherical charge distribution occupies a volume, and asking what the electric field inside the charge distribution is thus becomes relevant. In this case, the charge enclosed depends on the distance r of the field point relative to the radius of the charge distribution R, such as that shown in Figure $2.4.3$.

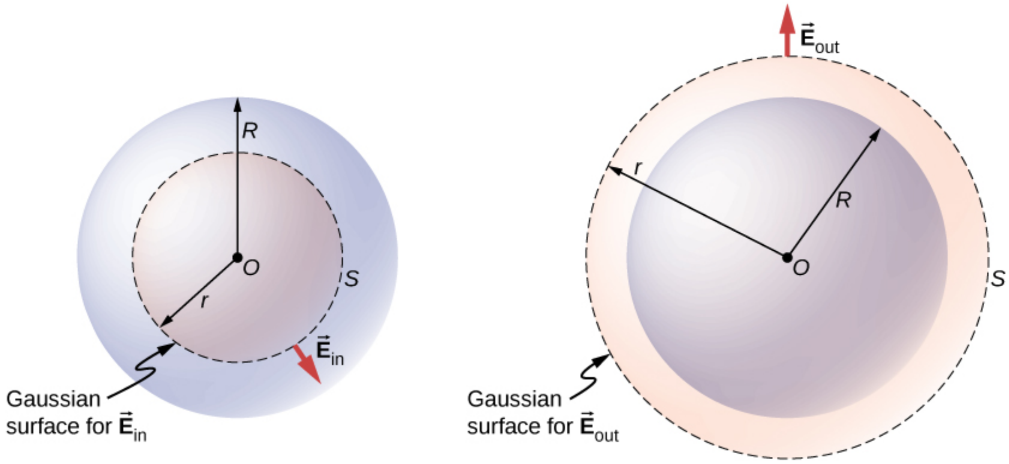


Figure 2.4.3: A spherically symmetrical charge distribution and the Gaussian surface used for finding the field (a) inside and (b) outside the distribution.

If point P is located outside the charge distribution—that is, if $r \geq R$ —then the Gaussian surface containing P encloses all charges in the sphere. In this case, q_{enc} equals the total charge in the sphere. On the other hand, if point P is within the spherical charge distribution, that is, if $r < R$, then the Gaussian surface encloses a smaller sphere than the sphere of charge distribution. In this case, q_{enc} is less than the total charge present in the sphere. Referring to Figure 2.4.3, we can write q_{enc} as

$$q_{enc} = q_{tot} \text{ (total charge) if } r \geq R$$

$$q_{enc} = q_{within\ r<R} \text{ (only charge within } r < R) \text{ if } r < R$$

The field at a point outside the charge distribution is also called $\vec{E}_{out}$, and the field at a point inside the charge distribution is called $\vec{E}_{in}$. Focusing on the two types of field points, either inside or outside the charge distribution, we can now write the magnitude of the electric field as

$$P \text{ outside sphere } E_{out} = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{r^2}$$

$$P \text{ inside sphere } E_{in} = \frac{1}{4\pi\epsilon_0}\frac{q_{\text{within } r<R}}{r^2}.$$

Note that the electric field outside a spherically symmetrical charge distribution is identical to that of a point charge at the center that has a charge equal to the total charge of the spherical charge distribution. This is remarkable since the charges are not located at the center only. We now work out specific examples of spherical charge distributions, starting with the case of a uniformly charged sphere.

✔ Uniformly Charged Sphere

A sphere of radius R, such as that shown in Figure 2.4.3, has a uniform volume charge density ρ_0. Find the electric field at a point outside the sphere and at a point inside the sphere.

Strategy

Apply the Gauss's law problem-solving strategy, where we have already worked out the flux calculation.

Solution

The charge enclosed by the Gaussian surface is given by

$$q_{enc} = \int \rho_0 dV = \int_0^r \rho_0 4\pi r'^2 dr' = \rho\left(\frac{4}{3}\pi r^3\right).$$

The answer for electric field amplitude can then be written down immediately for a point outside the sphere, labeled E_{out} and a point inside the sphere, labeled E_{in}.

$$E_{out} = \frac{1}{4\pi\epsilon_0}\frac{q_{tot}}{r^2},\ q_{tot} = \frac{4}{3}\pi R^3\,\rho_0,$$

$$E_{in} = \frac{q_{enc}}{4\pi\epsilon_0 r^2} = \frac{\rho_0 r}{3\epsilon_0},\ since\, q_{enc} = \frac{4}{3}\pi r^3 \rho_0.$$

It is interesting to note that the magnitude of the electric field increases inside the material as you go out, since the amount of charge enclosed by the Gaussian surface increases with the volume. Specifically, the charge enclosed grows $\propto r^3$, whereas the field from each infinitesimal element of charge drops off $\propto 1/r^2$ with the net result that the electric field within the distribution increases in strength linearly with the radius. The magnitude of the electric field outside the sphere decreases as you go away from the charges, because the included charge remains the same but the distance increases. Figure $2.4.4$ displays the variation of the magnitude of the electric field with distance from the center of a uniformly charged sphere.

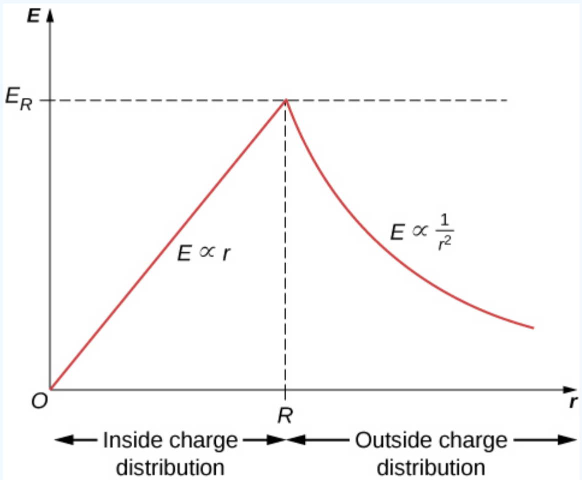


Figure $2.4.4$: Electric field of a uniformly charged, non-conducting sphere increases inside the sphere to a maximum at the surface and then decreases as $1/r^2$. Here, $E_R = \frac{\rho_0 R}{3\epsilon_0}$. The electric field is due to a spherical charge distribution of uniform charge density and total charge Q as a function of distance from the center of the distribution.

The direction of the electric field at any point P is radially outward from the origin if ρ_0 is positive, and inward (i.e., toward the center) if ρ_0 is negative. The electric field at some representative space points are displayed in Figure $2.4.5$ whose radial coordinates **r** are $r = R/2$, $r = R$, and $r = 2R$.

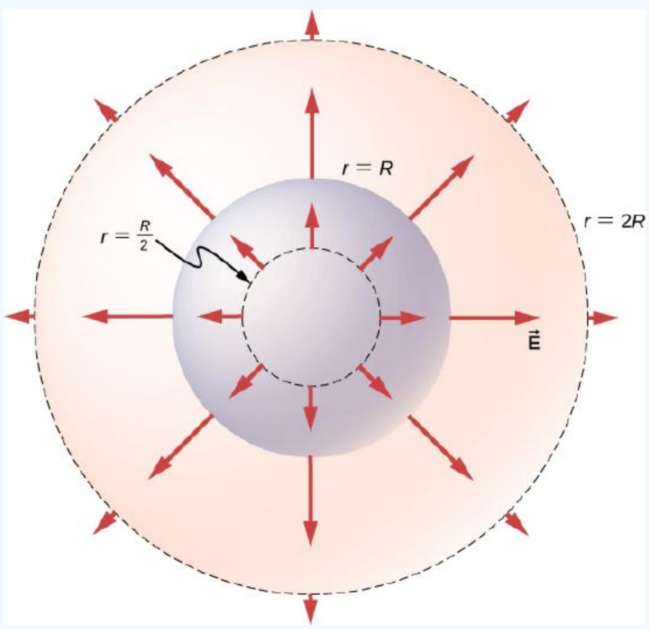


Figure $2.4.5$: Electric field vectors inside and outside a uniformly charged sphere.

Significance

Notice that E_{out} has the same form as the equation of the electric field of an isolated point charge. In determining the electric field of a uniform spherical charge distribution, we can therefore assume that all of the charge inside the appropriate spherical Gaussian surface is located at the center of the distribution.

✓ Non-Uniformly Charged Sphere

A non-conducting sphere of radius R has a non-uniform charge density that varies with the distance from its center as given by

$$\rho(r) = ar^n (r \leq R;\ n \geq 0),$$

where a is a constant. We require $n \geq 0$ so that the charge density is not undefined at $r = 0$. Find the electric field at a point outside the sphere and at a point inside the sphere.

Strategy

Apply the Gauss's law strategy given above, where we work out the enclosed charge integrals separately for cases inside and outside the sphere.

Solution

Since the given charge density function has only a radial dependence and no dependence on direction, we have a spherically symmetrical situation. Therefore, the magnitude of the electric field at any point is given above and the direction is radial. We just need to find the enclosed charge q_{enc}, which depends on the location of the field point.

A note about symbols: We use r' for locating charges in the charge distribution and **r** for locating the field point(s) at the Gaussian surface(s). The letter R is used for the radius of the charge distribution.

As charge density is not constant here, we need to integrate the charge density function over the volume enclosed by the Gaussian surface. Therefore, we set up the problem for charges in one spherical shell, say between r' and $r' + dr'$ as shown in Figure $2.4.6$. The volume of charges in the shell of infinitesimal width is equal to the product of the area of surface $4\pi r'^2$ and the thickness dr'. Multiplying the volume with the density at this location, which is ar'^n, gives the charge in the shell:

$$dq = ar'^n 4\pi r'^2 dr'.$$

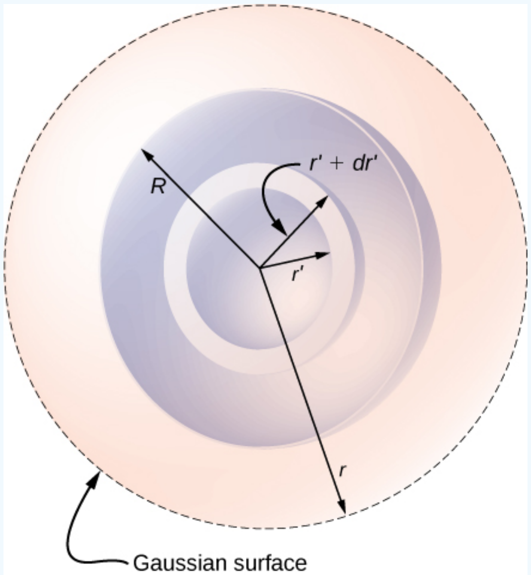


Figure $2.4.6$: Spherical symmetry with non-uniform charge distribution. In this type of problem, we need four radii: R is the radius of the charge distribution, r is the radius of the Gaussian surface, r' is the inner radius of the spherical shell, and $r' + dr'$ is the outer radius of the spherical shell. The spherical shell is used to calculate the charge enclosed within the Gaussian surface. The range for r' is from 0 to r for the field at a point inside the charge distribution and from 0 to R for the field at a point outside the charge distribution. If $r > R$, then the Gaussian surface encloses more volume than the charge distribution, but the additional volume does not contribute to q_{enc}.

(a) **Field at a point outside the charge distribution.** In this case, the Gaussian surface, which contains the field point P, has a radius r that is greater than the radius R of the charge distribution, $r > R$. Therefore, all charges of the charge distribution are enclosed within the Gaussian surface. Note that the space between $r' = R$ and $r' = r$ is empty of charges and therefore does not contribute to the integral over the volume enclosed by the Gaussian surface:

$$q_{enc} = \int dq = \int_0^R ar'^n 4\pi r'^2 dr' = \frac{4\pi a}{n+3} R^{n+3}.$$

This is used in the general result for E_{out} above to obtain the electric field at a point outside the charge distribution as

$$\vec{E}_{out} = \left[\frac{aR^{n+3}}{\epsilon_0(n+3)}\right] \frac{1}{r^2}\hat{r},$$

where $\hat{r}$ is a unit vector in the direction from the origin to the field point at the Gaussian surface.

(b) **Field at a point inside the charge distribution.** The Gaussian surface is now buried inside the charge distribution, with $r < R$. Therefore, only those charges in the distribution that are within a distance r of the center of the spherical charge distribution count in r_{enc}:

$$q_{enc} = \int_0^r ar'^n 4\pi r'^2 dr' = \frac{4\pi a}{n+3} r^{n+3}.$$

Now, using the general result above for $\vec{E}_{in}$, we find the electric field at a point that is a distance r from the center and lies within the charge distribution as

$$\vec{E}_{in} = \left[\frac{a}{\epsilon_0(n+3)}\right] r^{n+1}\hat{r},$$

where the direction information is included by using the unit radial vector.

? Exercise 2.4.1

Check that the electric fields for the sphere reduce to the correct values for a point charge.

Answer

In this case, there is only $\vec{E}_{out}$. So, yes.

Charge Distribution with Cylindrical Symmetry

A charge distribution has **cylindrical symmetry** if the charge density depends only upon the distance r from the axis of a cylinder and must not vary along the axis or with direction about the axis. In other words, if your system varies if you rotate it around the axis, or shift it along the axis, you do not have cylindrical symmetry.

Figure $2.4.7$ shows four situations in which charges are distributed in a cylinder. A uniform charge density ρ_0 in an infinite straight wire has a cylindrical symmetry, and so does an infinitely long cylinder with constant charge density ρ_0. An infinitely long cylinder that has different charge densities along its length, such as a charge density ρ_1 for $z > 0$ and $\rho_2 \neq \rho_1$ for $z < 0$, does not have a usable cylindrical symmetry for this course. Neither does a cylinder in which charge density varies with the direction, such as a charge density ρ_1 for $0 \leq \theta < \pi$ and $\rho_2 \neq \rho_1$ for $\pi \leq \theta < 2\pi$. A system with concentric cylindrical shells, each with uniform charge densities, albeit different in different shells, as in FiFigure $2.4.7d$, does have cylindrical symmetry if they are infinitely long. The infinite length requirement is due to the charge density changing along the axis of a finite cylinder. In real systems, we don't have infinite cylinders; however, if the cylindrical object is considerably longer than the radius from it that we are interested in, then the approximation of an infinite cylinder becomes useful.

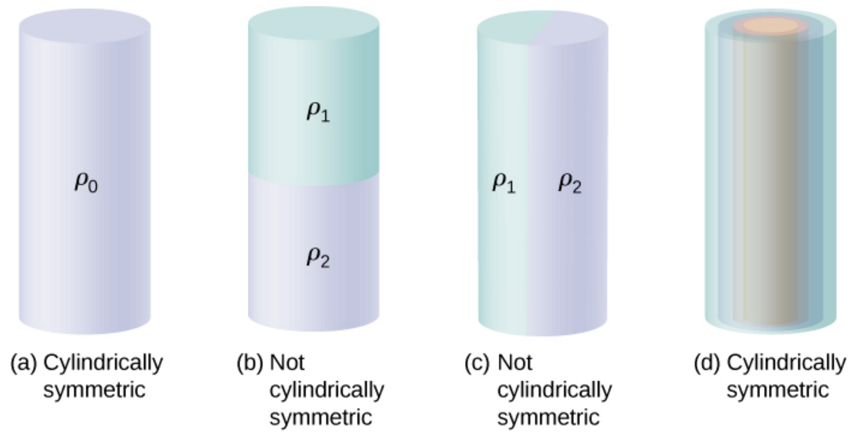


Figure 2.4.7: To determine whether a given charge distribution has cylindrical symmetry, look at the cross-section of an “infinitely long” cylinder. If the charge density does not depend on the polar angle of the cross-section or along the axis, then you have cylindrical symmetry. (a) Charge density is constant in the cylinder; (b) upper half of the cylinder has a different charge density from the lower half; (c) left half of the cylinder has a different charge density from the right half; (d) charges are constant in different cylindrical rings, but the density does not depend on the polar angle. Cases (a) and (d) have cylindrical symmetry, whereas (b) and (c) do not.

Consequences of symmetry

In all cylindrically symmetrical cases, the electric field E_p at any point P must also display cylindrical symmetry.

Cylindrical symmetry: $\vec{E}_p = E_p(r)\hat{r}$, where r is the distance from the axis and $\hat{r}$ is a unit vector directed perpendicularly away from the axis (Figure 2.4.8).

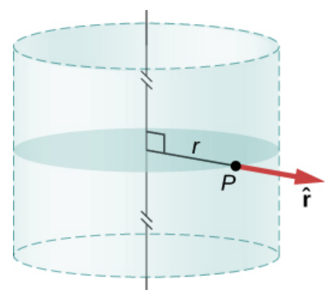


Figure 2.4.8: The electric field in a cylindrically symmetrical situation depends only on the distance from the axis. The direction of the electric field is pointed away from the axis for positive charges and toward the axis for negative charges.

Gaussian surface and flux calculation

To make use of the direction and functional dependence of the electric field, we choose a closed Gaussian surface in the shape of a cylinder with the same axis as the axis of the charge distribution. The flux through this surface of radius s and height L is easy to compute if we divide our task into two parts: (a) a flux through the flat ends and (b) a flux through the curved surface (Figure 2.4.9).

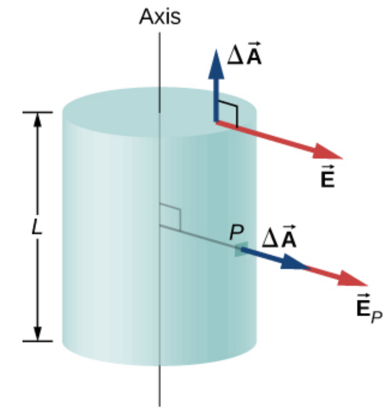


Figure 2.4.9: The Gaussian surface in the case of cylindrical symmetry. The electric field at a patch is either parallel or perpendicular to the normal to the patch of the Gaussian surface.

The electric field is perpendicular to the cylindrical side and parallel to the planar end caps of the surface. The flux through the cylindrical part is

$$\int_S \vec{E}\cdot\hat{n}\,dA = E\int_S dA = E(2\pi rL),$$

whereas the flux through the end caps is zero because $\vec{E}\cdot\hat{n}=0$ there. Thus, the flux is

$$\int_S \vec{E}\cdot\hat{n}\,dA = E(2\pi rL)+0+0 = 2\pi rLE.$$

Using Gauss's law

According to Gauss's law, the flux must equal the amount of charge within the volume enclosed by this surface, divided by the permittivity of free space. When you do the calculation for a cylinder of length L, you find that q_{enc} of Gauss's law is directly proportional to L. Let us write it as charge per unit length (λ_{enc}) times length L:

$$q_{enc} = \lambda_{enc}L.$$

Hence, Gauss's law for any cylindrically symmetrical charge distribution yields the following magnitude of the electric field a distance s away from the axis:

$$\text{Magnitude: } E(r) = \frac{\lambda_{enc}}{2\pi\epsilon_0}\frac{1}{r}.$$

The charge per unit length λ_{enc} depends on whether the field point is inside or outside the cylinder of charge distribution, just as we have seen for the spherical distribution.

Computing enclosed charge

Let R be the radius of the cylinder within which charges are distributed in a cylindrically symmetrical way. Let the field point P be at a distance s from the axis. (The side of the Gaussian surface includes the field point P.) When $r>R$ (that is, when P is outside the charge distribution), the Gaussian surface includes all the charge in the cylinder of radius R and length L. When $r<R$ (P is located inside the charge distribution), then only the charge within a cylinder of radius s and length L is enclosed by the Gaussian surface:

$\lambda_{enc} = (\text{total charge})\text{ if } r\geq R$

$\lambda_{enc} = (\text{only charge within } t<R)\text{ if } r<R$

✔ Uniformly Charged Cylindrical Shell

A very long non-conducting cylindrical shell of radius R has a uniform surface charge density σ_0 Find the electric field (a) at a point outside the shell and (b) at a point inside the shell.

Strategy

Apply the Gauss's law strategy given earlier, where we treat the cases inside and outside the shell separately.

Solution

a. **Electric field at a point outside the shell.** For a point outside the cylindrical shell, the Gaussian surface is the surface of a cylinder of radius $r > R$ and length L, as shown in Figure $2.4.10$. The charge enclosed by the Gaussian cylinder is equal to the charge on the cylindrical shell of length L. Therefore, λ_{enc} is given by

$$\lambda_{enc} = \frac{\sigma_0 2\pi RL}{L} = 2\pi R\sigma_0.$$

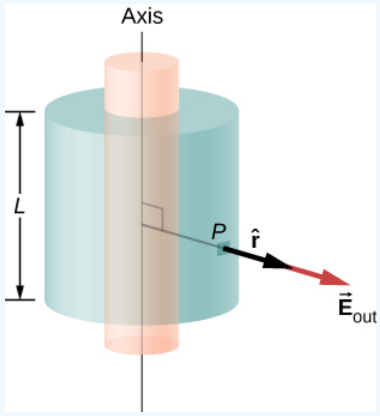


Figure 2.4.10: A Gaussian surface surrounding a cylindrical shell.

Hence, the electric field at a point P outside the shell at a distance r away from the axis is

$$\vec{E} = \frac{2\pi R\sigma_0}{2\pi\epsilon_0}\frac{1}{r}\hat{r} = \frac{R\sigma_0}{\epsilon_0}\frac{1}{r}\hat{r}\ (r > R)$$

where $\hat{r}$ is a unit vector, perpendicular to the axis and pointing away from it, as shown in the figure. The electric field at P points in the direction of $\hat{r}$ given in Figure $2.4.10$ if $\sigma_0 > 0$ and in the opposite direction to $\hat{r}$ if $\sigma_0 < 0$.

b. **Electric field at a point inside the shell.** For a point inside the cylindrical shell, the Gaussian surface is a cylinder whose radius r is less than R (Figure $2.4.11$). This means no charges are included inside the Gaussian surface:

$$\lambda_{enc} = 0.$$

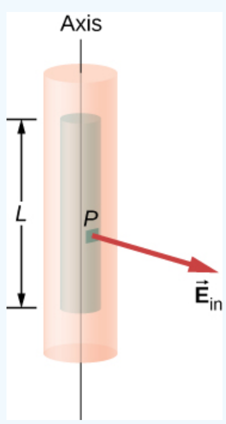


Figure 2.4.11: A Gaussian surface within a cylindrical shell.

This gives the following equation for the magnitude of the electric field E_{in} at a point whose r is less than R of the shell of charges.

$$E_{in} 2\pi rL = 0 (r < R),$$

This gives us

$$E_{in} = 0 (r < R).$$

Significance

Notice that the result inside the shell is exactly what we should expect: No enclosed charge means zero electric field. Outside the shell, the result becomes identical to a wire with uniform charge $R\sigma$.

Exercise 2.4.2

A thin straight wire has a uniform linear charge density λ_0. Find the electric field at a distance d from the wire, where d is much less than the length of the wire.

Answer

$\vec{E} = \frac{\lambda_0}{2\pi\epsilon_0}\frac{1}{d}\hat{r}$; This agrees with the calculation of Calculating Electric Fields of Charge Distributions where we found the electric field by integrating over the charged wire. Notice how much simpler the calculation of this electric field is with Gauss's law.

Charge Distribution with Planar Symmetry

A planar symmetry of charge density is obtained when charges are uniformly spread over a large flat surface. In planar symmetry, all points in a plane parallel to the plane of charge are identical with respect to the charges.

Consequences of symmetry

We take the plane of the charge distribution to be the xy-plane and we find the electric field at a space point P with coordinates (x, y, z). Since the charge density is the same at all (x, y)-coordinates in the $z = 0$ plane, by symmetry, the electric field at P cannot depend on the x- or y-coordinates of point P, as shown in Figure $2.4.12$. Therefore, the electric field at P can only depend on the distance from the plane and has a direction either toward the plane or away from the plane. That is, the electric field at P has only a nonzero z-component.

Uniform charges in xy plane: $\vec{E} = E(z)\hat{z}$ where z is the distance from the plane and $\hat{z}$ is the unit vector normal to the plane. Note that in this system, $E(z) = E(-z)$, although of course they point in opposite directions.

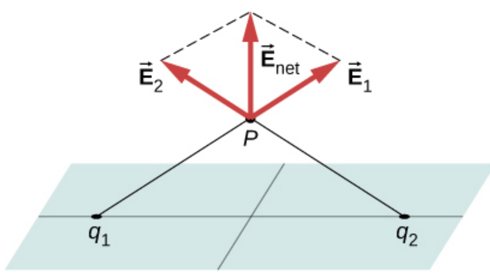


Figure $2.4.12$: The components of the electric field parallel to a plane of charges cancel out the two charges located symmetrically from the field point P. Therefore, the field at any point is pointed vertically from the plane of charges. For any point P and charge q_1, we can always find a q_2 with this effect.

Gaussian surface and flux calculation

In the present case, a convenient Gaussian surface is a box, since the expected electric field points in one direction only. To keep the Gaussian box symmetrical about the plane of charges, we take it to straddle the plane of the charges, such that one face containing the field point P is taken parallel to the plane of the charges. In Figure $2.4.13$, sides I and II of the Gaussian surface (the box) that are parallel to the infinite plane have been shaded. They are the only surfaces that give rise to nonzero flux because the electric field and the area vectors of the other faces are perpendicular to each other.

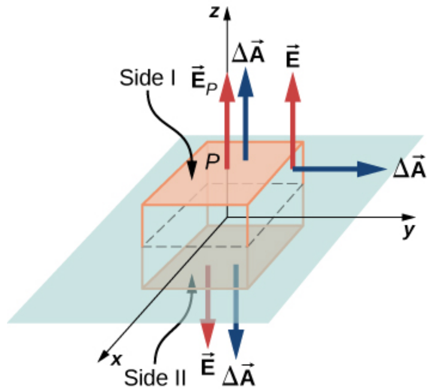


Figure 2.4.13: A thin charged sheet and the Gaussian box for finding the electric field at the field point P. The normal to each face of the box is from inside the box to outside. On two faces of the box, the electric fields are parallel to the area vectors, and on the other four faces, the electric fields are perpendicular to the area vectors.

Let A be the area of the shaded surface on each side of the plane and E_P be the magnitude of the electric field at point P. Since sides I and II are at the same distance from the plane, the electric field has the same magnitude at points in these planes, although the directions of the electric field at these points in the two planes are opposite to each other.

Magnitude at I or II: $E(z) = E_p$.

If the charge on the plane is positive, then the direction of the electric field and the area vectors are as shown in Figure $2.4.13$. Therefore, we find for the flux of electric field through the box

$$\Phi = \int_S \vec{E}_p \cdot \hat{n} dA = E_p A + E_p A + 0 + 0 + 0 + 0 = 2E_p A$$

where the zeros are for the flux through the other sides of the box. Note that if the charge on the plane is negative, the directions of electric field and area vectors for planes I and II are opposite to each other, and we get a negative sign for the flux. According to Gauss's law, the flux must equal q_{enc}/ϵ_0. From Figure $2.4.13$, we see that the charges inside the volume enclosed by the Gaussian box reside on an area A of the xy-plane. Hence,

$$q_{enc} = \sigma_0 A.$$

Using the equations for the flux and enclosed charge in Gauss's law, we can immediately determine the electric field at a point at height z from a uniformly charged plane in the xy-plane:

$$\vec{E}_p = \frac{\sigma_0}{2\epsilon_0} \hat{n}.$$

The direction of the field depends on the sign of the charge on the plane and the side of the plane where the field point P is located. Note that above the plane, $\hat{n} = +\hat{z}$, while below the plane, $\hat{n} = -\hat{z}$.

You may be surprised to note that the electric field does not actually depend on the distance from the plane; this is an effect of the assumption that the plane is infinite. In practical terms, the result given above is still a useful approximation for finite planes near the center.

2.5: Conductors in Electrostatic Equilibrium

Learning Objectives

By the end of this section, you will be able to:

- Describe the electric field within a conductor at equilibrium
- Describe the electric field immediately outside the surface of a charged conductor at equilibrium
- Explain why if the field is not as described in the first two objectives, the conductor is not at equilibrium

So far, we have generally been working with charges occupying a volume within an insulator. We now study what happens when free charges are placed on a conductor. Generally, in the presence of a (generally external) electric field, the free charge in a conductor redistributes and very quickly reaches electrostatic equilibrium. The resulting charge distribution and its electric field have many interesting properties, which we can investigate with the help of Gauss's law and the concept of electric potential.

The Electric Field inside a Conductor Vanishes

If an electric field is present inside a conductor, it exerts forces on the **free electrons** (also called conduction electrons), which are electrons in the material that are not bound to an atom. These free electrons then accelerate. However, moving charges by definition means nonstatic conditions, contrary to our assumption. Therefore, when electrostatic equilibrium is reached, the charge is distributed in such a way that the electric field inside the conductor vanishes.

If you place a piece of a metal near a positive charge, the free electrons in the metal are attracted to the external positive charge and migrate freely toward that region. The region the electrons move to then has an excess of electrons over the protons in the atoms and the region from where the electrons have migrated has more protons than electrons. Consequently, the metal develops a negative region near the charge and a positive region at the far end (Figure $2.5.1$). As we saw in the preceding chapter, this separation of equal magnitude and opposite type of electric charge is called **polarization**. If you remove the external charge, the electrons migrate back and neutralize the positive region.

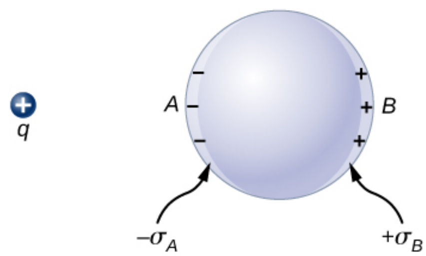


Figure $2.5.1$: Polarization of a metallic sphere by an external point charge $+q$. The near side of the metal has an opposite surface charge compared to the far side of the metal. The sphere is said to be polarized. When you remove the external charge, the polarization of the metal also disappears.

The polarization of the metal happens only in the presence of external charges. You can think of this in terms of electric fields. The external charge creates an external electric field. When the metal is placed in the region of this electric field, the electrons and protons of the metal experience electric forces due to this external electric field, but only the conduction electrons are free to move in the metal over macroscopic distances. The movement of the conduction electrons leads to the polarization, which creates an induced electric field in addition to the external electric field (Figure $2.5.2$). The net electric field is a vector sum of the fields of $+q$ and the surface charge densities $-\sigma_A$ and $+\sigma_B$. This means that the net field inside the conductor is different from the field outside the conductor.

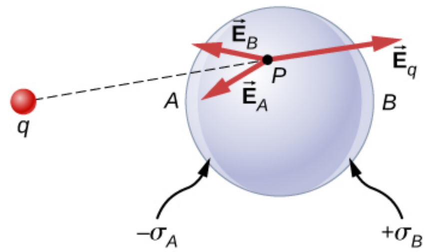


Figure $2.5.2$: In the presence of an external charge q, the charges in a metal redistribute. The electric field at any point has three contributions, from $+q$ and the induced charges $-\sigma_A$ and $+\sigma_B$. Note that the surface charge distribution will not be uniform in this case.

The redistribution of charges is such that the sum of the three contributions at any point P inside the conductor is

$$\vec{E}_p = \vec{E}_q + \vec{E}_B + \vec{E}_A = \vec{0}.$$

Now, thanks to Gauss's law, we know that there is no net charge enclosed by a Gaussian surface that is solely within the volume of the conductor at equilibrium. That is, $q_{enc} = 0$ and hence

$$\vec{E}_{net} = \vec{0} \text{ (at points inside a conductor).}$$

Charge on a Conductor

An interesting property of a conductor in static equilibrium is that extra charges on the conductor end up on the outer surface of the conductor, regardless of where they originate. Figure $2.5.3$ illustrates a system in which we bring an external positive charge inside the cavity of a metal and then touch it to the inside surface. Initially, the inside surface of the cavity is negatively charged and the outside surface of the conductor is positively charged. When we touch the inside surface of the cavity, the induced charge is neutralized, leaving the outside surface and the whole metal charged with a net positive charge.

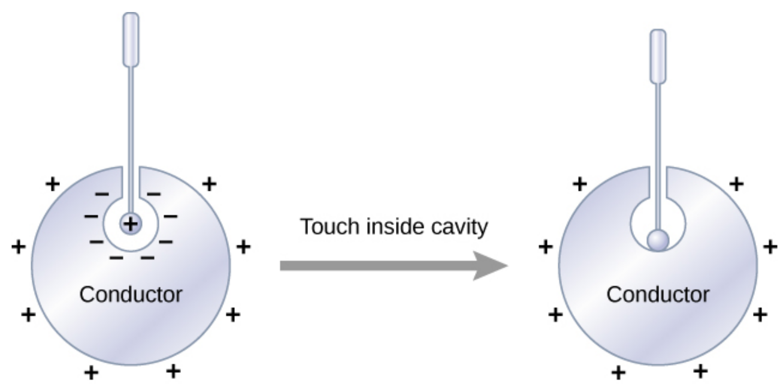


Figure $2.5.3$: Electric charges on a conductor migrate to the outside surface no matter where you put them initially.

To see why this happens, note that the Gaussian surface in iFigure $2.5.4$ (the dashed line) follows the contour of the actual surface of the conductor and is located an infinitesimal distance *within* it. Since $E = 0$ everywhere inside a conductor,

$$\oint \vec{E} \cdot \hat{n} dA = 0.$$

Thus, from Gauss' law, there is no net charge inside the Gaussian surface. But the Gaussian surface lies just below the actual surface of the conductor; consequently, there is no net charge inside the conductor. Any excess charge must lie on its surface.

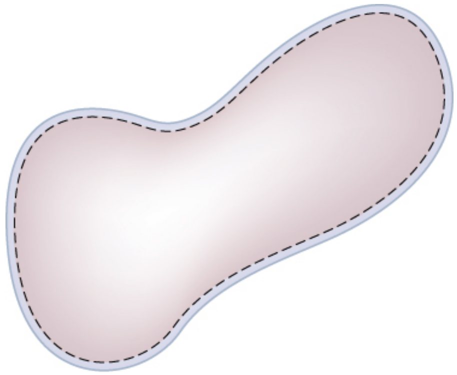


Figure $2.5.4$: The dashed line represents a Gaussian surface that is just beneath the actual surface of the conductor.

This particular property of conductors is the basis for an extremely accurate method developed by Plimpton and Lawton in 1936 to verify Gauss's law and, correspondingly, Coulomb's law. A sketch of their apparatus is shown in Figure $2.5.5$. Two spherical shells are connected to one another through an electrometer E, a device that can detect a very slight amount of charge flowing from one shell to the other. When switch S is thrown to the left, charge is placed on the outer shell by the battery B. Will charge flow through the electrometer to the inner shell?

No. Doing so would mean a violation of Gauss's law. Plimpton and Lawton did not detect any flow and, knowing the sensitivity of their electrometer, concluded that if the radial dependence in Coulomb's law were $1/r^{2+\delta}$, δ would be less than 2×10^{-9} [1]. More recent measurements place δ at less than 3×10^{-16} [2], a number so small that the validity of Coulomb's law seems indisputable.

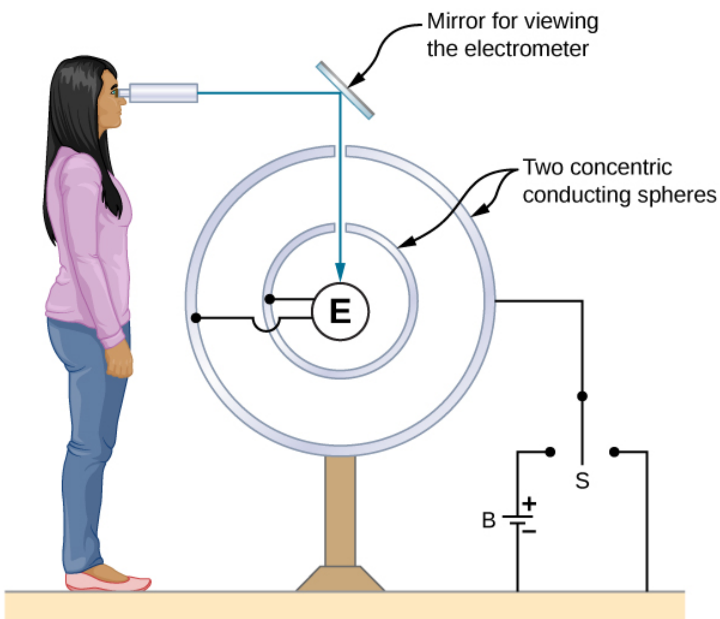


Figure $2.5.5$: A representation of the apparatus used by Plimpton and Lawton. Any transfer of charge between the spheres is detected by the electrometer E.

[1] S. Plimpton and W. Lawton. 1936. "A Very Accurate Test of Coulomb's Law of Force between Charges." Physical Review 50, No. 11:
1066, doi:10.1103/PhysRev.50.1066

[2] E. Williams, J. Faller, and H. Hill. 1971. "New Experimental Test of Coulomb's Law: A Laboratory Upper Limit on the Photon Rest
Mass." Physical Review Letters 26, No. 12: 721, doi:10.1103/PhysRevLett.26.721

The Electric Field at the Surface of a Conductor

If the electric field had a component parallel to the surface of a conductor, free charges on the surface would move, a situation contrary to the assumption of electrostatic equilibrium. Therefore, the electric field is always perpendicular to the surface of a conductor.

At any point just above the surface of a conductor, the surface charge density δ and the magnitude of the electric field E are related by

$$E = \frac{\sigma}{\epsilon_0}.$$

To see this, consider an infinitesimally small Gaussian cylinder that surrounds a point on the surface of the conductor, as in Figure $2.5.6$. The cylinder has one end face inside and one end face outside the surface. The height and cross-sectional area of the cylinder are δ and ΔA, respectively. The cylinder's sides are perpendicular to the surface of the conductor, and its end faces are parallel to the surface. Because the cylinder is infinitesimally small, the charge density σ is essentially constant over the surface enclosed, so the total charge inside the Gaussian cylinder is $\sigma\Delta A$. Now E is perpendicular to the surface of the conductor outside the conductor and vanishes within it, because otherwise, the charges would accelerate, and we would not be in equilibrium. Electric flux therefore crosses only the outer end face of the Gaussian surface and may be written as $E\Delta A$ since the cylinder is assumed to be small enough that E is approximately constant over that area. From Gauss' law,

$$E\Delta A = \frac{\sigma\Delta A}{\epsilon_0}.$$

Thus

$$E = \frac{\sigma}{\epsilon_0}.$$

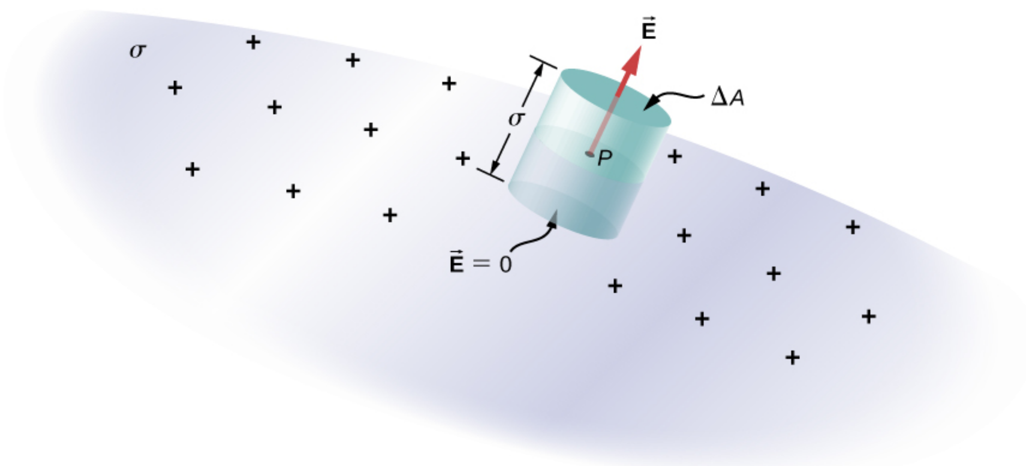


Figure $2.5.6$: An infinitesimally small cylindrical Gaussian surface surrounds point P, which is on the surface of the conductor. The field $\vec{E}$ is perpendicular to the surface of the conductor outside the conductor and vanishes within it.

✔ Electric Field of a Conducting Plate

The infinite conducting plate in Figure $2.5.7$ has a uniform surface charge density σ. Use Gauss' law to find the electric field outside the plate. Compare this result with that previously calculated directly.

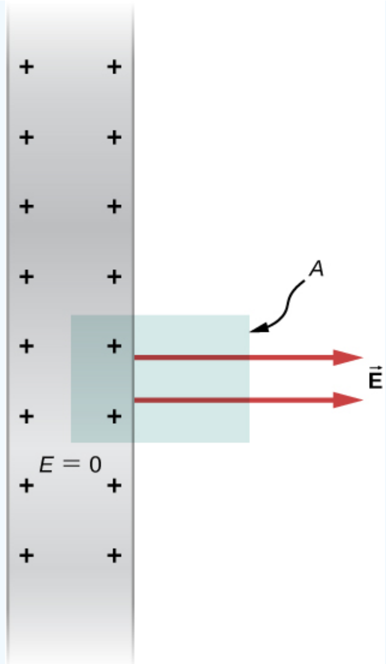


Figure 2.5.7: A side view of an infinite conducting plate and Gaussian cylinder with cross-sectional area A.

Strategy

For this case, we use a cylindrical Gaussian surface, a side view of which is shown.

Solution

The flux calculation is similar to that for an infinite sheet of charge from the previous chapter with one major exception: The left face of the Gaussian surface is inside the conductor where $\vec{E} = \vec{0}$, so the total flux through the Gaussian surface is EA rather than $2EA$. Then from Gauss' law,

$$EA = \frac{\sigma A}{\epsilon_0}$$

and the electric field outside the plate is

$$E = \frac{\sigma}{\epsilon_0}.$$

Significance

This result is in agreement with the result from the previous section, and consistent with the rule stated above.

Electric Field between Oppositely Charged Parallel Plates

Two large conducting plates carry equal and opposite charges, with a surface charge density σ of magnitude $6.81 \times 10^{-7}\,\mathrm{C/m^2}$, as shown in Figure $2.5.8$. The separation between the plates is $l = 6.50\,\mathrm{mm}$. What is the electric field between the plates?

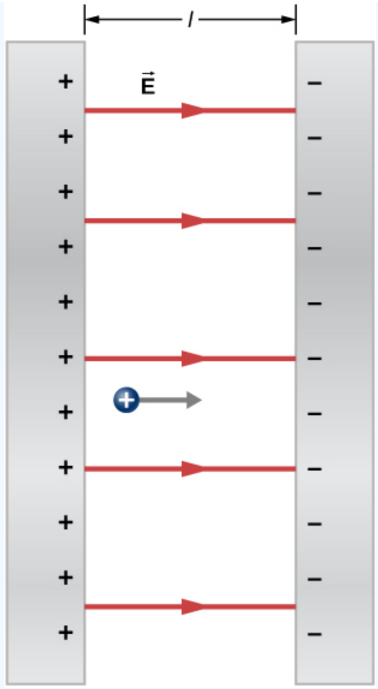


Figure 2.5.8: The electric field between oppositely charged parallel plates. A test charge is released at the positive plate.

Strategy Note that the electric field at the surface of one plate only depends on the charge on that plate. Thus, apply $E = \sigma/\epsilon_0$ with the given values.

Solution The electric field is directed from the positive to the negative plate, as shown in the figure, and its magnitude is given by

$$E = \frac{\sigma}{\epsilon_0} = \frac{6.81 \times 10^{-7}\,\mathrm{C/m^2}}{8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2}} = 7.69 \times 10^4\,\mathrm{N/C}$$

Significance

This formula is applicable to more than just a plate. Furthermore, two-plate systems will be important later.

✔ A Conducting Sphere

The isolated conducting sphere (Figure $2.5.9$) has a radius R and an excess charge q. What is the electric field both inside and outside the sphere?

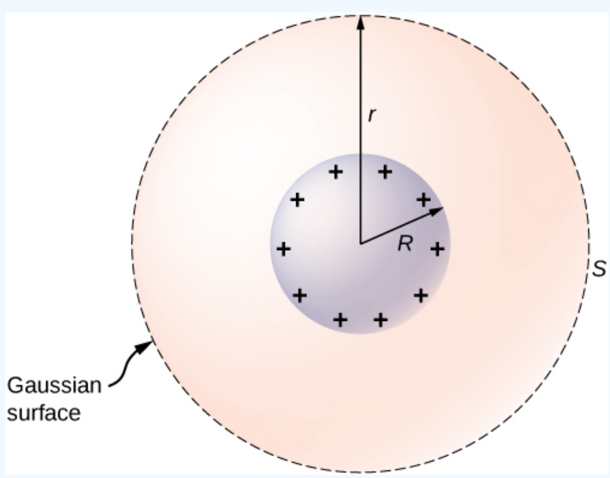


Figure 2.5.9: An isolated conducting sphere.

Strategy The sphere is isolated, so its surface change distribution and the electric field of that distribution are spherically symmetrical. We can therefore represent the field as $\vec{E} = E(r)\hat{r}$. To calculate $E(r)$, we apply Gauss's law over a closed spherical surface S of radius r that is concentric with the conducting sphere.

Solution

Since r is constant and $\hat{n} = \hat{r}$ on the sphere,

$$\oint_S \vec{E} \cdot \hat{n} dA = E(r) \oint_S dA = E(r) 4\pi r^2.$$

For $r < R$, S is within the conductor, so $q_{enc} = 0$, and Gauss's law gives

$$E(r) = 0,$$

as expected inside a conductor. If $r > R$, S encloses the conductor so $q_{enc} = q$. From Gauss's law,

$$E(r) 4\pi r^2 = \frac{q}{\epsilon_0}.$$

The electric field of the sphere may therefore be written as

$$\vec{E} = \vec{0} \; (r < R),$$

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r} \; (r \geq R).$$

Significance

Notice that in the region $r \geq R$, the electric field due to a charge q placed on an isolated conducting sphere of radius R is identical to the electric field of a point charge q located at the center of the sphere. The difference between the charged metal and a point charge occurs only at the space points inside the conductor. For a point charge placed at the center of the sphere, the electric field is not zero at points of space occupied by the sphere, but a conductor with the same amount of charge has a zero electric field at those points (Figure 2.5.10). However, there is no distinction at the outside points in space where $r > R$, and we can replace the isolated charged spherical conductor by a point charge at its center with impunity.

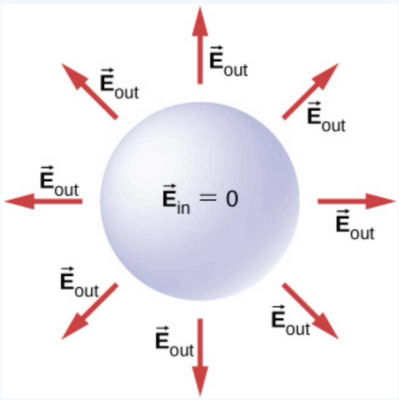


Figure 2.5.10: Electric field of a positively charged metal sphere. The electric field inside is zero, and the electric field outside is same as the electric field of a point charge at the center, although the charge on the metal sphere is at the surface.

? Exercise 2.5.1

How will the system above change if there are charged objects external to the sphere?

Answer

If there are other charged objects around, then the charges on the surface of the sphere will not necessarily be spherically symmetrical; there will be more in certain direction than in other directions.

For a conductor with a cavity, if we put a charge $+q$ inside the cavity, then the charge separation takes place in the conductor, with $-q$ amount of charge on the inside surface and a $+q$ amount of charge at the outside surface (Figure $2.5.11a$). For the same conductor with a charge $+q$ outside it, there is no excess charge on the inside surface; both the positive and negative induced charges reside on the outside surface (Figure $2.5.11b$).

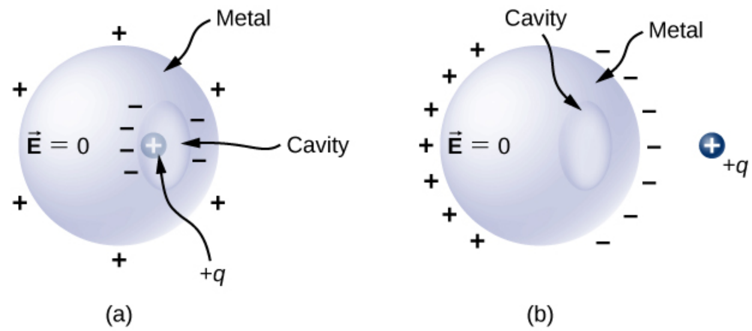


Figure $2.5.11$: (a) A charge inside a cavity in a metal. The distribution of charges at the outer surface does not depend on how the charges are distributed at the inner surface, since the E-field inside the body of the metal is zero. That magnitude of the charge on the outer surface does depend on the magnitude of the charge inside, however. (b) A charge outside a conductor containing an inner cavity. The cavity remains free of charge. The polarization of charges on the conductor happens at the surface.

If a conductor has two cavities, one of them having a charge $+q_a$ inside it and the other a charge $-q_b$ the polarization of the conductor results in $-q_a$ on the inside surface of the cavity a, $+q_b$ on the inside surface of the cavity b, and $q_a - q_b$ on the outside surface (Figure $2.5.12$). The charges on the surfaces may not be uniformly spread out; their spread depends upon the geometry. The only rule obeyed is that when the equilibrium has been reached, the charge distribution in a conductor is such that the electric field by the charge distribution in the conductor cancels the electric field of the external charges at all space points inside the body of the conductor.

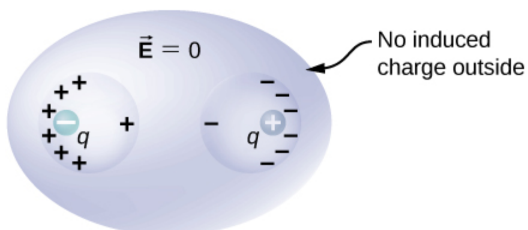


Figure $2.5.12$: The charges induced by two equal and opposite charges in two separate cavities of a conductor. If the net charge on the cavity is nonzero, the external surface becomes charged to the amount of the net charge.

2.6: Gauss's Law (Exercises)

Conceptual Questions

6.2 Electric Flux

1. Discuss how would orient a planar surface of area **A** in a uniform electric field of magnitude E_0 to obtain

(a) the maximum flux and

(b) the minimum flux through the area.

2. What are the maximum and minimum values of the flux in the preceding question?

3. The net electric flux crossing a closed surface is always zero. True or false?

4. The net electric flux crossing an open surface is never zero. True or false?

6.3 Explaining Gauss's Law

5. Two concentric spherical surfaces enclose a point charge **q**. The radius of the outer sphere is twice that of the inner one. Compare the electric fluxes crossing the two surfaces.

6. Compare the electric flux through the surface of a cube of side length a that has a charge **q** at its center to the flux through a spherical surface of radius a with a charge q at its center.

7. (a) If the electric flux through a closed surface is zero, is the electric field necessarily zero at all points on the surface?

(b) What is the net charge inside the surface?

8. Discuss how Gauss's law would be affected if the electric field of a point charge did not vary as $1/r^2$.

9. Discuss the similarities and differences between the gravitational field of a point mass **m** and the electric field of a point charge **q**.

10. Discuss whether Gauss's law can be applied to other forces, and if so, which ones.

11. Is the term $\vec{E}$ in Gauss's law the electric field produced by just the charge inside the Gaussian surface?

12. Reformulate Gauss's law by choosing the unit normal of the Gaussian surface to be the one directed inward.

6.4 Applying Gauss's Law

13. Would Gauss's law be helpful for determining the electric field of two equal but opposite charges a fixed distance apart?

14. Discuss the role that symmetry plays in the application of Gauss's law. Give examples of continuous charge distributions in which Gauss's law is useful and not useful in determining the electric field.

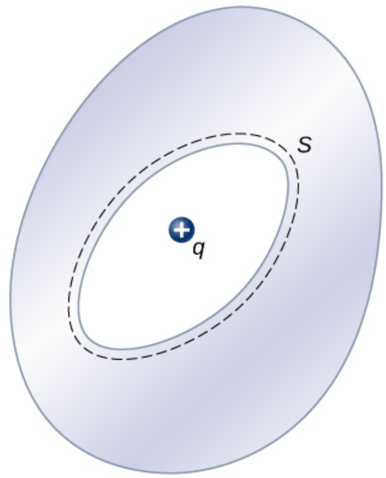
15. Discuss the restrictions on the Gaussian surface used to discuss planar symmetry. For example, is its length important? Does the cross-section have to be square? Must the end faces be on opposite sides of the sheet?

6.5 Conductors in Electrostatic Equilibrium

16. Is the electric field inside a metal always zero?

17. Under electrostatic conditions, the excess charge on a conductor resides on its surface. Does this mean that all the conduction electrons in a conductor are on the surface?

18. A charge **q** is placed in the cavity of a conductor as shown below. Will a charge outside the conductor experience an electric field due to the presence of **q**?



19. The conductor in the preceding figure has an excess charge of $-5.0 \mu C$. If a $2.0-\mu C$ point charge is placed in the cavity, what is the net charge on the surface of the cavity and on the outer surface of the conductor?

Problems

6.2 Electric Flux

20. A uniform electric field of magnitude $1.1 \times 10^4 N/C$ is perpendicular to a square sheet with sides 2.0 m long. What is the electric flux through the sheet?

21. Calculate the flux through the sheet of the previous problem if the plane of the sheet is at an angle of **60°** to the field. Find the flux for both directions of the unit normal to the sheet.

22. Find the electric flux through a rectangular area **3cm×2cm** between two parallel plates where there is a constant electric field of 30 N/C for the following orientations of the area: (a) parallel to the plates, (b) perpendicular to the plates, and (c) the normal to the area making a **30°** angle with the direction of the electric field. Note that this angle can also be given as **180° +30°**.

23. The electric flux through a square-shaped area of side 5 cm near a large charged sheet is found to be $3 \times 10^{-5} N \cdot m^2/C$. when the area is parallel to the plate. Find the charge density on the sheet.

24. Two large rectangular aluminum plates of area $150 cm^2$ face each other with a separation of 3 mm between them. The plates are charged with equal amount of opposite charges, **±20μC**. The charges on the plates face each other. Find the flux through a circle of radius 3 cm between the plates when the normal to the circle makes an angle of **5°** with a line perpendicular to the plates. Note that this angle can also be given as **180°+5°**.

25. A square surface of area $2 cm^2$ is in a space of uniform electric field of magnitude $10^3 N/C$. The amount of flux through it depends on how the square is oriented relative to the direction of the electric field. Find the electric flux through the square, when the normal to it makes the following angles with electric field: (a) **30°**, (b) **90°**, and (c) **0°**. Note that these angles can also be given as **180°+θ**.

26. A vector field is pointed along the **z**-axis, \(\displaystyle \vec{v} =\frac{α}{x^2+y^2}\hat{z}.

(a) Find the flux of the vector field through a rectangle in the xy-plane between a<x<ba<x<b and c<y<dc<y<d.

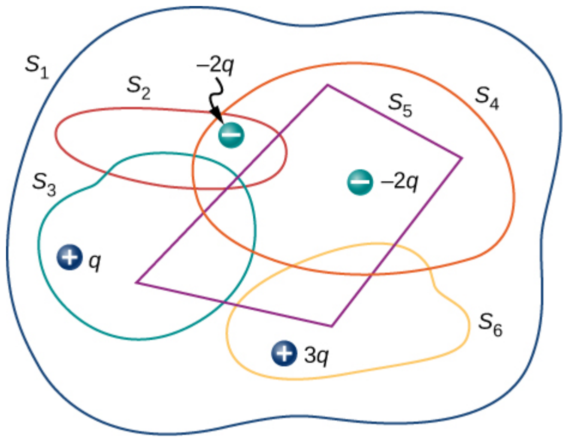
(b) Do the same through a rectangle in the yz-plane between a<z<ba<z<b and c<y<dc<y<d. (Leave your answer as an integral.)

27. Consider the uniform electric field $\vec{E} = (4.0\hat{j} + 3.0\hat{k}) \times 10^3 N/C$. What is its electric flux through a circular area of radius 2.0 m that lies in the **xy**-plane?

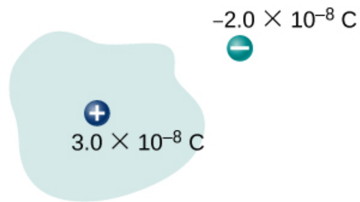
28. Repeat the previous problem, given that the circular area is (a) in the **yz**-plane and (b) **45°** above the **xy**-plane.

29. An infinite charged wire with charge per unit length λ lies along the central axis of a cylindrical surface of radius **r** and length **l**. What is the flux through the surface due to the electric field of the charged wire?

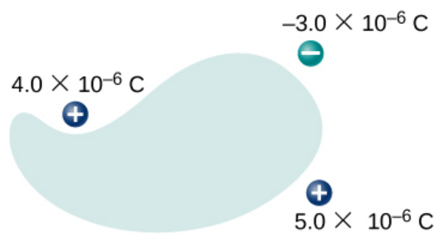
30. Determine the electric flux through each closed surface whose cross-section inside the surface is shown below.



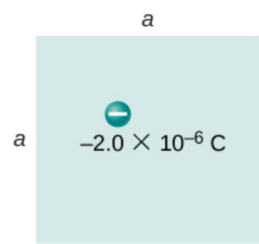
31. Find the electric flux through the closed surface whose cross-sections are shown below.



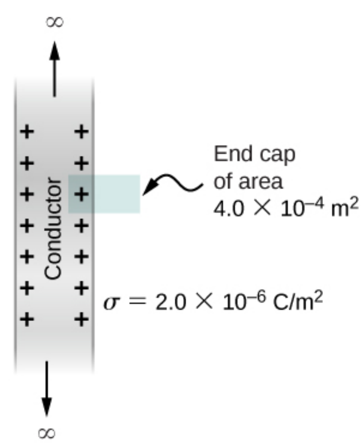
(a)



(b)



(c)



(d)

32. A point charge **q** is located at the center of a cube whose sides are of length **a**. If there are no other charges in this system, what is the electric flux through one face of the cube?

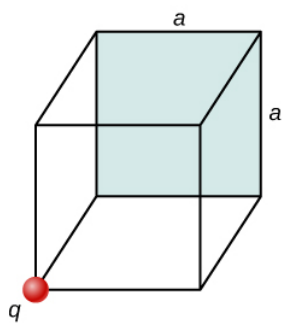
33. A point charge of $10\mu C$ is at an unspecified location inside a cube of side 2 cm. Find the net electric flux though the surfaces of the cube.

34. A net flux of $1.0 \times 10^4 N \cdot^m 2/C$ passes inward through the surface of a sphere of radius 5 cm.

(a) How much charge is inside the sphere?

(b) How precisely can we determine the location of the charge from this information?

35. A charge **q** is placed at one of the corners of a cube of side **a**, as shown below. Find the magnitude of the electric flux through the shaded face due to **q**. Assume $q > 0$.



36. The electric flux through a cubical box 8.0 cm on a side is $1.2 \times 10^3 N \cdot m^2/C$. What is the total charge enclosed by the box?

37. The electric flux through a spherical surface is $4.0 \times 10^4 N \cdot m^2/C$. What is the net charge enclosed by the surface?

38. A cube whose sides are of length **d** is placed in a uniform electric field of magnitude $E = 4.0 \times 10^3 N/C$ so that the field is perpendicular to two opposite faces of the cube. What is the net flux through the cube?

39. Repeat the previous problem, assuming that the electric field is directed along a body diagonal of the cube.

40. A total charge $5.0 \times 10^{-6} C$ is distributed uniformly throughout a cubical volume whose edges are 8.0 cm long.

(a) What is the charge density in the cube?

(b) What is the electric flux through a cube with 12.0-cm edges that is concentric with the charge distribution?

(c) Do the same calculation for cubes whose edges are 10.0 cm long and 5.0 cm long.

(d) What is the electric flux through a spherical surface of radius 3.0 cm that is also concentric with the charge distribution?

6.4 Applying Gauss's Law

41. Recall that in the example of a uniform charged sphere, $\rho_0 = Q/(\frac{4}{3}\pi R^3)$. Rewrite the answers in terms of the total charge **Q** on the sphere.

42. Suppose that the charge density of the spherical charge distribution shown in Figure 6.23 is $\rho(r) = \rho_0 r/R$ for $r \leq R$ and zero for $r > R$. Obtain expressions for the electric field both inside and outside the distribution.

43. A very long, thin wire has a uniform linear charge density of $50\mu C/m$. What is the electric field at a distance 2.0 cm from the wire?

44. A charge of $-30\mu C$ is distributed uniformly throughout a spherical volume of radius 10.0 cm. Determine the electric field due to this charge at a distance of

(a) 2.0 cm,

(b) 5.0 cm, and

(c) 20.0 cm from the center of the sphere.

45. Repeat your calculations for the preceding problem, given that the charge is distributed uniformly over the surface of a spherical conductor of radius 10.0 cm.

46. A total charge **Q** is distributed uniformly throughout a spherical shell of inner and outer radii r_1 and r_2, respectively. Show that the electric field due to the charge is

$$\vec{E} = \vec{0} \ (r \leq r_1);$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\left(\frac{r^3 - r_1^3}{r_2^3 - r_1^3}\right)\hat{r} \ \ (r_1 \leq r \leq r_2);$$

$$\vec{E} = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r} \ \ (r \geq r_2)\,.$$

47. When a charge is placed on a metal sphere, it ends up in equilibrium at the outer surface. Use this information to determine the electric field of $+3.0\mu C$ charge put on a 5.0-cm aluminum spherical ball at the following two points in space:

(a) a point 1.0 cm from the center of the ball (an inside point) and

(b) a point 10 cm from the center of the ball (an outside point).

48. A large sheet of charge has a uniform charge density of $10\mu C/m^2$. What is the electric field due to this charge at a point just above the surface of the sheet?

49. Determine if approximate cylindrical symmetry holds for the following situations. State why or why not.

(a) A 300-cm long copper rod of radius 1 cm is charged with +500 nC of charge and we seek electric field at a point 5 cm from the center of the rod.

(b) A 10-cm long copper rod of radius 1 cm is charged with +500 nC of charge and we seek electric field at a point 5 cm from the center of the rod.

(c) A 150-cm wooden rod is glued to a 150-cm plastic rod to make a 300-cm long rod, which is then painted with a charged paint so that one obtains a uniform charge density. The radius of each rod is 1 cm, and we seek an electric field at a point that is 4 cm from the center of the rod.

(d) Same rod as (c), but we seek electric field at a point that is 500 cm from the center of the rod.

50. A long silver rod of radius 3 cm has a charge of $-5\mu C/cm$ on its surface.

(a) Find the electric field at a point 5 cm from the center of the rod (an outside point).

(b) Find the electric field at a point 2 cm from the center of the rod (an inside point).

51. The electric field at 2 cm from the center of long copper rod of radius 1 cm has a magnitude 3 N/C and directed outward from the axis of the rod.

(a) How much charge per unit length exists on the copper rod?

(b) What would be the electric flux through a cube of side 5 cm situated such that the rod passes through opposite sides of the cube perpendicularly?

52. A long copper cylindrical shell of inner radius 2 cm and outer radius 3 cm surrounds concentrically a charged long aluminum rod of radius 1 cm with a charge density of 4 pC/m. All charges on the aluminum rod reside at its surface. The inner surface of the copper shell has exactly opposite charge to that of the aluminum rod while the outer surface of the copper shell has the same charge as the aluminum rod. Find the magnitude and direction of the electric field at points that are at the following distances from the center of the aluminum rod:

(a) 0.5 cm, (b) 1.5 cm, (c) 2.5 cm, (d) 3.5 cm, and (e) 7 cm.

53. Charge is distributed uniformly with a density ρ throughout an infinitely long cylindrical volume of radius **R**. Show that the field of this charge distribution is directed radially with respect to the cylinder and that

$$E = \frac{\rho r}{2\varepsilon_0} \ \ (r \leq R)\,;$$

$$E = \frac{\rho R^2}{2\varepsilon_0 r} \ \ (r \geq R)$$

54. Charge is distributed throughout a very long cylindrical volume of radius **R** such that the charge density increases with the distance **r** from the central axis of the cylinder according to $\rho = \alpha r$, where α is a constant. Show that the field of this charge distribution is directed radially with respect to the cylinder and that

$$E = \frac{\alpha r^2}{3\varepsilon_0} \ \ (r \leq R)\,;$$

$$E = \frac{\alpha R^3}{3\varepsilon_0 r} \ \ (r \geq R)\,.$$

55. The electric field 10.0 cm from the surface of a copper ball of radius 5.0 cm is directed toward the ball's center and has magnitude $4.0 \times 10^2 N/C$. How much charge is on the surface of the ball?

56. Charge is distributed throughout a spherical shell of inner radius r_1 and outer radius r_2 with a volume density given by $\rho = \rho_0 r_1/r$, where ρ_0 is a constant. Determine the electric field due to this charge as a function of **r**, the distance from the center of the shell.

57. Charge is distributed throughout a spherical volume of radius **R** with a density $\rho = \alpha r^2$, where αα is a constant. Determine the electric field due to the charge at points both inside and outside the sphere.

58. Consider a uranium nucleus to be sphere of radius $R = 7.4 \times 10^{-15} m$ with a charge of **92e** distributed uniformly throughout its volume. (a) What is the electric force exerted on an electron when it is $3.0 \times 10^{-15} m$ from the center of the nucleus? (b) What is the acceleration of the electron at this point?

59. The volume charge density of a spherical charge distribution is given by $\rho(r) = \rho_0 e^{-\alpha r}$, where ρ_0 and α are constants. What is the electric field produced by this charge distribution?

6.5 Conductors in Electrostatic Equilibrium

60. An uncharged conductor with an internal cavity is shown in the following figure. Use the closed surface S along with Gauss' law to show that when a charge **q** is placed in the cavity a total charge **–q** is induced on the inner surface of the conductor. What is the charge on the outer surface of the conductor?

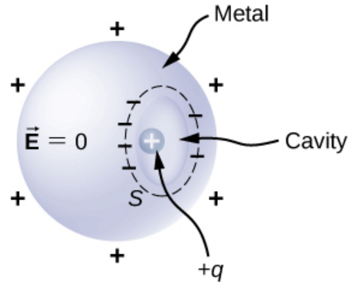
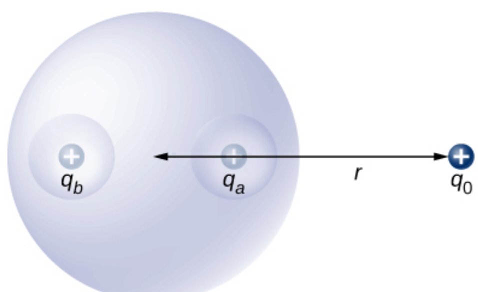


Figure 6.46: A charge inside a cavity of a metal. Charges at the outer surface do not depend on how the charges are distributed at the inner surface since E field inside the body of the metal is zero.

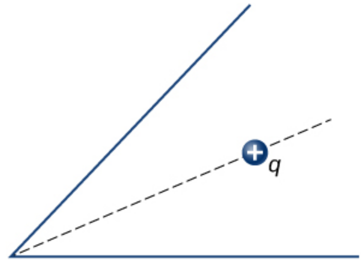
61. An uncharged spherical conductor **S** of radius **R** has two spherical cavities **A** and **B** of radii **a** and **b**, respectively as shown below. Two point charges $+q_a$ and $+q_b$ are placed at the center of the two cavities by using non-conducting supports. In addition, a point charge $+q_0$ is placed outside at a distance r from the center of the sphere.

(a) Draw approximate charge distributions in the metal although metal sphere has no net charge.

(b) Draw electric field lines. Draw enough lines to represent all distinctly different places.



62. A positive point charge is placed at the angle bisector of two uncharged plane conductors that make an angle of **45°**.See below. Draw the electric field lines.



63. A long cylinder of copper of radius 3 cm is charged so that it has a uniform charge per unit length on its surface of 3 C/m. (a) Find the electric field inside and outside the cylinder. (b) Draw electric field lines in a plane perpendicular to the rod.

64. An aluminum spherical ball of radius 4 cm is charged with **5μC** of charge. A copper spherical shell of inner radius 6 cm and outer radius 8 cm surrounds it. A total charge of **−8μC** is put on the copper shell.

(a) Find the electric field at all points in space, including points inside the aluminum and copper shell when copper shell and aluminum sphere are concentric.

(b) Find the electric field at all points in space, including points inside the aluminum and copper shell when the centers of copper shell and aluminum sphere are 1 cm apart.

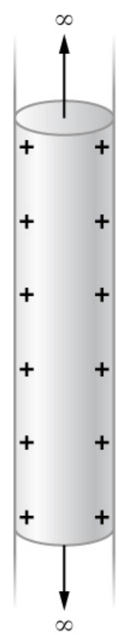
65. A long cylinder of aluminum of radius **R** meters is charged so that it has a uniform charge per unit length on its surface of λ. (a) Find the electric field inside and outside the cylinder. (b) Plot electric field as a function of distance from the center of the rod.

66. At the surface of any conductor in electrostatic equilibrium, $E = \sigma/\varepsilon_0$. Show that this equation is consistent with the fact that $E = kq/r^2$ at the surface of a spherical conductor.

67. Two parallel plates 10 cm on a side are given equal and opposite charges of magnitude $5.0 \times 10^{-9} C$. The plates are 1.5 mm apart. What is the electric field at the center of the region between the plates?

68. Two parallel conducting plates, each of cross-sectional area $400 cm^2$, are 2.0 cm apart and uncharged. If 1.0×10^{12} electrons are transferred from one plate to the other, what are (a) the charge density on each plate? (b) The electric field between the plates?

69. The surface charge density on a long straight metallic pipe is σ. What is the electric field outside and inside the pipe? Assume the pipe has a diameter of **2a**.



70. A point charge $q = -5.0 \times 10^{-12} C$ is placed at the center of a spherical conducting shell of inner radius 3.5 cm and outer radius 4.0 cm. The electric field just above the surface of the conductor is directed radially outward and has magnitude 8.0 N/C.

(a) What is the charge density on the inner surface of the shell?

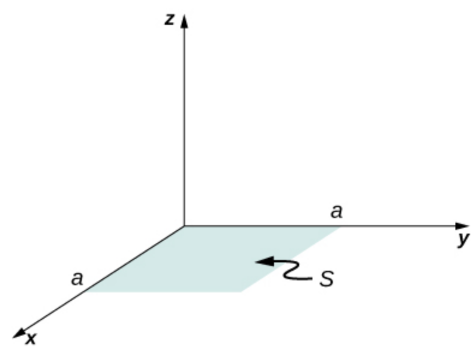
(b) What is the charge density on the outer surface of the shell?

(c) What is the net charge on the conductor?

71. A solid cylindrical conductor of radius a is surrounded by a concentric cylindrical shell of inner radius **b**. The solid cylinder and the shell carry charges **+Q** and **–Q**, respectively. Assuming that the length **L** of both conductors is much greater than a or b, determine the electric field as a function of r, the distance from the common central axis of the cylinders, for (a) $r < a$; (b) $a < r < b$; and (c) $r > b$.

Additional Problems

72. A vector field $\vec{E}$ (not necessarily an electric field; note units) is given by $\vec{E} = 3x^2\hat{k}$. Calculate $\int_S \vec{E} \cdot \hat{n} da$, where **S** is the area shown below. Assume that $\hat{n} = \hat{k}$.

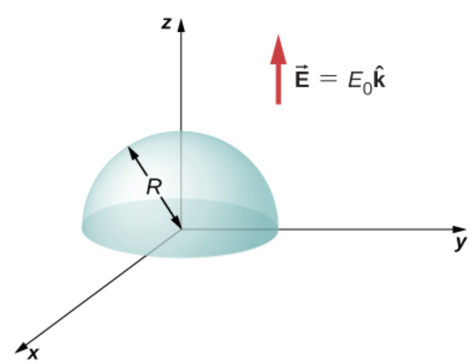


73. Repeat the preceding problem, with $\vec{E} = 2x\hat{i} + 3x^2\hat{k}$.

74. A circular area **S** is concentric with the origin, has radius a, and lies in the **yz**-plane. Calculate $\int_S \vec{E} \cdot \hat{n} dA$ for $\vec{E} = 3z^2\hat{i}$.

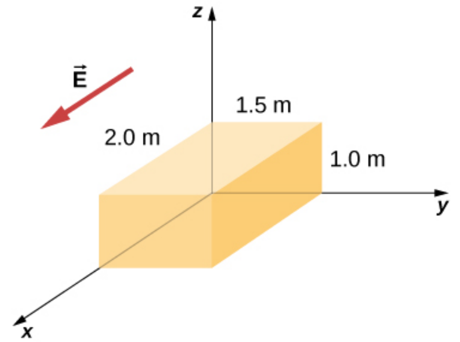
75. (a) Calculate the electric flux through the open hemispherical surface due to the electric field $\vec{E} = E_0\hat{k}$ (see below).

(b) If the hemisphere is rotated by **90°** around the **x**-axis, what is the flux through it?

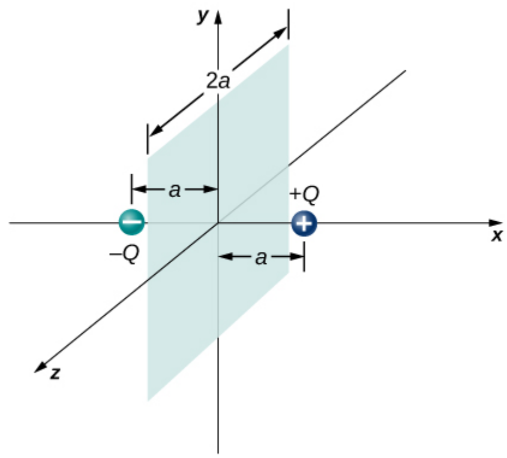


76. Suppose that the electric field of an isolated point charge were proportional to $1/r^{2+\sigma}$ rather than $1/r^2$. Determine the flux that passes through the surface of a sphere of radius **R** centered at the charge. Would Gauss's law remain valid?

77. The electric field in a region is given by $\vec{E} = a/(b+cx)\hat{i}$, where $a = 200 N \cdot m/C, b = 2.0 m$, and $c = 2.0$.What is the net charge enclosed by the shaded volume shown below?



78. Two equal and opposite charges of magnitude **Q** are located on the **x**-axis at the points **+a** and **–a**, as shown below. What is the net flux due to these charges through a square surface of side **2a** that lies in the **yz**-plane and is centered at the origin? (**Hint:** Determine the flux due to each charge separately, then use the principle of superposition. You may be able to make a symmetry argument.)



79. A fellow student calculated the flux through the square for the system in the preceding problem and got 0. What went wrong?

80. A **10cm×10cm** piece of aluminum foil of 0.1 mm thickness has a charge of $20\mu C$ that spreads on both wide side surfaces evenly. You may ignore the charges on the thin sides of the edges.

(a) Find the charge density.

(b) Find the electric field 1 cm from the center, assuming approximate planar symmetry.

81. Two **10cm×10cm** pieces of aluminum foil of thickness 0.1 mm face each other with a separation of 5 mm. One of the foils has a charge of $+30\mu C$ and the other has $-30\mu C$.

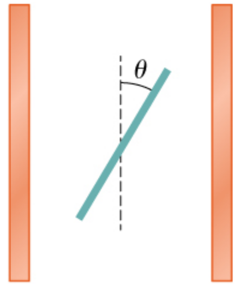
(a) Find the charge density at all surfaces, i.e., on those facing each other and those facing away.

(b) Find the electric field between the plates near the center assuming planar symmetry.

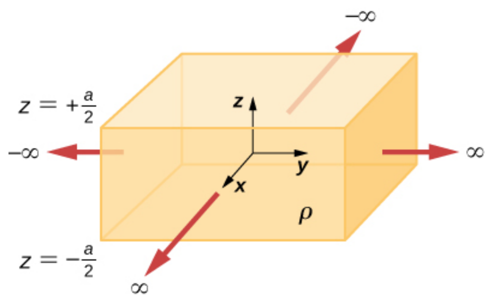
82. Two large copper plates facing each other have charge densities $\pm 4.0C/m^2$ on the surface facing the other plate, and zero in between the plates. Find the electric flux through a **3cm×4cm** rectangular area between the plates, as shown below, for the following orientations of the area.

(a) If the area is parallel to the plates, and

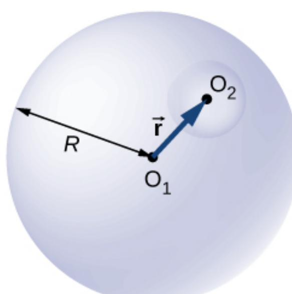
(b) if the area is tilted $\theta = 30^\circ$ from the parallel direction. Note, this angle can also be $\theta = 180^\circ + 30^\circ$.



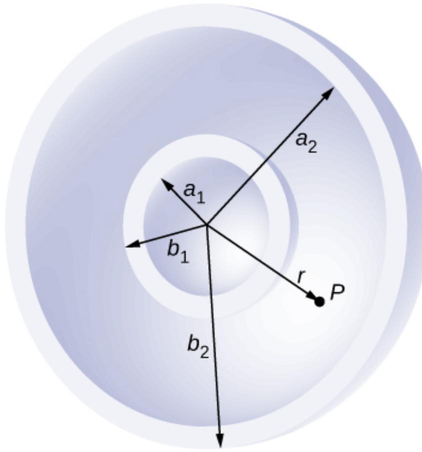
83. The infinite slab between the planes defined by $z=-a/2$ and $z=a/2$ contains a uniform volume charge density ρ(see below). What is the electric field produced by this charge distribution, both inside and outside the distribution?



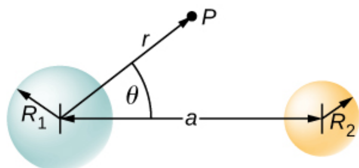
84. A total charge **Q** is distributed uniformly throughout a spherical volume that is centered at O_1 and has a radius **R**. Without disturbing the charge remaining, charge is removed from the spherical volume that is centered at O_2 (see below). Show that the electric field everywhere in the empty region is given by $\vec{E}=\frac{Q\vec{r}}{4\pi\varepsilon_0 R^3}$, where $\vec{r}$ is the displacement vector directed from O_1 to O_2.



85. A non-conducting spherical shell of inner radius a_1 and outer radius b_1 is uniformly charged with charged density ρ_1 inside another non-conducting spherical shell of inner radius a_2 and outer radius b_2 that is also uniformly charged with charge density ρ_2. See below. Find the electric field at space point **P** at a distance r from the common center such that (a) $r>b_2$, (b) $a_2<r<b_2$, (c) $b_1<r<a_2$, (d) $a_1<r<b_1$, and (e) $r<a_1$.

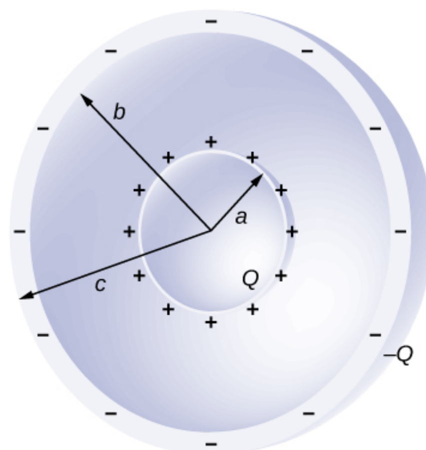


86. Two non-conducting spheres of radii R_1 and R_2 are uniformly charged with charge densities ρ_1 and ρ_2, respectively. They are separated at center-to-center distance a (see below). Find the electric field at point **P** located at a distance r from the center of sphere 1 and is in the direction θ from the line joining the two spheres assuming their charge densities are not affected by the presence of the other sphere. (**Hint:** Work one sphere at a time and use the superposition principle.)

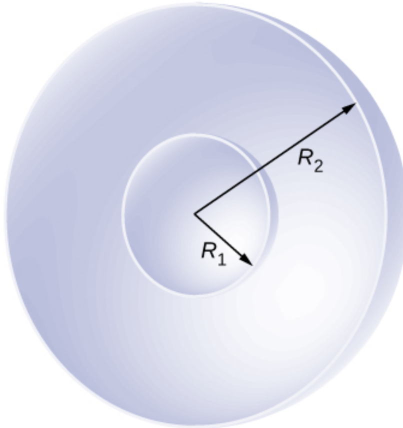


87. A disk of radius **R** is cut in a non-conducting large plate that is uniformly charged with charge density **σ** (coulomb per square meter). See below. Find the electric field at a height h above the center of the disk. ($h >> R, h << l$ or w). (**Hint:** Fill the hole with $\pm\sigma$.)

88. Concentric conducting spherical shells carry charges **Q** and **–Q**, respectively (see below). The inner shell has negligible thickness. Determine the electric field for (a) $r < a$; (b) $a < r < b$; (c) $b < r < c$; and (d) $r > c$.



89. Shown below are two concentric conducting spherical shells of radii R_1 and R_2, each of finite thickness much less than either radius. The inner and outer shell carry net charges q_1 and q_2, respectively, where both q_1 and q_2 are positive. What is the electric field for (a) $r < R_1$; (b) $R_1 < r < R_2$; and (c) $r > R_2$? (d) What is the net charge on the inner surface of the inner shell, the outer surface of the inner shell, the inner surface of the outer shell, and the outer surface of the outer shell?



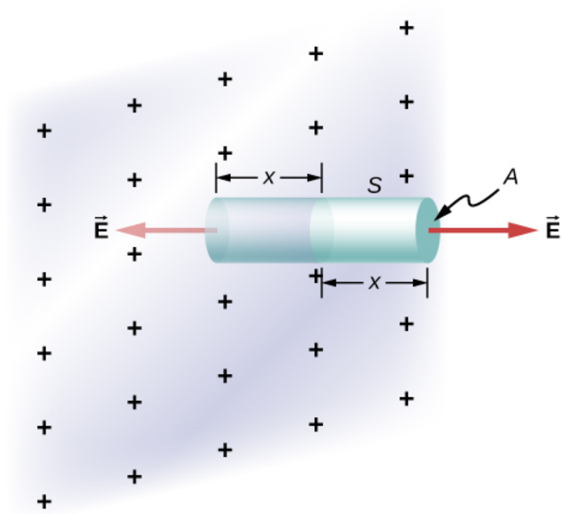
90. A point charge of q=5.0×10−8Cq=5.0×10−8C is placed at the center of an uncharged spherical conducting shell of inner radius 6.0 cm and outer radius 9.0 cm. Find the electric field at (a) r=4.0cmr=4.0cm, (b) r=8.0cmr=8.0cm, and (c) r=12.0cmr=12.0cm. (d) What are the charges induced on the inner and outer surfaces of the shell?

Challenge Problems

91. The Hubble Space Telescope can measure the energy flux from distant objects such as supernovae and stars. Scientists then use this data to calculate the energy emitted by that object. Choose an interstellar object which scientists have observed the flux at the Hubble with (for example, $Vega)$[3], find the distance to that object and the size of Hubble's primary mirror, and calculate the total energy flux. (**Hint**: The Hubble intercepts only a small part of the total flux.)

92. Re-derive Gauss's law for the gravitational field, with $\vec{g}$ directed positively outward.

93. An infinite plate sheet of charge of surface charge density σ is shown below. What is the electric field at a distance **x** from the sheet? Compare the result of this calculation with that of worked out in the text.



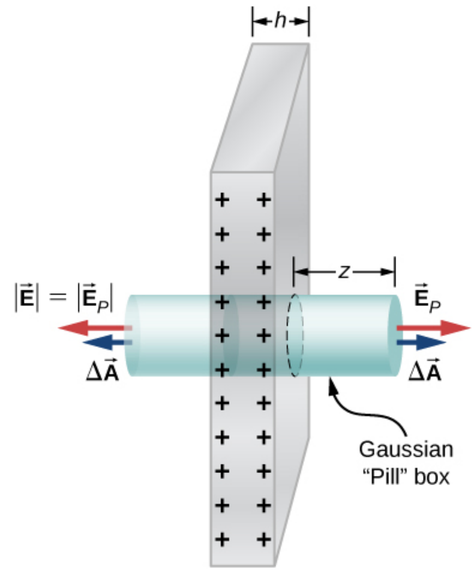
94. A spherical rubber balloon carries a total charge **Q** distributed uniformly over its surface. At $t = 0$, the radius of the balloon is **R**. The balloon is then slowly inflated until its radius reaches **2R** at the time t_0. Determine the electric field due to this charge as a function of time

(a) at the surface of the balloon,

(b) at the surface of radius **R**, and

(c) at the surface of radius **2R**. Ignore any effect on the electric field due to the material of the balloon and assume that the radius increases uniformly with time.

95. Find the electric field of a large conducting plate containing a net charge **q**. Let **A** be area of one side of the plate and **h** the thickness of the plate (see below). The charge on the metal plate will distribute mostly on the two planar sides and very little on the edges if the plate is thin.



CHAPTER OVERVIEW

3: Electric Potential

In this chapter, we examine the relationship between voltage and electrical energy, and begin to explore some of the many applications of electricity.

3.1: Prelude to Electric Potential
3.2: Electric Potential Energy
3.3: Electric Potential and Potential Difference
3.4: Calculations of Electric Potential
3.5: Determining Field from Potential
3.5.1: Calculating the Electric Field from the Electric Potential
3.6: Equipotential Surfaces and Conductors
3.7: Electric Potential (Exercises)

3.1: Prelude to Electric Potential

In Electric Charges and Fields, we just scratched the surface (or at least rubbed it) of electrical phenomena. Two terms commonly used to describe electricity are its energy and voltage, which we show in this chapter is directly related to the potential energy in a system. We know, for example, that great amounts of electrical energy can be stored in batteries, are transmitted cross-country via currents through power lines, and may jump from clouds to explode the sap of trees. In a similar manner, at the molecular level, ions cross cell membranes and transfer information.

Figure 3.1.1: The energy released in a lightning strike is an excellent illustration of the vast quantities of energy that may be stored and released by an electric potential difference. In this chapter, we calculate just how much energy can be released in a lightning strike and how this varies with the height of the clouds from the ground. (credit: Anthony Quintano)

We also know about voltages associated with electricity. Batteries are typically a few volts, the outlets in your home frequently produce 120 volts, and power lines can be as high as hundreds of thousands of volts. But energy and voltage are not the same thing. A motorcycle battery, for example, is small and would not be very successful in replacing a much larger car battery, yet each has the same voltage. In this chapter, we examine the relationship between voltage and electrical energy, and begin to explore some of the many applications of electricity.

3.2: Electric Potential Energy

> **Learning Objectives**
>
> By the end of this section, you will be able to:
>
> - Define the work done by an electric force
> - Define electric potential energy
> - Apply work and potential energy in systems with electric charges

When a free positive charge q is accelerated by an electric field, it is given kinetic energy (Figure $3.2.1$). The process is analogous to an object being accelerated by a gravitational field, as if the charge were going down an electrical hill where its electric potential energy is converted into kinetic energy, although of course the sources of the forces are very different. Let us explore the work done on a charge q by the electric field in this process, so that we may develop a definition of electric potential energy.

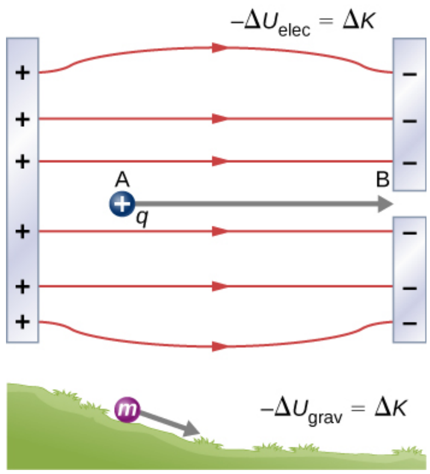


Figure $3.2.1$: A charge accelerated by an electric field is analogous to a mass going down a hill. In both cases, potential energy decreases as kinetic energy increases, $-\Delta U=\Delta K$. Work is done by a force, but since this force is conservative, we can write $W=-\Delta U$.

The electrostatic or Coulomb force is conservative, which means that the work done on q is independent of the path taken, as we will demonstrate later. This is exactly analogous to the gravitational force. When a force is conservative, it is possible to define a potential energy associated with the force. It is usually easier to work with the potential energy (because it depends only on position) than to calculate the work directly.

To show this explicitly, consider an electric charge $+q$ fixed at the origin and move another charge $+Q$ toward q in such a manner that, at each instant, the applied force $\vec{F}$ exactly balances the electric force $\vec{F}_e$ on Q (Figure $3.2.2$). The work done by the applied force $\vec{F}$ on the charge Q changes the potential energy of Q. We call this potential energy the **electrical potential energy** of Q.

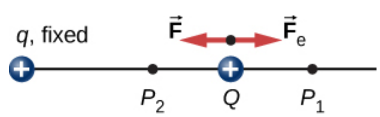


Figure $3.2.2$: Displacement of “test” charge Q in the presence of fixed “source” charge q.

The work W_{12} done by the applied force $\vec{F}$ when the particle moves from P_1 to P_2 may be calculated by

$$W_{12}=\int_{P_1}^{P_2}\vec{F}\cdot d\vec{l}.$$

Since the applied force $\vec{F}$ balances the electric force $\vec{F}_e$ on Q, the two forces have equal magnitude and opposite directions. Therefore, the applied force is

$$\vec{F}=-\vec{F}_e=-\frac{kqQ}{r^2}\hat{r},$$

where we have defined positive to be pointing away from the origin and r is the distance from the origin. The directions of both the displacement and the applied force in the system in Figure $3.2.2$ are parallel, and thus the work done on the system is positive.

We use the letter U to denote electric potential energy, which has units of joules (J). When a conservative force does negative work, the system gains potential energy. When a conservative force does positive work, the system loses potential energy, $\Delta U = -W$. In the system in Figure $3.2.2$, the Coulomb force acts in the opposite direction to the displacement; therefore, the work is negative. However, we have increased the potential energy in the two-charge system.

Example 3.2.1: Kinetic Energy of a Charged Particle

A $+3.0\text{ nC}$ charge Q is initially at rest a distance of 10 cm (r_1) from a $+5.0\text{ nC}$ charge q fixed at the origin (Figure $3.2.3$). Naturally, the Coulomb force accelerates Q away from q, eventually reaching 15 cm (r_2).

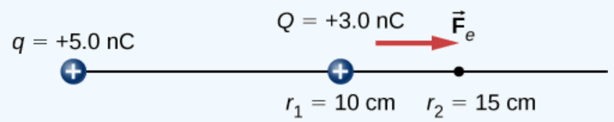


Figure $3.2.3$: The charge Q is repelled by q, thus having work done on it and gaining kinetic energy.

a. What is the work done by the electric field between r_1 and r_2?
b. How much kinetic energy does Q have at r_2?

Strategy

Calculate the work with the usual definition. Since Q started from rest, this is the same as the kinetic energy.

Solution

Integrating force over distance, we obtain

$$\begin{aligned} W_{12} &= \int_{r_1}^{r_2} \vec{F} \cdot d\vec{r} \\ &= \int_{r_1}^{r_2} \frac{kqQ}{r^2} dr \\ &= -\frac{kqQ}{r}\bigg|_{r_1}^{r_2} \\ &= kqQ\left[\frac{-1}{r_2} + \frac{1}{r_1}\right] \\ &= (8.99 \times 10^9\,\mathrm{N\,m^2/C^2})(5.0 \times 10^{-9}\,\mathrm{C})(3.0 \times 10^{-9}\,\mathrm{C})\left[\frac{-1}{0.15\,\mathrm{m}} + \frac{1}{0.10\,\mathrm{m}}\right] \\ &= 4.5 \times 10^{-7}\,\mathrm{J}. \end{aligned}$$

This is also the value of the kinetic energy at r_2.

Significance

Charge Q was initially at rest; the electric field of q did work on Q, so now Q has kinetic energy equal to the work done by the electric field.

Exercise 3.2.1

If Q has a mass of $4.00\,\mu\mathrm{g}$, what is the speed of Q at r_2?

Answer

$$K = \frac{1}{2}mv^2,\ v = \sqrt{2\frac{\mathrm{K}}{\mathrm{m}}} = \sqrt{2\frac{4.5 \times 10^{-7}\mathrm{J}}{4.00 \times 10^{-9}\mathrm{kg}}} = 15\,\mathrm{m/s}.$$

In this example, the work W done to accelerate a positive charge from rest is positive and results from a loss in U, or a negative ΔU. A value for U can be found at any point by taking one point as a reference and calculating the work needed to move a charge to the other point.

Electric Potential Energy

Work W done to accelerate a positive charge from rest is positive and results from a loss in U, or a negative ΔU. Mathematically,

$$W = -\Delta U. \tag{3.2.1}$$

Gravitational potential energy and electric potential energy are quite analogous. Potential energy accounts for work done by a conservative force and gives added insight regarding energy and energy transformation without the necessity of dealing with the force directly. It is much more common, for example, to use the concept of electric potential energy than to deal with the Coulomb force directly in real-world applications.

In polar coordinates with q at the origin and Q located at r, the displacement element vector is $d\vec{l} = \hat{r}\,dr$ and thus the work becomes

$$\begin{aligned} W_{12} &= kqQ \int_{r_1}^{r_2} \frac{1}{r^2}\hat{r}\cdot\hat{r}\,dr \\ &= \underbrace{kqQ\frac{1}{r_2}}_{\text{final point}} - \underbrace{kqQ\frac{1}{r_1}}_{\text{initial point}}. \end{aligned} \tag{3.2.2}$$

Notice that this result only depends on the endpoints and is otherwise independent of the path taken. To explore this further, compare path P_1 to P_2 with path $P_1P_3P_4P_2$ in Figure $3.2.4$.

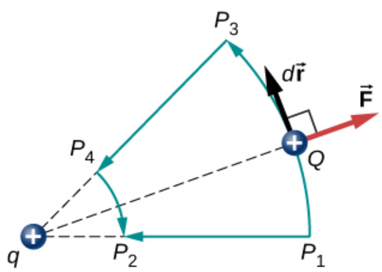


Figure $3.2.4$:Two paths for displacement P_1 to P_2. The work on segments P_1P_3 and P_4P_2 are zero due to the electrical force being perpendicular to the displacement along these paths. Therefore, work on paths P_1P_2 and $P_1P_3P_4P_2$ are equal.

The segments P_1P_3 and P_4P_2 are arcs of circles centered at q. Since the force on Q points either toward or away from q, no work is done by a force balancing the electric force, because it is perpendicular to the displacement along these arcs. Therefore, the only work done is along segment P_3P_4 which is identical to P_1P_2.

One implication of this work calculation is that if we were to go around the path $P_1P_3P_4P_2P_1$, the net work would be zero (Figure $3.2.5$). Recall that this is how we determine whether a force is conservative or not. Hence, because the electric force is related to the electric field by $\vec{F} = q\vec{E}$, the electric field is itself conservative. That is,

$$\oint \vec{E}\cdot d\vec{l} = 0.$$

Note that Q is a constant.

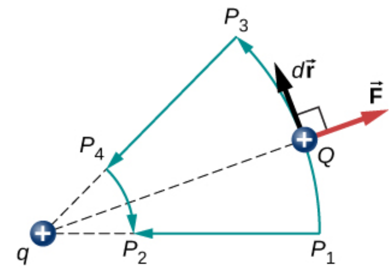


Figure 3.2.5: A closed path in an electric field. The net work around this path is zero.

Another implication is that we may define an electric potential energy. Recall that the work done by a conservative force is also expressed as the difference in the potential energy corresponding to that force. Therefore, the work W_{ref} to bring a charge from a reference point to a point of interest may be written as

$$W_{ref} = \int_{r_{\text{ref}}}^{r} \vec{F} \cdot d\vec{l}$$

and, by Equation 3.2.1, the difference in potential energy ($U_2 - U_1$) of the test charge Q between the two points is

$$\Delta U = -\int_{r_{\text{ref}}}^{r} \vec{F} \cdot d\vec{l}.$$

Therefore, we can write a general expression for the potential energy of two point charges (in spherical coordinates):

$$\Delta U = -\int_{r_{\text{ref}}}^{r} \frac{kqQ}{r^2} dr = -\left[-\frac{kqQ}{r}\right]_{r_{\text{ref}}}^{r} = kqQ\left[\frac{1}{r} - \frac{1}{r_{\text{ref}}}\right].$$

We may take the second term to be an arbitrary constant reference level, which serves as the zero reference:

$$U(r) = k\frac{qQ}{r} - U_{\text{ref}}.$$

A convenient choice of reference that relies on our common sense is that when the two charges are infinitely far apart, there is no interaction between them. (Recall the discussion of reference potential energy in Potential Energy and Conservation of Energy.) Taking the potential energy of this state to be zero removes the term U_{ref} from the equation (just like when we say the ground is zero potential energy in a gravitational potential energy problem), and the potential energy of Q when it is separated from q by a distance r assumes the form

$$\underbrace{U(r) = k\frac{qQ}{r}}_{\text{zero reference at } r=\infty}.$$

This formula is symmetrical with respect to q and Q, so it is best described as the potential energy of the two-charge system.

Example 3.2.2: Potential Energy of a Charged Particle

A $+3.0\text{ nC}$ charge Q is initially at rest a distance of 10 cm (r_1) from a $+5.0\text{ nC}$ charge q fixed at the origin (Figure 3.2.6). Naturally, the Coulomb force accelerates Q away from q, eventually reaching 15 cm (r_2).

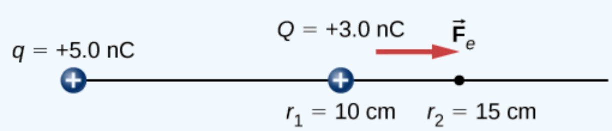


Figure 3.2.6: The charge Q is repelled by q, thus having work done on it and losing potential energy.

What is the change in the potential energy of the two-charge system from r_1 to r_2?

Strategy

Calculate the potential energy with the definition given above:

$\Delta U_{12} = -\int_{r_1}^{r_2} \vec{F} \cdot d\vec{r}$. Since Q started from rest, this is the same as the kinetic energy.

Solution

We have

$$\begin{aligned}\Delta U_{12} &= -\int_{r_1}^{r_2} \vec{F}\cdot d\vec{r} \\ &= -\int_{r_1}^{r_2} \frac{kqQ}{r^2}dr \\ &= -\left[-\frac{kqQ}{r}\right]_{r_1}^{r_2} \\ &= kqQ\left[\frac{1}{r_2}-\frac{1}{r_1}\right] \\ &= (8.99\times 10^9\,\mathrm{N\,m^2/C^2})(5.0\times 10^{-9}\,\mathrm{C})(3.0\times 10^{-9}\,\mathrm{C})\left[\frac{1}{0.15\,\mathrm{m}}-\frac{1}{0.10\,\mathrm{m}}\right] \\ &= -4.5\times 10^{-7}\,\mathrm{J}.\end{aligned} \tag{3.2.3}$$

Significance

The change in the potential energy is negative, as expected, and equal in magnitude to the change in kinetic energy in this system. Recall from Example 3.2.1 that the change in kinetic energy was positive.

Exercise 3.2.2

What is the potential energy of Q relative to the zero reference at infinity at r_2 in the above example?

Answer

It has kinetic energy of 4.5×10^{-7} J at point r_2 and potential energy of 9.0×10^{-7} J, which means that as Q approaches infinity, its kinetic energy totals three times the kinetic energy at r_2, since all of the potential energy gets converted to kinetic.

Due to Coulomb's law, the forces due to multiple charges on a test charge Q superimpose; they may be calculated individually and then added. This implies that the work integrals and hence the resulting potential energies exhibit the same behavior. To demonstrate this, we consider an example of assembling a system of four charges.

Example 3.2.3: Assembling Four Positive Charges

Find the amount of work an external agent must do in assembling four charges $+2.0\ \mu$C, $+3.0\ \mu$C, $+4.0\ \mu$C and $+5.0\ \mu$C at the vertices of a square of side 1.0 cm, starting each charge from infinity (Figure 3.2.7).

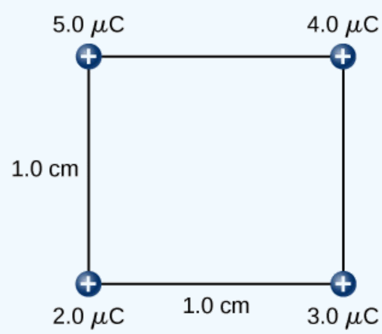


Figure 3.2.7: How much work is needed to assemble this charge configuration?

Strategy

We bring in the charges one at a time, giving them starting locations at infinity and calculating the work to bring them in from infinity to their final location. We do this in order of increasing charge.

Solution

Step 1. First bring the $+2.0\ \mu C$ charge to the origin. Since there are no other charges at a finite distance from this charge yet, no work is done in bringing it from infinity,

$$W_1 = 0.$$

Step 2. While keeping the $+2.0\ \mu C$ charge fixed at the origin, bring the $+3.0\ \mu C$ charge to $(x, y, z) = (1.0\text{ cm},\ 0,\ 0)$ (Figure 3.2.8). Now, the applied force must do work against the force exerted by the $+2.0\ \mu C$ charge fixed at the origin. The work done equals the change in the potential energy of the $+3.0\ \mu C$ charge:

$$\begin{aligned} W_2 &= k\frac{q_1q_2}{r12} \\ &= \left(9.0\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\frac{(2.0\times10^{-6}\text{C})(3.0\times10^{-6}\text{C})}{1.0\times10^{-2}\text{m}} \\ &= 5.4\text{ J}. \end{aligned}$$

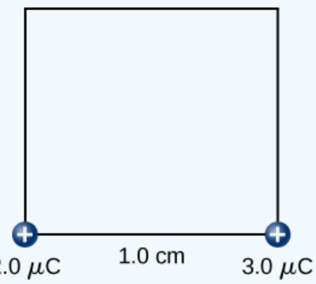


Figure 3.2.8: Step 2. Work W_2 to bring the $+3.0\ \mu C$ charge from infinity.

Step 3. While keeping the charges of $+2.0\ \mu C$ and $+3.0\ \mu C$ fixed in their places, bring in the $+4.0\ \mu C$ charge to $(x, y, z) = (1.0\text{ cm},\ 1.0\text{ cm},\ 0)$ (Figure 3.2.9). The work done in this step is

$$\begin{aligned} W_3 &= k\frac{q_1q_3}{r_{13}} + k\frac{q_2q_3}{r_{23}} \\ &= \left(9.0\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)\left[\frac{(2.0\times10^{-6}\text{C})(4.0\times10^{-6}\text{C})}{\sqrt{2}\times10^{-2}\text{m}} + \frac{(3.0\times10^{-6}\text{C})(4.0\times10^{-6}\text{C})}{1.0\times10^{-2}\text{m}}\right] \\ &= 15.9\text{ J}. \end{aligned}$$

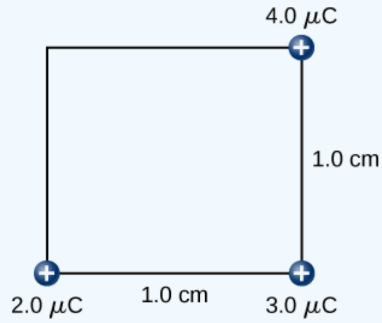


Figure 3.2.9: Step 3. The work W_3 to bring the $+4.0\ \mu C$ charge from infinity.

Step 4. Finally, while keeping the first three charges in their places, bring the $+5.0\ \mu C$ charge to $(x, y, z) = (0,\ 1.0\text{ cm},\ 0)$ (Figure 3.2.10). The work done here is

$$\begin{aligned} W_4 &= kq_4\left[\frac{q_1}{r_{14}} + \frac{q_2}{r_{24}} + \frac{q_3}{r_{34}}\right], \\ &= \left(9.0\times10^9\,\frac{\text{N}\cdot\text{m}^2}{\text{C}^2}\right)(5.0\times10^{-6}\text{C})\left[\frac{(2.0\times10^{-6}\text{C})}{1.0\times10^{-2}\text{m}} + \frac{(3.0\times10^{-6}\text{C})}{\sqrt{2}\times10^{-2}\text{m}} + \frac{(4.0\times10^{-6}\text{C})}{1.0\times10^{-2}\text{m}}\right] \\ &= 36.5\text{ J}. \end{aligned}$$

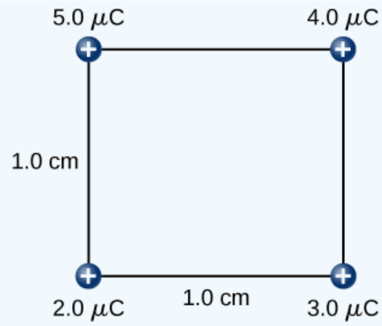


Figure 3.2.10: Step 4. The work W_4 to bring the $+5.0\ \mu\text{C}$ charge from infinity.

Hence, the total work done by the applied force in assembling the four charges is equal to the sum of the work in bringing each charge from infinity to its final position:

$$\begin{aligned} W_T &= W_1 + W_2 + W_3 + W_4 \\ &= 0 + 5.4\ \text{J} + 15.9\ \text{J} + 36.5\ \text{J} \\ &= 57.8\ \text{J}. \end{aligned}$$

Significance

The work on each charge depends only on its pairwise interactions with the other charges. No more complicated interactions need to be considered; the work on the third charge only depends on its interaction with the first and second charges, the interaction between the first and second charge does not affect the third.

Exercise 3.2.3

Is the electrical potential energy of two point charges positive or negative if the charges are of the same sign? Opposite signs? How does this relate to the work necessary to bring the charges into proximity from infinity?

Answer

Positive, negative, and these quantities are the same as the work you would need to do to bring the charges in from infinity

Note that the electrical potential energy is positive if the two charges are of the same type, either positive or negative, and negative if the two charges are of opposite types. This makes sense if you think of the change in the potential energy ΔU as you bring the two charges closer or move them farther apart. Depending on the relative types of charges, you may have to work on the system or the system would do work on you, that is, your work is either positive or negative. If you have to do positive work on the system (actually push the charges closer), then the energy of the system should increase. If you bring two positive charges or two negative charges closer, you have to do positive work on the system, which raises their potential energy. Since potential energy is proportional to $1/r$, the potential energy goes up when r goes down between two positive or two negative charges.

On the other hand, if you bring a positive and a negative charge nearer, you have to do negative work on the system (the charges are pulling you), which means that you take energy away from the system. This reduces the potential energy. Since potential energy is negative in the case of a positive and a negative charge pair, the increase in $1/r$ makes the potential energy more negative, which is the same as a reduction in potential energy.

The result from Example $3.2.2$ may be extended to systems with any arbitrary number of charges. In this case, it is most convenient to write the formula as

$$W_{12\ldots N} = \frac{k}{2}\sum_i^N \sum_j^N \frac{q_i q_j}{r_{ij}} \text{ for } i \neq j.$$

The factor of 1/2 accounts for adding each pair of charges twice.

3.3: Electric Potential and Potential Difference

Learning Objectives

By the end of this section, you will be able to:

- Define electric potential, voltage, and potential difference
- Define the electron-volt
- Calculate electric potential and potential difference from potential energy and electric field
- Describe systems in which the electron-volt is a useful unit
- Apply conservation of energy to electric systems

Recall that earlier we defined electric field to be a quantity independent of the test charge in a given system, which would nonetheless allow us to calculate the force that would result on an arbitrary test charge. (The default assumption in the absence of other information is that the test charge is positive.) We briefly defined a field for gravity, but gravity is always attractive, whereas the electric force can be either attractive or repulsive. Therefore, although potential energy is perfectly adequate in a gravitational system, it is convenient to define a quantity that allows us to calculate the work on a charge independent of the magnitude of the charge. Calculating the work directly may be difficult, since $W = \vec{F} \cdot \vec{d}$ and the direction and magnitude of $\vec{F}$ can be complex for multiple charges, for odd-shaped objects, and along arbitrary paths. But we do know that because $\vec{F}$, the work, and hence ΔU is proportional to the test charge q. To have a physical quantity that is independent of test charge, we define **electric potential** V (or simply potential, since electric is understood) to be the potential energy per unit charge:

Electric Potential

The electric potential energy per unit charge is

$$V = \frac{U}{q}. \tag{3.3.1}$$

Since U is proportional to q, the dependence on q cancels. Thus, V does not depend on q. The change in potential energy ΔU is crucial, so we are concerned with the difference in potential or potential difference ΔV between two points, where

Electric Potential Difference

The **electric potential difference** between points A and B, $V_B - V_A$ is defined to be the change in potential energy of a charge q moved from A to B, divided by the charge. Units of potential difference are joules per coulomb, given the name volt (V) after Alessandro Volta.

$$1\text{ V} = 1\text{ J/C} \tag{3.3.2}$$

The familiar term **voltage** is the common name for electric potential difference. Keep in mind that whenever a voltage is quoted, it is understood to be the potential difference between two points. For example, every battery has two terminals, and its voltage is the potential difference between them. More fundamentally, the point you choose to be zero volts is arbitrary. This is analogous to the fact that gravitational potential energy has an arbitrary zero, such as sea level or perhaps a lecture hall floor. It is worthwhile to emphasize the distinction between potential difference and electrical potential energy.

Potential Difference and Electrical Potential Energy

The relationship between potential difference (or voltage) and electrical potential energy is given by

$$\Delta V = \frac{\Delta U}{q} \tag{3.3.3}$$

or

$$\Delta U = q\Delta V. \tag{3.3.4}$$

Voltage is not the same as energy. Voltage is the energy per unit charge. Thus, a motorcycle battery and a car battery can both have the same voltage (more precisely, the same potential difference between battery terminals), yet one stores much more energy than the other because $\Delta U = q\Delta V$. The car battery can move more charge than the motorcycle battery, although both are 12-V batteries.

Example 3.3.1: Calculating Energy

You have a 12.0-V motorcycle battery that can move 5000 C of charge, and a 12.0-V car battery that can move 60,000 C of charge. How much energy does each deliver? (Assume that the numerical value of each charge is accurate to three significant figures.)

Strategy

To say we have a 12.0-V battery means that its terminals have a 12.0-V potential difference. When such a battery moves charge, it puts the charge through a potential difference of 12.0 V, and the charge is given a change in potential energy equal to $\Delta U = q\Delta V$. To find the energy output, we multiply the charge moved by the potential difference.

Solution

For the motorcycle battery, $q = 5000\,\text{C}$ and $\Delta V = 12.0\,\text{V}$. The total energy delivered by the motorcycle battery is

$$\Delta U_{cycle} = (5000\,\text{C})(12.0\,\text{V}) = (5000\,\text{C})(12.0\,\text{J/C}) = 6.00 \times 10^4\,\text{J}.$$

Similarly, for the car battery, $q = 60,000\,\text{C}$ and

$$\Delta U_{car} = (60,000\,\text{C})(12.0\,\text{V}) = 7.20 \times 10^5\,\text{J}.$$

Significance

Voltage and energy are related, but they are not the same thing. The voltages of the batteries are identical, but the energy supplied by each is quite different. A car battery has a much larger engine to start than a motorcycle. Note also that as a battery is discharged, some of its energy is used internally and its terminal voltage drops, such as when headlights dim because of a depleted car battery. The energy supplied by the battery is still calculated as in this example, but not all of the energy is available for external use.

Exercise 3.3.1

How much energy does a 1.5-V AAA battery have that can move 100 C?

Answer

$$\Delta U = q\Delta V = (100\,\text{C})(1.5\,\text{V}) = 150\,\text{J}$$

Note that the energies calculated in the previous example are absolute values. The change in potential energy for the battery is negative, since it loses energy. These batteries, like many electrical systems, actually move negative charge—electrons in particular. The batteries repel electrons from their negative terminals (A) through whatever circuitry is involved and attract them to their positive terminals (B), as shown in Figure 3.3.1. The change in potential is $\Delta V = V_B - V_A = +12\,V$ and the charge q is negative, so that $\Delta U = q\Delta V$ is negative, meaning the potential energy of the battery has decreased when q has moved from A to B.

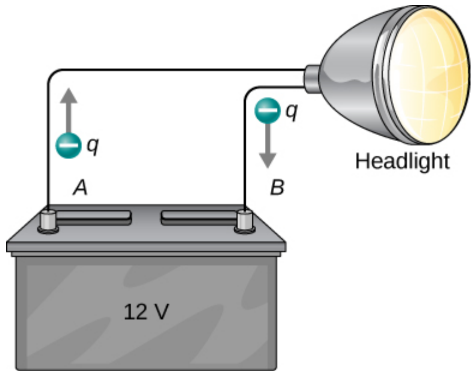


Figure 3.3.1: A battery moves negative charge from its negative terminal through a headlight to its positive terminal. Appropriate combinations of chemicals in the battery separate charges so that the negative terminal has an excess of negative charge, which is repelled by it and attracted to the excess positive charge on the other terminal. In terms of potential, the positive terminal is at a higher voltage than the negative terminal. Inside the battery, both positive and negative charges move.

Example 3.3.2: How Many Electrons Move through a Headlight Each Second?

When a 12.0-V car battery powers a single 30.0-W headlight, how many electrons pass through it each second?

Strategy

To find the number of electrons, we must first find the charge that moves in 1.00 s. The charge moved is related to voltage and energy through the equations $\Delta U = q\Delta V$. A 30.0-W lamp uses 30.0 joules per second. Since the battery loses energy, we have $\Delta U = -30\text{ J}$ and, since the electrons are going from the negative terminal to the positive, we see that $\Delta V = +12.0\,\text{V}$.

Solution

To find the charge q moved, we solve the equation $\Delta U = q\Delta V$:

$$q = \frac{\Delta U}{\Delta V}.$$

Entering the values for ΔU and ΔV, we get

$$q = \frac{-30.0\,\text{J}}{+12.0\,\text{V}} = \frac{-30.0\,\text{J}}{+12.0\,\text{J/C}} = -2.50\,\text{C}.$$

The number of electrons n_e is the total charge divided by the charge per electron. That is,

$$n_e = \frac{-2.50\,\text{C}}{-1.60\times10^{-19}\text{C}/e^-} = 1.56\times10^{19}\,\text{electrons}.$$

Significance

This is a very large number. It is no wonder that we do not ordinarily observe individual electrons with so many being present in ordinary systems. In fact, electricity had been in use for many decades before it was determined that the moving charges in many circumstances were negative. Positive charge moving in the opposite direction of negative charge often produces identical effects; this makes it difficult to determine which is moving or whether both are moving.

Exercise 3.3.2

How many electrons would go through a 24.0-W lamp?

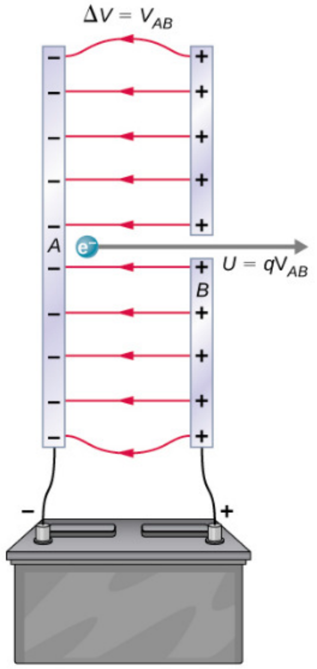
Answer

$-2.00\,C,\ n_e = 1.25\times10^{19}\,electrons$

The Electron-Volt

The energy per electron is very small in macroscopic situations like that in the previous example—a tiny fraction of a joule. But on a submicroscopic scale, such energy per particle (electron, proton, or ion) can be of great importance. For example, even a tiny fraction of a joule can be great enough for these particles to destroy organic molecules and harm living tissue. The particle may do its damage by direct collision, or it may create harmful X-rays, which can also inflict damage. It is useful to have an energy unit related to submicroscopic effects.

Figure $3.3.2$ shows a situation related to the definition of such an energy unit. An electron is accelerated between two charged metal plates, as it might be in an old-model television tube or oscilloscope. The electron gains kinetic energy that is later converted into another form—light in the television tube, for example. (Note that in terms of energy, "downhill" for the electron is "uphill" for a positive charge.) Since energy is related to voltage by $\Delta U = q\Delta V$, we can think of the joule as a coulomb-volt.



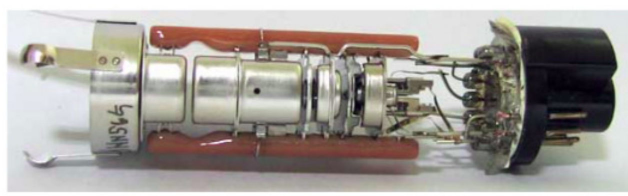


Figure $3.3.2$: A typical electron gun accelerates electrons using a potential difference between two separated metal plates. By conservation of energy, the kinetic energy has to equal the change in potential energy, so $KE = qV$. The energy of the electron in electron-volts is numerically the same as the voltage between the plates. For example, a 5000-V potential difference produces 5000-eV electrons. The conceptual construct, namely two parallel plates with a hole in one, is shown in (a), while a real electron gun is shown in (b).

The Electron-Volt Unit

On the submicroscopic scale, it is more convenient to define an energy unit called the **electron-volt** (eV), which is the energy given to a fundamental charge accelerated through a potential difference of 1 V. In equation form,

$$1\,\text{eV} = (1.60 \times 10^{-19}\,\text{C})(1\,\text{V}) = (1.60 \times 10^{-19}\,\text{C})(1\,\text{J/C}) = 1.60 \times 10^{-19}\,\text{J}.$$

An electron accelerated through a potential difference of 1 V is given an energy of 1 eV. It follows that an electron accelerated through 50 V gains 50 eV. A potential difference of 100,000 V (100 kV) gives an electron an energy of 100,000 eV (100 keV), and so on. Similarly, an ion with a double positive charge accelerated through 100 V gains 200 eV of energy. These simple relationships between accelerating voltage and particle charges make the electron-volt a simple and convenient energy unit in such circumstances.

The electron-volt is commonly employed in submicroscopic processes—chemical valence energies and molecular and nuclear binding energies are among the quantities often expressed in electron-volts. For example, about 5 eV of energy is required to break up certain organic molecules. If a proton is accelerated from rest through a potential difference of 30 kV, it acquires an energy of 30 keV (30,000 eV) and can break up as many as 6000 of these molecules $(30,000\,\text{eV} \div 5\,\text{eV per molecule} = 6000\,\text{molecules})$. Nuclear decay energies are on the order of 1 MeV (1,000,000 eV) per event and can thus produce significant biological damage.

Conservation of Energy

The total energy of a system is conserved if there is no net addition (or subtraction) due to work or heat transfer. For conservative forces, such as the electrostatic force, conservation of energy states that mechanical energy is a constant.

Mechanical energy is the sum of the kinetic energy and potential energy of a system; that is, $K+U=\text{constant}$. A loss of (\U\) for a charged particle becomes an increase in its (\K\). Conservation of energy is stated in equation form as

$$K+U=\text{constant}$$

or

$$K_i+U_i=K_f+U_f$$

where i and f stand for initial and final conditions. As we have found many times before, considering energy can give us insights and facilitate problem solving.

✓ Example 3.3.3: Electrical Potential Energy Converted into Kinetic Energy

Calculate the final speed of a free electron accelerated from rest through a potential difference of 100 V. (Assume that this numerical value is accurate to three significant figures.)

Strategy

We have a system with only conservative forces. Assuming the electron is accelerated in a vacuum, and neglecting the gravitational force (we will check on this assumption later), all of the electrical potential energy is converted into kinetic energy. We can identify the initial and final forms of energy to be

$K_i=0, K_f=\frac{1}{2}mv^2, U_i=qV, U_f=0.$

Solution

Conservation of energy states that

$$K_i+U_i=K_f+U_f.$$

Entering the forms identified above, we obtain

$$qV=\frac{mv^2}{2}.$$

We solve this for (\v\):

$$v=\sqrt{\frac{2qV}{m}}.$$

Entering values for (\q\), (\V\), and (\m\) gives

$$v=\sqrt{\frac{2(-1.60\times10^{-19}\text{C})(-100\,\text{J/C})}{9.11\times10^{-31}\text{kg}}}=5.93\times10^6\,\text{m/s}.$$

Significance

Note that both the charge and the initial voltage are negative, as in Figure $3.3.2$. From the discussion of electric charge and electric field, we know that electrostatic forces on small particles are generally very large compared with the gravitational force. The large final speed confirms that the gravitational force is indeed negligible here. The large speed also indicates how easy it is to accelerate electrons with small voltages because of their very small mass. Voltages much higher than the 100 V in this problem are typically used in electron guns. These higher voltages produce electron speeds so great that effects from special relativity must be taken into account and will be discussed elsewhere. That is why we consider a low voltage (accurately) in this example.

Exercise 3.3.3

How would this example change with a positron? A positron is identical to an electron except the charge is positive.

Answer

It would be going in the opposite direction, with no effect on the calculations as presented.

Voltage and Electric Field

So far, we have explored the relationship between voltage and energy. Now we want to explore the relationship between voltage and electric field. We will start with the general case for a non-uniform $\vec{E}$ field. Recall that our general formula for the potential energy of a test charge q at point P relative to reference point R is

$$U_p = -\int_R^p \vec{F} \cdot d\vec{l}.$$

When we substitute in the definition of electric field $(\vec{E} = \vec{F}/q)$, this becomes

$$U_p = -q\int_R^p \vec{E} \cdot d\vec{l}.$$

Applying our definition of potential $(V = U/q)$ to this potential energy, we find that, in general,

$$V_p = -\int_R^p \vec{E} \cdot d\vec{l}.$$

From our previous discussion of the potential energy of a charge in an electric field, the result is independent of the path chosen, and hence we can pick the integral path that is most convenient.

Consider the special case of a positive point charge q at the origin. To calculate the potential caused by q at a distance r from the origin relative to a reference of 0 at infinity (recall that we did the same for potential energy), let $P = r$ and $R = \infty$, with $d\vec{l} = d\vec{r} = \hat{r}dr$ and use $\vec{E} = \frac{kq}{r^2}\hat{r}$. When we evaluate the integral

$$V_p = -\int_R^p \vec{E} \cdot d\vec{l}$$

for this system, we have

$$V_r = -\int_\infty^r \frac{kq}{r^2}dr = \frac{kq}{r} - \frac{kq}{\infty} = \frac{kq}{r}.$$

This result,

$$V_r = \frac{kq}{r}$$

is the standard form of the potential of a point charge. This will be explored further in the next section.

To examine another interesting special case, suppose a uniform electric field $\vec{E}$ is produced by placing a potential difference (or voltage) ΔV across two parallel metal plates, labeled A and B (Figure $3.3.3$). Examining this situation will tell us what voltage is needed to produce a certain electric field strength. It will also reveal a more fundamental relationship between electric potential and electric field.

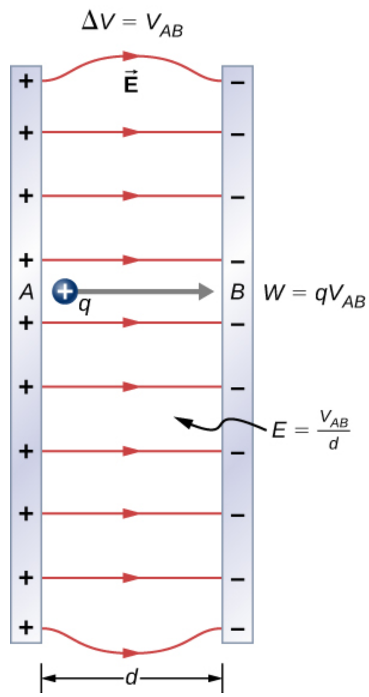


Figure $3.3.3$: The relationship between V and E for parallel conducting plates is $E = V/d$. (Note that $\Delta V = V_{AB}$ in magnitude. For a charge that is moved from plate A at higher potential to plate B at lower potential, a minus sign needs to be included as follows: $-\Delta V = V_A - V_B = V_{AB}$.)

From a physicist's point of view, either ΔV or $\vec{E}$ can be used to describe any interaction between charges. However, ΔV is a scalar quantity and has no direction, whereas $\vec{E}$ is a vector quantity, having both magnitude and direction. (Note that the magnitude of the electric field, a scalar quantity, is represented by E.) The relationship between ΔV and $\vec{E}$ is revealed by calculating the work done by the electric force in moving a charge from point A to point B. But, as noted earlier, arbitrary charge distributions require calculus. We therefore look at a uniform electric field as an interesting special case.

The work done by the electric field in Figure $3.3.3$ to move a positive charge q from A, the positive plate, higher potential, to B, the negative plate, lower potential, is

$$W = -\Delta U = -q\Delta V.$$

The potential difference between points A and B is

$$-\Delta V = -(V_B - V_A) = V_A - V_B = V_{AB}.$$

Entering this into the expression for work yields

$$W = qV_{AB}.$$

Work is $W = \vec{F} \cdot \vec{d} = Fd\cos\theta$: here $\cos\theta = 1$, since the path is parallel to the field. Thus, $W = Fd$. Since $F = qE$ we see that $W = qEd$.

Substituting this expression for work into the previous equation gives

$$qEd = qV_{AB}.$$

The charge cancels, so we obtain for the voltage between points A and B.

In uniform E-field only:

$$V_{AB} = Ed$$

$$E = \frac{V_{AB}}{d}$$

where d is the distance from A to B, or the distance between the plates in Figure $3.3.3$. Note that this equation implies that the units for electric field are volts per meter. We already know the units for electric field are newtons per coulomb; thus, the following relation among units is valid:

$$1\,\mathrm{N/C} = 1\,\mathrm{V/m}.$$

Furthermore, we may extend this to the integral form. Substituting Equation 3.3.3 into our definition for the potential difference between points A and B we obtain

$$V_{AB} = V_B - V_A = -\int_R^B \vec{E} \cdot d\vec{l} + \int_R^A \vec{E} \cdot d\vec{l}$$

which simplifies to

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}.$$

As a demonstration, from this we may calculate the potential difference between two points (A and B) equidistant from a point charge q at the origin, as shown in Figure $3.3.4$.

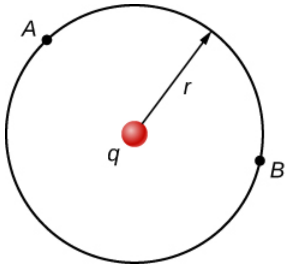


Figure $3.3.4$: The arc for calculating the potential difference between two points that are equidistant from a point charge at the origin.

To do this, we integrate around an arc of the circle of constant radius r between A and B, which means we let $d\vec{l} = r\hat{\varphi} d\varphi$, while using $\vec{E} = \frac{kq}{r^2}\hat{r}$. Thus,

$$\Delta V = V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}.$$

for this system becomes

$$V_B - V_A = -\int_A^B \frac{kq}{r^2}\hat{r} \cdot r\hat{\varphi} d\varphi.$$

However, $\hat{r} \cdot \hat{\varphi}$ and therefore

$$V_B - V_A = 0.$$

This result, that there is no difference in potential along a constant radius from a point charge, will come in handy when we map potentials.

Example $3.3.4A$: What Is the Highest Voltage Possible between Two Plates?

Dry air can support a maximum electric field strength of about $3.0 \times 10^6\ \mathrm{V/m}$. Above that value, the field creates enough ionization in the air to make the air a conductor. This allows a discharge or spark that reduces the field. What, then, is the maximum voltage between two parallel conducting plates separated by 2.5 cm of dry air?

Strategy

We are given the maximum electric field E between the plates and the distance d between them. We can use the equation $V_{AB} = Ed$ to calculate the maximum voltage.

Solution

The potential difference or voltage between the plates is

$$V_{AB} = Ed.$$

Entering the given values for E and d gives

$$V_{AB} = (3.0 \times 10^6\ \mathrm{V/m})(0.025\ \mathrm{m}) = 7.5 \times 10^4\ \mathrm{V}$$

or

$$V_{AB} = 75\ \mathrm{kV}.$$

(The answer is quoted to only two digits, since the maximum field strength is approximate.)

Significance

One of the implications of this result is that it takes about 75 kV to make a spark jump across a 2.5-cm (1-in.) gap, or 150 kV for a 5-cm spark. This limits the voltages that can exist between conductors, perhaps on a power transmission line. A smaller voltage can cause a spark if there are spines on the surface, since sharp points have larger field strengths than smooth surfaces. Humid air breaks down at a lower field strength, meaning that a smaller voltage will make a spark jump through humid air. The largest voltages can be built up with static electricity on dry days (Figure $3.3.5$).

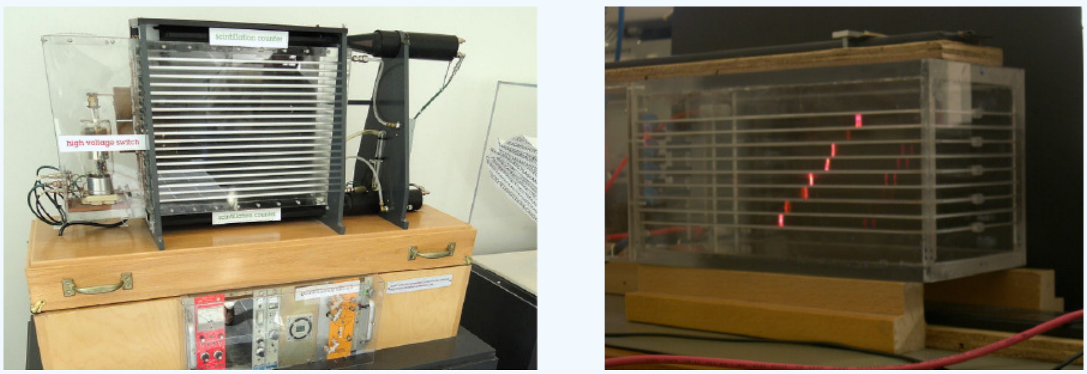


Figure $3.3.5$: A spark chamber is used to trace the paths of high-energy particles. Ionization created by the particles as they pass through the gas between the plates allows a spark to jump. The sparks are perpendicular to the plates, following electric field lines between them. The potential difference between adjacent plates is not high enough to cause sparks without the ionization produced by particles from accelerator experiments (or cosmic rays). This form of detector is now archaic and no longer in use except for demonstration purposes. (credit b: modification of work by Jack Collins)

✓ Example $3.3.1B$: Field and Force inside an Electron Gun

An electron gun (Figure $3.3.2$) has parallel plates separated by 4.00 cm and gives electrons 25.0 keV of energy. (a) What is the electric field strength between the plates? (b) What force would this field exert on a piece of plastic with a $0.500\ \mu\mathrm{C}$ charge that gets between the plates?

Strategy

Since the voltage and plate separation are given, the electric field strength can be calculated directly from the expression $E = \frac{V_{AB}}{d}$. Once we know the electric field strength, we can find the force on a charge by using $\vec{F} = q\vec{E}$. Since the electric field is in only one direction, we can write this equation in terms of the magnitudes, $F = qE$.

Solution

a. The expression for the magnitude of the electric field between two uniform metal plates is

$$E = \frac{V_{AB}}{d}.$$

Since the electron is a single charge and is given 25.0 keV of energy, the potential difference must be 25.0 kV. Entering this value for V_{AB} and the plate separation of 0.0400 m, we obtain

$$E = \frac{25.0\ \mathrm{kV}}{0.0400\ \mathrm{m}} = 6.25 \times 10^5\ \mathrm{V/m}.$$

b. The magnitude of the force on a charge in an electric field is obtained from the equation

$$F = qE.$$

Substituting known values gives

$$F = (0.500 \times 10^{-6}\,\mathrm{C})(6.25 \times 10^{5}\,\mathrm{V/m}) = 0.313\,\mathrm{N}.$$

Significance Note that the units are newtons, since $1\,\mathrm{V/m} = 1\,\mathrm{N/C}$. Because the electric field is uniform between the plates, the force on the charge is the same no matter where the charge is located between the plates.

Example $3.3.4C$: Calculating Potential of a Point Charge

Given a point charge $q = +2.0\,\mathrm{nC}$ at the origin, calculate the potential difference between point P_1 a distance $a = 4.0\,\mathrm{cm}$ from q, and P_2 a distance $b = 12.0\,\mathrm{cm}$ from q, where the two points have an angle of $\varphi = 24^\circ$ between them (Figure $3.3.6$).

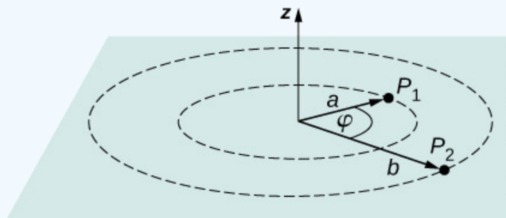


Figure $3.3.6$: Find the difference in potential between P_1 and P_2.

Strategy Do this in two steps. The first step is to use $V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$ and let $A = a = 4.0\,\mathrm{cm}$ and $B = b = 12.0\,\mathrm{cm}$, with $d\vec{l} = d\vec{r} = \hat{r}\,dr$ and $\vec{E} = \frac{kq}{r^2}\hat{r}$. Then perform the integral. The second step is to integrate $V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$ around an arc of constant radius r, which means we let $d\vec{l} = r\vec{\varphi}d\varphi$ with limits $0 \leq \varphi \leq 24^\circ$, still using $\vec{E} = \frac{kq}{r^2}\hat{r}$.

Then add the two results together.

Solution For the first part, $V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$ for this system becomes $V_b - V_a = -\int_a^b \frac{kq}{r^2}\hat{r}\cdot\hat{r}\,dr$ which computes to

$\Delta V = -\int_a^b \frac{kq}{r^2}dr = kq\left[\frac{1}{a} - \frac{1}{b}\right]$

$= (8.99 \times 10^9\,\mathrm{N\,m^2/C^2})(2.0 \times 10^{-9}\,\mathrm{C})\left[\frac{1}{0.040\,\mathrm{m}} - \frac{1}{0.12\,\mathrm{m}}\right] = 300\,\mathrm{V}$.

For the second step, $V_B - V_A = -\int_A^B \vec{E}\cdot d\vec{l}$ becomes $\Delta V = -\int_{0^o}^{24^\circ} \frac{kq}{r^2}\hat{r}\cdot r\hat{\varphi}d\varphi$, but $\hat{r}\cdot\hat{\varphi} = 0$ and therefore $\Delta V = 0$. Adding the two parts together, we get 300 V.

Significance

We have demonstrated the use of the integral form of the potential difference to obtain a numerical result. Notice that, in this particular system, we could have also used the formula for the potential due to a point charge at the two points and simply taken the difference.

Exercise $3.3.4$

From the examples, how does the energy of a lightning strike vary with the height of the clouds from the ground? Consider the cloud-ground system to be two parallel plates.

Answer

Given a fixed maximum electric field strength, the potential at which a strike occurs increases with increasing height above the ground. Hence, each electron will carry more energy. Determining if there is an effect on the total number of electrons lies in the future.

Before presenting problems involving electrostatics, we suggest a problem-solving strategy to follow for this topic.

Problem-Solving Strategy: Electrostatics

1. Examine the situation to determine if static electricity is involved; this may concern separated stationary charges, the forces among them, and the electric fields they create.
2. Identify the system of interest. This includes noting the number, locations, and types of charges involved.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful. Determine whether the Coulomb force is to be considered directly—if so, it may be useful to draw a free-body diagram, using electric field lines.
4. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). It is important to distinguish the Coulomb force F from the electric field E, for example.
5. Solve the appropriate equation for the quantity to be determined (the unknown) or draw the field lines as requested.
6. Examine the answer to see if it is reasonable: Does it make sense? Are units correct and the numbers involved reasonable?

3.4: Calculations of Electric Potential

Learning Objectives

By the end of this section, you will be able to:

- Calculate the potential due to a point charge
- Calculate the potential of a system of multiple point charges
- Describe an electric dipole
- Define dipole moment
- Calculate the potential of a continuous charge distribution

Point charges, such as electrons, are among the fundamental building blocks of matter. Furthermore, spherical charge distributions (such as charge on a metal sphere) create external electric fields exactly like a point charge. The electric potential due to a point charge is, thus, a case we need to consider.

We can use calculus to find the work needed to move a test charge q from a large distance away to a distance of r from a point charge q. Noting the connection between work and potential $W = -q\Delta V$, as in the last section, we can obtain the following result.

Electric Potential V of a Point Charge

The electric potential V of a point charge is given by

$$\underbrace{V = \frac{kq}{r}}_{\text{point charge}} \tag{3.4.1}$$

where k is a constant equal to $8.99 \times 10^9 \, \mathrm{N \cdot m^2/C^2}$.

The potential in Equation 3.4.1 at infinity is chosen to be zero. Thus, V for a point charge decreases with distance, whereas $\vec{E}$ for a point charge decreases with distance squared:

$$E = \frac{F}{q_t} = \frac{kq}{r^2}$$

Recall that the electric potential V is a scalar and has no direction, whereas the electric field $\vec{E}$ is a vector. To find the voltage due to a combination of point charges, you add the individual voltages as numbers. To find the total electric field, you must add the individual fields as vectors, taking magnitude and direction into account. This is consistent with the fact that V is closely associated with energy, a scalar, whereas $\vec{E}$ is closely associated with force, a vector.

Example 3.4.1: What Voltage Is Produced by a Small Charge on a Metal Sphere?

Charges in static electricity are typically in the nanocoulomb (nC) to microcoulomb $(\mu\mathrm{C})$ range. What is the voltage 5.00 cm away from the center of a 1-cm-diameter solid metal sphere that has a –3.00-nC static charge?

Strategy

As we discussed in Electric Charges and Fields, charge on a metal sphere spreads out uniformly and produces a field like that of a point charge located at its center. Thus, we can find the voltage using the equation $V = \frac{kq}{r}$.

Solution

Entering known values into the expression for the potential of a point charge (Equation 3.4.1), we obtain

$$\begin{aligned} V &= k\frac{q}{r} \\ &= (9.00\times10^{9}\,\mathrm{N\cdot m^2/C^2})\left(\frac{-3.00\times10^{-9}\,\mathrm{C}}{5.00\times10^{-2}\,\mathrm{m}}\right) \\ &= -539\,\mathrm{V}. \end{aligned}$$

Significance

The negative value for voltage means a positive charge would be attracted from a larger distance, since the potential is lower (more negative) than at larger distances. Conversely, a negative charge would be repelled, as expected.

✔ Example 3.4.2: What Is the Excess Charge on a Van de Graaff Generator?

A demonstration **Van de Graaff generator** has a 25.0-cm-diameter metal sphere that produces a voltage of 100 kV near its surface (Figure). What excess charge resides on the sphere? (Assume that each numerical value here is shown with three significant figures.)

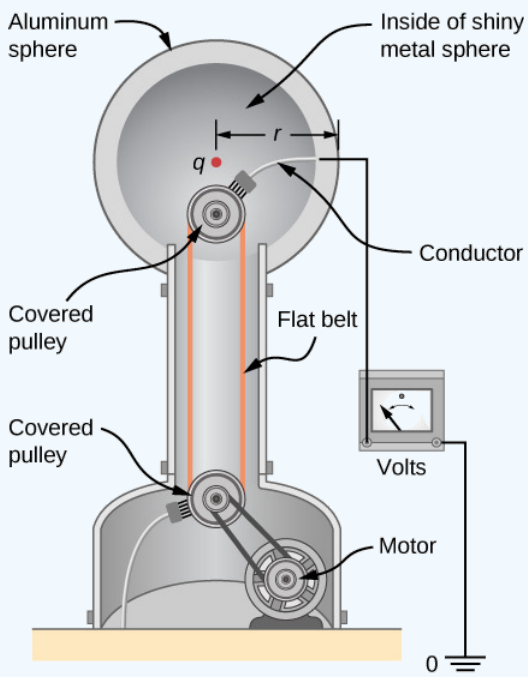


Figure 3.4.1: The voltage of this demonstration Van de Graaff generator is measured between the charged sphere and ground. Earth's potential is taken to be zero as a reference. The potential of the charged conducting sphere is the same as that of an equal point charge at its center.

Strategy

The potential on the surface is the same as that of a point charge at the center of the sphere, 12.5 cm away. (The radius of the sphere is 12.5 cm.) We can thus determine the excess charge using Equation 3.4.1

$$V = \frac{kq}{r}.$$

Solution

Solving for q and entering known values gives

$$
\begin{aligned}
q &= \frac{rV}{k} \\
&= \frac{(0.125\text{ m})(100 \times 10^3\text{ V})}{8.99 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2} \\
&= 1.39 \times 10^{-6}\text{ C} \\
&= 1.39\,\mu\text{C}.
\end{aligned}
$$

Significance

This is a relatively small charge, but it produces a rather large voltage. We have another indication here that it is difficult to store isolated charges.

? Exercise 3.4.1

What is the potential inside the metal sphere in Example 3.4.1?

Answer

$$
\begin{aligned}
V &= k\frac{q}{r} \\
&= (8.99 \times 10^9\text{ N}\cdot\text{m}^2/\text{C}^2)\left(\frac{-3.00 \times 10^{-9}\text{ C}}{5.00 \times 10^{-3}\text{ m}}\right) \\
&= -5390\text{ V}
\end{aligned}
$$

Recall that the electric field inside a conductor is zero. Hence, any path from a point on the surface to any point in the interior will have an integrand of zero when calculating the change in potential, and thus the potential in the interior of the sphere is identical to that on the surface.

The voltages in both of these examples could be measured with a meter that compares the measured potential with ground potential. Ground potential is often taken to be zero (instead of taking the potential at infinity to be zero). It is the potential difference between two points that is of importance, and very often there is a tacit assumption that some reference point, such as Earth or a very distant point, is at zero potential. As noted earlier, this is analogous to taking sea level as $h = 0$ when considering gravitational potential energy $U_g = mgh$.

Systems of Multiple Point Charges

Just as the electric field obeys a superposition principle, so does the electric potential. Consider a system consisting of N charges $q_1, q_2, \ldots, q_N$. What is the net electric potential V at a space point P from these charges? Each of these charges is a source charge that produces its own electric potential at point P, independent of whatever other changes may be doing. Let $V_1, V_2, \ldots, V_N$ be the electric potentials at P produced by the charges $q_1, q_2, \ldots, q_N$, respectively. Then, the net electric potential V_p at that point is equal to the sum of these individual electric potentials. You can easily show this by calculating the potential energy of a test charge when you bring the test charge from the reference point at infinity to point P:

$$V_p = V_1 + V_2 + \ldots + V_N = \sum_1^N V_i.$$

Note that electric potential follows the same principle of superposition as electric field and electric potential energy. To show this more explicitly, note that a test charge q_i at the point P in space has distances of $r_1, r_2, \ldots, r_N$ from the N charges fixed in space above, as shown in Figure 3.4.2. Using our formula for the potential of a point charge for each of these (assumed to be point) charges, we find that

$$V_p = \sum_1^N k\frac{q_i}{r_i} = k\sum_1^N \frac{q_i}{r_i}. \tag{3.4.2}$$

Therefore, the electric potential energy of the test charge is

$$U_p = q_t V_p = q_t k \sum_1^N \frac{q_i}{r_i},$$

which is the same as the work to bring the test charge into the system, as found in the first section of the chapter.

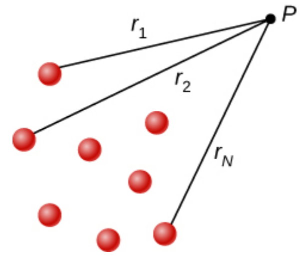


Figure 3.4.2: Notation for direct distances from charges to a space point P.

1The Electric Dipole

An **electric dipole** is a system of two equal but opposite charges a fixed distance apart. This system is used to model many real-world systems, including atomic and molecular interactions. One of these systems is the water molecule, under certain circumstances. These circumstances are met inside a microwave oven, where electric fields with alternating directions make the water molecules change orientation. This vibration is the same as heat at the molecular level.

Example 3.4.3: Electric Potential of a Dipole

Consider the dipole in Figure 3.4.3 with the charge magnitude of $q = 3.0\,\mu\text{C}$ and separation distance $d = 4.0\,\text{cm}$. What is the potential at the following locations in space? (a) (0, 0, 1.0 cm); (b) (0, 0, –5.0 cm); (c) (3.0 cm, 0, 2.0 cm).

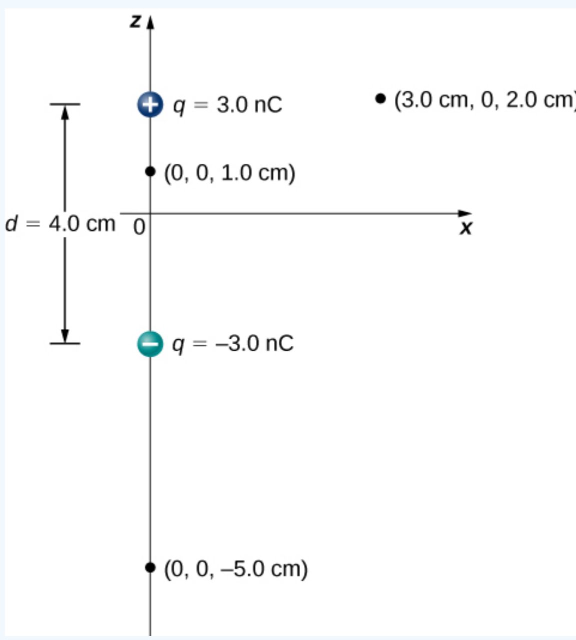


Figure 3.4.3: A general diagram of an electric dipole, and the notation for the distances from the individual charges to a point P in space.

Strategy

Apply $V_p = k \sum_1^N \frac{q_i}{r_i}$ to each of these three points.

Solution

a. $V_p = k \sum_1^N \frac{q_i}{r_i} = (9.0 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)\left(\frac{3.0\,\text{nC}}{0.010\,\text{m}} - \frac{3.0\,\text{nC}}{0.030\,\text{m}}\right) = 1.8 \times 10^3\,\text{V}$

b. $V_p = k\sum_1^N \frac{q_i}{r_i} = (9.0\times10^9\,\mathrm{N\cdot m^2/C^2})\left(\frac{3.0\,\mathrm{nC}}{0.070\,\mathrm{m}} - \frac{3.0\,\mathrm{nC}}{0.030\,\mathrm{m}}\right) = -5.1\times10^2\,\mathrm{V}$

c. $V_p = k\sum_1^N \frac{q_i}{r_i} = (9.0\times10^9\,\mathrm{N\cdot m^2/C^2})\left(\frac{3.0\,\mathrm{nC}}{0.030\,\mathrm{m}} - \frac{3.0\,\mathrm{nC}}{0.050\,\mathrm{m}}\right) = 3.6\times10^2\,\mathrm{V}$

Significance

Note that evaluating potential is significantly simpler than electric field, due to potential being a scalar instead of a vector.

? Exercise 3.4.1

What is the potential on the x-axis? The z-axis?

Answer

The x-axis the potential is zero, due to the equal and opposite charges the same distance from it. On the z-axis, we may superimpose the two potentials; we will find that for $z >> d$, again the potential goes to zero due to cancellation.

Now let us consider the special case when the distance of the point P from the dipole is much greater than the distance between the charges in the dipole, $r >> d$; for example, when we are interested in the electric potential due to a polarized molecule such as a water molecule. This is not so far (infinity) that we can simply treat the potential as zero, but the distance is great enough that we can simplify our calculations relative to the previous example.

We start by noting that in Figure 3.4.4 the potential is given by

$$V_p = V_+ + V_- = k\left(\frac{q}{r_+} - \frac{q}{r_-}\right)$$

where

$$r_\pm = \sqrt{x^2 + \left(z \pm \frac{d}{2}\right)^2}.$$

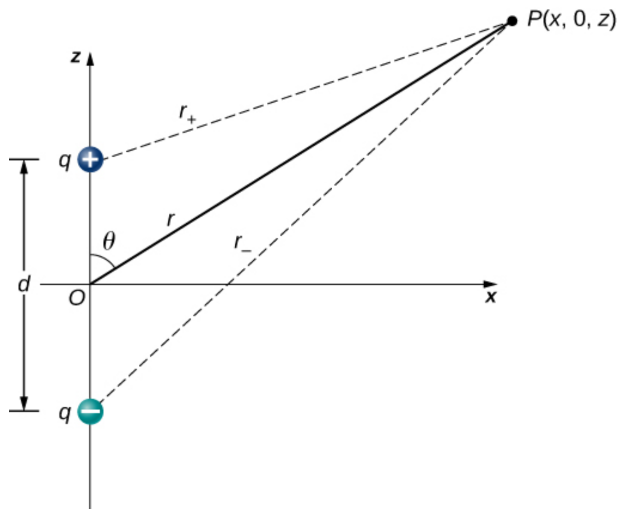


Figure 3.4.4: A general diagram of an electric dipole, and the notation for the distances from the individual charges to a point P in space.

This is still the exact formula. To take advantage of the fact that $r \gg d$, we rewrite the radii in terms of polar coordinates, with $x = r\sin\theta$ and $z = r\cos\theta$. This gives us

$$r_\pm = \sqrt{r^2\sin^2\theta + \left(r\cos\theta \pm \frac{d}{2}\right)^2}.$$

We can simplify this expression by pulling r out of the root,

$$r_{\pm} = \sqrt{\sin^2 \theta + \left(r \cos \theta \pm \frac{d}{2}\right)^2}$$

and then multiplying out the parentheses

$$r_{\pm} = r\sqrt{\sin^2 \theta + \cos^2 \theta \pm \cos\theta \frac{d}{r} + \left(\frac{d}{2r}\right)^2} = r\sqrt{1 \pm \cos\theta \frac{d}{r} + \left(\frac{d}{2r}\right)^2}.$$

The last term in the root is small enough to be negligible (remember $r >> d$, and hence $(d/r)^2$ is extremely small, effectively zero to the level we will probably be measuring), leaving us with

$$r_{\pm} = r\sqrt{1 \pm \cos\theta \frac{d}{r}}.$$

Using the **binomial approximation** (a standard result from the mathematics of series, when a is small)

$$\frac{1}{\sqrt{1 \pm a}} \approx 1 \pm \frac{a}{2}$$

and substituting this into our formula for V_p, we get

$$V_p = k\left[\frac{q}{r}\left(1 + \frac{d\cos\theta}{2r}\right) - \frac{q}{r}\left(1 - \frac{d\cos\theta}{2r}\right)\right] = k\frac{qd\cos\theta}{r^2}.$$

This may be written more conveniently if we define a new quantity, the **electric dipole moment**,

$$\vec{p} = q\vec{d},$$

where these vectors point from the negative to the positive charge. Note that this has magnitude qd. This quantity allows us to write the potential at point P due to a dipole at the origin as

$$V_p = k\frac{\vec{p} \cdot \hat{r}}{r^2}.$$

A diagram of the application of this formula is shown in Figure 3.4.5.

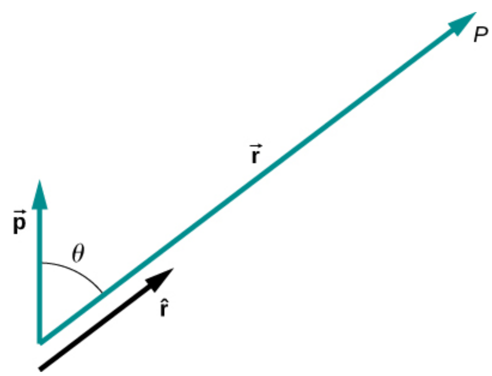


Figure 3.4.5: The geometry for the application of the potential of a dipole.

There are also higher-order moments, for quadrupoles, octupoles, and so on. You will see these in future classes.

Potential of Continuous Charge Distributions

We have been working with point charges a great deal, but what about continuous charge distributions? Recall from Equation 3.4.2 that

$$V_p = k\sum \frac{q_i}{r_i}.$$

We may treat a continuous charge distribution as a collection of infinitesimally separated individual points. This yields the integral

$$V_p = \int \frac{dq}{r}$$

for the potential at a point P. Note that r is the distance from each individual point in the charge distribution to the point P. As we saw in Electric Charges and Fields, the infinitesimal charges are given by

$$\underbrace{dq = \lambda\, dl}_{\text{one dimension}}$$

$$\underbrace{dq = \sigma\, dA}_{\text{two dimensions}}$$

$$\underbrace{dq = \rho\, dV}_{\text{three dimensions}}$$

where λ is linear charge density, σ is the charge per unit area, and ρ is the charge per unit volume.

Example $3.4.4$: Potential of a Line of Charge

Find the electric potential of a uniformly charged, nonconducting wire with linear density λ (coulomb/meter) and length L at a point that lies on a line that divides the wire into two equal parts.

Strategy

To set up the problem, we choose Cartesian coordinates in such a way as to exploit the symmetry in the problem as much as possible. We place the origin at the center of the wire and orient the y-axis along the wire so that the ends of the wire are at $y = \pm L/2$. The field point P is in the xy-plane and since the choice of axes is up to us, we choose the x-axis to pass through the field point P, as shown in Figure $3.4.6$.

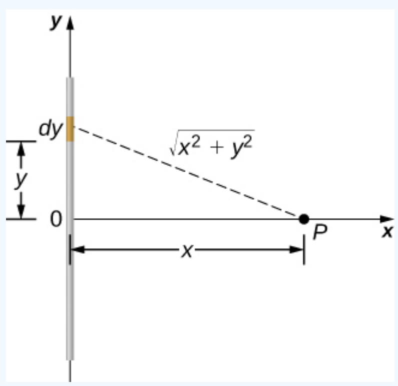


Figure $3.4.6$: We want to calculate the electric potential due to a line of charge.

Solution

Consider a small element of the charge distribution between y and $y + dy$. The charge in this cell is $dq = \lambda\, dy$ and the distance from the cell to the field point P is $\sqrt{x^2 + y^2}$. Therefore, the potential becomes

$$
\begin{aligned}
V_p &= k\int \frac{dq}{r} \\
&= k\int_{-L/2}^{L/2} \frac{\lambda\, dy}{\sqrt{x^2+y^2}} \\
&= k\lambda\left[\ln\left(y+\sqrt{y^2+x^2}\right)\right]_{-L/2}^{L/2} \\
&= k\lambda\left[\ln\left(\left(\frac{L}{2}\right)+\sqrt{\left(\frac{L}{2}\right)^2+x^2}\right)-\ln\left(\left(-\frac{L}{2}\right)+\sqrt{\left(-\frac{L}{2}\right)^2+x^2}\right)\right] \\
&= k\lambda \ln\left[\frac{L+\sqrt{L^2+4x^2}}{-L+\sqrt{L^2+4x^2}}\right].
\end{aligned}
$$

Significance

Note that this was simpler than the equivalent problem for electric field, due to the use of scalar quantities. Recall that we expect the zero level of the potential to be at infinity, when we have a finite charge. To examine this, we take the limit of the above potential as x approaches infinity; in this case, the terms inside the natural log approach one, and hence the potential approaches zero in this limit. Note that we could have done this problem equivalently in cylindrical coordinates; the only effect would be to substitute r for x and z for y.

Example $3.4.5$: Potential Due to a Ring of Charge

A ring has a uniform charge density λ, with units of coulomb per unit meter of arc. Find the electric potential at a point on the axis passing through the center of the ring.

Strategy

We use the same procedure as for the charged wire. The difference here is that the charge is distributed on a circle. We divide the circle into infinitesimal elements shaped as arcs on the circle and use cylindrical coordinates shown in Figure $3.4.7$.

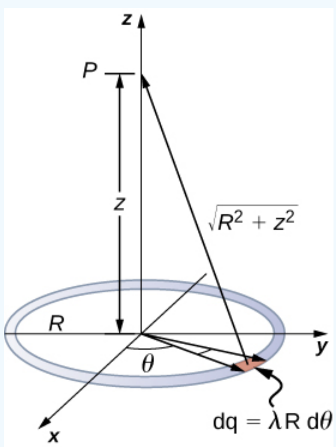


Figure $3.4.7$: We want to calculate the electric potential due to a ring of charge.

Solution

A general element of the arc between θ and $\theta+d\theta$ is of length $Rd\theta$ and therefore contains a charge equal to $\lambda Rd\theta$. The element is at a distance of $\sqrt{z^2+R^2}$ from P, and therefore the potential is

$$
\begin{aligned}
V_p &= k \int \frac{dq}{r} \\
&= k \int_0^{2\pi} \frac{\lambda R d\theta}{\sqrt{z^2 + R^2}} \\
&= \frac{k\lambda R}{\sqrt{z^2 + R^2}} \int_0^{2\pi} d\theta \\
&= \frac{2\pi k \lambda R}{\sqrt{z^2 + R^2}} \\
&= k \frac{q_{tot}}{\sqrt{z^2 + R^2}}.
\end{aligned}
$$

Significance

This result is expected because every element of the ring is at the same distance from point P. The net potential at P is that of the total charge placed at the common distance, $\sqrt{z^2 + R^2}$.

Example $3.4.6$: Potential Due to a Uniform Disk of Charge

A disk of radius R has a uniform charge density σ with units of coulomb meter squared. Find the electric potential at any point on the axis passing through the center of the disk.

Strategy

We divide the disk into ring-shaped cells, and make use of the result for a ring worked out in the previous example, then integrate over $\mathbf{r}$ in addition to θ. This is shown in Figure $3.4.8$.

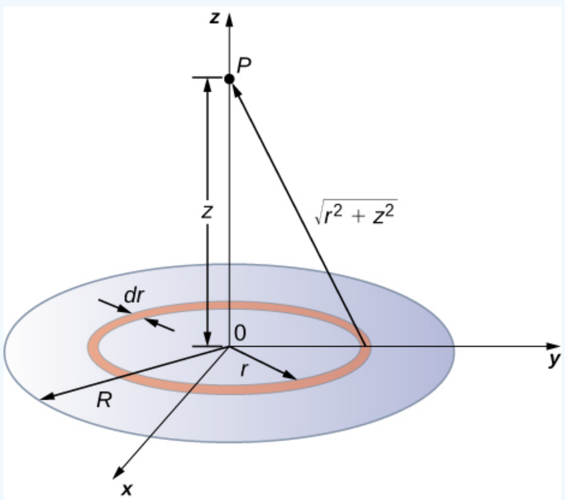


Figure $3.4.8$: We want to calculate the electric potential due to a disk of charge.

Solution

An infinitesimal width cell between cylindrical coordinates r and $r + dr$ shown in Figure $3.4.8$ will be a ring of charges whose electric potential dV_p at the field point has the following expression

$$dV_p = k \frac{dq}{\sqrt{z^2 + r^2}}$$

where

$$dq = \sigma 2\pi r dr.$$

The superposition of potential of all the infinitesimal rings that make up the disk gives the net potential at point P. This is accomplished by integrating from $r = 0$ to $r = R$:

$$V_p = \int dV_p = k2\pi\sigma \int_0^R \frac{r\,dr}{\sqrt{z^2+r^2}},$$
$$= k2\pi\sigma(\sqrt{z^2+R^2} - \sqrt{z^2}).$$

Significance

The basic procedure for a disk is to first integrate around and then over r. This has been demonstrated for uniform (constant) charge density. Often, the charge density will vary with r, and then the last integral will give different results.

Example 3.4.7: Potential Due to an Infinite Charged Wire

Find the electric potential due to an infinitely long uniformly charged wire.

Strategy

Since we have already worked out the potential of a finite wire of length L in Example 3.4.4, we might wonder if taking $L \to \infty$ in our previous result will work:

$$V_p = \lim_{L\to\infty} k\lambda \ln\left(\frac{L+\sqrt{L^2+4x^2}}{-L+\sqrt{L^2+4x^2}}\right).$$

However, this limit does not exist because the argument of the logarithm becomes $[2/0]$ as $L \to \infty$, so this way of finding V of an infinite wire does not work. The reason for this problem may be traced to the fact that the charges are not localized in some space but continue to infinity in the direction of the wire. Hence, our (unspoken) assumption that zero potential must be an infinite distance from the wire is no longer valid.

To avoid this difficulty in calculating limits, let us use the definition of potential by integrating over the electric field from the previous section, and the value of the electric field from this charge configuration from the previous chapter.

Solution

We use the integral

$$V_p = -\int_R^p \vec{E}\cdot d\vec{l}$$

where R is a finite distance from the line of charge, as shown in Figure 3.4.9.

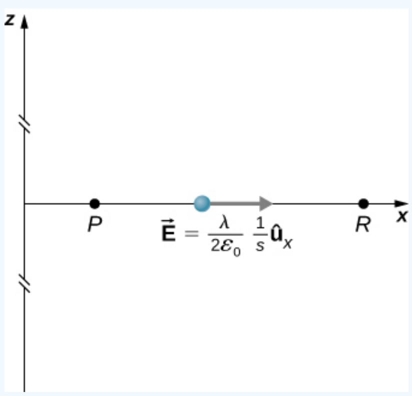


Figure 3.4.9: Points of interest for calculating the potential of an infinite line of charge.

With this setup, we use $\vec{E}_p = 2k\lambda\frac{1}{s}\hat{s}$ and $d\vec{l} = d\vec{s}$ to obtain

$$V_p - V_R = -\int_R^p 2k\lambda\frac{1}{s}ds$$
$$= -2k\lambda\ln\frac{s_p}{s_R}.$$

Now, if we define the reference potential $V_R = 0$ at $s_R = 1\,m$, this simplifies to

$$V_p = -2k\lambda \ln s_p.$$

Note that this form of the potential is quite usable; it is 0 at 1 m and is undefined at infinity, which is why we could not use the latter as a reference.

Significance

Although calculating potential directly can be quite convenient, we just found a system for which this strategy does not work well. In such cases, going back to the definition of potential in terms of the electric field may offer a way forward.

Exercise 3.4.7

What is the potential on the axis of a nonuniform ring of charge, where the charge density is $\lambda(\theta) = \lambda \cos\theta$?

Solution

It will be zero, as at all points on the axis, there are equal and opposite charges equidistant from the point of interest. Note that this distribution will, in fact, have a dipole moment.

3.5: Determining Field from Potential

Learning Objectives

By the end of this section, you will be able to:

- Explain how to calculate the electric field in a system from the given potential
- Calculate the electric field in a given direction from a given potential
- Calculate the electric field throughout space from a given potential

Recall that we were able, in certain systems, to calculate the potential by integrating over the electric field. As you may already suspect, this means that we may calculate the electric field by taking derivatives of the potential, although going from a scalar to a vector quantity introduces some interesting wrinkles. We frequently need $\vec{E}$ to calculate the force in a system; since it is often simpler to calculate the potential directly, there are systems in which it is useful to calculate V and then derive $\vec{E}$ from it.

In general, regardless of whether the electric field is uniform, it points in the direction of decreasing potential, because the force on a positive charge is in the direction of $\vec{E}$ and also in the direction of lower potential V. Furthermore, the magnitude of $\vec{E}$ equals the rate of decrease of V with distance. The faster V decreases over distance, the greater the electric field. This gives us the following result.

Relationship between Voltage and Uniform Electric Field

In equation form, the relationship between voltage and uniform electric field is

$$E = -\frac{\Delta V}{\Delta s}$$

where Δs is the distance over which the change in potential ΔV takes place. The minus sign tells us that E points in the direction of decreasing potential. The electric field is said to be the gradient (as in grade or slope) of the electric potential.

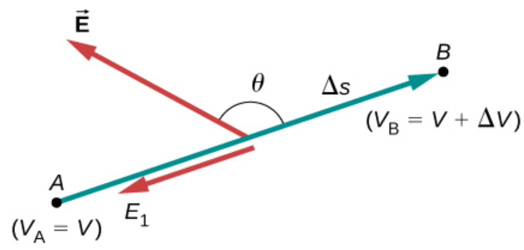


Figure $3.5.1$: The electric field component along the displacement Δs is given by $E = -\frac{\Delta V}{\Delta s}$. Note that A and B are assumed to be so close together that the field is constant along Δs.

For continually changing potentials, ΔV and Δs become infinitesimals, and we need differential calculus to determine the electric field. As shown in Figure $3.5.1$, if we treat the distance Δs as very small so that the electric field is essentially constant over it, we find that

$$E_s = -\frac{dV}{ds}.$$

Therefore, the electric field components in the Cartesian directions are given by

$$E_x = -\frac{\partial V}{\partial x},\ E_y = -\frac{\partial V}{\partial y},\ E_z = -\frac{\partial V}{\partial z}.$$

This allows us to define the "grad" or "del" vector operator, which allows us to compute the gradient in one step. In Cartesian coordinates, it takes the form

$$\vec{\nabla} = \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}.$$

With this notation, we can calculate the electric field from the potential with

$$\vec{E} = -\vec{\nabla} V, \tag{3.5.1}$$

a process we call **calculating the gradient of the potential**.

If we have a system with either cylindrical or spherical symmetry, we only need to use the del operator in the appropriate coordinates:

$$\vec{\nabla}_{cyl} = \underbrace{\hat{r}\frac{\partial}{\partial r} + \hat{\varphi}\frac{1}{r}\frac{\partial}{\partial \varphi} + \hat{z}\frac{\partial}{\partial z}}_{\text{Cylindrical}} \tag{3.5.2}$$

$$\vec{\nabla}_{sph} = \underbrace{\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}}_{\text{Spherical}} \tag{3.5.3}$$

✓ Example 3.5.1: Electric Field of a Point Charge

Calculate the electric field of a point charge from the potential.

Strategy

The potential is known to be $V = k\frac{q}{r}$, which has a spherical symmetry. Therefore, we use the spherical del operator (Equation 3.5.3) into Equation 3.5.1:

$$\vec{E} = -\vec{\nabla}_{sph} V.$$

Solution

Performing this calculation gives us

$$\begin{aligned} \vec{E} &= -\left(\hat{r}\frac{\partial}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\right) k\frac{q}{r} \\ &= -kq\left(\hat{r}\frac{\partial}{\partial r}\frac{1}{r} + \hat{\theta}\frac{1}{r}\frac{\partial}{\partial \theta}\frac{1}{r} + \hat{\varphi}\frac{1}{r\sin\theta}\frac{\partial}{\partial \varphi}\frac{1}{r}\right). \end{aligned}$$

This equation simplifies to

$$\vec{E} = -kq\left(\hat{r}\frac{-1}{r^2} + \hat{\theta}0 + \hat{\varphi}0\right) = k\frac{q}{r^2}\hat{r}$$

as expected.

Significance

We not only obtained the equation for the electric field of a point particle that we've seen before, we also have a demonstration that $\vec{E}$ points in the direction of decreasing potential, as shown in Figure 3.5.2.

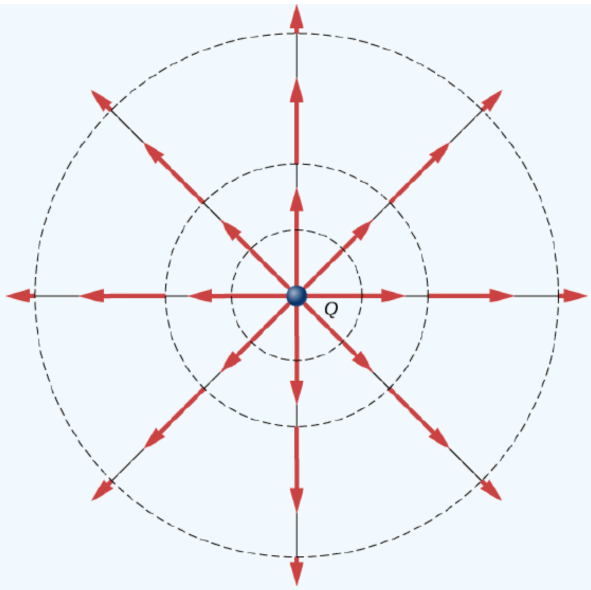


Figure $3.5.2$: Electric field vectors inside and outside a uniformly charged sphere.

✓ Example $3.5.2$: Electric Field of a Ring of Charge

Use the potential found previously to calculate the electric field along the axis of a ring of charge (Figure $3.5.3$).

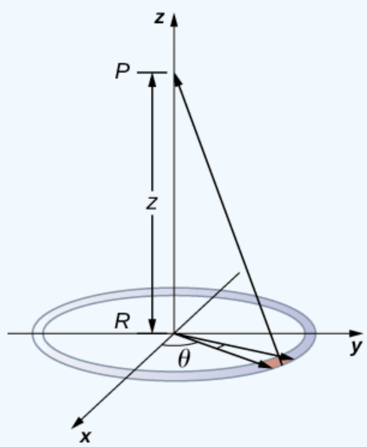


Figure $3.5.3$: We want to calculate the electric field from the electric potential due to a ring charge.

Strategy

In this case, we are only interested in one dimension, the z-axis. Therefore, we use

$$E_z = -\frac{\partial V}{\partial z}$$

with the potential

$$V = k\frac{q_{tot}}{\sqrt{z^2+R^2}}$$

found previously.

Solution

Taking the derivative of the potential yields

$$E_z = -\frac{\partial}{\partial z}\frac{kq_{tot}}{\sqrt{z^2+R^2}}$$
$$= k\frac{q_{tot}z}{(z^2+R^2)^{3/2}}.$$

Significance

Again, this matches the equation for the electric field found previously. It also demonstrates a system in which using the full del operator is not necessary.

? Exercise 3.5.1

Which coordinate system would you use to calculate the electric field of a dipole?

Answer

Any, but cylindrical is closest to the symmetry of a dipole.

3.5.1: Calculating the Electric Field from the Electric Potential

The plan here is to develop a relation between the electric field and the corresponding electric potential that allows you to calculate the electric field from the electric potential.

The electric field is the force-per-charge associated with empty points in space that have a forceper- charge because they are in the vicinity of a source charge or some source charges. The electric potential is the potential energy-per-charge associated with the same empty points in space. Since the electric field is the force-per-charge, and the electric potential is the potential energy-per-charge, the relation between the electric field and its potential is essentially a special case of the relation between any force and its associated potential energy. So, I'm going to start by developing the more general relation between a force and its potential energy, and then move on to the special case in which the force is the electric field times the charge of the victim and the potential energy is the electric potential times the charge of the victim.

The idea behind potential energy was that it represented an easy way of getting the work done by a force on a particle that moves from point A to point B under the influence of the force. By definition, the work done is the force along the path times the length of the path. If the force along the path varies along the path, then we take the force along the path at a particular point on the path, times the length of an infinitesimal segment of the path at that point, and repeat, for every infinitesimal segment of the path, adding the results as we go along. The final sum is the work. The potential energy idea represents the assignment of a value of potential energy to every point in space so that, rather than do the path integral just discussed, we simply subtract the value of the potential energy at point A from the value of the potential energy at point B. This gives us the change in the potential energy experienced by the particle in moving from point A to point B. Then, the work done is the negative of the change in potential energy. For this to be the case, the assignment of values of potential energy values to points in space must be done just right. For things to work out on a macroscopic level, we must ensure that they are correct at an infinitesimal level. We can do this by setting:

$$\text{Work as Change in Potential Energy} = \text{Work as Force-Along-Path times Path Length}$$

$$-dU = \vec{F} \cdot \overrightarrow{ds}$$

where:

- dU is an infinitesimal change in potential energy,
- $\vec{F}$ is a force, and
- $\overrightarrow{ds}$ is the infinitesimal displacement-along-the-path vector.

In Cartesian unit vector notation, $\overrightarrow{ds}$ can be expressed as $\overrightarrow{ds} = dx\,\hat{i} + dy\,\hat{j} + dz\hat{k}$, and $\vec{F}$ can be expressed as $\vec{F} = F_x\,\hat{i} + F_y\,\hat{j} + F_z\hat{k}$. Substituting these two expressions into our expression $-dU = \vec{F} \cdot \overrightarrow{ds}$, we obtain:

$$-dU = (F_x\,\hat{i} + F_y\,\hat{j} + F_z\hat{k}) \cdot (dx\,\hat{i} + dy\,\hat{j} + dz\hat{k})$$

$$-dU = F_x dx + F_y dy + F_z dz$$

Now check this out. If we hold y and z constant (in other words, if we consider dy and dz to be zero) then:

$$\underbrace{-dU = F_x dx}_{\text{when y and z are held constant}}$$

Dividing both sides by dx and switching sides yields:

$$\underbrace{F_x = -\frac{dU}{dx}}_{\text{when y and z are held contsant}}$$

That is, if you have the potential energy as a function of x, y, and z; and; you take the negative of the derivative with respect to x while holding y and z constant, you get the x component of the force that is characterized by the potential energy function. Taking the derivative of U with respect to x while holding the other variables constant is called taking the partial derivative of U with respect to x and written

$$\frac{\partial U}{\partial x}$$

Alternatively, one writes

$$\left.\frac{\partial U}{\partial x}\right|_{y,z}$$

to be read, "the partial derivative of U with respect to x holding y and z constant." This latter expression makes it more obvious to the reader just what is being held constant. Rewriting our expression for F_x with the partial derivative notation, we have:

$$F_x = -\frac{\partial U}{\partial x}$$

Returning to our expression $-dU = F_x dx + F_y dy + F_z dz$, if we hold x and z constant we get:

$$F_y = -\frac{\partial U}{\partial y}$$

and, if we hold x and y constant we get,

$$F_z = -\frac{\partial U}{\partial z}$$

Substituting these last three results into the force vector expressed in unit vector notation:

$$\vec{F} = F_x \hat{i} + F_y \hat{j} + F_z \hat{k}$$

yields

$$\vec{F} = -\frac{\partial U}{\partial x}\hat{i} - \frac{\partial U}{\partial y}\hat{j} - \frac{\partial U}{\partial z}\hat{k}$$

which can be written:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$$

Okay, now, this business of:

- taking the partial derivative of U with respect to x and multiplying the result by the unit vector $\hat{i}$ and then,
- taking the partial derivative of U with respect to y and multiplying the result by the unit vector $\hat{j}$ and then,
- taking the partial derivative of U with respect to z and multiplying the result by the unit vector $\hat{k}$, and then,
- adding all three partial-derivative-times-unit-vector quantities up,

is called "taking the gradient of U " and is written ∇U. "Taking the gradient" is something that you do to a scalar function, but, the result is a vector. In terms of our gradient notation, we can write our expression for the force as,

$$\vec{F} = -\nabla U \tag{3.5.1.1}$$

Check this out for the gravitational potential near the surface of the earth. Define a Cartesian coordinate system with, for instance, the origin at sea level, and, with the x-y plane being horizontal and the $+z$ direction being upward. Then, the potential energy of a particle of mass m is given as:

$$U = mgz$$

Now, suppose you knew this to be the potential but you didn't know the force. You can calculate the force using $\vec{F} = -\nabla U$, which, as you know, can be written:

$$\vec{F} = -\left(\frac{\partial U}{\partial x}\hat{i} + \frac{\partial U}{\partial y}\hat{j} + \frac{\partial U}{\partial z}\hat{k}\right)$$

Substituting $U = mgz$ in for U we have

$$\vec{F}=-\left(\frac{\partial}{\partial x}(mgz)\hat{i}+\frac{\partial}{\partial y}(mgz)\hat{j}+\frac{\partial}{\partial z}(mgz)\hat{k}\right)$$

Now remember, when we take the partial derivative with respect to x we are supposed to hold y and z constant. (There is no y.) But, if we hold z constant, then the whole thing (mgz) is constant. And, the derivative of a constant, with respect to x, is 0. In other words, $\frac{\partial}{\partial x}(mgz)=0$. Likewise, $\frac{\partial}{\partial y}(mgz)=0$. In fact, the only non zero partial derivative in our expression for the force is $\frac{\partial}{\partial z}(mgz)=mg$. So:

$$\vec{F}=-(0\hat{i}+0\hat{j}+mg\hat{k})$$

In other words:

$$\vec{F}=-mg\hat{k}$$

That is to say that, based on the gravitational potential $U=mgz$, the gravitational force is in the $\hat{k}$direction (downward), and, is of magnitude mg. Of course, you knew this in advance, the gravitational force in question is just the weight force. The example was just meant to familiarize you with the gradient operator and the relation between force and potential energy.

Okay, as important as it is that you realize that we are talking about a general relationship between force and potential energy, it is now time to narrow the discussion to the case of the electric force and the electric potential energy, and, from there, to derive a relation between the electric field and electric potential (which is electric potential-energy-per-charge).

Starting with $\vec{F}=-\nabla U$ written out the long way:

$$\vec{F}=-\left(\frac{\partial U}{\partial x}\hat{i}+\frac{\partial U}{\partial y}\hat{j}+\frac{\partial U}{\partial z}\hat{k}\right)$$

we apply it to the case of a particle with charge q in an electric field $\vec{E}$ (caused to exist in the region of space in question by some unspecified source charge or distribution of source charge). The electric field exerts a force $\vec{F}=q\vec{E}$ on the particle, and, the particle has electric potential energy $U=q\varphi$ where φ is the electric potential at the point in space at which the charged particle is located. Plugging these into $\vec{F}=-\left(\frac{\partial U}{\partial x}\hat{i}+\frac{\partial U}{\partial y}\hat{j}+\frac{\partial U}{\partial z}\hat{k}\right)$ yields:

$$q\vec{E}=-\left(\frac{\partial(q\varphi)}{\partial x}\hat{i}+\frac{\partial(q\varphi)}{\partial y}\hat{j}+\frac{\partial(q\varphi)}{\partial z}\hat{k}\right)$$

which I copy here for your convenience:

$$q\vec{E}=-\left(\frac{\partial(q\varphi)}{\partial x}\hat{i}+\frac{\partial(q\varphi)}{\partial y}\hat{j}+\frac{\partial(q\varphi)}{\partial z}\hat{k}\right)$$

The q inside each of the partial derivatives is a constant so we can factor it out of each partial derivative.

$$q\vec{E}=-\left(q\frac{\partial\varphi}{\partial x}\hat{i}+q\frac{\partial\varphi}{\partial y}\hat{j}+q\frac{\partial\varphi}{\partial z}\hat{k}\right)$$

Then, since q appears in every term, we can factor it out of the sum:

$$q\vec{E}=-q\left(\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j}+\frac{\partial\varphi}{\partial z}\hat{k}\right)$$

Dividing both sides by the charge of the victim yields the desired relation between the electric field and the electric potential:

$$\vec{E}=-\left(\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j}+\frac{\partial\varphi}{\partial z}\hat{k}\right)$$

We see that the electric field $\vec{E}$ is just the gradient of the electric potential φ. This result can be expressed more concisely by means of the gradient operator as:

$$\vec{E}=-\nabla\varphi \tag{3.5.1.2}$$

In Example 31-1, we found that the electric potential due to a pair of particles, one of charge $+q$ at $(0, d/2)$ and the other of charge $-q$ at $(0, -d/2)$, is given by:

$$\varphi=\frac{kq}{\sqrt{x^2+(y-\frac{d}{2})^2}}-\frac{kq}{\sqrt{x^2+(y+\frac{d}{2})^2}}$$

Such a pair of charges is called an electric dipole. Find the electric field of the dipole, valid for any point on the x axis.

Solution: We can use a symmetry argument and our conceptual understanding of the electric field due to a point charge to deduce that the x component of the electric field has to be zero, and, the y component has to be negative. But, let's use the gradient method to do that, and, to get an expression for the y component of the electric field. I do argue, however that, from our conceptual understanding of the electric field due to a point charge, neither particle's electric field has a z component in the x-y plane, so we are justified in neglecting the z component altogether. As such our gradient operator expression for the electric field

$$\vec{E}=-\nabla\varphi$$

becomes

$$\vec{E}=-\Big(\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j}\Big)$$

Let's work on the $\frac{\partial\varphi}{\partial x}$ part:

$$\frac{\partial\varphi}{\partial x}=\frac{\partial}{\partial x}\Big(\frac{kq}{\sqrt{x^2+(y-\frac{d}{2})^2}}-\frac{kq}{\sqrt{x^2+(y+\frac{d}{2})^2}}\Big)$$

$$\frac{\partial\varphi}{\partial x}=kq\frac{\partial}{\partial x}\Big(\Big[x^2+(y-\frac{d}{2})^2\Big]^{-\frac{1}{2}}-\Big[x^2+(y+\frac{d}{2})^2\Big]^{-\frac{1}{2}}\Big)$$

$$\frac{\partial\varphi}{\partial x}=kq\Big(-\frac{1}{2}\Big[x^2+(y-\frac{d}{2})^2\Big]^{-\frac{3}{2}}2x-\quad-\frac{1}{2}\Big[x^2+(y+\frac{d}{2})^2\Big]^{-\frac{3}{2}}2x\Big)$$

$$\frac{\partial\varphi}{\partial x}=kqx\Big(\Big[x^2+(y+\frac{d}{2})^2\Big]^{-\frac{3}{2}}-\Big[x^2+(y-\frac{d}{2})^2\Big]^{-\frac{3}{2}}\Big)$$

$$\frac{\partial\varphi}{\partial x}=\frac{kqx}{\Big[x^2+(y+\frac{d}{2})^2\Big]^{\frac{3}{2}}}-\frac{kqx}{\Big[x^2+(y-\frac{d}{2})^2\Big]^{\frac{3}{2}}}$$

We were asked to find the electric field on the x axis, so, we evaluate this expression at $y=0$:

$$\frac{\partial\varphi}{\partial x}=\frac{kqx}{\Big[x^2+(0+\frac{d}{2})^2\Big]^{\frac{3}{2}}}-\frac{kqx}{\Big[x^2+(0-\frac{d}{2})^2\Big]^{\frac{3}{2}}}\qquad \frac{\partial\varphi}{\partial x}\Big|_{y=0}=0$$

To continue with our determination of $\vec{E}=-(\frac{\partial\varphi}{\partial x}\hat{i}+\frac{\partial\varphi}{\partial y}\hat{j})$, we next solve for $\frac{\partial\varphi}{\partial y}$.

$$\frac{\partial\varphi}{\partial y}=\frac{\partial}{\partial y}\Big(\frac{kq}{\sqrt{x^2+(y-\frac{d}{2})^2}}-\frac{kq}{\sqrt{x^2+(y+\frac{d}{2})^2}}\Big)$$

$$\frac{\partial \varphi}{\partial y}=kq\frac{\partial}{\partial y}\Big(\Big[x^2+(y-\frac{d}{2})^2\Big]^{-\frac{1}{2}}-\Big[x^2+(y+\frac{d}{2})^2\Big]^{-\frac{1}{2}}\Big)$$

$$\frac{\partial \varphi}{\partial y}=kq\Big(-\frac{1}{2}\Big[x^2+(y-\frac{d}{2})^2\Big]^{-\frac{3}{2}}2(y-\frac{d}{2})-\ -\frac{1}{2}\Big[x^2+(y+\frac{d}{2})^2\Big]^{-\frac{3}{2}}2(y+\frac{d}{2})$$

$$\frac{\partial \varphi}{\partial y}=kq\Big(\Big[x^2+(y+\frac{d}{2})^2\Big]^{-\frac{3}{2}}(y+\frac{d}{2})-\Big[x^2+(y-\frac{d}{2})^2\Big]^{-\frac{3}{2}}(y-\frac{d}{2})\Big)$$

$$\frac{\partial \varphi}{\partial y}=\frac{kq(y+\frac{d}{2})}{\Big[x^2+(y+\frac{d}{2})^2\Big]^{\frac{3}{2}}}-\frac{kq(y-\frac{d}{2})}{\Big[x^2+(y-\frac{d}{2})^2\Big]^{\frac{3}{2}}}$$

Again, we were asked to find the electric field on the x axis, so, we evaluate this expression at $y=0$:

$$\frac{\partial \varphi}{\partial y}\Big|_{y=0}=\frac{kq\Big(0+\frac{d}{2}\Big)}{\Big[x^2+\Big(0+\frac{d}{2}\Big)^2\Big]^{\frac{3}{2}}}-\frac{kq\Big(0-\frac{d}{2}\Big)}{\Big[x^2+\Big(0-\frac{d}{2}\Big)^2\Big]^{\frac{3}{2}}} \quad \frac{\partial \varphi}{\partial y}\Big|_{y=0}=\frac{kqd}{\Big[x^2+\frac{d^2}{4}\Big]^{\frac{3}{2}}}$$

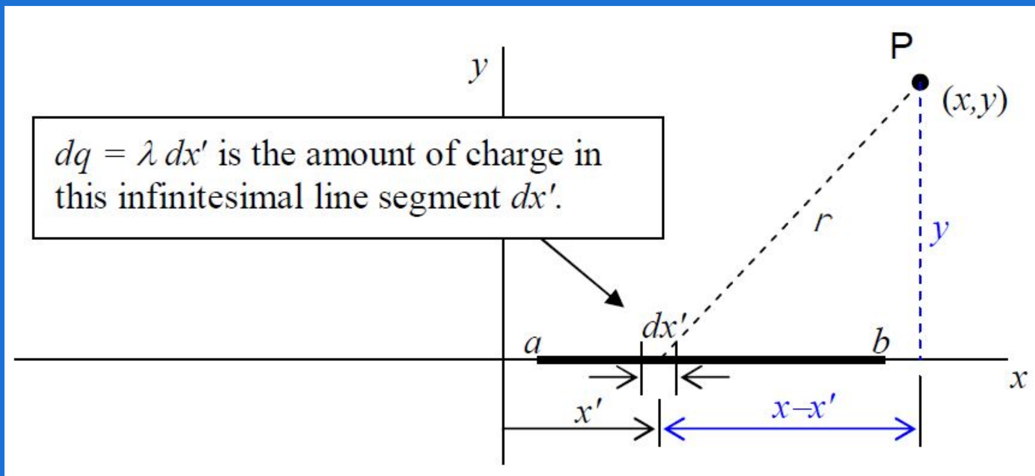
Plugging $\frac{\partial \varphi}{\partial x}\Big|_{y=0}=0$ and $\frac{\partial \varphi}{\partial y}\Big|_{y=0}=\frac{kqd}{\Big[x^2+\frac{d^2}{4}\Big]^{\frac{3}{2}}}$ into $\vec{E}=-\Big(\frac{\partial \varphi}{\partial x}\hat{i}+\frac{\partial \varphi}{\partial y}\hat{j}\Big)$ yields:

$$\vec{E}=-\Big(0\hat{i}+\frac{kqd}{[x^2+\frac{d^2}{4}]^{\frac{3}{2}}}\hat{j}\Big) \quad \vec{E}=-\frac{kqd}{[x^2+\frac{d^2}{4}]^{\frac{3}{2}}}\hat{j}$$

As expected, $\vec{E}$ is in the –y direction. Note that to find the electric field on the x axis, you have to take the derivatives first, and then evaluate at $y=0$.

A line of charge extends along the x axis from $x = a$ to $x = b$. On that line segment, the linear charge density λ is a constant. Find the electric potential as a function of position (x and y) due to that charge distribution on the x-y plane, and then, from the electric potential, determine the electric field on the x axis.

Solution: First, we need to use the methods of chapter 31 to get the potential for the specified charge distribution (a linear charge distribution with a constant linear charge density λ).



$$d\varphi = \frac{k\,dq}{r}$$

$$dq = \lambda dx' \quad \text{and} \quad r = \sqrt{(r - x')^2 + y^2} \quad d\varphi = \frac{k\lambda(x')dx'}{\sqrt{(x - x')^2 + y^2}} \quad \int d\varphi = \int_a^b \frac{k\lambda dx'}{\sqrt{(x - x')^2 + y^2}}$$

$$\varphi = k\lambda \int_a^b \frac{dx'}{\sqrt{(x - x')^2 + y^2}}$$

To carry out the integration, we use the variable substitution:

$$u = x - x' \quad du = -dx' \Rightarrow dx' = du$$

Lower Integration Limit: When

$$x' = a, u = x - a$$

Upper Integration Limit: When

$$x' = b, u = x - b$$

Making these substitutions, we obtain:

$$\varphi = k\lambda \int_{x-a}^{x-b} \frac{-du}{\sqrt{u^2 + y^2}}$$

which I copy here for your convenience:

$$\varphi = k\lambda \int_{x-a}^{x-b} \frac{-du}{\sqrt{u^2 + y^2}}$$

Using the minus sign to interchange the limits of integration, we have:

$$\varphi = k\lambda \int_{x-a}^{x-b} \frac{du}{\sqrt{u^2 + y^2}}$$

Using the appropriate integration formula from the formula sheet we obtain:

$$\varphi = k\lambda \ln(u + \sqrt{u^2 + y^2})\Big|_{x-b}^{x-a}$$

$$\varphi = k\lambda \Big\{ \ln[x - a + \sqrt{(x - a)^2 + y^2}\ ] - \ln\left[x - b + \sqrt{(x - b)^2 + y^2}\ \right] \Big\}$$

Okay, that's the potential. Now we have to take the gradient of it and evaluate the result at $y = 0$ to get the electric field on the x axis. We need to find

$\vec{E} = -\bigtriangledown\varphi$ which, in the absence of any z dependence, can be written as: $\vec{E} = -\left(\frac{\partial\varphi}{\partial x}\hat{i} + \frac{\partial\varphi}{\partial y}\hat{j}\right)$

We start by finding $\frac{\partial\varphi}{\partial x}$:

$$\frac{\partial\varphi}{\partial x} = \frac{\partial}{\partial x}\left(k\lambda\left\{\ln\left[x-a+\sqrt{(x-a)^2+y^2}\,\right] - \ln\left[x-b+\sqrt{(x-b)^2+y^2}\,\right]\right\}\right)$$

$$\frac{\partial\varphi}{\partial x} = k\lambda\left\{\frac{\partial}{\partial x}\ln\left[x-a+((x-a)^2+y^2)^{\frac{1}{2}}\right] - \frac{\partial}{\partial x}\ln\left[x-b+((x-b)^2+y^2)^{\frac{1}{2}}\right]\right\}$$

$$\frac{\partial\varphi}{\partial x} = k\lambda\left\{\frac{1+\frac{1}{2}\left((x-a)^2+y^2\right)^{-\frac{1}{2}}2(x-a)}{x-a+\left((x-a)^2+y^2\right)^{\frac{1}{2}}} - \frac{1+\frac{1}{2}\left((x-b)^2+y^2\right)^{-\frac{1}{2}}2(x-b)}{x-b+\left((x-b)^2+y^2\right)^{\frac{1}{2}}}\right\}$$

$$\frac{\partial\varphi}{\partial x} = k\lambda\left\{\frac{1+(x-a)\left((x-a)^2+y^2\right)^{-\frac{1}{2}}}{x-a+\left((x-a)^2+y^2\right)^{\frac{1}{2}}} - \frac{1+(x-b)\left((x-b)^2+y^2\right)^{-\frac{1}{2}}}{x-b+\left((x-b)^2+y^2\right)^{\frac{1}{2}}}\right\}$$

Evaluating this at $y = 0$ yields:

$$\frac{\partial\varphi}{\partial x}\Big|_{y=0} = k\lambda\left(\frac{1}{x-a} - \frac{1}{x-b}\right)$$

Now, let's work on getting $\frac{\partial\varphi}{\partial y}$. I'll copy our result for φ from above and then take the partial derivative with respect to y (holding x constant):

$$\varphi = k\lambda\left\{\ln\left[x-a+\sqrt{(x-a)^2+y^2}\,\right] - \ln\left[x-b+\sqrt{(x-b)^2+y^2}\,\right]\right\}$$

$$\frac{\partial\varphi}{\partial y} = \frac{\partial}{\partial y}\left(k\lambda\left\{\ln\left[x-a+\sqrt{(x-a)^2+y^2}\,\right] - \ln\left[x-b+\sqrt{(x-b)^2+y^2}\,\right]\right\}\right)$$

$$\frac{\partial\varphi}{\partial y} = k\lambda\left\{\frac{\partial}{\partial y}\ln\left[x-a+\left((x-a)^2+y^2\right)^{\frac{1}{2}}\right] - \frac{\partial}{\partial x}\ln\left[x-b+\left((x-b)^2+y^2\right)^{\frac{1}{2}}\right]\right\}$$

$$\frac{\partial \varphi}{\partial y}=k\lambda\left\{\frac{\frac{1}{2}\left((x-a)^2+y^2\right)^{-\frac{1}{2}}2y}{x-a+\left((x-a)^2+y^2\right)^{\frac{1}{2}}}-\frac{\frac{1}{2}\left((x-b)^2+y^2\right)^{-\frac{1}{2}}2y}{x-b+\left((x-b)^2+y^2\right)^{\frac{1}{2}}}\right\}$$

$$\frac{\partial \varphi}{\partial y}=k\lambda\left\{\frac{y\left((x-a)^2+y^2\right)^{-\frac{1}{2}}}{x-a+\left((x-a)^2+y^2\right)^{\frac{1}{2}}}-\frac{y\left((x-b)^2+y^2\right)^{-\frac{1}{2}}}{x-b+\left((x-b)^2+y^2\right)^{\frac{1}{2}}}\right\}$$

Evaluating this at $y=0$ yields:

$$\left.\frac{\partial \varphi}{\partial y}\right|_{y=0}=0$$

Plugging $\left.\frac{\partial \varphi}{\partial x}\right|_{y=0}=k\lambda\left(\frac{1}{x-a}-\frac{1}{x-b}\right)$ and $\left.\frac{\partial \varphi}{\partial y}\right|_{y=0}=0$ into $\vec{E}=-\left(\frac{\partial \varphi}{\partial x}\hat{i}+\frac{\partial \varphi}{\partial y}\hat{j}\right)$ yields:

$$\vec{E}=-\left(k\lambda\left(\frac{1}{x-a}-\frac{1}{x-b}\right)\hat{i}+0\hat{j}\right) \quad \vec{E}=k\lambda\left(\frac{1}{x-b}-\frac{1}{x-a}\right)\hat{i}$$

3.6: Equipotential Surfaces and Conductors

Learning Objectives

By the end of this section, you will be able to:

- Define equipotential surfaces and equipotential lines
- Explain the relationship between equipotential lines and electric field lines
- Map equipotential lines for one or two point charges
- Describe the potential of a conductor
- Compare and contrast equipotential lines and elevation lines on topographic maps

We can represent electric potentials (voltages) pictorially, just as we drew pictures to illustrate electric fields. This is not surprising, since the two concepts are related. Consider Figure $3.6.1$, which shows an isolated positive point charge and its electric field lines, which radiate out from a positive charge and terminate on negative charges. We use blue arrows to represent the magnitude and direction of the electric field, and we use green lines to represent places where the electric potential is constant. These are called **equipotential surfaces** in three dimensions, or **equipotential lines** in two dimensions. The term **equipotential** is also used as a noun, referring to an equipotential line or surface. The potential for a point charge is the same anywhere on an imaginary sphere of radius **r** surrounding the charge. This is true because the potential for a point charge is given by $V = kq/r$ and thus has the same value at any point that is a given distance **r** from the charge. An equipotential sphere is a circle in the two-dimensional view of Figure $3.6.1$. Because the electric field lines point radially away from the charge, they are perpendicular to the equipotential lines.

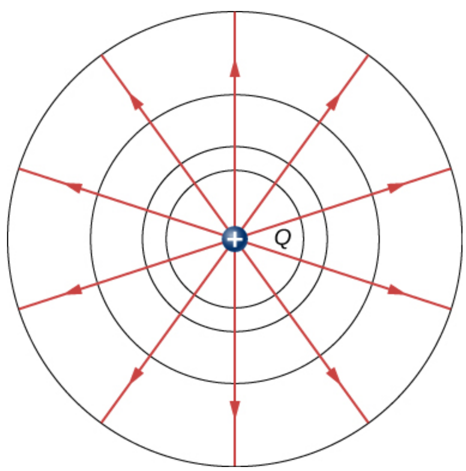


Figure $3.6.1$: An isolated point charge Q with its electric field lines in blue and equipotential lines in green. The potential is the same along each equipotential line, meaning that no work is required to move a charge anywhere along one of those lines. Work is needed to move a charge from one equipotential line to another. Equipotential lines are perpendicular to electric field lines in every case. For a three-dimensional version, explore the first media link.

It is important to note that *equipotential lines are always perpendicular to electric field lines*. No work is required to move a charge along an equipotential, since $\Delta V = 0$. Thus, the work is

$$W = -\Delta U = -q\Delta V = 0.$$

Work is zero if the direction of the force is perpendicular to the displacement. Force is in the same direction as E, so motion along an equipotential must be perpendicular to E. More precisely, work is related to the electric field by

$$\begin{aligned} W &= \vec{F} \cdot \vec{d} \qquad (3.6.1) \\ &= q\vec{E} \cdot \vec{d} \qquad (3.6.2) \\ &= qEd \cos \theta \\ &= 0. \end{aligned}$$

Note that in Equation 3.6.1, E and F symbolize the magnitudes of the electric field and force, respectively. Neither q nor E is zero and d is also not zero. So cos θ must be 0, meaning θ must be 90^o. In other words, motion along an equipotential is perpendicular to **E**.

One of the rules for static electric fields and conductors is that the electric field must be perpendicular to the surface of any conductor. This implies that a *conductor is an equipotential surface in static situations*. There can be no voltage difference across the surface of a conductor, or charges will flow. One of the uses of this fact is that a conductor can be fixed at what we consider zero volts by connecting it to the earth with a good conductor—a process called **grounding**. Grounding can be a useful safety tool. For example, grounding the metal case of an electrical appliance ensures that it is at zero volts relative to Earth.

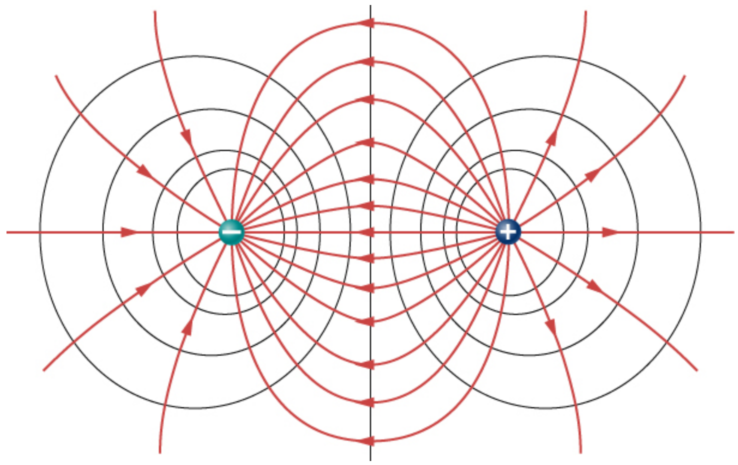


Figure $3.6.2$: The electric field lines and equipotential lines for two equal but opposite charges. The equipotential lines can be drawn by making them perpendicular to the electric field lines, if those are known. Note that the potential is greatest (most positive) near the positive charge and least (most negative) near the negative charge. For a three-dimensional version, explore the first media link.

Because a conductor is an equipotential, it can replace any equipotential surface. For example, in Figure $3.6.2$, a charged spherical conductor can replace the point charge, and the electric field and potential surfaces outside of it will be unchanged, confirming the contention that a spherical charge distribution is equivalent to a point charge at its center.

Figure $3.6.2$ shows the electric field and equipotential lines for two equal and opposite charges. Given the electric field lines, the equipotential lines can be drawn simply by making them perpendicular to the electric field lines. Conversely, given the equipotential lines, as in Figure $3.6.2a$, the electric field lines can be drawn by making them perpendicular to the equipotentials, as in Figure $3.6.2b$.

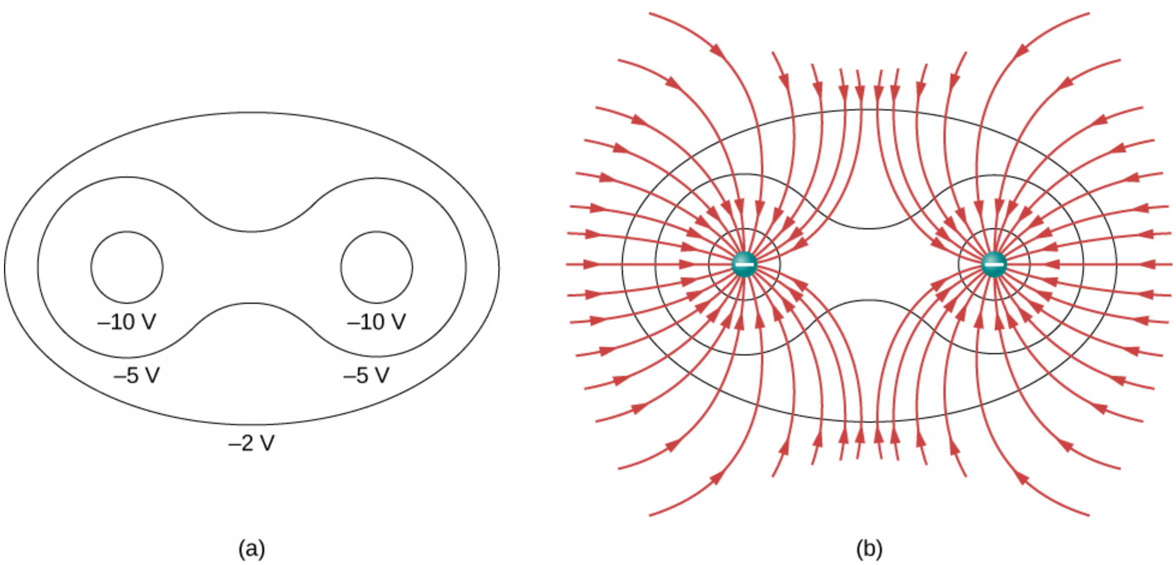


Figure $3.6.3$: (a) These equipotential lines might be measured with a voltmeter in a laboratory experiment. (b) The corresponding electric field lines are found by drawing them perpendicular to the equipotentials. Note that these fields are consistent with two equal negative charges. For a three-dimensional version, play with the first media link.

To improve your intuition, we show a three-dimensional variant of the potential in a system with two opposing charges. Figure $3.6.4$ displays a three-dimensional map of electric potential, where lines on the map are for equipotential surfaces. The hill is at the positive charge, and the trough is at the negative charge. The potential is zero far away from the charges. Note that the cut off at a particular potential implies that the charges are on conducting spheres with a finite radius.

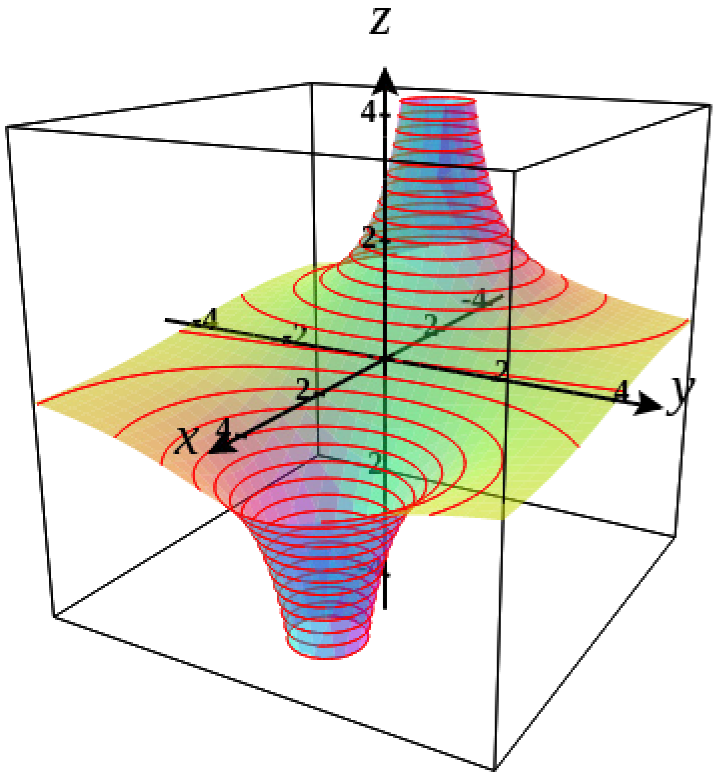


Figure $3.6.4$: Electric potential map of two opposite charges of equal magnitude on conducting spheres. The potential is negative near the negative charge and positive near the positive charge. This dynamic image is powered by CalcPlot3D and can be viewed here.

A two-dimensional map of the cross-sectional plane that contains both charges is shown in Figure $3.6.5$. The line that is equidistant from the two opposite charges corresponds to zero potential, since at the points on the line, the positive potential from the positive charge cancels the negative potential from the negative charge. Equipotential lines in the cross-sectional plane are closed loops, which are not necessarily circles, since at each point, the net potential is the sum of the potentials from each charge.

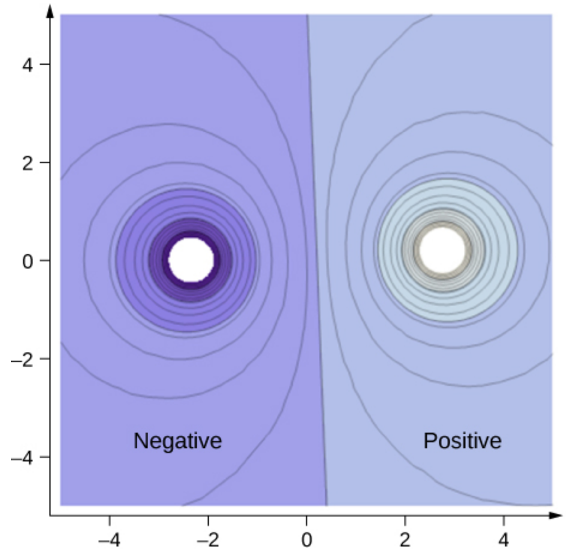


Figure 3.6.5: A cross-section of the electric potential map of two opposite charges of equal magnitude. The potential is negative near the negative charge and positive near the positive charge.

Note

View this simulation to observe and modify the equipotential surfaces and electric fields for many standard charge configurations. There's a lot to explore.

One of the most important cases is that of the familiar parallel conducting plates shown in Figure 3.6.6. Between the plates, the equipotentials are evenly spaced and parallel. The same field could be maintained by placing conducting plates at the equipotential lines at the potentials shown.

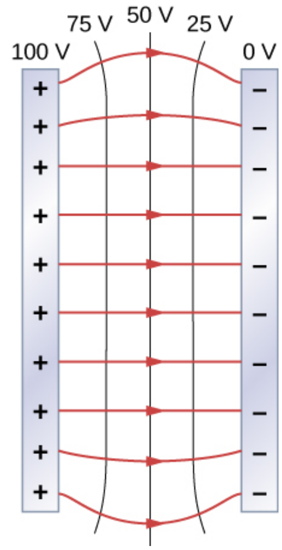
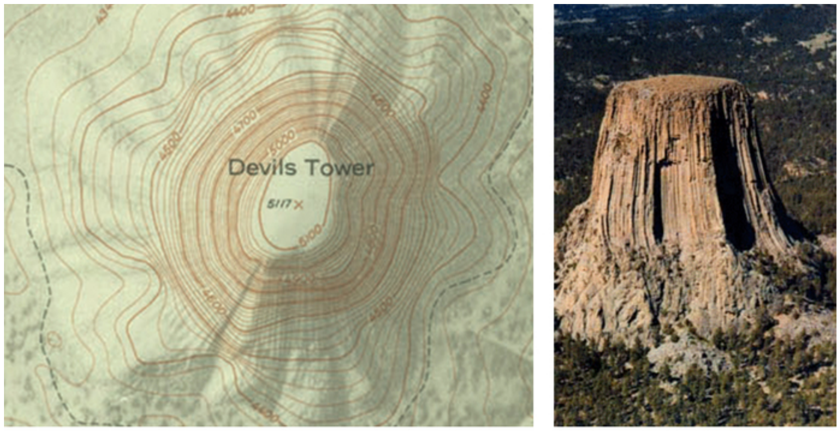


Figure 3.6.6: The electric field and equipotential lines between two metal plates. Note that the electric field is perpendicular to the equipotentials and hence normal to the plates at their surface as well as in the center of the region between them.

Consider the parallel plates Figure 3.6.6. These have equipotential lines that are parallel to the plates in the space between and evenly spaced. An example of this (with sample values) is given in Figure 3.6.6. We could draw a similar set of equipotential isolines for gravity on hills . If the hill has any extent at the same slope, the isolines along that extent would be parallel to each other. Furthermore, in regions of constant slope, the isolines would be evenly spaced. An example of real topographic lines is shown in Figure 3.6.7.



(a) (b)

Figure 3.6.6. (a) A topographical map of Devil's Tower, Wyoming. Lines that are close together indicate very steep terrain. (b) A perspective photo of Devil's Tower shows just how steep its sides are. Notice the top of the tower has the same shape as the center of the topographical map.

Example 3.6.1: Calculating Equipotential Lines

You have seen the equipotential lines of a point charge in Figure 3.6.1. How do we calculate them? For example, if we have a $+10\text{-nC}$ charge at the origin, what are the equipotential surfaces at which the potential is (a) 100 V, (b) 50 V, (c) 20 V, and (d) 10 V?

Strategy

Set the equation for the potential of a point charge equal to a constant and solve for the remaining variable(s). Then calculate values as needed.

Solution

In $V = k\frac{q}{r}$, let V be a constant. The only remaining variable is r; hence, $r = k\frac{q}{V} = \text{constant}$. Thus, the equipotential surfaces are spheres about the origin. Their locations are:

a. $r = k\frac{q}{V} = (8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)\frac{(10 \times 10^{-9}\,\text{C})}{100\,\text{V}} = 0.90\,\text{m}$;
b. $r = k\frac{q}{V} = (8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)\frac{(10 \times 10^{-9}\,\text{C})}{50\,\text{V}} = 1.8\,\text{m}$;
c. $r = k\frac{q}{V} = (8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)\frac{(10 \times 10^{-9}\,\text{C})}{20\,\text{V}} = 4.5\,\text{m}$;
d. $r = k\frac{q}{V} = (8.99 \times 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2)\frac{(10 \times 10^{-9}\,\text{C})}{10\,\text{V}} = 9.0\,\text{m}$.

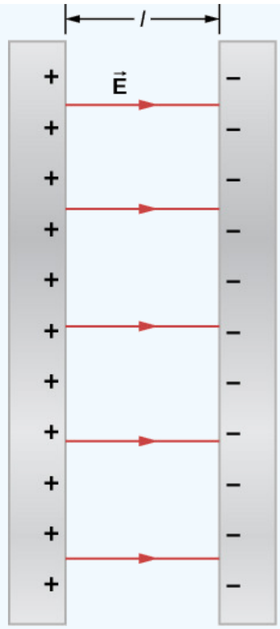
Significance

This means that equipotential surfaces around a point charge are spheres of constant radius, as shown earlier, with well-defined locations.

Example 3.6.2: Potential Difference between Oppositely Charged Parallel Plates

Two large conducting plates carry equal and opposite charges, with a surface charge density σ of magnitude $6.81 \times 10^{-7}\,\text{C/m}$, as shown in Figure 3.6.8. The separation between the plates is $l = 6.50\,\text{mm}$.

a. What is the electric field between the plates?
b. What is the potential difference between the plates?
c. What is the distance between equipotential planes which differ by 100 V?



Strategy

1. Since the plates are described as “large” and the distance between them is not, we will approximate each of them as an infinite plane, and apply the result from Gauss’s law in the previous chapter.
2. Use $\Delta V_{AB} = -\int_A^B \vec{E} \cdot d\vec{l}$.
3. Since the electric field is constant, find the ratio of 100 V to the total potential difference; then calculate this fraction of the distance.

Solution

a. The electric field is directed from the positive to the negative plate as shown in the figure, and its magnitude is given by

$$\begin{aligned} E &= \frac{\sigma}{\epsilon_0} \\ &= \frac{6.81 \times 10^{-7}\,\mathrm{C/m^2}}{8.85 \times 10^{-12}\,\mathrm{C^2/N \cdot m^2}} \\ &= 7.69 \times 10^4\,\mathrm{V/m}. \end{aligned}$$

b. To find the potential difference ΔV between the plates, we use a path from the negative to the positive plate that is directed against the field. The displacement vector $d\vec{l}$ and the electric field $\vec{E}$ are antiparallel so $\vec{E} \cdot d\vec{l} = -E\,dl$. The potential difference between the positive plate and the negative plate is then

$$\begin{aligned} \Delta V &= -\int E \cdot dl \\ &= E \int dl \\ &= El \\ &= (7.69 \times 10^4\,\mathrm{V/m})(6.50 \times 10^{-3}\,\mathrm{m}) \\ &= 500\,\mathrm{V} \end{aligned}$$

c. The total potential difference is 500 V, so 1/5 of the distance between the plates will be the distance between 100-V potential differences. The distance between the plates is 6.5 mm, so there will be 1.3 mm between 100-V potential differences.

Significance

You have now seen a numerical calculation of the locations of equipotentials between two charged parallel plates.

Exercise 3.6.1

What are the equipotential surfaces for an infinite line charge?

Answer

infinite cylinders of constant radius, with the line charge as the axis

Distribution of Charges on Conductors

In Example 3.6.1 with a point charge, we found that the equipotential surfaces were in the form of spheres, with the point charge at the center. Given that a conducting sphere in electrostatic equilibrium is a spherical equipotential surface, we should expect that we could replace one of the surfaces in Example 3.6.2 with a conducting sphere and have an identical solution outside the sphere. Inside will be rather different, however.

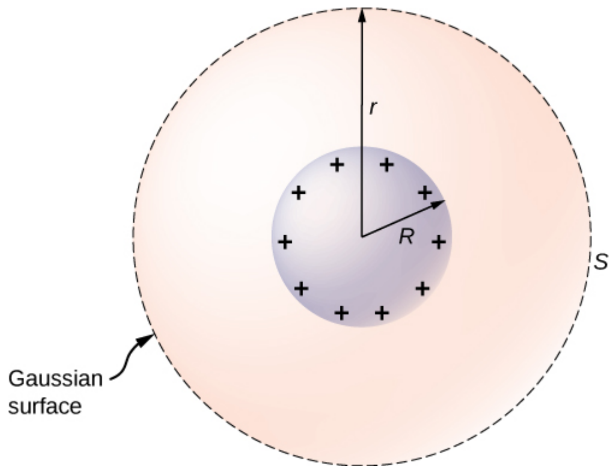


Figure 3.6.9: An isolated conducting sphere.

To investigate this, consider the isolated conducting sphere of Figure 3.6.9 that has a radius R and an excess charge q. To find the electric field both inside and outside the sphere, note that the sphere is isolated, so its surface change distribution and the electric field of that distribution are spherically symmetric. We can therefore represent the field as $\vec{E} = E(r)\hat{r}$. To calculate $E(r)$, we apply Gauss's law over a closed spherical surface S of radius r that is concentric with the conducting sphere. Since r is constant and $\hat{n} = \hat{r}$ on the sphere,

$$\oint \vec{E} \cdot \hat{n}\, da = E(r) \oint da \tag{3.6.3}$$

$$= E(r) 4\pi r^2. \tag{3.6.4}$$

For $r < R$, S is within the conductor, so recall from our previous study of Gauss's law that $q_{enc} = 0$ and Gauss's law gives $E(r) = 0$, as expected inside a conductor at equilibrium. If $r > R$, S encloses the conductor so $q_{enc} = q$. From Gauss's law,

$$E(r) 4\pi r^2 = \frac{q}{\epsilon_0}.$$

The electric field of the sphere may therefore be written as

$$E = 0\ (r < R),$$

and

$$E = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \hat{r}\ (r \geq R).$$

As expected, in the region $r \geq R$, the electric field due to a charge q placed on an isolated conducting sphere of radius R is identical to the electric field of a point charge q located at the center of the sphere.

To find the electric potential inside and outside the sphere, note that for $r \geq R$, the potential must be the same as that of an isolated point charge q located at $r = 0$,

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{r} (r \geq R)$$

simply due to the similarity of the electric field.

For $r < R$, $E = 0$, so $V(r)$ is constant in this region. Since $V(R) = q/4\pi\epsilon_0 R$,

$$V(r) = \frac{1}{4\pi\epsilon_0} \frac{q}{R} (r < R).$$

We will use this result to show that

$$\sigma_1 R_1 = \sigma_2 R_2,$$

for two conducting spheres of radii R_1 and R_2, with surface charge densities σ_1 and σ_2 respectively, that are connected by a thin wire, as shown in Figure $3.6.10$. The spheres are sufficiently separated so that each can be treated as if it were isolated (aside from the wire). Note that the connection by the wire means that this entire system must be an equipotential.

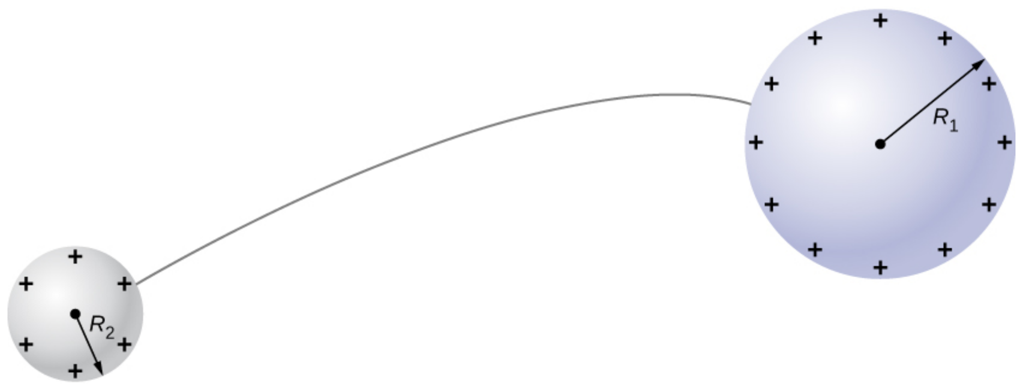


Figure $3.6.10$: Two conducting spheres are connected by a thin conducting wire.

We have just seen that the electrical potential at the surface of an isolated, charged conducting sphere of radius R is

$$V = \frac{1}{4\pi\epsilon_0} \frac{q}{R}.$$

Now, the spheres are connected by a conductor and are therefore at the same potential; hence

$$\frac{1}{4\pi\epsilon_0} \frac{q_1}{R_1} = \frac{1}{4\pi r\epsilon_0} \frac{q_2}{R_2},$$

and

$$\frac{q_1}{R_1} = \frac{q_2}{R_2}.$$

The net charge on a conducting sphere and its surface charge density are related by $q = \sigma(4\pi R^2)$. Substituting this equation into the previous one, we find

$$\sigma_1 R_1 = \sigma_2 R_2.$$

Obviously, two spheres connected by a thin wire do not constitute a typical conductor with a variable radius of curvature. Nevertheless, this result does at least provide a qualitative idea of how charge density varies over the surface of a conductor. The equation indicates that where the radius of curvature is large (points B and D in $3.6.11$), σ and E are small.

Similarly, the charges tend to be denser where the curvature of the surface is greater, as demonstrated by the charge distribution on oddly shaped metal (Figure $3.6.11$). The surface charge density is higher at locations with a small radius of curvature than at locations with a large radius of curvature.

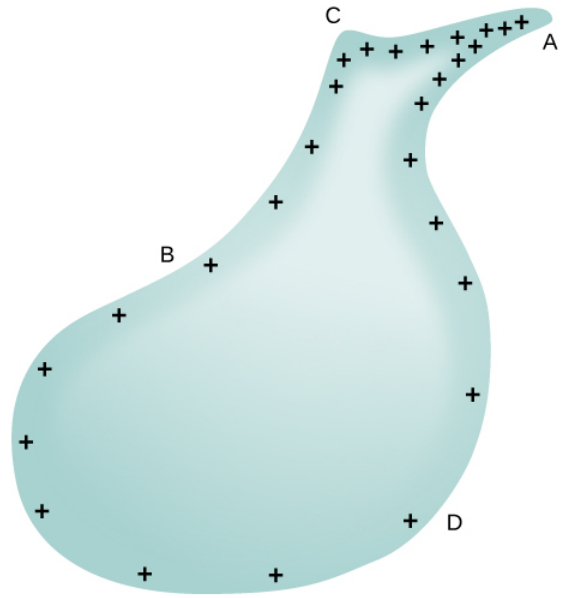


Figure 3.6.11: The surface charge density and the electric field of a conductor are greater at regions with smaller radii of curvature.

A practical application of this phenomenon is the **lightning rod**, which is simply a grounded metal rod with a sharp end pointing upward. As positive charge accumulates in the ground due to a negatively charged cloud overhead, the electric field around the sharp point gets very large. When the field reaches a value of approximately $3.0 \times 10^6\,\mathrm{N/C}$ (the **dielectric strength** of the air), the free ions in the air are accelerated to such high energies that their collisions with air molecules actually ionize the molecules. The resulting free electrons in the air then flow through the rod to Earth, thereby neutralizing some of the positive charge. This keeps the electric field between the cloud and the ground from getting large enough to produce a lightning bolt in the region around the rod.

An important application of electric fields and equipotential lines involves the heart. The heart relies on electrical signals to maintain its rhythm. The movement of electrical signals causes the chambers of the heart to contract and relax. When a person has a heart attack, the movement of these electrical signals may be disturbed. An artificial pacemaker and a defibrillator can be used to initiate the rhythm of electrical signals. The equipotential lines around the heart, the thoracic region, and the axis of the heart are useful ways of monitoring the structure and functions of the heart. An electrocardiogram (ECG) measures the small electric signals being generated during the activity of the heart.

PheT

Play around with this simulation to move point charges around on the playing field and then view the electric field, voltages, equipotential lines, and more.

3.7: Electric Potential (Exercises)

Conceptual Questions

7.2: Electric Potential Energy

1. Would electric potential energy be meaningful if the electric field were not conservative?

2. Why do we need to be careful about work done **on** the system versus work done **by** the system in calculations?

3. Does the order in which we assemble a system of point charges affect the total work done?

7.3: Electric Potential and Potential Difference

4. Discuss how potential difference and electric field strength are related. Give an example.

5. What is the strength of the electric field in a region where the electric potential is constant?

6. If a proton is released from rest in an electric field, will it move in the direction of increasing or decreasing potential? Also answer this question for an electron and a neutron. Explain why.

7. Voltage is the common word for potential difference. Which term is more descriptive, voltage or potential difference?

8. If the voltage between two points is zero, can a test charge be moved between them with zero net work being done? Can this necessarily be done without exerting a force? Explain.

9. What is the relationship between voltage and energy? More precisely, what is the relationship between potential difference and electric potential energy?

10. Voltages are always measured between two points. Why?

11. How are units of volts and electron-volts related? How do they differ?

12. Can a particle move in a direction of increasing electric potential, yet have its electric potential energy decrease? Explain

7.4: Calculations of Electric Potential

13. Compare the electric dipole moments of charges $\pm Q$ separated by a distance d and charges $\pm Q/2$ separated by a distance **d/2.**

14. Would Gauss's law be helpful for determining the electric field of a dipole? Why?

15. In what region of space is the potential due to a uniformly charged sphere the same as that of a point charge? In what region does it differ from that of a point charge?

16. Can the potential of a nonuniformly charged sphere be the same as that of a point charge? Explain.

7.5: Determining Field from Potential

17. If the electric field is zero throughout a region, must the electric potential also be zero in that region?

18. Explain why knowledge of $\vec{E}(x, y, z)$ is not sufficient to determine $V(x, y, z)$. What about the other way around?

7.6: Equipotential Surfaces and Conductors

19. If two points are at the same potential, are there any electric field lines connecting them?

20. Suppose you have a map of equipotential surfaces spaced 1.0 V apart. What do the distances between the surfaces in a particular region tell you about the strength of the $\vec{E}$ in that region?

21. Is the electric potential necessarily constant over the surface of a conductor?

22. Under electrostatic conditions, the excess charge on a conductor resides on its surface. Does this mean that all of the conduction electrons in a conductor are on the surface?

23. Can a positively charged conductor be at a negative potential? Explain.

24. Can equipotential surfaces intersect?

7.7: Applications of Electrostatics

25. Why are the metal support rods for satellite network dishes generally grounded?

26. (a) Why are fish reasonably safe in an electrical storm?

(b) Why are swimmers nonetheless ordered to get out of the water in the same circumstance?

27. What are the similarities and differences between the processes in a photocopier and an electrostatic precipitator?

28. About what magnitude of potential is used to charge the drum of a photocopy machine? A web search for “xerography” may be of use.

Problems

7.2 Electric Potential Energy

29. Consider a charge $Q_1(+5.0\mu C)$ fixed at a site with another charge Q_2 (charge $+3.0\mu C$, mass $6.0\mu g$) moving in the neighboring space. (a) Evaluate the potential energy of Q_2 when it is 4.0 cm from Q_1. (b) If Q_2 starts from rest from a point 4.0 cm from Q_1, what will be its speed when it is 8.0 cm from Q_1? (**Note:** Q_1 is held fixed in its place.)

30. Two charges $Q_1(+2.00\mu C)$ and $Q_2(+2.00\mu C)$ are placed symmetrically along the **x**-axis at $x = \pm 3.00cm$. Consider a charge Q_3 of charge $+4.00\mu C$ and mass 10.0 mg moving along the **y**-axis. If Q_3 starts from rest at $y = 2.00cm$, what is its speed when it reaches $y = 4.00cm$?

31. To form a hydrogen atom, a proton is fixed at a point and an electron is brought from far away to a distance of $0.529 \times 10^{-10}m$, the average distance between proton and electron in a hydrogen atom. How much work is done?

32. (a) What is the average power output of a heart defibrillator that dissipates 400 J of energy in 10.0 ms? (b) Considering the high-power output, why doesn’t the defibrillator produce serious burns?

7.3 Electric Potential and Potential Difference

33. Find the ratio of speeds of an electron and a negative hydrogen ion (one having an extra electron) accelerated through the same voltage, assuming non-relativistic final speeds. Take the mass of the hydrogen ion to be $1.67 \times 10^{-27}kg$.

34. An evacuated tube uses an accelerating voltage of 40 kV to accelerate electrons to hit a copper plate and produce X-rays. Non-relativistically, what would be the maximum speed of these electrons?

35. Show that units of **V/m** and **N/C** for electric field strength are indeed equivalent.

36. What is the strength of the electric field between two parallel conducting plates separated by 1.00 cm and having a potential difference (voltage) between them of $1.50 \times 10^4 V$?

37. The electric field strength between two parallel conducting plates separated by 4.00 cm is $7.50 \times 10^4 V$.

(a) What is the potential difference between the plates?

(b) The plate with the lowest potential is taken to be zero volts. What is the potential 1.00 cm from that plate and 3.00 cm from the other?

38. The voltage across a membrane forming a cell wall is 80.0 mV and the membrane is 9.00 nm thick. What is the electric field strength? (The value is surprisingly large, but correct.) You may assume a uniform electric field.

39. Two parallel conducting plates are separated by 10.0 cm, and one of them is taken to be at zero volts.

(a) What is the electric field strength between them, if the potential 8.00 cm from the zero volt plate (and 2.00 cm from the other) is 450 V?

(b) What is the voltage between the plates?

40. Find the maximum potential difference between two parallel conducting plates separated by 0.500 cm of air, given the maximum sustainable electric field strength in air to be $3.0 \times 10^6 V/m$.

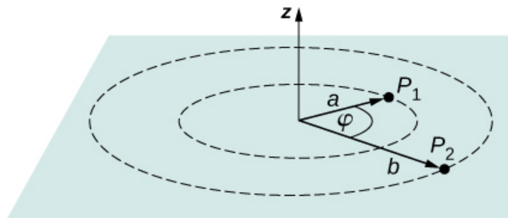
41. An electron is to be accelerated in a uniform electric field having a strength of $2.00 \times 10^6 V/m$.

(a) What energy in keV is given to the electron if it is accelerated through 0.400 m?

(b) Over what distance would it have to be accelerated to increase its energy by 50.0 GeV?

42. Use the definition of potential difference in terms of electric field to deduce the formula for potential difference between $r = r_a$ and $r = r_b$ for a point charge located at the origin. Here r is the spherical radial coordinate.

43. The electric field in a region is pointed away from the **z**-axis and the magnitude depends upon the distance s from the axis. The magnitude of the electric field is given as $E = \frac{\alpha}{s}$ where α is a constant. Find the potential difference between points P_1 and P_2, explicitly stating the path over which you conduct the integration for the line integral.



44. Singly charged gas ions are accelerated from rest through a voltage of 13.0 V. At what temperature will the average kinetic energy of gas molecules be the same as that given these ions?

7.4 Calculations of Electric Potential

45. A 0.500-cm-diameter plastic sphere, used in a static electricity demonstration, has a uniformly distributed 40.0-pC charge on its surface. What is the potential near its surface?

46. How far from a $1.00 - \mu C$ point charge is the potential 100 V? At what distance is it $2.00 \times 10^2 V$?

47. If the potential due to a point charge is $5.00 \times 10^2 V$ at a distance of 15.0 m, what are the sign and magnitude of the charge?

48. In nuclear fission, a nucleus splits roughly in half. (a) What is the potential $2.00 \times 10^{-14} m$ from a fragment that has 46 protons in it? (b) What is the potential energy in MeV of a similarly charged fragment at this distance?

49. A research Van de Graaff generator has a 2.00-m-diameter metal sphere with a charge of 5.00 mC on it. Assume the potential energy is zero at a reference point infinitely far away from the Van de Graaff.

(a) What is the potential near its surface?

(b) At what distance from its center is the potential 1.00 MV?

(c) An oxygen atom with three missing electrons is released near the Van de Graaff generator. What is its kinetic energy in MeV when the atom is at the distance found in part b?

50. An electrostatic paint sprayer has a 0.200-m-diameter metal sphere at a potential of 25.0 kV that repels paint droplets onto a grounded object.

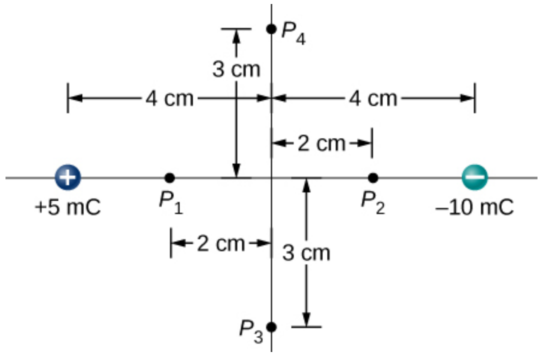
(a) What charge is on the sphere?

(b) What charge must a 0.100-mg drop of paint have to arrive at the object with a speed of 10.0 m/s?

51. (a) What is the potential between two points situated 10 cm and 20 cm from a $3.0 - \mu C$ point charge?

(b) To what location should the point at 20 cm be moved to increase this potential difference by a factor of two?

52. Find the potential at points P_1, P_2, P_3, and P_4 in the diagram due to the two given charges.



53. Two charges $-2.0\mu C$ and $+2.0\mu C$ are separated by 4.0 cm on the **z**-axis symmetrically about origin, with the positive one uppermost. Two space points of interest P_1 and P_2 are located 3.0 cm and 30 cm from origin at an angle **30°** with respect to the **z**-axis. Evaluate electric potentials at P_1 and P_2 in two ways:

(a) Using the exact formula for point charges, and

(b) using the approximate dipole potential formula.

54. (a) Plot the potential of a uniformly charged 1-m rod with 1 C/m charge as a function of the perpendicular distance from the center. Draw your graph from $s=0.1m$ to $s=1.0m$.

(b) On the same graph, plot the potential of a point charge with a 1-C charge at the origin.

(c) Which potential is stronger near the rod? (d) What happens to the difference as the distance increases? Interpret your result.

7.5 Determining Field from Potential

55. Throughout a region, equipotential surfaces are given by $z=constant$. The surfaces are equally spaced with $V=100V$ for $z=0.00m$, $V=200V$ for $z=0.50m$, $V=300V$ for $z=1.00m$. What is the electric field in this region?

56. In a particular region, the electric potential is given by $V=-xy^2z+4xy$. What is the electric field in this region?

57. Calculate the electric field of an infinite line charge, throughout space.

7.6 Equipotential Surfaces and Conductors

58. Two very large metal plates are placed 2.0 cm apart, with a potential difference of 12 V between them. Consider one plate to be at 12 V, and the other at 0 V. (a) Sketch the equipotential surfaces for 0, 4, 8, and 12 V.

(b) Next sketch in some electric field lines, and confirm that they are perpendicular to the equipotential lines.

59. A very large sheet of insulating material has had an excess of electrons placed on it to a surface charge density of $-3.00nC/m^2$.

(a) As the distance from the sheet increases, does the potential increase or decrease? Can you explain why without any calculations? Does the location of your reference point matter?

(b) What is the shape of the equipotential surfaces?

(c) What is the spacing between surfaces that differ by 1.00 V?

60. A metallic sphere of radius 2.0 cm is charged with $+5.0-\mu C$ charge, which spreads on the surface of the sphere uniformly. The metallic sphere stands on an insulated stand and is surrounded by a larger metallic spherical shell, of inner radius 5.0 cm and outer radius 6.0 cm. Now, a charge of $-5.0-\mu C$ is placed on the inside of the spherical shell, which spreads out uniformly on the inside surface of the shell. If potential is zero at infinity, what is the potential of

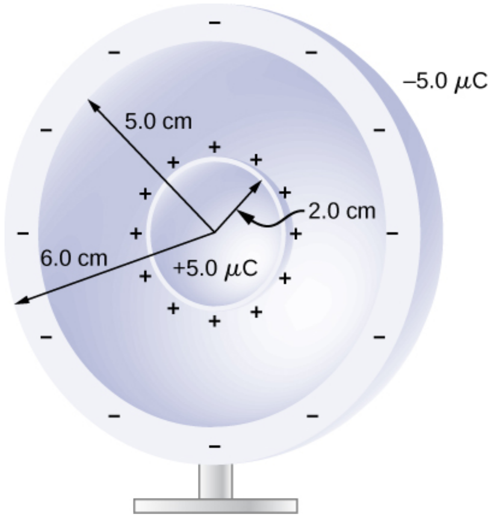
(a) the spherical shell,

(b) the sphere,

(c) the space between the two,

(d) inside the sphere, and

(e) outside the shell?



61. Two large charged plates of charge density $\pm 30\mu C/m^2$ face each other at a separation of 5.0 mm.

(a) Find the electric potential everywhere.

(b) An electron is released from rest at the negative plate; with what speed will it strike the positive plate?

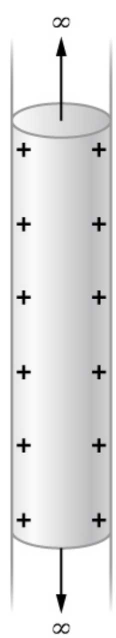
62. A long cylinder of aluminum of radius R meters is charged so that it has a uniform charge per unit length on its surface of λ.

(a) Find the electric field inside and outside the cylinder.

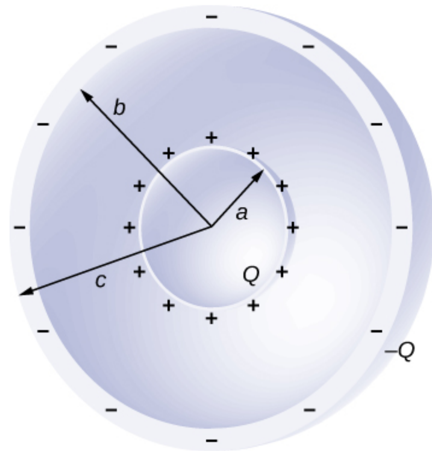
(b) Find the electric potential inside and outside the cylinder. (c) Plot electric field and electric potential as a function of distance from the center of the rod.

63. Two parallel plates 10 cm on a side are given equal and opposite charges of magnitude $5.0 \times 10^{-9} C$.The plates are 1.5 mm apart. What is the potential difference between the plates?

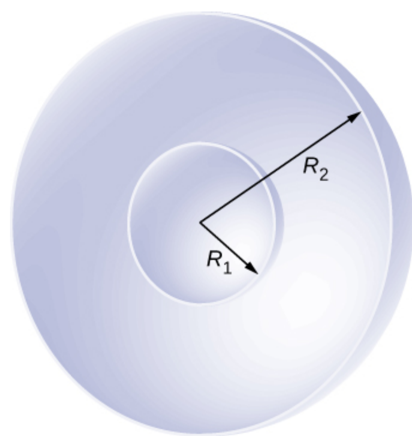
64. The surface charge density on a long straight metallic pipe is σ. What is the electric potential outside and inside the pipe? Assume the pipe has a diameter of **2a**.



65. Concentric conducting spherical shells carry charges **Q** and **–Q**, respectively. The inner shell has negligible thickness. What is the potential difference between the shells?



66. Shown below are two concentric spherical shells of negligible thicknesses and radii R_1 and R_2. The inner and outer shell carry net charges q_1 and q_2, respectively, where both q_1 and q_2 are positive. What is the electric potential in the regions (a) $r < R_1$, (b) $R_1 < r < R_2$, and (c) $r > R_2$?



67. A solid cylindrical conductor of radius a is surrounded by a concentric cylindrical shell of inner radius **b**. The solid cylinder and the shell carry charges **Q** and **–Q**, respectively. Assuming that the length **L** of both conductors is much greater than **a** or **b**, what is the potential difference between the two conductors?

7.7 Applications of Electrostatics

68. (a) What is the electric field 5.00 m from the center of the terminal of a Van de Graaff with a 3.00-mC charge, noting that the field is equivalent to that of a point charge at the center of the terminal?

(b) At this distance, what force does the field exert on a $2.00 - \mu C$ charge on the Van de Graaff's belt?

69. (a) What is the direction and magnitude of an electric field that supports the weight of a free electron near the surface of Earth?

(b) Discuss what the small value for this field implies regarding the relative strength of the gravitational and electrostatic forces.

70. A simple and common technique for accelerating electrons is shown in Figure $3.7.1$, where there is a uniform electric field between two plates. Electrons are released, usually from a hot filament, near the negative plate, and there is a small hole in the positive plate that allows the electrons to continue moving.

(a) Calculate the acceleration of the electron if the field strength is $2.50 \times 10^4 N/C$.

(b) Explain why the electron will not be pulled back to the positive plate once it moves through the hole.

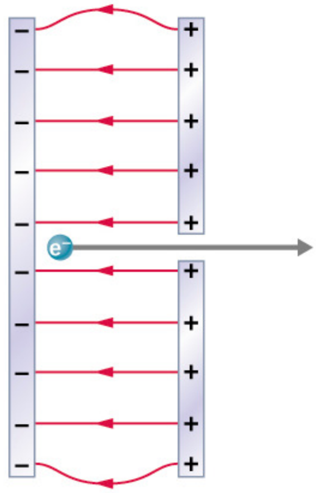


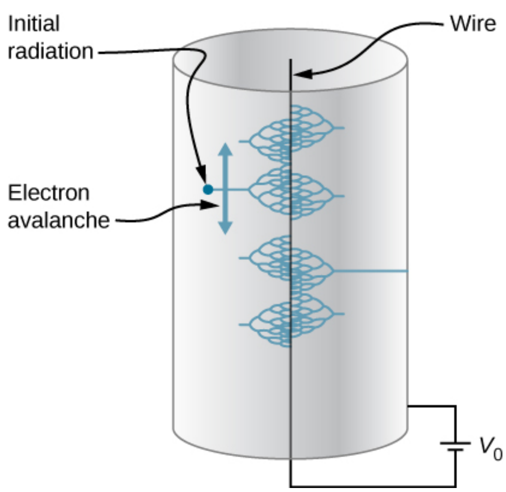
Figure 3.7.1: Parallel conducting plates with opposite charges on them create a relatively uniform electric field used to accelerate electrons to the right. Those that go through the hole can be used to make a TV or computer screen glow or to produce X-rays.

71. In a Geiger counter, a thin metallic wire at the center of a metallic tube is kept at a high voltage with respect to the metal tube. Ionizing radiation entering the tube knocks electrons off gas molecules or sides of the tube that then accelerate towards the center wire, knocking off even more electrons. This process eventually leads to an avalanche that is detectable as a current. A particular Geiger counter has a tube of radius **R** and the inner wire of radius a is at a potential of V_0 volts with respect to the outer metal tube. Consider a point P at a distance s from the center wire and far away from the ends.

(a) Find a formula for the electric field at a point **P** inside using the infinite wire approximation.

(b) Find a formula for the electric potential at a point **P** inside.

(c) Use $V_0 = 900 V, a = 3.00 mm, R = 2.00 cm$, and find the value of the electric field at a point 1.00 cm from the center.



72. The practical limit to an electric field in air is about $3.00 \times 10^6 N/C$. Above this strength, sparking takes place because air begins to ionize.

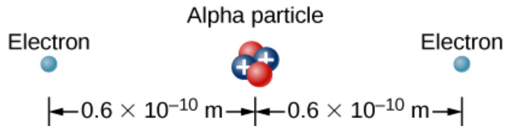
(a) At this electric field strength, how far would a proton travel before hitting the speed of light (ignore relativistic effects)?

(b) Is it practical to leave air in particle accelerators?

73. To form a helium atom, an alpha particle that contains two protons and two neutrons is fixed at one location, and two electrons are brought in from far away, one at a time. The first electron is placed at $0.600 \times 10^{-10} m$ from the alpha particle and held there while the second electron is brought to $0.600 \times 10^{-10} m$ from the alpha particle on the other side from the first electron. See the final configuration below.

(a) How much work is done in each step?

(b) What is the electrostatic energy of the alpha particle and two electrons in the final configuration?



74. Find the electrostatic energy of eight equal charges $(+3\mu C)$ each fixed at the corners of a cube of side 2 cm.

75. The probability of fusion occurring is greatly enhanced when appropriate nuclei are brought close together, but mutual Coulomb repulsion must be overcome. This can be done using the kinetic energy of high-temperature gas ions or by accelerating the nuclei toward one another.

(a) Calculate the potential energy of two singly charged nuclei separated by $1.00 \times 10^{-12} m$.

(b) At what temperature will atoms of a gas have an average kinetic energy equal to this needed electrical potential energy?

76. A bare helium nucleus has two positive charges and a mass of $6.64 \times 10^{-27} kg$.

(a) Calculate its kinetic energy in joules at 2.00 of the speed of light.

(b) What is this in electron-volts?

(c) What voltage would be needed to obtain this energy?

77. An electron enters a region between two large parallel plates made of aluminum separated by a distance of 2.0 cm and kept at a potential difference of 200 V. The electron enters through a small hole in the negative plate and moves toward the positive plate. At the time the electron is near the negative plate, its speed is $4.0 \times 10^5 m/s$. Assume the electric field between the plates to be uniform, and find the speed of electron at

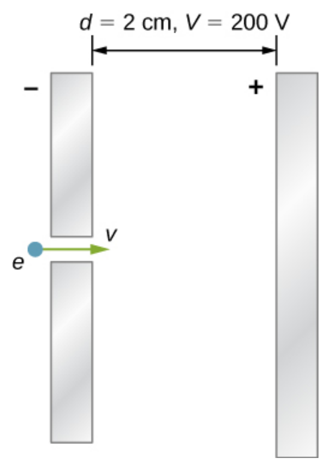
(a) 0.10 cm,

(b) 0.50 cm,

(c) 1.0 cm, and

(d) 1.5 cm from the negative plate, and

(e) immediately before it hits the positive plate.



78. How far apart are two conducting plates that have an electric field strength of $4.50 \times 10^3 V/m$ between them, if their potential difference is 15.0 kV?

79. (a) Will the electric field strength between two parallel conducting plates exceed the breakdown strength of dry air, which is $3.00 \times 10^6 V/m$, if the plates are separated by 2.00 mm and a potential difference of $5.0 \times 10^3 V$ is applied?

(b) How close together can the plates be with this applied voltage?

80. Membrane walls of living cells have surprisingly large electric fields across them due to separation of ions. What is the voltage across an 8.00-nm-thick membrane if the electric field strength across it is 5.50 MV/m? You may assume a uniform electric field.

81. A double charged ion is accelerated to an energy of 32.0 keV by the electric field between two parallel conducting plates separated by 2.00 cm. What is the electric field strength between the plates?

82. The temperature near the center of the Sun is thought to be 15 million degrees Celsius $(1.5 \times 10^{7} {}^\circ C)$ (or kelvin). Through what voltage must a singly charged ion be accelerated to have the same energy as the average kinetic energy of ions at this temperature?

83. A lightning bolt strikes a tree, moving 20.0 C of charge through a potential difference of $1.00 \times 1^0 2MV$.

(a) What energy was dissipated?

(b) What mass of water could be raised from **15°C** to the boiling point and then boiled by this energy?

(c) Discuss the damage that could be caused to the tree by the expansion of the boiling steam.

84. What is the potential $0.530 \times 10^{-10} m$ from a proton (the average distance between the proton and electron in a hydrogen atom)?

85. (a) A sphere has a surface uniformly charged with 1.00 C. At what distance from its center is the potential 5.00 MV? (b) What does your answer imply about the practical aspect of isolating such a large charge?

86. What are the sign and magnitude of a point charge that produces a potential of –2.00 V at a distance of 1.00 mm?

87. In one of the classic nuclear physics experiments at the beginning of the twentieth century, an alpha particle was accelerated toward a gold nucleus, and its path was substantially deflected by the Coulomb interaction. If the energy of the doubly charged alpha nucleus was 5.00 MeV, how close to the gold nucleus (79 protons) could it come before being deflected?

Additional Problems

88. A 12.0-V battery-operated bottle warmer heats 50.0 g of glass, $2.50 \times 10^2 g$ of baby formula, and $2.00 \times 10^2 g$ of aluminum from **20.0°C** to **90.0°C**.

(a) How much charge is moved by the battery?

(b) How many electrons per second flow if it takes 5.00 min to warm the formula? (Hint: Assume that the specific heat of baby formula is about the same as the specific heat of water.)

89. A battery-operated car uses a 12.0-V system. Find the charge the batteries must be able to move in order to accelerate the 750 kg car from rest to 25.0 m/s, make it climb a $2.00 \times 10^2 - m$ high hill, and finally cause it to travel at a constant 25.0 m/s while climbing with $5.00 \times 10^2 - N$ force for an hour.

90. (a) Find the voltage near a 10.0 cm diameter metal sphere that has 8.00 C of excess positive charge on it.

(b) What is unreasonable about this result?

(c) Which assumptions are responsible?

91. A uniformly charged half-ring of radius 10 cm is placed on a nonconducting table. It is found that 3.0 cm above the center of the half-ring the potential is –3.0 V with respect to zero potential at infinity. How much charge is in the half-ring?

92. A glass ring of radius 5.0 cm is painted with a charged paint such that the charge density around the ring varies continuously given by the following function of the polar angle $\theta, \lambda = (3.0 \times 10^{-6} C/m) cos^2\theta.$. Find the potential at a point 15 cm above the center.

93. A CD disk of radius $(R = 3.0 cm)$ is sprayed with a charged paint so that the charge varies continually with radial distance rfrom the center in the following manner: $\sigma = -(6.0 C/m) r/R$ Find the potential at a point 4 cm above the center.

94. (a) What is the final speed of an electron accelerated from rest through a voltage of 25.0 MV by a negatively charged Van de Graff terminal? (b) What is unreasonable about this result? (c) Which assumptions are responsible?

95. A large metal plate is charged uniformly to a density of $\sigma = 2.0 \times 10^{-9} C/m^2$. How far apart are the equipotential surfaces that represent a potential difference of 25 V?

96. Your friend gets really excited by the idea of making a lightning rod or maybe just a sparking toy by connecting two spheres as shown in Figure 7.39, and making R_2 so small that the electric field is greater than the dielectric strength of air, just from the usual 150 V/m electric field near the surface of the Earth. If R_1 is 10 cm, how small does R_2 need to be, and does this seem practical? (**Hint**: recall the calculation for electric field at the surface of a conductor from Gauss's Law.)

97. (a) Find $x >> L$ limit of the potential of a finite uniformly charged rod and show that it coincides with that of a point charge formula. (b) Why would you expect this result?

98. A small spherical pith ball of radius 0.50 cm is painted with a silver paint and then $-10\mu C$ of charge is placed on it. The charged pith ball is put at the center of a gold spherical shell of inner radius 2.0 cm and outer radius 2.2 cm.

(a) Find the electric potential of the gold shell with respect to zero potential at infinity.

(b) How much charge should you put on the gold shell if you want to make its potential 100 V?

99. Two parallel conducting plates, each of cross-sectional area $400 cm^2$, are 2.0 cm apart and uncharged. If 1.0×10^{12} electrons are transferred from one plate to the other,

(a) what is the potential difference between the plates?

(b) What is the potential difference between the positive plate and a point 1.25 cm from it that is between the plates?

100. A point charge of $q = 5.0 \times 10^{-8} C$ is placed at the center of an uncharged spherical conducting shell of inner radius 6.0 cm and outer radius 9.0 cm. Find the electric potential at

(a) $r = 4.0 cm$,

(b) $r = 8.0 cm$,

(c) $r = 12.0 cm$.

101. Earth has a net charge that produces an electric field of approximately 150 N/C downward at its surface.

(a) What is the magnitude and sign of the excess charge, noting the electric field of a conducting sphere is equivalent to a point charge at its center?

(b) What acceleration will the field produce on a free electron near Earth's surface?

(c) What mass object with a single extra electron will have its weight supported by this field?

102. Point charges of $25.0\mu C$ and $45.0\mu C$ are placed 0.500 m apart.

(a) At what point along the line between them is the electric field zero?

(b) What is the electric field halfway between them?

103. What can you say about two charges q_1 and q_2, if the electric field one-fourth of the way from q_1 to q_2 is zero?

104. Calculate the angular velocity ω of an electron orbiting a proton in the hydrogen atom, given the radius of the orbit is $0.530 \times 10^{-10} m$. You may assume that the proton is stationary and the centripetal force is supplied by Coulomb attraction.

105. An electron has an initial velocity of $5.00 \times 10^6 m/s$ in a uniform $2.00 \times 10^5 - N/C$ electric field. The field accelerates the electron in the direction opposite to its initial velocity.

(a) What is the direction of the electric field?

(b) How far does the electron travel before coming to rest?

(c) How long does it take the electron to come to rest?

(d) What is the electron's velocity when it returns to its starting point?

Challenge Problems

106. Three Na^+ and three Cl^- ions are placed alternately and equally spaced around a circle of radius 50 nm. Find the electrostatic energy stored.

107. Look up (presumably online, or by dismantling an old device and making measurements) the magnitude of the potential deflection plates (and the space between them) in an ink jet printer. Then look up the speed with which the ink comes out the nozzle. Can you calculate the typical mass of an ink drop?

108. Use the electric field of a finite sphere with constant volume charge density to calculate the electric potential, throughout space. Then check your results by calculating the electric field from the potential.

109. Calculate the electric field of a dipole throughout space from the potential.

CHAPTER OVERVIEW

4: Capacitance

Capacitors are important components of electrical circuits in many electronic devices, including pacemakers, cell phones, and computers. In this chapter, we study their properties, and, over the next few chapters, we examine their function in combination with other circuit elements. By themselves, capacitors are often used to store electrical energy and release it when needed; with other circuit components, capacitors often act as part of a filter that allows some electrical signals to pass while blocking others. You can see why capacitors are considered one of the fundamental components of electrical circuits.

4.1: Prelude to Capacitance
4.2: Capacitors and Capacitance
4.3: Energy Stored in a Capacitor
4.4: Capacitor with a Dielectric
4.5: Molecular Model of a Dielectric
4.6: Capacitance (Exercises)

4.1: Prelude to Capacitance

Capacitors are important components of electrical circuits in many electronic devices, including pacemakers, cell phones, and computers. In this chapter, we study their properties, and, over the next few chapters, we examine their function in combination with other circuit elements. By themselves, capacitors are often used to store electrical energy and release it when needed; with other circuit components, capacitors often act as part of a filter that allows some electrical signals to pass while blocking others. You can see why capacitors are considered one of the fundamental components of electrical circuits.

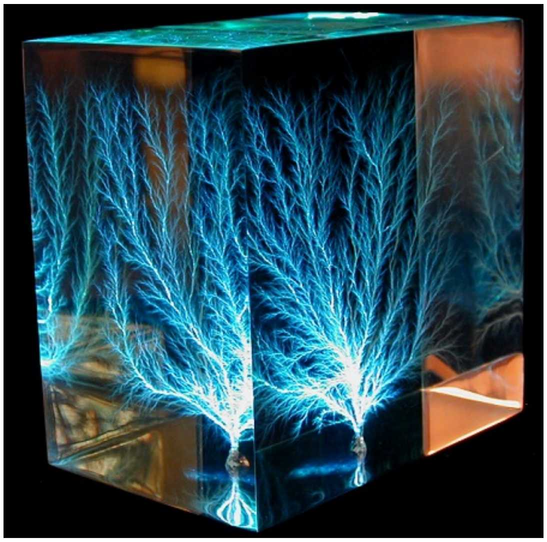


Figure 4.1.1: The tree-like branch patterns in this clear acrylic block are created by irradiating the block with an electron beam. This tree is known as a Lichtenberg figure, named for the German physicist Georg Christof Lichtenberg (1742–1799), who was the first to study these patterns. The “branches” are created by the dielectric breakdown produced by a strong electric field. (Bert Hickman).

4.2: Capacitors and Capacitance

Learning Objectives

By the end of this section, you will be able to:

- Explain the concepts of a capacitor and its capacitance
- Describe how to evaluate the capacitance of a system of conductors

A **capacitor** is a device used to store electrical charge and electrical energy. It consists of at least two electrical conductors separated by a distance. (Note that such electrical conductors are sometimes referred to as "electrodes," but more correctly, they are "capacitor plates.") The space between capacitors may simply be a vacuum, and, in that case, a capacitor is then known as a "vacuum capacitor." However, the space is usually filled with an insulating material known as a **dielectric**. (You will learn more about dielectrics in the sections on dielectrics later in this chapter.) The amount of storage in a capacitor is determined by a property called **capacitance**, which you will learn more about a bit later in this section.

Capacitors have applications ranging from filtering static from radio reception to energy storage in heart defibrillators. Typically, commercial capacitors have two conducting parts close to one another but not touching, such as those in Figure $4.2.1$. Most of the time, a dielectric is used between the two plates. When battery terminals are connected to an initially uncharged capacitor, the battery potential moves a small amount of charge of magnitude Q from the positive plate to the negative plate. The capacitor remains neutral overall, but with charges $+Q$ and $-Q$ residing on opposite plates.

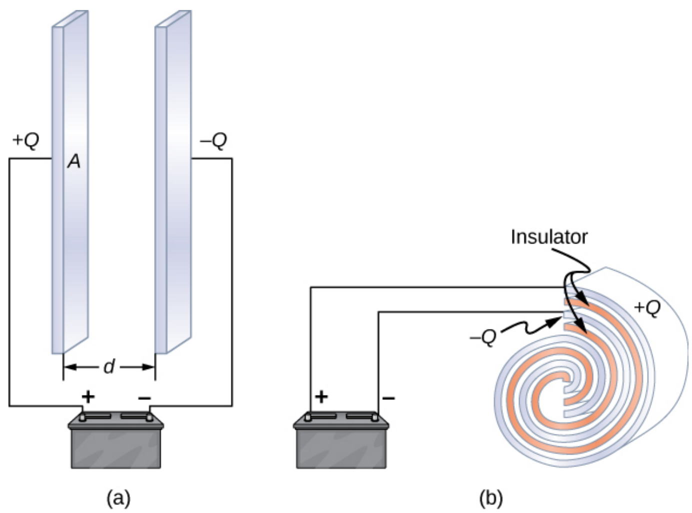


Figure $4.2.1$: Both capacitors shown here were initially uncharged before being connected to a battery. They now have charges of $+Q$ and $-Q$ (respectively) on their plates. (a) A parallel-plate capacitor consists of two plates of opposite charge with area A separated by distance d. (b) A rolled capacitor has a dielectric material between its two conducting sheets (plates).

A system composed of two identical parallel-conducting plates separated by a distance is called a **parallel-plate capacitor** (Figure $4.2.2$). The magnitude of the electrical field in the space between the parallel plates is $E = \sigma/\epsilon_0$, where σ denotes the surface charge density on one plate (recall that σ is the charge **Q** per the surface area **A**). Thus, the magnitude of the field is directly proportional to **Q**.

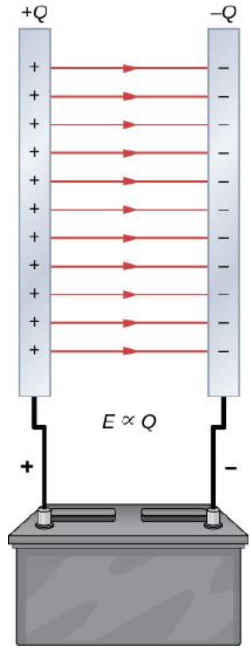


Figure 4.2.2: The charge separation in a capacitor shows that the charges remain on the surfaces of the capacitor plates. Electrical field lines in a parallel-plate capacitor begin with positive charges and end with negative charges. The magnitude of the electrical field in the space between the plates is in direct proportion to the amount of charge on the capacitor.

Capacitors with different physical characteristics (such as shape and size of their plates) store different amounts of charge for the same applied voltage V across their plates. The **capacitance** C of a capacitor is defined as the ratio of the maximum charge Q that can be stored in a capacitor to the applied voltage V across its plates. In other words, capacitance is the largest amount of charge per volt that can be stored on the device:

$$C = \frac{Q}{V} \tag{4.2.1}$$

The SI unit of capacitance is the **farad** (F), named after Michael Faraday (1791–1867). Since capacitance is the charge per unit voltage, one farad is one coulomb per one volt, or

$$1\, F = \frac{1\, C}{1\, V}.$$

By definition, a 1.0-F capacitor is able to store 1.0 C of charge (a very large amount of charge) when the potential difference between its plates is only 1.0 V. One farad is therefore a very large capacitance. Typical capacitance values range from picofarads ($1\, pF = 10 - 12 F$) to millifarads $(1\, mF = 10^{-3} F)$, which also includes microfarads $(1\, \mu C = 10^{-6} F)$.. Capacitors can be produced in various shapes and sizes (Figure 4.2.3).

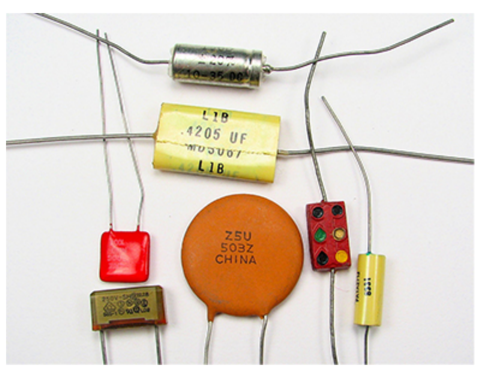


Figure 4.2.3: These are some typical capacitors used in electronic devices. A capacitor’s size is not necessarily related to its capacitance value.

Calculation of Capacitance

We can calculate the capacitance of a pair of conductors with the standard approach that follows.

Problem-Solving Strategy: Calculating Capacitance

1. Assume that the capacitor has a charge Q.
2. Determine the electrical field $\vec{E}$ between the conductors. If symmetry is present in the arrangement of conductors, you may be able to use Gauss's law for this calculation.
3. Find the potential difference between the conductors from

$$V_B - V_A = -\int_A^B \vec{E} \cdot d\vec{l}, \tag{4.2.2}$$

where the path of integration leads from one conductor to the other. The magnitude of the potential difference is then $V = |V_B - V_A|$.
4. With V known, obtain the capacitance directly from Equation 4.2.1.

To show how this procedure works, we now calculate the capacitances of parallel-plate, spherical, and cylindrical capacitors. In all cases, we assume vacuum capacitors (empty capacitors) with no dielectric substance in the space between conductors.

Parallel-Plate Capacitor

The parallel-plate capacitor (Figure $4.2.4$) has two identical conducting plates, each having a surface area A, separated by a distance d. When a voltage V is applied to the capacitor, it stores a charge Q, as shown. We can see how its capacitance may depend on A and d by considering characteristics of the Coulomb force. We know that force between the charges increases with charge values and decreases with the distance between them. We should expect that the bigger the plates are, the more charge they can store. Thus, C should be greater for a larger value of A. Similarly, the closer the plates are together, the greater the attraction of the opposite charges on them. Therefore, C should be greater for a smaller d.

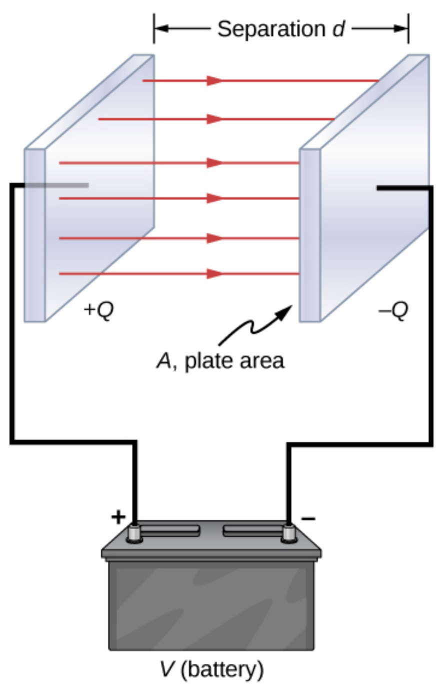


Figure 4.2.4: In a parallel-plate capacitor with plates separated by a distance \(**d**\), each plate has the same surface area \(**A**\).

We define the surface charge density σ on the plates as

$$\sigma = \frac{Q}{A}.$$

We know from previous chapters that when d is small, the electrical field between the plates is fairly uniform (ignoring edge effects) and that its magnitude is given by

$$E = \frac{\sigma}{\epsilon_0},$$

where the constant ϵ_0 is the permittivity of free space, $\epsilon_0 = 8.85 \times 10^{-12} F/m$. The SI unit of F/m is equivalent to $C^2/N \cdot m^2$. Since the electrical field $\vec{E}$ between the plates is uniform, the potential difference between the plates is

$$V = Ed = \frac{\sigma d}{\epsilon_0} = \frac{Qd}{\epsilon_0 A}.$$

Therefore Equation 4.2.1 gives the capacitance of a parallel-plate capacitor as

$$C = \frac{Q}{V} = \frac{Q}{Qd/\epsilon_0 A} = \epsilon_0 \frac{A}{d}. \tag{4.2.3}$$

Notice from this equation that capacitance is a function **only of the geometry** and what material fills the space between the plates (in this case, vacuum) of this capacitor. In fact, this is true not only for a parallel-plate capacitor, but for all capacitors: The capacitance is independent of Q or V. If the charge changes, the potential changes correspondingly so that Q/V remains constant.

Example 4.2.1A: Capacitance and Charge Stored in a Parallel-Plate Capacitor

a. What is the capacitance of an empty parallel-plate capacitor with metal plates that each have an area of $1.00\, m^2$, separated by 1.00 mm?
b. How much charge is stored in this capacitor if a voltage of $3.00 \times 10^3 V$ is applied to it?

Strategy

Finding the capacitance C is a straightforward application of Equation 4.2.3. Once we find C, we can find the charge stored by using Equation 4.2.1.

Solution

1. Entering the given values into Equation 4.2.3 yields

$$C = \epsilon_0 \frac{A}{d} = \left(8.85 \times 10^{-12} \frac{F}{m}\right) \frac{1.00\, m^2}{1.00 \times 10^{-3} m} = 8.85 \times 10^{-9} F = 8.85\, nF.$$

   This small capacitance value indicates how difficult it is to make a device with a large capacitance.

2. Inverting Equation 4.2.1 and entering the known values into this equation gives

$$Q = CV = (8.85 \times 10^{-9} F)(3.00 \times 10^3 V) = 26.6\, \mu C.$$

Significance

This charge is only slightly greater than those found in typical static electricity applications. Since air breaks down (becomes conductive) at an electrical field strength of about 3.0 MV/m, no more charge can be stored on this capacitor by increasing the voltage.

Example 4.2.1B: A 1-F Parallel-Plate Capacitor

Suppose you wish to construct a parallel-plate capacitor with a capacitance of 1.0 F. What area must you use for each plate if the plates are separated by 1.0 mm?

Solution

Rearranging Equation 4.2.3, we obtain

$$A = \frac{Cd}{\epsilon_0} = \frac{(1.0\, F)(1.0 \times 10^{-3} m)}{8.85 \times 10^{-12} F/m} = 1.1 \times 10^8\, m^2.$$

Each square plate would have to be 10 km across. It used to be a common prank to ask a student to go to the laboratory stockroom and request a 1-F parallel-plate capacitor, until stockroom attendants got tired of the joke.

? Exercise $4.2.1A$

The capacitance of a parallel-plate capacitor is 2.0 pF. If the area of each plate is $2.4\,cm^2$, what is the plate separation?

Answer

$1.1 \times 10^{-3} m$

? Exercise $4.2.1B$

Verify that σ/V and ϵ_0/d have the same physical units.

Spherical Capacitor

A spherical capacitor is another set of conductors whose capacitance can be easily determined (Figure $4.2.5$). It consists of two concentric conducting spherical shells of radii R_1 (inner shell) and R_2 (outer shell). The shells are given equal and opposite charges $+Q$ and $-Q$, respectively. From symmetry, the electrical field between the shells is directed radially outward. We can obtain the magnitude of the field by applying Gauss's law over a spherical Gaussian surface of radius **r** concentric with the shells. The enclosed charge is $+Q$; therefore we have

$$\oint_S \vec{E} \cdot \hat{n} dA = E(4\pi r^2) = \frac{Q}{\epsilon_0}.$$

Thus, the electrical field between the conductors is

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}.$$

We substitute this $\vec{E}$ into Equation 4.2.2 and integrate along a radial path between the shells:

$$V = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{l} = \int_{R_1}^{R_2} \left(\frac{1}{4\pi\epsilon_0}\frac{Q}{r^2}\hat{r}\right) \cdot (\hat{r}\,dr) = \frac{Q}{4\pi\epsilon_0}\int_{R_1}^{R_2} \frac{dr}{r^2} = \frac{Q}{4\pi\epsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

In this equation, the potential difference between the plates is

$$V = -(V_2 - V_1) = V_1 - V_2.$$

We substitute this result into Equation 4.2.1 to find the capacitance of a spherical capacitor:

$$C = \frac{Q}{V} = 4\pi\epsilon_0 \frac{R_1 R_2}{R_2 - R_1}. \tag{4.2.4}$$

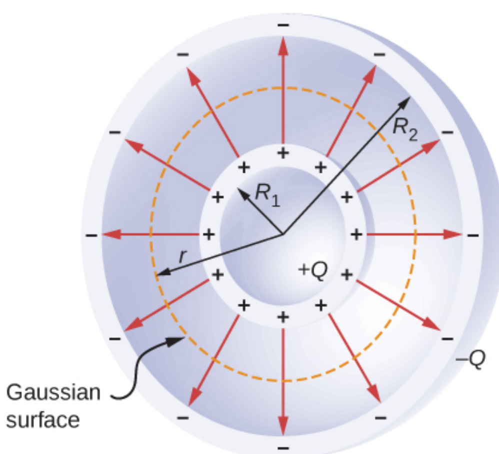


Figure $4.2.5$: A spherical capacitor consists of two concentric conducting spheres. Note that the charges on a conductor reside on its surface.

Example 4.2.2: Capacitance of an Isolated Sphere

Calculate the capacitance of a single isolated conducting sphere of radius R_1 and compare it with Equation 4.2.4 in the limit as $R_2 \to \infty$.

Strategy

We assume that the charge on the sphere is Q, and so we follow the four steps outlined earlier. We also assume the other conductor to be a concentric hollow sphere of infinite radius.

Solution

On the outside of an isolated conducting sphere, the electrical field is given by Equation 4.2.2. The magnitude of the potential difference between the surface of an isolated sphere and infinity is

$$\begin{aligned} V &= \int_{R_1}^{+\infty} \vec{E} \cdot d\vec{l} \\ &= \frac{Q}{4\pi\epsilon_0} \int_{R_1}^{+\infty} \frac{1}{r^2} \hat{r} \cdot (\hat{r}\, dr) \\ &= \frac{Q}{4\pi\epsilon_0} \int_{R_1}^{+\infty} \frac{dr}{r^2} \\ &= \frac{1}{4\pi\epsilon_0} \frac{Q}{R_1} \end{aligned}$$

The capacitance of an isolated sphere is therefore

$$C = \frac{Q}{V} = Q \frac{4\pi\epsilon_0 R_1}{Q} = 4\pi\epsilon_0 R_1.$$

Significance

The same result can be obtained by taking the limit of Equation 4.2.4 as $R_2 \to \infty$. A single isolated sphere is therefore equivalent to a spherical capacitor whose outer shell has an infinitely large radius.

Exercise 4.2.2

The radius of the outer sphere of a spherical capacitor is five times the radius of its inner shell. What are the dimensions of this capacitor if its capacitance is 5.00 pF?

Answer

3.59 cm, 17.98 cm

Cylindrical Capacitor

A cylindrical capacitor consists of two concentric, conducting cylinders (Figure 4.2.6). The inner cylinder, of radius R_1, may either be a shell or be completely solid. The outer cylinder is a shell of inner radius R_2. We assume that the length of each cylinder is **l** and that the excess charges $+Q$ and $-Q$ reside on the inner and outer cylinders, respectively.

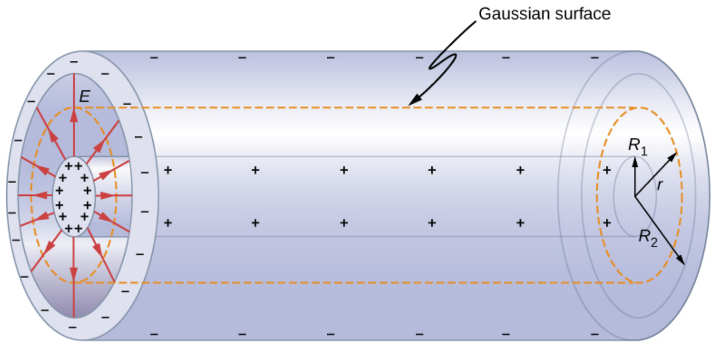


Figure 4.2.6: A cylindrical capacitor consists of two concentric, conducting cylinders. Here, the charge on the outer surface of the inner cylinder is positive (indicated by $+$) and the charge on the inner surface of the outer cylinder is negative (indicated by $-$).

With edge effects ignored, the electrical field between the conductors is directed radially outward from the common axis of the cylinders. Using the Gaussian surface shown in Figure 4.2.6, we have

$$\oint_S \vec{E} \cdot \hat{n} dA = E(2\pi r l) = \frac{Q}{\epsilon_0}.$$

Therefore, the electrical field between the cylinders is

$$\vec{E} = \frac{1}{2\pi\epsilon_0} \frac{Q}{r\,l} \hat{r}.$$

where $\hat{r}$ is the unit radial vector along the radius of the cylinder. We can substitute into Equation 4.2.2 and find the potential difference between the cylinders:

$$V = \int_{R_1}^{R_2} \vec{E} \cdot d\vec{l}_p = \frac{Q}{2\pi\epsilon_0 l} \int_{R_1}^{R_2} \frac{1}{r} \hat{r} \cdot (\hat{r}\, dr) = \frac{Q}{2\pi\epsilon_0 l} \int_{R_1}^{R_2} \frac{dr}{r} = \frac{Q}{2\pi\epsilon_0 l} \ln r \Big|_{R_1}^{R_2} = \frac{Q}{2\pi\epsilon_0 l} \frac{R_2}{R_1}.$$

Thus, the capacitance of a cylindrical capacitor is

$$C = \frac{Q}{V} = \frac{2\pi\epsilon_0\, l}{\ln(R_2/R_1)}. \tag{4.2.5}$$

As in other cases, this capacitance depends only on the geometry of the conductor arrangement. An important application of Equation 4.2.5 is the determination of the capacitance per unit length of a **coaxial cable**, which is commonly used to transmit time-varying electrical signals. A coaxial cable consists of two concentric, cylindrical conductors separated by an insulating material. (Here, we assume a vacuum between the conductors, but the physics is qualitatively almost the same when the space between the conductors is filled by a dielectric.) This configuration shields the electrical signal propagating down the inner conductor from stray electrical fields external to the cable. Current flows in opposite directions in the inner and the outer conductors, with the outer conductor usually grounded. Now, from Equation 4.2.5, the capacitance per unit length of the coaxial cable is given by

$$\frac{C}{l} = \frac{2\pi\epsilon_0}{\ln(R_2/R_1)}.$$

In practical applications, it is important to select specific values of C/l. This can be accomplished with appropriate choices of radii of the conductors and of the insulating material between them.

? Exercise 4.2.3

When a cylindrical capacitor is given a charge of 0.500 nC, a potential difference of 20.0 V is measured between the cylinders.

a. What is the capacitance of this system?
b. If the cylinders are 1.0 m long, what is the ratio of their radii?

Answer

a. 25.0 pF

b. 9.2

Several types of practical capacitors are shown in Figure $4.2.3$. Common capacitors are often made of two small pieces of metal foil separated by two small pieces of insulation (Figure $4.2.1b$). The metal foil and insulation are encased in a protective coating, and two metal leads are used for connecting the foils to an external circuit. Some common insulating materials are mica, ceramic, paper, and Teflon™ non-stick coating.

Another popular type of capacitor is an **electrolytic capacitor**. It consists of an oxidized metal in a conducting paste. The main advantage of an electrolytic capacitor is its high capacitance relative to other common types of capacitors. For example, capacitance of one type of aluminum electrolytic capacitor can be as high as 1.0 F. However, you must be careful when using an electrolytic capacitor in a circuit, because it only functions correctly when the metal foil is at a higher potential than the conducting paste. When reverse polarization occurs, electrolytic action destroys the oxide film. This type of capacitor cannot be connected across an alternating current source, because half of the time, ac voltage would have the wrong polarity, as an alternating current reverses its polarity (see Alternating-Current Circuits on alternating-current circuits).

A **variable air capacitor** (Figure $4.2.7$) has two sets of parallel plates. One set of plates is fixed (indicated as "stator"), and the other set of plates is attached to a shaft that can be rotated (indicated as "rotor"). By turning the shaft, the cross-sectional area in the overlap of the plates can be changed; therefore, the capacitance of this system can be tuned to a desired value. Capacitor tuning has applications in any type of radio transmission and in receiving radio signals from electronic devices. Any time you tune your car radio to your favorite station, think of capacitance.

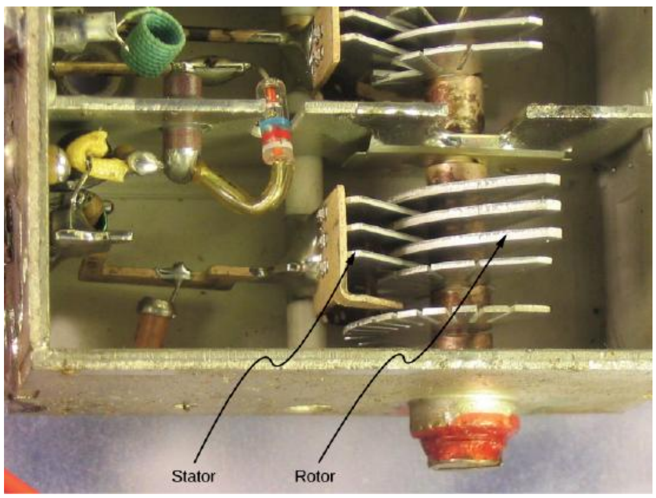


Figure $4.2.7$: In a variable air capacitor, capacitance can be tuned by changing the effective area of the plates. (credit: modification of work by Robbie Sproule)

The symbols shown in Figure $4.2.8$ are circuit representations of various types of capacitors. We generally use the symbol shown in Figure $4.2.8a$. The symbol in Figure $4.2.8c$ represents a variable-capacitance capacitor. Notice the similarity of these symbols to the symmetry of a parallel-plate capacitor. An electrolytic capacitor is represented by the symbol in part Figure $4.2.8b$, where the curved plate indicates the negative terminal.

Figure a shows two vertical lines. Figure b shows a vertical line to the left and another, slightly curved vertical line to the right. Figure c shows two vertical lines and an arrow cutting across them diagonally. In all figures, each line is connected to a horizontal line on the outside.

Figure $4.2.8$: This shows three different circuit representations of capacitors. The symbol in (a) is the most commonly used one. The symbol in (b) represents an electrolytic capacitor. The symbol in (c) represents a variable-capacitance capacitor.

An interesting applied example of a capacitor model comes from cell biology and deals with the electrical potential in the plasma membrane of a living cell (Figure $4.2.9$). **Cell membranes** separate cells from their surroundings, but allow some selected ions to pass in or out of the cell. The potential difference across a membrane is about 70 mV. The cell membrane may be 7 to 10 nm thick. Treating the cell membrane as a nano-sized capacitor, the estimate of the smallest electrical field strength across its 'plates' yields the value

$$E=\frac{V}{d}=\frac{70\times 10^{-3}V}{10\times 10^{-9}m}=7\times 10^{6}V/m>3\,MV/m.$$

This magnitude of electrical field is great enough to create an electrical spark in the air.

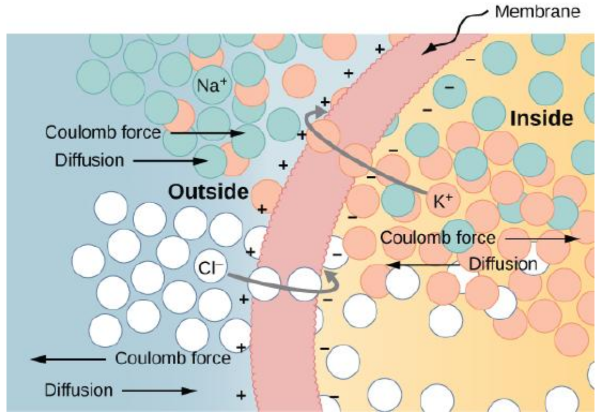


Figure 4.2.9: The semipermeable membrane of a biological cell has different concentrations of ions on its interior surface than on its exterior. Diffusion moves the K^+ (potassium) and Cl^- (chloride) ions in the directions shown, until the Coulomb force halts further transfer. In this way, the exterior of the membrane acquires a positive charge and its interior surface acquires a negative charge, creating a potential difference across the membrane. The membrane is normally impermeable to Na^+ (sodium ions).

Simulation

Visit the PhET Explorations: Capacitor Lab to explore how a capacitor works. Change the size of the plates and add a dielectric to see the effect on capacitance. Change the voltage and see charges built up on the plates. Observe the electrical field in the capacitor. Measure the voltage and the electrical field.

4.3: Energy Stored in a Capacitor

Learning Objectives

By the end of this section, you will be able to:

- Explain how energy is stored in a capacitor
- Use energy relations to determine the energy stored in a capacitor network

Most of us have seen dramatizations of medical personnel using a defibrillator to pass an electrical current through a patient's heart to get it to beat normally. Often realistic in detail, the person applying the shock directs another person to "make it 400 joules this time." The energy delivered by the defibrillator is stored in a capacitor and can be adjusted to fit the situation. SI units of joules are often employed. Less dramatic is the use of capacitors in microelectronics to supply energy when batteries are charged (Figure $4.3.1$). Capacitors are also used to supply energy for flash lamps on cameras.

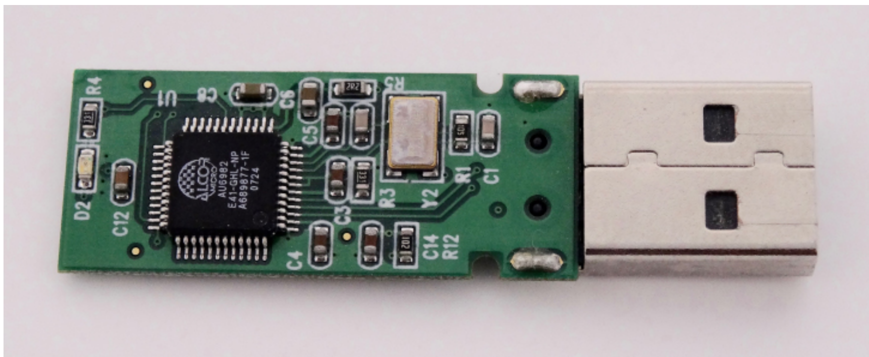


Figure $4.3.1$: The capacitors on the circuit board for an electronic device follow a labeling convention that identifies each one with a code that begins with the letter "C."

The energy U_C stored in a capacitor is electrostatic potential energy and is thus related to the charge **Q** and voltage **V** between the capacitor plates. A charged capacitor stores energy in the electrical field between its plates. As the capacitor is being charged, the electrical field builds up. When a charged capacitor is disconnected from a battery, its energy remains in the field in the space between its plates.

To gain insight into how this energy may be expressed (in terms of **Q** and **V**), consider a charged, empty, parallel-plate capacitor; that is, a capacitor without a dielectric but with a vacuum between its plates. The space between its plates has a volume **Ad**, and it is filled with a uniform electrostatic field **E**. The total energy U_C of the capacitor is contained within this space. The **energy density** u_E in this space is simply U_C divided by the volume **Ad**. If we know the energy density, the energy can be found as $U_C = u_E(Ad)$. We will learn in Electromagnetic Waves (after completing the study of Maxwell's equations) that the energy density u_E in a region of free space occupied by an electrical field **E** depends only on the magnitude of the field and is

$$u_E = \frac{1}{2}\epsilon_0 E^2.$$

If we multiply the energy density by the volume between the plates, we obtain the amount of energy stored between the plates of a parallel-plate capacitor $U_C = u_E(Ad) = \frac{1}{2}\epsilon_0 E^2 Ad = \frac{1}{2}\epsilon_0 \frac{V^2}{d^2} Ad = \frac{1}{2}V^2 \epsilon_0 \frac{A}{d} = \frac{1}{2}V^2 C$.

In this derivation, we used the fact that the electrical field between the plates is uniform so that $E = V/d$ and $C = \epsilon_0 A/d$. Because $C = Q/V$, we can express this result in other equivalent forms:

$$U_C = \frac{1}{2}V^2 C = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}QV. \tag{4.3.1}$$

The expression in Equation 4.3.1 for the energy stored in a parallel-plate capacitor is generally valid for all types of capacitors. To see this, consider any uncharged capacitor (not necessarily a parallel-plate type). At some instant, we connect it across a battery, giving it a potential difference $V = q/C$ between its plates. Initially, the charge on the plates is $Q = 0$. As the capacitor is being charged, the charge gradually builds up on its plates, and after some time, it reaches the value **Q**. To move an infinitesimal charge

dq from the negative plate to the positive plate (from a lower to a higher potential), the amount of work **dW** that must be done on **dq** is $dW = W\,dq = \frac{q}{C}dq$.

This work becomes the energy stored in the electrical field of the capacitor. In order to charge the capacitor to a charge **Q**, the total work required is

$$W = \int_0^{W(Q)} dW = \int_0^Q \frac{q}{C}dq = \frac{1}{2}\frac{Q^2}{C}.$$

Since the geometry of the capacitor has not been specified, this equation holds for any type of capacitor. The total work **W** needed to charge a capacitor is the electrical potential energy U_C stored in it, or $U_C = W$. When the charge is expressed in coulombs, potential is expressed in volts, and the capacitance is expressed in farads, this relation gives the energy in joules.

Knowing that the energy stored in a capacitor is $U_C = Q^2/(2C)$, we can now find the energy density u_E stored in a vacuum between the plates of a charged parallel-plate capacitor. We just have to divide U_C by the volume **Ad** of space between its plates and take into account that for a parallel-plate capacitor, we have $E = \sigma/\epsilon_0$ and $C = \epsilon_0 A/d$. Therefore, we obtain

$$u_E = \frac{U_C}{Ad} = \frac{1}{2}\frac{Q^2}{C}\frac{1}{Ad} = \frac{1}{2}\frac{Q^2}{\epsilon_0 A/d}\frac{1}{Ad} = \frac{1}{2}\frac{1}{\epsilon_0}\left(\frac{Q}{A}\right)^2 = \frac{\sigma^2}{2\epsilon_0} = \frac{(E\epsilon_0)^2}{2\epsilon_0} = \frac{\epsilon_0}{2}E^2$$

We see that this expression for the density of energy stored in a parallel-plate capacitor is in accordance with the general relation expressed in Equation ???. We could repeat this calculation for either a spherical capacitor or a cylindrical capacitor—or other capacitors—and in all cases, we would end up with the general relation given by Equation ???.

Energy Stored in a Capacitor

Calculate the energy stored in the capacitor network in Figure 8.3.4a when the capacitors are fully charged and when the capacitances are $C_1 = 12.0\,\mu F$, $C_2 = 2.0\,\mu F$, and $C_3 = 4.0\,\mu F$, respectively.

Strategy

We use Equation 4.3.1 to find the energy U_1, U_2, and U_3 stored in capacitors 1, 2, and 3, respectively. The total energy is the sum of all these energies.

Solution We identify $C_1 = 12.0\,\mu F$ and $V_1 = 4.0\,V$, $C_2 = 2.0\,\mu F$ and $V_2 = 8.0\,V$, $C_3 = 4.0\,\mu F$ and $V_3 = 8.0\,V$. The energies stored in these capacitors are

$$U_1 = \frac{1}{2}C_1V_1^2 = \frac{1}{2}(12.0\,\mu F)(4.0\,V)^2 = 96\,\mu J,$$

$$U_2 = \frac{1}{2}C_2V_2^2 = \frac{1}{2}(2.0\,\mu F)(8.0\,V)^2 = 64\,\mu J,$$

$$U_3 = \frac{1}{2}C_3V_3^2 = \frac{1}{2}(4.0\,\mu F)(8.0\,V)^2 = 130\,\mu J,$$

The total energy stored in this network is

$$U_C = U_1 + U_2 + U_3 = 96\,\mu J + 64\,\mu J + 130\,\mu J = 0.29\,mJ.$$

Significance

We can verify this result by calculating the energy stored in the single $4.0-\mu F$ capacitor, which is found to be equivalent to the entire network. The voltage across the network is 12.0 V. The total energy obtained in this way agrees with our previously obtained result, $U_C = \frac{1}{2}CV^2 = \frac{1}{2}(4.0\,\mu F)(12.0\,V)^2 = 0.29\,mJ$

Exercise 4.3.1

The potential difference across a 5.0-pF capacitor is 0.40 V. (a) What is the energy stored in this capacitor? (b) The potential difference is now increased to 1.20 V. By what factor is the stored energy increased?

Answer

a. $4.0 \times 10^{-13} J$; b. 9 times

In a cardiac emergency, a portable electronic device known as an automated external defibrillator (AED) can be a lifesaver. A **defibrillator** (Figure $4.3.2$) delivers a large charge in a short burst, or a shock, to a person's heart to correct abnormal heart rhythm (an arrhythmia). A heart attack can arise from the onset of fast, irregular beating of the heart—called cardiac or ventricular fibrillation. Applying a large shock of electrical energy can terminate the arrhythmia and allow the body's natural pacemaker to resume its normal rhythm. Today, it is common for ambulances to carry AEDs. AEDs are also found in many public places. These are designed to be used by lay persons. The device automatically diagnoses the patient's heart rhythm and then applies the shock with appropriate energy and waveform. CPR (cardiopulmonary resuscitation) is recommended in many cases before using a defibrillator.

Figure $4.3.2$: Automated external defibrillators are found in many public places. These portable units provide verbal instructions for use in the important first few minutes for a person suffering a cardiac attack.

Example $4.3.2$: Capacitance of a Heart Defibrillator

A heart defibrillator delivers $4.00 \times 10^2 J$ of energy by discharging a capacitor initially at $1.00 \times 10^4 V$. What is its capacitance?

Strategy

We are given U_C and **V**, and we are asked to find the capacitance **C**. We solve Equation $4.3.1$ for **C** and substitute.

Solution

Solving this expression for **C** and entering the given values yields $C = 2\frac{U_C}{V^2} = 2\frac{4.00 \times 10^2 J}{(1.00 \times 10^4 V)^2} = 8.00\,\mu F$.

4.4: Capacitor with a Dielectric

Learning Objectives

By the end of this section, you will be able to:

- Describe the effects a dielectric in a capacitor has on capacitance and other properties
- Calculate the capacitance of a capacitor containing a dielectric

As we discussed earlier, an insulating material placed between the plates of a capacitor is called a dielectric. Inserting a dielectric between the plates of a capacitor affects its capacitance. To see why, let's consider an experiment described in Figure 4.4.1. Initially, a capacitor with capacitance C_0 when there is air between its plates is charged by a battery to voltage V_0. When the capacitor is fully charged, the battery is disconnected. A charge Q_0 then resides on the plates, and the potential difference between the plates is measured to be V_0. Now, suppose we insert a dielectric that **totally** fills the gap between the plates. If we monitor the voltage, we find that the voltmeter reading has dropped to a **smaller** value V. We write this new voltage value as a fraction of the original voltage V_0, with a positive number κ, $\kappa > 1$.

$$V = \frac{1}{\kappa} V_0.$$

The constant κ in this equation is called the **dielectric constant** of the material between the plates, and its value is characteristic for the material. A detailed explanation for why the dielectric reduces the voltage is given in the next section. Different materials have different dielectric constants (a table of values for typical materials is provided in the next section). Once the battery becomes disconnected, there is no path for a charge to flow to the battery from the capacitor plates. Hence, the insertion of the dielectric has no effect on the charge on the plate, which remains at a value of Q_0. Therefore, we find that the capacitance of the capacitor with a dielectric is

$$C = \frac{Q_0}{V} = \frac{Q_0}{V_0/\kappa} = \kappa \frac{Q_0}{V_0} = \kappa C_0. \tag{4.4.1}$$

This equation tells us that the **capacitance C_0 of an empty (vacuum) capacitor can be increased by a factor of κ when we insert a dielectric material to completely fill the space between its plates**. Note that Equation 4.4.1 can also be used for an empty capacitor by setting $\kappa = 1$. In other words, we can say that the dielectric constant of the vacuum is 1, which is a reference value.

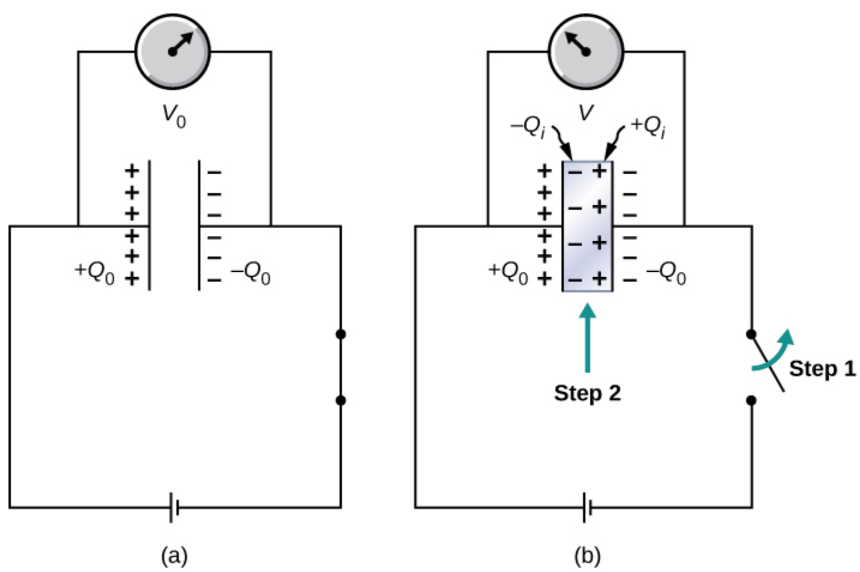


Figure 4.4.1: (a) When fully charged, a vacuum capacitor has a voltage V_0 and charge Q_0 (the charges remain on plate's inner surfaces; the schematic indicates the sign of charge on each plate). (b) In step 1, the battery is disconnected. Then, in step 2, a dielectric (that is electrically neutral) is inserted into the charged capacitor. When the voltage across the capacitor is now measured, it is found that the voltage value has decreased to $V = V_0/\kappa$. The schematic indicates the sign of the induced charge that is now present on the surfaces of the dielectric material between the plates.

The principle expressed by Equation 4.4.1 is widely used in the construction industry (Figure 4.4.2). Metal plates in an electronic stud finder act effectively as a capacitor. You place a stud finder with its flat side on the wall and move it continually in the horizontal direction. When the finder moves over a wooden stud, the capacitance of its plates changes, because wood has a different dielectric constant than a gypsum wall. This change triggers a signal in a circuit, and thus the stud is detected.

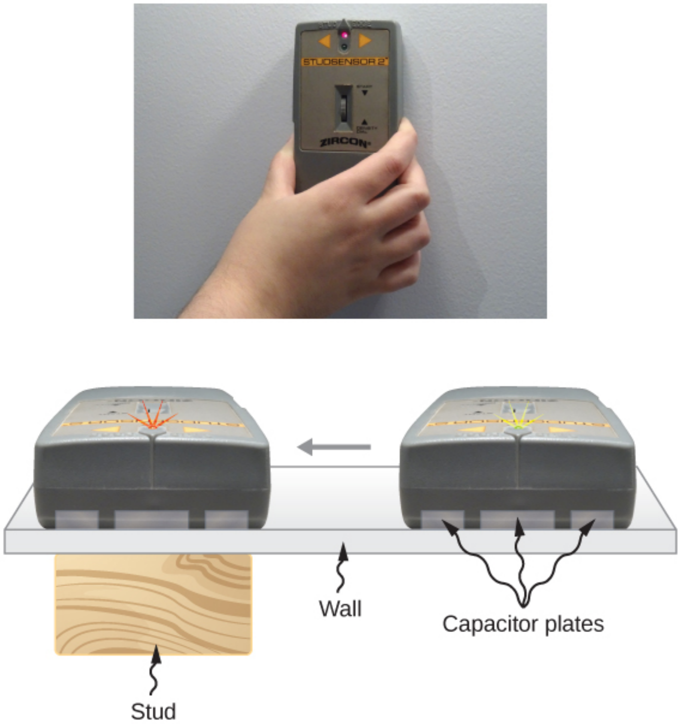


Figure 4.4.2: An electronic stud finder is used to detect wooden studs behind drywall.

The electrical energy stored by a capacitor is also affected by the presence of a dielectric. When the energy stored in an empty capacitor is U_0, the energy U stored in a capacitor with a dielectric is smaller by a factor of κ.

$$U = \frac{1}{2}\frac{Q^2}{C} = \frac{1}{2}\frac{Q_0^2}{\kappa C_0} = \frac{1}{\kappa}U_0. \tag{4.4.2}$$

As a dielectric material sample is brought near an empty charged capacitor, the sample reacts to the electrical field of the charges on the capacitor plates. Just as we learned in Electric Charges and Fields on electrostatics, there will be the induced charges on the surface of the sample; however, they are not free charges like in a conductor, because a perfect insulator does not have freely moving charges. These induced charges on the dielectric surface are of an opposite sign to the free charges on the plates of the capacitor, and so they are attracted by the free charges on the plates. Consequently, the dielectric is "pulled" into the gap, and the work to polarize the dielectric material between the plates is done at the expense of the stored electrical energy, which is reduced, in accordance with Equation 4.4.2.

Example 4.4.1: Inserting a Dielectric into an Isolated Capacitor

An empty 20.0-pF capacitor is charged to a potential difference of 40.0 V. The charging battery is then disconnected, and a piece of Teflon™ with a dielectric constant of 2.1 is inserted to completely fill the space between the capacitor plates (see Figure 4.4.1). What are the values of:

a. the capacitance,
b. the charge of the plate,
c. the potential difference between the plates, and
d. the energy stored in the capacitor with and without dielectric?

Strategy

We identify the original capacitance $C_0 = 20.0\, pF$ and the original potential difference $V_0 = 40.0\, V$ between the plates. We combine Equation 4.4.1 with other relations involving capacitance and substitute.

Solution

a. The capacitance increases to

$$C = \kappa C_0 = 2.1(20.0\, pF) = 42.0\, pF.$$

b. Without dielectric, the charge on the plates is

$$Q_0 = C_0 V_0 = (20.0\, pF)(40.0\, V) = 0.8\, nC.$$

Since the battery is disconnected before the dielectric is inserted, the plate charge is unaffected by the dielectric and remains at 0.8 nC.

c. With the dielectric, the potential difference becomes

$$V = \frac{1}{\kappa} V_0 = \frac{1}{2.1} 40.0\, V = 19.0\, V.$$

d. The stored energy without the dielectric is

$$U_0 = \frac{1}{2} C_0 V_0^2 = \frac{1}{2}(20.0\, pF)(40.0\, V)^2 = 16.0\, nJ.$$

With the dielectric inserted, we use Equation 4.4.2 to find that the stored energy decreases to

$$U = \frac{1}{\kappa} U_0 = \frac{1}{2.1} 16.0\, nJ = 7.6\, nJ.$$

Significance

Notice that the effect of a dielectric on the capacitance of a capacitor is a drastic increase of its capacitance. This effect is far more profound than a mere change in the geometry of a capacitor.

? Exercise 4.4.1

When a dielectric is inserted into an isolated and charged capacitor, the stored energy decreases to 33% of its original value.

a. What is the dielectric constant?
b. How does the capacitance change?

Answer

a. 3.0; b. $C = 3.0\, C_0$

4.5: Molecular Model of a Dielectric

> **Learning Objectives**
>
> By the end of this section, you will be able to:
>
> - Explain the polarization of a dielectric in a uniform electrical field
> - Describe the effect of a polarized dielectric on the electrical field between capacitor plates
> - Explain dielectric breakdown

We can understand the effect of a dielectric on capacitance by looking at its behavior at the molecular level. As we have seen in earlier chapters, in general, all molecules can be classified as either **polar** or **nonpolar**. There is a net separation of positive and negative charges in an isolated polar molecule, whereas there is no charge separation in an isolated nonpolar molecule (Figure $4.5.1$). In other words, polar molecules have permanent **electric-dipole moments** and nonpolar molecules do not. For example, a molecule of water is polar, and a molecule of oxygen is nonpolar. Nonpolar molecules can become polar in the presence of an external electrical field, which is called **induced polarization**.

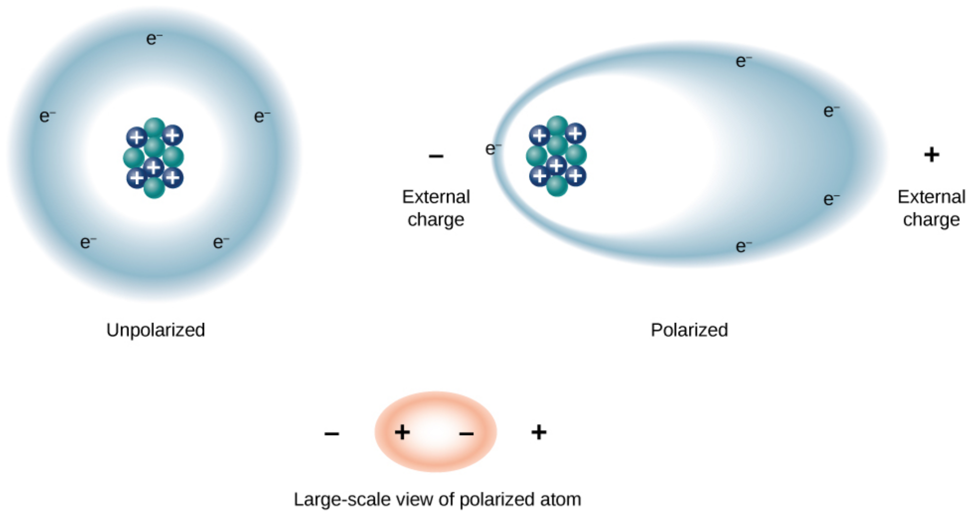


Figure $4.5.1$: The concept of polarization: In an unpolarized atom or molecule, a negatively charged electron cloud is evenly distributed around positively charged centers, whereas a polarized atom or molecule has an excess of negative charge at one side so that the other side has an excess of positive charge. However, the entire system remains electrically neutral. The charge polarization may be caused by an external electrical field. Some molecules and atoms are permanently polarized (electric dipoles) even in the absence of an external electrical field (polar molecules and atoms).

Let's first consider a dielectric composed of polar molecules. In the absence of any external electrical field, the electric dipoles are oriented randomly, as illustrated in Figure $4.5.2a$. However, if the dielectric is placed in an external electrical field $\vec{E}_0$, the polar molecules align with the external field, as shown in $4.5.2b$ of the figure. Opposite charges on adjacent dipoles within the volume of dielectric neutralize each other, so there is no net charge within the dielectric (see the dashed circles in part (b)). However, this is not the case very close to the upper and lower surfaces that border the dielectric (the region enclosed by the dashed rectangles in $4.5.2b$, where the alignment does produce a net charge. Since the external electrical field merely aligns the dipoles, the dielectric as a whole is neutral, and the surface charges induced on its opposite faces are equal and opposite. These **induced surface charges** $+Q_i$ and $-Q_i$ produce an additional electrical field $\vec{E}_i$ (an **induced electrical field**), which **opposes** the external field $\vec{E}_0$, as illustrated in part (c).

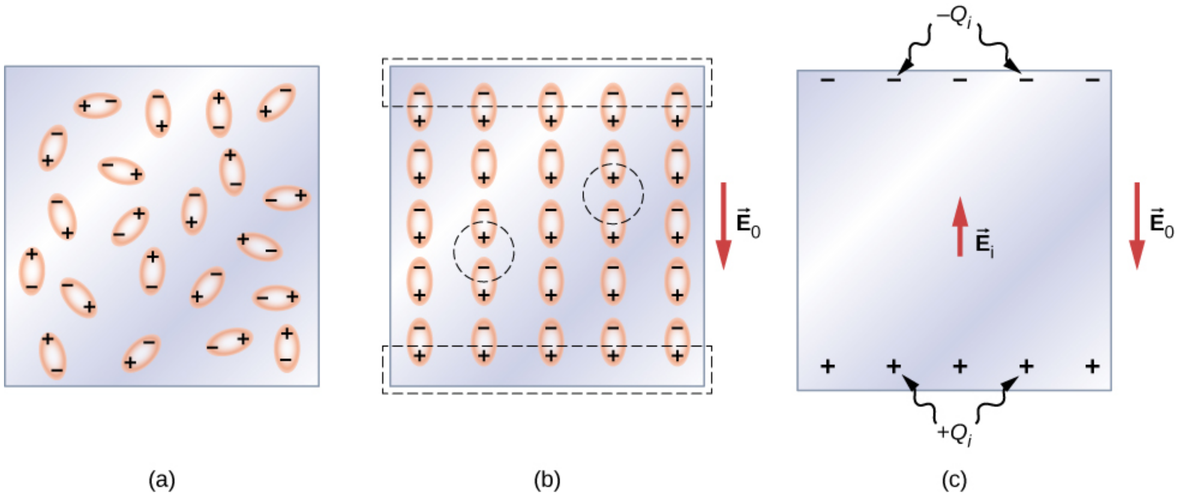


Figure 4.5.2: A dielectric with polar molecules: (a) In the absence of an external electrical field; (b) in the presence of an external electrical field $\vec{E}_0$. The dashed lines indicate the regions immediately adjacent to the capacitor plates. (c) The induced electrical field $\vec{E}_i$ inside the dielectric produced by the induced surface charge Q_i of the dielectric. Note that, in reality, the individual molecules are not perfectly aligned with an external field because of thermal fluctuations; however, the **average** alignment is along the field lines as shown.

The same effect is produced when the molecules of a dielectric are nonpolar. In this case, a nonpolar molecule acquires an induced **electric-dipole moment** because the external field $\vec{E}_0$ causes a separation between its positive and negative charges. The induced dipoles of the nonpolar molecules align with $\vec{E}_0$ in the same way as the permanent dipoles of the polar molecules are aligned (shown in part (b)). Hence, the electrical field within the dielectric is weakened regardless of whether its molecules are polar or nonpolar.

Therefore, when the region between the parallel plates of a charged capacitor, such as that shown in Figure $4.5.3a$, is filled with a dielectric, within the dielectric there is an electrical field $\vec{E}_0$ due to the **free charge** $\boldsymbol{Q_0}$ on the capacitor plates and an electrical field $\vec{E}_i$ due to the induced charge Q_i on the surfaces of the dielectric. Their vector sum gives the net electrical field $\vec{E}$ within the dielectric between the capacitor plates (shown in part (b) of the figure):

$$\vec{E} = \vec{E}_0 + \vec{E}_i. \tag{4.5.1}$$

This net field can be considered to be the field produced by an **effective charge** $\boldsymbol{Q_0 - Q_i}$ on the capacitor.

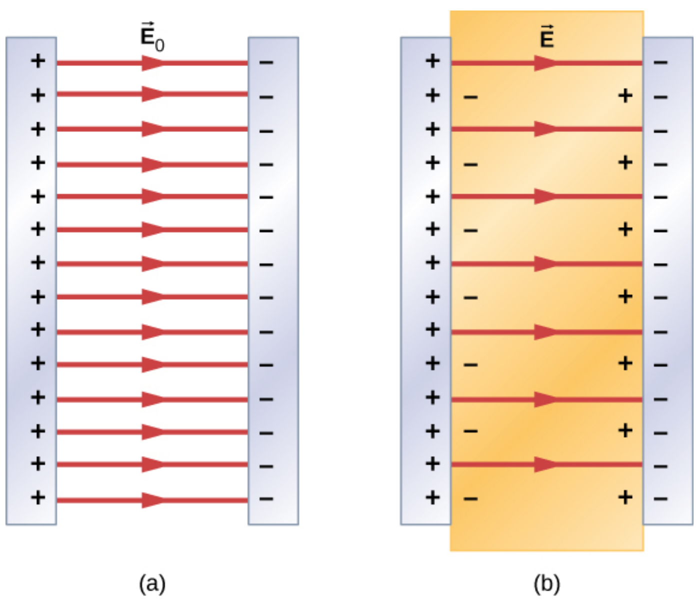


Figure 4.5.3: Electrical field: (a) In an empty capacitor, electrical field $\vec{E}_0$ (b) In a dielectric-filled capacitor, electrical field $\vec{E}$.

In most dielectrics, the net electrical field $\vec{E}$ is proportional to the field $\vec{E}_0$ produced by the free charge. In terms of these two electrical fields, the dielectric constant κ of the material is defined as

$$\kappa = \frac{E_0}{E}. \tag{4.5.2}$$

Since $\vec{E}_0$ and $\vec{E}_i$ point in opposite directions, the magnitude **E** is smaller than the magnitude $\vec{E}_0$ and therefore $\kappa > 1$. Combining Equations 4.5.1 with 4.5.2, and rearranging the terms, yields the following expression for the induced electrical field in a dielectric:

$$\vec{E}_i = \left(\frac{1}{\kappa} - 1\right) \vec{E}_0. \tag{4.5.3}$$

When the magnitude of an external electrical field becomes too large, the molecules of dielectric material start to become ionized. A molecule or an atom is ionized when one or more electrons are removed from it and become free electrons, no longer bound to the molecular or atomic structure. When this happens, the material can conduct, thereby allowing charge to move through the dielectric from one capacitor plate to the other. This phenomenon is called **dielectric breakdown**. (Figure 8.1.1 shows typical random-path patterns of electrical discharge during dielectric breakdown.) The critical value, E_c of the electrical field at which the molecules of an insulator become ionized is called the **dielectric strength** of the material. The dielectric strength imposes a limit on the voltage that can be applied for a given plate separation in a capacitor. For example, the dielectric strength of air is $E_c = 3.0\, MV/m$, so for an air-filled capacitor with a plate separation of $d = 1.00\, mm$, the limit on the potential difference that can be safely applied across its plates without causing dielectric breakdown is $V = E_c d = (3.0 \times 10^6 V/m)(1.00 \times 10^{-3} m) = 3.0\, kV$.

However, this limit becomes 60.0 kV when the same capacitor is filled with Teflon™, whose dielectric strength is about $60.0\, MV/m$. Because of this limit imposed by the dielectric strength, the amount of charge that an air-filled capacitor can store is only $Q_0 = \kappa_{air} C_0 (3.0\, kV)$ and the charge stored on the same Teflon™-filled capacitor can be as much as

$$\begin{aligned} Q &= \kappa_{teflon} C_0 (60.0\, kV) \\ &= \kappa_{teflon} \frac{Q_0}{\kappa_{air}(3.0\, kV)} (60.0\, kV) \\ &= 20 \frac{\kappa_{teflon}}{\kappa_{air}} Q_0 = 20 \frac{2.1}{1.00059} Q_0 \\ &\cong 42 Q_0. \end{aligned} \tag{4.5.4}$$

which is about 42 times greater than a charge stored on an air-filled capacitor. Typical values of dielectric constants and dielectric strengths for various materials are given in Table 4.5.1. Notice that the dielectric constant κ is exactly 1.0 for a vacuum (the empty space serves as a reference condition) and very close to 1.0 for air under normal conditions (normal pressure at room temperature). These two values are so close that, in fact, the properties of an air-filled capacitor are essentially the same as those of an empty capacitor.

Table 4.5.1: Representative Values of Dielectric Constants and Dielectric Strengths of Various Materials at Room Temperature

Material	Dielectric constant κ	Dielectric strength $E_c [\times 10^6\, V/m]$
Vacuum	1	∞
Dry air (1 atm)	1.00059	3.0
Teflon™	2.1	60 to 173
Paraffin	2.3	11
Silicon oil	2.5	10 to 15
Polystyrene	2.56	19.7
Nylon	3.4	14
Paper	3.7	16

Material	Dielectric constant κ	Dielectric strength $E_c [\times 10^6\ V/m]$
Fused quartz	3.78	8
Glass	4 to 6	9.8 to 13.8
Concrete	4.5	–
Bakelite	4.9	24
Diamond	5.5	2,000
Pyrex glass	5.6	14
Mica	6.0	118
Neoprene rubber	6.7	15.7 to 26.7
Water	80	-
Sulfuric acid	84 to 100	-
Titanium dioxide	86 to 173	–
Strontium titanate	310	8
Barium titanate	1,200 to 10,000	–
Calcium copper titanate	> 250,000	–

Not all substances listed in the table are good insulators, despite their high dielectric constants. Water, for example, consists of polar molecules and has a large dielectric constant of about 80. In a water molecule, electrons are more likely found around the oxygen nucleus than around the hydrogen nuclei. This makes the oxygen end of the molecule slightly negative and leaves the hydrogens end slightly positive, which makes the molecule easy to align along an external electrical field, and thus water has a large dielectric constant. However, the polar nature of water molecules also makes water a good solvent for many substances, which produces undesirable effects, because any concentration of free ions in water conducts electricity.

✔ Example 4.5.1: Electrical Field and Induced Surface Charge

Suppose that the distance between the plates of the capacitor in Example 8.5.1 is 2.0 mm and the area of each plate is $4.5 \times 10^{-3} m^2$. Determine:

a. the electrical field between the plates before and after the Teflon™ is inserted, and
b. the surface charge induced on the Teflon™ surfaces.

Strategy

In part (a), we know that the voltage across the empty capacitor is $V_0 = 40\ V$, so to find the electrical fields we use the relation $V = Ed$ and Equation 4.5.3. In part (b), knowing the magnitude of the electrical field, we use the expression for the magnitude of electrical field near a charged plate $E = \sigma/\epsilon_0$, where σ is a uniform surface charge density caused by the surface charge. We use the value of free charge $Q_0 = 8.0 \times 10^{-10} C$ obtained in Example 8.5.1.

Solution

1. The electrical field E_0 between the plates of an empty capacitor is

$$E_0 = \frac{V_0}{d} = \frac{40\ V}{2.0 \times 10^{-3} m} = 2.010^4 V/m.$$

The electrical field E with the Teflon™ in place is

$$E = \frac{1}{\kappa} E_0 = \frac{1}{2.1} 2.0 \times 10^4 V/m = 9.5 \times 10^3\ V/m.$$

2. The effective charge on the capacitor is the difference between the free charge Q_0 and the induced charge Q_i. The electrical field in the Teflon™ is caused by this effective charge. Thus

$$\begin{aligned} E &= \frac{1}{\epsilon_0}\sigma \\ &= \frac{1}{\epsilon_0}\frac{Q_0 - Q_i}{A}. \end{aligned}$$

We invert this equation to obtain Q_i, which yields

$$\begin{aligned} Q_i &= Q_0 - \epsilon_0 AE \\ &= 8.0 \times 10^{-10} C - \left(8.85 \times 10^{-12} \frac{C^2}{N \cdot m^2}\right)(4.5 \times 10^{-3} m^2)\left(9.5 \times 10^3 \frac{V}{m}\right) \\ &= 4.2 \times 10^{-10} C = 0.42\, nC. \end{aligned}$$

Example 4.5.2: Inserting a Dielectric into a Capacitor Connected to a Battery

When a battery of voltage V_0 is connected across an empty capacitor of capacitance C_0, the charge on its plates is Q_0, and the electrical field between its plates is E_0. A dielectric of dielectric constant κ is inserted between the plates **while the battery remains in place**, as shown in Figure 4.5.4.

a. Find the capacitance C, the voltage V across the capacitor, and the electrical field E between the plates after the dielectric is inserted.
b. Obtain an expression for the free charge Q on the plates of the filled capacitor and the induced charge Q_i on the dielectric surface in terms of the original plate charge Q_0.

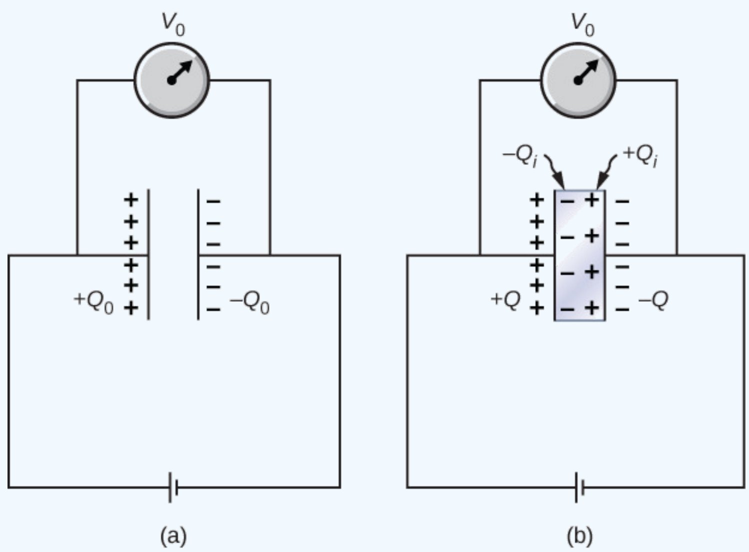


Figure 4.5.4: A dielectric is inserted into the charged capacitor while the capacitor remains connected to the battery.

Strategy

We identify the known values:V_0, C_0, E_o, κ, and Q_0. Our task is to express the unknown values in terms of these known values.

Solution

(a) The capacitance of the filled capacitor is $C = \kappa C_0$. Since the battery is always connected to the capacitor plates, the potential difference between them does not change; hence, $V = V_0$. Because of that, the electrical field in the filled capacitor is the same as the field in the empty capacitor, so we can obtain directly that

$$E = \frac{V}{d} = \frac{V_0}{d} = E_0.$$

(b) For the filled capacitor, the free charge on the plates is

$$\begin{aligned} Q &= CV \\ &= (\kappa C_0)V_0 \\ &= \kappa(C_0V_0) \\ &= \kappa Q_0. \end{aligned}$$

The electrical field E in the filled capacitor is due to the effective charge $Q - Q_i$ (Figure $4.5.4b$). Since $E = E_0$, we have

$$\frac{Q - Q_i}{\epsilon_0 A} = \frac{Q_0}{\epsilon_0 A}.$$

Solving this equation for Q_i, we obtain for the induced charge

$$\begin{aligned} Q_i &= Q - Q_0 \\ &= \kappa Q_0 - Q_0 \\ &= (\kappa - 1)Q_0. \end{aligned}$$

Significance

Notice that for materials with dielectric constants larger than 2 (see Table $4.5.1$), the induced charge on the surface of dielectric is larger than the charge on the plates of a vacuum capacitor. The opposite is true for gasses like air whose dielectric constant is smaller than 2.

? Exercise $4.5.1$

Continuing with Example $4.5.2$, show that when the battery is connected across the plates the energy stored in dielectric-filled capacitor is $U = \kappa U_0$ (larger than the energy U_0 of an empty capacitor kept at the same voltage). Compare this result with the result $U = U_0/\kappa$ found previously for an isolated, charged capacitor.

? Exercise $4.5.2$

Repeat the calculations of previous example for the case in which the battery remains connected while the dielectric is placed in the capacitor.

Answer

a. $C_0 = 20\,pF$, $C = 42\,pF$;

b. $Q_0 = 0.8\,nC$, $Q = 1.7\,nC$:

c. $V_0 = V = 40\,V$;

d. $U_0 = 16\,nJ$, $U = 34\,nJ$

4.6: Capacitance (Exercises)

Conceptual Questions

8.2 Capacitors and Capacitance

1. Does the capacitance of a device depend on the applied voltage? Does the capacitance of a device depend on the charge residing on it?

2. Would you place the plates of a parallel-plate capacitor closer together or farther apart to increase their capacitance?

3. The value of the capacitance is zero if the plates are not charged. True or false?

4. If the plates of a capacitor have different areas, will they acquire the same charge when the capacitor is connected across a battery?

5. Does the capacitance of a spherical capacitor depend on which sphere is charged positively or negatively?

8.3 Capacitors in Series and in Parallel

6. If you wish to store a large amount of charge in a capacitor bank, would you connect capacitors in series or in parallel? Explain.

7. What is the maximum capacitance you can get by connecting three **1.0-μF** capacitors? What is the minimum capacitance?

8.4 Energy Stored in a Capacitor

8. If you wish to store a large amount of energy in a capacitor bank, would you connect capacitors in series or parallel? Explain.

8.5 Capacitor with a Dielectric

9. Discuss what would happen if a conducting slab rather than a dielectric were inserted into the gap between the capacitor plates.

10. Discuss how the energy stored in an empty but charged capacitor changes when a dielectric is inserted if (a) the capacitor is isolated so that its charge does not change; (b) the capacitor remains connected to a battery so that the potential difference between its plates does not change.

8.6 Molecular Model of a Dielectric

11. Distinguish between dielectric strength and dielectric constant.

12. Water is a good solvent because it has a high dielectric constant. Explain.

13. Water has a high dielectric constant. Explain why it is then not used as a dielectric material in capacitors.

14. Elaborate on why molecules in a dielectric material experience net forces on them in a non-uniform electrical field but not in a uniform field.

15. Explain why the dielectric constant of a substance containing permanent molecular electric dipoles decreases with increasing temperature.

16. Give a reason why a dielectric material increases capacitance compared with what it would be with air between the plates of a capacitor. How does a dielectric material also allow a greater voltage to be applied to a capacitor? (The dielectric thus increases **C** and permits a greater **V**.)

17. Elaborate on the way in which the polar character of water molecules helps to explain water's relatively large dielectric constant.

18. Sparks will occur between the plates of an air-filled capacitor at a lower voltage when the air is humid than when it is dry. Discuss why, considering the polar character of water molecules.

Problems

8.2 Capacitors and Capacitance

19. What charge is stored in a **180.0-μF** capacitor when 120.0 V is applied to it?

20. Find the charge stored when 5.50 V is applied to an 8.00-pF capacitor.

21. Calculate the voltage applied to a **2.00-μF** capacitor when it holds **3.10μC** of charge.

22. What voltage must be applied to an 8.00-nF capacitor to store 0.160 mC of charge?

23. What capacitance is needed to store **3.00μC** of charge at a voltage of 120 V?

24. What is the capacitance of a large Van de Graaff generator's terminal, given that it stores 8.00 mC of charge at a voltage of 12.0 MV?

25. The plates of an empty parallel-plate capacitor of capacitance 5.0 pF are 2.0 mm apart. What is the area of each plate?

26. A 60.0-pF vacuum capacitor has a plate area of $0.010 m^2$. What is the separation between its plates?

27. A set of parallel plates has a capacitance of **5.0μF**. How much charge must be added to the plates to increase the potential difference between them by 100 V?

28. Consider Earth to be a spherical conductor of radius 6400 km and calculate its capacitance.

29. If the capacitance per unit length of a cylindrical capacitor is 20 pF/m, what is the ratio of the radii of the two cylinders?

30. An empty parallel-plate capacitor has a capacitance of **20μF**. How much charge must leak off its plates before the voltage across them is reduced by 100 V?

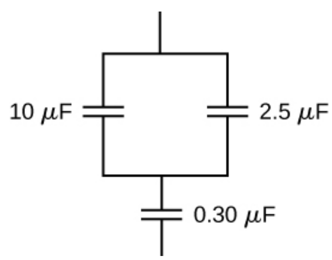
8.3 Capacitors in Series and in Parallel

31. A 4.00-pF is connected in series with an 8.00-pF capacitor and a 400-V potential difference is applied across the pair. (a) What is the charge on each capacitor?

(b) What is the voltage across each capacitor?

32. Three capacitors, with capacitances of $C_1 = 2.0 \mu F, C_2 = 3.0 \mu F$, and $C_3 = 6.0 \mu F$, respectively, are connected in parallel. A 500-V potential difference is applied across the combination. Determine the voltage across each capacitor and the charge on each capacitor.

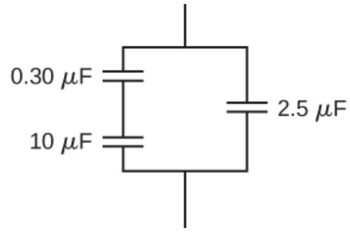
33. Find the total capacitance of this combination of series and parallel capacitors shown below.



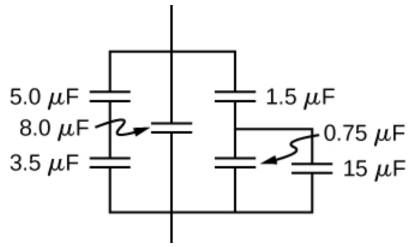
34. Suppose you need a capacitor bank with a total capacitance of 0.750 F but you have only 1.50-mF capacitors at your disposal. What is the smallest number of capacitors you could connect together to achieve your goal, and how would you connect them?

35. What total capacitances can you make by connecting a **5.00-μF** and a **8.00-μF** capacitor?

36. Find the equivalent capacitance of the combination of series and parallel capacitors shown below.



37. Find the net capacitance of the combination of series and parallel capacitors shown below.



38. A 40-pF capacitor is charged to a potential difference of 500 V. Its terminals are then connected to those of an uncharged 10-pF capacitor. Calculate:

(a) the original charge on the 40-pF capacitor;

(b) the charge on each capacitor after the connection is made; and

(c) the potential difference across the plates of each capacitor after the connection.

39. A **2.0-μF** capacitor and a **4.0-μF** capacitor are connected in series across a 1.0-kV potential. The charged capacitors are then disconnected from the source and connected to each other with terminals of like sign together. Find the charge on each capacitor and the voltage across each capacitor.

8.4 Energy Stored in a Capacitor

40. How much energy is stored in an **8.00-μF** capacitor whose plates are at a potential difference of 6.00 V?

41. A capacitor has a charge of **2.5μC** when connected to a 6.0-V battery. How much energy is stored in this capacitor?

42. How much energy is stored in the electrical field of a metal sphere of radius 2.0 m that is kept at a 10.0-V potential?

43. (a) What is the energy stored in the **10.0-μF** capacitor of a heart defibrillator charged to $9.00 \times 10^3 V$?

(b) Find the amount of the stored charge.

44. In open-heart surgery, a much smaller amount of energy will defibrillate the heart.

(a) What voltage is applied to the **8.00-μF** capacitor of a heart defibrillator that stores 40.0 J of energy?

(b) Find the amount of the stored charge.

45. A **165-μF** capacitor is used in conjunction with a dc motor. How much energy is stored in it when 119 V is applied?

46. Suppose you have a 9.00-V battery, a **2.00-μF** capacitor, and a **7.40-μF** capacitor.

(a) Find the charge and energy stored if the capacitors are connected to the battery in series.

(b) Do the same for a parallel connection.

47. An anxious physicist worries that the two metal shelves of a wood frame bookcase might obtain a high voltage if charged by static electricity, perhaps produced by friction.

(a) What is the capacitance of the empty shelves if they have area $1.00 \times 10^2 m$ and are 0.200 m apart?

(b) What is the voltage between them if opposite charges of magnitude 2.00 nC are placed on them?

(c) To show that this voltage poses a small hazard, calculate the energy stored.

(d) The actual shelves have an area 100 times smaller than these hypothetical shelves. Are his fears justified?

48. A parallel-plate capacitor is made of two square plates 25 cm on a side and 1.0 mm apart. The capacitor is connected to a 50.0-V battery. With the battery still connected, the plates are pulled apart to a separation of 2.00 mm. What are the energies stored in the capacitor before and after the plates are pulled farther apart? Why does the energy decrease even though work is done in separating the plates?

49. Suppose that the capacitance of a variable capacitor can be manually changed from 100 pF to 800 pF by turning a dial, connected to one set of plates by a shaft, from **0°** to **180°.** With the dial set at 180°180° (corresponding to **C=800pF**), the capacitor is connected to a 500-V source. After charging, the capacitor is disconnected from the source, and the dial is turned to **0°**. If friction is negligible, how much work is required to turn the dial from **180°** to **0°**?

8.5 Capacitor with a Dielectric

50. Show that for a given dielectric material, the maximum energy a parallel-plate capacitor can store is directly proportional to the volume of dielectric.

51. An air-filled capacitor is made from two flat parallel plates 1.0 mm apart. The inside area of each plate is 8.0cm28.0cm2.

(a) What is the capacitance of this set of plates?

(b) If the region between the plates is filled with a material whose dielectric constant is 6.0, what is the new capacitance?

52. A capacitor is made from two concentric spheres, one with radius 5.00 cm, the other with radius 8.00 cm.

(a) What is the capacitance of this set of conductors?

(b) If the region between the conductors is filled with a material whose dielectric constant is 6.00, what is the capacitance of the system?

53. A parallel-plate capacitor has charge of magnitude **9.00μC** on each plate and capacitance **3.00μF** when there is air between the plates. The plates are separated by 2.00 mm. With the charge on the plates kept constant, a dielectric with $\kappa = 5$ is inserted between the plates, completely filling the volume between the plates.

(a) What is the potential difference between the plates of the capacitor, before and after the dielectric has been inserted?

(b) What is the electrical field at the point midway between the plates before and after the dielectric is inserted?

54. Some cell walls in the human body have a layer of negative charge on the inside surface. Suppose that the surface charge densities are $\pm 0.50 \times 10^{-3} C/m^2$, the cell wall is $5.0 \times 10^{-9} m$ thick, and the cell wall material has a dielectric constant of **κ=5.4.**

(a) Find the magnitude of the electric field in the wall between two charge layers.

(b) Find the potential difference between the inside and the outside of the cell. Which is at higher potential?

(c) A typical cell in the human body has volume $10^{-16} m^3$. Estimate the total electrical field energy stored in the wall of a cell of this size when assuming that the cell is spherical. (**Hint:** Calculate the volume of the cell wall.)

55. A parallel-plate capacitor with only air between its plates is charged by connecting the capacitor to a battery. The capacitor is then disconnected from the battery, without any of the charge leaving the plates.

(a) A voltmeter reads 45.0 V when placed across the capacitor. When a dielectric is inserted between the plates, completely filling the space, the voltmeter reads 11.5 V. What is the dielectric constant of the material?

(b) What will the voltmeter read if the dielectric is now pulled away out so it fills only one-third of the space between the plates?

8.6 Molecular Model of a Dielectric

56. Two flat plates containing equal and opposite charges are separated by material 4.0 mm thick with a dielectric constant of 5.0. If the electrical field in the dielectric is 1.5 MV/m, what are

(a) the charge density on the capacitor plates, and

(b) the induced charge density on the surfaces of the dielectric?

57. For a Teflon™-filled, parallel-plate capacitor, the area of the plate is $50.0 cm^2$ and the spacing between the plates is 0.50 mm. If the capacitor is connected to a 200-V battery, find

(a) the free charge on the capacitor plates,

(b) the electrical field in the dielectric, and

(c) the induced charge on the dielectric surfaces.

58. Find the capacitance of a parallel-plate capacitor having plates with a surface area of $5.00 m^2$ and separated by 0.100 mm of Teflon™.

59. (a) What is the capacitance of a parallel-plate capacitor with plates of area $1.50 m^2$ that are separated by 0.0200 mm of neoprene rubber?

(b) What charge does it hold when 9.00 V is applied to it?

60. Two parallel plates have equal and opposite charges. When the space between the plates is evacuated, the electrical field is $E = 3.20 \times 10^5 V/m$. When the space is filled with dielectric, the electrical field is $E = 2.50 \times 10^5 V/m$.

(a) What is the surface charge density on each surface of the dielectric?

(b) What is the dielectric constant?

61. The dielectric to be used in a parallel-plate capacitor has a dielectric constant of 3.60 and a dielectric strength of $1.60 \times 10^7 V/m$. The capacitor has to have a capacitance of 1.25 nF and must be able to withstand a maximum potential difference 5.5 kV. What is the minimum area the plates of the capacitor may have?

62. When a 360-nF air capacitor is connected to a power supply, the energy stored in the capacitor is **18.5μJ**. While the capacitor is connected to the power supply, a slab of dielectric is inserted that completely fills the space between the plates. This increases the stored energy by **23.2μJ**.

(a) What is the potential difference between the capacitor plates?

(b) What is the dielectric constant of the slab?

63. A parallel-plate capacitor has square plates that are 8.00 cm on each side and 3.80 mm apart. The space between the plates is completely filled with two square slabs of dielectric, each 8.00 cm on a side and 1.90 mm thick. One slab is Pyrex glass and the other slab is polystyrene. If the potential difference between the plates is 86.0 V, find how much electrical energy can be stored in this capacitor.

Additional Problems

64. A capacitor is made from two flat parallel plates placed 0.40 mm apart. When a charge of **0.020μC** is placed on the plates the potential difference between them is 250 V.

(a) What is the capacitance of the plates?

(b) What is the area of each plate?

(c) What is the charge on the plates when the potential difference between them is 500 V?

(d) What maximum potential difference can be applied between the plates so that the magnitude of electrical fields between the plates does not exceed 3.0 MV/m?

65. An air-filled (empty) parallel-plate capacitor is made from two square plates that are 25 cm on each side and 1.0 mm apart. The capacitor is connected to a 50-V battery and fully charged. It is then disconnected from the battery and its plates are pulled apart to a separation of 2.00 mm.

(a) What is the capacitance of this new capacitor?

(b) What is the charge on each plate?

(c) What is the electrical field between the plates?

66. Suppose that the capacitance of a variable capacitor can be manually changed from 100 to 800 pF by turning a dial connected to one set of plates by a shaft, from **0°** to **180°**. With the dial set at **180°** (corresponding to **C=800pF**), the capacitor is connected to a 500-V source. After charging, the capacitor is disconnected from the source, and the dial is turned to **0°**. (a) What is the charge on the capacitor? (b) What is the voltage across the capacitor when the dial is set to **0°**?

67. Earth can be considered as a spherical capacitor with two plates, where the negative plate is the surface of Earth and the positive plate is the bottom of the ionosphere, which is located at an altitude of approximately 70 km. The potential difference between Earth's surface and the ionosphere is about 350,000 V.

(a) Calculate the capacitance of this system.

(b) Find the total charge on this capacitor.

(c) Find the energy stored in this system.

68. A **4.00-μF** capacitor and a **6.00-μF** capacitor are connected in parallel across a 600-V supply line.

(a) Find the charge on each capacitor and voltage across each.

(b) The charged capacitors are disconnected from the line and from each other. They are then reconnected to each other with terminals of unlike sign together. Find the final charge on each capacitor and the voltage across each.

69. Three capacitors having capacitances of 8.40, 8.40, and 4.20 **μF**, respectively, are connected in series across a 36.0-V potential difference.

(a) What is the charge on the **4.20-μF** capacitor?

(b) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other with the positively charged plates connected together. What is the voltage across each capacitor in the parallel combination?

70. A parallel-plate capacitor with capacitance **5.0μF** is charged with a 12.0-V battery, after which the battery is disconnected. Determine the minimum work required to increase the separation between the plates by a factor of 3.

71. (a) How much energy is stored in the electrical fields in the capacitors (in total) shown below?

(b) Is this energy equal to the work done by the 400-V source in charging the capacitors?

72. Three capacitors having capacitances 8.4, 8.4, and 4.2 **μF** are connected in series across a 36.0-V potential difference.

(a) What is the total energy stored in all three capacitors?

(b) The capacitors are disconnected from the potential difference without allowing them to discharge. They are then reconnected in parallel with each other with the positively charged plates connected together. What is the total energy now stored in the capacitors?

73. (a) An **8.00-μF** capacitor is connected in parallel to another capacitor, producing a total capacitance of **5.00μF**. What is the capacitance of the second capacitor?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

74. (a) On a particular day, it takes $9.60 \times 10^3 J$ of electrical energy to start a truck's engine. Calculate the capacitance of a capacitor that could store that amount of energy at 12.0 V.

(b) What is unreasonable about this result?

(c) Which assumptions are responsible?

75. (a) A certain parallel-plate capacitor has plates of area $4.00 m^2$, separated by 0.0100 mm of nylon, and stores 0.170 C of charge. What is the applied voltage?

(b) What is unreasonable about this result?

(c) Which assumptions are responsible or inconsistent?

76. A prankster applies 450 V to an **80.0-μF** capacitor and then tosses it to an unsuspecting victim. The victim's finger is burned by the discharge of the capacitor through 0.200 g of flesh. Estimate, what is the temperature increase of the flesh? Is it reasonable to assume that no thermodynamic phase change happened?

Challenge Problems

77. A spherical capacitor is formed from two concentric spherical conducting spheres separated by vacuum. The inner sphere has radius 12.5 cm and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor.

(a) What is the capacitance of the capacitor?

(b) What is the magnitude of the electrical field at **r=12.6cm**, just outside the inner sphere?

(c) What is the magnitude of the electrical field at **r=14.7cm**, just inside the outer sphere?

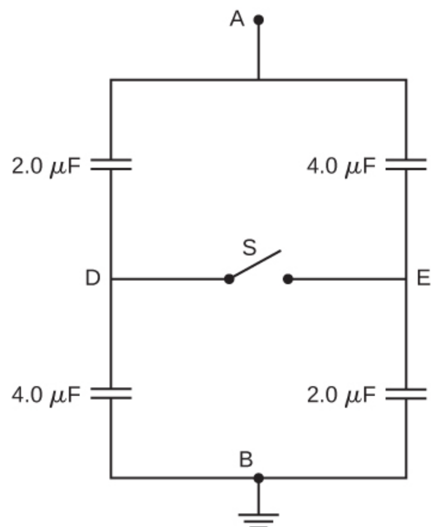
(d) For a parallel-plate capacitor the electrical field is uniform in the region between the plates, except near the edges of the plates. Is this also true for a spherical capacitor?

78. The network of capacitors shown below are all uncharged when a 300-V potential is applied between points A and B with the switch S open.

(a) What is the potential difference $V_E - V_D$?

(b) What is the potential at point E after the switch is closed?

(c) How much charge flows through the switch after it is closed?



79. Electronic flash units for cameras contain a capacitor for storing the energy used to produce the flash. In one such unit the flash lasts for 1/675 fraction of a second with an average light power output of 270 kW.

(a) If the conversion of electrical energy to light is 95% efficient (because the rest of the energy goes to thermal energy), how much energy must be stored in the capacitor for one flash?

(b) The capacitor has a potential difference between its plates of 125 V when the stored energy equals the value stored in part (a). What is the capacitance?

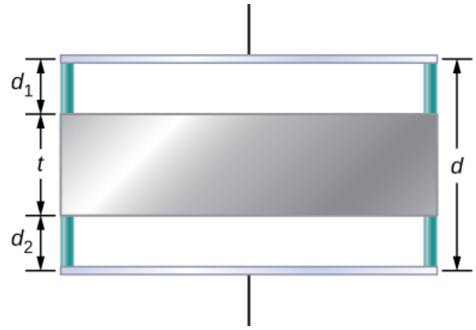
80. A spherical capacitor is formed from two concentric spherical conducting shells separated by a vacuum. The inner sphere has radius 12.5 cm and the outer sphere has radius 14.8 cm. A potential difference of 120 V is applied to the capacitor.

(a) What is the energy density at **r=12.6cm**, just outside the inner sphere?

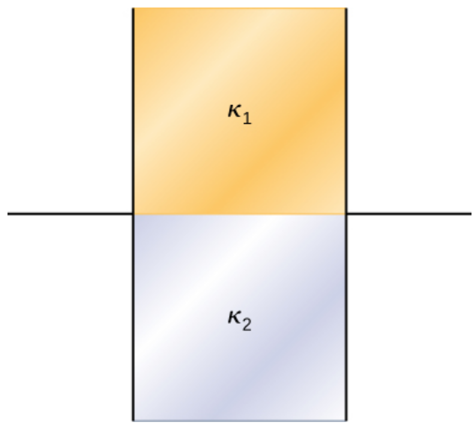
(b) What is the energy density at **r=14.7cm**, just inside the outer sphere?

(c) For the parallel-plate capacitor the energy density is uniform in the region between the plates, except near the edges of the plates. Is this also true for the spherical capacitor?

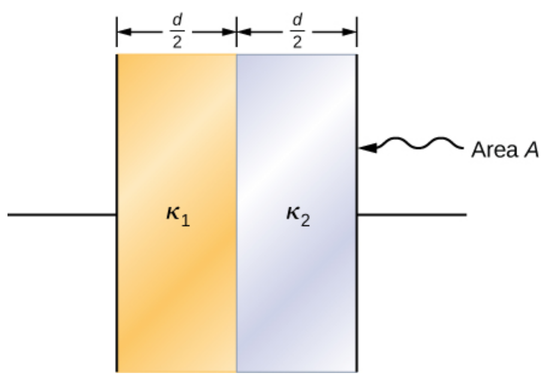
81. A metal plate of thickness t is held in place between two capacitor plates by plastic pegs, as shown below. The effect of the pegs on the capacitance is negligible. The area of each capacitor plate and the area of the top and bottom surfaces of the inserted plate are all A. What is the capacitance of this system?



82. A parallel-plate capacitor is filled with two dielectrics, as shown below. When the plate area is A and separation between plates is d, show that the capacitance is given by $C = \varepsilon_0 \frac{A}{d} \frac{\kappa_1 + \kappa_2}{2}$.



83. A parallel-plate capacitor is filled with two dielectrics, as shown below. Show that the capacitance is given by $C = 2\varepsilon_0 \frac{A}{d} \frac{\kappa_1 \kappa_2}{\kappa_1 + \kappa_2}$.



84. A capacitor has parallel plates of area $12 cm^2$ separated by 2.0 mm. The space between the plates is filled with polystyrene.

(a) Find the maximum permissible voltage across the capacitor to avoid dielectric breakdown.

(b) When the voltage equals the value found in part (a), find the surface charge density on the surface of the dielectric.

CHAPTER OVERVIEW

5: Current and Resistance

In this chapter, we study the electrical current through a material, where the electrical current is the rate of flow of charge. We also examine a characteristic of materials known as the resistance. Resistance is a measure of how much a material impedes the flow of charge, and it will be shown that the resistance depends on temperature. In general, a good conductor, such as copper, gold, or silver, has very low resistance. Some materials, called superconductors, have zero resistance at very low temperatures.

5.1: Prelude to Current and Resistance
5.2: Electrical Current
5.3: Model of Conduction in Metals
5.4: Resistivity and Resistance
5.5: Ohm's Law
5.6: Electrical Energy and Power
5.7: Superconductors
5.8: Current and Resistance (Exercises)

5.1: Prelude to Current and Resistance

In this chapter, we study the electrical current through a material, where the electrical current is the rate of flow of charge. We also examine a characteristic of materials known as the resistance. Resistance is a measure of how much a material impedes the flow of charge, and it will be shown that the resistance depends on temperature. In general, a good conductor, such as copper, gold, or silver, has very low resistance. Some materials, called superconductors, have zero resistance at very low temperatures.

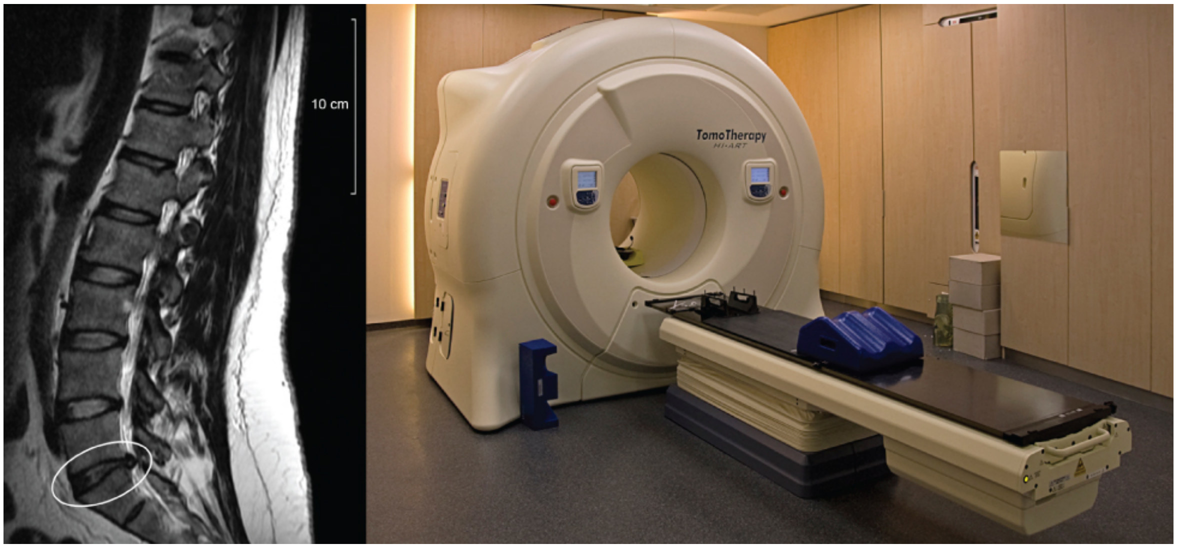


Figure 5.1.1: Magnetic resonance imaging (MRI) uses superconducting magnets and produces high-resolution images without the danger of radiation. The image on the left shows the spacing of vertebrae along a human spinal column, with the circle indicating where the vertebrae are too close due to a ruptured disc. On the right is a picture of the MRI instrument, which surrounds the patient on all sides. A large amount of electrical current is required to operate the electromagnets (credit right: modification of work by "digital cat"/Flickr).

High currents are required for the operation of electromagnets. Superconductors can be used to make electromagnets that are 10 times stronger than the strongest conventional electromagnets. These superconducting magnets are used in the construction of magnetic resonance imaging (MRI) devices that can be used to make high-resolution images of the human body. The chapter-opening picture shows an MRI image of the vertebrae of a human subject and the MRI device itself. Superconducting magnets have many other uses. For example, superconducting magnets are used in the Large Hadron Collider (LHC) to curve the path of protons in the ring.

5.2: Electrical Current

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe an electrical current
- Define the unit of electrical current
- Explain the direction of current flow

Up to now, we have considered primarily static charges. When charges did move, they were accelerated in response to an electrical field created by a voltage difference. The charges lost potential energy and gained kinetic energy as they traveled through a potential difference where the electrical field did work on the charge.

Although charges do not require a material to flow through, the majority of this chapter deals with understanding the movement of charges through a material. The rate at which the charges flow past a location—that is, the amount of charge per unit time—is known as the **electrical current**. When charges flow through a medium, the current depends on the voltage applied, the material through which the charges flow, and the state of the material. Of particular interest is the motion of charges in a conducting wire. In previous chapters, charges were accelerated due to the force provided by an electrical field, losing potential energy and gaining kinetic energy. In this chapter, we discuss the situation of the force provided by an electrical field in a conductor, where charges lose kinetic energy to the material reaching a constant velocity, known as the "**drift velocity**." This is analogous to an object falling through the atmosphere and losing kinetic energy to the air, reaching a constant terminal velocity.

If you have ever taken a course in first aid or safety, you may have heard that in the event of electric shock, it is the current, not the voltage, which is the important factor on the severity of the shock and the amount of damage to the human body. Current is measured in units called amperes; you may have noticed that circuit breakers in your home and fuses in your car are rated in amps (or amperes). But what is the ampere and what does it measure?

Defining Current and the Ampere

Electrical current is defined to be the rate at which charge flows. When there is a large current present, such as that used to run a refrigerator, a large amount of charge moves through the wire in a small amount of time. If the current is small, such as that used to operate a handheld calculator, a small amount of charge moves through the circuit over a long period of time.

Electrical Current

The average electrical current I is the rate at which charge flows,

$$I_{ave} = \frac{\Delta Q}{\Delta t}, \tag{5.2.1}$$

where ΔQ is the amount of net charge passing through a given cross-sectional area in time Δt (Figure $5.2.1$). The SI unit for current is the **ampere** (A), named for the French physicist André-Marie Ampère (1775–1836). Since $I = \frac{\Delta Q}{\Delta t}$, we see that an ampere is defined as one coulomb of charge passing through a given area per second:

$$1A \equiv 1\frac{C}{s}.$$

The instantaneous electrical current, or simply the **electrical current**, is the time derivative of the charge that flows and is found by taking the limit of the average electrical current as $\Delta t \rightarrow 0$.

$$I = \lim_{\Delta t \rightarrow 0} \frac{\Delta Q}{\Delta t} = \frac{dQ}{dt}.$$

Most electrical appliances are rated in amperes (or amps) required for proper operation, as are fuses and circuit breakers.

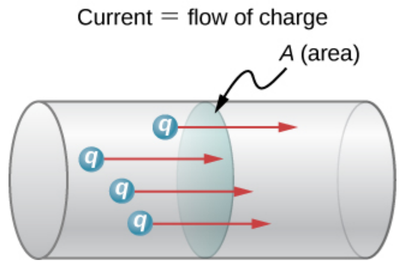


Figure 5.2.1: The rate of flow of charge is current. An ampere is the flow of one coulomb of charge through an area in one second. A current of one amp would result from 6.25×10^{18} electrons flowing through the area A each second.

✔ culating the Average Current

The main purpose of a battery in a car or truck is to run the electric **starter motor**, which starts the engine. The operation of starting the vehicle requires a large current to be supplied by the battery. Once the engine starts, a device called an alternator takes over supplying the electric power required for running the vehicle and for charging the battery.

a. What is the average current involved when a truck battery sets in motion 720 C of charge in 4.00 s while starting an engine?
b. How long does it take 1.00 C of charge to flow from the battery?

Strategy

We can use the definition of the average current in Equation 5.2.1 to find the average current in part (a), since charge and time are given. For part (b), once we know the average current, we use Equation 5.2.1 to find the time required for 1.00 C of charge to flow from the battery.

Solution

a. Entering the given values for charge and time into the definition of current gives

$$\begin{aligned} I &= \frac{\Delta Q}{\Delta t} \\ &= \frac{720\,C}{4.00\,s} \\ &= 180\,C/s \\ &= 180\,A. \end{aligned}$$

b. Solving the relationship $I = \frac{\Delta Q}{\Delta t}$ for time Δt and entering the known values for charge and current gives

$$\begin{aligned} \Delta t &= \frac{\Delta Q}{I} \\ &= \frac{1.00\,C}{180\,C/s} \\ &= 5.56 \times 10^{-3}\,s \\ &= 5.56\,ms. \end{aligned}$$

Significance

a. This large value for current illustrates the fact that a large charge is moved in a small amount of time. The currents in these “starter motors” are fairly large to overcome the inertia of the engine.
b. A high current requires a short time to supply a large amount of charge. This large current is needed to supply the large amount of energy needed to start the engine.

Calculating Instantaneous Currents

Consider a charge moving through a cross-section of a wire where the charge is modeled as $Q(t) = Q_M(1 - e^{-t/\tau})$. Here, Q_M is the charge after a long period of time, as time approaches infinity, with units of coulombs, and τ is a time constant with units of seconds (Figure $5.2.2$). What is the current through the wire?

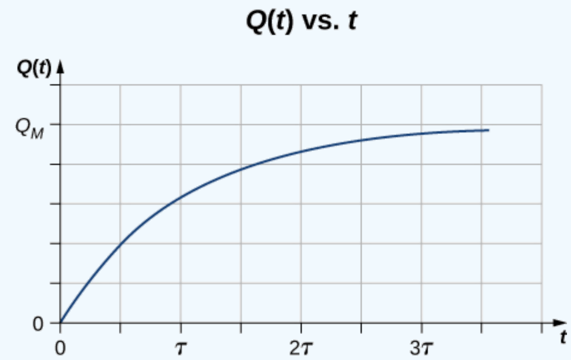


Figure $5.2.2$: A graph of the charge moving through a cross-section of a wire over time.

Strategy

The current through the cross-section can be found from $I = \frac{dQ}{dt}$. Notice from the figure that the charge increases to Q_M and the derivative decreases, approaching zero, as time increases (Figure $5.2.2$).

Solution

The derivative can be found using $\frac{d}{dx}e_u = e^u\frac{du}{dx}$.

$$\begin{aligned} I &= \frac{dQ}{dt} \\ &= \frac{d}{dt}\left[Q_M\left(1 - e^{-t/\tau}\right)\right] \\ &= \frac{Q_M}{\tau}e^{-t/\tau}. \end{aligned}$$

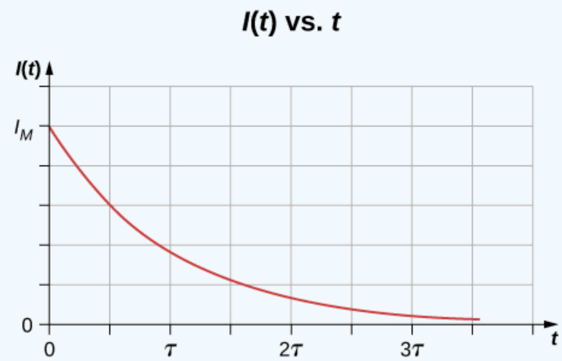


Figure $5.2.3$: A graph of the current flowing through the wire over time.

Significance

The current through the wire in question decreases exponentially, as shown in Figure $5.2.3$. In later chapters, it will be shown that a time-dependent current appears when a capacitor charges or discharges through a resistor. Recall that a capacitor is a device that stores charge. You will learn about the resistor in Model of Conduction in Metals.

? Exercise $5.2.1A$

Handheld calculators often use small solar cells to supply the energy required to complete the calculations needed to complete your next physics exam. The current needed to run your calculator can be as small as 0.30 mA. How long would it take for 1.00 C of charge to flow from the solar cells? Can solar cells be used, instead of batteries, to start traditional internal combustion engines presently used in most cars and trucks?

Answer

The time for 1.00 C of charge to flow would be

$$\Delta t = \frac{\Delta Q}{I} = \frac{1.00\,C}{0.300 \times 10^{-3} C/s} = 3.33 \times 10^{3}\ s.$$

This is slightly less than an hour. This is quite different from the 5.55 ms for the truck battery. The calculator takes a very small amount of energy to operate, unlike the truck's starter motor. There are several reasons that vehicles use batteries and not solar cells. Aside from the obvious fact that a light source to run the solar cells for a car or truck is not always available, the large amount of current needed to start the engine cannot easily be supplied by present-day solar cells. Solar cells can possibly be used to charge the batteries. Charging the battery requires a small amount of energy when compared to the energy required to run the engine and the other accessories such as the heater and air conditioner. Present day solar-powered cars are powered by solar panels, which may power an electric motor, instead of an internal combustion engine.

? Exercise $5.2.1B$

Circuit breakers in a home are rated in amperes, normally in a range from 10 amps to 30 amps, and are used to protect the residents from harm and their appliances from damage due to large currents. A single 15-amp circuit breaker may be used to protect several outlets in the living room, whereas a single 20-amp circuit breaker may be used to protect the refrigerator in the kitchen. What can you deduce from this about current used by the various appliances?

Answer

The total current needed by all the appliances in the living room (a few lamps, a television, and your laptop) draw less current and require less power than the refrigerator.

Current in a Circuit

In the previous paragraphs, we defined the current as the charge that flows through a cross-sectional area per unit time. In order for charge to flow through an appliance, such as the headlight shown in Figure $5.2.4$, there must be a complete path (or **circuit**) from the positive terminal to the negative terminal. Consider a simple circuit of a car battery, a switch, a headlight lamp, and wires that provide a current path between the components. In order for the lamp to light, there must be a complete path for current flow. In other words, a charge must be able to leave the positive terminal of the battery, travel through the component, and back to the negative terminal of the battery. The switch is there to control the circuit. Part (a) of the figure shows the simple circuit of a car battery, a switch, a conducting path, and a headlight lamp. Also shown is the **schematic** of the circuit [part (b)]. A schematic is a graphical representation of a circuit and is very useful in visualizing the main features of a circuit. Schematics use standardized symbols to represent the components in a circuits and solid lines to represent the wires connecting the components. The battery is shown as a series of long and short lines, representing the historic voltaic pile. The lamp is shown as a circle with a loop inside, representing the filament of an incandescent bulb. The switch is shown as two points with a conducting bar to connect the two points and the wires connecting the components are shown as solid lines. The schematic in part (c) shows the direction of current flow when the switch is closed.

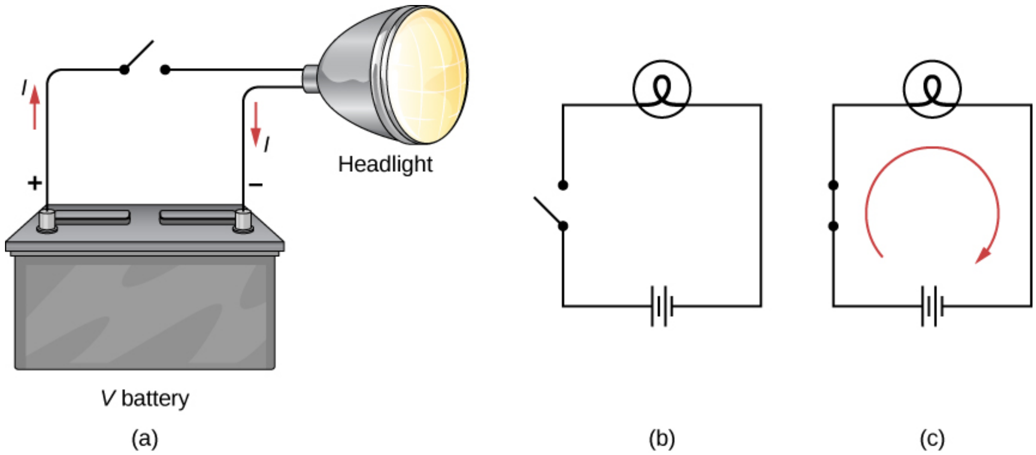


Figure 5.2.4: (a) A simple electric circuit of a headlight (lamp), a battery, and a switch. When the switch is closed, an uninterrupted path for current to flow through is supplied by conducting wires connecting a load to the terminals of a battery. (b) In this schematic, the battery is represented by parallel lines, which resemble plates in the original design of a battery. The longer lines indicate the positive terminal. The conducting wires are shown as solid lines. The switch is shown, in the open position, as two terminals with a line representing a conducting bar that can make contact between the two terminals. The lamp is represented by a circle encompassing a filament, as would be seen in an incandescent light bulb. (c) When the switch is closed, the circuit is complete and current flows from the positive terminal to the negative terminal of the battery.

When the switch is closed in Figure 5.2.4c, there is a complete path for charges to flow, from the positive terminal of the battery, through the switch, then through the headlight and back to the negative terminal of the battery. Note that the direction of current flow is from positive to negative. The direction of conventional current is always represented in the direction that positive charge would flow, from the positive terminal to the negative terminal.

The conventional current flows from the positive terminal to the negative terminal, but depending on the actual situation, positive charges, negative charges, or both may move. In metal wires, for example, current is carried by electrons—that is, negative charges move. In ionic solutions, such as salt water, both positive and negative charges move. This is also true in nerve cells. A Van de Graaff generator, used for nuclear research, can produce a current of pure positive charges, such as protons. In the Tevatron Accelerator at Fermilab, before it was shut down in 2011, beams of protons and antiprotons traveling in opposite directions were collided. The protons are positive and therefore their current is in the same direction as they travel. The antiprotons are negativity charged and thus their current is in the opposite direction that the actual particles travel.

A closer look at the current flowing through a wire is shown in Figure 5.2.5. The figure illustrates the movement of charged particles that compose a current. The fact that conventional current is taken to be in the direction that positive charge would flow can be traced back to American scientist and statesman Benjamin Franklin in the 1700s. Having no knowledge of the particles that make up the atom (namely the proton, electron, and neutron), Franklin believed that electrical current flowed from a material that had more of an “electrical fluid” and to a material that had less of this “electrical fluid.” He coined the term **positive** for the material that had more of this electrical fluid and **negative** for the material that lacked the electrical fluid. He surmised that current would flow from the material with more electrical fluid—the positive material—to the negative material, which has less electrical fluid. Franklin called this direction of current a positive current flow. This was pretty advanced thinking for a man who knew nothing about the atom.

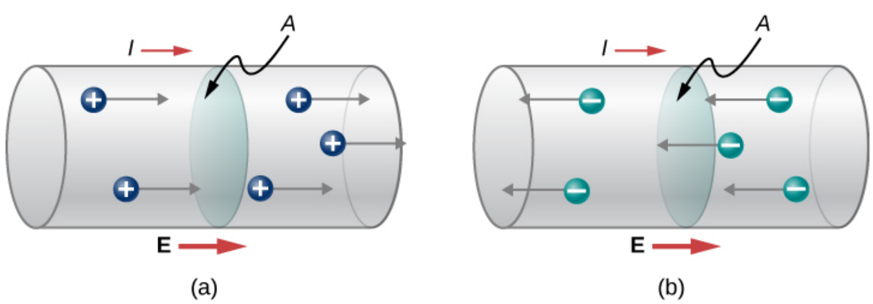


Figure 5.2.5: Current **I** is the rate at which charge moves through an area **A**, such as the cross-section of a wire. Conventional current is defined to move in the direction of the electrical field. (a) Positive charges move in the direction of the electrical field, which is the same direction as conventional current. (b) Negative charges move in the direction opposite to the electrical field. Conventional current is in the direction opposite to the movement of negative charge. The flow of electrons is sometimes referred to as electronic flow.

We now know that a material is positive if it has a greater number of protons than electrons, and it is negative if it has a greater number of electrons than protons. In a conducting metal, the current flow is due primarily to electrons flowing from the negative material to the positive material, but for historical reasons, we consider the positive current flow and the current is shown to flow from the positive terminal of the battery to the negative terminal.

It is important to realize that an electrical field is present in conductors and is responsible for producing the current (Figure $5.2.5$). In previous chapters, we considered the static electrical case, where charges in a conductor quickly redistribute themselves on the surface of the conductor in order to cancel out the external electrical field and restore equilibrium. In the case of an electrical circuit, the charges are prevented from ever reaching equilibrium by an external source of electric potential, such as a battery. The energy needed to move the charge is supplied by the electric potential from the battery.

Although the electrical field is responsible for the motion of the charges in the conductor, the work done on the charges by the electrical field does not increase the kinetic energy of the charges. We will show that the electrical field is responsible for keeping the electric charges moving at a "drift velocity."

5.3: Model of Conduction in Metals

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Define the drift velocity of charges moving through a metal
- Define the vector current density
- Describe the operation of an incandescent lamp

When electrons move through a conducting wire, they do not move at a constant velocity, that is, the electrons do not move in a straight line at a constant speed. Rather, they interact with and collide with atoms and other free electrons in the conductor. Thus, the electrons move in a zig-zag fashion and drift through the wire. We should also note that even though it is convenient to discuss the direction of current, current is a scalar quantity. When discussing the velocity of charges in a current, it is more appropriate to discuss the current density. We will come back to this idea at the end of this section.

Drift Velocity

Electrical signals move very rapidly. Telephone conversations carried by currents in wires cover large distances without noticeable delays. Lights come on as soon as a light switch is moved to the 'on' position. Most electrical signals carried by currents travel at speeds on the order of $10^8 m/s$ a significant fraction of the speed of light. Interestingly, the individual charges that make up the current move much slower on average, typically drifting at speeds on the order of $10^{-4} m/s$. How do we reconcile these two speeds, and what does it tell us about standard conductors?

The high speed of electrical signals results from the fact that the force between charges acts rapidly at a distance. Thus, when a free charge is forced into a wire, as in Figure $5.3.1$, the incoming charge pushes other charges ahead of it due to the repulsive force between like charges. These moving charges push on charges farther down the line. The density of charge in a system cannot easily be increased, so the signal is passed on rapidly. The resulting electrical shock wave moves through the system at nearly the speed of light. To be precise, this fast-moving signal, or shock wave, is a rapidly propagating change in the electrical field.

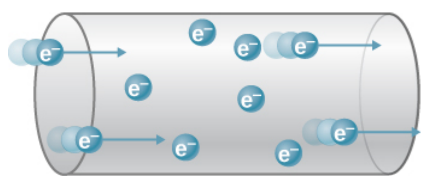


Figure $5.3.1$: When charged particles are forced into this volume of a conductor, an equal number are quickly forced to leave. The repulsion between like charges makes it difficult to increase the number of charges in a volume. Thus, as one charge enters, another leaves almost immediately, carrying the signal rapidly forward.

Good conductors have large numbers of free charges. In metals, the free charges are free electrons. (In fact, good electrical conductors are often good heat conductors too, because large numbers of free electrons can transport thermal energy as well as carry electrical current.) Figure $5.3.2$ shows how free electrons move through an ordinary conductor. The distance that an individual electron can move between collisions with atoms or other electrons is quite small. The electron paths thus appear nearly random, like the motion of atoms in a gas. But there is an electrical field in the conductor that causes the electrons to drift in the direction shown (opposite to the field, since they are negative). The **drift velocity** $\vec{v}_d$ is the average velocity of the free charges. Drift velocity is quite small, since there are so many free charges. If we have an estimate of the density of free electrons in a conductor, we can calculate the drift velocity for a given current. The larger the density, the lower the velocity required for a given current.

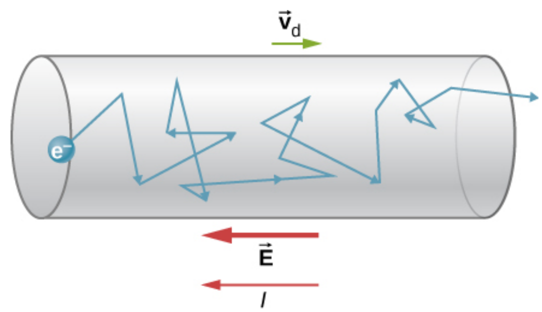


Figure 5.3.2: Free electrons moving in a conductor make many collisions with other electrons and other particles. A typical path of one electron is shown. The average velocity of the free charges is called the drift velocity $\vec{v}_d$ and for electrons, it is in the direction opposite to the electrical field. The collisions normally transfer energy to the conductor, requiring a constant supply of energy to maintain a steady current.

Free-electron collisions transfer energy to the atoms of the conductor. The electrical field does work in moving the electrons through a distance, but that work does not increase the kinetic energy (nor speed) of the electrons. The work is transferred to the conductor's atoms, often increasing temperature. Thus, a continuous power input is required to keep a current flowing. (An exception is superconductors, for reasons we shall explore in a later chapter. Superconductors can have a steady current without a continual supply of energy—a great energy savings.) For a conductor that is not a superconductor, the supply of energy can be useful, as in an incandescent light bulb filament (Figure 5.3.3). The supply of energy is necessary to increase the temperature of the tungsten filament, so that the filament glows.

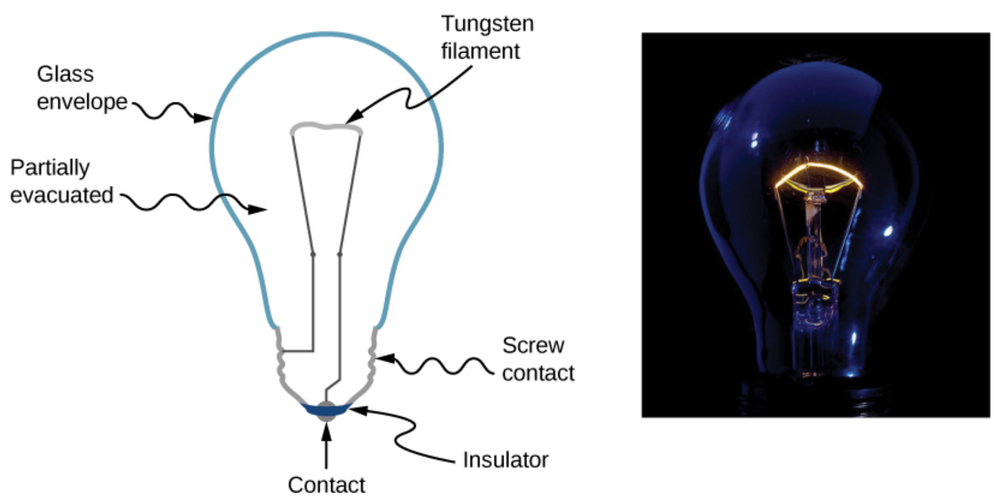


Figure 5.3.3: The incandescent lamp is a simple design. A tungsten filament is placed in a partially evacuated glass envelope. One end of the filament is attached to the screw base, which is made out of a conducting material. The second end of the filament is attached to a second contact in the base of the bulb. The two contacts are separated by an insulating material. Current flows through the filament, and the temperature of the filament becomes large enough to cause the filament to glow and produce light. However, these bulbs are not very energy efficient, as evident from the heat coming from the bulb. In the year 2012, the United States, along with many other countries, began to phase out incandescent lamps in favor of more energy-efficient lamps, such as light-emitting diode (LED) lamps and compact fluorescent lamps (CFL) (credit right: modification of work by Serge Saint).

We can obtain an expression for the relationship between current and drift velocity by considering the number of free charges in a segment of wire, as illustrated in Figure 5.3.4. The number of free charges per unit volume, or the number density of free charges, is given the symbol n where

$$n = \frac{\text{number of charges}}{\text{volume}}.$$

The value of n depends on the material. The shaded segment has a volume $Av_d\,dt$, so that the number of free charges in the volume is $nAv_d\,dt$. The charge dQ in this segment is thus $qnAv_d\,dt$, where q is the amount of charge on each carrier. (The magnitude of the charge of electrons is $q = 1.60 \times 10^{-19}\,C$.) Current is charge moved per unit time; thus, if all the original charges move out of this segment in time **dt**, the current is

$$I = \frac{dQ}{dt} = qnAv_d.$$

Rearranging terms gives

$$v_d = \frac{I}{nqA}$$

where

- v_d is the drift velocity,
- $\boldsymbol{n}$ is the free charge density,
- $\boldsymbol{A}$ is the cross-sectional area of the wire, and
- I is the current through the wire.

The carriers of the current each have charge **q** and move with a drift velocity of magnitude v_d.

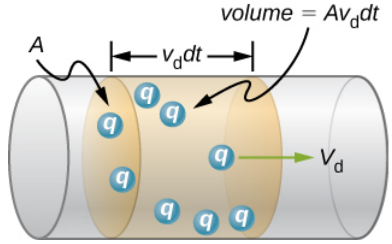


Figure $5.3.4$: All the charges in the shaded volume of this wire move out in a time **dt**, having a drift velocity of magnitude v_d.

Note that simple drift velocity is not the entire story. The speed of an electron is sometimes much greater than its drift velocity. In addition, not all of the electrons in a conductor can move freely, and those that do move might move somewhat faster or slower than the drift velocity. So what do we mean by free electrons?

Atoms in a metallic conductor are packed in the form of a lattice structure. Some electrons are far enough away from the atomic nuclei that they do not experience the attraction of the nuclei as strongly as the inner electrons do. These are the free electrons. They are not bound to a single atom but can instead move freely among the atoms in a "sea" of electrons. When an electrical field is applied, these free electrons respond by accelerating. As they move, they collide with the atoms in the lattice and with other electrons, generating thermal energy, and the conductor gets warmer. In an insulator, the organization of the atoms and the structure do not allow for such free electrons.

As you know, electric power is usually supplied to equipment and appliances through round wires made of a conducting material (copper, aluminum, silver, or gold) that are stranded or solid. The diameter of the wire determines the current-carrying capacity—the larger the diameter, the greater the current-carrying capacity. Even though the current-carrying capacity is determined by the diameter, wire is not normally characterized by the diameter directly. Instead, wire is commonly sold in a unit known as "gauge." Wires are manufactured by passing the material through circular forms called "drawing dies." In order to make thinner wires, manufacturers draw the wires through multiple dies of successively thinner diameter. Historically, the gauge of the wire was related to the number of drawing processes required to manufacture the wire. For this reason, the larger the gauge, the smaller the diameter. In the United States, the American Wire Gauge (AWG) was developed to standardize the system. Household wiring commonly consists of 10-gauge (2.588-mm diameter) to 14-gauge (1.628-mm diameter) wire. A device used to measure the gauge of wire is shown in Figure $5.3.5$.

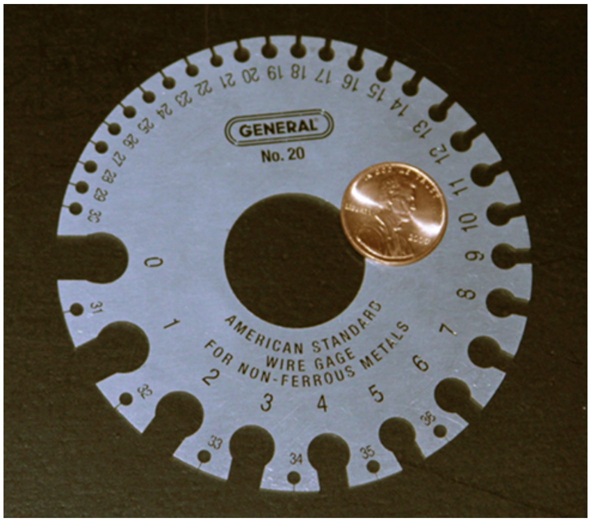


Figure 5.3.5: A device for measuring the gauge of electrical wire. As you can see, higher gauge numbers indicate thinner wires.

✔ Example 5.3.1: Calculating Drift Velocity in a Common Wire

Calculate the drift velocity of electrons in a copper wire with a diameter of 2.053 mm (12-gauge) carrying a 20.0-A current, given that there is one free electron per copper atom. (Household wiring often contains 12-gauge copper wire, and the maximum current allowed in such wire is usually 20.0 A.) The density of copper is $8.80 \times 10^3 kg/m^3$ and the atomic mass of copper is 63.54 g/mol.

Strategy

We can calculate the drift velocity using the equation $I = nqAv_d$. The current is $I = 20.00\,A$ and $q = 1.60 \times 10{-}19C$ is the charge of an electron. We can calculate the area of a cross-section of the wire using the formula $A = \pi r^2$, where r is one-half the diameter. The given diameter is 2.053 mm, so **r** is 1.0265 mm. We are given the density of copper, $8.80 \times 10^3 kg/m^3$, and the atomic mass of copper is $63.54\,g/mol$. We can use these two quantities along with Avogadro's number, $6.02 \times 10^{23}\,atoms/mol$, to determine **n**, the number of free electrons per cubic meter.

Solution

First, we calculate the density of free electrons in copper. There is one free electron per copper atom. Therefore, the number of free electrons is the same as the number of copper atoms per m^3. We can now find **n** as follows:

$$\begin{aligned} n &= \frac{1\,e^-}{atom} \times \frac{6.02 \times 10^{23} atoms}{mol} \times \frac{1\,mol}{63.54\,g} \times \frac{1000\,g}{kg} \times \frac{8.80 \times 10^3 kg}{1\,m^3} \\ &= 8.34 \times 10^{28} e^-/m^3. \end{aligned}$$

The cross-sectional area of the wire is

$$\begin{aligned} A &= \pi r^2 \\ &= \pi \left(\frac{2.05 \times 10^{-3} m}{2} \right)^2 \\ &= 3.30 \times 10^{-6} m^2. \end{aligned}$$

Rearranging $I = nqAv_d$ to isolate drift velocity gives

$$\begin{aligned} v_d &= \frac{I}{nqA} \\ &= \frac{20.00\,A}{(8.34 \times 10^{28}/m^3)(-1.60 \times 10^{-19} C)(3.30 \times 10^{-6} m^2)} \\ &= -4.54 \times 10^{-4} m/s. \end{aligned}$$

Significance

The minus sign indicates that the negative charges are moving in the direction opposite to conventional current. The small value for drift velocity (on the order of $10^{-4} m/s$ confirms that the signal moves on the order of 10^{12} times faster (about $10^{8} m/s$ than the charges that carry it.

Exercise 5.3.1

In Example 5.3.1, the drift velocity was calculated for a 2.053-mm diameter (12-gauge) copper wire carrying a 20-amp current. Would the drift velocity change for a 1.628-mm diameter (14-gauge) wire carrying the same 20-amp current?

Answer

The diameter of the 14-gauge wire is smaller than the diameter of the 12-gauge wire. Since the drift velocity is inversely proportional to the cross-sectional area, the drift velocity in the 14-gauge wire is larger than the drift velocity in the 12-gauge wire carrying the same current. The number of electrons per cubic meter will remain constant.

Current Density

Although it is often convenient to attach a negative or positive sign to indicate the overall direction of motion of the charges, current is a scalar quantity, $I = \frac{dQ}{dt}$. It is often necessary to discuss the details of the motion of the charge, instead of discussing the overall motion of the charges. In such cases, it is necessary to discuss the current density, $\vec{J}$, a vector quantity. The **current density** is the flow of charge through an infinitesimal area, divided by the area. The current density must take into account the local magnitude and direction of the charge flow, which varies from point to point. The unit of current density is ampere per meter squared, and the direction is defined as the direction of net flow of positive charges through the area.

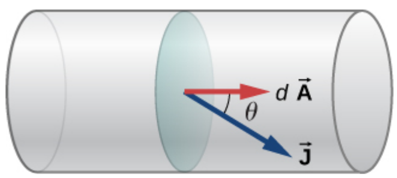


Figure 5.3.6: The current density $\vec{J}$ is defined as the current passing through an infinitesimal cross-sectional area divided by the area. The direction of the current density is the direction of the net flow of positive charges and the magnitude is equal to the current divided by the infinitesimal area.

The relationship between the current and the current density can be seen in Figure 5.3.6. The differential current flow through the area $d\vec{A}$ is found as

$$dI = \vec{J} \cdot d\vec{A} = JdA \cos \theta,$$

where θ is the angle between the area and the current density. The total current passing through area $d\vec{A}$ can be found by integrating over the area,

$$I = \iint_{area} \vec{J} \cdot d\vec{A}.$$

Consider the magnitude of the current density, which is the current divided by the area:

$$J = \frac{I}{A} = \frac{n|q|Av_d}{A} = n|q|v_d.$$

Thus, the current density is $\vec{J} = nq\vec{v}_d$. If **q** is positive, $\vec{v}_d$ is in the same direction as the electrical field $\vec{E}$. If **q** is negative, $\vec{v}_d$ is in the opposite direction of $\vec{E}$. Either way, the direction of the current density $\vec{J}$ is in the direction of the electrical field $\vec{E}$.

✓ Example 5.3.2: Calculating the Current Density in a Wire

The current supplied to a lamp with a 100-W light bulb is 0.87 amps. The lamp is wired using a copper wire with diameter 2.588 mm (10-gauge). Find the magnitude of the current density.

Strategy

The current density is the current moving through an infinitesimal cross-sectional area divided by the area. We can calculate the magnitude of the current density using $J = \frac{I}{A}$. The current is given as 0.87 A. The cross-sectional area can be calculated to be $A = 5.26\, mm^2$.

Solution

Calculate the current density using the given current $I = 0.87\, A$ and the area, found to be $A = 5.26\, mm^2$.

$$J = \frac{I}{A} = \frac{0.87\, A}{5.26 \times 10^{-6} m^2} = 1.65 \times 10^5 \frac{A}{m^2}.$$

Significance

The current density in a conducting wire depends on the current through the conducting wire and the cross-sectional area of the wire. For a given current, as the diameter of the wire increases, the charge density decreases.

? Exercise 5.3.2

The current density is proportional to the current and inversely proportional to the area. If the current density in a conducting wire increases, what would happen to the drift velocity of the charges in the wire?

Answer

The current density in a conducting wire increases due to an increase in current. The drift velocity is inversely proportional to the current $\left(v_d = \frac{nqA}{I}\right)$, so the drift velocity would decrease.

What is the significance of the current density? The current density is proportional to the current, and the current is the number of charges that pass through a cross-sectional area per second. The charges move through the conductor, accelerated by the electric force provided by the electrical field. The electrical field is created when a voltage is applied across the conductor. In Ohm's Law, we will use this relationship between the current density and the electrical field to examine the relationship between the current through a conductor and the voltage applied.

5.4: Resistivity and Resistance

Learning Objectives

By the end of this section, you will be able to:

- Differentiate between resistance and resistivity
- Define the term conductivity
- Describe the electrical component known as a resistor
- State the relationship between resistance of a resistor and its length, cross-sectional area, and resistivity
- State the relationship between resistivity and temperature

What drives current? We can think of various devices—such as batteries, generators, wall outlets, and so on—that are necessary to maintain a current. All such devices create a potential difference and are referred to as voltage sources. When a voltage source is connected to a conductor, it applies a potential difference **V** that creates an electrical field. The electrical field, in turn, exerts force on free charges, causing current. The amount of current depends not only on the magnitude of the voltage, but also on the characteristics of the material that the current is flowing through. The material can resist the flow of the charges, and the measure of how much a material resists the flow of charges is known as the **resistivity**. This resistivity is crudely analogous to the friction between two materials that resists motion.

Resistivity

When a voltage is applied to a conductor, an electrical field $\vec{E}$ is created, and charges in the conductor feel a force due to the electrical field. The current density $\vec{J}$ that results depends on the electrical field and the properties of the material. This dependence can be very complex. In some materials, including metals at a given temperature, the current density is approximately proportional to the electrical field. In these cases, the current density can be modeled as

$$\vec{J} = \sigma \vec{E},$$

where σ is the **electrical conductivity**. The electrical conductivity is analogous to thermal conductivity and is a measure of a material's ability to conduct or transmit electricity. Conductors have a higher electrical conductivity than insulators. Since the electrical conductivity is $\sigma = J/E$, the units are

$$\sigma = \frac{|J|}{|E|} = \frac{A/m^2}{V/m} = \frac{A}{V \cdot m}.$$

Here, we define a unit named the **ohm** with the Greek symbol uppercase omega, Ω. The unit is named after Georg Simon Ohm, whom we will discuss later in this chapter. The Ω is used to avoid confusion with the number 0. One ohm equals one volt per amp: $1\,\Omega = 1\,V/A$. The units of electrical conductivity are therefore $(\Omega \cdot m)^{-1}$.

Conductivity is an intrinsic property of a material. Another intrinsic property of a material is the resistivity, or electrical **resistivity**. The resistivity of a material is a measure of how strongly a material opposes the flow of electrical current. The symbol for resistivity is the lowercase Greek letter rho, ρ, and resistivity is the reciprocal of electrical conductivity:

$$\rho = \frac{1}{\sigma}.$$

The unit of resistivity in SI units is the ohm-meter $(\Omega \cdot m$. We can define the resistivity in terms of the electrical field and the current density.

$$\rho = \frac{E}{J}.$$

The greater the resistivity, the larger the field needed to produce a given current density. The lower the resistivity, the larger the current density produced by a given electrical field. Good conductors have a high conductivity and low resistivity. Good insulators have a low conductivity and a high resistivity. Table $5.4.1$ lists resistivity and conductivity values for various materials.

Table $5.4.1$: Resistivities and Conductivities of Various Materials at 20 °C[1] Values depend strongly on amounts and types of impurities.

Material	Conductivity, σ $(\Omega \cdot m)^{-1}$	Resistivity, ρ $(\Omega \cdot m)$	Temperature Coefficient α $(^oC)^{-1}$
Conductors			
Silver	6.29×10^7	1.59×10^{-8}	0.0038
Copper	5.95×10^7	1.68×10^{-8}	0.0039
Gold	4.10×10^7	2.44×10^{-8}	0.0034
Aluminum	3.77×10^7	2.65×10^{-8}	0.0039
Tungsten	1.79×10^7	5.60×10^{-8}	0.0045
Iron	1.03×10^7	9.71×10^{-8}	0.0065
Platinum	0.94×10^7	10.60×10^{-8}	0.0039
Steel	0.50×10^7	20.00×10^{-8}	
Lead	0.45×10^7	22.00×10^{-8}	
Manganin (Cu, Mn. Ni alloy)	0.21×10^7	48.20×10^{-8}	0.000002
Constantan (Cu, Ni alloy)	0.20×10^7	49.00×10^{-8}	0.00003
Mercury	0.10×10^7	98.00×10^{-8}	0.0009
Nichrome (Ni, Fe, Cr alloy)	0.10×10^7	100.00×10^{-8}	0.0004
Semiconductors [1]			
Carbon (pure)	2.86×10^4	3.50×10^{-5}	-0.0005
Carbon	$(2.86 - 1.67) \times 10^{-6}$	$(3.5 - 60) \times 10^{-5}$	-0.0005
Germanium (pure)		600×10^{-3}	-0.048
Germanium		$(1 - 600) \times 10^{-3}$	-0.050
Silicon (pure)		2300	-0.075
Silicon		0.1 - 2300	-0.07
Insulators			
Amber	2.00×10^{-15}	5×10^{14}	
Glass	$10^{-9} - 19^{-14}$	$10^9 - 10^{14}$	
Lucite	$< 10^{-13}$	$> 10^{13}$	
Mica	$10^{-11} - 10^{-15}$	$10^{11} - 10^{15}$	
Quartz (fused)	1.33×10^{-18}	75×10^{16}	
Rubber (hard)	$10^{-13} - 10^{-16}$	$10^{13} - 10^{16}$	
Sulfur	10^{-15}	10^{15}	
Teflon™	$< 10^{-13}$	$> 10^{13}$	
Wood	$10^{-8} - 10^{-11}$	$10^8 - 10^{11}$	

The materials listed in the table are separated into categories of conductors, semiconductors, and insulators, based on broad groupings of resistivity. Conductors have the smallest resistivity, and insulators have the largest; semiconductors have intermediate resistivity. Conductors have varying but large, free charge densities, whereas most charges in insulators are bound to atoms and are not free to move. Semiconductors are intermediate, having far fewer free charges than conductors, but having properties that make the number of free charges depend strongly on the type and amount of impurities in the semiconductor. These unique properties of semiconductors are put to use in modern electronics, as we will explore in later chapters.

Example 5.4.1: Current Density, Resistance, and Electrical field for a Current-Carrying Wire

Calculate the current density, resistance, and electrical field of a 5-m length of copper wire with a diameter of 2.053 mm (12-gauge) carrying a current of $I - 10\,mA$.

Strategy

We can calculate the current density by first finding the cross-sectional area of the wire, which is $A = 3.31\,mm^2$, and the definition of current density $J = \frac{I}{A}$. The resistance can be found using the length of the wire $L = 5.00\,m$, the area, and the resistivity of copper $\rho = 1.68 \times 10^{-8}\Omega \cdot m$, where $R = \rho\frac{L}{A}$. The resistivity and current density can be used to find the electrical field.

Solution

First, we calculate the current density:

$$\begin{aligned} J &= \frac{I}{A} \\ &= \frac{10 \times 10^{-3}\,A}{3.31 \times 10^{-6}\,m^2} \\ &= 3.02 \times 10^{3}\,\frac{A}{m^2}. \end{aligned}$$

The resistance of the wire is

$$\begin{aligned} R &= \rho\frac{L}{A} \\ &= (1.68 \times 10^{-8}\Omega \cdot m)\frac{5.00\,m}{3.31 \times 10^{-6}\,m^2} \\ &= 0.025\,\Omega. \end{aligned}$$

Finally, we can find the electrical field:

$$\begin{aligned} E &= \rho J \\ &= 1.68 \times 10^{-8}\Omega \cdot m\left(3.02 \times 10^{3}\,\frac{A}{m^2}\right) \\ &= 5.07 \times 10^{-5}\,\frac{V}{m}. \end{aligned}$$

Significance

From these results, it is not surprising that copper is used for wires for carrying current because the resistance is quite small. Note that the current density and electrical field are independent of the length of the wire, but the voltage depends on the length.

Exercise 5.4.1

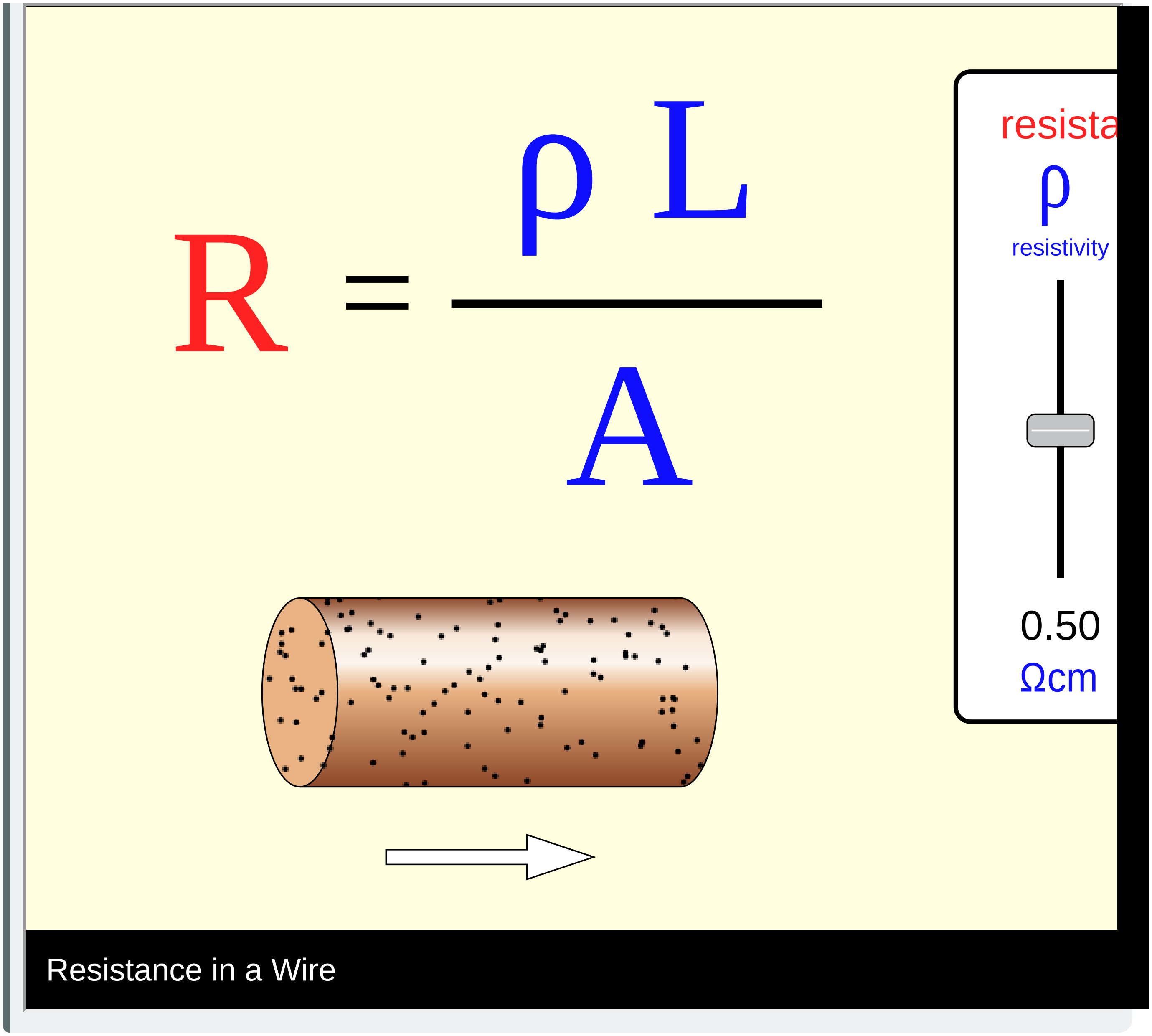
Copper wires use routinely used for extension cords and house wiring for several reasons. Copper has the highest electrical conductivity rating, and therefore the lowest resistivity rating, of all nonprecious metals. Also important is the tensile strength, where the tensile strength is a measure of the force required to pull an object to the point where it breaks. The tensile strength of a material is the maximum amount of tensile stress it can take before breaking. Copper has a high tensile strength, $2 \times 10^{8}\,\frac{N}{m^2}$. A third important characteristic is ductility. Ductility is a measure of a material's ability to be drawn into wires and a measure of the flexibility of the material, and copper has a high ductility. Summarizing, for a conductor to be a suitable candidate for making wire, there are at least three important characteristics: low resistivity, high tensile strength, and high ductility. What other materials are used for wiring and what are the advantages and disadvantages?

Answer

Silver, gold, and aluminum are all used for making wires. All four materials have a high conductivity, silver having the highest. All four can easily be drawn into wires and have a high tensile strength, though not as high as copper. The obvious disadvantage of gold and silver is the cost, but silver and gold wires are used for special applications, such as speaker wires. Gold does not oxidize, making better connections between components. Aluminum wires do have their drawbacks. Aluminum has a higher resistivity than copper, so a larger diameter is needed to match the resistance per length of copper wires, but aluminum is cheaper than copper, so this is not a major drawback. Aluminum wires do not have as high of a ductility and tensile strength as copper, but the ductility and tensile strength is within acceptable levels. There are a few concerns that must be addressed in using aluminum and care must be used when making connections. Aluminum has a higher rate of thermal expansion than copper, which can lead to loose connections and a possible fire hazard. The oxidation of aluminum does not conduct and can cause problems. Special techniques must be used when using aluminum wires and components, such as electrical outlets, must be designed to accept aluminum wires.

PhET

View this interactive simulation to see what the effects of the cross-sectional area, the length, and the resistivity of a wire are on the resistance of a conductor. Adjust the variables using slide bars and see if the resistance becomes smaller or larger.



Temperature Dependence of Resistivity

Looking back at Table 5.4.1, you will see a column labeled "Temperature Coefficient." The resistivity of some materials has a strong temperature dependence. In some materials, such as copper, the resistivity increases with increasing temperature. In fact, in most conducting metals, the resistivity increases with increasing temperature. The increasing temperature causes increased vibrations of the atoms in the lattice structure of the metals, which impede the motion of the electrons. In other materials, such as carbon, the resistivity decreases with increasing temperature. In many materials, the dependence is approximately linear and can be modeled using a linear equation:

$$\rho \approx \rho_0[1+\alpha(T-T_0)],$$

where ρ is the resistivity of the material at temperature **T**, α is the temperature coefficient of the material, and ρ_0 is the resistivity at T_0, usually taken as $T_0 = 20.00^o C$.

Note also that the temperature coefficient α is negative for the semiconductors listed in Table 5.4.1, meaning that their resistivity decreases with increasing temperature. They become better conductors at higher temperature, because increased thermal agitation increases the number of free charges available to carry current. This property of decreasing ρ with temperature is also related to the type and amount of impurities present in the semiconductors.

Resistance

We now consider the resistance of a wire or component. The resistance is a measure of how difficult it is to pass current through a wire or component. Resistance depends on the resistivity. The resistivity is a characteristic of the material used to fabricate a wire or other electrical component, whereas the resistance is a characteristic of the wire or component.

To calculate the resistance, consider a section of conducting wire with cross-sectional area **A**, length **L**, and resistivity ρ. A battery is connected across the conductor, providing a potential difference ΔV across it (Figure $5.4.1$). The potential difference produces an electrical field that is proportional to the current density, according to $\vec{E} = \rho \vec{J}$.

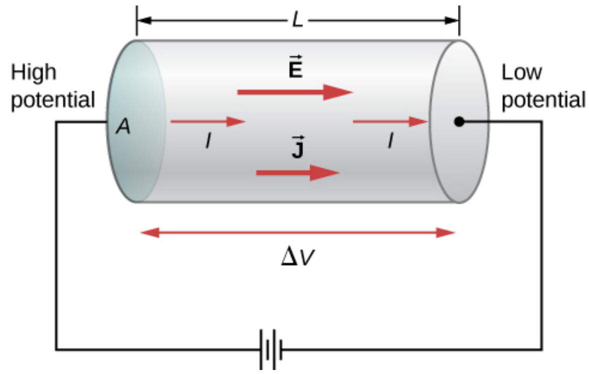


Figure $5.4.1$: A potential provided by a battery is applied to a segment of a conductor with a cross-sectional area A and a length L.

The magnitude of the electrical field across the segment of the conductor is equal to the voltage divided by the length, $E = V/L$, and the magnitude of the current density is equal to the current divided by the cross-sectional area, $J = I/A$. Using this information and recalling that the electrical field is proportional to the resistivity and the current density, we can see that the voltage is proportional to the current:

$$\begin{aligned} E &= \rho J \\ \frac{V}{L} &= \rho \frac{I}{A} \\ V &= \left(\rho \frac{L}{A}\right) I. \end{aligned}$$

Definition: Resistance

The ratio of the voltage to the current is defined as the **resistance** R:

$$R \equiv \frac{V}{I}.$$

The resistance of a cylindrical segment of a conductor is equal to the resistivity of the material times the length divided by the area:

$$R \equiv \frac{V}{I} = \rho \frac{L}{A}.$$

The unit of resistance is the ohm, Ω. For a given voltage, the higher the resistance, the lower the current.

Resistors

A common component in electronic circuits is the resistor. The resistor can be used to reduce current flow or provide a voltage drop. Figure $5.4.2$ shows the symbols used for a resistor in schematic diagrams of a circuit. Two commonly used standards for circuit diagrams are provided by the American National Standard Institute (ANSI, pronounced "AN-see") and the International Electrotechnical Commission (IEC). Both systems are commonly used. We use the ANSI standard in this text for its visual recognition, but we note that for larger, more complex circuits, the IEC standard may have a cleaner presentation, making it easier to read.

Figure 5.4.2: Symbols for a resistor used in circuit diagrams. (a) The ANSI symbol; (b) the IEC symbol.

Material and shape dependence of resistance

A resistor can be modeled as a cylinder with a cross-sectional area **A** and a length **L**, made of a material with a resistivity ρ (Figure 5.4.3). The resistance of the resistor is $R = \rho \frac{L}{A}$

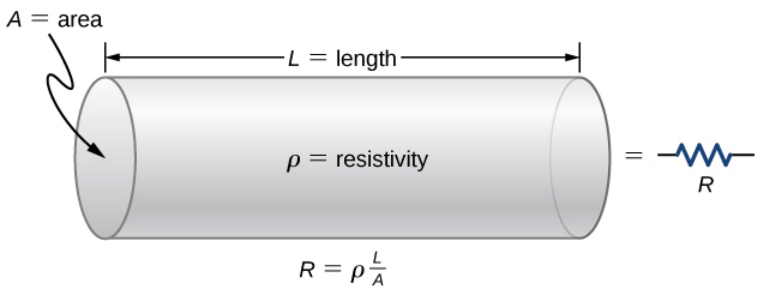


Figure 5.4.3: A model of a resistor as a uniform cylinder of length **L** and cross-sectional area **A**. Its resistance to the flow of current is analogous to the resistance posed by a pipe to fluid flow. The longer the cylinder, the greater its resistance. The larger its cross-sectional area **A**, the smaller its resistance.

The most common material used to make a resistor is carbon. A carbon track is wrapped around a ceramic core, and two copper leads are attached. A second type of resistor is the metal film resistor, which also has a ceramic core. The track is made from a metal oxide material, which has semiconductive properties similar to carbon. Again, copper leads are inserted into the ends of the resistor. The resistor is then painted and marked for identification. A resistor has four colored bands, as shown in Figure 5.4.4.

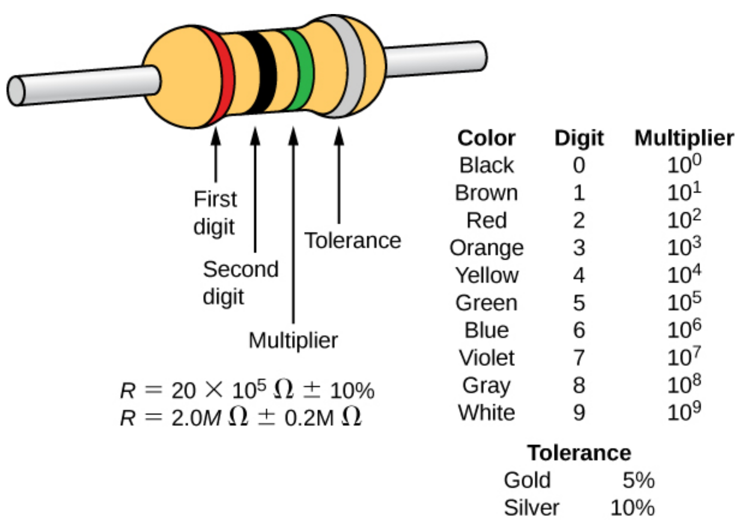


Figure 5.4.4: Many resistors resemble the figure shown above. The four bands are used to identify the resistor. The first two colored bands represent the first two digits of the resistance of the resistor. The third color is the multiplier. The fourth color represents the tolerance of the resistor. The resistor shown has a resistance of $20 \times 10^5\ \Omega \pm 10\%$

Resistances range over many orders of magnitude. Some ceramic insulators, such as those used to support power lines, have resistances of $10^{12}\ \Omega$ or more. A dry person may have a hand-to-foot resistance of $10^5\ \Omega$ whereas the resistance of the human heart is about $10^3\ \Omega$ A meter-long piece of large-diameter copper wire may have a resistance of $10^{-5}\ \Omega$, and superconductors have no resistance at all at low temperatures. As we have seen, resistance is related to the shape of an object and the material of which it is composed.

The resistance of an object also depends on temperature, since R_0 is directly proportional to ρ. For a cylinder, we know $R = \rho \frac{L}{A}$, so if **L** and **A** do not change greatly with temperature, **R** has the same temperature dependence as ρ. (Examination of the coefficients of linear expansion shows them to be about two orders of magnitude less than typical temperature coefficients of resistivity, so the effect of temperature on **L** and **A** is about two orders of magnitude less than on ρ.) Thus,

$$R = R_0(1 + \alpha \Delta T) \tag{5.4.1}$$

is the temperature dependence of the resistance of an object, where R_0 is the original resistance (usually taken to be $T = 20.00^o C$ and **R** is the resistance after a temperature change ΔT. The color code gives the resistance of the resistor at a temperature of $T = 20.00^o C$.

Numerous thermometers are based on the effect of temperature on resistance (Figure 5.4.5). One of the most common thermometers is based on the thermistor, a semiconductor crystal with a strong temperature dependence, the resistance of which is measured to obtain its temperature. The device is small, so that it quickly comes into thermal equilibrium with the part of a person it touches.

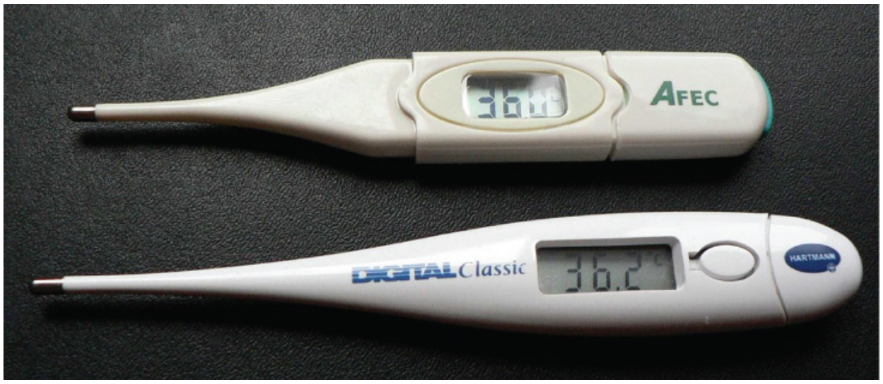


Figure 5.4.5: These familiar thermometers are based on the automated measurement of a thermistor's temperature-dependent resistance.

Example 5.4.2: Calculating Resistance

Although caution must be used in applying $\rho = \rho_0(1 + \alpha \Delta T)$ and $R = R_0(1 + \alpha \Delta T)$ for temperature changes greater than $100^o C$, for tungsten, the equations work reasonably well for very large temperature changes. A tungsten filament at $20^o C$ has a resistance of $0.350\,\Omega$. What would the resistance be if the temperature is increased to $2850^o C$?

Strategy

This is a straightforward application of Equation 5.4.1, since the original resistance of the filament is given as $R_0 = 0.350\,\Omega$ and the temperature change is $\Delta T = 2830^o C$.

Solution

The resistance of the hotter filament R is obtained by entering known values into the above equation:

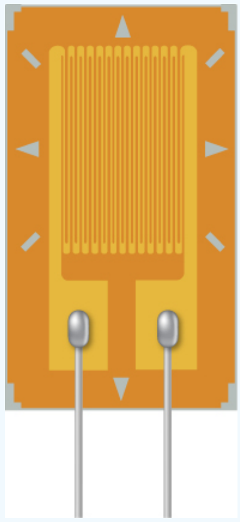
$$\begin{aligned} R &= R_0(1 + \alpha \Delta T) \\ &= (0.350\,\Omega)\left(1 + \left(\frac{4.5 \times 10^{-3}}{^oC}\right)(2830^o C)\right) \\ &= 4.8\,\Omega \end{aligned}$$

Significance

Notice that the resistance changes by more than a factor of 10 as the filament warms to the high temperature and the current through the filament depends on the resistance of the filament and the voltage applied. If the filament is used in an incandescent light bulb, the initial current through the filament when the bulb is first energized will be higher than the current after the filament reaches the operating temperature.

Exercise 5.4.2

A strain gauge is an electrical device to measure strain, as shown below. It consists of a flexible, insulating backing that supports a conduction foil pattern. The resistance of the foil changes as the backing is stretched. How does the strain gauge resistance change? Is the strain gauge affected by temperature changes?



Answer

The foil pattern stretches as the backing stretches, and the foil tracks become longer and thinner. Since the resistance is calculated as $R = \rho \frac{L}{A}$, the resistance increases as the foil tracks are stretched. When the temperature changes, so does the resistivity of the foil tracks, changing the resistance. One way to combat this is to use two strain gauges, one used as a reference and the other used to measure the strain. The two strain gauges are kept at a constant temperature

The Resistance of Coaxial Cable

Long cables can sometimes act like antennas, picking up electronic noise, which are signals from other equipment and appliances. Coaxial cables are used for many applications that require this noise to be eliminated. For example, they can be found in the home in cable TV connections or other audiovisual connections. Coaxial cables consist of an inner conductor of radius r_i surrounded by a second, outer concentric conductor with radius r_0 (Figure 5.4.6). The space between the two is normally filled with an insulator such as polyethylene plastic. A small amount of radial leakage current occurs between the two conductors. Determine the resistance of a coaxial cable of length **L**.

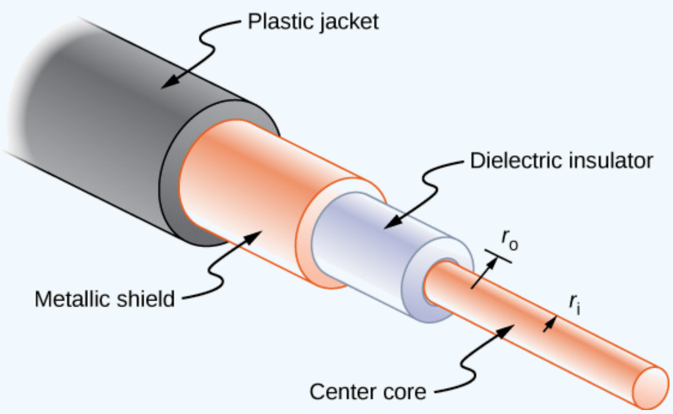


Figure 5.4.6: Coaxial cables consist of two concentric conductors separated by insulation. They are often used in cable TV or other audiovisual connections.

Strategy

We cannot use the equation $R = \rho \frac{L}{A}$ directly. Instead, we look at concentric cylindrical shells, with thickness **dr**, and integrate.

Solution

We first find an expression for dR and then integrate from r_i to r_0,

$$\begin{aligned} dR &= \frac{\rho}{A} dr \\ &= \frac{\rho}{2\pi r L} dr, \end{aligned}$$

Integrating both sides

$$\begin{aligned} R &= \int_{r_i}^{r_0} dR \\ &= \int_{r_i}^{r_0} \frac{\rho}{2\pi r L} dr \\ &= \frac{\rho}{2\pi L} \int_{r_i}^{r_0} \frac{1}{r} dr \\ &= \frac{\rho}{2\pi L} \ln \frac{r_0}{r_i}. \end{aligned}$$

Significance

The resistance of a coaxial cable depends on its length, the inner and outer radii, and the resistivity of the material separating the two conductors. Since this resistance is not infinite, a small leakage current occurs between the two conductors. This leakage current leads to the attenuation (or weakening) of the signal being sent through the cable.

Exercise 5.4.3

The resistance between the two conductors of a coaxial cable depends on the resistivity of the material separating the two conductors, the length of the cable and the inner and outer radius of the two conductor. If you are designing a coaxial cable, how does the resistance between the two conductors depend on these variables?

Answer

The longer the length, the smaller the resistance. The greater the resistivity, the higher the resistance. The larger the difference between the outer radius and the inner radius, that is, the greater the ratio between the two, the greater the resistance. If you are attempting to maximize the resistance, the choice of the values for these variables will depend on the application. For example, if the cable must be flexible, the choice of materials may be limited.

Phet: Battery-Resistor Circuit

View this simulation to see how the voltage applied and the resistance of the material the current flows through affects the current through the material. You can visualize the collisions of the electrons and the atoms of the material effect the temperature of the material.

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5.5: Ohm's Law

Learning Objectives

By the end of this section, you will be able to:

- Describe Ohm's law
- Recognize when Ohm's law applies and when it does not

We have been discussing three electrical properties so far in this chapter: current, voltage, and resistance. It turns out that many materials exhibit a simple relationship among the values for these properties, known as Ohm's law. Many other materials do not show this relationship, so despite being called Ohm's law, it is not considered a law of nature, like Newton's laws or the laws of thermodynamics. But it is very useful for calculations involving materials that do obey Ohm's law.

Description of Ohm's Law

The current that flows through most substances is directly proportional to the voltage V applied to it. The German physicist Georg Simon Ohm (1787–1854) was the first to demonstrate experimentally that the current in a metal wire is **directly proportional to the voltage applied**:

$$I \propto V.$$

This important relationship is the basis for **Ohm's law**. It can be viewed as a cause-and-effect relationship, with voltage the cause and current the effect. This is an empirical law, which is to say that it is an experimentally observed phenomenon, like friction. Such a linear relationship doesn't always occur. Any material, component, or device that obeys Ohm's law, where the current through the device is proportional to the voltage applied, is known as an ohmic material or **ohmic** component. Any material or component that does not obey Ohm's law is known as a **nonohmic** material or nonohmic component.

Ohm's Experiment

In a paper published in 1827, Georg Ohm described an experiment in which he measured voltage across and current through various simple electrical circuits containing various lengths of wire. A similar experiment is shown in Figure $5.5.1$. This experiment is used to observe the current through a resistor that results from an applied voltage. In this simple circuit, a resistor is connected in series with a battery. The voltage is measured with a voltmeter, which must be placed across the resistor (in parallel with the resistor). The current is measured with an ammeter, which must be in line with the resistor (in series with the resistor).

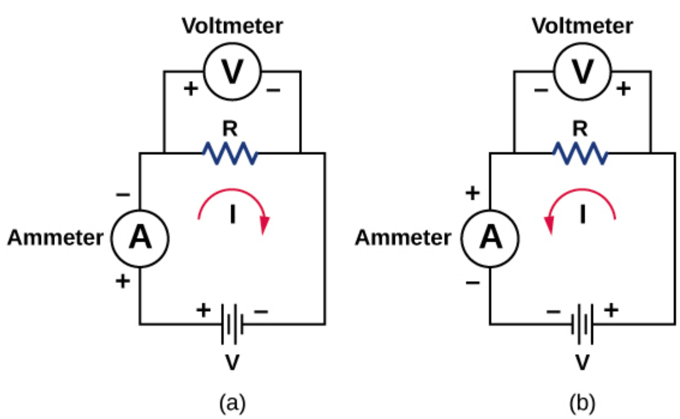


Figure $5.5.1$: The experimental set-up used to determine if a resistor is an ohmic or nonohmic device. (a) When the battery is attached, the current flows in the clockwise direction and the voltmeter and ammeter have positive readings. (b) When the leads of the battery are switched, the current flows in the counterclockwise direction and the voltmeter and ammeter have negative readings.

In this updated version of Ohm's original experiment, several measurements of the current were made for several different voltages. When the battery was hooked up as in Figure $5.5.1a$, the current flowed in the clockwise direction and the readings of the voltmeter and ammeter were positive. Does the behavior of the current change if the current flowed in the opposite direction? To get the current to flow in the opposite direction, the leads of the battery can be switched. When the leads of the battery were switched, the readings of the voltmeter and ammeter readings were negative because the current flowed in the opposite direction, in this case, counterclockwise. Results of a similar experiment are shown in Figure 5.5.2.

I(A)	V(V)
–2.74	–10.00
–2.25	–9.00
–2.00	–8.00
–1.75	–7.00
–1.50	–6.00
–1.49	–5.00
–1.00	–4.00
–0.51	–3.00
–0.74	–2.00
–0.49	–1.00
+0.00	+0.00
+0.49	+1.00
+0.50	+2.00
+0.99	+3.00
+0.76	+4.00
+1.01	+5.00
+1.74	+6.00
+1.75	+7.00
+2.00	+8.00
+2.49	+9.00
+2.50	+10.00

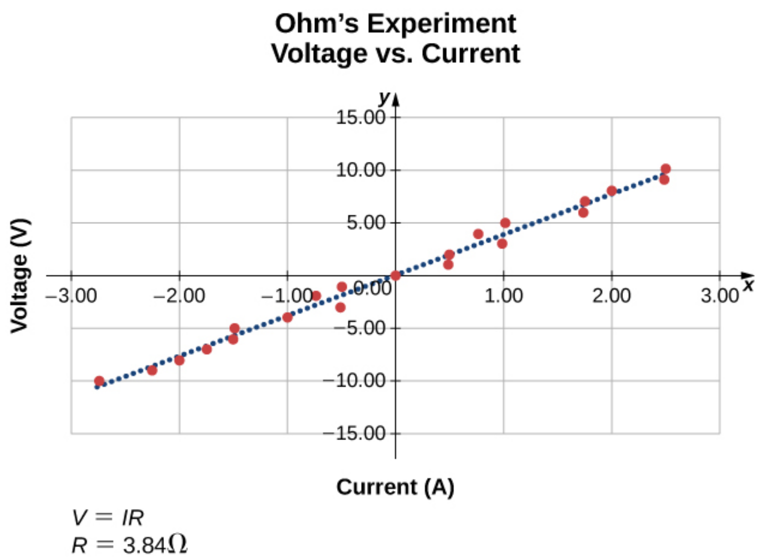


Figure 5.5.1: A resistor is placed in a circuit with a battery. The voltage applied varies from −10.00 V to +10.00 V, increased by 1.00-V increments. A plot shows values of the voltage versus the current typical of what a casual experimenter might find.

In this experiment, the voltage applied across the resistor varies from −10.00 to +10.00 V, by increments of 1.00 V. The current through the resistor and the voltage across the resistor are measured. A plot is made of the voltage versus the current, and the result is approximately linear. The slope of the line is the resistance, or the voltage divided by the current. This result is known as **Ohm's law**:

$$V = IR \tag{5.5.1}$$

where V is the voltage measured in volts across the object in question, I is the current measured through the object in amps, and R is the resistance in units of ohms. As stated previously, any device that shows a linear relationship between the voltage and the current is known as an ohmic device. A resistor is therefore an ohmic device.

Example 5.5.1: Measuring Resistance

A carbon resistor at room temperature (20^oC) is attached to a 9.00-V battery and the current measured through the resistor is 3.00 mA.

a. What is the resistance of the resistor measured in ohms?
b. If the temperature of the resistor is increased to 60^oC by heating the resistor, what is the current through the resistor?

Strategy

a. The resistance can be found using Ohm's law. Ohm's law states that $V = IR$, so the resistance can be found using $R = V/I$.
b. First, the resistance is temperature dependent so the new resistance after the resistor has been heated can be found using $R = R_0(1+\alpha\Delta T)$. The current can be found using Ohm's law in the form $I = V/R$.

Solution

a. Using Ohm's law and solving for the resistance yields the resistance at room temperature:

$$R = \frac{V}{I} = \frac{9.00\,V}{3.00\times 10^{-3}\,A} = 3.00\times 10^3\,\Omega = 3.00 k\Omega$$

b. The resistance at 60^oC can be found using $R = R_0(1+\alpha\Delta T)$ where the temperature coefficient for carbon is $\alpha = -0.0005$.

$$R = R_0(1+\alpha\Delta T) = 3.00\times 10^3(1-0.0005(60^oC-20^oC)) = 2.94\,k\Omega.$$

The current through the heated resistor is

$$I = \frac{V}{R} = \frac{9.00\,V}{2.94 \times 10^3\,\Omega} = 3.06 \times 10^{-3}\,A = 3.06\,mA.$$

Significance

A change in temperature of $40^o C$ resulted in a 2.00% change in current. This may not seem like a very great change, but changing electrical characteristics can have a strong effect on the circuits. For this reason, many electronic appliances, such as computers, contain fans to remove the heat dissipated by components in the electric circuits.

? Exercise 5.5.1

The voltage supplied to your house varies as $V(t) = V_{max} sin\,(2\pi\,ft)$. If a resistor is connected across this voltage, will Ohm's law $V = IR$ still be valid?

Answer

Yes, Ohm's law is still valid. At every point in time the current is equal to $I(t) = V(t)/R$, so the current is also a function of time, $I(t) = \frac{V_{max}}{R} sin\,(2\pi\,ft)$.

Simulation: PhET

See how Ohm's law (Equation 5.5.1) relates to a simple circuit. Adjust the voltage and resistance, and see the current change according to Ohm's law. The sizes of the symbols in the equation change to match the circuit diagram.

Nonohmic devices do not exhibit a linear relationship between the voltage and the current. One such device is the semiconducting circuit element known as a **diode**. A diode is a circuit device that allows current flow in only one direction. A diagram of a simple circuit consisting of a battery, a diode, and a resistor is shown in Figure 5.5.3. Although we do not cover the theory of the diode in this section, the diode can be tested to see if it is an ohmic or a nonohmic device.

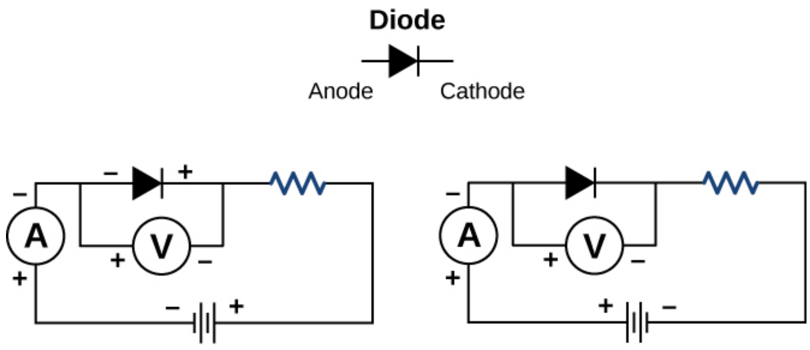


Figure 5.5.3: A diode is a semiconducting device that allows current flow only if the diode is forward biased, which means that the anode is positive and the cathode is negative.

A plot of current versus voltage is shown in Figure 5.5.4. Note that the behavior of the diode is shown as current versus voltage, whereas the resistor operation was shown as voltage versus current. A diode consists of an anode and a cathode. When the anode is at a negative potential and the cathode is at a positive potential, as shown in part (a), the diode is said to have reverse bias. With reverse bias, the diode has an extremely large resistance and there is very little current flow—essentially zero current—through the diode and the resistor. As the voltage applied to the circuit increases, the current remains essentially zero, until the voltage reaches the breakdown voltage and the diode conducts current. When the battery and the potential across the diode are reversed, making the anode positive and the cathode negative, the diode conducts and current flows through the diode if the voltage is greater than 0.7 V. The resistance of the diode is close to zero. (This is the reason for the resistor in the circuit; if it were not there, the current would become very large.) You can see from the graph in Figure 5.5.4 that the voltage and the current do not have a linear relationship. Thus, the diode is an example of a nonohmic device.

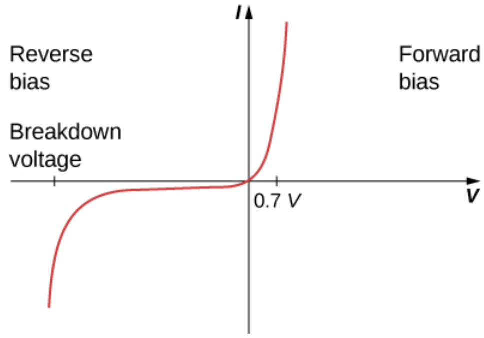


Figure 5.5.4: When the voltage across the diode is negative and small, there is very little current flow through the diode. As the voltage reaches the breakdown voltage, the diode conducts. When the voltage across the diode is positive and greater than 0.7 V (the actual voltage value depends on the diode), the diode conducts. As the voltage applied increases, the current through the diode increases, but the voltage across the diode remains approximately 0.7 V.

Ohm's law is commonly stated as $V = IR$, but originally it was stated as a microscopic view, in terms of the current density, the conductivity, and the electrical field. This microscopic view suggests the proportionality $V \propto I$ comes from the drift velocity of the free electrons in the metal that results from an applied electrical field. As stated earlier, the current density is proportional to the applied electrical field. The reformulation of Ohm's law is credited to Gustav Kirchhoff, whose name we will see again in the next chapter.

5.6: Electrical Energy and Power

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Express electrical power in terms of the voltage and the current
- Describe the power dissipated by a resistor in an electric circuit
- Calculate the energy efficiency and cost effectiveness of appliances and equipment

In an electric circuit, electrical energy is continuously converted into other forms of energy. For example, when a current flows in a conductor, electrical energy is converted into thermal energy within the conductor. The electrical field, supplied by the voltage source, accelerates the free electrons, increasing their kinetic energy for a short time. This increased kinetic energy is converted into thermal energy through collisions with the ions of the lattice structure of the conductor. Previously, we defined power as the rate at which work is done by a force measured in watts. Power can also be defined as the rate at which energy is transferred. In this section, we discuss the time rate of energy transfer, or power, in an electric circuit.

Power in Electric Circuits

Power is associated by many people with electricity. Power transmission lines might come to mind. We also think of light bulbs in terms of their power ratings in watts. What is the expression for **electric power**?

Let us compare a 25-W bulb with a 60-W bulb (Figure $5.6.1a$). The 60-W bulb glows brighter than the 25-W bulb. Although it is not shown, a 60-W light bulb is also warmer than the 25-W bulb. The heat and light is produced by from the conversion of electrical energy. The kinetic energy lost by the electrons in collisions is converted into the internal energy of the conductor and radiation. How are voltage, current, and resistance related to electric power?

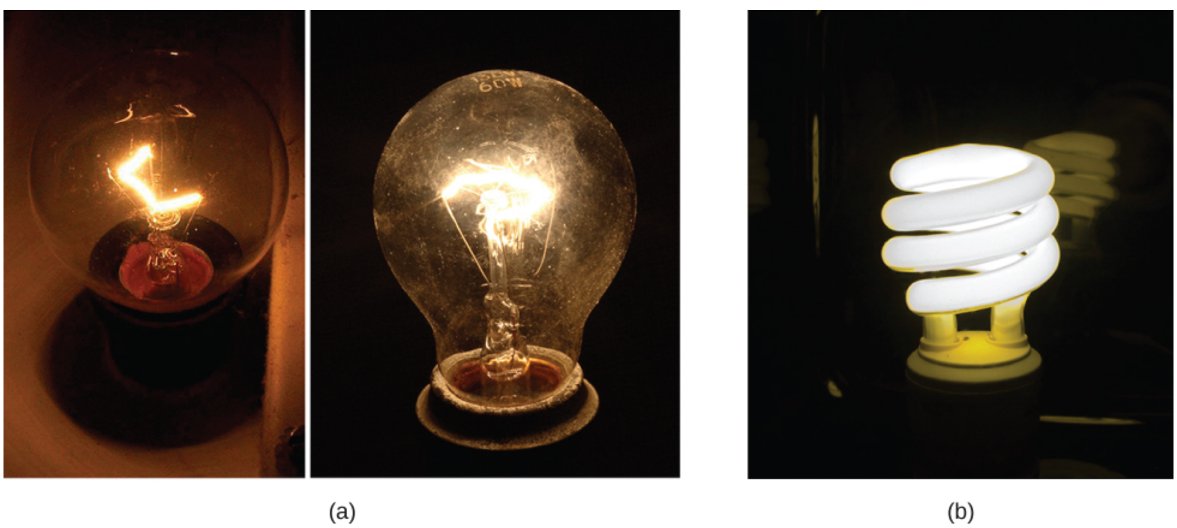


Figure $5.6.1$: (a) Pictured above are two incandescent bulbs: a 25-W bulb (left) and a 60-W bulb (right). The 60-W bulb provides a higher intensity light than the 25-W bulb. The electrical energy supplied to the light bulbs is converted into heat and light. (b) This compact fluorescent light (CFL) bulb puts out the same intensity of light as the 60-W bulb, but at 1/4 to 1/10 the input power. (credit a: modification of works by “Dickbauch”/Wikimedia Commons and Greg Westfall; credit b: modification of work by “dbgg1979”/Flickr)

To calculate electric power, consider a voltage difference existing across a material (Figure $5.6.2$). The electric potential V_1 is higher than the electric potential at V_2, and the voltage difference is negative $V = V_2 - V_1$. As discussed in Electric Potential, an electrical field exists between the two potentials, which points from the higher potential to the lower potential. Recall that the electrical potential is defined as the potential energy per charge, $V = \Delta U/q$, and the charge ΔQ loses potential energy moving through the potential difference.

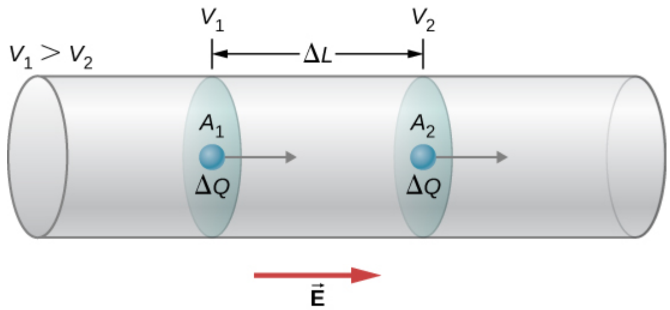


Figure 5.6.2: When there is a potential difference across a conductor, an electrical field is present that points in the direction from the higher potential to the lower potential.

If the charge is positive, the charge experiences a force due to the electrical field $\vec{F} = m\vec{a} = \Delta Q\vec{E}$. This force is necessary to keep the charge moving. This force does not act to accelerate the charge through the entire distance ΔL because of the interactions of the charge with atoms and free electrons in the material. The speed, and therefore the kinetic energy, of the charge do not increase during the entire trip across ΔL, and charge passing through area A_2 has the same drift velocity v_d as the charge that passes through area A_1. However, work is done on the charge, by the electrical field, which changes the potential energy. Since the change in the electrical potential difference is negative, the electrical field is found to be

$$E = -\frac{(V_2 - V_1)}{\Delta L} = \frac{V}{\Delta L}.$$

The work done on the charge is equal to the electric force times the length at which the force is applied,

$$W = F\Delta L = (\Delta QE)\Delta L = \left(\Delta Q\frac{V}{\Delta L}\right)\Delta L = \Delta QV = \Delta U.$$

The charge moves at a drift velocity v_d so the work done on the charge results in a loss of potential energy, but the average kinetic energy remains constant. The lost electrical potential energy appears as thermal energy in the material. On a microscopic scale, the energy transfer is due to collisions between the charge and the molecules of the material, which leads to an increase in temperature in the material. The loss of potential energy results in an increase in the temperature of the material, which is dissipated as radiation. In a resistor, it is dissipated as heat, and in a light bulb, it is dissipated as heat and light.

The power dissipated by the material as heat and light is equal to the time rate of change of the work:

$$P = IV = I(IR) = I^2R$$

or

$$P = IV = \left(\frac{V}{R}\right)V = \frac{V^2}{R}.$$

If a resistor is connected to a battery, the power dissipated as radiant energy by the wires and the resistor is equal to

$$P = IV = I^2R = \frac{V^2}{R}.$$

The power supplied from the battery is equal to current times the voltage, $P = IV$.

Definition: Electric Power

The electric power gained or lost by any device has the form

$$P = IV.$$

The power dissipated by a resistor has the form

$$P = I^2R = \frac{V^2}{R}.$$

Different insights can be gained from the three different expressions for electric power. For example, $P = V^2/R$ implies that the lower the resistance connected to a given voltage source, the greater the power delivered. Furthermore, since voltage is squared in $P = V^2/R$, the effect of applying a higher voltage is perhaps greater than expected. Thus, when the voltage is doubled to a 25-W bulb, its power nearly quadruples to about 100 W, burning it out. If the bulb's resistance remained constant, its power would be exactly 100 W, but at the higher temperature, its resistance is higher, too.

Example 5.6.1: Calculating Power in Electric Devices

A DC winch motor is rated at 20.00 A with a voltage of 115 V. When the motor is running at its maximum power, it can lift an object with a weight of 4900.00 N a distance of 10.00 m, in 30.00 s, at a constant speed.

a. What is the power consumed by the motor?
b. What is the power used in lifting the object? Ignore air resistance. (c) Assuming that the difference in the power consumed by the motor and the power used lifting the object are dissipated as heat by the resistance of the motor, estimate the resistance of the motor?

Strategy

a. The power consumed by the motor can be found using $P = IV$.
b. The power used in lifting the object at a constant speed can be found using $P = Fv$, where the speed is the distance divided by the time. The upward force supplied by the motor is equal to the weight of the object because the acceleration is zero. (c) The resistance of the motor can be found using $P = I^2R$.

Solution

1. The power consumed by the motor is equal to $P = IV$ and the current is given as 20.00 A and the voltage is 115.00 V:

$$P = IV = (20.00\,A)115.00\,V = 2300.00\,W.$$

2. The power used lifting the object is equal to $P = Fv$ where the force is equal to the weight of the object (1960 N) and the magnitude of the velocity is

$$v = \frac{10.00\,m}{30.00\,s} = 0.33\frac{m}{s}$$

$$P = Fv = (4900\,N)0.33\,m/s = 1633.33\,W.$$

3. The difference in the power equals $2300.00\,W - 1633.33\,W = 666.67\,W$and the resistance can be found using $P = I^2R$:

$$R = \frac{P}{I^2} = \frac{666.67\,W}{(20.00\,A)^2} = 1.67\,\Omega.$$

Significance The resistance of the motor is quite small. The resistance of the motor is due to many windings of copper wire. The power dissipated by the motor can be significant since the thermal power dissipated by the motor is proportional to the square of the current $(P = I^2R)$.

Exercise 5.6.1

Electric motors have a reasonably high efficiency. A 100-hp motor can have an efficiency of 90% and a 1-hp motor can have an efficiency of 80%. Why is it important to use high-performance motors?

Answer

Even though electric motors are highly efficient 10–20% of the power consumed is wasted, not being used for doing useful work. Most of the 10–20% of the power lost is transferred into heat dissipated by the copper wires used to make the coils of the motor. This heat adds to the heat of the environment and adds to the demand on power plants providing the power. The demand on the power plant can lead to increased greenhouse gases, particularly if the power plant uses coal or gas as fuel.

A fuse (Figure 5.6.3) is a device that protects a circuit from currents that are too high. A fuse is basically a short piece of wire between two contacts. As we have seen, when a current is running through a conductor, the kinetic energy of the charge carriers is converted into thermal energy in the conductor. The piece of wire in the fuse is under tension and has a low melting point. The wire is designed to heat up and break at the rated current. The fuse is destroyed and must be replaced, but it protects the rest of the circuit. Fuses act quickly, but there is a small time delay while the wire heats up and breaks.

Figure $5.6.3$: A fuse consists of a piece of wire between two contacts. When a current passes through the wire that is greater than the rated current, the wire melts, breaking the connection. Pictured is a "blown" fuse where the wire broke protecting a circuit (credit: modification of work by "Shardayyy"/Flickr).

Circuit breakers are also rated for a maximum current, and open to protect the circuit, but can be reset. Circuit breakers react much faster. The operation of circuit breakers is not within the scope of this chapter and will be discussed in later chapters. Another method of protecting equipment and people is the ground fault circuit interrupter (GFCI), which is common in bathrooms and kitchens. The GFCI outlets respond very quickly to changes in current. These outlets open when there is a change in magnetic field produced by current-carrying conductors, which is also beyond the scope of this chapter and is covered in a later chapter.

The Cost of Electricity

The more electric appliances you use and the longer they are left on, the higher your electric bill. This familiar fact is based on the relationship between energy and power. You pay for the energy used. Since $P = \frac{dE}{dt}$, we see that

$$E = \int P dt$$

is the energy used by a device using power **P** for a time interval **t**. If power is delivered at a constant rate, then then the energy can be found by $E = Pt$. For example, the more light bulbs burning, the greater **P** used; the longer they are on, the greater **t** is.

The energy unit on electric bills is the kilowatt-hour $(kW \cdot h)$, consistent with the relationship $E = Pt$. It is easy to estimate the cost of operating electrical appliances if you have some idea of their power consumption rate in watts or kilowatts, the time they are on in hours, and the cost per kilowatt-hour for your electric utility. Kilowatt-hours, like all other specialized energy units such as food calories, can be converted into joules. You can prove to yourself that $1\, kW \cdot h = 3.6 \times 10^6\, J$.

The electrical energy (**E**) used can be reduced either by reducing the time of use or by reducing the power consumption of that appliance or fixture. This not only reduces the cost but also results in a reduced impact on the environment. Improvements to lighting are some of the fastest ways to reduce the electrical energy used in a home or business. About 20% of a home's use of energy goes to lighting, and the number for commercial establishments is closer to 40%. Fluorescent lights are about four times more efficient than incandescent lights—this is true for both the long tubes and the compact fluorescent lights (CFLs), e.g., Figure $5.6.1b$. Thus, a 60-W incandescent bulb can be replaced by a 15-W CFL, which has the same brightness and color. CFLs have a bent tube inside a globe or a spiral-shaped tube, all connected to a standard screw-in base that fits standard incandescent light sockets. (Original problems with color, flicker, shape, and high initial investment for CFLs have been addressed in recent years.)

The heat transfer from these CFLs is less, and they last up to 10 times longer than incandescent bulbs. The significance of an investment in such bulbs is addressed in the next example. New white LED lights (which are clusters of small LED bulbs) are even more efficient (twice that of CFLs) and last five times longer than CFLs.

Example \(\PageIndex{2\): Calculating the Cost Effectiveness of LED Bulb

The typical replacement for a 100-W incandescent bulb is a 20-W LED bulb. The 20-W LED bulb can provide the same amount of light output as the 100-W incandescent light bulb. What is the cost savings for using the LED bulb in place of the incandescent bulb for one year, assuming $0.10 per kilowatt-hour is the average energy rate charged by the power company? Assume that the bulb is turned on for three hours a day.

Strategy

a. Calculate the energy used during the year for each bulb, using $E = Pt$.
b. Multiply the energy by the cost.

Solution

a. Calculate the power for each bulb.

$$E_{Incandescent} = Pt = 100\,W\left(\frac{1\,kW}{1000\,W}\right)\left(\frac{3\,h}{day}\right)(365\,days) = 109.5\,kW \cdot h$$

$$E_{LED} = Pt = 20\,W\left(\frac{1\,kW}{1000\,W}\right)\left(\frac{3\,h}{day}\right)(365\,days) = 21.9\,kW \cdot h$$

b. Calculate the cost for each.

$$cost_{Incandescent} = 109.5\,kW \cdot h\left(\frac{\$0.10}{kW \cdot h}\right) = \$10.95$$

$$cost_{LED} = 21.90\,kW \cdot h\left(\frac{\$0.10}{kW \cdot h}\right) = \$2.19$$

Significance

A LED bulb uses 80% less energy than the incandescent bulb, saving $8.76 over the incandescent bulb for one year. The LED bulb can cost $20.00 and the 100-W incandescent bulb can cost $0.75, which should be calculated into the computation. A typical lifespan of an incandescent bulb is 1200 hours and is 50,000 hours for the LED bulb. The incandescent bulb would last 1.08 years at 3 hours a day and the LED bulb would last 45.66 years. The initial cost of the LED bulb is high, but the cost to the home owner will be $0.69 for the incandescent bulbs versus $0.44 for the LED bulbs per year. (Note that the LED bulbs are coming down in price.) The cost savings per year is approximately $8.50, and that is just for one bulb.

Exercise 5.6.2

Is the efficiency of the various light bulbs the only consideration when comparing the various light bulbs?

Answer

No, the efficiency is a very important consideration of the light bulbs, but there are many other considerations. As mentioned above, the cost of the bulbs and the life span of the bulbs are important considerations. For example, CFL bulbs contain mercury, a neurotoxin, and must be disposed of as hazardous waste. When replacing incandescent bulbs that are being controlled by a dimmer switch with LED, the dimmer switch may need to be replaced. The dimmer switches for LED lights are comparably priced to the incandescent light switches, but this is an initial cost which should be considered. The spectrum of light should also be considered, but there is a broad range of color temperatures available, so you should be able to find one that fits your needs. None of these considerations mentioned are meant to discourage the use of LED or CFL light bulbs, but they are considerations.

Changing light bulbs from incandescent bulbs to CFL or LED bulbs is a simple way to reduce energy consumption in homes and commercial sites. CFL bulbs operate with a much different mechanism than do incandescent lights. The mechanism is complex and beyond the scope of this chapter, but here is a very general description of the mechanism. CFL bulbs contain argon and mercury vapor housed within a spiral-shaped tube. The CFL bulbs use a "ballast" that increases the voltage used by the CFL bulb. The ballast produce an electrical current, which passes through the gas mixture and excites the gas molecules. The excited gas molecules produce ultraviolet (UV) light, which in turn stimulates the fluorescent coating on the inside of the tube. This coating fluoresces in the visible spectrum, emitting visible light. Traditional fluorescent tubes and CFL bulbs had a short time delay of up to a few seconds while the mixture was being "warmed up" and the molecules reached an excited state. It should be noted that these bulbs do contain mercury, which is poisonous, but if the bulb is broken, the mercury is never released. Even if the bulb is broken, the mercury tends to remain in the fluorescent coating. The amount is also quite small and the advantage of the energy saving may outweigh the disadvantage of using mercury.

The CFL light bulbs are being replaced with LED light bulbs, where LED stands for “light-emitting diode.” The diode was briefly discussed as a nonohmic device, made of semiconducting material, which essentially permits current flow in one direction. LEDs are a special type of diode made of semiconducting materials infused with impurities in combinations and concentrations that enable the extra energy from the movement of the electrons during electrical excitation to be converted into visible light. Semiconducting devices will be explained in greater detail in Condensed Matter Physics.

Commercial LEDs are quickly becoming the standard for commercial and residential lighting, replacing incandescent and CFL bulbs. They are designed for the visible spectrum and are constructed from gallium doped with arsenic and phosphorous atoms. The color emitted from an LED depends on the materials used in the semiconductor and the current. In the early years of LED development, small LEDs found on circuit boards were red, green, and yellow, but LED light bulbs can now be programmed to produce millions of colors of light as well as many different hues of white light.

Comparison of Incandescent, CFL, and LED Light Bulbs

The energy savings can be significant when replacing an incandescent light bulb or a CFL light bulb with an LED light. Light bulbs are rated by the amount of power that the bulb consumes, and the amount of light output is measured in lumens. The lumen (lm) is the SI -derived unit of luminous flux and is a measure of the total quantity of visible light emitted by a source. A 60-W incandescent light bulb can be replaced with a 13- to 15-W CFL bulb or a 6- to 8-W LED bulb, all three of which have a light output of approximately 800 lm. A table of light output for some commonly used light bulbs appears in Table $5.6.1$.

The life spans of the three types of bulbs are significantly different. An LED bulb has a life span of 50,000 hours, whereas the CFL has a lifespan of 8000 hours and the incandescent lasts a mere 1200 hours. The LED bulb is the most durable, easily withstanding rough treatment such as jarring and bumping. The incandescent light bulb has little tolerance to the same treatment since the filament and glass can easily break. The CFL bulb is also less durable than the LED bulb because of its glass construction. The amount of heat emitted is 3.4 btu/h for the 8-W LED bulb, 85 btu/h for the 60-W incandescent bulb, and 30 btu/h for the CFL bulb. As mentioned earlier, a major drawback of the CFL bulb is that it contains mercury, a neurotoxin, and must be disposed of as hazardous waste. From these data, it is easy to understand why the LED light bulb is quickly becoming the standard in lighting.

Table $5.6.1$: Light Output of LED, Incandescent, and CFL Light Bulbs

Light Output (lumens)	LED Light Bulbs (watts)	Incandescent Light Bulbs (watts)	CFL Light Bulbs (watts)
450	4−5	40	9−13
800	6−8	60	13−15
1100	9−13	75	18−25
1600	16−20	100	23−30
2600	25−28	150	30−55

Summary of Relationships

In this chapter, we have discussed relationships between voltages, current, resistance, and power. Figure $5.6.4$ shows a summary of the relationships between these measurable quantities for ohmic devices. (Recall that ohmic devices follow Ohm’s law $V = IR$.) For example, if you need to calculate the power, use the pink section, which shows that $P = VI$, $P = \frac{V^2}{R}$, and $P = I^2R$.

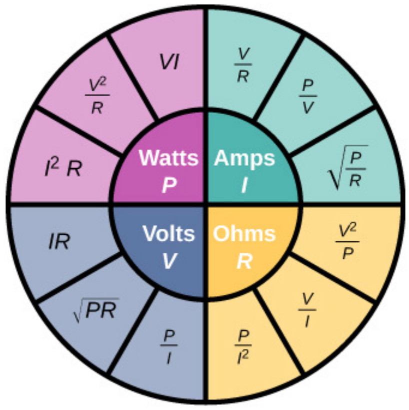


Figure 5.6.4: This circle shows a summary of the equations for the relationships between power, current, voltage, and resistance.

Which equation you use depends on what values you are given, or you measure. For example if you are given the current and the resistance, use $P = I^2R$. Although all the possible combinations may seem overwhelming, don't forget that they all are combinations of just two equations, Ohm's law $(V = IR)$ and power $(P = IV)$.

5.7: Superconductors

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Describe the phenomenon of superconductivity
- List applications of superconductivity

Touch the power supply of your laptop computer or some other device. It probably feels slightly warm. That heat is an unwanted byproduct of the process of converting household electric power into a current that can be used by your device. Although electric power is reasonably efficient, other losses are associated with it. As discussed in the section on power and energy, transmission of electric power produces I^2R line losses. These line losses exist whether the power is generated from conventional power plants (using coal, oil, or gas), nuclear plants, solar plants, hydroelectric plants, or wind farms. These losses can be reduced, but not eliminated, by transmitting using a higher voltage. It would be wonderful if these line losses could be eliminated, but that would require transmission lines that have zero resistance. In a world that has a global interest in not wasting energy, the reduction or elimination of this unwanted thermal energy would be a significant achievement. Is this possible?

The Resistance of Mercury

In 1911, Heike **Kamerlingh Onnes** of Leiden University, a Dutch physicist, was looking at the temperature dependence of the resistance of the element mercury. He cooled the sample of mercury and noticed the familiar behavior of a linear dependence of resistance on temperature; as the temperature decreased, the resistance decreased. Kamerlingh Onnes continued to cool the sample of mercury, using liquid helium. As the temperature approached $4.2\,K\,(-269.2^oC)$, the resistance abruptly went to zero (Figure $5.7.1$). This temperature is known as the **critical temperature** T_c for mercury. The sample of mercury entered into a phase where the resistance was absolutely zero. This phenomenon is known as **superconductivity**. (**Note:** If you connect the leads of a three-digit ohmmeter across a conductor, the reading commonly shows up as $0.00\,\Omega$. The resistance of the conductor is not actually zero, it is less than $0.01\,\Omega$.) There are various methods to measure very small resistances, such as the four-point method, but an ohmmeter is not an acceptable method to use for testing resistance in superconductivity.

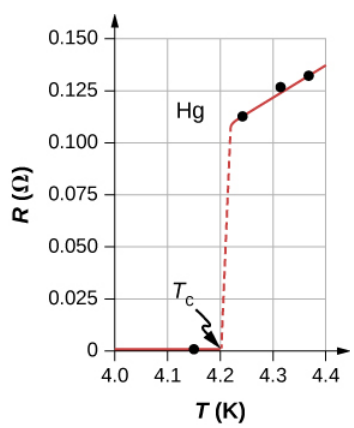


Figure $5.7.1$: The resistance of a sample of mercury is zero at very low temperatures—it is a superconductor up to the temperature of about 4.2 K. Above that critical temperature, its resistance makes a sudden jump and then increases nearly linearly with temperature.

Other Superconducting Materials

As research continued, several other materials were found to enter a superconducting phase, when the temperature reached near absolute zero. In 1941, an alloy of niobium-nitride was found that could become superconducting at $T_c = 16\,K\,(-257^oC)$ and in 1953, vanadium-silicon was found to become superconductive at $T_c = 17.5\,K\,(-255.7^oC)$. The temperatures for the transition into superconductivity were slowly creeping higher. Strangely, many materials that make good conductors, such as copper, silver, and gold, do not exhibit superconductivity. Imagine the energy savings if transmission lines for electric power-generating stations could be made to be superconducting at temperatures near room temperature! A resistance of zero ohms means no I^2R losses and a great boost to reducing energy consumption. The problem is that $T_c = 17.5\,K$ is still very cold and in the range of liquid helium temperatures. At this temperature, it is not cost effective to transmit electrical energy because of the cooling requirements.

A large jump was seen in 1986, when a team of researchers, headed by Dr. Ching Wu Chu of Houston University, fabricated a brittle, ceramic compound with a transition temperature of $T_c = 92\,K\,(-181^oC)$. The ceramic material, composed of yttrium barium copper oxide (YBCO), was an insulator at room temperature. Although this temperature still seems quite cold, it is near the boiling point of liquid nitrogen, a liquid commonly used in refrigeration. You may have noticed refrigerated trucks traveling down the highway labeled as "Liquid Nitrogen Cooled."

YBCO ceramic is a material that could be useful for transmitting electrical energy because the cost saving of reducing the I^2R losses are larger than the cost of cooling the superconducting cable, making it financially feasible. There were and are many engineering problems to overcome. For example, unlike traditional electrical cables, which are flexible and have a decent tensile strength, ceramics are brittle and would break rather than stretch under pressure. Processes that are rather simple with traditional cables, such as making connections, become difficult when working with ceramics. The problems are difficult and complex, and material scientists and engineers are coming up with innovative solutions.

An interesting consequence of the resistance going to zero is that once a current is established in a superconductor, it persists without an applied voltage source. Current loops in a superconductor have been set up and the current loops have been observed to persist for years without decaying.

Zero resistance is not the only interesting phenomenon that occurs as the materials reach their transition temperatures. A second effect is the exclusion of magnetic fields. This is known as the **Meissner effect** (Figure $5.7.2$). A light, permanent magnet placed over a superconducting sample will levitate in a stable position above the superconductor. High-speed trains have been developed that levitate on strong superconducting magnets, eliminating the friction normally experienced between the train and the tracks. In Japan, the Yamanashi Maglev test line opened on April 3, 1997. In April 2015, the MLX01 test vehicle attained a speed of 374 mph (603 km/h).

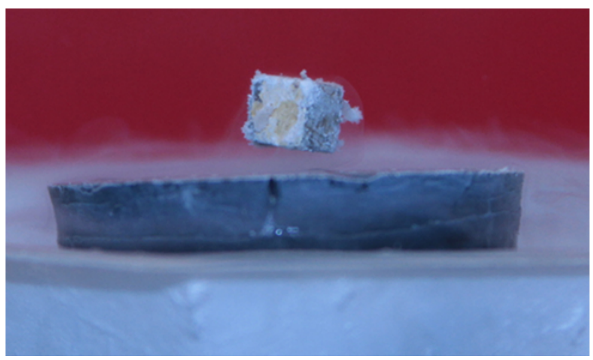


Figure $5.7.1$: A small, strong magnet levitates over a superconductor cooled to liquid nitrogen temperature. The magnet levitates because the superconductor excludes magnetic fields.

Table $5.7.1$ shows a select list of elements, compounds, and high-temperature superconductors, along with the critical temperatures for which they become superconducting. Each section is sorted from the highest critical temperature to the lowest. Also listed is the critical magnetic field for some of the materials. This is the strength of the magnetic field that destroys superconductivity. Finally, the type of the superconductor is listed.

There are two types of superconductors. There are 30 pure metals that exhibit zero resistivity below their critical temperature and exhibit the Meissner effect, the property of excluding magnetic fields from the interior of the superconductor while the superconductor is at a temperature below the critical temperature. These metals are called Type I superconductors. The superconductivity exists only below their critical temperatures and below a critical magnetic field strength. Type I superconductors are well described by the BCS theory (described next). Type I superconductors have limited practical applications because the strength of the critical magnetic field needed to destroy the superconductivity is quite low.

Type II superconductors are found to have much higher critical magnetic fields and therefore can carry much higher current densities while remaining in the superconducting state. A collection of various ceramics containing barium-copper-oxide have much higher critical temperatures for the transition into a superconducting state. Superconducting materials that belong to this subcategory of the Type II superconductors are often categorized as high-temperature superconductors.

Introduction to BCS Theory

Type I superconductors, along with some Type II superconductors can be modeled using the **BCS theory**, proposed by John **Bardeen**, Leon **Cooper**, and Robert **Schrieffer**. Although the theory is beyond the scope of this chapter, a short summary of the theory is provided here. (More detail is provided in Condensed Matter Physics.) The theory considers pairs of electrons and how they are coupled together through lattice-vibration interactions. Through the interactions with the crystalline lattice, electrons near the Fermi energy level feel a small attractive force and form pairs (**Cooper pairs**), and the coupling is known as a phonon interaction. Single electrons are fermions, which are particles that obey the Pauli exclusion principle. The Pauli exclusion principle in quantum mechanics states that two identical fermions (particles with half-integer spin) cannot occupy the same quantum state simultaneously. Each electron has four quantum numbers (n, ℓ, m_ℓ, m_s). The principal quantum number (n) describes the energy of the electron, the orbital angular momentum quantum number (ℓ) indicates the most probable distance from the nucleus, the magnetic quantum number m_ℓ describes the energy levels in the subshell, and the electron spin quantum number m_s describes the orientation of the spin of the electron, either up or down. As the material enters a superconducting state, pairs of electrons act more like bosons, which can condense into the same energy level and need not obey the Pauli exclusion principle. The electron pairs have a slightly lower energy and leave an energy gap above them on the order of 0.001 eV. This energy gap inhibits collision interactions that lead to ordinary resistivity. When the material is below the critical temperature, the thermal energy is less than the band gap and the material exhibits zero resistivity.

Table 5.7.1: Superconductor Critical Temperatures

Material	Symbol or Formula	Critical Temperature $T_c(K)$	Critical Magnetic Field $\boldsymbol{H_c(T)}$	Type
Elements				
Lead	Pb	7.19	0.08	I
Lanthanum	La	(α) 4.90 - (β) 6.30		I
Tantalum	Ta	4.48	0.09	I
Mercury	Hg	(α) 4.15 - (β) 3.95	0.04	I
Tin	Sn	3.72	0.03	I
Indium	In	3.40	0.03	I
Thallium	Tl	2.39	0.03	I
Rhenium	Re	2.40	0.03	I
Thorium	Th	1.37	0.013	I
Protactinium	Pa	1.40		I
Aluminum	Al	1.20	0.01	I
Gallium	Ga	1.10	0.005	I
Zinc	Zn	0.86	0.014	I
Titanium	Ti	0.39	0.01	I
Uranium	U	(α) 0.68 - (β) 1.80		I
Cadmium	Cd	11.4	4.00	I
Compounds				
Niobium-germanium	Nb_3Ge	23.20	37.00	II
Niobium-tin	Nb_3Sn	18.30	30.00	II
Niobium-nitrite	NbN	16.00		II
Niobium-titanium	NbTi	10.00	15.00	II
High-Temperature Oxides				

Material	Symbol or Formula	Critical Temperature $T_c(K)$	Critical Magnetic Field $H_c(T)$	Type
	$HgBa_2CaCu_2O_8$	134.00		II
	$Ti_2Ba_2Ca_2Cu_3O_{10}$	125.00		II
	$YBa_2Cu_3O_7$	92.00	120.00	II

Applications of Superconductors

Superconductors can be used to make superconducting magnets. These magnets are 10 times stronger than the strongest electromagnets. These magnets are currently in use in magnetic resonance imaging (MRI), which produces high-quality images of the body interior without dangerous radiation.

Another interesting application of superconductivity is the **SQUID** (superconducting quantum interference device). A SQUID is a very sensitive magnetometer used to measure extremely subtle magnetic fields. The operation of the SQUID is based on superconducting loops containing Josephson junctions. A **Josephson junction** is the result of a theoretical prediction made by B. D. Josephson in an article published in 1962. In the article, Josephson described how a supercurrent can flow between two pieces of superconductor separated by a thin layer of insulator. This phenomenon is now called the **Josephson effect**. The SQUID consists of a superconducting current loop containing two Josephson junctions, as shown in Figure $5.7.3$. When the loop is placed in even a very weak magnetic field, there is an interference effect that depends on the strength of the magnetic field.

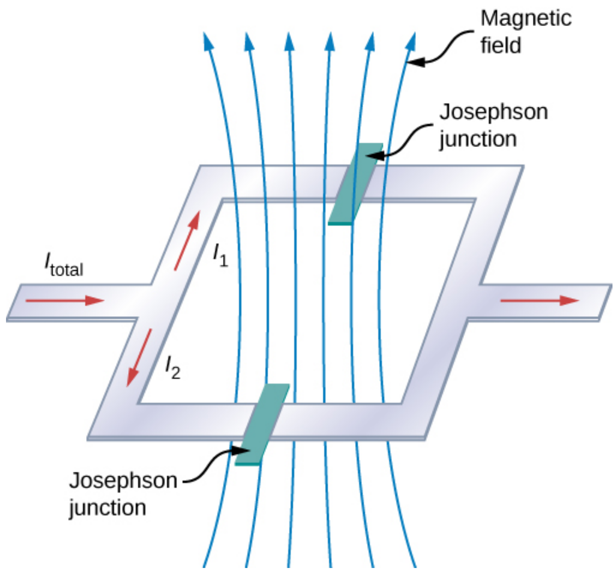


Figure $5.7.3$: The SQUID (superconducting quantum interference device) uses a superconducting current loop and two Josephson junctions to detect magnetic fields as low as 10^{-14} (Earth's magnet field is on the order of $0.3 \times 10^{-5} T$).

Superconductivity is a fascinating and useful phenomenon. At critical temperatures near the boiling point of liquid nitrogen, superconductivity has special applications in MRIs, particle accelerators, and high-speed trains. Will we reach a state where we can have materials enter the superconducting phase at near room temperatures? It seems a long way off, but if scientists in 1911 were asked if we would reach liquid-nitrogen temperatures with a ceramic, they might have thought it implausible.

5.8: Current and Resistance (Exercises)

Conceptual Questions

9.2 Electrical Current

1. Can a wire carry a current and still be neutral—that is, have a total charge of zero? Explain.

2. Car batteries are rated in ampere-hours (**A·h**). To what physical quantity do ampere-hours correspond (voltage, current, charge, energy, power,…)?

3. When working with high-power electric circuits, it is advised that whenever possible, you work “one-handed” or “keep one hand in your pocket.” Why is this a sensible suggestion?

9.3 Model of Conduction in Metals

4. Incandescent light bulbs are being replaced with more efficient LED and CFL light bulbs. Is there any obvious evidence that incandescent light bulbs might not be that energy efficient? Is energy converted into anything but visible light?

5. It was stated that the motion of an electron appears nearly random when an electrical field is applied to the conductor. What makes the motion nearly random and differentiates it from the random motion of molecules in a gas?

6. Electric circuits are sometimes explained using a conceptual model of water flowing through a pipe. In this conceptual model, the voltage source is represented as a pump that pumps water through pipes and the pipes connect components in the circuit. Is a conceptual model of water flowing through a pipe an adequate representation of the circuit? How are electrons and wires similar to water molecules and pipes? How are they different?

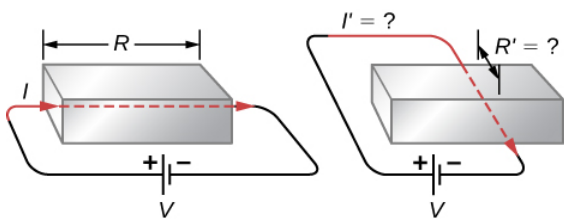
7. An incandescent light bulb is partially evacuated. Why do you suppose that is?

9.4 Resistivity and Resistance

8. The IR drop across a resistor means that there is a change in potential or voltage across the resistor. Is there any change in current as it passes through a resistor? Explain.

9. Do impurities in semiconducting materials listed in Table 9.1 supply free charges? (Hint: Examine the range of resistivity for each and determine whether the pure semiconductor has the higher or lower conductivity.)

10. Does the resistance of an object depend on the path current takes through it? Consider, for example, a rectangular bar—is its resistance the same along its length as across its width?

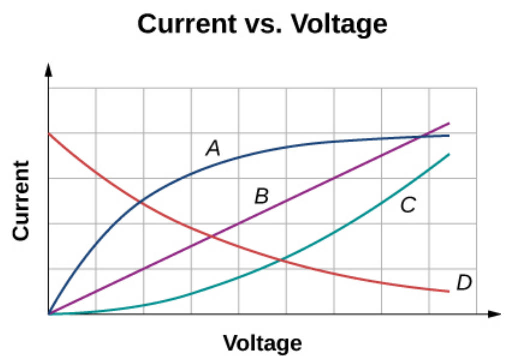


11. If aluminum and copper wires of the same length have the same resistance, which has the larger diameter? Why?

9.5 Ohm's Law

12. In Determining Field from Potential, resistance was defined as $R \equiv \frac{V}{I}$. In this section, we presented Ohm’s law, which is commonly expressed as **V=IR.** The equations look exactly alike. What is the difference between Ohm’s law and the definition of resistance?

13. Shown below are the results of an experiment where four devices were connected across a variable voltage source. The voltage is increased and the current is measured. Which device, if any, is an ohmic device?



14. The current I is measured through a sample of an ohmic material as a voltage V is applied. (a) What is the current when the voltage is doubled to 2V (assume the change in temperature of the material is negligible)? (b) What is the voltage applied is the current measured is 0.2I (assume the change in temperature of the material is negligible)? What will happen to the current if the material if the voltage remains constant, but the temperature of the material increases significantly?

9.6 Electrical Energy and Power

15. Common household appliances are rated at 110 V, but power companies deliver voltage in the kilovolt range and then step the voltage down using transformers to 110 V to be used in homes. You will learn in later chapters that transformers consist of many turns of wire, which warm up as current flows through them, wasting some of the energy that is given off as heat. This sounds inefficient. Why do the power companies transport electric power using this method?

16. Your electric bill gives your consumption in units of kilowatt-hour (kW ·· h). Does this unit represent the amount of charge, current, voltage, power, or energy you buy?

17. Resistors are commonly rated at $\frac{1}{8}W$, $\frac{1}{4}W$, $\frac{1}{2}W$, 1 W and 2 W for use in electrical circuits. If a current of **I=2.00A** is accidentally passed through a **R=1.00Ω** resistor rated at 1 W, what would be the most probable outcome? Is there anything that can be done to prevent such an accident?

18. An immersion heater is a small appliance used to heat a cup of water for tea by passing current through a resistor. If the voltage applied to the appliance is doubled, will the time required to heat the water change? By how much? Is this a good idea?

9.7 Superconductors

19. What requirement for superconductivity makes current superconducting devices expensive to operate?

20. Name two applications for superconductivity listed in this section and explain how superconductivity is used in the application. Can you think of a use for superconductivity that is not listed?

Problems

9.2 Electrical Current

21. A Van de Graaff generator is one of the original particle accelerators and can be used to accelerate charged particles like protons or electrons. You may have seen it used to make human hair stand on end or produce large sparks. One application of the Van de Graaff generator is to create X-rays by bombarding a hard metal target with the beam. Consider a beam of protons at 1.00 keV and a current of 5.00 mA produced by the generator.

(a) What is the speed of the protons?

(b) How many protons are produced each second?

22. A cathode ray tube (CRT) is a device that produces a focused beam of electrons in a vacuum. The electrons strike a phosphor-coated glass screen at the end of the tube, which produces a bright spot of light. The position of the bright spot of light on the screen can be adjusted by deflecting the electrons with electrical fields, magnetic fields, or both. Although the CRT tube was once commonly found in televisions, computer displays, and oscilloscopes, newer appliances use a liquid crystal display (LCD) or plasma screen. You still may come across a CRT in your study of science. Consider a CRT with an electron beam average current of 25.00μA25.00μA. How many electrons strike the screen every minute?

23. How many electrons flow through a point in a wire in 3.00 s if there is a constant current of **I=4.00A**?

24. A conductor carries a current that is decreasing exponentially with time. The current is modeled as $I = I_0 e^{-t/\tau}$, where $I_0 = 3.00A$ is the current at time **t=0.00s** and **τ=0.50s** is the time constant. How much charge flows through the conductor between **t=0.00s** and **t=3τ**?

25. The quantity of charge through a conductor is modeled as $Q = 4.00\frac{C}{s^4}t^4 - 1.00\frac{C}{s}t + 6.00mC$. What is the current at time **t=3.00s**?

26. The current through a conductor is modeled as $I(t) = I_m sin(2\pi[60Hz]t)$. Write an equation for the charge as a function of time.

27. The charge on a capacitor in a circuit is modeled as $Q(t) = Q_{max} cos(\omega t + \phi)$. What is the current through the circuit as a function of time?

9.3 Model of Conduction in Metals

28. An aluminum wire 1.628 mm in diameter (14-gauge) carries a current of 3.00 amps.

(a) What is the absolute value of the charge density in the wire?

(b) What is the drift velocity of the electrons?

(c) What would be the drift velocity if the same gauge copper were used instead of aluminum? The density of copper is $8.96g/cm^3$ and the density of aluminum is $2.70g/cm^3$. The molar mass of aluminum is 26.98 g/mol and the molar mass of copper is 63.5 g/mol. Assume each atom of metal contributes one free electron.

29. The current of an electron beam has a measured current of **I=50.00μA** with a radius of 1.00 mm. What is the magnitude of the current density of the beam?

30. A high-energy proton accelerator produces a proton beam with a radius of **r=0.90mm**. The beam current is **I=9.00μA** and is constant. The charge density of the beam is $n = 6.00 \times 10^{11}$ protons per cubic meter.

(a) What is the current density of the beam?

(b) What is the drift velocity of the beam?

(c) How much time does it take for 1.00×10^{10} protons to be emitted by the accelerator?

31. Consider a wire of a circular cross-section with a radius of **R=3.00mm**. The magnitude of the current density is modeled as $J = cr^2 = 5.00 \times 10^6 \frac{A}{m^4} r^2$. What is the current through the inner section of the wire from the center to **r=0.5R**?

32. A cylindrical wire has a current density from the center of the wire's cross section as $J(r) = Cr^2$ where r is in meters, J is in amps per square meter, and $C = 10^3$ A/m^4. This current density continues to the end of the wire at a radius of 1.0 mm. Calculate the current just outside of this wire.

33. The current supplied to an air conditioner unit is 4.00 amps. The air conditioner is wired using a 10-gauge (diameter 2.588 mm) wire. The charge density is $n = 8.48 \times 10^{28} \frac{electrons}{m^3}$. Find the magnitude of

(a) current density and

(b) the drift velocity.

9.4 Resistivity and Resistance

34. What current flows through the bulb of a 3.00-V flashlight when its hot resistance is **3.60Ω**?

35. Calculate the effective resistance of a pocket calculator that has a 1.35-V battery and through which 0.200 mA flows.

36. How many volts are supplied to operate an indicator light on a DVD player that has a resistance of **140Ω**, given that 25.0 mA passes through it?

37. What is the resistance of a 20.0-m-long piece of 12-gauge copper wire having a 2.053-mm diameter?

38. The diameter of 0-gauge copper wire is 8.252 mm. Find the resistance of a 1.00-km length of such wire used for power transmission.

39. If the 0.100-mm-diameter tungsten filament in a light bulb is to have a resistance of **0.200Ω** at **20.0°C**, how long should it be?

40. A lead rod has a length of 30.00 cm and a resistance of **5.00μΩ**. What is the radius of the rod?

41. Find the ratio of the diameter of aluminum to copper wire, if they have the same resistance per unit length (as they might in household wiring).

42. What current flows through a 2.54-cm-diameter rod of pure silicon that is 20.0 cm long, when 1.00×10^3 is applied to it? (Such a rod may be used to make nuclear-particle detectors, for example.)

43. (a) To what temperature must you raise a copper wire, originally at **20.0°C**, to double its resistance, neglecting any changes in dimensions? (b) Does this happen in household wiring under ordinary circumstances?

44. A resistor made of nichrome wire is used in an application where its resistance cannot change more than 1.00% from its value at **20.0°C**. Over what temperature range can it be used?

45. Of what material is a resistor made if its resistance is 40.0% greater at **100.0°C** than at **20.0°C**?

46. An electronic device designed to operate at any temperature in the range from **−10.0°C** to **55.0°C** contains pure carbon resistors. By what factor does their resistance increase over this range?**47.**

(a) Of what material is a wire made, if it is 25.0 m long with a diameter of 0.100 mm and has a resistance of **77.7Ω** at **20.0°C**? (b) What is its resistance at **150.0°C**?

48. Assuming a constant temperature coefficient of resistivity, what is the maximum percent decrease in the resistance of a constantan wire starting at **20.0°C**?

49. A copper wire has a resistance of **0.500Ω** at **20.0°C**, and an iron wire has a resistance of **0.525Ω** at the same temperature. At what temperature are their resistances equal?

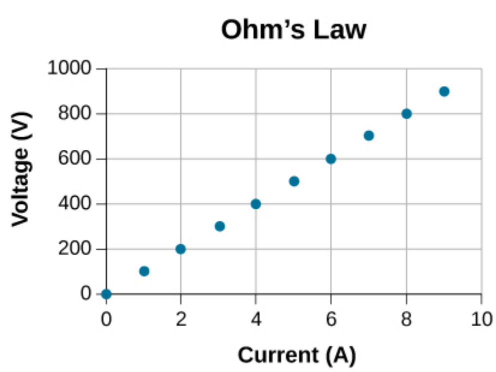
9.5 Ohm's Law

50. A **2.2-kΩ** resistor is connected across a D cell battery (1.5 V). What is the current through the resistor?

51. A resistor rated at **250kΩ** is connected across two D cell batteries (each 1.50 V) in series, with a total voltage of 3.00 V. The manufacturer advertises that their resistors are within 5% of the rated value. What are the possible minimum current and maximum current through the resistor?

52. A resistor is connected in series with a power supply of 20.00 V. The current measure is 0.50 A. What is the resistance of the resistor?

53. A resistor is placed in a circuit with an adjustable voltage source. The voltage across and the current through the resistor and the measurements are shown below. Estimate the resistance of the resistor.



54. The following table show the measurements of a current through and the voltage across a sample of material. Plot the data, and assuming the object is an ohmic device, estimate the resistance.

Table: Measurements of current through and the voltage across a sample of material

I(A)	V(V)
0	3

2	23
4	39
6	58
8	77
10	100
12	119
14	142
16	162

9.6 Electrical Energy and Power

55. A **20.00-V** battery is used to supply current to a **10-kΩ** resistor. Assume the voltage drop across any wires used for connections is negligible.

(a) What is the current through the resistor?

(b) What is the power dissipated by the resistor?

(c) What is the power input from the battery, assuming all the electrical power is dissipated by the resistor?

(d) What happens to the energy dissipated by the resistor?

56. What is the maximum voltage that can be applied to a **20-kΩ** resistor rated at $\frac{1}{4}W$?

57. A heater is being designed that uses a coil of 14-gauge nichrome wire to generate 300 W using a voltage of **V=110V.** How long should the engineer make the wire?

58. An alternative to CFL bulbs and incandescent bulbs are light-emitting diode (LED) bulbs. A 100-W incandescent bulb can be replaced by a 16-W LED bulb. Both produce 1600 lumens of light. Assuming the cost of electricity is $0.10 per kilowatt-hour, how much does it cost to run the bulb for one year if it runs for four hours a day?

59. The power dissipated by a resistor with a resistance of **R=100Ω** is **P=2.0W**. What are the current through and the voltage drop across the resistor?

60. Running late to catch a plane, a driver accidentally leaves the headlights on after parking the car in the airport parking lot. During takeoff, the driver realizes the mistake. Having just replaced the battery, the driver knows that the battery is a 12-V automobile battery, rated at 100 **A·h**. The driver, knowing there is nothing that can be done, estimates how long the lights will shine, assuming there are two 12-V headlights, each rated at 40 W. What did the driver conclude?

61. A physics student has a single-occupancy dorm room. The student has a small refrigerator that runs with a current of 3.00 A and a voltage of 110 V, a lamp that contains a 100-W bulb, an overhead light with a 60-W bulb, and various other small devices adding up to 3.00 W.

(a) Assuming the power plant that supplies 110 V electricity to the dorm is 10 km away and the two aluminum transmission cables use 0-gauge wire with a diameter of 8.252 mm, estimate the percentage of the total power supplied by the power company that is lost in the transmission.

(b) What would be the result is the power company delivered the electric power at 110 kV?

62. A 0.50-W, **220-Ω** resistor carries the maximum current possible without damaging the resistor. If the current were reduced to half the value, what would be the power consumed?

9.7 Superconductors

63. Consider a power plant is located 60 km away from a residential area uses 0-gauge $(A=42.40mm^2)$ wire of copper to transmit power at a current of **I=100.00A**. How much more power is dissipated in the copper wires than it would be in superconducting wires?

64. A wire is drawn through a die, stretching it to four times its original length. By what factor does its resistance increase?

65. Digital medical thermometers determine temperature by measuring the resistance of a semiconductor device called a thermistor (which has **α=−0.06/°C**) when it is at the same temperature as the patient. What is a patient's temperature if the thermistor's resistance at that temperature is 82.0% of its value at **37°C** (normal body temperature)?

66. Electrical power generators are sometimes "load tested" by passing current through a large vat of water. A similar method can be used to test the heat output of a resistor. A **R=30Ω** resistor is connected to a 9.0-V battery and the resistor leads are waterproofed and the resistor is placed in 1.0 kg of room temperature water (**T=20°C**). Current runs through the resistor for 20 minutes. Assuming all the electrical energy dissipated by the resistor is converted to heat, what is the final temperature of the water?

67. A 12-guage gold wire has a length of 1 meter.

(a) What would be the length of a silver 12-gauge wire with the same resistance?

(b) What are their respective resistances at the temperature of boiling water?

68. What is the change in temperature required to decrease the resistance for a carbon resistor by 10%?

Additional Problems

69. A coaxial cable consists of an inner conductor with radius $r_i = 0.25cm$ and an outer radius of $r_o = 0.5cm$ and has a length of 10 meters. Plastic, with a resistivity of $\rho = 2.00 \times 10^{13} \Omega \cdot m$, separates the two conductors. What is the resistance of the cable?

70. A 10.00-meter long wire cable that is made of copper has a resistance of 0.051 ohms.

(a) What is the weight if the wire was made of copper?

(b) What is the weight of a 10.00-meter-long wire of the same gauge made of aluminum?

(c)What is the resistance of the aluminum wire? The density of copper is $8960kg/m^3$ and the density of aluminum is $2760kg/m^3$.

71. A nichrome rod that is 3.00 mm long with a cross-sectional area of $1.00mm^2$ is used for a digital thermometer.

(a) What is the resistance at room temperature?

(b) What is the resistance at body temperature?

72. The temperature in Philadelphia, PA can vary between **68.00°F** and **100.00°F** in one summer day. By what percentage will an aluminum wire's resistance change during the day?

73. When 100.0 V is applied across a 5-gauge (diameter 4.621 mm) wire that is 10 m long, the magnitude of the current density is $2.0 \times 10^8 A/m^2$. What is the resistivity of the wire?

74. A wire with a resistance of **5.0Ω** is drawn out through a die so that its new length is twice times its original length. Find the resistance of the longer wire. You may assume that the resistivity and density of the material are unchanged.

75. What is the resistivity of a wire of 5-gauge wire ($A = 16.8 \times 10^{-6} m^2$), 5.00 m length, and **5.10mΩ** resistance?

76. Coils are often used in electrical and electronic circuits. Consider a coil which is formed by winding 1000 turns of insulated 20-gauge copper wire (area $0.52mm^2$) in a single layer on a cylindrical non-conducting core of radius 2.0 mm. What is the resistance of the coil? Neglect the thickness of the insulation.

77. Currents of approximately 0.06 A can be potentially fatal. Currents in that range can make the heart fibrillate (beat in an uncontrolled manner). The resistance of a dry human body can be approximately **100kΩ**.

(a) What voltage can cause 0.2 A through a dry human body?

(b) When a human body is wet, the resistance can fall to **100Ω**. What voltage can cause harm to a wet body?

78. A 20.00-ohm, 5.00-watt resistor is placed in series with a power supply.

(a) What is the maximum voltage that can be applied to the resistor without harming the resistor?

(b) What would be the current through the resistor?

79. A battery with an emf of 24.00 V delivers a constant current of 2.00 mA to an appliance. How much work does the battery do in three minutes?

80. A 12.00-V battery has an internal resistance of a tenth of an ohm.

(a) What is the current if the battery terminals are momentarily shorted together?

(b) What is the terminal voltage if the battery delivers 0.25 amps to a circuit?

Challenge Problems

81. A 10-gauge copper wire has a cross-sectional area $A = 5.26 mm^2$ and carries a current of **I=5.00A**. The density of copper is $\rho = 89.50 g/cm^3$. One mole of copper atoms ($6.02 \times 10^{23} atoms$) has a mass of approximately 63.50 g. What is the magnitude of the drift velocity of the electrons, assuming that each copper atom contributes one free electron to the current?

82. The current through a 12-gauge wire is given as **I(t)=(5.00A)sin(2π60Hzt)**. What is the current density at time 15.00 ms?

83. A particle accelerator produces a beam with a radius of 1.25 mm with a current of 2.00 mA. Each proton has a kinetic energy of 10.00 MeV.

(a) What is the velocity of the protons?

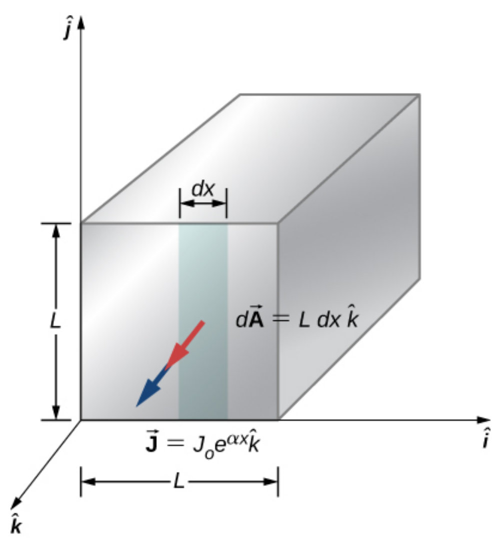
(b) What is the number (n) of protons per unit volume?

(c) How many electrons pass a cross sectional area each second?

84. In this chapter, most examples and problems involved direct current (DC). DC circuits have the current flowing in one direction, from positive to negative. When the current was changing, it was changed linearly from $I = -I_{max}$ to $I = +I_{max}$ and the voltage changed linearly from $V = -V_{max}$ to $V = +V_{max}$, where $V_{max} = I_{max} R$. Suppose a voltage source is placed in series with a resistor of **R=10Ω** that supplied a current that alternated as a sine wave, for example, $I(t) = (3.00 A) sin(\frac{2\pi}{4.00 s} t)$. (a) What would a graph of the voltage drop across the resistor **V(t)** versus time look like? (b) What would a plot of **V(t)** versus **I(t)** for one period look like? (Hint: If you are not sure, try plotting **V(t)** versus **I(t)** using a spreadsheet.)

85. A current of **I=25A** is drawn from a 100-V battery for 30 seconds. By how much is the chemical energy reduced?

86. Consider a square rod of material with sides of length **L=3.00cm** with a current density of $J = J_0 e^{\alpha x} \hat{k} = (0.35 \frac{A}{m^2})^{e(2.1 \times 10^{-3} m^{-1}) x} \hat{k}$ as shown below. Find the current that passes through the face of



87. A resistor of an unknown resistance is placed in an insulated container filled with 0.75 kg of water. A voltage source is connected in series with the resistor and a current of 1.2 amps flows through the resistor for 10 minutes. During this time, the temperature of the water is measured and the temperature change during this time is **ΔT=10.00°C**.

(a) What is the resistance of the resistor?

(b) What is the voltage supplied by the power supply?

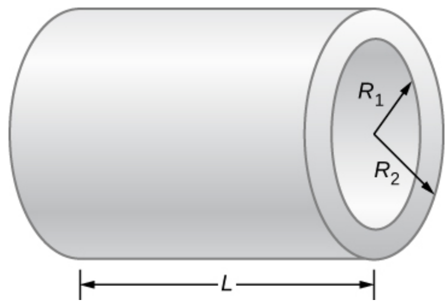
88. The charge that flows through a point in a wire as a function of time is modeled as $q(t) = q_0 e^{-t/T} = 10.0Ce^{-t/5s}$.

(a) What is the initial current through the wire at time **t=0.00s**?

(b) Find the current at time $t = \frac{1}{2}T$.

(c) At what time t will the current be reduced by one-half $I = \frac{1}{2}I_0$?

89. Consider a resistor made from a hollow cylinder of carbon as shown below. The inner radius of the cylinder is Ri=0.20mmRi=0.20mm and the outer radius is $R_0 = 0.30mm$. The length of the resistor is **L=0.90mm**. The resistivity of the carbon is $\rho = 3.5 \times 10^{-5} \Omega \cdot m$. (a) Prove that the resistance perpendicular from the axis is $R = \frac{\rho}{2\pi L} ln(\frac{R_0}{R_i})$. (b) What is the resistance?



90. What is the current through a cylindrical wire of radius **R=0.1mm** if the current density is $J = \frac{J_0}{R} r$, where $J_0 = 32000 \frac{A}{m^2}$?

91. A student uses a 100.00-W, 115.00-V radiant heater to heat the student’s dorm room, during the hours between sunset and sunrise, 6:00 p.m. to 7:00 a.m.

(a) What current does the heater operate at?

(b) How many electrons move through the heater?

(c) What is the resistance of the heater?

(d) How much heat was added to the dorm room?

92. A 12-V car battery is used to power a 20.00-W, 12.00-V lamp during the physics club camping trip/star party. The cable to the lamp is 2.00 meters long, 14-gauge copper wire with a charge density of $n = 9.50 \times 10^{28} m^{-3}$.

(a) What is the current draw by the lamp?

(b) How long would it take an electron to get from the battery to the lamp?

93. A physics student uses a 115.00-V immersion heater to heat 400.00 grams (almost two cups) of water for herbal tea. During the two minutes it takes the water to heat, the physics student becomes bored and decides to figure out the resistance of the heater. The student starts with the assumption that the water is initially at the temperature of the room $T_i = 25.00^\circ C$ and reaches $T_f = 100.00^\circ C$. The specific heat of the water is $c = 4180 \frac{J}{kg}$. What is the resistance of the heater?

CHAPTER OVERVIEW

6: Direct-Current Circuits

In the preceding few chapters, we discussed electric components, including capacitors, resistors, and diodes. In this chapter, we use these electric components in circuits. A circuit is a collection of electrical components connected to accomplish a specific task. The second section of this chapter covers the analysis of series and parallel circuits that consist of resistors. Later in this chapter, we introduce the basic equations and techniques to analyze any circuit, including those that are not reducible through simplifying parallel and series elements. But first, we need to understand how to power a circuit.

6.1: Prelude to Direct-Current Circuits

6.2: Electromotive Force

6.2.1: Electromotive Force and Internal Resistance

6.3: Resistors in Series and Parallel

6.4: Kirchhoff's Rules

6.5: Capacitors in Series and in Parallel

6.6: RC Circuits

6.7: Direct-Current Circuits (Exercise)

6.1: Prelude to Direct-Current Circuits

In the preceding few chapters, we discussed electric components, including capacitors, resistors, and diodes. In this chapter, we use these electric components in circuits. A circuit is a collection of electrical components connected to accomplish a specific task. Figure 6.1.1 shows an amplifier circuit, which takes a small-amplitude signal and amplifies it to power the speakers in earbuds. Although the circuit looks complex, it actually consists of a set of series, parallel, and series-parallel circuits. The second section of this chapter covers the analysis of series and parallel circuits that consist of resistors. Later in this chapter, we introduce the basic equations and techniques to analyze any circuit, including those that are not reducible through simplifying parallel and series elements. But first, we need to understand how to power a circuit.

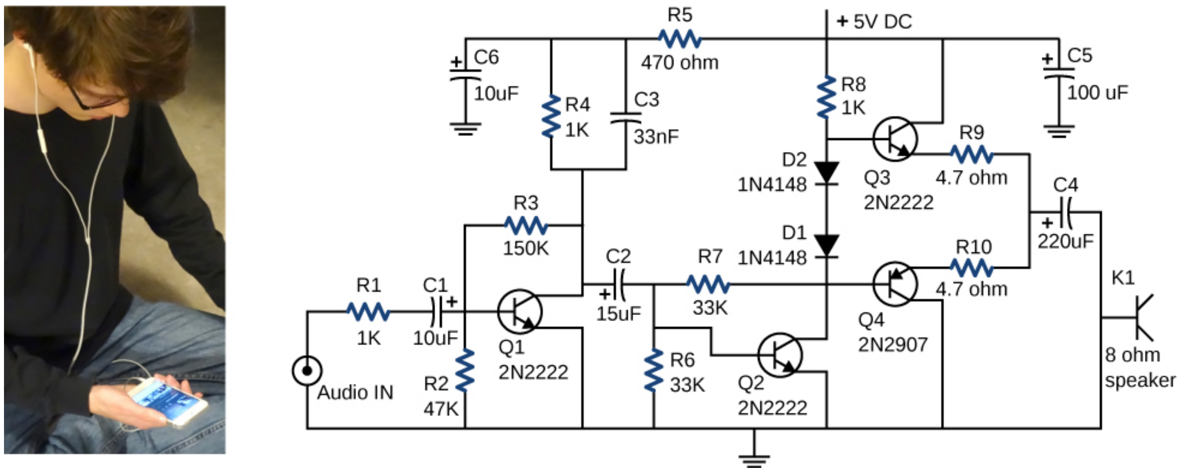


Figure 6.1.1: This circuit shown is used to amplify small signals and power the earbud speakers attached to a cellular phone. This circuit's components include resistors, capacitors, and diodes, all of which have been covered in previous chapters, as well as transistors, which are semi-conducting devices covered in Condensed Matter Physics. Circuits using similar components are found in all types of equipment and appliances you encounter in everyday life, such as alarm clocks, televisions, computers, and refrigerators. (credit: Jane Whitney)

6.2: Electromotive Force

Learning Objectives

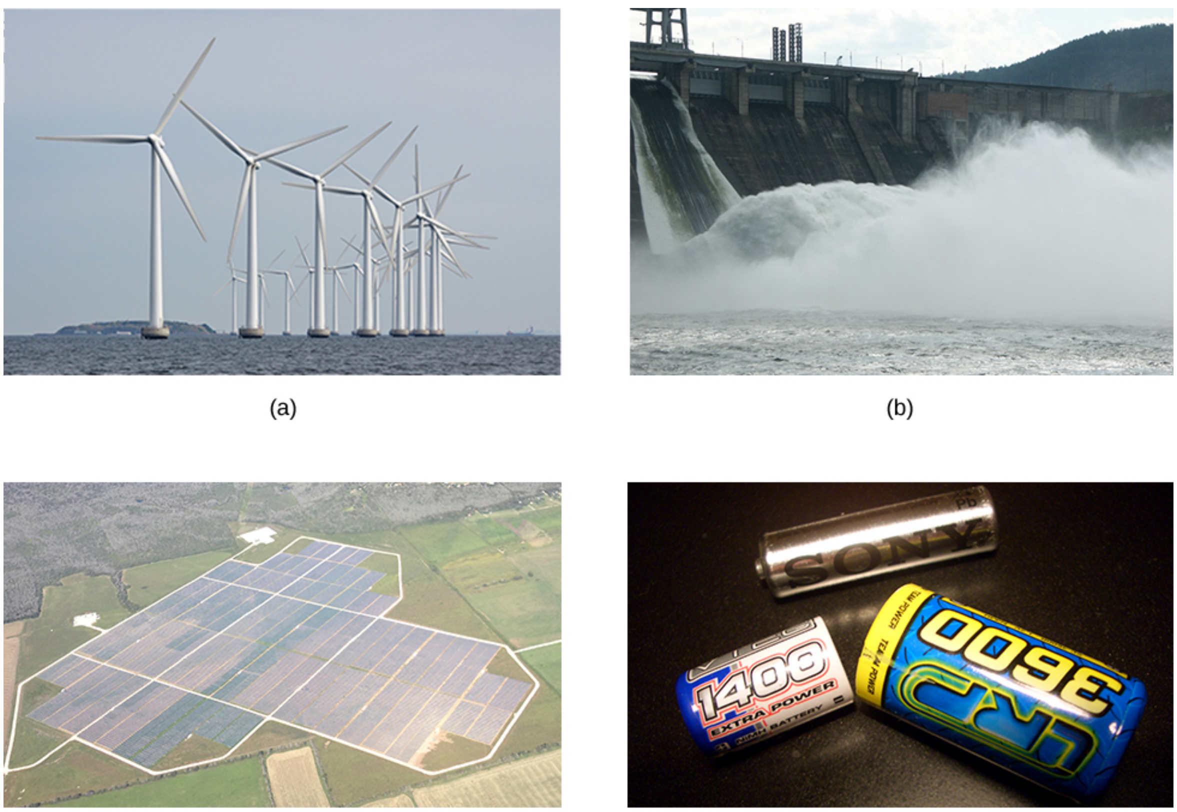
By the end of the section, you will be able to:

- Describe the electromotive force (emf) and the internal resistance of a battery
- Explain the basic operation of a battery

If you forget to turn off your car lights, they slowly dim as the battery runs down. Why don't they suddenly blink off when the battery's energy is gone? Their gradual dimming implies that the battery output voltage decreases as the battery is depleted. The reason for the decrease in output voltage for depleted batteries is that all voltage sources have two fundamental parts—a source of electrical energy and an internal resistance. In this section, we examine the energy source and the internal resistance.

Introduction to Electromotive Force

Voltage has many sources, a few of which are shown in Figure $6.2.2$. All such devices create a **potential difference** and can supply current if connected to a circuit. A special type of potential difference is known as **electromotive force (emf)**. The emf is not a force at all, but the term 'electromotive force' is used for historical reasons. It was coined by Alessandro Volta in the 1800s, when he invented the first battery, also known as the **voltaic pile**. Because the electromotive force is not a force, it is common to refer to these sources simply as sources of emf (pronounced as the letters "ee-em-eff"), instead of sources of electromotive force.



(c) (d)

Figure $6.2.1$: A variety of voltage sources. (a) The Brazos Wind Farm in Fluvanna, Texas; (b) the Krasnoyarsk Dam in Russia; (c) a solar farm; (d) a group of nickel metal hydride batteries. The voltage output of each device depends on its construction and load. The voltage output equals emf only if there is no load. (credit a: modification of work by "Leaflet"/Wikimedia Commons; credit b: modification of work by Alex Polezhaev; credit c: modification of work by US Department of Energy; credit d: modification of work by Tiaa Monto)

If the electromotive force is not a force at all, then what is the emf and what is a source of emf? To answer these questions, consider a simple circuit of a 12-V lamp attached to a 12-V battery, as shown in Figure $6.2.2$. The **battery** can be modeled as a two-terminal device that keeps one terminal at a higher electric potential than the second terminal. The higher electric potential is sometimes called the positive terminal and is labeled with a plus sign. The lower-potential terminal is sometimes called the negative terminal and labeled with a minus sign. This is the source of the emf.

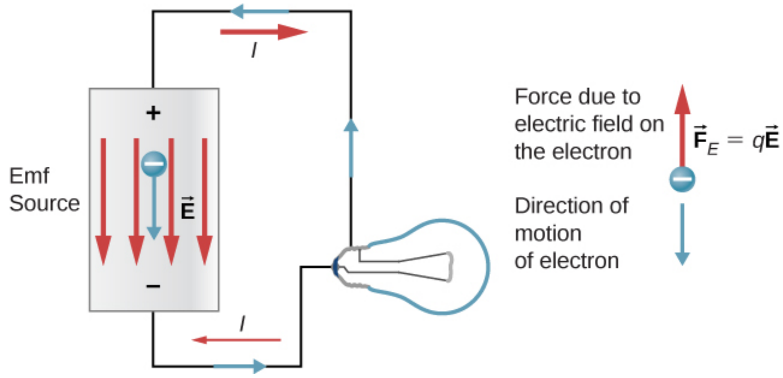


Figure 6.2.2: A source of emf maintains one terminal at a higher electric potential than the other terminal, acting as a source of current in a circuit.

When the emf source is not connected to the lamp, there is no net flow of charge within the emf source. Once the battery is connected to the lamp, charges flow from one terminal of the battery, through the lamp (causing the lamp to light), and back to the other terminal of the battery. If we consider positive (conventional) current flow, positive charges leave the positive terminal, travel through the lamp, and enter the negative terminal.

Positive current flow is useful for most of the circuit analysis in this chapter, but in metallic wires and resistors, electrons contribute the most to current, flowing in the opposite direction of positive current flow. Therefore, it is more realistic to consider the movement of electrons for the analysis of the circuit in Figure $6.2.2$. The electrons leave the negative terminal, travel through the lamp, and return to the positive terminal. In order for the emf source to maintain the potential difference between the two terminals, negative charges (electrons) must be moved from the positive terminal to the negative terminal. The emf source acts as a charge pump, moving negative charges from the positive terminal to the negative terminal to maintain the potential difference. This increases the potential energy of the charges and, therefore, the electric potential of the charges.

The force on the negative charge from the electric field is in the opposite direction of the electric field, as shown in Figure $6.2.2$. In order for the negative charges to be moved to the negative terminal, work must be done on the negative charges. This requires energy, which comes from chemical reactions in the battery. The potential is kept high on the positive terminal and low on the negative terminal to maintain the potential difference between the two terminals. The emf is equal to the work done on the charge per unit charge $\left(\epsilon = \frac{dW}{dq}\right)$ when there is no current flowing. Since the unit for work is the joule and the unit for charge is the coulomb, the unit for emf is the volt $(1\,V = 1\,J/C)$.

The **terminal voltage** $V_{terminal}$ of a battery is voltage measured across the terminals of the battery when there is no load connected to the terminal. An ideal battery is an emf source that maintains a constant terminal voltage, independent of the current between the two terminals. An ideal battery has no internal resistance, and the terminal voltage is equal to the emf of the battery. In the next section, we will show that a real battery does have internal resistance and the terminal voltage is always less than the emf of the battery.

The Origin of Battery Potential

The combination of chemicals and the makeup of the terminals in a battery determine its emf. The **lead acid battery** used in cars and other vehicles is one of the most common combinations of chemicals. Figure $6.2.3$ shows a single cell (one of six) of this battery. The cathode (positive) terminal of the cell is connected to a lead oxide plate, whereas the anode (negative) terminal is connected to a lead plate. Both plates are immersed in sulfuric acid, the electrolyte for the system.

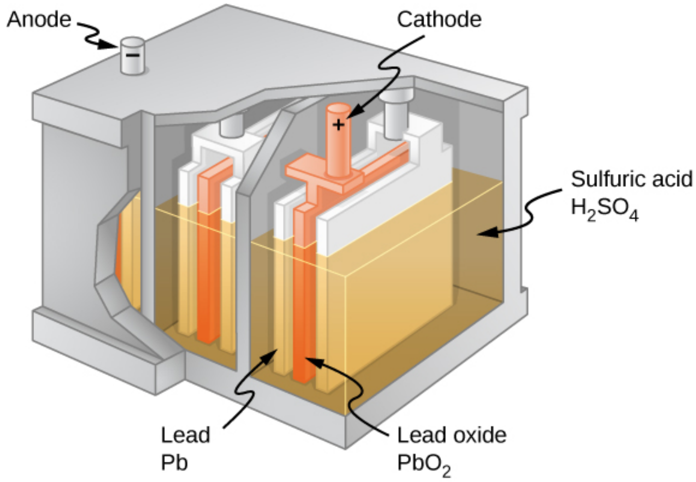


Figure $6.2.3$: Chemical reactions in a lead-acid cell separate charge, sending negative charge to the anode, which is connected to the lead plates. The lead oxide plates are connected to the positive or cathode terminal of the cell. Sulfuric acid conducts the charge, as well as participates in the chemical reaction.

Knowing a little about how the chemicals in a lead-acid battery interact helps in understanding the potential created by the battery. Figure $6.2.4$ shows the result of a single chemical reaction. Two electrons are placed on the **anode**, making it negative, provided that the **cathode** supplies two electrons. This leaves the cathode positively charged, because it has lost two electrons. In short, a separation of charge has been driven by a chemical reaction.

Note that the reaction does not take place unless there is a complete circuit to allow two electrons to be supplied to the cathode. Under many circumstances, these electrons come from the anode, flow through a resistance, and return to the cathode. Note also that since the chemical reactions involve substances with resistance, it is not possible to create the emf without an internal resistance.

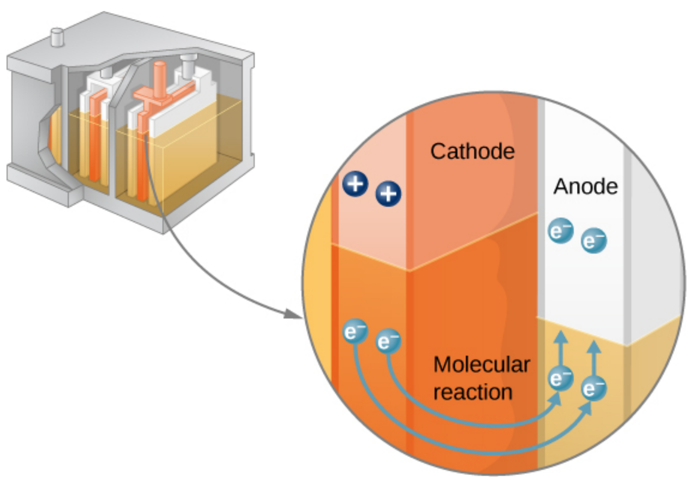


Figure $6.2.4$: In a lead-acid battery, two electrons are forced onto the anode of a cell, and two electrons are removed from the cathode of the cell. The chemical reaction in a lead-acid battery places two electrons on the anode and removes two from the cathode. It requires a closed circuit to proceed, since the two electrons must be supplied to the cathode.

Internal Resistance and Terminal Voltage

The amount of resistance to the flow of current within the voltage source is called the **internal resistance**. The internal resistance **r** of a battery can behave in complex ways. It generally increases as a battery is depleted, due to the oxidation of the plates or the reduction of the acidity of the electrolyte. However, internal resistance may also depend on the magnitude and direction of the current through a voltage source, its temperature, and even its history. The internal resistance of rechargeable nickel-cadmium cells, for example, depends on how many times and how deeply they have been depleted. A simple model for a battery consists of an idealized emf source ϵ and an internal resistance **r** (Figure $6.2.5$).

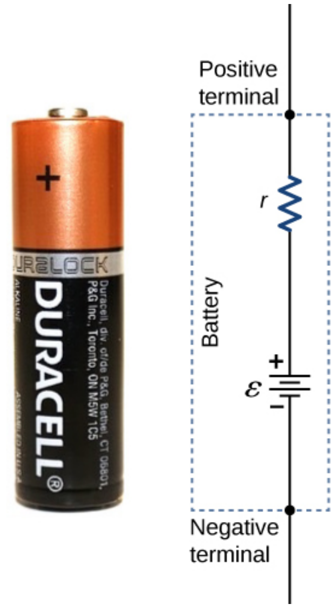


Figure 6.2.5: A battery can be modeled as an idealized emf (ϵ) with an internal resistance (**r**). The terminal voltage of the battery is $V_{terminal} = \epsilon - Ir$.

Suppose an external resistor, known as the load resistance **R**, is connected to a voltage source such as a battery, as in Figure 6.2.6. The figure shows a model of a battery with an emf , an internal resistance **r**, and a load resistor **R** connected across its terminals. Using conventional current flow, positive charges leave the positive terminal of the battery, travel through the resistor, and return to the negative terminal of the battery. The terminal voltage of the battery depends on the emf, the internal resistance, and the current, and is equal to

Note

$$V_{terminal} = \epsilon - Ir$$

For a given emf and internal resistance, the terminal voltage decreases as the current increases due to the potential drop **Ir** of the internal resistance.

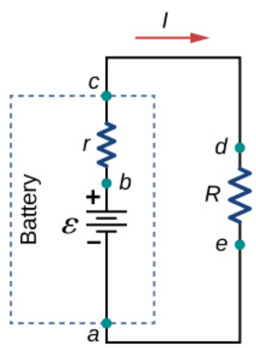


Figure 6.2.6: Schematic of a voltage source and its load resistor **R**. Since the internal resistance **r** is in series with the load, it can significantly affect the terminal voltage and the current delivered to the load.

A graph of the potential difference across each element the circuit is shown in Figure 6.2.7. A current **I** runs through the circuit, and the potential drop across the internal resistor is equal to **Ir**. The terminal voltage is equal to $\epsilon - Ir$, which is equal to the **potential drop** across the load resistor $IR = \epsilon - Ir$. As with potential energy, it is the change in voltage that is important. When the term "voltage" is used, we assume that it is actually the change in the potential, or ΔV. However, Δ is often omitted for convenience.

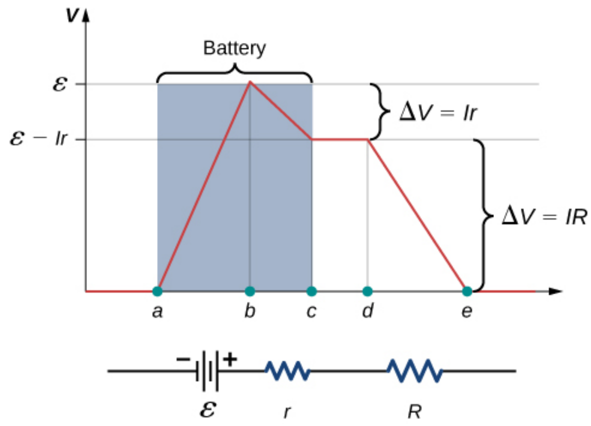


Figure 6.2.7: A graph of the voltage through the circuit of a battery and a load resistance. The electric potential increases the emf of the battery due to the chemical reactions doing work on the charges. There is a decrease in the electric potential in the battery due to the internal resistance. The potential decreases due to the internal resistance $-Ir$, making the terminal voltage of the battery equal to $(\epsilon - Ir)$. The voltage then decreases by (**IR**). The current is equal to $I = \frac{\epsilon}{r+R}$.

The current through the load resistor is $I = \frac{\epsilon}{r+R}$. We see from this expression that the smaller the internal resistance **r**, the greater the current the voltage source supplies to its load **R**. As batteries are depleted, **r** increases. If **r** becomes a significant fraction of the load resistance, then the current is significantly reduced, as the following example illustrates.

Example 6.2.1: Analyzing a Circuit with a Battery and a Load

A given battery has a 12.00-V emf and an internal resistance of $0.100\,\Omega$ (a) Calculate its terminal voltage when connected to a $10.00\,\Omega$ load. (b) What is the terminal voltage when connected to a $0.500\,\Omega$ load? (c) What power does the $0.500\,\Omega$ load dissipate? (d) If the internal resistance grows to $0.500\,\Omega$, find the current, terminal voltage, and power dissipated by a $0.500\,\Omega$ load.

Strategy

The analysis above gave an expression for current when internal resistance is taken into account. Once the current is found, the terminal voltage can be calculated by using the equation $V_{terminal} = \epsilon - Ir$. Once current is found, we can also find the power dissipated by the resistor.

Solution

1. Entering the given values for the emf, load resistance, and internal resistance into the expression above yields

$$I = \frac{\epsilon}{R+r} = \frac{12.00\,V}{10.10\,\Omega} = 1.188\,A.$$

Enter the known values into the equation $V_{terminal} = \epsilon - Ir$ to get the terminal voltage:

$$V_{terminal} = \epsilon - Ir = 12.00\,V - (1.188\,A)(0.100\,\Omega) = 11.90\,V.$$

The terminal voltage here is only slightly lower than the emf, implying that the current drawn by this light load is not significant.

2. Similarly, with $R_{load} = 0.500\,\Omega$, the current is

$$I = \frac{\epsilon}{R+r} = \frac{12.00\,V}{0.600\,\Omega} = 20.00\,A.$$

The terminal voltage is now

$$V_{terminal} = \epsilon - Ir = 12.00\,V - (20.00\,A)(0.100\,\Omega) = 10.00\,V.$$

The terminal voltage exhibits a more significant reduction compared with emf, implying $0.500\,\Omega$ is a heavy load for this battery. A “heavy load” signifies a larger draw of current from the source but not a larger resistance.

3. The power dissipated by the $0.500\,\Omega$ load can be found using the formula $P = I^2R$. Entering the known values gives

$$P = I^2R = (20.0\,A)^2(0.500\,\Omega) = 2.00 \times 10^2\,W.$$

Note that this power can also be obtained using the expression $\frac{V^2}{R}$ or IV, where **V** is the terminal voltage (10.0 V in this case).

4. Here, the internal resistance has increased, perhaps due to the depletion of the battery, to the point where it is as great as the load resistance. As before, we first find the current by entering the known values into the expression, yielding

$$I=\frac{\epsilon}{R+r}=\frac{12.00\,V}{1.00\,\Omega}=12.00\,A.$$

Now the terminal voltage is

$$V_{terminal}=\epsilon-Ir=12.00\,V-(12.00\,A)(0.500\,\Omega)=6.00\,V,$$

and the power dissipated by the load is

$$P=I^2R=(12.00\,A)^2(0.500\,\Omega)=72.00\,W.$$

We see that the increased internal resistance has significantly decreased the terminal voltage, current, and power delivered to a load.

Significance

The internal resistance of a battery can increase for many reasons. For example, the internal resistance of a rechargeable battery increases as the number of times the battery is recharged increases. The increased internal resistance may have two effects on the battery. First, the terminal voltage will decrease. Second, the battery may overheat due to the increased power dissipated by the internal resistance.

? Exercise 6.2.1

If you place a wire directly across the two terminal of a battery, effectively shorting out the terminals, the battery will begin to get hot. Why do you suppose this happens?

Solution

If a wire is connected across the terminals, the load resistance is close to zero, or at least considerably less than the internal resistance of the battery. Since the internal resistance is small, the current through the circuit will be large, $I=\frac{\epsilon}{R+r}=\frac{\epsilon}{0+r}=\frac{\epsilon}{r}$. The large current causes a high power to be dissipated by the internal resistance $(P=I^2r)$. The power is dissipated as heat.

Battery Testers

Battery testers, such as those in Figure $6.2.8$, use small load resistors to intentionally draw current to determine whether the terminal potential drops below an acceptable level. Although it is difficult to measure the internal resistance of a battery, battery testers can provide a measurement of the internal resistance of the battery. If internal resistance is high, the battery is weak, as evidenced by its low terminal voltage.

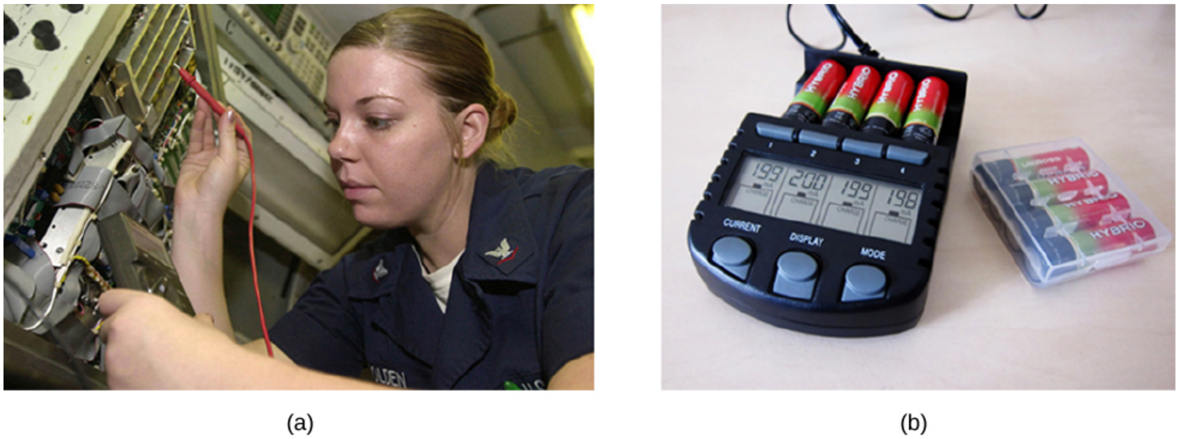


Figure 6.2.8: Battery testers measure terminal voltage under a load to determine the condition of a battery. (a) A US Navy electronics technician uses a battery tester to test large batteries aboard the aircraft carrier USS **Nimitz**. The battery tester she uses has a small resistance that can dissipate large amounts of power. (b) The small device shown is used on small batteries and has a digital display to indicate the acceptability of the terminal voltage. (credit a: modification of work by Jason A. Johnston; credit b: modification of work by Keith Williamson)

Some batteries can be recharged by passing a current through them in the direction opposite to the current they supply to an appliance. This is done routinely in cars and in batteries for small electrical appliances and electronic devices (Figure 6.2.9). The voltage output of the battery charger must be greater than the emf of the battery to reverse the current through it. This causes the terminal voltage of the battery to be greater than the emf, since $V = \epsilon - Ir$ and **I** is now negative.

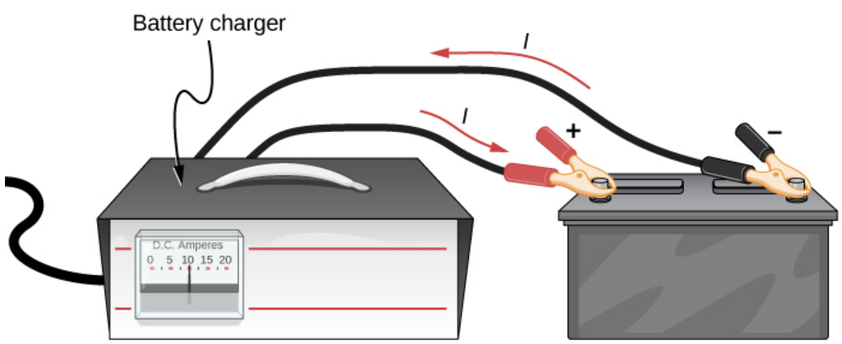


Figure 6.2.9: A car battery charger reverses the normal direction of current through a battery, reversing its chemical reaction and replenishing its chemical potential.

It is important to understand the consequences of the internal resistance of emf sources, such as batteries and solar cells, but often, the analysis of circuits is done with the terminal voltage of the battery, as we have done in the previous sections. The terminal voltage is referred to as simply as **V**, dropping the subscript "terminal." This is because the internal resistance of the battery is difficult to measure directly and can change over time.

6.2.1: Electromotive Force and Internal Resistance

The reader is reminded of the following definition from section 4.1:

Definition. The potential difference across the poles of a cell when no current is being taken from it is called the *electromotive force* (EMF) of the cell.

I shall use the symbol E for EMF.

Question. A 4 Ω resistance is connected across a cell of EMF 2 V. What current flows?

The immediate answer is 0.5 A – but this is likely to be wrong. The reason is that a cell has a resistance of its own – its *internal resistance*. The internal resistance of a lead-acid cell is typically quite small, but most dry cells have an appreciable internal resistance. If the external resistance is R and the internal resistance is r, the total resistance of the circuit is $R+r$, so that the current that flows is $E/(R+r)$.

Whenever a current is taken from a cell (or battery) the potential difference across its poles *drops* to a value less than its EMF. We can think of a cell as an EMF in series with an internal resistance:

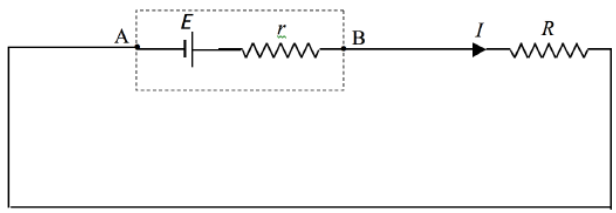


FIGURE IV.4

If we take the point A as having zero potential, we see that the potential of the point B will be E - Ir, and this, then, is the potential difference across the poles of the cell when a current I is being taken from it.

? Show that this can also be written as $\frac{ER}{R+r}$.

6.3: Resistors in Series and Parallel

Learning Objectives

By the end of the section, you will be able to:

- Define the term equivalent resistance
- Calculate the equivalent resistance of resistors connected in series
- Calculate the equivalent resistance of resistors connected in parallel

In Current and Resistance, we described the term 'resistance' and explained the basic design of a resistor. Basically, a resistor limits the flow of charge in a circuit and is an ohmic device where $V = IR$. Most circuits have more than one resistor. If several resistors are connected together and connected to a battery, the current supplied by the battery depends on the **equivalent resistance** of the circuit.

The equivalent resistance of a combination of resistors depends on both their individual values and how they are connected. The simplest combinations of resistors are series and parallel connections (Figure $6.3.1$). In a **series circuit**, the output current of the first resistor flows into the input of the second resistor; therefore, the current is the same in each resistor. In a **parallel circuit**, all of the resistor leads on one side of the resistors are connected together and all the leads on the other side are connected together. In the case of a parallel configuration, each resistor has the same potential drop across it, and the currents through each resistor may be different, depending on the resistor. The sum of the individual currents equals the current that flows into the parallel connections.

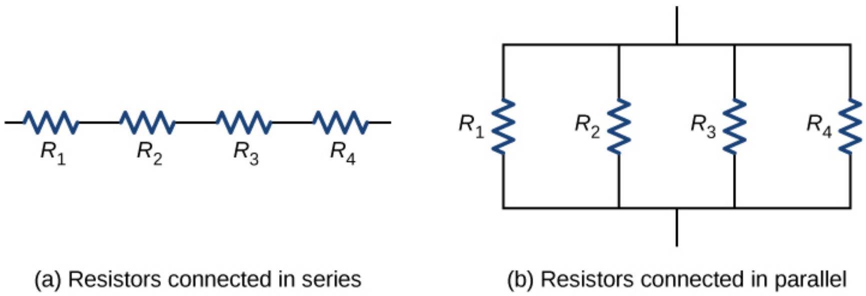


Figure $6.3.1$: (a) For a series connection of resistors, the current is the same in each resistor. (b) For a parallel connection of resistors, the voltage is the same across each resistor.

Resistors in Series

Resistors are said to be in series whenever the current flows through the resistors sequentially. Consider Figure $6.3.2$, which shows three resistors in series with an applied voltage equal to V_{ab}. Since there is only one path for the charges to flow through, the current is the same through each resistor. The equivalent resistance of a set of resistors in a series connection is equal to the algebraic sum of the individual resistances.

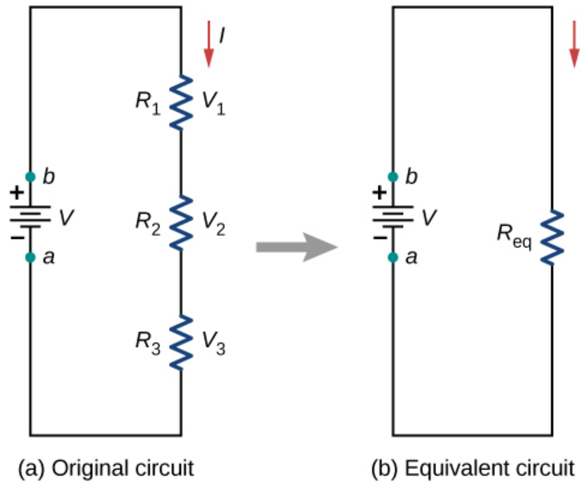


Figure 6.3.2: (a) Three resistors connected in series to a voltage source. (b) The original circuit is reduced to an equivalent resistance and a voltage source.

In Figure 6.3.2, the current coming from the voltage source flows through each resistor, so the current through each resistor is the same. The current through the circuit depends on the voltage supplied by the voltage source and the resistance of the resistors. For each resistor, a potential drop occurs that is equal to the loss of electric potential energy as a current travels through each resistor. According to Ohm's law, the potential drop V across a resistor when a current flows through it is calculated using the equation $V = IR$, where I is the current in amps (A) and R is the resistance in ohms (Ω). Since energy is conserved, and the voltage is equal to the potential energy per charge, the sum of the voltage applied to the circuit by the source and the potential drops across the individual resistors around a loop should be equal to zero:

$$\sum_{i=1}^{N} V_i = 0.$$

This equation is often referred to as Kirchhoff's loop law, which we will look at in more detail later in this chapter. For Figure 6.3.2, the sum of the potential drop of each resistor and the voltage supplied by the voltage source should equal zero:

$$\begin{aligned} V - V_1 - V_2 - V_3 &= 0, \\ V &= V_1 + V_2 + V_3, \\ &= IR_1 + IR_2 + IR_3, \end{aligned}$$

Solving for I

$$\begin{aligned} I &= \frac{V}{R_1 + R_2 + R_3} \\ &= \frac{V}{R_S}. \end{aligned}$$

Since the current through each component is the same, the equality can be simplified to an equivalent resistance (R_S), which is just the sum of the resistances of the individual resistors.

Equivalent Resistance in Series Circuits

Any number of resistors can be connected in series. If N resistors are connected in series, the **equivalent resistance** is

$$R_S = R_1 + R_2 + R_3 + \ldots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i. \tag{6.3.1}$$

One result of components connected in a series circuit is that if something happens to one component, it affects all the other components. For example, if several lamps are connected in series and one bulb burns out, all the other lamps go dark.

Example 6.3.1: Equivalent Resistance, Current, and Power in a Series Circuit

A battery with a terminal voltage of 9 V is connected to a circuit consisting of four $20\,\Omega$ and one $10\,\Omega$ resistors all in series (Figure 6.3.3). Assume the battery has negligible internal resistance.

a. Calculate the equivalent resistance of the circuit.
b. Calculate the current through each resistor.
c. Calculate the potential drop across each resistor.
d. Determine the total power dissipated by the resistors and the power supplied by the battery.

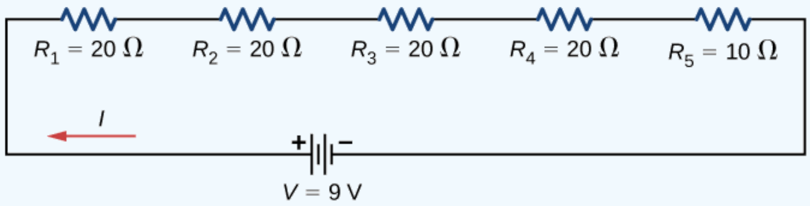


Figure 6.3.3: A simple series circuit with five resistors.

Strategy

In a series circuit, the equivalent resistance is the algebraic sum of the resistances. The current through the circuit can be found from Ohm's law and is equal to the voltage divided by the equivalent resistance. The potential drop across each resistor can be found using Ohm's law. The power dissipated by each resistor can be found using $P = I^2R$, and the total power dissipated by the resistors is equal to the sum of the power dissipated by each resistor. The power supplied by the battery can be found using $P = I\epsilon$.

Solution

1. The equivalent resistance is the algebraic sum of the resistances (Equation 6.3.1):

$$\begin{aligned} R_S &= R_1 + R_2 + R_3 + R_4 + R_5 \\ &= 20\,\Omega + 20\,\Omega + 20\,\Omega + 20\,\Omega + 10\,\Omega = 90\,\Omega. \end{aligned}$$

2. The current through the circuit is the same for each resistor in a series circuit and is equal to the applied voltage divided by the equivalent resistance:

$$I = \frac{V}{R_S} = \frac{9\,V}{90\,\Omega} = 0.1\,A.$$

Note that the sum of the potential drops across each resistor is equal to the voltage supplied by the battery.

3. The power dissipated by a resistor is equal to $P = I^2R$, and the power supplied by the battery is equal to $P = I\epsilon$.

$$P_1 = P_2 = P_3 = P_4 = (0.1\,A)^2(20\,\Omega) = 0.2\,W,$$

$$P_5 = (0.1\,A)^2(10\,\Omega) = 0.1\,W,$$

$$P_{dissipated} = 0.2\,W + 0.2\,W + 0.2\,W + 0.2\,W + 0.1\,W = 0.9\,W,$$

$$P_{source} = I\epsilon = (0.1\,A)(9\,V) = 0.9\,W.$$

Significance

There are several reasons why we would use multiple resistors instead of just one resistor with a resistance equal to the equivalent resistance of the circuit. Perhaps a resistor of the required size is not available, or we need to dissipate the heat generated, or we want to minimize the cost of resistors. Each resistor may cost a few cents to a few dollars, but when multiplied by thousands of units, the cost saving may be appreciable.

? Exercise 6.3.1

Some strings of miniature holiday lights are made to short out when a bulb burns out. The device that causes the short is called a shunt, which allows current to flow around the open circuit. A "short" is like putting a piece of wire across the component.

The bulbs are usually grouped in series of nine bulbs. If too many bulbs burn out, the shunts eventually open. What causes this?

Answer

The equivalent resistance of nine bulbs connected in series is 9**R**. The current is $I = V/9\,R$. If one bulb burns out, the equivalent resistance is 8**R**, and the voltage does not change, but the current increases $(I = V/8\,R$. As more bulbs burn out, the current becomes even higher. Eventually, the current becomes too high, burning out the shunt.

Let's briefly summarize the major features of resistors in series:

1. Series resistances add together to get the equivalent resistance (Equation 6.3.1):

$$R_S = R_1 + R_2 + R_3 + \ldots + R_{N-1} + R_N = \sum_{i=1}^{N} R_i.$$

2. The same current flows through each resistor in series.
3. Individual resistors in series do not get the total source voltage, but divide it. The total potential drop across a series configuration of resistors is equal to the sum of the potential drops across each resistor.

Resistors in Parallel

Figure $6.3.4$ shows resistors in parallel, wired to a voltage source. Resistors are in parallel when one end of all the resistors are connected by a continuous wire of negligible resistance and the other end of all the resistors are also connected to one another through a continuous wire of negligible resistance. The potential drop across each resistor is the same. Current through each resistor can be found using Ohm's law $I = V/R$, where the voltage is constant across each resistor. For example, an automobile's headlights, radio, and other systems are wired in parallel, so that each subsystem utilizes the full voltage of the source and can operate completely independently. The same is true of the wiring in your house or any building.

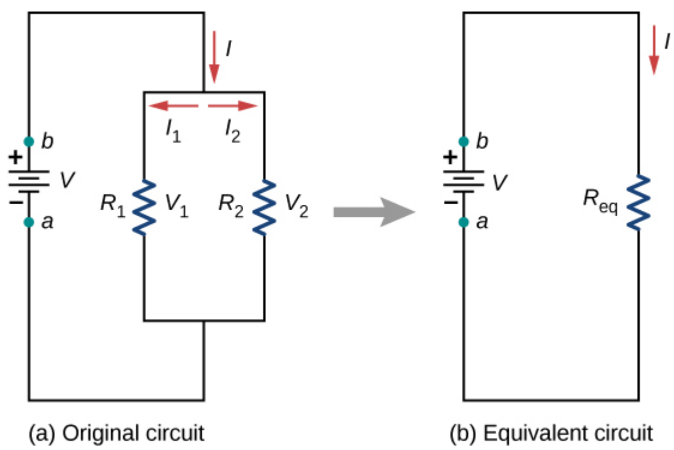


Figure $6.3.4$: Two resistors connected in parallel to a voltage source. (b) The original circuit is reduced to an equivalent resistance and a voltage source.

The current flowing from the voltage source in Figure $6.3.4$ depends on the voltage supplied by the voltage source and the equivalent resistance of the circuit. In this case, the current flows from the voltage source and enters a junction, or node, where the circuit splits flowing through resistors R_1 and R_2. As the charges flow from the battery, some go through resistor R_1 and some flow through resistor R_2. The sum of the currents flowing into a junction must be equal to the sum of the currents flowing out of the junction:

$$\sum I_{in} = \sum I_{out}.$$

This equation is referred to as **Kirchhoff's junction rule** and will be discussed in detail in the next section. In Figure $6.3.4$, the junction rule gives $I = I_1 + I_2$. There are two loops in this circuit, which leads to the equations $V = I_1 R_1$ and $I_1 R_1 = I_2 R_2$. Note the voltage across the resistors in parallel are the same ($V = V_1 = V_2$) and the current is additive:

$$\begin{aligned} I &= I_1 + I_2 \\ &= \frac{V_1}{R_1} + \frac{V_2}{R_2} \\ &= \frac{V}{R_1} + \frac{V}{R_2} \\ &= V\left(\frac{1}{R_1} + \frac{1}{R_2}\right) = \frac{V}{R_P} \end{aligned}$$

Solving for the R_P

$$R_P = \left(\frac{1}{R_1} + \frac{1}{R_2}\right)^{-1}.$$

Equivalent Resistance in Parallel Circuits

Generalizing to any number of N resistors, the equivalent resistance R_P of a parallel connection is related to the individual resistances by

$$R_P = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots + \frac{1}{R_{N-1}} + \frac{1}{R_N}\right)^{-1} = \left(\sum_{i=1}^{N} \frac{1}{R_i}\right)^{-1}. \tag{6.3.2}$$

This relationship results in an equivalent resistance R_P that is less than the smallest of the individual resistances. When resistors are connected in parallel, more current flows from the source than would flow for any of them individually, so the total resistance is lower.

Example 6.3.2: Analysis of a parallel circuit

Three resistors $R_1 = 1.00\,\Omega$, $R_2 = 2.00\,\Omega$, and $R_3 = 2.00\,\Omega$, are connected in parallel. The parallel connection is attached to a $V = 3.00\,V$ voltage source.

a. What is the equivalent resistance?
b. Find the current supplied by the source to the parallel circuit.
c. Calculate the currents in each resistor and show that these add together to equal the current output of the source.
d. Calculate the power dissipated by each resistor.
e. Find the power output of the source and show that it equals the total power dissipated by the resistors.

Strategy

(a) The total resistance for a parallel combination of resistors is found using Equation 6.3.2. (Note that in these calculations, each intermediate answer is shown with an extra digit.)

(b) The current supplied by the source can be found from Ohm's law, substituting R_P for the total resistance $I = \frac{V}{R_P}$.

(c) The individual currents are easily calculated from Ohm's law $\left(I_i = \frac{V_i}{R_i}\right)$, since each resistor gets the full voltage. The total current is the sum of the individual currents:

$$I = \sum_i I_i.$$

(d) The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three are known. Let us use $P_i = V^2/R_i$, since each resistor gets full voltage.

(e) The total power can also be calculated in several ways, use $P = IV$.

Solution

1. The total resistance for a parallel combination of resistors is found using Equation 6.3.2. Entering known values gives

$$R_P = \left(\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = \left(\frac{1}{1.00\,\Omega} + \frac{1}{2.00\,\Omega} + \frac{1}{2.00\,\Omega}\right)^{-1} = 0.50\,\Omega.$$

The total resistance with the correct number of significant digits is $R_{eq} = 0.50\,\Omega$. As predicted, R_P is less than the smallest individual resistance.

2. The total current can be found from Ohm's law, substituting R_P for the total resistance. This gives

$$I = \frac{V}{R_P} = \frac{3.00\,V}{0.50\,\Omega} = 6.00\,A.$$

Current **I** for each device is much larger than for the same devices connected in series (see the previous example). A circuit with parallel connections has a smaller total resistance than the resistors connected in series.

3. The individual currents are easily calculated from Ohm's law, since each resistor gets the full voltage. Thus,

$$I_1 = \frac{V}{R_1} = \frac{3.00\,V}{1.00\,\Omega} = 3.00\,A.$$

Similarly,

$$I_2 = \frac{V}{R_2} = \frac{3.00\,V}{2.00\,\Omega} = 1.50\,A$$

and

$$I_3 = \frac{V}{R_3} = \frac{3.00\,V}{2.00\,\Omega} = 1.50\,A.$$

The total current is the sum of the individual currents:

$$I_1 + I_2 + I_3 = 6.00\,A.$$

4. The power dissipated by each resistor can be found using any of the equations relating power to current, voltage, and resistance, since all three are known. Let us use $P = V^2/R$, since each resistor gets full voltage. Thus,

$$P_1 = \frac{V^2}{R_1} = \frac{(3.00\,V)^2}{1.00\,\Omega} = 9.00\,W.$$

Similarly,

$$P_2 = \frac{V^2}{R_2} = \frac{(3.00\,V)^2}{2.00\,\Omega} = 4.50\,W.$$

and

$$P_3 = \frac{V^2}{R_3} = \frac{(3.00\,V)^2}{2.00\,\Omega} = 4.50\,W.$$

5. The total power can also be calculated in several ways. Choosing $P = IV$ and entering the total current yields

$$P = IV = (6.00\,A)(3.00\,V) = 18.00\,W.$$

Significance

Total power dissipated by the resistors is also 18.00 W:

$$P_1 + P_2 + P_3 = 9.00\,W + 4.50\,W + 4.50\,W = 18.00\,W.$$

Notice that the total power dissipated by the resistors equals the power supplied by the source.

? Exercise $6.3.2A$

Consider the same potential difference $(V = 3.00\,V)$ applied to the same three resistors connected in series. Would the equivalent resistance of the series circuit be higher, lower, or equal to the three resistor in parallel? Would the current through the series circuit be higher, lower, or equal to the current provided by the same voltage applied to the parallel circuit? How would the power dissipated by the resistor in series compare to the power dissipated by the resistors in parallel?

Solution

The equivalent of the series circuit would be $R_{eq} = 1.00\,\Omega + 2.00\,\Omega + 2.00\,\Omega = 5.00\,\Omega$, which is higher than the equivalent resistance of the parallel circuit $R_{eq} = 0.50\,\Omega$. The equivalent resistor of any number of resistors is always higher than the equivalent resistance of the same resistors connected in parallel. The current through for the series circuit would be $I = \frac{3.00\,V}{5.00\,\Omega} = 0.60\,A$, which is lower than the sum of the currents through each resistor in the parallel circuit, $I = 6.00\,A$. This is not surprising since the equivalent resistance of the series circuit is higher. The current through a series connection of any number of resistors will always be lower than the current into a parallel connection of the same resistors, since the equivalent resistance of the series circuit will be higher than the parallel circuit. The power dissipated by the resistors in series would be $P = 1.800\,W$, which is lower than the power dissipated in the parallel circuit $P = 18.00\,W$.

? Exercise $6.3.2B$

How would you use a river and two waterfalls to model a parallel configuration of two resistors? How does this analogy break down?

Solution

A river, flowing horizontally at a constant rate, splits in two and flows over two waterfalls. The water molecules are analogous to the electrons in the parallel circuits. The number of water molecules that flow in the river and falls must be equal to the number of molecules that flow over each waterfall, just like sum of the current through each resistor must be equal to the current flowing into the parallel circuit. The water molecules in the river have energy due to their motion and height. The potential energy of the water molecules in the river is constant due to their equal heights. This is analogous to the constant change in voltage across a parallel circuit. Voltage is the potential energy across each resistor.

The analogy quickly breaks down when considering the energy. In the waterfall, the potential energy is converted into kinetic energy of the water molecules. In the case of electrons flowing through a resistor, the potential drop is converted into heat and light, not into the kinetic energy of the electrons.

Let us summarize the major features of resistors in parallel:

1. Equivalent resistance is found from Equation $6.3.2$ and is smaller than any individual resistance in the combination.
2. The potential drop across each resistor in parallel is the same.
3. Parallel resistors do not each get the total current; they divide it. The current entering a parallel combination of resistors is equal to the sum of the current through each resistor in parallel.

In this chapter, we introduced the equivalent resistance of resistors connect in series and resistors connected in parallel. You may recall from the Section on Capacitance, we introduced the equivalent capacitance of capacitors connected in series and parallel. Circuits often contain both capacitors and resistors. Table $6.3.1$ summarizes the equations used for the equivalent resistance and equivalent capacitance for series and parallel connections.

Table $6.3.1$: Summary for Equivalent Resistance and Capacitance in Series and Parallel Combinations

	Series combination	Parallel combination
Equivalent capacitance	$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \ldots$	$C_P = C_1 + C_2 + C_3 + \ldots$
Equivalent resistance	$R_S = R_1 + R_2 + R_3 + \ldots = \sum_{i=1}^{N} R_i$	$\frac{1}{R_P} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3} + \ldots$

Combinations of Series and Parallel

More complex connections of resistors are often just combinations of series and parallel connections. Such combinations are common, especially when wire resistance is considered. In that case, wire resistance is in series with other resistances that are in parallel.

Combinations of series and parallel can be reduced to a single equivalent resistance using the technique illustrated in Figure $6.3.5$. Various parts can be identified as either series or parallel connections, reduced to their equivalent resistances, and then further reduced until a single equivalent resistance is left. The process is more time consuming than difficult. Here, we note the equivalent resistance as R_{eq}.

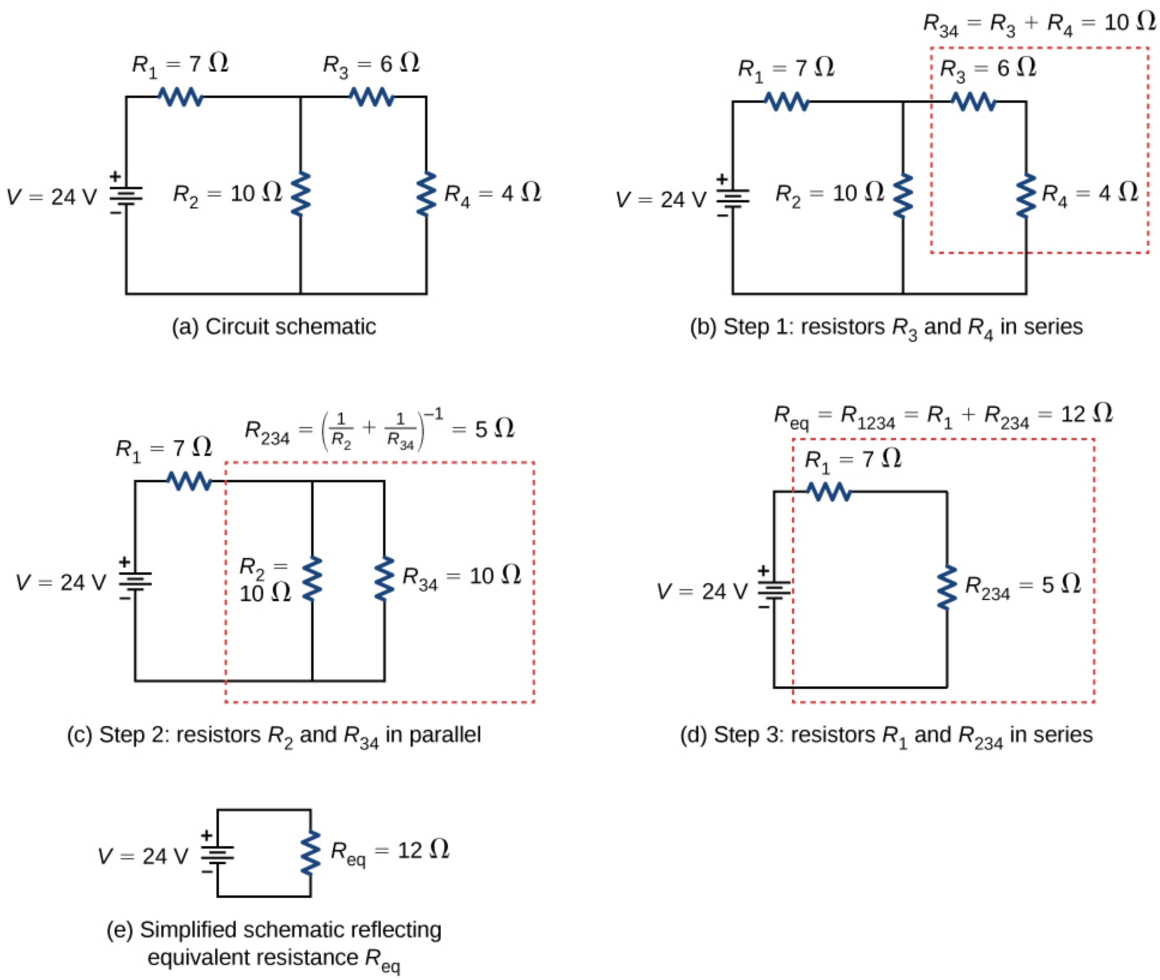


Figure $6.3.5$: (a) The original circuit of four resistors. (b) Step 1: The resistors R_3 and R_4 are in series and the equivalent resistance is $R_{34} = 10\,\Omega$ (c) Step 2: The reduced circuit shows resistors R_2 and R_{34} are in parallel, with an equivalent resistance of $R_{234} = 5\,\Omega$. (d) Step 3: The reduced circuit shows that R_1 and R_{234} are in series with an equivalent resistance of $R_{1234} = 12\,\Omega$ which is the equivalent resistance R_{eq}. (e) The reduced circuit with a voltage source of $V = 24\,V$ with an equivalent resistance of $R_{eq} = 12\,\Omega$. This results in a current of $I = 2\,A$ from the voltage source.

Notice that resistors R_3 and R_4 are in series. They can be combined into a single equivalent resistance. One method of keeping track of the process is to include the resistors as subscripts. Here the equivalent resistance of R_3 and R_4 is

$$R_{34} = R_3 + R_4 = 6\,\Omega + 4\,\Omega = 10\,\Omega.$$

The circuit now reduces to three resistors, shown in Figure $6.3.5c$. Redrawing, we now see that resistors R_2 and R_{34} constitute a parallel circuit. Those two resistors can be reduced to an equivalent resistance:

$$R_{234} = \left(\frac{1}{R_2} + \frac{1}{R_{34}}\right)^{-1} = \left(\frac{1}{10\,\Omega} + \frac{1}{10\,\Omega}\right)^{-1} = 5\,\Omega.$$

This step of the process reduces the circuit to two resistors, shown in in Figure $6.3.5d$. Here, the circuit reduces to two resistors, which in this case are in series. These two resistors can be reduced to an equivalent resistance, which is the equivalent resistance of the circuit:

$$R_{eq} = R_{1234} = R_1 + R_{234} = 7\,\Omega + 5\Omega = 12\,\Omega.$$

The main goal of this circuit analysis is reached, and the circuit is now reduced to a single resistor and single voltage source.

Now we can analyze the circuit. The current provided by the voltage source is $I = \frac{V}{R_{eq}} = \frac{24\,V}{12\,\Omega} = 2\,A$. This current runs through resistor R_1 and is designated as I_1. The potential drop across R_1 can be found using Ohm's law:

$$V_1 = I_1 R_1 = (2\,A)(7\,\Omega) = 14\,V.$$

Looking at Figure $6.3.5c$, this leaves $24\,V - 14\,V = 10\,V$ to be dropped across the parallel combination of R_2 and R_{34}. The current through R_2 can be found using Ohm's law:

$$I_2 = \frac{V_2}{R_2} = \frac{10\,V}{10\,\Omega} = 1\,A.$$

The resistors R_3 and R_4 are in series so the currents I_3 and I_4 are equal to

$$I_3 = I_4 = I - I_2 = 2\,A - 1\,A = 1\,A.$$

Using Ohm's law, we can find the potential drop across the last two resistors. The potential drops are $V_3 = I_3 R_3 = 6\,V$ and $V_4 = I_4 R_4 = 4\,V$. The final analysis is to look at the power supplied by the voltage source and the power dissipated by the resistors. The power dissipated by the resistors is

$$\begin{aligned} P_1 &= I_1^2 R_1 = (2\,A)^2(7\,\Omega) = 28\,W, \\ P_2 &= I_2^2 R_2 = (1\,A)^2(10\,\Omega) = 10\,W, \\ P_3 &= I_3^2 R_3 = (1\,A)^2(6\,\Omega) = 6\,W, \\ P_4 &= I_4^2 R_4 = (1\,A)^2(4\,\Omega) = 4\,W, \\ P_{dissipated} &= P_1 + P_2 + P_3 + P_4 = 48\,W. \end{aligned}$$

The total energy is constant in any process. Therefore, the power supplied by the voltage source is

$$\begin{aligned} P_s &= IV \\ &= (2\,A)(24\,V) = 48\,W \end{aligned}$$

Analyzing the power supplied to the circuit and the power dissipated by the resistors is a good check for the validity of the analysis; they should be equal.

✓ Example 6.3.3: Combining Series and parallel circuits

Figure $6.3.6$ shows resistors wired in a combination of series and parallel. We can consider R_1 to be the resistance of wires leading to R_2 and R_3.

a. Find the equivalent resistance of the circuit.
b. What is the potential drop V_1 across resistor R_1?
c. Find the current I_2 through resistor R_2.
d. What power is dissipated by R_2?

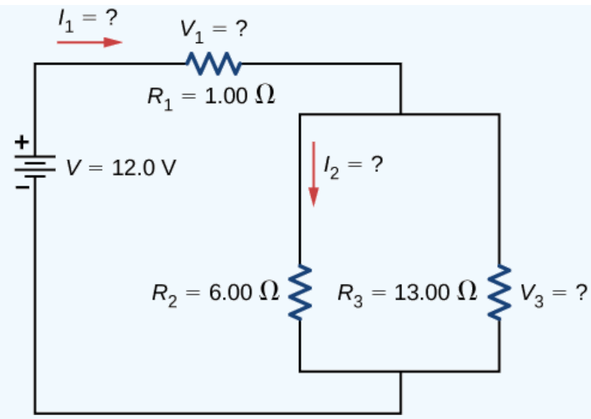


Figure $6.3.6$: These three resistors are connected to a voltage source so that R_2 and R_3 are in parallel with one another and that combination is in series with R_1.

Strategy

(a) To find the equivalent resistance, first find the equivalent resistance of the parallel connection of R_2 and R_3. Then use this result to find the equivalent resistance of the series connection with R_1.

(b) The current through R_1 can be found using Ohm's law and the voltage applied. The current through R_1 is equal to the current from the battery. The potential drop V_1 across the resistor R_1 (which represents the resistance in the connecting wires) can be found using Ohm's law.

(c) The current through R_2 can be found using Ohm's law $I_2 = \frac{V_2}{R_2}$. The voltage across R_2 can be found using $V_2 = V - V_1$.

(d) Using Ohm's law $(V_2 = I_2 R_2)$, the power dissipated by the resistor can also be found using $P_2 = I_2^2 R_2 = \frac{V_2^2}{R_2}$.

Solution

1. To find the equivalent resistance of the circuit, notice that the parallel connection of R_2 and R_3 is in series with R_1, so the equivalent resistance is

$$R_{eq} = R_1 + \left(\frac{1}{R_2} + \frac{1}{R_3}\right)^{-1} = 1.00\,\Omega + \left(\frac{1}{6.00\,\Omega} + \frac{1}{13.00\,\Omega}\right)^{-1} = 5.10\,\Omega.$$

   The total resistance of this combination is intermediate between the pure series and pure parallel values ($20.0\,\Omega$ and $0.804\,\Omega$, respectively).

2. The current through R_1 is equal to the current supplied by the battery:

$$I_1 = I = \frac{V}{R_{eq}} = \frac{12.0\,V}{5.10\,\Omega} = 2.35\,A.$$

   The voltage across R_1 is

$$V_1 = I_1 R_1 = (2.35\,A)(1\,\Omega) = 2.35\,V.$$

   The voltage applied to R_2 and R_3 is less than the voltage supplied by the battery by an amount V_1. When wire resistance is large, it can significantly affect the operation of the devices represented by R_2 and R_3.

3. To find the current through R_2, we must first find the voltage applied to it. The voltage across the two resistors in parallel is the same:

$$V_2 = V_3 = V - V_1 = 12.0\,V - 2.35\,V = 9.65\,V.$$

   Now we can find the current I_2 through resistance R_2 using Ohm's law:

$$I_2 = \frac{V_2}{R_2} = \frac{9.65\,V}{6.00\,\Omega} = 1.61\,A.$$

   The current is less than the 2.00 A that flowed through R_2 when it was connected in parallel to the battery in the previous parallel circuit example.

4. The power dissipated by R_2 is given by

$$P_2 = I_2^2 R_2 = (1.61\,A)^2 (6.00\,\Omega) = 15.5\,W.$$

Significance

The analysis of complex circuits can often be simplified by reducing the circuit to a voltage source and an equivalent resistance. Even if the entire circuit cannot be reduced to a single voltage source and a single equivalent resistance, portions of the circuit may be reduced, greatly simplifying the analysis.

? Exercise 6.3.3

Consider the electrical circuits in your home. Give at least two examples of circuits that must use a combination of series and parallel circuits to operate efficiently.

Solution

All the overhead lighting circuits are in parallel and connected to the main supply line, so when one bulb burns out, all the overhead lighting does not go dark. Each overhead light will have at least one switch in series with the light, so you can turn it on and off.

A refrigerator has a compressor and a light that goes on when the door opens. There is usually only one cord for the refrigerator to plug into the wall. The circuit containing the compressor and the circuit containing the lighting circuit are in parallel, but there is a switch in series with the light. A thermostat controls a switch that is in series with the compressor to control the temperature of the refrigerator.

Practical Implications

One implication of this last example is that resistance in wires reduces the current and power delivered to a resistor. If wire resistance is relatively large, as in a worn (or a very long) extension cord, then this loss can be significant. If a large current is drawn, the **IR** drop in the wires can also be significant and may become apparent from the heat generated in the cord.

For example, when you are rummaging in the **refrigerator** and the motor comes on, the refrigerator light dims momentarily. Similarly, you can see the passenger compartment light dim when you start the engine of your car (although this may be due to resistance inside the battery itself).

What is happening in these high-current situations is illustrated in Figure 6.3.7. The device represented by R_3 has a very low resistance, so when it is switched on, a large current flows. This increased current causes a larger **IR** drop in the wires represented by R_1, reducing the voltage across the light bulb (which is R_2), which then dims noticeably.

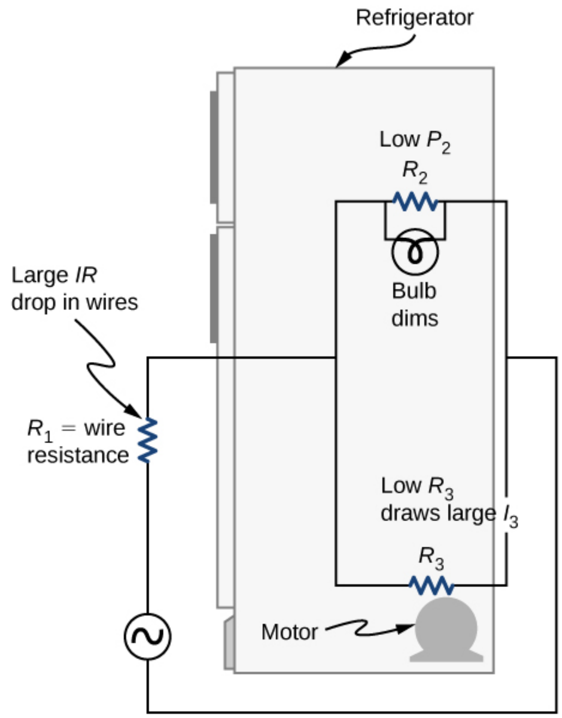


Figure 6.3.7: Why do lights dim when a large appliance is switched on? The answer is that the large current the appliance motor draws causes a significant **IR** drop in the wires and reduces the voltage across the light.

Problem-Solving Strategy: Series and Parallel Resistors

1. Draw a clear circuit diagram, labeling all resistors and voltage sources. This step includes a list of the known values for the problem, since they are labeled in your circuit diagram.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
3. Determine whether resistors are in series, parallel, or a combination of both series and parallel. Examine the circuit diagram to make this assessment. Resistors are in series if the same current must pass sequentially through them.
4. Use the appropriate list of major features for series or parallel connections to solve for the unknowns. There is one list for series and another for parallel.
5. Check to see whether the answers are reasonable and consistent.

Example 6.3.4: Combining Series and Parallel circuits

Two resistors connected in series (R_1, R_2) are connected to two resistors that are connected in parallel (R_3, R_4). The series-parallel combination is connected to a battery. Each resistor has a resistance of 10.00 Ohms. The wires connecting the resistors and battery have negligible resistance. A current of 2.00 Amps runs through resistor R_1. What is the voltage supplied by the voltage source?

Strategy

Use the steps in the preceding problem-solving strategy to find the solution for this example.

Solution

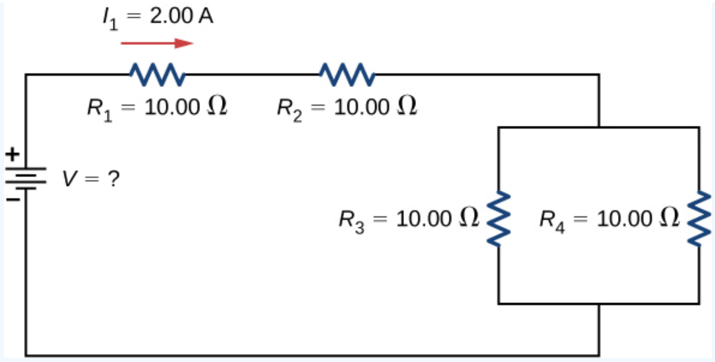


Figure 6.3.8: To find the unknown voltage, we must first find the equivalent resistance of the circuit.

1. Draw a clear circuit diagram (Figure 6.3.8).
2. The unknown is the voltage of the battery. In order to find the voltage supplied by the battery, the equivalent resistance must be found.
3. In this circuit, we already know that the resistors R_1 and R_2 are in series and the resistors R_3 and R_4 are in parallel. The equivalent resistance of the parallel configuration of the resistors R_3 and R_4 is in series with the series configuration of resistors R_1 and R_2.
4. The voltage supplied by the battery can be found by multiplying the current from the battery and the equivalent resistance of the circuit. The current from the battery is equal to the current through R_1 and is equal to 2.00 A. We need to find the equivalent resistance by reducing the circuit. To reduce the circuit, first consider the two resistors in parallel. The equivalent resistance is

$$R_{34} = \left(\frac{1}{10.00\,\Omega} + \frac{1}{10.00\,\Omega}\right)^{-1} = 5.00\,\Omega.$$

   This parallel combination is in series with the other two resistors, so the equivalent resistance of the circuit is $R_{eq} = R_1 + R_2 + R_{34} = (25.00\,\Omega$. The voltage supplied by the battery is therefore $V = IR_{eq} = 2.00\,A(25.00\,\Omega) = 50.00\,V.$
5. One way to check the consistency of your results is to calculate the power supplied by the battery and the power dissipated by the resistors. The power supplied by the battery is $P_{batt} = IV = 100.00\,W$.

Since they are in series, the current through R_2 equals the current through R_1. Since $R_3 = R_4$, the current through each will be 1.00 Amps. The power dissipated by the resistors is equal to the sum of the power dissipated by each resistor:

$$\begin{aligned} P &= I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + I_4^2 R_4 \\ &= 40.00\,W + 40.00\,W + 10.00\,W + 10.00\,W = 100.\,W. \end{aligned}$$

Since the power dissipated by the resistors equals the power supplied by the battery, our solution seems consistent.

Significance

If a problem has a combination of series and parallel, as in this example, it can be reduced in steps by using the preceding problem-solving strategy and by considering individual groups of series or parallel connections. When finding R_{eq} for a parallel connection, the reciprocal must be taken with care. In addition, units and numerical results must be reasonable. Equivalent series resistance should be greater, whereas equivalent parallel resistance should be smaller, for example. Power should be greater for the same devices in parallel compared with series, and so on.

6.4: Kirchhoff's Rules

Learning Objectives

By the end of the section, you will be able to:

- State Kirchhoff’s junction rule
- State Kirchhoff’s loop rule
- Analyze complex circuits using Kirchhoff’s rules

We have just seen that some circuits may be analyzed by reducing a circuit to a single voltage source and an equivalent resistance. Many complex circuits cannot be analyzed with the series-parallel techniques developed in the preceding sections. In this section, we elaborate on the use of Kirchhoff’s rules to analyze more complex circuits. For example, the circuit in Figure $6.4.1$ is known as a **multi-loop circuit**, which consists of junctions. A junction, also known as a node, is a connection of three or more wires. In this circuit, the previous methods cannot be used, because not all the resistors are in clear series or parallel configurations that can be reduced. Give it a try. The resistors R_1 and R_2 are in series and can be reduced to an equivalent resistance. The same is true of resistors R_4 and R_5. But what do you do then?

Even though this circuit cannot be analyzed using the methods already learned, two circuit analysis rules can be used to analyze any circuit, simple or complex. The rules are known as **Kirchhoff’s rules**, after their inventor Gustav **Kirchhoff** (1824–1887).

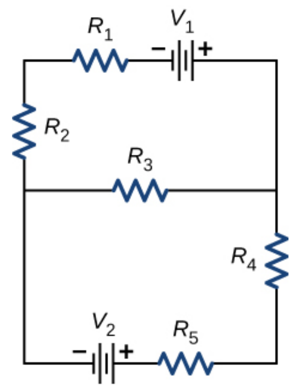


Figure $6.4.1$: This circuit cannot be reduced to a combination of series and parallel connections. However, we can use Kirchhoff’s rules to analyze it.

Kirchhoff’s Rules

- Kirchhoff’s first rule—the junction rule. The sum of all currents entering a junction must equal the sum of all currents leaving the junction:

$$\sum I_{in} = \sum I_{out}.$$

- Kirchhoff’s second rule—the loop rule. The algebraic sum of changes in potential around any closed circuit path (loop) must be zero:

$$\sum V = 0.$$

We now provide explanations of these two rules, followed by problem-solving hints for applying them and a worked example that uses them.

Kirchhoff’s First Rule

Kirchhoff’s first rule (the **junction rule**) applies to the charge entering and leaving a junction (Figure $6.4.2$). As stated earlier, a junction, or node, is a connection of three or more wires. Current is the flow of charge, and charge is conserved; thus, whatever charge flows into the junction must flow out.

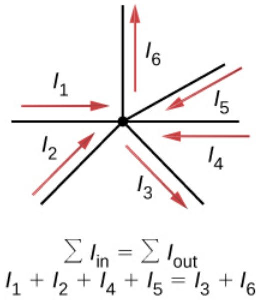


Figure 6.4.2: Charge must be conserved, so the sum of currents into a junction must be equal to the sum of currents out of the junction.

Although it is an over-simplification, an analogy can be made with water pipes connected in a plumbing junction. If the wires in Figure 6.4.2 were replaced by water pipes, and the water was assumed to be incompressible, the volume of water flowing into the junction must equal the volume of water flowing out of the junction.

Kirchhoff's Second Rule

Kirchhoff's second rule (the **loop rule**) applies to potential differences. The loop rule is stated in terms of potential **V** rather than potential energy, but the two are related since $U = qV$. In a closed loop, whatever energy is supplied by a voltage source, the energy must be transferred into other forms by the devices in the loop, since there are no other ways in which energy can be transferred into or out of the circuit. Kirchhoff's loop rule states that the algebraic sum of potential differences, including voltage supplied by the voltage sources and resistive elements, in any loop must be equal to zero. For example, consider a simple loop with no junctions, as in Figure 6.4.3.

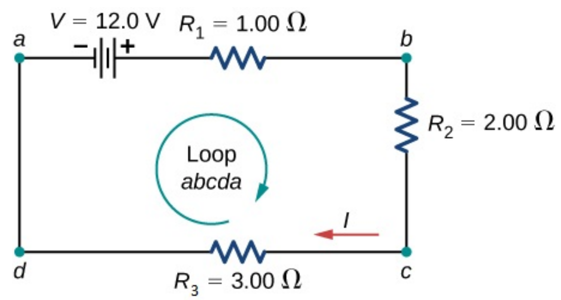


Figure 6.4.3: A simple loop with no junctions. Kirchhoff's loop rule states that the algebraic sum of the voltage differences is equal to zero.

The circuit consists of a voltage source and three external load resistors. The labels **a**, **b**, **c**, and **d** serve as references, and have no other significance. The usefulness of these labels will become apparent soon. The loop is designated as Loop **abcda**, and the labels help keep track of the voltage differences as we travel around the circuit. Start at point **a** and travel to point **b**. The voltage of the voltage source is added to the equation and the potential drop of the resistor R_1 is subtracted. From point **b** to **c**, the potential drop across R_2 is subtracted. From **c** to **d**, the potential drop across R_3 is subtracted. From points **d** to **a**, nothing is done because there are no components.

Figure 6.4.4 shows a graph of the voltage as we travel around the loop. Voltage increases as we cross the battery, whereas voltage decreases as we travel across a resistor. The **potential drop**, or change in the electric potential, is equal to the current through the resistor times the resistance of the resistor. Since the wires have negligible resistance, the voltage remains constant as we cross the wires connecting the components.

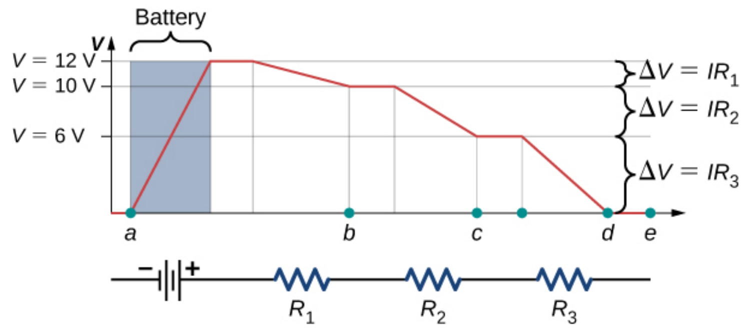


Figure 6.4.4: A voltage graph as we travel around the circuit. The voltage increases as we cross the battery and decreases as we cross each resistor. Since the resistance of the wire is quite small, we assume that the voltage remains constant as we cross the wires connecting the components.

Then Kirchhoff's loop rule states

$$V - IR_1 - IR_2 - IR_3 = 0.$$

The loop equation can be used to find the current through the loop:

$$I = \frac{V}{R_1 + R_2 + R_3} = \frac{12.00\,V}{1.00\,\Omega + 2.00\,\Omega + 3.00\,\Omega} = 2.00\,A.$$

This loop could have been analyzed using the previous methods, but we will demonstrate the power of Kirchhoff's method in the next section.

Applying Kirchhoff's Rules

By applying Kirchhoff's rules, we generate a set of linear equations that allow us to find the unknown values in circuits. These may be currents, voltages, or resistances. Each time a rule is applied, it produces an equation. If there are as many independent equations as unknowns, then the problem can be solved.

Using Kirchhoff's method of analysis requires several steps, as listed in the following procedure.

✔ Problem-Solving Strategy: Kirchhoff's Rules

1. Label points in the circuit diagram using lowercase letters **a**, **b**, **c**, …. These labels simply help with orientation.
2. Locate the junctions in the circuit. The junctions are points where three or more wires connect. Label each junction with the currents and directions into and out of it. Make sure at least one current points into the junction and at least one current points out of the junction.
3. Choose the loops in the circuit. Every component must be contained in at least one loop, but a component may be contained in more than one loop.
4. Apply the junction rule. Again, some junctions should not be included in the analysis. You need only use enough nodes to include every current.
5. Apply the loop rule. Use the map in Figure 6.4.5.

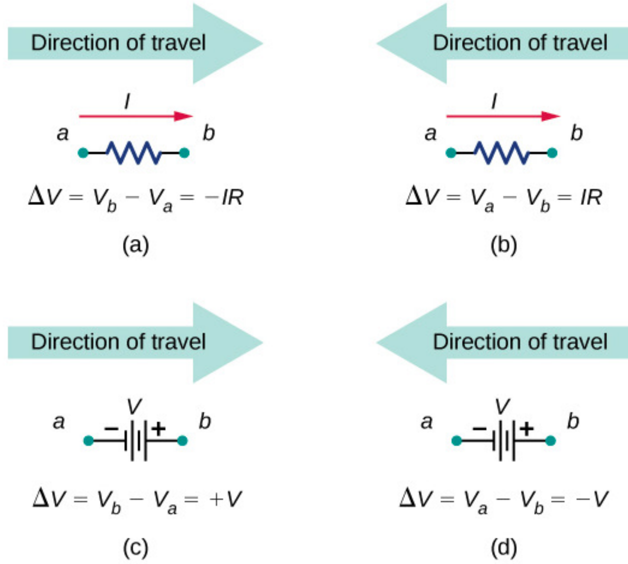


Figure $6.4.5$: Each of these resistors and voltage sources is traversed from **a** to **b**. (a) When moving across a resistor in the same direction as the current flow, subtract the potential drop. (b) When moving across a resistor in the opposite direction as the current flow, add the potential drop. (c) When moving across a voltage source from the negative terminal to the positive terminal, add the potential drop. (d) When moving across a voltage source from the positive terminal to the negative terminal, subtract the potential drop.

Let's examine some steps in this procedure more closely. When locating the junctions in the circuit, do not be concerned about the direction of the currents. If the direction of current flow is not obvious, choosing any direction is sufficient as long as at least one current points into the junction and at least one current points out of the junction. If the arrow is in the opposite direction of the conventional current flow, the result for the current in question will be negative but the answer will still be correct.

The number of nodes depends on the circuit. Each current should be included in a node and thus included in at least one junction equation. Do not include nodes that are not linearly independent, meaning nodes that contain the same information.

Consider Figure $6.4.6$. There are two junctions in this circuit: Junction **b** and Junction **e**. Points **a**, **c**, **d**, and **f** are not junctions, because a junction must have three or more connections. The equation for Junction **b** is $I_1 = I_2 + I_3$, and the equation for Junction **e** is $I_2 + I_3 = I_1$. These are equivalent equations, so it is necessary to keep only one of them.

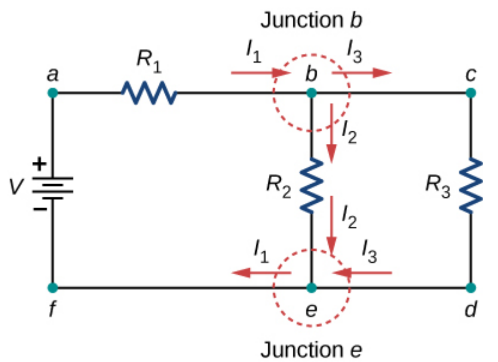


Figure $6.4.6$: At first glance, this circuit contains two junctions, Junction **b** and Junction **e**, but only one should be considered because their junction equations are equivalent.

When choosing the loops in the circuit, you need enough loops so that each component is covered once, without repeating loops. Figure $6.4.7$ shows four choices for loops to solve a sample circuit; choices (a), (b), and (c) have a sufficient amount of loops to solve the circuit completely. Option (d) reflects more loops than necessary to solve the circuit.

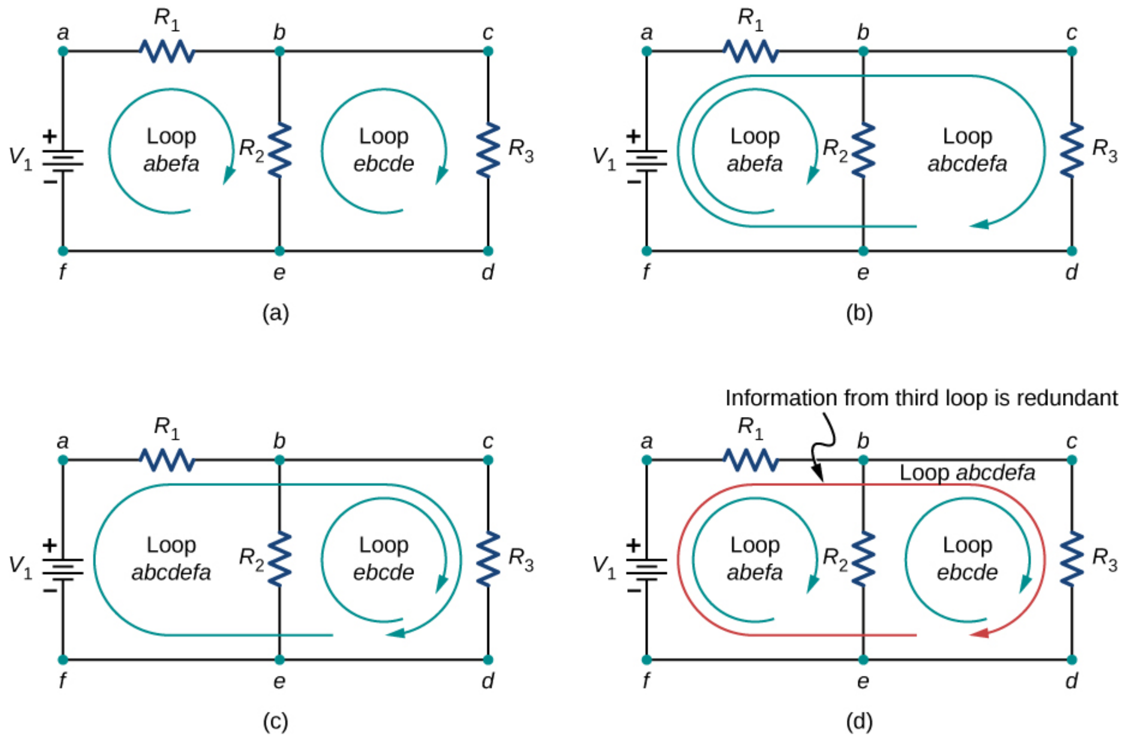


Figure 6.4.7: Panels (a)–(c) are sufficient for the analysis of the circuit. In each case, the two loops shown contain all the circuit elements necessary to solve the circuit completely. Panel (d) shows three loops used, which is more than necessary. Any two loops in the system will contain all information needed to solve the circuit. Adding the third loop provides redundant information.

Consider the circuit in Figure $6.4.8a$. Let us analyze this circuit to find the current through each resistor. First, label the circuit as shown in part (b).

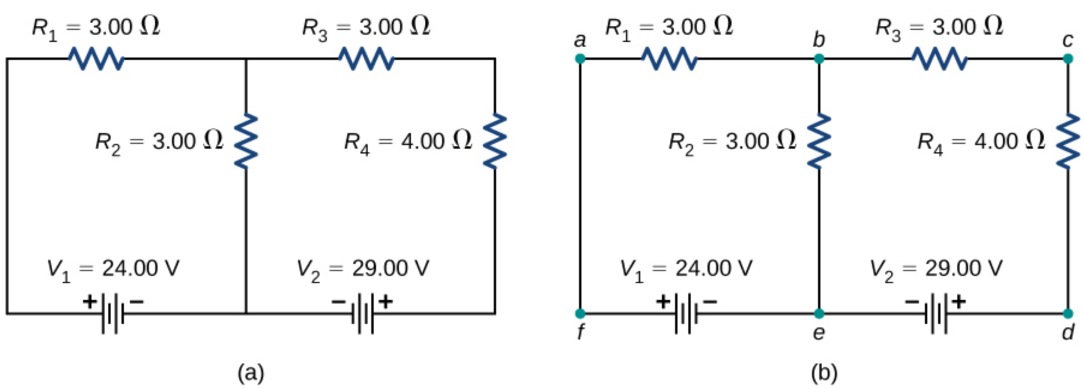


Figure 6.4.8: (a) A multi-loop circuit. (b) Label the circuit to help with orientation.

Next, determine the junctions. In this circuit, points **b** and **e** each have three wires connected, making them junctions. Start to apply Kirchhoff's junction rule $\left(\sum I_{in} = \sum I_{out}\right)$ by drawing arrows representing the currents and labeling each arrow, as shown in Figure $6.4.9$. Junction **b** shows that $I_1 = I_2 + I_3$ and Junction **e** shows that $I_2 + I_3 = I_1$. Since Junction **e** gives the same information of Junction **b**, it can be disregarded. This circuit has three unknowns, so we need three linearly independent equations to analyze it.

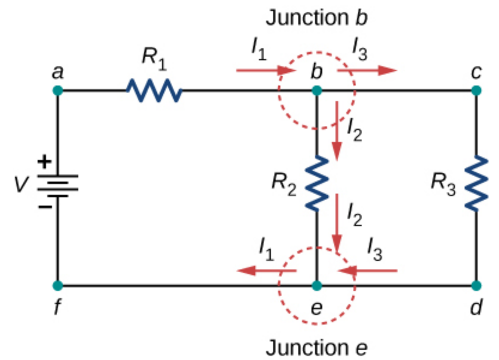


Figure 6.4.9: (a) This circuit has two junctions, labeled b and e, but only node b is used in the analysis. (b) Labeled arrows represent the currents into and out of the junctions.

Next we need to choose the loops. In Figure 6.4.10, Loop **abefa** includes the voltage source V_1 and resistors R_1 and R_2. The loop starts at point **a**, then travels through points **b**, **e**, and **f**, and then back to point **a**. The second loop, Loop **ebcde**, starts at point **e** and includes resistors R_2 and R_3, and the voltage source V_2.

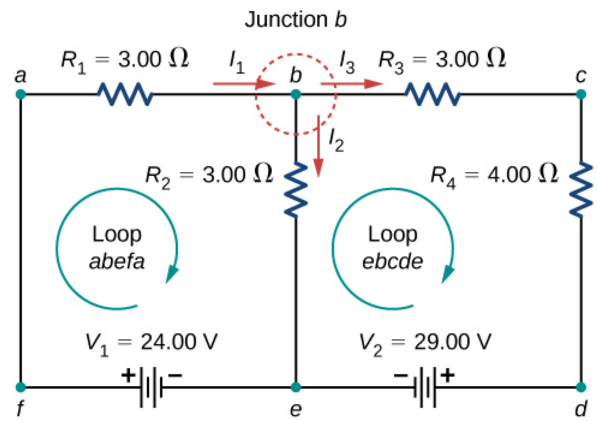


Figure 6.4.10: Choose the loops in the circuit.

Now we can apply Kirchhoff's loop rule, using the map in Figure 6.4.5. Starting at point **a** and moving to point **b**, the resistor R_1 is crossed in the same direction as the current flow I_1, so the potential drop I_1R_1 is subtracted. Moving from point **b** to point **e**, the resistor R_2 is crossed in the same direction as the current flow I_2 so the potential drop I_2R_2 is subtracted. Moving from point **e** to point **f**, the voltage source V_1 is crossed from the negative terminal to the positive terminal, so V_1 is added. There are no components between points **f** and **a.** The sum of the voltage differences must equal zero:

$$Loop\ abefa:\ -I_1R_1 - I_2R_2 + V_1 = 0\ or\ V_1 = I_1R_1 + I_2R_2.$$

Finally, we check loop **ebcde**. We start at point **e** and move to point **b**, crossing R_2 in the opposite direction as the current flow I_2. The potential drop I_2R_2 is added. Next, we cross R_3 and R_4 in the same direction as the current flow I_3 and subtract the potential drops I_3R_3 and I_3R_4. Note that the current is the same through resistors R_3 and R_4,because they are connected in series. Finally, the voltage source is crossed from the positive terminal to the negative terminal, and the voltage source V_2 is subtracted. The sum of these voltage differences equals zero and yields the loop equation

$$Loop\ ebcde:\ I_2R_2 - I_3(R_3 + R_4) - V_2 = 0.$$

We now have three equations, which we can solve for the three unknowns.

$$\text{Junction b: } I_1 - I_2 - I_3 = 0. \tag{6.4.1}$$

$$\text{Loop abefa: } I_1R_1 + I_2R_2 = V_1. \tag{6.4.2}$$

$$\text{Loop ebcde: } I_2R_2 - I_3(R_3 + R_4) = V_2. \tag{6.4.3}$$

To solve the three equations for the three unknown currents, start by eliminating current I_2. First add Equation 6.4.1 times R_2 to Equation 6.4.2. The result is Equation 6.4.4:

$$(R_1 + R_2)I_1 - R_2 I_3 = V_1.$$

$$6\,\Omega I_1 - 3\Omega I_3 = 24\,V. \tag{6.4.4}$$

Next, subtract Equation 6.4.3 from Equation 6.4.2. The result is Equation 6.4.5:

$$I_1 R_1 + I_3(R_3 + R_4) = V_1 - V_2.$$

$$3\Omega I_1 + 7\Omega I_3 = -5\,V. \tag{6.4.5}$$

We can solve Equations 6.4.4 and 6.4.5 for current I_1. Adding seven times Equation 6.4.4 and three times Equation 6.4.5 results in $51\,\Omega I_1 = 153\,V$, or $I_1 = 3.00\,A$. Using Equation 6.4.4 results in $I_3 = -2.00\,A$. Finally, Equation 6.4.1 yields $I_2 = I_1 - I_3 = 5.00\,A$. One way to check that the solutions are consistent is to check the power supplied by the voltage sources and the power dissipated by the resistors:

$$P_{in} = I_1 V_1 + I_3 V_2 = 130\,W,$$

$$P_{out} = I_1^2 R_1 + I_2^2 R_2 + I_3^2 R_3 + I_3^2 R_4 = 130\,W.$$

Note that the solution for the current I_3 is negative. This is the correct answer, but suggests that the arrow originally drawn in the junction analysis is the direction opposite of conventional current flow. The power supplied by the second voltage source is 58 W and not −58 W.

Example 6.4.1: Calculating Current by Using Kirchhoff's Rules

Find the currents flowing in the circuit in Figure 6.4.11.

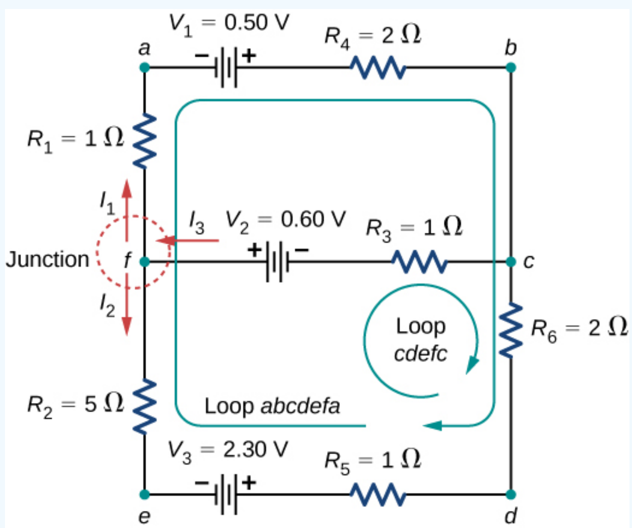


Figure 6.4.11: This circuit is combination of series and parallel configurations of resistors and voltage sources. This circuit cannot be analyzed using the techniques discussed in Electromotive Force but can be analyzed using Kirchhoff's rules.

Strategy

This circuit is sufficiently complex that the currents cannot be found using Ohm's law and the series-parallel techniques—it is necessary to use Kirchhoff's rules. Currents have been labeled I_1, I_2, and I_3 in the figure, and assumptions have been made about their directions. Locations on the diagram have been labeled with letters **a** through **h**. In the solution, we apply the junction and loop rules, seeking three independent equations to allow us to solve for the three unknown currents.

Solution

Applying the junction and loop rules yields the following three equations. We have three unknowns, so three equations are required.

$$Junction\,c:\; I_1 + I_2 = I_3.$$

$$Loop\,abcdefa:\; I_1(R_1 + R_4) - I_2(R_2 + R_5 + R_6) = V_1 - V_3.$$

$$Loop\, cdefc: \; I_2(R_2 + R_5 + R_6) + I_3R_3 = V_2 + V_3.$$

Simplify the equations by placing the unknowns on one side of the equations.

$$Junction\, c: \; I_1 + I_2 - I_3 = 0.$$

$$Loop\, abcdefa: \; I_1(3\Omega) - I_2(8\Omega) = 0.5\,V - 2.30\,V.$$

$$Loop\, cdefc: \; I_2(8\Omega) + I_3(1\Omega) = 0.6\,V + 2.30\,V.$$

Simplify the equations. The first loop equation can be simplified by dividing both sides by 3.00. The second loop equation can be simplified by dividing both sides by 6.00.

$$Junction\, c: \; I_1 + I_2 - I_3 = 0.$$

$$Loop\, abcdefa: \; I_1(3\Omega) - I_2(8\Omega) = -1.8\,V.$$

$$Loop\, cdefc: \; I_2(8\Omega) + I_3(1\Omega) = 2.90\,V.$$

The results are

$$I_1 = 0.20\,A,\; I_2 = 0.30\,A,\; I_3 = 0.50\,A.$$

Significance

A method to check the calculations is to compute the power dissipated by the resistors and the power supplied by the voltage sources:

$$P_{R_1} = I_1^2R_1 = 0.04\,W.$$

$$P_{R_2} = I_2^2R_2 = 0.45\,W.$$

$$P_{R_3} = I_3^2R_3 = 0.25\,W.$$

$$P_{R_4} = I_1^2R_4 = 0.08\,W.$$

$$P_{R_5} = I_2^2R_5 = 0.09\,W.$$

$$P_{R_6} = I_2^2R_1 = 0.18\,W.$$

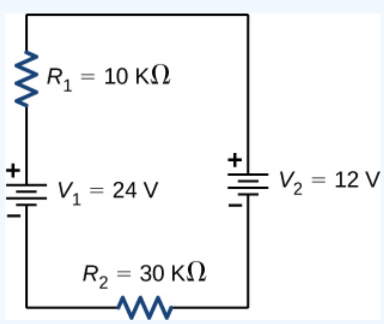
$$P_{dissipated} = 1.09\,W.$$

$$P_{source} = I_1V_1 + I_2V_3 + I_3V_2 = 0.10 + 0.69\,W + 0.30\,W = 1.09\,W.$$

The power supplied equals the power dissipated by the resistors.

? Exercise 6.4.1

In considering the following schematic and the power supplied and consumed by a circuit, will a voltage source always provide power to the circuit, or can a voltage source consume power?



Answer

The circuit can be analyzed using Kirchhoff's loop rule. The first voltage source supplies power: $P_{in} = IV_1 = 7.20\,mW$. The second voltage source consumes power: $P_{out} = IV_2 + I^2R_1 + I^2R_2 = 7.2\,mW$.

Example 6.4.2: Calculating Current by Using Kirchhoff's Rules

Find the current flowing in the circuit in Figure 6.4.12.

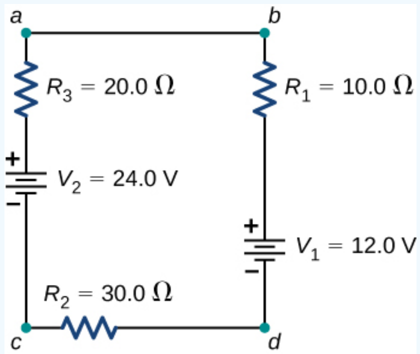


Figure 6.4.12: This circuit consists of three resistors and two batteries connected in series. Note that the batteries are connected with opposite polarities.

Strategy

This circuit can be analyzed using Kirchhoff's rules. There is only one loop and no nodes. Choose the direction of current flow. For this example, we will use the clockwise direction from point **a** to point **b**. Consider Loop **abcda** and use Figure 6.4.5 to write the loop equation. Note that according to Figure 6.4.5, battery V_1 will be added and battery V_2 will be subtracted.

Solution

Applying the junction rule yields the following three equations. We have one unknown, so one equation is required:

$$Loop\, abcda:\; -IR_1 - V_1 - IR_2 + V_2 - IR_3 = 0.$$

Simplify the equations by placing the unknowns on one side of the equations. Use the values given in the figure.

$$I(R_1 + R_2 + R_3) = V_2 - V_1.$$

$$I = \frac{V_2 - V_1}{R_1 + R_2 + R_3} = \frac{24\,V - 12\,V}{10.0\,\Omega + 30.0\,\Omega + 10.0\,\Omega} = 0.20\,A.$$

Significance

The power dissipated or consumed by the circuit equals the power supplied to the circuit, but notice that the current in the battery V_1 is flowing through the battery from the positive terminal to the negative terminal and consumes power.

$$P_{R_1} = I^2R_1 = 0.40\,W$$

$$P_{R_2} = I^2R_2 = 1.20\,W$$

$$P_{R_3} = I^2R_3 = 0.80\,W$$

$$P_{V_1} = IV_1 = 2.40\,W$$

$$P_{dissipated} = 4.80\,W$$

$$P_{source} = IV_2 = 4.80\,W$$

The power supplied equals the power dissipated by the resistors and consumed by the battery V_1.

Exercise 6.4.2

When using Kirchhoff's laws, you need to decide which loops to use and the direction of current flow through each loop. In analyzing the circuit in Example 6.4.2, the direction of current flow was chosen to be clockwise, from point **a** to point **b**. How would the results change if the direction of the current was chosen to be counterclockwise, from point **b** to point **a**?

Answer

The current calculated would be equal to $I=-0.20\ A$ instead of $I=0.20\ A$. The sum of the power dissipated and the power consumed would still equal the power supplied.

Multiple Voltage Sources

Many devices require more than one battery. Multiple voltage sources, such as batteries, can be connected in series configurations, parallel configurations, or a combination of the two.

In series, the positive terminal of one battery is connected to the negative terminal of another battery. Any number of voltage sources, including batteries, can be connected in series. Two batteries connected in series are shown in Figure 6.4.13. Using Kirchhoff's loop rule for the circuit in part (b) gives the result

$$\epsilon_1 - Ir_1 + \epsilon_2 - Ir_2 - IR = 0,$$

$$[(\epsilon_1+\epsilon_2) - I(r_1+r_2)] - IR = 0.$$

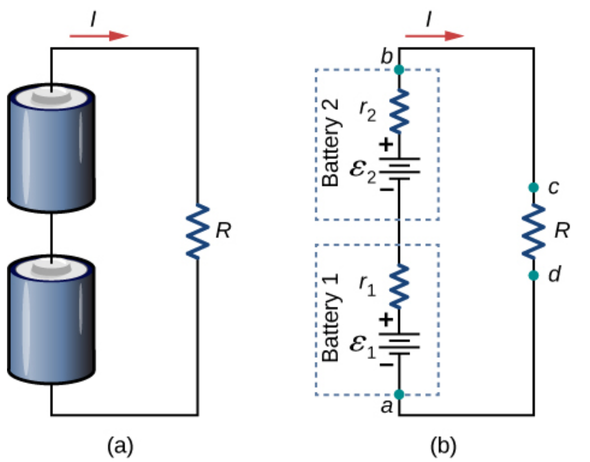


Figure 6.4.13: (a) Two batteries connected in series with a load resistor. (b) The circuit diagram of the two batteries and the load resistor, with each battery modeled as an idealized emf source and an internal resistance.

When voltage sources are in series, their internal resistances can be added together and their emfs can be added together to get the total values. Series connections of voltage sources are common—for example, in flashlights, toys, and other appliances. Usually, the cells are in series in order to produce a larger total emf. In Figure 6.4.13, the terminal voltage is

$$V_{terminal} = (\epsilon_1 - Ir_1) + (\epsilon_2 - Ir_2) = [(\epsilon_1+\epsilon_2) - I(r_1+r_2) - I(r_1+r_2)] = (\epsilon_1+\epsilon_2) + Ir_{eq}.$$

Note that the same current **I** is found in each battery because they are connected in series. The disadvantage of series connections of cells is that their internal resistances are additive.

Batteries are connected in series to increase the voltage supplied to the circuit. For instance, an LED flashlight may have two AAA cell batteries, each with a terminal voltage of 1.5 V, to provide 3.0 V to the flashlight.

Any number of batteries can be connected in series. For **N** batteries in series, the terminal voltage is equal to

Note

$$V_{terminal} = (\epsilon_1 + \epsilon_2 + \ldots + \epsilon_{N-1} + \epsilon_N) - I(r_1 + r_2 + \ldots + r_{N-1} + r_N) = \sum_{i=1}^{N} \epsilon_i - Ir_{eq}$$

where the equivalent resistance is

$$r_{eq} = \sum_{i=1}^{N} r_i$$

When a load is placed across voltage sources in series, as in Figure $6.4.14$, we can find the current:

$$(\epsilon_1 - Ir_1) + (\epsilon_2 - Ir_2) = IR,$$

$$Ir_1 + Ir_2 + IR = \epsilon_1 + \epsilon_2,$$

$$I = \frac{\epsilon_1 + \epsilon_2}{r_1 + r_2 + R}.$$

As expected, the internal resistances increase the equivalent resistance.

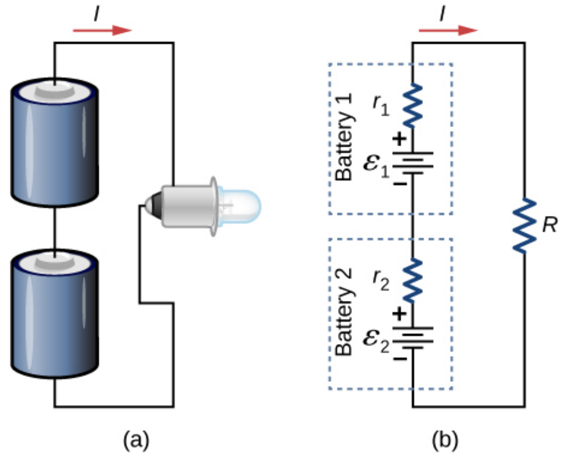


Figure 6.4.14: Two batteries connect in series to an LED bulb, as found in a flashlight.

Voltage sources, such as batteries, can also be connected in parallel. Figure $6.4.15$ shows two batteries with identical emfs in parallel and connected to a load resistance. When the batteries are connect in parallel, the positive terminals are connected together and the negative terminals are connected together, and the load resistance is connected to the positive and negative terminals. Normally, voltage sources in parallel have identical emfs. In this simple case, since the voltage sources are in parallel, the total emf is the same as the individual emfs of each battery.

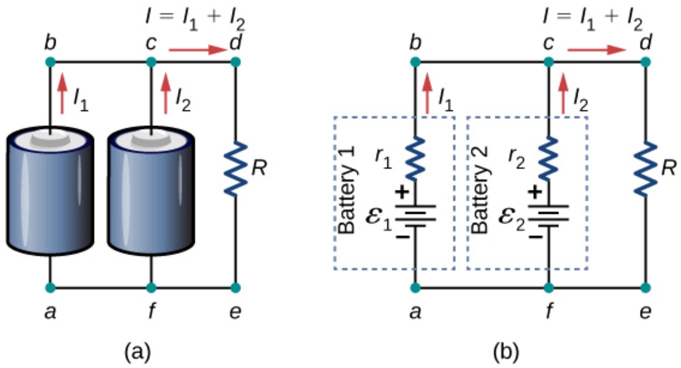


Figure 6.4.15: (a) Two batteries connect in parallel to a load resistor. (b) The circuit diagram shows the shows battery as an emf source and an internal resistor. The two emf sources have identical emfs (each labeled by ϵ) connected in parallel that produce the same emf.

Consider the Kirchhoff analysis of the circuit in Figure $6.4.15b$. There are two loops and a node at point **b** and $\epsilon = \epsilon_1 = \epsilon_2$.

Node b: $I_1 + I_2 - I = 0$.

Loop abcfa: $\epsilon_2 - I_1 r_1 + I_2 r_2 - \epsilon = 0,\ I_1 r_1 = I_2 r_2$.

Loop fcdef: $\epsilon_2 - I_2 r_2 - IR = 0,\ \ \epsilon - I_2 r_2 - IR = 0.$

Solving for the current through the load resistor results in $I=\frac{\epsilon}{r_{eq}+R}$, where $r_{eq}=\left(\frac{1}{r_1}+\frac{1}{r_2}\right)^{-1}$. The terminal voltage is equal to the potential drop across the load resistor $IR=\left(\frac{\epsilon}{r_{eq}+R}\right)$.

The parallel connection reduces the internal resistance and thus can produce a larger current.

Any number of batteries can be connected in parallel. For **N** batteries in parallel, the terminal voltage is equal to

Note

$$V_{terminal}=\epsilon-I\left(\frac{1}{r_1}+\frac{1}{r_2}+\ldots+\frac{1}{r_{N-1}}+\frac{1}{r_N}\right)^{-1}=\epsilon-Ir_{eq}$$

where the equivalent resistance is

$$r_{eq}=\left(\sum_{i=1}^{N}\frac{1}{r_i}\right)^{-1}$$

As an example, some diesel trucks use two 12-V batteries in parallel; they produce a total emf of 12 V but can deliver the larger current needed to start a diesel engine.

In summary, the terminal voltage of batteries in series is equal to the sum of the individual emfs minus the sum of the internal resistances times the current. When batteries are connected in parallel, they usually have equal emfs and the terminal voltage is equal to the emf minus the equivalent internal resistance times the current, where the equivalent internal resistance is smaller than the individual internal resistances. Batteries are connected in series to increase the terminal voltage to the load. Batteries are connected in parallel to increase the current to the load.

Solar Cell Arrays

Another example dealing with multiple voltage sources is that of combinations of **solar cells** - wired in both series and parallel combinations to yield a desired voltage and current. Photovoltaic generation, which is the conversion of sunlight directly into electricity, is based upon the photoelectric effect. The photoelectric effect is beyond the scope of this chapter and is covered in Photons and Matter Waves, but in general, photons hitting the surface of a solar cell create an electric current in the cell.

Most solar cells are made from pure silicon. Most single cells have a voltage output of about 0.5 V, while the current output is a function of the amount of sunlight falling on the cell (the incident solar radiation known as the insolation). Under bright noon sunlight, a current per unit area of about 100 mA/cm^2 of cell surface area is produced by typical single-crystal cells.

Individual solar cells are connected electrically in modules to meet electrical energy needs. They can be wired together in series or in parallel - connected like the batteries discussed earlier. A solar-cell array or module usually consists of between 36 and 72 cells, with a power output of 50 W to 140 W.

Solar cells, like batteries, provide a direct current (dc) voltage. Current from a dc voltage source is unidirectional. Most household appliances need an alternating current (ac) voltage.

6.5: Capacitors in Series and in Parallel

Learning Objectives

By the end of this section, you will be able to:

- Explain how to determine the equivalent capacitance of capacitors in series and in parallel combinations
- Compute the potential difference across the plates and the charge on the plates for a capacitor in a network and determine the net capacitance of a network of capacitors

Several capacitors can be connected together to be used in a variety of applications. Multiple connections of capacitors behave as a single equivalent capacitor. The total capacitance of this equivalent single capacitor depends both on the individual capacitors and how they are connected. Capacitors can be arranged in two simple and common types of connections, known as **series** and **parallel**, for which we can easily calculate the total capacitance. These two basic combinations, series and parallel, can also be used as part of more complex connections.

The Series Combination of Capacitors

Figure $6.5.1$ illustrates a series combination of three capacitors, arranged in a row within the circuit. As for any capacitor, the capacitance of the combination is related to both charge and voltage:

$$C = \frac{Q}{V}.$$

When this series combination is connected to a battery with voltage **V**, each of the capacitors acquires an identical charge **Q**. To explain, first note that the charge on the plate connected to the positive terminal of the battery is $+Q$ and the charge on the plate connected to the negative terminal is $-Q$. Charges are then induced on the other plates so that the sum of the charges on all plates, and the sum of charges on any pair of capacitor plates, is zero. However, the potential drop $V_1 = Q/C_1$ on one capacitor may be different from the potential drop $V_2 = Q/C_2$ on another capacitor, because, generally, the capacitors may have different capacitances. The series combination of two or three capacitors resembles a single capacitor with a smaller capacitance. Generally, any number of capacitors connected in series is equivalent to one capacitor whose capacitance (called the **equivalent capacitance**) is smaller than the smallest of the capacitances in the series combination. Charge on this equivalent capacitor is the same as the charge on any capacitor in a series combination: That is, **all capacitors of a series combination have the same charge**. This occurs due to the conservation of charge in the circuit. When a charge **Q** in a series circuit is removed from a plate of the first capacitor (which we denote as $-Q$), it must be placed on a plate of the second capacitor (which we denote as $+Q$), and so on.

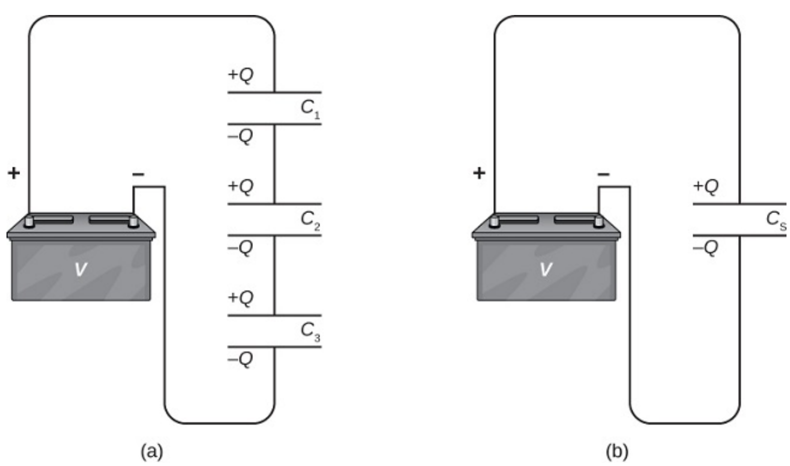


Figure $6.5.1$: (a) Three capacitors are connected in series. The magnitude of the charge on each plate is Q. (b) The network of capacitors in (a) is equivalent to one capacitor that has a smaller capacitance than any of the individual capacitances in (a), and the charge on its plates is Q.

We can find an expression for the total (equivalent) capacitance by considering the voltages across the individual capacitors. The potentials across capacitors 1, 2, and 3 are, respectively, $V_1 = Q/C_1$, $V_2 = Q/C_2$, and $V_3 = Q/C_3$. These potentials must sum up to the voltage of the battery, giving the following potential balance:

$$V = V_1 + V_2 + V_3.$$

Potential V is measured across an equivalent capacitor that holds charge Q and has an equivalent capacitance C_S. Entering the expressions for V_1, V_2, and V_3, we get

$$\frac{Q}{C_S} = \frac{Q}{C_1} + \frac{Q}{C_2} + \frac{Q}{C_3}.$$

Canceling the charge **Q**, we obtain an expression containing the equivalent capacitance, C_S, of three capacitors connected in series:

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3}.$$

This expression can be generalized to any number of capacitors in a series network.

Series Combination

For capacitors connected in a series combination, the reciprocal of the equivalent capacitance is the sum of reciprocals of individual capacitances:

$$\frac{1}{C_S} = \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} + \dots \tag{6.5.1}$$

Example 6.5.1: Equivalent Capacitance of a Series Network

Find the total capacitance for three capacitors connected in series, given their individual capacitances are $1.000 \mu F$, $5.000 \mu F$, and $8.000 \mu F$.

Strategy

Because there are only three capacitors in this network, we can find the equivalent capacitance by using Equation 6.5.1 with three terms.

Solution

We enter the given capacitances into Equation 6.5.1:

$$\begin{aligned} \frac{1}{C_S} &= \frac{1}{C_1} + \frac{1}{C_2} + \frac{1}{C_3} \\ &= \frac{1}{1.000 \mu F} + \frac{1}{5.000 \mu F} + \frac{1}{8.000 \mu F} \\ &= \frac{1.325}{\mu F}. \end{aligned}$$

Now we invert this result and obtain

$$\begin{aligned} C_S &= \frac{\mu F}{1.325} \\ &= 0.755 \mu F. \end{aligned}$$

Significance

Note that in a series network of capacitors, the equivalent capacitance is always less than the smallest individual capacitance in the network.

The Parallel Combination of Capacitors

A parallel combination of three capacitors, with one plate of each capacitor connected to one side of the circuit and the other plate connected to the other side, is illustrated in Figure $6.5.2a$. Since the capacitors are connected in parallel, **they all have the same voltage V across their plates**. However, each capacitor in the parallel network may store a different charge. To find the equivalent capacitance C_p of the parallel network, we note that the total charge **Q** stored by the network is the sum of all the individual charges:

$$Q = Q_1 + Q_2 + Q_3.$$

On the left-hand side of this equation, we use the relation $Q = C_pV$, which holds for the entire network. On the right-hand side of the equation, we use the relations $Q_1 = C_1V$, $Q_2 = C_2V$, and $Q_3 = C_3V$ for the three capacitors in the network. In this way we obtain

$$C_pV = C_1V + C_2V + C_3V.$$

This equation, when simplified, is the expression for the equivalent capacitance of the parallel network of three capacitors:

$$C_p = C_1 + C_2 + C_3.$$

This expression is easily generalized to any number of capacitors connected in parallel in the network.

Parallel Combination

For capacitors connected in a parallel combination, the equivalent (net) capacitance is the sum of all individual capacitances in the network,

$$C_p = C_1 + C_2 + C_3 + \ldots \tag{6.5.2}$$

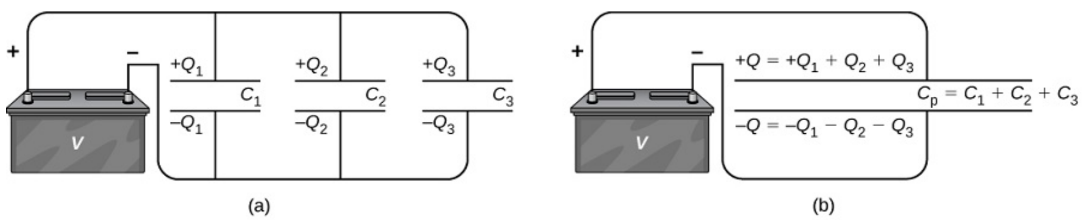


Figure 6.5.2: (a) Three capacitors are connected in parallel. Each capacitor is connected directly to the battery. (b) The charge on the equivalent capacitor is the sum of the charges on the individual capacitors.

Example 6.5.2: Equivalent Capacitance of a Parallel Network

Find the net capacitance for three capacitors connected in parallel, given their individual capacitances are $1.0 \mu F$, $5.0 \mu F$, and $8.0 \mu F$.

Strategy
Because there are only three capacitors in this network, we can find the equivalent capacitance by using Equation 6.5.2 with three terms.

Solution
Entering the given capacitances into Equation 6.5.2 yields

$$\begin{aligned} C_p &= C_1 + C_2 + C_3 \\ &= 1.0 \mu F + 5.0 \mu F + 8.0 \mu F \\ &= 14.0 \mu F. \end{aligned}$$

Significance
Note that in a parallel network of capacitors, the equivalent capacitance is always larger than any of the individual capacitances in the network.

Capacitor networks are usually some combination of series and parallel connections, as shown in Figure 6.5.3. To find the net capacitance of such combinations, we identify parts that contain only series or only parallel connections, and find their equivalent capacitances. We repeat this process until we can determine the equivalent capacitance of the entire network. The following example illustrates this process.

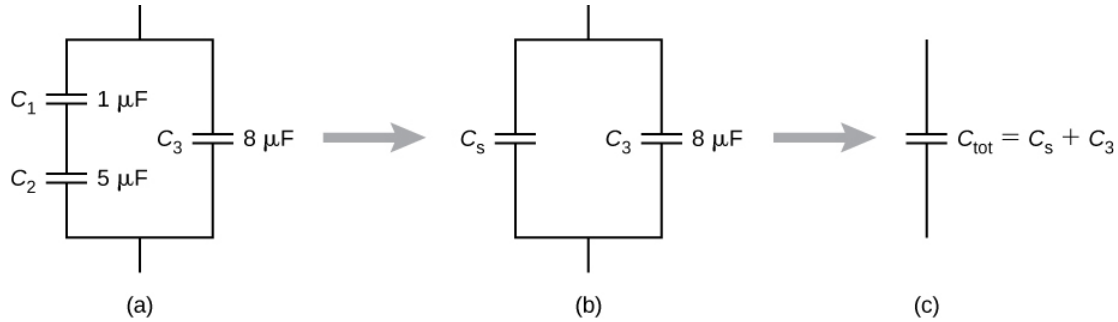


Figure 6.5.3: (a) This circuit contains both series and parallel connections of capacitors. (b) C_1 and C_2 are in series; their equivalent capacitance is C_S c) The equivalent capacitance C_S is connected in parallel with C_3. Thus, the equivalent capacitance of the entire network is the sum of C_S and C_3.

✓ Example 6.5.3: Equivalent Capacitance of a Network

Find the total capacitance of the combination of capacitors shown in Figure 6.5.3. Assume the capacitances are known to three decimal places ($C_1 = 1.000\mu F, C_2 = 5.000\mu F, C_3 = 8.000\mu F$). Round your answer to three decimal places.

Strategy

We first identify which capacitors are in series and which are in parallel. Capacitors C_1 and C_2 are in series. Their combination, labeled C_S is in parallel with C_3.

Solution

Since C_1 and C_2 are in series, their equivalent capacitance C_S is obtained with Equation 6.5.1:

$$\begin{aligned}\frac{1}{C_S} &= \frac{1}{C_1} + \frac{1}{C_2} \\ &= \frac{1}{1.000\mu F} + \frac{1}{5.000\mu F} \\ &= \frac{1.200}{\mu F}\end{aligned}$$

Therefor

$$C_S = 0.833\mu F.$$

Capacitance C_S is connected in parallel with the third capacitance C_3, so we use Equation 6.5.2 find the equivalent capacitance **C** of the entire network:

$$\begin{aligned}C &= C_S + C_3 \\ &= 0.833\mu F + 8.000\mu F \\ &= 8.833\mu F.\end{aligned}$$

✓ Network of Capacitors

Determine the net capacitance **C** of the capacitor combination shown in Figure 6.5.4 when the capacitances are $C_1 = 12.0\mu F, C_2 = 2.0\mu F$, and $C_3 = 4.0\mu F$. When a 12.0-V potential difference is maintained across the combination, find the charge and the voltage across each capacitor.

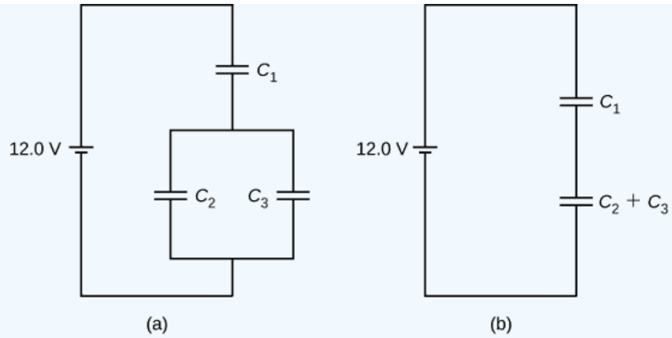


Figure 6.5.4: (a) A capacitor combination. (b) An equivalent two-capacitor combination.

Strategy We first compute the net capacitance C_{23} of the parallel connection C_2 and C_3. Then **C** is the net capacitance of the series connection C_1 and C_{23}. We use the relation $C = Q/V$ to find the charges Q_1, Q_2, and Q_3, and the voltages V_1, V_2, and V_3 across capacitors 1, 2, and 3, respectively.

Solution The equivalent capacitance for C_2 and C_3 is

$$C_{23} = C_2 + C_3 = 2.0\mu F + 4.0\mu F = 6.0\mu F.$$

The entire three-capacitor combination is equivalent to two capacitors in series,

$$\frac{1}{C} = \frac{1}{12.0\mu F} + \frac{1}{6.0\mu F} = \frac{1}{4.0\mu F} \Rightarrow C = 4.0\mu F.$$

Consider the equivalent two-capacitor combination in Figure $6.5.2b$. Since the capacitors are in series, they have the same charge, $Q_1 = Q_{23}$. Also, the capacitors share the 12.0-V potential difference, so

$$12.0V = V_1 + V_{23} = \frac{Q_1}{C_1} + \frac{Q_{23}}{C_{23}} = \frac{Q_1}{12.0\mu F} + \frac{Q_1}{6.0\mu F} \Rightarrow Q_1 = 48.0\mu C.$$

Now the potential difference across capacitor 1 is

$$V_1 = \frac{Q_1}{C_1} = \frac{48.0\mu C}{12.0\mu F} = 4.0V.$$

Because capacitors 2 and 3 are connected in parallel, they are at the same potential difference:

$$V_2 = V_3 = 12.0V - 4.0V = 8.0V.$$

Hence, the charges on these two capacitors are, respectively,

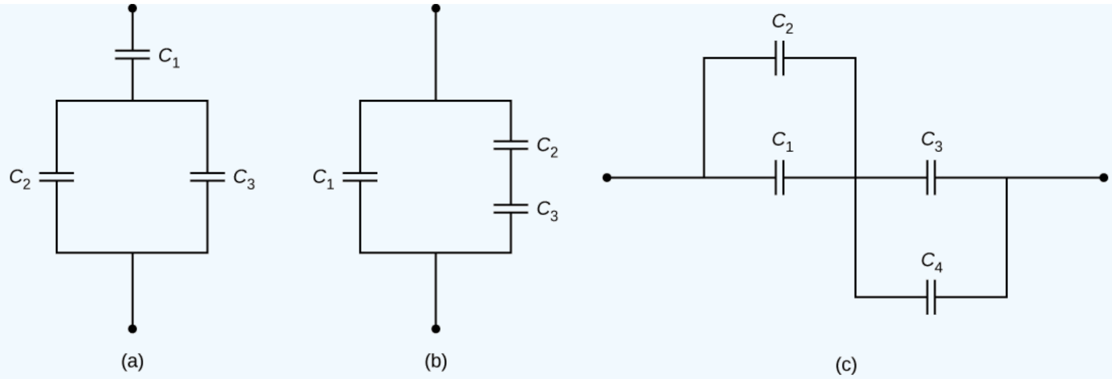
$$Q_2 = C_2V_2 = (2.0\mu F)(8.0V) = 16.0\mu C,$$

$$Q_3 = C_3V_3 = (4.0\mu F)(8.0V) = 32.0\mu C.$$

Significance As expected, the net charge on the parallel combination of C_2 and C_3 is $Q_{23} = Q_2 + Q_3 = 48.0\mu C$.

? Exercise 6.5.1

Determine the net capacitance **C** of each network of capacitors shown below. Assume that $C_1 = 1.0pF, C_2 = 2.0pF, C_3 = 4.0pF$, and $C_4 = 5.0pF$. Find the charge on each capacitor, assuming there is a potential difference of 12.0 V across each network.



Answer a

$C = 0.86pF, Q_1 = 10pC, Q_2 = 3.4pC, Q_3 = 6.8pC$

Answer b

$C = 2.3pF, Q_1 = 12pC, Q_2 = Q_3 = 16pC$

Answer c

$C = 2.3pF, Q_1 = 9.0pC, Q_2 = 18pC, Q_3 = 12pC, Q_4 = 15pC$

6.6: RC Circuits

Learning Objectives

By the end of the section, you will be able to:

- Describe the charging process of a capacitor
- Describe the discharging process of a capacitor
- List some applications of RC circuits

When you use a flash camera, it takes a few seconds to charge the capacitor that powers the flash. The light flash discharges the capacitor in a tiny fraction of a second. Why does charging take longer than discharging? This question and several other phenomena that involve charging and discharging capacitors are discussed in this module.

Circuits with Resistance and Capacitance

An **RC circuit** is a circuit containing resistance and capacitance. As presented in Capacitance, the capacitor is an electrical component that stores electric charge, storing energy in an electric field.

Figure $6.6.1a$ shows a simple **RC** circuit that employs a dc (direct current) voltage source ε, a resistor R, a capacitor C, and a two-position switch. The circuit allows the capacitor to be charged or discharged, depending on the position of the switch. When the switch is moved to position \(**A**\), the capacitor charges, resulting in the circuit in Figure $6.6.1b$. When the switch is moved to position **B**, the capacitor discharges through the resistor.

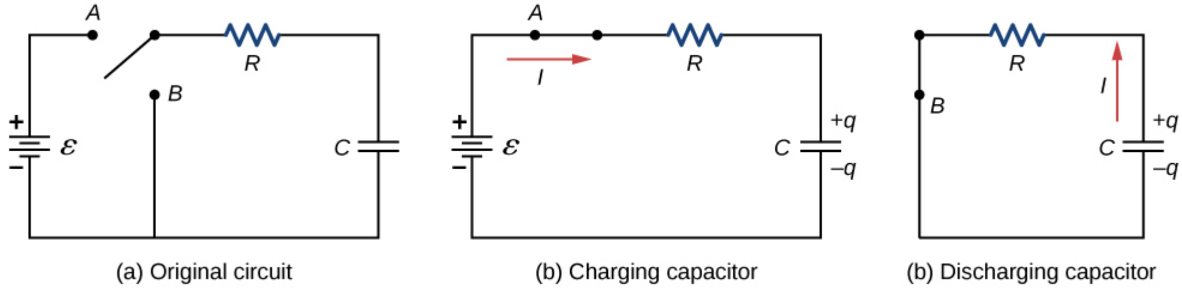


Figure 6.6.1: (a) An **RC** circuit with a two-pole switch that can be used to charge and discharge a capacitor. (b) When the switch is moved to position **A**, the circuit reduces to a simple series connection of the voltage source, the resistor, the capacitor, and the switch. (c) When the switch is moved to position **B**, the circuit reduces to a simple series connection of the resistor, the capacitor, and the switch. The voltage source is removed from the circuit.

Charging a Capacitor

We can use Kirchhoff's loop rule to understand the charging of the capacitor. This results in the equation $\epsilon - V_R - V_C = 0$. This equation can be used to model the charge as a function of time as the capacitor charges. Capacitance is defined as $C = q/V$, so the voltage across the capacitor is $V_C = \frac{q}{C}$. Using Ohm's law, the potential drop across the resistor is $V_R = IR$, and the current is defined as $I = dq/dt$.

$$\epsilon - V_R - V_C = 0,$$

$$\epsilon - IR - \frac{q}{C} = 0,$$

$$\epsilon - R\frac{dq}{dt} - \frac{q}{C} = 0.$$

This differential equation can be integrated to find an equation for the charge on the capacitor as a function of time.

$$\epsilon - R\frac{dq}{dt} - \frac{q}{C} = 0.$$

$$\frac{dq}{dt} = \frac{\epsilon C - q}{RC},$$

$$\int_0^q \frac{dq}{\epsilon C - q} = \frac{1}{RC}\int_0^t dt.$$

Let $u = \epsilon C - q$, then $du = -dq$. The result is

$$-\int_0^q \frac{du}{u} = \frac{1}{RC}\int_0^t dt,$$

$$\ln\left(\frac{\epsilon C - q}{\epsilon C}\right) = -\frac{1}{RC}t.$$

$$\frac{\epsilon C - q}{\epsilon C} = e^{-t/RC}.$$

Simplifying results in an equation for the charge on the charging capacitor as a function of time:

$$q(t) = C\epsilon\left(1 - e^{-\frac{t}{RC}}\right) = Q\left(1 - e^{-\frac{t}{\tau}}\right).$$

A graph of the charge on the capacitor versus time is shown in Figure $6.6.2a$. First note that as time approaches infinity, the exponential goes to zero, so the charge approaches the maximum charge $Q = C\epsilon$ and has units of coulombs. The units of **RC** are seconds, units of time. This quantity is known as the **time constant**:

$$\tau = RC.$$

At time $t = \tau = RC$, the charge equal to $1 - e^{-1} = 1 - 0.368 = 0.632$ of the maximum charge $Q = C\epsilon$. Notice that the time rate change of the charge is the slope at a point of the charge versus time plot. The slope of the graph is large at time $t - 0.0\ s$ and approaches zero as time increases.

As the charge on the capacitor increases, the current through the resistor decreases, as shown in Figure $6.6.2b$. The current through the resistor can be found by taking the time derivative of the charge.

$$\begin{aligned} I(t) &= \frac{dq}{dt} \\ &= \frac{d}{dt}\left[C\epsilon\left(1 - e^{-\frac{t}{RC}}\right)\right], \\ &= C\epsilon\left(\frac{1}{RC}\right)e^{-\frac{t}{RC}} \\ &= \frac{\epsilon}{R}e^{-\frac{t}{RC}} \\ &= I_0 e^{-\frac{t}{RC}}, \end{aligned}$$

$$I(t) = I_0 e^{-t/\tau}.$$

At time $t = 0.0\ s$, the current through the resistor is $I_0 = \frac{\epsilon}{R}$. As time approaches infinity, the current approaches zero. At time $t = \tau$, the current through the resistor is $I(t = \tau) = I_0 e^{-1} = 0.368 I_0$.

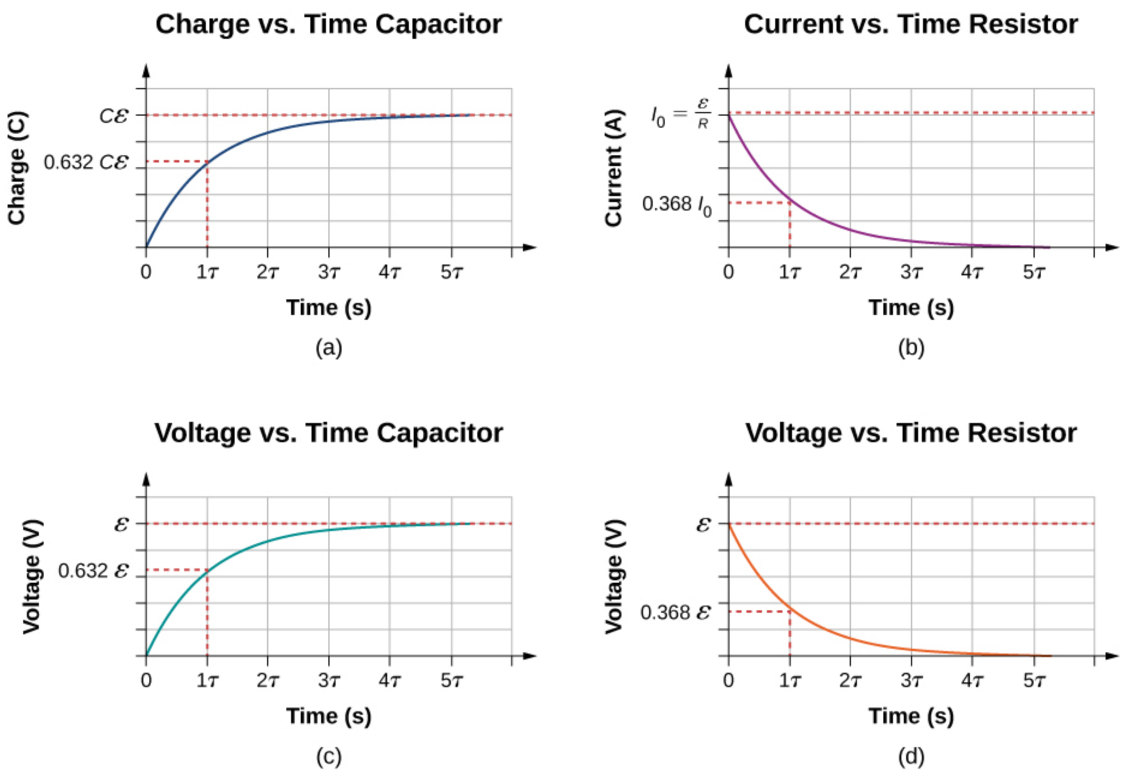


Figure $6.6.2$: (a) Charge on the capacitor versus time as the capacitor charges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

Figures $6.6.2c$ and Figure $6.6.2d$ show the voltage differences across the capacitor and the resistor, respectively. As the charge on the capacitor increases, the current decreases, as does the voltage difference across the resistor $V_R(t) = (I_0 R)e^{-t/\tau} = \epsilon e^{-t/\tau}$. The voltage difference across the capacitor increases as $V_C(t) = \epsilon(1 - e^{-t/\tau})$.

Discharging a Capacitor

When the switch in Figure $6.6.3a$ is moved to position **B**, the circuit reduces to the circuit in part (c), and the charged capacitor is allowed to discharge through the resistor. A graph of the charge on the capacitor as a function of time is shown in Figure $6.6.3a$. Using Kirchhoff's loop rule to analyze the circuit as the capacitor discharges results in the equation $-V_R - V_C = 0$, which simplifies to $IR + \frac{q}{C} = 0$. Using the definition of current $\frac{dq}{dt}R = -\frac{q}{C}$ and integrating the loop equation yields an equation for the charge on the capacitor as a function of time:

$$q(t) = Qe^{-t/\tau}.$$

Here, **Q** is the initial charge on the capacitor and $\tau = RC$ is the time constant of the circuit. As shown in the graph, the charge decreases exponentially from the initial charge, approaching zero as time approaches infinity.

The current as a function of time can be found by taking the time derivative of the charge:

$$I(t) = -\frac{Q}{RC}e^{-t/\tau}.$$

The negative sign shows that the current flows in the opposite direction of the current found when the capacitor is charging. Figure $6.6.3b$ shows an example of a plot of charge versus time and current versus time. A plot of the voltage difference across the capacitor and the voltage difference across the resistor as a function of time are shown in Figures $6.6.3c$ and $6.6.3d$. Note that the magnitudes of the charge, current, and voltage all decrease exponentially, approaching zero as time increases.

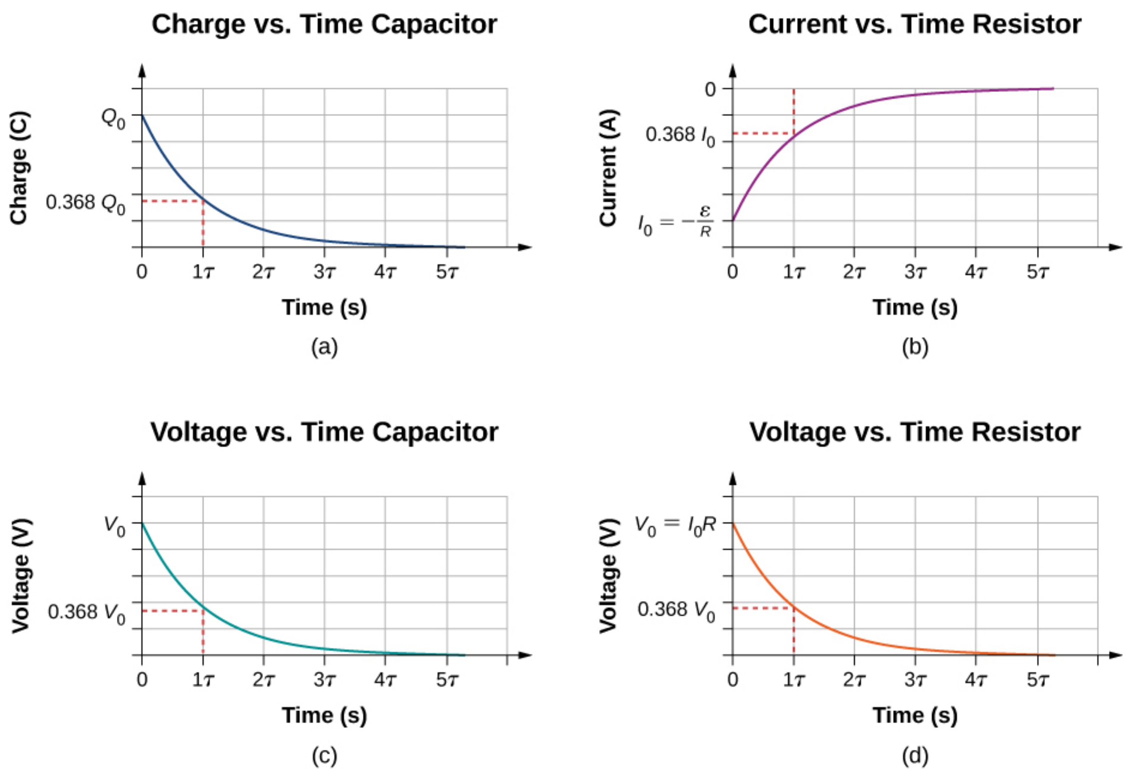
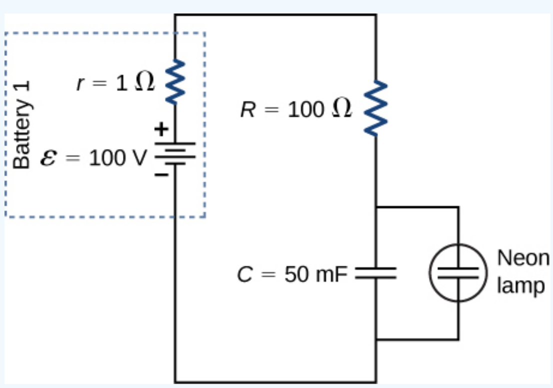


Figure 6.6.3: (a) Charge on the capacitor versus time as the capacitor discharges. (b) Current through the resistor versus time. (c) Voltage difference across the capacitor. (d) Voltage difference across the resistor.

Now we can explain why the **flash camera** mentioned at the beginning of this section takes so much longer to charge than discharge: The resistance while charging is significantly greater than while discharging. The internal resistance of the battery accounts for most of the resistance while charging. As the battery ages, the increasing internal resistance makes the charging process even slower.

Example 6.6.2: The Relaxation Oscillator

One application of an **RC** circuit is the relaxation oscillator, as shown below. The relaxation oscillator consists of a voltage source, a resistor, a capacitor, and a neon lamp. The neon lamp acts like an open circuit (infinite resistance) until the potential difference across the neon lamp reaches a specific voltage. At that voltage, the lamp acts like a short circuit (zero resistance), and the capacitor discharges through the neon lamp and produces light. In the relaxation oscillator shown, the voltage source charges the capacitor until the voltage across the capacitor is 80 V. When this happens, the neon in the lamp breaks down and allows the capacitor to discharge through the lamp, producing a bright flash. After the capacitor fully discharges through the neon lamp, it begins to charge again, and the process repeats. Assuming that the time it takes the capacitor to discharge is negligible, what is the time interval between flashes?



Strategy

The time period can be found from considering the equation $V_C(t) = \epsilon(1 - e^{=t/\tau})$. where $\tau = (R + r)C$.

Solution

The neon lamp flashes when the voltage across the capacitor reaches 80 V. The **RC** time constant is equal to $\tau = (R + r) = (101\,\Omega)(50 \times 10^{-3}\,F) = 5.05\,s$. We can solve the voltage equation for the time it takes the capacitor to reach 80 V:

$$V_C(t) = \epsilon(1 - e^{-t/\tau}),$$

$$e^{-t/\tau} = 1 - \frac{V_C(t)}{\epsilon},$$

$$ln(e^{-t/\tau}) = ln\left(1 - \frac{V_C(t)}{\epsilon}\right),$$

$$t = -\tau ln\left(1 - \frac{V_C(t)}{\epsilon}\right) = -5.05\,s \cdot ln\left(1 - \frac{80\,V}{100\,V}\right) = 8.13\,s.$$

Significance

One application of the relaxation oscillator is for controlling indicator lights that flash at a frequency determined by the values for **R** and **C**. In this example, the neon lamp will flash every 8.13 seconds, a frequency of $f = \frac{1}{T} = \frac{1}{8.13\,s} = 0.55\,Hz$. The relaxation oscillator has many other practical uses. It is often used in electronic circuits, where the neon lamp is replaced by a transistor or a device known as a tunnel diode. The description of the transistor and tunnel diode is beyond the scope of this chapter, but you can think of them as voltage controlled switches. They are normally open switches, but when the right voltage is applied, the switch closes and conducts. The “switch” can be used to turn on another circuit, turn on a light, or run a small motor. A relaxation oscillator can be used to make the turn signals of your car blink or your cell phone to vibrate.

RC circuits have many applications. They can be used effectively as timers for applications such as intermittent windshield wipers, pace makers, and strobe lights. Some models of intermittent windshield wipers use a variable resistor to adjust the interval between sweeps of the wiper. Increasing the resistance increases the **RC** time constant, which increases the time between the operation of the wipers.

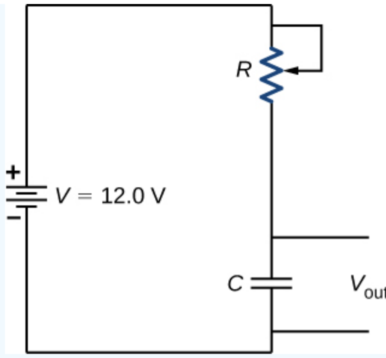
Another application is the **pacemaker**. The heart rate is normally controlled by electrical signals, which cause the muscles of the heart to contract and pump blood. When the heart rhythm is abnormal (the heartbeat is too high or too low), pace makers can be used to correct this abnormality. Pacemakers have sensors that detect body motion and breathing to increase the heart rate during physical activities, thus meeting the increased need for blood and oxygen, and an **RC** timing circuit can be used to control the time between voltage signals to the heart.

Looking ahead to the study of ac circuits (Alternating-Current Circuits), ac voltages vary as sine functions with specific frequencies. Periodic variations in voltage, or electric signals, are often recorded by scientists. These voltage signals could come from music recorded by a microphone or atmospheric data collected by radar. Occasionally, these signals can contain unwanted frequencies known as “noise.” **RC** filters can be used to filter out the unwanted frequencies.

In the study of electronics, a popular device known as a 555 timer provides timed voltage pulses. The time between pulses is controlled by an **RC** circuit. These are just a few of the countless applications of **RC** circuits.

✔ Example 6.6.2: Intermittent Windshield Wipers

A relaxation oscillator is used to control a pair of windshield wipers. The relaxation oscillator consists of a 10.00-mF capacitor and a $10.00\,k\Omega$ variable resistor known as a rheostat. A knob connected to the variable resistor allows the resistance to be adjusted from $0.00\,\Omega$ to $10.00\,k\Omega$. The output of the capacitor is used to control a voltage-controlled switch. The switch is normally open, but when the output voltage reaches 10.00 V, the switch closes, energizing an electric motor and discharging the capacitor. The motor causes the windshield wipers to sweep once across the windshield and the capacitor begins to charge again. To what resistance should the rheostat be adjusted for the period of the wiper blades be 10.00 seconds?



Strategy

The resistance considers the equation $V_{out}(t) = V(1 - e^{-t/\tau})$, where $\tau = RC$. The capacitance, output voltage, and voltage of the battery are given. We need to solve this equation for the resistance.

Solution

The output voltage will be 10.00 V and the voltage of the battery is 12.00 V. The capacitance is given as 10.00 mF. Solving for the resistance yields

$$V_{out}(t) = V(1 - e^{-t/\tau})$$

$$e^{-t/RC} = 1 - \frac{V_{out}(t)}{V},$$

$$ln(e^{-t/RC}) = ln\left(1 - \frac{V_{out}(t)}{V}\right),$$

$$-\frac{t}{RC} = ln\left(1 - \frac{V_{out}(t)}{V}\right),$$

$$R = \frac{-t}{C\,ln\left(1 - \frac{V_C(t)}{V}\right)} = \frac{-10.00\,s}{10 \times 10^{-3}\,F\,ln\left(1 - \frac{10\,V}{12\,V}\right)} = 558.11\,\Omega.$$

Significance

Increasing the resistance increases the time delay between operations of the windshield wipers. When the resistance is zero, the windshield wipers run continuously. At the maximum resistance, the period of the operation of the wipers is:

$$t = -RCln\left(1 - \frac{V_{out}(t)}{V}\right) = -(10 \times 10^{-3}\,F)(10 \times 10^{3}\,\Omega)ln\left(1 - \frac{10\,V}{12\,V}\right) = 179.18\,s = 2.98\,min.$$

The **RC** circuit has thousands of uses and is a very important circuit to study. Not only can it be used to time circuits, it can also be used to filter out unwanted frequencies in a circuit and used in power supplies, like the one for your computer, to help turn ac voltage to dc voltage.

6.7: Direct-Current Circuits (Exercise)

Conceptual Questions

10.2 Electromotive Force

1. What effect will the internal resistance of a rechargeable battery have on the energy being used to recharge the battery?

2. A battery with an internal resistance of r and an emf of 10.00 V is connected to a load resistor **R=r**. As the battery ages, the internal resistance triples. How much is the current through the load resistor reduced?

3. Show that the power dissipated by the load resistor is maximum when the resistance of the load resistor is equal to the internal resistance of the battery.

10.3 Resistors in Series and Parallel

4. A voltage occurs across an open switch. What is the power dissipated by the open switch?

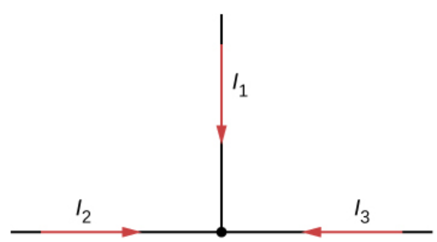
5. The severity of a shock depends on the magnitude of the current through your body. Would you prefer to be in series or in parallel with a resistance, such as the heating element of a toaster, if you were shocked by it? Explain.

6. Suppose you are doing a physics lab that asks you to put a resistor into a circuit, but all the resistors supplied have a larger resistance than the requested value. How would you connect the available resistances to attempt to get the smaller value asked for?

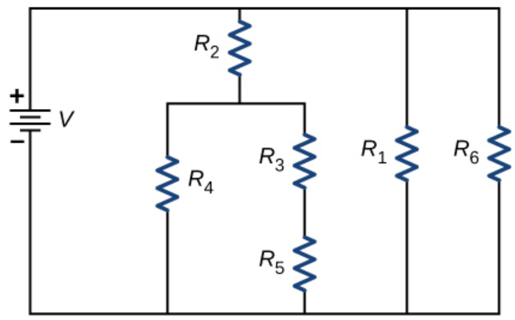
7. Some light bulbs have three power settings (not including zero), obtained from multiple filaments that are individually switched and wired in parallel. What is the minimum number of filaments needed for three power settings?

10.4 Kirchhoff's Rules

8. Can all of the currents going into the junction shown below be positive? Explain.



9. Consider the circuit shown below. Does the analysis of the circuit require Kirchhoff's method, or can it be redrawn to simplify the circuit? If it is a circuit of series and parallel connections, what is the equivalent resistance?



10. Do batteries in a circuit always supply power to a circuit, or can they absorb power in a circuit? Give an example.

11. What are the advantages and disadvantages of connecting batteries in series? In parallel?

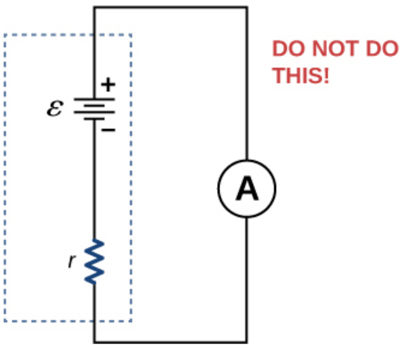
12. Semi-tractor trucks use four large 12-V batteries. The starter system requires 24 V, while normal operation of the truck's other electrical components utilizes 12 V. How could the four batteries be connected to produce 24 V? To produce 12 V? Why is 24 V better than 12 V for starting the truck's engine (a very heavy load)?

10.5 Electrical Measuring Instruments

13. What would happen if you placed a voltmeter in series with a component to be tested?

14. What is the basic operation of an ohmmeter as it measures a resistor?

15. Why should you not connect an ammeter directly across a voltage source as shown below?



10.6 RC Circuits

16. A battery, switch, capacitor, and lamp are connected in series. Describe what happens to the lamp when the switch is closed.

17. When making an ECG measurement, it is important to measure voltage variations over small time intervals. The time is limited by the RC constant of the circuit—it is not possible to measure time variations shorter than RC. How would you manipulate R and C in the circuit to allow the necessary measurements?

10.6 Household Wiring and Electrical Safety

18. Why isn't a short circuit necessarily a shock hazard?

19. We are often advised to not flick electric switches with wet hands, dry your hand first. We are also advised to never throw water on an electric fire. Why?

Problems

10.2 Electromotive Force

20. A car battery with a 12-V emf and an internal resistance of **0.050Ω** is being charged with a current of 60 A. Note that in this process, the battery is being charged.

(a) What is the potential difference across its terminals?

(b) At what rate is thermal energy being dissipated in the battery?

(c) At what rate is electric energy being converted into chemical energy?

21. The label on a battery-powered radio recommends the use of rechargeable nickel-cadmium cells (nicads), although they have a 1.25-V emf, whereas alkaline cells have a 1.58-V emf. The radio has a **3.20Ω** resistance.

(a) Draw a circuit diagram of the radio and its batteries. Now, calculate the power delivered to the radio

(b) when using nicad cells, each having an internal resistance of **0.0400Ω**, and

(c) when using alkaline cells, each having an internal resistance of **0.200Ω**.

(d) Does this difference seem significant, considering that the radio's effective resistance is lowered when its volume is turned up?

22. An automobile starter motor has an equivalent resistance of **0.0500Ω** and is supplied by a 12.0-V battery with a **0.0100-Ω** internal resistance.

(a) What is the current to the motor?

(b) What voltage is applied to it?

(c) What power is supplied to the motor?

(d) Repeat these calculations for when the battery connections are corroded and add **0.0900Ω** to the circuit. (Significant problems are caused by even small amounts of unwanted resistance in low-voltage, high-current applications.)

23. (a) What is the internal resistance of a voltage source if its terminal potential drops by 2.00 V when the current supplied increases by 5.00 A?

(b) Can the emf of the voltage source be found with the information supplied?

24. A person with body resistance between his hands of **10.0kΩ** accidentally grasps the terminals of a 20.0-kV power supply. (Do NOT do this!)

(a) Draw a circuit diagram to represent the situation.

(b) If the internal resistance of the power supply is **2000Ω**, what is the current through his body?

(c) What is the power dissipated in his body?

(d) If the power supply is to be made safe by increasing its internal resistance, what should the internal resistance be for the maximum current in this situation to be 1.00 mA or less?

(e) Will this modification compromise the effectiveness of the power supply for driving low-resistance devices? Explain your reasoning.

25. A 12.0-V emf automobile battery has a terminal voltage of 16.0 V when being charged by a current of 10.0 A.

(a) What is the battery's internal resistance?

(b) What power is dissipated inside the battery?

(c) At what rate (in **°C/min**) will its temperature increase if its mass is 20.0 kg and it has a specific heat of **0.300kcal/kg·°C**, assuming no heat escapes?

10.3 Resistors in Series and Parallel

26. (a) What is the resistance of a $1.00 \times 10^2 - \Omega,$ a $2.50 - k\Omega,$ and a $4.00 - k\Omega$ resistor connected in series?

(b) In parallel?

27. What are the largest and smallest resistances you can obtain by connecting a **36.0-Ω**, a **50.0-Ω**, and a **700-Ω** resistor together?

28. An 1800-W toaster, a 1400-W speaker, and a 75-W lamp are plugged into the same outlet in a 15-A fuse and 120-V circuit. (The three devices are in parallel when plugged into the same socket.)

(a) What current is drawn by each device?

(b) Will this combination blow the 15-A fuse?

29. Your car's 30.0-W headlight and 2.40-kW starter are ordinarily connected in parallel in a 12.0-V system. What power would one headlight and the starter consume if connected in series to a 12.0-V battery? (Neglect any other resistance in the circuit and any change in resistance in the two devices.)

30. (a) Given a 48.0-V battery and **24.0-Ω** and **96.0-Ω**9 resistors, find the current and power for each when connected in series.

(b) Repeat when the resistances are in parallel.

31. Referring to the example combining series and parallel circuits and Figure 10.16, calculate I_3 in the following two different ways:

(a) from the known values of I and I_2;

(b) using Ohm's law for R_3. In both parts, explicitly show how you follow the steps in the Figure 10.17.

32. Referring to Figure 10.16,

(a) Calculate P_3 and note how it compares with P_3 found in the first two example problems in this module.

(b) Find the total power supplied by the source and compare it with the sum of the powers dissipated by the resistors.

33. Refer to Figure 10.17 and the discussion of lights dimming when a heavy appliance comes on.

(a) Given the voltage source is 120 V, the wire resistance is **0.800Ω**, and the bulb is nominally 75.0 W, what power will the bulb dissipate if a total of 15.0 A passes through the wires when the motor comes on? Assume negligible change in bulb resistance.

(b) What power is consumed by the motor?

34. Show that if two resistors R_1 and R_2 are combined and one is much greater than the other ($R_1 \gg R_2$),

(a) their series resistance is very nearly equal to the greater resistance R_1 and

(b) their parallel resistance is very nearly equal to the smaller resistance R_2.

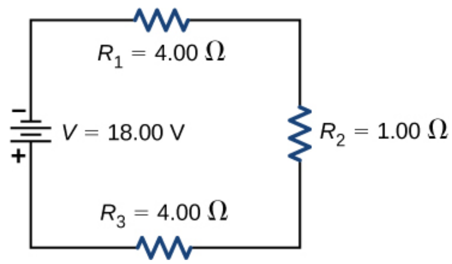
35. Consider the circuit shown below. The terminal voltage of the battery is **V=18.00V**.

(a) Find the equivalent resistance of the circuit.

(b) Find the current through each resistor.

(c) Find the potential drop across each resistor.

(d) Find the power dissipated by each resistor. (e) Find the power supplied by the battery.

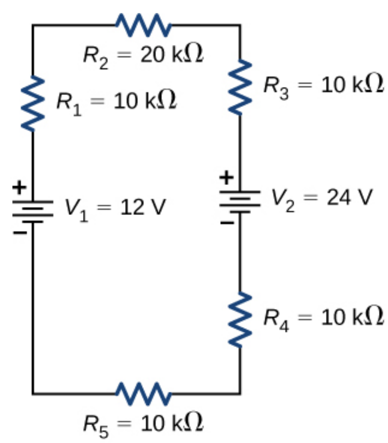


10.4 Kirchhoff's Rules

36. Consider the circuit shown below.

(a) Find the voltage across each resistor.

(b)What is the power supplied to the circuit and the power dissipated or consumed by the circuit?



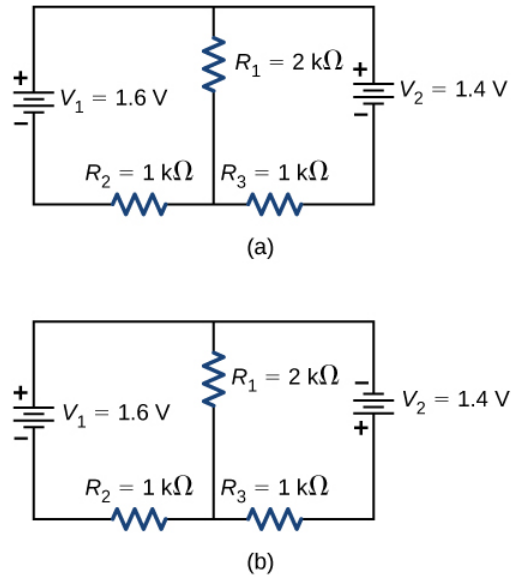
37. Consider the circuits shown below.

(a) What is the current through each resistor in part (a)?

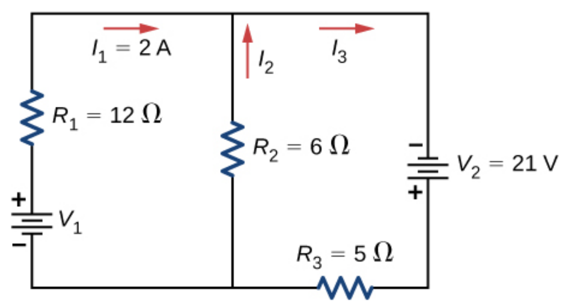
(b) What is the current through each resistor in part (b)?

(c) What is the power dissipated or consumed by each circuit?

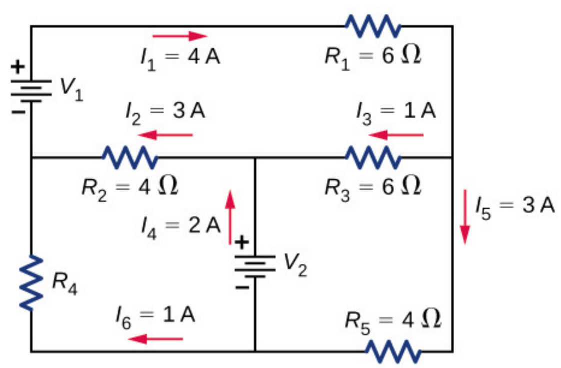
(d) What is the power supplied to each circuit?



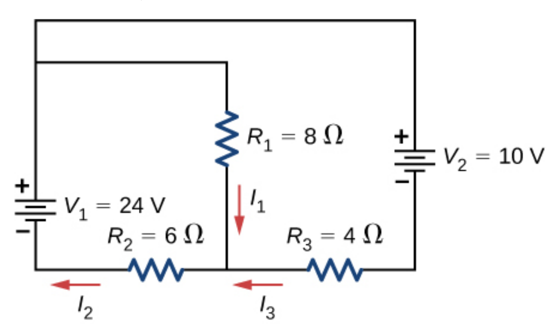
38. Consider the circuit shown below. Find V_1, I_2, and I_3.



39. Consider the circuit shown below. Find V_1, V_2, and R_4.



40. Consider the circuit shown below. Find I_1, I_2, and I_3.

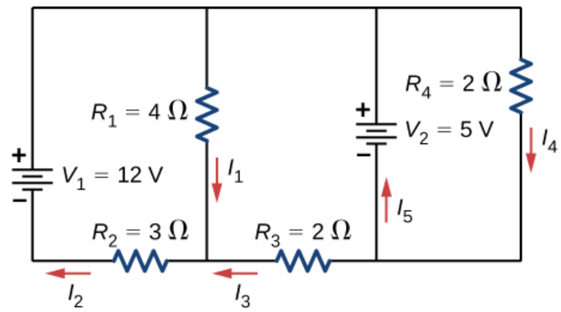


41. Consider the circuit shown below.

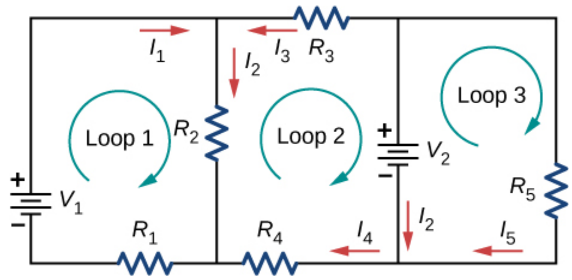
(a) Find I_1, I_2, I_3, I_4, and I_5.

(b) Find the power supplied by the voltage sources.

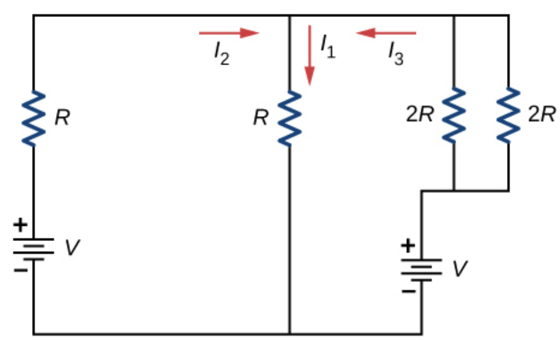
(c) Find the power dissipated by the resistors



42. Consider the circuit shown below. Write the three loop equations for the loops shown.



43. Consider the circuit shown below. Write equations for the three currents in terms of **R** and **V**.



44. Consider the circuit shown in the preceding problem. Write equations for the power supplied by the voltage sources and the power dissipated by the resistors in terms of R and V.

45. A child's electronic toy is supplied by three 1.58-V alkaline cells having internal resistances of **0.0200Ω** in series with a 1.53-V carbon-zinc dry cell having a **0.100-Ω** internal resistance. The load resistance is **10.0Ω**.

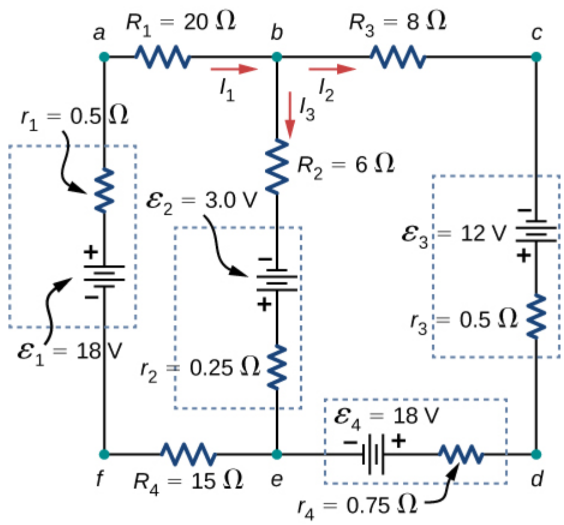
(a) Draw a circuit diagram of the toy and its batteries.

(b) What current flows?

(c) How much power is supplied to the load?

(d) What is the internal resistance of the dry cell if it goes bad, resulting in only 0.500 W being supplied to the load?

46. Apply the junction rule to Junction b shown below. Is any new information gained by applying the junction rule at e?



47. Apply the loop rule to Loop afedcba in the preceding problem.

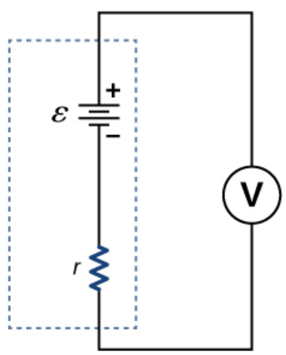
10.5 Electrical Measuring Instruments

48. Suppose you measure the terminal voltage of a 1.585-V alkaline cell having an internal resistance of **0.100Ω** by placing a **1.00-kΩ** voltmeter across its terminals (see below).

(a) What current flows?

(b) Find the terminal voltage.

(c) To see how close the measured terminal voltage is to the emf, calculate their ratio.



10.6 RC Circuits

49. The timing device in an automobile's intermittent wiper system is based on an RC time constant and utilizes a **0.500-μF** capacitor and a variable resistor. Over what range must R be made to vary to achieve time constants from 2.00 to 15.0 s?

50. A heart pacemaker fires 72 times a minute, each time a 25.0-nF capacitor is charged (by a battery in series with a resistor) to 0.632 of its full voltage. What is the value of the resistance?

51. The duration of a photographic flash is related to an RC time constant, which is **0.100μs** for a certain camera.

(a) If the resistance of the flash lamp is **0.0400Ω** during discharge, what is the size of the capacitor supplying its energy?

(b) What is the time constant for charging the capacitor, if the charging resistance is **800kΩ**?

52. A 2.00- and a **7.50-μF** capacitor can be connected in series or parallel, as can a 25.0- and a **100-kΩ** resistor. Calculate the four RC time constants possible from connecting the resulting capacitance and resistance in series.

53. A **500-Ω** resistor, an uncharged **1.50-μF** capacitor, and a 6.16-V emf are connected in series.

(a) What is the initial current?

(b) What is the RC time constant?

(c) What is the current after one time constant? (d) What is the voltage on the capacitor after one time constant?

54. A heart defibrillator being used on a patient has an RC time constant of 10.0 ms due to the resistance of the patient and the capacitance of the defibrillator.

(a) If the defibrillator has a capacitance of **8.00μF,** what is the resistance of the path through the patient? (You may neglect the capacitance of the patient and the resistance of the defibrillator.)

(b) If the initial voltage is 12.0 kV, how long does it take to decline to $6.00 \times 10^2 V$?

55. An ECG monitor must have an RC time constant less than $1.00 \times 10^2 \mu s$ to be able to measure variations in voltage over small time intervals.

(a) If the resistance of the circuit (due mostly to that of the patient's chest) is **1.00kΩ**, what is the maximum capacitance of the circuit?

(b) Would it be difficult in practice to limit the capacitance to less than the value found in (a)?

56. Using the exact exponential treatment, determine how much time is required to charge an initially uncharged 100-pF capacitor through a **75.0-MΩ** resistor to **90.0%** of its final voltage.

57. If you wish to take a picture of a bullet traveling at 500 m/s, then a very brief flash of light produced by an RC discharge through a flash tube can limit blurring. Assuming 1.00 mm of motion during one RC constant is acceptable, and given that the flash is driven by a **600-μF** capacitor, what is the resistance in the flash tube?

10.7 Household Wiring and Electrical Safety

58. (a) How much power is dissipated in a short circuit of 240-V ac through a resistance of **0.250Ω**? (b) What current flows?

59. What voltage is involved in a 1.44-kW short circuit through a **0.100-Ω** resistance?

60. Find the current through a person and identify the likely effect on her if she touches a 120-V ac source:

(a) if she is standing on a rubber mat and offers a total resistance of **300kΩ**;

(b) if she is standing barefoot on wet grass and has a resistance of only **4000kΩ**.

61. While taking a bath, a person touches the metal case of a radio. The path through the person to the drainpipe and ground has a resistance of **4000Ω**. What is the smallest voltage on the case of the radio that could cause ventricular fibrillation?

62. A man foolishly tries to fish a burning piece of bread from a toaster with a metal butter knife and comes into contact with 120-V ac. He does not even feel it since, luckily, he is wearing rubber-soled shoes. What is the minimum resistance of the path the current follows through the person?

63. (a) During surgery, a current as small as **20.0μA** applied directly to the heart may cause ventricular fibrillation. If the resistance of the exposed heart is **300Ω**, what is the smallest voltage that poses this danger?

(b) Does your answer imply that special electrical safety precautions are needed?

64. (a) What is the resistance of a 220-V ac short circuit that generates a peak power of 96.8 kW?

(b) What would the average power be if the voltage were 120 V ac?

65. A heart defibrillator passes 10.0 A through a patient's torso for 5.00 ms in an attempt to restore normal beating.

(a) How much charge passed?

(b) What voltage was applied if 500 J of energy was dissipated?

(c) What was the path's resistance? (d) Find the temperature increase caused in the 8.00 kg of affected tissue.

66. A short circuit in a 120-V appliance cord has a **0.500-Ω** resistance. Calculate the temperature rise of the 2.00 g of surrounding materials, assuming their specific heat capacity is **0.200cal/g·°C** and that it takes 0.0500 s for a circuit breaker to interrupt the current. Is this likely to be damaging?

Additional Problems

67. A circuit contains a D cell battery, a switch, a **20-Ω** resistor, and four 20-mF capacitors connected in series.

(a) What is the equivalent capacitance of the circuit?

(b) What is the RC time constant?

(c) How long before the current decreases to **50%** of the initial value once the switch is closed?

68. A circuit contains a D-cell battery, a switch, a **20-Ω** resistor, and three 20-mF capacitors. The capacitors are connected in parallel, and the parallel connection of capacitors are connected in series with the switch, the resistor and the battery.

(a) What is the equivalent capacitance of the circuit?

(b) What is the RC time constant?

(c) How long before the current decreases to **50%** of the initial value once the switch is closed?

69. Consider the circuit below. The battery has an emf of **ε=30.00V** and an internal resistance of **r=1.00Ω**.

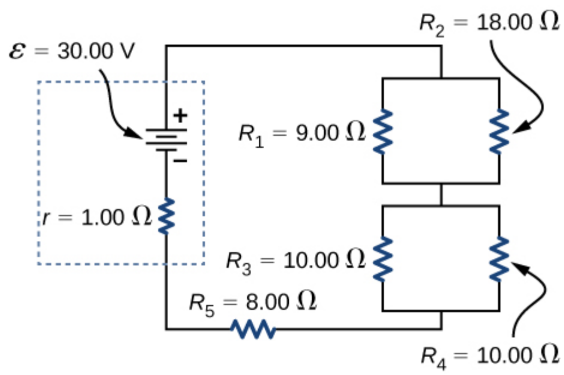
(a) Find the equivalent resistance of the circuit and the current out of the battery.

(b) Find the current through each resistor.

(c) Find the potential drop across each resistor.

(d) Find the power dissipated by each resistor.

(e) Find the total power supplied by the batteries.



70. A homemade capacitor is constructed of 2 sheets of aluminum foil with an area of 2.00 square meters, separated by paper, 0.05 mm thick, of the same area and a dielectric constant of 3.7. The homemade capacitor is connected in series with a **100.00-Ω** resistor, a switch, and a 6.00-V voltage source.

(a) What is the RC time constant of the circuit?

(b) What is the initial current through the circuit, when the switch is closed? (c) How long does it take the current to reach one third of its initial value?

71. A student makes a homemade resistor from a graphite pencil 5.00 cm long, where the graphite is 0.05 mm in diameter. The resistivity of the graphite is $\rho = 1.38 \times 10^{-5} \Omega/m$. The homemade resistor is place in series with a switch, a 10.00-mF capacitor and a 0.50-V power source.

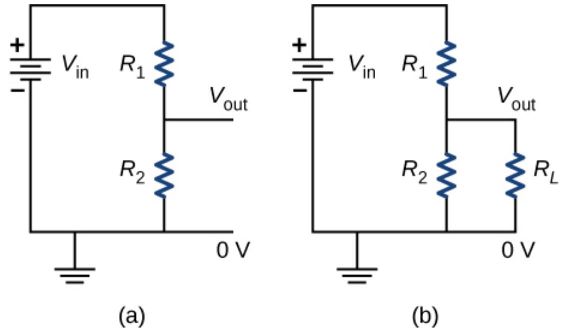
(a) What is the RC time constant of the circuit?

(b) What is the potential drop across the pencil 1.00 s after the switch is closed?

72. The rather simple circuit shown below is known as a voltage divider. The symbol consisting of three horizontal lines is represents "ground" and can be defined as the point where the potential is zero. The voltage divider is widely used in circuits and a single voltage source can be used to provide reduced voltage to a load resistor as shown in the second part of the figure. (a) What is the output voltage V_{out} of circuit

(a) in terms of R_1, R_2, and V_{in}?

(b) What is the output voltage V_{out} of circuit (b) in terms of R_1, R_2, R_L, and V_{in}?



73. Three **300-Ω** resistors are connect in series with an AAA battery with a rating of 3 AmpHours. (a) How long can the battery supply the resistors with power? (b) If the resistors are connected in parallel, how long can the battery last?

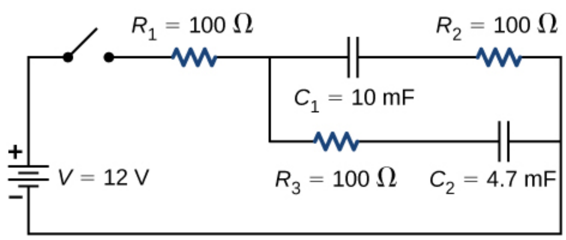
74. Consider a circuit that consists of a real battery with an emf εε and an internal resistance of r connected to a variable resistor R.

(a) In order for the terminal voltage of the battery to be equal to the emf of the battery, what should the resistance of the variable resistor be adjusted to?

(b) In order to get the maximum current from the battery, what should the resistance of the variable resistor be adjusted to?

(c) In order for the maximum power output of the battery to be reached, what should the resistance of the variable resistor be set to?

75. Consider the circuit shown below. What is the energy stored in each capacitor after the switch has been closed for a very long time?



76. Consider a circuit consisting of a battery with an emf εε and an internal resistance of r connected in series with a resistorR and a capacitor C. Show that the total energy supplied by the battery while charging the battery is equal to $\varepsilon^2 C$.

77. Consider the circuit shown below. The terminal voltages of the batteries are shown.

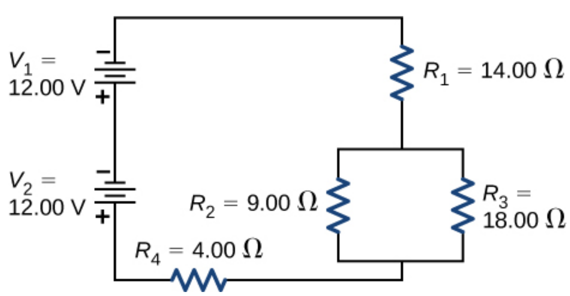
(a) Find the equivalent resistance of the circuit and the current out of the battery.

(b) Find the current through each resistor.

(c) Find the potential drop across each resistor.

(d) Find the power dissipated by each resistor.

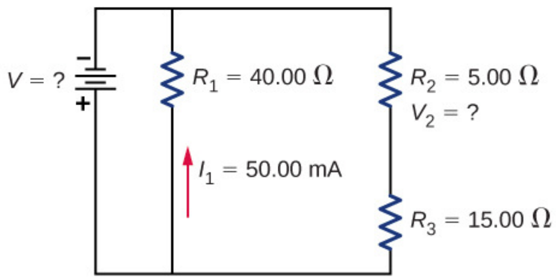
(e) Find the total power supplied by the batteries.



78. Consider the circuit shown below.

(a) What is the terminal voltage of the battery?

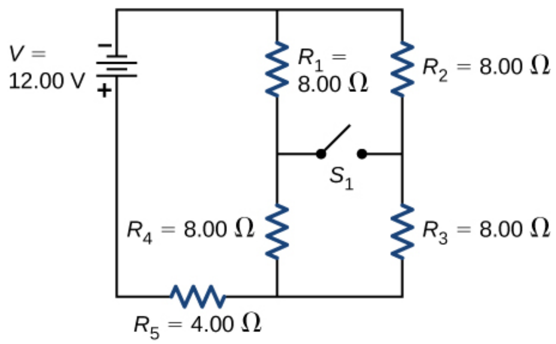
(b) What is the potential drop across resistor R_2?



79. Consider the circuit shown below.

(a)Determine the equivalent resistance and the current from the battery with switch S_1 open.

(b) Determine the equivalent resistance and the current from the battery with switch S_1 closed.



80. Two resistors, one having a resistance of **145Ω**, are connected in parallel to produce a total resistance of **150Ω**.

(a) What is the value of the second resistance?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

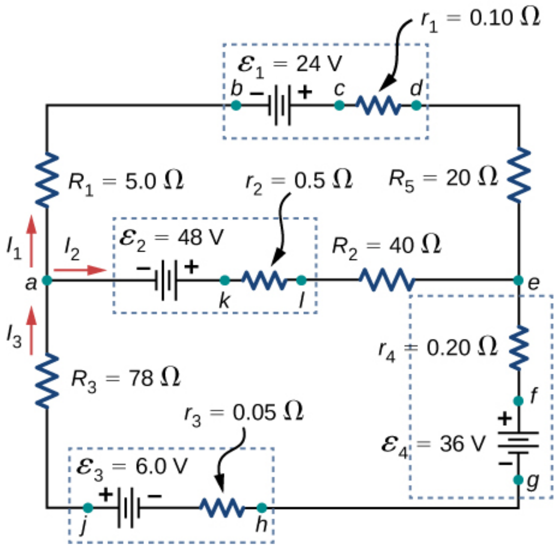
81. Two resistors, one having a resistance of **900kΩ**, are connected in series to produce a total resistance of **0.500MΩ**.

(a) What is the value of the second resistance?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

82. Apply the junction rule at point a shown below.



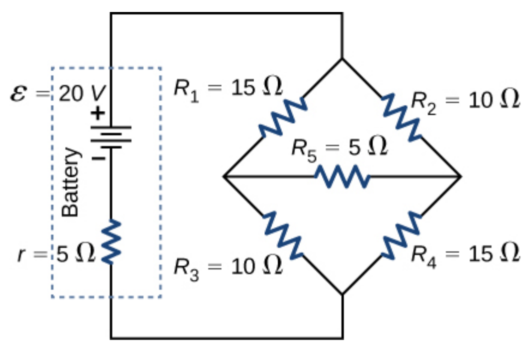
83. Apply the loop rule to Loop akledcba in the preceding problem.

84. Find the currents flowing in the circuit in the preceding problem. Explicitly show how you follow the steps in the Problem-Solving Strategy: Series and Parallel Resistors.

85. Consider the circuit shown below.

(a) Find the current through each resistor.

(b) Check the calculations by analyzing the power in the circuit.



86. A flashing lamp in a Christmas earring is based on an RC discharge of a capacitor through its resistance. The effective duration of the flash is 0.250 s, during which it produces an average 0.500 W from an average 3.00 V.

(a) What energy does it dissipate?

(b) How much charge moves through the lamp?

(c) Find the capacitance.

(d) What is the resistance of the lamp? (Since average values are given for some quantities, the shape of the pulse profile is not needed.)

87. A **160-μF** capacitor charged to 450 V is discharged through a **31.2-kΩ** resistor.

(a) Find the time constant.

(b) Calculate the temperature increase of the resistor, given that its mass is 2.50 g and its specific heat is **1.67kJ/kg·°C**, noting that most of the thermal energy is retained in the short time of the discharge.

(c) Calculate the new resistance, assuming it is pure carbon.

(d) Does this change in resistance seem significant?

Challenge Problems

88. Some camera flashes use flash tubes that require a high voltage. They obtain a high voltage by charging capacitors in parallel and then internally changing the connections of the capacitors to place them in series. Consider a circuit that uses four AAA batteries connected in series to charge six 10-mF capacitors through an equivalent resistance of **100Ω**. The connections are then switched internally to place the capacitors in series. The capacitors discharge through a lamp with a resistance of **100Ω**.

(a) What is the RC time constant and the initial current out of the batteries while they are connected in parallel?

(b) How long does it take for the capacitors to charge to **90%** of the terminal voltages of the batteries?

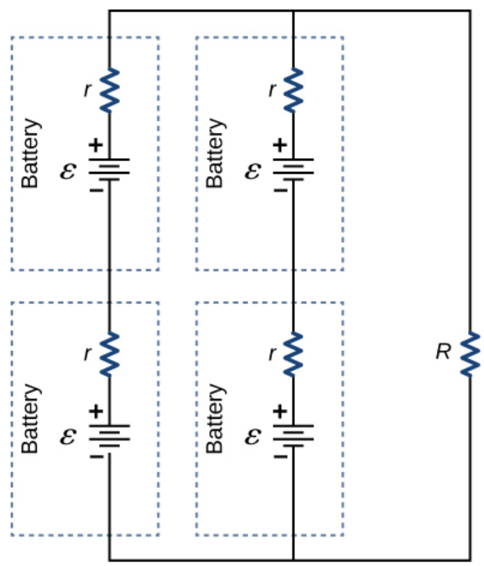
(c) What is the RC time constant and the initial current of the capacitors connected in series assuming it discharges at 90%90% of full charge?

(d) How long does it take the current to decrease to **10%** of the initial value?

89. Consider the circuit shown below. Each battery has an emf of 1.50 V and an internal resistance of **1.00Ω**.

(a) What is the current through the external resistor, which has a resistance of 10.00 ohms?

(b) What is the terminal voltage of each battery?

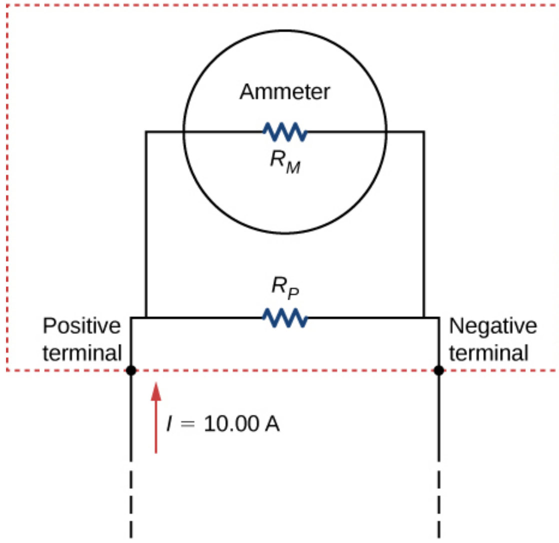


90. Analog meters use a galvanometer, which essentially consists of a coil of wire with a small resistance and a pointer with a scale attached. When current runs through the coil, the pointer turns; the amount the pointer turns is proportional to the amount of current running through the coil. Galvanometers can be used to make an ammeter if a resistor is placed in parallel with the galvanometer. Consider a galvanometer that has a resistance of **25.00Ω** and gives a full scale reading when a **50-μA** current runs through it. The galvanometer is to be used to make an ammeter that has a full scale reading of 10.00 A, as shown below. Recall that an ammeter is connected in series with the circuit of interest, so all 10 A must run through the meter.

(a) What is the current through the parallel resistor in the meter?

(b) What is the voltage across the parallel resistor?

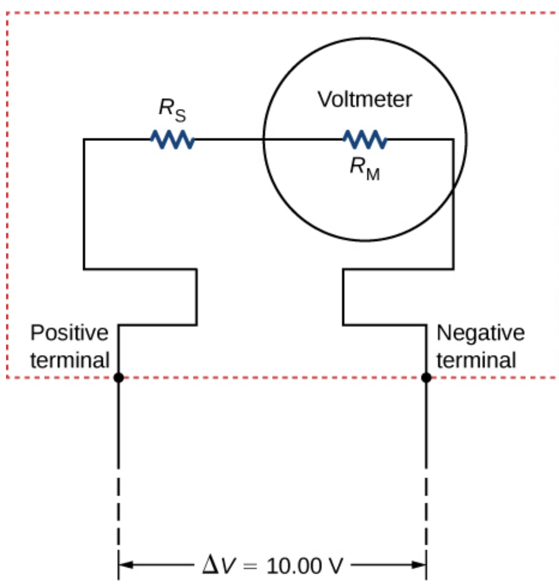
(c) What is the resistance of the parallel resistor?



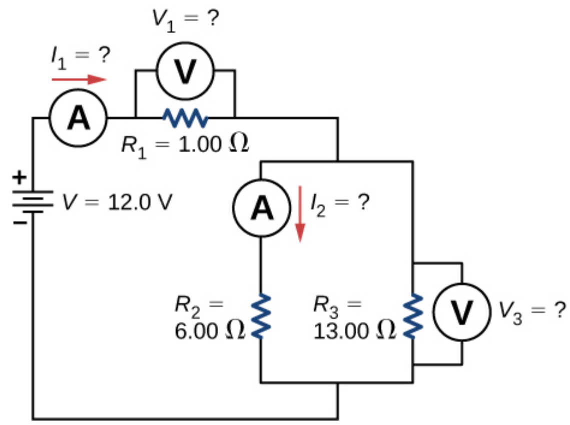
91. Analog meters use a galvanometer, which essentially consists of a coil of wire with a small resistance and a pointer with a scale attached. When current runs through the coil, the point turns; the amount the pointer turns is proportional to the amount of current running through the coil. Galvanometers can be used to make a voltmeter if a resistor is placed in series with the galvanometer. Consider a galvanometer that has a resistance of **25.00Ω** and gives a full scale reading when a **50-μA** current runs through it. The galvanometer is to be used to make an voltmeter that has a full scale reading of 10.00 V, as shown below. Recall that a voltmeter is connected in parallel with the component of interest, so the meter must have a high resistance or it will change the current running through the component.

(a) What is the potential drop across the series resistor in the meter?

(b) What is the resistance of the parallel resistor?



92. Consider the circuit shown below. Find I_1, V_1, I_2, and V_3.

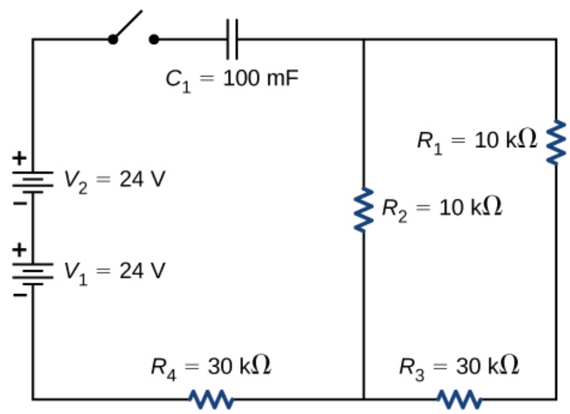


93. Consider the circuit below.

(a) What is the RC time constant of the circuit?

(b) What is the initial current in the circuit once the switch is closed?

(c) How much time passes between the instant the switch is closed and the time the current has reached half of the initial current?



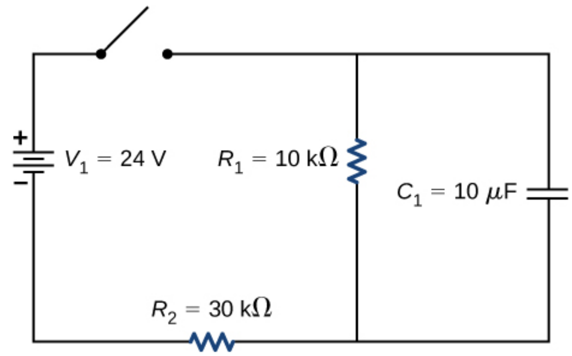
94. Consider the circuit below.

(a) What is the initial current through resistor R_2 when the switch is closed?

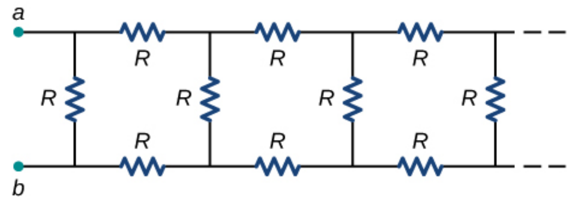
(b) What is the current through resistor R_2 when the capacitor is fully charged, long after the switch is closed?

(c) What happens if the switch is opened after it has been closed for some time?

(d) If the switch has been closed for a time period long enough for the capacitor to become fully charged, and then the switch is opened, how long before the current through resistor R_1 reaches half of its initial value?



95. Consider the infinitely long chain of resistors shown below. What is the resistance between terminals **a** and **b**?



96. Consider the circuit below. The capacitor has a capacitance of 10 mF. The switch is closed and after a long time the capacitor is fully charged.

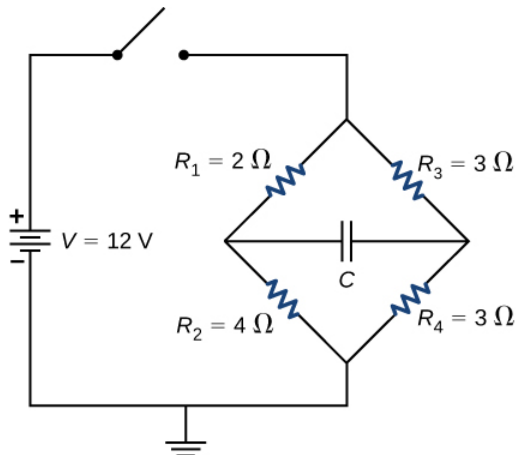
(a) What is the current through each resistor a long time after the switch is closed?

(b) What is the voltage across each resistor a long time after the switch is closed?

(c) What is the voltage across the capacitor a long time after the switch is closed?

(d) What is the charge on the capacitor a long time after the switch is closed?

(e) The switch is then opened. The capacitor discharges through the resistors. How long from the time before the current drops to one fifth of the initial value?



97. A 120-V immersion heater consists of a coil of wire that is placed in a cup to boil the water. The heater can boil one cup of **20.00°C** water in 180.00 seconds. You buy one to use in your dorm room, but you are worried that you will overload the circuit and trip the 15.00-A, 120-V circuit breaker, which supplies your dorm room. In your dorm room, you have four 100.00-W incandescent lamps and a 1500.00-W space heater.

(a) What is the power rating of the immersion heater?

(b) Will it trip the breaker when everything is turned on?

(c) If it you replace the incandescent bulbs with 18.00-W LED, will the breaker trip when everything is turned on?

98. Find the resistance that must be placed in series with a **25.0-Ω** galvanometer having a **50.0-μA** sensitivity (the same as the one discussed in the text) to allow it to be used as a voltmeter with a 3000-V full-scale reading. Include a circuit diagram with your solution.

99. Find the resistance that must be placed in parallel with a **60.0-Ω** galvanometer having a 1.00-mA sensitivity (the same as the one discussed in the text) to allow it to be used as an ammeter with a 25.0-A full-scale reading. Include a circuit diagram with your solution.

CHAPTER OVERVIEW

7: Magnetic Forces and Fields

For the past few chapters, we have been studying electrostatic forces and fields, which are caused by electric charges at rest. These electric fields can move other free charges, such as producing a current in a circuit; however, the electrostatic forces and fields themselves come from other static charges. In this chapter, we see that when an electric charge moves, it generates other forces and fields. These additional forces and fields are what we commonly call magnetism.

7.1: Prelude to Magnetic Forces and Fields
7.2: Magnetism and Its Historical Discoveries
7.3: Magnetic Fields and Lines
7.4: Motion of a Charged Particle in a Magnetic Field
7.5: Magnetic Force on a Current-Carrying Conductor
7.5.1: Force and Torque on a Current Loop
7.5.2: The Hall Effect
7.6: Magnetic Forces and Fields (Exercise)

7.1: Prelude to Magnetic Forces and Fields

For the past few chapters, we have been studying electrostatic forces and fields, which are caused by electric charges at rest. These electric fields can move other free charges, such as producing a current in a circuit; however, the electrostatic forces and fields themselves come from other static charges. In this chapter, we see that when an electric charge moves, it generates other forces and fields. These additional forces and fields are what we commonly call magnetism.

Figure 7.1.1: An industrial electromagnet is capable of lifting thousands of pounds of metallic waste. (credit: modification of work by "BedfordAl"/Flickr)

Before we examine the origins of magnetism, we first describe what it is and how magnetic fields behave. Once we are more familiar with magnetic effects, we can explain how they arise from the behavior of atoms and molecules, and how magnetism is related to electricity. The connection between electricity and magnetism is fascinating from a theoretical point of view, but it is also immensely practical, as shown by an industrial electromagnet that can lift thousands of pounds of metal.

7.2: Magnetism and Its Historical Discoveries

Learning Objectives

By the end of this section, you will be able to:

- Explain attraction and repulsion by magnets
- Describe the historical and contemporary applications of magnetism

Magnetism has been known since the time of the ancient Greeks, but it has always been a bit mysterious. You can see electricity in the flash of a lightning bolt, but when a compass needle points to magnetic north, you can't see any force causing it to rotate. People learned about magnetic properties gradually, over many years, before several physicists of the nineteenth century connected magnetism with electricity. In this section, we review the basic ideas of magnetism and describe how they fit into the picture of a magnetic field.

Brief History of Magnetism

Magnets are commonly found in everyday objects, such as toys, hangers, elevators, doorbells, and computer devices. Experimentation on these magnets shows that all magnets have two poles: One is labeled north (N) and the other is labeled south (S). Magnetic poles repel if they are alike (both N or both S), they attract if they are opposite (one N and the other S), and both poles of a magnet attract unmagnetized pieces of iron. An important point to note here is that you cannot isolate an individual magnetic pole. Every piece of a magnet, no matter how small, which contains a north pole must also contain a south pole.

Note

Visit this website for an interactive demonstration of magnetic north and south poles.

An example of a magnet is a **compass needle**. It is simply a thin bar magnet suspended at its center, so it is free to rotate in a horizontal plane. Earth itself also acts like a very large bar magnet, with its south-seeking pole near the geographic North Pole (Figure 7.2.1). The north pole of a compass is attracted toward Earth's geographic North Pole because the magnetic pole that is near the geographic North Pole is actually a south magnetic pole. Confusion arises because the geographic term "North Pole" has come to be used (incorrectly) for the magnetic pole that is near the North Pole. Thus, "**north magnetic pole**" is actually a misnomer—it should be called the **south magnetic pole**. [Note that the orientation of Earth's magnetic field is not permanent but changes ("flips") after long time intervals. Eventually, Earth's north magnetic pole may be located near its geographic North Pole.]

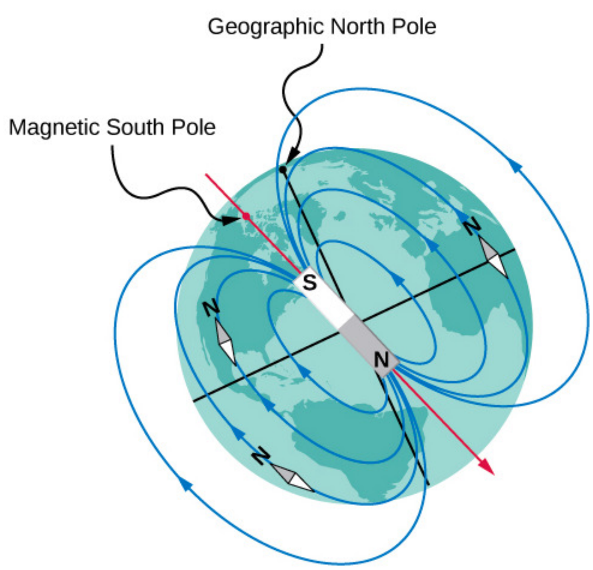


Figure 7.2.1: The north pole of a compass needle points toward the south pole of a magnet, which is how today's magnetic field is oriented from inside Earth. It also points toward Earth's geographic North Pole because the geographic North Pole is near the magnetic south pole.

Back in 1819, the Danish physicist Hans **Oersted** was performing a lecture demonstration for some students and noticed that a compass needle moved whenever current flowed in a nearby wire. Further investigation of this phenomenon convinced Oersted that an electric current could somehow cause a magnetic force. He reported this finding to an 1820 meeting of the French Academy of Science.

Soon after this report, Oersted's investigations were repeated and expanded upon by other scientists. Among those whose work was especially important were Jean-Baptiste **Biot** and Felix **Savart**, who investigated the forces exerted on magnets by currents; André Marie **Ampère**, who studied the forces exerted by one current on another; François **Arago**, who found that iron could be magnetized by a current; and Humphry **Davy**, who discovered that a magnet exerts a force on a wire carrying an electric current. Within 10 years of Oersted's discovery, Michael **Faraday** found that the relative motion of a magnet and a metallic wire induced current in the wire. This finding showed not only that a current has a magnetic effect, but that a magnet can generate electric current. You will see later that the names of Biot, Savart, Ampère, and Faraday are linked to some of the fundamental laws of electromagnetism.

The evidence from these various experiments led Ampère to propose that electric current is the source of all magnetic phenomena. To explain permanent magnets, he suggested that matter contains microscopic current loops that are somehow aligned when a material is magnetized. Today, we know that permanent magnets are actually created by the alignment of spinning electrons, a situation quite similar to that proposed by Ampère. This model of permanent magnets was developed by Ampère almost a century before the atomic nature of matter was understood. (For a full quantum mechanical treatment of magnetic spins, see Quantum Mechanics and Atomic Structure.)

Contemporary Applications of Magnetism

Today, magnetism plays many important roles in our lives. Physicists' understanding of magnetism has enabled the development of technologies that affect both individuals and society. The electronic tablet in your purse or backpack, for example, wouldn't have been possible without the applications of magnetism and electricity on a small scale (Figure $7.2.2$). Weak changes in a magnetic field in a thin film of iron and chromium were discovered to bring about much larger changes in resistance, called **giant magnetoresistance**. Information can then be recorded magnetically based on the direction in which the iron layer is magnetized. As a result of the discovery of giant magnetoresistance and its applications to digital storage, the 2007 Nobel Prize in Physics was awarded to Albert Fert from France and Peter Grunberg from Germany.

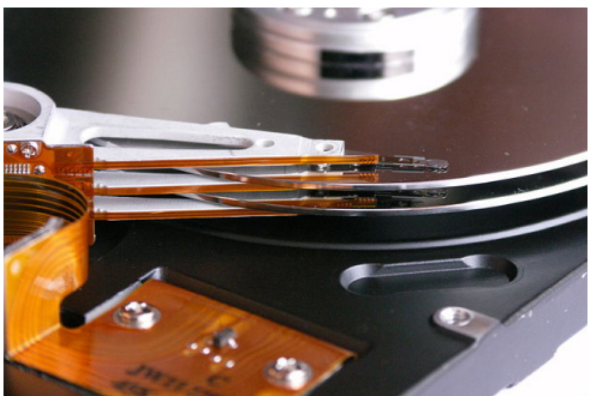


Figure $7.2.2$: Engineering technology like computer storage would not be possible without a deep understanding of magnetism. (credit: Klaus Eifert)

All electric motors—with uses as diverse as powering refrigerators, starting cars, and moving elevators—contain magnets. Generators, whether producing hydroelectric power or running bicycle lights, use magnetic fields. Recycling facilities employ magnets to separate iron from other refuse. Research into using magnetic containment of fusion as a future energy source has been continuing for several years. Magnetic resonance imaging (MRI) has become an important diagnostic tool in the field of medicine, and the use of magnetism to explore brain activity is a subject of contemporary research and development. The list of applications also includes computer hard drives, tape recording, detection of inhaled asbestos, and levitation of high-speed trains. Magnetism is involved in the structure of atomic energy levels, as well as the motion of cosmic rays and charged particles trapped in the Van Allen belts around Earth. Once again, we see that all these disparate phenomena are linked by a small number of underlying physical principles.

7.3: Magnetic Fields and Lines

Learning Objectives

By the end of this section, you will be able to:

- Define the magnetic field based on a moving charge experiencing a force
- Apply the right-hand rule to determine the direction of a magnetic force based on the motion of a charge in a magnetic field
- Sketch magnetic field lines to understand which way the magnetic field points and how strong it is in a region of space

We have outlined the properties of magnets, described how they behave, and listed some of the applications of magnetic properties. Even though there are no such things as isolated magnetic charges, we can still define the attraction and repulsion of magnets as based on a field. In this section, we define the magnetic field, determine its direction based on the right-hand rule, and discuss how to draw magnetic field lines.

Defining the Magnetic Field

A magnetic field is defined by the force that a charged particle experiences moving in this field, after we account for the gravitational and any additional electric forces possible on the charge. The magnitude of this force is proportional to the amount of charge **q**, the speed of the charged particle **v**, and the magnitude of the applied magnetic field. The direction of this force is perpendicular to both the direction of the moving charged particle and the direction of the applied magnetic field. Based on these observations, we define the magnetic field strength **B** based on the **magnetic force** $\vec{F}$ on a charge **q** moving at velocity as the cross product of the velocity and magnetic field, that is,

$$\vec{F} = q\vec{v} \times \vec{B}. \tag{7.3.1}$$

In fact, this is how we define the magnetic field $\vec{B}$ - in terms of the force on a charged particle moving in a magnetic field. The magnitude of the force is determined from the definition of the cross product as it relates to the magnitudes of each of the vectors. In other words, the magnitude of the force satisfies

$$F = qvB\sin\theta \tag{7.3.2}$$

where **θ** is the angle between the velocity and the magnetic field.

The SI unit for magnetic field strength B is called the tesla (T) after the eccentric, but brilliant inventor Nikola Tesla (1856–1943), where

$$1\,T = \frac{1\,N}{A \cdot m}.$$

A smaller unit, called the **gauss** (G) is sometimes used, where

$$1\,G = 10^{-4}\,T$$

The strongest permanent magnets have fields near 2 T; superconducting electromagnets may attain 10 T or more. Earth's magnetic field on its surface is only about $5 \times 10^{-5}\,T$ or $0.5\,G$.

Problem-Solving Strategy: Direction of the Magnetic Field by the Right-Hand Rule

The direction of the magnetic force $\vec{F}$ is perpendicular to the plane formed by $\vec{v}$ and $\vec{B}$ as determined by the **right-hand rule-1** (or RHR-1), which is illustrated in Figure $7.3.1$.

1. Orient your right hand so that your fingers curl in the plane defined by the velocity and magnetic field vectors.
2. Using your right hand, sweep from the velocity toward the magnetic field with your fingers through the smallest angle possible.
3. The magnetic force is directed where your thumb is pointing.
4. If the charge was negative, reverse the direction found by these steps.

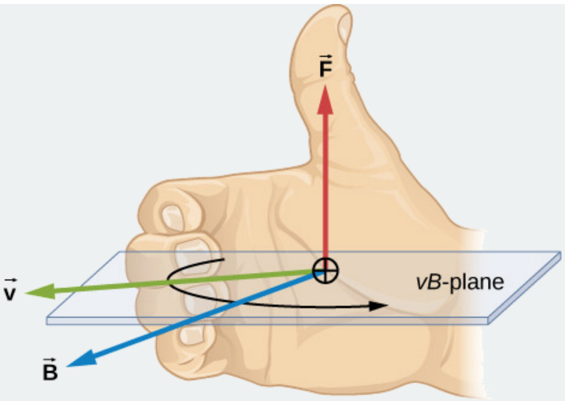


Figure 7.3.1: Magnetic fields exert forces on moving charges. The direction of the magnetic force on a moving charge is perpendicular to the plane formed by b$\vec{v}$ and $\vec{B}$ and follows the right-hand rule-1 (RHR-1) as shown. The magnitude of the force is proportional to q, v, B, and the sine of the angle between $\vec{v}$ and $\vec{B}$.

Note

Visit this website for additional practice with the direction of magnetic fields.

There is no magnetic force on static charges. However, there is a magnetic force on charges moving at an angle to a magnetic field. When charges are stationary, their electric fields do not affect magnets. However, when charges move, they produce magnetic fields that exert forces on other magnets. When there is relative motion, a connection between electric and magnetic forces emerges - each affects the other.

Example 7.3.1: An Alpha-Particle Moving in a Magnetic Field

An alpha-particle $(q = 3.2 \times 10^{-19} C)$ moves through a uniform magnetic field whose magnitude is 1.5 T. The field is directly parallel to the positive **z**-axis of the rectangular coordinate system of Figure 7.3.2. What is the magnetic force on the alpha-particle when it is moving (a) in the positive **x**-direction with a speed of $5.0 \times 10^4 m/s$? (b) in the negative **y**-direction with a speed of $5.0 \times 10^4 m/s$? (c) in the positive **z**-direction with a speed of $5.0 \times 10^4 m/s$? (d) with a velocity $\vec{v} = \left(2.0\hat{i} - 3.0\hat{j} + 1.0\hat{k}\right) \times 10^4 m/s$?

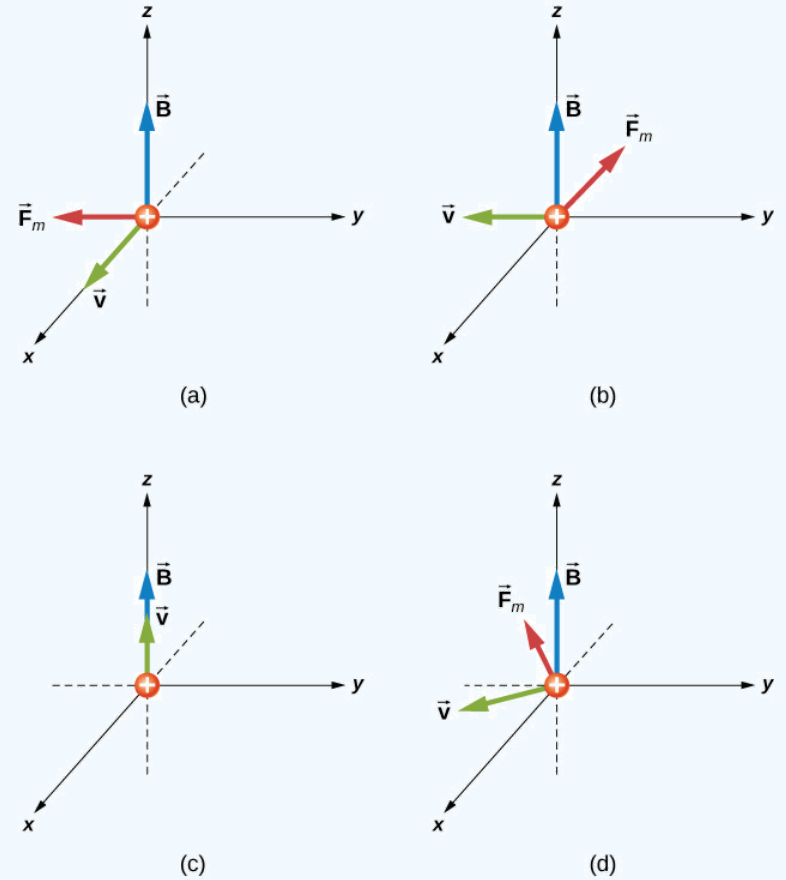


Figure 7.3.2: The magnetic forces on an alpha-particle moving in a uniform magnetic field. The field is the same in each drawing, but the velocity is different.

Strategy

We are given the charge, its velocity, and the magnetic field strength and direction. We can thus use the equation $\vec{F} = q\vec{v} \times \vec{B}$ or $F = qvB sin\,\theta$ to calculate the force. The direction of the force is determined by RHR-1.

Solution

1. First, to determine the direction, start with your fingers pointing in the positive **x**-direction. Sweep your fingers upward in the direction of magnetic field. Your thumb should point in the negative **y**-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:

$$\vec{F} = q\vec{v} \times \vec{B} = (3.2 \times 10^{-19} C)(5.0 \times 10^4 m/s\,\hat{i}) \times (1.5\,T\,\hat{k}) = -2.4 \times 10^{-14} N\,\hat{j}$$

2. First, to determine the directionality, start with your fingers pointing in the negative **y**-direction. Sweep your fingers upward in the direction of magnetic field as in the previous problem. Your thumb should be open in the negative **x**-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:

$$\vec{F} = q\vec{v} \times \vec{B} = (3.2 \times 10^{-19} C)(-5.0 \times 10^4 m/s\,\hat{i}) \times (1.5\,T\,\hat{k}) = -2.4 \times 10^{-14} N\,\hat{i}$$

An alternative approach is to use Equation 7.3.2 to find the magnitude of the force. This applies for both parts (a) and (b). Since the velocity is perpendicular to the magnetic field, the angle between them is 90 degrees. Therefore, the magnitude of the force is:

$$F = qvB\sin\,\theta = (3.2 \times 10^{-19} C)(5.0 \times 10^4 m/s)(1.5\,T) sin(90^o)) = 2.4 \times 10^{-14} N.$$

3. Since the velocity and magnetic field are parallel to each other, there is no orientation of your hand that will result in a force direction. Therefore, the force on this moving charge is zero. This is confirmed by the cross product. When you cross two vectors pointing in the same direction, the result is equal to zero.
4. First, to determine the direction, your fingers could point in any orientation; however, you must sweep your fingers upward in the direction of the magnetic field. As you rotate your hand, notice that the thumb can point in any **x**- or **y**-direction possible, but not in the **z**-direction. This should match the mathematical answer. To calculate the force, we use the given charge, velocity, and magnetic field and the definition of the magnetic force in cross-product form to calculate:

$$\vec{F} = q\vec{v} \times \vec{B} = (3.2 \times 10^{-19} C)((2.0\hat{i} - 3.0\hat{j} + 1.0\hat{k}) \times 10^4 m/s) \times (1.5\, T\hat{k})$$

$$(-14.4\hat{i} - 9.6\hat{j}) \times 10^{-15} N.$$

This solution can be rewritten in terms of a magnitude and angle in the **xy**-plane:

$$|\vec{F}| = \sqrt{F_x^2 + F_y^2} = \sqrt{(-14.4)^2 + (-9.6)^2} \times 10^{-15} N = 1.7 \times 10^{-14} N$$

$$\theta = tan^{-1}\left(\frac{F_y}{F_x}\right) = tan^{-1}\left(\frac{-9.6 \times 10^{-15} N}{-14.4 \times 10^{-15} N}\right) = 34^o.$$

The magnitude of the force can also be calculated using Equation 7.3.2. The velocity in this question, however, has three components. The **z**-component of the velocity can be neglected, because it is parallel to the magnetic field and therefore generates no force. The magnitude of the velocity is calculated from the **x**- and **y**-components. The angle between the velocity in the **xy**-plane and the magnetic field in the **z**-plane is 90 degrees. Therefore, the force is calculated to be:

$$|\vec{v}| = \sqrt{(2)^2 + (-3)^2} \times 10^4 \frac{m}{s} = 3.6 \times 10^4 \frac{m}{s}$$

$$F = qvBsin\,\theta = (3.2 \times 10^{-19} C)(3.6 \times 10^4 m/s)(1.5\, T)sin(90^o) = 1.7 \times 10^{-14} N$$

This is the same magnitude of force calculated by unit vectors.

Significance

The cross product in this formula results in a third vector that must be perpendicular to the other two. Other physical quantities, such as angular momentum, also have three vectors that are related by the cross product. Note that typical force values in magnetic force problems are much larger than the gravitational force. Therefore, for an isolated charge, the magnetic force is the dominant force governing the charge’s motion.

? Exercise 7.3.1

Repeat the previous problem with the magnetic field in the **x**-direction rather than in the **z**-direction. Check your answers with RHR-1.

Answer a

0 N

Answer b

$2.4 \times 10^{-14}\hat{k}N$

Answer c

$2.4 \times 10^{-14}\hat{j}N$

Answer d

$7.2\hat{j} + 2.2\hat{k}) \times 10^{-15} N$

Representing Magnetic Fields

The representation of magnetic fields by **magnetic field lines** is very useful in visualizing the strength and direction of the magnetic field. As shown in Figure $7.3.3$, each of these lines forms a closed loop, even if not shown by the constraints of the space available for the figure. The field lines emerge from the north pole (N), loop around to the south pole (S), and continue through the bar magnet back to the north pole.

Magnetic field lines have several hard-and-fast rules:

1. The direction of the magnetic field is tangent to the field line at any point in space. A small compass will point in the direction of the field line.
2. The strength of the field is proportional to the closeness of the lines. It is exactly proportional to the number of lines per unit area perpendicular to the lines (called the areal density).
3. Magnetic field lines can never cross, meaning that the field is unique at any point in space.
4. Magnetic field lines are continuous, forming closed loops without a beginning or end. They are directed from the north pole to the south pole.

The last property is related to the fact that the north and south poles cannot be separated. It is a distinct difference from electric field lines, which generally begin on positive charges and end on negative charges or at infinity. If isolated magnetic charges (referred to as **magnetic monopoles**) existed, then magnetic field lines would begin and end on them.

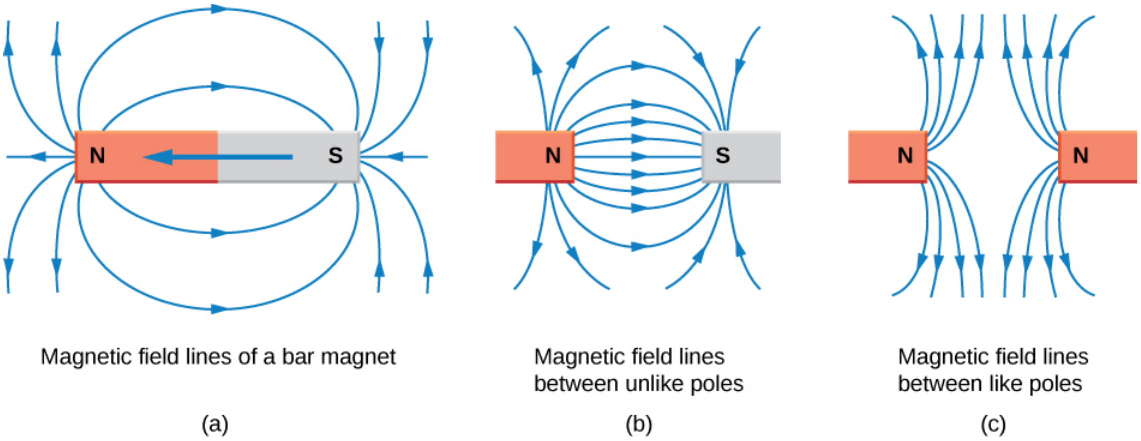


Figure $7.3.3$: Magnetic field lines are defined to have the direction in which a small compass points when placed at a location in the field. The strength of the field is proportional to the closeness (or density) of the lines. If the interior of the magnet could be probed, the field lines would be found to form continuous, closed loops. To fit in a reasonable space, some of these drawings may not show the closing of the loops; however, if enough space were provided, the loops would be closed.

7.4: Motion of a Charged Particle in a Magnetic Field

Learning Objectives

By the end of this section, you will be able to:

- Explain how a charged particle in an external magnetic field undergoes circular motion
- Describe how to determine the radius of the circular motion of a charged particle in a magnetic field

A charged particle experiences a force when moving through a magnetic field. What happens if this field is uniform over the motion of the charged particle? What path does the particle follow? In this section, we discuss the circular motion of the charged particle as well as other motion that results from a charged particle entering a magnetic field.

The simplest case occurs when a charged particle moves perpendicular to a uniform **B**-field (Figure $7.4.1$). If the field is in a vacuum, the magnetic field is the dominant factor determining the motion. Since the magnetic force is perpendicular to the direction of travel, a charged particle follows a curved path in a magnetic field. The particle continues to follow this curved path until it forms a complete circle. Another way to look at this is that the magnetic force is always perpendicular to velocity, so that it does no work on the charged particle. The particle's kinetic energy and speed thus remain constant. The direction of motion is affected but not the speed.

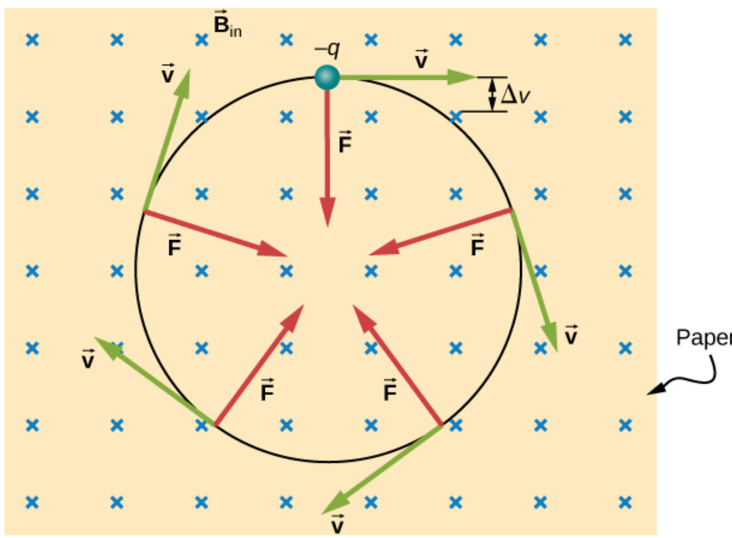


Figure $7.4.1$: A negatively charged particle moves in the plane of the paper in a region where the magnetic field is perpendicular to the paper (represented by the small X's - like the tails of arrows). The magnetic force is perpendicular to the velocity, so velocity changes in direction but not magnitude. The result is uniform circular motion. (Note that because the charge is negative, the force is opposite in direction to the prediction of the right-hand rule.)

In this situation, the magnetic force supplies the centripetal force $F_C = \frac{mv^2}{r}$. Noting that the velocity is perpendicular to the magnetic field, the magnitude of the magnetic force is reduced to $F = qvB$. Because the magnetic force **F** supplies the centripetal force F_C, we have

$$qvB = \frac{mv^2}{r}.$$

Solving for **r** yields

$$r = \frac{mv}{qB}. \tag{7.4.1}$$

Here, **r** is the radius of curvature of the path of a charged particle with mass **m** and charge **q**, moving at a speed **v** that is perpendicular to a magnetic field of strength **B**. The time for the charged particle to go around the circular path is defined as the period, which is the same as the distance traveled (the circumference) divided by the speed. Based on this and Equation, we can derive the period of motion as

$$T = \frac{2\pi r}{v} = \frac{2\pi}{v}\frac{mv}{qB} = \frac{2\pi m}{qB}. \tag{7.4.2}$$

If the velocity is not perpendicular to the magnetic field, then we can compare each component of the velocity separately with the magnetic field. The component of the velocity perpendicular to the magnetic field produces a magnetic force perpendicular to both this velocity and the field:

$$v_{perp} = v \sin\theta \tag{7.4.3}$$

$$v_{para} = v \cos\theta. \tag{7.4.4}$$

where θ is the angle between **v** and **B**. The component parallel to the magnetic field creates constant motion along the same direction as the magnetic field, also shown in Equation. The parallel motion determines the **pitch p** of the helix, which is the distance between adjacent turns. This distance equals the parallel component of the velocity times the period:

$$p = v_{para} T. \tag{7.4.5}$$

The result is a **helical motion**, as shown in the following figure.

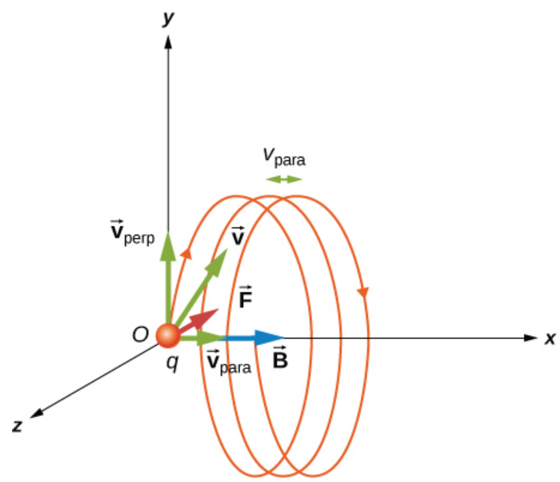


Figure 7.4.2: A charged particle moving with a velocity not in the same direction as the magnetic field. The velocity component perpendicular to the magnetic field creates circular motion, whereas the component of the velocity parallel to the field moves the particle along a straight line. The pitch is the horizontal distance between two consecutive circles. The resulting motion is helical.

While the charged particle travels in a helical path, it may enter a region where the magnetic field is not uniform. In particular, suppose a particle travels from a region of strong magnetic field to a region of weaker field, then back to a region of stronger field. The particle may reflect back before entering the stronger magnetic field region. This is similar to a wave on a string traveling from a very light, thin string to a hard wall and reflecting backward. If the reflection happens at both ends, the particle is trapped in a so-called magnetic bottle.

Trapped particles in magnetic fields are found in the Van Allen radiation belts around Earth, which are part of Earth's magnetic field. These belts were discovered by James Van Allen while trying to measure the flux of cosmic rays on Earth (high-energy particles that come from outside the solar system) to see whether this was similar to the flux measured on Earth. Van Allen found that due to the contribution of particles trapped in Earth's magnetic field, the flux was much higher on Earth than in outer space. Aurorae, like the famous aurora borealis (northern lights) in the Northern Hemisphere (Figure 7.4.3), are beautiful displays of light emitted as ions recombine with electrons entering the atmosphere as they spiral along magnetic field lines. (The ions are primarily oxygen and nitrogen atoms that are initially ionized by collisions with energetic particles in Earth's atmosphere.) Aurorae have also been observed on other planets, such as Jupiter and Saturn.

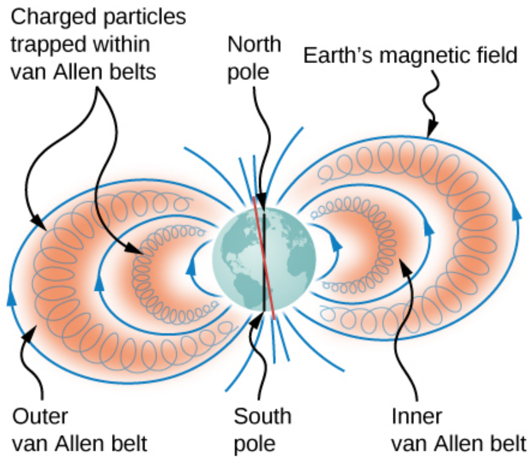


Figure 7.4.3: (a) The Van Allen radiation belts around Earth trap ions produced by cosmic rays striking Earth's atmosphere. (b) The magnificent spectacle of the aurora borealis, or northern lights, glows in the northern sky above Bear Lake near Eielson Air Force Base, Alaska. Shaped by Earth's magnetic field, this light is produced by glowing molecules and ions of oxygen and nitrogen. (credit b: modification of work by USAF Senior Airman Joshua Strang)

✓ Example 7.4.1: Beam Deflector

A research group is investigating short-lived radioactive isotopes. They need to design a way to transport alpha-particles (helium nuclei) from where they are made to a place where they will collide with another material to form an isotope. The beam of alpha-particles $(m = 6.64 \times 10^{-27} kg, \, q = 3.2 \times 10^{-19} C)$ bends through a 90-degree region with a uniform magnetic field of 0.050 T (Figure 7.4.4). (a) In what direction should the magnetic field be applied? (b) How much time does it take the alpha-particles to traverse the uniform magnetic field region?

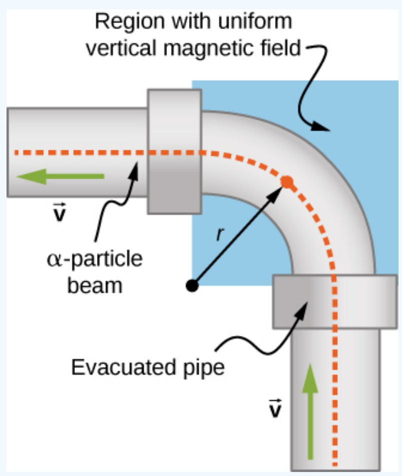


Figure 7.4.4: Top view of the beam deflector setup.

Strategy

1. The direction of the magnetic field is shown by the RHR-1. Your fingers point in the direction of **v**, and your thumb needs to point in the direction of the force, to the left. Therefore, since the alpha-particles are positively charged, the magnetic field must point down.
2. The period of the alpha-particle going around the circle is

$$T = \frac{2\pi m}{qB}.$$

Because the particle is only going around a quarter of a circle, we can take 0.25 times the period to find the time it takes to go around this path.

Solution

1. Let's start by focusing on the alpha-particle entering the field near the bottom of the picture. First, point your thumb up the page. In order for your palm to open to the left where the centripetal force (and hence the magnetic force) points, your fingers need to change orientation until they point into the page. This is the direction of the applied magnetic field.
2. The period of the charged particle going around a circle is calculated by using the given mass, charge, and magnetic field in the problem. This works out to be

$$T = \frac{2\pi m}{qB} = \frac{2\pi(6.64\times 10^{-27} kg)}{(3.2\times 10^{-19} C)(0.050\, T)} = 2.6\times 10^{-6} s.$$

   However, for the given problem, the alpha-particle goes around a quarter of the circle, so the time it takes would be

$$t = 0.25\times 2.61\times 10^{-6} s = 6.5\times 10^{-7} s.$$

Significance

This time may be quick enough to get to the material we would like to bombard, depending on how short-lived the radioactive isotope is and continues to emit alpha-particles. If we could increase the magnetic field applied in the region, this would shorten the time even more. The path the particles need to take could be shortened, but this may not be economical given the experimental setup.

? Exercise 7.4.1

A uniform magnetic field of magnitude 1.5 T is directed horizontally from west to east. (a) What is the magnetic force on a proton at the instant when it is moving vertically downward in the field with a speed of $4\times 10^7\ m/s$? (b) Compare this force with the weight **w** of a proton.

Solution

a. $9.6\times 10^{-12} N$ toward the south;

b. $\frac{w}{F_m} = 1.7\times 10^{-15}$

✓ Example 7.4.2: Helical Motion in a Magnetic Field

A proton enters a uniform magnetic field of $1.0\times 10^{-4} T$ with a speed of $5\times 10^5\ m/s$. At what angle must the magnetic field be from the velocity so that the pitch of the resulting helical motion is equal to the radius of the helix?

Strategy

The pitch of the motion relates to the parallel velocity times the period of the circular motion, whereas the radius relates to the perpendicular velocity component. After setting the radius and the pitch equal to each other, solve for the angle between the magnetic field and velocity or θ.

Solution

The pitch is given by Equation 7.4.5, the period is given by Equation 7.4.2, and the radius of circular motion is given by Equation 7.4.1. Note that the velocity in the radius equation is related to only the perpendicular velocity, which is where the circular motion occurs. Therefore, we substitute the sine component of the overall velocity into the radius equation to equate the pitch and radius

$$p = r$$

$$v_{\parallel} T = \frac{mv}{qB}$$

$$v\cos\theta \frac{2\pi m}{qB} = \frac{mv\sin\theta}{qB}$$

$$2\pi = \tan\theta$$

$$\theta = 81.0^o.$$

Significance

If this angle were 0^o, only parallel velocity would occur and the helix would not form, because there would be no circular motion in the perpendicular plane. If this angle were 90^o only circular motion would occur and there would be no movement of the circles perpendicular to the motion. That is what creates the helical motion.

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7.5: Magnetic Force on a Current-Carrying Conductor

Learning Objectives

By the end of this section, you will be able to:

- Determine the direction in which a current-carrying wire experiences a force in an external magnetic field
- Calculate the force on a current-carrying wire in an external magnetic field

Moving charges experience a force in a magnetic field. If these moving charges are in a wire—that is, if the wire is carrying a current—the wire should also experience a force. However, before we discuss the force exerted on a current by a magnetic field, we first examine the magnetic field generated by an electric current. We are studying two separate effects here that interact closely: A current-carrying wire generates a magnetic field and the magnetic field exerts a force on the current-carrying wire.

Magnetic Fields Produced by Electrical Currents

When discussing historical discoveries in magnetism, we mentioned Oersted's finding that a wire carrying an electrical current caused a nearby compass to deflect. A connection was established that electrical currents produce magnetic fields. (This connection between electricity and magnetism is discussed in more detail in Sources of Magnetic Fields.)

The compass needle near the wire experiences a force that aligns the needle tangent to a circle around the wire. Therefore, a current-carrying wire produces circular loops of magnetic field. To determine the direction of the magnetic field generated from a wire, we use a second right-hand rule. In RHR-2, your thumb points in the direction of the current while your fingers wrap around the wire, pointing in the direction of the magnetic field produced (Figure $7.5.1$). If the magnetic field were coming at you or out of the page, we represent this with a dot. If the magnetic field were going into the page, we represent this with an

These symbols come from considering a vector arrow: An arrow pointed toward you, from your perspective, would look like a dot or the tip of an arrow. An arrow pointed away from you, from your perspective, would look like a cross or an A composite sketch of the magnetic circles is shown in Figure $7.5.1$, where the field strength is shown to decrease as you get farther from the wire by loops that are farther separated.

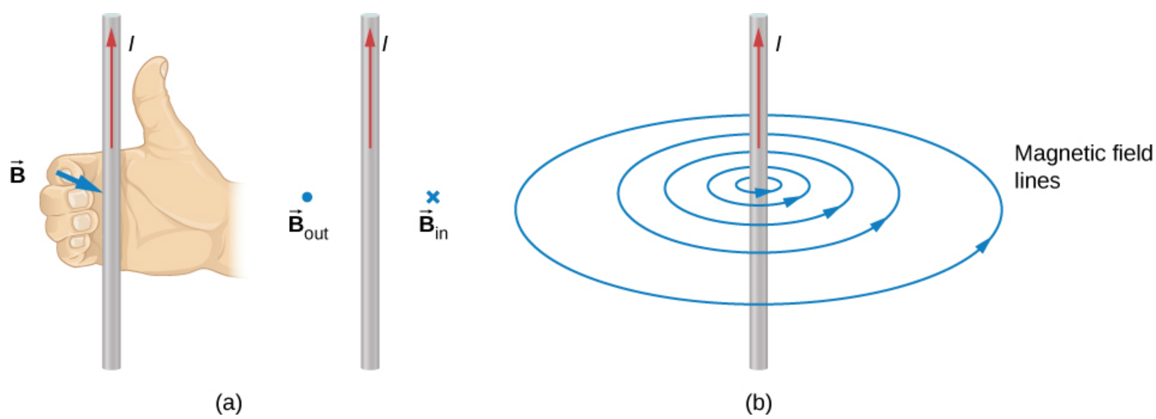


Figure $7.5.1$: (a) When the wire is in the plane of the paper, the field is perpendicular to the paper. Note the symbols used for the field pointing inward (like the tail of an arrow) and the field pointing outward (like the tip of an arrow). (b) A long and straight wire creates a field with magnetic field lines forming circular loops.

Calculating the Magnetic Force

Electric current is an ordered movement of charge. A current-carrying wire in a magnetic field must therefore experience a force due to the field. To investigate this force, let's consider the infinitesimal section of wire as shown in Figure $7.5.3$. The length and cross-sectional area of the section are **dl** and **A**, respectively, so its volume is $V = A \cdot dl$. The wire is formed from material that contains **n** charge carriers per unit volume, so the number of charge carriers in the section is $nA \cdot dl$. If the charge carriers move with drift velocity $\vec{v}_d$ the current **I** in the wire is (from Current and Resistance)

$$I = neAv_d.$$

The magnetic force on any single charge carrier is $e\vec{v}_d \times \vec{B}$, so the total magnetic force $d\vec{F}$ on the $nA \cdot dl$ charge carriers in the section of wire is

$$d\vec{F} = (nA \cdot dl)e\vec{v}_d \times \vec{B}.$$

We can define **dl** to be a vector of length **dl** pointing along $\vec{v}_d$, which allows us to rewrite this equation as

$$d\vec{F} = neAv_d d\vec{l} \times \vec{B},$$

or

$$d\vec{F} = Id\vec{l} \times \vec{B}. \tag{7.5.1}$$

This is the magnetic force on the section of wire. Note that it is actually the net force exerted by the field on the charge carriers themselves. The direction of this force is given by RHR-1, where you point your fingers in the direction of the current and curl them toward the field. Your thumb then points in the direction of the force.

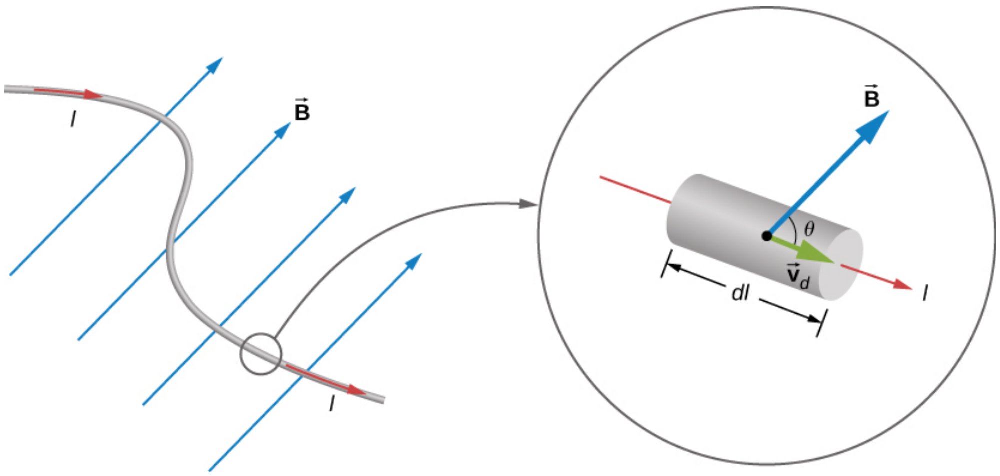


Figure 7.5.2: An infinitesimal section of current-carrying wire in a magnetic field.

To determine the magnetic force $\vec{F}$ on a wire of arbitrary length and shape, we must integrate Equation 7.5.1 over the entire wire. If the wire section happens to be straight and **B** is uniform, the equation differentials become absolute quantities, giving us

$$\vec{F} = I\vec{l} \times \vec{B}.$$

This is the force on a straight, current-carrying wire in a uniform magnetic field.

Example 7.5.1: Balancing the Gravitational and Magnetic Forces on a Current-Carrying Wire

A wire of length 50 cm and mass 10 g is suspended in a horizontal plane by a pair of flexible leads (Figure 7.5.3). The wire is then subjected to a constant magnetic field of magnitude 0.50 T, which is directed as shown. What are the magnitude and direction of the current in the wire needed to remove the tension in the supporting leads?

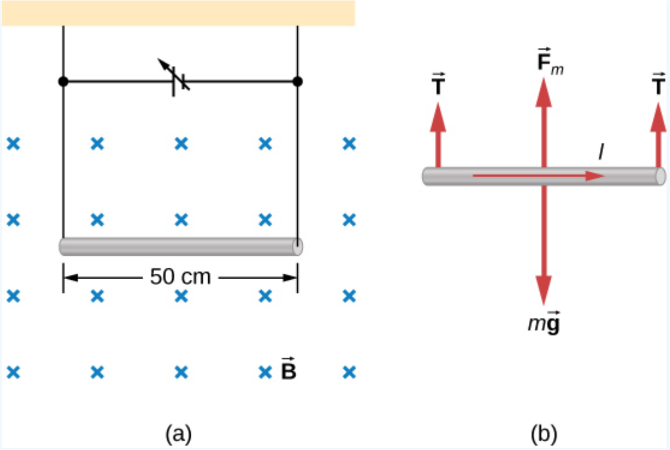


Figure 7.5.3: (a) A wire suspended in a magnetic field. (b) The free-body diagram for the wire.

Strategy

From the free-body diagram in the figure, the tensions in the supporting leads go to zero when the gravitational and magnetic forces balance each other. Using the RHR-1, we find that the magnetic force points up. We can then determine the current **I** by equating the two forces.

Solution

Equate the two forces of weight and magnetic force on the wire:

$$mg = IlB.$$

Thus,

$$I = \frac{mg}{lB} = \frac{(0.010\,kg}{9.8\,m/s^2)}(0.50\,m)(0.50\,T) = 0.39\,A.$$

Significance

This large magnetic field creates a significant force on a length of wire to counteract the weight of the wire.

✓ Example 7.5.2: Calculating Magnetic Force on a Current-Carrying Wire

A long, rigid wire lying along the **y**-axis carries a 5.0-A current flowing in the positive **y**-direction. (a) If a constant magnetic field of magnitude 0.30 T is directed along the positive **x**-axis, what is the magnetic force per unit length on the wire? (b) If a constant magnetic field of 0.30 T is directed 30 degrees from the +**x**-axis towards the +**y**-axis, what is the magnetic force per unit length on the wire?

Strategy

The magnetic force on a current-carrying wire in a magnetic field is given by $\vec{F} = I\vec{l} \times \vec{B}$. For part a, since the current and magnetic field are perpendicular in this problem, we can simplify the formula to give us the magnitude and find the direction through the RHR-1. The angle **θ** is 90 degrees, which means $sin\,\theta = 1$. Also, the length can be divided over to the left-hand side to find the force per unit length. For part b, the current times length is written in unit vector notation, as well as the magnetic field. After the cross product is taken, the directionality is evident by the resulting unit vector.

Solution

1. We start with the general formula for the magnetic force on a wire. We are looking for the force per unit length, so we divide by the length to bring it to the left-hand side. We also set $sin\,\theta$. The solution therefore is

$$F = IlB\,sin\,\theta$$

$$\frac{F}{l} = (5.0\,A)(0.30\,T)$$

$$\frac{F}{l} = 1.5\, N/m.$$

Directionality: Point your fingers in the positive **y**-direction and curl your fingers in the positive **x**-direction. Your thumb will point in the $-\vec{k}$ direction. Therefore, with directionality, the solution is

$$\frac{\vec{F}}{l} = -1.5\vec{k}\, N/m.$$

2. The current times length and the magnetic field are written in unit vector notation. Then, we take the cross product to find the force:

$$\vec{F} = I\vec{l} \times \vec{B} = (5.0A)l\hat{j} \times (0.30T\, cos(30^o)\hat{i}$$

$$\vec{F}/l = -1.30\hat{k}\, N/m.$$

Significance

This large magnetic field creates a significant force on a small length of wire. As the angle of the magnetic field becomes more closely aligned to the current in the wire, there is less of a force on it, as seen from comparing parts a and b.

Exercise 7.5.1

A straight, flexible length of copper wire is immersed in a magnetic field that is directed into the page. (a) If the wire's current runs in the +**x**-direction, which way will the wire bend? (b) Which way will the wire bend if the current runs in the –**x**-direction?

Solution

a. bends upward; b. bends downward

Example 7.5.3: Force on a Circular Wire

A circular current loop of radius **R** carrying a current **I** is placed in the **xy**-plane. A constant uniform magnetic field cuts through the loop parallel to the **y**-axis (Figure 7.5.4). Find the magnetic force on the upper half of the loop, the lower half of the loop, and the total force on the loop.

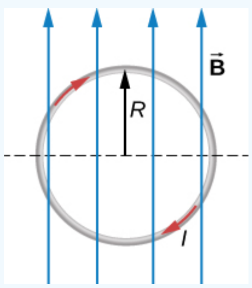


Figure 7.5.4: A loop of wire carrying a current in a magnetic field.

Strategy

The magnetic force on the upper loop should be written in terms of the differential force acting on each segment of the loop. If we integrate over each differential piece, we solve for the overall force on that section of the loop. The force on the lower loop is found in a similar manner, and the total force is the addition of these two forces.

Solution

A differential force on an arbitrary piece of wire located on the upper ring is:

$$dF = IB\, sin\, \theta\, dl,$$

where θ is the angle between the magnetic field direction (+**y**) and the segment of wire. A differential segment is located at the same radius, so using an arc-length formula, we have:

$$dl = Rd\theta$$

$$dF = IBR\,sin\,\theta\,d\theta.$$

In order to find the force on a segment, we integrate over the upper half of the circle, from 0 to π. This results in:

$$F = IBR\int_0^\pi sin\,\theta\,d\theta = IBR(-cos\pi + cos0) = 2IBR.$$

The lower half of the loop is integrated from π to zero, giving us:

$$F = IBR\int_\pi^0 sin\,\theta\,d\theta = IBR(-cos0 + cos\pi) = -2IBR.$$

The net force is the sum of these forces, which is zero.

Significance

The total force on any closed loop in a uniform magnetic field is zero. Even though each piece of the loop has a force acting on it, the net force on the system is zero. (Note that there is a net torque on the loop, which we consider in the next section.)

7.5.1: Force and Torque on a Current Loop

Learning Objectives

By the end of this section, you will be able to:

- Evaluate the net force on a current loop in an external magnetic field
- Evaluate the net torque on a current loop in an external magnetic field
- Define the magnetic dipole moment of a current loop

Motors are the most common application of magnetic force on current-carrying wires. Motors contain loops of wire in a magnetic field. When current is passed through the loops, the magnetic field exerts torque on the loops, which rotates a shaft. Electrical energy is converted into mechanical work in the process. Once the loop's surface area is aligned with the magnetic field, the direction of current is reversed, so there is a continual torque on the loop (Figure 7.5.1.1). This reversal of the current is done with commutators and brushes. The commutator is set to reverse the current flow at set points to keep continual motion in the motor. A basic commutator has three contact areas to avoid and dead spots where the loop would have zero instantaneous torque at that point. The brushes press against the commutator, creating electrical contact between parts of the commutator during the spinning motion.

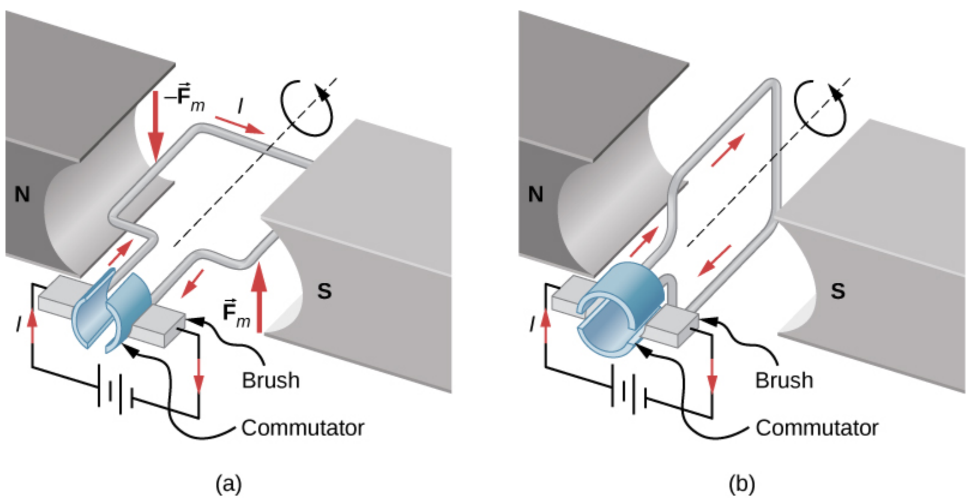


Figure 7.5.1.1: A simplified version of a dc electric motor. (a) The rectangular wire loop is placed in a magnetic field. The forces on the wires closest to the magnetic poles (N and S) are opposite in direction as determined by the right-hand rule-1. Therefore, the loop has a net torque and rotates to the position shown in (b). (b) The brushes now touch the commutator segments so that no current flows through the loop. No torque acts on the loop, but the loop continues to spin from the initial velocity given to it in part (a). By the time the loop flips over, current flows through the wires again but now in the opposite direction, and the process repeats as in part (a). This causes continual rotation of the loop.

In a uniform magnetic field, a current-carrying loop of wire, such as a loop in a motor, experiences both forces and torques on the loop. Figure 7.5.1.1 shows a rectangular loop of wire that carries a current **I** and has sides of lengths **a** and **b**. The loop is in a uniform magnetic field: $\vec{B} = B\hat{j}$. The magnetic force on a straight current-carrying wire of length **l** is given by $I\vec{l} \times \vec{B}$. To find the net force on the loop, we have to apply this equation to each of the four sides. The force on side 1 is

$$\vec{F}_1 = IaB\sin(90^o - \theta)\hat{i} = IaB\cos\theta\hat{i}$$

where the direction has been determined with the RHR-1. The current in side 3 flows in the opposite direction to that of side 1, so

$$\vec{F}_3 = -IaB\sin(90^o + \theta)\hat{i} = -IaB\cos\theta\hat{i}$$

The currents in sides 2 and 4 are perpendicular to $\vec{B}$ and the forces on these sides are

$$\vec{F}_2 = IbB\hat{k}$$

$$\vec{F}_4 = -IbB\hat{k}.$$

We can now find the net force on the loop:

$$\sum \vec{F}_{net} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4 = 0.$$

Although this result $(\sum F = 0)$ has been obtained for a rectangular loop, it is far more general and holds for current-carrying loops of arbitrary shapes; that is, there is no net force on a current loop in a uniform magnetic field.

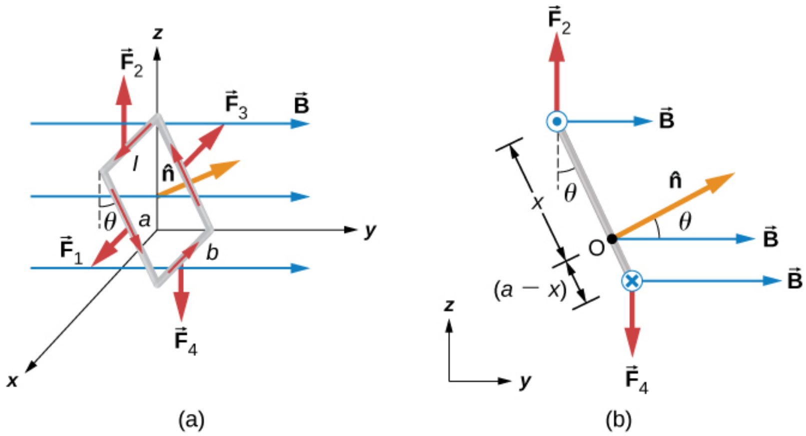


Figure $7.5.1.2$: (a) A rectangular current loop in a uniform magnetic field is subjected to a net torque but not a net force. (b) A side view of the coil.

To find the net torque on the current loop shown in Figure $7.5.1.2a$, we first consider F_1 and F_3. Since they have the same line of action and are equal and opposite, the sum of their torques about any axis is zero (see Fixed-Axis Rotation). Thus, if there is any torque on the loop, it must be furnished by F_2 and F_4. Let's calculate the torques around the axis that passes through point **O** of Figure $7.5.1.2b$(a side view of the coil) and is perpendicular to the plane of the page. The point **O** is a distance **x** from side 2 and a distance $(a-x)$ from side 4 of the loop. The moment arms of F_2 and F_4 are $x\,sin\,\theta$ and $(a-x)\,sin\,\theta$, respectively, so the net torque on the loop is

$$\sum \vec{\tau} = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 + \vec{\tau}_4 = F_2 x\,sin\,\theta \hat{i} - F_4(a-x)\,sin\,(\theta)\hat{i}$$

$$-IbBx\,sin\,\theta \hat{i} - IbB(a-x) sin\,\theta \hat{i}.$$

This simplifies to

$$\vec{\tau} = -IAB\,sin\,\theta \hat{i}$$

where $A = ab$ is the area of the loop.

Notice that this torque is independent of **x**; it is therefore independent of where point **O** is located in the plane of the current loop. Consequently, the loop experiences the same torque from the magnetic field about any axis in the plane of the loop and parallel to the **x**-axis.

A closed-current loop is commonly referred to as a **magnetic dipole** and the term **IA** is known as its **magnetic dipole moment** $\boldsymbol{\mu}$. Actually, the magnetic dipole moment is a vector that is defined as

$$\vec{\mu} = IA\hat{n}$$

where $\hat{n}$ is a unit vector directed perpendicular to the plane of the loop (see Figure $7.5.1.2$). The direction of $\hat{n}$ is obtained with the RHR-2—if you curl the fingers of your right hand in the direction of current flow in the loop, then your thumb points along $\hat{n}$. If the loop contains **N** turns of wire, then its magnetic dipole moment is given by

$$\vec{\mu} = NIA\hat{n}.$$

In terms of the magnetic dipole moment, the torque on a current loop due to a uniform magnetic field can be written simply as

$$\vec{\tau} = \vec{\mu} \times \vec{B}.$$

This equation holds for a current loop in a two-dimensional plane of arbitrary shape.

Using a calculation analogous to that found in Capacitance for an electric dipole, the potential energy of a magnetic dipole is

$$U = -\vec{\mu} \cdot \vec{B}.$$

Example 7.5.1.1: Forces and Torques on Current-Carrying Loops

A circular current loop of radius 2.0 cm carries a current of 2.0 mA. (a) What is the magnitude of its magnetic dipole moment? (b) If the dipole is oriented at 30 degrees to a uniform magnetic field of magnitude 0.50 T, what is the magnitude of the torque it experiences and what is its potential energy?

Strategy

The dipole moment is defined by the current times the area of the loop. The area of the loop can be calculated from the area of the circle. The torque on the loop and potential energy are calculated from identifying the magnetic moment, magnetic field, and angle oriented in the field.

Solution

1. The magnetic moment **μ** is calculated by the current times the area of the loop or πr^2.

$$\mu = IA = (2.0 \times 10^{-3} A)(\pi (0.02\, m)^2) = 2.5 \times 10^{-6} A \cdot m^2$$

2. The torque and potential energy are calculated by identifying the magnetic moment, magnetic field, and the angle between these two vectors. The calculations of these quantities are:

$$\tau = \vec{\mu} \times \vec{B} = \mu B \sin\theta = (2.5 \times 10^{-6} A \cdot m^2)(0.50 T) sin(30^o) = 6.3 \times 10^{-7} N \cdot m$$

$$U = -\vec{\mu} \cdot \vec{B} = -\mu B cos\theta = -(2.5 \times 10^{-6} A \cdot m^2)(0.50 T) cos(30^o) = -1.1 \times 10^{-6} J.$$

Significance

The concept of magnetic moment at the atomic level is discussed in the next chapter. The concept of aligning the magnetic moment with the magnetic field is the functionality of devices like magnetic motors, whereby switching the external magnetic field results in a constant spinning of the loop as it tries to align with the field to minimize its potential energy.

? Exercise 7.5.1.1

In what orientation would a magnetic dipole have to be to produce (a) a maximum torque in a magnetic field? (b) A maximum energy of the dipole?

Solution

a. aligned or anti-aligned; b. perpendicular

7.5.2: The Hall Effect

Learning Objectives

By the end of this section, you will be able to:

- Explain a scenario where the magnetic and electric fields are crossed and their forces balance each other as a charged particle moves through a velocity selector
- Compare how charge carriers move in a conductive material and explain how this relates to the Hall effect

In 1879, E.H. Hall devised an experiment that can be used to identify the sign of the predominant charge carriers in a conducting material. From a historical perspective, this experiment was the first to demonstrate that the charge carriers in most metals are negative.

Visit this website to find more information about the Hall effect.

We investigate the **Hall effect** by studying the motion of the free electrons along a metallic strip of width **l** in a constant magnetic field (Figure $7.5.2.1$). The electrons are moving from left to right, so the magnetic force they experience pushes them to the bottom edge of the strip. This leaves an excess of positive charge at the top edge of the strip, resulting in an electric field **E** directed from top to bottom. The charge concentration at both edges builds up until the electric force on the electrons in one direction is balanced by the magnetic force on them in the opposite direction. Equilibrium is reached when:

$$eE = ev_dB \tag{7.5.2.1}$$

where **e** is the magnitude of the electron charge, v_d is the drift speed of the electrons, and **E** is the magnitude of the electric field created by the separated charge. Solving this for the drift speed results in

$$v_d = \frac{E}{B}. \tag{7.5.2.2}$$

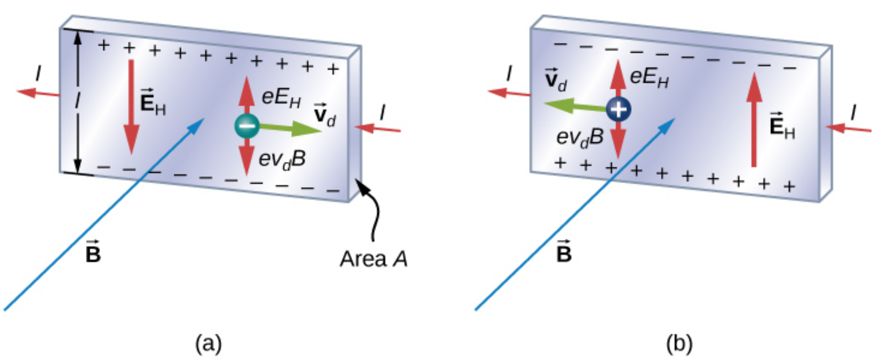


Figure $7.5.2.1$: In the Hall effect, a potential difference between the top and bottom edges of the metal strip is produced when moving charge carriers are deflected by the magnetic field. (a) Hall effect for negative charge carriers; (b) Hall effect for positive charge carriers.

A scenario where the electric and magnetic fields are perpendicular to one another is called a crossed-field situation. If these fields produce equal and opposite forces on a charged particle with the velocity that equates the forces, these particles are able to pass through an apparatus, called a **velocity selector**, undeflected. This velocity is represented in Equation $7.5.2.3$. Any other velocity of a charged particle sent into the same fields would be deflected by the magnetic force or electric force.

Going back to the Hall effect, if the current in the strip is **I**, then from Current and Resistance, we know that

$$I = nev_dA \tag{7.5.2.3}$$

where **n** is the number of charge carriers per volume and **A** is the cross-sectional area of the strip. Combining the equations for v_d and **I** results in

$$I = ne\left(\frac{E}{B}\right)A. \tag{7.5.2.4}$$

The field **E** is related to the potential difference **V** between the edges of the strip by

$$E = \frac{V}{l}. \tag{7.5.2.5}$$

The quantity V is called the **Hall potential** and can be measured with a voltmeter. Finally, combining the equations for **I** and **E** gives us

$$V = \frac{IBl}{neA} \tag{7.5.2.6}$$

where the upper edge of the strip in Figure $7.5.2.1$is positive with respect to the lower edge.

We can also combine Equation 7.5.2.1 and Equation 7.5.2.5 to get an expression for the Hall voltage in terms of the magnetic field:

$$V = Blv_d.$$

What if the charge carriers are positive, as in Figure $7.5.2.1$? For the same current **I**, the magnitude of **V** is still given by Equation 7.5.2.6. However, the upper edge is now negative with respect to the lower edge. Therefore, by simply measuring the sign of **V**, we can determine the sign of the majority charge carriers in a metal.

Hall potential measurements show that electrons are the dominant charge carriers in most metals. However, Hall potentials indicate that for a few metals, such as tungsten, beryllium, and many semiconductors, the majority of charge carriers are positive. It turns out that conduction by positive charge is caused by the migration of missing electron sites (called holes) on ions. Conduction by holes is studied later in Condensed Matter Physics.

The Hall effect can be used to measure magnetic fields. If a material with a known density of charge carriers **n** is placed in a magnetic field and **V** is measured, then the field can be determined from Equation ???. In research laboratories where the fields of electromagnets used for precise measurements have to be extremely steady, a “Hall probe” is commonly used as part of an electronic circuit that regulates the field.

Example 7.5.2.1: Velocity Selector

An electron beam enters a crossed-field velocity selector with magnetic and electric fields of 2.0 mT and $6.0 \times 10^3\ N/C$, respectively. (a) What must the velocity of the electron beam be to traverse the crossed fields undeflected? If the electric field is turned off, (b) what is the acceleration of the electron beam and (c) what is the radius of the circular motion that results?

Strategy

The electron beam is not deflected by either of the magnetic or electric fields if these forces are balanced. Based on these balanced forces, we calculate the velocity of the beam. Without the electric field, only the magnetic force is used in Newton’s second law to find the acceleration. Lastly, the radius of the path is based on the resulting circular motion from the magnetic force.

Solution

1. The velocity of the unperturbed beam of electrons with crossed fields is calculated by Equation 7.5.2.2:

$$v_d = \frac{E}{B} = \frac{6 \times 10^3 N/C}{2 \times 10^{-3} T} = 3 \times 10^6 m/s.$$

2. The acceleration is calculated from the net force from the magnetic field, equal to mass times acceleration. The magnitude of the acceleration is:

$$ma = qvB$$

$$a = \frac{qvB}{m} = \frac{(1.6 \times 10^{-19} C)(3 \times 10^6 m/s)(2 \times 10^{-3} T)}{0.1 \times 10^{-31} kg} = 1.1 \times 10^{15} m/s^2.$$

3. The radius of the path comes from a balance of the circular and magnetic forces, or Equation 7.5.2.2:

$$r = \frac{mv}{qB} = \frac{(9.1 \times 10^{-31} kg)(3 \times 10^{6} m/s)}{(1.6 \times 10^{-19} C)(2 \times 10^{-3} T)} = 8.5 \times 10^{-3} m.$$

Significance

If electrons in the beam had velocities above or below the answer in part (a), those electrons would have a stronger net force exerted by either the magnetic or electric field. Therefore, only those electrons at this specific velocity would make it through.

The Hall Potential in a Silver Ribbon

Figure $7.5.2.2$shows a silver ribbon whose cross section is 1.0 cm by 0.20 cm. The ribbon carries a current of 100 A from left to right, and it lies in a uniform magnetic field of magnitude 1.5 T. Using a density value of $n = 5.9 \times 10^{28}$ electrons per cubic meter for silver, find the Hall potential between the edges of the ribbon.

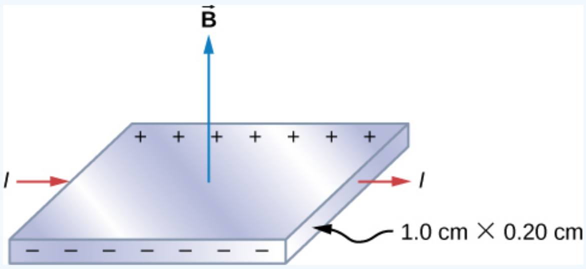


Figure $7.5.2.2$: Finding the Hall potential in a silver ribbon in a magnetic field is shown.

Strategy

Since the majority of charge carriers are electrons, the polarity of the Hall voltage is that indicated in the figure. The value of the Hall voltage is calculated using Equation $7.5.2.6$.

Solution

When calculating the Hall voltage, we need to know the current through the material, the magnetic field, the length, the number of charge carriers, and the area. Since all of these are given, the Hall voltage is calculated as:

$$\begin{aligned} v &= \frac{IBl}{neA} \\ &= \frac{(100\,A)(1.5\,T)(1.0 \times 10^{-2} m)}{(5.9 \times 10^{28}/m^3)(1.6 \times 10^{-19} C)(2.0 \times 10^{-5} m^2)} \\ &= 7.9 \times 10^{-6} V. \end{aligned}$$

Significance

As in this example, the Hall potential is generally very small, and careful experimentation with sensitive equipment is required for its measurement.

Exercise $7.5.2.1$

A Hall probe consists of a copper strip, $n = 8.5 \times 10^{28}$ electrons per cubic meter, which is 2.0 cm wide and 0.10 cm thick. What is the magnetic field when **I** = 50 A and the Hall potential is

a. $4.0\,\mu V$ and
b. $6.0\,\mu V$?

Answer a

1.1 T

Answer b

1.6 T

7.6: Magnetic Forces and Fields (Exercise)

Conceptual Questions

11.3 Magnetic Fields and Lines

1. Discuss the similarities and differences between the electrical force on a charge and the magnetic force on a charge.

2. (a) Is it possible for the magnetic force on a charge moving in a magnetic field to be zero?

(b) Is it possible for the electric force on a charge moving in an electric field to be zero?

(c) Is it possible for the resultant of the electric and magnetic forces on a charge moving simultaneously through both fields to be zero?

11.4 Motion of a Charged Particle in a Magnetic Field

3. At a given instant, an electron and a proton are moving with the same velocity in a constant magnetic field. Compare the magnetic forces on these particles. Compare their accelerations.

4. Does increasing the magnitude of a uniform magnetic field through which a charge is traveling necessarily mean increasing the magnetic force on the charge? Does changing the direction of the field necessarily mean a change in the force on the charge?

5. An electron passes through a magnetic field without being deflected. What do you conclude about the magnetic field?

6. If a charged particle moves in a straight line, can you conclude that there is no magnetic field present?

7. How could you determine which pole of an electromagnet is north and which pole is south?

11.5 Magnetic Force on a Current-Carrying Conductor

8. Describe the error that results from accidently using your left rather than your right hand when determining the direction of a magnetic force.

9. Considering the magnetic force law, are the velocity and magnetic field always perpendicular? Are the force and velocity always perpendicular? What about the force and magnetic field?

10. Why can a nearby magnet distort a cathode ray tube television picture?

11. A magnetic field exerts a force on the moving electrons in a current carrying wire. What exerts the force on a wire?

12. There are regions where the magnetic field of earth is almost perpendicular to the surface of Earth. What difficulty does this cause in the use of a compass?

11.7 The Hall Effect

13. Hall potentials are much larger for poor conductors than for good conductors. Why?

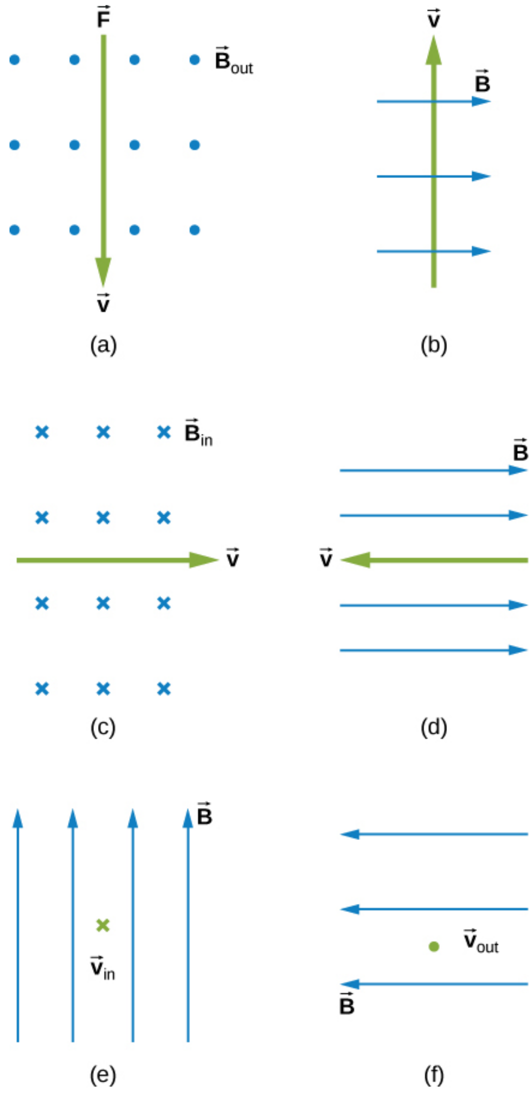
11.8 Applications of Magnetic Forces and Fields

14. Describe the primary function of the electric field and the magnetic field in a cyclotron.

Problems

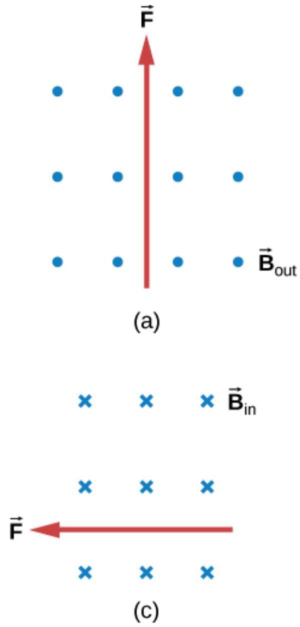
11.3 Magnetic Fields and Lines

15. What is the direction of the magnetic force on a positive charge that moves as shown in each of the six cases?



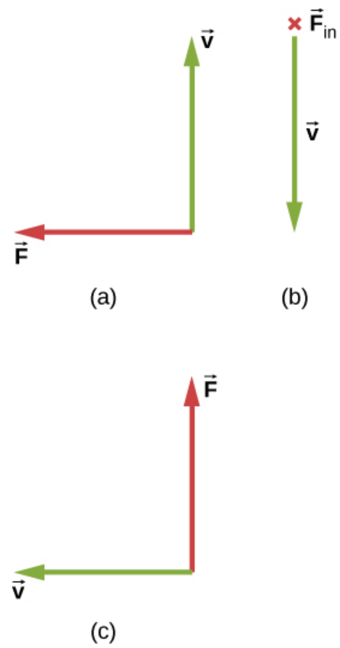
16. Repeat previous exercise for a negative charge.

17. What is the direction of the velocity of a negative charge that experiences the magnetic force shown in each of the three cases, assuming it moves perpendicular to B?



18. Repeat previous exercise for a positive charge.

19. What is the direction of the magnetic field that produces the magnetic force on a positive charge as shown in each of the three cases, assuming $\vec{B}$ is perpendicular to $\vec{v}$?



20. Repeat previous exercise for a negative charge.

21. (a) Aircraft sometimes acquire small static charges. Suppose a supersonic jet has a 0.500-μC charge and flies due west at a speed of 660. m/s over Earth's south magnetic pole, where the *[Math Processing Error]* magnetic field points straight up. What are the direction and the magnitude of the magnetic force on the plane?

(b) Discuss whether the value obtained in part (a) implies this is a significant or negligible effect.

22. (a) A cosmic ray proton moving toward Earth at *[Math Processing Error]* experiences a magnetic force of *[Math Processing Error]*. What is the strength of the magnetic field if there is a 45° angle between it and the proton's velocity?

(b) Is the value obtained in part a. consistent with the known strength of Earth's magnetic field on its surface? Discuss.

23. An electron moving at *[Math Processing Error]* in a 1.25-T magnetic field experiences a magnetic force of *[Math Processing Error]*. What angle does the velocity of the electron make with the magnetic field? There are two answers.

24. (a) A physicist performing a sensitive measurement wants to limit the magnetic force on a moving charge in her equipment to less than *[Math Processing Error]*. What is the greatest the charge can be if it moves at a maximum speed of 30.0 m/s in Earth's field?

(b) Discuss whether it would be difficult to limit the charge to less than the value found in (a) by comparing it with typical static electricity and noting that static is often absent.

11.4 Motion of a Charged Particle in a Magnetic Field

25. A cosmic-ray electron moves at *[Math Processing Error]* perpendicular to Earth's magnetic field at an altitude where the field strength is *[Math Processing Error]*. What is the radius of the circular path the electron follows?

26. (a) Viewers of Star Trek have heard of an antimatter drive on the Starship Enterprise. One possibility for such a futuristic energy source is to store antimatter charged particles in a vacuum chamber, circulating in a magnetic field, and then extract them as needed. Antimatter annihilates normal matter, producing pure energy. What strength magnetic field is needed to hold antiprotons, moving at *[Math Processing Error]* in a circular path 2.00 m in radius? Antiprotons have the same mass as protons but the opposite (negative) charge.

(b) Is this field strength obtainable with today's technology or is it a futuristic possibility?

27. (a) An oxygen-16 ion with a mass of *[Math Processing Error]* travels at *[Math Processing Error]* perpendicular to a 1.20-T magnetic field, which makes it move in a circular arc with a 0.231-m radius. What positive charge is on the ion?

(b) What is the ratio of this charge to the charge of an electron? (c) Discuss why the ratio found in (b) should be an integer.

28. An electron in a TV CRT moves with a speed of *[Math Processing Error]*, in a direction perpendicular to Earth's field, which has a strength of *[Math Processing Error]*. (a) What strength electric field must be applied perpendicular to the Earth's field to make the electron moves in a straight line? (b) If this is done between plates separated by 1.00 cm, what is the voltage applied? (Note that TVs are usually surrounded by a ferromagnetic material to shield against external magnetic fields and avoid the need for such a correction.)

29. (a) At what speed will a proton move in a circular path of the same radius as the electron in the previous exercise?

(b) What would the radius of the path be if the proton had the same speed as the electron?

(c) What would the radius be if the proton had the same kinetic energy as the electron?

(d) The same momentum?

30. (a) What voltage will accelerate electrons to a speed of *[Math Processing Error]*? (b) Find the radius of curvature of the path of a proton accelerated through this potential in a 0.500-T field and compare this with the radius of curvature of an electron accelerated through the same potential.

31. An alpha-particle (*[Math Processing Error]*) travels in a circular path of radius 25 cm in a uniform magnetic field of magnitude 1.5 T.

(a) What is the speed of the particle?

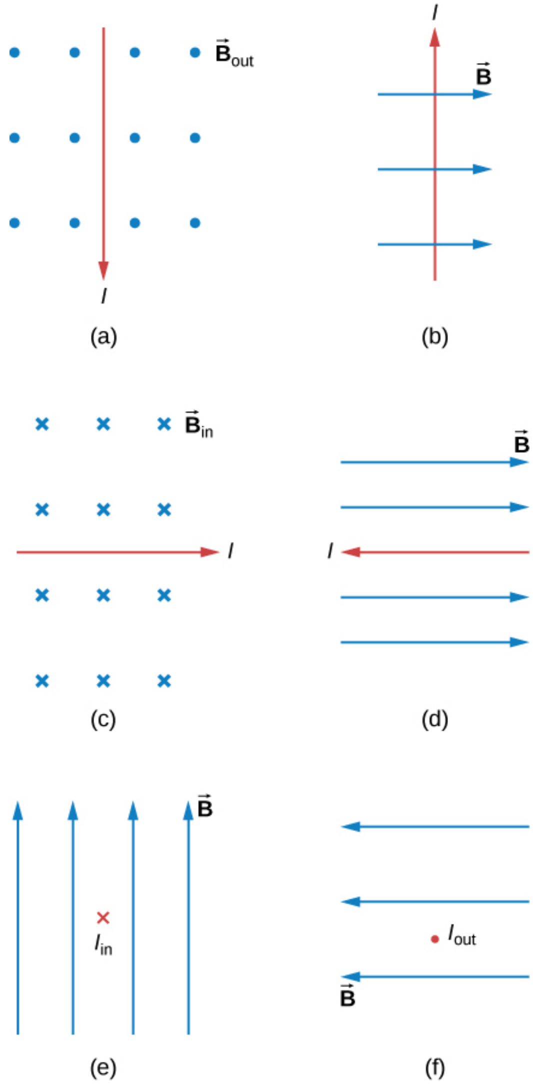
(b) What is the kinetic energy in electron-volts?

(c) Through what potential difference must the particle be accelerated in order to give it this kinetic energy?

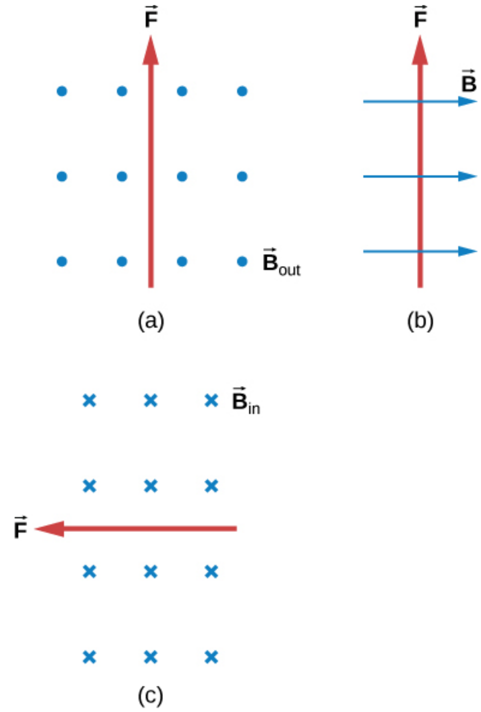
32. A particle of charge **q** and mass **m** is accelerated from rest through a potential difference **V**, after which it encounters a uniform magnetic field **B**. If the particle moves in a plane perpendicular to **B**, what is the radius of its circular orbit?

11.5 Magnetic Force on a Current-Carrying Conductor

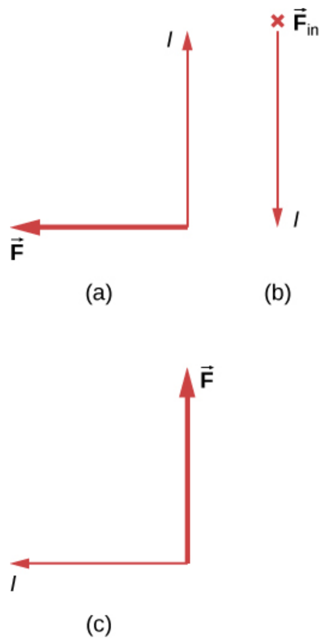
33. What is the direction of the magnetic force on the current in each of the six cases?



34. What is the direction of a current that experiences the magnetic force shown in each of the three cases, assuming the current runs perpendicular to $\vec{B}$?



35. What is the direction of the magnetic field that produces the magnetic force shown on the currents in each of the three cases, assuming $\vec{B}$ is perpendicular to **I**?



36. (a) What is the force per meter on a lightning bolt at the equator that carries 20,000 A perpendicular to Earth's *[Math Processing Error]* field? (b) What is the direction of the force if the current is straight up and Earth's field direction is due north, parallel to the ground?

37. (a) A dc power line for a light-rail system carries 1000 A at an angle of 30.0º to Earth's *[Math Processing Error]* field. What is the force on a 100-m section of this line?

(b) Discuss practical concerns this presents, if any.

38. A wire carrying a 30.0-A current passes between the poles of a strong magnet that is perpendicular to its field and experiences a 2.16-N force on the 4.00 cm of wire in the field. What is the average field strength?

11.6 Force and Torque on a Current Loop

39. (a) By how many percent is the torque of a motor decreased if its permanent magnets lose 5.0% of their strength?

(b) How many percent would the current need to be increased to return the torque to original values?

40. (a) What is the maximum torque on a 150-turn square loop of wire 18.0 cm on a side that carries a 50.0-A current in a 1.60-T field?

(b) What is the torque when $\boldsymbol{\theta}$ is 10.9º?

41. Find the current through a loop needed to create a maximum torque of $\mathbf{9.0 N \cdot m}$. The loop has 50 square turns that are 15.0 cm on a side and is in a uniform 0.800-T magnetic field.

42. Calculate the magnetic field strength needed on a 200-turn square loop 20.0 cm on a side to create a maximum torque of 300 N · m if the loop is carrying 25.0 A.

43. Since the equation for torque on a current-carrying loop is $\boldsymbol{\tau} = \mathbf{NIAB} \sin \boldsymbol{\theta}$, the units of N · m must equal units of *[Math Processing Error]*. Verify this.

44. (a) At what angle θ is the torque on a current loop 90.0% of maximum?

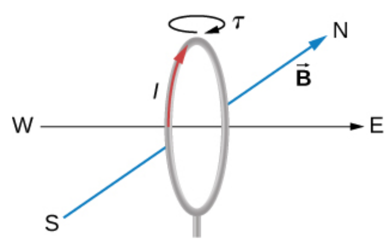
(b) 50.0% of maximum?

(c) 10.0% of maximum?

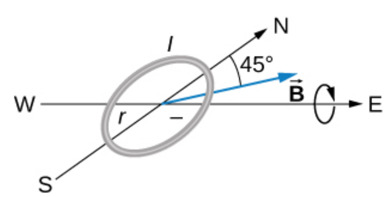
45. A proton has a magnetic field due to its spin. The field is similar to that created by a circular current loop *[Math Processing Error]* in radius with a current of *[Math Processing Error]*. Find the maximum torque on a proton in a 2.50-T field. (This is a significant torque on a small particle.)

46. (a) A 200-turn circular loop of radius 50.0 cm is vertical, with its axis on an east-west line. A current of 100 A circulates clockwise in the loop when viewed from the east. Earth's field here is due north, parallel to the ground, with a strength of *[Math Processing Error]*. What are the direction and magnitude of the torque on the loop?

(b) Does this device have any practical applications as a motor?



47. Repeat the previous problem, but with the loop lying flat on the ground with its current circulating counterclockwise (when viewed from above) in a location where Earth's field is north, but at an angle 45.0° below the horizontal and with a strength of *[Math Processing Error]*.



11.7 The Hall Effect

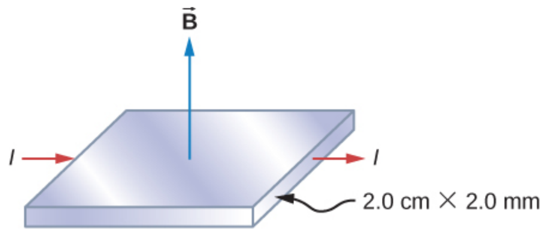
48. A strip of copper is placed in a uniform magnetic field of magnitude 2.5 T. The Hall electric field is measured to be *[Math Processing Error]*.

(a) What is the drift speed of the conduction electrons?

(b) Assuming that *[Math Processing Error]*electrons per cubic meter and that the cross-sectional area of the strip is *[Math Processing Error]*, calculate the current in the strip.

(c) What is the Hall coefficient 1/nq?

49. The cross-sectional dimensions of the copper strip shown are 2.0 cm by 2.0 mm. The strip carries a current of 100 A, and it is placed in a magnetic field of magnitude B = 1.5 T. What are the value and polarity of the Hall potential in the copper strip?



50. The magnitudes of the electric and magnetic fields in a velocity selector are *[Math Processing Error]* and 0.080 T, respectively.

(a) What speed must a proton have to pass through the selector?

(b) Also calculate the speeds required for an alpha-particle and a singly ionized *[Math Processing Error]* atom to pass through the selector.

51. A charged particle moves through a velocity selector at constant velocity. In the selector, *[Math Processing Error]* and **B** = 0.250 T. When the electric field is turned off, the charged particle travels in a circular path of radius 3.33 mm. Determine the charge-to-mass ratio of the particle.

52. A Hall probe gives a reading of **1.5μV** for a current of 2 A when it is placed in a magnetic field of 1 T. What is the magnetic field in a region where the reading is **2μV** for 1.7 A of current?

11.8 Applications of Magnetic Forces and Fields

53. A physicist is designing a cyclotron to accelerate protons to one-tenth the speed of light. The magnetic field will have a strength of 1.5 T. Determine

(a) the rotational period of the circulating protons and

(b) the maximum radius of the protons' orbit.

54. The strengths of the fields in the velocity selector of a Bainbridge mass spectrometer are **B = 0.500 T** and *[Math Processing Error]*, and the strength of the magnetic field that separates the ions is *[Math Processing Error]*. A stream of singly charged Li ions is found to bend in a circular arc of radius 2.32 cm. What is the mass of the Li ions?

55. The magnetic field in a cyclotron is 1.25 T, and the maximum orbital radius of the circulating protons is 0.40 m.

(a) What is the kinetic energy of the protons when they are ejected from the cyclotron?

(b) What is this energy in MeV?

(c) Through what potential difference would a proton have to be accelerated to acquire this kinetic energy?

(d) What is the period of the voltage source used to accelerate the protons?

(e) Repeat the calculations for alpha-particles.

56. A mass spectrometer is being used to separate common oxygen-16 from the much rarer oxygen-18, taken from a sample of old glacial ice. (The relative abundance of these oxygen isotopes is related to climatic temperature at the time the ice was deposited.) The ratio of the masses of these two ions is 16 to 18, the mass of oxygen-16 is *[Math Processing Error]*, and they are singly charged and travel at *[Math Processing Error]* in a 1.20-T magnetic field. What is the separation between their paths when they hit a target after traversing a semicircle?

57. (a) Triply charged uranium-235 and uranium-238 ions are being separated in a mass spectrometer. (The much rarer uranium-235 is used as reactor fuel.) The masses of the ions are *[Math Processing Error]* and *[Math Processing Error]*,respectively, and they travel at *[Math Processing Error]* in a 0.250-T field. What is the separation between their paths when they hit a target after traversing a semicircle? (b) Discuss whether this distance between their paths seems to be big enough to be practical in the separation of uranium-235 from uranium-238.

Additional Problems

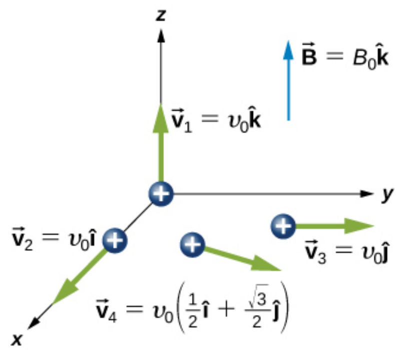
58. Calculate the magnetic force on a hypothetical particle of charge *[Math Processing Error]* moving with a velocity of *[Math Processing Error]* in a magnetic field of *[Math Processing Error]*.

59. Repeat the previous problem with a new magnetic field of *[Math Processing Error]*.

60. An electron is projected into a uniform magnetic field *[Math Processing Error]* with a velocity of *[Math Processing Error]*. What is the magnetic force on the electron?

61. The mass and charge of a water droplet are *[Math Processing Error]* and *[Math Processing Error]*, respectively. If the droplet is given an initial horizontal velocity of *[Math Processing Error]*, what magnetic field will keep it moving in this direction? Why must gravity be considered here?

62. Four different proton velocities are given. For each case, determine the magnetic force on the proton in terms of *[Math Processing Error]*, and *[Math Processing Error]*.



63. An electron of kinetic energy 2000 eV passes between parallel plates that are 1.0 cm apart and kept at a potential difference of 300 V. What is the strength of the uniform magnetic field B that will allow the electron to travel undeflected through the plates? Assume E and B are perpendicular.

64. An alpha-particle *[Math Processing Error]* moving with a velocity *[Math Processing Error]* enters a region where *[Math Processing Error]* and *[Math Processing Error]*. What is the initial force on it?

65. An electron moving with a velocity *[Math Processing Error]* enters a region where there is a uniform electric field and a uniform magnetic field. The magnetic field is given by *[Math Processing Error]*. If the electron travels through a region without being deflected, what is the electric field?

66. At a particular instant, an electron is traveling west to east with a kinetic energy of 10 keV. Earth's magnetic field has a horizontal component of *[Math Processing Error]* north and a vertical component of *[Math Processing Error]* down. (a) What is the path of the electron? (b) What is the radius of curvature of the path?

67. What is the (a) path of a proton and (b) the magnetic force on the proton that is traveling west to east with a kinetic energy of 10 keV in Earth's magnetic field that has a horizontal component of 1.8×10^{-5} T north and a vertical component of 5.0×10^{-5} T down?

68. What magnetic field is required in order to confine a proton moving with a speed of *[Math Processing Error]* to a circular orbit of radius 10 cm?

69. An electron and a proton move with the same speed in a plane perpendicular to a uniform magnetic field. Compare the radii and periods of their orbits.

70. A proton and an alpha-particle have the same kinetic energy and both move in a plane perpendicular to a uniform magnetic field. Compare the periods of their orbits.

71. A singly charged ion takes *[Math Processing Error]* to complete eight revolutions in a uniform magnetic field of magnitude *[Math Processing Error]*. What is the mass of the ion?

72. A particle moving downward at a speed of *[Math Processing Error]* enters a uniform magnetic field that is horizontal and directed from east to west.

(a) If the particle is deflected initially to the north in a circular arc, is its charge positive or negative?

(b) If B = 0.25 T and the charge-to-mass ratio (q/m) of the particle is *[Math Processing Error]*, what is the radius of the path?

(c) What is the speed of the particle after it has moved in the field for *[Math Processing Error]*? for 2.0 s?

73. A proton, deuteron, and an alpha-particle are all accelerated through the same potential difference. They then enter the same magnetic field, moving perpendicular to it. Compute the ratios of the radii of their circular paths. Assume that *[Math Processing Error]* and *[Math Processing Error]*.

74. A singly charged ion is moving in a uniform magnetic field of *[Math Processing Error]* completes 10 revolutions in *[Math Processing Error]*. Identify the ion.

75. Two particles have the same linear momentum, but particle A has four times the charge of particle B. If both particles move in a plane perpendicular to a uniform magnetic field, what is the ratio *[Math Processing Error]* of the radii of their circular orbits?

76. A uniform magnetic field of magnitude *[Math Processing Error]* is directed parallel to the z-axis. A proton enters the field with a velocity *[Math Processing Error]* and travels in a helical path with a radius of 5.0 cm.

(a) What is the value of B?

(b) What is the time required for one trip around the helix?

(c) Where is the proton *[Math Processing Error]* after entering the field?

77. An electron moving at *[Math Processing Error]* enters a magnetic field that makes a *[Math Processing Error]* angle with the x-axis of magnitude 0.20 T. Calculate the

(a) pitch and

(b) radius of the trajectory.

78. (a) A 0.750-m-long section of cable carrying current to a car starter motor makes an angle of 60º with Earth's *[Math Processing Error]* field. What is the current when the wire experiences a force of *[Math Processing Error]*?

(b) If you run the wire between the poles of a strong horseshoe magnet, subjecting 5.00 cm of it to a 1.75-T field, what force is exerted on this segment of wire?

79. (a) What is the angle between a wire carrying an 8.00-A current and the 1.20-T field it is in if 50.0 cm of the wire experiences a magnetic force of 2.40 N?

(b) What is the force on the wire if it is rotated to make an angle of 90º with the field?

80. A 1.0-m-long segment of wire lies along the x-axis and carries a current of 2.0 A in the positive x-direction. Around the wire is the magnetic field of *[Math Processing Error]*. Find the magnetic force on this segment.

81. A 5.0-m section of a long, straight wire carries a current of 10 A while in a uniform magnetic field of magnitude *[Math Processing Error]*. Calculate the magnitude of the force on the section if the angle between the field and the direction of the current is

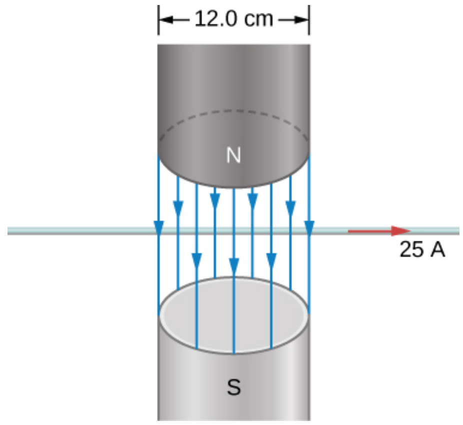
(a) 45°;

(b) 90°;

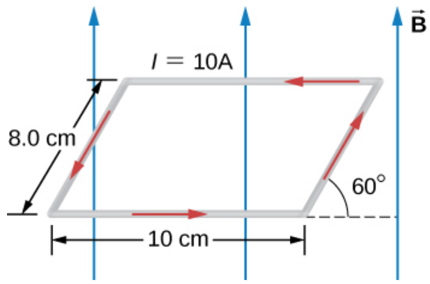
(c) 0°; or

(d) 180°.

82. An electromagnet produces a magnetic field of magnitude 1.5 T throughout a cylindrical region of radius 6.0 cm. A straight wire carrying a current of 25 A passes through the field as shown in the accompanying figure. What is the magnetic force on the wire?



83. The current loop shown in the accompanying figure lies in the plane of the page, as does the magnetic field. Determine the net force and the net torque on the loop if **I = 10 A** and **B = 1.5 T.**



84. A circular coil of radius 5.0 cm is wound with five turns and carries a current of 5.0 A. If the coil is placed in a uniform magnetic field of strength 5.0 T, what is the maximum torque on it?

85. A circular coil of wire of radius 5.0 cm has 20 turns and carries a current of 2.0 A. The coil lies in a magnetic field of magnitude 0.50 T that is directed parallel to the plane of the coil.

(a) What is the magnetic dipole moment of the coil?

(b) What is the torque on the coil?

86. A current-carrying coil in a magnetic field experiences a torque that is 75% of the maximum possible torque. What is the angle between the magnetic field and the normal to the plane of the coil?

87. A 4.0-cm by 6.0-cm rectangular current loop carries a current of 10 A. What is the magnetic dipole moment of the loop?

88. A circular coil with 200 turns has a radius of 2.0 cm.

(a) What current through the coil results in a magnetic dipole moment of 3.0 Am2?

(b) What is the maximum torque that the coil will experience in a uniform field of strength *[Math Processing Error]*?

(c) If the angle between μ and B is 45°, what is the magnitude of the torque on the coil?

(d) What is the magnetic potential energy of coil for this orientation?

89. The current through a circular wire loop of radius 10 cm is 5.0 A.

(a) Calculate the magnetic dipole moment of the loop.

(b) What is the torque on the loop if it is in a uniform 0.20-T magnetic field such that **μ** and B are directed at **30°** to each other?

(c) For this position, what is the potential energy of the dipole?

90. A wire of length 1.0 m is wound into a single-turn planar loop. The loop carries a current of 5.0 A, and it is placed in a uniform magnetic field of strength 0.25 T.

(a) What is the maximum torque that the loop will experience if it is square?

(b) If it is circular?

(c) At what angle relative to B would the normal to the circular coil have to be oriented so that the torque on it would be the same as the maximum torque on the square coil?

91. Consider an electron rotating in a circular orbit of radius r. Show that the magnitudes of the magnetic dipole moment μ and the angular momentum L of the electron are related by:

[Math Processing Error].

92. The Hall effect is to be used to find the sign of charge carriers in a semiconductor sample. The probe is placed between the poles of a magnet so that magnetic field is pointed up. A current is passed through a rectangular sample placed horizontally. As current is passed through the sample in the east direction, the north side of the sample is found to be at a higher potential than the south side. Decide if the number density of charge carriers is positively or negatively charged.

93. The density of charge carriers for copper is *[Math Processing Error]* electrons per cubic meter. What will be the Hall voltage reading from a probe made up of **3cm×2 cm×1 cm(L×W×T)** copper plate when a current of **1.5 A** is passed through it in a magnetic field of **2.5 T** perpendicular to the **3cm×2 cm.**

94. The Hall effect is to be used to find the density of charge carriers in an unknown material. A Hall voltage 40 **μV** for 3-A current is observed in a 3-T magnetic field for a rectangular sample with length 2 cm, width 1.5 cm, and height 0.4 cm. Determine the density of the charge carriers.

95. Show that the Hall voltage across wires made of the same material, carrying identical currents, and subjected to the same magnetic field is inversely proportional to their diameters. (Hint: Consider how drift velocity depends on wire diameter.)

96. A velocity selector in a mass spectrometer uses a 0.100-T magnetic field.

(a) What electric field strength is needed to select a speed of *[Math Processing Error]*?

(b) What is the voltage between the plates if they are separated by 1.00 cm?

97. Find the radius of curvature of the path of a 25.0-MeV proton moving perpendicularly to the 1.20-T field of a cyclotron.

98. Unreasonable results To construct a non-mechanical water meter, a 0.500-T magnetic field is placed across the supply water pipe to a home and the Hall voltage is recorded.

(a) Find the flow rate through a 3.00-cm-diameter pipe if the Hall voltage is 60.0 mV.

(b) What would the Hall voltage be for the same flow rate through a 10.0-cm-diameter pipe with the same field applied?

99. Unreasonable results A charged particle having mass *[Math Processing Error]* (that of a helium atom) moving at 8.70×105m/s perpendicular to a 1.50-T magnetic field travels in a circular path of radius 16.0 mm.

(a) What is the charge of the particle?

(b) What is unreasonable about this result?

(c) Which assumptions are responsible?

100. Unreasonable results An inventor wants to generate 120-V power by moving a 1.00-m-long wire perpendicular to Earth's *[Math Processing Error]* field.

(a) Find the speed with which the wire must move.

(b) What is unreasonable about this result? (c) Which assumption is responsible?

101. Unreasonable results Frustrated by the small Hall voltage obtained in blood flow measurements, a medical physicist decides to increase the applied magnetic field strength to get a 0.500-V output for blood moving at 30.0 cm/s in a 1.50-cm-diameter vessel.

(a) What magnetic field strength is needed?

(b) What is unreasonable about this result?

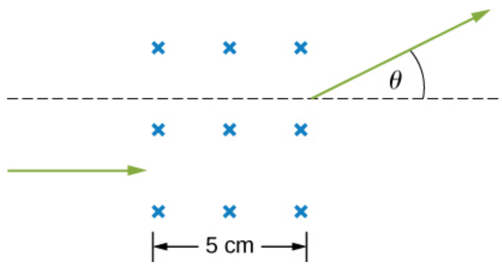
(c) Which premise is responsible?

Challenge Problems

102. A particle of charge **+q** and mass **m** moves with velocity *[Math Processing Error]* pointed in the **+y**-direction as it crosses the **x**-axis at **x = R** at a particular time. There is a negative charge **–Q** fixed at the origin, and there exists a uniform magnetic field *[Math Processing Error]* pointed in the **+z**-direction. It is found that the particle describes a circle of radius **R** about **–Q.** Find *[Math Processing Error]* in terms of the given quantities.

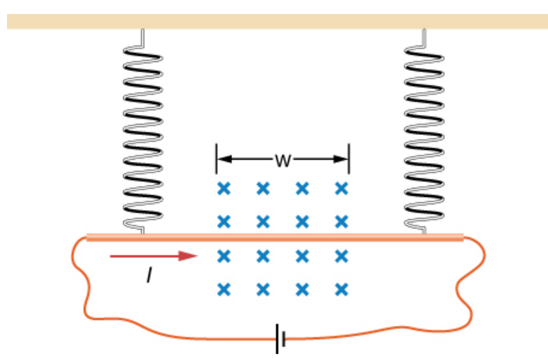
103. A proton of speed *[Math Processing Error]* enters a region of uniform magnetic field of **B = 0.5 T** at an angle of *[Math Processing Error]* to the magnetic field. In the region of magnetic field proton describes a helical path with radius R and pitch p (distance between loops). Find **R** and **p**.

104. A particle's path is bent when it passes through a region of non-zero magnetic field although its speed remains unchanged. This is very useful for "beam steering" in particle accelerators. Consider a proton of speed *[Math Processing Error]* entering a region of uniform magnetic field 0.2 T over a 5-cm-wide region. Magnetic field is perpendicular to the velocity of the particle. By how much angle will the path of the proton be bent? (Hint: The particle comes out tangent to a circle.)

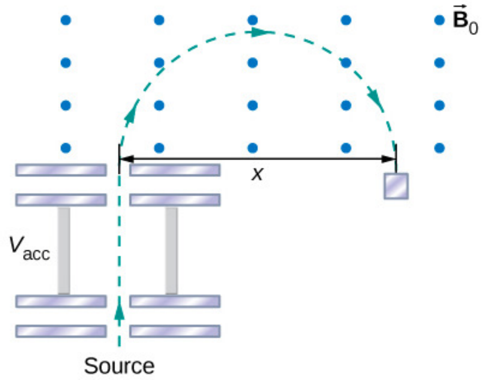


105. In a region a non-uniform magnetic field exists such that *[Math Processing Error]* and *[Math Processing Error]*, where a is a constant. At some time **t**, a wire of length **L** is carrying a current **I** is located along the **x-axis** from origin to **x = L**. Find the magnetic force on the wire at this instant in time.

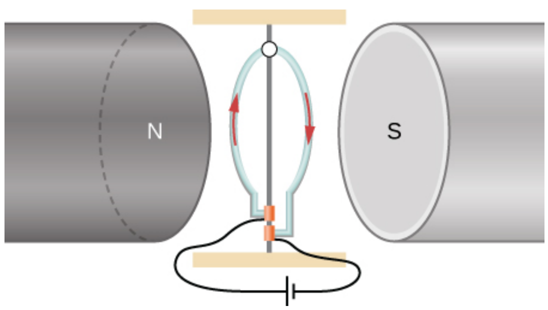
106. A copper rod of mass **m** and length **L** is hung from the ceiling using two springs of spring constant **k**. A uniform magnetic field of magnitude *[Math Processing Error]* pointing perpendicular to the rod and spring (coming out of the page in the figure) exists in a region of space covering a length **w** of the copper rod. The ends of the rod are then connected by flexible copper wire across the terminals of a battery of voltage **V**. Determine the change in the length of the springs when a current I runs through the copper rod in the direction shown in figure. (Ignore any force by the flexible wire.)



107. The accompanied figure shows an arrangement for measuring mass of ions by an instrument called the mass spectrometer. An ion of mass **m** and charge **+q** is produced essentially at rest in source **S**, a chamber in which a gas discharge is taking place. The ion is accelerated by a potential difference *[Math Processing Error]* and allowed to enter a region of constant magnetic field *[Math Processing Error]*. In the uniform magnetic field region, the ion moves in a semicircular path striking a photographic plate at a distance x from the entry point. Derive a formula for mass m in terms of *[Math Processing Error]*, and *[Math Processing Error]*.



108. A wire is made into a circular shape of radius **R** and pivoted along a central support. The two ends of the wire are touching a brush that is connected to a dc power source. The structure is between the poles of a magnet such that we can assume there is a uniform magnetic field on the wire. In terms of a coordinate system with origin at the center of the ring, magnetic field is *[Math Processing Error]*, and the ring rotates about the **z-axis**. Find the torque on the ring when it is not in the xz-plane.



109. A long-rigid wire lies along the **x-axis** and carries a current of **2.5 A** in the positive x-direction. Around the wire is the magnetic field *[Math Processing Error]*, with **x** in meters and **B** in millitesla. Calculate the magnetic force on the segment of wire between **x = 2.0 m** and **x = 4.0 m**.

110. A circular loop of wire of area 10 *[Math Processing Error]* carries a current of 25 A. At a particular instant, the loop lies in the **xy**-plane and is subjected to a magnetic field *[Math Processing Error]*. As viewed from above the **xy**-plane, the current is circulating clockwise.

(a) What is the magnetic dipole moment of the current loop?

(b) At this instant, what is the magnetic torque on the loop?

CHAPTER OVERVIEW

8: Sources of Magnetic Fields

In this chapter, we examine how magnetic fields are created by arbitrary distributions of electric current, using the Biot-Savart law. Then we look at how current-carrying wires create magnetic fields and deduce the forces that arise between two current-carrying wires due to these magnetic fields. We also study the torques produced by the magnetic fields of current loops. We then generalize these results to an important law of electromagnetism, called Ampère's law.

8.1: Prelude to Sources of Magnetic Fields
8.2: The Biot-Savart Law
8.2.1: Magnetic Field due to a Thin Straight Wire
8.2.2: Magnetic Field of a Current Loop
8.3: Magnetic Force between Two Parallel Currents
8.4: Ampère's Law
8.5: Solenoids and Toroids
8.6: Magnetism in Matter
8.6.1: Magnets
8.7: Sources of Magnetic Fields (Exercise)

8.1: Prelude to Sources of Magnetic Fields

In the preceding chapter, we saw that a moving charged particle produces a magnetic field. This connection between electricity and magnetism is exploited in electromagnetic devices, such as a computer hard drive. In fact, it is the underlying principle behind most of the technology in modern society, including telephones, television, computers, and the internet.

Figure *[Math Processing Error]*: An external hard drive attached to a computer works by magnetically encoding information that can be stored or retrieved quickly. A key idea in the development of digital devices is the ability to produce and use magnetic fields in this way. (credit: modification of work by “Miss Karen”/Flickr)

In this chapter, we examine how magnetic fields are created by arbitrary distributions of electric current, using the Biot-Savart law. Then we look at how current-carrying wires create magnetic fields and deduce the forces that arise between two current-carrying wires due to these magnetic fields. We also study the torques produced by the magnetic fields of current loops. We then generalize these results to an important law of electromagnetism, called Ampère’s law.

We examine some devices that produce magnetic fields from currents in geometries based on loops, known as solenoids and toroids. Finally, we look at how materials behave in magnetic fields and categorize materials based on their responses to magnetic fields.

8.2: The Biot-Savart Law

Learning Objectives

By the end of this section, you will be able to:

- Explain how to derive a magnetic field from an arbitrary current in a line segment
- Calculate magnetic field from the Biot-Savart law in specific geometries, such as a current in a line and a current in a circular arc

We have seen that mass produces a gravitational field and also interacts with that field. Charge produces an electric field and also interacts with that field. Since moving charge (that is, current) interacts with a magnetic field, we might expect that it also creates that field—and it does.

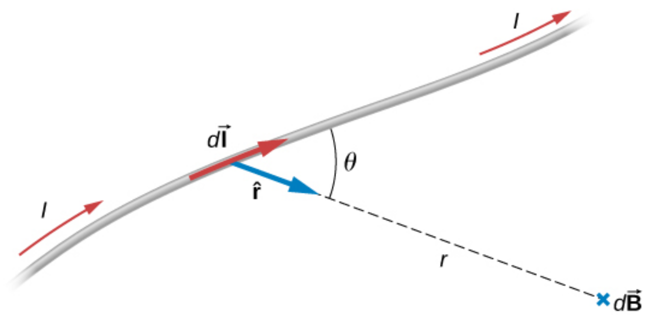


Figure *[Math Processing Error]*: A current element *[Math Processing Error]* produces a magnetic field at point *[Math Processing Error]* given by the Biot-Savart law (Equation *[Math Processing Error]*).

The equation used to calculate the magnetic field produced by a current is known as the Biot-Savart law. It is an empirical law named in honor of two scientists who investigated the interaction between a straight, current-carrying wire and a permanent magnet. This law enables us to calculate the magnitude and direction of the magnetic field produced by a current in a wire. The **Biot-Savart law** states that at any point *[Math Processing Error]* (Figure *[Math Processing Error]*), the magnetic field *[Math Processing Error]* due to an element *[Math Processing Error]* of a current-carrying wire is given by

[Math Processing Error]

The constant *[Math Processing Error]* is known as the **permeability of free space** and is exactly

[Math Processing Error]

in the SI system. The infinitesimal wire segment *[Math Processing Error]* is in the same direction as the current *[Math Processing Error]* (assumed positive), *[Math Processing Error]* is the distance from *[Math Processing Error]* to *[Math Processing Error]* and *[Math Processing Error]* is a unit vector that points from *[Math Processing Error]* to *[Math Processing Error]*, as shown in Figure *[Math Processing Error]*. The direction of *[Math Processing Error]* is determined by applying the right-hand rule to the vector product *[Math Processing Error]*. The magnitude of *[Math Processing Error]* is

[Math Processing Error]

where *[Math Processing Error]* is the angle between *[Math Processing Error]* and *[Math Processing Error]*. Notice that if *[Math Processing Error]*, then *[Math Processing Error]*. The field produced by a current element *[Math Processing Error]* has no component parallel to *[Math Processing Error]*.

The magnetic field due to a finite length of current-carrying wire is found by integrating Equation *[Math Processing Error]* along the wire, giving us the usual form of the Biot-Savart law.

Biot-Savart law

The magnetic field *[Math Processing Error]* due to an element *[Math Processing Error]* of a current-carrying wire is given by

[Math Processing Error]

Since this is a vector integral, contributions from different current elements may not point in the same direction. Consequently, the integral is often difficult to evaluate, even for fairly simple geometries. The following strategy may be helpful.

Problem-Solving Strategy: Solving Biot-Savart Problems

To solve Biot-Savart law problems, the following steps are helpful:

1. Identify that the Biot-Savart law is the chosen method to solve the given problem. If there is symmetry in the problem comparing *[Math Processing Error]* and *[Math Processing Error]*, Ampère's law may be the preferred method to solve the question.
2. Draw the current element length *[Math Processing Error]* and the unit vector *[Math Processing Error]* noting that *[Math Processing Error]* points in the direction of the current and *[Math Processing Error]* points from the current element toward the point where the field is desired.
3. Calculate the cross product *[Math Processing Error]*.The resultant vector gives the direction of the magnetic field according to the Biot-Savart law.
4. Use Equation *[Math Processing Error]* and substitute all given quantities into the expression to solve for the magnetic field. Note all variables that remain constant over the entire length of the wire may be factored out of the integration.
5. Use the right-hand rule to verify the direction of the magnetic field produced from the current or to write down the direction of the magnetic field if only the magnitude was solved for in the previous part.

Calculating Magnetic Fields of Short Current Segments

A short wire of length 1.0 cm carries a current of 2.0 A in the vertical direction (Figure *[Math Processing Error]*). The rest of the wire is shielded so it does not add to the magnetic field produced by the wire. Calculate the magnetic field at point **P**, which is 1 meter from the wire in the **x**-direction.

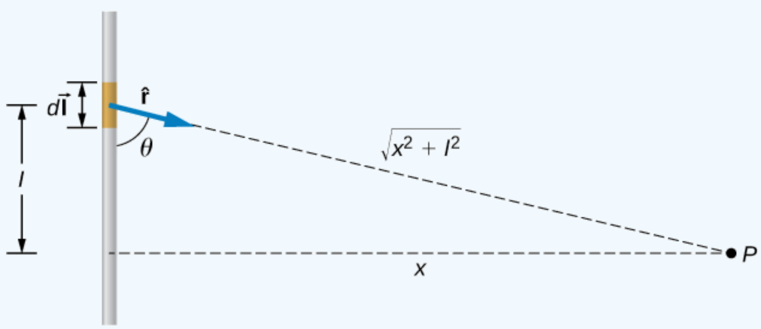


Figure *[Math Processing Error]*: A small line segment carries a current *[Math Processing Error]* in the vertical direction. What is the magnetic field at a distance **x** from the segment?

Strategy

We can determine the magnetic field at point *[Math Processing Error]* using the Biot-Savart law. Since the current segment is much smaller than the distance **x**, we can drop the integral from the expression. The integration is converted back into a summation, but only for small *[Math Processing Error]*, which we now write as *[Math Processing Error]*. Another way to think about it is that each of the radius values is nearly the same, no matter where the current element is on the line segment, if *[Math Processing Error]* is small compared to **x**. The angle *[Math Processing Error]* is calculated using a tangent function. Using the numbers given, we can calculate the magnetic field at *[Math Processing Error]*.

Solution

The angle between *[Math Processing Error]* and *[Math Processing Error]* is calculated from trigonometry, knowing the distances **l** and **x** from the problem:

[Math Processing Error]

The magnetic field at point *[Math Processing Error]* is calculated by the Biot-Savart law (Equation *[Math Processing Error]*):

[Math Processing Error]

From the right-hand rule and the Biot-Savart law, the field is directed into the page.

Significance

This approximation is only good if the length of the line segment is very small compared to the distance from the current element to the point. If not, the integral form of the Biot-Savart law must be used over the entire line segment to calculate the magnetic field.

Exercise [Math Processing Error]

Using Example [Math Processing Error], at what distance would **P** have to be to measure a magnetic field half of the given answer?

Solution

1.41 meters

Example [Math Processing Error]: Calculating Magnetic Field of a Circular Arc of Wire

A wire carries a current **I** in a circular arc with radius **R** swept through an arbitrary angle [Math Processing Error] (Figure [Math Processing Error]). Calculate the magnetic field at the center of this arc at point **P**.

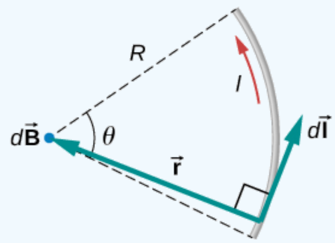


Figure [Math Processing Error]: A wire segment carrying a current **I**. The path [Math Processing Error] and radial direction [Math Processing Error] are indicated.

Strategy

We can determine the magnetic field at point **P** using the Biot-Savart law. The radial and path length directions are always at a right angle, so the cross product turns into multiplication. We also know that the distance along the path **dl** is related to the radius times the angle [Math Processing Error] (in radians). Then we can pull all constants out of the integration and solve for the magnetic field.

Solution

The Biot-Savart law starts with the following equation:

[Math Processing Error]

As we integrate along the arc, all the contributions to the magnetic field are in the same direction (out of the page), so we can work with the magnitude of the field. The cross product turns into multiplication because the path [Math Processing Error] and the radial direction are perpendicular. We can also substitute the arc length formula, [Math Processing Error]:

[Math Processing Error]

The current and radius can be pulled out of the integral because they are the same regardless of where we are on the path. This leaves only the integral over the angle,

[Math Processing Error]

The angle varies on the wire from 0 to [Math Processing Error]; hence, the result is

[Math Processing Error]

Significance

The direction of the magnetic field at point [Math Processing Error] is determined by the right-hand rule, as shown in the previous chapter. If there are other wires in the diagram along with the arc, and you are asked to find the net magnetic field, find each contribution from a wire or arc and add the results by superposition of vectors. Make sure to pay attention to the direction of each contribution. Also note that in a symmetric situation, like a straight or circular wire, contributions from opposite sides of point [Math Processing Error] cancel each other.

Exercise [Math Processing Error]

The wire loop forms a full circle of radius **R** and current **I**. What is the magnitude of the magnetic field at the center?

Solution

[Math Processing Error]

8.2.1: Magnetic Field due to a Thin Straight Wire

Learning Objectives

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a thin, straight wire.
- Determine the dependence of the magnetic field from a thin, straight wire based on the distance from it and the current flowing in the wire.
- Sketch the magnetic field created from a thin, straight wire by using the second right-hand rule.

How much current is needed to produce a significant magnetic field, perhaps as strong as Earth's field? Surveyors will tell you that overhead electric power lines create magnetic fields that interfere with their compass readings. Indeed, when Oersted discovered in 1820 that a current in a wire affected a compass needle, he was not dealing with extremely large currents. How does the shape of wires carrying current affect the shape of the magnetic field created? We noted in Chapter 28 that a current loop created a magnetic field similar to that of a bar magnet, but what about a straight wire? We can use the Biot-Savart law to answer all of these questions, including determining the magnetic field of a long straight wire.

Figure *[Math Processing Error]* shows a section of an infinitely long, straight wire that carries a current **I**. What is the magnetic field at a point **P**, located a distance **R** from the wire?

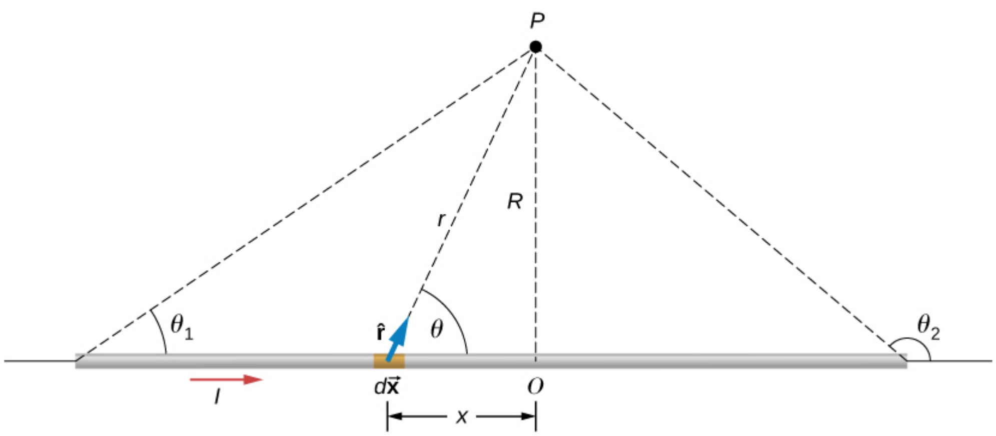


Figure *[Math Processing Error]*: A section of a thin, straight current-carrying wire. The independent variable *[Math Processing Error]* has the limits *[Math Processing Error]* and *[Math Processing Error]*.

Let's begin by considering the magnetic field due to the current element *[Math Processing Error]* located at the position **x**. Using the right-hand rule 1 from the previous chapter, *[Math Processing Error]* points out of the page for any element along the wire. At point *[Math Processing Error]*, therefore, the magnetic fields due to all current elements have the same direction. This means that we can calculate the net field there by evaluating the scalar sum of the contributions of the elements. With

[Math Processing Error]

we have from the **Biot-Savart law**

[Math Processing Error]

The wire is symmetrical about point *[Math Processing Error]*, so we can set the limits of the integration from zero to infinity and double the answer, rather than integrate from negative infinity to positive infinity. Based on the picture and trigonometry, we can write expressions for *[Math Processing Error]* and *[Math Processing Error]* in terms of **x** and **R**, namely:

[Math Processing Error]

[Math Processing Error]

Substituting these expressions into Equation *[Math Processing Error]*, the magnetic field integration becomes

[Math Processing Error]

Evaluating the integral yields

[Math Processing Error]

Substituting the limits gives us the solution

[Math Processing Error]

The magnetic field lines of the infinite wire are circular and centered at the wire (Figure *[Math Processing Error]*), and they are identical in every plane perpendicular to the wire. Since the field decreases with distance from the wire, the spacing of the field lines must increase correspondingly with distance. The direction of this magnetic field may be found with a second form of the **right-hand rule** (Figure *[Math Processing Error]*). If you hold the wire with your right hand so that your thumb points along the current, then your fingers wrap around the wire in the same sense as *[Math Processing Error]*.

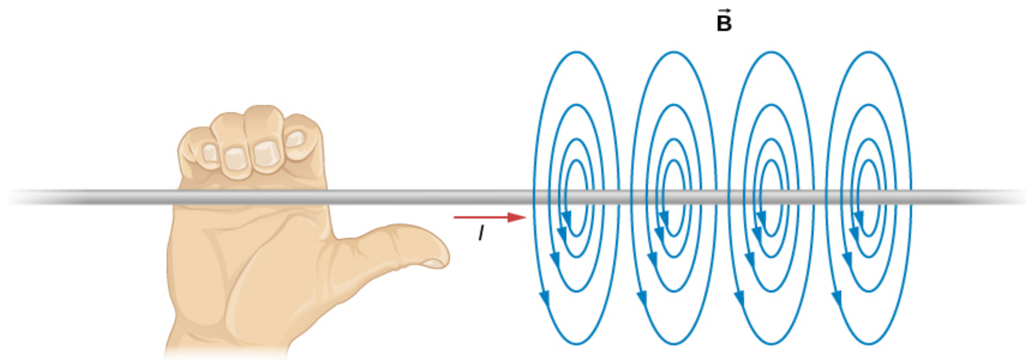


Figure *[Math Processing Error]*:. Some magnetic field lines of an infinite wire. The direction of *[Math Processing Error]* can be found with a form of the right-hand rule.

The direction of the field lines can be observed experimentally by placing several small compass needles on a circle near the wire, as illustrated in Figure *[Math Processing Error]*. When there is no current in the wire, the needles align with Earth's magnetic field. However, when a large current is sent through the wire, the compass needles all point tangent to the circle. Iron filings sprinkled on a horizontal surface also delineate the field lines, as shown in Figure *[Math Processing Error]*.

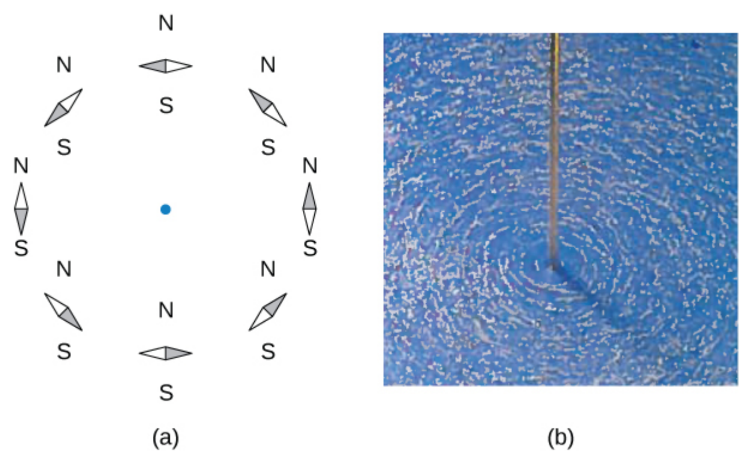


Figure *[Math Processing Error]*: The shape of the magnetic field lines of a long wire can be seen using (a) small compass needles and (b) iron filings.

✔ Example *[Math Processing Error]*: Calculating Magnetic Field Due to Three Wires

Three wires sit at the corners of a square, all carrying currents of 2 amps into the page as shown in Figure *[Math Processing Error]*. Calculate the magnitude of the magnetic field at the other corner of the square, point **P**, if the length of each side of the square is 1 cm.

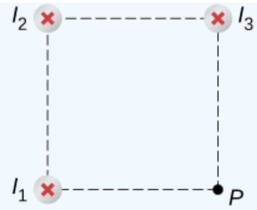


Figure *[Math Processing Error]*: Three wires have current flowing into the page. The magnetic field is determined at the fourth corner of the square.

Strategy

The magnetic field due to each wire at the desired point is calculated. The diagonal distance is calculated using the Pythagorean theorem. Next, the direction of each magnetic field's contribution is determined by drawing a circle centered at the point of the wire and out toward the desired point. The direction of the magnetic field contribution from that wire is tangential to the curve. Lastly, working with these vectors, the resultant is calculated.

Solution

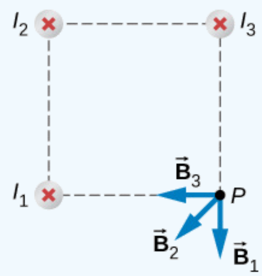
Wires 1 and 3 both have the same magnitude of magnetic field contribution at point **P**:

[Math Processing Error]

Wire 2 has a longer distance and a magnetic field contribution at point **P** of:

[Math Processing Error]

The vectors for each of these magnetic field contributions are shown.



The magnetic field in the **x**-direction has contributions from wire 3 and the **x**-component of wire 2:

[Math Processing Error]

The **y**-component is similarly the contributions from wire 1 and the **y**-component of wire 2:

[Math Processing Error]

Therefore, the net magnetic field is the resultant of these two components:

[Math Processing Error]

Significance

The geometry in this problem results in the magnetic field contributions in the **x**- and **y**-directions having the same magnitude. This is not necessarily the case if the currents were different values or if the wires were located in different positions. Regardless of the numerical results, working on the components of the vectors will yield the resulting magnetic field at the point in need.

? Exercise *[Math Processing Error]*

Using Example *[Math Processing Error]*, keeping the currents the same in wires 1 and 3, what should the current be in wire 2 to counteract the magnetic fields from wires 1 and 3 so that there is no net magnetic field at point *[Math Processing Error]*?

Solution

4 amps flowing out of the page

8.2.2: Magnetic Field of a Current Loop

Learning Objectives

By the end of this section, you will be able to:

- Explain how the Biot-Savart law is used to determine the magnetic field due to a current in a loop of wire at a point along a line perpendicular to the plane of the loop.
- Determine the magnetic field of an arc of current.

The circular loop of Figure *[Math Processing Error]* has a radius **R**, carries a current **I**, and lies in the **xz**-plane. What is the magnetic field due to the current at an arbitrary point **P** along the axis of the loop?

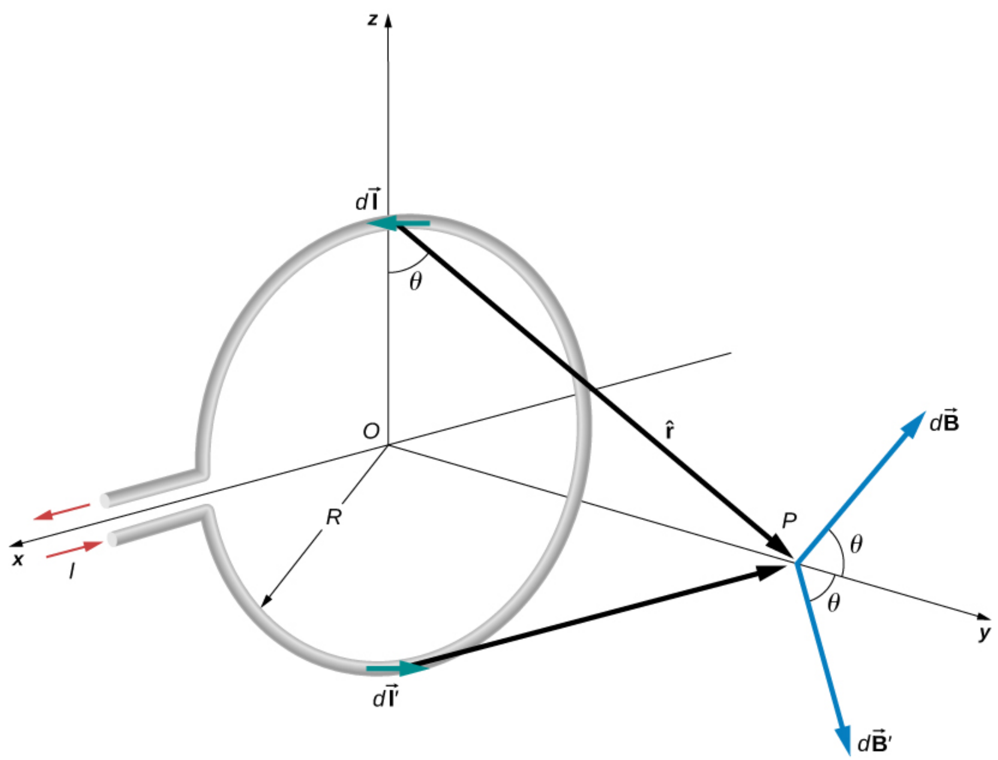


Figure *[Math Processing Error]*: Determining the magnetic field at point **P** along the axis of a current-carrying loop of wire.

We can use the Biot-Savart law to find the magnetic field due to a current. We first consider arbitrary segments on opposite sides of the loop to qualitatively show by the vector results that the net magnetic field direction is along the central axis from the loop. From there, we can use the Biot-Savart law to derive the expression for magnetic field.

Let **P** be a distance **y** from the center of the loop. From the right-hand rule, the magnetic field *[Math Processing Error]* at **P**, produced by the current element *[Math Processing Error]* is directed at an angle *[Math Processing Error]* above the **y**-axis as shown. Since *[Math Processing Error]* is parallel along the **x**-axis and *[Math Processing Error]* is in the **yz**-plane, the two vectors are perpendicular, so we have

[Math Processing Error] where we have used *[Math Processing Error]*.

Now consider the magnetic field *[Math Processing Error]* due to the current element *[Math Processing Error]*, which is directly opposite *[Math Processing Error]* on the loop. The magnitude of *[Math Processing Error]* is also given by Equation *[Math Processing Error]*, but it is directed at an angle below the **y**-axis. The components of *[Math Processing Error]* and *[Math Processing Error]* perpendicular to the **y**-axis therefore cancel, and in calculating the net magnetic field, only the components along the **y**-axis need to be considered. The components perpendicular to the axis of the loop sum to zero in pairs. Hence at point **P**:

[Math Processing Error]

For all elements *[Math Processing Error]* on the wire, **y**, **R**, and *[Math Processing Error]* are constant and are related by

[Math Processing Error]

Now from Equation [Math Processing Error], the magnetic field at **P** is

[Math Processing Error] where we have used [Math Processing Error]. As discussed in the previous chapter, the closed current loop is a magnetic dipole of moment [Math Processing Error]. For this example, [Math Processing Error] and [Math Processing Error], so the magnetic field at **P** can also be written as

[Math Processing Error]

By setting [Math Processing Error] in Equation [Math Processing Error], we obtain the magnetic field at the center of the loop:

Note

[Math Processing Error]

This equation becomes [Math Processing Error] for a flat coil of **n** loops per length. It can also be expressed as

[Math Processing Error]

If we consider [Math Processing Error] in Equation [Math Processing Error], the expression reduces to an expression known as the magnetic field from a dipole:

[Math Processing Error]

The calculation of the magnetic field due to the circular current loop at points off-axis requires rather complex mathematics, so we'll just look at the results. The magnetic field lines are shaped as shown in Figure [Math Processing Error]. Notice that one field line follows the axis of the loop. This is the field line we just found. Also, very close to the wire, the field lines are almost circular, like the lines of a long straight wire.

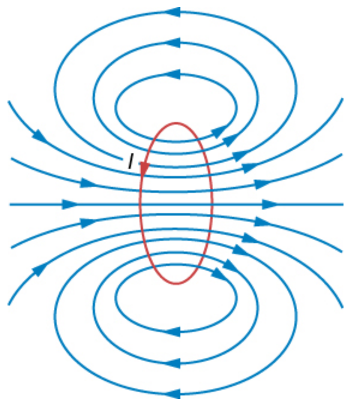


Figure [Math Processing Error]: Sketch of the magnetic field lines of a circular current loop.

Magnetic Field between Two Loops

Two loops of wire carry the same current of 10 mA, but flow in opposite directions as seen in Figure [Math Processing Error]. One loop is measured to have a radius of [Math Processing Error] while the other loop has a radius of [Math Processing Error]. The distance from the first loop to the point where the magnetic field is measured is 0.25 m, and the distance from that point to the second loop is 0.75 m. What is the magnitude of the net magnetic field at point **P**?

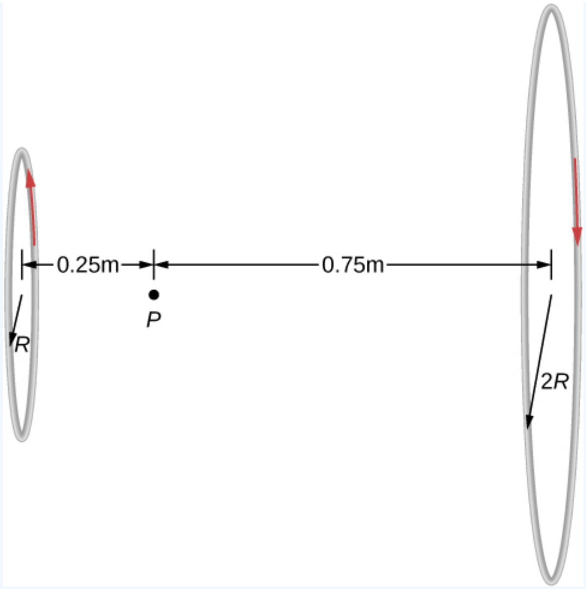


Figure *[Math Processing Error]*: Two loops of different radii have the same current but flowing in opposite directions. The magnetic field at point **P** is measured to be zero.

Strategy

The magnetic field at point **P** has been determined in Equation *[Math Processing Error]*. Since the currents are flowing in opposite directions, the net magnetic field is the difference between the two fields generated by the coils. Using the given quantities in the problem, the net magnetic field is then calculated.

Solution

Solving for the net magnetic field using Equation *[Math Processing Error]* and the given quantities in the problem yields

[Math Processing Error]

[Math Processing Error]

[Math Processing Error] to the right.

Significance

Helmholtz coils typically have loops with equal radii with current flowing in the same direction to have a strong uniform field at the midpoint between the loops. A similar application of the magnetic field distribution created by Helmholtz coils is found in a magnetic bottle that can temporarily trap charged particles. See Magnetic Forces and Fields for a discussion on this.

? Exercise *[Math Processing Error]*

Using Example *[Math Processing Error]*, at what distance would you have to move the first coil to have zero measurable magnetic field at point **P**?

Solution

0.608 meters

8.3: Magnetic Force between Two Parallel Currents

Learning Objectives

By the end of this section, you will be able to:

- Explain how parallel wires carrying currents can attract or repel each other
- Define the ampere and describe how it is related to current-carrying wires
- Calculate the force of attraction or repulsion between two current-carrying wires

You might expect that two current-carrying wires generate significant forces between them, since ordinary currents produce magnetic fields and these fields exert significant forces on ordinary currents. But you might not expect that the force between wires is used to define the ampere. It might also surprise you to learn that this force has something to do with why large circuit breakers burn up when they attempt to interrupt large currents.

The force between two long, straight, and parallel conductors separated by a distance **r** can be found by applying what we have developed in the preceding sections. Figure *[Math Processing Error]* shows the wires, their currents, the field created by one wire, and the consequent force the other wire experiences from the created field. Let us consider the field produced by wire 1 and the force it exerts on wire 2 (call the force *[Math Processing Error]*). The field due to *[Math Processing Error]* at a distance **r** is

[Math Processing Error]

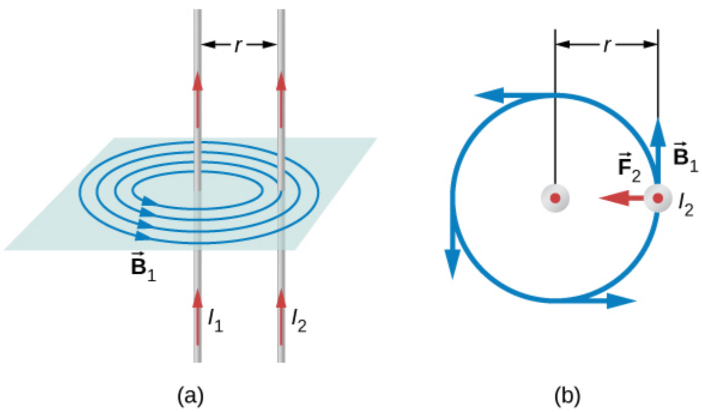


Figure *[Math Processing Error]*: (a) The magnetic field produced by a long straight conductor is perpendicular to a parallel conductor, as indicated by right-hand rule (RHR)-2. (b) A view from above of the two wires shown in (a), with one magnetic field line shown for wire 1. RHR-1 shows that the force between the parallel conductors is attractive when the currents are in the same direction. A similar analysis shows that the force is repulsive between currents in opposite directions.

This field is uniform from the wire 1 and perpendicular to it, so the force *[Math Processing Error]* it exerts on a length **l** of wire 2 is given by *[Math Processing Error]* with *[Math Processing Error]*:

[Math Processing Error]

The forces on the wires are equal in magnitude, so we just write **F** for the magnitude of *[Math Processing Error]* (Note that *[Math Processing Error]*.) Since the wires are very long, it is convenient to think in terms of **F/l**, the force per unit length. Substituting the expression for *[Math Processing Error]* into Equation *[Math Processing Error]* and rearranging terms gives

Note

[Math Processing Error]

The ratio **F/l** is the force per unit length between two parallel currents *[Math Processing Error]* and *[Math Processing Error]* separated by a distance **r**. The force is attractive if the currents are in the same direction and repulsive if they are in opposite directions.

This force is responsible for the **pinch effect** in electric arcs and other plasmas. The force exists whether the currents are in wires or not. It is only apparent if the overall charge density is zero; otherwise, the Coulomb repulsion overwhelms the magnetic attraction.

In an electric arc, where charges are moving parallel to one another, an attractive force squeezes currents into a smaller tube. In large circuit breakers, such as those used in neighborhood power distribution systems, the pinch effect can concentrate an arc between plates of a switch trying to break a large current, burn holes, and even ignite the equipment. Another example of the pinch effect is found in the solar plasma, where jets of ionized material, such as solar flares, are shaped by magnetic forces.

The definition of the **ampere** is based on the force between current-carrying wires. Note that for long, parallel wires separated by 1 meter with each carrying 1 ampere, the force per meter is

[Math Processing Error]

Since *[Math Processing Error]* is exactly *[Math Processing Error]* by definition, and because *[Math Processing Error]*, the force per meter is exactly *[Math Processing Error]*. This is the basis of the definition of the ampere.

Infinite-length wires are impractical, so in practice, a current balance is constructed with coils of wire separated by a few centimeters. Force is measured to determine current. This also provides us with a method for measuring the coulomb. We measure the charge that flows for a current of one ampere in one second. That is, *[Math Processing Error]*. For both the ampere and the coulomb, the method of measuring force between conductors is the most accurate in practice.

Example *[Math Processing Error]*: Calculating Forces on Wires

Two wires, both carrying current out of the page, have a current of magnitude 5.0 mA. The first wire is located at (0.0 cm, 3.0 cm) while the other wire is located at (4.0 cm, 0.0 cm) as shown in Figure *[Math Processing Error]*. What is the magnetic force per unit length of the first wire on the second and the second wire on the first?

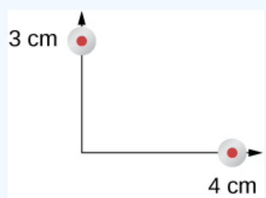


Figure *[Math Processing Error]*: Two current-carrying wires at given locations with currents out of the page.

Strategy

Each wire produces a magnetic field felt by the other wire. The distance along the hypotenuse of the triangle between the wires is the radial distance used in the calculation to determine the force per unit length. Since both wires have currents flowing in the same direction, the direction of the force is toward each other.

Solution

The distance between the wires results from finding the hypotenuse of a triangle:

[Math Processing Error]

The force per unit length can then be calculated using the known currents in the wires:

[Math Processing Error]

The force from the first wire pulls the second wire. The angle between the radius and the **x**-axis is

[Math Processing Error]

The unit vector for this is calculated by

[Math Processing Error]

Therefore, the force per unit length from wire one on wire 2 is

[Math Processing Error]

The force per unit length from wire 2 on wire 1 is the negative of the previous answer:

[Math Processing Error]

Significance

These wires produced magnetic fields of equal magnitude but opposite directions at each other's locations. Whether the fields are identical or not, the forces that the wires exert on each other are always equal in magnitude and opposite in direction (Newton's third law).

Exercise [Math Processing Error]

Two wires, both carrying current out of the page, have a current of magnitude 2.0 mA and 3.0 mA, respectively. The first wire is located at (0.0 cm, 5.0 cm) while the other wire is located at (12.0 cm, 0.0 cm). What is the magnitude of the magnetic force per unit length of the first wire on the second and the second wire on the first?

Answer

Both have a force per unit length of *[Math Processing Error]*

8.4: Ampère’s Law

Learning Objectives

By the end of this section, you will be able to:

- Explain how Ampère’s law relates the magnetic field produced by a current to the value of the current
- Calculate the magnetic field from a long straight wire, either thin or thick, by Ampère’s law

A fundamental property of a static magnetic field is that, unlike an electrostatic field, it is not conservative. A conservative field is one that does the same amount of work on a particle moving between two different points regardless of the path chosen. Magnetic fields do not have such a property. Instead, there is a relationship between the magnetic field and its source, electric current. It is expressed in terms of the line integral of *[Math Processing Error]* and is known as **Ampère’s law**. This law can also be derived directly from the Biot-Savart law. We now consider that derivation for the special case of an infinite, straight wire.

Figure *[Math Processing Error]* shows an arbitrary plane perpendicular to an infinite, straight wire whose current **I** is directed out of the page. The magnetic field lines are circles directed counterclockwise and centered on the wire. To begin, let’s consider *[Math Processing Error]* over the closed paths **M** and **N**. Notice that one path (**M**) encloses the wire, whereas the other (**N**) does not. Since the field lines are circular, *[Math Processing Error]* is the product of **B** and the projection of **dl** onto the circle passing through *[Math Processing Error]*. If the radius of this particular circle is **r**, the projection is *[Math Processing Error]*, and

[Math Processing Error]

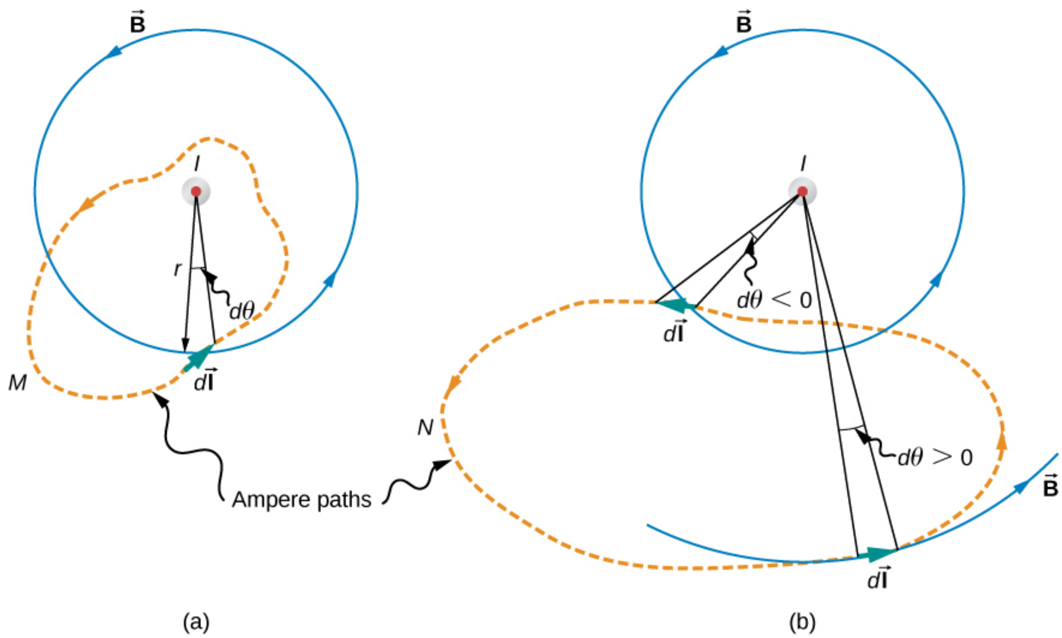


Figure *[Math Processing Error]*: The current I of a long, straight wire is directed out of the page. The integral *[Math Processing Error]* equals *[Math Processing Error]* and 0, respectively, for paths M and N.

With *[Math Processing Error]* given by Equation 12.4.1,

[Math Processing Error]

For path **M**, which circulates around the wire, *[Math Processing Error]* and

[Math Processing Error]

Path **N**, on the other hand, circulates through both positive (counterclockwise) and negative (clockwise) *[Math Processing Error]* (see Figure *[Math Processing Error]*), and since it is closed, *[Math Processing Error]*. Thus for path **N**,

[Math Processing Error]

The extension of this result to the general case is Ampère’s law.

Ampere's Law

Over an arbitrary closed path,

[Math Processing Error]

where **I** is the total current passing through any open surface **S** whose perimeter is the path of integration. Only currents inside the path of integration need be considered.

To determine whether a specific current **I** is positive or negative, curl the fingers of your right hand in the direction of the path of integration, as shown in Figure [Math Processing Error]. If **I** passes through **S** in the same direction as your extended thumb, **I** is positive; if **I** passes through **S** in the direction opposite to your extended thumb, it is negative.

Problem-Solving Strategy: Ampère's Law

To calculate the magnetic field created from current in wire(s), use the following steps:

1. Identify the symmetry of the current in the wire(s). If there is no symmetry, use the Biot-Savart law to determine the magnetic field.
2. Determine the direction of the magnetic field created by the wire(s) by right-hand rule 2.
3. Chose a path loop where the magnetic field is either constant or zero.
4. Calculate the current inside the loop.
5. Calculate the line integral [Math Processing Error] around the closed loop.
6. Equate [Math Processing Error] with [Math Processing Error] with [Math Processing Error] and solve for [Math Processing Error].

Using Ampère's Law to Calculate the Magnetic Field Due to a Wire

Use Ampère's law to calculate the magnetic field due to a steady current **I** in an infinitely long, thin, straight wire as shown in Figure [Math Processing Error].

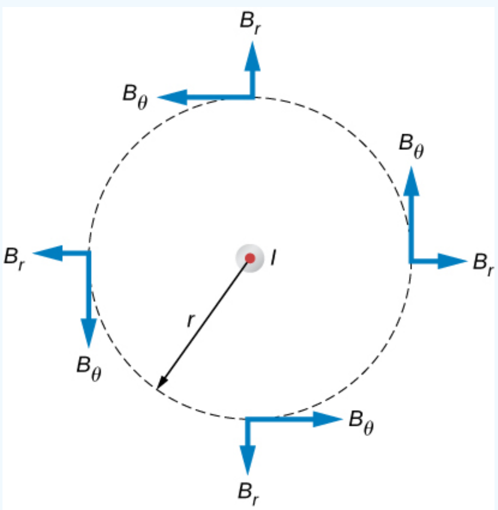


Figure [Math Processing Error]: The possible components of the magnetic field **B** due to a current **I**, which is directed out of the page. The radial component is zero because the angle between the magnetic field and the path is at a right angle.

Strategy

Consider an arbitrary plane perpendicular to the wire, with the current directed out of the page. The possible magnetic field components in this plane, [Math Processing Error] and [Math Processing Error] are shown at arbitrary points on a circle of radius **r** centered on the wire. Since the field is cylindrically symmetric, neither [Math Processing Error] nor [Math Processing Error] varies with the position on this circle. Also from symmetry, the radial lines, if they exist, must be directed either all inward or all outward from the wire. This means, however, that there must be a net magnetic flux across an arbitrary cylinder concentric with the wire. The radial component of the magnetic field must be zero because *[Math Processing Error]*. Therefore, we can apply Ampère's law to the circular path as shown.

Solution

Over this path *[Math Processing Error]* is constant and parallel to *[Math Processing Error]*, so

[Math Processing Error]

Thus Ampère's law reduces to

[Math Processing Error]

Finally, since *[Math Processing Error]* is the only component of *[Math Processing Error]*, we can drop the subscript and write

[Math Processing Error]

This agrees with the Biot-Savart calculation above.

Significance

Ampère's law works well if you have a path to integrate over which *[Math Processing Error]* has results that are easy to simplify. For the infinite wire, this works easily with a path that is circular around the wire so that the magnetic field factors out of the integration. If the path dependence looks complicated, you can always go back to the Biot-Savart law and use that to find the magnetic field.

Example *[Math Processing Error]*: Calculating the Magnetic Field of a Thick Wire with Ampère's Law

The radius of the long, straight wire of Figure *[Math Processing Error]* is **a**, and the wire carries a current *[Math Processing Error]* that is distributed uniformly over its cross-section. Find the magnetic field both inside and outside the wire.

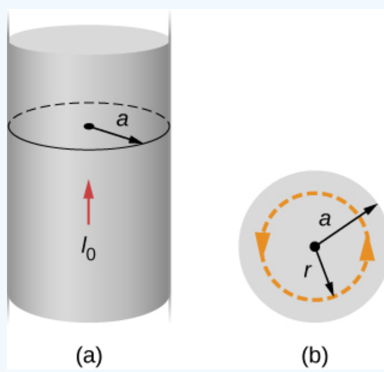


Figure *[Math Processing Error]*: (a) A model of a current-carrying wire of radius **a** and current *[Math Processing Error]*. (b) A cross-section of the same wire showing the radius **a** and the Ampère's loop of radius **r**.

Strategy

This problem has the same geometry as Example *[Math Processing Error]*, but the enclosed current changes as we move the integration path from outside the wire to inside the wire, where it doesn't capture the entire current enclosed (see Figure *[Math Processing Error]*).

Solution

For any circular path of radius **r** that is centered on the wire,

[Math Processing Error]

From Ampère's law, this equals the total current passing through any surface bounded by the path of integration.

Consider first a circular path that is inside the wire *[Math Processing Error]* such as that shown in part (a) of Figure *[Math Processing Error]*. We need the current **I** passing through the area enclosed by the path. It's equal to the current density **J** times the area enclosed. Since the current is uniform, the current density inside the path equals the current density in the whole wire, which is *[Math Processing Error]*. Therefore the current **I** passing through the area enclosed by the path is

[Math Processing Error]

We can consider this ratio because the current density **J** is constant over the area of the wire. Therefore, the current density of a part of the wire is equal to the current density in the whole area. Using Ampère's law, we obtain

[Math Processing Error]

and the magnetic field inside the wire is

[Math Processing Error]

Outside the wire, the situation is identical to that of the infinite thin wire of the previous example; that is,

[Math Processing Error]

The variation of **B** with **r** is shown in Figure *[Math Processing Error]*.

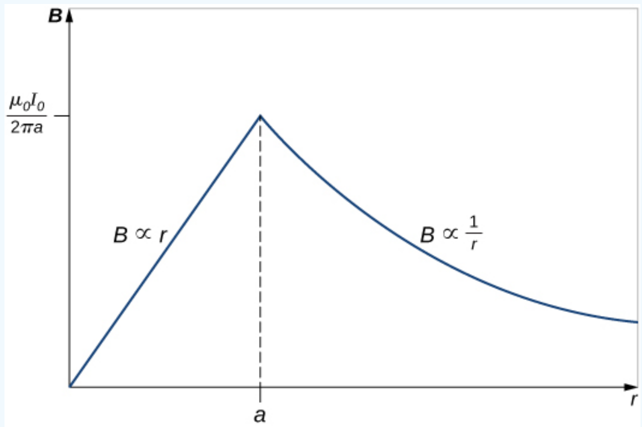


Figure *[Math Processing Error]*: Variation of the magnetic field produced by a current *[Math Processing Error]* in a long, straight wire of radius **a**.

Significance

The results show that as the radial distance increases inside the thick wire, the magnetic field increases from zero to a familiar value of the magnetic field of a thin wire. Outside the wire, the field drops off regardless of whether it was a thick or thin wire.

This result is similar to how Gauss's law for electrical charges behaves inside a uniform charge distribution, except that Gauss's law for electrical charges has a uniform volume distribution of charge, whereas Ampère's law here has a uniform area of current distribution. Also, the drop-off outside the thick wire is similar to how an electric field drops off outside of a linear charge distribution, since the two cases have the same geometry and neither case depends on the configuration of charges or currents once the loop is outside the distribution.

✔ Using Ampère's Law with Arbitrary Paths

Use Ampère's law to evaluate *[Math Processing Error]* for the current configurations and paths in Figure *[Math Processing Error]*.

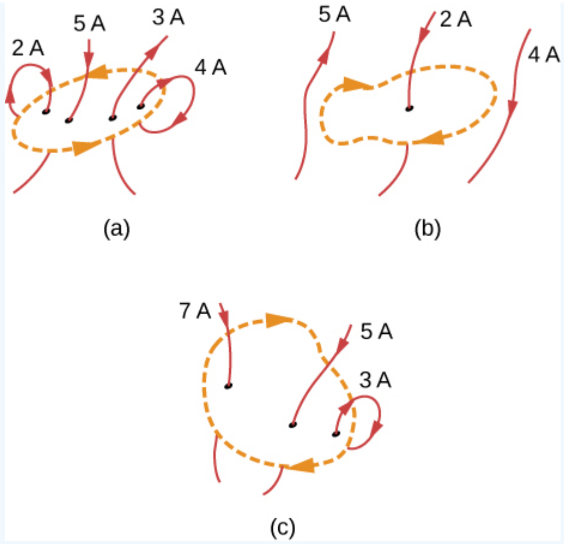


Figure [Math Processing Error]: Current configurations and paths for Example [Math Processing Error].

Strategy

Ampère's law states that [Math Processing Error] where **I** is the total current passing through the enclosed loop. The quickest way to evaluate the integral is to calculate [Math Processing Error] by finding the net current through the loop. Positive currents flow with your right-hand thumb if your fingers wrap around in the direction of the loop. This will tell us the sign of the answer.

Solution

(a) The current going downward through the loop equals the current going out of the loop, so the net current is zero. Thus, [Math Processing Error]

(b) The only current to consider in this problem is 2A because it is the only current inside the loop. The right-hand rule shows us the current going downward through the loop is in the positive direction. Therefore, the answer is [Math Processing Error]

(c) The right-hand rule shows us the current going downward through the loop is in the positive direction. There are [Math Processing Error] of current going downward and –3 A going upward. Therefore, the total current is 9 A and [Math Processing Error].

Significance

If the currents all wrapped around so that the same current went into the loop and out of the loop, the net current would be zero and no magnetic field would be present. This is why wires are very close to each other in an electrical cord. The currents flowing toward a device and away from a device in a wire equal zero total current flow through an Ampère loop around these wires. Therefore, no stray magnetic fields can be present from cords carrying current.

? Exercise [Math Processing Error]

Consider using Ampère's law to calculate the magnetic fields of a finite straight wire and of a circular loop of wire. Why is it not useful for these calculations?

Answer

In these cases the integrals around the Ampèrian loop are very difficult because there is no symmetry, so this method would not be useful.

8.5: Solenoids and Toroids

Learning Objectives

By the end of this section, you will be able to:

- Establish a relationship for how the magnetic field of a solenoid varies with distance and current by using both the Biot-Savart law and Ampère's law
- Establish a relationship for how the magnetic field of a toroid varies with distance and current by using Ampère's law

Two of the most common and useful electromagnetic devices are called solenoids and toroids. In one form or another, they are part of numerous instruments, both large and small. In this section, we examine the magnetic field typical of these devices.

Solenoids

A long wire wound in the form of a helical coil is known as a **solenoid**. Solenoids are commonly used in experimental research requiring magnetic fields. A solenoid is generally easy to wind, and near its center, its magnetic field is quite uniform and directly proportional to the current in the wire.

Figure *[Math Processing Error]* shows a solenoid consisting of **N** turns of wire tightly wound over a length **L**. A current **I** is flowing along the wire of the solenoid. The number of turns per unit length is **N**/**L**; therefore, the number of turns in an infinitesimal length **dy** are (**N**/**L**)**dy** turns. This produces a current

[Math Processing Error]

We first calculate the magnetic field at the point **P** of Figure *[Math Processing Error]*. This point is on the central axis of the solenoid. We are basically cutting the solenoid into thin slices that are **dy** thick and treating each as a current loop. Thus, **dI** is the current through each slice. The magnetic field *[Math Processing Error]* due to the current **dI** in **dy** can be found with the help of Equation 12.5.3 and Equation *[Math Processing Error]*:

[Math Processing Error]

where we used Equation *[Math Processing Error]* to replace **dI**. The resultant field at **P** is found by integrating *[Math Processing Error]* along the entire length of the solenoid. It's easiest to evaluate this integral by changing the independent variable from **y** to *[Math Processing Error]*. From inspection of Figure *[Math Processing Error]*, we have:

[Math Processing Error]

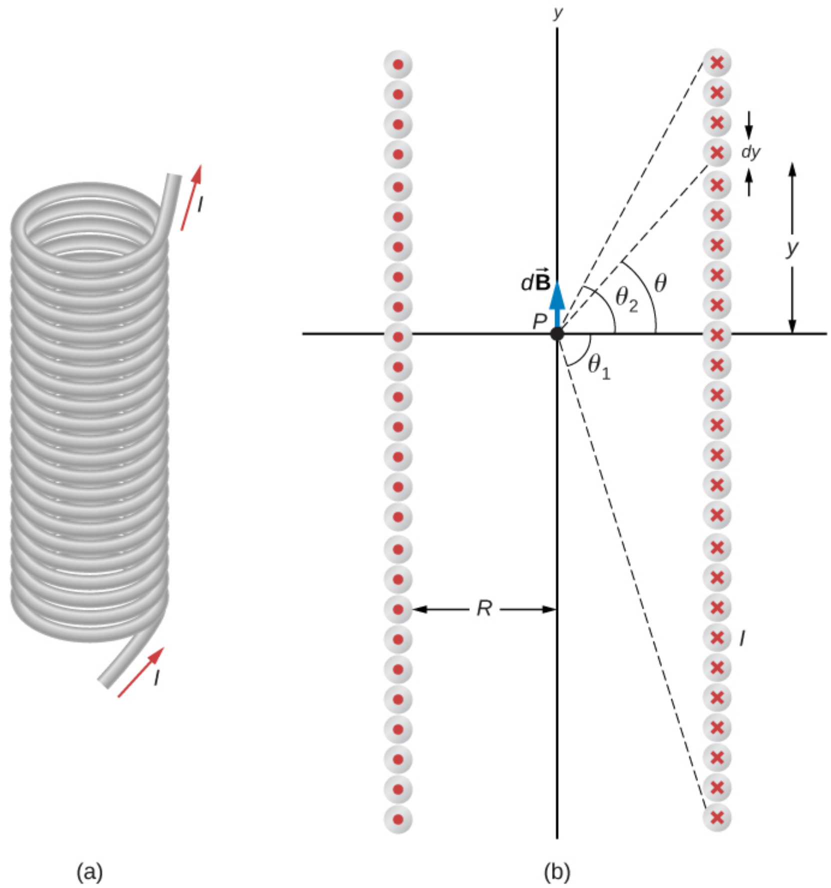


Figure [Math Processing Error]: (a) A solenoid is a long wire wound in the shape of a helix. (b) The magnetic field at the point **P** on the axis of the solenoid is the net field due to all of the current loops.

Taking the differential of both sides of this equation, we obtain

[Math Processing Error]

[Math Processing Error]

When this is substituted into the equation for [Math Processing Error], we have

[Math Processing Error]

which is the magnetic field along the central axis of a finite solenoid.

Of special interest is the infinitely long solenoid, for which [Math Processing Error]. From a practical point of view, the infinite solenoid is one whose length is much larger than its radius [Math Processing Error]. In this case, [Math Processing Error] and [Math Processing Error]. Then from Equation [Math Processing Error], the magnetic field along the central axis of an infinite solenoid is

[Math Processing Error] or

[Math Processing Error]

where **n** is the number of turns per unit length. You can find the direction of [Math Processing Error] with a right-hand rule: Curl your fingers in the direction of the current, and your thumb points along the magnetic field in the interior of the solenoid.

We now use these properties, along with Ampère's law, to calculate the magnitude of the magnetic field at any location inside the infinite solenoid. Consider the closed path of Figure [Math Processing Error]. Along segment 1, [Math Processing Error] is uniform and parallel to the path. Along segments 2 and 4, [Math Processing Error] is perpendicular to part of the path and vanishes over the rest of it. Therefore, segments 2 and 4 do not contribute to the line integral in Ampère's law. Along segment 3, [Math Processing Error] because the magnetic field is zero outside the solenoid. If you consider an Ampère's law loop outside of the solenoid, the current flows in opposite directions on different segments of wire. Therefore, there is no enclosed current and no magnetic field according to Ampère’s law. Thus, there is no contribution to the line integral from segment 3. As a result, we find

[Math Processing Error]

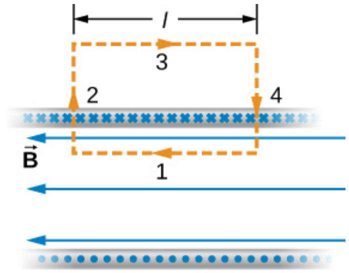


Figure *[Math Processing Error]*: The path of integration used in Ampère’s law to evaluate the magnetic field of an infinite solenoid.

The solenoid has **n** turns per unit length, so the current that passes through the surface enclosed by the path is **nlI**. Therefore, from Ampère’s law,

[Math Processing Error] and

Note

[Math Processing Error]

within the solenoid. This agrees with what we found earlier for **B** on the central axis of the solenoid. Here, however, the location of segment 1 is arbitrary, so we have found that this equation gives the magnetic field everywhere inside the infinite solenoid.

Outside the solenoid, one can draw an Ampère’s law loop around the entire solenoid. This would enclose current flowing in both directions. Therefore, the net current inside the loop is zero. According to Ampère’s law, if the net current is zero, the magnetic field must be zero. Therefore, for locations outside of the solenoid’s radius, the magnetic field is zero.

When a patient undergoes a **magnetic resonance imaging** (MRI) scan, the person lies down on a table that is moved into the center of a large solenoid that can generate very large magnetic fields. The solenoid is capable of these high fields from high currents flowing through superconducting wires. The large magnetic field is used to change the spin of protons in the patient’s body. The time it takes for the spins to align or relax (return to original orientation) is a signature of different tissues that can be analyzed to see if the structures of the tissues is normal (Figure *[Math Processing Error]*).

Figure *[Math Processing Error]*: **.** In an MRI machine, a large magnetic field is generated by the cylindrical solenoid surrounding the patient. (credit: Liz West)

Example [Math Processing Error]: Magnetic Field Inside a SOlenoid

A solenoid has 300 turns wound around a cylinder of diameter 1.20 cm and length 14.0 cm. If the current through the coils is 0.410 A, what is the magnitude of the magnetic field inside and near the middle of the solenoid?

Strategy

We are given the number of turns and the length of the solenoid so we can find the number of turns per unit length. Therefore, the magnetic field inside and near the middle of the solenoid is given by Equation [Math Processing Error]. Outside the solenoid, the magnetic field is zero.

Solution

The number of turns per unit length is

[Math Processing Error]

The magnetic field produced inside the solenoid is

[Math Processing Error]

[Math Processing Error]

Significance

This solution is valid only if the length of the solenoid is reasonably large compared with its diameter. This example is a case where this is valid.

Exercise [Math Processing Error]

What is the ratio of the magnetic field produced from using a finite formula over the infinite approximation for an angle [Math Processing Error] of (a) [Math Processing Error]? (b) [Math Processing Error]? The solenoid has 1000 turns in 50 cm with a current of 1.0 A flowing through the coils

Solution

a. 1.00382; b. 1.00015

Toroids

A toroid is a donut-shaped coil closely wound with one continuous wire, as illustrated in part (a) of Figure [Math Processing Error]. If the toroid has **N** windings and the current in the wire is **I**, what is the magnetic field both inside and outside the toroid?

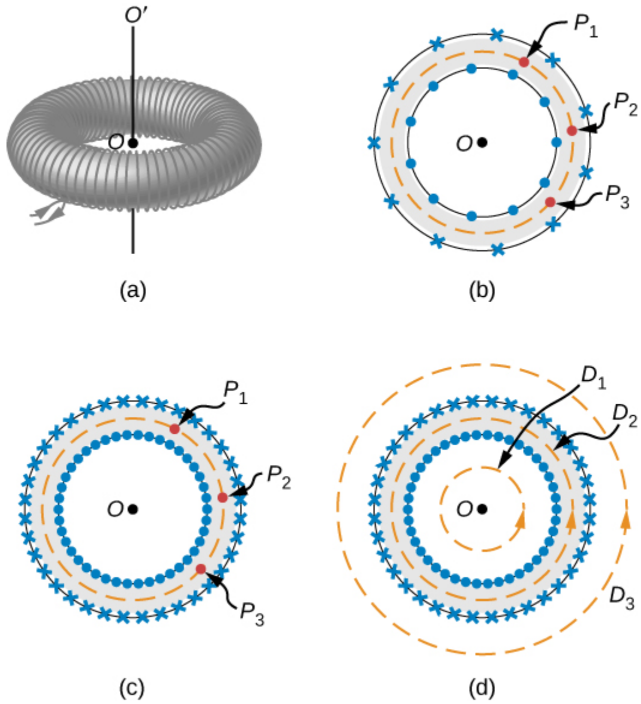


Figure *[Math Processing Error]*: (a) A toroid is a coil wound into a donut-shaped object. (b) A loosely wound toroid does not have cylindrical symmetry. (c) In a tightly wound toroid, cylindrical symmetry is a very good approximation. (d) Several paths of integration for Ampère's law.

We begin by assuming cylindrical symmetry around the axis **OO**'. Actually, this assumption is not precisely correct, for as part (b) of Figure *[Math Processing Error]* shows, the view of the toroidal coil varies from point to point (for example, *[Math Processing Error]* and *[Math Processing Error]*) on a circular path centered around **OO**'. However, if the toroid is tightly wound, all points on the circle become essentially equivalent [part (c) of Figure *[Math Processing Error]*], and cylindrical symmetry is an accurate approximation.

With this symmetry, the magnetic field must be tangent to and constant in magnitude along any circular path centered on **OO**'. This allows us to write for each of the paths *[Math Processing Error]* and *[Math Processing Error]* shown in part (d) of Figure *[Math Processing Error]*,

[Math Processing Error]

Ampère's law relates this integral to the net current passing through any surface bounded by the path of integration. For a path that is external to the toroid, either no current passes through the enclosing surface (path *[Math Processing Error]*), or the current passing through the surface in one direction is exactly balanced by the current passing through it in the opposite direction (path *[Math Processing Error]*). In either case, there is no net current passing through the surface, so

[Math Processing Error] and

[Math Processing Error]

The turns of a toroid form a helix, rather than circular loops. As a result, there is a small field external to the coil; however, the derivation above holds if the coils were circular.

For a circular path within the toroid (path *[Math Processing Error]*), the current in the wire cuts the surface **N** times, resulting in a net current **NI** through the surface. We now find with Ampère's law,

[Math Processing Error] and

Note

[Math Processing Error]

The magnetic field is directed in the counterclockwise direction for the windings shown. When the current in the coils is reversed, the direction of the magnetic field also reverses.

The magnetic field inside a toroid is not uniform, as it varies inversely with the distance **r** from the axis **OO'**. However, if the central radius **R** (the radius midway between the inner and outer radii of the toroid) is much larger than the cross-sectional diameter of the coils **r**, the variation is fairly small, and the magnitude of the magnetic field may be calculated by Equation *[Math Processing Error]* where *[Math Processing Error]*.

8.6: Magnetism in Matter

> **Learning Objectives**
>
> By the end of this section, you will be able to:
>
> - Classify magnetic materials as paramagnetic, diamagnetic, or ferromagnetic, based on their response to a magnetic field
> - Sketch how magnetic dipoles align with the magnetic field in each type of substance
> - Define hysteresis and magnetic susceptibility, which determines the type of magnetic material

Why are certain materials magnetic and others not? And why do certain substances become magnetized by a field, whereas others are unaffected? To answer such questions, we need an understanding of magnetism on a microscopic level.

Within an atom, every electron travels in an orbit and spins on an internal axis. Both types of motion produce current loops and therefore magnetic dipoles. For a particular atom, the net magnetic dipole moment is the vector sum of the magnetic dipole moments. Values of μ for several types of atoms are given in Table $8.6.1$. Notice that some atoms have a zero net dipole moment and that the magnitudes of the nonvanishing moments are typically $10^{23}\ A\cdot m^2$.

Table $8.6.1$: Magnetic Moments of Some Atoms

Atom	Magnetic Moment $(10^{-24}\ A\cdot m^2)$
H	9.27
He	0
Li	9.27
O	13.9
Na	9.27
S	13.9

A handful of matter has approximately 10^{26} atoms and ions, each with its magnetic dipole moment. If no external magnetic field is present, the magnetic dipoles are randomly oriented—as many are pointed up as down, as many are pointed east as west, and so on. Consequently, the net magnetic dipole moment of the sample is zero. However, if the sample is placed in a magnetic field, these dipoles tend to align with the field, and this alignment determines how the sample responds to the field. On the basis of this response, a material is said to be either paramagnetic, ferromagnetic, or diamagnetic.

In a **paramagnetic material**, only a small fraction (roughly one-third) of the magnetic dipoles are aligned with the applied field. Since each dipole produces its own magnetic field, this alignment contributes an extra magnetic field, which enhances the applied field. When a **ferromagnetic material** is placed in a magnetic field, its magnetic dipoles also become aligned; furthermore, they become locked together so that a permanent magnetization results, even when the field is turned off or reversed. This permanent magnetization happens in ferromagnetic materials but not paramagnetic materials. **Diamagnetic materials** are composed of atoms that have no net magnetic dipole moment. However, when a diamagnetic material is placed in a magnetic field, a magnetic dipole moment is directed opposite to the applied field and therefore produces a magnetic field that opposes the applied field. We now consider each type of material in greater detail.

Paramagnetic Materials

For simplicity, we assume our sample is a long, cylindrical piece that completely fills the interior of a long, tightly wound solenoid. When there is no current in the solenoid, the magnetic dipoles in the sample are randomly oriented and produce no net magnetic field. With a solenoid current, the magnetic field due to the solenoid exerts a torque on the dipoles that tends to align them with the field. In competition with the aligning torque are thermal collisions that tend to randomize the orientations of the dipoles. The relative importance of these two competing processes can be estimated by comparing the energies involved. The energy difference between a magnetic dipole aligned with and against a magnetic field is $U_B = 2\mu B$. If $\mu = 9.3\times 10^{-24} A\cdot m^2$ (the value of atomic hydrogen) and **B** = 1.0 T, then

$$U_B = 1.9 \times 10^{-23} J.$$

At a room temperature of $27^o C$ the thermal energy per atom is

$$U_T \approx kT = (1.38 \times 10^{-23} J/K)(300\,K) = 4.1 \times 10^{-21} J,$$

which is about 220 times greater than U_B. Clearly, energy exchanges in thermal collisions can seriously interfere with the alignment of the magnetic dipoles. As a result, only a small fraction of the dipoles is aligned at any instant.

The four sketches of Figure $8.6.1$ furnish a simple model of this alignment process. In part (a), before the field of the solenoid (not shown) containing the paramagnetic sample is applied, the magnetic dipoles are randomly oriented and there is no net magnetic dipole moment associated with the material. With the introduction of the field, a partial alignment of the dipoles takes place, as depicted in part (b). The component of the net magnetic dipole moment that is perpendicular to the field vanishes. We may then represent the sample by part (c), which shows a collection of magnetic dipoles completely aligned with the field. By treating these dipoles as current loops, we can picture the dipole alignment as equivalent to a current around the surface of the material, as in part (d). This fictitious surface current produces its own magnetic field, which enhances the field of the solenoid.

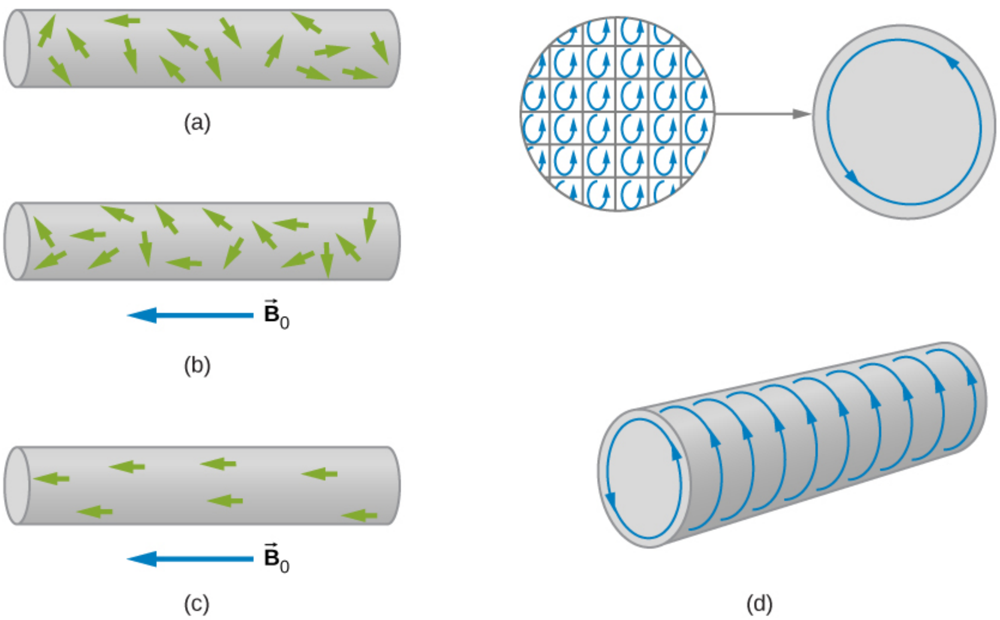


Figure $8.6.1$: The alignment process in a paramagnetic material filling a solenoid (not shown). (a) Without an applied field, the magnetic dipoles are randomly oriented. (b) With a field, partial alignment occurs. (c) An equivalent representation of part (b). (d) The internal currents cancel, leaving an effective surface current that produces a magnetic field similar to that of a finite solenoid.

We can express the total magnetic field $\vec{B}$ in the material as

$$\vec{B} = \vec{B}_0 + \vec{B}_m, \tag{8.6.1}$$

where $\vec{B}_0$ is the field due to the current I_0 in the solenoid and $\vec{B}_m$ is the field due to the surface current I_m around the sample. Now $\vec{B}_m$ is usually proportional to $\vec{B}_0$ a fact we express by

$$\vec{B}_m = \chi \vec{B}_0, \tag{8.6.2}$$

where χ is a dimensionless quantity called the magnetic susceptibility. Values of χ for some paramagnetic materials are given in Table $8.6.2$. Since the alignment of magnetic dipoles is so weak, χ is very small for paramagnetic materials. By combining Equation $8.6.1$ and Equation $8.6.2$, we obtain:

$$\vec{B} = \vec{B}_0 + \chi \vec{B}_0 = (1+\chi)\vec{B}_0. \tag{8.6.3}$$

For a sample within an infinite solenoid, this becomes

$$B = (1+\chi)\mu_0 n I. \tag{8.6.4}$$

This expression tells us that the insertion of a paramagnetic material into a solenoid increases the field by a factor of $(1+\chi)$. However, since χ is so small, the field isn't enhanced very much.

The quantity

$$\mu=(1+\chi)\mu_0. \tag{8.6.5}$$

is called the **magnetic permeability of a material**. In terms of μ, Equation 8.6.4 can be written as

$$B=\mu nI \tag{8.6.6}$$

for the filled solenoid.

Table 8.6.2: Magnetic Susceptibilities*Note: Unless otherwise specified, values given are for room temperature.

Paramagnetic Materials	χ	Diamagnetic Materials	χ
Aluminum	2.2×10^{-5}	Bismuth	-1.7×10^{-5}
Calcium	1.4×10^{-5}	Carbon (diamond)	-2.2×10^{-5}
Chromium	3.1×10^{-4}	Copper	-9.7×10^{-6}
Magnesium	1.2×10^{-5}	Lead	-1.8×10^{-5}
Oxygen gas (1 atm)	1.8×10^{-6}	Mercury	-2.8×10^{-5}
Oxygen liquid (90 K)	3.5×10^{-3}	Hydrogen gas (1 atm)	-2.2×10^{-9}
Tungsten	6.8×10^{-5}	Nitrogen gas (1 atm)	-6.7×10^{-9}
Air (1 atm)	3.6×10^{-7}	Water	-9.1×10^{-6}

Diamagnetic Materials

A magnetic field always induces a magnetic dipole in an atom. This induced dipole points opposite to the applied field, so its magnetic field is also directed opposite to the applied field. In paramagnetic and ferromagnetic materials, the induced magnetic dipole is masked by much stronger permanent magnetic dipoles of the atoms. However, in diamagnetic materials, whose atoms have no permanent magnetic dipole moments, the effect of the induced dipole is observable.

We can now describe the magnetic effects of diamagnetic materials with the same model developed for paramagnetic materials. In this case, however, the fictitious surface current flows opposite to the solenoid current, and the magnetic susceptibility χ is negative. Values of χ for some diamagnetic materials are also given in Table 8.6.2.

> Water is a common diamagnetic material. Animals are mostly composed of water. Experiments have been performed on frogs and mice in diverging magnetic fields. The water molecules are repelled from the applied magnetic field against gravity until the animal reaches an equilibrium. The result is that the animal is levitated by the magnetic field.

Ferromagnetic Materials

Common magnets are made of a ferromagnetic material such as iron or one of its alloys. Experiments reveal that a ferromagnetic material consists of tiny regions known as **magnetic domains**. Their volumes typically range from 10^{-12} to $10^{-8} m^3$, and they contain about 10^{17} to 10^{21} atoms. Within a domain, the magnetic dipoles are rigidly aligned in the same direction by coupling among the atoms. This coupling, which is due to quantum mechanical effects, is so strong that even thermal agitation at room temperature cannot break it. The result is that each domain has a net dipole moment. Some materials have weaker coupling and are ferromagnetic only at lower temperatures.

If the domains in a ferromagnetic sample are randomly oriented, as shown in Figure 8.6.1a, the sample has no net magnetic dipole moment and is said to be unmagnetized. Suppose that we fill the volume of a solenoid with an unmagnetized ferromagnetic sample. When the magnetic field $\vec{B}_0$ of the solenoid is turned on, the dipole moments of the domains rotate so that they align somewhat with the field, as depicted in Figure 8.6.1b. In addition, the aligned domains tend to increase in size at the expense of unaligned ones. The net effect of these two processes is the creation of a net magnetic dipole moment for the ferromagnet that is directed along the applied magnetic field. This net magnetic dipole moment is much larger than that of a paramagnetic sample, and the domains, with their large numbers of atoms, do not become misaligned by thermal agitation. Consequently, the field due to the alignment of the domains is quite large.

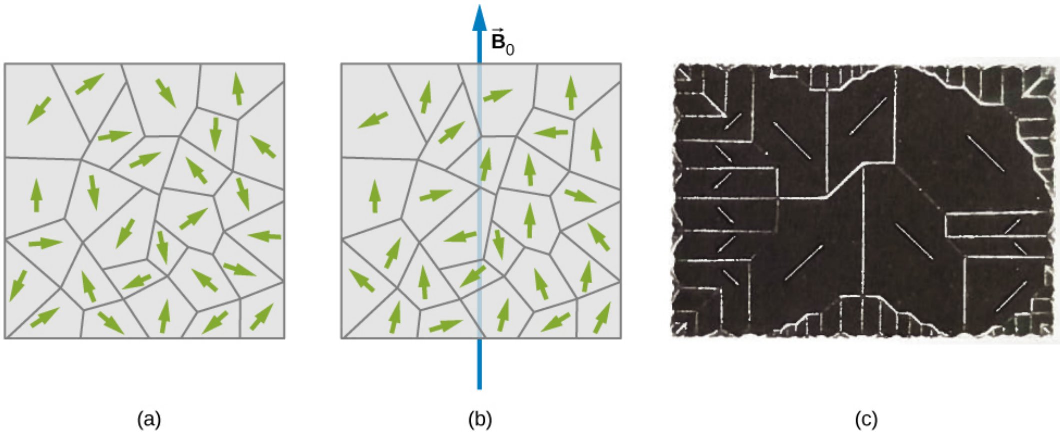


Figure 8.6.2: (a) Domains are randomly oriented in an unmagnetized ferromagnetic sample such as iron. The arrows represent the orientations of the magnetic dipoles within the domains. (b) In an applied magnetic field, the domains align somewhat with the field. (c) The domains of a single crystal of nickel. The white lines show the boundaries of the domains. These lines are produced by iron oxide powder sprinkled on the crystal.

Besides iron, only four elements contain the magnetic domains needed to exhibit ferromagnetic behavior: cobalt, nickel, gadolinium, and dysprosium. Many alloys of these elements are also ferromagnetic. Ferromagnetic materials can be described using Equation 8.6.3 through Equation 8.6.6, the paramagnetic equations. However, the value of χ for ferromagnetic material is usually on the order of 10^3 to 10^4, and it also depends on the history of the magnetic field to which the material has been subject. A typical plot of **B** (the total field in the material) versus B_0 (the applied field) for an initially unmagnetized piece of iron is shown in Figure 8.6.2c. Some sample numbers are (1) for $B_0 = 1.0 \times 10^{-4} T$, $B = 0.60\,T$, and $\chi = \left({}^{0.60}/_{1.0\times10^{-4}}\right) - 1 \approx 6.0 \times 10^3$; for (2) for $B_0 = 6.0 \times 10^{-4} T$, $B = 1.5\,T$, and $\chi = \left({}^{1.5}/_{6.0\times10^{-4}}\right) - 1 \approx 2.5 \times 10^3$.

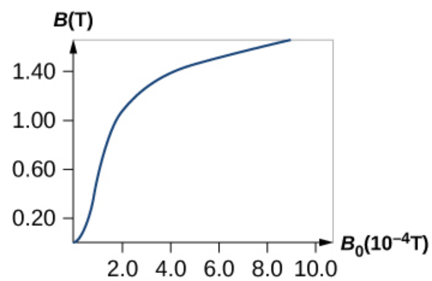


Figure 8.6.3: (a) The magnetic field B in annealed iron as a function of the applied field B_0.

When B_0 is varied over a range of positive and negative values, **B** is found to behave as shown in Figure 8.6.3. Note that the same B_0 (corresponding to the same current in the solenoid) can produce different values of **B** in the material. The magnetic field **B** produced in a ferromagnetic material by an applied field B_0 depends on the magnetic history of the material. This effect is called **hysteresis**, and the curve of Figure 8.6.4 is called a hysteresis loop. Notice that **B** does not disappear when $B_0 = 0$ (i.e., when the current in the solenoid is turned off). The iron stays magnetized, which means that it has become a permanent magnet.

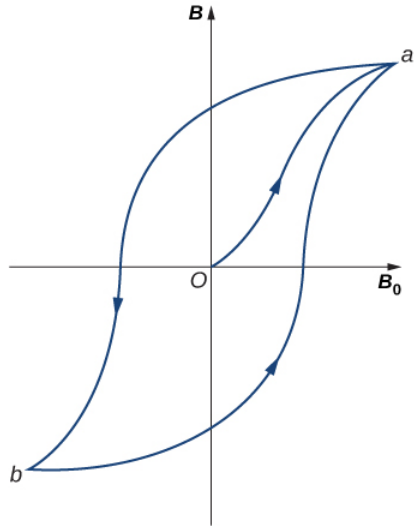


Figure 8.6.4: A typical hysteresis loop for a ferromagnet. When the material is first magnetized, it follows a curve from 0 to **a**. When B_0 is reversed, it takes the path shown from **a** to **b**. If B_0 is reversed again, the material follows the curve from **b** to **a**.

Like the paramagnetic sample of Figure 8.6.2, the partial alignment of the domains in a ferromagnet is equivalent to a current flowing around the surface. A bar magnet can therefore be pictured as a tightly wound solenoid with a large current circulating through its coils (the surface current). You can see in Figure 8.6.5 that this model fits quite well. The fields of the bar magnet and the finite solenoid are strikingly similar. The figure also shows how the poles of the bar magnet are identified. To form closed loops, the field lines outside the magnet leave the north (N) pole and enter the south (S) pole, whereas inside the magnet, they leave S and enter N.

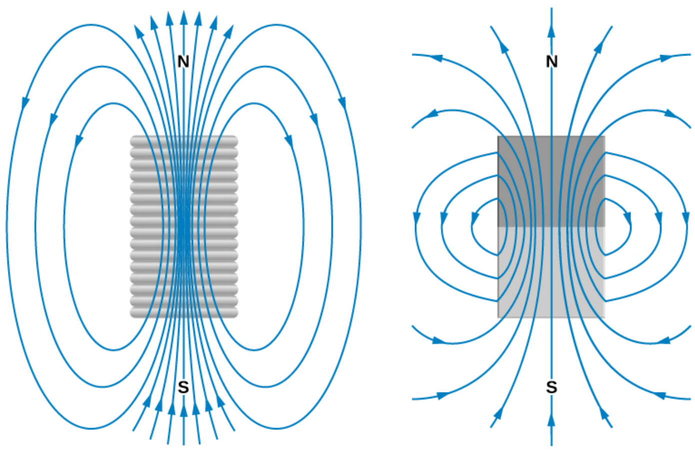


Figure 8.6.5: Comparison of the magnetic fields of a finite solenoid and a bar magnet.

Ferromagnetic materials are found in computer hard disk drives and permanent data storage devices (Figure 8.6.6). A material used in your hard disk drives is called a spin valve, which has alternating layers of ferromagnetic (aligning with the external magnetic field) and antiferromagnetic (each atom is aligned opposite to the next) metals. It was observed that a significant change in resistance was discovered based on whether an applied magnetic field was on the spin valve or not. This large change in resistance creates a quick and consistent way for recording or reading information by an applied current.

Figure $8.6.6$: The inside of a hard disk drive. The silver disk contains the information, whereas the thin stylus on top of the disk reads and writes information to the disk.

Example $8.6.1$: Iron Core in a Coil

A long coil is tightly wound around an iron cylinder whose magnetization curve is shown in Figure $8.6.3$. (a) If $n = 20$ turns per centimeter, what is the applied field B_0 when $I_0 = 0.20\ A$? (b) What is the net magnetic field for this same current? (c) What is the magnetic susceptibility in this case?

Strategy

(a) The magnetic field of a solenoid is calculated using $\vec{B} = \mu_0 n I \hat{j}$. (b) The graph is read to determine the net magnetic field for this same current. (c) The magnetic susceptibility is calculated using Equation $8.6.4$.

Solution

1. The applied field B_0 of the coil is

$$B_0 = \mu_0 n I_0 = (4\pi \times 10^{-7} T \cdot m/A)(2000/m)(0.20\ A)$$
$$B_0 = 5.0 \times 10^{-4} T.$$

2. From inspection of the magnetization curve of Figure $8.6.3$, we see that, for this value of B_0, $B = 1.4\ T$. Notice that the internal field of the aligned atoms is much larger than the externally applied field.
3. The magnetic susceptibility is calculated to be

$$\chi = \frac{B}{B_0} - 1 = \frac{1.4\ T}{5.0 \times 10^{-4} T} - 1 = 2.8 \times 10^3.$$

Significance

Ferromagnetic materials have susceptibilities in the range of 10^3 which compares well to our results here. Paramagnetic materials have fractional susceptibilities, so their applied field of the coil is much greater than the magnetic field generated by the material.

Exercise $8.6.1$

Repeat the calculations from the previous example for $I_0 = 0.040\ A$.

Answer

a. $1.0 \times 10^{-4} T$; b. 0.60 T; c. 6.0×10^3

8.6.1: Magnets

learning objectives

- Identify two types of magnets

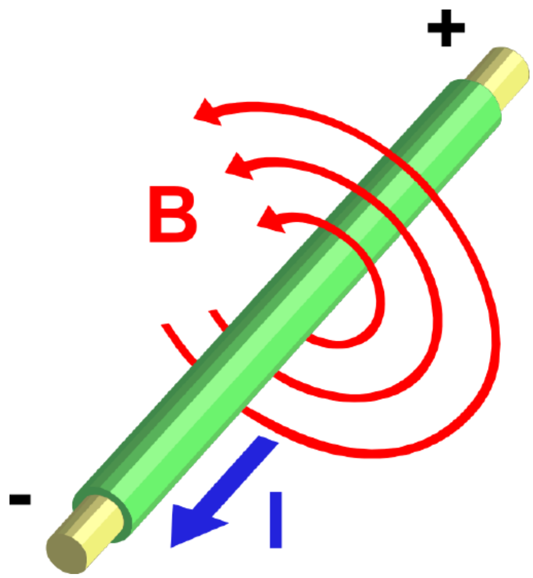
Ferromagnets and Electromagnets

In common language it is often understood that ‘magnet’ refers to a permanent magnet, like one that might adorn a family’s refrigerator, or function as the needle in a hiker’s compass. Such magnets are called ferromagnets. In the second class of magnets—known as electromagnets—the magnetic field is generated through the use of electric current. These magnets can be found in all types of electronic devices. We’ll explore these two types of magnets below.

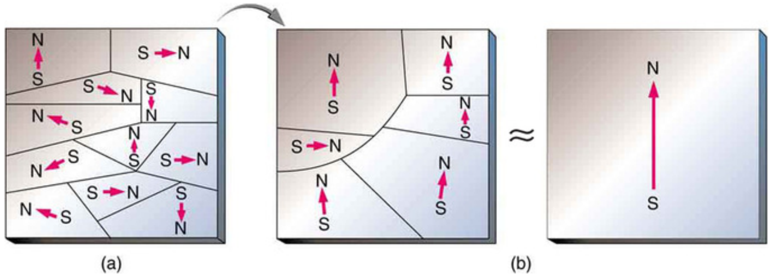
Ferromagnets

Only certain materials (e.g., iron, cobalt, nickel, and gadolinium) exhibit strong magnetic effects. These materials are called ferromagnetic, after the Latin word *ferrum* (iron). A group of materials made from the alloys of the rare earth elements are also used as strong and permanent magnets (neodymium is a common one). Other materials exhibit weak magnetic effects detectable only with sensitive instruments. Not only do ferromagnetic materials respond strongly to magnets (the way iron is attracted to magnets), they can also be magnetized themselves—that is, they can be induced to become magnetic or made into permanent magnets.

When a magnet is brought near a previously unmagnetized ferromagnetic material, it causes local magnetization of the material with unlike poles closest, as in. This results in the attraction of the previously unmagnetized material to the magnet as diagramed in. When current produces a magnetic field on a microscopic scale, as illustrated in, the regions within the material called magnetic domainsact like small bar magnets. Within domains, the poles of individual atoms are aligned, and each atom acts like a tiny bar magnet. In an unmagnetized ferromagnetic object, domains are small and randomly oriented. In response to an external magnetic field, the domains may grow to millimeter size, aligning themselves as shown in part (b) of the second figure. This induced magnetization can become permanent if the material is heated and then cooled, or simply tapped in the presence of other magnets.



Current Produces a Magnetic Field: Current (I) through a wire produces a magnetic field (B). The field is oriented according to the right-hand rule.



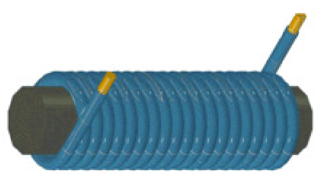
Unmagnetized to Magnetized Iron: (a) An unmagnetized piece of iron (or other ferromagnetic material) has randomly oriented domains. (b) When magnetized by an external field, the domains show greater alignment, and some grow at the expense of others. Individual atoms are aligned within domains; each atom acts like a tiny bar magnet.

Conversely, a permanent magnet can be demagnetized by hard blows or by heating it in the absence of another magnet. Increased thermal motion at higher temperature can disrupt and randomize the orientation and the size of the domains. There is a well-defined temperature for ferromagnetic materials called the Curie temperature, above which they cannot be magnetized. The Curie temperature for iron is well above room temperature at 1043 K (770ºC). Several elements and alloys have Curie temperatures much lower than room temperature, and are ferromagnetic only below those temperatures.

Electromagnets

In an electromagnet the magnetic field is produced by the flow of electric current. If the current disappears, the magnetic field is turned off. Electromagnets are widely used as components of electrical devices, such as motors, generators, relays, loudspeakers, hard disks, MRI machines, scientific instruments and magnetic separation equipment; they are also employed as industrial lifting electromagnets for picking up and moving heavy iron objects like scrap iron.

An electric current flowing in a wire creates a magnetic field around the wire. To concentrate the magnetic field, the wire is wound into a coil with many turns of wire lying side by side. The magnetic field from all the turns of wire passes through the center of the coil creating a strong magnetic field there. The coil forming the shape of a straight tube (a helix) is called a solenoid, as shown in. Much stronger magnetic fields can be produced if a "core" of ferromagnetic material (such as soft iron) is placed inside the coil. Due to the high magnetic permeability μ of the ferromagnetic material, the ferromagnetic core increases the magnetic field to thousands of times the strength of the field of the coil alone. This is called a ferromagnetic-core or iron-core electromagnet.



Electromagnet (Solenoid): A simple electromagnet consisting of a coil of insulated wire wrapped around an iron core. The strength of magnetic field generated is proportional to the amount of current.

The direction of the magnetic field through a coil of wire can be likend to a form of the right-hand rule. If the fingers of the right hand are curled around the coil in the direction of current flow (conventional current, flow of positive charge) through the windings, the thumb points in the direction of the field inside the coil. The side of the magnet from which the field lines emerge is defined as the north pole.The main advantage of an electromagnet over a permanent magnet is that the magnetic field can be rapidly manipulated over a wide range by controlling the amount of electric current; a continuous supply of electrical energy is required to maintain the field.

Key Points

- Only certain materials, such as iron, cobalt, nickel, and gadolinium, exhibit strong magnetic effects. These materials are called ferromagnetic. Ferromagnetic materials will respond strongly to magnets and can also be magnetized themselves.
- Regions of uniform called magnetic domains are randomly oriented in unmagnetized ferromagnetic material, but may become aligned under the influence of an external magnetic field. This process may become permanent if heated and cooled in the presence of a magnetic field.
- A ferromagnetic will lose its magnetism if heated about its Curie temperature.

- Electromagnets are a type of magnet in which the magnetic field is produced by the flow of current.
- A strong electromagnet called a solenoid may be produced by wrapping wires into a coil and passing a current through them. The magnetic field of all the turns of wire passes through the center of the coil, creating a strong magnetic field there.

Key Terms

- **magnetic domain**: A region within a magnetic material which has uniform magnetization. This means that the individual magnetic moments of the atoms are aligned with one another and they point in the same direction.
- **Curie temperature**: The temperature above which a material will lose its magnetism.
- **solenoid**: A coil of wire that acts as a magnet when an electric current flows through it.

8.7: Sources of Magnetic Fields (Exercise)

Conceptual Questions

12.2 The Biot-Savart Law

1. For calculating magnetic fields, what are the advantages and disadvantages of the Biot-Savart law?

2. Describe the magnetic field due to the current in two wires connected to the two terminals of a source of emf and twisted tightly around each other.

3. How can you decide if a wire is infinite?

4. Identical currents are carried in two circular loops; however, one loop has twice the diameter as the other loop. Compare the magnetic fields created by the loops at the center of each loop.

12.3 Magnetic Field Due to a Thin Straight Wire

5. How would you orient two long, straight, current-carrying wires so that there is no net magnetic force between them? (**Hint:** What orientation would lead to one wire not experiencing a magnetic field from the other?)

12.4 Magnetic Force between Two Parallel Currents

6. Compare and contrast the electric field of an infinite line of charge and the magnetic field of an infinite line of current.

7. Is $\vec{B}$ constant in magnitude for points that lie on a magnetic field line?

12.5 Magnetic Field of a Current Loop

8. Is the magnetic field of a current loop uniform?

9. What happens to the length of a suspended spring when a current passes through it?

10. Two concentric circular wires with different diameters carry currents in the same direction. Describe the force on the inner wire.

12.6 Ampère’s Law

11. Is Ampère’s law valid for all closed paths? Why isn’t it normally useful for calculating a magnetic field?

12.7 Solenoids and Toroids

12. Is the magnetic field inside a toroid completely uniform? Almost uniform?

13. Explain why $\vec{B}=0$ inside a long, hollow copper pipe that is carrying an electric current parallel to the axis. Is $\vec{B}=0$ outside the pipe?

12.8 Magnetism in Matter

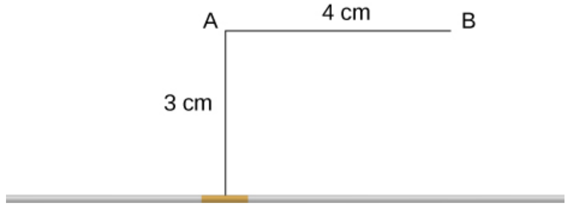
14. A diamagnetic material is brought close to a permanent magnet. What happens to the material?

15. If you cut a bar magnet into two pieces, will you end up with one magnet with an isolated north pole and another magnet with an isolated south pole? Explain your answer.

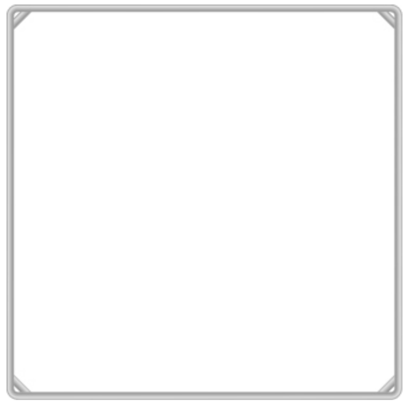
Problems

12.2 The Biot-Savart Law

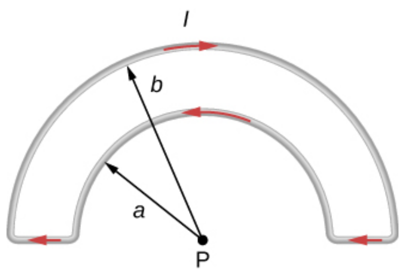
16. A 10-A current flows through the wire shown. What is the magnitude of the magnetic field due to a 0.5-mm segment of wire as measured at (a) point A and (b) point B?



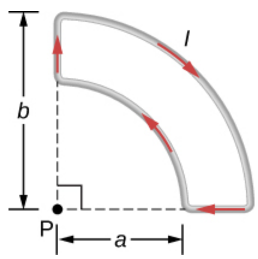
17. Ten amps flow through a square loop where each side is 20 cm in length. At each corner of the loop is a 0.01-cm segment that connects the longer wires as shown. Calculate the magnitude of the magnetic field at the center of the loop.



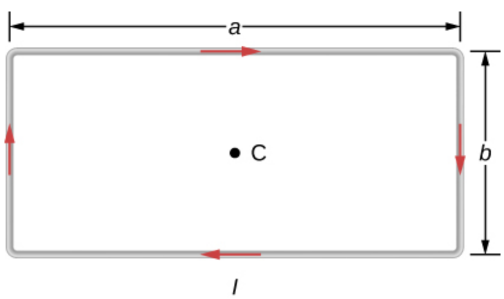
18. What is the magnetic field at P due to the current I in the wire shown?



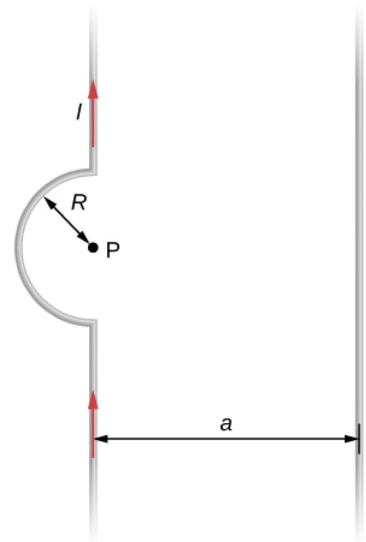
19. The accompanying figure shows a current loop consisting of two concentric circular arcs and two perpendicular radial lines. Determine the magnetic field at point P.



20. Find the magnetic field at the center C of the rectangular loop of wire shown in the accompanying figure.



21. Two long wires, one of which has a semicircular bend of radius R, are positioned as shown in the accompanying figure. If both wires carry a current **I**, how far apart must their parallel sections be so that the net magnetic field at P is zero? Does the current in the straight wire flow up or down?



12.3 Magnetic Field Due to a Thin Straight Wire

22. A typical current in a lightning bolt is 10^4 A. Estimate the magnetic field 1 m from the bolt.

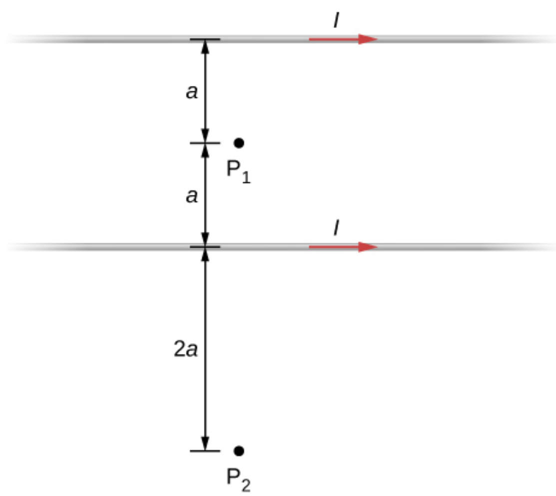
23. The magnitude of the magnetic field 50 cm from a long, thin, straight wire is **8.0μT**. What is the current through the long wire?

24. A transmission line strung 7.0 m above the ground carries a current of 500 A. What is the magnetic field on the ground directly below the wire? Compare your answer with the magnetic field of Earth.

25. A long, straight, horizontal wire carries a left-to-right current of 20 A. If the wire is placed in a uniform magnetic field of magnitude $4.0 \times 10^{-5} T$ that is directed vertically downward, what is the resultant magnitude of the magnetic field 20 cm above the wire? 20 cm below the wire?

26. The two long, parallel wires shown in the accompanying figure carry currents in the same direction. If $I_1 = 10A$ and $I_2 = 20A$, what is the magnetic field at point P?

27. The accompanying figure shows two long, straight, horizontal wires that are parallel and a distance 2a apart. If both wires carry current I in the same direction, (a) what is the magnetic field at P_1? (b) P_2?



28. Repeat the calculations of the preceding problem with the direction of the current in the lower wire reversed.

29. Consider the area between the wires of the preceding problem. At what distance from the top wire is the net magnetic field a minimum? Assume that the currents are equal and flow in opposite directions.

12.4 Magnetic Force between Two Parallel Currents

30. Two long, straight wires are parallel and 25 cm apart.

(a) If each wire carries a current of 50 A in the same direction, what is the magnetic force per meter exerted on each wire?

(b) Does the force pull the wires together or push them apart?

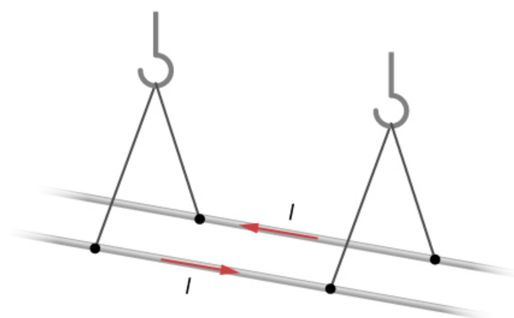
(c) What happens if the currents flow in opposite directions?

31. Two long, straight wires are parallel and 10 cm apart. One carries a current of 2.0 A, the other a current of 5.0 A.

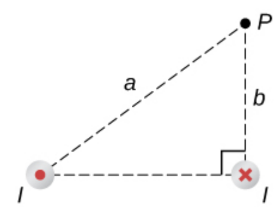
(a) If the two currents flow in opposite directions, what is the magnitude and direction of the force per unit length of one wire on the other?

(b) What is the magnitude and direction of the force per unit length if the currents flow in the same direction?

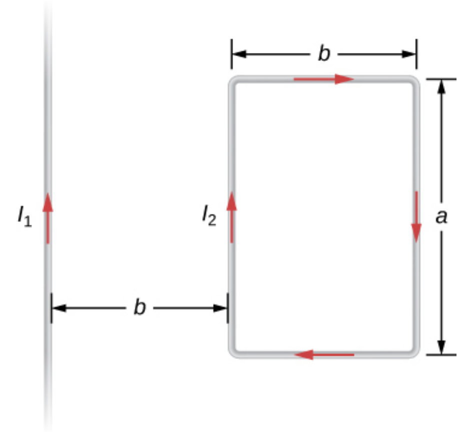
32. Two long, parallel wires are hung by cords of length 5.0 cm, as shown in the accompanying figure. Each wire has a mass per unit length of 30 g/m, and they carry the same current in opposite directions. What is the current if the cords hang at **6.0°** with respect to the vertical?



33. A circuit with current **I** has two long parallel wire sections that carry current in opposite directions. Find magnetic field at a point **P** near these wires that is a distance a from one wire and b from the other wire as shown in the figure.



34. The infinite, straight wire shown in the accompanying figure carries a current I_1. The rectangular loop, whose long sides are parallel to the wire, carries a current I_2. What are the magnitude and direction of the force on the rectangular loop due to the magnetic field of the wire?



12.5 Magnetic Field of a Current Loop

35. When the current through a circular loop is 6.0 A, the magnetic field at its center is $2.0 \times 10^{-4} T$. What is the radius of the loop?

36. How many turns must be wound on a flat, circular coil of radius 20 cm in order to produce a magnetic field of magnitude $4.0 \times 10^{-5} T$ at the center of the coil when the current through it is 0.85 A?

37. A flat, circular loop has 20 turns. The radius of the loop is 10.0 cm and the current through the wire is 0.50 A. Determine the magnitude of the magnetic field at the center of the loop.

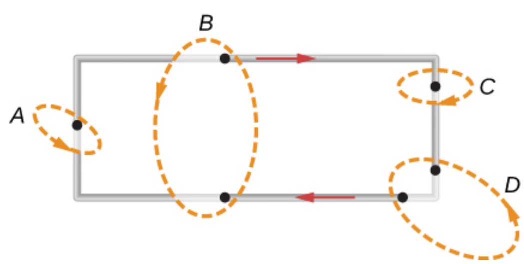
38. A circular loop of radius **R** carries a current **I**. At what distance along the axis of the loop is the magnetic field one-half its value at the center of the loop?

39. Two flat, circular coils, each with a radius **R** and wound with **N** turns, are mounted along the same axis so that they are parallel a distance **d** apart. What is the magnetic field at the midpoint of the common axis if a current **I** flows in the same direction through each coil?

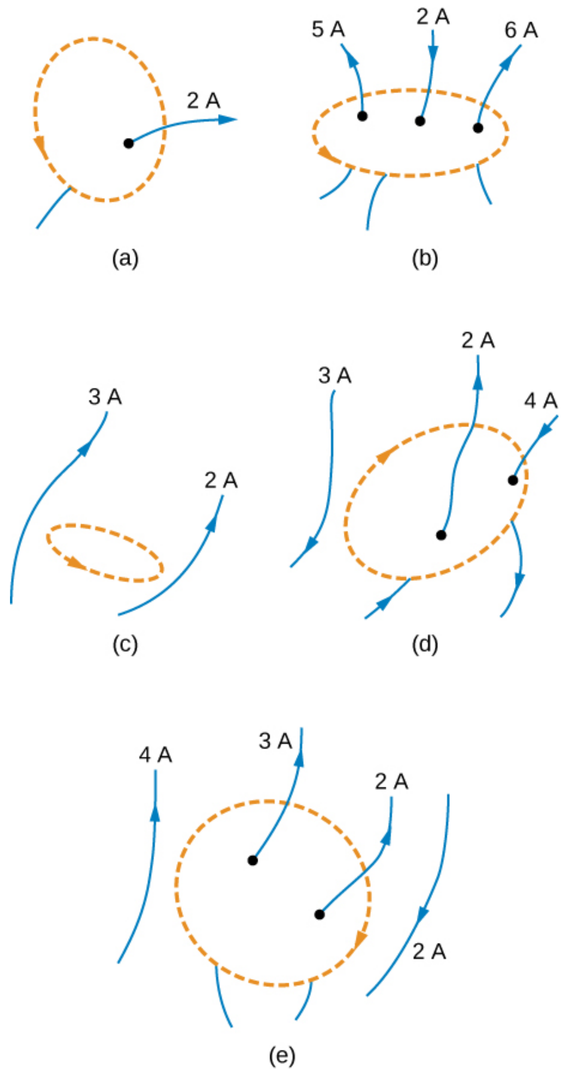
40. For the coils in the preceding problem, what is the magnetic field at the center of either coil?

12.6 Ampère's Law

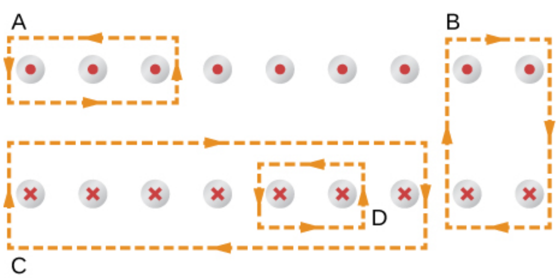
41. A current **I** flows around the rectangular loop shown in the accompanying figure. Evaluate $\oint \vec{B} \cdot \vec{dl}$ for the paths **A, B,C**, and **D.**



42. Evaluate $\oint \vec{B} \cdot \vec{dl}$ for each of the cases shown in the accompanying figure.



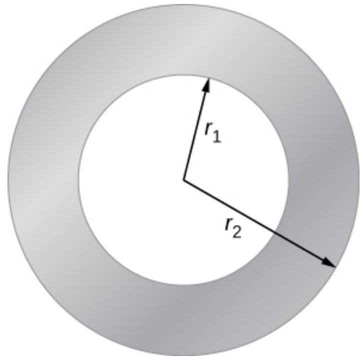
43. The coil whose lengthwise cross section is shown in the accompanying figure carries a current I and has N evenly spaced turns distributed along the length l. Evaluate $\oint \vec{B} \cdot \overrightarrow{dl}$ for the paths indicated.



44. A superconducting wire of diameter 0.25 cm carries a current of 1000 A. What is the magnetic field just outside the wire?

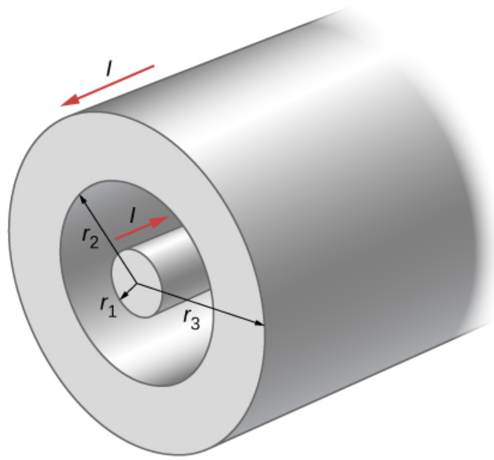
45. A long, straight wire of radius R carries a current I that is distributed uniformly over the cross-section of the wire. At what distance from the axis of the wire is the magnitude of the magnetic field a maximum?

46. The accompanying figure shows a cross-section of a long, hollow, cylindrical conductor of inner radius $r_1 = 3.0 cm$ and outer radius $r_2 = 5.0 cm$. A 50-A current distributed uniformly over the cross-section flows into the page. Calculate the magnetic field at $r = 2.0 cm$, $r = 4.0 cm$, and $r = 6.0 cm$.



47. A long, solid, cylindrical conductor of radius 3.0 cm carries a current of 50 A distributed uniformly over its cross-section. Plot the magnetic field as a function of the radial distance r from the center of the conductor.

48. A portion of a long, cylindrical **coaxial cable** is shown in the accompanying figure. A current I flows down the center conductor, and this current is returned in the outer conductor. Determine the magnetic field in the regions (a) $r \leq r_1$, (b) $r_2 \geq r \geq r_1$, (c) $r_3 \geq r \geq r_2$, and (d) $r \geq r_3$. Assume that the current is distributed uniformly over the cross sections of the two parts of the cable.



12.7 Solenoids and Toroids

49. A solenoid is wound with 2000 turns per meter. When the current is 5.2 A, what is the magnetic field within the solenoid?

50. A solenoid has 12 turns per centimeter. What current will produce a magnetic field of $2.0 \times 10^{-2} T$ within the solenoid?

51. If a current is 2.0 A, how many turns per centimeter must be wound on a solenoid in order to produce a magnetic field of $2.0 \times 10^{-3} T$ within it?

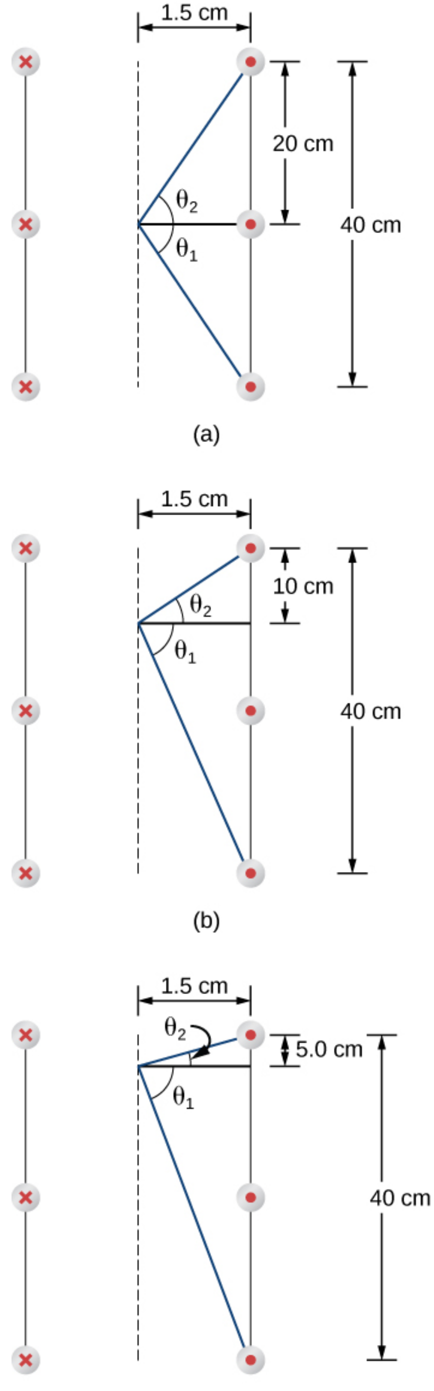
52. A solenoid is 40 cm long, has a diameter of 3.0 cm, and is wound with 500 turns. If the current through the windings is 4.0 A, what is the magnetic field at a point on the axis of the solenoid that is

(a) at the center of the solenoid,

(b) 10.0 cm from one end of the solenoid, and

(c) 5.0 cm from one end of the solenoid?

(d) Compare these answers with the infinite-solenoid case.



53. Determine the magnetic field on the central axis at the opening of a semi-infinite solenoid. (That is, take the opening to be at **x=0** and the other end to be at $x = \infty$)

54. By how much is the approximation $B = \mu_0 nI$ in error at the center of a solenoid that is 15.0 cm long, has a diameter of 4.0 cm, is wrapped with **n** turns per meter, and carries a current **I**?

55. A solenoid with 25 turns per centimeter carries a current I. An electron moves within the solenoid in a circle that has a radius of 2.0 cm and is perpendicular to the axis of the solenoid. If the speed of the electron is $2.0 \times 10^5 m/s$, what is I?

56. A toroid has 250 turns of wire and carries a current of 20 A. Its inner and outer radii are 8.0 and 9.0 cm. What are the values of its magnetic field at **r=8.1, 8.5,** and **8.9cm**?

57. A toroid with a square cross section 3.0 cm × 3.0 cm has an inner radius of 25.0 cm. It is wound with 500 turns of wire, and it carries a current of 2.0 A. What is the strength of the magnetic field at the center of the square cross section?

12.8 Magnetism in Matter

58. The magnetic field in the core of an air-filled solenoid is 1.50 T. By how much will this magnetic field decrease if the air is pumped out of the core while the current is held constant?

59. A solenoid has a ferromagnetic core, **n = 1000** turns per meter, and **I = 5.0 A**. If B inside the solenoid is 2.0 T, what is **χ** for the core material?

60. A 20-A current flows through a solenoid with 2000 turns per meter. What is the magnetic field inside the solenoid if its core is (a) a vacuum and (b) filled with liquid oxygen at 90 K?

61. The magnetic dipole moment of the iron atom is about $2.1 \times 10^{-23} A \cdot m^2$.

(a) Calculate the maximum magnetic dipole moment of a domain consisting of 10^{19} iron atoms.

(b) What current would have to flow through a single circular loop of wire of diameter 1.0 cm to produce this magnetic dipole moment?

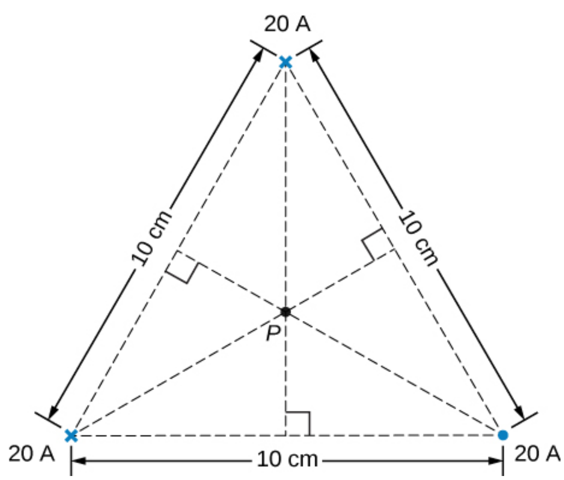
62. Suppose you wish to produce a 1.2-T magnetic field in a toroid with an iron core for which $\chi = 4.0 \times 10^3$. The toroid has a mean radius of 15 cm and is wound with 500 turns. What current is required?

63. A current of 1.5 A flows through the windings of a large, thin toroid with 200 turns per meter and a radius of 1 meter. If the toroid is filled with iron for which $\chi = 3.0 \times 10^3$, what is the magnetic field within it?

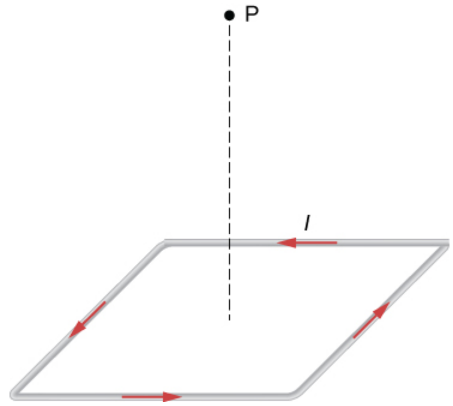
64. A solenoid with an iron core is 25 cm long and is wrapped with 100 turns of wire. When the current through the solenoid is 10 A, the magnetic field inside it is 2.0 T. For this current, what is the permeability of the iron? If the current is turned off and then restored to 10 A, will the magnetic field necessarily return to 2.0 T?

Additional Problems

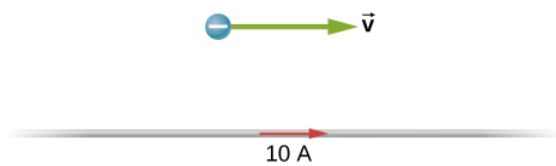
65. Three long, straight, parallel wires, all carrying 20 A, are positioned as shown in the accompanying figure. What is the magnitude of the magnetic field at the point **P**?



66. A current **I** flows around a wire bent into the shape of a square of side a. What is the magnetic field at the point P that is a distance **z** above the center of the square (see the accompanying figure)?



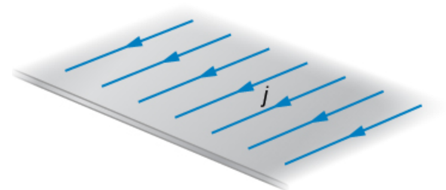
67. The accompanying figure shows a long, straight wire carrying a current of 10 A. What is the magnetic force on an electron at the instant it is 20 cm from the wire, traveling parallel to the wire with a speed of $2.0 \times 10^5 m/s$? Describe qualitatively the subsequent motion of the electron.



68. Current flows along a thin, infinite sheet as shown in the accompanying figure. The current per unit length along the sheet is J in amperes per meter.

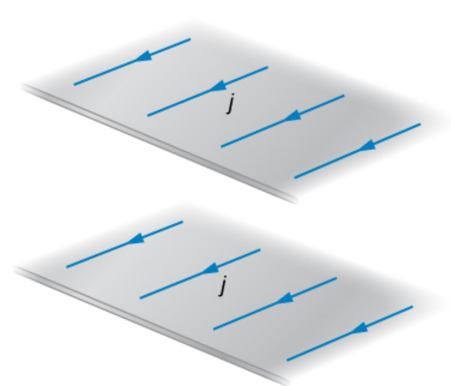
(a) Use the Biot-Savart law to show that $B = \mu_0 J/2$ on either side of the sheet. What is the direction of $\vec{B}$ on each side?

(b) Now use Ampère’s law to calculate the field.

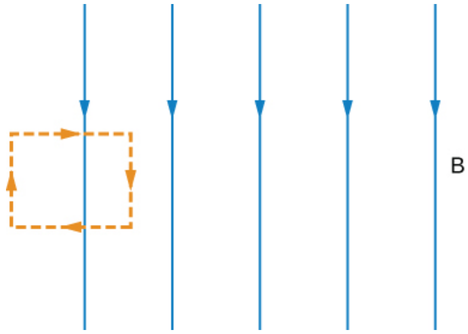


69. (a) Use the result of the previous problem to calculate the magnetic field between, above, and below the pair of infinite sheets shown in the accompanying figure.

(b) Repeat your calculations if the direction of the current in the lower sheet is reversed.



70. We often assume that the magnetic field is uniform in a region and zero everywhere else. Show that in reality it is impossible for a magnetic field to drop abruptly to zero, as illustrated in the accompanying figure. (**Hint**: Apply Ampère’s law over the path shown.)



71. How is the fractional change in the strength of the magnetic field across the face of the toroid related to the fractional change in the radial distance from the axis of the toroid?

72. Show that the expression for the magnetic field of a toroid reduces to that for the field of an infinite solenoid in the limit that the central radius goes to infinity.

73. A toroid with an inner radius of 20 cm and an outer radius of 22 cm is tightly wound with one layer of wire that has a diameter of 0.25 mm.

(a) How many turns are there on the toroid?

(b) If the current through the toroid windings is 2.0 A, what is the strength of the magnetic field at the center of the toroid?

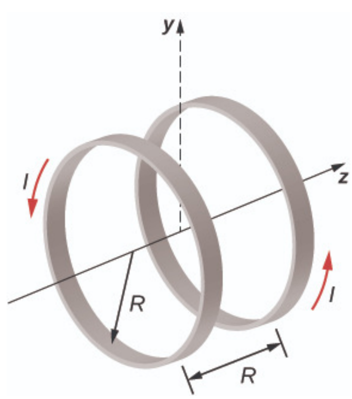
74. A wire element has $vecdl, I\vec{dl} = JAdl = Jdv$, where A and dv are the cross-sectional area and volume of the element, respectively. Use this, the Biot-Savart law, and $J = nev$ to show that the magnetic field of a moving point charge q is given by:

$$\vec{B} = \frac{\mu_0}{4\pi}\frac{qv \times \hat{r}}{r^2}.$$

75. A reasonably uniform magnetic field over a limited region of space can be produced with the Helmholtz coil, which consists of two parallel coils centered on the same axis. The coils are connected so that they carry the same current I. Each coil has N turns and radius R, which is also the distance between the coils.

(a) Find the magnetic field at any point on the z-axis shown in the accompanying figure.

(b) Show that dB/dz and d^2B_dz2 are both zero at **z = 0.** (These vanishing derivatives demonstrate that the magnetic field varies only slightly near **z = 0**.)



76. A charge of **4.0μC** is distributed uniformly around a thin ring of insulating material. The ring has a radius of 0.20 m and rotates at $2.0 \times 10^4 rev/min$ around the axis that passes through its center and is perpendicular to the plane of the ring. What is the magnetic field at the center of the ring?

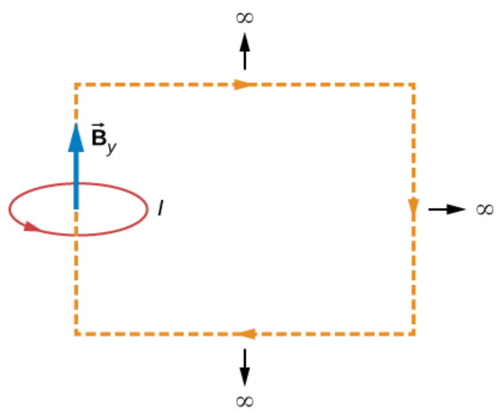
77. A thin, nonconducting disk of radius **R** is free to rotate around the axis that passes through its center and is perpendicular to the face of the disk. The disk is charged uniformly with a total charge **q**. If the disk rotates at a constant angular velocity **ω**, what is the magnetic field at its center?

78. Consider the disk in the previous problem. Calculate the magnetic field at a point on its central axis that is a distance **y** above the disk.

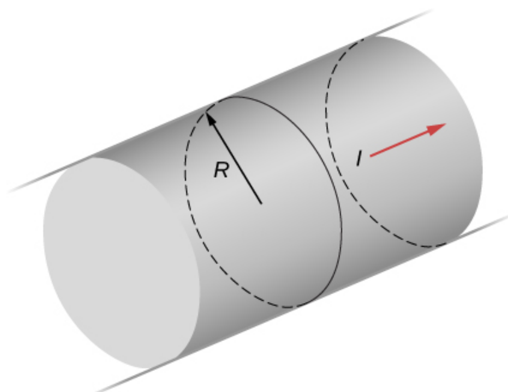
79. Consider the axial magnetic field $B_y = \mu_0 I R^2 / 2(y^2 + R^2)^{3/2}$ of the circular current loop shown below.

(a) Evaluate $\int_{-a}^{a} B_y dy$. Also so show that $\lim_{a \to \infty} \int_{-a}^{a} B_y dy = \mu_0 I$.

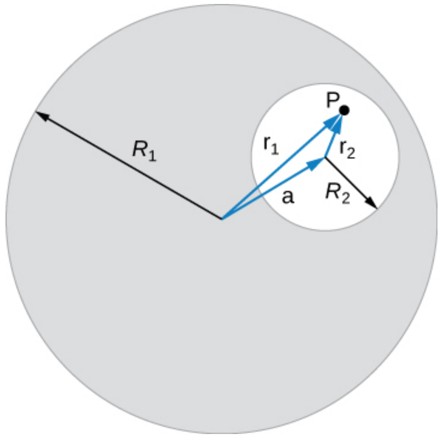
(b) Can you deduce this limit without evaluating the integral? (**Hint:**See the accompanying figure.)



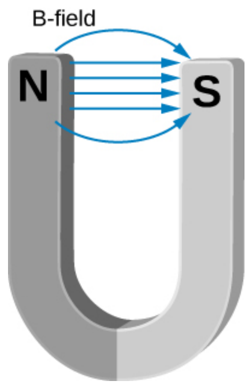
80. The current density in the long, cylindrical wire shown in the accompanying figure varies with distance r from the center of the wire according to $J = cr$, where c is a constant. (a) What is the current through the wire? (b) What is the magnetic field produced by this current for $r \leq R$? For $r \geq R$?



81. A long, straight, cylindrical conductor contains a cylindrical cavity whose axis is displaced by a from the axis of the conductor, as shown in the accompanying figure. The current density in the conductor is given by $\hat{J} = J_0 \hat{k}$, where J_0 is a constant and $\hat{k}$ is along the axis of the conductor. Calculate the magnetic field at an arbitrary point P in the cavity by superimposing the field of a solid cylindrical conductor with radius R_1 and current density $\vec{J}$ onto the field of a solid cylindrical conductor with radius R_2 and current density $-\vec{J}$. Then use the fact that the appropriate azimuthal unit vectors can be expressed as $\hat{\theta_1} = \hat{k} \times \hat{r_1}$ and $\hat{\theta_2} = \hat{k} \times \hat{r_2}$ to show that everywhere inside the cavity the magnetic field is given by the constant $\vec{B} = \frac{1}{2}\mu_0 J_0 k \times a$, where $a = r_1 - r_2$ and $r_1 = r_1 \hat{r_1}$ is the position of P relative to the center of the conductor and $2_{=} r_2 \overrightarrow{r_2}$ is the position of **P** relative to the center of the cavity.

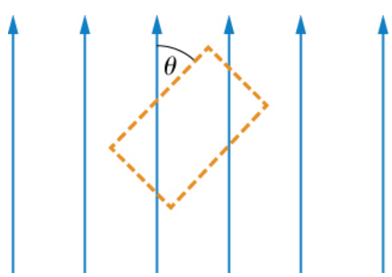


82. Between the two ends of a horseshoe magnet the field is uniform as shown in the diagram. As you move out to outside edges, the field bends. Show by Ampère's law that the field must bend and thereby the field weakens due to these bends.



83. Show that the magnetic field of a thin wire and that of a current loop are zero if you are infinitely far away.

84. An Ampère loop is chosen as shown by dashed lines for a parallel constant magnetic field as shown by solid arrows. Calculate $\vec{B} \cdot \vec{dl}$ for each side of the loop then find the entire $\oint \vec{B} \cdot \vec{dl}$. Can you think of an Ampère loop that would make the problem easier? Do those results match these?



85. A very long, thick cylindrical wire of radius R carries a current density J that varies across its cross-section. The magnitude of the current density at a point a distance **r** from the center of the wire is given by $J = J_0 \frac{r}{R}$, where J_0 is a constant. Find the magnetic field

(a) at a point outside the wire and

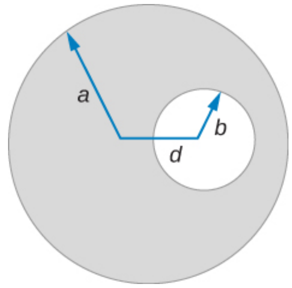
(b) at a point inside the wire. Write your answer in terms of the net current I through the wire.

86. A very long, cylindrical wire of radius a has a circular hole of radius **b** in it at a distance d from the center. The wire carries a uniform current of magnitude **I** through it. The direction of the current in the figure is out of the paper. Find the magnetic field

(a) at a point at the edge of the hole closest to the center of the thick wire,

(b) at an arbitrary point inside the hole, and

(c) at an arbitrary point outside the wire. (Hint: Think of the hole as a sum of two wires carrying current in the opposite directions.)

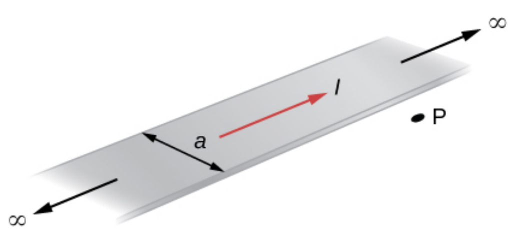


87. Magnetic field inside a torus. Consider a torus of rectangular cross-section with inner radius a and outer radius b. N turns of an insulated thin wire are wound evenly on the torus tightly all around the torus and connected to a battery producing a steady current I in the wire. Assume that the current on the top and bottom surfaces in the figure is radial, and the current on the inner and outer radii surfaces is vertical. Find the magnetic field inside the torus as a function of radial distance **r** from the axis.

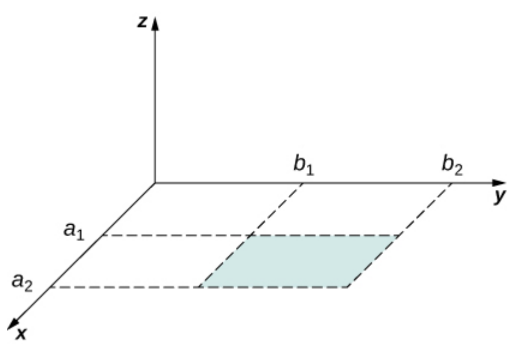
88. Two long coaxial copper tubes, each of length **L**, are connected to a battery of voltage **V**. The inner tube has inner radius **a** and outer radius **b**, and the outer tube has inner radius c and outer radius **d**. The tubes are then disconnected from the battery and rotated in the same direction at angular speed of **ω** radians per second about their common axis. Find the magnetic field (a) at a point inside the space enclosed by the inner tube **r<a**, and (b) at a point between the tubes **b<r<c**, and (c) at a point outside the tubes **r>d**. (Hint: Think of copper tubes as a capacitor and find the charge density based on the voltage applied, **Q=VC,** \(\displaystyle C=\frac{2πε_0L}{ln(c/b)}.)

Challenge Problems

89. The accompanying figure shows a flat, infinitely long sheet of width a that carries a current I uniformly distributed across it. Find the magnetic field at the point P, which is in the plane of the sheet and at a distance **x** from one edge. Test your result for the limit **a→0**.



90. A hypothetical current flowing in the z-direction creates the field $vecB=C[(x/y^2)\hat{i}+(1/y)\hat{j}]$ in the rectangular region of the **xy**-plane shown in the accompanying figure. Use Ampère's law to find the current through the rectangle.



91. A nonconducting hard rubber circular disk of radius **R** is painted with a uniform surface charge density **σ**. It is rotated about its axis with angular speed **ω**. (a) Find the magnetic field produced at a point on the axis a distance h meters from the center of the disk. (b) Find the numerical value of magnitude of the magnetic field when $\sigma = 1C/m^2$, $R = 20cm$, $h = 2cm$, and $\omega = 400rad/sec$,, and compare it with the magnitude of magnetic field of Earth, which is about 1/2 Gauss.

CHAPTER OVERVIEW

9: Electromagnetic Induction

In this and the next several chapters, you will see a wonderful symmetry in the behavior exhibited by time-varying electric and magnetic fields. Mathematically, this symmetry is expressed by an additional term in Ampère's law and by another key equation of electromagnetism called Faraday's law. We also discuss how moving a wire through a magnetic field produces an emf or voltage.

- 9.1: Prelude to Electromagnetic Induction
- 9.2: Faraday's Law
- 9.3: Lenz's Law
- 9.4: Motional Emf
- 9.5: Induced Electric Fields
- 9.6: Eddy Currents
- 9.7: Induction, Transformers, and Generators
- 9.8: Electromagnetic Induction (Exercises)
- 9.9: Mutual Inductance
- 9.10: Self-Inductance and Inductors
- 9.11: Energy in a Magnetic Field
- 9.12: RL Circuits
  - 9.12.1: Oscillations in an LC Circuit
  - 9.12.2: RLC Series Circuits
- 9.13: Inductance (Exercise)

9.1: Prelude to Electromagnetic Induction

We have been considering electric fields created by fixed charge distributions and magnetic fields produced by constant currents, but electromagnetic phenomena are not restricted to these stationary situations. Most of the interesting applications of electromagnetism are, in fact, time-dependent. To investigate some of these applications, we now remove the time-independent assumption that we have been making and allow the fields to vary with time. In this and the next several chapters, you will see a wonderful symmetry in the behavior exhibited by time-varying electric and magnetic fields. Mathematically, this symmetry is expressed by an additional term in Ampère's law and by another key equation of electromagnetism called Faraday's law. We also discuss how moving a wire through a magnetic field produces an emf or voltage. Lastly, we describe applications of these principles, such as the card reader shown above.

Figure 9.1.1: The black strip found on the back of credit cards and driver's licenses is a very thin layer of magnetic material with information stored on it. Reading and writing the information on the credit card is done with a swiping motion. The physical reason why this is necessary is called electromagnetic induction and is discussed in this chapter.

9.2: Faraday's Law

Learning Objectives

By the end of this section, you will be able to:

- Determine the magnetic flux through a surface, knowing the strength of the magnetic field, the surface area, and the angle between the normal to the surface and the magnetic field
- Use Faraday's law to determine the magnitude of induced emf in a closed loop due to changing magnetic flux through the loop

The first productive experiments concerning the effects of time-varying magnetic fields were performed by Michael **Faraday** in 1831. One of his early experiments is represented in Figure $9.2.1$. An **emf** is induced when the magnetic field in the coil is changed by pushing a bar magnet into or out of the coil. Emfs of opposite signs are produced by motion in opposite directions, and the directions of emfs are also reversed by reversing poles. The same results are produced if the coil is moved rather than the magnet—it is the relative motion that is important. The faster the motion, the greater the emf, and there is no emf when the magnet is stationary relative to the coil.

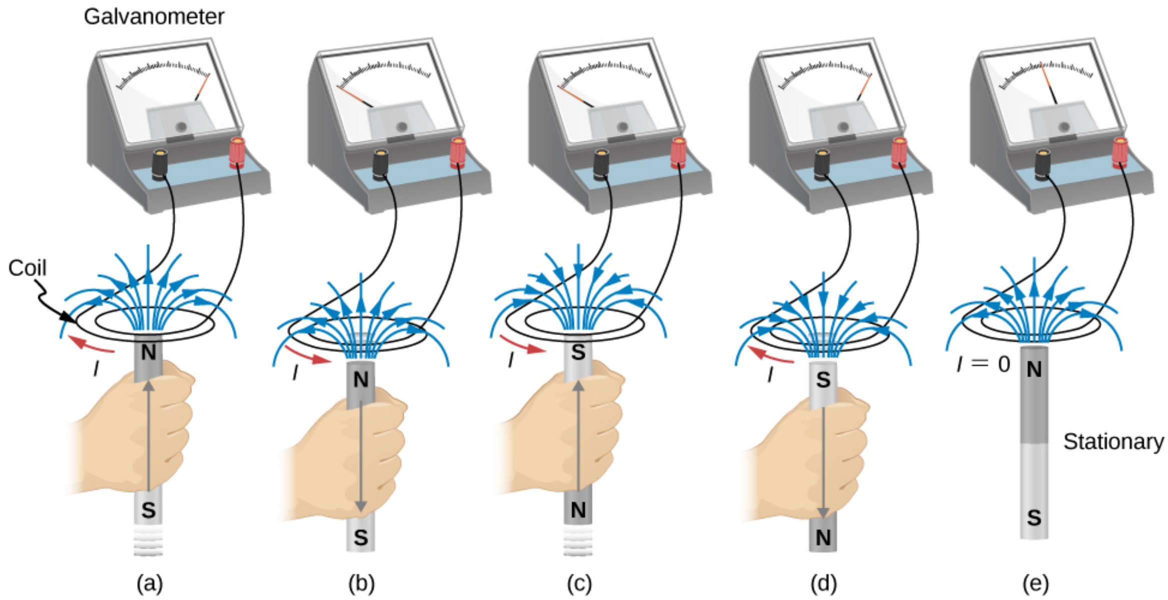


Figure $9.2.1$: Movement of a magnet relative to a coil produces emfs as shown (a–d). The same emfs are produced if the coil is moved relative to the magnet. This short-lived emf is only present during the motion. The greater the speed, the greater the magnitude of the emf, and the emf is zero when there is no motion, as shown in (e).

Faraday also discovered that a similar effect can be produced using two circuits—a changing current in one circuit induces a current in a second, nearby circuit. For example, when the switch is closed in circuit 1 of Figure $9.2.1a$, the ammeter needle of circuit 2 momentarily deflects, indicating that a short-lived current surge has been induced in that circuit. The ammeter needle quickly returns to its original position, where it remains. However, if the switch of circuit 1 is now suddenly opened, another short-lived current surge in the direction opposite from before is observed in circuit 2.

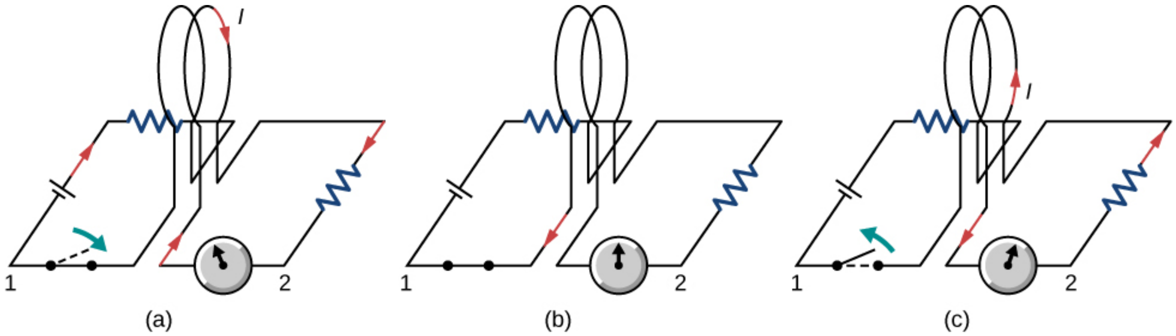


Figure 9.2.2: (a) Closing the switch of circuit 1 produces a short-lived current surge in circuit 2. (b) If the switch remains closed, no current is observed in circuit 2. (c) Opening the switch again produces a short-lived current in circuit 2 but in the opposite direction from before.

Faraday realized that in both experiments, a current flowed in the circuit containing the ammeter only when the magnetic field in the region occupied by that circuit was **changing**. As the magnet of the figure was moved, the strength of its magnetic field at the loop changed; and when the current in circuit 1 was turned on or off, the strength of its magnetic field at circuit 2 changed. Faraday was eventually able to interpret these and all other experiments involving magnetic fields that vary with time in terms of the following law.

Faraday's Law

The emf ϵ induced is the negative change in the magnetic flux Φ_m per unit time. Any change in the magnetic field or change in orientation of the area of the coil with respect to the magnetic field induces a voltage (emf).

The magnetic flux is a measurement of the amount of magnetic field lines through a given surface area, as seen in Figure 9.2.3. This definition is similar to the electric flux studied earlier. This means that if we have

$$\Phi_m = \int_S \vec{B} \cdot \hat{n} dA,$$

then the **induced emf** or the voltage generated by a conductor or coil moving in a magnetic field is

$$\epsilon = -\frac{d}{dt} \int_S \vec{B} \cdot \hat{n} dA = -\frac{d\Phi_m}{dt}.$$

The negative sign describes the direction in which the induced emf drives current around a circuit. However, that direction is most easily determined with a rule known as Lenz's law, which we will discuss shortly.

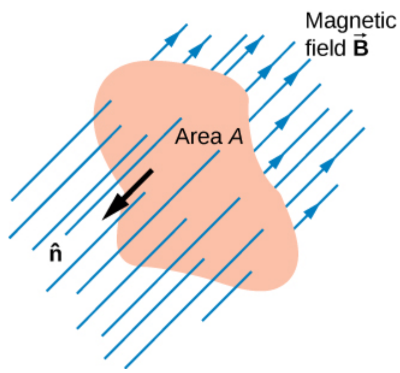


Figure 9.2.3: The magnetic flux is the amount of magnetic field lines cutting through a surface area **A** defined by the unit area vector $\hat{n}$. If the angle between the unit area $\hat{n}$ and magnetic field vector $\vec{B}$ are parallel or antiparallel, as shown in the diagram, the magnetic flux is the highest possible value given the values of area and magnetic field.

$9.2.1a$ depicts a circuit and an arbitrary surface **S** that it bounds. Notice that **S** is an **open surface**. It can be shown that **any** open surface bounded by the circuit in question can be used to evaluate Φ_m. For example, Φ_m is the same for the various surfaces $S_1, S_2, \ldots$ of part (b) of the figure.

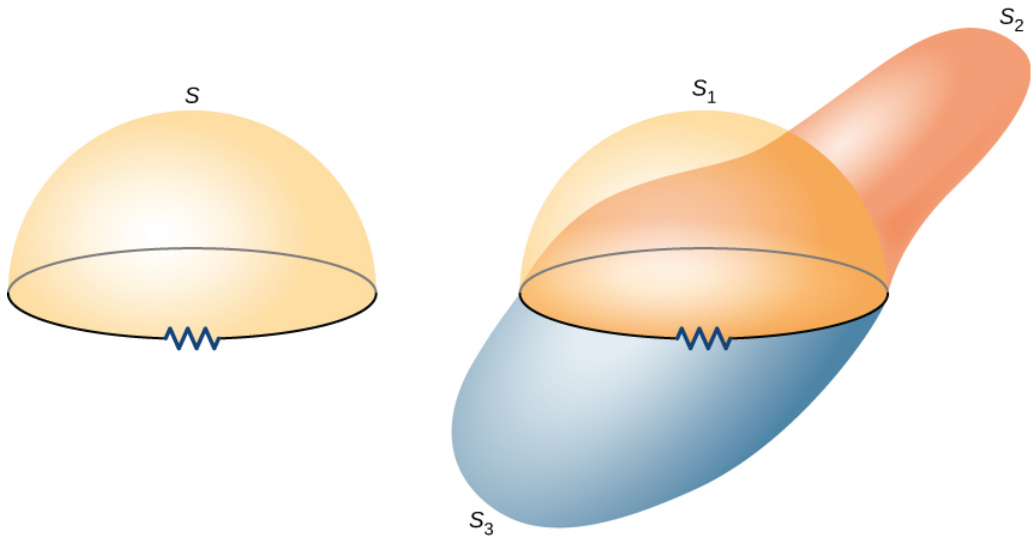


Figure 9.2.4: (a) A circuit bounding an arbitrary open surface **S**. The planar area bounded by the circuit is not part of **S**. (b) Three arbitrary open surfaces bounded by the same circuit. The value of Φ_m is the same for all these surfaces.

The SI unit for magnetic flux is the weber (Wb),

$$1\,Wb = 1\,T \cdot m^2.$$

Occasionally, the magnetic field unit is expressed as webers per square meter (Wb/m^2) instead of teslas, based on this definition. In many practical applications, the circuit of interest consists of a number **N** of tightly wound turns (Figure 9.2.5). Each turn experiences the same magnetic flux. Therefore, the net magnetic flux through the circuits is **N** times the flux through one turn, and Faraday's law is written as

$$\epsilon = -\frac{d}{dt}(N\Phi_m) = -N\frac{d\Phi_m}{dt}.$$

A Square Coil in a Changing Magnetic Field

The square coil of Figure 9.2.1 has sides $l = 0.25\,m$ long and is tightly wound with $N = 200$ turns of wire. The resistance of the coil is $R = 5.0\,\Omega$ The coil is placed in a spatially uniform magnetic field that is directed perpendicular to the face of the coil and whose magnitude is decreasing at a rate $dB/dt = -0.040\,T/s$. (a) What is the magnitude of the emf induced in the coil? (b) What is the magnitude of the current circulating through the coil?

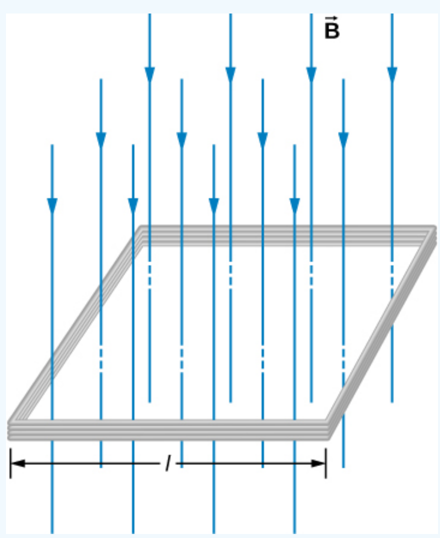


Figure 9.2.5: A square coil with **N** turns of wire with uniform magnetic field $\vec{B}$ directed in the downward direction, perpendicular to the coil.

Strategy

The area vector, or $\hat{n}$ direction, is perpendicular to area covering the loop. We will choose this to be pointing downward so that $\vec{B}$ is parallel to $\hat{n}$ and that the flux turns into multiplication of magnetic field times area. The area of the loop is not changing in time, so it can be factored out of the time derivative, leaving the magnetic field as the only quantity varying in time. Lastly, we can apply Ohm's law once we know the induced emf to find the current in the loop.

Solution

1. The flux through one turn is

$$\Phi_m = BA = Bt^2,$$

so we can calculate the magnitude of the emf from Faraday's law. The sign of the emf will be discussed in the next section, on Lenz's law:

$$|\epsilon| = \left|-N\frac{d\Phi_m}{dt}\right| = Nl^2\frac{dB}{dt}$$

$$= (200)(0.25\,m)^2(0.040\,T/s) = 0.50\,V.$$

- The magnitude of the current induced in the coil is

$$I = \frac{\epsilon}{R} = \frac{0.50\,V}{5.0\,\Omega} = 0.10\,A.$$

Significance

If the area of the loop were changing in time, we would not be able to pull it out of the time derivative. Since the loop is a closed path, the result of this current would be a small amount of heating of the wires until the magnetic field stops changing. This may increase the area of the loop slightly as the wires are heated.

? Exercise 9.2.1

A closely wound coil has a radius of 4.0 cm, 50 turns, and a total resistance of $40\,\Omega$. At what rate must a magnetic field perpendicular to the face of the coil change in order to produce Joule heating in the coil at a rate of 2.0 mW?

Solution

1.1 T/s

9.3: Lenz's Law

Learning Objectives

By the end of this section, you will be able to:

- Use Lenz's law to determine the direction of induced emf whenever a magnetic flux changes
- Use Faraday's law with Lenz's law to determine the induced emf in a coil and in a solenoid

The direction in which the induced emf drives current around a wire loop can be found through the negative sign. However, it is usually easier to determine this direction with Lenz's law, named in honor of its discoverer, Heinrich Lenz (1804–1865). (Faraday also discovered this law, independently of Lenz.) We state Lenz's law as follows:

Lenz's Law

The direction of the induced emf drives current around a wire loop to always **oppose** the change in magnetic flux that causes the emf.

Lenz's law can also be considered in terms of conservation of energy. If pushing a magnet into a coil causes current, the energy in that current must have come from somewhere. If the induced current causes a magnetic field opposing the increase in field of the magnet we pushed in, then the situation is clear. We pushed a magnet against a field and did work on the system, and that showed up as current. If it were not the case that the induced field opposes the change in the flux, the magnet would be pulled in produce a current without anything having done work. Electric potential energy would have been created, violating the conservation of energy.

To determine an induced emf ϵ, you first calculate the magnetic flux Φ_m and then obtain $d\Phi_m/dt$. The magnitude of ϵ is given by

$$\epsilon = \left| \frac{d\Phi_m}{dt} \right|.$$

Finally, you can apply Lenz's law to determine the sense of ϵ. This will be developed through examples that illustrate the following problem-solving strategy.

Problem-Solving Strategy: Lenz's Law

To use Lenz's law to determine the directions of induced magnetic fields, currents, and emfs:

- Make a sketch of the situation for use in visualizing and recording directions.
- Determine the direction of the applied magnetic field $\vec{B}$.
- Determine whether its magnetic flux is increasing or decreasing.
- Now determine the direction of the induced magnetic field $\vec{B}$. The induced magnetic field tries to reinforce a magnetic flux that is decreasing or opposes a magnetic flux that is increasing. Therefore, the induced magnetic field adds or subtracts to the applied magnetic field, depending on the change in magnetic flux.
- Use **right-hand rule 2** (RHR-2; see Magnetic Forces and Fields) to determine the direction of the induced current **I** that is responsible for the induced magnetic field $\vec{B}$.
- The direction (or polarity) of the induced emf can now drive a conventional current in this direction.

Let's apply Lenz's law to the system of Figure $9.3.1a$. We designate the "front" of the closed conducting loop as the region containing the approaching bar magnet, and the "back" of the loop as the other region. As the north pole of the magnet moves toward the loop, the flux through the loop due to the field of the magnet increases because the strength of field lines directed from the front to the back of the loop is increasing. A current is therefore induced in the loop. By Lenz's law, the direction of the induced current must be such that its own magnetic field is directed in a way to **oppose** the changing flux caused by the field of the approaching magnet. Hence, the induced current circulates so that its magnetic field lines through the loop are directed from the back to the front of the loop. By RHR-2, place your thumb pointing against the magnetic field lines, which is toward the bar magnet. Your fingers wrap in a counterclockwise direction as viewed from the bar magnet. Alternatively, we can determine the direction of the induced current by treating the current loop as an electromagnet that **opposes** the approach of the north pole of the bar magnet. This occurs when the induced current flows as shown, for then the face of the loop nearer the approaching magnet is also a north pole.

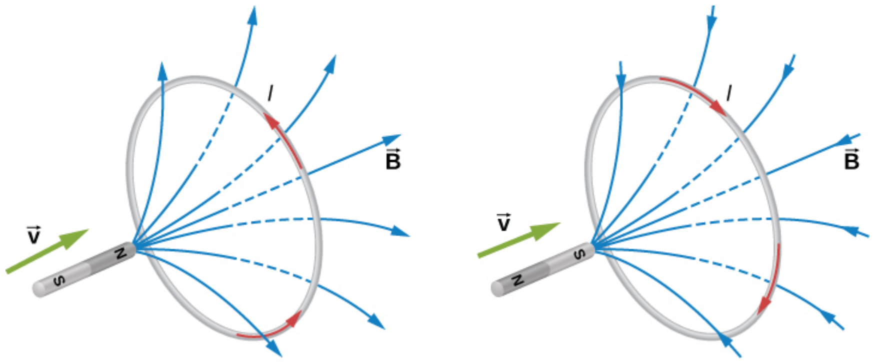


Figure 9.3.1: The change in magnetic flux caused by the approaching magnet induces a current in the loop. (a) An approaching north pole induces a counterclockwise current with respect to the bar magnet. (b) An approaching south pole induces a clockwise current with respect to the bar magnet.

Part (b) of the figure shows the south pole of a magnet moving toward a conducting loop. In this case, the flux through the loop due to the field of the magnet increases because the number of field lines directed from the back to the front of the loop is increasing. To oppose this change, a current is induced in the loop whose field lines through the loop are directed from the front to the back. Equivalently, we can say that the current flows in a direction so that the face of the loop nearer the approaching magnet is a south pole, which then repels the approaching south pole of the magnet. By RHR-2, your thumb points away from the bar magnet. Your fingers wrap in a clockwise fashion, which is the direction of the induced current.

Another example illustrating the use of Lenz's law is shown in Figure $9.3.2$. When the switch is opened, the decrease in current through the solenoid causes a decrease in magnetic flux through its coils, which induces an emf in the solenoid. This emf must oppose the change (the termination of the current) causing it. Consequently, the induced emf has the polarity shown and drives in the direction of the original current. This may generate an arc across the terminals of the switch as it is opened.

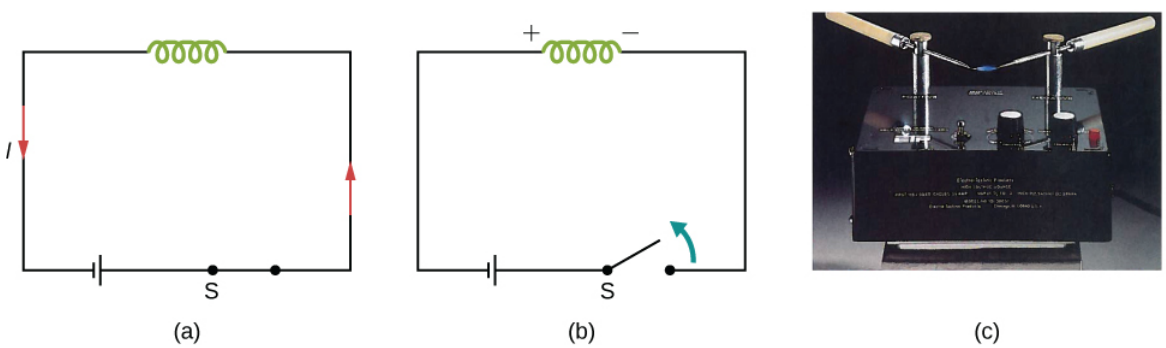


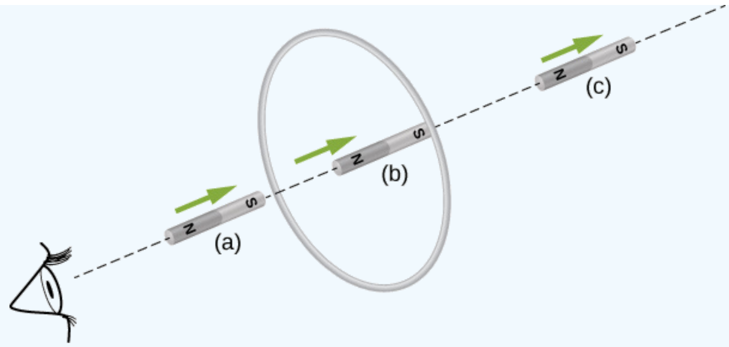
Figure 9.3.2: (a) A solenoid connected to a source of emf. (b) Opening switch S terminates the current, which in turn induces an emf in the solenoid. (c) A potential difference between the ends of the sharply pointed rods is produced by inducing an emf in a coil. This potential difference is large enough to produce an arc between the sharp points.

Exercise $9.3.1A$

Find the direction of the induced current in the wire loop shown below as the magnet enters, passes through, and leaves the loop.

Solution

To the observer shown, the current flows clockwise as the magnet approaches, decreases to zero when the magnet is centered in the plane of the coil, and then flows counterclockwise as the magnet leaves the coil.



? Exercise $9.3.1B$

Verify the directions of the induced currents in Figure 13.2.2.

✓ Example $9.3.1A$: A Circular Coil in a Changing Magnetic Field

A magnetic field $\vec{B}$ is directed outward perpendicular to the plane of a circular coil of radius $r = 0.50\,m$ (Figure $9.3.3$). The field is cylindrically symmetrical with respect to the center of the coil, and its magnitude decays exponentially according to $B = (1.5T)e^{(5.0s^{-1})t}$, where **B** is in teslas and **t** is in seconds.

a. Calculate the emf induced in the coil at the times $t_1 = 0$, $t_2 = 5.0 \times 10^{-2}\,s$, and $t_3 = 1.0\,s$.
b. Determine the current in the coil at these three times if its resistance is $10\,\Omega$.

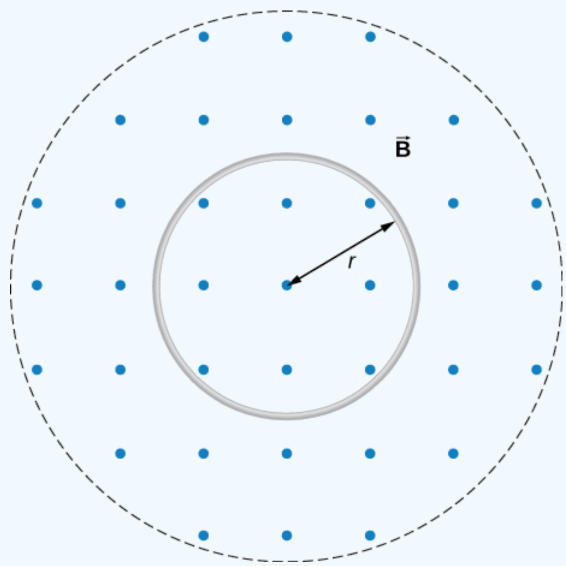


Figure $9.3.3$: A circular coil in a decreasing magnetic field.

Strategy

Since the magnetic field is perpendicular to the plane of the coil and constant over each spot in the coil, the dot product of the magnetic field $\vec{B}$ and normal to the area unit vector $\hat{n}$ turns into a multiplication. The magnetic field can be pulled out of the integration, leaving the flux as the product of the magnetic field times area. We need to take the time derivative of the exponential function to calculate the emf using Faraday's law. Then we use Ohm's law to calculate the current.

Solution

1. Since $\vec{B}$ is perpendicular to the plane of the coil, the magnetic flux is given by

$$\Phi_m = B\pi r^2 = (1.5e^{-5.0t}T)\pi(0.50\,m)^2$$
$$= 1.2e^{-(5.0s^{-1})t}Wb.$$

From Faraday's law, the magnitude of the induced emf is

$$\epsilon = \left|\frac{d\Phi_m}{dt}\right| = \left|\frac{d}{dt}(1.2e^{-(5.0s^{-1})t}Wb)\right| = 6.0e^{-(5.0s^{-1})t}V.$$

Since $\vec{B}$ is directed out of the page and is decreasing, the induced current must flow counterclockwise when viewed from above so that the magnetic field it produces through the coil also points out of the page. For all three times, the sense of ε is counterclockwise; its magnitudes are

$$\epsilon(t_1) = 6.0V;\ \epsilon(t_2) = 4.7\,V;\ \epsilon(t_3) = 0040\,V.$$

2. From Ohm's law, the respective currents are

$$I(t_1) = \frac{\epsilon(t_1)}{R} = \frac{6.0\,V}{10\,\Omega} = 0.60\,A;$$
$$I(t_2) = \frac{4.7\,V}{10\,\Omega} = 0.47\,A;$$

and

$$I(t_3) = \frac{0.040\,V}{10\,\Omega} = 4.0\times 10^{-3}\,A.$$

Significance

An emf voltage is created by a changing magnetic flux over time. If we know how the magnetic field varies with time over a constant area, we can take its time derivative to calculate the induced emf.

Example $9.3.1B$: Changing Magnetic Field Inside a Solenoid

The current through the windings of a solenoid with $n = 2000$ turns per meter is changing at a rate $dI/dt = 3.0\,A/s$. (See Sources of Magnetic Fields for a discussion of solenoids.) The solenoid is 50-cm long and has a cross-sectional diameter of 3.0 cm. A small coil consisting of $N = 20$ closely wound turns wrapped in a circle of diameter 1.0 cm is placed in the middle of the solenoid such that the plane of the coil is perpendicular to the central axis of the solenoid. Assuming that the infinite-solenoid approximation is valid at the location of the small coil, determine the magnitude of the emf induced in the coil.

Strategy

The magnetic field in the middle of the solenoid is a uniform value of $\mu_0 nI$. This field is producing a maximum magnetic flux through the coil as it is directed along the length of the solenoid. Therefore, the magnetic flux through the coil is the product of the solenoid's magnetic field times the area of the coil. Faraday's law involves a time derivative of the magnetic flux. The only quantity varying in time is the current, the rest can be pulled out of the time derivative. Lastly, we include the number of turns in the coil to determine the induced emf in the coil.

Solution

Since the field of the solenoid is given by $B = \mu_0 nI$, the flux through each turn of the small coil is

$$\Phi_m = \mu_0 nI\left(\frac{\pi d^2}{4}\right),$$

where d is the diameter of the coil. Now from Faraday's law, the magnitude of the emf induced in the coil is

$$\begin{aligned}\epsilon &= \left|N\frac{d\Phi_m}{dt}\right| \\ &= \left|N\mu_0 n\frac{\pi d^2}{4}\frac{dI}{dt}\right| \\ &= 20(4\pi\times 10^{-7}T\cdot m/s)(2000\,m^{-1})\frac{\pi(0.010\,m)^2}{4}(3.0\,A/s) \\ &= 1.2\times 10^{-5}\,V.\end{aligned}$$

Significance

When the current is turned on in a vertical solenoid, as shown in Figure $9.3.4$, the ring has an induced emf from the solenoid's changing magnetic flux that opposes the change. The result is that the ring is fired vertically into the air.

Figure $9.3.4$: The jumping ring. When a current is turned on in the vertical solenoid, a current is induced in the metal ring. The stray field produced by the solenoid causes the ring to jump off the solenoid.

Note

A demonstration of the jumping ring from MIT.

9.4: Motional Emf

Learning Objectives

By the end of this section, you will be able to:

- Determine the magnitude of an induced emf in a wire moving at a constant speed through a magnetic field
- Discuss examples that use motional emf, such as a rail gun and a tethered satellite

Magnetic flux depends on three factors: the strength of the magnetic field, the area through which the field lines pass, and the orientation of the field with the surface area. If any of these quantities varies, a corresponding variation in magnetic flux occurs. So far, we've only considered flux changes due to a changing field. Now we look at another possibility: a changing area through which the field lines pass including a change in the orientation of the area.

Two examples of this type of flux change are represented in Figure $9.4.1$. In part (a), the flux through the rectangular loop increases as it moves into the magnetic field, and in part (b), the flux through the rotating coil varies with the angle θ.

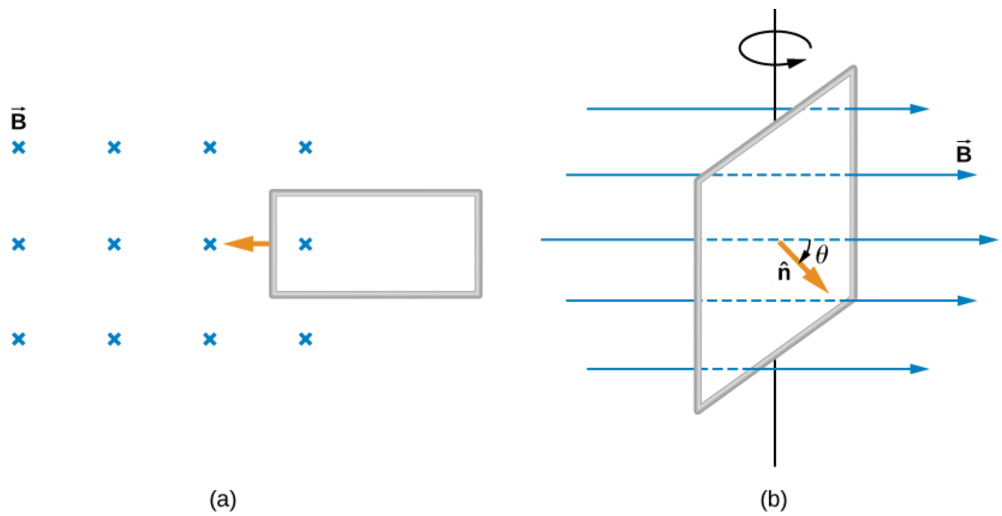


Figure $9.4.1$: (a) Magnetic flux changes as a loop moves into a magnetic field; (b) magnetic flux changes as a loop rotates in a magnetic field.

It's interesting to note that what we perceive as the cause of a particular flux change actually depends on the frame of reference we choose. For example, if you are at rest relative to the moving coils of Figure $9.4.1b$, you would see the flux vary because of a changing magnetic field—in part (a), the field moves from left to right in your reference frame, and in part (b), the field is rotating. It is often possible to describe a flux change through a coil that is moving in one particular reference frame in terms of a changing magnetic field in a second frame, where the coil is stationary. However, reference-frame questions related to magnetic flux are beyond the level of this textbook. We'll avoid such complexities by always working in a frame at rest relative to the laboratory and explain flux variations as due to either a changing field or a changing area.

Now let's look at a conducting rod pulled in a circuit, changing magnetic flux. The area enclosed by the circuit 'MNOP' of Figure $9.4.2$ is **lx** and is perpendicular to the magnetic field, so we can simplify the integration of $\Phi_\mathrm{m} = \int_S \overrightarrow{\mathbf{B}} \cdot \hat{\mathbf{n}} dA$ into a multiplication of magnetic field and area. The magnetic flux through the open surface is therefore

$$\Phi_m = B\,l\,x.$$

Since B and l are constant and the velocity of the rod is $v = dx/dt$, we can now restate Faraday's law, Equation 13.2.2, for the magnitude of the emf in terms of the moving conducting rod as

$$\epsilon = \frac{d\Phi_m}{dt} = Bl\frac{dx}{dt} = Blv.$$

The current induced in the circuit is the emf divided by the resistance or

$$I = \frac{Blv}{R}.$$

Furthermore, the direction of the induced emf satisfies Lenz's law, as you can verify by inspection of the figure.

This calculation of motionally induced emf is not restricted to a rod moving on conducting rails. With $\vec{F} = q\vec{v} \times \vec{B}$ as the starting point, it can be shown that $\epsilon = -d\Phi_m/dt$ holds for any change in flux caused by the motion of a conductor. We saw in Faraday's Law that the emf induced by a time-varying magnetic field obeys this same relationship, which is Faraday's law. Thus Faraday's law **holds for all flux changes**, whether they are produced by a changing magnetic field, by motion, or by a combination of the two.

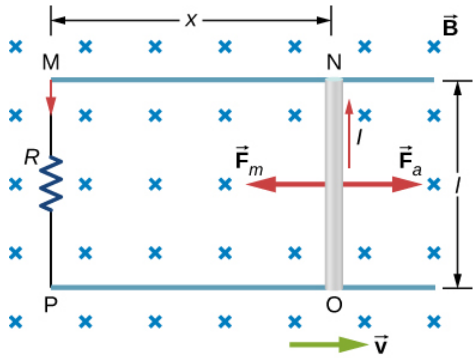


Figure $9.4.2$: A conducting rod is pushed to the right at constant velocity. The resulting change in the magnetic flux induces a current in the circuit.

From an energy perspective, $\vec{F}_a$ produces power $F_a v$, and the resistor dissipates power I^2R. Since the rod is moving at constant velocity, the applied force F_a must balance the magnetic force $F_m = IlB$ on the rod when it is carrying the induced current **I**. Thus the power produced is

$$F_a v = IlBv = \frac{Blv}{R} \cdot lBv = \frac{l^2B^2v^2}{R}.$$

The power dissipated is

$$P = I^2R = \left(\frac{Blv}{R}\right)^2 R = \frac{l^2B^2v^2}{R}.$$

In satisfying the principle of energy conservation, the produced and dissipated powers are equal.

This principle can be seen in the operation of a rail gun. A **rail gun** is an electromagnetic projectile launcher that uses an apparatus similar to Figure $9.4.2$ and is shown in schematic form in Figure $9.4.3$. The conducting rod is replaced with a projectile or weapon to be fired. So far, we've only heard about how motion causes an emf. In a rail gun, the optimal shutting off/ramping down of a magnetic field decreases the flux in between the rails, causing a current to flow in the rod (armature) that holds the projectile. This current through the armature experiences a magnetic force and is propelled forward. Rail guns, however, are not used widely in the military due to the high cost of production and high currents: Nearly one million amps is required to produce enough energy for a rail gun to be an effective weapon.

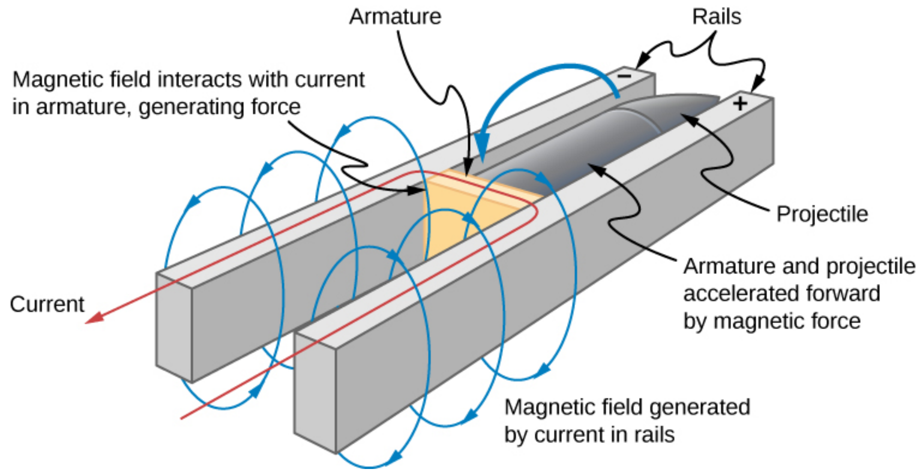


Figure 9.4.3: Current through two rails drives a conductive projectile forward by the magnetic force created.

We can calculate a motionally induced emf with Faraday's law **even when an actual closed circuit is not present**. We simply imagine an enclosed area whose boundary includes the moving conductor, calculate Φ_m, and then find the emf from Faraday's law. For example, we can let the moving rod of Figure 9.4.4 be one side of the imaginary rectangular area represented by the dashed lines. The area of the rectangle is **lx**, so the magnetic flux through it is $\Phi_m = Blx$. Differentiating this equation, we obtain

$$\frac{d\Phi_m}{dt} = Bl\frac{dx}{dt} = Blv,$$

which is identical to the potential difference between the ends of the rod that we determined earlier.

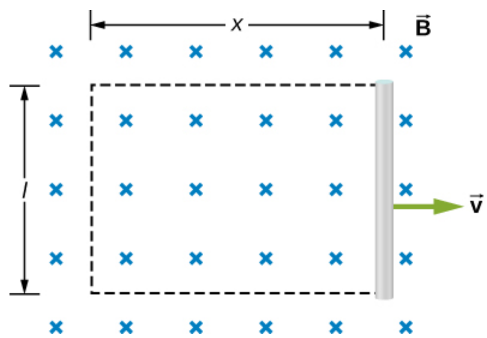


Figure 9.4.4: With the imaginary rectangle shown, we can use Faraday's law to calculate the induced emf in the moving rod.

Motional emfs in Earth's weak magnetic field are not ordinarily very large, or we would notice voltage along metal rods, such as a screwdriver, during ordinary motions. For example, a simple calculation of the motional emf of a 1.0-m rod moving at 3.0 m/s perpendicular to the Earth's field gives

$$emf = Blv = (5.0 \times 10^{-5} T)(1.0\, m)(3.0\, m/s) = 150\, \mu V.$$

This small value is consistent with experience. There is a spectacular exception, however. In 1992 and 1996, attempts were made with the space shuttle to create large motional emfs. The tethered satellite was to be let out on a 20-km length of wire, as shown in Figure 9.4.5, to create a 5-kV emf by moving at orbital speed through Earth's field. This emf could be used to convert some of the shuttle's kinetic and potential energy into electrical energy if a complete circuit could be made. To complete the circuit, the stationary ionosphere was to supply a return path through which current could flow. (The ionosphere is the rarefied and partially ionized atmosphere at orbital altitudes. It conducts because of the ionization. The ionosphere serves the same function as the stationary rails and connecting resistor in Figure 9.4.3, without which there would not be a complete circuit.) Drag on the current in the cable due to the magnetic force $F = IlB\sin\theta$ does the work that reduces the shuttle's kinetic and potential energy, and allows it to be converted into electrical energy. Both tests were unsuccessful. In the first, the cable hung up and could only be extended a couple of hundred meters; in the second, the cable broke when almost fully extended. Example 9.4.1 indicates feasibility in principle.

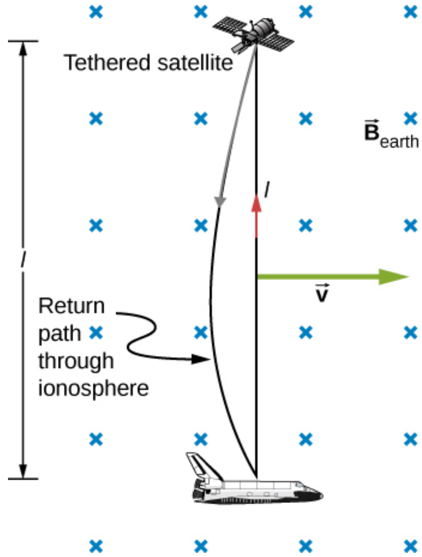


Figure 9.4.5: Motional emf as electrical power conversion for the space shuttle was the motivation for the tethered satellite experiment. A 5-kV emf was predicted to be induced in the 20-km tether while moving at orbital speed in Earth's magnetic field. The circuit is completed by a return path through the stationary ionosphere.

Example 9.4.1: Calculating the Large Motional Emf of an Object in Orbit

Calculate the motional emf induced along a 20.0-km conductor moving at an orbital speed of 7.80 km/s perpendicular to Earth's $5.00 \times 10^{-5} T$ magnetic field.

Strategy

This is a great example of using the equation motional $\epsilon = B\,l\,v$.

Solution

Entering the given values into $\epsilon = Blv$ gives

$$\begin{aligned}\epsilon &= Blv \\ &= (5.00 \times 10^{-5} T)(2.00 \times 10^{4}\, m)(7.80 \times 10^{3}\, m/s) \\ &= 7.80 \times 10^{3}\, V.\end{aligned}$$

Significance

The value obtained is greater than the 5-kV measured voltage for the shuttle experiment, since the actual orbital motion of the tether is not perpendicular to Earth's field. The 7.80-kV value is the maximum emf obtained when $\theta = 90^{o}$ and so $sin\,\theta = 1$.

Example 9.4.2: A Metal Rod Rotating in a Magnetic Field

Part (a) of Figure 9.4.6 shows a metal rod **OS** that is rotating in a horizontal plane around point **O**. The rod slides along a wire that forms a circular arc **PST** of radius r. The system is in a constant magnetic field $\vec{B}$ that is directed out of the page.

a. If you rotate the rod at a constant angular velocity ω, what is the current I in the closed loop **OPSO**? Assume that the resistor R furnishes all of the resistance in the closed loop.
b. Calculate the work per unit time that you do while rotating the rod and show that it is equal to the power dissipated in the resistor.

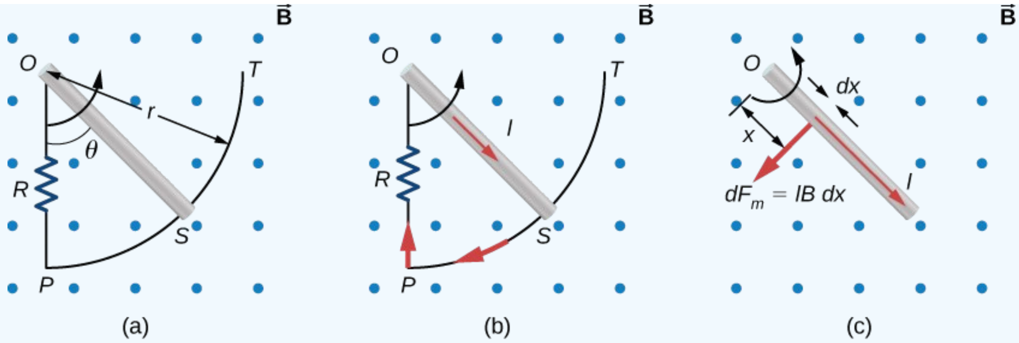


Figure 9.4.6: (a) The end of a rotating metal rod slides along a circular wire in a horizontal plane. (b) The induced current in the rod. (c) The magnetic force on an infinitesimal current segment.

Strategy

The magnetic flux is the magnetic field times the area of the quarter circle or $A = r^2\theta/2$. When finding the emf through Faraday's law, all variables are constant in time but θ, with $\omega = d\theta/dt$. To calculate the work per unit time, we know this is related to the torque times the angular velocity. The torque is calculated by knowing the force on a rod and integrating it over the length of the rod.

Solution

1. From geometry, the area of the loop **OPSO** is $A = \frac{r^2\theta}{2}$. Hence, the magnetic flux through the loop is

$$\Phi_m = BA = B\frac{r^2\theta}{2}.$$

Differentiating with respect to time and using $\omega = d\theta/dt$, we have

$$\epsilon = \left|\frac{d\Phi_m}{dt}\right| = \frac{Br^2\omega}{2}.$$

When divided by the resistance **R** of the loop, this yields for the magnitude of the induced current

$$I = \frac{\epsilon}{R} = \frac{Br^2\omega}{2R}.$$

As θ increases, so does the flux through the loop due to $\vec{B}$. To counteract this increase, the magnetic field due to the induced current must be directed into the page in the region enclosed by the loop. Therefore, as part (b) of Figure 9.4.6 illustrates, the current circulates clockwise.

2. You rotate the rod by exerting a torque on it. Since the rod rotates at constant angular velocity, this torque is equal and opposite to the torque exerted on the current in the rod by the original magnetic field. The magnetic force on the infinitesimal segment of length **dx** shown in part (c) of Figure 9.4.6 is $dF_m = IBdx$, so the magnetic torque on this segment is

$$d\tau_m = x \cdot dF_m = IBxdx.$$

The net magnetic torque on the rod is then

$$\tau_m = \int_0^r d\tau_m = IB\int_0^r x\,dx = \frac{1}{2}IBr^2.$$

The torque τ that you exert on the rod is equal and opposite to τ_m, and the work that you do when the rod rotates through an angle $d\theta$ is $dW = rd\theta$. Hence, the work per unit time that you do on the rod is

$$\frac{dW}{dt} = \tau\frac{d\theta}{dt} = \frac{1}{2}IBr^2\frac{d\theta}{dt} = \frac{1}{2}\left(\frac{Br^2\omega}{2R}\right)Br^2\omega = \frac{B^2r^4\omega^2}{4R},$$

where we have substituted for **I**. The power dissipated in the resister is $P = IR^2$, which can be written as

$$P = \left(\frac{Br^2\omega}{2R}\right)^2 R = \frac{B^2r^4\omega^2}{4R}.$$

Therefore, we see that

$$P = \frac{dW}{dt}.$$

Hence, the power dissipated in the resistor is equal to the work per unit time done in rotating the rod.

Significance

An alternative way of looking at the induced emf from Faraday's law is to integrate in space instead of time. The solution, however, would be the same. The motional emf is

$$|\epsilon| = \int B\, v\, dl.$$

The velocity can be written as the angular velocity times the radius and the differential length written as dr. Therefore,

$$\epsilon| = B \int vdr = B\omega \int_0^l rdr = \frac{1}{2} B\omega l^2,$$

which is the same solution as before.

✔ Example 9.4.3: A Rectangular Coil Rotating in a Magnetic Field

A rectangular coil of area **A** and **N** turns is placed in a uniform magnetic field $\vec{B} = B\hat{j}$, as shown in Figure 9.4.7. The coil is rotated about the **z**-axis through its center at a constant angular velocity ω. Obtain an expression for the induced emf in the coil.

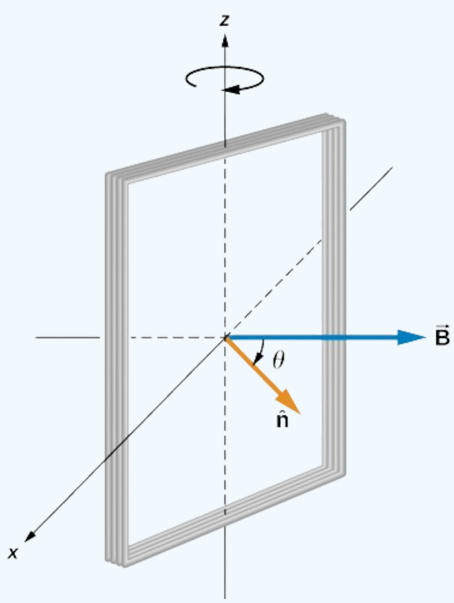


Figure 9.4.7: A rectangular coil rotating in a uniform magnetic field.

Strategy

According to the diagram, the angle between the perpendicular to the surface ($\hat{n}$) and the magnetic field ($\vec{B}$) is θ. The dot product of $B \cdot \hat{n}$ simplifies to only the cos θ component of the magnetic field, namely where the magnetic field projects onto the unit area vector $\hat{n}$. The magnitude of the magnetic field and the area of the loop are fixed over time, which makes the integration simplify quickly. The induced emf is written out using Faraday's law.

Solution

When the coil is in a position such that its normal vector $\hat{n}$ makes an angle θ with the magnetic field $\vec{B}$ the magnetic flux through a single turn of the coil is

$$\Phi_m = \int_S \vec{B} \cdot \hat{n} dA = BA \cos \theta.$$

From Faraday's law, the emf induced in the coil is

$$\epsilon = -N\frac{d\Phi_m}{dt} = NBA\sin\theta\frac{d\theta}{dt}.$$

The constant angular velocity is $\omega = d\theta/dt$. The angle θ represents the time evolution of the angular velocity or ωt. This is changes the function to time space rather than θ. The induced emf therefore varies sinusoidally with time according to

$$\epsilon = \epsilon_0 \sin\omega t,$$

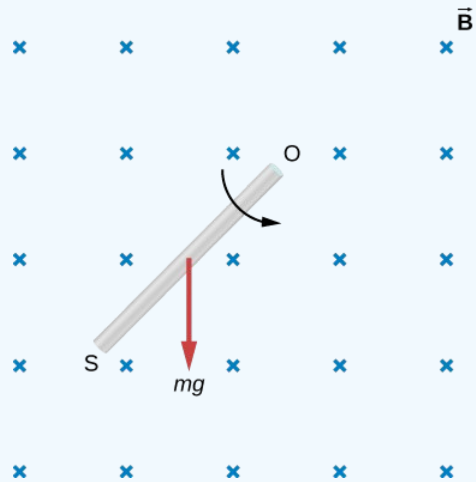
where $\epsilon_0 = NBA\omega$.

Significance

If the magnetic field strength or area of the loop were also changing over time, these variables wouldn't be able to be pulled out of the time derivative to simply the solution as shown. This example is the basis for an electric generator, as we will give a full discussion in Applications of Newton's Law.

Exercise 9.4.1

Shown below is a rod of length **l** that is rotated counterclockwise around the axis through **O** by the torque due to $m\vec{g}$. Assuming that the rod is in a uniform magnetic field $\vec{B}$, what is the emf induced between the ends of the rod when its angular velocity is ω? Which end of the rod is at a higher potential?



Answer

$\epsilon = Bl^2\omega/2$, with **O** at a higher potential than **S**

Exercise 9.4.2

A rod of length 10 cm moves at a speed of 10 m/s perpendicularly through a 1.5-T magnetic field. What is the potential difference between the ends of the rod?

Answer

1.5 V

9.5: Induced Electric Fields

Learning Objectives

By the end of this section, you will be able to:

- Connect the relationship between an induced emf from Faraday's law to an electric field, thereby showing that a changing magnetic flux creates an electric field
- Solve for the electric field based on a changing magnetic flux in time

The fact that emfs are induced in circuits implies that work is being done on the conduction electrons in the wires. What can possibly be the source of this work? We know that it's neither a battery nor a magnetic field, for a battery does not have to be present in a circuit where current is induced, and magnetic fields never do work on moving charges. The answer is that the source of the work is an electric field $\vec{E}$ that is induced in the wires. The work done by $\vec{E}$ in moving a unit charge completely around a circuit is the induced emf ε; that is,

$$\epsilon = \oint \vec{E} \cdot d\vec{l},$$

where $\oint$ represents the line integral around the circuit. Faraday's law can be written in terms of the induced electric field as

$$\oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt}.$$

There is an important distinction between the electric field induced by a changing magnetic field and the electrostatic field produced by a fixed charge distribution. Specifically, the induced electric field is nonconservative because it does net work in moving a charge over a closed path, whereas the electrostatic field is conservative and does no net work over a closed path. Hence, electric potential can be associated with the electrostatic field, but not with the induced field. The following equations represent the distinction between the two types of electric field:

$$\underbrace{\oint \vec{E} \cdot d\vec{l} \neq 0}_{\text{Induced Electric Field}}$$

$$\underbrace{\oint \vec{E} \cdot d\vec{l} = 0}_{\text{Electrostatic Electric Fields}} \quad .$$

Our results can be summarized by combining these equations:

$$\epsilon = \oint \vec{E} \cdot d\vec{l} = -\frac{d\Phi_m}{dt}. \tag{9.5.1}$$

Example 9.5.1: Induced Electric Field in a Circular Coil

What is the induced electric field in the circular coil of Example 13.3.1A (and Figure 13.3.3) at the three times indicated?

Strategy

Using cylindrical symmetry, the electric field integral simplifies into the electric field times the circumference of a circle. Since we already know the induced emf, we can connect these two expressions by Faraday's law to solve for the induced electric field.

Solution

The induced electric field in the coil is constant in magnitude over the cylindrical surface, similar to how Ampere's law problems with cylinders are solved. Since $\vec{E}$ is tangent to the coil,

$$\oint \vec{E} \cdot d\vec{l} = \oint E dl = 2\pi r E.$$

When combined with Equation 9.5.1, this gives

$$E = \frac{\epsilon}{2\pi r}.$$

The direction of ϵ is counterclockwise, and $\vec{E}$ circulates in the same direction around the coil. The values of **E** are

$$E(t_1) = \frac{6.0\,V}{2\pi\,(0.50\,m)} = 1.9\,V/m;$$

$$E(t_2) = \frac{4.7\,V}{2\pi\,(0.50\,m)} = 1.5\,V/m;$$

$$E(t_3) = \frac{0.040\,V}{2\pi\,(0.50\,m)} = 0.013\,V/m;$$

Significance

When the magnetic flux through a circuit changes, a nonconservative electric field is induced, which drives current through the circuit. But what happens if $dB/dt \neq 0$ in free space where there isn't a conducting path? The answer is that this case can be treated **as if a conducting path were present**; that is, nonconservative electric fields are induced wherever $dB/dt \neq 0$ whether or not there is a conducting path present.

These nonconservative electric fields always satisfy Equation 9.5.1. For example, if the circular coil were removed, an electric field **in free space** at $r = 0.50\,m$ would still be directed counterclockwise, and its magnitude would still be 1.9 V/m at $t = 0$. 1.5 V/m at $t = 5.0 \times 10^{-2}\,s$, etc. The existence of induced electric fields is certainly **not** restricted to wires in circuits.

Example $9.5.2$: Electric Field Induced by the Changing Magnetic Field of a Solenoid

Figure $9.5.1a$ shows a long solenoid with radius R and n turns per unit length; its current decreases with time according to $I = I_0 e^{-\alpha t}$. What is the magnitude of the induced electric field at a point a distance **r** from the central axis of the solenoid (a) when $r > R$ and (b) when $r < R$ [Figure $9.5.1b$]. (c) What is the direction of the induced field at both locations? Assume that the infinite-solenoid approximation is valid throughout the regions of interest.

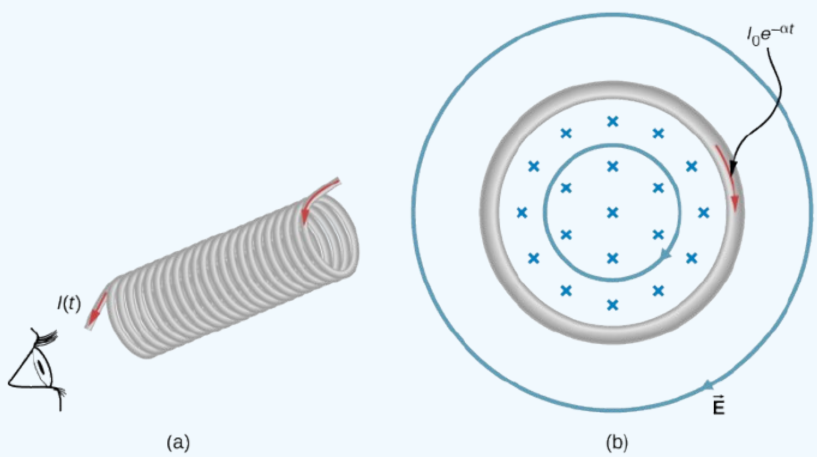


Figure $9.5.1$: (a) The current in a long solenoid is decreasing exponentially. (b) A cross-sectional view of the solenoid from its left end. The cross-section shown is near the middle of the solenoid. An electric field is induced both inside and outside the solenoid.

Strategy

Using the formula for the magnetic field inside an infinite solenoid and Faraday's law, we calculate the induced emf. Since we have cylindrical symmetry, the electric field integral reduces to the electric field times the circumference of the integration path. Then we solve for the electric field.

Solution

a. The magnetic field is confined to the interior of the solenoid where

$$B = \mu_0 n I = \mu_0 n I_0 e^{-\alpha t}.$$

Thus, the magnetic flux through a circular path whose radius r is greater than R, the solenoid radius, is

$$\Phi_m = BA = \mu_0 n I_0 \pi R^2 e^{-\alpha t}.$$

The induced field $\vec{E}$ is tangent to this path, and because of the cylindrical symmetry of the system, its magnitude is constant on the path. Hence, we have

$$\left|\oint \vec{E} \cdot d\vec{l}\right| = \left|\frac{d\Phi_m}{dt}\right|,$$

$$E(2\pi r) = \left|\frac{d}{dt}(\mu_0 n I_0 \pi R^2 e^{-\alpha t})\right| = \alpha \mu_0 n I_0 \pi R^2 e^{-\alpha t},$$

$$E = \frac{\alpha \mu_0 n I_0 R^2}{2r} e^{-\alpha t} \ (r > R).$$

b. For a path of radius **r** inside the solenoid, $\Phi_m = B\pi r^2$, so

$$E(2\pi r) = \left|\frac{d}{dt}(\mu_0 n I_0 \pi r^2 e^{-\alpha t})\right| = \alpha \mu_0 n I_0 \pi r^2 e^{-\alpha t},$$

and the induced field is

$$E = \frac{\alpha \mu_0 n I_0 r}{2} e^{-\alpha t} \ (r < R).$$

c. The magnetic field points into the page as shown in part (b) and is decreasing. If either of the circular paths were occupied by conducting rings, the currents induced in them would circulate as shown, in conformity with Lenz's law. The induced electric field must be so directed as well.

Significance

In part (b), note that $|\vec{E}|$ increases with r inside and decreases as $1/r$ outside the solenoid, as shown in Figure $9.5.2$.

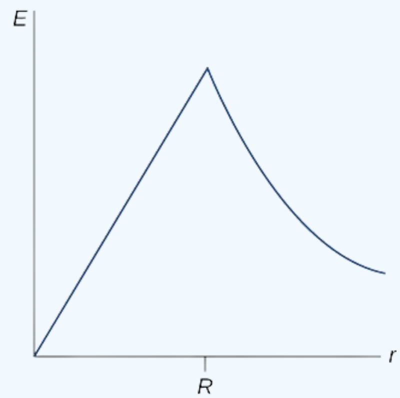


Figure $9.5.2$: The electric field vs. distance **r**. When $r < R$, the electric field rises linearly, whereas when $r > R$, the electric field falls of proportional to 1/**r**.

? Exercise $9.5.1$

Suppose that the coil of Example 13.3.1A is a square rather than circular. Can Equation $9.5.1$ be used to calculate (a) the induced emf and (b) the induced electric field?

Answer

a. yes; b. Yes; however there is a lack of symmetry between the electric field and coil, making $\oint \vec{E} \cdot d\vec{l}$ a more complicated relationship that can't be simplified as shown in the example.

Exercise 9.5.2

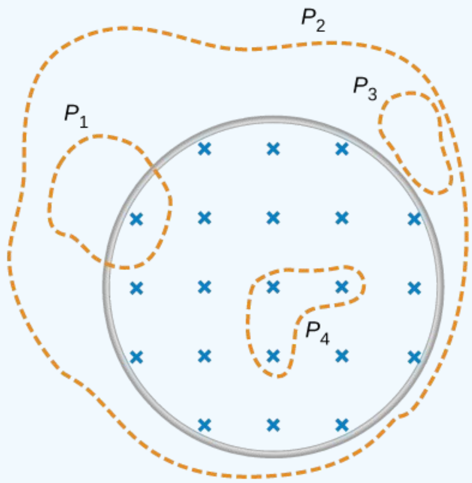
What is the magnitude of the induced electric field in Example 9.5.2 at $t=0$ if $r=6.0\,cm$, $R=2.0\,cm$, $n=2000$ turns per meter, $I_0=2.0\,A$, and $\alpha=200\,s^{-1}$?

Answer

$3.4\times10^{-3}\,V/m$

Exercise 9.5.3

The magnetic field shown below is confined to the cylindrical region shown and is changing with time. Identify those paths for which $\epsilon=\oint\vec{E}\cdot d\vec{l}\neq 0$.



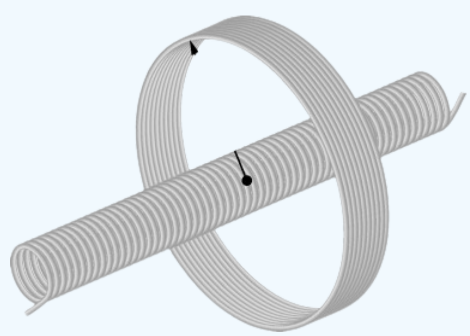
Answer

$P_1,\ P_2,\ P_4$

Exercise 9.5.1

A long solenoid of cross-sectional area $5.0\,cm^2$ is wound with 25 turns of wire per centimeter. It is placed in the middle of a closely wrapped coil of 10 turns and radius 25 cm, as shown below.

a. What is the emf induced in the coil when the current through the solenoid is decreasing at a rate $dI/dt=-0.20\,A/s$?
b. What is the electric field induced in the coil?



Answer

a. $3.1\times10^{-6}V$; b. $2.0\times10^{-7}\,V/m$

9.6: Eddy Currents

Learning Objectives

By the end of this section, you will be able to:

- Explain how eddy currents are created in metals
- Describe situations where eddy currents are beneficial and where they are not helpful

As discussed two sections earlier, a motional emf is induced when a conductor moves in a magnetic field or when a magnetic field moves relative to a conductor. If motional emf can cause a current in the conductor, we refer to that current as an **eddy current**.

Magnetic Damping

Eddy currents can produce significant drag, called **magnetic damping**, on the motion involved. Consider the apparatus shown in Figure $9.6.1$, which swings a pendulum bob between the poles of a strong magnet. (This is another favorite physics demonstration.) If the bob is metal, significant drag acts on the bob as it enters and leaves the field, quickly damping the motion. If, however, the bob is a slotted metal plate, as shown in part (b) of the figure, the magnet produces a much smaller effect. There is no discernible effect on a bob made of an insulator. Why does drag occur in both directions, and are there any uses for magnetic drag?

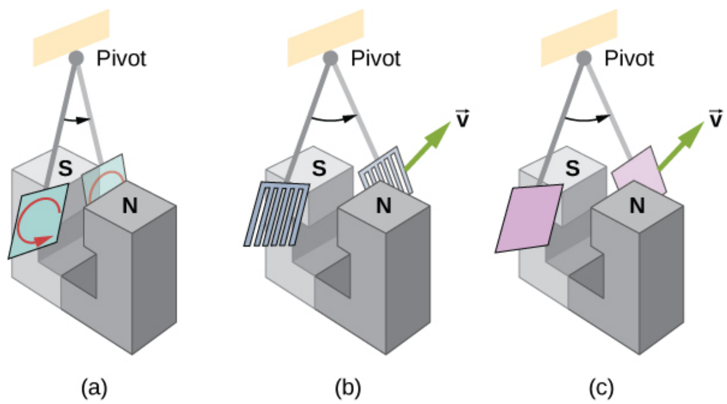


Figure $9.6.1$: A common physics demonstration device for exploring eddy currents and magnetic damping. (a) The motion of a metal pendulum bob swinging between the poles of a magnet is quickly damped by the action of eddy currents. (b) There is little effect on the motion of a slotted metal bob, implying that eddy currents are made less effective. (c) There is also no magnetic damping on a nonconducting bob, since the eddy currents are extremely small.

Figure $9.6.2$ shows what happens to the metal plate as it enters and leaves the magnetic field. In both cases, it experiences a force opposing its motion. As it enters from the left, flux increases, setting up an eddy current (Faraday's law) in the counterclockwise direction (Lenz's law), as shown. Only the right-hand side of the current loop is in the field, so an unopposed force acts on it to the left (RHR-1). When the metal plate is completely inside the field, there is no eddy current if the field is uniform, since the flux remains constant in this region. But when the plate leaves the field on the right, flux decreases, causing an eddy current in the clockwise direction that, again, experiences a force to the left, further slowing the motion. A similar analysis of what happens when the plate swings from the right toward the left shows that its motion is also damped when entering and leaving the field.

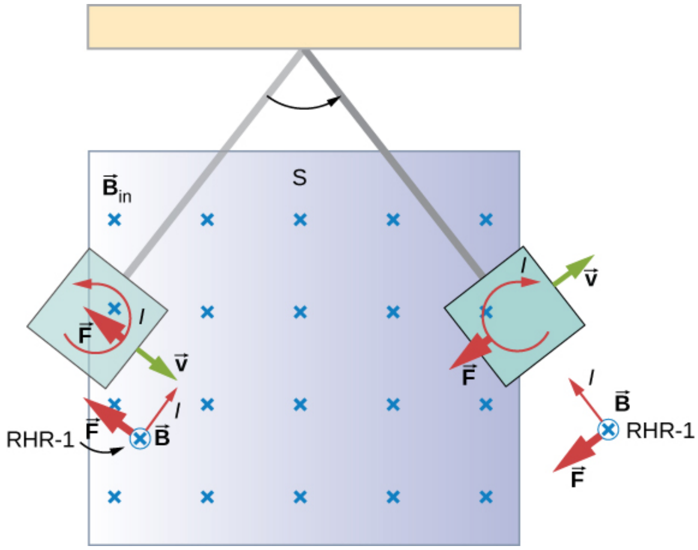


Figure 9.6.2: A more detailed look at the conducting plate passing between the poles of a magnet. As it enters and leaves the field, the change in flux produces an eddy current. Magnetic force on the current loop opposes the motion. There is no current and no magnetic drag when the plate is completely inside the uniform field.

When a slotted metal plate enters the field (Figure 9.6.3), an emf is induced by the change in flux, but it is less effective because the slots limit the size of the current loops. Moreover, adjacent loops have currents in opposite directions, and their effects cancel. When an insulating material is used, the eddy current is extremely small, so magnetic damping on insulators is negligible. If eddy currents are to be avoided in conductors, then they must be slotted or constructed of thin layers of conducting material separated by insulating sheets.

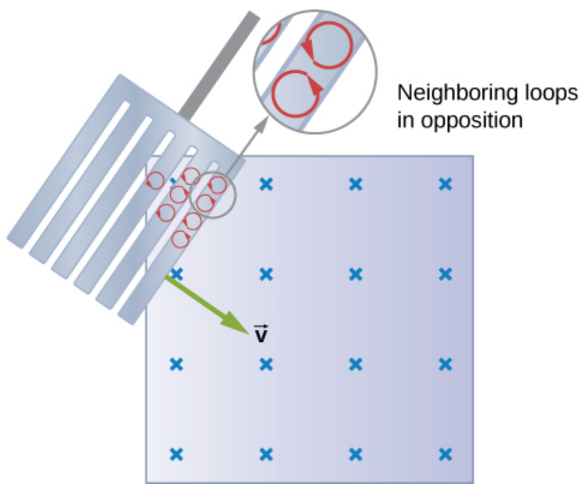


Figure 9.6.3: Eddy currents induced in a slotted metal plate entering a magnetic field form small loops, and the forces on them tend to cancel, thereby making magnetic drag almost zero.

Applications of Magnetic Damping

One use of magnetic damping is found in sensitive laboratory balances. To have maximum sensitivity and accuracy, the balance must be as friction-free as possible. But if it is friction-free, then it will oscillate for a very long time. Magnetic damping is a simple and ideal solution. With magnetic damping, drag is proportional to speed and becomes zero at zero velocity. Thus, the oscillations are quickly damped, after which the damping force disappears, allowing the balance to be very sensitive (Figure 9.6.4). In most balances, magnetic damping is accomplished with a conducting disc that rotates in a fixed field.

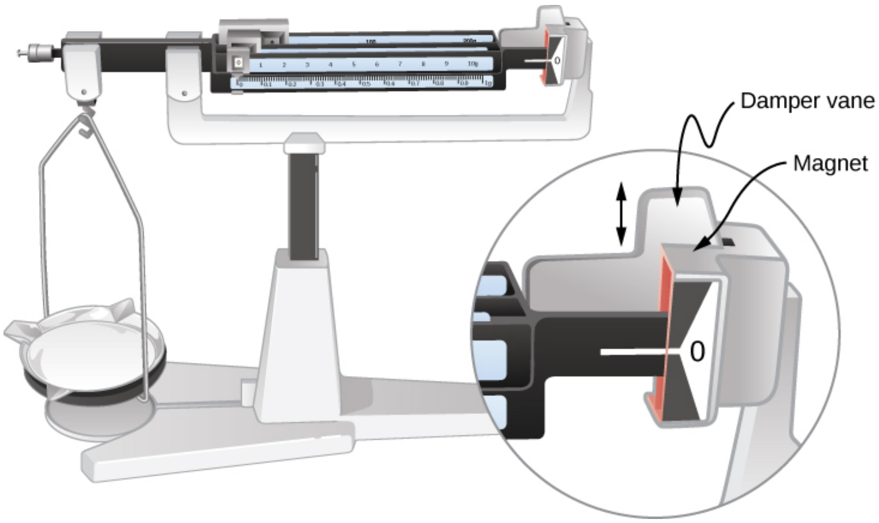


Figure 9.6.4: Magnetic damping of this sensitive balance slows its oscillations. Since Faraday's law of induction gives the greatest effect for the most rapid change, damping is greatest for large oscillations and goes to zero as the motion stops.

Since eddy currents and magnetic damping occur only in conductors, recycling centers can use magnets to separate metals from other materials. Trash is dumped in batches down a ramp, beneath which lies a powerful magnet. Conductors in the trash are slowed by magnetic damping while nonmetals in the trash move on, separating from the metals (Figure 9.6.5). This works for all metals, not just ferromagnetic ones. A magnet can separate out the ferromagnetic materials alone by acting on stationary trash.

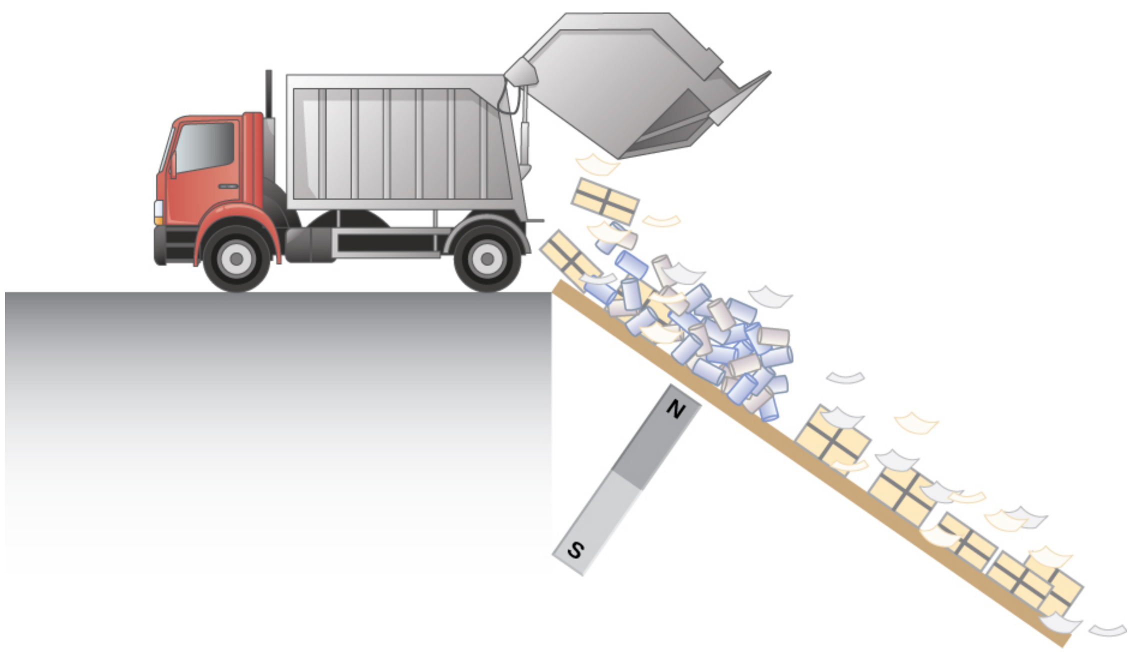


Figure 9.6.5: Metals can be separated from other trash by magnetic drag. Eddy currents and magnetic drag are created in the metals sent down this ramp by the powerful magnet beneath it. Nonmetals move on.

Other major applications of eddy currents appear in **metal detectors** and **braking systems** in trains and roller coasters. Portable metal detectors (Figure 9.6.6) consist of a primary coil carrying an alternating current and a secondary coil in which a current is induced. An eddy current is induced in a piece of metal close to the detector, causing a change in the induced current within the secondary coil. This can trigger some sort of signal, such as a shrill noise.

Figure $9.6.6$: A soldier in Iraq uses a metal detector to search for explosives and weapons. (credit: U.S. Army)

Braking using eddy currents is safer because factors such as rain do not affect the braking and the braking is smoother. However, eddy currents cannot bring the motion to a complete stop, since the braking force produced decreases as speed is reduced. Thus, speed can be reduced from say 20 m/s to 5 m/s, but another form of braking is needed to completely stop the vehicle. Generally, powerful rare-earth magnets such as neodymium magnets are used in roller coasters. Figure $9.6.7$ shows rows of magnets in such an application. The vehicle has metal fins (normally containing copper) that pass through the magnetic field, slowing the vehicle down in much the same way as with the pendulum bob shown in Figure $9.6.1$.

Figure $9.6.7$: The rows of rare-earth magnets (protruding horizontally) are used for magnetic braking in roller coasters. (credit: Stefan Scheer)

Induction cooktops have electromagnets under their surface. The magnetic field is varied rapidly, producing eddy currents in the base of the pot, causing the pot and its contents to increase in temperature. Induction cooktops have high efficiencies and good response times but the base of the pot needs to be conductors, such as iron or steel, for induction to work.

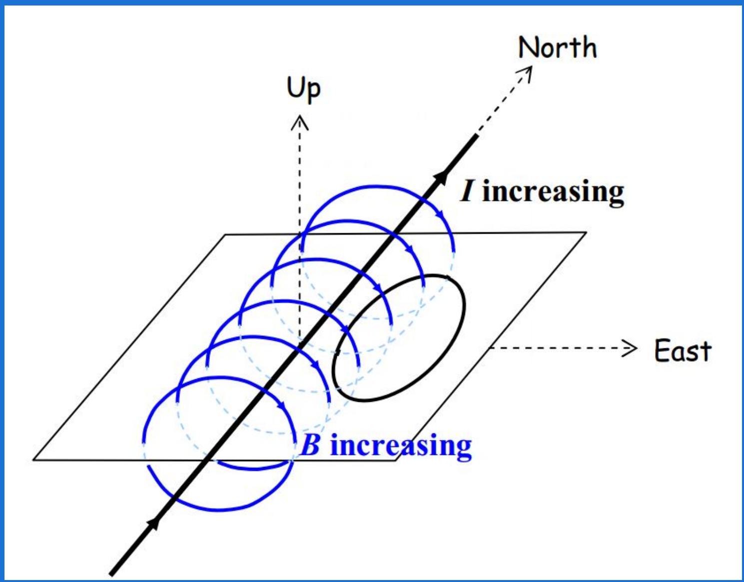
9.7: Induction, Transformers, and Generators

In this chapter we provide examples chosen to further familiarize you with Faraday's Law of Induction and Lenz's Law. The last example is the generator, the device used in the world's power plants to convert mechanical energy into electrical energy.

A straight wire carries a current due northward. Due east of the straight wire, at the same elevation as the straight wire, is a horizontal loop of wire. The current in the straight wire is increasing. Which way is the current, induced in the loop by the changing magnetic field of the straight wire, directed around the loop?

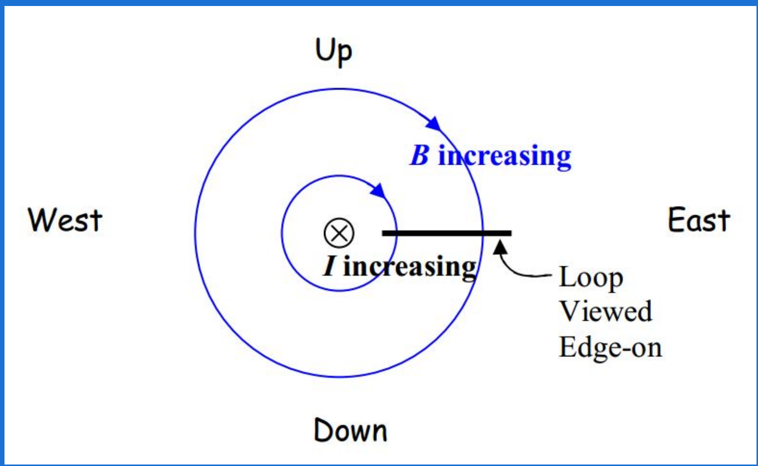
Solution

I’m going to draw the given situation from a few different viewpoints, just to help you get used to visualizing this kind of situation. As viewed from above-and-to-the-southeast, the configuration (aside from the fact that magnetic field lines are invisible) appears as:



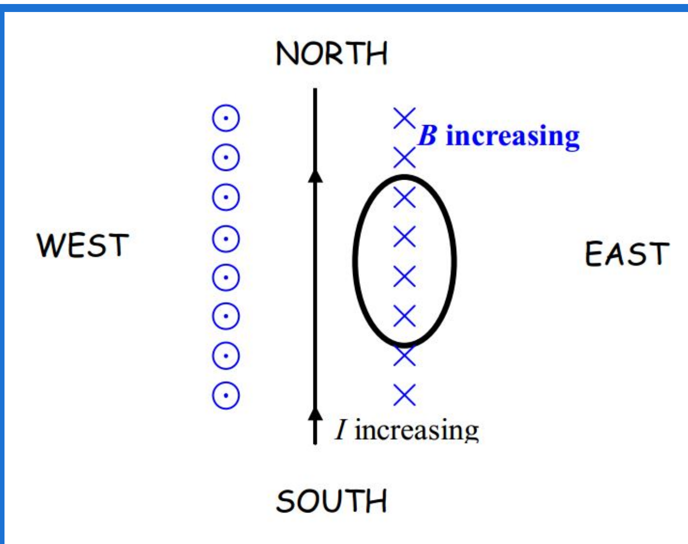
where I included a sheet of paper in the diagram to help you visualize things.

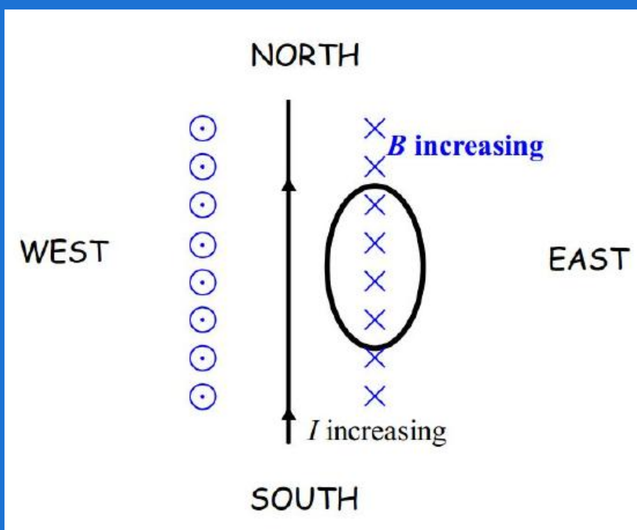
Here’s a view of the same configuration from the south, looking due north:



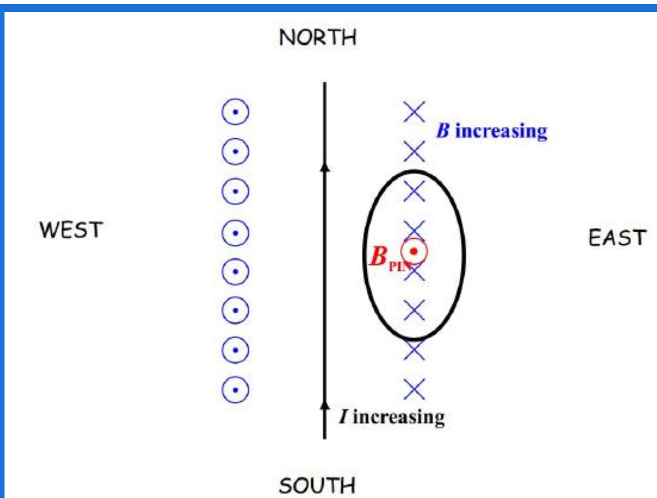
Both diagrams make it clear that we have an increasing number of downward-directed magnetic field lines through the loop. It is important to keep in mind that a field diagram is a diagrammatic manner of conveying information about an infinite set of vectors. There is no such thing as a curved vector. A vector is always directed along a straight line. The magnetic field vector is tangent to the magnetic field lines characterizing that vector. At the location of the loop, every magnetic field vector depicted in the diagram above is straight downward. While it is okay to say that we have an increasing number of magnetic field lines directed downward through the loop, please keep in mind that the field lines characterize vectors.

In presenting my solution to the example question, “What is the direction of the current induced in a horizontal loop that is due east of a straight wire carrying an increasing current due north?” I wouldn’t draw either one of the diagrams above. The first one takes too long to draw and there is no good way to show the direction of the current in the loop in the second one. The view from above is the most convenient one:

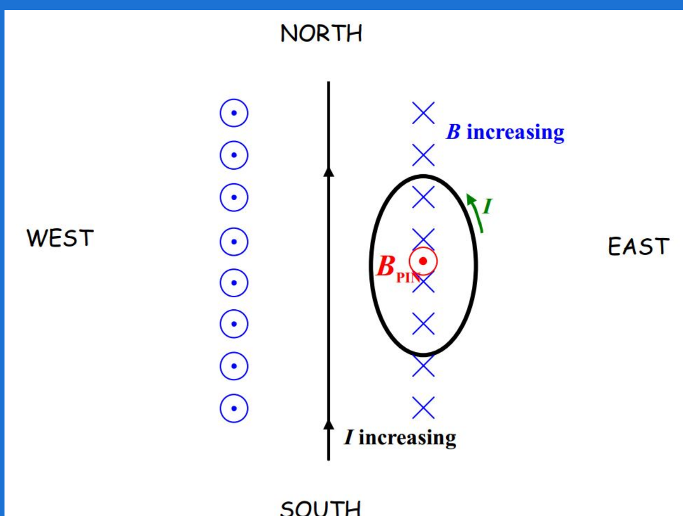




In this view (in which the downward direction is into the page) it is easy to see that what we have is an increasing number of downward-directed magnetic field lines through the loop (more specifically, through the region enclosed by the loop). In its futile attempt to keep the number of magnetic field lines directed downward through the loop the same as what it was, $\vec{B}_{PIN}$ must be directed upward in order to cancel out the newly-appearing downward-directed magnetic field lines. [Recall the sequence: The changing number of magnetic field lines induces (by Faraday's Law) a current in the loop. That current produces (by Ampere's Law) a magnetic field ($\vec{B}_{PIN}$) of its own. Lenz's Law relates the end product ($\vec{B}_{PIN}$) to the original change (increasing number of downward-through-the-loop magnetic field lines).]

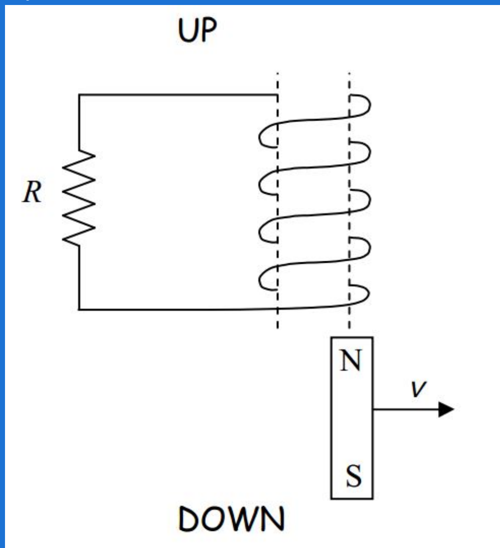


That's interesting. We know the direction of the magnetic field produced by the induced current before we even known the direction of the induced current itself. So, what must the direction of the induced current be in order to produce an upward-directed magnetic field ($\vec{B}_{PIN}$)? Well, by the right-hand rule for something curly something straight, the current must be counterclockwise, as viewed from above.

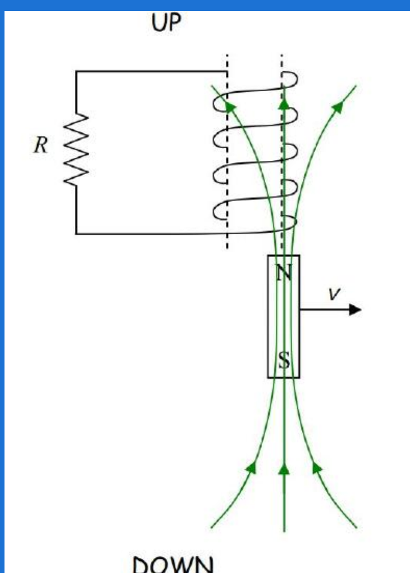


Hey. That's the answer to the question. We're done with that example. Here's another one:

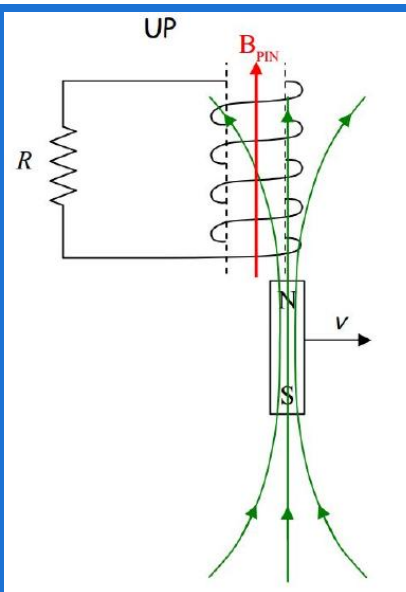
A person is moving a bar magnet, aligned north pole up, out from under a coil of wire, as depicted below. What is the direction of the current in the resistor?



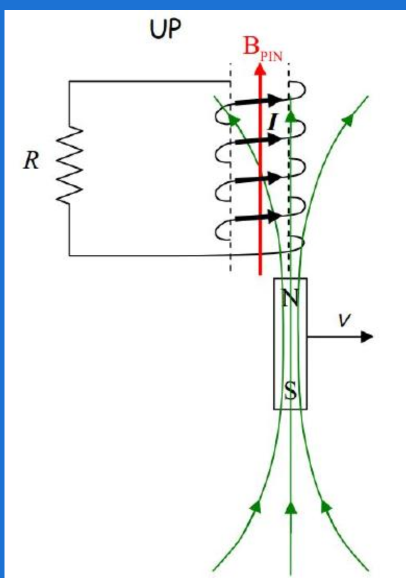
The magnetic field of the bar magnet extends upward through the coil.



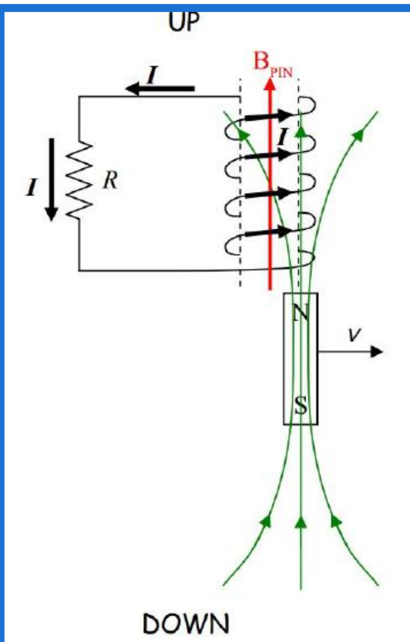
As the magnet moves out from under the coil, it takes its magnetic field with it. So, as regards the coil, what we have is a decreasing number of upward-directed magnetic field lines through the coil. By Faraday's Law, this induces a current in the coil. By Ampere's Law, the current produces a magnetic field, $\vec{B}_{PIN}$. By Lenz's Law $\vec{B}_{PIN}$ is upward, to make up for the departing upward-directed magnetic field lines through the coil.



So, what is the direction of the current that is causing $\vec{B}_{PIN}$? The right-hand rule will tell us that. Point the thumb of your cupped right hand in the direction of $\vec{B}_{PIN}$. Your fingers will then be curled around (counterclockwise as viewed from above) in the direction of the current.

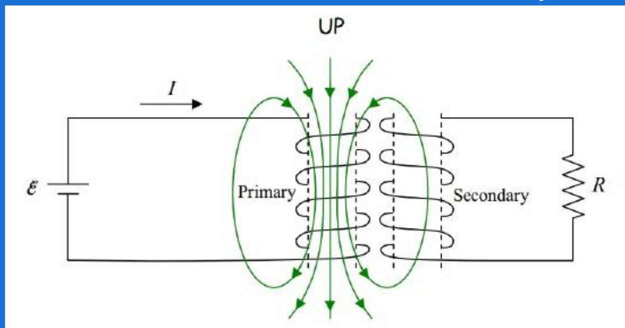


Because of the way the coil is wound, such a current will be directed out the top of the coil downward through the resistor.

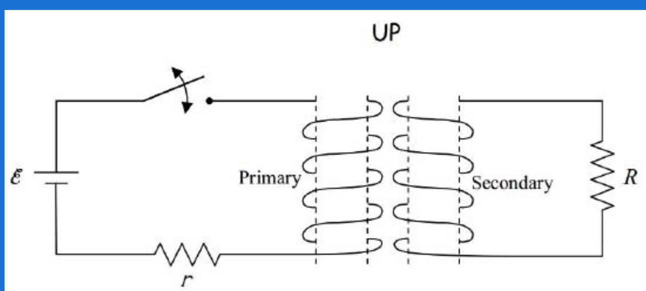


That’s the answer to the question posed in the example. (What is the direction of the current in the resistor?)

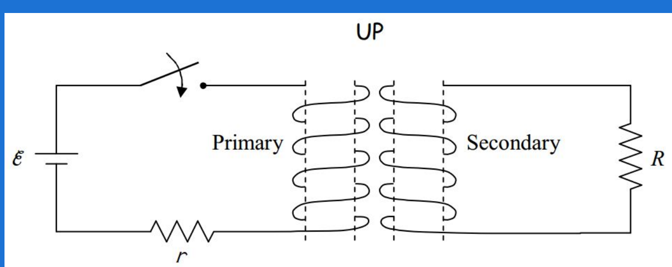
When you put two coils of wire near each other, such that when you create a magnetic field by using a seat of EMF to cause a current in one coil, that magnetic field extends through the region encircled by the other coil, you create a transformer. Let's call the coil in which you initially cause the current, the primary coil, and the other one, the secondary coil.



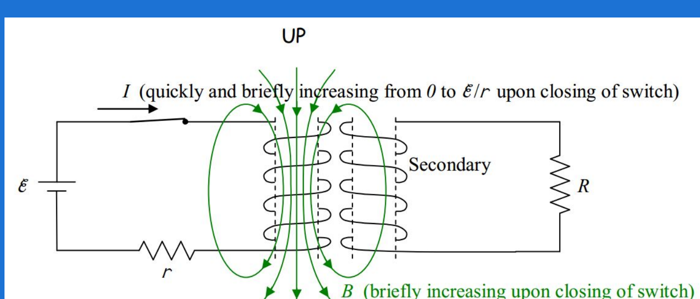
If you cause the current in the primary coil to be changing, then the magnetic field produced by that coil is changing. Thus, the flux through the secondary coil is changing, and, by Faraday's Law of Induction, a current will be induced in the secondary coil. One way to cause the current in the primary coil to be changing would be to put a switch in the primary circuit (the circuit in which the primary coil is wired) and to repeatedly open and close it.



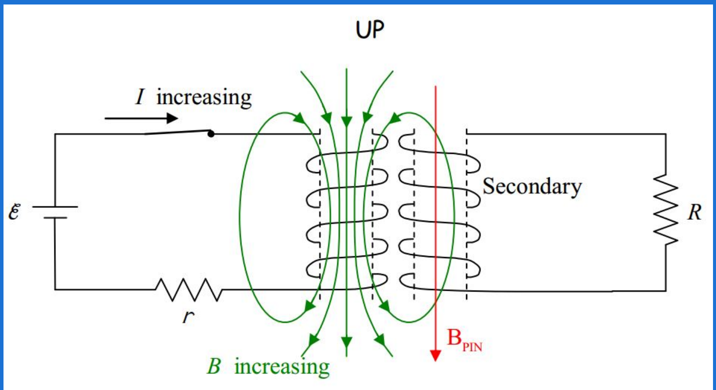
Okay, enough preamble, here's the question: What is the direction of the transient current induced in the circuit above when the switch is closed?



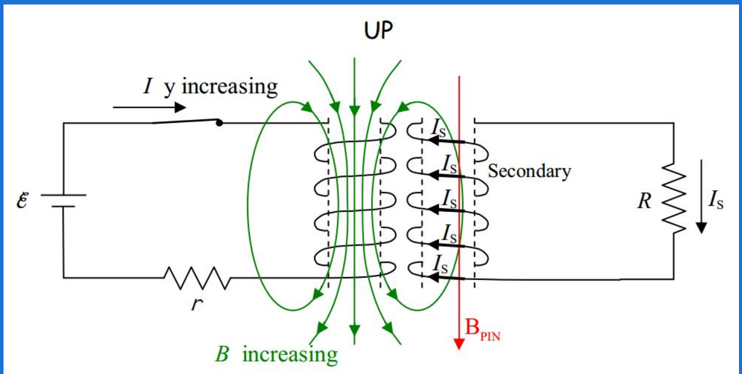
Upon closing the switch, the current in the primary circuit very quickly builds up to ε/r. While the time that it takes for the current to build up to ε/r is very short, it is during this time interval that the current is changing. Hence, it is on this time interval that we must focus our attention in order to answer the question about the direction of the transient current in resistor R in the secondary circuit. The current in the primary causes a magnetic field. Because the current is increasing, the magnetic field vector at each point in space is increasing in magnitude.



The increasing magnetic field causes upward-directed magnetic field lines in the region encircled by the secondary coil. There were no magnetic field lines through that coil before the switch was closed, so clearly, what we have here is an increasing number of upward-directed magnetic field lines through the secondary coil. By Faraday's Law this will induce a current in the coil. By Ampere's law, the current induced in the secondary will produce a magnetic field of its own, one that I like to call $\vec{B}_{PIN}$ for "The Magnetic field Produced by the Induced Current." By Lenz's Law, $\vec{B}_{PIN}$ must be downward to cancel out some of the newly-appearing upward-directed magnetic field lines through the secondary. (I hope it is clear that what I call the magnetic field lines through the secondary, are the magnetic field lines passing through the region encircled by the secondary coil.)



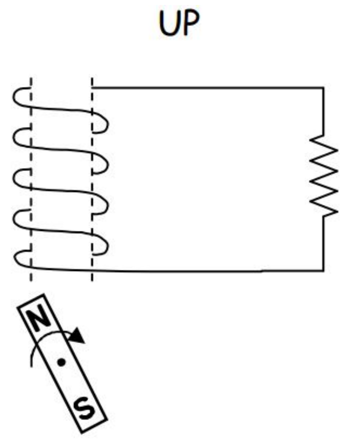
Okay. Now the question is, which way must the current be directed around the coil in order to create the downward-directed magnetic field $\vec{B}_{PIN}$ that we have deduced it does create. As usual, the right-hand rule for something curly something straight reveals the answer. We point the thumb of the cupped right hand in the direction of $\vec{B}_{PIN}$ and cannot fail to note that the fingers curl around in a direction that can best be described as "clockwise as viewed from above."



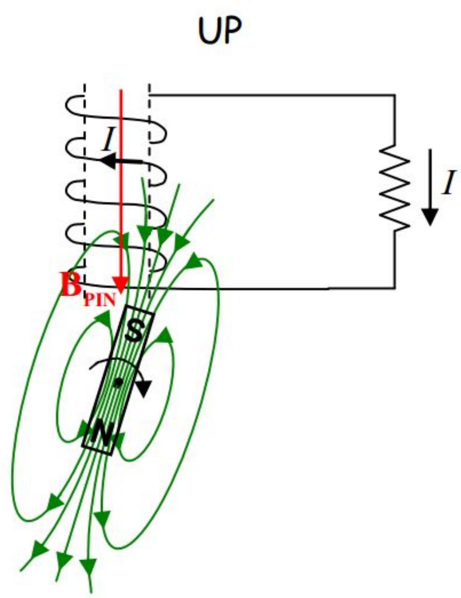
Because of the way the secondary coil is wound, such a current will be directed out of the secondary at the top of the coil and downward through resistor R. This is the answer to the question posed in the example.

An Electric Generator

Consider a magnet that is caused to rotate in the vicinity of a coil of wire as depicted below.

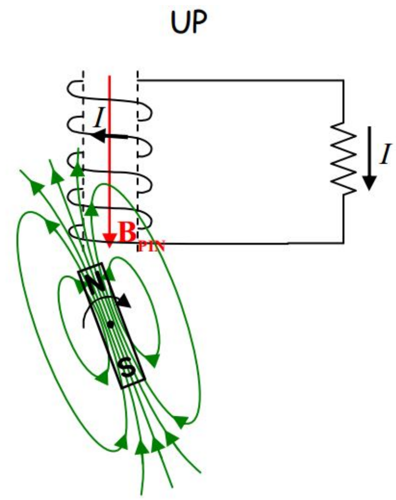


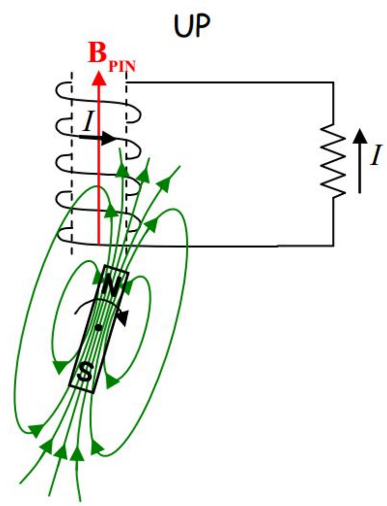
As a result of the rotating magnet, the number and direction of the magnetic field lines through the coil is continually changing. This induces a current in the coil, which, as it turns out, is also changing. Check it out in the case of magnet that is, from our viewpoint, rotating clockwise. In the orientation of the rotating magnet depicted here:

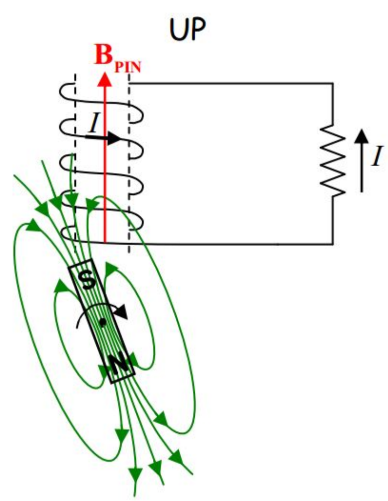


as the magnet rotates, the number of its magnetic field lines extending downward through the coil is decreasing. In accord with Faraday's Law, this induces a current in the coil which, in accord with Ampere's Law, produces a magnetic field of its own. By Lenz's Law, the field ($\vec{B}_{PIN}$) produced by the induced current must be downward to make up for the loss of downwarddirected magnetic field lines through the coil. To produce $\vec{B}_{PIN}$ downward, the induced current must be clockwise, as viewed from above. Based on the way the wire is wrapped and the coil is connected in the circuit, a current that is clockwise as viewed from above, in the coil, is directed out of the coil at the top of the coil and downward through the resistor.

In the following diagrams we show the magnet in each of several successive orientations. Keep in mind that someone or something is spinning the magnet by mechanical means. You can assume for instance that a person is turning the magnet with her hand. As the magnet turns the number of magnetic field lines is changing in a specific manner for each of the orientations depicted. You the reader are asked to apply Lenz's Law and the Right Hand Rule for Something Curley, Something Straight to verify that the current (caused by the spinning magnet) through the resistor is in the direction depicted:





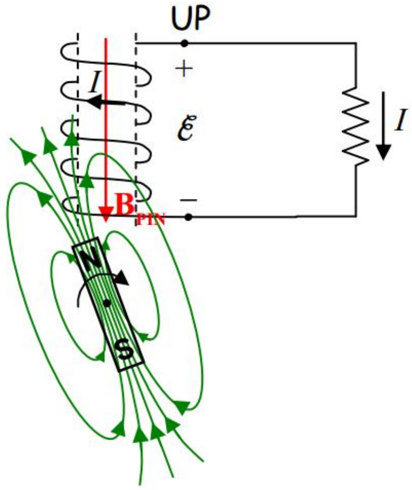


As the magnet continues to rotate clockwise, the next orientation it achieves is our starting point and the process repeats itself over and over again.

Recapping and extrapolating, the current through the resistor in the series of diagrams above, is:

downward, downward, upward, upward, downward, downward, upward, upward, …

For half of each rotation, the current is downward, and for the other half of each rotation, the current is upward. In quantifying this behavior, one focuses on the EMF induced in the coil:

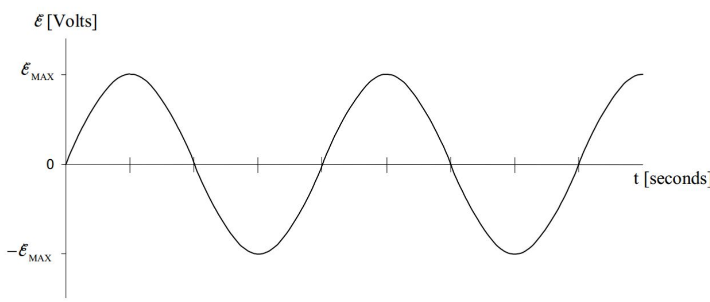


The EMF across the coil varies sinusoidally with time as:

$$\varepsilon=\varepsilon_{MAX}\sin(2\pi ft) \tag{9.7.1}$$

where:

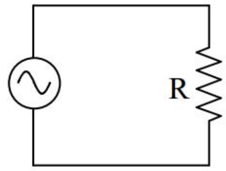
- ε which stands for EMF, is the time-varying electric potential difference between the terminals of a coil in close proximity to a magnet that is rotating relative to the coil as depicted in the diagrams above. This potential difference is caused to exist, and to vary the way it does, by the changing magnetic flux through the coil.
- ε_{MAX} is the maximum value of the EMF of the coil.
- f is the frequency of oscillations of the EMF across the coil. It is exactly equal to the rotation rate of the magnet expressed in rotations per second, a unit that is equivalent to hertz.



The device that we have been discussing (coil-plus-rotating magnet) is called a generator, or more specifically, an electric generator. A generator is a seat of EMF that causes there to be a potential difference between its terminals that varies sinusoidally with time. The schematic representation of such a time-varying seat of EMF is:

It takes work to spin the magnet. The magnetic field caused by the current induced in the coil exerts a torque on the magnet that always tends to slow it down. So, to keep the magnet spinning, one must continually exert a torque on the magnet in the direction in which it is spinning. The generator is the main component of any electrical power plant. It converts mechanical energy to electrical energy. The kind of power plant you are dealing with is determined by what your power company uses to spin the magnet. If moving water is used to spin the magnet, we call the power plant a hydroelectric plant. If a steam turbine is used to spin the magnet, then the power plant is designated by its method of heating and vaporizing water. For instance, if one heats and vaporizes the water by means of burning coal, one calls the power plant a coal-fired power plant. If one heats and vaporizes the water by means of a nuclear reactor, one calls the power plant a nuclear power plant.

Consider a “device which causes a potential difference between its terminals that varies sinusoidally with time” in a simple circuit:

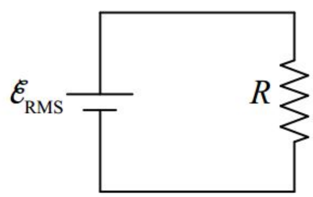


The time-varying seat of EMF causes a potential difference across the resistor, in this simple circuit, equal, at any instant in time, to the voltage across the time-varying seat of EMF. As a result, there is a current in the resistor. The current is given by $I=\frac{V}{R}$, our defining equation for resistance, solved for the current I. Because the algebraic sign of the potential difference across the resistor is continually alternating, the direction of the current in the resistor is continually alternating. Such a current is called an alternating current (AC). It has become traditional to use the abbreviation AC) to the extent that we do so in a redundant fashion, often referring to an alternating current as an AC current. (When we need to distinguish it from AC, we call the “oneway” kind of current that, say, a battery causes in a circuit, direct current, abbreviated DC.)

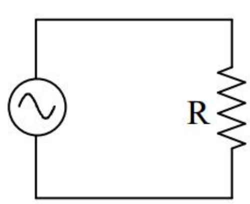
A device that causes current in a resistor, whether that current is alternating or not, is delivering energy to the resistor at a rate that we call power. The power delivered to a resistor can be expressed as $P=IV$ where I is the current through the resistor and V is the voltage across the resistor. Using the defining equation of resistance, $V=IR$, the power can be expressed as $P=I^2R$. A “device which causes a potential difference between its terminals that varies sinusoidally with time”, what I have been referring to as a “time-varying seat of EMF” is typically referred to as an AC power source. An AC power source is typically referred to in terms of the frequency of oscillations, and, the voltage that a DC power source, an ordinary seat of EMF, would have to maintain across its terminals to cause the same average power in any resistor that might be connected across the terminals of the AC power source. The voltage in question is typically referred to as ε_{RMS} or V_{RMS} where the reasoning behind the name of the subscript will become evident shortly.

Since the power delivered by an ordinary seat of EMF is a constant, its average power is the value it always has.

Here’s the fictitious circuit



that would cause the same resistor power as the AC power source in question. The average power (which is just the power in the case of a DC circuit) is given by $P_{AVG}=I\varepsilon_{RMS}$, which, by means of our defining equation of resistance solved for I, $I=V/R$, (where the voltage across the resistor is, by inspection, ε_{RMS}) can be written $P_{AVG}=\frac{\varepsilon_{RMS}^2}{R}$. So far, this is old stuff, with an unexplained name for the EMF voltage. Now let’s consider the AC circuit:



The power is $P=\frac{\varepsilon^2}{R}=\frac{[\varepsilon_{MAX}\sin(2\pi ft)]^2}{R}=\frac{\varepsilon_{MAX}^2[\sin(2\pi ft)]^2}{R}$. The average value of the square of the sine function is $\frac{1}{2}$. So the average power is $P_{AVG}=\frac{1}{2}\frac{\varepsilon_{MAX}^2}{R}$. Combining this with our expression $P_{AVG}=\frac{\varepsilon_{RMS}^2}{R}$ from above yields:

$$\frac{\varepsilon_{RMS}^2}{R} = \frac{1}{2}\frac{\varepsilon_{MAX}^2}{R}$$

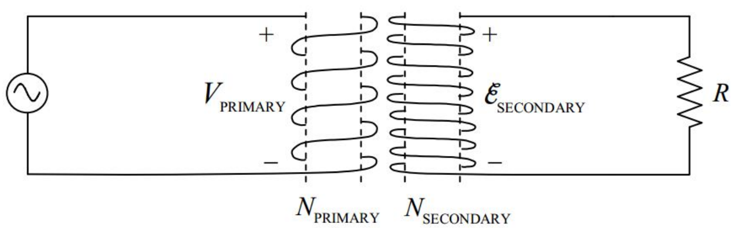
$$\varepsilon_{RMS} = \sqrt{\frac{1}{2}}\varepsilon_{MAX} \tag{9.7.2}$$

Now we are in a position to explain why we called the equivalent EMF, ε_{RMS}. In our expression $P_{AVG} = \frac{1}{2}\frac{\varepsilon_{MAX}^2}{R}$, we can consider $\frac{\varepsilon_{MAX}^2}{2}$ to be the average value of the square of our timevarying EMF $\varepsilon = \varepsilon_{MAX} = \sin(2\pi f t)$. Another name for "average" is "mean" so we can consider $\frac{\varepsilon_{MAX}^2}{2}$ to be the mean value of ε^2. On the right side of our expression for our equivalent EMF, $\varepsilon_{RMS} = \frac{1}{\sqrt{2}}\varepsilon_{MAX}$, we have the square root of $\frac{\varepsilon_{MAX}^2}{2}$, that is, we have the square root of the mean of the square of the EMF ε. And indeed the subscript "RMS" stands for "root mean squared." RMS values are convenient for circuits consisting of resistors and AC power sources in that, one can analyze such circuits using RMS values the same way one analyzes DC circuits.

More on the Transformer

When the primary coil of a transformer is driven by an AC power source, it creates a magnetic field which varies sinusoidally in such a manner as to cause a sinusoidal EMF, of the same frequency as the source, to be induced in the secondary coil. The RMS value of the EMF induced in the secondary coil is directly proportional to the RMS value of the sinusoidal potential difference imposed across the primary. The constant of proportionality is the ratio of the number of turns in the secondary to the number of turns in the primary.

$$\varepsilon_{SECONDARY} = \frac{N_{SECONDARY}}{N_{PRIMARY}} V_{PRIMARY} \tag{9.7.3}$$



When the number of windings in the secondary coil is greater than the number of windings in the primary coil, the transformer is said to be a step-up transformer and the secondary voltage is greater than the primary voltage. When the number of windings in the secondary coil is less than the number of windings in the primary coil, the transformer is said to be a step-down transformer and the secondary voltage is less than the primary voltage.

The Electrical Power in Your House

When you plug your toaster into a wall outlet, you bring the prongs of the plug into contact with two conductors between which there is a time-varying potential difference characterized as 115 volts 60 Hz AC. The 60 Hz is the frequency of oscillations of the potential difference resulting from a magnet completing 60 rotations per second, back at the power plant. A step-up transformer is used near the power plant to step the power plant output up to a high voltage. Transmission lines at a very high potential, with respect to each other, provide a conducting path to a transformer near your home where the voltage is stepped down. Power lines at a much lower potential provide the conducting path to the wires in your home. 115 volts is the RMS value of the potential difference between the two conductors in each pair of slots in your wall outlets. Since $\varepsilon_{RMS} = \frac{1}{\sqrt{2}}\varepsilon_{MAX}$, we have $\varepsilon_{MAX} = \sqrt{2}\,\varepsilon_{RMS}$, so $\varepsilon_{MAX} = \sqrt{2}(115 \text{ volts})$, or $\varepsilon_{MAX} = 163$ volts. Thus,

$$\varepsilon = (163 \text{ volts}) \sin[2\pi(60 Hz)t]$$

which can be written as,

$$\varepsilon = (163 \text{ volts}) \sin[(377 \frac{\text{rad}}{s} t]$$

9.8: Electromagnetic Induction (Exercises)

Conceptual Questions

13.2 Faraday's Law

1. A stationary coil is in a magnetic field that is changing with time. Does the emf induced in the coil depend on the actual values of the magnetic field?

2. In Faraday's experiments, what would be the advantage of using coils with many turns?

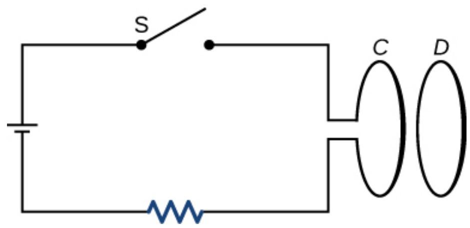
3. A copper ring and a wooden ring of the same dimensions are placed in magnetic fields so that there is the same change in magnetic flux through them. Compare the induced electric fields and currents in the rings.

4. Discuss the factors determining the induced emf in a closed loop of wire.

5. (a) Does the induced emf in a circuit depend on the resistance of the circuit?

(b) Does the induced current depend on the resistance of the circuit?

6. How would changing the radius of loop **D** shown below affect its emf, assuming **C** and **D** are much closer together compared to their radii?



7. Can there be an induced emf in a circuit at an instant when the magnetic flux through the circuit is zero?

8. Does the induced emf always act to decrease the magnetic flux through a circuit?

9. How would you position a flat loop of wire in a changing magnetic field so that there is no induced emf in the loop?

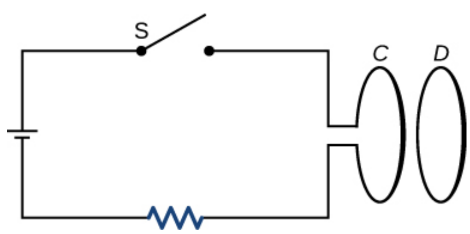
10. The normal to the plane of a single-turn conducting loop is directed at an angle **θ** to a spatially uniform magnetic field $vecB$. It has a fixed area and orientation relative to the magnetic field. Show that the emf induced in the loop is given by $\varepsilon = (dB/dt)(A cos\theta)$,where A is the area of the loop.

13.3 Lenz's Law

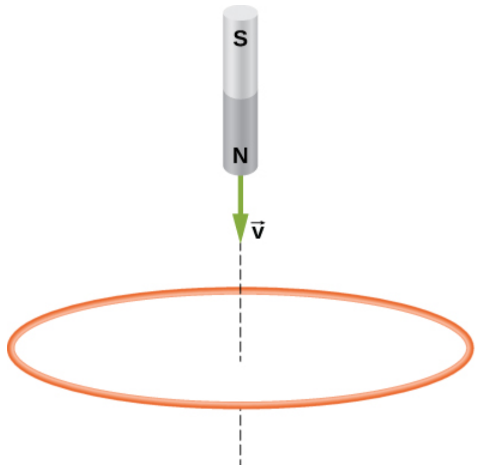
11. The circular conducting loops shown in the accompanying figure are parallel, perpendicular to the plane of the page, and coaxial.

(a) When the switch S is closed, what is the direction of the current induced in **D**?

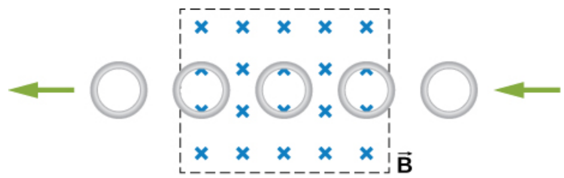
(b) When the switch is opened, what is the direction of the current induced in loop **D**?



12. The north pole of a magnet is moved toward a copper loop, as shown below. If you are looking at the loop from above the magnet, will you say the induced current is circulating clockwise or counterclockwise?

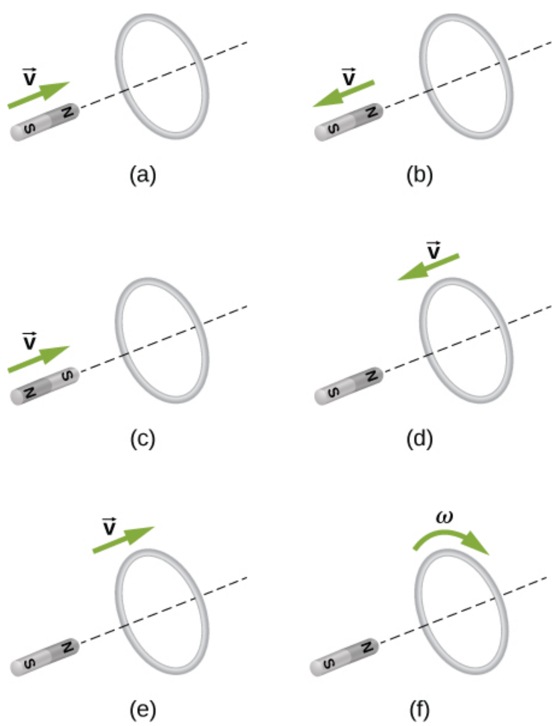


13. The accompanying figure shows a conducting ring at various positions as it moves through a magnetic field. What is the sense of the induced emf for each of those positions?



14. Show that **ε** and $d\Phi_m/dt$ have the same units.

15. State the direction of the induced current for each case shown below, observing from the side of the magnet.



13.4 Motional Emf

16. A bar magnet falls under the influence of gravity along the axis of a long copper tube. If air resistance is negligible, will there be a force to oppose the descent of the magnet? If so, will the magnet reach a terminal velocity?

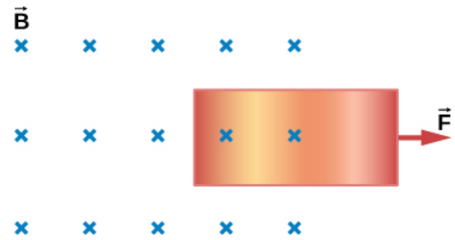
17. Around the geographic North Pole (or magnetic South Pole), Earth's magnetic field is almost vertical. If an airplane is flying northward in this region, which side of the wing is positively charged and which is negatively charged?

18. A wire loop moves translationally (no rotation) in a uniform magnetic field. Is there an emf induced in the loop?

13.5 Induced Electric Fields

19. Is the work required to accelerate a rod from rest to a speed v in a magnetic field greater than the final kinetic energy of the rod? Why?

20. The copper sheet shown below is partially in a magnetic field. When it is pulled to the right, a resisting force pulls it to the left. Explain. What happen if the sheet is pushed to the left?



13.6 Eddy Currents

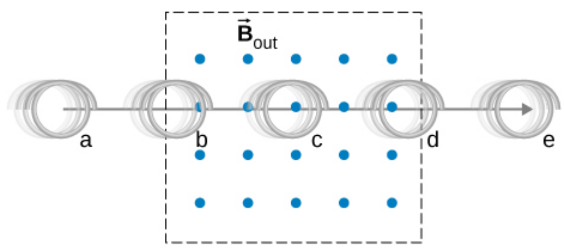
21. A conducting sheet lies in a plane perpendicular to a magnetic field $\vec{B}$ that is below the sheet. If $\vec{B}$ oscillates at a high frequency and the conductor is made of a material of low resistivity, the region above the sheet is effectively shielded from $\vec{B}$. Explain why. Will the conductor shield this region from static magnetic fields?

22. Electromagnetic braking can be achieved by applying a strong magnetic field to a spinning metal disk attached to a shaft.

(a) How can a magnetic field slow the spinning of a disk?

(b) Would the brakes work if the disk was made of plastic instead of metal?

23. A coil is moved through a magnetic field as shown below. The field is uniform inside the rectangle and zero outside. What is the direction of the induced current and what is the direction of the magnetic force on the coil at each position shown?



Problems

13.2 Faraday's Law

24. A 50-turn coil has a diameter of 15 cm. The coil is placed in a spatially uniform magnetic field of magnitude 0.50 T so that the face of the coil and the magnetic field are perpendicular. Find the magnitude of the emf induced in the coil if the magnetic field is reduced to zero uniformly in

(a) 0.10 s,

(b) 1.0 s, and

(c) 60 s.

25. Repeat your calculations of the preceding problem's time of 0.1 s with the plane of the coil making an angle of

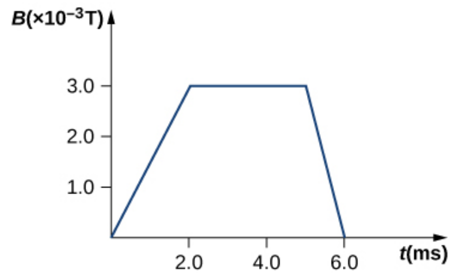
(a) **30°**,

(b) **60°,** and

(c) **90°** with the magnetic field.

26. A square loop whose sides are 6.0-cm long is made with copper wire of radius 1.0 mm. If a magnetic field perpendicular to the loop is changing at a rate of 5.0 mT/s, what is the current in the loop?

27. The magnetic field through a circular loop of radius 10.0 cm varies with time as shown below. The field is perpendicular to the loop. Plot the magnitude of the induced emf in the loop as a function of time.



28. The accompanying figure shows a single-turn rectangular coil that has a resistance of 2.0Ω.2.0Ω. The magnetic field at all points inside the coil varies according to $B=B_0e^{-\alpha t}$, where $B_0=0.25T$ and **α=200Hz**. What is the current induced in the coil at

(a) **t=0.001s**,

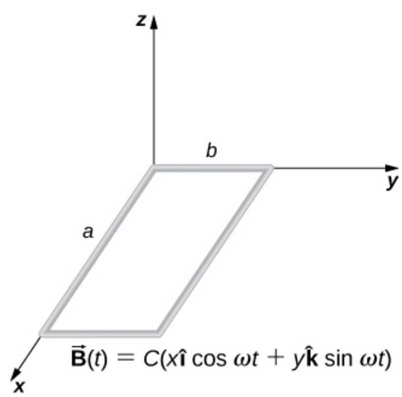
(b) **0.002 s**,

(c) **2.0 s**?

29. How would the answers to the preceding problem change if the coil consisted of 20 closely spaced turns?

30. A long solenoid with **n=10** turns per centimeter has a cross-sectional area of $5.0cm^2$ and carries a current of 0.25 A. A coil with five turns encircles the solenoid. When the current through the solenoid is turned off, it decreases to zero in 0.050 s. What is the average emf induced in the coil?

31. A rectangular wire loop with length a and width b lies in the xy-plane, as shown below. Within the loop there is a time-dependent magnetic field given by $\vec{B}(t)=C((xcos\omega t)\hat{i}+(ysin\omega t)\hat{k})$, with $\vec{B}(t)$ in tesla. Determine the emf induced in the loop as a function of time.



32. The magnetic field perpendicular to a single wire loop of diameter 10.0 cm decreases from 0.50 T to zero. The wire is made of copper and has a diameter of 2.0 mm and length 1.0 cm. How much charge moves through the wire while the field is changing?

13.3 Lenz's Law

33. A single-turn circular loop of wire of radius 50 mm lies in a plane perpendicular to a spatially uniform magnetic field. During a 0.10-s time interval, the magnitude of the field increases uniformly from 200 to 300 mT.

(a) Determine the emf induced in the loop.

(b) If the magnetic field is directed out of the page, what is the direction of the current induced in the loop?

34. When a magnetic field is first turned on, the flux through a 20-turn loop varies with time according to $\Phi_m = 5.0t^2 - 2.0t$, where Φ_m is in milliwebers, t is in seconds, and the loop is in the plane of the page with the unit normal pointing outward.

(a) What is the emf induced in the loop as a function of time? What is the direction of the induced current at

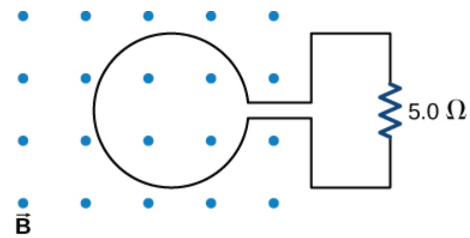
(b) t = 0,

(c) 0.10,

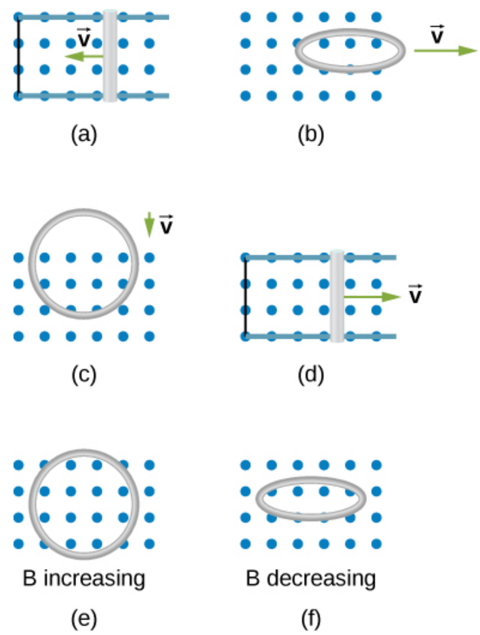
(d) 1.0, and

(e) 2.0 s?

35. The magnetic flux through the loop shown in the accompanying figure varies with time according to $\Phi_m = 2.00e^{-3t} sin(120\pi t)$,where Φ_m is in milliwebers. What are the direction and magnitude of the current through the **5.00-Ω** resistor at (a) **t=0**t=0; (b) $t = 2.17 \times 10^{-2} s$, and (c) **t=3.00s**?



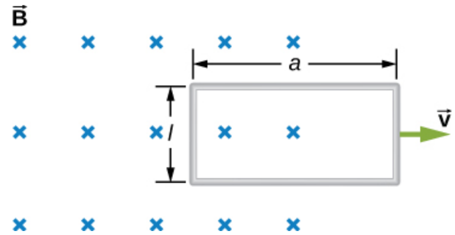
36. Use Lenz's law to determine the direction of induced current in each case.



13.4 Motional Emf

37. An automobile with a radio antenna 1.0 m long travels at 100.0 km/h in a location where the Earth's horizontal magnetic field is $5.5 \times 10^{-5} T$. What is the maximum possible emf induced in the antenna due to this motion?

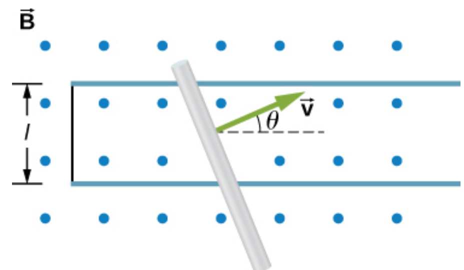
38. The rectangular loop of N turns shown below moves to the right with a constant velocity $\vec{v}$ while leaving the poles of a large electromagnet. (a) Assuming that the magnetic field is uniform between the pole faces and negligible elsewhere, determine the induced emf in the loop. (b) What is the source of work that produces this emf?



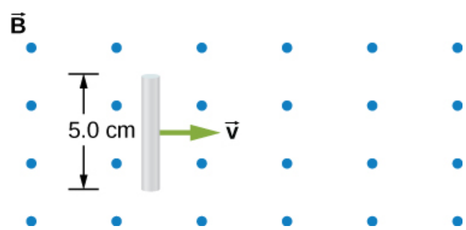
39. Suppose the magnetic field of the preceding problem oscillates with time according to $B = B_0 \sin \omega t$. What then is the emf induced in the loop when its trailing side is a distance d from the right edge of the magnetic field region?

40. A coil of 1000 turns encloses an area of$25 cm^2$. It is rotated in 0.010 s from a position where its plane is perpendicular to Earth's magnetic field to one where its plane is parallel to the field. If the strength of the field is $6.0 \times 10^{-5} T$, what is the average emf induced in the coil?

41. In the circuit shown in the accompanying figure, the rod slides along the conducting rails at a constant velocity $\vec{v}$. The velocity is in the same plane as the rails and directed at an angle θθ to them. A uniform magnetic field $\vec{B}$ is directed out of the page. What is the emf induced in the rod?



42. The rod shown in the accompanying figure is moving through a uniform magnetic field of strength $B = 0.50 T$ with a constant velocity of magnitude $v = 8.0 m/s.$. What is the potential difference between the ends of the rod? Which end of the rod is at a higher potential?



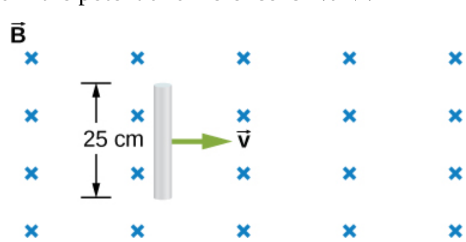
43. A 25-cm rod moves at 5.0 m/s in a plane perpendicular to a magnetic field of strength 0.25 T. The rod, velocity vector, and magnetic field vector are mutually perpendicular, as indicated in the accompanying figure. Calculate

(a) the magnetic force on an electron in the rod,

(b) the electric field in the rod, and

(c) the potential difference between the ends of the rod.

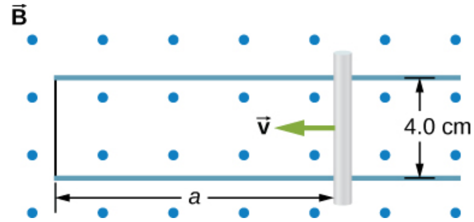
(d) What is the speed of the rod if the potential difference is 1.0 V?



44. In the accompanying figure, the rails, connecting end piece, and rod all have a resistance per unit length of **2.0Ω/cm**. The rod moves to the left at **v=3.0m/s**. If **B=0.75T** everywhere in the region, what is the current in the circuit

(a) when **a=8.0cm**?

(b) when **a=5.0cm**? Specify also the sense of the current flow.



45. The rod shown below moves to the right on essentially zero-resistance rails at a speed of **v=3.0m/s**. If **B=0.75T** everywhere in the region, what is the current through the **5.0-Ω** resistor? Does the current circulate clockwise or counterclockwise?

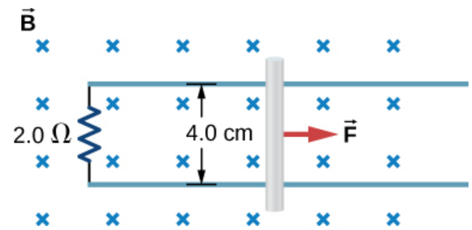
46. Shown below is a conducting rod that slides along metal rails. The apparatus is in a uniform magnetic field of strength 0.25 T, which is directly into the page. The rod is pulled to the right at a constant speed of 5.0 m/s by a force $\vec{F}$. The only significant resistance in the circuit comes from the **2.0-Ω** resistor shown.

(a) What is the emf induced in the circuit?

(b) What is the induced current? Does it circulate clockwise or counter clockwise?

(c) What is the magnitude of $\vec{F}$?

(d) What are the power output of $\vec{F}$ and the power dissipated in the resistor?



13.5 Induced Electric Fields

47. Calculate the induced electric field in a 50-turn coil with a diameter of 15 cm that is placed in a spatially uniform magnetic field of magnitude 0.50 T so that the face of the coil and the magnetic field are perpendicular. This magnetic field is reduced to zero in 0.10 seconds. Assume that the magnetic field is cylindrically symmetric with respect to the central axis of the coil.

48. The magnetic field through a circular loop of radius 10.0 cm varies with time as shown in the accompanying figure. The field is perpendicular to the loop. Assuming cylindrical symmetry with respect to the central axis of the loop, plot the induced electric field in the loop as a function of time.

49. The current I through a long solenoid with n turns per meter and radius R is changing with time as given by dI/dt. Calculate the induced electric field as a function of distance r from the central axis of the solenoid.

50. Calculate the electric field induced both inside and outside the solenoid of the preceding problem if $I = I_0 sin\omega t.$.

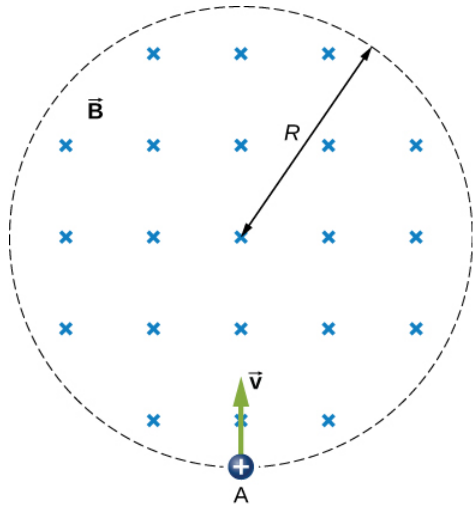
51. Over a region of radius R, there is a spatially uniform magnetic field $\vec{B}$. (See below.) At **t=0**, **B=1.0T**, after which it decreases at a constant rate to zero in 30 s.

(a) What is the electric field in the regions where $r \leq R$ and $r \geq R$ during that 30-s interval?

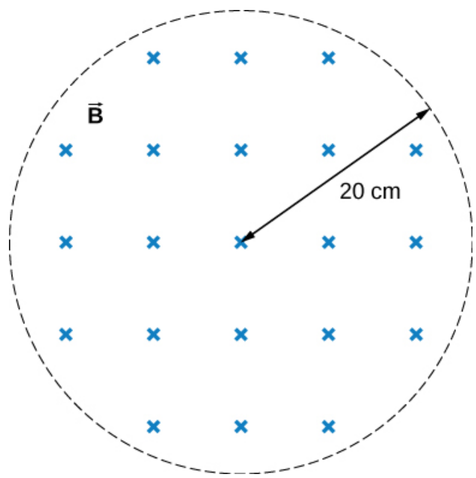
(b) Assume that **R=10.0cm**. How much work is done by the electric field on a proton that is carried once clock wise around a circular path of radius 5.0 cm?

(c) How much work is done by the electric field on a proton that is carried once counterclockwise around a circular path of any radius $r \geq R$?

(d) At the instant when **B=0.50T**, a proton enters the magnetic field at A, moving a velocity $\vec{v}(v = 5.0 \times 10^6 m/s)$ as shown. What are the electric and magnetic forces on the proton at that instant?



52. The magnetic field at all points within the cylindrical region whose cross-section is indicated in the accompanying figure starts at 1.0 T and decreases uniformly to zero in 20 s. What is the electric field (both magnitude and direction) as a function of **r**, the distance from the geometric center of the region?



53. The current in a long solenoid of radius 3 cm is varied with time at a rate of 2 A/s. A circular loop of wire of radius 5 cm and resistance **2Ω** surrounds the solenoid. Find the electrical current induced in the loop.

54. The current in a long solenoid of radius 3 cm and 20 turns/cm is varied with time at a rate of 2 A/s. Find the electric field at a distance of 4 cm from the center of the solenoid.

13.7 Electric Generators and Back Emf

55. Design a current loop that, when rotated in a uniform magnetic field of strength 0.10 T, will produce an emf $\varepsilon = \varepsilon_0 sin\omega t,$, where $\varepsilon_0 = 110V$ and $\omega = 120\pi rad/s$.

56. A flat, square coil of 20 turns that has sides of length 15.0 cm is rotating in a magnetic field of strength 0.050 T. If the maximum emf produced in the coil is 30.0 mV, what is the angular velocity of the coil?

57. A 50-turn rectangular coil with dimensions **0.15m×0.40m** rotates in a uniform magnetic field of magnitude 0.75 T at 3600 rev/min.

(a) Determine the emf induced in the coil as a function of time.

(b) If the coil is connected to a **1000-Ω** resistor, what is the power as a function of time required to keep the coil turning at 3600 rpm?

(c) Answer part (b) if the coil is connected to a **2000-Ω** resistor.

58. The square armature coil of an alternating current generator has 200 turns and is 20.0 cm on side. When it rotates at 3600 rpm, its peak output voltage is 120 V.

(a) What is the frequency of the output voltage?

(b) What is the strength of the magnetic field in which the coil is turning?

59. A flip coil is a relatively simple device used to measure a magnetic field. It consists of a circular coil of N turns wound with fine conducting wire. The coil is attached to a ballistic galvanometer, a device that measures the total charge that passes through it. The coil is placed in a magnetic field $\vec{B}$ such that its face is perpendicular to the field. It is then flipped through 180°,180°, and the total chargeQ that flows through the galvanometer is measured.

(a) If the total resistance of the coil and galvanometer is R, what is the relationship between B and Q? Because the coil is very small, you can assume that$\vec{B}$ is uniform over it.

(b) How can you determine whether or not the magnetic field is perpendicular to the face of the coil?

60. The flip coil of the preceding problem has a radius of 3.0 cm and is wound with 40 turns of copper wire. The total resistance of the coil and ballistic galvanometer is **0.20Ω**. When the coil is flipped through **180°** in a magnetic field $\vec{B}$, a change of 0.090 C flows through the ballistic galvanometer.

(a) Assuming that $\vec{B}$ and the face of the coil are initially perpendicular, what is the magnetic field?

(b) If the coil is flipped through **90°,** what is the reading of the galvanometer?

61. A 120-V, series-wound motor has a field resistance of 80 **Ω** and an armature resistance of 10 **Ω**. When it is operating at full speed, a back emf of 75 V is generated.

(a) What is the initial current drawn by the motor? When the motor is operating at full speed, where are

(b) the current drawn by the motor,

(c) the power output of the source,

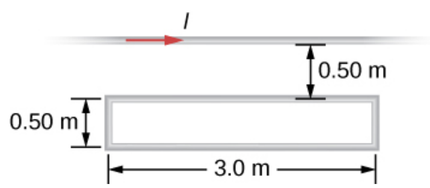
(d) the power output of the motor, and

(e) the power dissipated in the two resistances?

62. A small series-wound dc motor is operated from a 12-V car battery. Under a normal load, the motor draws 4.0 A, and when the armature is clamped so that it cannot turn, the motor draws 24 A. What is the back emf when the motor is operating normally?

Additional Problems

63. Shown in the following figure is a long, straight wire and a single-turn rectangular loop, both of which lie in the plane of the page. The wire is parallel to the long sides of the loop and is 0.50 m away from the closer side. At an instant when the emf induced in the loop is 2.0 V, what is the time rate of change of the current in the wire?



64. A metal bar of mass 500 g slides outward at a constant speed of 1.5 cm/s over two parallel rails separated by a distance of 30 cm which are part of a U-shaped conductor. There is a uniform magnetic field of magnitude 2 T pointing out of the page over the entire area. The railings and metal bar have an equivalent resistance of **150Ω**.

(a) Determine the induced current, both magnitude and direction.

(b) Find the direction of the induced current if the magnetic field is pointing into the page.

(c) Find the direction of the induced current if the magnetic field is pointed into the page and the bar moves inwards.

65. A current is induced in a circular loop of radius 1.5 cm between two poles of a horseshoe electromagnet when the current in the electromagnet is varied. The magnetic field in the area of the loop is perpendicular to the area and has a uniform magnitude. If the rate of change of magnetic field is 10 T/s, find the magnitude and direction of the induced current if resistance of the loop is **25Ω**.

66. A metal bar of length 25 cm is placed perpendicular to a uniform magnetic field of strength 3 T.

(a) Determine the induced emf between the ends of the rod when it is not moving.

(b) Determine the emf when the rod is moving perpendicular to its length and magnetic field with a speed of 50 cm/s.

67. A coil with 50 turns and area 10 cm^2 is oriented with its plane perpendicular to a 0.75-T magnetic field. If the coil is flipped over (rotated through **180°**) in 0.20 s, what is the average emf induced in it?

68. A 2-turn planer loop of flexible wire is placed inside a long solenoid of n turns per meter that carries a constant current I_0. The areaA of the loop is changed by pulling on its sides while ensuring that the plane of the loop always remains perpendicular to the axis of the solenoid. If **n=500turns** per meter, $I_0 = 20A$, and $A = 20cm^2$, what is the emf induced in the loop when **dA/dt=100**?

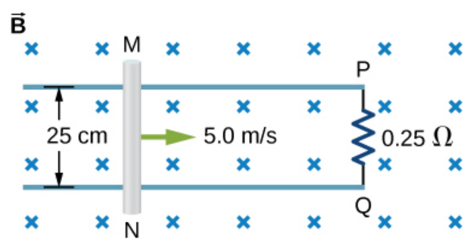
69. The conducting rod shown in the accompanying figure moves along parallel metal rails that are 25-cm apart. The system is in a uniform magnetic field of strength 0.75 T, which is directed into the page. The resistances of the rod and the rails are negligible, but the section PQ has a resistance of **0.25Ω**.

(a) What is the emf (including its sense) induced in the rod when it is moving to the right with a speed of 5.0 m/s?

(b) What force is required to keep the rod moving at this speed?

(c) What is the rate at which work is done by this force?

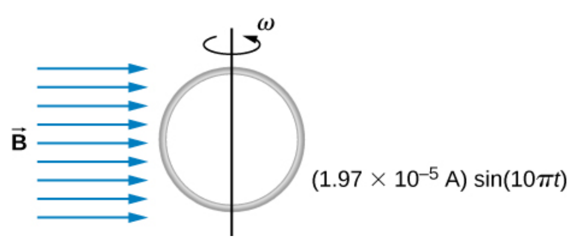
(d) What is the power dissipated in the resistor?



70. A circular loop of wire of radius 10 cm is mounted on a vertical shaft and rotated at a frequency of 5 cycles per second in a region of uniform magnetic field of 2 Gauss perpendicular to the axis of rotation.

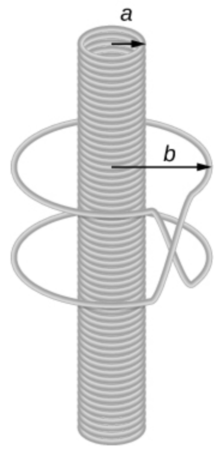
(a) Find an expression for the time-dependent flux through the ring.

(b) Determine the time-dependent current through the ring if it has a resistance of 10 **Ω**.



71. The magnetic field between the poles of a horseshoe electromagnet is uniform and has a cylindrical symmetry about an axis from the middle of the South Pole to the middle of the North Pole. The magnitude of the magnetic field changes as a rate of dB/dt due to the changing current through the electromagnet. Determine the electric field at a distance r from the center.

72. A long solenoid of radius a with n turns per unit length is carrying a time-dependent current $I(t) = I_0 sin(\omega t)$, where I_0 and **ω** are constants. The solenoid is surrounded by a wire of resistance **R** that has two circular loops of radius **b** with **b>a** (see the following figure). Find the magnitude and direction of current induced in the outer loops at time **t=0**.



73. A 120-V, series-wound dc motor draws 0.50 A from its power source when operating at full speed, and it draws 2.0 A when it starts. The resistance of the armature coils is **10Ω**.

(a) What is the resistance of the field coils?

(b) What is the back emf of the motor when it is running at full speed?

(c) The motor operates at a different speed and draws 1.0 A from the source. What is the back emf in this case?

74. The armature and field coils of a series-wound motor have a total resistance of **3.0Ω**. When connected to a 120-V source and running at normal speed, the motor draws 4.0 A.

(a) How large is the back emf?

(b) What current will the motor draw just after it is turned on? Can you suggest a way to avoid this large initial current?

Challenge Problems

75. A copper wire of length L is fashioned into a circular coil with **N** turns. When the magnetic field through the coil changes with time, for what value of N is the induced emf a maximum?

76. A 0.50-kg copper sheet drops through a uniform horizontal magnetic field of 1.5 T, and it reaches a terminal velocity of 2.0 m/s.

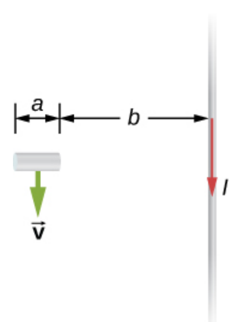
(a) What is the net magnetic force on the sheet after it reaches terminal velocity?

(b) Describe the mechanism responsible for this force.

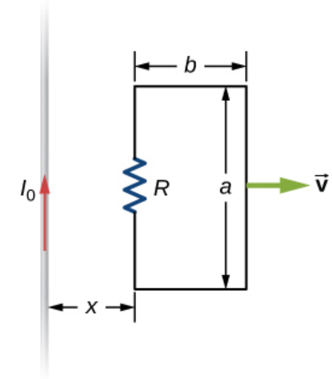
(c) How much power is dissipated as Joule heating while the sheet moves at terminal velocity?

77. A circular copper disk of radius 7.5 cm rotates at 2400 rpm around the axis through its center and perpendicular to its face. The disk is in a uniform magnetic field $\vec{B}$ of strength 1.2 T that is directed along the axis. What is the potential difference between the rim and the axis of the disk?

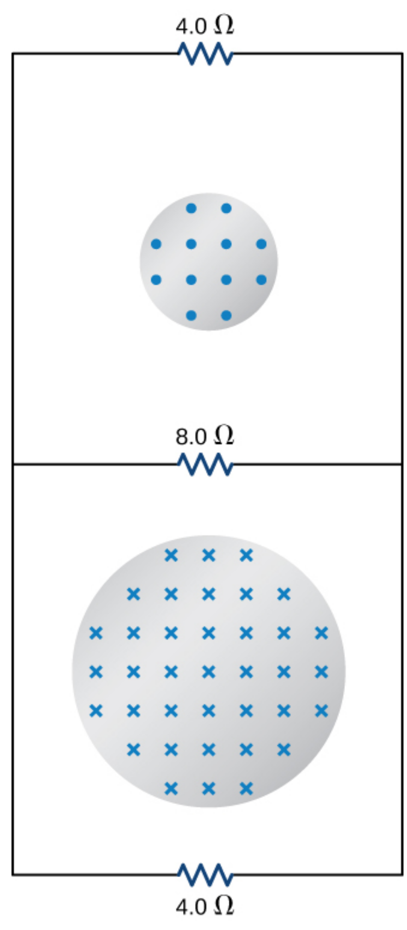
78. A short rod of length a moves with its velocity $\vec{v}$ parallel to an infinite wire carrying a current **I** (see below). If the end of the rod nearer the wire is a distance b from the wire, what is the emf induced in the rod?



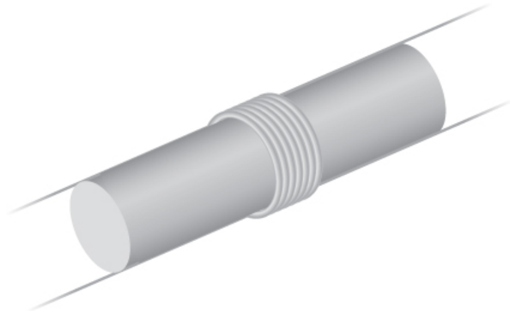
79. A rectangular circuit containing a resistance R is pulled at a constant velocity $\vec{v}$ away from a long, straight wire carrying a current I_0 (see below). Derive an equation that gives the current induced in the circuit as a function of the distance x between the near side of the circuit and the wire.



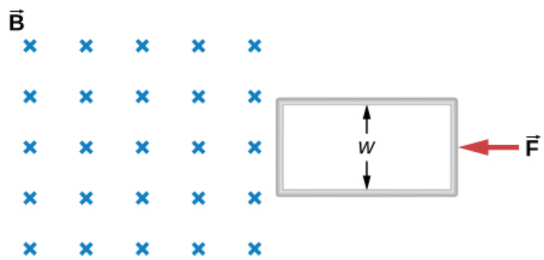
80. Two infinite solenoids cross the plane of the circuit as shown below. The radii of the solenoids are 0.10 and 0.20 m, respectively, and the current in each solenoid is changing such that **dB/dt=50.0T/s.** What are the currents in the resistors of the circuit?



81. An eight-turn coil is tightly wrapped around the outside of the long solenoid as shown below. The radius of the solenoid is 2.0 cm and it has 10 turns per centimeter. The current through the solenoid increases according to $I = I_0(1 - e^{-\alpha t})$, where $I_0 = 4.0A$ and $\alpha = 2.0 \times 10^{-2} s^{-1}$. What is the emf induced in the coil when (a) $t = 0$, (b) $t = 1.0 \times 10^2 s$, and (c) $t \to \infty$?



82. Shown below is a long rectangular loop of width **w**, length **l**, mass **m**, and resistance **R**. The loop starts from rest at the edge of a uniform magnetic field $\vec{B}$ and is pushed into the field by a constant force $\vec{F}$. Calculate the speed of the loop as a function of time.



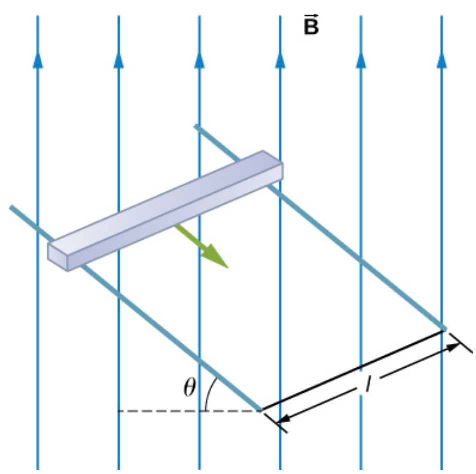
83. A square bar of mass **m** and resistance **R** is sliding without friction down very long, parallel conducting rails of negligible resistance (see below). The two rails are a distance l apart and are connected to each other at the bottom of the incline by a zero-resistance wire. The rails are inclined at an angle **θ**, and there is a uniform vertical magnetic field $\vec{B}$ throughout the region.

(a) Show that the bar acquires a terminal velocity given by $v = \dfrac{mgRsin\theta}{B^2 l^2 cos^2\theta}$.

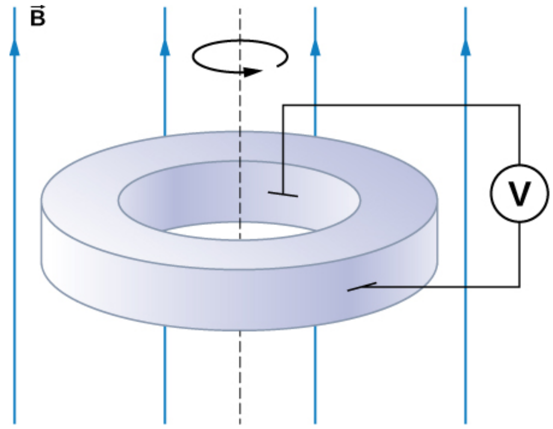
(b) Calculate the work per unit time done by the force of gravity.

(c) Compare this with the power dissipated in the Joule heating of the bar.

(d) What would happen if $\vec{B}$ were reversed?



84. The accompanying figure shows a metal disk of inner radius r_1 and other radius r_2 rotating at an angular velocity $\vec{\omega}$ while in a uniform magnetic field directed parallel to the rotational axis. The brush leads of a voltmeter are connected to the dark's inner and outer surfaces as shown. What is the reading of the voltmeter?



85. A long solenoid with 10 turns per centimeter is placed inside a copper ring such that both objects have the same central axis. The radius of the ring is 10.0 cm, and the radius of the solenoid is 5.0 cm.

(a) What is the emf induced in the ring when the current I through the solenoid is 5.0 A and changing at a rate of 100 A/s?

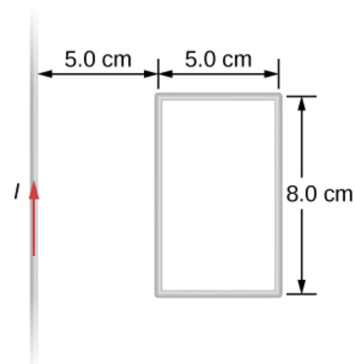
(b) What is the emf induced in the ring when **I=2.0A** and **dI/dt=100A/s**?

(c) What is the electric field inside the ring for these two cases?

(d) Suppose the ring is moved so that its central axis and the central axis of the solenoid are still parallel but no longer coincide. (You should assume that the solenoid is still inside the ring.) Now what is the emf induced in the ring?

(e) Can you calculate the electric field in the ring as you did in part (c)?

86. The current in the long, straight wire shown in the accompanying figure is given by $I = I_0 sin\omega t$, where $I_0 = 15A$ and $\omega = 120\pi rad/s$. What is the current induced in the rectangular loop at (a) **t=0** and (b) $t = 2.1 \times 10^{-3} s$? The resistance of the loop is **2.0Ω**.



87. A 500-turn coil with a $0.250 - m^2$ area is spun in Earth's $5.00 \times 10^{-5} T$ magnetic field, producing a 12.0-kV maximum emf.

(a) At what angular velocity must the coil be spun?

(b) What is unreasonable about this result?

(c) Which assumption or premise is responsible?

88. A circular loop of wire of radius 10 cm is mounted on a vertical shaft and rotated at a frequency of 5 cycles per second in a region of uniform magnetic field of $2 \times 10^{-4} T$ perpendicular to the axis of rotation.

(a) Find an expression for the time-dependent flux through the ring

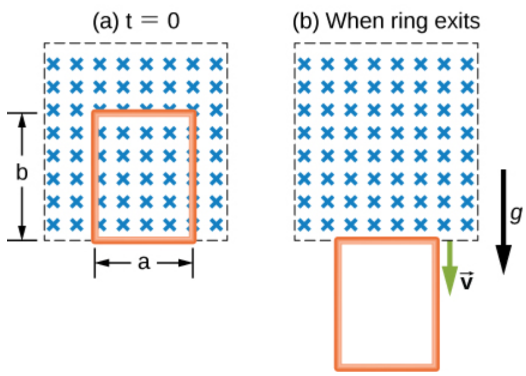
(b) Determine the time-dependent current through the ring if it has a resistance of **10Ω**.

89. A long solenoid of radius aa with nn turns per unit length is carrying a time-dependent current $I(t) = I_0 sin\omega t$ where I_0 and ωω are constants. The solenoid is surrounded by a wire of resistance R that has two circular loops of radius **b** with **b>a**. Find the magnitude and direction of current induced in the outer loops at time **t=0**.

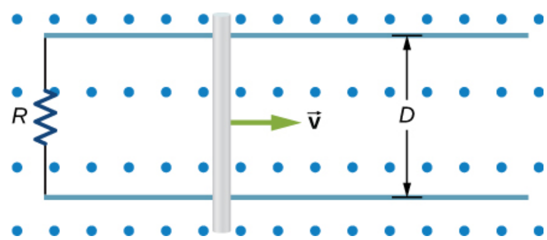
90. A rectangular copper loop of mass 100 g and resistance **0.2Ω** is in a region of uniform magnetic field that is perpendicular to the area enclosed by the ring and horizontal to Earth's surface (see below). The loop is let go from rest when it is at the edge of the nonzero magnetic field region.

(a) Find an expression for the speed when the loop just exits the region of uniform magnetic field.

(b) If it was let go at t=0t=0, what is the time when it exits the region of magnetic field for the following values: $a = 25cm, b = 50cm, B = 3T, g = 9.8m/s^2$? Assume that the magnetic field of the induced current is negligible compared to 3 T.



91. A metal bar of mass **m** slides without friction over two rails a distance **D** apart in the region that has a uniform magnetic field of magnitude B_0 and direction perpendicular to the rails (see below). The two rails are connected at one end to a resistor whose resistance is much larger than the resistance of the rails and the bar. The bar is given an initial speed of v_0. It is found to slow down. How far does the bar go before coming to rest? Assume that the magnetic field of the induced current is negligible compared to B_0.



92. A time-dependent uniform magnetic field of magnitude **B(t)** is confined in a cylindrical region of radius **R**. A conducting rod of length **2D** is placed in the region, as shown below. Show that the emf between the ends of the rod is given by $\frac{dB}{dt} D\sqrt{R^2 - D^2}$. (**Hint**: To find the emf between the ends, we need to integrate the electric field from one end to the other. To find the electric field, use Faraday's law as "Ampère's law for **E**.")

9.9: Mutual Inductance

LEARNING OBJECTIVES

By the end of this section, you will be able to:

- Correlate two nearby circuits that carry time-varying currents with the emf induced in each circuit
- Describe examples in which mutual inductance may or may not be desirable

Inductance is the property of a device that tells us how effectively it induces an emf in another device. In other words, it is a physical quantity that expresses the effectiveness of a given device.

When two circuits carrying time-varying currents are close to one another, the magnetic flux through each circuit varies because of the changing current **I** in the other circuit. Consequently, an emf is induced in each circuit by the changing current in the other. This type of emf is therefore called a **mutually induced emf**, and the phenomenon that occurs is known as **mutual inductance (M)**. As an example, let's consider two tightly wound coils (Figure 9.9.1). Coils 1 and 2 have N_1 and N_2 turns and carry currents I_1 and I_2 respectively. The flux through a single turn of coil 2 produced by the magnetic field of the current in coil 1 is Φ_{12}, whereas the flux through a single turn of coil 1 due to the magnetic field of I_2 is Φ_{12}.

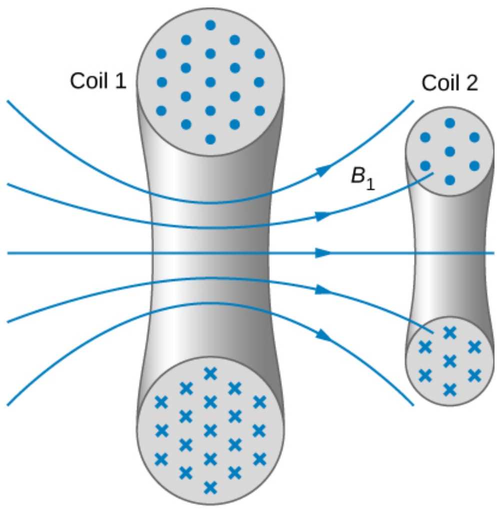


Figure 9.9.1: Some of the magnetic field lines produced by the current in coil 1 pass through coil 2.

The mutual inductance M_{21} of coil 2 with respect to coil 1 is the ratio of the flux through the N_2 turns of coil 2 produced by the magnetic field of the current in coil 1, divided by that current, that is,

$$M_{21} = \frac{N_2\Phi_{21}}{I_1}. \tag{9.9.1}$$

Similarly, the mutual inductance of coil 1 with respect to coil 2 is

$$M_{12} = \frac{N_1\Phi_{12}}{I_2}. \tag{9.9.2}$$

Like capacitance, mutual inductance is a geometric quantity. It depends on the shapes and relative positions of the two coils, and it is independent of the currents in the coils. The SI unit for mutual inductance **M** is called the **henry (H)** in honor of Joseph **Henry** (1799–1878), an American scientist who discovered induced emf independently of Faraday. Thus, we have $1\,H = 1\,V\cdot s/A$. From Equations 9.9.1 and 9.9.2, we can show that $M_{21} = M_{12}$, so we usually drop the subscripts associated with mutual inductance and write

$$M = \frac{N_2\Phi_{21}}{I_1} = \frac{N_1\Phi_{12}}{I_2}. \tag{9.9.3}$$

The emf developed in either coil is found by combining Faraday's law and the definition of mutual inductance. Since $N_2\Phi_{21}$ is the total flux through coil 2 due to I_1, we obtain

$$\epsilon_2 = -\frac{d}{dt}(N_2\Phi_{21}) \tag{9.9.4}$$

$$= -\frac{d}{dt}(MI_1) \tag{9.9.5}$$

$$= -M\frac{dI_1}{dt} \tag{9.9.6}$$

where we have used the fact that M is a time-independent constant because the geometry is time-independent. Similarly, we have

$$\epsilon_1 = -M\frac{dI_2}{dt}. \tag{9.9.7}$$

In Equation 9.9.7, we can see the significance of the earlier description of mutual inductance (M) as a geometric quantity. The value of M neatly encapsulates the physical properties of circuit elements and allows us to separate the physical layout of the circuit from the dynamic quantities, such as the emf and the current. Equation 9.9.7 defines the mutual inductance in terms of properties in the circuit, whereas the previous definition of mutual inductance in Equation 9.9.1 is defined in terms of the magnetic flux experienced, regardless of circuit elements. You should be careful when using Equations 9.9.6 and 9.9.6 because ϵ_1 and ϵ_2 do not necessarily represent the total emfs in the respective coils. Each coil can also have an emf induced in it because of its **self-inductance** (self-inductance will be discussed in more detail in a later section).

A large mutual inductance **M** may or may not be desirable. We want a transformer to have a large mutual inductance. But an appliance, such as an electric clothes dryer, can induce a dangerous emf on its metal case if the mutual inductance between its coils and the case is large. One way to reduce mutual inductance is to counter-wind coils to cancel the magnetic field produced (Figure 9.9.2).

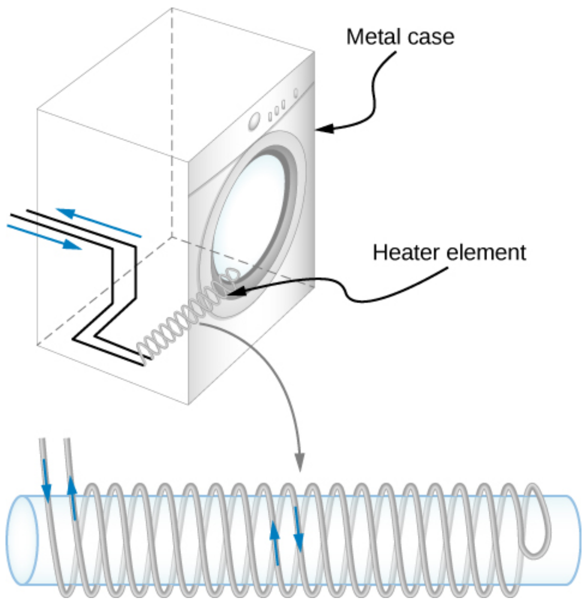


Figure 9.9.2: The heating coils of an electric clothes dryer can be counter-wound so that their magnetic fields cancel one another, greatly reducing the mutual inductance with the case of the dryer.

Digital signal processing is another example in which mutual inductance is reduced by counter-winding coils. The rapid on/off emf representing 1s and 0s in a digital circuit creates a complex time-dependent magnetic field. An emf can be generated in neighboring conductors. If that conductor is also carrying a digital signal, the induced emf may be large enough to switch 1s and 0s, with consequences ranging from inconvenient to disastrous.

Example 9.9.1: Mutual Inductance

Figure 9.9.3 shows a coil of N_2 turns and radius R_2 surrounding a long solenoid of length l_1, radius R_1, and N_1 turns.

a. What is the mutual inductance of the two coils?
b. If $N_1 = 500\,turns$, $N_2 = 10\,turns$, $R_1 = 3.10\,cm$, $l_1 = 75.0\,cm$, and the current in the solenoid is changing at a rate of 200 A/s, what is the emf induced in the surrounding coil?

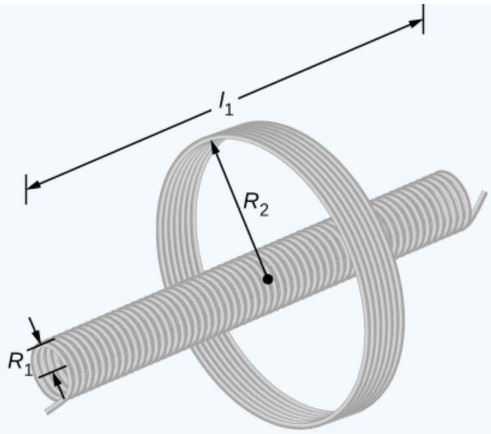


Figure 9.9.3: A solenoid surrounded by a coil.

Strategy

There is no magnetic field outside the solenoid, and the field inside has magnitude $B_1 = \mu_0(N_1/l_1)I_1$ and is directed parallel to the solenoid's axis. We can use this magnetic field to find the magnetic flux through the surrounding coil and then use this flux to calculate the mutual inductance for part (a), using Equation 9.9.3. We solve part (b) by calculating the mutual inductance from the given quantities and using Equation 9.9.6 to calculate the induced emf.

Solution

1. The magnetic flux Φ_{21} through the surrounding coil is

$$\begin{aligned}\Phi_{21} &= B_1 \pi R_1^2 \\ &= \frac{\mu_0 N_1 I_1}{l_1} \pi R_1^2.\end{aligned}$$

Now from Equation 9.9.3, the mutual inductance is

$$\begin{aligned}M &= \frac{N_2 \Phi_{21}}{I_1} \\ &= \left(\frac{N_2}{I_1}\right)\left(\frac{\mu_0 N_1 I_1}{l_1}\right)\pi R_1^2 \\ &= \frac{\mu_0 N_1 N_2 \pi R_1^2}{l_1}.\end{aligned}$$

2. Using the previous expression and the given values, the mutual inductance is

$$\begin{aligned}M &= \frac{(4\pi \times 10^{-7}\, T \cdot m/A)(500)(10)\pi(0.0310\, m)^2}{0.750\, m} \\ &= 2.53 \times 10^{-5}\, H.\end{aligned}$$

Thus, from Equation 9.9.6, the emf induced in the surrounding coil is

$$\begin{aligned}\epsilon_2 &= -M\frac{dI_1}{dt} \\ &= -(2.53 \times 10^{-5} H)(200\, A/s) \\ &= -5.06 \times 10^{-3} V.\end{aligned}$$

Significance

Notice that **M** in part (a) is independent of the radius R_2 of the surrounding coil because the solenoid's magnetic field is confined to its interior. In principle, we can also calculate **M** by finding the magnetic flux through the solenoid produced by the current in the surrounding coil. This approach is much more difficult because Φ_{12} is so complicated. However, since $M_{12} = M_{21}$, we do know the result of this calculation.

Exercise $9.9.1$

A current $I(t) = (5.0\ A)\ sin\left((120\pi\ rad/s)t\right)$ flows through the solenoid of part (b) of Example $9.9.1$. What is the maximum emf induced in the surrounding coil?

Solution

$4.77 \times 10^{-2}\ V$

9.10: Self-Inductance and Inductors

Learning Objectives

By the end of this section, you will be able to:

- Correlate the rate of change of current to the induced emf created by that current in the same circuit
- Derive the self-inductance for a cylindrical solenoid
- Derive the self-inductance for a rectangular toroid

Mutual inductance arises when a current in one circuit produces a changing magnetic field that induces an emf in another circuit. But can the magnetic field affect the current in the original circuit that produced the field? The answer is yes, and this is the phenomenon called **self-inductance**.

Inductors

Figure $9.10.1$ shows some of the magnetic field lines due to the current in a circular loop of wire. If the current is constant, the magnetic flux through the loop is also constant. However, if the current **I** were to vary with time—say, immediately after switch S is closed—then the magnetic flux Φ_m would correspondingly change. Then Faraday's law tells us that an emf ϵ would be induced in the circuit, where

$$\epsilon = -\frac{d\Phi_m}{dt}. \tag{9.10.1}$$

Since the magnetic field due to a current-carrying wire is directly proportional to the current, the flux due to this field is also proportional to the current; that is,

$$\Phi_m \propto I. \tag{9.10.2}$$

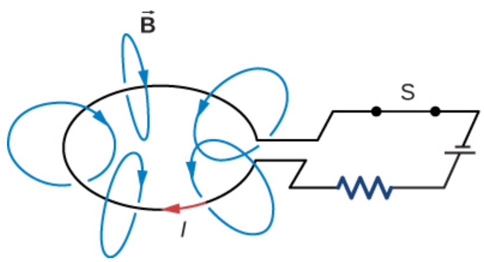


Figure 9.10.1: A magnetic field is produced by the current **I** in the loop. If **I** were to vary with time, the magnetic flux through the loop would also vary and an emf would be induced in the loop.

This can also be written as

$$\Phi_m = LI \tag{9.10.3}$$

where the constant of proportionality **L** is known as the **self-inductance** of the wire loop. If the loop has **N** turns, this equation becomes

$$\boxed{N\Phi_m = LI} \tag{9.10.4}$$

By convention, the positive sense of the normal to the loop is related to the current by the right-hand rule, so in Figure $9.10.1$, the normal points downward. With this convention, Φ_m is positive in Equation 9.10.4, so **L always has a positive value**.

For a loop with **N** turns, $\epsilon = -N d\Phi_m/dt$, so the induced emf may be written in terms of the self-inductance as

$$\boxed{\epsilon = -L\frac{dI}{dt}.} \tag{9.10.5}$$

When using this equation to determine **L**, it is easiest to ignore the signs of ϵ and dI/dt, and calculate **L** as

$$L = \frac{|\epsilon|}{|dI/dt|}.$$

Since self-inductance is associated with the magnetic field produced by a current, any configuration of conductors possesses self-inductance. For example, besides the wire loop, a long, straight wire has self-inductance, as does a coaxial cable. A coaxial cable is most commonly used by the cable television industry and may also be found connecting to your cable modem. Coaxial cables are used due to their ability to transmit electrical signals with minimal distortions. Coaxial cables have two long cylindrical conductors that possess current and a self-inductance that may have undesirable effects.

Figure $9.10.2$: Symbol used to represent an inductor in a circuit.

A circuit element used to provide self-inductance is known as an **inductor**. It is represented by the symbol shown in Figure $9.10.2$, which resembles a coil of wire, the basic form of the inductor. Figure $9.10.3$ shows several types of inductors commonly used in circuits.

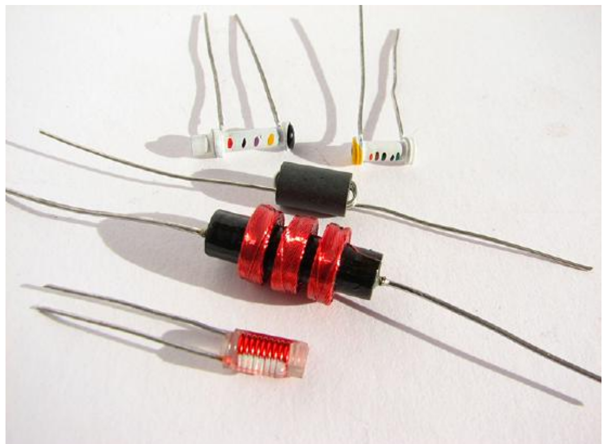


Figure $9.10.3$: A variety of inductors. Whether they are encapsulated like the top three shown or wound around in a coil like the bottom-most one, each is simply a relatively long coil of wire. (credit: Windell Oskay)

In accordance with Lenz's law, the negative sign in Equation $9.10.5$ indicates that the induced emf across an inductor always has a polarity that **opposes** the change in the current. For example, if the current flowing from **A** to **B** in Figure $9.10.4a$ were increasing, the induced emf (represented by the imaginary battery) would have the polarity shown in order to oppose the increase. If the current from **A** to **B** were decreasing, then the induced emf would have the opposite polarity, again to oppose the change in current (Figure $9.10.4b$). Finally, if the current through the inductor were constant, no emf would be induced in the coil.

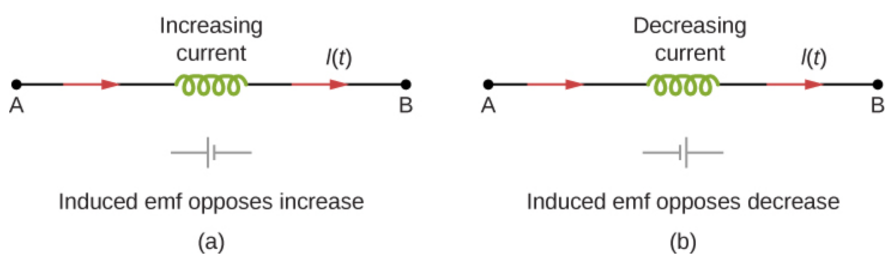


Figure $9.10.4$: The induced emf across an inductor always acts to oppose the change in the current. This can be visualized as an imaginary battery causing current to flow to oppose the change in (a) and reinforce the change in (b).

One common application of inductance is to allow **traffic signals** to sense when vehicles are waiting at a street intersection. An electrical circuit with an inductor is placed in the road underneath the location where a waiting car will stop. The body of the car increases the inductance and the circuit changes, sending a signal to the traffic lights to change colors. Similarly, **metal detectors** used for airport security employ the same technique. A coil or inductor in the metal detector frame acts as both a transmitter and a receiver. The pulsed signal from the transmitter coil induces a signal in the receiver. The self-inductance of the circuit is affected by any metal object in the path (Figure $9.10.5$). Metal detectors can be adjusted for sensitivity and can also sense the presence of metal on a person.

Figure 9.10.5: The familiar security gate at an airport not only detects metals, but can also indicate their approximate height above the floor. (credit: “Alexbuirds”/Wikimedia Commons)

Large induced voltages are found in **camera flashes**. Camera flashes use a battery, two inductors that function as a transformer, and a switching system or **oscillator** to induce large voltages. Recall from Oscillations on oscillations that “oscillation” is defined as the fluctuation of a quantity, or repeated regular fluctuations of a quantity, between two extreme values around an average value. Also recall (from Electromagnetic Induction on electromagnetic induction) that we need a changing magnetic field, brought about by a changing current, to induce a voltage in another coil. The oscillator system does this many times as the battery voltage is boosted to over 1000 volts. (You may hear the high-pitched whine from the transformer as the capacitor is being charged.) A capacitor stores the high voltage for later use in powering the flash.

Example 9.10.1: Self-Inductance of a Coil

An induced emf of 2.0 V is measured across a coil of 50 closely wound turns while the current through it increases uniformly from 0.0 to 5.0 A in 0.10 s. (a) What is the self-inductance of the coil? (b) With the current at 5.0 A, what is the flux through each turn of the coil?

Strategy

Both parts of this problem give all the information needed to solve for the self-inductance in part (a) or the flux through each turn of the coil in part (b). The equations needed are Equation 9.10.5 for part (a) and Equation 9.10.4 for part (b).

Solution

1. Ignoring the negative sign and using magnitudes, we have, from Equation 9.10.5,

$$L = \frac{\epsilon}{dI/dt} = \frac{2.0\,V}{5.0\,A/0.10\,s} = 4.0 \times 10^{-2}\,H.$$

2. From Equation 9.10.4, the flux is given in terms of the current by $\Phi_m = LI/N$, so

$$\Phi_m = \frac{(4.0 \times 10^{-2}\,H)(5.0\,A)}{50\,turns)} = 4.0 \times 10^{-3}\,Wb.$$

Significance

The self-inductance and flux calculated in parts (a) and (b) are typical values for coils found in contemporary devices. If the current is not changing over time, the flux is not changing in time, so no emf is induced.

Exercise 9.10.1

Current flows through the inductor in Figure 9.10.4 from **B** to **A** instead of from **A** to **B** as shown. Is the current increasing or decreasing in order to produce the emf given in diagram (a)? In diagram (b)?

Answer

a. decreasing; b. increasing; Since the current flows in the opposite direction of the diagram, in order to get a positive emf on the left-hand side of diagram (a), we need to decrease the current to the left, which creates a reinforced emf where the positive end is on the left-hand side. To get a positive emf on the right-hand side of diagram (b), we need to increase the current to the left, which creates a reinforced emf where the positive end is on the right-hand side.

? Exercise 9.10.2

A changing current induces an emf of 10 V across a 0.25-H inductor. What is the rate at which the current is changing?

Answer

40 A/s

A good approach for calculating the self-inductance of an inductor consists of the following steps:

Problem-Solving Strategy: Self-Inductance

1. Assume a current **I** is flowing through the inductor.
2. Determine the magnetic field $\vec{B}$ produced by the current. If there is appropriate symmetry, you may be able to do this with Ampère's law.
3. Obtain the magnetic flux, Φ_m.
4. With the flux known, the self-inductance can be found from Equation 9.10.4, $L = N\Phi_m/I$.

To demonstrate this procedure, we now calculate the self-inductances of two inductors.

Cylindrical Solenoid

Consider a long, cylindrical solenoid with length **l**, cross-sectional area **A**, and **N** turns of wire. We assume that the length of the solenoid is so much larger than its diameter that we can take the magnetic field to be $B = \mu_0 nI$ throughout the interior of the solenoid, that is, we ignore end effects in the solenoid. With a current **I** flowing through the coils, the magnetic field produced within the solenoid is

$$B = \mu_0 \left(\frac{N}{l}\right) I, \tag{9.10.6}$$

so the magnetic flux through one turn is

$$\Phi_m = BA = \frac{\mu_0 NA}{l} I. \tag{9.10.7}$$

Using Equation 9.10.4, we find for the self-inductance of the solenoid,

✓ Note

$$L_{solenoid} = \frac{N\Phi_m}{I} = \frac{\mu_0 N^2 A}{l}. \tag{9.10.8}$$

If $n = N/l$ is the number of turns per unit length of the solenoid, we may write Equation 9.10.8 as

$$L = \mu_0 \left(\frac{N}{l}\right)^2 Al = \mu_0 n^2 Al = \mu_0 n^2 (V), \tag{9.10.9}$$

where $V = Al$ is the volume of the solenoid. Notice that **the self-inductance of a long solenoid depends only on its physical properties** (such as the number of turns of wire per unit length and the volume), and not on the magnetic field or the current. This is true for inductors in general.

Rectangular Toroid

A toroid with a rectangular cross-section is shown in Figure 9.10.6. The inner and outer radii of the toroid are R_1 and R_2, and h is the height of the toroid. Applying Ampère's law in the same manner as we did in Example 13.5.2 for a toroid with a circular cross-section, we find the magnetic field inside a rectangular toroid is also given by

$$B = \frac{\mu_0 NI}{2\pi r},$$

where **r** is the distance from the central axis of the toroid. Because the field changes within the toroid, we must calculate the flux by integrating over the toroid's cross-section. Using the infinitesimal cross-sectional area element $da = h\,dr$ shown in Figure 9.10.6, we obtain

$$\Phi_m = \int B\,da = \int_{R_1}^{R_2} \left(\frac{\mu_0 NI}{2\pi r}\right)(hdr) = \frac{\mu_0 NhI}{2\pi} ln \frac{R_2}{R_1}. \tag{9.10.10}$$

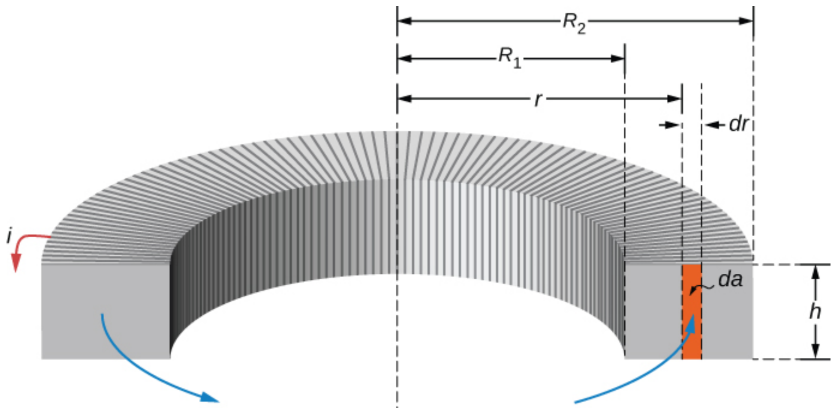


Figure 9.10.6: Calculating the self-inductance of a rectangular toroid.

Now from Equation 9.10.10, we obtain for the self-inductance of a rectangular toroid

Note

$$L = \frac{N\Phi_m}{I} = \frac{\mu_0 N^2 h}{2\pi} ln \frac{R_2}{R_1}. \tag{9.10.11}$$

As expected, the self-inductance is a constant determined by only the physical properties of the toroid.

Exercise 9.10.3

(a) Calculate the self-inductance of a solenoid that is tightly wound with wire of diameter 0.10 cm, has a cross-sectional area of $0.90\,cm^2$, and is 40 cm long. (b) If the current through the solenoid decreases uniformly from 10 to 0 A in 0.10 s, what is the emf induced between the ends of the solenoid?

Answer

a. $4.5 \times 10^{-5} H$; b. $4.5 \times 10^{-3} V$

Exercise 9.10.4

(a) What is the magnetic flux through one turn of a solenoid of self-inductance $8.0 \times 10^{-5} H$ when a current of 3.0 A flows through it? Assume that the solenoid has 1000 turns and is wound from wire of diameter 1.0 mm. (b) What is the cross-sectional area of the solenoid?

Answer

a. $2.4 \times 10^{-7} Wb$; b. $6.4 \times 10^{-5} m^2$

9.11: Energy in a Magnetic Field

Learning Objectives

By the end of this section, you will be able to:

- Explain how energy can be stored in a magnetic field
- Derive the equation for energy stored in a coaxial cable given the magnetic energy density

The energy of a capacitor is stored in the electric field between its plates. Similarly, an inductor has the capability to store energy, but in its magnetic field. This energy can be found by integrating the **magnetic energy density**,

$$u_m = \frac{B^2}{2\mu_0}$$

over the appropriate volume. To understand where this formula comes from, let's consider the long, cylindrical solenoid of the previous section. Again using the infinite solenoid approximation, we can assume that the magnetic field is essentially constant and given by $B = \mu_0 n I$ everywhere inside the solenoid. Thus, the energy stored in a solenoid or the magnetic energy density times volume is equivalent to

$$U = u_m(V) = \frac{(\mu_0 n I)^2}{2\mu_0}(Al) = \frac{1}{2}(\mu_0 n^2 Al)I^2. \tag{9.11.1}$$

With the substitution of Equation 14.3.12, this becomes

$$\boxed{U = \frac{1}{2}LI^2.}$$

Although derived for a special case, this equation gives the energy stored in the magnetic field of **any** inductor. We can see this by considering an arbitrary inductor through which a changing current is passing. At any instant, the magnitude of the induced emf is $\epsilon = Ldi/dt$, where i is the induced current at that instance. Therefore, the power absorbed by the inductor is

$$P = \epsilon i = L\frac{di}{dt}i.$$

The total energy stored in the magnetic field when the current increases from 0 to **I** in a time interval from 0 to **t** can be determined by integrating this expression:

$$U = \int_0^t P dt' = \int_0^t L\frac{di}{dt'}idt' = L\int_0^l idi = \frac{1}{2}LI^2. \tag{9.11.2}$$

Example 9.11.1: Self-Inductance of a Coaxial Cable

Figure $9.11.1$ shows two long, concentric cylindrical shells of radii R_1 and R_2. As discussed in Capacitance on capacitance, this configuration is a simplified representation of a coaxial cable. The capacitance per unit length of the cable has already been calculated. Now (a) determine the magnetic energy stored per unit length of the coaxial cable and (b) use this result to find the self-inductance per unit length of the cable.

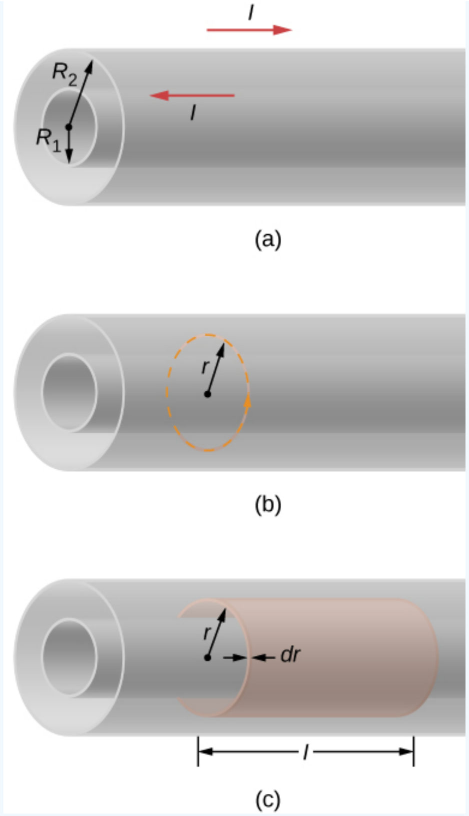


Figure 9.11.1: (a) A coaxial cable is represented here by two hollow, concentric cylindrical conductors along which electric current flows in opposite directions. (b) The magnetic field between the conductors can be found by applying Ampère's law to the dashed path. (c) The cylindrical shell is used to find the magnetic energy stored in a length **l** of the cable.

Strategy

The magnetic field both inside and outside the coaxial cable is determined by Ampère's law. Based on this magnetic field, we can use Equation 9.11.2 to calculate the energy density of the magnetic field. The magnetic energy is calculated by an integral of the magnetic energy density times the differential volume over the cylindrical shell. After the integration is carried out, we have a closed-form solution for part (a). The self-inductance per unit length is determined based on this result and Equation 9.11.2.

Solution

1. We determine the magnetic field between the conductors by applying Ampère's law to the dashed circular path shown in Figure 9.11.1b. Because of the cylindrical symmetry, $\vec{B}$ is constant along the path, and

$$\oint \vec{B} \cdot d\vec{l} = B(2\pi r) = \mu_0 I.$$

This gives us

$$B = \frac{\mu_0 I}{2\pi r}.$$

In the region outside the cable, a similar application of Ampère's law shows that $B = 0$, since no net current crosses the area bounded by a circular path where $r > R_2$. This argument also holds when $r < R_1$; that is, in the region within the inner cylinder. All the magnetic energy of the cable is therefore stored between the two conductors. Since the energy density of the magnetic field is

$$u_m = \frac{B^2}{2\mu_0}$$

the energy stored in a cylindrical shell of inner radius **r**, outer radius $r + dr$ and length **l** (see part (c) of the figure) is

$$u_m = \frac{\mu_0 I^2}{8\pi^2 r^2}.$$

Thus, the total energy of the magnetic field in a length **l** of the cable is

$$U = \int_{R_1}^{R_2} dU = \int_{R_1}^{R_2} \frac{\mu_0 I^2}{8\pi^2 r^2}(2\pi r l)dr = \frac{\mu_0 I^2 l}{4\pi} ln \frac{R_2}{R_1},$$

and the energy per unit length is $(\mu_0 I^2/4\pi) ln(R_2/R_1)$.

2. From Equation 9.11.2,

$$U = \frac{1}{2} L I^2,$$

where **L** is the self-inductance of a length **l** of the coaxial cable. Equating the previous two equations, we find that the self-inductance per unit length of the cable is

$$\frac{L}{l} = \frac{\mu_0}{2\pi} \ln \frac{R_2}{R_1}.$$

Significance

The inductance per unit length depends only on the inner and outer radii as seen in the result. To increase the inductance, we could either increase the outer radius (R_2) or decrease the inner radius (R_1). In the limit as the two radii become equal, the inductance goes to zero. In this limit, there is no coaxial cable. Also, the magnetic energy per unit length from part (a) is proportional to the square of the current.

Exercise 9.11.1

How much energy is stored in the inductor of Example 14.3.1 after the current reaches its maximum value?

Solution

0.50 J

9.12: RL Circuits

Learning Objectives

By the end of this section, you will be able to:

- Analyze circuits that have an inductor and resistor in series
- Describe how current and voltage exponentially grow or decay based on the initial conditions

A circuit with resistance and self-inductance is known as an **RL circuit**. Figure $9.12.1a$ shows an **RL** circuit consisting of a resistor, an inductor, a constant source of emf, and switches S_1 and S_2. When S_1 is closed, the circuit is equivalent to a single-loop circuit consisting of a resistor and an inductor connected across a source of emf (Figure $9.12.1b$). When S_1 is opened and S_2 is closed, the circuit becomes a single-loop circuit with only a resistor and an inductor (Figure $9.12.1c$).

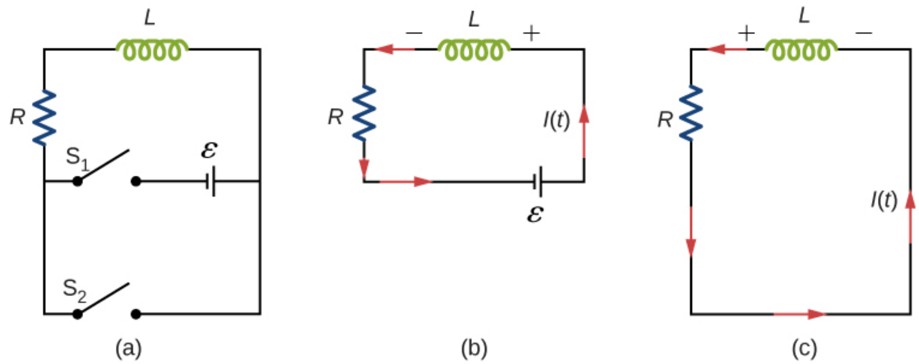


Figure 9.12.1: (a) An RL circuit with switches S_1 and S_2. (b) The equivalent circuit with S_1 closed and S_2 open. (c) The equivalent circuit after S_1 is opened and S_2 is closed.

We first consider the **RL** circuit of Figure $9.12.1b$. Once S_1 is closed and S_2 is open, the source of emf produces a current in the circuit. If there were no self-inductance in the circuit, the current would rise immediately to a steady value of ϵ/R. However, from Faraday's law, the increasing current produces an emf $V_L = -L(dI/dt)$ across the inductor. In accordance with Lenz's law, the induced emf counteracts the increase in the current and is directed as shown in the figure. As a result, $I(t)$ starts at zero and increases asymptotically to its final value.

Applying Kirchhoff's loop rule to this circuit, we obtain

$$\epsilon - L\frac{dI}{dt} - IR = 0, \tag{9.12.1}$$

which is a first-order differential equation for $I(t)$. Notice its similarity to the equation for a capacitor and resistor in series (see **RC Circuits**). Similarly, the solution to Equation 9.12.1 can be found by making substitutions in the equations relating the capacitor to the inductor. This gives

$$I(t) = \frac{\epsilon}{R}(1 - e^{-Rt/L}) \tag{9.12.2}$$

$$= \frac{\epsilon}{R}(1 - e^{-t/\tau_L}), \tag{9.12.3}$$

where

$$\tau_L = L/R \tag{9.12.4}$$

is the **inductive time constant** of the circuit.

The current $I(t)$ is plotted in Figure $9.12.2a$. It starts at zero, and as $t \rightarrow \infty$, $I(t)$ approaches ϵ/R asymptotically. The induced emf $V_L(t)$ is directly proportional to dI/dt, or the slope of the curve. Hence, while at its greatest immediately after the switches are thrown, the induced emf decreases to zero with time as the current approaches its final value of ϵ/R. The circuit then becomes equivalent to a resistor connected across a source of emf.

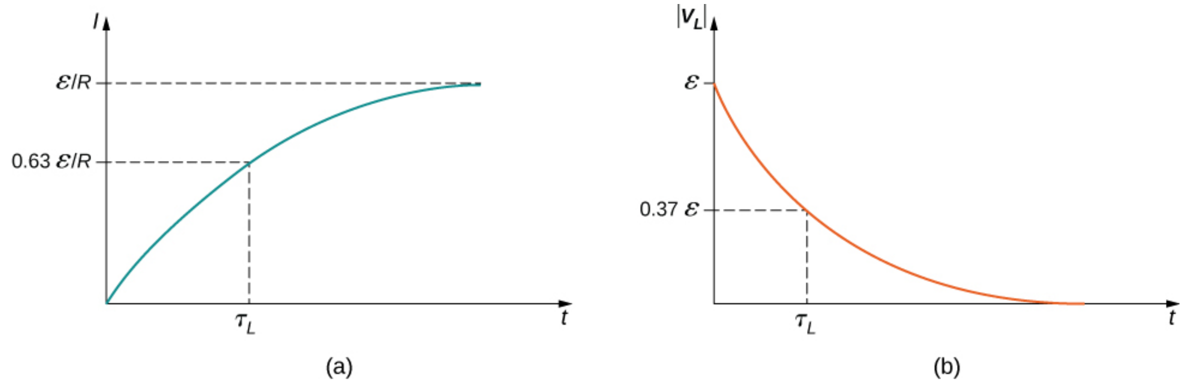


Figure 9.12.2: Time variation of (a) the electric current and (b) the magnitude of the induced voltage across the coil in the circuit of Figure 9.12.1b.

The energy stored in the magnetic field of an inductor is

$$U_L = \frac{1}{2}LI^2.$$

Thus, as the current approaches the maximum current ϵ/R, the stored energy in the inductor increases from zero and asymptotically approaches a maximum of $L(\epsilon/R)^2/2$.

The time constant τ_L tells us how rapidly the current increases to its final value. At $t = \tau_L$, the current in the circuit is, from Equation 9.12.3,

$$I(\tau_L) = \frac{\epsilon}{R}(1 - e^{-1}) = 0.63\frac{\epsilon}{R},$$

which is 63% of the final ϵ/R. The smaller the inductive time constant $\tau_L = L/R$, the more rapidly the current approaches ϵ/R.

We can find the time dependence of the induced voltage across the inductor in this circuit by using $V_L(t) = -L(dI/dt)$ and Equation 9.12.3:

$$V_L(t) = -L\frac{dI}{dt} = -\epsilon e^{-t/\tau_L}.$$

The magnitude of this function is plotted in Figure $9.12.2b$. The greatest value of $L(dI/dt)$ is ϵ; it occurs when dI/dt is greatest, which is immediately after S_1 is closed and S_2 is opened. In the approach to steady state, dI/dt decreases to zero. As a result, the voltage across the inductor also vanishes as $t \to \infty$.

The time constant τ_L also tells us how quickly the induced voltage decays. At $t = \tau_L$ the magnitude of the induced voltage is

$$|V_L(\tau_L)| = \epsilon e^{-1} = 0.37\epsilon = 0.37V(0).$$

The voltage across the inductor therefore drops to about 37% of its initial value after one time constant. The shorter the time constant τ_L, the more rapidly the voltage decreases.

After enough time has elapsed so that the current has essentially reached its final value, the positions of the switches in Figure $9.12.1a$ are reversed, giving us the circuit in part (c). At $t = 0$, the current in the circuit is $I(0) = \epsilon/R$. With Kirchhoff's loop rule, we obtain

$$IR + L\frac{dI}{dt} = 0.$$

The solution to this equation is similar to the solution of the equation for a discharging capacitor, with similar substitutions. The current at time **t** is then

$$I(t) = \frac{\epsilon}{R}e^{-t/\tau_L}.$$

The current starts at $I(0) = \epsilon/R$ and decreases with time as the energy stored in the inductor is depleted (Figure $9.12.3$).

The time dependence of the voltage across the inductor can be determined from $V_L = -L(dI/dt)$:

$$V_L(0) = \epsilon e^{-t/\tau_L}. \tag{9.12.5}$$

This voltage is initially $V_L(0) = \epsilon$, and it decays to zero like the current. The energy stored in the magnetic field of the inductor, $LI^2/2$, also decreases exponentially with time, as it is dissipated by Joule heating in the resistance of the circuit.

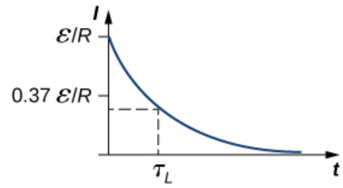


Figure 9.12.3: Time variation of electric current in the *RL* circuit of Figure $9.12.1c$. The induced voltage across the coil also decays exponentially.

Example 9.12.1: An RL Circuit with a Source of emf

In the circuit of Figure $9.12.1a$, let $\epsilon = 2.0\,V$, $R = 4.0\,\Omega$, and $L = 4.0\,H$. With S_1 closed and S_2 open (Figure $9.12.1b$), (a) what is the time constant of the circuit? (b) What are the current in the circuit and the magnitude of the induced emf across the inductor at $t = 0$, at $t = 2.0\tau_L$, and as $t \to \infty$?

Strategy

The time constant for an inductor and resistor in a series circuit is calculated using Equation 9.12.4. The current through and voltage across the inductor are calculated by the scenarios detailed from Equation 9.12.3 and Equation 9.12.5.

Solution

1. The inductive time constant is

$$\tau_L = \frac{L}{R} = \frac{4.0\,H}{4.0\,\Omega} = 1.0\,s.$$

2. The current in the circuit of Figure $9.12.1b$ increases according to Equation 9.12.3:

$$I(t) = \frac{\epsilon}{R}(1 - e^{-t/\tau_L}).$$

At $t = 0$,

$$(1 - e^{-t/\tau_L}) = (1 - 1) = 0;\ so\, I(0) = 0.$$

At $t = 2.0\tau_L$ and $t \to \infty$, we have, respectively,

$$I(2.0\tau_L) = \frac{\epsilon}{R}(1 - e^{-2.0}) = (0.50\,A)(0.86) = 0.43\,A$$

and

$$I(\infty) = \frac{\epsilon}{R} = 0.50\,A.$$

From Equation 9.12.5, the magnitude of the induced emf decays as

$$|V_L(t)| = \epsilon e^{-t/\tau_L}.$$

At $t = 0$, $t = 2.0\tau_L$, and as $t \to \infty$, we obtain

$$|V_L(0)| = \epsilon = V,$$

$$|V_L(2.0\tau_L)| = (2.0\,V)e^{-2.0} = 0.27\,V$$

and

$$|V_L(\infty)| = 0.$$

Significance

If the time of the measurement were much larger than the time constant, we would not see the decay or growth of the voltage across the inductor or resistor. The circuit would quickly reach the asymptotic values for both of these (Figure $9.12.4$).

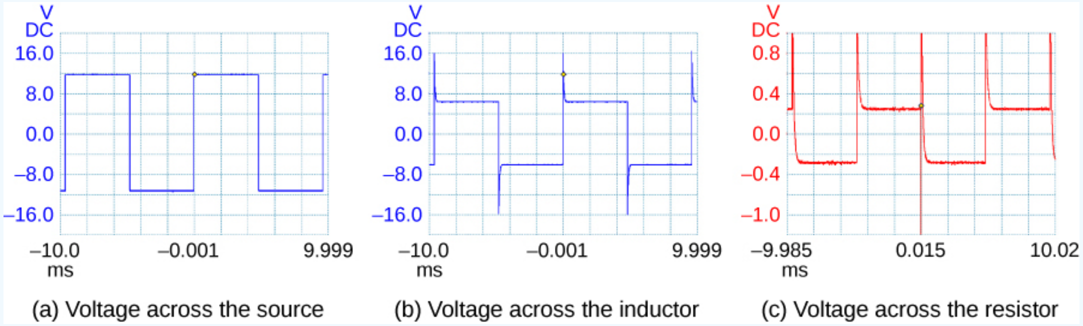


Figure $9.12.4$: A generator in an **RL** circuit produces a square-pulse output in which the voltage oscillates between zero and some set value. These oscilloscope traces show (a) the voltage across the source; (b) the voltage across the inductor; (c) the voltage across the resistor.

Example $9.12.1$: An RL Circuit without a Source of emf

After the current in the **RL** circuit of Example $9.12.1$ has reached its final value, the positions of the switches are reversed so that the circuit becomes the one shown in Figure $9.12.1c$.

a. How long does it take the current to drop to half its initial value?
b. How long does it take before the energy stored in the inductor is reduced to 1.0% of its maximum value?

Strategy

The current in the inductor will now decrease as the resistor dissipates this energy. Therefore, the current falls as an exponential decay. We can also use that same relationship as a substitution for the energy in an inductor formula to find how the energy decreases at different time intervals.

Solution

1. With the switches reversed, the current decreases according to

$$I(t) = \frac{\epsilon}{R}e^{-t/\tau_L} = I(0)e^{-t/\tau_L}.$$

At a time **t** when the current is one-half its initial value, we have

$$I(t) = 0.50 I(0) \; so \, e^{-t/\tau_L} = 0.50,$$

and

$$t = -[\ln(0.50)]\tau_L = 0.69(1.0\,s) = 0.69\,s$$

where we have used the inductive time constant found in Example $9.12.1$.

2. The energy stored in the inductor is given by

$$U_L(t) = \frac{1}{2}L[I(t)]^2 = \frac{1}{2}L\left(\frac{\epsilon}{R}e^{-t/\tau_L}\right)^2 = \frac{L\epsilon^2}{2R^2}e^{-2t/\tau_L}.$$

If the energy drops to 1.0% of its initial value at a time **t**, we have

$$U_L(t) = (0.010)U_L(0) \; or \; \frac{L\epsilon^2}{2R^2}e^{-2t/\tau_L} = (0.010)\frac{L\epsilon^2}{2R^2}.$$

Upon canceling terms and taking the natural logarithm of both sides, we obtain

$$-\frac{2t}{\tau_L} = \ln(0.010),$$

so

$$t = -\frac{1}{2}\tau_L \ln(0.010).$$

Since $\tau_L = 1.0\ s$, the time it takes for the energy stored in the inductor to decrease to 1.0% of its initial value is

$$t = -\frac{1}{2}(1.0\ s)\ln(0.010) = 2.3\ s.$$

Significance

This calculation only works if the circuit is at maximum current in situation (b) prior to this new situation. Otherwise, we start with a lower initial current, which will decay by the same relationship.

? Exercise 9.12.1

Verify that **RC** and **L/R** have the dimensions of time

? Exercise 9.12.2

a. If the current in the circuit of in Figure $9.12.1b$ increases to 90% of its final value after 5.0 s, what is the inductive time constant?
b. If $R = 20\ \Omega$, what is the value of the self-inductance?
c. If the $R = 20\ \Omega$ resistor is replaced with a $R = 100\ \Omega$ resistor, what is the time taken for the current to reach 90% of its final value?

Answer

a. 2.2 s; b. 43 H; c. 1.0 s

? Exercise 9.12.3

For the circuit of in Figure $9.12.1b$, show that when steady state is reached, the difference in the total energies produced by the battery and dissipated in the resistor is equal to the energy stored in the magnetic field of the coil

9.12.1: Oscillations in an LC Circuit

Learning Objectives

By the end of this section, you will be able to:

- Explain why charge or current oscillates between a capacitor and inductor, respectively, when wired in series
- Describe the relationship between the charge and current oscillating between a capacitor and inductor wired in series

It is worth noting that both capacitors and inductors store energy, in their electric and magnetic fields, respectively. A circuit containing both an inductor (**L**) and a capacitor (**C**) can oscillate without a source of emf by shifting the energy stored in the circuit between the electric and magnetic fields. Thus, the concepts we develop in this section are directly applicable to the exchange of energy between the electric and magnetic fields in electromagnetic waves, or light. We start with an idealized circuit of zero resistance that contains an inductor and a capacitor, an **LC circuit**.

An **LC** circuit is shown in Figure $9.12.1.1$. If the capacitor contains a charge q_0 before the switch is closed, then all the energy of the circuit is initially stored in the electric field of the capacitor (Figure $9.12.1.1a$). This energy is

$$U_C = \frac{1}{2}\frac{q_0^2}{C}.$$

When the switch is closed, the capacitor begins to discharge, producing a current in the circuit. The current, in turn, creates a magnetic field in the inductor. The net effect of this process is a transfer of energy from the capacitor, with its diminishing electric field, to the inductor, with its increasing magnetic field.

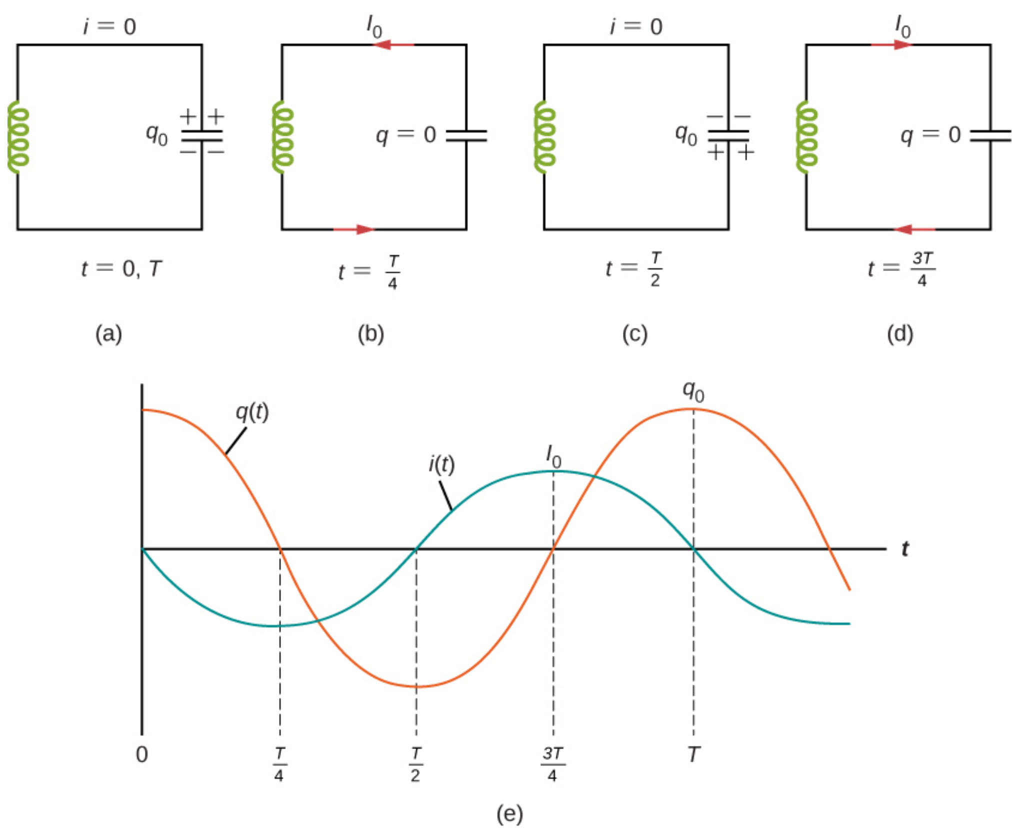


Figure 9.12.1.1: (a–d) The oscillation of charge storage with changing directions of current in an **LC** circuit. (e) The graphs show the distribution of charge and current between the capacitor and inductor.

In Figure $9.12.1.1b$, the capacitor is completely discharged and all the energy is stored in the magnetic field of the inductor. At this instant, the current is at its maximum value I_0 and the energy in the inductor is

$$U_L = \frac{1}{2}LI_0^2.$$

Since there is no resistance in the circuit, no energy is lost through Joule heating; thus, the maximum energy stored in the capacitor is equal to the maximum energy stored at a later time in the inductor:

$$\frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}LI_0^2.$$

At an arbitrary time when the capacitor charge is **q(t)** and the current is **i(t)**, the total energy **U** in the circuit is given by

$$\frac{q^2(t)}{2C} + \frac{Li^2}{2}.$$

Because there is no energy dissipation,

$$U = \frac{1}{2}\frac{q^2}{C} + \frac{1}{2}Li^2 = \frac{1}{2}\frac{q_0^2}{C} = \frac{1}{2}LI_0^2.$$

After reaching its maximum I_0, the current **i(t)** continues to transport charge between the capacitor plates, thereby recharging the capacitor. Since the inductor resists a change in current, current continues to flow, even though the capacitor is discharged. This continued current causes the capacitor to charge with opposite polarity. The electric field of the capacitor increases while the magnetic field of the inductor diminishes, and the overall effect is a transfer of energy from the inductor **back** to the capacitor. From the law of energy conservation, the maximum charge that the capacitor re-acquires is q_0. However, as Figure $9.12.1.1c$ shows, the capacitor plates are charged **opposite** to what they were initially.

When fully charged, the capacitor once again transfers its energy to the inductor until it is again completely discharged, as shown in Figure $9.12.1.1d$ Then, in the last part of this cyclic process, energy flows back to the capacitor, and the initial state of the circuit is restored.

We have followed the circuit through one complete cycle. Its electromagnetic oscillations are analogous to the mechanical oscillations of a mass at the end of a spring. In this latter case, energy is transferred back and forth between the mass, which has kinetic energy $mv^2/2$, and the spring, which has potential energy $kx^2/2$. With the absence of friction in the mass-spring system, the oscillations would continue indefinitely. Similarly, the oscillations of an **LC** circuit with no resistance would continue forever if undisturbed; however, this ideal zero-resistance **LC** circuit is not practical, and any **LC** circuit will have at least a small resistance, which will radiate and lose energy over time.

The frequency of the oscillations in a resistance-free **LC** circuit may be found by analogy with the mass-spring system. For the circuit, $i(t) = dq(t)/dt$, the total electromagnetic energy **U** is

$$U = \frac{1}{2}Li^2 + \frac{1}{2}\frac{q^2}{C}.$$

For the mass-spring system, $v(t) = dx(t)/dt$, the total mechanical energy **E** is

$$E = \frac{1}{2}mv^2 + \frac{1}{2}kx^2.$$

The equivalence of the two systems is clear. To go from the mechanical to the electromagnetic system, we simply replace **m** by **L**, **v** by **i**, **k** by 1/**C**, and **x** by **q**. Now **x(t)** is given by

$$x(t) = A\,cos(\omega t + \phi)$$

where $\omega = \sqrt{k/m}$. Hence, the charge on the capacitor in an **LC** circuit is given by

$$q(t) = q_0\,cos(\omega t + \phi) \tag{9.12.1.1}$$

where the angular frequency of the oscillations in the circuit is

$$\omega = \sqrt{\frac{1}{LC}}. \tag{9.12.1.2}$$

Finally, the current in the **LC** circuit is found by taking the time derivative of **q(t)**:

$$i(t) = \frac{dq(t)}{dt} = -\omega q_0\,sin(\omega t + \phi).$$

The time variations of **q** and **I** are shown in Figure 9.12.1.1efor $\phi = 0$.

An LC Circuit

In an **LC** circuit, the self-inductance is 2.0×10^{-2} H and the capacitance is 8.0×10^{-6} F. At $t = 0$ all of the energy is stored in the capacitor, which has charge 1.2×10^{-5} C. (a) What is the angular frequency of the oscillations in the circuit? (b) What is the maximum current flowing through circuit? (c) How long does it take the capacitor to become completely discharged? (d) Find an equation that represents **q(t)**.

Strategy

The angular frequency of the **LC** circuit is given by Equation 9.12.1.2. To find the maximum current, the maximum energy in the capacitor is set equal to the maximum energy in the inductor. The time for the capacitor to become discharged if it is initially charged is a quarter of the period of the cycle, so if we calculate the period of the oscillation, we can find out what a quarter of that is to find this time. Lastly, knowing the initial charge and angular frequency, we can set up a cosine equation to find **q**(**t**).

Solution

1. From Equation 9.12.1.2, the angular frequency of the oscillations is
$$\omega = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(2.0 \times 10^{-2}\ H)(8.0 \times 10^{-6}\ F)}} = 2.5 \times 10^{3}\ rad/s.$$
2. The current is at its maximum I_0 when all the energy is stored in the inductor. From the law of energy conservation,
$$\frac{1}{2}LI_0^2 = \frac{1}{2}\frac{q_0^2}{C},$$
so
$$I_0 = \sqrt{\frac{1}{LC}}q_0 = (2.5 \times 10^{3}\ rad/s)(1.2 \times 10^{-5} C) = 3.0 \times 10^{-2} A.$$
This result can also be found by an analogy to simple harmonic motion, where current and charge are the velocity and position of an oscillator.
3. The capacitor becomes completely discharged in one-fourth of a cycle, or during a time **T**/4, where **T** is the period of the oscillations. Since
$$T = \frac{2\pi}{\omega} = \frac{2\pi}{2.5 \times 10^{3}\ rad/s} = 2.5 \times 10^{-3} s,$$
the time taken for the capacitor to become fully discharged is $(2.5 \times 10^{-3} s)/4 = 6.3 \times 10^{-4} s$.
4. The capacitor is completely charged at $t = 0$, so $q(0) = q_0$. Using 9.12.1.1, we obtain
$$q(0) = q_0 = q_0\ cos\,\phi.$$
Thus, $\phi = 0$, and
$$q(t) = (1.2 \times 10^{-5} C) cos(2.5 \times 10^{3} t).$$

Significance

The energy relationship set up in part (b) is not the only way we can equate energies. At most times, some energy is stored in the capacitor and some energy is stored in the inductor. We can put both terms on each side of the equation. By examining the circuit only when there is no charge on the capacitor or no current in the inductor, we simplify the energy equation.

Exercise 9.12.1.1

The angular frequency of the oscillations in an **LC** circuit is 2.0×10^{3} rad/s. (a) If $L = 0.10\ H$, what is **C**? (b) Suppose that at $t = 0$ all the energy is stored in the inductor. What is the value of ϕ? (c) A second identical capacitor is connected in parallel with the original capacitor. What is the angular frequency of this circuit?

Solution

a. $2.5\ \mu F$; b. $\pi/2$ rad or $3\pi/2$ rad; c. 1.4×10^3 rad/s

9.12.2: RLC Series Circuits

Learning Objectives

By the end of this section, you will be able to:

- Determine the angular frequency of oscillation for a resistor, inductor, capacitor (**RLC**) series circuit
- Relate the **RLC** circuit to a damped spring oscillation

When the switch is closed in the **RLC** circuit of Figure $9.12.2.1a$, the capacitor begins to discharge and electromagnetic energy is dissipated by the resistor at a rate i^2R. With **U** given by Equation 14.4.2, we have

$$\frac{dU}{dt} = \frac{q}{C}\frac{dq}{dt} + Li\frac{di}{dt} = -i^2R$$

where **i** and **q** are time-dependent functions. This reduces to

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = 0. \tag{9.12.2.1}$$

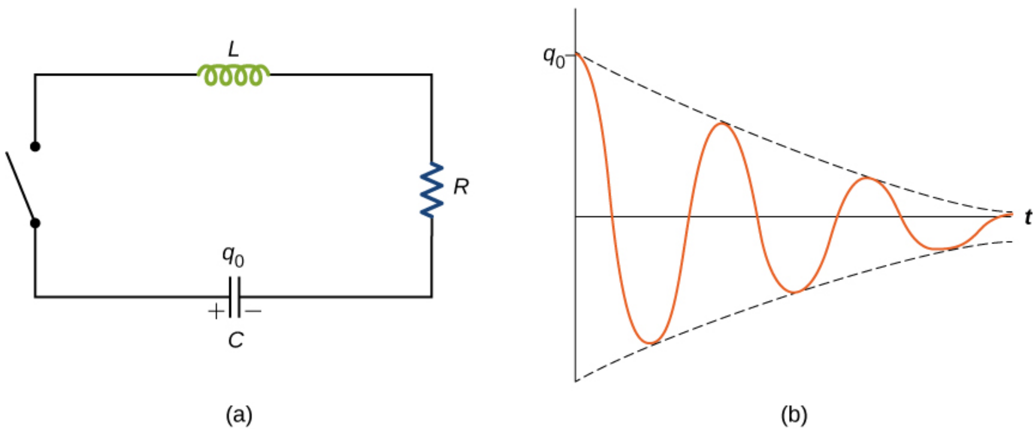


Figure 9.12.2.1: (a) An **RLC** circuit. Electromagnetic oscillations begin when the switch is closed. The capacitor is fully charged initially. (b) Damped oscillations of the capacitor charge are shown in this curve of charge versus time, or **q** versus **t**. The capacitor contains a charge q_0 before the switch is closed.

This equation is analogous to

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0,$$

which is the equation of motion for a **damped mass-spring system** (you first encountered this equation in Oscillations). As we saw in that chapter, it can be shown that the solution to this differential equation takes three forms, depending on whether the angular frequency of the undamped spring is greater than, equal to, or less than **b**/2**m**. Therefore, the result can be underdamped $(\sqrt{k/m} > b/2m)$, critically damped $(\sqrt{k/m} = b/2m)$, or overdamped $(\sqrt{k/m} < b/2m)$. By analogy, the solution **q(t)** to the **RLC** differential equation has the same feature. Here we look only at the case of under-damping. By replacing **m** by **L**, **b** by **R**, **k** by 1/**C**, and **x** by **q** in Equation 9.12.2.1, and assuming $\sqrt{1/LC} > R/2L$, we obtain

Note

$$q(t) = q_0 e^{-Rt/2L} cos(\omega' t + \phi) \tag{9.12.2.2}$$

where the angular frequency of the oscillations is given by

Note

$$\omega' = \sqrt{\frac{1}{LC} - \left(\frac{R}{2L}\right)^2} \tag{9.12.2.3}$$

This underdamped solution is shown in Figure $9.12.2.1b$ Notice that the amplitude of the oscillations decreases as energy is dissipated in the resistor. Equation 9.12.2.2 can be confirmed experimentally by measuring the voltage across the capacitor as a function of time. This voltage, multiplied by the capacitance of the capacitor, then gives **q(t)**.

Note

Try an interactive circuit construction kit that allows you to graph current and voltage as a function of time. You can add inductors and capacitors to work with any combination of **R**, **L**, and **C** circuits with both dc and ac sources.

Note

Try out a circuit-based java applet website that has many problems with both dc and ac sources that will help you practice circuit problems.

Exercise 9.12.2.1

In an **RLC** circuit, $L = 5.0\,mH$, $C = 6.0\,\mu F$, and $R = 200\,\Omega$. (a) Is the circuit underdamped, critically damped, or overdamped? (b) If the circuit starts oscillating with a charge of $3.0 \times 10^{-3} C$ on the capacitor, how much energy has been dissipated in the resistor by the time the oscillations cease?

Answer

a. overdamped; b. 0.75 J

9.13: Inductance (Exercise)

Conceptual Questions

14.2 Mutual Inductance

1. Show that $N\Phi_m/I$ and $\varepsilon/(dI/dt)$, which are both expressions for self-inductance, have the same units.

2. A 10-H inductor carries a current of 20 A. Describe how a 50-V emf can be induced across it.

3. The ignition circuit of an automobile is powered by a 12-V battery. How are we able to generate large voltages with this power source?

4. When the current through a large inductor is interrupted with a switch, an arc appears across the open terminals of the switch. Explain.

14.3 Self-Inductance and Inductors

5. Does self-inductance depend on the value of the magnetic flux? Does it depend on the current through the wire? Correlate your answers with the equation $N\Phi_m = LI$.

6. Would the self-inductance of a 1.0 m long, tightly wound solenoid differ from the self-inductance per meter of an infinite, but otherwise identical, solenoid?

7. Discuss how you might determine the self-inductance per unit length of a long, straight wire.

8. The self-inductance of a coil is zero if there is no current passing through the windings. True or false?

9. How does the self-inductance per unit length near the center of a solenoid (away from the ends) compare with its value near the end of the solenoid?

14.4 Energy in a Magnetic Field

10. Show that $LI^2/2$ has units of energy.

14.5 RL Circuits

11. Use Lenz's law to explain why the initial current in the RL circuit of Figure 14.12(b) is zero.

12. When the current in the **RL** circuit of Figure 14.12(b) reaches its final value ε/R, what is the voltage across the inductor? Across the resistor?

13. Does the time required for the current in an **RL** circuit to reach any fraction of its steady-state value depend on the emf of the battery?

14. An inductor is connected across the terminals of a battery. Does the current that eventually flows through the inductor depend on the internal resistance of the battery? Does the time required for the current to reach its final value depend on this resistance?

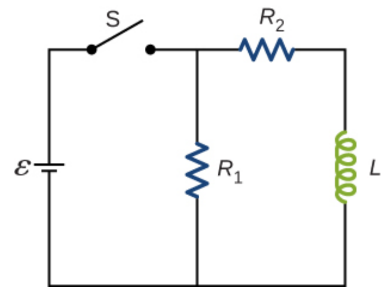
15. At what time is the voltage across the inductor of the **RL** circuit of Figure 14.12(b) a maximum?

16. In the simple **RL** circuit of Figure 14.12(b), can the emf induced across the inductor ever be greater than the emf of the battery used to produce the current?

17. If the emf of the battery of Figure 14.12(b) is reduced by a factor of 2, by how much does the steady-state energy stored in the magnetic field of the inductor change?

18. A steady current flows through a circuit with a large inductive time constant. When a switch in the circuit is opened, a large spark occurs across the terminals of the switch. Explain.

19. Describe how the currents through R_1 and R_2 shown below vary with time after switch S is closed.



20. Discuss possible practical applications of **RL** circuits.

14.6 Oscillations in an LC Circuit

21. Do Kirchhoff's rules apply to circuits that contain inductors and capacitors?

22. Can a circuit element have both capacitance and inductance?

23. In an LC circuit, what determines the frequency and the amplitude of the energy oscillations in either the inductor or capacitor?

14.7 RLC Series Circuits

24. When a wire is connected between the two ends of a solenoid, the resulting circuit can oscillate like an **RLC** circuit. Describe what causes the capacitance in this circuit.

25. Describe what effect the resistance of the connecting wires has on an oscillating **LC** circuit.

26. Suppose you wanted to design an **LC** circuit with a frequency of 0.01 Hz. What problems might you encounter?

27. A radio receiver uses an **RLC** circuit to pick out particular frequencies to listen to in your house or car without hearing other unwanted frequencies. How would someone design such a circuit?

Problems

14.2 Mutual Inductance

28. When the current in one coil changes at a rate of 5.6 A/s, an emf of $6.3 \times 10^{-3} V$. is induced in a second, nearby coil. What is the mutual inductance of the two coils?

29. An **emf** of $9.7 \times 10^{-3} V$ is induced in a coil while the current in a nearby coil is decreasing at a rate of 2.7 A/s. What is the mutual inductance of the two coils?

30. Two coils close to each other have a mutual inductance of 32 mH. If the current in one coil decays according to $I = I_0 e^{-\alpha t}$, where $I_0 = 5.0 A$ and $\alpha = 2.0 \times 10^3 s^{-1}$, what is the emf induced in the second coil immediately after the current starts to decay? At $t = 1.0 \times 10^{-3} s$?

31. A coil of 40 turns is wrapped around a long solenoid of cross-sectional area $7.5 \times 10^{-3} m^2$. The solenoid is 0.50 m long and has 500 turns.

(a) What is the mutual inductance of this system?

(b) The outer coil is replaced by a coil of 40 turns whose radius is three times that of the solenoid. What is the mutual inductance of this configuration?

32. A 600-turn solenoid is 0.55 m long and 4.2 cm in diameter. Inside the solenoid, a small (1.1cm×1.4cm), single-turn rectangular coil is fixed in place with its face perpendicular to the long axis of the solenoid. What is the mutual inductance of this system?

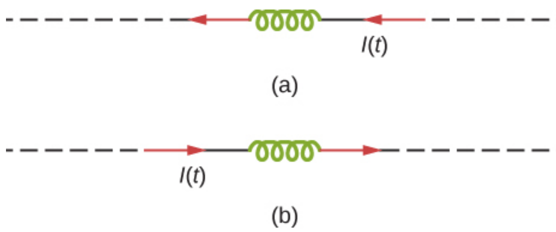
33. A toroidal coil has a mean radius of 16 cm and a cross-sectional area of $0.25 cm^2$; it is wound uniformly with 1000 turns. A second toroidal coil of 750 turns is wound uniformly over the first coil. Ignoring the variation of the magnetic field within a toroid, determine the mutual inductance of the two coils.

34. A solenoid of N_1 turns has length l_1 and radius R_1, and a second smaller solenoid of N_2 turns has length l_2 and radius R_2. The smaller solenoid is placed completely inside the larger solenoid so that their long axes coincide. What is the mutual inductance of the two solenoids?

14.3 Self-Inductance and Inductors

35. An **emf** of 0.40 V is induced across a coil when the current through it changes uniformly from 0.10 to 0.60 A in 0.30 s. What is the self-inductance of the coil?

36. The current shown in part (a) below is increasing, whereas that shown in part (b) is decreasing. In each case, determine which end of the inductor is at the higher potential.



37. What is the rate at which the current though a 0.30-H coil is changing if an emf of 0.12 V is induced across the coil?

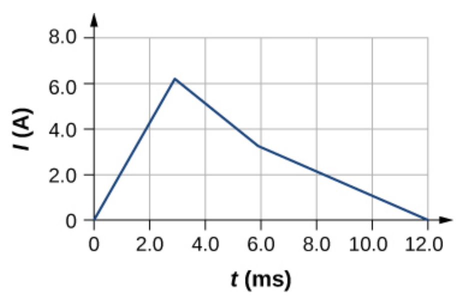
38. When a camera uses a flash, a fully charged capacitor discharges through an inductor. In what time must the 0.100-A current through a 2.00-mH inductor be switched on or off to induce a 500-V emf?

39. A coil with a self-inductance of 2.0 H carries a current that varies with time according to $I(t) = (2.0A) sin 120\pi t$. Find an expression for the emf induced in the coil.

40. A solenoid 50 cm long is wound with 500 turns of wire. The cross-sectional area of the coil is $2.0 cm^2$ What is the self-inductance of the solenoid?

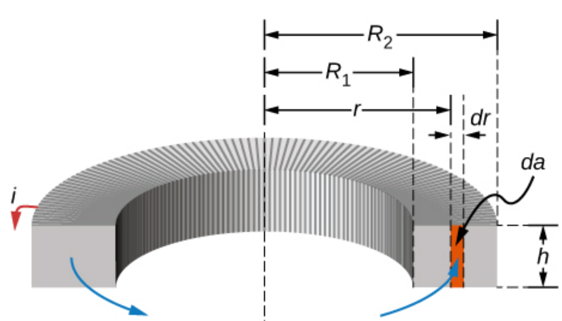
41. A coil with a self-inductance of 3.0 H carries a current that decreases at a uniform rate $dI/dt = -0.050A/s$. What is the **emf** induced in the coil? Describe the polarity of the induced **emf.**

42. The current $I(t)$ through a 5.0-mH inductor varies with time, as shown below. The resistance of the inductor is **5.0Ω**. Calculate the voltage across the inductor at $t = 2.0ms, t = 4.0ms$, and $t = 8.0ms$.



43. A long, cylindrical solenoid with 100 turns per centimeter has a radius of 1.5 cm. (a) Neglecting end effects, what is the self-inductance per unit length of the solenoid? (b) If the current through the solenoid changes at the rate 5.0 A/s, what is the emf induced per unit length?

44. Suppose that a rectangular toroid has 2000 windings and a self-inductance of 0.040 H. If h=0.10mh=0.10m, what is the ratio of its outer radius to its inner radius?



45. What is the self-inductance per meter of a coaxial cable whose inner radius is 0.50 mm and whose outer radius is 4.00 mm?

14.4 Energy in a Magnetic Field

46. At the instant a current of 0.20 A is flowing through a coil of wire, the energy stored in its magnetic field is $6.0 \times 10^{-3} J$. What is the self-inductance of the coil?

47. Suppose that a rectangular toroid has 2000 windings and a self-inductance of 0.040 H. If $h = 0.10m$, what is the current flowing through a rectangular toroid when the energy in its magnetic field is $2.0 \times 10^{-6} J$?

48. Solenoid **A** is tightly wound while solenoid **B** has windings that are evenly spaced with a gap equal to the diameter of the wire. The solenoids are otherwise identical. Determine the ratio of the energies stored per unit length of these solenoids when the same current flows through each.

49. A 10-H inductor carries a current of 20 A. How much ice at **0°C** could be melted by the energy stored in the magnetic field of the inductor? (Hint: Use the value $L_f = 334 J/g$ for ice.)

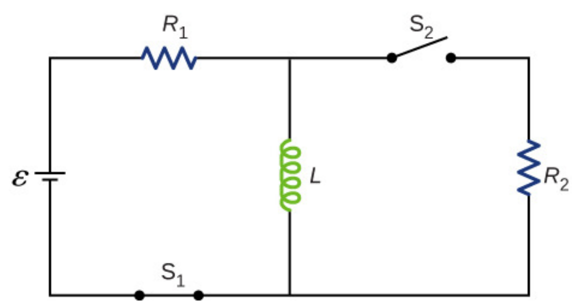
50. A coil with a self-inductance of 3.0 H and a resistance of **100Ω** carries a steady current of 2.0 A. (a) What is the energy stored in the magnetic field of the coil? (b) What is the energy per second dissipated in the resistance of the coil?

51. A current of 1.2 A is flowing in a coaxial cable whose outer radius is five times its inner radius. What is the magnetic field energy stored in a 3.0-m length of the cable?

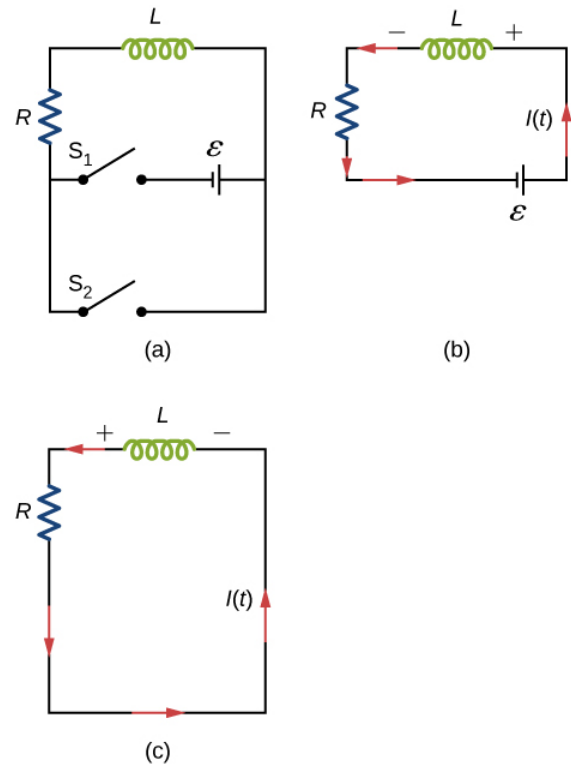
14.5 RL Circuits

52. In Figure 14.12, $\varepsilon = 12V$, $L = 20mH$, and $R = 5.0\Omega$. Determine (a) the time constant of the circuit, (b) the initial current through the resistor, (c) the final current through the resistor, (d) the current through the resistor when $t = 2\tau_L$, and (e) the voltages across the inductor and the resistor when $t = 2\tau_L$.

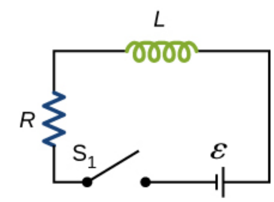
53. For the circuit shown below, $\varepsilon = 20V$, $L = 4.0mH$, and $R = 5.0\Omega$. After steady state is reached with S_1 closed and S_2 open, S_2 is closed and immediately thereafter (at $t = 0$) S_1 is opened. Determine (a) the current through L at $t = 0$, (b) the current through L at $t = 4.0 \times 10^{-4} s$, and (c) the voltages across L and R at $t = 4.0 \times 10^{-4} s$. $R_1 = R_2 = R$.



54. The current in the **RL** circuit shown here increases to **40%** of its steady-state value in 2.0 s. What is the time constant of the circuit?



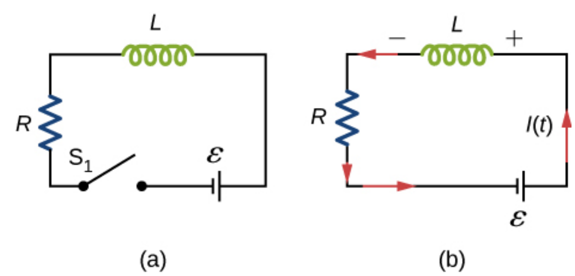
55. How long after switch S_1 is thrown does it take the current in the circuit shown to reach half its maximum value? Express your answer in terms of the time constant of the circuit.



56. Examine the circuit shown below in part (a). Determine dI/dt at the instant after the switch is thrown in the circuit of

(a), thereby producing the circuit of

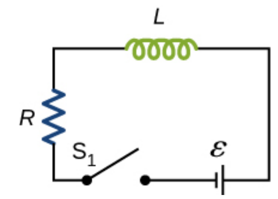
(b). Show that if I were to continue to increase at this initial rate, it would reach its maximum ε/R in one time constant.



57. The current in the **RL** circuit shown below reaches half its maximum value in 1.75 ms after the switch S_1 is thrown. Determine

(a) the time constant of the circuit and

(b) the resistance of the circuit if $L = 250mH$.

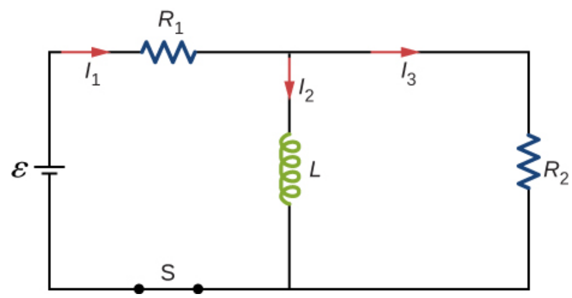


58. Consider the circuit shown below. Find I_1, I_2, and I_3 when

(a) the switch S is first closed,

(b) after the currents have reached steady-state values, and

(c) at the instant the switch is reopened (after being closed for a long time).



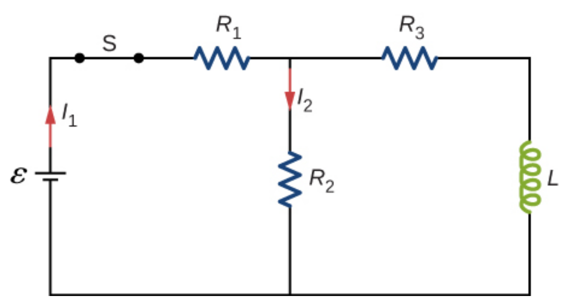
59. For the circuit shown below, $\varepsilon = 50V$, $R_1 = 10\Omega$, $R_2 = R_3 = 19.4\Omega$ and $L = 2.0mH$. Find the values of I_1 and I_2

(a) immediately after switch S is closed,

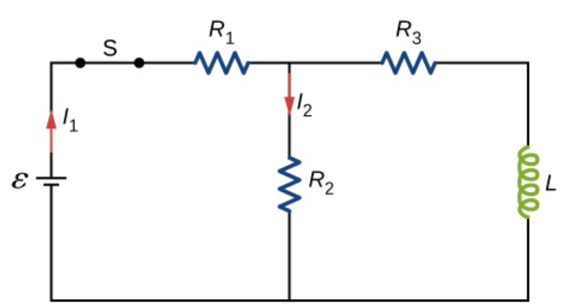
(b) a long time after S is closed,

(c) immediately after S is reopened, and

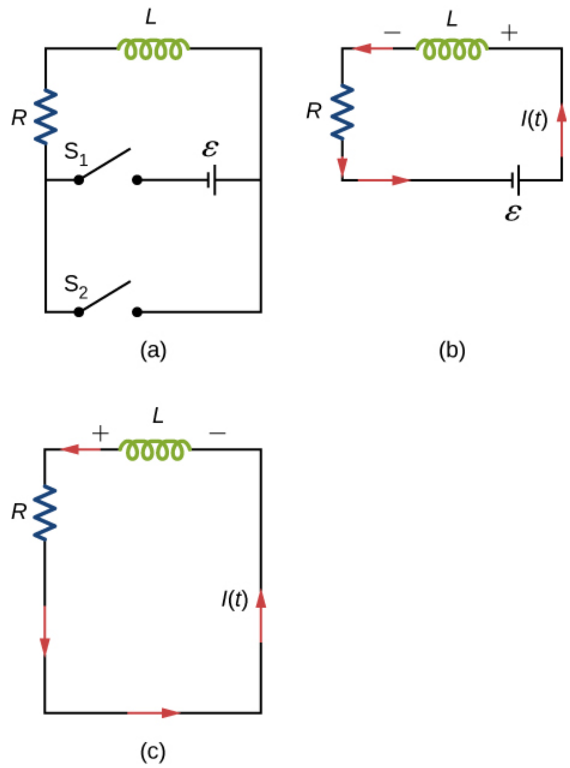
(d) a long time after S is reopened.



60. For the circuit shown below, find the current through the inductor $2.0 \times 10^{-5}s$ after the switch is reopened.



61. Show that for the circuit shown below, the initial energy stored in the inductor, $LI^2(0)/2$, is equal to the total energy eventually dissipated in the resistor, $\int_0^\infty I^2(t)Rdt$.



14.6 Oscillations in an LC Circuit

62. A 5000-pF capacitor is charged to 100 V and then quickly connected to an 80-mH inductor. Determine

(a) the maximum energy stored in the magnetic field of the inductor,

(b) the peak value of the current, and

(c) the frequency of oscillation of the circuit.

63. The self-inductance and capacitance of an **LC** circuit are 0.20 mH and 5.0 pF. What is the angular frequency at which the circuit oscillates?

64. What is the self-inductance of an **LC** circuit that oscillates at 60 Hz when the capacitance is $10\mu F$?

65. In an oscillating **LC** circuit, the maximum charge on the capacitor is $2.0 \times 10^{-6} C$ and the maximum current through the inductor is 8.0 mA.

(a) What is the period of the oscillations?

(b) How much time elapses between an instant when the capacitor is uncharged and the next instant when it is fully charged?

66. The self-inductance and capacitance of an oscillating **LC** circuit are $L = 20 mH$ and $C = 1.0\mu F$, respectively.

(a) What is the frequency of the oscillations?

(b) If the maximum potential difference between the plates of the capacitor is 50 V, what is the maximum current in the circuit?

67. In an oscillating **LC** circuit, the maximum charge on the capacitor is q_m. Determine the charge on the capacitor and the current through the inductor when energy is shared equally between the electric and magnetic fields. Express your answer in terms of q_m, L, and C.

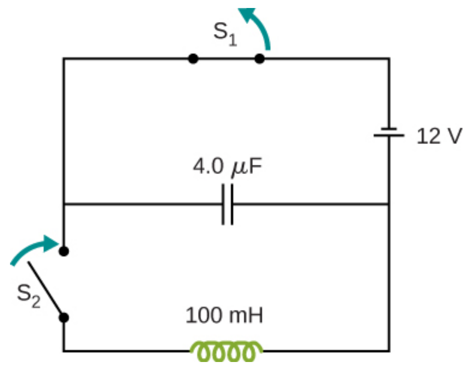
68. In the circuit shown below, S_1 is opened and S_2 is closed simultaneously. Determine

(a) the frequency of the resulting oscillations,

(b) the maximum charge on the capacitor,

(c) the maximum current through the inductor, and

(d) the electromagnetic energy of the oscillating circuit.



69. An **LC** circuit in an AM tuner (in a car stereo) uses a coil with an inductance of 2.5 mH and a variable capacitor. If the natural frequency of the circuit is to be adjustable over the range 540 to 1600 kHz (the AM broadcast band), what range of capacitance is required?

14.7 RLC Series Circuits

70. In an oscillating **RLC** circuit, $R = 5.0\Omega, L = 5.0mH$, and $C = 500\mu F$. What is the angular frequency of the oscillations?

71. In an oscillating **RLC** circuit with $L = 10mH, C = 1.5\mu F$, and $R = 2.0\Omega$, how much time elapses before the amplitude of the oscillations drops to half its initial value?

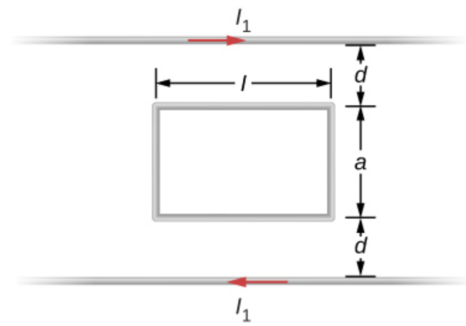
72. What resistance **R** must be connected in series with a 200-mH inductor of the resulting **RLC** oscillating circuit is to decay to **50%** of its initial value of charge in 50 cycles? To **0.10%** of its initial value in 50 cycles?

Additional Problems

73. Show that the self-inductance per unit length of an infinite, straight, thin wire is infinite.

74. Two long, parallel wires carry equal currents in opposite directions. The radius of each wire is a, and the distance between the centers of the wires is d. Show that if the magnetic flux within the wires themselves can be ignored, the self-inductance of a length l of such a pair of wires is $L = \frac{\mu_0 l}{\pi} ln \frac{d-a}{a}$. (Hint: Calculate the magnetic flux through a rectangle of length l between the wires and then use $L = N\Phi/I$.)

75. A small, rectangular single loop of wire with dimensions l, and a is placed, as shown below, in the plane of a much larger, rectangular single loop of wire. The two short sides of the larger loop are so far from the smaller loop that their magnetic fields over the smaller fields over the smaller loop can be ignored. What is the mutual inductance of the two loops?



76. Suppose that a cylindrical solenoid is wrapped around a core of iron whose magnetic susceptibility is x. Using Equation 14.9, show that the self-inductance of the solenoid is given by $L = \frac{(1+x)\mu_0 N^2 A}{l}$, where **I** is its length, **A** its cross-sectional area, and **N** its total number of turns.

77. The solenoid of the preceding problem is wrapped around an iron core whose magnetic susceptibility is 4.0×10^3.

(a) If a current of 2.0 A flows through the solenoid, what is the magnetic field in the iron core?

(b) What is the effective surface current formed by the aligned atomic current loops in the iron core?

(c) What is the self-inductance of the filled solenoid?

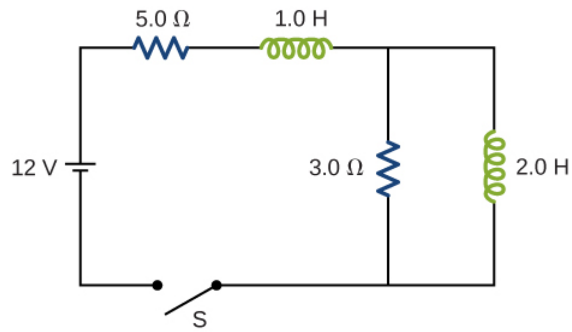
78. A rectangular toroid with inner radius $R_1 = 7.0 cm$, outer radius $R_2 = 9.0 cm$, height $h = 3.0$, and $N = 3000$ turns is filled with an iron core of magnetic susceptibility 5.2×10^3.

(a) What is the self-inductance of the toroid?

(b) If the current through the toroid is 2.0 A, what is the magnetic field at the center of the core?

(c) For this same 2.0-A current, what is the effective surface current formed by the aligned atomic current loops in the iron core?

79. The switch S of the circuit shown below is closed at $t = 0$. Determine (a) the initial current through the battery and (b) the steady-state current through the battery.



80. In an oscillating **RLC** circuit, $R = 7.0\Omega, L = 10 mH$, and $C = 3.0 \mu F$. Initially, the capacitor has a charge of $8.0 \mu C$ and the current is zero. Calculate the charge on the capacitor

(a) five cycles later and

(b) 50 cycles later.

81. A 25.0-H inductor has 100 A of current turned off in 1.00 ms.

(a) What voltage is induced to oppose this?

(b) What is unreasonable about this result?

(c) Which assumption or premise is responsible?

Challenge Problems

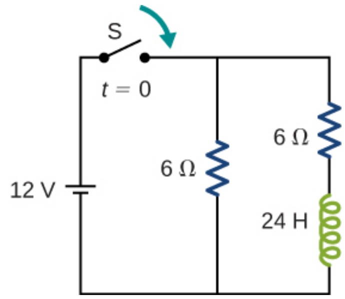
82. A coaxial cable has an inner conductor of radius a, and outer thin cylindrical shell of radius b. A current **I** flows in the inner conductor and returns in the outer conductor. The self-inductance of the structure will depend on how the current in the inner cylinder tends to be distributed. Investigate the following two extreme cases.

(a) Let current in the inner conductor be distributed only on the surface and find the self-inductance.

(b) Let current in the inner cylinder be distributed uniformly over its cross-section and find the self-inductance. Compare with your results in (a).

83. In a damped oscillating circuit the energy is dissipated in the resistor. The **Q**-factor is a measure of the persistence of the oscillator against the dissipative loss. (a) Prove that for a lightly damped circuit the energy, **U**, in the circuit decreases according to the following equation. $\frac{dU}{dt} = -2\beta U$, where $\beta = \frac{R}{2L}$. (b) Using the definition of the **Q**-factor as energy divided by the loss over the next cycle, prove that **Q**-factor of a lightly damped oscillator as defined in this problem is $Q \equiv \frac{U_{begin}}{\Delta U_{one\ cycle}} = \frac{1}{2\pi R}\sqrt{\frac{L}{C}}$. (Hint: For (b), to obtain **Q**, divide **E** at the beginning of one cycle by the change ΔE over the next cycle.)

84. The switch in the circuit shown below is closed at $t = 0s$. Find currents through (a) R_1, (b) R_2, and (c) the battery as function of time.



85. A square loop of side 2 cm is placed 1 cm from a long wire carrying a current that varies with time at a constant rate of 3 A/s as shown below.

(a) Use Ampère's law and find the magnetic field as a function of time from the current in the wire.

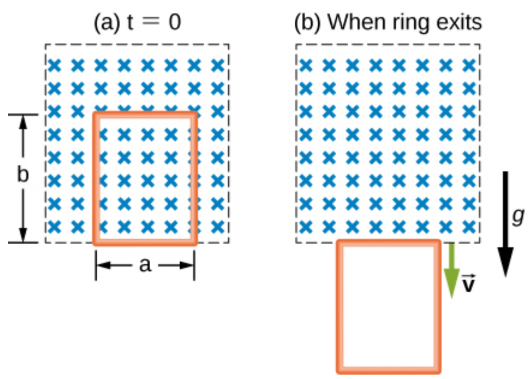
(b) Determine the magnetic flux through the loop.

(c) If the loop has a resistance of **3Ω**, how much induced current flows in the loop?

86. A rectangular copper ring, of mass 100 g and resistance **0.2Ω**, is in a region of uniform magnetic field that is perpendicular to the area enclosed by the ring and horizontal to Earth's surface. The ring is let go from rest when it is at the edge of the nonzero magnetic field region (see below).

(a) Find its speed when the ring just exits the region of uniform magnetic field.

(b) If it was let go at t=0t=0, what is the time when it exits the region of magnetic field for the following values: $a=25cm, b=50cm, B=3T,$ and $g=9.8m/s^2$? Assume the magnetic field of the induced current is negligible compared to 3 T.



CHAPTER OVERVIEW

10: Alternating-Current Circuits

In this chapter, we use Kirchhoff's laws to analyze four simple circuits in which ac flows. We have discussed the use of the resistor, capacitor, and inductor in circuits with batteries. These components are also part of ac circuits. However, because ac is required, the constant source of emf supplied by a battery is replaced by an ac voltage source, which produces an oscillating emf.

10.1: Prelude to Alternating-Current Circuits
10.2: AC Sources
10.3: Simple AC Circuits
10.4: RLC Series Circuits with AC
10.5: Power in an AC Circuit
10.6: Resonance in an AC Circuit
10.7: Transformers
10.8: Alternating-Current Circuits (Exercise)

10.1: Prelude to Alternating-Current Circuits

Electric power is delivered to our homes by alternating current (ac) through high-voltage transmission lines. As explained in Transformers, transformers can then change the amplitude of the alternating potential difference to a more useful form. This lets us transmit power at very high voltages, minimizing resistive heating losses in the lines, and then furnish that power to homes at lower, safer voltages. Because constant potential differences are unaffected by transformers, this capability is more difficult to achieve with direct-current transmission.

Figure 10.1.1: The current we draw into our houses is an alternating current (ac). Power lines transmit ac to our neighborhoods, where local power stations and transformers distribute it to our homes. In this chapter, we discuss how a transformer works and how it allows us to transmit power at very high voltages and minimal heating losses across the lines.

In this chapter, we use Kirchhoff's laws to analyze four simple circuits in which ac flows. We have discussed the use of the resistor, capacitor, and inductor in circuits with batteries. These components are also part of ac circuits. However, because ac is required, the constant source of emf supplied by a battery is replaced by an ac voltage source, which produces an oscillating emf.

10.2: AC Sources

Learning Objectives

By the end of the section, you will be able to:

- Explain the differences between direct current (dc) and alternating current (ac)
- Define characteristic features of alternating current and voltage, such as the amplitude or peak and the frequency

Most examples dealt with so far in this book, particularly those using batteries, have constant-voltage sources. Thus, once the current is established, it is constant. **Direct current (dc)** is the flow of electric charge in only one direction. It is the steady state of a constant-voltage circuit.

Most well-known applications, however, use a time-varying voltage source. **Alternating current (ac)** is the flow of electric charge that periodically reverses direction. An ac is produced by an alternating emf, which is generated in a power plant, as described in Induced Electric Fields. If the ac source varies periodically, particularly sinusoidally, the circuit is known as an ac circuit. Examples include the commercial and residential power that serves so many of our needs.

The ac voltages and frequencies commonly used in businesses and homes vary around the world. In a typical house, the potential difference between the two sides of an electrical outlet alternates sinusoidally with a frequency of 60 or 50 Hz and an amplitude of 170 or 311 V, depending on whether you live in the United States or Europe, respectively. Most people know the potential difference for electrical outlets is 120 V or 220 V in the US or Europe, but as explained later in the chapter, these voltages are not the peak values given here but rather are related to the common voltages we see in our electrical outlets. Figure $10.2.1$ shows graphs of voltage and current versus time for typical dc and ac power in the United States.

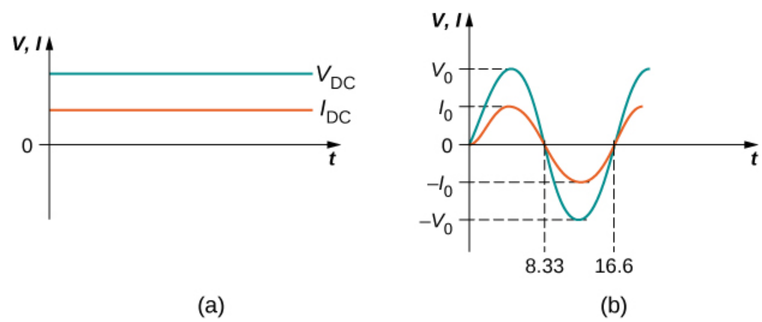


Figure 10.2.1: (a) The dc voltage and current are constant in time, once the current is established. (b) The voltage and current versus time are quite different for ac power. In this example, which shows 60-Hz ac power and time t in milliseconds, voltage and current are sinusoidal and are in phase for a simple resistance circuit. The frequencies and peak voltages of ac sources differ greatly.

Suppose we hook up a resistor to an ac voltage source and determine how the voltage and current vary in time across the resistor. Figure $10.2.2$ shows a schematic of a simple circuit with an ac voltage source. The voltage fluctuates sinusoidally with time at a fixed frequency, as shown, on either the battery terminals or the resistor. Therefore, the **ac voltage**, or the "voltage at a plug," can be given by

$$v(t) = V_0 \sin \omega t,$$

where

- v is the voltage at time t,
- V_0 is the peak voltage, and
- ω is the angular frequency in radians per second.

For a typical house in the United States, $V_0 = 156\, V$ and $\omega = 120\pi\, rad/s$, whereas in Europe, $V_0 = 311\, V$ and $\omega = 100\pi\, rad/s$.

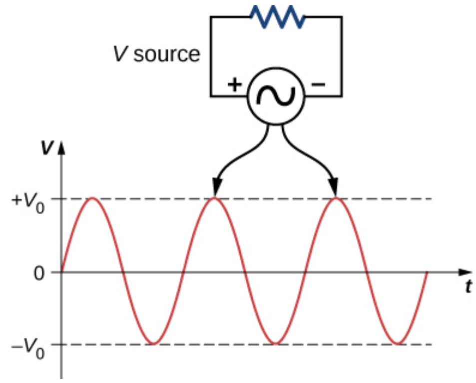


Figure 10.2.2: The potential difference **V** between the terminals of an ac voltage source fluctuates, so the source and the resistor have ac sine waves on top of each other. The mathematical expression for **v** is given by $v = V_0\, sin\, \omega t$.

For this simple resistance circuit, $I = V/R$, so the **ac current**, meaning the current that fluctuates sinusoidally with time at a fixed frequency, is

$$i(t) = I_0 \sin \omega t,$$

where

- $i(t)$ is the current at time t and
- I_0 is the peak current and is equal to V_0/R.

For this example, the voltage and current are said to be in phase, meaning that their sinusoidal functional forms have peaks, troughs, and nodes in the same place. They oscillate in sync with each other, as shown in Figure $10.2.1b$ In these equations, and throughout this chapter, we use lowercase letters (such as i) to indicate instantaneous values and capital letters (such as I) to indicate maximum, or peak, values.

Current in the resistor alternates back and forth just like the driving voltage, since $I = V/R$. If the resistor is a fluorescent light bulb, for example, it brightens and dims 120 times per second as the current repeatedly goes through zero. A 120-Hz flicker is too rapid for your eyes to detect, but if you wave your hand back and forth between your face and a fluorescent light, you will see the stroboscopic effect of ac.

? Exercise 10.2.1

If a European ac voltage source is considered, what is the time difference between the zero crossings on an ac voltage-versus-time graph?

Solution

10 ms

10.3: Simple AC Circuits

Learning Objectives

By the end of the section, you will be able to:

- Interpret phasor diagrams and apply them to ac circuits with resistors, capacitors, and inductors
- Define the reactance for a resistor, capacitor, and inductor to help understand how current in the circuit behaves compared to each of these devices

In this section, we study simple models of ac voltage sources connected to three circuit components: (1) a resistor, (2) a capacitor, and (3) an inductor. The power furnished by an ac voltage source has an emf given by

$$v(t) = V_0 \, sin\, \omega t,$$

as shown in Figure $10.3.1$. This sine function assumes we start recording the voltage when it is $v = 0\,V$ at a time of $t = 0\,s$. A phase constant may be involved that shifts the function when we start measuring voltages, similar to the phase constant in the waves we studied in Waves. However, because we are free to choose when we start examining the voltage, we can ignore this phase constant for now. We can measure this voltage across the circuit components using one of two methods: (1) a quantitative approach based on our knowledge of circuits, or (2) a graphical approach that is explained in the coming sections.

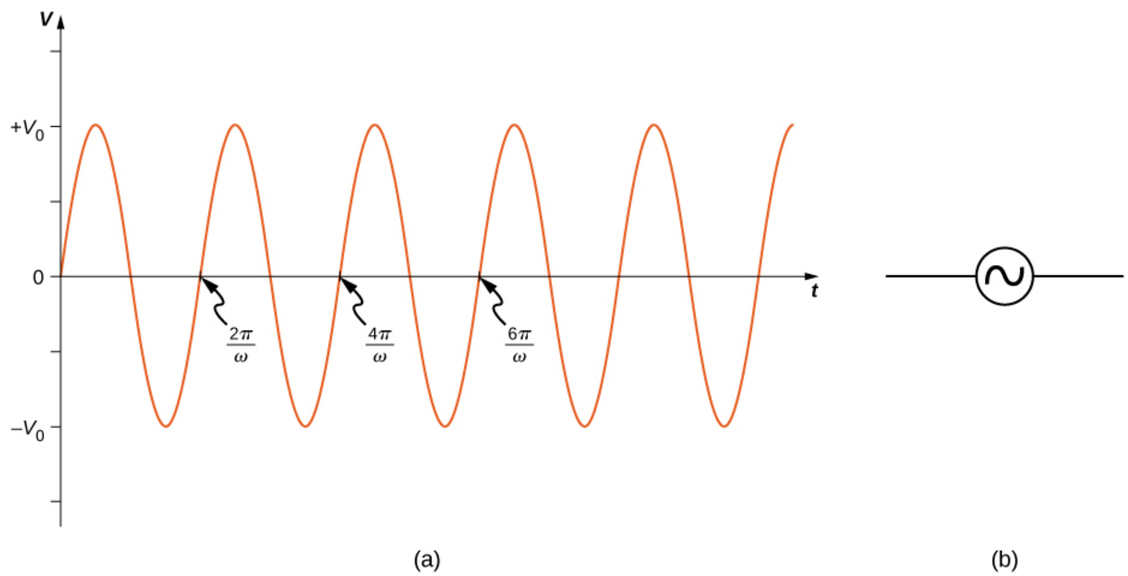


Figure 10.3.1: (a) The output $v(t) = V_0 \, sin\, \omega t$ of an ac generator. (b) Symbol used to represent an ac voltage source in a circuit diagram.

Resistor

First, consider a **resistor** connected across an ac voltage source. From Kirchhoff's loop rule, the instantaneous voltage across the resistor of Figure $10.3.2a$ is

$$v_R(t) = V_0 \, sin\, \omega t$$

and the instantaneous current through the resistor is

$$i_R(t) = \frac{v_R(t)}{R} = \frac{V_0}{R} \, sin\, \omega t = I_0 \, sin\, \omega t.$$

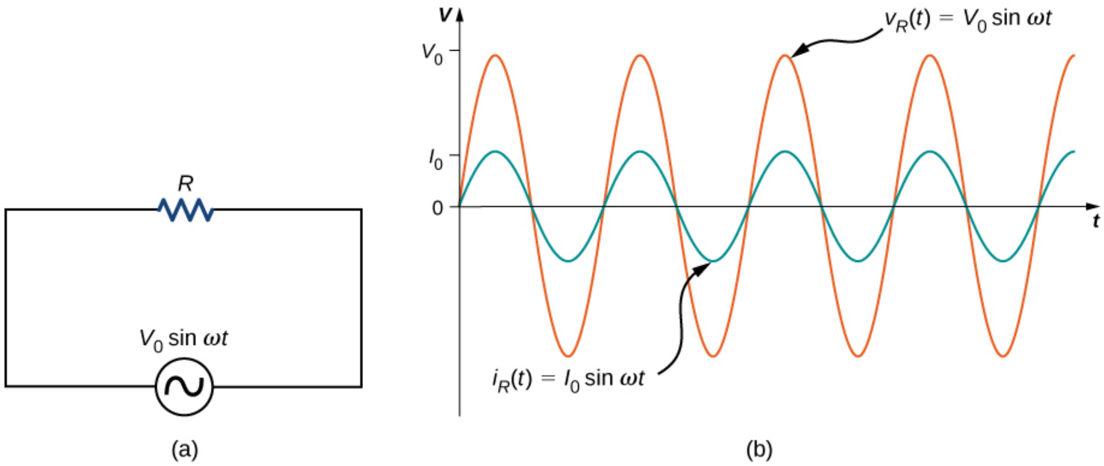


Figure 10.3.2: (a) A resistor connected across an ac voltage source. (b) The current $i_R(t)$ through the resistor and the voltage $v_R(t)$ across the resistor. The two quantities are in phase.

Here, $I_0 = V_0/R$ is the amplitude of the time-varying current. Plots of $i_R(t)$ and $v_R(t)$ are shown in Figure $10.3.2b$ Both curves reach their maxima and minima at the same times, that is, the current through and the voltage across the resistor are in phase.

Graphical representations of the phase relationships between current and voltage are often useful in the analysis of ac circuits. Such representations are called **phasor diagrams**. The phasor diagram for $i_R(t)$ is shown in Figure $10.3.3a$, with the current on the vertical axis. The arrow (or phasor) is rotating counterclockwise at a constant angular frequency ω, so we are viewing it at one instant in time. If the length of the arrow corresponds to the current amplitude I_0, the projection of the rotating arrow onto the vertical axis is $i_R(t) = I_0 \, sin \, \omega t$, which is the instantaneous current.

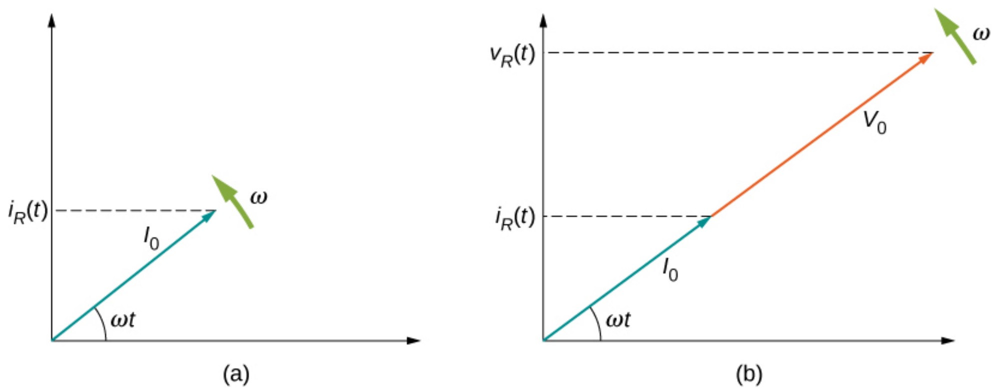


Figure 10.3.3: (a) The phasor diagram representing the current through the resistor of Figure 10.3.2 (b) The phasor diagram representing both $i_R(t)$ and $v_R(t)$.

The vertical axis on a phasor diagram could be either the voltage or the current, depending on the phasor that is being examined. In addition, several quantities can be depicted on the same phasor diagram. For example, both the current $i_R(t)$ and the voltage $v_R(t)$ are shown in the diagram of Figure $10.3.3b$ Since they have the same frequency and are in phase, their phasors point in the same direction and rotate together. The relative lengths of the two phasors are arbitrary because they represent different quantities; however, the ratio of the lengths of the two phasors can be represented by the resistance, since one is a voltage phasor and the other is a current phasor.

Capacitor

Now let's consider a **capacitor** connected across an ac voltage source. From Kirchhoff's loop rule, the instantaneous voltage across the capacitor of Figure $10.3.4a$ is

$$v_C(t) = V_0 \, sin \, \omega t.$$

Recall that the charge in a capacitor is given by $Q = CV$. This is true at any time measured in the ac cycle of voltage. Consequently, the instantaneous charge on the capacitor is

$$q(t) = Cv_C(t) = CV_0 \, sin \, \omega t.$$

Since the current in the circuit is the rate at which charge enters (or leaves) the capacitor,

$$i_C(t) = \frac{dq(t)}{dt} = \omega C V_0 \cos \omega t = I_0 \cos \omega t,$$

where $I_0 = \omega C V_0$ is the current amplitude. Using the trigonometric relationship $\cos \omega t = \sin(\omega t + \pi/2)$, we may express the instantaneous current as

$$i_C(t) = I_0 \sin\left(\omega t + \frac{\pi}{2}\right).$$

Dividing V_0 by I_0, we obtain an equation that looks similar to Ohm's law:

$$\frac{V_0}{I_0} = \frac{1}{\omega C} = X_C. \tag{10.3.1}$$

The quantity X_C is analogous to resistance in a dc circuit in the sense that both quantities are a ratio of a voltage to a current. As a result, they have the same unit, the ohm. Keep in mind, however, that a capacitor stores and discharges electric energy, whereas a resistor dissipates it. The quantity X_C is known as the capacitive reactance of the capacitor, or the opposition of a capacitor to a change in current. It depends inversely on the frequency of the ac source—high frequency leads to low capacitive reactance.

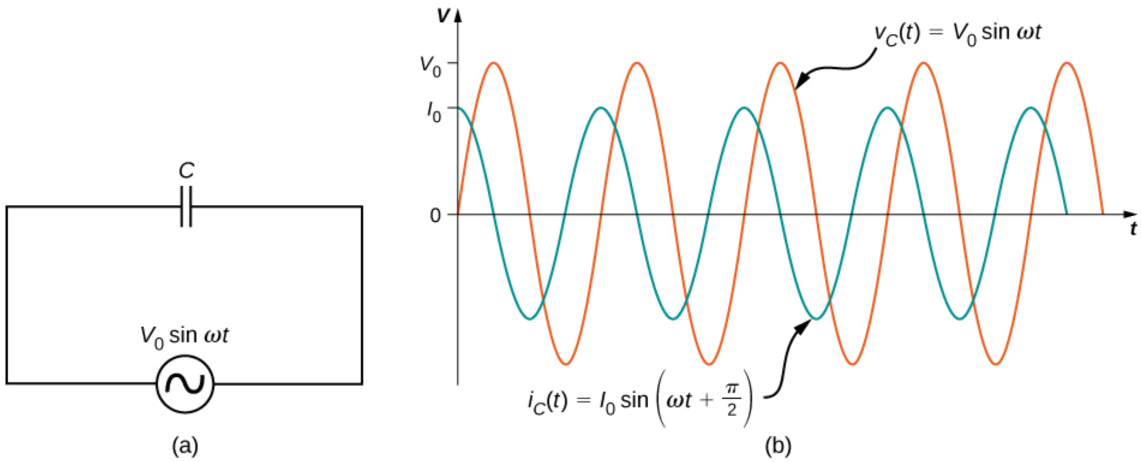


Figure 10.3.4: (a) A capacitor connected across an ac generator. (b) The current $i_C(t)$ through the capacitor and the voltage $v_C(t)$ across the capacitor. Notice that $i_C(t)$ leads $v_C(t)$ by $\pi/2$ rad.

A comparison of the expressions for $v_C(t)$ and $i_C(t)$ shows that there is a phase difference of $\pi/2$ rad between them. When these two quantities are plotted together, the current peaks a quarter cycle (or $\pi/2$ rad) ahead of the voltage, as illustrated in Figure $10.3.4b$. The current through a capacitor leads the voltage across a capacitor by $\pi/2$ rad, or a quarter of a cycle.

The corresponding phasor diagram is shown in Figure $10.3.5$. Here, the relationship between $i_C(t)$ and $v_C(t)$ is represented by having their phasors rotate at the same angular frequency, with the current phasor leading by $\pi/2$ rad.

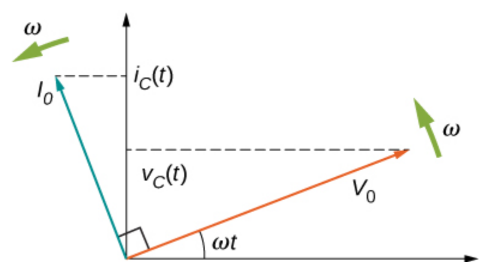


Figure 10.3.5: The current phasor leads the voltage phasor by $\pi/2$ rad as they both rotate with the same angular frequency.

To this point, we have exclusively been using peak values of the current or voltage in our discussion, namely, I_0 and V_0. However, if we average out the values of current or voltage, these values are zero. Therefore, we often use a second convention called the root mean square value, or rms value, in discussions of current and voltage. The rms operates in reverse of the terminology. First, you square the function, next, you take the mean, and then, you find the square root. As a result, the rms values of current and voltage are not zero. Appliances and devices are commonly quoted with rms values for their operations, rather than peak values. We indicate rms values with a subscript attached to a capital letter (such as I_{rms}).

Although a capacitor is basically an open circuit, an **rms current**, or the root mean square of the current, appears in a circuit with an ac voltage applied to a capacitor. Consider that

Note

$$I_{rms} = \frac{I_0}{\sqrt{2}},$$

where I_0 is the peak current in an ac system. The **rms voltage**, or the root mean square of the voltage, is

Note

$$V_{rms} = \frac{V_0}{\sqrt{2}},$$

where V_0 is the peak voltage in an ac system. The rms current appears because the voltage is continually reversing, charging, and discharging the capacitor. If the frequency goes to zero, which would be a dc voltage, X_C tends to infinity, and the current is zero once the capacitor is charged. At very high frequencies, the capacitor's reactance tends to zero—it has a negligible reactance and does not impede the current (it acts like a simple wire).

Inductor

Lastly, let's consider an **inductor** connected to an ac voltage source. From Kirchhoff's loop rule, the voltage across the inductor **L** of Figure $10.3.6a$ is

$$v_L(t) = V_0 \, sin \, \omega t. \tag{10.3.2}$$

The emf across an inductor is equal to $\epsilon = -L(di_L/dt)$; however, the potential difference across the inductor is $v_L(t) = L di_L(t)/dt$, because if we consider that the voltage around the loop must equal zero, the voltage gained from the ac source must dissipate through the inductor. Therefore, connecting this with the ac voltage source, we have

$$\frac{di_L(t)}{dt} = \frac{V_0}{L} sin \, \omega t.$$

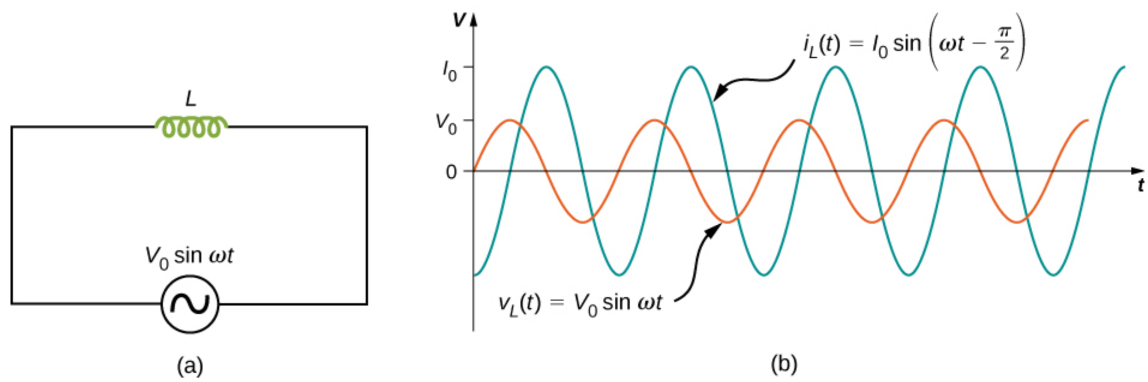


Figure 10.3.6: (a) An inductor connected across an ac generator. (b) The current $i_L(t)$ through the inductor and the voltage $v_L(t)$ across the inductor. Here $i_L(t)$ lags $v_L(t)$ by $\pi/2$ rad.

The current $i_L(t)$ is found by integrating this equation. Since the circuit does not contain a source of constant emf, there is no steady current in the circuit. Hence, we can set the constant of integration, which represents the steady current in the circuit, equal to zero, and we have

$$i_L(t) = -\frac{V_0}{\omega L} cos \, \omega t = \frac{V_0}{\omega L} sin \left(\omega t - \frac{\pi}{2}\right) = I_0 \, sin \left(\omega t - \frac{\pi}{2}\right), \tag{10.3.3}$$

where $I_0 = V_0/\omega L$. The relationship between V_0 and I_0 may also be written in a form analogous to Ohm's law:

✓ Note

$$\frac{V_0}{I_0} = \omega L = X_L. \tag{10.3.4}$$

The quantity X_L is known as the **inductive reactance** of the inductor, or the opposition of an inductor to a change in current; its unit is also the ohm. Note that X_L varies directly as the frequency of the ac source—high frequency causes high inductive reactance.

A phase difference of $\pi/2$ rad occurs between the current through and the voltage across the inductor. From Equation 10.3.2 and Equation 10.3.3, the current through an inductor lags the potential difference across an inductor by $\pi/2$ rad, or a quarter of a cycle. The phasor diagram for this case is shown in Figure 10.3.7.

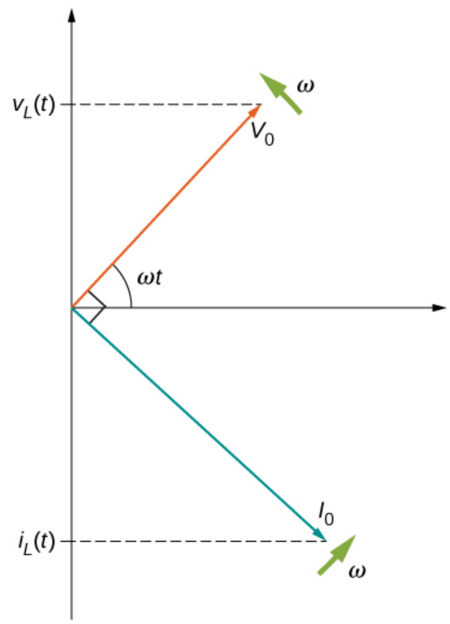


Figure 10.3.7: The current phasor lags the voltage phasor by $\pi/2$ rad as they both rotate with the same angular frequency.

✓ Note

An animation from the University of New South Wales AC Circuits illustrates some of the concepts we discuss in this chapter. They also include wave and phasor diagrams that evolve over time so that you can get a better picture of how each changes over time.

✓ Example 10.3.1: Simple AC CIrcuits

An ac generator produces an emf of amplitude 10 V at a frequency $f = 60\,Hz$. Determine the voltages across and the currents through the circuit elements when the generator is connected to (a) a $100\,\Omega$ resistor, (b) a $10\,\mu F$ capacitor, and (c) a 15-mH inductor.

Strategy

The entire AC voltage across each device is the same as the source voltage. We can find the currents by finding the reactance **X** of each device and solving for the peak current using $I_0 = V_0/X$.

Solution

The voltage across the terminals of the source is

$$v(t) = V_0\, sin\, \omega t = (10\,V) sin\, 120\pi t,$$

where $\omega = 2\pi f = 120\pi\, rad/s$ is the angular frequency. Since **v(t)** is also the voltage across each of the elements, we have

$$v(t) = v_R(t) = v_C(t) = v_L(t) = (10\,V) sin\,120\pi t.$$

a. When $R = 100\,\Omega$, the amplitude of the current through the resistor is

$$I_0 = V_0/R = 10\,V/100\,\Omega = 0.10\,A,$$

so

$$i_R(t) = (0.10\,A) sin\,120\pi t.$$

b. From Equation 10.3.1, the capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{1}{(120\pi\,rad/s)(10 \times 10^{-6}F)} = 265\,\Omega,$$

so the maximum value of the current is

$$I_0 = \frac{V_0}{X_C} = \frac{10\,V}{265\,\Omega} = 3.8 \times 10^{-2}\,A$$

and the instantaneous current is given by

$$i_C(t) = (3.8 \times 10^{-2}\,A)\,sin\,\left(120\pi t + \frac{\pi}{2}\right).$$

c. From Equation 10.3.4, the inductive reactance is

$$X_L = \omega L = (120\pi\,rad/s)(15 \times 10^{-3}H) = 5.7\,\Omega.$$

The maximum current is therefore

$$I_0 = \frac{10\,V}{5.7\,\Omega} = 1.8\,A$$

and the instantaneous current is

$$i_L(t) = (1.8\,A)\,sin\,\left(120\pi t - \frac{\pi}{2}\right).$$

Significance

Although the voltage across each device is the same, the peak current has different values, depending on the reactance. The reactance for each device depends on the values of resistance, capacitance, or inductance.

? Exercise 10.3.1

Repeat Example 10.3.1 for an ac source of amplitude 20 V and frequency 100 Hz.

Answer

a. $(20\,V)\,sin\,200\pi t$; $(0.20\,A)\,sin\,200\pi t$

b. $(20\,V)\,sin\,200\pi t$; $(0.13\,A)\,sin\,(200\pi t + \pi/2)$

c. $(20\,V)\,sin\,200\pi t$; $(2.1\,A)\,sin\,(200\pi t - \pi/2)$

10.4: RLC Series Circuits with AC

Learning Objectives

By the end of the section, you will be able to:

- Describe how the current varies in a resistor, a capacitor, and an inductor while in series with an ac power source
- Use phasors to understand the phase angle of a resistor, capacitor, and inductor ac circuit and to understand what that phase angle means
- Calculate the impedance of a circuit

The ac circuit shown in Figure 10.4.1, called an **RLC** series circuit, is a series combination of a resistor, capacitor, and inductor connected across an ac source. It produces an emf of

$$v(t) = V_0 \sin \omega t.$$

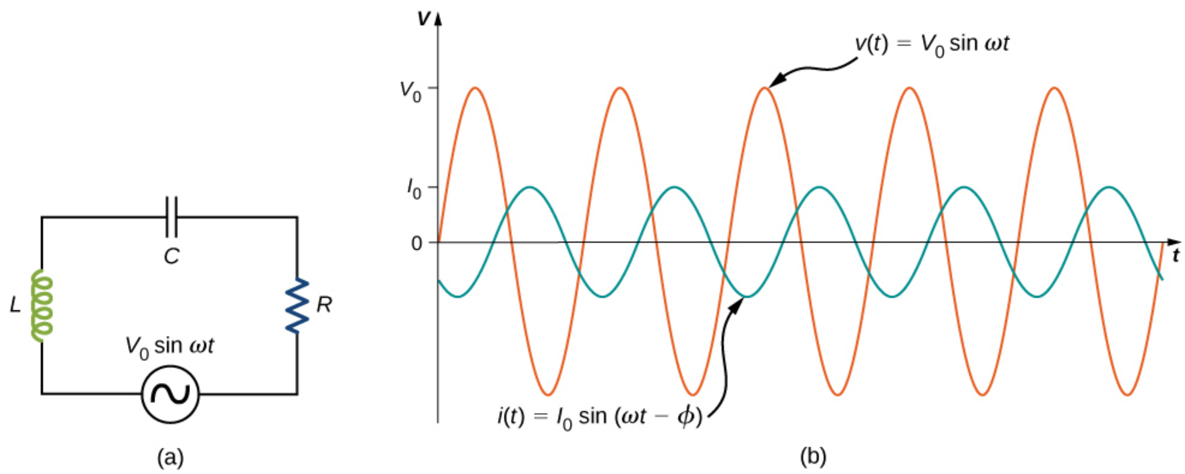


Figure 10.4.1: (a) An RLC series circuit. (b) A comparison of the generator output voltage and the current. The value of the phase difference ϕ depends on the values of R, C, and L.

Since the elements are in series, the same current flows through each element at all points in time. The relative phase between the current and the emf is not obvious when all three elements are present. Consequently, we represent the current by the general expression

$$i(t) = I_0 \sin(\omega t - \phi),$$

where I_0 is the current amplitude and ϕ is the phase angle between the current and the applied voltage. The phase angle is thus the amount by which the voltage and current are out of phase with each other in a circuit. Our task is to find I_0 and ϕ.

A phasor diagram involving $i(t)$, $v_R(t)$, $v_C(t)$, and $v_L(t)$ is helpful for analyzing the circuit. As shown in Figure 10.4.2, the phasor representing $v_R(t)$ points in the same direction as the phasor for $i(t)$; its amplitude is $V_R = I_0 R$. The $v_C(t)$ phasor lags the **i(t)** phasor by $\pi/2$ rad and has the amplitude $V_C = I_0 X_C$. The phasor for $v_L(t)$ leads the **i(t)** phasor by $\pi/2$ rad and has the amplitude $V_L = I_0 X_L$.

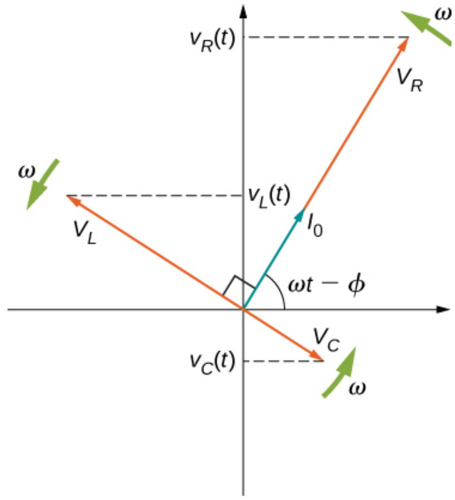


Figure 10.4.1.

At any instant, the voltage across the **RLC** combination is $v_R(t)+v_L(t)+v_C(t)=v(t)$, the emf of the source. Since a component of a sum of vectors is the sum of the components of the individual vectors—for example, $(A+B)_y = A_y + B_y$ - the projection of the vector sum of phasors onto the vertical axis is the sum of the vertical projections of the individual phasors. Hence, if we add vectorially the phasors representing $v_R(t)$, $v_L(t)$, and $v_C(t)$ and then find the projection of the resultant onto the vertical axis, we obtain

$$v_R(t)+v_L(t)+v_C(t)=v(t)=V_0 \, sin \, \omega t.$$

The vector sum of the phasors is shown in Figure $10.4.3$. The resultant phasor has an amplitudeV_0 and is directed at an angle ϕ with respect to the $v_R(t)$, or **i(t)**, phasor. The projection of this resultant phasor onto the vertical axis is $v(t)=V_0 \, sin \, \omega t$. We can easily determine the unknown quantities I_0 and ϕ from the geometry of the phasor diagram. For the phase angle,

$$\phi = tan^{-1}\frac{V_L - V_C}{V_R} = tan^{-1}\frac{I_0 X_L - I_0 X_C}{I_0 R},$$

and after cancellation of I_0, this becomes

$$\phi = tan^{-1}\frac{X_L - X_C}{R}. \tag{10.4.1}$$

Furthermore, from the Pythagorean theorem,

$$V_0 = \sqrt{V_R^2 + (V_L - V_C)^2} = \sqrt{(I_0 R)^2 + (I_0 X_L - I_) X_C)^2} = I_0 \sqrt{R^2 + (X_L - X_C)^2}.$$

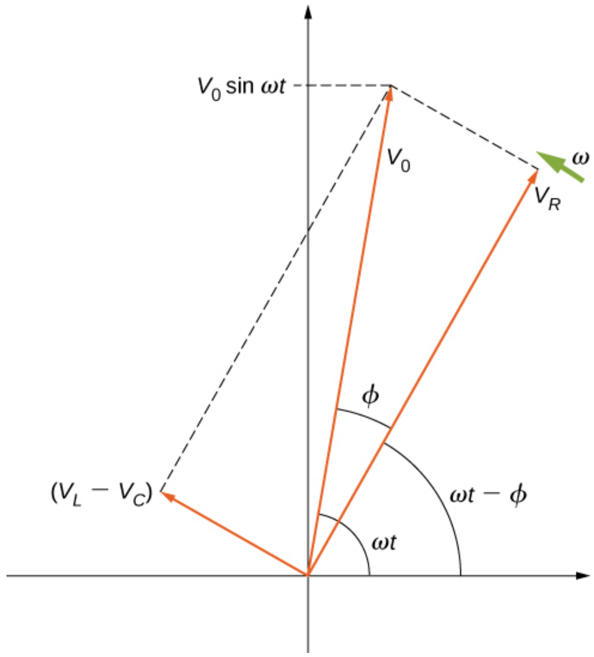


Figure 10.4.3: The resultant of the phasors for $v_L(t)$, $v_C(t)$, and $v_R(t)$ is equal to the phasor for $v_R(t) = V_0 \, sin \, \omega t$. The **i(t)** phasor (not shown) is aligned with the $v_R(t)$ phasor.

The current amplitude is therefore the ac version of Ohm's law:

$$I_0 = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{V_0}{Z}, \tag{10.4.2}$$

where

$$Z = \sqrt{R^2 + (X_L - X_C)^2} \tag{10.4.3}$$

is known as the impedance of the circuit. Its unit is the ohm, and it is the ac analog to resistance in a dc circuit, which measures the combined effect of resistance, capacitive reactance, and inductive reactance (Figure 10.4.4).

Figure 10.4.4: Power capacitors are used to balance the impedance of the effective inductance in transmission lines.

The **RLC** circuit is analogous to the wheel of a car driven over a corrugated road (Figure 10.4.5). The regularly spaced bumps in the road drive the wheel up and down; in the same way, a voltage source increases and decreases. The shock absorber acts like the resistance of the **RLC** circuit, damping and limiting the amplitude of the oscillation. Energy within the wheel system goes back and forth between kinetic and potential energy stored in the car spring, analogous to the shift between a maximum current, with energy stored in an inductor, and no current, with energy stored in the electric field of a capacitor. The amplitude of the wheel's motion is at a maximum if the bumps in the road are hit at the resonant frequency, which we describe in more detail in Resonance in an AC Circuit.

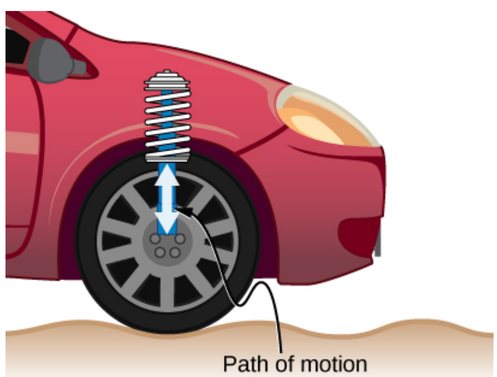


Figure 10.4.5: On a car, the shock absorber damps motion and dissipates energy. This is much like the resistance in an RLC circuit. The mass and spring determine the resonant frequency.

Problem-Solving Strategy: AC Circuits

To analyze an ac circuit containing resistors, capacitors, and inductors, it is helpful to think of each device's reactance and find the equivalent reactance using the rules we used for equivalent resistance in the past. Phasors are a great method to determine whether the emf of the circuit has positive or negative phase (namely, leads or lags other values). A mnemonic device of "ELI the ICE man" is sometimes used to remember that the emf (E) leads the current (I) in an inductor (L) and the current (I) leads the emf (E) in a capacitor (C).

Use the following steps to determine the emf of the circuit by phasors:

1. Draw the phasors for voltage across each device: resistor, capacitor, and inductor, including the phase angle in the circuit.
2. If there is both a capacitor and an inductor, find the net voltage from these two phasors, since they are antiparallel.
3. Find the equivalent phasor from the phasor in step 2 and the resistor's phasor using trigonometry or components of the phasors. The equivalent phasor found is the emf of the circuit.

Example 10.4.1: An RLC Series Circuit

The output of an ac generator connected to an **RLC** series combination has a frequency of 200 Hz and an amplitude of 0.100 V. If $R = 4.00\,\Omega$, $L = 3.00 \times 10^{-3} H$, and $C = 8.00 \times 10^{-4} F$, what are (a) the capacitive reactance, (b) the inductive reactance, (c) the impedance, (d) the current amplitude, and (e) the phase difference between the current and the emf of the generator?

Strategy

The reactances and impedance in (a)–(c) are found by substitutions into Equation 15.3.8, Equation 15.3.14, and Equation 10.4.2, respectively. The current amplitude is calculated from the peak voltage and the impedance. The phase difference between the current and the emf is calculated by the inverse tangent of the difference between the reactances divided by the resistance.

Solution

1. From Equation 15.3.8, the capacitive reactance is

$$X_C = \frac{1}{\omega C} = \frac{1}{2\pi(200\,Hz)(8.00 \times 10^{-4} F)} = 0.995\,\Omega.$$

2. From Equation 15.3.14, the inductive reactance is

$$X_L = \omega L = 2\pi(200\,Hz)(3.00 \times 10^{-3} H) = 3.77\,\Omega.$$

3. Substituting the values of **R**, X_C, and X_L into Equation 10.4.2, we obtain for the impedance

$$Z = \sqrt{(4.00\,)^2 + (3.77\,\Omega - 0.995\,\Omega)^2} = 4.87\,\Omega.$$

4. The current amplitude is

$$I_0 = \frac{V_0}{Z} = \frac{0.100\,V}{4.87\,\Omega} = 2.05 \times 10^{-2} A.$$

5. From Equation 10.4.1, the phase difference between the current and the emf is

$$\phi = tan^{-1}\frac{X_L - X_C}{R} = tan^{-1}\frac{2.77\,\Omega}{4.00\,\Omega} = 0.607\,rad.$$

Significance

The phase angle is positive because the reactance of the inductor is larger than the reactance of the capacitor.

Exercise 10.4.1

Find the voltages across the resistor, the capacitor, and the inductor in the circuit of Figure 10.4.1 using $v(t) = V_0\, sin\,\omega t$ as the output of the ac generator.

Solution

$v_R = (V_0 R/Z)\, sin\,(\omega t - \phi)$; $v_C = (V_0 X_C/Z)\, sin\,(\omega t - \phi + \pi 2/) = -(V_0 X_C/Z)\, cos\,(\omega t - \phi)$;
$v_L = (V_0 X_L/Z)\, sin\,(\omega t - \phi - \pi/2) = (V_0 X_L/Z)\, cos\,(\omega t - \phi)$

10.5: Power in an AC Circuit

Learning Objectives

By the end of the section, you will be able to:

- Describe how average power from an ac circuit can be written in terms of peak current and voltage and of rms current and voltage
- Determine the relationship between the phase angle of the current and voltage and the average power, known as the power factor

A circuit element dissipates or produces power according to $P = IV$, where **I** is the current through the element and V is the voltage across it. Since the current and the voltage both depend on time in an ac circuit, the instantaneous power $p(t) = i(t)v(t)$ is also time dependent. A plot of $p(t)$ for various circuit elements is shown in Figure $10.5.1$. For a resistor, $i(t)$ and $v(t)$ are in phase and therefore always have the same sign. For a capacitor or inductor, the relative signs of $i(t)$ and $v(t)$ vary over a cycle due to their phase differences. Consequently, $p(t)$ is positive at some times and negative at others, indicating that capacitive and inductive elements produce power at some instants and absorb it at others.

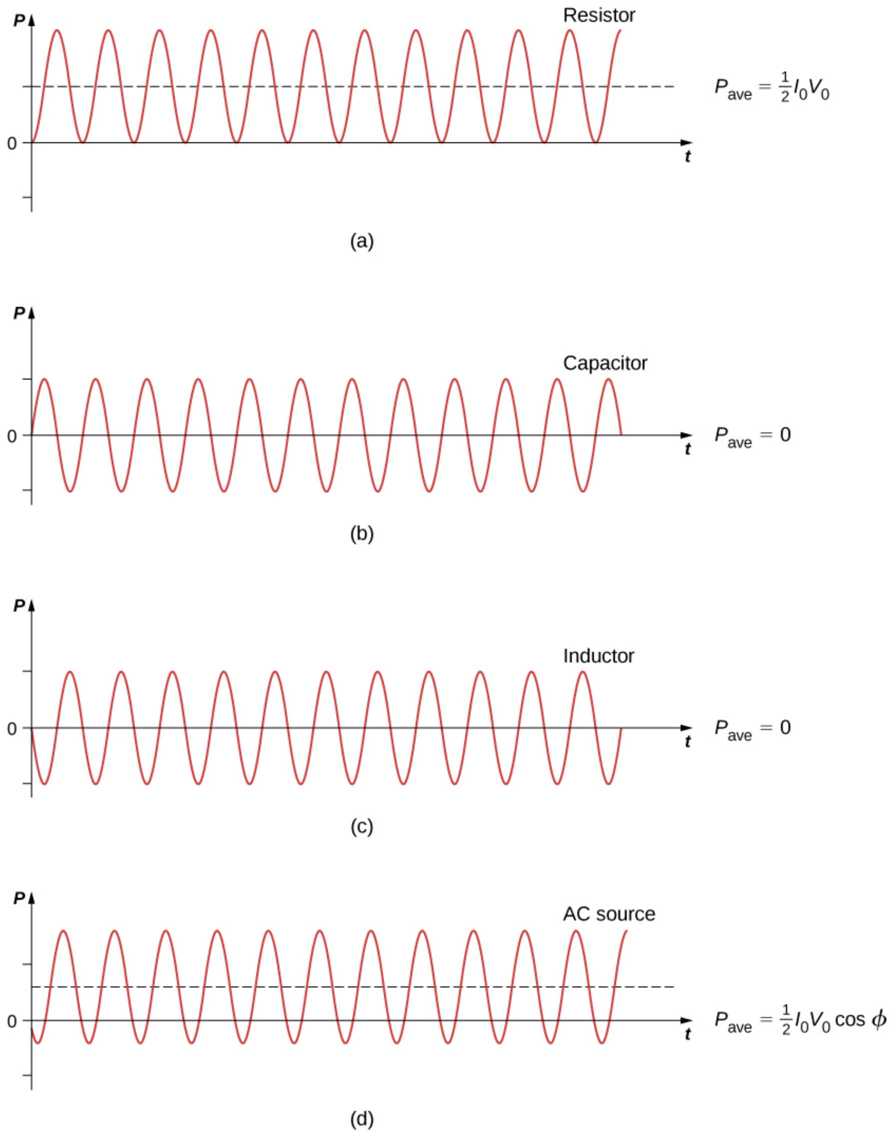


Figure 10.5.1: Graph of instantaneous power for various circuit elements. (a) For the resistor, $P_{ave} = I_0V_0/2$, whereas for (b) the capacitor and (c) the inductor, $P_{ave} = 0$. (d) For the source, $P_{ave} = I_0V_0(cos\ \phi)/2$, which may be positive, negative, or zero, depending on ϕ.

Because instantaneous power varies in both magnitude and sign over a cycle, it seldom has any practical importance. What we're almost always concerned with is the power averaged over time, which we refer to as the **average power**. It is defined by the time average of the instantaneous power over one cycle:

$$P_{ave} = \frac{1}{T}\int_0^T p(t)dt, \tag{10.5.1}$$

where $T = 2\pi/\omega$ is the period of the oscillations. With the substitutions $v(t) = V_0\ \sin\ \omega t$ and $i(t) = I_0\ \sin\left(\omega t - \phi\right)$, Equation 10.5.1 becomes

$$P_{ave} = \frac{I_0V_0}{T}\int_0^T \sin\left(\omega t - \phi\right)\ \sin\ \omega t\ dt.$$

Using the trigonometric difference identity

$$\sin\left(A - B\right) = \sin\ A\ \cos\ B - \sin\ B\ \cos\ A$$

we obtain

$$P_{ave} = \frac{I_0 V_0 \cos \phi}{T} \int_0^T \sin^2 \omega t \, dt - \frac{I_0 V_0 \sin \phi}{T} \int_0^T \sin \omega t \cos \omega t \, dt.$$

Evaluation of these two integrals yields

$$\frac{1}{T} \int_0^T \sin^2 \omega t \, dt = \frac{1}{2}$$

and

$$\frac{1}{T} \int_0^T \sin \omega t \cos \omega t \, dt = 0.$$

Hence, the average power associated with a circuit element is given by

$$\boxed{P_{\text{ave}} = \frac{1}{2} I_0 V_0 \cos \phi.} \tag{10.5.2}$$

In engineering applications, $\cos \phi$ is known as the **power factor**, which is the amount by which the power delivered in the circuit is less than the theoretical maximum of the circuit due to voltage and current being out of phase. For a resistor, $\phi = 0$, so the average power dissipated is

$$P_{ave} = \frac{1}{2} I_0 V_0. \tag{10.5.3}$$

A comparison of $p(t)$ and P_{ave} is shown in Figure $10.5.1d$. To make Equation 10.5.3 look like its dc counterpart, we use the rms values I_{rms} and V_{rms} of the current and the voltage. By definition, these are

$$I_{rms} = \sqrt{i^2_{ave}}$$

and

$$V_{rms} = \sqrt{v^2_{ave}},$$

where

$$t^2_{ave} = \frac{1}{T} \int_0^T i^2(t) dt$$

and

$$v^2_{ave} = \frac{1}{T} \int_0^T v^2(t) dt.$$

With $i(t) = I_0 \sin(\omega t - \phi)$ and $v(t) = V_0 \sin \omega t$, we obtain

$$I_{rms} = \frac{1}{\sqrt{2}} I_0$$

and

$$V_{rms} = \frac{1}{\sqrt{2}} V_0.$$

We may then write for the average power dissipated by a resistor,

$$P_{\text{ave}} = \frac{1}{2} I_0 V_0 = I_{\text{rms}} V_{\text{rms}} = I^2_{\text{rms}} R. \tag{10.5.4}$$

This equation further emphasizes why the rms value is chosen in discussion rather than peak values. Both Equations 10.5.2 and 10.5.4 are correct for average power, but the rms values in the formula give a cleaner representation, so the extra factor of 1/2 is not necessary.

Alternating voltages and currents are usually described in terms of their rms values. For example, the 110 V from a household outlet is an rms value. The amplitude of this source is $110\sqrt{2}\,V = 156\,V$. Because most ac meters are calibrated in terms of rms values, a typical ac voltmeter placed across a household outlet will read 110 V.

For a capacitor and an inductor, $\phi = \pi/2$ and $-\pi/2\,rad$, respectively. Since cos $\pi/2 = cos(-\pi/2) = 0$, we find from Equation 10.5.2 that the average power dissipated by either of these elements is $P_{ave} = 0$. Capacitors and inductors absorb energy from the circuit during one half-cycle and then discharge it back to the circuit during the other half-cycle. This behavior is illustrated in the plots of Figures $10.5.1b$and $10.5.1c$which show $p(t)$ oscillating sinusoidally about zero.

The phase angle for an ac generator may have any value. If $cos\,\phi > 0$, the generator produces power; if $cos\,\phi < 0$, it absorbs power. In terms of rms values, the average power of an ac generator is written as

$$P_{ave} = I_{rms}V_{rms}\ \cos\,\phi.$$

For the generator in an **RLC** circuit,

$$tan\,\phi = \frac{X_L - X_C}{R}$$

and

$$cos\,\phi = \frac{R}{\sqrt{R^2 + (X_L - X_C)^2}} = \frac{R}{Z}.$$

Hence the average power of the generator is

$$P_{ave} = I_{rms}V_{rms}\ \cos\,\phi = \frac{V_{rms}}{Z}V_{rms}\frac{R}{Z} = \frac{V_{rms}^2 R}{Z^2}. \tag{10.5.5}$$

This can also be written as

$$P_{ave} = I_{rms}^2 R,$$

which designates that the power produced by the generator is dissipated in the resistor. As we can see, Ohm's law for the rms ac is found by dividing the rms voltage by the impedance.

Example 10.5.1: Power Output of a Generator

An ac generator whose emf is given by

$$v(t) = (4.00\,V)\ \sin\left[(1.00 \times 10^4\,rad/s)t\right]$$

is connected to an **RLC** circuit for which $L = 2.00 \times 10^{-3}\,H$, $C = 4.00 \times 10^{-6}\,F$, and $R = 5.00\,\Omega$.

a. What is the rms voltage across the generator?
b. What is the impedance of the circuit?
c. What is the average power output of the generator?

Strategy

The rms voltage is the amplitude of the voltage times $1/\sqrt{2}$. The impedance of the circuit involves the resistance and the reactances of the capacitor and the inductor. The average power is calculated by Equation 10.5.5 because we have the impedance of the circuit Z, the rms voltage V_{rms}, and the resistance R.

Solution

1. Since $V_0 = 4.00\,V$, the rms voltage across the generator is

$$V_{rms} = \frac{1}{\sqrt{2}}(4.00\,V) = 2.83\,V.$$

2. The impedance of the circuit is

$$Z = \sqrt{r^2 + (x_l - x_c)^2}$$

$$= \sqrt{(5.00\,\Omega)^2 + \left[(1.00 \times 10^4\,rad/s)(2.00 \times 10^{-3}\,H) - \frac{1}{(1.00 \times 10^4\,rad/s)(4.00 \times 10^{-6}\,F)}\right]^2}$$

$$= 7.07\,\Omega.$$

3. From Equation 10.5.5, the average power transferred to the circuit is

$$P_{ave} = \frac{V_{rms}^2 R}{Z^2} = \frac{(2.83\,V)^2(5.00\,\Omega)}{(7.07\,\Omega)^2} = 0.801\,W.$$

Significance

If the resistance is much larger than the reactance of the capacitor or inductor, the average power is a dc circuit equation of $P = V^2/R$, where **V** replaces the rms voltage.

Exercise $10.5.1A$

An ac voltmeter attached across the terminals of a 45-Hz ac generator reads 7.07 V. Write an expression for the emf of the generator.

Answer

$v(t) = (10.0\,V)\ \sin 90\pi t$

Exercise $10.5.1B$

Show that the rms voltages across a resistor, a capacitor, and an inductor in an ac circuit where the rms current is I_{rms} are given by $I_{rms}R$, $I_{rms}X_C$, and $I_{rms}X_L$, respectively. Determine these values for the components of the **RLC** circuit of Equation 10.5.2.

Answer

2.00 V; 10.01 V; 8.01 V

10.6: Resonance in an AC Circuit

Learning Objectives

By the end of the section, you will be able to:

- Determine the peak ac resonant angular frequency for a RLC circuit
- Explain the width of the average power versus angular frequency curve and its significance using terms like bandwidth and quality factor

In the **RLC** series circuit of Figure 15.4.1, the current amplitude is, from Equation 15.4.7,

$$I_0 = \frac{V_0}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}. \tag{10.6.1}$$

If we can vary the frequency of the ac generator while keeping the amplitude of its output voltage constant, then the current changes accordingly. A plot of I_0 versus ω is shown in Figure 10.6.1.

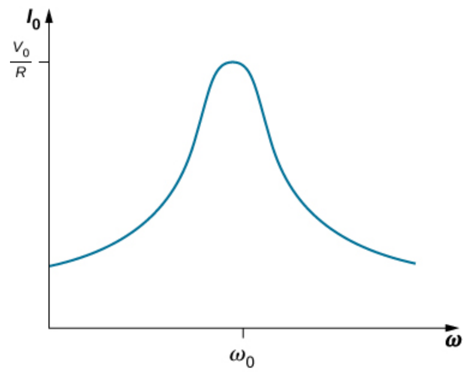


Figure 10.6.1: At an RLC circuit's resonant frequency, $\omega_0 = \sqrt{1/LC}$, the current amplitude is at its maximum value.

In Oscillations, we encountered a similar graph where the amplitude of a damped harmonic oscillator was plotted against the angular frequency of a sinusoidal driving force (see Forced Oscillations). This similarity is more than just a coincidence, as shown earlier by the application of Kirchhoff's loop rule to the circuit of Figure 15.4.1. This yields

$$L\frac{di}{dt} + iR + \frac{q}{C} = V_0 \sin \omega t, \tag{10.6.2}$$

or

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V_0 \sin \omega t,$$

where we substituted **dq**(t)/**dt** for **i**(t). A comparison of Equation 10.6.2 and, from Oscillations, Damped Oscillations for damped harmonic motion clearly demonstrates that the driven **RLC** series circuit is the electrical analog of the driven damped harmonic oscillator.

The **resonant frequency** f_0 of the **RLC** circuit is the frequency at which the amplitude of the current is a maximum and the circuit would oscillate if not driven by a voltage source. By inspection, this corresponds to the angular frequency $\omega_0 = 2\pi f_0$ at which the impedance **Z** in Equation 10.6.1 is a minimum, or when

$$\omega_0 L = \frac{1}{\omega_0 C} \tag{10.6.3}$$

and

$$\omega_0 = \sqrt{\frac{1}{LC}}. \tag{10.6.4}$$

This is the resonant angular frequency of the circuit. Substituting ω_0 into Equation 15.4.5, Equation 15.4.7, and Equation 15.4.8, we find that at resonance,

$$\phi = tan^{-1}(0) = 0,\ I_0 = V_0/R,\ and\ Z = R.$$

Therefore, at resonance, an **RLC** circuit is purely resistive, with the applied emf and current in phase.

What happens to the power at resonance? Equation 15.5.18 tells us how the average power transferred from an ac generator to the **RLC** combination varies with frequency. In addition, P_{ave} reaches a maximum when Z, which depends on the frequency, is a minimum, that is, when $X_L = X_C$ and $Z = R$. Thus, at resonance, the average power output of the source in an **RLC** series circuit is a maximum. From Equation 15.5.18, this maximum is V_{rms}^2/R.

Figure $10.6.2$ is a typical plot of P_{ave} versus ω in the region of maximum power output. The **bandwidth** $\Delta\omega$ of the resonance peak is defined as the range of angular frequencies ω over which the average power P_{ave} is greater than one-half the maximum value of P_{ave}. The sharpness of the peak is described by a dimensionless quantity known as the **quality factor Q** of the circuit. By definition,

$$Q = \frac{\omega_0}{\Delta\omega}, \tag{10.6.5}$$

where ω_0 is the resonant angular frequency. A high **Q** indicates a sharp resonance peak. We can give **Q** in terms of the circuit parameters as

$$Q = \frac{\omega_0 L}{R}. \tag{10.6.6}$$

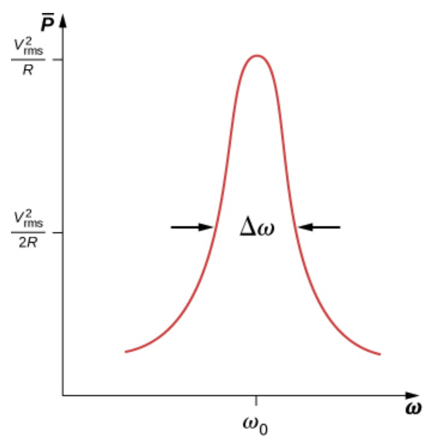


Figure 10.6.2: Like the current, the average power transferred from an ac generator to an **RLC** circuit peaks at the resonant frequency.

Resonant circuits are commonly used to pass or reject selected frequency ranges. This is done by adjusting the value of one of the elements and hence "tuning" the circuit to a particular resonant frequency. For example, in radios, the receiver is tuned to the desired station by adjusting the resonant frequency of its circuitry to match the frequency of the station. If the tuning circuit has a high **Q**, it will have a small bandwidth, so signals from other stations at frequencies even slightly different from the resonant frequency encounter a high impedance and are not passed by the circuit. Cell phones work in a similar fashion, communicating with signals of around 1 GHz that are tuned by an inductor-capacitor circuit. One of the most common applications of capacitors is their use in ac-timing circuits, based on attaining a resonant frequency. A metal detector also uses a shift in resonance frequency in detecting metals (Figure $10.6.3$).

Figure 10.6.3: When a metal detector comes near a piece of metal, the self-inductance of one of its coils changes. This causes a shift in the resonant frequency of a circuit containing the coil. That shift is detected by the circuitry and transmitted to the diver by means of the headphones.

Example 10.6.1: Resonance in an RLC Series Circuit

a. What is the resonant frequency of the circuit of Example 15.3.1?
b. If the ac generator is set to this frequency without changing the amplitude of the output voltage, what is the amplitude of the current?

Strategy

The resonant frequency for a **RLC** circuit is calculated from Equation 10.6.4, which comes from a balance between the reactances of the capacitor and the inductor. Since the circuit is at resonance, the impedance is equal to the resistor. Then, the peak current is calculated by the voltage divided by the resistance.

Solution

1. The resonant frequency is found from Equation 10.6.4:

$$\begin{aligned} f_0 &= \frac{1}{2\pi}\sqrt{\frac{1}{LC}} \\ &= \frac{1}{2\pi}\sqrt{\frac{1}{(3.00 \times 10^{-3}\,H)(8.00 \times 10^{-4}\,F)}} \\ &= 1.03 \times 10^2\ Hz. \end{aligned}$$

2. At resonance, the impedance of the circuit is purely resistive, and the current amplitude is

$$I_0 = \frac{0.100\,V}{4.00\,\Omega} = 2.50 \times 10^{-2}\,A.$$

Significance

If the circuit were not set to the resonant frequency, we would need the impedance of the entire circuit to calculate the current.

Example 10.6.2: Power Transfer in an RLC Series Circuit at Resonance

a. What is the resonant angular frequency of an **RLC** circuit with $R = 0.200\,\Omega$, $L = 4.00 \times 10^{-3}\,H$, and $C = 2.00 \times 10^{-6}\,F$?
b. If an ac source of constant amplitude 4.00 V is set to this frequency, what is the average power transferred to the circuit?
c. Determine **Q** and the bandwidth of this circuit.

Strategy

The resonant angular frequency is calculated from Equation 10.6.4. The average power is calculated from the rms voltage and the resistance in the circuit. The quality factor is calculated from Equation 10.6.6 and by knowing the resonant frequency. The bandwidth is calculated from Equation 10.6.5 and by knowing the quality factor.

Solution

1. The resonant angular frequency is

$$\begin{aligned}\omega_0 &= \sqrt{\frac{1}{LC}}\\ &= \sqrt{\frac{1}{(4.00\times 10^{-3}\,H)(2.00\times 10^{-6}\,F)}}\\ &= 1.12\times 10^4\,rad/s.\end{aligned}$$

2. At this frequency, the average power transferred to the circuit is a maximum. It is

$$P_{ave} = \frac{V_{rms}^2}{R} = \frac{[(1/\sqrt{2})(4.00\,V)]^2}{0.200\,\Omega} = 40.0\,W.$$

3. The quality factor of the circuit is

$$Q = \frac{\omega_0 L}{R} = \frac{(1.12\times 10^4\,rad/s)(4.00\times 10^{-3}\,H)}{0.200\,\Omega} = 224.$$

We then find for the bandwidth

$$\Delta\omega = \frac{\omega_0}{Q} = \frac{1.12\times 10^4\,rad/s}{224} = 50.0\,rad/s.$$

Significance

If a narrower bandwidth is desired, a lower resistance or higher inductance would help. However, a lower resistance increases the power transferred to the circuit, which may not be desirable, depending on the maximum power that could possibly be transferred.

? Exercise 10.6.1

In the circuit of Figure 15.4.1, $L = 2.0\times 10^{-3}\,H$, $C = 5.0\times 10^{-4}\,F$, and $R = 40\,\Omega$.

a. What is the resonant frequency?
b. What is the impedance of the circuit at resonance?
c. If the voltage amplitude is 10 V, what is **i(t)** at resonance?
d. The frequency of the AC generator is now changed to 200 Hz. Calculate the phase difference between the current and the emf of the generator.

Answer

a. 160 Hz; b. 40Ω; c. $(0.25A)$ sin $10^3 t$; d. 0.023 rad

? Exercise 10.6.2

What happens to the resonant frequency of an **RLC** series circuit when the following quantities are increased by a factor of 4: (a) the capacitance, (b) the self-inductance, and (c) the resistance?

Answer

a. halved; b. halved; c. same

Exercise 10.6.3

The resonant angular frequency of an **RLC** series circuit is $4.0 \times 10^2\, rad/s$. An ac source operating at this frequency transfers an average power of $2.0 \times 10^{-2} W$ to the circuit. The resistance of the circuit is $0.50\, \Omega$. Write an expression for the emf of the source.

Answer

$$v(t) = (0.14\, V)\, sin\, (4.0 \times 10^2 t)$$

10.7: Transformers

Learning Objectives

By the end of the section, you will be able to:

- Explain why power plants transmit electricity at high voltages and low currents and how they do this
- Develop relationships among current, voltage, and the number of windings in step-up and step-down transformers

Although ac electric power is produced at relatively low voltages, it is sent through transmission lines at very high voltages (as high as 500 kV). The same power can be transmitted at different voltages because power is the product $I_{rms}V_{rms}$. (For simplicity, we ignore the phase factor cos ϕ.) A particular power requirement can therefore be met with a low voltage and a high current or with a high voltage and a low current. The advantage of the high-voltage/low-current choice is that it results in lower $I^2_{rms}R$ ohmic losses in the transmission lines, which can be significant in lines that are many kilometers long (Figure $10.7.1$).

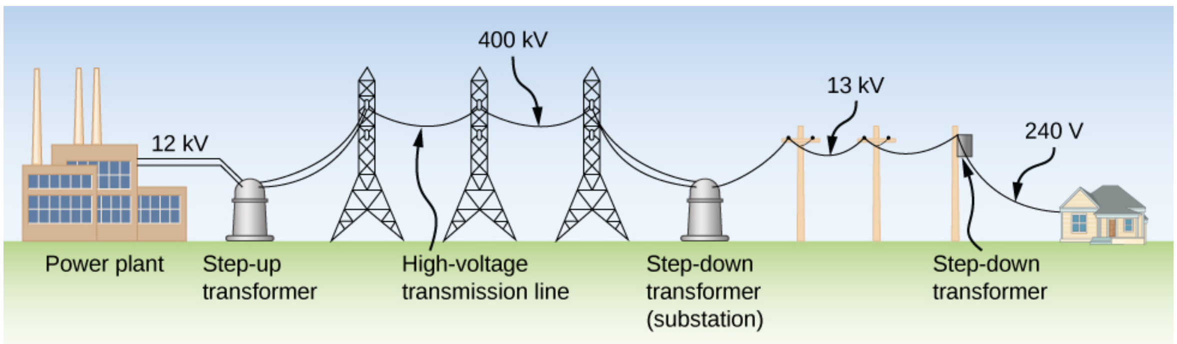


Figure 10.7.1: The rms voltage from a power plant eventually needs to be stepped down from 12 kV to 240 V so that it can be safely introduced into a home. A high-voltage transmission line allows a low current to be transmitted via a substation over long distances.

Typically, the alternating emfs produced at power plants are “stepped up” to very high voltages before being transmitted through power lines; then, they must be “stepped down” to relatively safe values (110 or 220 V rms) before they are introduced into homes. The device that transforms voltages from one value to another using induction is the **transformer** (Figure $10.7.2$).

Figure 10.7.2: Transformers are used to step down the high voltages in transmission lines to the 110 to 220 V used in homes. (credit: modification of work by "Fortyseven"/Flickr)

As Figure 10.7.3 illustrates, a transformer basically consists of two separated coils, or windings, wrapped around a soft iron core. The primary winding has N_p loops, or turns, and is connected to an alternating voltage $v_p(t)$. The secondary winding has N_s turns and is connected to a load resistor R_s. We assume the ideal case for which all magnetic field lines are confined to the core so that the same magnetic flux permeates each turn of both the primary and the secondary windings. We also neglect energy losses to magnetic hysteresis, to ohmic heating in the windings, and to ohmic heating of the induced eddy currents in the core. A good transformer can have losses as low as 1% of the transmitted power, so this is not a bad assumption.

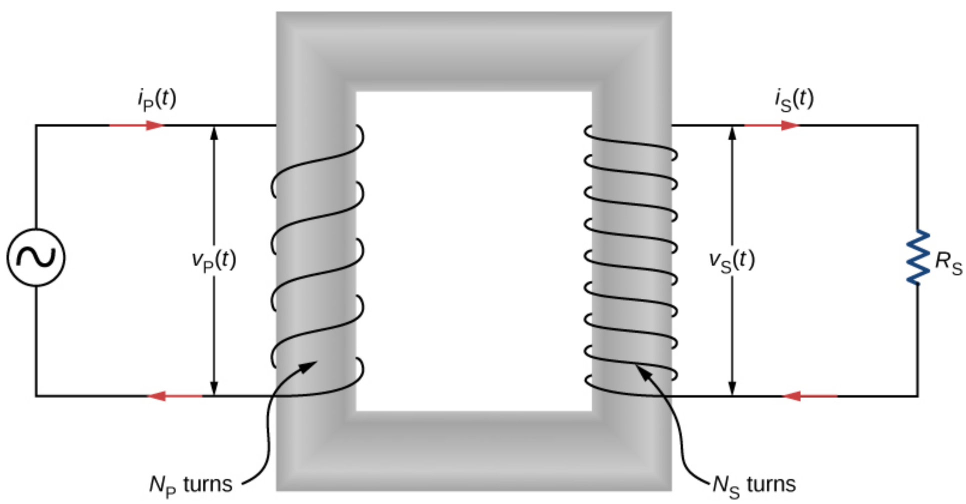


Figure 10.7.3: A step-up transformer (more turns in the secondary winding than in the primary winding). The two windings are wrapped around a soft iron core.

To analyze the transformer circuit, we first consider the primary winding. The input voltage $v_p(t)$ is equal to the potential difference induced across the primary winding. From Faraday's law, the induced potential difference is $-N_p(d\Phi/dt)$, where Φ is the flux through one turn of the primary winding. Thus,

$$v_p(t) = -N_p \frac{d\Phi}{dt}.$$

Similarly, the output voltage $v_s(t)$ delivered to the load resistor must equal the potential difference induced across the secondary winding. Since the transformer is ideal, the flux through every turn of the secondary winding is also Φ and

$$v_s(t) = -N_s \frac{d\Phi}{dt}.$$

Combining the last two equations, we have

$$v_s(t) = \frac{N_s}{N_p} v_p(t). \tag{10.7.1}$$

Hence, with appropriate values for N_s and N_p, the input voltage $v_p(t)$ may be "stepped up" $(N_s > N_p)$ or "stepped down" $(N_s < N_p)$ to $v_s(t)$, the output voltage. This is often abbreviated as the **transformer equation**,

$$\frac{V_s}{V_p} = \frac{N_s}{N_p}, \tag{10.7.2}$$

which shows that the ratio of the secondary to primary voltages in a transformer equals the ratio of the number of turns in their windings. For a **step-up transformer**, which increases voltage and decreases current, this ratio is greater than one; for a **step-down transformer**, which decreases voltage and increases current, this ratio is less than one.

From the law of energy conservation, the power introduced at any instant by $v_p(t)$ to the primary winding must be equal to the power dissipated in the resistor of the secondary circuit; thus,

$$i_p(t) v_p(t) = i_s(t) v_s(t).$$

When combined with Equation 10.7.2, this gives

$$i_s(t) = \frac{N_p}{N_s} i_p(t). \tag{10.7.3}$$

If the voltage is stepped up, the current is stepped down, and vice versa.

Finally, we can use $i_s(t) = v_s(t)/R_s$, along with Equation 10.7.1 and Equation 10.7.3, to obtain

$$v_p(t) = i_p \left[\left(\frac{N_p}{N_s} \right)^2 R_s \right],$$

which tells us that the input voltage $v_p(t)$ "sees" not a resistance R_s but rather a resistance

$$R_p = \left(\frac{N_p}{N_s} \right)^2 R_s.$$

Our analysis has been based on instantaneous values of voltage and current. However, the resulting equations are not limited to instantaneous values; they hold also for maximum and rms values.

Example 10.7.1: A Step-Down Transformer

A transformer on a utility pole steps the rms voltage down from 12 kV to 240 V.

a. What is the ratio of the number of secondary turns to the number of primary turns?
b. If the input current to the transformer is 2.0 A, what is the output current?
c. Determine the power loss in the transmission line if the total resistance of the transmission line is 200 Ω.
d. What would the power loss have been if the transmission line was at 240 V the entire length of the line, rather than providing voltage at 12 kV? What does this say about transmission lines?

Strategy

The number of turns related to the voltages is found from Equation 10.7.1. The output current is calculated using Equation 10.7.3.

Solution

a. Using Equation 10.7.1 with rms values V_p and V_s we have

$$\frac{N_s}{N_p} = \frac{240\,V}{12 \times 10^3 V} = \frac{1}{50},$$

so the primary winding has 50 times the number of turns in the secondary winding.

b. From Equation 10.7.3, the output rms current I_s is found using the transformer equation with current

$$I_S = \frac{N_p}{N_S} I_p \tag{10.7.4}$$

such that

$$I_s = \frac{N_p}{N_s} I_p = (50)(2.0\,A) = 100\,A.$$

c. The power loss in the transmission line is calculated to be

$$P_{loss} = I_p^2 R = (2.0\,A)^2 (200\,\Omega) = 800\,W.$$

d. If there were no transformer, the power would have to be sent at 240 V to work for these houses, and the power loss would be

$$P_{loss} = I_s^2 R = (100\,A)^2 (200\,\Omega) = 2 \times 10^6\,W.$$

Therefore, when power needs to be transmitted, we want to avoid power loss. Thus, lines are sent with high voltages and low currents and adjusted with a transformer before power is sent into homes.

Significance

This application of a step-down transformer allows a home that uses 240-V outlets to have 100 A available to draw upon. This can power many devices in the home.

Exercise 10.7.1

A transformer steps the line voltage down from 110 to 9.0 V so that a current of 0.50 A can be delivered to a doorbell.

a. What is the ratio of the number of turns in the primary and secondary windings?
b. What is the current in the primary winding?
c. What is the resistance seen by the 110-V source?

Answer a

12:1

Answer b

0.042 A

Answer c

$2.6 \times 10^3\ \Omega$

10.8: Alternating-Current Circuits (Exercise)

Conceptual Questions

15.2 AC Sources

1. What is the relationship between frequency and angular frequency?

15.3 Simple AC Circuits

2. Explain why at high frequencies a capacitor acts as an ac short, whereas an inductor acts as an open circuit.

15.4 RLC Series Circuits with AC

3. In an **RLC** series circuit, can the voltage measured across the capacitor be greater than the voltage of the source? Answer the same question for the voltage across the inductor.

15.5 Power in an AC Circuit

4. For what value of the phase angle ϕ between the voltage output of an ac source and the current is the average power output of the source a maximum?

5. Discuss the differences between average power and instantaneous power.

6. The average ac current delivered to a circuit is zero. Despite this, power is dissipated in the circuit. Explain.

7. Can the instantaneous power output of an ac source ever be negative? Can the average power output be negative?

8. The power rating of a resistor used in ac circuits refers to the maximum average power dissipated in the resistor. How does this compare with the maximum instantaneous power dissipated in the resistor?

15.7 Transformers

9. Why do transmission lines operate at very high voltages while household circuits operate at fairly small voltages?

10. How can you distinguish the primary winding from the secondary winding in a step-up transformer?

11. Battery packs in some electronic devices are charged using an adapter connected to a wall socket. Speculate as to the purpose of the adapter.

12. Will a transformer work if the input is a dc voltage?

13. Why are the primary and secondary coils of a transformer wrapped around the same closed loop of iron?

Problems

15.2 AC Sources

14. Write an expression for the output voltage of an ac source that has an amplitude of 12 V and a frequency of 200 Hz.

15.3 Simple AC Circuits

15. Calculate the reactance of a **5.0-μF** capacitor at

(a) 60 Hz,

(b) 600 Hz, and

(c) 6000 Hz.

16. What is the capacitance of a capacitor whose reactance is **10Ω** at 60 Hz?

17. Calculate the reactance of a 5.0-mH inductor at

(a) 60 Hz,

(b) 600 Hz, and

(c) 6000 Hz.

18. What is the self-inductance of a coil whose reactance is **10Ω** at 60 Hz?

19. At what frequency is the reactance of a **20-μF** capacitor equal to that of a 10-mH inductor?

20. At 1000 Hz, the reactance of a 5.0-mH inductor is equal to the reactance of a particular capacitor. What is the capacitance of the capacitor?

21. A **50-Ω** resistor is connected across the **emf** $v(t) = (160V)sin(120\pi t)$. Write an expression for the current through the resistor.

22. A **25-μF** capacitor is connected to an emf given by $v(t) = (160V)sin(120\pi t)$.

(a) What is the reactance of the capacitor?

(b) Write an expression for the current output of the source.

23. A 100-mH inductor is connected across the emf of the preceding problem.

(a) What is the reactance of the inductor?

(b) Write an expression for the current through the inductor.

15.4 RLC Series Circuits with AC

24. What is the impedance of a series combination of a **50-Ω** resistor, a 5.0-μF5.0-μF capacitor, and a 10-μF10-μF capacitor at a frequency of 2.0 kHz?

25. A resistor and capacitor are connected in series across an ac generator. The emf of the generator is given by $v(t) = V0cos\omega t,$ where $V0 = 120V, \omega = 120\pi rad/s, R = 400\Omega$, and $C = 4.0\mu F$.

(a) What is the impedance of the circuit?

(b) What is the amplitude of the current through the resistor?

(c) Write an expression for the current through the resistor.

(d) Write expressions representing the voltages across the resistor and across the capacitor.

26. A resistor and inductor are connected in series across an ac generator. The emf of the generator is given by $v(t) = V0cos\omega t,$ where $V0 = 120V$ and $\omega = 120\pi rad/s$; also, $R = 400\Omega$ and $L = 1.5H$.

(a) What is the impedance of the circuit?

(b) What is the amplitude of the current through the resistor?

(c) Write an expression for the current through the resistor.

(d) Write expressions representing the voltages across the resistor and across the inductor.

27. In an **RLC** series circuit, the voltage amplitude and frequency of the source are 100 V and 500 Hz, respectively, an $R = 500\Omega, L = 0.20H,$ and $C = 2.0\mu F$.

(a) What is the impedance of the circuit?

(b) What is the amplitude of the current from the source?

(c) If the emf of the source is given by $v(t) = (100V)sin 1000\pi t$, how does the current vary with time?

(d) Repeat the calculations with i changed to **0.20μF**.

28. An **RLC** series circuit with $R = 600\Omega, L = 30mH$, and $C = 0.050\mu F$ is driven by an ac source whose frequency and voltage amplitude are 500 Hz and 50 V, respectively.

(a) What is the impedance of the circuit?

(b) What is the amplitude of the current in the circuit?

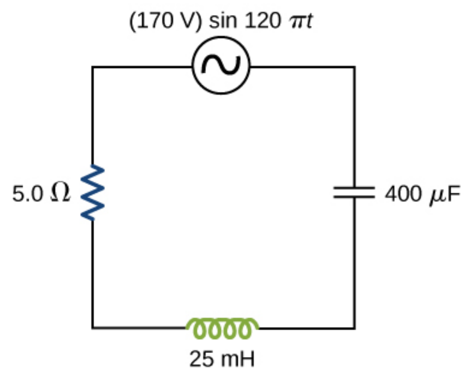
(c) What is the phase angle between the emf of the source and the current?

29. For the circuit shown below, what are

(a) the total impedance and

(b) the phase angle between the current and the emf?

(c) Write an expression for $i(t)$.



15.5 Power in an AC Circuit

30. The emf of an ac source is given by $v(t)=V0 sin\omega t,$ where $V0=100V$ and $\omega=200\pi rad/s.$. Calculate the average power output of the source if it is connected across

(a) a **20-μF** capacitor,

(b) a 20-mH inductor, and

(c) a **50-Ω** resistor.

31. Calculate the rms currents for an ac source is given by $v(t)=V_0 \sin \omega t,$ where $V_0=100V$ and $\omega=200\pi\ rad/s$ when connected across

(a) a **20-μF** capacitor,

(b) a 20-mH inductor, and

(c) a **50-Ω** resistor.

32. A 40-mH inductor is connected to a 60-Hz AC source whose voltage amplitude is 50 V. If an AC voltmeter is placed across the inductor, what does it read?

33. For an **RLC** series circuit, the voltage amplitude and frequency of the source are 100 V and 500 Hz, respectively; **R=500Ω**; and **L=0.20H**. Find the average power dissipated in the resistor for the following values for the capacitance:

(a) **C=2.0μF** and

(b) **C=0.20μF.**

34. An ac source of voltage amplitude 10 V delivers electric energy at a rate of 0.80 W when its current output is 2.5 A. What is the phase angle **ϕ** between the emf and the current?

35. An **RLC** series circuit has an impedance of **60Ω** and a power factor of 0.50, with the voltage lagging the current. (a) Should a capacitor or an inductor be placed in series with the elements to raise the power factor of the circuit? (b) What is the value of the reactance across the inductor that will raise the power factor to unity?

15.6 Resonance in an AC Circuit

36. (a) Calculate the resonant angular frequency of an **RLC** series circuit for which $R=20\Omega, L=75mH,$ and **C=4.0μF**. (b) If **R** is changed to **300Ω**, what happens to the resonant angular frequency?

37. The resonant frequency of an **RLC** series circuit is $2.0\times10^3 Hz$. If the self-inductance in the circuit is 5.0 mH, what is the capacitance in the circuit?

38. (a) What is the resonant frequency of an **RLC** series circuit with **R=20Ω**, **L=2.0mH**, and **C=4.0μF**?

(b) What is the impedance of the circuit at resonance?

39. For an **RLC** series circuit, **R=100Ω**, **L=150mH**, and **C=0.25μF**.

(a) If an ac source of variable frequency is connected to the circuit, at what frequency is maximum power dissipated in the resistor?

(b) What is the quality factor of the circuit?

40. An ac source of voltage amplitude 100 V and variable frequency **f** drives an **RLC** series circuit with **R=10Ω**, **L=2.0mH**, and **C=25μF.**

(a) Plot the current through the resistor as a function of the frequency f.

(b) Use the plot to determine the resonant frequency of the circuit.

41. (a) What is the resonant frequency of a resistor, capacitor, and inductor connected in series if **R=100Ω**, **L=2.0H,** and **C=5.0μF**?

(b) If this combination is connected to a 100-V source operating at the constant frequency, what is the power output of the source?

(c) What is the **Q** of the circuit?

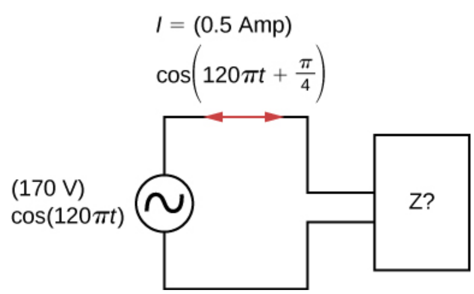
(d) What is the bandwidth of the circuit?

42. Suppose a coil has a self-inductance of 20.0 H and a resistance of **200Ω**. What

(a) capacitance and

(b) resistance must be connected in series with the coil to produce a circuit that has a resonant frequency of 100 Hz and a **Q** of 10?

43. An ac generator is connected to a device whose internal circuits are not known. We only know current and voltage outside the device, as shown below. Based on the information given, what can you infer about the electrical nature of the device and its power usage?



15.7 Transformers

44. A step-up transformer is designed so that the output of its secondary winding is 2000 V (rms) when the primary winding is connected to a 110-V (rms) line voltage.

(a) If there are 100 turns in the primary winding, how many turns are there in the secondary winding?

(b) If a resistor connected across the secondary winding draws an rms current of 0.75 A, what is the current in the primary winding?

45. A step-up transformer connected to a 110-V line is used to supply a hydrogen-gas discharge tube with 5.0 kV (rms). The tube dissipates 75 W of power.

(a) What is the ratio of the number of turns in the secondary winding to the number of turns in the primary winding?

(b) What are the rms currents in the primary and secondary windings?

(c) What is the effective resistance seen by the 110-V source?

46. An ac source of emf delivers 5.0 mW of power at an rms current of 2.0 mA when it is connected to the primary coil of a transformer. The rms voltage across the secondary coil is 20 V.

(a) What are the voltage across the primary coil and the current through the secondary coil?

(b) What is the ratio of secondary to primary turns for the transformer?

47. A transformer is used to step down 110 V from a wall socket to 9.0 V for a radio.

(a) If the primary winding has 500 turns, how many turns does the secondary winding have?

(b) If the radio operates at a current of 500 mA, what is the current through the primary winding?

48. A transformer is used to supply a 12-V model train with power from a 110-V wall plug. The train operates at 50 W of power.

(a) What is the rms current in the secondary coil of the transformer?

(b) What is the rms current in the primary coil?

(c) What is the ratio of the number of primary to secondary turns?

(d) What is the resistance of the train?

(e) What is the resistance seen by the 110-V source?

Additional Problems

49. The emf of an dc source is given by $v(t) = V_0 sin\omega t,$ where $V_0 = 100V$ and $\omega = 200\pi rad/s$. Find an expression that represents the output current of the source if it is connected across

(a) a **20-μF** capacitor,

(b) a 20-mH inductor, and

(c) a **50-Ω** resistor.

50. A **700-pF** capacitor is connected across an ac source with a voltage amplitude of 160 V and a frequency of 20 kHz.

(a) Determine the capacitive reactance of the capacitor and the amplitude of the output current of the source.

(b) If the frequency is changed to 60 Hz while keeping the voltage amplitude at 160 V, what are the capacitive reactance and the current amplitude?

51. A 20-mH inductor is connected across an AC source with a variable frequency and a constant-voltage amplitude of 9.0 V.

(a) Determine the reactance of the circuit and the maximum current through the inductor when the frequency is set at 20 kHz.

(b) Do the same calculations for a frequency of 60 Hz.

52. A **30-μF** capacitor is connected across a 60-Hz ac source whose voltage amplitude is 50 V.

(a) What is the maximum charge on the capacitor?

(b) What is the maximum current into the capacitor?

(c) What is the phase relationship between the capacitor charge and the current in the circuit?

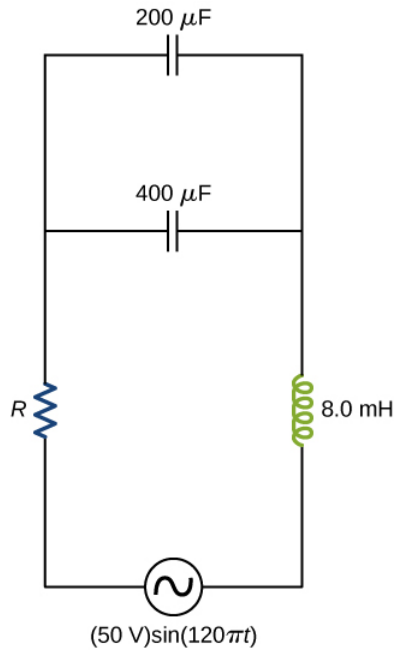
53. A 7.0-mH inductor is connected across a 60-Hz ac source whose voltage amplitude is 50 V.

(a) What is the maximum current through the inductor?

(b) What is the phase relationship between the current through and the potential difference across the inductor?

54. What is the impedance of an **RLC** series circuit at the resonant frequency?

55. What is the resistance **R** in the circuit shown below if the amplitude of the ac through the inductor is 4.24 A?



56. An ac source of voltage amplitude 100 V and frequency 1.0 kHz drives an **RLC** series circuit with **R=20Ω**, **L=4.0mH**, and **C=50μF**.

(a) Determine the rms current through the circuit.

(b) What are the rms voltages across the three elements?

(c) What is the phase angle between the emf and the current?

(d) What is the power output of the source?

(e) What is the power dissipated in the resistor?

57. In an RLC series circuit, $R = 200\Omega, L = 1.0H, C = 50\mu F, V_0 = 120V$, and $f = 50Hz$. What is the power output of the source?

58. A power plant generator produces 100 A at 15 kV (rms). A transformer is used to step up the transmission line voltage to 150 kV (rms).

(a) What is rms current in the transmission line?

(b) If the resistance per unit length of the line is $8.6 \times 10^{-8}\Omega/m$, what is the power loss per meter in the line?

(c) What would the power loss per meter be if the line voltage were 15 kV (rms)?

59. Consider a power plant located 25 km outside a town delivering 50 MW of power to the town. The transmission lines are made of aluminum cables with a $7cm^2$ cross-sectional area. Find the loss of power in the transmission lines if it is transmitted at

(a) 200 kV (rms) and

(b) 120 V (rms).

60. Neon signs require 12-kV for their operation. A transformer is to be used to change the voltage from 220-V (rms) ac to 12-kV (rms) ac.

(a) What must the ratio be of turns in the secondary winding to the turns in the primary winding?

(b) What is the maximum rms current the neon lamps can draw if the fuse in the primary winding goes off at 0.5 A?

(c) How much power is used by the neon sign when it is drawing the maximum current allowed by the fuse in the primary winding?

Challenge Problems

61. The 335-kV ac electricity from a power transmission line is fed into the primary winding of a transformer. The ratio of the number of turns in the secondary winding to the number in the primary winding is $N_s/N_p = 1000$.

(a) What voltage is induced in the secondary winding?

(b) What is unreasonable about this result?

(c) Which assumption or premise is responsible?

62. A **1.5-kΩ** resistor and 30-mH inductor are connected in series, as shown below, across a 120-V (rms) ac power source oscillating at 60-Hz frequency.

(a) Find the current in the circuit.

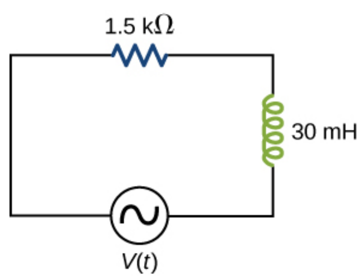
(b) Find the voltage drops across the resistor and inductor.

(c) Find the impedance of the circuit.

(d) Find the power dissipated in the resistor.

(e) Find the power dissipated in the inductor.

(f) Find the power produced by the source.



63. A **20-Ω** resistor, **50-μF** capacitor, and 30-mH inductor are connected in series with an ac source of amplitude 10 V and frequency 125 Hz.

(a) What is the impedance of the circuit?

(b) What is the amplitude of the current in the circuit?

(c) What is the phase constant of the current? Is it leading or lagging the source voltage?

(d) Write voltage drops across the resistor, capacitor, and inductor and the source voltage as a function of time.

(e) What is the power factor of the circuit? (f) How much energy is used by the resistor in 2.5 s?

64. A **200-Ω** resistor, **150-μF** capacitor, and 2.5-H inductor are connected in series with an ac source of amplitude 10 V and variable angular frequency ω.

(a) What is the value of the resonance frequency ω_R?

(b) What is the amplitude of the current if $\omega = \omega_R$?

(c) What is the phase constant of the current when $\omega = \omega_R$? Is it leading or lagging the source voltage, or is it in phase?

(d) Write an equation for the voltage drop across the resistor as a function of time when $\omega = \omega_R$.

(e) What is the power factor of the circuit when $\omega = \omega_R$?

(f) How much energy is used up by the resistor in 2.5 s when $\omega = \omega_R$?

65. Find the reactances of the following capacitors and inductors in ac circuits with the given frequencies in each case:

(a) 2-mH inductor with a frequency 60-Hz of the ac circuit;

(b) 2-mH inductor with a frequency 600-Hz of the ac circuit;

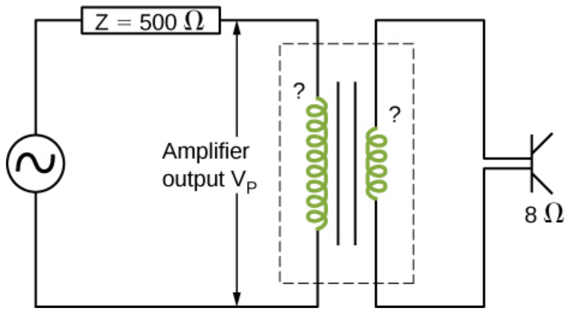
(c) 20-mH inductor with a frequency 6-Hz of the ac circuit;

(d) 20-mH inductor with a frequency 60-Hz of the ac circuit;

(e) 2-mF capacitor with a frequency 60-Hz of the ac circuit; and

(f) 2-mF capacitor with a frequency 600-Hz of the AC circuit.

66. An output impedance of an audio amplifier has an impedance of **500Ω** and has a mismatch with a low-impedance **8-Ω** loudspeaker. You are asked to insert an appropriate transformer to match the impedances. What turns ratio will you use, and why? Use the simplified circuit shown below.



67. Show that the SI unit for capacitive reactance is the ohm. Show that the SI unit for inductive reactance is also the ohm.

68. A coil with a self-inductance of 16 mH and a resistance of **6.0Ω** is connected to an ac source whose frequency can be varied. At what frequency will the voltage across the coil lead the current through the coil by **45°?**

69. An **RLC** series circuit consists of a **50-Ω** resistor, a **200-μF** capacitor, and a 120-mH inductor whose coil has a resistance of **20Ω**. The source for the circuit has an rms emf of 240 V at a frequency of 60 Hz. Calculate the rms voltages across the

(a) resistor,

(b) capacitor, and

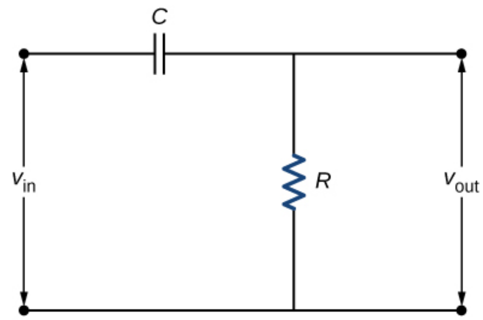
(c) inductor.

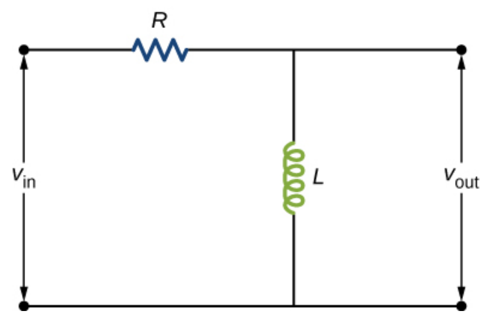
70. An **RLC** series circuit consists of a **10-Ω** resistor, an **8.0-μF** capacitor, and a 50-mH inductor. A 110-V (rms) source of variable frequency is connected across the combination. What is the power output of the source when its frequency is set to one-half the resonant frequency of the circuit?

71. Shown below are two circuits that act as crude high-pass filters. The input voltage to the circuits is v_{in}, and the output voltage is vout.vout.

(a) Show that for the capacitor circuit, $\frac{v_{out}}{v_{in}} = \frac{1}{\sqrt{1+1/\omega^2 R^2 C^2}}$, and for the inductor circuit, $\frac{v_{out}}{v_{in}} = \frac{\omega L}{\sqrt{R^2+\omega^2 L^2}}$.

(b) Show that for high frequencies, $v_{out} \approx v_{in}$, but for low frequencies, $v_{out} \approx 0$.

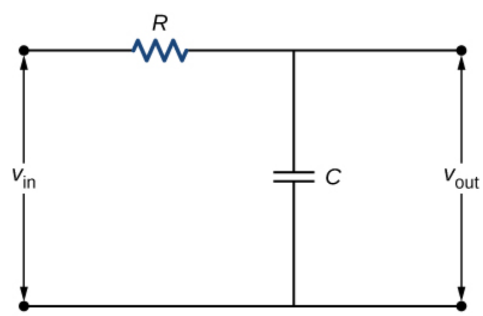


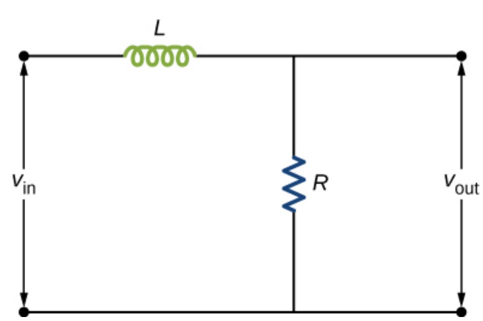


72. The two circuits shown below act as crude low-pass filters. The input voltage to the circuits is v_{in}, and the output voltage is v_{out}.

(a) Show that for the capacitor circuit, $\frac{v_{out}}{v_{in}} = \frac{1}{\sqrt{1+\omega^2R^2C^2}}$, and for the inductor circuit, $\frac{v_{out}}{v_{in}} = \frac{R}{\sqrt{R^2+\omega^2L^2}}$.

(b) Show that for low frequencies, $v_{out} \approx v_{in}$, but for high frequencies, $v_{out} \approx 0$.





CHAPTER OVERVIEW

11: Electromagnetic Waves

In this chapter, we explain Maxwell's theory and show how it leads to his prediction of electromagnetic waves. We use his theory to examine what electromagnetic waves are, how they are produced, and how they transport energy and momentum. We conclude by summarizing some of the many practical applications of electromagnetic waves.

11.1: Prelude to Electromagnetic Waves
11.2: Maxwell's Equations and Electromagnetic Waves
11.3: The Electromagnetic Spectrum
11.4: The Propagation of Light
11.5: The Law of Reflection
11.6: Refraction
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11.10.1: Prelude to Geometric Optics and Image Formation
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11.1: Prelude to Electromagnetic Waves

Our view of objects in the sky at night, the warm radiance of sunshine, the sting of sunburn, our cell phone conversations, and the X-rays revealing a broken bone—all are brought to us by electromagnetic waves. It would be hard to overstate the practical importance of electromagnetic waves, through their role in vision, through countless technological applications, and through their ability to transport the energy from the Sun through space to sustain life and almost all of its activities on Earth.

Figure 11.1.16: The pressure from sunlight predicted by Maxwell's equations helped produce the tail of Comet McNaught. (credit: modification of work by Sebastian Deiries—ESO)

Theory predicted the general phenomenon of electromagnetic waves before anyone realized that light is a form of an electromagnetic wave. In the mid-nineteenth century, James Clerk Maxwell formulated a single theory combining all the electric and magnetic effects known at that time. Maxwell's equations, summarizing this theory, predicted the existence of electromagnetic waves that travel at the speed of light. His theory also predicted how these waves behave, and how they carry both energy and momentum. The tails of comets, such as Comet McNaught in Figure 16.1, provide a spectacular example. Energy carried by light from the Sun warms the comet to release dust and gas. The momentum carried by the light exerts a weak force that shapes the dust into a tail of the kind seen here. The flux of particles emitted by the Sun, called the solar wind, typically produces an additional, second tail, as described in detail in this chapter.

In this chapter, we explain Maxwell's theory and show how it leads to his prediction of electromagnetic waves. We use his theory to examine what electromagnetic waves are, how they are produced, and how they transport energy and momentum. We conclude by summarizing some of the many practical applications of electromagnetic waves.

11.2: Maxwell's Equations and Electromagnetic Waves

Learning Objectives

By the end of this section, you will be able to:

- Explain Maxwell's correction of Ampère's law by including the displacement current
- State and apply Maxwell's equations in integral form
- Describe how the symmetry between changing electric and changing magnetic fields explains Maxwell's prediction of electromagnetic waves
- Describe how Hertz confirmed Maxwell's prediction of electromagnetic waves

James Clerk **Maxwell** (1831–1879) was one of the major contributors to physics in the nineteenth century (Figure $11.2.1$). Although he died young, he made major contributions to the development of the kinetic theory of gases, to the understanding of color vision, and to the nature of Saturn's rings. He is probably best known for having combined existing knowledge of the laws of electricity and of magnetism with insights of his own into a complete overarching electromagnetic theory, represented by **Maxwell's equations**.

Figure 11.2.1: James Clerk Maxwell, a nineteenth-century physicist, developed a theory that explained the relationship between electricity and magnetism, and correctly predicted that visible light consists of electromagnetic waves.

Maxwell's Correction to the Laws of Electricity and Magnetism

The four basic laws of electricity and magnetism had been discovered experimentally through the work of physicists such as Oersted, Coulomb, Gauss, and Faraday. Maxwell discovered logical inconsistencies in these earlier results and identified the incompleteness of Ampère's law as their cause.

Recall that according to Ampère's law, the integral of the magnetic field around a closed loop **C** is proportional to the current **I** passing through any surface whose boundary is loop **C** itself:

$$\oint \vec{B} \cdot d\vec{s} = \mu_0 I. \tag{11.2.1}$$

There are infinitely many surfaces that can be attached to any loop, and Ampère's law stated in Equation 11.2.1 is independent of the choice of surface.

Consider the set-up in Figure $11.2.2$. A source of emf is abruptly connected across a parallel-plate capacitor so that a time-dependent current **I** develops in the wire. Suppose we apply Ampère's law to loop **C** shown at a time before the capacitor is fully charged, so that $I \neq 0$. Surface S_1 gives a nonzero value for the enclosed current **I**, whereas surface S_2 gives zero for the enclosed current because no current passes through it:

$$\underbrace{\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I}_{\text{if surface } S_1 \text{is used}}$$

$$\underbrace{=0}_{\text{if surface } S_2 \text{is used}}$$

Clearly, Ampère's law in its usual form does not work here. This may not be surprising, because Ampère's law as applied in earlier chapters required a steady current, whereas the current in this experiment is changing with time and is not steady at all.

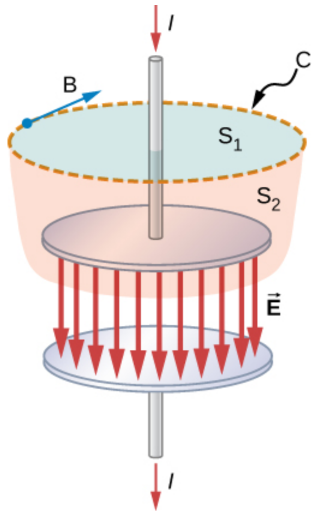


Figure 11.2.2: The currents through surface S_1 and surface S_2 are unequal, despite having the same boundary loop **C**.

How can Ampère's law be modified so that it works in all situations? Maxwell suggested including an additional contribution, called the displacement current I_d, to the real current **I**,

$$\boxed{\oint_S \vec{B}\cdot d\vec{s} = \mu_0 (I + I_d)} \tag{11.2.2}$$

where the displacement current is defined to be

$$\boxed{I_d = \epsilon_0 \frac{d\Phi_E}{dt}.} \tag{11.2.3}$$

Here ϵ_0 is the **permittivity of free space** and Φ_E is the electric flux, defined as

$$\Phi_E = \iint_{Surface\ S} \vec{E}\cdot d\vec{A}.$$

The **displacement current** is analogous to a real current in Ampère's law, entering into Ampère's law in the same way. It is produced, however, by a changing electric field. It accounts for a changing electric field producing a magnetic field, just as a real current does, but the displacement current can produce a magnetic field even where no real current is present. When this extra term is included, the modified Ampère's law equation becomes

$$\oint_C \vec{B}\cdot d\vec{s} = \mu_0 I + \epsilon_0 \mu_0 \frac{d\Phi_E}{dt}$$

and is independent of the surface **S** through which the current **I** is measured.

We can now examine this modified version of Ampère's law to confirm that it holds independent of whether the surface S_1 or the surface S_2 in Figure 11.2.2 is chosen. The electric field $\vec{E}$ corresponding to the flux Φ_E in Equation 11.2.3 is between the capacitor plates. Therefore, the $\vec{E}$ field and the displacement current through the surface S_1 are both zero, and Equation 11.2.2 takes the form

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 I. \tag{11.2.4}$$

We must now show that for surface S_2, through which no actual current flows, the displacement current leads to the same value $\mu_0 I$ for the right side of the Ampère's law equation. For surface S_2 the equation becomes

$$\oint_C \vec{B} \cdot d\vec{s} = \mu_0 \frac{d}{dt}\left[\epsilon_0 \iint_{Surface\ S_2} \vec{E} \cdot d\vec{A}\right].$$

Gauss's law for electric charge requires a closed surface and cannot ordinarily be applied to a surface like S_1 alone or S_2 alone. But the two surfaces S_1 and S_2 form a closed surface in Figure $11.2.2$ and can be used in Gauss's law. Because the electric field is zero on S_1, the flux contribution through S_1 is zero. This gives us

$$\oint_{Surface\ S_1+S_2} \vec{E} \cdot d\vec{A} = \iint_{Surface\ S_1} \vec{E} \cdot d\vec{A} + \iint_{Surface\ S_2} \vec{E} \cdot d\vec{A} \tag{11.2.5}$$

$$= 0 + \iint_{Surface\ S_2} \vec{E} \cdot d\vec{A} \tag{11.2.6}$$

$$= \iint_{Surface\ S_2} \vec{E} \cdot d\vec{A}. \tag{11.2.7}$$

Therefore, we can replace the integral over S_2 in Equation 11.2.4 with the closed Gaussian surface $S_1 + S_2$ and apply Gauss's law to obtain

$$\oint_{S_1} \vec{B} \cdot d\vec{s} = \mu_0 \frac{dQ_{in}}{dt} = \mu_0 I.$$

Thus, the modified Ampère's law equation is the same using surface S_2, where the right-hand side results from the displacement current, as it is for the surface S_1, where the contribution comes from the actual flow of electric charge.

Displacement current in a charging capacitor

A parallel-plate capacitor with capacitance **C** whose plates have area **A** and separation distance **d** is connected to a resistor **R** and a battery of voltage **V**. The current starts to flow at $t = 0$.

a. Find the displacement current between the capacitor plates at time **t**.
b. From the properties of the capacitor, find the corresponding real current $I = \frac{dQ}{dt}$, and compare the answer to the expected current in the wires of the corresponding **RC** circuit.

Strategy

We can use the equations from the analysis of an **RC** circuit (Alternating-Current Circuits) plus Maxwell's version of Ampère's law.

Solution

a. The voltage between the plates at time **t** is given by

$$V_C = \frac{1}{C} Q(t) = V_0 \left(1 - e^{-t/RC}\right).$$

Let the **z**-axis point from the positive plate to the negative plate. Then the **z**-component of the electric field between the plates as a function of time **t** is

$$E_z(t) = \frac{V_0}{d}\left(1 - e^{-t/RC}\right).$$

Therefore, the z-component of the displacement current I_d between the plates is

$$I_d(t) = \epsilon_0 A \frac{\partial E_z(t)}{\partial t} = \epsilon_0 A \frac{V_0}{d} \times \frac{1}{RC} e^{-t/RC} = \frac{V_0}{R} e^{-t/RC},$$

where we have used $C = \epsilon_0 \frac{A}{d}$ for the capacitance.

b. From the expression for V_C the charge on the capacitor is

$$Q(t) = CV_C = CV_0\left(1 - e^{-t/RC}\right).$$

The current into the capacitor after the circuit is closed, is therefore

$$I = \frac{dQ}{dt} = \frac{V_0}{R}e^{-t/RC}.$$

This current is the same as I_d found in (a).

Maxwell's Equations

With the correction for the displacement current, Maxwell's equations take the form

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{in}}{\epsilon_0}\ (\text{Gauss's law}) \tag{11.2.8}$$

$$\oint \vec{B}\cdot d\vec{A} = 0\ (\text{Gauss's law for magnetism}) \tag{11.2.9}$$

$$\oint \vec{E}\cdot d\vec{s} = -\frac{d\Phi_m}{dt}\ (\text{Faraday's law}) \tag{11.2.10}$$

$$\oint \vec{B}\cdot d\vec{s} = \mu_0 I + \epsilon_0 I + \epsilon_0\mu_0\frac{d\Phi_E}{dt}\ (\text{Ampere-Maxwell law}). \tag{11.2.11}$$

Once the fields have been calculated using these four equations, the **Lorentz force equation**

$$\vec{F} = q\vec{E} + q\vec{v}\times\vec{B}$$

gives the force that the fields exert on a particle with charge **q** moving with velocity $\vec{v}$. The Lorentz force equation combines the force of the electric field and of the magnetic field on the moving charge. The magnetic and electric forces have been examined in earlier modules. These four Maxwell's equations are, respectively:

Maxwell's Equations

1. **Gauss's law**

The electric flux through any closed surface is equal to the electric charge Q_{in} enclosed by the surface. Gauss's law (Equation 11.2.8) describes the relation between an electric charge and the electric field it produces. This is often pictured in terms of electric field lines originating from positive charges and terminating on negative charges, and indicating the direction of the electric field at each point in space.

2. **Gauss's law for magnetism**

The magnetic field flux through any closed surface is zero (Equation 11.2.9). This is equivalent to the statement that magnetic field lines are continuous, having no beginning or end. Any magnetic field line entering the region enclosed by the surface must also leave it. No magnetic monopoles, where magnetic field lines would terminate, are known to exist (see section on Magnetic Fields and Lines).

3. **Faraday's law**

A changing magnetic field induces an electromotive force (emf) and, hence, an electric field. The direction of the emf opposes the change. Equation 11.2.10 is Faraday's law of induction and includes Lenz's law. The electric field from a changing magnetic field has field lines that form closed loops, without any beginning or end.

4. **Ampère-Maxwell law**

Magnetic fields are generated by moving charges or by changing electric fields. This fourth of Maxwell's equations, Equation 11.2.11, encompasses Ampère's law and adds another source of magnetic fields, namely changing electric fields.

Maxwell's equations and the Lorentz force law together encompass all the laws of electricity and magnetism. The symmetry that Maxwell introduced into his mathematical framework may not be immediately apparent. Faraday's law describes how changing magnetic fields produce electric fields. The displacement current introduced by Maxwell results instead from a changing electric field and accounts for a changing electric field producing a magnetic field. The equations for the effects of both changing electric fields and changing magnetic fields differ in form only where the absence of magnetic monopoles leads to missing terms. This symmetry between the effects of changing magnetic and electric fields is essential in explaining the nature of electromagnetic waves.

Later application of Einstein's theory of relativity to Maxwell's complete and symmetric theory showed that electric and magnetic forces are not separate but are different manifestations of the same thing—the electromagnetic force. The electromagnetic force and weak nuclear force are similarly unified as the electroweak force. This unification of forces has been one motivation for attempts to unify all of the four basic forces in nature—the gravitational, electrical, strong, and weak nuclear forces (see Particle Physics and Cosmology).

The Mechanism of Electromagnetic Wave Propagation

To see how the symmetry introduced by Maxwell accounts for the existence of combined electric and magnetic waves that propagate through space, imagine a time-varying magnetic field $\vec{B}_0(t)$ produced by the high-frequency alternating current seen in Figure $11.2.3$. We represent $\vec{B}_0(t)$ in the diagram by one of its field lines. From Faraday's law, the changing magnetic field through a surface induces a time-varying electric field $\vec{E}_0(t)$ at the boundary of that surface. The displacement current source for the electric field, like the Faraday's law source for the magnetic field, produces only closed loops of field lines, because of the mathematical symmetry involved in the equations for the induced electric and induced magnetic fields. A field line representation of $\vec{E}_0(t)$ is shown. In turn, the changing electric field $\vec{E}_0(t)$ creates a magnetic field $\vec{B}_1(t)$ according to the modified Ampère's law. This changing field induces $\vec{E}_1(t)$ which induces $\vec{B}_2(t)$ and so on. We then have a self-continuing process that leads to the creation of time-varying electric and magnetic fields in regions farther and farther away from **O**. This process may be visualized as the propagation of an electromagnetic wave through space.

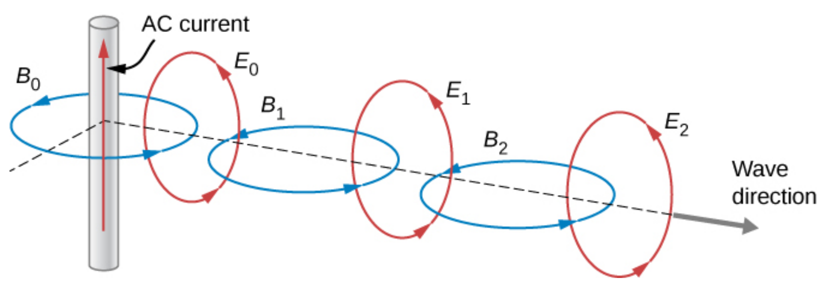


Figure 11.2.3: How changing $\vec{E}$ and $\vec{B}$ fields propagate through space.

In the next section, we show in more precise mathematical terms how Maxwell's equations lead to the prediction of electromagnetic waves that can travel through space without a material medium, implying a speed of electromagnetic waves equal to the speed of light.

Prior to Maxwell's work, experiments had already indicated that light was a wave phenomenon, although the nature of the waves was yet unknown. In 1801, Thomas Young (1773–1829) showed that when a light beam was separated by two narrow slits and then recombined, a pattern made up of bright and dark fringes was formed on a screen. Young explained this behavior by assuming that light was composed of waves that added constructively at some points and destructively at others (see Interference). Subsequently, Jean Foucault (1819–1868), with measurements of the speed of light in various media, and Augustin Fresnel (1788–1827), with detailed experiments involving interference and diffraction of light, provided further conclusive evidence that light was a wave. So, light was known to be a wave, and Maxwell had predicted the existence of electromagnetic waves that traveled at the speed of light. The conclusion seemed inescapable: Light must be a form of electromagnetic radiation. But Maxwell's theory showed that other wavelengths and frequencies than those of light were possible for electromagnetic waves. He showed that electromagnetic radiation with the same fundamental properties as visible light should exist at any frequency. It remained for others to test, and confirm, this prediction.

? Exercise 11.2.1

When the emf across a capacitor is turned on and the capacitor is allowed to charge, when does the magnetic field induced by the displacement current have the greatest magnitude?

Solution

It is greatest immediately after the current is switched on. The displacement current and the magnetic field from it are proportional to the rate of change of electric field between the plates, which is greatest when the plates first begin to charge.

Hertz's Observations

The German physicist Heinrich Hertz (1857–1894) was the first to generate and detect certain types of electromagnetic waves in the laboratory. Starting in 1887, he performed a series of experiments that not only confirmed the existence of electromagnetic waves but also verified that they travel at the speed of light.

Hertz used an alternating-current **RLC** (resistor-inductor-capacitor) circuit that resonates at a known frequency $f_0 = \frac{1}{2\pi\sqrt{LC}}$ and connected it to a loop of wire, as shown in Figure $11.2.4$. High voltages induced across the gap in the loop produced sparks that were visible evidence of the current in the circuit and helped generate electromagnetic waves.

Across the laboratory, Hertz placed another loop attached to another **RLC** circuit, which could be tuned (as the dial on a radio) to the same resonant frequency as the first and could thus be made to receive electromagnetic waves. This loop also had a gap across which sparks were generated, giving solid evidence that electromagnetic waves had been received.

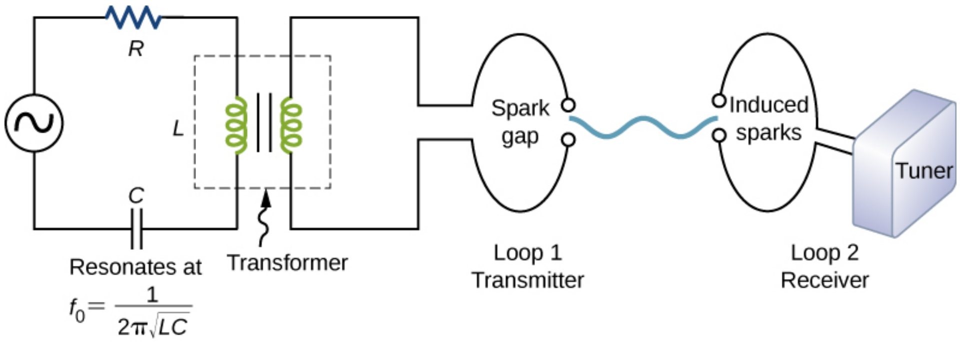


Figure 11.2.4: The apparatus used by Hertz in 1887 to generate and detect electromagnetic waves.

Hertz also studied the reflection, refraction, and interference patterns of the electromagnetic waves he generated, confirming their wave character. He was able to determine the wavelengths from the interference patterns, and knowing their frequencies, he could calculate the propagation speed using the equation $v = f\lambda$, where **v** is the speed of a wave, **f** is its frequency, and λ is its wavelength. Hertz was thus able to prove that electromagnetic waves travel at the speed of light. The SI unit for frequency, the hertz $(1\ Hz = 1\ cycle/second)$, is named in his honor.

? Exercise 11.2.2

Could a purely electric field propagate as a wave through a vacuum without a magnetic field? Justify your answer.

Solution

No. The changing electric field according to the modified version of Ampère's law would necessarily induce a changing magnetic field.

11.3: The Electromagnetic Spectrum

Learning Objectives

By the end of this section, you will be able to:

- Explain how electromagnetic waves are divided into different ranges, depending on wavelength and corresponding frequency
- Describe how electromagnetic waves in different categories are produced
- Describe some of the many practical everyday applications of electromagnetic waves

Electromagnetic waves have a vast range of practical everyday applications that includes such diverse uses as communication by cell phone and radio broadcasting, WiFi, cooking, vision, medical imaging, and treating cancer. In this module, we discuss how electromagnetic waves are classified into categories such as radio, infrared, ultraviolet, and so on. We also summarize some of the main applications for each range.

The different categories of electromagnetic waves differ in their wavelength range, or equivalently, in their corresponding frequency ranges. Their properties change smoothly from one frequency range to the next, with different applications in each range. A brief overview of the production and utilization of electromagnetic waves is found in Table $11.3.1$.

Table 11.3.1: Electromagnetic Waves

Type of wave	Production	Applications	Issues
Radio	Accelerating charges	Communications, Remote controls, MRI	Requires control for band use
Microwaves	Accelerating charges and thermal agitation	Communications, Ovens, Radar, Cell phone use	
Infrared	Thermal agitation and electronic transitions	Thermal imaging, Heating	Absorbed by atmosphere, Greenhouse effect
Visible light	Thermal agitation and electronic transitions	Photosynthesis, Human vision	
Ultraviolet	Thermal agitation and electronic transitions	Sterilization, Vitamin D production	Ozone depletion, Cancer causing
X-rays	Inner electronic transitions and fast collisions	Security, Medical diagnosis, Cancer therapy	Cancer causing
Gamma rays	Nuclear decay	Nuclear medicine, Security, Medical diagnosis, Cancer therapy	Cancer causing, Radiation damage

The relationship $c = f\lambda$ between frequency **f** and wavelength λ applies to all waves and ensures that greater frequency means smaller wavelength. Figure $11.3.2$ shows how the various types of electromagnetic waves are categorized according to their wavelengths and frequencies - that is, it shows the electromagnetic spectrum.

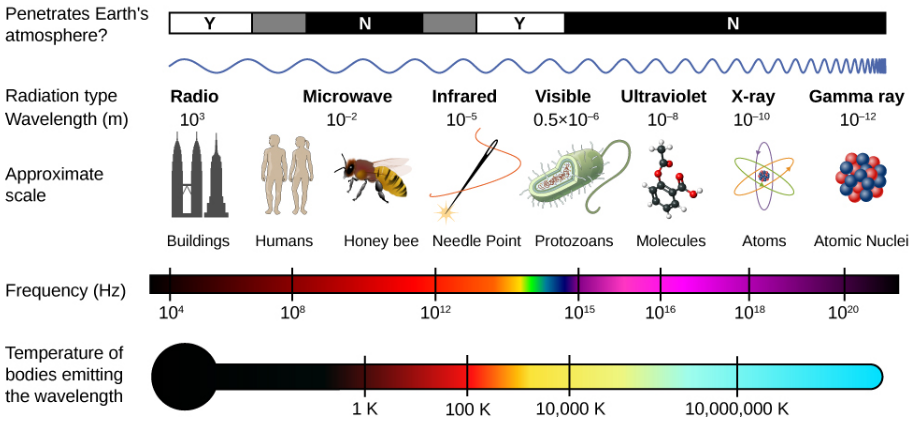


Figure 11.3.1: The electromagnetic spectrum, showing the major categories of electromagnetic waves.

Radio Waves

The term **radio waves** refers to electromagnetic radiation with wavelengths greater than about 0.1 m. Radio waves are commonly used for audio communications (i.e., for radios), but the term is used for electromagnetic waves in this range regardless of their application. Radio waves typically result from an alternating current in the wires of a broadcast antenna. They cover a very broad wavelength range and are divided into many subranges, including microwaves, electromagnetic waves used for AM and FM radio, cellular telephones, and TV signals.

There is no lowest frequency of radio waves, but ELF waves, or "extremely low frequency" are among the lowest frequencies commonly encountered, from 3 Hz to 3 kHz. The accelerating charge in the ac currents of electrical power lines produce electromagnetic waves in this range. ELF waves are able to penetrate sea water, which strongly absorbs electromagnetic waves of higher frequency, and therefore are useful for submarine communications.

In order to use an electromagnetic wave to transmit information, the amplitude, frequency, or phase of the wave is **modulated**, or varied in a controlled way that encodes the intended information into the wave. In AM radio transmission, the amplitude of the wave is modulated to mimic the vibrations of the sound being conveyed. Fourier's theorem implies that the modulated AM wave amounts to a superposition of waves covering some narrow frequency range. Each AM station is assigned a specific carrier frequency that, by international agreement, is allowed to vary by $\pm 5\, kHz$. In FM radio transmission, the frequency of the wave is modulated to carry this information, as illustrated in Figure $11.3.2$, and the frequency of each station is allowed to use 100 kHz on each side of its carrier frequency. The electromagnetic wave produces a current in a receiving antenna, and the radio or television processes the signal to produce the sound and any image. The higher the frequency of the radio wave used to carry the data, the greater the detailed variation of the wave that can be carried by modulating it over each time unit, and the more data that can be transmitted per unit of time. The assigned frequencies for AM broadcasting are 540 to 1600 kHz, and for FM are 88 MHz to108 MHz.

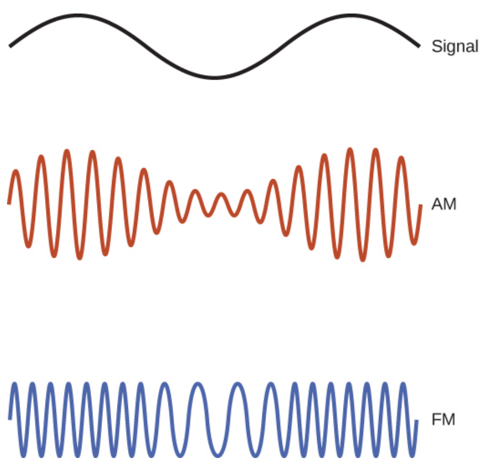


Figure 11.3.2: Electromagnetic waves are used to carry communications signals by varying the wave's amplitude (AM), its frequency (FM), or its phase.

Cell phone conversations, and **television** voice and video images are commonly transmitted as digital data, by converting the signal into a sequence of binary ones and zeros. This allows clearer data transmission when the signal is weak, and allows using computer algorithms to compress the digital data to transmit more data in each frequency range. Computer data as well is transmitted as a sequence of binary ones and zeros, each one or zero constituting one bit of data.

Microwaves

Microwaves are the highest-frequency electromagnetic waves that can be produced by currents in macroscopic circuits and devices. Microwave frequencies range from about $10^9 Hz$ to nearly $10^{12} Hz$. Their high frequencies correspond to short wavelengths compared with other radio waves—hence the name "microwave." Microwaves also occur naturally as the cosmic background radiation left over from the origin of the universe. Along with other ranges of electromagnetic waves, they are part of the radiation that any object above absolute zero emits and absorbs because of **thermal agitation**, that is, from the thermal motion of its atoms and molecules.

Most satellite-transmitted information is carried on **microwaves**. **Radar** is a common application of microwaves. By detecting and timing microwave echoes, radar systems can determine the distance to objects as diverse as clouds, aircraft, or even the surface of Venus.

Microwaves of 2.45 GHz are commonly used in microwave ovens. The electrons in a water molecule tend to remain closer to the oxygen nucleus than the hydrogen nuclei (Figure $11.3.3$). This creates two separated centers of equal and opposite charges, giving the molecule a dipole moment. The oscillating electric field of the microwaves inside the oven exerts a torque that tends to align each molecule first in one direction and then in the other, with the motion of each molecule coupled to others around it. This pumps energy into the continual thermal motion of the water to heat the food. The plate under the food contains no water, and remains relatively unheated.

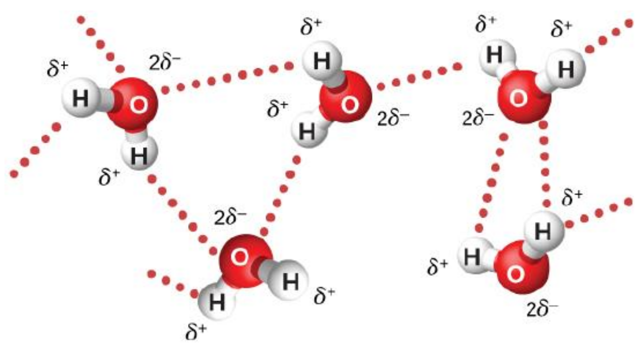


Figure 11.3.3: The oscillating electric field in a microwave oven exerts a torque on water molecules because of their dipole moment, and the torque reverses direction 4.90×10^9 times per second. Interactions between the molecules distributes the energy being pumped into them. The δ^+ and δ^- denote the charge distribution on the molecules.

The microwaves in a microwave oven reflect off the walls of the oven, so that the superposition of waves produces standing waves, similar to the standing waves of a vibrating guitar or violin string (Normal Modes of a Standing Sound Wave). A rotating fan acts as a stirrer by reflecting the microwaves in different directions, and food turntables, help spread out the hot spots.

✔ Example 11.3.1: Why Microwave Ovens Heat Unevenly

How far apart are the hotspots in a 2.45-GHz microwave oven?

Strategy

Consider the waves along one direction in the oven, being reflected at the opposite wall from where they are generated.

Solution

The antinodes, where maximum intensity occurs, are half the wavelength apart, with separation

$$\begin{aligned} d &= \frac{1}{2}\lambda && (11.3.1)\\ &= \frac{1}{2}\frac{c}{f} && (11.3.2)\\ &= \frac{3.00\times 10^8 m/s}{2(2.45\times 10^9\ Hz)} && (11.3.3)\\ &= 6.02\ cm. && (11.3.4)\end{aligned}$$

Significance

The distance between the hot spots in a microwave oven are determined by the wavelength of the microwaves.

A cell phone has a radio receiver and a weak radio transmitter, both of which can quickly tune to hundreds of specifically assigned microwave frequencies. The low intensity of the transmitted signal gives it an intentionally limited range. A ground-based system links the phone to only to the broadcast tower assigned to the specific small area, or cell, and smoothly transitions its connection to the next cell when the signal reception there is the stronger one. This enables a cell phone to be used while changing location.

Microwaves also provide the **WiFi** that enables owners of cell phones, laptop computers, and similar devices to connect wirelessly to the Internet at home and at coffee shops and airports. A wireless WiFi router is a device that exchanges data over the Internet through the cable or another connection, and uses microwaves to exchange the data wirelessly with devices such as cell phones and computers. The term WiFi itself refers to the standards followed in modulating and analyzing the microwaves so that wireless routers and devices from different manufacturers work compatibly with one another. The computer data in each direction consist of sequences of binary zeros and ones, each corresponding to a binary bit. The microwaves are in the range of 2.4 GHz to 5.0 GHz range.

Other wireless technologies also use microwaves to provide everyday communications between devices. **Bluetooth** developed alongside WiFi as a standard for radio communication in the 2.4-GHz range between nearby devices, for example, to link to headphones and audio earpieces to devices such as radios, or a driver's cell phone to a hands-free device to allow answering phone calls without fumbling directly with the cell phone.

Microwaves find use also in radio tagging, using RFID (radio frequency identification) technology. Examples are RFID tags attached to store merchandize, transponder for toll booths use attached to the windshield of a car, or even a chip embedded into a pet's skin. The device responds to a microwave signal by emitting a signal of its own with encoded information, allowing stores to quickly ring up items at their cash registers, drivers to charge tolls to their account without stopping, and lost pets to be reunited with their owners. NFC (near field communication) works similarly, except it is much shorter range. Its mechanism of interaction is the induced magnetic field at microwave frequencies between two coils. Cell phones that have NFC capability and the right software can supply information for purchases using the cell phone instead of a physical credit card. The very short range of the data transfer is a desired security feature in this case.

Infrared Radiation

The boundary between the microwave and infrared regions of the electromagnetic spectrum is not well defined (Figure $11.3.1$). **Infrared radiation** is generally produced by thermal motion, and the vibration and rotation of atoms and molecules. Electronic transitions in atoms and molecules can also produce **infrared radiation**. About half of the solar energy arriving at Earth is in the infrared region, with most of the rest in the visible part of the spectrum. About 23% of the solar energy is absorbed in the atmosphere, about 48% is absorbed at Earth's surface, and about 29% is reflected back into space.

The range of infrared frequencies extends up to the lower limit of visible light, just below red. In fact, infrared means "below red." Water molecules rotate and vibrate particularly well at infrared frequencies. Reconnaissance satellites can detect buildings, vehicles, and even individual humans by their infrared emissions, whose power radiation is proportional to the fourth power of the absolute temperature. More mundanely, we use infrared lamps, including those called **quartz heaters**, to preferentially warm us because we absorb infrared better than our surroundings.

The familiar handheld "remotes" for changing channels and settings on television sets often transmit their signal by modulating an infrared beam. If you try to use a TV remote without the infrared emitter being in direct line of sight with the infrared detector, you may find the television not responding. Some remotes use Bluetooth instead and reduce this annoyance.

Visible Light

Visible light is the narrow segment of the electromagnetic spectrum between about 400 nm and about 750 nm to which the normal human eye responds. Visible light is produced by vibrations and rotations of atoms and molecules, as well as by electronic transitions within atoms and molecules. The receivers or detectors of light largely utilize electronic transitions.

Red light has the lowest frequencies and longest wavelengths, whereas violet has the highest frequencies and shortest wavelengths (Figure $11.3.4$). Blackbody radiation from the Sun peaks in the visible part of the spectrum but is more intense in the red than in the violet, making the sun yellowish in appearance.

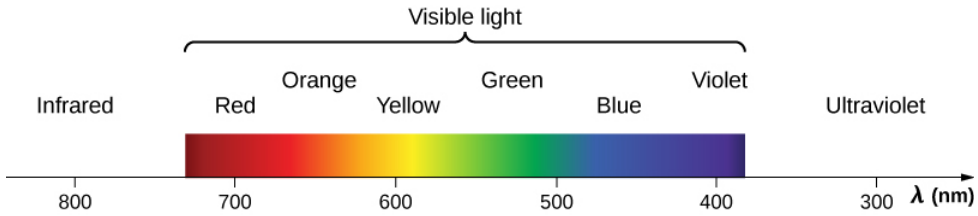


Figure 11.3.4. A small part of the electromagnetic spectrum that includes its visible components. The divisions between infrared, visible, and ultraviolet are not perfectly distinct, nor are those between the seven rainbow colors.

Living things - plants and animals - have evolved to utilize and respond to parts of the electromagnetic spectrum in which they are embedded. We enjoy the beauty of nature through visible light. Plants are more selective. Photosynthesis uses parts of the visible spectrum to make sugars.

Ultraviolet Radiation

Ultraviolet means “above violet.” The electromagnetic frequencies of **ultraviolet radiation (UV)** extend upward from violet, the highest-frequency visible light. The highest-frequency ultraviolet overlaps with the lowest-frequency X-rays. The wavelengths of ultraviolet extend from 400 nm down to about 10 nm at its highest frequencies. Ultraviolet is produced by atomic and molecular motions and electronic transitions.

UV radiation from the Sun is broadly subdivided into three wavelength ranges: UV-A (320–400 nm) is the lowest frequency, then UV-B (290–320 nm) and UV-C (220–290 nm). Most UV-B and all UV-C are absorbed by ozone (O_3) molecules in the upper atmosphere. Consequently, 99% of the solar UV radiation reaching Earth’s surface is UV-A.

Sunburn is caused by large exposures to UV-B and UV-C, and repeated exposure can increase the likelihood of skin cancer. The tanning response is a defense mechanism in which the body produces pigments in inert skin layers to reduce exposure of the living cells below.

As examined in a later chapter, the shorter the wavelength of light, the greater the energy change of an atom or molecule that absorbs the light in an electronic transition. This makes short-wavelength ultraviolet light damaging to living cells. It also explains why ultraviolet radiation is better able than visible light to cause some materials to glow, or **fluoresce**.

Besides the adverse effects of ultraviolet radiation, there are also benefits of exposure in nature and uses in technology. Vitamin D production in the skin results from exposure to UV-B radiation, generally from sunlight. Several studies suggest vitamin D deficiency is associated with the development of a range of cancers (prostate, breast, colon), as well as osteoporosis. Low-intensity ultraviolet has applications such as providing the energy to cause certain dyes to fluoresce and emit visible light, for example, in printed money to display hidden watermarks as counterfeit protection.

X-Rays

X-rays have wavelengths from about $10^{-8} m$ to $10^{-12} m$. They have shorter wavelengths, and higher frequencies, than ultraviolet, so that the energy they transfer at an atomic level is greater. As a result, X-rays have adverse effects on living cells similar to those of ultraviolet radiation, but they are more penetrating. Cancer and genetic defects can be induced by X-rays. Because of their effect on rapidly dividing cells, X-rays can also be used to treat and even cure cancer.

The widest use of X-rays is for imaging objects that are opaque to visible light, such as the human body or aircraft parts. In humans, the risk of cell damage is weighed carefully against the benefit of the diagnostic information obtained.

Gamma Rays

Soon after nuclear radioactivity was first detected in 1896, it was found that at least three distinct types of radiation were being emitted, and these were designated as alpha, beta, and gamma rays. The most penetrating nuclear radiation, the **gamma ray** γ -ray) was later found to be an extremely high-frequency electromagnetic wave.

The lower end of the γ-ray frequency range overlaps the upper end of the X-ray range. Gamma rays have characteristics identical to X-rays of the same frequency—they differ only in source. The name "gamma rays" is generally used for electromagnetic radiation emitted by a nucleus, while X-rays are generally produced by bombarding a target with energetic electrons in an X-ray tube. At higher frequencies, γ-rays are more penetrating and more damaging to living tissue. They have many of the same uses as X-rays, including cancer therapy. Gamma radiation from radioactive materials is used in nuclear medicine.

Use this simulation to explore how light interacts with molecules in our atmosphere.

- Explore how light interacts with molecules in our atmosphere.
- Identify that absorption of light depends on the molecule and the type of light.
- Relate the energy of the light to the resulting motion.
- Identify that energy increases from microwave to ultraviolet.
- Predict the motion of a molecule based on the type of light it absorbs.

Exercise 11.3.1

How do the electromagnetic waves for the different kinds of electromagnetic radiation differ?

Answer

They fall into different ranges of wavelength, and therefore also different corresponding ranges of frequency.

11.4: The Propagation of Light

Learning Objectives

By the end of this section, you will be able to:

- Determine the index of refraction, given the speed of light in a medium
- List the ways in which light travels from a source to another location

The Speed of Light: Early Measurements

The first measurement of the speed of light was made by the Danish astronomer Ole Roemer (1644–1710) in 1675. He studied the orbit of Io, one of the four large moons of Jupiter, and found that it had a period of revolution of 42.5 h around Jupiter. He also discovered that this value fluctuated by a few seconds, depending on the position of Earth in its orbit around the Sun. Roemer realized that this fluctuation was due to the finite speed of light and could be used to determine **c**.

Roemer found the period of revolution of Io by measuring the time interval between successive eclipses by Jupiter. Figure $11.4.1a$ shows the planetary configurations when such a measurement is made from Earth in the part of its orbit where it is receding from Jupiter. When Earth is at point **A**, Earth, Jupiter, and Io are aligned. The next time this alignment occurs, Earth is at point **B**, and the light carrying that information to Earth must travel to that point. Since **B** is farther from Jupiter than **A**, light takes more time to reach Earth when Earth is at **B**. Now imagine it is about 6 months later, and the planets are arranged as in Figure $11.4.1b$. The measurement of Io's period begins with Earth at point A' and Io eclipsed by Jupiter. The next eclipse then occurs when Earth is at point B', to which the light carrying the information of this eclipse must travel. Since B' is closer to Jupiter than A', light takes less time to reach Earth when it is at B'. This time interval between the successive eclipses of Io seen at A' and B' is therefore less than the time interval between the eclipses seen at **A** and **B**. By measuring the difference in these time intervals and with appropriate knowledge of the distance between Jupiter and Earth, Roemer calculated that the speed of light was $2.0 \times 10^8\ m/s$, which is only 33% below the value accepted today.

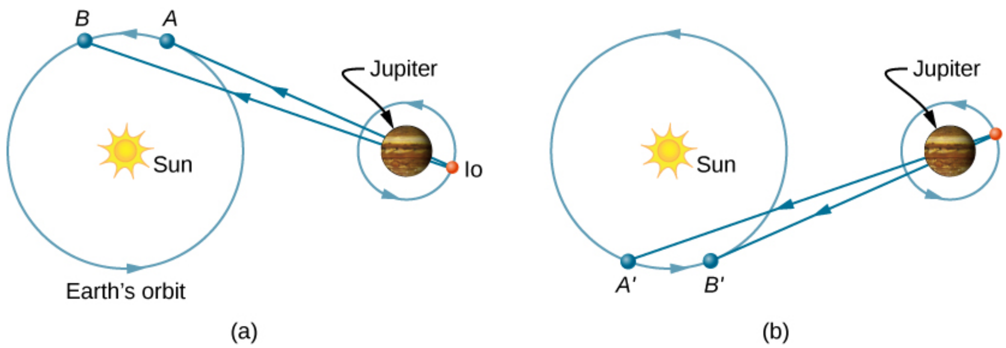


Figure 11.4.1: Roemer's astronomical method for determining the speed of light. Measurements of Io's period done with the configurations of parts (a) and (b) differ, because the light path length and associated travel time increase from A to B (a) but decrease from A'A′ to B'B′ (b).

The first successful terrestrial measurement of the speed of light was made by Armand Fizeau (1819–1896) in 1849. He placed a toothed wheel that could be rotated very rapidly on one hilltop and a mirror on a second hilltop 8 km away (Figure $11.4.2$). An intense light source was placed behind the wheel, so that when the wheel rotated, it chopped the light beam into a succession of pulses. The speed of the wheel was then adjusted until no light returned to the observer located behind the wheel. This could only happen if the wheel rotated through an angle corresponding to a displacement of (n+½) teeth, while the pulses traveled down to the mirror and back. Knowing the rotational speed of the wheel, the number of teeth on the wheel, and the distance to the mirror, Fizeau determined the speed of light to be $3.15 \times 10^8\ m/s$, which is only 5% too high.

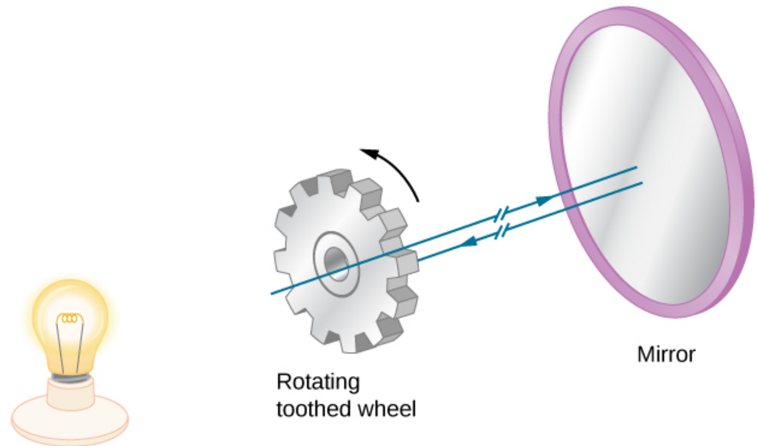


Figure 11.4.2: Fizeau's method for measuring the speed of light. The teeth of the wheel block the reflected light upon return when the wheel is rotated at a rate that matches the light travel time to and from the mirror.

The French physicist Jean Bernard Léon Foucault (1819–1868) modified Fizeau's apparatus by replacing the toothed wheel with a rotating mirror. In 1862, he measured the speed of light to be 2.98×10^8m/s, which is within 0.6% of the presently accepted value. Albert Michelson (1852–1931) also used Foucault's method on several occasions to measure the speed of light. His first experiments were performed in 1878; by 1926, he had refined the technique so well that he found **c** to be $(2.99796\pm4)\times10^8$m/s.

Today, the speed of light is known to great precision. In fact, the speed of light in a vacuum **c** is so important that it is accepted as one of the basic physical quantities and has the value

$$c=2.99792458\times10^8 m/s \equiv 3.00\times10^8 m/s \tag{11.4.1}$$

where the approximate value of 3.00×10^8m/s is used whenever three-digit accuracy is sufficient.

Speed of Light in Matter

The speed of light through matter is less than it is in a vacuum, because light interacts with atoms in a material. The speed of light depends strongly on the type of material, since its interaction varies with different atoms, crystal lattices, and other substructures. We can define a constant of a material that describes the speed of light in it, called the index of refraction **n**:

$$n=\frac{c}{v} \tag{11.4.2}$$

where v is the observed speed of light in the material.

Since the speed of light is always less than **c** in matter and equals **c** only in a vacuum, the index of refraction is always greater than or equal to one; that is, $n\geq1$. Table 11.4.1 gives the indices of refraction for some representative substances. The values are listed for a particular wavelength of light, because they vary slightly with wavelength. (This can have important effects, such as colors separated by a prism, as we will see in Dispersion.) Note that for gases, **n** is close to 1.0. This seems reasonable, since atoms in gases are widely separated, and light travels at **c** in the vacuum between atoms. It is common to take $n=1$ for gases unless great precision is needed. Although the speed of light **v** in a medium varies considerably from its value **c** in a vacuum, it is still a large speed.

Figure 11.4.1: Index of Refraction in Various MediaFor light with a wavelength of 589 nm in a vacuum

Medium	**n**
Gases at 0°C, 1 atm	
Air	1.000293
Carbon dioxide	1.00045
Hydrogen	1.000139
Oxygen	1.000271

Medium	n
Liquids at 20°C	
Benzene	1.501
Carbon disulfide	1.628
Carbon tetrachloride	1.461
Ethanol	1.361
Glycerine	1.473
Water, fresh	1.333
Solids at 20°C	
Diamond	2.419
Fluorite	1.434
Glass, crown	1.52
Glass, flint	1.66
Ice (at 0°C)0°C)	1.309
Polystyrene	1.49
Plexiglas	1.51
Quartz, crystalline	1.544
Quartz, fused	1.458
Sodium chloride	1.544
Zircon	1.923

Example 11.4.1: Speed of Light in Jewelry

Calculate the speed of light in zircon, a material used in jewelry to imitate diamond.

Strategy

We can calculate the speed of light in a material v from the index of refraction n of the material, using Equation \red{index}

Solution

Rearranging Equation 11.4.2 for v gives us

$$v = \frac{c}{n}.$$

The index of refraction for zircon is given as 1.923 in Table 11.4.1, and c is given in Equation 11.4.1. Entering these values in the equation gives

$$\begin{aligned} v &= \frac{3.00 \times 10^8 m/s}{1.923} \\ &= 1.56 \times 10^8 m/s. \end{aligned}$$

Significance

This speed is slightly larger than half the speed of light in a vacuum and is still high compared with speeds we normally experience. The only substance listed in Table 11.4.1 that has a greater index of refraction than zircon is diamond. We shall see later that the large index of refraction for zircon makes it sparkle more than glass, but less than diamond.

? Exercise 11.4.1

Table 11.4.1 shows that ethanol and fresh water have very similar indices of refraction. By what percentage do the speeds of light in these liquids differ?

Answer

2.1% (to two significant figures)

The Ray Model of Light

You have already studied some of the wave characteristics of light in the previous chapter on Electromagnetic Waves. In this chapter, we start mainly with the ray characteristics. There are three ways in which light can travel from a source to another location (Figure 11.4.1). It can come directly from the source through empty space, such as from the Sun to Earth. Or light can travel through various media, such as air and glass, to the observer. Light can also arrive after being reflected, such as by a mirror. In all of these cases, we can model the path of light as a straight line called a ray.

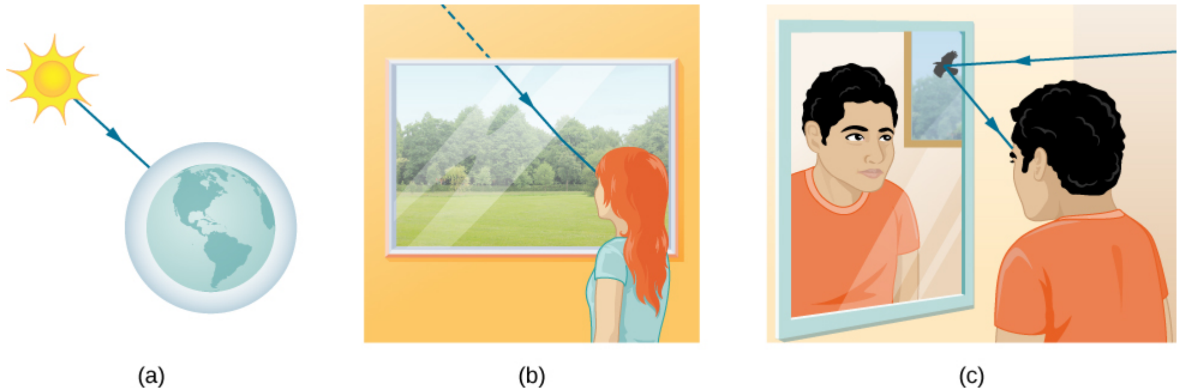


Figure 11.4.3: Three methods for light to travel from a source to another location. (a) Light reaches the upper atmosphere of Earth, traveling through empty space directly from the source. (b) Light can reach a person by traveling through media like air and glass. (c) Light can also reflect from an object like a mirror. In the situations shown here, light interacts with objects large enough that it travels in straight lines, like a ray.

Experiments show that when light interacts with an object several times larger than its wavelength, it travels in straight lines and acts like a ray. Its wave characteristics are not pronounced in such situations. Since the wavelength of visible light is less than a micron (a thousandth of a millimeter), it acts like a ray in the many common situations in which it encounters objects larger than a micron. For example, when visible light encounters anything large enough that we can observe it with unaided eyes, such as a coin, it acts like a ray, with generally negligible wave characteristics.

In all of these cases, we can model the path of light as straight lines. Light may change direction when it encounters objects (such as a mirror) or in passing from one material to another (such as in passing from air to glass), but it then continues in a straight line or as a ray. The word “ray” comes from mathematics and here means a straight line that originates at some point. It is acceptable to visualize light rays as laser rays. The **ray model of light** describes the path of light as straight lines.

Since light moves in straight lines, changing directions when it interacts with materials, its path is described by geometry and simple trigonometry. This part of optics, where the ray aspect of light dominates, is therefore called geometric optics. Two laws govern how light changes direction when it interacts with matter. These are the **law of reflection**, for situations in which light bounces off matter, and the **law of refraction**, for situations in which light passes through matter. We will examine more about each of these laws in upcoming sections of this chapter.

11.5: The Law of Reflection

Learning Objectives

By the end of this section, you will be able to:

- Explain the reflection of light from polished and rough surfaces
- Describe the principle and applications of corner reflectors

Whenever we look into a mirror, or squint at sunlight glinting from a lake, we are seeing a reflection. When you look at a piece of white paper, you are seeing light scattered from it. Large telescopes use reflection to form an image of stars and other astronomical objects.

The **law of reflection** states that the angle of reflection equals the angle of incidence:

$$\theta_r = \theta_i \tag{11.5.1}$$

The law of reflection is illustrated in Figure $11.5.1$, which also shows how the angle of incidence and angle of reflection are measured relative to the perpendicular to the surface at the point where the light ray strikes.

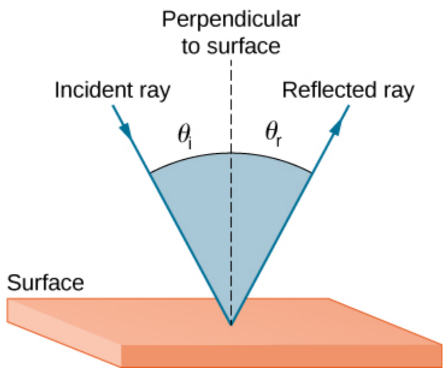


Figure 11.5.1: The law of reflection states that the angle of reflection equals the angle of incidence—$\theta_r=\theta_i$. The angles are measured relative to the perpendicular to the surface at the point where the ray strikes the surface.

We expect to see reflections from smooth surfaces, but Figure $11.5.2$ illustrates how a rough surface reflects light. Since the light strikes different parts of the surface at different angles, it is reflected in many different directions, or diffused. Diffused light is what allows us to see a sheet of paper from any angle, as shown in Figure $11.5.1a$.

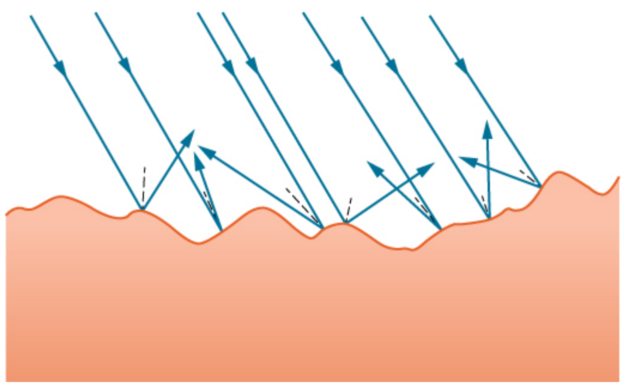


Figure 11.5.2: Light is diffused when it reflects from a rough surface. Here, many parallel rays are incident, but they are reflected at many different angles, because the surface is rough.

People, clothing, leaves, and walls all have rough surfaces and can be seen from all sides. A mirror, on the other hand, has a smooth surface (compared with the wavelength of light) and reflects light at specific angles, as illustrated in Figure $11.5.3b$. When the Moon reflects from a lake, as shown in Figure $11.5.1c$, a combination of these effects takes place.

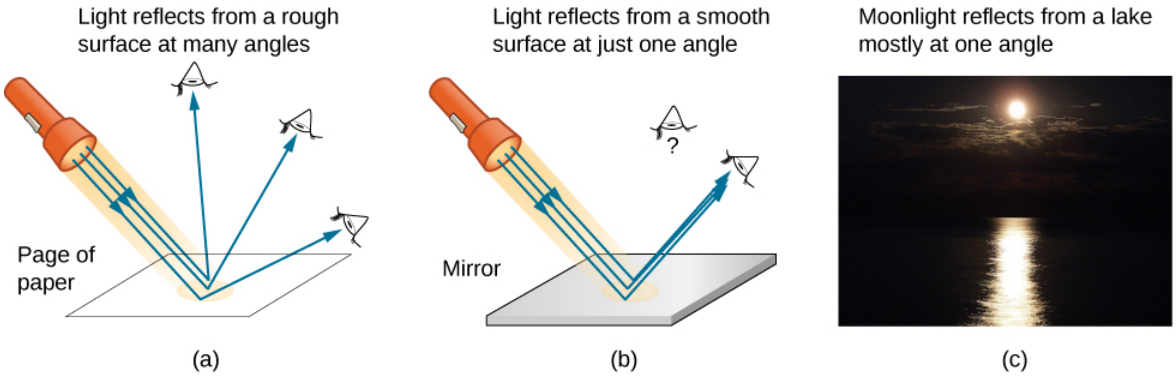


Figure 11.5.3: (a) When a sheet of paper is illuminated with many parallel incident rays, it can be seen at many different angles, because its surface is rough and diffuses the light. (b) A mirror illuminated by many parallel rays reflects them in only one direction, because its surface is very smooth. Only the observer at a particular angle sees the reflected light. (c) Moonlight is spread out when it is reflected by the lake, because the surface is shiny but uneven. (credit c: modification of work by Diego Torres Silvestre)

When you see yourself in a mirror, it appears that the image is actually behind the mirror (Figure 11.5.4). We see the light coming from a direction determined by the law of reflection. The angles are such that the image is exactly the same distance behind the mirror as you stand in front of the mirror. If the mirror is on the wall of a room, the images in it are all behind the mirror, which can make the room seem bigger. Although these mirror images make objects appear to be where they cannot be (like behind a solid wall), the images are not figments of your imagination. Mirror images can be photographed and videotaped by instruments and look just as they do with our eyes (which are optical instruments themselves). The precise manner in which images are formed by mirrors and lenses is discussed in an upcoming chapter on Geometric Optics and Image Formation.

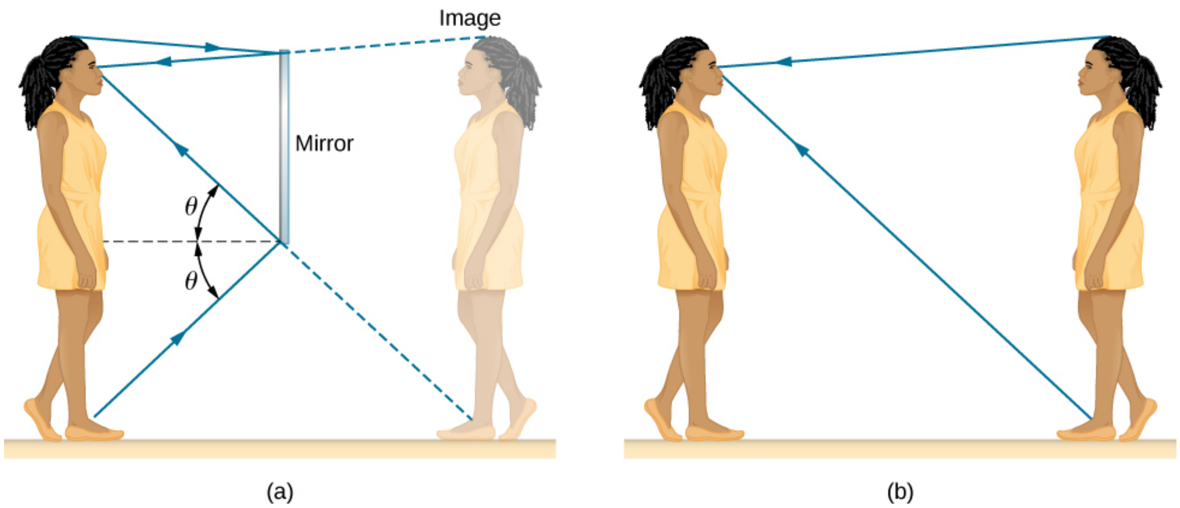


Figure 11.5.4: (a) Your image in a mirror is behind the mirror. The two rays shown are those that strike the mirror at just the correct angles to be reflected into the eyes of the person. The image appears to be behind the mirror at the same distance away as (b) if you were looking at your twin directly, with no mirror.

Corner Reflectors (Retroreflectors)

A light ray that strikes an object consisting of two mutually perpendicular reflecting surfaces is reflected back exactly parallel to the direction from which it came (Figure 11.5.5). This is true whenever the reflecting surfaces are perpendicular, and it is independent of the angle of incidence. Such an object is called a **corner reflector**, since the light bounces from its inside corner. Corner reflectors are a subclass of retroreflectors, which all reflect rays back in the directions from which they came. Although the geometry of the proof is much more complex, corner reflectors can also be built with three mutually perpendicular reflecting surfaces and are useful in three-dimensional applications.

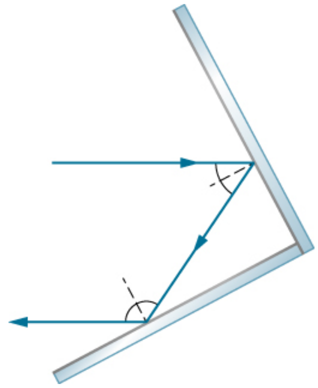
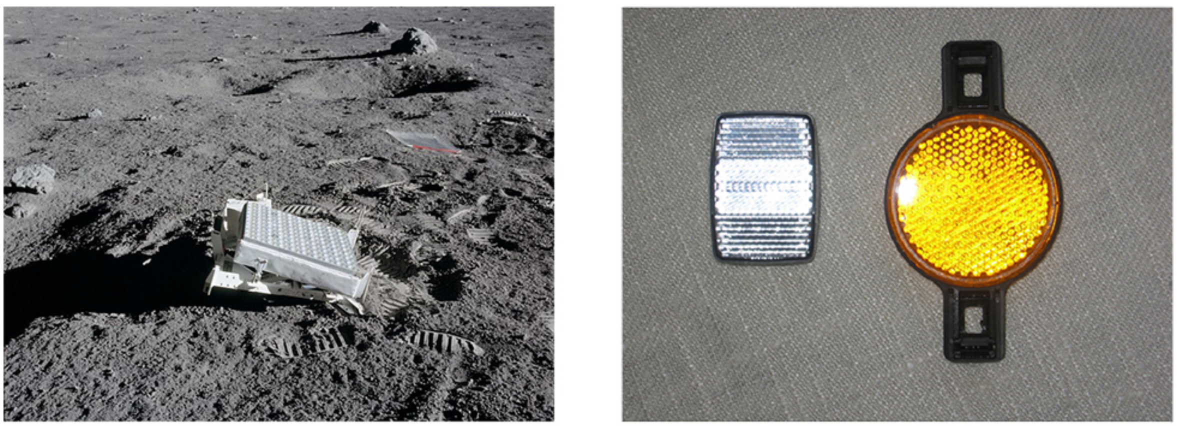


Figure 11.5.5: A light ray that strikes two mutually perpendicular reflecting surfaces is reflected back exactly parallel to the direction from which it came.

Many inexpensive reflector buttons on bicycles, cars, and warning signs have corner reflectors designed to return light in the direction from which it originated. Rather than simply reflecting light over a wide angle, retroreflection ensures high visibility if the observer and the light source are located together, such as a car's driver and headlights. The Apollo astronauts placed a true corner reflector on the Moon (Figure 11.5.6). Laser signals from Earth can be bounced from that corner reflector to measure the gradually increasing distance to the Moon of a few centimeters per year.



(a) (b)

Figure 11.5.6: (a) Astronauts placed a corner reflector on the Moon to measure its gradually increasing orbital distance. (b) The bright spots on these bicycle safety reflectors are reflections of the flash of the camera that took this picture on a dark night. (credit a: modification of work by NASA; credit b: modification of work by "Julo"/Wikimedia Commons)

Working on the same principle as these optical reflectors, corner reflectors are routinely used as radar reflectors (Figure 11.5.7) for radio-frequency applications. Under most circumstances, small boats made of fiberglass or wood do not strongly reflect radio waves emitted by radar systems. To make these boats visible to radar (to avoid collisions, for example), radar reflectors are attached to boats, usually in high places.

Figure 11.5.7: A radar reflector hoisted on a sailboat is a type of corner reflector. (credit: Tim Sheerman-Chase)

As a counterexample, if you are interested in building a stealth airplane, radar reflections should be minimized to evade detection. One of the design considerations would then be to avoid building 90°90° corners into the airframe.

11.6: Refraction

Learning Objectives

By the end of this section, you will be able to:

- Describe how rays change direction upon entering a medium
- Apply the law of refraction in problem solving

You may often notice some odd things when looking into a fish tank. For example, you may see the same fish appearing to be in two different places (Figure 11.6.1). This happens because light coming from the fish to you changes direction when it leaves the tank, and in this case, it can travel two different paths to get to your eyes. The changing of a light ray's direction (loosely called bending) when it passes through substances of different refractive indices is called **refraction** and is related to changes in the speed of light, $v = c/n$. Refraction is responsible for a tremendous range of optical phenomena, from the action of lenses to data transmission through optical fibers.

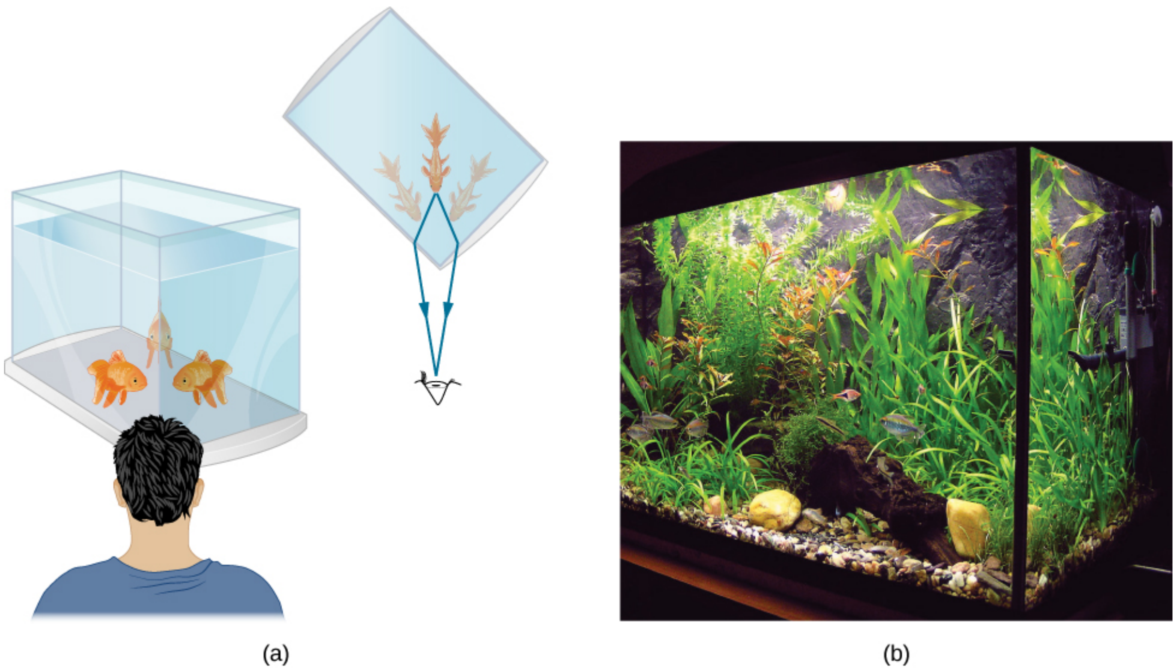


Figure 11.6.1: (a) Looking at the fish tank as shown, we can see the same fish in two different locations, because light changes directions when it passes from water to air. In this case, the light can reach the observer by two different paths, so the fish seems to be in two different places. This bending of light is called refraction and is responsible for many optical phenomena. (b) This image shows refraction of light from a fish near the top of a fish tank.

Figure 11.6.2 shows how a ray of light changes direction when it passes from one medium to another. As before, the angles are measured relative to a perpendicular to the surface at the point where the light ray crosses it. (Some of the incident light is reflected from the surface, but for now we concentrate on the light that is transmitted.) The change in direction of the light ray depends on the relative values of the indices of refraction of the two media involved. In the situations shown, medium 2 has a greater index of refraction than medium 1. Note that as shown in Figure 11.6.1a, the direction of the ray moves closer to the perpendicular when it progresses from a medium with a lower index of refraction to one with a higher index of refraction. Conversely, as shown in Figure 11.6.1b, the direction of the ray moves away from the perpendicular when it progresses from a medium with a higher index of refraction to one with a lower index of refraction. The path is exactly reversible.

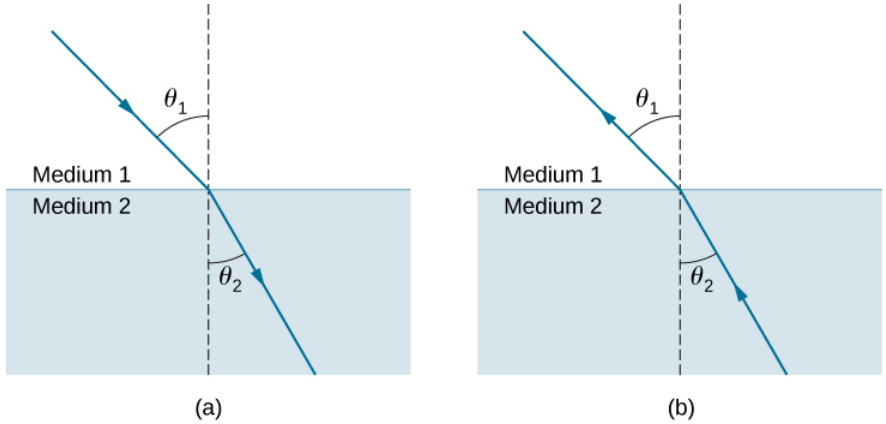


Figure 11.6.2: The change in direction of a light ray depends on how the index of refraction changes when it crosses from one medium to another. In the situations shown here, the index of refraction is greater in medium 2 than in medium 1. (a) A ray of light moves closer to the perpendicular when entering a medium with a higher index of refraction. (b) A ray of light moves away from the perpendicular when entering a medium with a lower index of refraction.

The amount that a light ray changes its direction depends both on the incident angle and the amount that the speed changes. For a ray at a given incident angle, a large change in speed causes a large change in direction and thus a large change in angle. The exact mathematical relationship is the law of refraction, or Snell's law, after the Dutch mathematician Willebrord Snell (1591–1626), who discovered it in 1621. The law of refraction is stated in equation form as

$$n_1 \sin \theta_1 = n_2 \sin \theta_2. \tag{11.6.1}$$

Here (n_1\) and n_2 are the indices of refraction for media 1 and 2, and θ_1 and θ_2 are the angles between the rays and the perpendicular in media 1 and 2. The incoming ray is called the incident ray, the outgoing ray is called the refracted ray, and the associated angles are the incident angle and the refracted angle, respectively.

Snell's experiments showed that the law of refraction is obeyed and that a characteristic index of refraction n could be assigned to a given medium and its value measured. Snell was not aware that the speed of light varied in different media, a key fact used when we derive the law of refraction theoretically using Huygens's Principle.

Example 11.6.1: Determining the Index of Refraction

Find the index of refraction for medium 2 in Figure 11.6.1a, assuming medium 1 is air and given that the incident angle is 30.0° and the angle of refraction is 22.0°.

Strategy

The index of refraction for air is taken to be 1 in most cases (and up to four significant figures, it is 1.000). Thus, $n_1 = 1.00$ here. From the given information, $\theta_1 = 30.0^\circ$ and $\theta_2 = 22.0^\circ$. With this information, the only unknown in Snell's law is n_2, so we can use Snell's law (Equation 11.6.1) to find it.

Solution

From Snell's law (Equation 11.6.1), we have

$$\begin{aligned} n_1 \sin\theta_1 &= n_2 \sin\theta_2 \\ n_2 &= n_1 \frac{\sin\theta_1}{\sin\theta_2}. \end{aligned}$$

Entering known values,

$$\begin{aligned} n_2 &= 1.00 \frac{\sin 30.0^\circ}{\sin 22.0^\circ} \\ &= \frac{0.500}{0.375} \\ &= 1.33. \end{aligned}$$

Significance

This is the index of refraction for water, and Snell could have determined it by measuring the angles and performing this calculation. He would then have found 1.33 to be the appropriate index of refraction for water in all other situations, such as when a ray passes from water to glass. Today, we can verify that the index of refraction is related to the speed of light in a medium by measuring that speed directly.

Explore bending of light between two media with different indices of refraction. Use the “Intro” simulation and see how changing from air to water to glass changes the bending angle. Use the protractor tool to measure the angles and see if you can recreate the configuration in Example 11.6.1. Also by measurement, confirm that the angle of reflection equals the angle of incidence.

Example 11.6.2: A Larger Change in Direction

Suppose that in a situation like that in Example 11.6.1, light goes from air to diamond and that the incident angle is 30.0°. Calculate the angle of refraction θ_2 in the diamond.

Strategy

Again, the index of refraction for air is taken to be $n_1=1.00$, and we are given $\theta_1=30.0°$. We can look up the index of refraction for diamond, finding $n_2=2.419$. The only unknown in Snell’s law is θ_2, which we wish to determine.

Solution

Solving Snell’s law (Equation 11.6.1) for $\sin\theta_2$ yields

$$\sin\theta_2 = \frac{n_1}{n_2}\sin\theta_1.$$

Entering known values,

$$\sin\theta_2 = \frac{1.00}{2.419}\sin 30.0^\circ = (0.413)(0.500) = 0.207.$$

The angle is thus

$$\theta_2 = \sin^{-1}(0.207) = 11.9^\circ.$$

Significance

For the same 30.0° angle of incidence, the angle of refraction in diamond is significantly smaller than in water (11.9° rather than 22.0°—see Example 11.6.2). This means there is a larger change in direction in diamond. The cause of a large change in direction is a large change in the index of refraction (or speed). In general, the larger the change in speed, the greater the effect on the direction of the ray.

Exercise 11.6.1: Zircon

The solid with the next highest index of refraction after diamond is zircon. If the diamond in Example 11.6.2 were replaced with a piece of zircon, what would be the new angle of refraction?

Answer

15.1°

11.6.1: Total Internal Reflection

Learning Objectives

By the end of this section, you will be able to:

- Explain the phenomenon of total internal reflection
- Describe the workings and uses of optical fibers
- Analyze the reason for the sparkle of diamonds

A good-quality mirror may reflect more than 90% of the light that falls on it, absorbing the rest. But it would be useful to have a mirror that reflects all of the light that falls on it. Interestingly, we can produce total reflection using an aspect of refraction.

Consider what happens when a ray of light strikes the surface between two materials, as shown in Figure $11.6.1.1a$ Part of the light crosses the boundary and is refracted; the rest is reflected. If, as shown in the figure, the index of refraction for the second medium is less than for the first, the ray bends away from the perpendicular. (Since $n_1 > n_2$, the angle of refraction is greater than the angle of incidence—that is, $\theta_1 > \theta_2$.) Now imagine what happens as the incident angle increases. This causes θ_2 to increase also. The largest the angle of refraction θ_2 can be is 90°, as shown in Figure $11.6.1.1b$

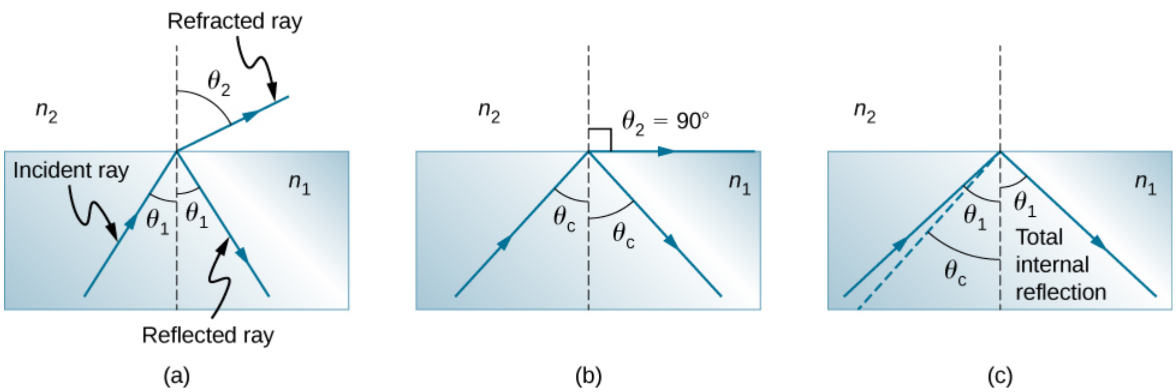


Figure 11.6.1.1: (a) A ray of light crosses a boundary where the index of refraction decreases. That is, $n_2 < n_1$. The ray bends away from the perpendicular. (b) The critical angle θ_c is the angle of incidence for which the angle of refraction is 90°. (c) Total internal reflection occurs when the incident angle is greater than the critical angle.

The **critical angle** θ_c for a combination of materials is defined to be the incident angle θ_1 that produces an angle of refraction of 90°. That is, θ_c is the incident angle for which $\theta_2 = 90^\circ$. If the incident angle θ_1 is greater than the critical angle, as shown in Figure $11.6.1.1c$ then all of the light is reflected back into medium 1, a condition called **total internal reflection**. (As Figure $11.6.1.1$ shows, the reflected rays obey the **law of reflection** so that the angle of reflection is equal to the angle of incidence in all three cases.)

Snell's law states the relationship between angles and indices of refraction. It is given by

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

When the incident angle equals the critical angle ($\theta_1 = \theta_c$), the angle of refraction is 90° ($\theta_2 = 90^\circ$). Noting that $\sin 90^\circ = 1$, Snell's law in this case becomes

$$n_1 \sin\,\theta_1 = n_2.$$

The critical angle θ_c for a given combination of materials is thus

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right) \tag{11.6.1.1}$$

for $n_1 > n_2$.

Total internal reflection occurs for any incident angle greater than the critical angle θ_c, and it can only occur when the second medium has an index of refraction less than the first. Note that this equation is written for a light ray that travels in medium 1 and reflects from medium 2, as shown in Figure $11.6.1.1$

Example 11.6.1.1: Determining a Critical Angle

What is the critical angle for light traveling in a polystyrene (a type of plastic) pipe surrounded by air? The index of refraction for polystyrene is 1.49.

Strategy

The index of refraction of air can be taken to be 1.00, as before. Thus, the condition that the second medium (air) has an index of refraction less than the first (plastic) is satisfied, and we can use the equation

$$\theta_c = \sin^{-1}\left(\frac{n_2}{n_1}\right)$$

to find the critical angle θ_c, where $n_2 = 1.00$ and $n_1 = 1.49$.

Solution

Substituting the identified values gives

$$\begin{aligned}\theta_c &= \sin^{-1}\left(\frac{1.00}{1.49}\right)\\ &= \sin^{-1}(0.671)\\ &= 42.2^\circ.\end{aligned}$$

Significance

This result means that any ray of light inside the plastic that strikes the surface at an angle greater than 42.2° is totally reflected. This makes the inside surface of the clear plastic a perfect mirror for such rays, without any need for the silvering used on common mirrors. Different combinations of materials have different critical angles, but any combination with $n_1 > n_2$ can produce total internal reflection. The same calculation as made here shows that the critical angle for a ray going from water to air is 48.6°, whereas that from diamond to air is 24.4°, and that from flint glass to crown glass is 66.3°.

Exercise 11.6.1.1

At the surface between air and water, light rays can go from air to water and from water to air. For which ray is there no possibility of total internal reflection?

Answer

air to water, because the condition that the second medium must have a smaller index of refraction is not satisfied

In the photo that opens this chapter, the image of a swimmer underwater is captured by a camera that is also underwater. The swimmer in the upper half of the photograph, apparently facing upward, is, in fact, a reflected image of the swimmer below. The circular ripple near the photograph's center is actually on the water surface. The undisturbed water surrounding it makes a good reflecting surface when viewed from below, thanks to total internal reflection. However, at the very top edge of this photograph, rays from below strike the surface with incident angles less than the critical angle, allowing the camera to capture a view of activities on the pool deck above water.

Fiber Optics: Endoscopes to Telephones

Fiber optics is one application of total internal reflection that is in wide use. In communications, it is used to transmit telephone, internet, and cable TV signals. Fiber optics employs the transmission of light down fibers of plastic or glass. Because the fibers are thin, light entering one is likely to strike the inside surface at an angle greater than the critical angle and, thus, be totally reflected (Figure 11.6.1.2). The index of refraction outside the fiber must be smaller than inside. In fact, most fibers have a varying refractive index to allow more light to be guided along the fiber through total internal refraction. Rays are reflected around corners as shown, making the fibers into tiny light pipes.

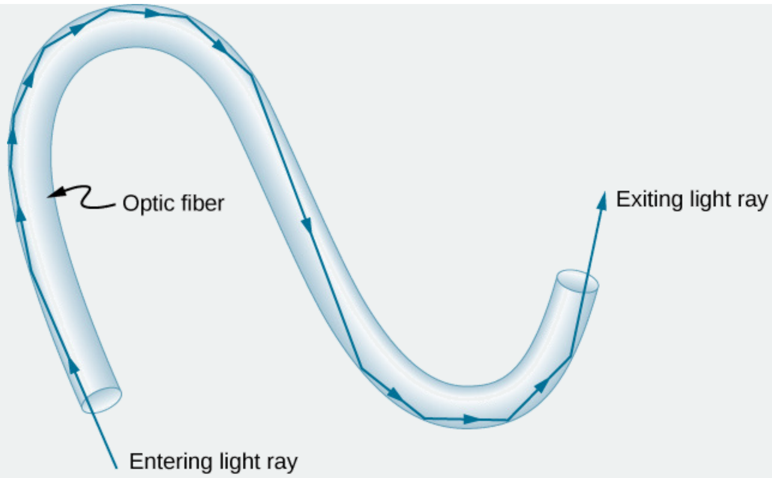


Figure 11.6.1.2: Light entering a thin optic fiber may strike the inside surface at large or grazing angles and is completely reflected if these angles exceed the critical angle. Such rays continue down the fiber, even following it around corners, since the angles of reflection and incidence remain large.

Bundles of fibers can be used to transmit an image without a lens, as illustrated in Figure 11.6.1.3. The output of a device called an endoscope is shown in Figure 11.6.1.1b. Endoscopes are used to explore the interior of the body through its natural orifices or minor incisions. Light is transmitted down one fiber bundle to illuminate internal parts, and the reflected light is transmitted back out through another bundle to be observed.

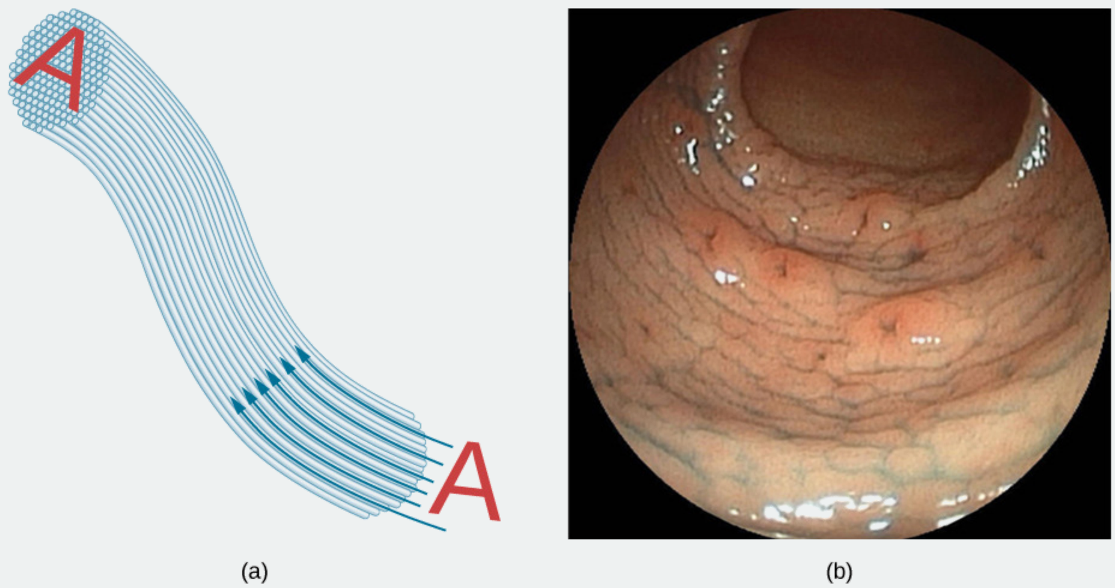


Figure 11.6.1.3: (a) An image "A" is transmitted by a bundle of optical fibers. (b) An endoscope is used to probe the body, both transmitting light to the interior and returning an image such as the one shown of a human epiglottis (a structure at the base of the tongue). (credit b: modification of work by "Med_Chaos"/Wikimedia Commons)

Fiber optics has revolutionized surgical techniques and observations within the body, with a host of medical diagnostic and therapeutic uses. Surgery can be performed, such as arthroscopic surgery on a knee or shoulder joint, employing cutting tools attached to and observed with the endoscope. Samples can also be obtained, such as by lassoing an intestinal polyp for external examination. The flexibility of the fiber optic bundle allows doctors to navigate it around small and difficult-to-reach regions in the body, such as the intestines, the heart, blood vessels, and joints. Transmission of an intense laser beam to burn away obstructing plaques in major arteries, as well as delivering light to activate chemotherapy drugs, are becoming commonplace. Optical fibers have in fact enabled microsurgery and remote surgery where the incisions are small and the surgeon's fingers do not need to touch the diseased tissue.

Optical fibers in bundles are surrounded by a **cladding** material that has a lower index of refraction than the core (Figure 11.6.1.4). The cladding prevents light from being transmitted between fibers in a bundle. Without cladding, light could pass between fibers in contact, since their indices of refraction are identical. Since no light gets into the cladding (there is total internal reflection back into the core), none can be transmitted between clad fibers that are in contact with one another. Instead, the light is propagated along the length of the fiber, minimizing the loss of signal and ensuring that a quality image is formed at the other end. The cladding and an additional protective layer make optical fibers durable as well as flexible.

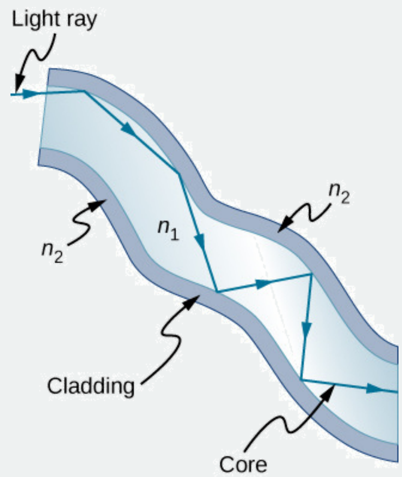


Figure 11.6.1.4: Fibers in bundles are clad by a material that has a lower index of refraction than the core to ensure total internal reflection, even when fibers are in contact with one another.

Special tiny lenses that can be attached to the ends of bundles of fibers have been designed and fabricated. Light emerging from a fiber bundle can be focused through such a lens, imaging a tiny spot. In some cases, the spot can be scanned, allowing quality imaging of a region inside the body. Special minute optical filters inserted at the end of the fiber bundle have the capacity to image the interior of organs located tens of microns below the surface without cutting the surface—an area known as nonintrusive diagnostics. This is particularly useful for determining the extent of cancers in the stomach and bowel.

In another type of application, optical fibers are commonly used to carry signals for telephone conversations and internet communications. Extensive optical fiber cables have been placed on the ocean floor and underground to enable optical communications. Optical fiber communication systems offer several advantages over electrical (copper)-based systems, particularly for long distances. The fibers can be made so transparent that light can travel many kilometers before it becomes dim enough to require amplification—much superior to copper conductors. This property of optical fibers is called low loss. Lasers emit light with characteristics that allow far more conversations in one fiber than are possible with electric signals on a single conductor. This property of optical fibers is called high bandwidth. Optical signals in one fiber do not produce undesirable effects in other adjacent fibers. This property of optical fibers is called reduced crosstalk. We shall explore the unique characteristics of laser radiation in a later chapter.

Corner Reflectors and Diamonds

Corner reflectors are perfectly efficient when the conditions for total internal reflection are satisfied. With common materials, it is easy to obtain a critical angle that is less than 45°. One use of these perfect mirrors is in binoculars, as shown in Figure 11.6.1.5. Another use is in periscopes found in submarines.

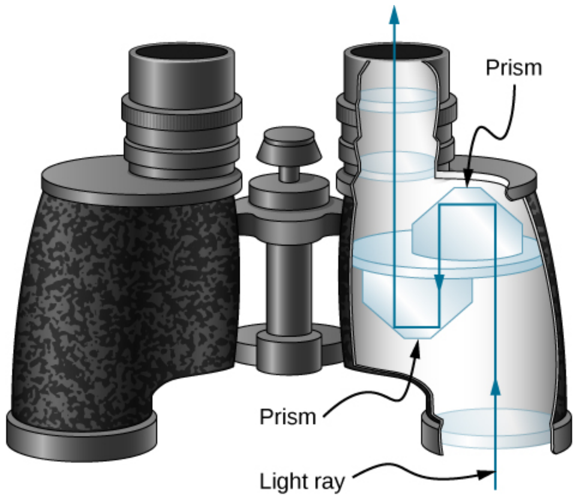


Figure 11.6.1.5: These binoculars employ corner reflectors (prisms) with total internal reflection to get light to the observer's eyes.

Total internal reflection, coupled with a large index of refraction, explains why **diamonds** sparkle more than other materials. The critical angle for a diamond-to-air surface is only **24.4°**, so when light enters a diamond, it has trouble getting back out (Figure 11.6.1.6). Although light freely enters the diamond, it can exit only if it makes an angle less than 24.4°. Facets on diamonds are specifically intended to make this unlikely. Good diamonds are very clear, so that the light makes many internal reflections and is concentrated before exiting—hence the bright sparkle. (Zircon is a natural gemstone that has an exceptionally large index of refraction, but it is not as large as diamond, so it is not as highly prized. Cubic zirconia is manufactured and has an even higher index of refraction (≈**2.17**), but it is still less than that of diamond.) The colors you see emerging from a clear diamond are not due to the diamond's color, which is usually nearly colorless, but result from dispersion. Colored diamonds get their color from structural defects of the crystal lattice and the inclusion of minute quantities of graphite and other materials. The Argyle Mine in Western Australia produces around 90% of the world's pink, red, champagne, and cognac diamonds, whereas around 50% of the world's clear diamonds come from central and southern Africa.

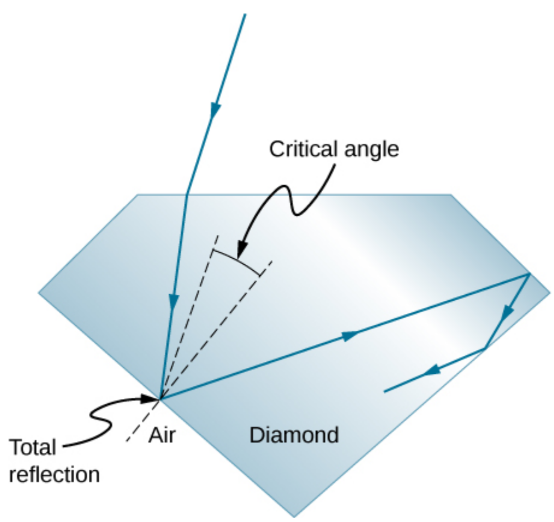


Figure 11.6.1.6: Light cannot easily escape a diamond, because its critical angle with air is so small. Most reflections are total, and the facets are placed so that light can exit only in particular ways—thus concentrating the light and making the diamond sparkle brightly.

Explore refraction and reflection of light between two media with different indices of refraction. Try to make the refracted ray disappear with total internal reflection. Use the protractor tool to measure the critical angle and compare with the prediction from Equation 11.6.1.1.

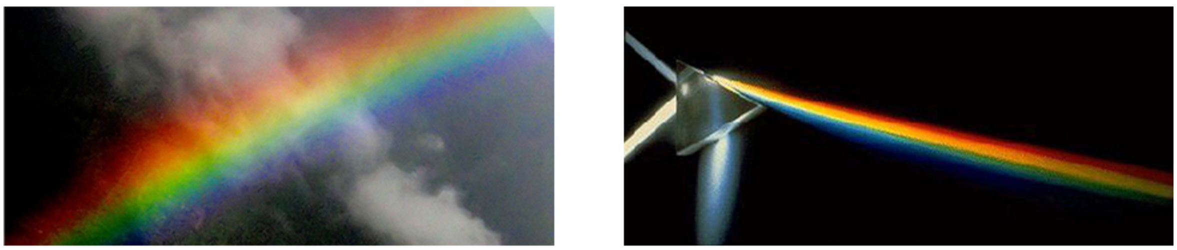
11.6.2: Dispersion

Learning Objectives

By the end of this section, you will be able to:

- Explain the cause of dispersion in a prism
- Describe the effects of dispersion in producing rainbows
- Summarize the advantages and disadvantages of dispersion

Everyone enjoys the spectacle of a rainbow glimmering against a dark stormy sky. How does sunlight falling on clear drops of rain get broken into the rainbow of colors we see? The same process causes white light to be broken into colors by a clear glass prism or a diamond (Figure $11.6.2.1$).



(a) (b)

Figure 11.6.2.1: The colors of the rainbow (a) and those produced by a prism (b) are identical. (credit a: modification of work by "Alfredo55"/Wikimedia Commons; credit b: modification of work by NASA)

We see about six colors in a rainbow—red, orange, yellow, green, blue, and violet; sometimes indigo is listed, too. These colors are associated with different wavelengths of light, as shown in Figure $11.6.2.2$ When our eye receives pure-wavelength light, we tend to see only one of the six colors, depending on wavelength. The thousands of other hues we can sense in other situations are our eye's response to various mixtures of wavelengths. White light, in particular, is a fairly uniform mixture of all visible wavelengths. Sunlight, considered to be white, actually appears to be a bit yellow, because of its mixture of wavelengths, but it does contain all visible wavelengths. The sequence of colors in rainbows is the same sequence as the colors shown in the figure. This implies that white light is spread out in a rainbow according to wavelength. Dispersion is defined as the spreading of white light into its full spectrum of wavelengths. More technically, dispersion occurs whenever the propagation of light depends on wavelength.

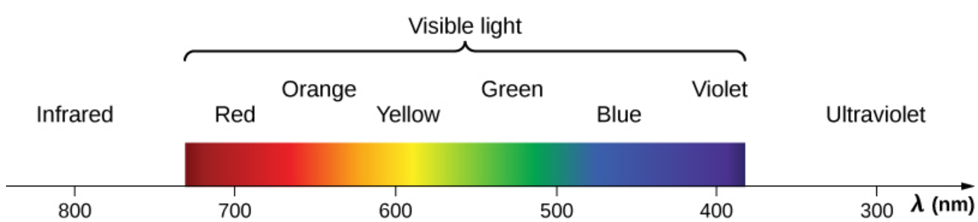


Figure 11.6.2.2: Even though rainbows are associated with six colors, the rainbow is a continuous distribution of colors according to wavelengths.

Any type of wave can exhibit dispersion. For example, sound waves, all types of electromagnetic waves, and water waves can be dispersed according to wavelength. Dispersion may require special circumstances and can result in spectacular displays such as in the production of a rainbow. This is also true for sound, since all frequencies ordinarily travel at the same speed. If you listen to sound through a long tube, such as a vacuum cleaner hose, you can easily hear it dispersed by interaction with the tube. Dispersion, in fact, can reveal a great deal about what the wave has encountered that disperses its wavelengths. The dispersion of electromagnetic radiation from outer space, for example, has revealed much about what exists between the stars—the so-called interstellar medium.

Nick Moore's video discusses dispersion of a pulse as he taps a long spring. Follow his explanation as Moore replays the high-speed footage showing high frequency waves outrunning the lower frequency waves. https://www.youtube.com/watch?v=KbmOcT5sX7I

Refraction is responsible for dispersion in rainbows and many other situations. The angle of refraction depends on the index of refraction, as we know from Snell's law. We know that the index of refraction **n** depends on the medium. But for a given medium, **n** also depends on wavelength (Table $11.6.2.1$).

Table 11.6.2.1: Index of Refraction (n) in Selected Media at Various Wavelengths

Medium	Red (660 nm)	Orange (610 nm)	Yellow (580 nm)	Green (550 nm)	Blue (470 nm)	Violet (410 nm)
Water	1.331	1.332	1.333	1.335	1.338	1.342
Diamond	2.410	2.415	2.417	2.426	2.444	2.458
Glass, crown	1.512	1.514	1.518	1.519	1.524	1.530
Glass, flint	1.662	1.665	1.667	1.674	1.684	1.698
Polystyrene	1.488	1.490	1.492	1.493	1.499	1.506
Quartz, fused	1.455	1.456	1.458	1.459	1.462	1.468

Note that for a given medium, **n** increases as wavelength decreases and is greatest for violet light. Thus, violet light is bent more than red light, as shown for a prism in Figure $11.6.2.3b$ White light is dispersed into the same sequence of wavelengths as seen in Figures $11.6.2.1$and $11.6.2.2$

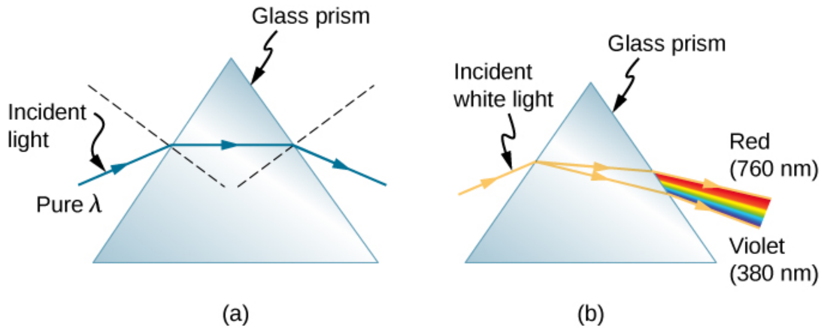
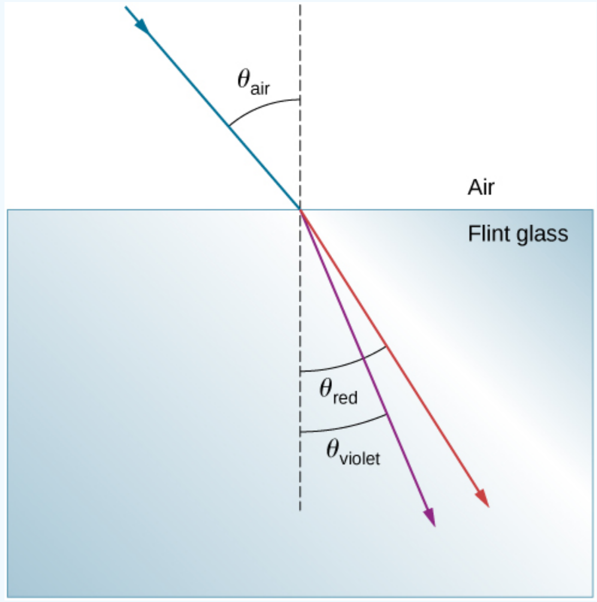


Figure 11.6.2.3: (a) A pure wavelength of light falls onto a prism and is refracted at both surfaces. (b) White light is dispersed by the prism (shown exaggerated). Since the index of refraction varies with wavelength, the angles of refraction vary with wavelength. A sequence of red to violet is produced, because the index of refraction increases steadily with decreasing wavelength.

Example $11.6.2.1$: Dispersion of White Light by Flint Glass

A beam of white light goes from air into flint glass at an incidence angle of 43.2°. What is the angle between the red (660 nm) and violet (410 nm) parts of the refracted light?



Strategy

Values for the indices of refraction for flint glass at various wavelengths are listed in Table 11.6.2.1. Use these values for calculate the angle of refraction for each color and then take the difference to find the dispersion angle.

Solution

Applying the law of refraction for the red part of the beam

$$n_{air}\sin\theta_{air} = n_{red}\sin\theta_{red},$$

we can solve for the angle of refraction as

$$\theta_{red} = \sin^{-1}\left(\frac{n_{air}\sin\theta_{air}}{n_{red}}\right) = \sin^{-1}\left[\frac{(1.000)\sin 43.2^\circ}{(1.512)}\right] = 27.0^\circ.$$

Similarly, the angle of incidence for the violet part of the beam is

$$\theta_{violet} = \sin^{-1}\left(\frac{n_{air}\, sin\theta_{air}}{n_{violet}}\right) = \sin^{-1}\left[\frac{(1.000)\sin 43.2^\circ}{(1.530)}\right] = 26.4^\circ.$$

The difference between these two angles is

$$\theta_{red} - \theta_{violet} = 27.0^\circ - 26.4^\circ = 0.6^\circ.$$

Significance

Although 0.6° may seem like a negligibly small angle, if this beam is allowed to propagate a long enough distance, the dispersion of colors becomes quite noticeable.

? Exercise 11.6.2.1

In the preceding example, how much distance inside the block of flint glass would the red and the violet rays have to progress before they are separated by 1.0 mm?

Answer

9.3 cm

Rainbows are produced by a combination of refraction and reflection. You may have noticed that you see a rainbow only when you look away from the Sun. Light enters a drop of water and is reflected from the back of the drop (Figure $11.6.2.4$).

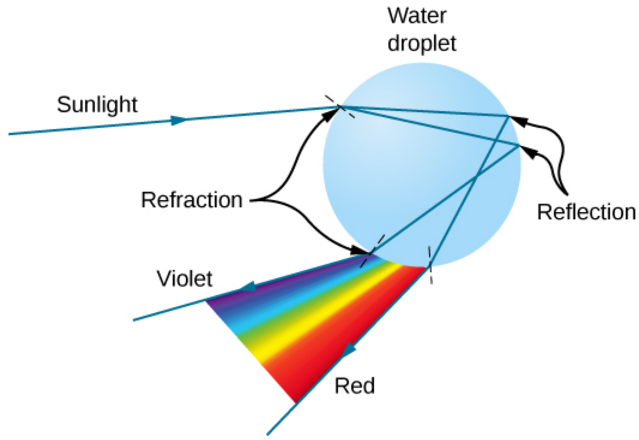


Figure 11.6.2.4: A ray of light falling on this water drop enters and is reflected from the back of the drop. This light is refracted and dispersed both as it enters and as it leaves the drop.

The light is refracted both as it enters and as it leaves the drop. Since the index of refraction of water varies with wavelength, the light is dispersed, and a rainbow is observed (Figure $11.6.2.4a$). (No dispersion occurs at the back surface, because the law of reflection does not depend on wavelength.) The actual rainbow of colors seen by an observer depends on the myriad rays being refracted and reflected toward the observer's eyes from numerous drops of water. The effect is most spectacular when the background is dark, as in stormy weather, but can also be observed in waterfalls and lawn sprinklers. The arc of a rainbow comes from the need to be looking at a specific angle relative to the direction of the Sun, as illustrated in Figure $11.6.2.4b$ If two reflections of light occur within the water drop, another "secondary" rainbow is produced. This rare event produces an arc that lies above the primary rainbow arc, as in Figure $11.6.2.4c$ and produces colors in the reverse order of the primary rainbow, with red at the lowest angle and violet at the largest angle.

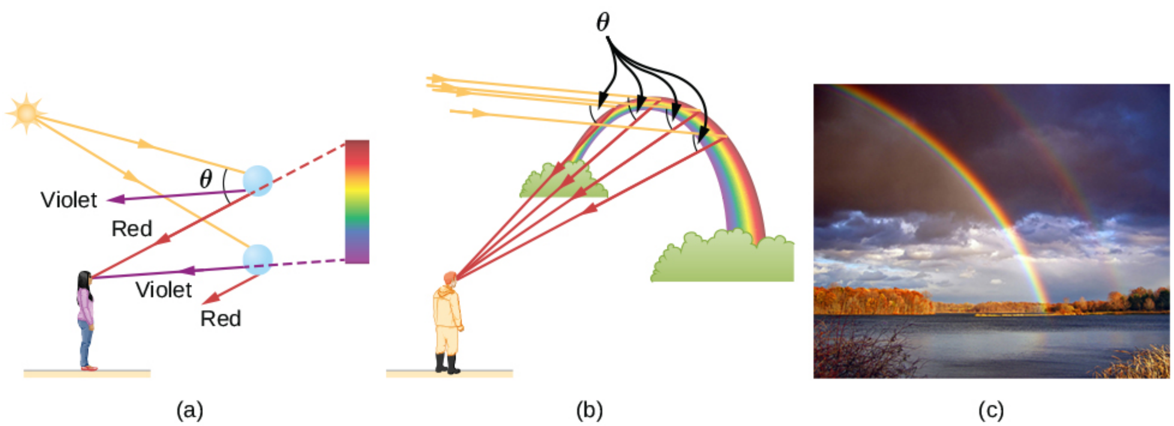


Figure 11.6.2.5: (a) Different colors emerge in different directions, and so you must look at different locations to see the various colors of a rainbow. (b) The arc of a rainbow results from the fact that a line between the observer and any point on the arc must make the correct angle with the parallel rays of sunlight for the observer to receive the refracted rays. (c) Double rainbow. (credit c: modification of work by "Nicholas"/Wikimedia Commons)

Dispersion may produce beautiful rainbows, but it can cause problems in optical systems. White light used to transmit messages in a fiber is dispersed, spreading out in time and eventually overlapping with other messages. Since a laser produces a nearly pure wavelength, its light experiences little dispersion, an advantage over white light for transmission of information. In contrast, dispersion of electromagnetic waves coming to us from outer space can be used to determine the amount of matter they pass through.

11.7: Polarization

Learning Objectives

By the end of this section, you will be able to:

- Explain the change in intensity as polarized light passes through a polarizing filter
- Calculate the effect of polarization by reflection and Brewster's angle
- Describe the effect of polarization by scattering
- Explain the use of polarizing materials in devices such as LCDs

Polarizing sunglasses are familiar to most of us. They have a special ability to cut the glare of light reflected from water or glass (Figure $11.7.1$). They have this ability because of a wave characteristic of light called polarization. What is polarization? How is it produced? What are some of its uses? The answers to these questions are related to the wave character of light.

(a) (b)

Figure 11.7.1: These two photographs of a river show the effect of a polarizing filter in reducing glare in light reflected from the surface of water. Part (b) of this figure was taken with a polarizing filter and part (a) was not. As a result, the reflection of clouds and sky observed in part (a) is not observed in part (b). Polarizing sunglasses are particularly useful on snow and water. (credit a and credit b: modifications of work by "Amithshs"/Wikimedia Commons)

Malus's Law

Light is one type of electromagnetic (EM) wave. EM waves are *transverse waves* consisting of varying electric and magnetic fields that oscillate perpendicular to the direction of propagation (Figure $11.7.2$). However, in general, there are no specific directions for the oscillations of the electric and magnetic fields; they vibrate in any randomly oriented plane perpendicular to the direction of propagation. Polarization is the attribute that a wave's oscillations do have a definite direction relative to the direction of propagation of the wave. (This is not the same type of polarization as that discussed for the separation of charges.) Waves having such a direction are said to be polarized. For an EM wave, we define the direction of polarization to be the direction parallel to the electric field. Thus, we can think of the electric field arrows as showing the direction of polarization, as in Figure $11.7.2$.

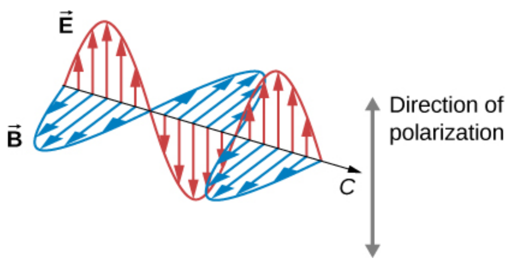


Figure 11.7.2: An EM wave, such as light, is a transverse wave. The electric $\vec{E}$ and magnetic $\vec{B}$ fields are perpendicular to the direction of propagation. The direction of polarization of the wave is the direction of the electric field.

To examine this further, consider the transverse waves in the ropes shown in Figure 11.7.3. The oscillations in one rope are in a vertical plane and are said to be vertically polarized. Those in the other rope are in a horizontal plane and are horizontally polarized. If a vertical slit is placed on the first rope, the waves pass through. However, a vertical slit blocks the horizontally polarized waves. For EM waves, the direction of the electric field is analogous to the disturbances on the ropes.

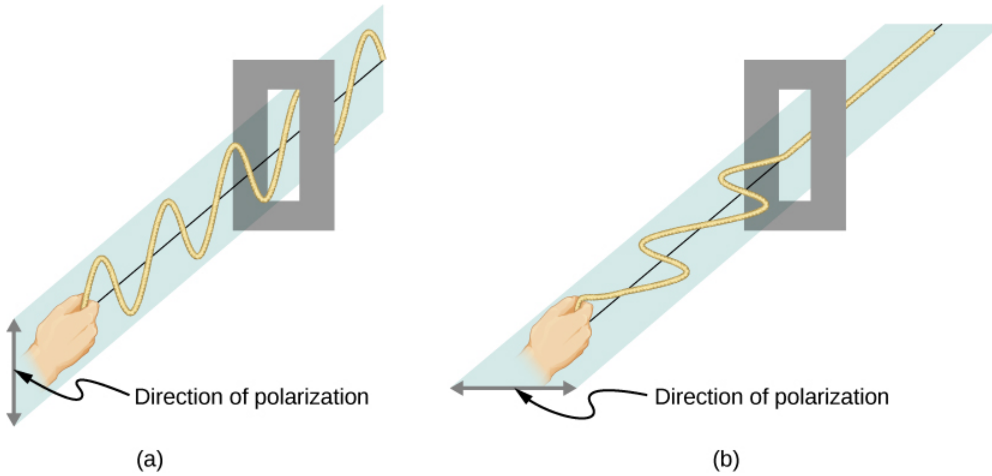


Figure 11.7.3: The transverse oscillations in one rope (a) are in a vertical plane, and those in the other rope (b) are in a horizontal plane. The first is said to be vertically polarized, and the other is said to be horizontally polarized. Vertical slits pass vertically polarized waves and block horizontally polarized waves.

The Sun and many other light sources produce waves that have the electric fields in random directions (Figure 11.7.1a). Such light is said to be unpolarized, because it is composed of many waves with all possible directions of polarization. Polaroid materials—which were invented by the founder of the Polaroid Corporation, Edwin Land—act as a polarizing slit for light, allowing only polarization in one direction to pass through. Polarizing filters are composed of long molecules aligned in one direction. If we think of the molecules as many slits, analogous to those for the oscillating ropes, we can understand why only light with a specific polarization can get through. The axis of a polarizing filter is the direction along which the filter passes the electric field of an EM wave.

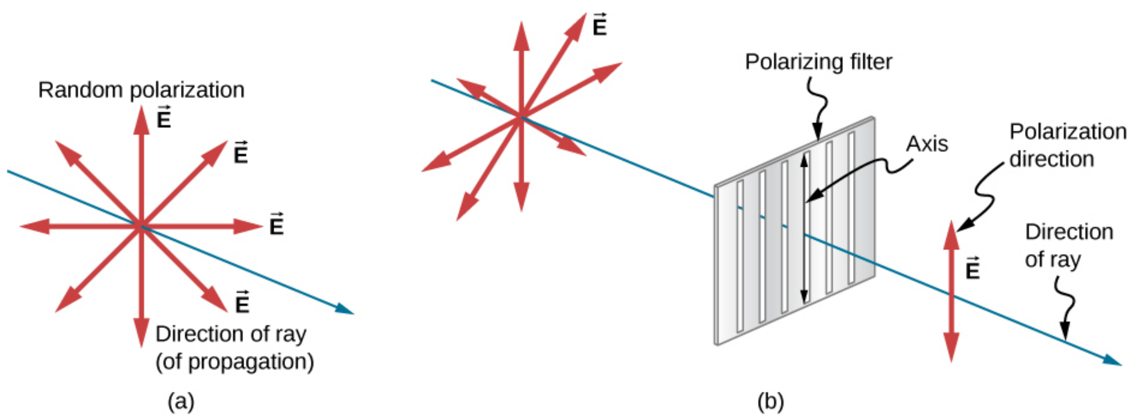


Figure 11.7.4: The slender arrow represents a ray of unpolarized light. The bold arrows represent the direction of polarization of the individual waves composing the ray. (a) If the light is unpolarized, the arrows point in all directions. (b) A polarizing filter has a polarization axis that acts as a slit passing through electric fields parallel to its direction. The direction of polarization of an EM wave is defined to be the direction of its electric field.

Figure 11.7.5 shows the effect of two polarizing filters on originally unpolarized light. The first filter polarizes the light along its axis. When the axes of the first and second filters are aligned (parallel), then all of the polarized light passed by the first filter is also passed by the second filter. If the second polarizing filter is rotated, only the component of the light parallel to the second filter's axis is passed. When the axes are perpendicular, no light is passed by the second filter.

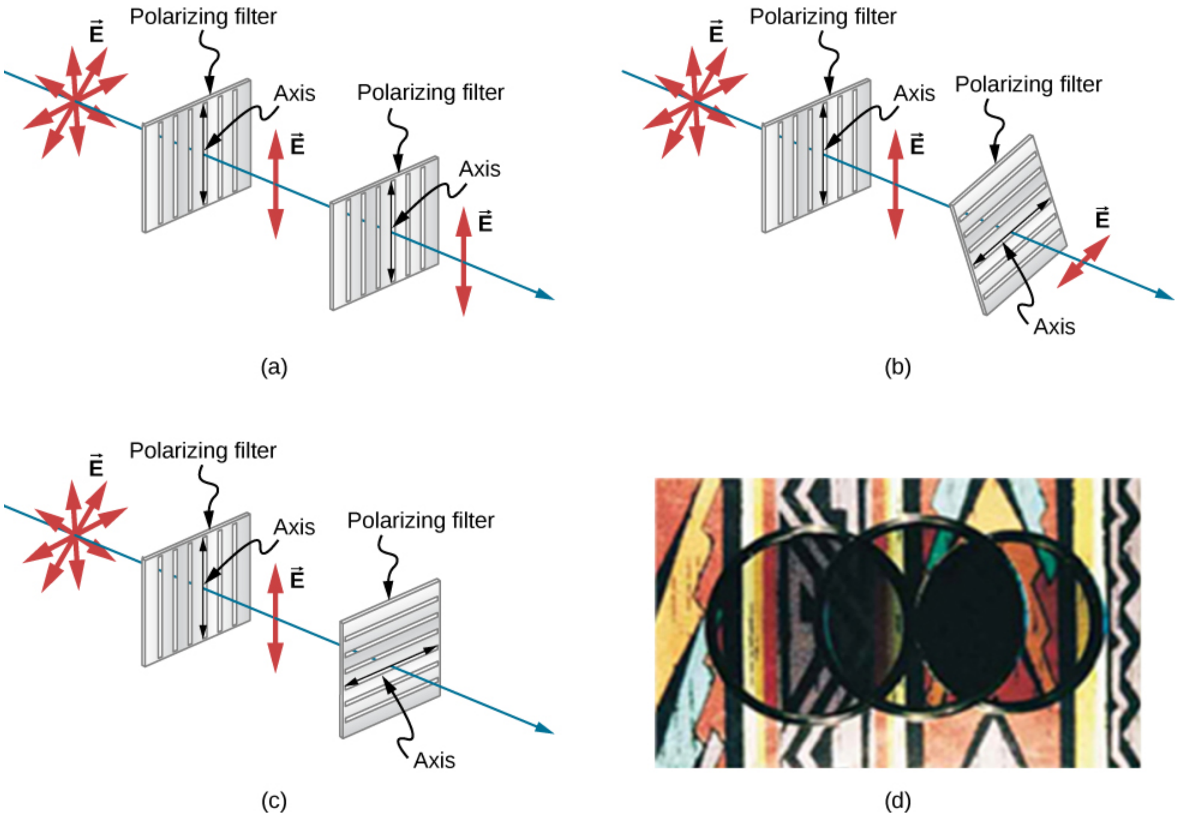


Figure 11.7.5: The effect of rotating two polarizing filters, where the first polarizes the light. (a) All of the polarized light is passed by the second polarizing filter, because its axis is parallel to the first. (b) As the second filter is rotated, only part of the light is passed. (c) When the second filter is perpendicular to the first, no light is passed. (d) In this photograph, a polarizing filter is placed above two others. Its axis is perpendicular to the filter on the right (dark area) and parallel to the filter on the left (lighter area). (credit d: modification of work by P.P. Urone)

Only the component of the EM wave parallel to the axis of a filter is passed. Let us call the angle between the direction of polarization and the axis of a filter θ. If the electric field has an amplitude **E**, then the transmitted part of the wave has an amplitude $E\cos\theta$ (Figure 11.7.6). Since the intensity of a wave is proportional to its amplitude squared, the intensity **I** of the transmitted wave is related to the incident wave by

$$I = I_0 \cos^2\theta$$

where I_0 is the intensity of the polarized wave before passing through the filter. This equation is known as *Malus's law*.

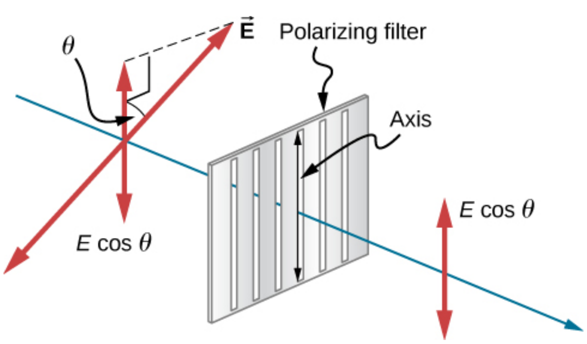


Figure 11.7.6: A polarizing filter transmits only the component of the wave parallel to its axis, reducing the intensity of any light not polarized parallel to its axis.

This Open Source Physics animation helps you visualize the electric field vectors as light encounters a polarizing filter. You can rotate the filter—note that the angle displayed is in radians. You can also rotate the animation for 3D visualization.

Example 11.7.1: Calculating Intensity Reduction by a Polarizing Filter

What angle is needed between the direction of polarized light and the axis of a polarizing filter to reduce its intensity by **90.0%**?

Strategy

When the intensity is reduced by **90.0%**, it is **10.0%** or 0.100 times its original value. That is, $\mathbf{I=0.100I_0}$. Using this information, the equation $\mathbf{I=I_0\cos^2\theta}$ can be used to solve for the needed angle.

Solution

Solving Malus's law (Equation ???) for $\cos\theta$ and substituting with the relationship between **I** and $\mathbf{I_0}$ gives

$$\cos\theta=\frac{I}{I_0}=\frac{0.100I_0}{I_0}=0.3162.$$

Solving for θ yields

$$\theta=\cos^{-1}0.3162=71.6^\circ.$$

Significance

A fairly large angle between the direction of polarization and the filter axis is needed to reduce the intensity to 10.0% of its original value. This seems reasonable based on experimenting with polarizing films. It is interesting that at an angle of 45°, the intensity is reduced to 50% of its original value. Note that 71.6° is 18.4° from reducing the intensity to zero, and that at an angle of 18.4°, the intensity is reduced to 90.0% of its original value, giving evidence of symmetry.

? Exercise 11.7.1

Although we did not specify the direction in Example 11.7.1, let's say the polarizing filter was rotated clockwise by 71.6° to reduce the light intensity by 90.0%. What would be the intensity reduction if the polarizing filter were rotated counterclockwise by 71.6°?

Answer

also 90.0%

Polarization by Reflection

By now, you can probably guess that polarizing sunglasses cut the glare in reflected light, because that light is polarized. You can check this for yourself by holding polarizing sunglasses in front of you and rotating them while looking at light reflected from water or glass. As you rotate the sunglasses, you will notice the light gets bright and dim, but not completely black. This implies the reflected light is partially polarized and cannot be completely blocked by a polarizing filter.

Figure 11.7.7 illustrates what happens when unpolarized light is reflected from a surface. Vertically polarized light is preferentially refracted at the surface, so the reflected light is left more horizontally polarized. The reasons for this phenomenon are beyond the scope of this text, but a convenient mnemonic for remembering this is to imagine the polarization direction to be like an arrow. Vertical polarization is like an arrow perpendicular to the surface and is more likely to stick and not be reflected. Horizontal polarization is like an arrow bouncing on its side and is more likely to be reflected. Sunglasses with vertical axes thus block more reflected light than unpolarized light from other sources.

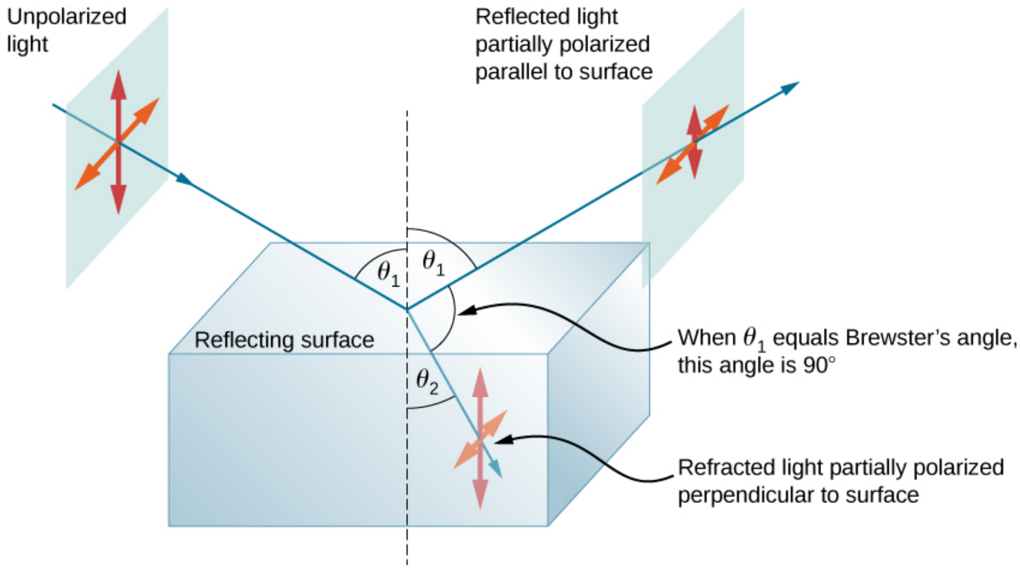


Figure 11.7.7: Polarization by reflection. Unpolarized light has equal amounts of vertical and horizontal polarization. After interaction with a surface, the vertical components are preferentially absorbed or refracted, leaving the reflected light more horizontally polarized. This is akin to arrows striking on their sides and bouncing off, whereas arrows striking on their tips go into the surface.

Since the part of the light that is not reflected is refracted, the amount of polarization depends on the indices of refraction of the media involved. It can be shown that reflected light is completely polarized at an angle of reflection θ_b given by

$$tan\,\theta_b = \frac{n_2}{n_1}$$

where n_1 is the medium in which the incident and reflected light travel and n_2 is the index of refraction of the medium that forms the interface that reflects the light. This equation is known as Brewster's law and θ_b is known as Brewster's angle, named after the nineteenth-century Scottish physicist who discovered them.

This Open Source Physics animation shows incident, reflected, and refracted light as rays and EM waves. Try rotating the animation for 3D visualization and also change the angle of incidence. Near Brewster's angle, the reflected light becomes highly polarized.

Example 11.7.2: Calculating Polarization by Reflection

(a) At what angle will light traveling in air be completely polarized horizontally when reflected from water? (b) From glass?

Strategy

All we need to solve these problems are the indices of refraction. Air has n_1=1.00, water has n_2=1.333, and crown glass has n'_2=1.520. The equation $tan\,\theta_b = \frac{n_2}{n_1}$ can be directly applied to find θ_b in each case.

Solution

a. Putting the known quantities into the equation

$$\tan\,\theta_b = \frac{n_2}{n_1}$$

gives

$$\tan\,\theta_b = \frac{n_2}{n_1} = \frac{1.333}{1.00} = 1.333.$$

Solving for the angle θ_b yields

$$\theta_b = tan^{-1} 1.333 = 53.1^\circ.$$

b. Similarly, for crown glass and air,

$$tan\,\theta'_b = \frac{n'_2}{n_1} = \frac{1.520}{1.00} = 1.52.$$

Thus,

$$\theta'_b = tan^{-1}1.52 = 56.7^{\circ}.$$

Significance

Light reflected at these angles could be completely blocked by a good polarizing filter held with its axis vertical. Brewster's angle for water and air are similar to those for glass and air, so that sunglasses are equally effective for light reflected from either water or glass under similar circumstances. Light that is not reflected is refracted into these media. Therefore, at an incident angle equal to Brewster's angle, the refracted light is slightly polarized vertically. It is not completely polarized vertically, because only a small fraction of the incident light is reflected, so a significant amount of horizontally polarized light is refracted.

? Exercise 11.7.2

What happens at Brewster's angle if the original incident light is already 100% vertically polarized?

Answer

There will be only refraction but no reflection.

Atomic Explanation of Polarizing Filters

Polarizing filters have a polarization axis that acts as a slit. This slit passes EM waves (often visible light) that have an electric field parallel to the axis. This is accomplished with long molecules aligned perpendicular to the axis, as shown in Figure 11.7.8.

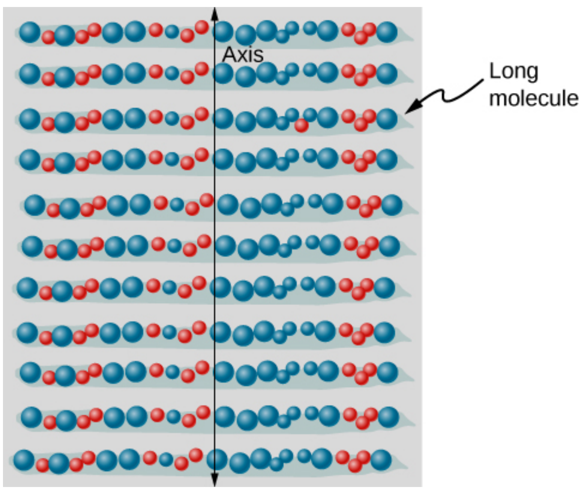


Figure 11.7.8: Long molecules are aligned perpendicular to the axis of a polarizing filter. In an EM wave, the component of the electric field perpendicular to these molecules passes through the filter, whereas the component parallel to the molecules is absorbed.

Figure 11.7.9 illustrates how the component of the electric field parallel to the long molecules is absorbed. An EM wave is composed of oscillating electric and magnetic fields. The electric field is strong compared with the magnetic field and is more effective in exerting force on charges in the molecules. The most affected charged particles are the electrons, since electron masses are small. If an electron is forced to oscillate, it can absorb energy from the EM wave. This reduces the field in the wave and, hence, reduces its intensity. In long molecules, electrons can more easily oscillate parallel to the molecule than in the perpendicular direction. The electrons are bound to the molecule and are more restricted in their movement perpendicular to the molecule. Thus, the electrons can absorb EM waves that have a component of their electric field parallel to the molecule. The electrons are much less responsive to electric fields perpendicular to the molecule and allow these fields to pass. Thus, the axis of the polarizing filter is perpendicular to the length of the molecule.

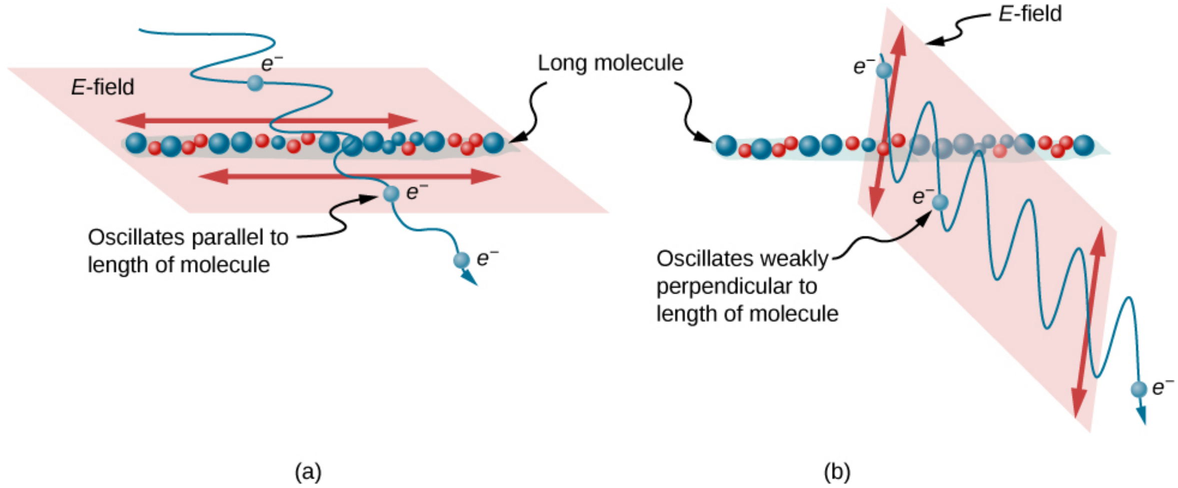


Figure 11.7.9: Diagram of an electron in a long molecule oscillating parallel to the molecule. The oscillation of the electron absorbs energy and reduces the intensity of the component of the EM wave that is parallel to the molecule.

Polarization by Scattering

If you hold your polarizing sunglasses in front of you and rotate them while looking at blue sky, you will see the sky get bright and dim. This is a clear indication that light scattered by air is partially polarized. Figure 11.7.10 helps illustrate how this happens. Since light is a transverse EM wave, it vibrates the electrons of air molecules perpendicular to the direction that it is traveling. The electrons then radiate like small antennae. Since they are oscillating perpendicular to the direction of the light ray, they produce EM radiation that is polarized perpendicular to the direction of the ray. When viewing the light along a line perpendicular to the original ray, as in the figure, there can be no polarization in the scattered light parallel to the original ray, because that would require the original ray to be a longitudinal wave. Along other directions, a component of the other polarization can be projected along the line of sight, and the scattered light is only partially polarized. Furthermore, multiple scattering can bring light to your eyes from other directions and can contain different polarizations.

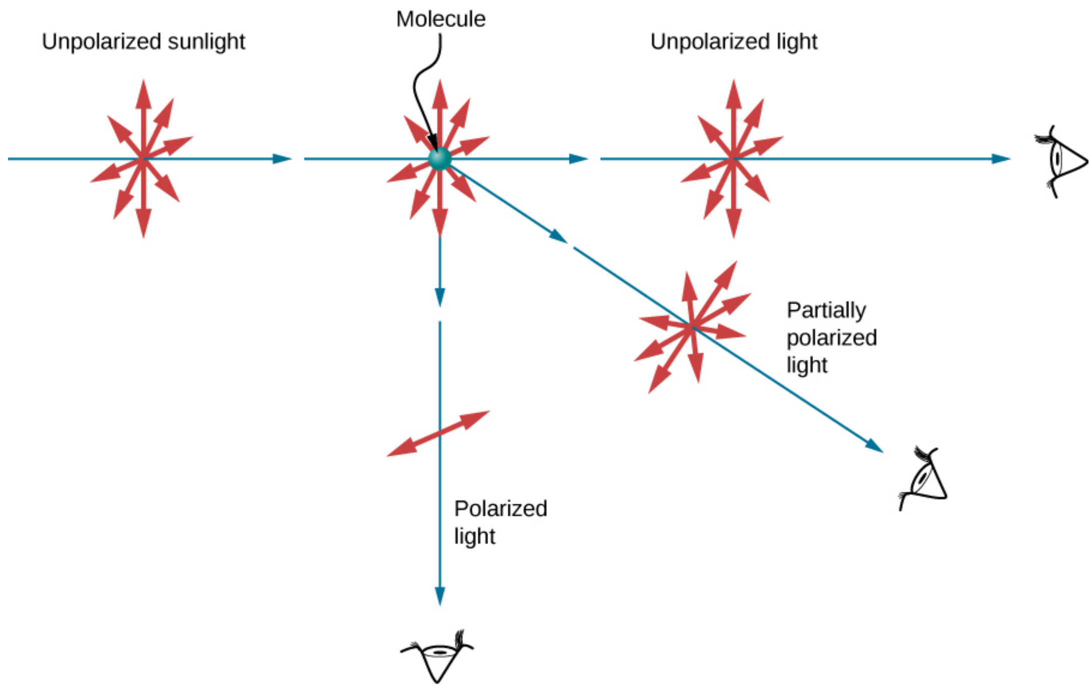


Figure 11.7.10: Polarization by scattering. Unpolarized light scattering from air molecules shakes their electrons perpendicular to the direction of the original ray. The scattered light therefore has a polarization perpendicular to the original direction and none parallel to the original direction.

Photographs of the sky can be darkened by polarizing filters, a trick used by many photographers to make clouds brighter by contrast. Scattering from other particles, such as smoke or dust, can also polarize light. Detecting polarization in scattered EM waves can be a useful analytical tool in determining the scattering source.

A range of optical effects are used in sunglasses. Besides being polarizing, sunglasses may have colored pigments embedded in them, whereas others use either a nonreflective or reflective coating. A recent development is photochromic lenses, which darken in the sunlight and become clear indoors. Photochromic lenses are embedded with organic microcrystalline molecules that change their properties when exposed to UV in sunlight, but become clear in artificial lighting with no UV.

Liquid Crystals and Other Polarization Effects in Materials

Although you are undoubtedly aware of liquid crystal displays (LCDs) found in watches, calculators, computer screens, cellphones, flat screen televisions, and many other places, you may not be aware that they are based on polarization. Liquid crystals are so named because their molecules can be aligned even though they are in a liquid. Liquid crystals have the property that they can rotate the polarization of light passing through them by 90°. Furthermore, this property can be turned off by the application of a voltage, as illustrated in Figure 11.7.11. It is possible to manipulate this characteristic quickly and in small, well-defined regions to create the contrast patterns we see in so many LCD devices.

In flat screen LCD televisions, a large light is generated at the back of the TV. The light travels to the front screen through millions of tiny units called pixels (picture elements). One of these is shown in Figure 11.7.11. Each unit has three cells, with red, blue, or green filters, each controlled independently. When the voltage across a liquid crystal is switched off, the liquid crystal passes the light through the particular filter. We can vary the picture contrast by varying the strength of the voltage applied to the liquid crystal.

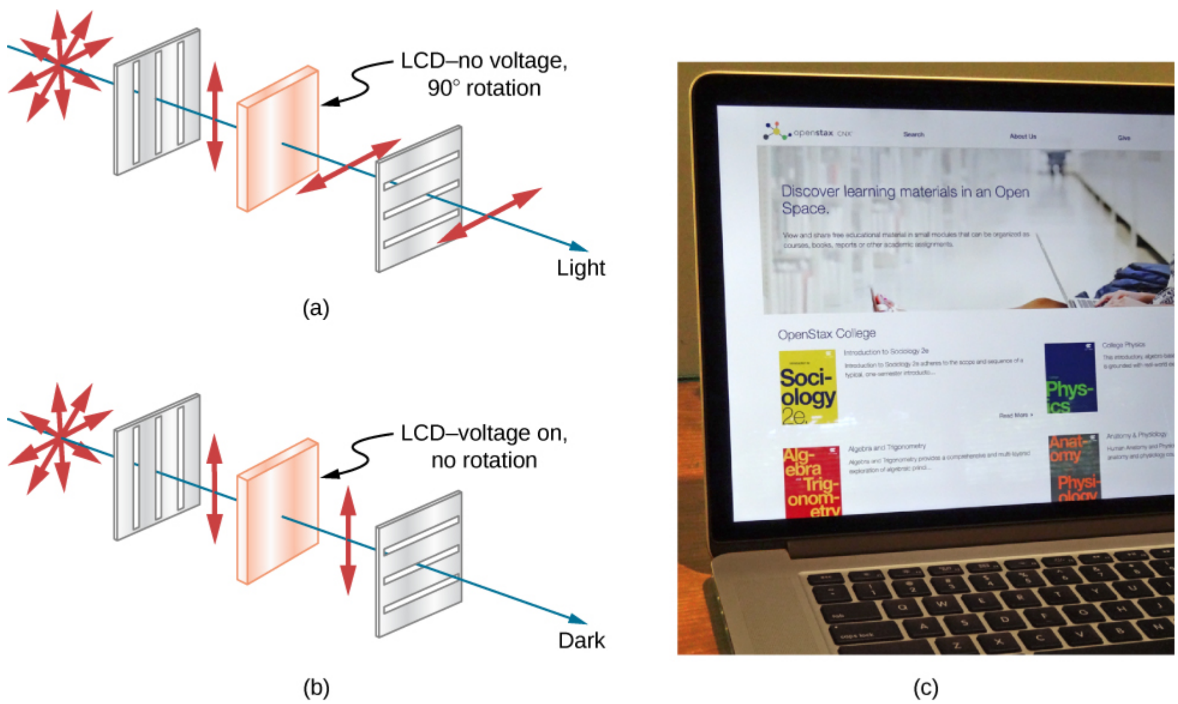


Figure 11.7.11: (a) Polarized light is rotated 90° by a liquid crystal and then passed by a polarizing filter that has its axis perpendicular to the direction of the original polarization. (b) When a voltage is applied to the liquid crystal, the polarized light is not rotated and is blocked by the filter, making the region dark in comparison with its surroundings. (c) LCDs can be made color specific, small, and fast enough to use in laptop computers and TVs.

Many crystals and solutions rotate the plane of polarization of light passing through them. Such substances are said to be optically active. Examples include sugar water, insulin, and collagen (Figure 11.7.11). In addition to depending on the type of substance, the amount and direction of rotation depend on several other factors. Among these is the concentration of the substance, the distance the light travels through it, and the wavelength of light. Optical activity is due to the asymmetrical shape of molecules in the substance, such as being helical. Measurements of the rotation of polarized light passing through substances can thus be used to measure concentrations, a standard technique for sugars. It can also give information on the shapes of molecules, such as proteins, and factors that affect their shapes, such as temperature and pH.

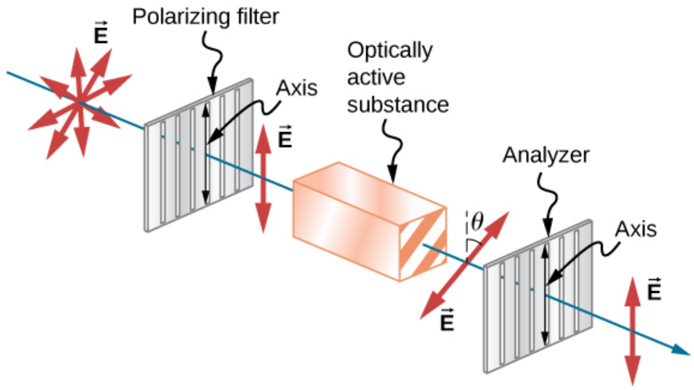


Figure 11.7.11. Optical activity is the ability of some substances to rotate the plane of polarization of light passing through them. The rotation is detected with a polarizing filter or analyzer.

Glass and plastic become optically active when stressed: the greater the stress, the greater the effect. Optical stress analysis on complicated shapes can be performed by making plastic models of them and observing them through crossed filters, as seen in Figure $11.7.12$ It is apparent that the effect depends on wavelength as well as stress. The wavelength dependence is sometimes also used for artistic purposes.

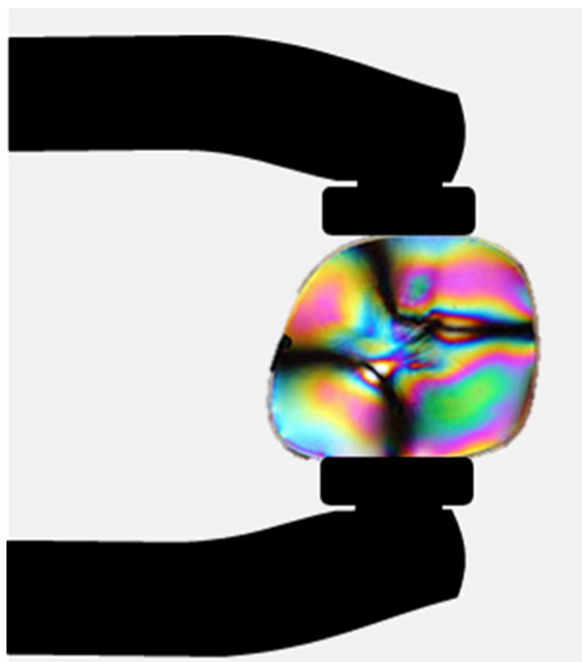


Figure 11.7.13: Optical stress analysis of a plastic lens placed between crossed polarizers. (credit: “Infopro”/Wikimedia Commons)

Another interesting phenomenon associated with polarized light is the ability of some crystals to split an unpolarized beam of light into two polarized beams. This occurs because the crystal has one value for the index of refraction of polarized light but a different value for the index of refraction of light polarized in the perpendicular direction, so that each component has its own angle of refraction. Such crystals are said to be birefringent, and, when aligned properly, two perpendicularly polarized beams will emerge from the crystal (Figure $11.7.14$). Birefringent crystals can be used to produce polarized beams from unpolarized light. Some birefringent materials preferentially absorb one of the polarizations. These materials are called dichroic and can produce polarization by this preferential absorption. This is fundamentally how polarizing filters and other polarizers work.

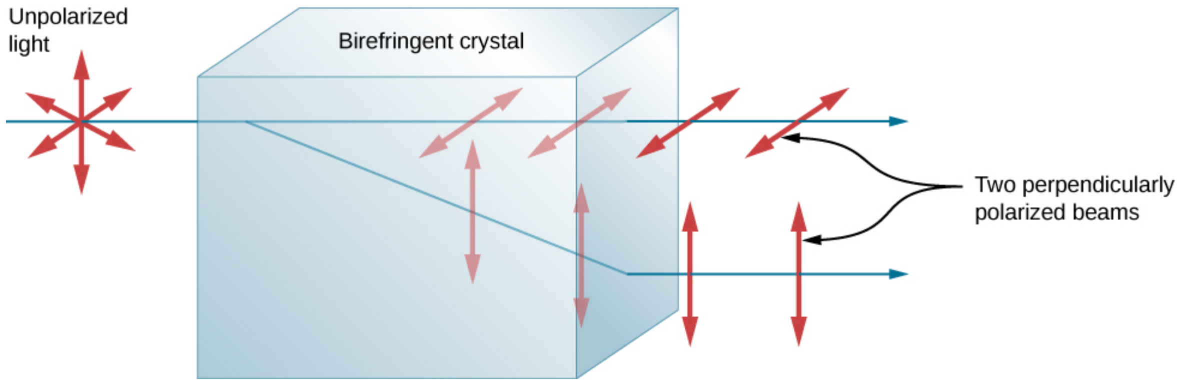


Figure 11.7.14: Birefringent materials, such as the common mineral calcite, split unpolarized beams of light into two with two different values of index of refraction.

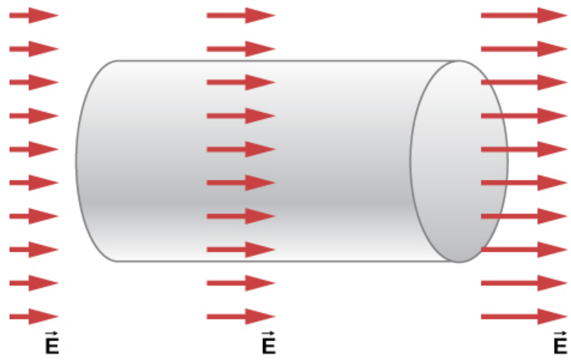
11.8: Electromagnetic Waves (Exercises)

Conceptual Questions

16.2 Maxwell's Equations and Electromagnetic Waves

1. Explain how the displacement current maintains the continuity of current in a circuit containing a capacitor.

2. Describe the field lines of the induced magnetic field along the edge of the imaginary horizontal cylinder shown below if the cylinder is in a spatially uniform electric field that is horizontal, pointing to the right, and increasing in magnitude.

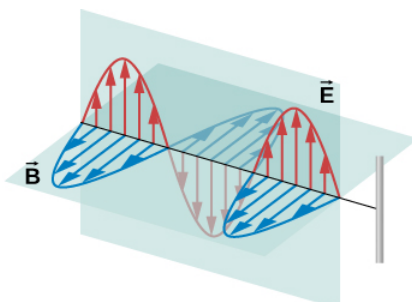


3. Why is it much easier to demonstrate in a student lab that a changing magnetic field induces an electric field than it is to demonstrate that a changing electric field produces a magnetic field?

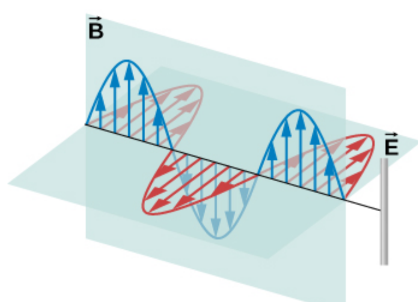
16.3 Plane Electromagnetic Waves

4. If the electric field of an electromagnetic wave is oscillating along the z-axis and the magnetic field is oscillating along the x-axis, in what possible direction is the wave traveling?

5. In which situation shown below will the electromagnetic wave be more successful in inducing a current in the wire? Explain.

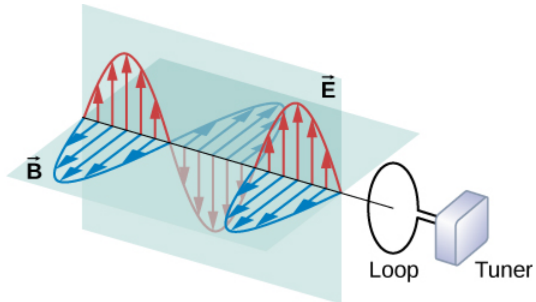


(a)

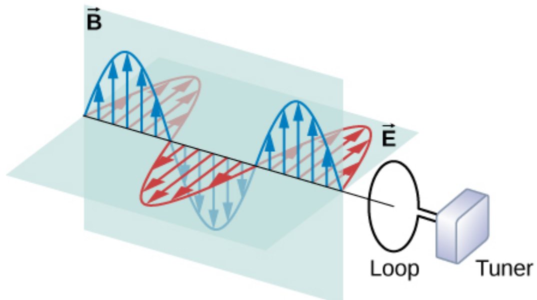


(b)

6. In which situation shown below will the electromagnetic wave be more successful in inducing a current in the loop? Explain.



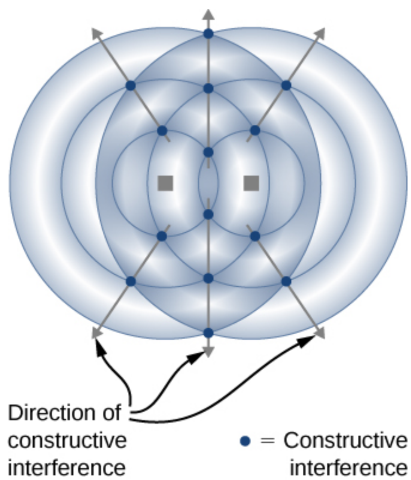
(a)



(b)

7. Under what conditions might wires in a circuit where the current flows in only one direction emit electromagnetic waves?

8. Shown below is the interference pattern of two radio antennas broadcasting the same signal. Explain how this is analogous to the interference pattern for sound produced by two speakers. Could this be used to make a directional antenna system that broadcasts preferentially in certain directions? Explain.



16.4 Energy Carried by Electromagnetic Waves

9. When you stand outdoors in the sunlight, why can you feel the energy that the sunlight carries, but not the momentum it carries?

10. How does the intensity of an electromagnetic wave depend on its electric field? How does it depend on its magnetic field?

11. What is the physical significance of the Poynting vector?

12. A 2.0-mW helium-neon laser transmits a continuous beam of red light of cross-sectional area $0.25 cm^2$. If the beam does not diverge appreciably, how would its rms electric field vary with distance from the laser? Explain.

16.5 Momentum and Radiation Pressure

13. Why is the radiation pressure of an electromagnetic wave on a perfectly reflecting surface twice as large as the pressure on a perfectly absorbing surface?

14. Why did the early Hubble Telescope photos of Comet Ison approaching Earth show it to have merely a fuzzy coma around it, and not the pronounced double tail that developed later (see below)?

(credit: ESA, Hubble)

15. (a) If the electric field and magnetic field in a sinusoidal plane wave were interchanged, in which direction relative to before would the energy propagate?

(b) What if the electric and the magnetic fields were both changed to their negatives?

16.6 The Electromagnetic Spectrum

16. Compare the speed, wavelength, and frequency of radio waves and X-rays traveling in a vacuum.

17. Accelerating electric charge emits electromagnetic radiation. How does this apply in each case: (a) radio waves, (b) infrared radiation.

18. Compare and contrast the meaning of the prefix “micro” in the names of SI units in the term microwaves.

19. Part of the light passing through the air is scattered in all directions by the molecules comprising the atmosphere. The wavelengths of visible light are larger than molecular sizes, and the scattering is strongest for wavelengths of light closest to sizes of molecules.

(a) Which of the main colors of light is scattered the most?

(b) Explain why this would give the sky its familiar background color at midday.

20. When a bowl of soup is removed from a microwave oven, the soup is found to be steaming hot, whereas the bowl is only warm to the touch. Discuss the temperature changes that have occurred in terms of energy transfer.

21. Certain orientations of a broadcast television antenna give better reception than others for a particular station. Explain.

22. What property of light corresponds to loudness in sound?

23. Is the visible region a major portion of the electromagnetic spectrum?

24. Can the human body detect electromagnetic radiation that is outside the visible region of the spectrum?

25. Radio waves normally have their **E** and **B** fields in specific directions, whereas visible light usually has its **E** and **B** fields in random and rapidly changing directions that are perpendicular to each other and to the propagation direction. Can you explain why?

26. Give an example of resonance in the reception of electromagnetic waves.

27. Illustrate that the size of details of an object that can be detected with electromagnetic waves is related to their wavelength, by comparing details observable with two different types (for example, radar and visible light).

28. In which part of the electromagnetic spectrum are each of these waves:

(a) $f = 10.0 kHz$,

(b) $f = \lambda = 750 nm$,

(c) $f = 1.25 \times 10^8 Hz$,

(d) $0.30 nm$

29. In what range of electromagnetic radiation are the electromagnetic waves emitted by power lines in a country that uses 50-Hz ac current?

30. If a microwave oven could be modified to merely tune the waves generated to be in the infrared range instead of using microwaves, how would this affect the uneven heating of the oven?

31. A leaky microwave oven in a home can sometimes cause interference with the homeowner's WiFi system. Why?

32. When a television news anchor in a studio speaks to a reporter in a distant country, there is sometimes a noticeable lag between when the anchor speaks in the studio and when the remote reporter hears it and replies. Explain what causes this delay.

Problems

16.2 Maxwell's Equations and Electromagnetic Waves

33. Show that the magnetic field at a distance **r** from the axis of two circular parallel plates, produced by placing charge **Q(t)** on the plates is

$$B_{ind} = \frac{\mu_0}{2\pi r} \frac{dQ(t)}{dt}$$

34. Express the displacement current in a capacitor in terms of the capacitance and the rate of change of the voltage across the capacitor.

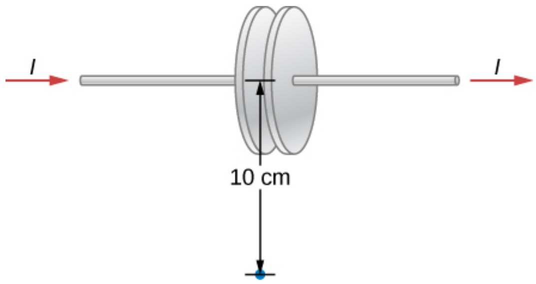
35. A potential difference $V(t) = V_0 sin\omega t$ is maintained across a parallel-plate capacitor with capacitance **C** consisting of two circular parallel plates. A thin wire with resistance **R** connects the centers of the two plates, allowing charge to leak between plates while they are charging.

(a) Obtain expressions for the leakage current $I_{res}(t)$ in the thin wire. Use these results to obtain an expression for the current $I_{real}(t)$ in the wires connected to the capacitor.

(b) Find the displacement current in the space between the plates from the changing electric field between the plates.

(c) Compare $I_{real}(t)$ with the sum of the displacement current $I_d(t)$ and resistor current $I_{res}(t)$ between the plates, and explain why the relationship you observe would be expected.

36. Suppose the parallel-plate capacitor shown below is accumulating charge at a rate of 0.010 C/s. What is the induced magnetic field at a distance of 10 cm from the capacitator?



37. The potential difference **V(t)** between parallel plates shown above is instantaneously increasing at a rate of $10^7 V/s$. What is the displacement current between the plates if the separation of the plates is 1.00 cm and they have an area of $0.200m^2$?

38. A parallel-plate capacitor has a plate area of $A=0.250m^2$ and a separation of 0.0100 m. What must be must be the angular frequency ω for a voltage $V(t)=V_0 sin\omega t$ with $V_0=100V$ to produce a maximum displacement induced current of 1.00 A between the plates?

39. The voltage across a parallel-plate capacitor with area $A=800cm^2$ and separation $d=2mm$ varies sinusoidally as $V=(15mV)cos(150t)$, where **t** is in seconds. Find the displacement current between the plates.

40. The voltage across a parallel-plate capacitor with area **A** and separation **d** varies with time **t** as $V=at^2$, where a is a constant. Find the displacement current between the plates.

16.3 Plane Electromagnetic Waves

41. If the Sun suddenly turned off, we would not know it until its light stopped coming. How long would that be, given that the Sun is $1.496\times 10^{11}m$ away?

42. What is the maximum electric field strength in an electromagnetic wave that has a maximum magnetic field strength of $5.00\times 10^{-4}T$ (about 10 times Earth's magnetic field)?

43. An electromagnetic wave has a frequency of 12 MHz. What is its wavelength in vacuum?

44. If electric and magnetic field strengths vary sinusoidally in time at frequency 1.00 GHz, being zero at $t=0$, then $E=E_0 sin 2\pi ft$ and $B=B_0 sin 2\pi ft$.

(a) When are the field strengths next equal to zero?

(b) When do they reach their most negative value? (c) How much time is needed for them to complete one cycle?

45. The electric field of an electromagnetic wave traveling in vacuum is described by the following wave function:

$$\vec{E}=(5.00V/m)cos[kx-(6.00\times 10^9 s^{-1})t+0.40]\hat{j}$$

where **k** is the wavenumber in rad/m, **x** is in m, **t** is in s.

Find the following quantities:

(a) amplitude

(b) frequency

(c) wavelength

(d) the direction of the travel of the wave

(e) the associated magnetic field wave

46. A plane electromagnetic wave of frequency 20 GHz moves in the positive y-axis direction such that its electric field is pointed along the z-axis. The amplitude of the electric field is 10 V/m. The start of time is chosen so that at $t=0$, the electric field has a value 10 V/m at the origin.

(a) Write the wave function that will describe the electric field wave.

(b) Find the wave function that will describe the associated magnetic field wave.

47. The following represents an electromagnetic wave traveling in the direction of the positive **y**-axis:

$$E_x=0; E_y=E_0 cos(kx-\omega t); E_z=0$$

$$B_x=0; B_y=0; B_z=B_0 cos(kx-\omega t)$$

The wave is passing through a wide tube of circular cross-section of radius **R** whose axis is along the **y**-axis. Find the expression for the displacement current through the tube.

48. While outdoors on a sunny day, a student holds a large convex lens of radius 4.0 cm above a sheet of paper to produce a bright spot on the paper that is 1.0 cm in radius, rather than a sharp focus. By what factor is the electric field in the bright spot of light related to the electric field of sunlight leaving the side of the lens facing the paper?

49. A plane electromagnetic wave travels northward. At one instant, its electric field has a magnitude of 6.0 V/m and points eastward. What are the magnitude and direction of the magnetic field at this instant?

50. The electric field of an electromagnetic wave is given by

$$E = (6.0 \times 10^{-3} V/m) sin[2\pi(\frac{x}{18m} - \frac{t}{6.0 \times 10^{-8} s})]\hat{j}\ .$$

Write the equations for the associated magnetic field and Poynting vector.

51. A radio station broadcasts at a frequency of 760 kHz. At a receiver some distance from the antenna, the maximum magnetic field of the electromagnetic wave detected is $2.15 \times 10^{-11} T$.

(a) What is the maximum electric field?

(b) What is the wavelength of the electromagnetic wave?

52. The filament in a clear incandescent light bulb radiates visible light at a power of 5.00 W. Model the glass part of the bulb as a sphere of radius $r_0 = 3.00cm$ and calculate the amount of electromagnetic energy from visible light inside the bulb.

53. At what distance does a 100-W lightbulb produce the same intensity of light as a 75-W lightbulb produces 10 m away? (Assume both have the same efficiency for converting electrical energy in the circuit into emitted electromagnetic energy.)

54. An incandescent light bulb emits only 2.6 W of its power as visible light. What is the rms electric field of the emitted light at a distance of 3.0 m from the bulb?

55. A 150-W lightbulb emits 5% of its energy as electromagnetic radiation. What is the magnitude of the average Poynting vector 10 m from the bulb?

56. A small helium-neon laser has a power output of 2.5 mW. What is the electromagnetic energy in a 1.0-m length of the beam?

57. At the top of Earth's atmosphere, the time-averaged Poynting vector associated with sunlight has a magnitude of about $1.4kW/m^2$.

(a) What are the maximum values of the electric and magnetic fields for a wave of this intensity?

(b) What is the total power radiated by the sun? Assume that the Earth is $1.5 \times 10^{11} m$ from the Sun and that sunlight is composed of electromagnetic plane waves.

58. The magnetic field of a plane electromagnetic wave moving along the z axis is given by

$\vec{B} = B_0(coskz + \omega t)\hat{j}$, where $B_0 = 5.00 \times 10^{-10} T$ and $k = 3.14 \times 10^{-2} m^{-1}$.

(a) Write an expression for the electric field associated with the wave.

(b) What are the frequency and the wavelength of the wave?

(c) What is its average Poynting vector?

59. What is the intensity of an electromagnetic wave with a peak electric field strength of 125 V/m?

60. Assume the helium-neon lasers commonly used in student physics laboratories have power outputs of 0.500 mW.

(a) If such a laser beam is projected onto a circular spot 1.00 mm in diameter, what is its intensity?

(b) Find the peak magnetic field strength.

(c) Find the peak electric field strength.

61. An AM radio transmitter broadcasts 50.0 kW of power uniformly in all directions. (a) Assuming all of the radio waves that strike the ground are completely absorbed, and that there is no absorption by the atmosphere or other objects, what is the intensity 30.0 km away? (**Hint:** Half the power will be spread over the area of a hemisphere.) (b) What is the maximum electric field strength at this distance?

62. Suppose the maximum safe intensity of microwaves for human exposure is taken to be $1.00 W/m^2$.

(a) If a radar unit leaks 10.0 W of microwaves (other than those sent by its antenna) uniformly in all directions, how far away must you be to be exposed to an intensity considered to be safe? Assume that the power spreads uniformly over the area of a sphere with no complications from absorption or reflection.

(b) What is the maximum electric field strength at the safe intensity? (Note that early radar units leaked more than modern ones do. This caused identifiable health problems, such as cataracts, for people who worked near them.)

63. A 2.50-m-diameter university communications satellite dish receives TV signals that have a maximum electric field strength (for one channel) of **7.50μV/m** (see below). (a) What is the intensity of this wave? (b) What is the power received by the antenna? (c) If the orbiting satellite broadcasts uniformly over an area of $1.50 \times 10^{13} m^2$ (a large fraction of North America), how much power does it radiate?

64. Lasers can be constructed that produce an extremely high intensity electromagnetic wave for a brief time—called pulsed lasers. They are used to initiate nuclear fusion, for example. Such a laser may produce an electromagnetic wave with a maximum electric field strength of $1.00 \times 10^{11} V/m$ for a time of 1.00 ns.

(a) What is the maximum magnetic field strength in the wave?

(b) What is the intensity of the beam?

(c) What energy does it deliver on an $1.00 - mm^2$ area?

16.5 Momentum and Radiation Pressure

65. A 150-W lightbulb emits 5% of its energy as electromagnetic radiation. What is the radiation pressure on an absorbing sphere of radius 10 m that surrounds the bulb?

66. What pressure does light emitted uniformly in all directions from a 100-W incandescent light bulb exert on a mirror at a distance of 3.0 m, if 2.6 W of the power is emitted as visible light?

67. A microscopic spherical dust particle of radius **2μm** and mass **10μg** is moving in outer space at a constant speed of 30 cm/sec. A wave of light strikes it from the opposite direction of its motion and gets absorbed. Assuming the particle decelerates uniformly to zero speed in one second, what is the average electric field amplitude in the light?

68. A Styrofoam spherical ball of radius 2 mm and mass **20μg** is to be suspended by the radiation pressure in a vacuum tube in a lab. How much intensity will be required if the light is completely absorbed the ball?

69. Suppose that $\vec{S}_{avg}$ for sunlight at a point on the surface of Earth is $900W/m^2$.

(a) If sunlight falls perpendicularly on a kite with a reflecting surface of area 0.75m20.75m2, what is the average force on the kite due to radiation pressure?

(b) How is your answer affected if the kite material is black and absorbs all sunlight?

70. Sunlight reaches the ground with an intensity of about $1.0kW/m^2$. A sunbather has a body surface area of 0.8m20.8m2 facing the sun while reclining on a beach chair on a clear day.

(a) how much energy from direct sunlight reaches the sunbather's skin per second?

(b) What pressure does the sunlight exert if it is absorbed?

71. Suppose a spherical particle of mass m and radius **R** in space absorbs light of intensity **I** for time **t**.

(a) How much work does the radiation pressure do to accelerate the particle from rest in the given time it absorbs the light?

(b) How much energy carried by the electromagnetic waves is absorbed by the particle over this time based on the radiant energy incident on the particle?

16.6 The Electromagnetic Spectrum

72. How many helium atoms, each with a radius of about 31 pm, must be placed end to end to have a length equal to one wavelength of 470 nm blue light?

73. If you wish to detect details of the size of atoms (about 0.2 nm) with electromagnetic radiation, it must have a wavelength of about this size.

(a) What is its frequency?

(b) What type of electromagnetic radiation might this be?

74. Find the frequency range of visible light, given that it encompasses wavelengths from 380 to 760 nm.

75. (a) Calculate the wavelength range for AM radio given its frequency range is 540 to 1600 kHz.

(b) Do the same for the FM frequency range of 88.0 to 108 MHz.

76. Radio station WWVB, operated by the National Institute of Standards and Technology (NIST) from Fort Collins, Colorado, at a low frequency of 60 kHz, broadcasts a time synchronization signal whose range covers the entire continental US. The timing of the synchronization signal is controlled by a set of atomic clocks to an accuracy of $1\times10^{-12}s$, and repeats every 1 minute. The signal is used for devices, such as radio-controlled watches, that automatically synchronize with it at preset local times. WWVB's long wavelength signal tends to propagate close to the ground.

(a) Calculate the wavelength of the radio waves from WWVB.

(b) Estimate the error that the travel time of the signal causes in synchronizing a radio controlled watch in Norfolk, Virginia, which is 1570 mi (2527 km) from Fort Collins, Colorado.

77. An outdoor WiFi unit for a picnic area has a 100-mW output and a range of about 30 m. What output power would reduce its range to 12 m for use with the same devices as before? Assume there are no obstacles in the way and that microwaves into the ground are simply absorbed.

78. The prefix "mega" (M) and "kilo" (k), when referring to amounts of computer data, refer to factors of 1024 or 210210rather than 1000 for the prefix **kilo**, and $1024^2=2^{20}$ rather than 1,000,000 for the prefix **Mega** (M). If a wireless (WiFi) router transfers 150 Mbps of data, how many bits per second is that in decimal arithmetic?

79. A computer user finds that his wireless router transmits data at a rate of 75 Mbps (megabits per second). Compare the average time to transmit one bit of data with the time difference between the wifi signal reaching an observer's cell phone directly and by bouncing back to the observer from a wall 8.00 m past the observer.

80. (a) The ideal size (most efficient) for a broadcast antenna with one end on the ground is one-fourth the wavelength (**λ/4**) of the electromagnetic radiation being sent out. If a new radio station has such an antenna that is 50.0 m high, what frequency does it broadcast most efficiently? Is this in the AM or FM band?

(b) Discuss the analogy of the fundamental resonant mode of an air column closed at one end to the resonance of currents on an antenna that is one-fourth their wavelength.

81. What are the wavelengths of (a) X-rays of frequency $2.0 \times 10^{17} Hz$?

(b) Yellow light of frequency $5.1 \times 10^{14} Hz$?

(c) Gamma rays of frequency $1.0 \times 10^{23} Hz$?

82. For red light of **λ=660nm**, what are f, ω and k?

83. A radio transmitter broadcasts plane electromagnetic waves whose maximum electric field at a particular location is $1.55 \times 10^{-3} V/m$. What is the maximum magnitude of the oscillating magnetic field at that location? How does it compare with Earth's magnetic field?

84. (a) Two microwave frequencies authorized for use in microwave ovens are: 915 and 2450 MHz. Calculate the wavelength of each.

(b) Which frequency would produce smaller hot spots in foods due to interference effects?

85. During normal beating, the heart creates a maximum 4.00-mV potential across 0.300 m of a person's chest, creating a 1.00-Hz electromagnetic wave.

(a) What is the maximum electric field strength created?

(b) What is the corresponding maximum magnetic field strength in the electromagnetic wave?

(c) What is the wavelength of the electromagnetic wave?

86. Distances in space are often quoted in units of light-years, the distance light travels in 1 year.

(a) How many meters is a light-year?

(b) How many meters is it to Andromeda, the nearest large galaxy, given that it is 2.54×10^6 ly away?

(c) The most distant galaxy yet discovered is 13.4×10^9 ly away. How far is this in meters?

87. A certain 60.0-Hz ac power line radiates an electromagnetic wave having a maximum electric field strength of 13.0 kV/m.

(a) What is the wavelength of this very-low-frequency electromagnetic wave?

(b) What type of electromagnetic radiation is this wave

(c) What is its maximum magnetic field strength?

88. (a) What is the frequency of the 193-nm ultraviolet radiation used in laser eye surgery? (b) Assuming the accuracy with which this electromagnetic radiation can ablate (reshape) the cornea is directly proportional to wavelength, how much more accurate can this UV radiation be than the shortest visible wavelength of light?

Additional Problems

89. In a region of space, the electric field is pointed along the x-axis, but its magnitude changes as described by

$$E_x = (10 N/C) sin(20x - 500t)$$

$$E_y = E_z = 0$$

where **t** is in nanoseconds and **x** is in cm. Find the displacement current through a circle of radius 3 cm in the $x = 0$ plane at $t = 0$.

90. A microwave oven uses electromagnetic waves of frequency $f = 2.45 \times 10^9 Hz$ to heat foods. The waves reflect from the inside walls of the oven to produce an interference pattern of standing waves whose antinodes are hot spots that can leave observable pit marks in some foods. The pit marks are measured to be 6.0 cm apart. Use the method employed by Heinrich Hertz to calculate the speed of electromagnetic waves this implies.

Use the Appendix D for the next two exercises

91. Galileo proposed measuring the speed of light by uncovering a lantern and having an assistant a known distance away uncover his lantern when he saw the light from Galileo's lantern, and timing the delay. How far away must the assistant be for the delay to equal the human reaction time of about 0.25 s?

92. Show that the wave equation in one dimension

$$\frac{\partial^2 f}{\partial x^2}=\frac{1}{v^2}\frac{\partial^2 f}{\partial t^2}$$

is satisfied by any doubly differentiable function of either the form $f(x-vt)$ or $f(x+vt)$.

93. On its highest power setting, a microwave oven increases the temperature of 0.400 kg of spaghetti by **45.0°C** in 120 s.

(a) What was the rate of energy absorption by the spaghetti, given that its specific heat is $3.76\times10^3 J/kg\cdot{}^\circ C$? Assume the spaghetti is perfectly absorbing.

(b) Find the average intensity of the microwaves, given that they are absorbed over a circular area 20.0 cm in diameter.

(c) What is the peak electric field strength of the microwave?

(d) What is its peak magnetic field strength?

94. A certain microwave oven projects 1.00 kW of microwaves onto a 30-cm-by-40-cm area.

(a) What is its intensity in W/m^2?

(b) Calculate the maximum electric field strength E_0 in these waves.

(c) What is the maximum magnetic field strength B_0 ?

95. Electromagnetic radiation from a 5.00-mW laser is concentrated on a $1.00-mm^2$ area.

(a) What is the intensity in W/m^2?

(b) Suppose a 2.00-nC electric charge is in the beam. What is the maximum electric force it experiences?

(c) If the electric charge moves at 400 m/s, what maximum magnetic force can it feel?

96. A 200-turn flat coil of wire 30.0 cm in diameter acts as an antenna for FM radio at a frequency of 100 MHz. The magnetic field of the incoming electromagnetic wave is perpendicular to the coil and has a maximum strength of $1.00\times10^{-12}T$.

(a) What power is incident on the coil?

(b) What average emf is induced in the coil over one-fourth of a cycle?

(c) If the radio receiver has an inductance of **2.50μH**, what capacitance must it have to resonate at 100 MHz?

97. Suppose a source of electromagnetic waves radiates uniformly in all directions in empty space where there are no absorption or interference effects.

(a) Show that the intensity is inversely proportional to r^2, the distance from the source squared.

(b) Show that the magnitudes of the electric and magnetic fields are inversely proportional to r.

98. A radio station broadcasts its radio waves with a power of 50,000 W. What would be the intensity of this signal if it is received on a planet orbiting Proxima Centuri, the closest star to our Sun, at 4.243 ly away?

99. The Poynting vector describes a flow of energy whenever electric and magnetic fields are present. Consider a long cylindrical wire of radius **r** with a current **I** in the wire, with resistance **R** and voltage **V**. From the expressions for the electric field along the wire and the magnetic field around the wire, obtain the magnitude and direction of the Poynting vector at the surface. Show that it accounts for an energy flow into the wire from the fields around it that accounts for the Ohmic heating of the wire.

100. The Sun's energy strikes Earth at an intensity of $1.37kW/m^2$. Assume as a model approximation that all of the light is absorbed. (Actually, about 30% of the light intensity is reflected out into space.)

(a) Calculate the total force that the Sun's radiation exerts on Earth.

(b) Compare this to the force of gravity between the Sun and Earth.

Note: Earth's mass is $5.972 \times 10^{24} kg$.

101. If a **Lightsail** spacecraft were sent on a Mars mission, by what fraction would its propulsion force be reduced when it reached Mars?

102. Lunar astronauts placed a reflector on the Moon's surface, off which a laser beam is periodically reflected. The distance to the Moon is calculated from the round-trip time.

(a) To what accuracy in meters can the distance to the Moon be determined, if this time can be measured to 0.100 ns?

(b) What percent accuracy is this, given the average distance to the Moon is 384,400 km?

103. Radar is used to determine distances to various objects by measuring the round-trip time for an echo from the object.

(a) How far away is the planet Venus if the echo time is 1000 s?

(b) What is the echo time for a car 75.0 m from a highway police radar unit?

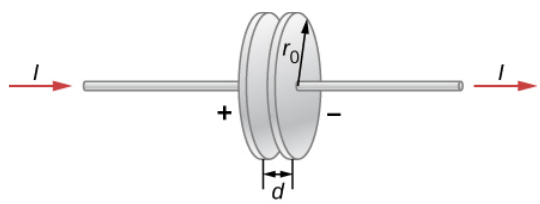
(c) How accurately (in nanoseconds) must you be able to measure the echo time to an airplane 12.0 km away to determine its distance within 10.0 m?

104. Calculate the ratio of the highest to lowest frequencies of electromagnetic waves the eye can see, given the wavelength range of visible light is from 380 to 760 nm. (Note that the ratio of highest to lowest frequencies the ear can hear is 1000.)

105. How does the wavelength of radio waves for an AM radio station broadcasting at 1030 KHz compare with the wavelength of the lowest audible sound waves (of 20 Hz). The speed of sound in air at **20°C** is about 343 m/s.

Challenge Problems

106. A parallel-plate capacitor with plate separation **d** is connected to a source of emf that places a time-dependent voltage **V(t)** across its circular plates of radius r_0 and area $A = \pi r_0^2$ (see below).



(a) Write an expression for the time rate of change of energy inside the capacitor in terms of **V(t)** and **dV(t)/dt**.

(b) Assuming that **V(t)** is increasing with time, identify the directions of the electric field lines inside the capacitor and of the magnetic field lines at the edge of the region between the plates, and then the direction of the Poynting vector $\vec{S}$ at this location.

(c) Obtain expressions for the time dependence of **E(t)**, for **B(t)** from the displacement current, and for the magnitude of the Poynting vector at the edge of the region between the plates.

(d) From $\vec{S}$, obtain an expression in terms of **V(t)** and **dV(t)/dt** for the rate at which electromagnetic field energy enters the region between the plates.

(e) Compare the results of parts (a) and (d) and explain the relationship between them.

107. A particle of cosmic dust has a density $\rho = 2.0 g/cm^3$.

(a) Assuming the dust particles are spherical and light absorbing, and are at the same distance as Earth from the Sun, determine the particle size for which radiation pressure from sunlight is equal to the Sun's force of gravity on the dust particle.

(b) Explain how the forces compare if the particle radius is smaller.

(c) Explain what this implies about the sizes of dust particle likely to be present in the inner solar system compared with outside the Oort cloud.

11.9: The Nature of Light (Exercises)

Conceptual Questions

1.1 The Propagation of Light

1. Under what conditions can light be modeled like a ray? Like a wave?

2. Why is the index of refraction always greater than or equal to 1?

3. Does the fact that the light flash from lightning reaches you before its sound prove that the speed of light is extremely large or simply that it is greater than the speed of sound? Discuss how you could use this effect to get an estimate of the speed of light.

4. Speculate as to what physical process might be responsible for light traveling more slowly in a medium than in a vacuum.

1.2 The Law of Reflection

5. Using the law of reflection, explain how powder takes the shine off of a person's nose. What is the name of the optical effect?

1.3 Refraction

6. Diffusion by reflection from a rough surface is described in this chapter. Light can also be diffused by refraction. Describe how this occurs in a specific situation, such as light interacting with crushed ice.

7. Will light change direction toward or away from the perpendicular when it goes from air to water? Water to glass? Glass to air?

8. Explain why an object in water always appears to be at a depth shallower than it actually is?

9. Explain why a person's legs appear very short when wading in a pool. Justify your explanation with a ray diagram showing the path of rays from the feet to the eye of an observer who is out of the water.

10. Explain why an oar that is partially submerged in water appears bent.

1.4 Total Internal Reflection

11. A ring with a colorless gemstone is dropped into water. The gemstone becomes invisible when submerged. Can it be a diamond? Explain.

12. The most common type of mirage is an illusion that light from faraway objects is reflected by a pool of water that is not really there. Mirages are generally observed in deserts, when there is a hot layer of air near the ground. Given that the refractive index of air is lower for air at higher temperatures, explain how mirages can be formed.

13. How can you use total internal reflection to estimate the index of refraction of a medium?

1.5 Dispersion

14. Is it possible that total internal reflection plays a role in rainbows? Explain in terms of indices of refraction and angles, perhaps referring to that shown below. Some of us have seen the formation of a double rainbow; is it physically possible to observe a triple rainbow? A photograph of a double rainbow.

15. A high-quality diamond may be quite clear and colorless, transmitting all visible wavelengths with little absorption. Explain how it can sparkle with flashes of brilliant color when illuminated by white light.

1.6 Huygens's Principle

16. How do wave effects depend on the size of the object with which the wave interacts? For example, why does sound bend around the corner of a building while light does not?

17. Does Huygens's principle apply to all types of waves?

18. If diffraction is observed for some phenomenon, it is evidence that the phenomenon is a wave. Does the reverse hold true? That is, if diffraction is not observed, does that mean the phenomenon is not a wave?

1.7 Polarization

19. Can a sound wave in air be polarized? Explain.

20. No light passes through two perfect polarizing filters with perpendicular axes. However, if a third polarizing filter is placed between the original two, some light can pass. Why is this? Under what circumstances does most of the light pass?

21. Explain what happens to the energy carried by light that it is dimmed by passing it through two crossed polarizing filters.

22. When particles scattering light are much smaller than its wavelength, the amount of scattering is proportional to $\frac{1}{\lambda}$. Does this mean there is more scattering for small λ than large λ? How does this relate to the fact that the sky is blue?

23. Using the information given in the preceding question, explain why sunsets are red.

24. When light is reflected at Brewster's angle from a smooth surface, it is 100 polarized parallel to the surface. Part of the light will be refracted into the surface. Describe how you would do an experiment to determine the polarization of the refracted light. What direction would you expect the polarization to have and would you expect it to be 100?

25. If you lie on a beach looking at the water with your head tipped slightly sideways, your polarized sunglasses do not work very well. Why not?

Problems

1.1 The Propagation of Light

26. What is the speed of light in water? In glycerine?

27. What is the speed of light in air? In crown glass?

28. Calculate the index of refraction for a medium in which the speed of light is $2.012 \times 10^8 m/s$, and identify the most likely substance based on Table 1.1.

29. In what substance in Table 1.1 is the speed of light $2.290 \times 10^8 m/s$?

30. There was a major collision of an asteroid with the Moon in medieval times. It was described by monks at Canterbury Cathedral in England as a red glow on and around the Moon. How long after the asteroid hit the Moon, which is $3.84 \times 10^5 km$ away, would the light first arrive on Earth?

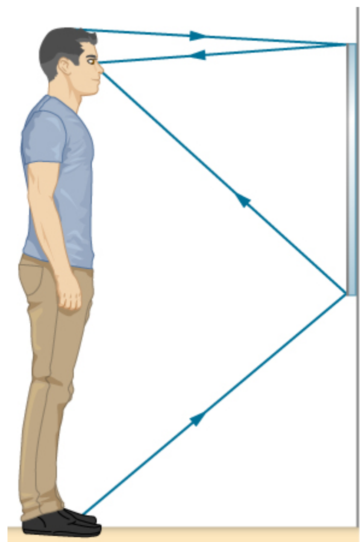
31. Components of some computers communicate with each other through optical fibers having an index of refraction $n = 1.55$. What time in nanoseconds is required for a signal to travel 0.200 m through such a fiber?

32. Compare the time it takes for light to travel 1000 m on the surface of Earth and in outer space.

33. How far does light travel underwater during a time interval of $1.50 \times 10^{-6}\,s$?

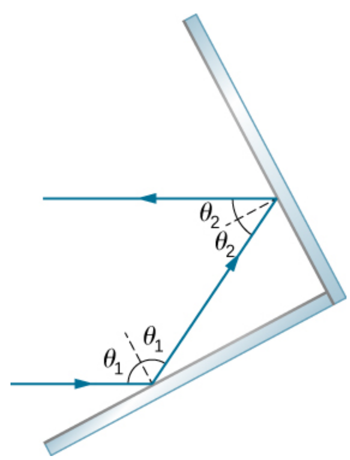
1.2 The Law of Reflection

34. Suppose a man stands in front of a mirror as shown below. His eyes are 1.65 m above the floor and the top of his head is 0.13 m higher. Find the height above the floor of the top and bottom of the smallest mirror in which he can see both the top of his head and his feet. How is this distance related to the man's height?



The figure is a drawing of a man standing in front of a mirror and looking at his image. The mirror is about half as tall as the man, with the top of the mirror above his eyes but below the top of his head. The light rays from his feet reach the bottom of the mirror and reflect to his eyes. The rays from the top of his head reach the top of the mirror and reflect to his eyes.

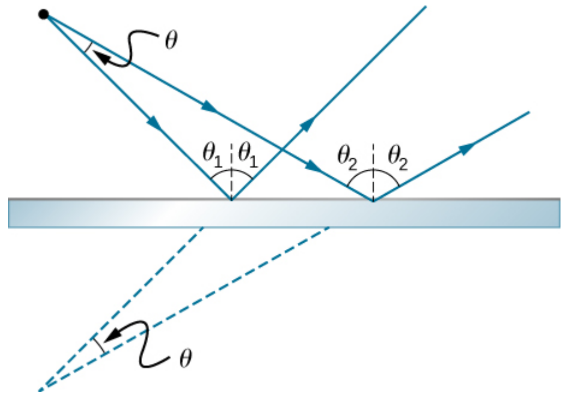
35. Show that when light reflects from two mirrors that meet each other at a right angle, the outgoing ray is parallel to the incoming ray, as illustrated below.



Two mirrors meet each other at a right angle. An incoming ray of light hits one mirror at an angle of theta one to the normal, is reflected at the same angle of theta one on the other side of the normal, then hits the other mirror at an angle of theta two to the normal and reflects at the same angle of theta two on the other side of the normal, such that the outgoing ray is parallel to the incoming ray.

36. On the Moon's surface, lunar astronauts placed a corner reflector, off which a laser beam is periodically reflected. The distance to the Moon is calculated from the round-trip time. What percent correction is needed to account for the delay in time due to the slowing of light in Earth's atmosphere? Assume the distance to the Moon is precisely $3.84 \times 10^8 m$ and Earth's atmosphere (which varies in density with altitude) is equivalent to a layer 30.0 km thick with a constant index of refraction $n = 1.000293$.

37. A flat mirror is neither converging nor diverging. To prove this, consider two rays originating from the same point and diverging at an angle θ (see below). Show that after striking a plane mirror, the angle between their directions remains θ.



Light rays diverging from a point at an angle theta are incident on a mirror at two different places and their reflected rays diverge. One ray hits at an angle theta one from the normal, and reflects at the same angle theta one on the other side of the normal. The other ray hits at a larger angle theta two from the normal, and reflects at the same angle theta two on the other side of the normal. When the reflected rays are extended backwards from their points of reflection, they meet at a point behind the mirror, at the same angle theta with which they left the source.

1.3 Refraction

Unless otherwise specified, for problems 1 through 10, the indices of refraction of glass and water should be taken to be 1.50 and 1.333, respectively.

38. A light beam in air has an angle of incidence of 35° at the surface of a glass plate. What are the angles of reflection and refraction?

39. A light beam in air is incident on the surface of a pond, making an angle of 20°20° with respect to the surface. What are the angles of reflection and refraction?

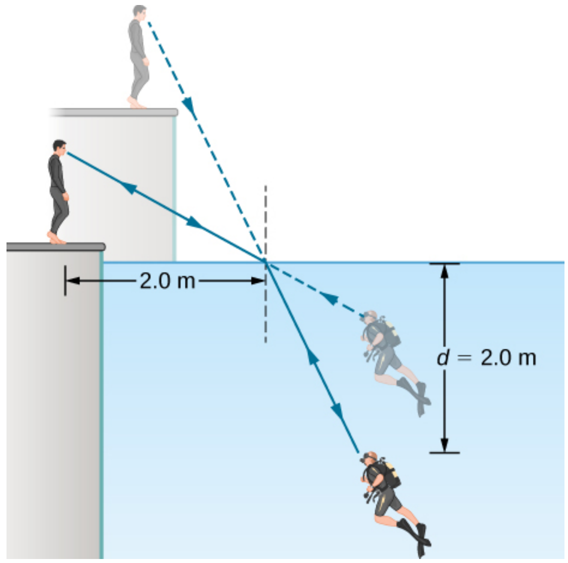
40. When a light ray crosses from water into glass, it emerges at an angle of 30° with respect to the normal of the interface. What is its angle of incidence?

41. A pencil flashlight submerged in water sends a light beam toward the surface at an angle of incidence of 30°. What is the angle of refraction in air?

42. Light rays from the Sun make a 30° angle to the vertical when seen from below the surface of a body of water. At what angle above the horizon is the Sun?

43. The path of a light beam in air goes from an angle of incidence of 35° to an angle of refraction of 22° when it enters a rectangular block of plastic. What is the index of refraction of the plastic?

44. A scuba diver training in a pool looks at his instructor as shown below. What angle does the ray from the instructor's face make with the perpendicular to the water at the point where the ray enters? The angle between the ray in the water and the perpendicular to the water is 25.0°.



A scuba diver and his trainer look at each other. They see each other at the locations given by straight line extrapolations of the rays reaching their eyes. To the trainer, the scuba diver appears less deep than he actually is, and to the diver, the trainer appears higher than he actually is. To the trainer, the scuba diver's feet appear to be at a depth of two point zero meters. The incident ray from the trainer strikes the water surface at a horizontal distance of two point zero meters from the trainer. The diver's head is a vertical distance of d equal to two point zero meters below the surface of the water.

45. (a) Using information in the preceding problem, find the height of the instructor's head above the water, noting that you will first have to calculate the angle of incidence.

(b) Find the apparent depth of the diver's head below water as seen by the instructor.

1.4 Total Internal Reflection

46. Verify that the critical angle for light going from water to air is 48.6°, as discussed at the end of Example 1.4, regarding the critical angle for light traveling in a polystyrene (a type of plastic) pipe surrounded by air.

47. (a) At the end of Example 1.4, it was stated that the critical angle for light going from diamond to air is 24.4°. Verify this.

(b) What is the critical angle for light going from zircon to air?

48. An optical fiber uses flint glass clad with crown glass. What is the critical angle?

49. At what minimum angle will you get total internal reflection of light traveling in water and reflected from ice?

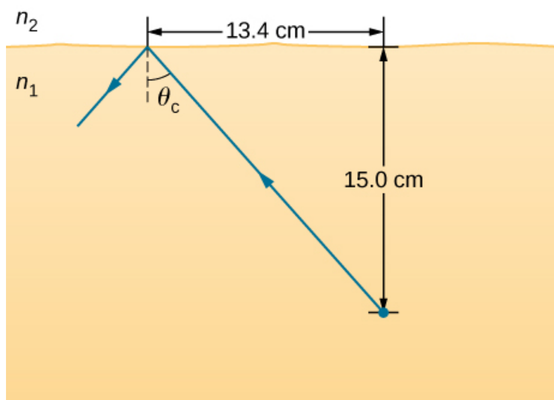
50. Suppose you are using total internal reflection to make an efficient corner reflector. If there is air outside and the incident angle is 45.0°, what must be the minimum index of refraction of the material from which the reflector is made?

51. You can determine the index of refraction of a substance by determining its critical angle.

(a) What is the index of refraction of a substance that has a critical angle of 68.4° when submerged in water? What is the substance, based on Table 1.1?

(b) What would the critical angle be for this substance in air?

52. A ray of light, emitted beneath the surface of an unknown liquid with air above it, undergoes total internal reflection as shown below. What is the index of refraction for the liquid and its likely identification?

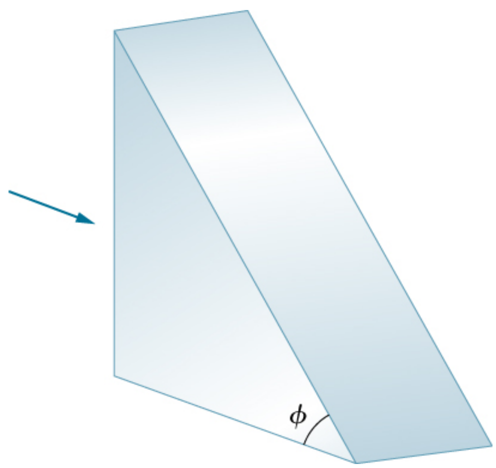


A light ray travels from an object placed in a medium n 1 at 15.0 centimeters below the horizontal interface with medium n 2. This ray gets totally internally reflected with theta c as critical angle. The horizontal distance between the object and the point of incidence is 13.4 centimeters.

53. Light rays fall normally on the vertical surface of the glass prism ($n = 1.50$ shown below.

(a) What is the largest value for ϕ such that the ray is totally reflected at the slanted face?

(b) Repeat the calculation of part (a) if the prism is immersed in water.



A right angle triangular prism has a horizontal base and a vertical side. The hypotenuse of the triangle makes an angle of phi with the horizontal base. A horizontal light rays is incident normally on the vertical surface of the prism.

1.5 Dispersion

54. (a) What is the ratio of the speed of red light to violet light in diamond, based on Table 1.2?

(b) What is this ratio in polystyrene?

(c) Which is more dispersive?

55. A beam of white light goes from air into water at an incident angle of 75.0°. At what angles are the red (660 nm) and violet (410 nm) parts of the light refracted?

56. By how much do the critical angles for red (660 nm) and violet (410 nm) light differ in a diamond surrounded by air?

57. (a) A narrow beam of light containing yellow (580 nm) and green (550 nm) wavelengths goes from polystyrene to air, striking the surface at a 30.0° incident angle. What is the angle between the colors when they emerge?

(b) How far would they have to travel to be separated by 1.00 mm?

58. A parallel beam of light containing orange (610 nm) and violet (410 nm) wavelengths goes from fused quartz to water, striking the surface between them at a 60.0° incident angle. What is the angle between the two colors in water?

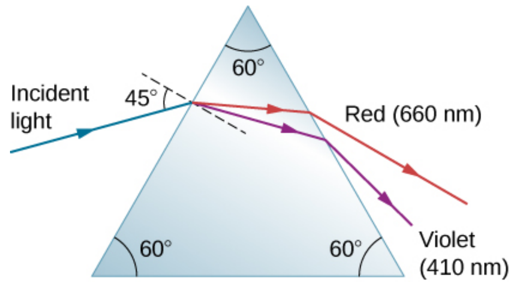
59. A ray of 610-nm light goes from air into fused quartz at an incident angle of 55.0°. At what incident angle must 470 nm light enter flint glass to have the same angle of refraction?

60. A narrow beam of light containing red (660 nm) and blue (470 nm) wavelengths travels from air through a 1.00-cm-thick flat piece of crown glass and back to air again. The beam strikes at a 30.0° incident angle.

(a) At what angles do the two colors emerge?

(b) By what distance are the red and blue separated when they emerge?

61. A narrow beam of white light enters a prism made of crown glass at a 45.0° incident angle, as shown below. At what angles, θ_R and θ_V, do the red (660 nm) and violet (410 nm) components of the light emerge from the prism?



A blue incident light ray at an angle of incidence equal to 45 degrees to the normal falls on an equilateral triangular prism whose corners are all at angles equal to 60 degrees. At the first surface, the ray refracts and splits into red and violet rays. These rays hit the second surface and emerge from the prism. The red light with 660 nanometers bends less than the violet light with 410 nanometers.

1.7 Polarization

62. What angle is needed between the direction of polarized light and the axis of a polarizing filter to cut its intensity in half?

63. The angle between the axes of two polarizing filters is 45.0°. By how much does the second filter reduce the intensity of the light coming through the first?

64. Two polarizing sheets P_1 and P_2 are placed together with their transmission axes oriented at an angle θ to each other. What is θ when only 25 of the maximum transmitted light intensity passes through them?

65. Suppose that in the preceding problem the light incident on P_1 is unpolarized. At the determined value of θ, what fraction of the incident light passes through the combination?

66. If you have completely polarized light of intensity $150 W/m^2$, what will its intensity be after passing through a polarizing filter with its axis at an 89.0° angle to the light's polarization direction?

67. What angle would the axis of a polarizing filter need to make with the direction of polarized light of intensity $1.00 kW/m^2$ to reduce the intensity to $10.0 W/m^2$?

68. At the end of Example 1.7, it was stated that the intensity of polarized light is reduced to 90.0 of its original value by passing through a polarizing filter with its axis at an angle of 18.4° to the direction of polarization. Verify this statement.

69. Show that if you have three polarizing filters, with the second at an angle of 45.0° to the first and the third at an angle of 90.0° to the first, the intensity of light passed by the first will be reduced to 25.0 of its value. (This is in contrast to having only the first and third, which reduces the intensity to zero, so that placing the second between them increases the intensity of the transmitted light.)

70. Three polarizing sheets are placed together such that the transmission axis of the second sheet is oriented at 25.0° to the axis of the first, whereas the transmission axis of the third sheet is oriented at 40.0° (in the same sense) to the axis of the first. What fraction of the intensity of an incident unpolarized beam is transmitted by the combination?

71. In order to rotate the polarization axis of a beam of linearly polarized light by 90.0°, a student places sheets P_1 and P_2 with their transmission axes at 45.0° and 90.0°, respectively, to the beam's axis of polarization.

(a) What fraction of the incident light passes through P_1 and

(b) through the combination?

(c) Repeat your calculations for part (b) for transmission-axis angles of 30.0° and 90.0°, respectively.

72. It is found that when light traveling in water falls on a plastic block, Brewster's angle is 50.0°. What is the refractive index of the plastic?

73. At what angle will light reflected from diamond be completely polarized?

74. What is Brewster's angle for light traveling in water that is reflected from crown glass?

75. A scuba diver sees light reflected from the water's surface. At what angle will this light be completely polarized?

Additional Problems

76. From his measurements, Roemer estimated that it took 22 min for light to travel a distance equal to the diameter of Earth's orbit around the Sun.

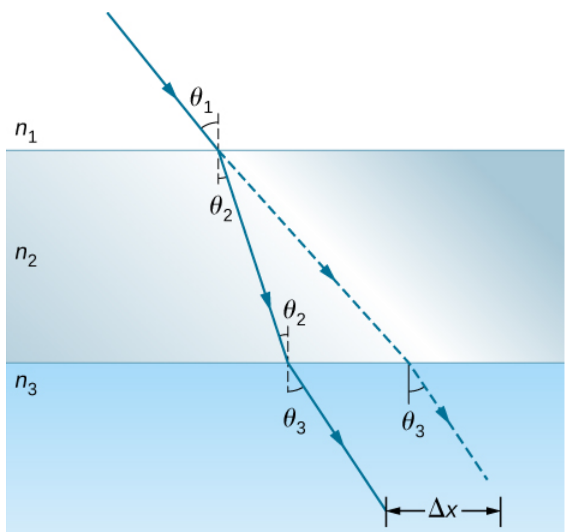
(a) Use this estimate along with the known diameter of Earth's orbit to obtain a rough value of the speed of light.

(b) Light actually takes 16.5 min to travel this distance. Use this time to calculate the speed of light.

77. Cornu performed Fizeau's measurement of the speed of light using a wheel of diameter 4.00 cm that contained 180 teeth. The distance from the wheel to the mirror was 22.9 km. Assuming he measured the speed of light accurately, what was the angular velocity of the wheel?

78. Suppose you have an unknown clear substance immersed in water, and you wish to identify it by finding its index of refraction. You arrange to have a beam of light enter it at an angle of 45.0°, and you observe the angle of refraction to be 40.3°. What is the index of refraction of the substance and its likely identity?

79. Shown below is a ray of light going from air through crown glass into water, such as going into a fish tank. Calculate the amount the ray is displaced by the glass (Δx), given that the incident angle is 40.0°. and the glass is 1.00 cm thick.



The figure illustrates refraction occurring when light travels from medium n_1 to n_3 through an intermediate medium n_2. The incident ray makes an angle θ_1 with a perpendicular drawn at the point of incidence at the interface between n_1 and n_2. The light ray entering n_2 bends towards the perpendicular line making an angle θ_2 with it on the n_2 side. The ray arrives at the interface between n_2 and n_3 at an angle of θ_2 to a perpendicular drawn at the point of incidence at this interface, and the transmitted ray bends away from the perpendicular, making an angle of theta three to the perpendicular on the n_3 side. A straight line extrapolation of the original incident ray is shown as a dotted line. This line is parallel to the refracted ray in the third medium, n_3, and is shifted a distance delta x from the refracted ray. The extrapolated ray is at the same angle theta three to the perpendicular in medium n_3 as the refracted ray.

80. Considering the previous problem, show that θ_3 is the same as it would be if the second medium were not present.

81. At what angle is light inside crown glass completely polarized when reflected from water, as in a fish tank?

82. Light reflected at 55.6° from a window is completely polarized. What is the window's index of refraction and the likely substance of which it is made?

83. (a) Light reflected at 62.5° from a gemstone in a ring is completely polarized. Can the gem be a diamond?

(b) At what angle would the light be completely polarized if the gem was in water?

84. If θ_b is Brewster's angle for light reflected from the top of an interface between two substances, and θ'_b is Brewster's angle for light reflected from below, prove that $\theta_b + \theta'_b = 90.0^\circ$.

85. Unreasonable results Suppose light travels from water to another substance, with an angle of incidence of 10.0° and an angle of refraction of 14.9°.

(a) What is the index of refraction of the other substance?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

86. Unreasonable results Light traveling from water to a gemstone strikes the surface at an angle of 80.0° and has an angle of refraction of 15.2°.

(a) What is the speed of light in the gemstone?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

87. If a polarizing filter reduces the intensity of polarized light to 50.0 of its original value, by how much are the electric and magnetic fields reduced?

88. Suppose you put on two pairs of polarizing sunglasses with their axes at an angle of 15.0°. How much longer will it take the light to deposit a given amount of energy in your eye compared with a single pair of sunglasses? Assume the lenses are clear except for their polarizing characteristics.

89. (a) On a day when the intensity of sunlight is $1.00 kW/m^2$, a circular lens 0.200 m in diameter focuses light onto water in a black beaker. Two polarizing sheets of plastic are placed in front of the lens with their axes at an angle of 20.0°. Assuming the sunlight is unpolarized and the polarizers are 100 efficient, what is the initial rate of heating of the water in $^\circ C/s$, assuming it is 80.0 absorbed? The aluminum beaker has a mass of 30.0 grams and contains 250 grams of water.

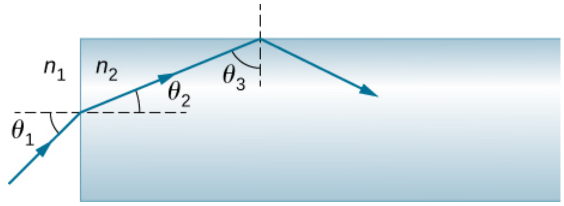
(b) Do the polarizing filters get hot? Explain.

Challenge Problems

90. Light shows staged with lasers use moving mirrors to swing beams and create colorful effects. Show that a light ray reflected from a mirror changes direction by 2θ when the mirror is rotated by an angle θ.

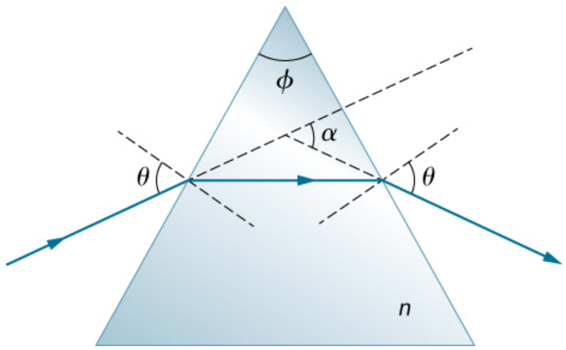
91. Consider sunlight entering Earth's atmosphere at sunrise and sunset—that is, at a 90.0°. incident angle. Taking the boundary between nearly empty space and the atmosphere to be sudden, calculate the angle of refraction for sunlight. This lengthens the time the Sun appears to be above the horizon, both at sunrise and sunset. Now construct a problem in which you determine the angle of refraction for different models of the atmosphere, such as various layers of varying density. Your instructor may wish to guide you on the level of complexity to consider and on how the index of refraction varies with air density.

92. A light ray entering an optical fiber surrounded by air is first refracted and then reflected as shown below. Show that if the fiber is made from crown glass, any incident ray will be totally internally reflected.



The figure shows light traveling from n_1 and incident onto the left face of a rectangular block of material n_2. The ray is incident at an angle of incidence θ_1, measured relative to the normal to the surface where the ray enters. The angle of refraction is θ_2, again, relative to the normal to the surface. The refracted ray falls onto the upper face of the block and gets totally internally reflected with θ_3 as the angle of incidence.

93. A light ray falls on the left face of a prism (see below) at the angle of incidence θ for which the emerging beam has an angle of refraction θ at the right face. Show that the index of refraction n of the glass prism is given by $n = \frac{sin\frac{1}{2}(\alpha+\phi)}{sin\frac{1}{2}\phi}$ where ϕ is the vertex angle of the prism and α is the angle through which the beam has been deviated. If $\alpha = 37.0^\circ$ and the base angles of the prism are each 50.0°, what is **n**?



A light ray falls on the left face of a triangular prism whose upper vertex has an angle of phi and whose index of refraction is n. The angle of incidence of the ray relative to the normal to the left face is theta. The ray refracts in the prism. The refracted ray is horizontal, parallel to the base of the prism. The refracted ray reaches the right face of the prism and refracts as it emerges out of the prism. The emerging ray makes an angle of theta with the normal to the right face.

94. If the apex angle ϕ in the previous problem is 20.0° and $n = 1.50$, what is the value of α?

95. The light incident on polarizing sheet P_1 is linearly polarized at an angle of 30.0° with respect to the transmission axis of P_1. Sheet P_2 is placed so that its axis is parallel to the polarization axis of the incident light, that is, also at 30.0° with respect to P_1.

(a) What fraction of the incident light passes through P_1?

(b) What fraction of the incident light is passed by the combination?

(c) By rotating P_2, a maximum in transmitted intensity is obtained. What is the ratio of this maximum intensity to the intensity of transmitted light when P_2 is at 30.0° with respect to P_1?

96. Prove that if **I** is the intensity of light transmitted by two polarizing filters with axes at an angle θ and I' is the intensity when the axes are at an angle $90.0^\circ - \theta$, then $I + I' = I_0$, the original intensity. (Hint: Use the trigonometric identities $cos 90.0^\circ - \theta = sin\theta$ and $cos^2\theta + sin^2\theta = 1$.)

SECTION OVERVIEW

11.10: Geometric Optics and Image Formation

This chapter introduces the major ideas of geometric optics, which describe the formation of images due to reflection and refraction. It is called “geometric” optics because the images can be characterized using geometric constructions, such as ray diagrams. We have seen that visible light is an electromagnetic wave; however, its wave nature becomes evident only when light interacts with objects with dimensions comparable to the wavelength (about 500 nm for visible light). Therefore, the laws of geometric optics only apply to light interacting with objects much larger than the wavelength of the light.

11.10.1: Prelude to Geometric Optics and Image Formation

11.10.2: Images Formed by Plane Mirrors

11.10.3: Spherical Mirrors

11.10.4: Images Formed by Refraction

11.10.5: Thin Lenses

11.10.6: The Eye

11.10.7: The Camera

11.10.8: The Simple Magnifier

11.10.9: Microscopes and Telescopes

11.10.E: Geometric Optics and Image Formation (Exercises)

11.10.1: Prelude to Geometric Optics and Image Formation

This chapter introduces the major ideas of geometric optics, which describe the formation of images due to reflection and refraction. It is called “geometric” optics because the images can be characterized using geometric constructions, such as ray diagrams. We have seen that visible light is an electromagnetic wave; however, its wave nature becomes evident only when light interacts with objects with dimensions comparable to the wavelength (about 500 nm for visible light). Therefore, the laws of geometric optics only apply to light interacting with objects much larger than the wavelength of the light.

Figure 11.10.1.1: Cloud Gate is a public sculpture by Anish Kapoor located in Millennium Park in Chicago. Its stainless steel plates reflect and distort images around it, including the Chicago skyline. Dedicated in 2006, it has become a popular tourist attraction, illustrating how art can use the principles of physical optics to startle and entertain. (credit: modification of work by Dhilung Kirat)

11.10.2: Images Formed by Plane Mirrors

Learning Objectives

By the end of this section, you will be able to:

- Describe how an image is formed by a plane mirror.
- Distinguish between real and virtual images.
- Find the location and characterize the orientation of an image created by a plane mirror.

You only have to look as far as the nearest bathroom to find an example of an image formed by a mirror. Images in a plane mirror are the same size as the object, are located behind the mirror, and are oriented in the same direction as the object (i.e., "upright").

To understand how this happens, consider Figure $11.10.2.1$ Two rays emerge from point P, strike the mirror, and reflect into the observer's eye. Note that we use the law of reflection to construct the reflected rays. If the reflected rays are extended backward behind the mirror (see dashed lines), they seem to originate from point Q. This is where the image of point P is located. If we repeat this process for point $P'P'$, we obtain its image at point Q'. You should convince yourself by using basic geometry that the image height (the distance from Q to Q') is the same as the object height (the distance from P to P'). By forming images of all points of the object, we obtain an upright image of the object behind the mirror.

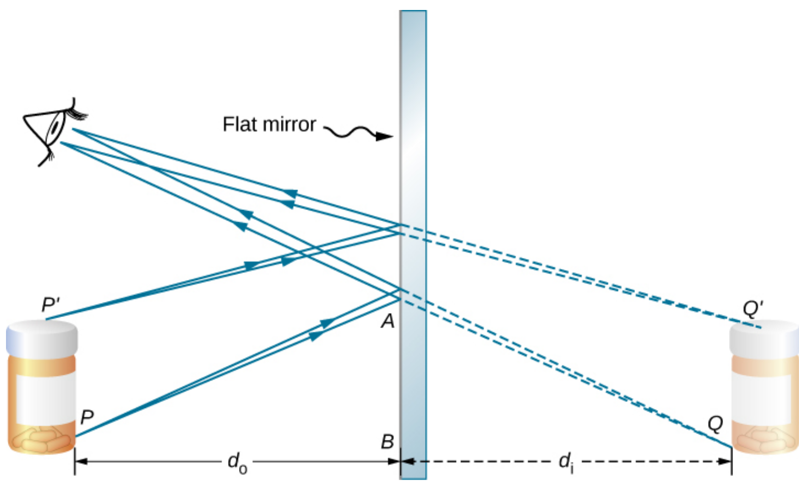


Figure 11.10.2.1. Two light rays originating from point P on an object are reflected by a flat mirror into the eye of an observer. The reflected rays are obtained by using the law of reflection. Extending these reflected rays backward, they seem to come from point Q behind the mirror, which is where the virtual image is located. Repeating this process for point P′ gives the image point Q′. The image height is thus the same as the object height, the image is upright, and the object distance d_o is the same as the image distance d_i. (credit: modification of work by Kevin Dufendach)

Notice that the reflected rays appear to the observer to come directly from the image behind the mirror. In reality, these rays come from the points on the mirror where they are reflected. The image behind the mirror is called a virtual image because it cannot be projected onto a screen—the rays only appear to originate from a common point behind the mirror. If you walk behind the mirror, you cannot see the image, because the rays do not go there. However, in front of the mirror, the rays behave exactly as if they come from behind the mirror, so that is where the virtual image is located.

Later in this chapter, we discuss real images; a real image can be projected onto a screen because the rays physically go through the image. You can certainly see both real and virtual images. The difference is that a virtual image cannot be projected onto a screen, whereas a real image can.

Locating an Image in a Plane Mirror

The law of reflection tells us that the angle of incidence is the same as the angle of reflection. Applying this to triangles PAB and QAB in Figure $11.10.2.1$and using basic geometry shows that they are congruent triangles. This means that the distance PB from the object to the mirror is the same as the distance BQ from the mirror to the image. The object distance (denoted d_o) is the distance from the mirror to the object (or, more generally, from the center of the optical element that creates its image). Similarly, the image distance (denoted d_i) is the distance from the mirror to the image (or, more generally, from the center of the optical element that creates it). If we measure distances from the mirror, then the object and image are in opposite directions, so for a plane mirror, the object and image distances should have the opposite signs:

$$d_o = -d_i.$$

An extended object such as the container in Figure $11.10.2.1$can be treated as a collection of points, and we can apply the method above to locate the image of each point on the extended object, thus forming the extended image.

Multiple Images

If an object is situated in front of two mirrors, you may see images in both mirrors. In addition, the image in the first mirror may act as an object for the second mirror, so the second mirror may form an image of the image. If the mirrors are placed parallel to each other and the object is placed at a point other than the midpoint between them, then this process of image-of-an-image continues without end, as you may have noticed when standing in a hallway with mirrors on each side. This is shown in Figure $11.10.2.2$, which shows three images produced by the blue object. Notice that each reflection reverses front and back, just like pulling a right-hand glove inside out produces a left-hand glove (this is why a reflection of your right hand is a left hand). Thus, the fronts and backs of images 1 and 2 are both inverted with respect to the object, and the front and back of image 3 is inverted with respect to image 2, which is the object for image 3.

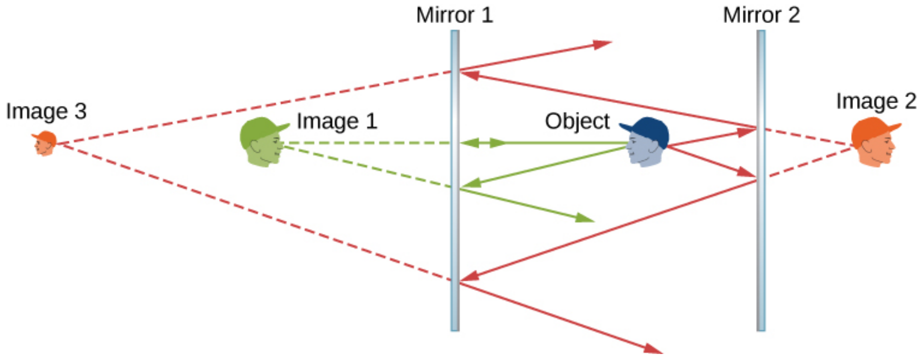


Figure 11.10.2.2. Two parallel mirrors can produce, in theory, an infinite number of images of an object placed off center between the mirrors. Three of these images are shown here. The front and back of each image is inverted with respect to its object. Note that the colors are only to identify the images. For normal mirrors, the color of an image is essentially the same as that of its object.

You may have noticed that image 3 is smaller than the object, whereas images 1 and 2 are the same size as the object. The ratio of the image height with respect to the object height is called magnification. More will be said about magnification in the next section.

Infinite reflections may terminate. For instance, two mirrors at right angles form three images, as shown in Figure $11.10.2.3a$ Images 1 and 2 result from rays that reflect from only a single mirror, but image 1,2 is formed by rays that reflect from both mirrors. This is shown in the ray-tracing diagram in (\PageIndex{3b}\). To find image 1,2, you have to look behind the corner of the two mirrors.

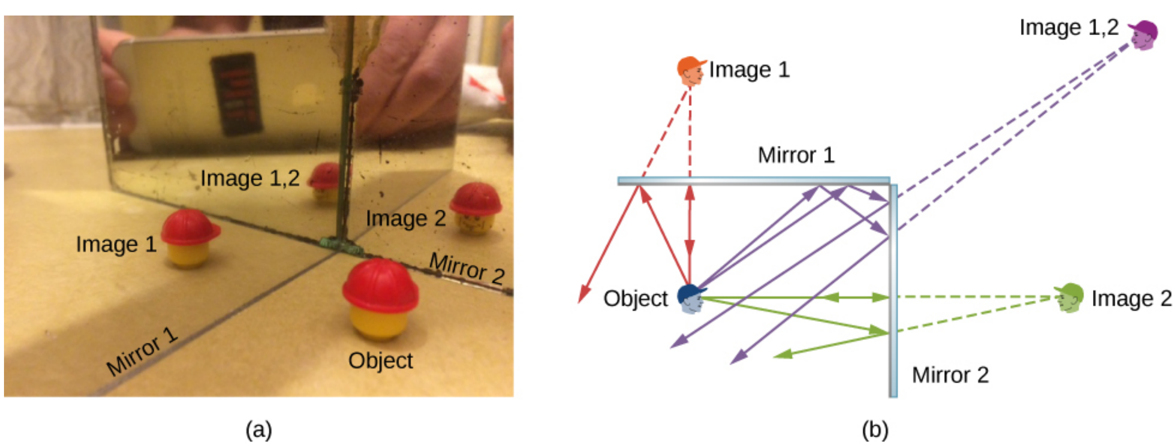


Figure 11.10.2.3. Two mirrors can produce multiple images. (a) Three images of a plastic head are visible in the two mirrors at a right angle. (b) A single object reflecting from two mirrors at a right angle can produce three images, as shown by the green, purple, and red images.

11.10.3: Spherical Mirrors

Learning Objectives

By the end of this section, you will be able to:

- Describe image formation by spherical mirrors.
- Use ray diagrams and the mirror equation to calculate the properties of an image in a spherical mirror.

The image in a plane mirror has the same size as the object, is upright, and is the same distance behind the mirror as the object is in front of the mirror. A curved mirror, on the other hand, can form images that may be larger or smaller than the object and may form either in front of the mirror or behind it. In general, any curved surface will form an image, although some images make be so distorted as to be unrecognizable (think of fun house mirrors). Because curved mirrors can create such a rich variety of images, they are used in many optical devices that find many uses. We will concentrate on spherical mirrors for the most part, because they are easier to manufacture than mirrors such as parabolic mirrors and so are more common.

Curved Mirrors

We can define two general types of spherical mirrors. If the reflecting surface is the outer side of the sphere, the mirror is called a **convex mirror**. If the inside surface is the reflecting surface, it is called a **concave mirror**.

Symmetry is one of the major hallmarks of many optical devices, including mirrors and lenses. The symmetry axis of such optical elements is often called the principal axis or optical axis. For a spherical mirror, the optical axis passes through the mirror's center of curvature and the mirror's vertex, as shown in Figure $11.10.3.1$

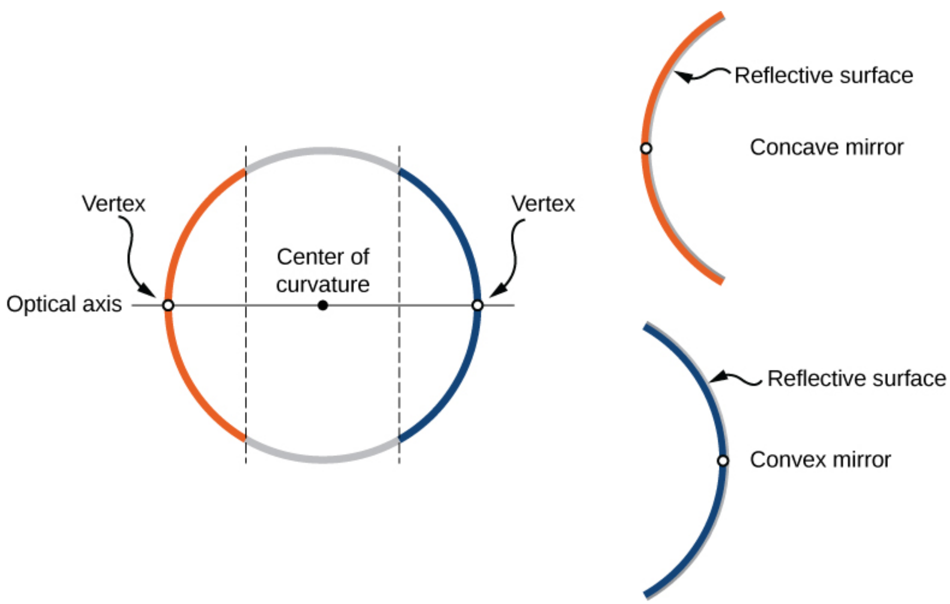


Figure 11.10.3.1. A spherical mirror is formed by cutting out a piece of a sphere and silvering either the inside or outside surface. A concave mirror has silvering on the interior surface (think "cave"), and a convex mirror has silvering on the exterior surface.

Consider rays that are parallel to the optical axis of a parabolic mirror, as shown in Figure $11.10.3.2a$ Following the law of reflection, these rays are reflected so that they converge at a point, called the **focal point**. Figure $11.10.3.2b$shows a spherical mirror that is large compared with its radius of curvature. For this mirror, the reflected rays do not cross at the same point, so the mirror does not have a well-defined focal point. This is called **spherical aberration** and results in a blurred image of an extended object. Figure $11.10.3.2c$ shows a spherical mirror that is small compared to its radius of curvature. This mirror is a good approximation of a parabolic mirror, so rays that arrive parallel to the optical axis are reflected to a well-defined focal point. The distance along the optical axis from the mirror to the focal point is called the focal length of the mirror.

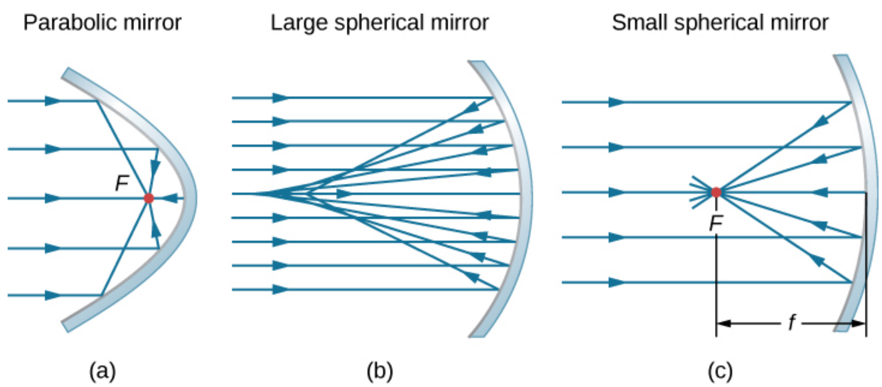


Figure 11.10.3.2: (a) Parallel rays reflected from a parabolic mirror cross at a single point called the focal point F. (b) Parallel rays reflected from a large spherical mirror do not cross at a common point. (c) If a spherical mirror is small compared with its radius of curvature, it better approximates the central part of a parabolic mirror, so parallel rays essentially cross at a common point. The distance along the optical axis from the mirror to the focal point is the focal length f of the mirror.

A convex spherical mirror also has a focal point, as shown in Figure $11.10.3.3$ Incident rays parallel to the optical axis are reflected from the mirror and seem to originate from point F at focal length f behind the mirror. Thus, the focal point is virtual because no real rays actually pass through it; they only appear to originate from it.

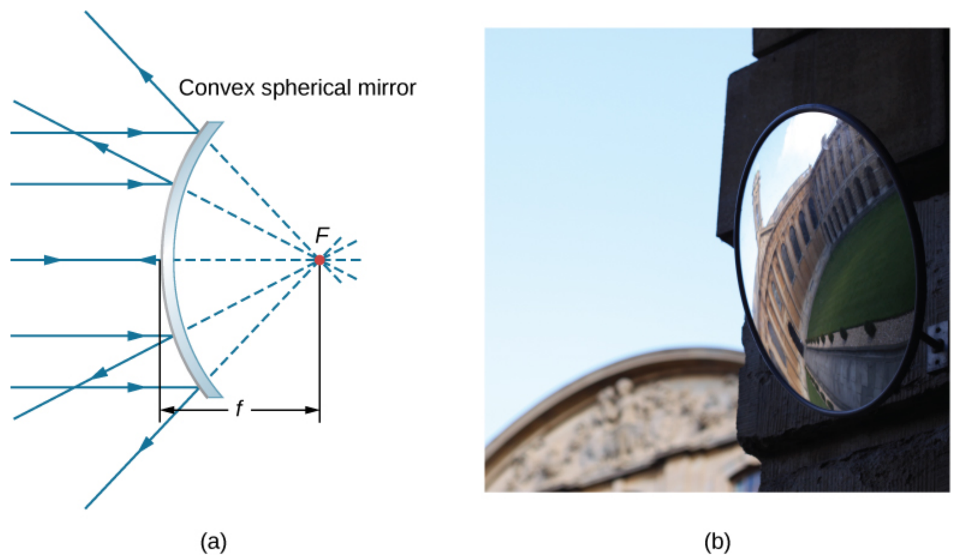


Figure 11.10.3.3: (a) Rays reflected by a convex spherical mirror: Incident rays of light parallel to the optical axis are reflected from a convex spherical mirror and seem to originate from a well-defined focal point at focal distance f on the opposite side of the mirror. The focal point is virtual because no real rays pass through it. (b) Photograph of a virtual image formed by a convex mirror. (credit b: modification of work by Jenny Downing)

How does the focal length of a mirror relate to the mirror's radius of curvature? Figure $11.10.3.4$ shows a single ray that is reflected by a spherical concave mirror. The incident ray is parallel to the optical axis. The point at which the reflected ray crosses the optical axis is the focal point. Note that all incident rays that are parallel to the optical axis are reflected through the focal point —we only show one ray for simplicity. We want to find how the focal length FP (denoted by f) relates to the radius of curvature of the mirror, R, whose length is

$$R = CF + FP. \tag{11.10.3.1}$$

The law of reflection tells us that angles $\angle OXC$ and $\angle CXF$ are the same, and because the incident ray is parallel to the optical axis, angles $\angle OXC$ and $\angle XCP$ are also the same. Thus, triangle CXF is an isosceles triangle with $CF = FX$. If the angle θ is small then

$$\sin\theta \approx \theta \tag{11.10.3.2}$$

which is called the "**small-angle approximation**"), then $FX \approx FP$ or $CF \approx FP$. Inserting this into Equation 11.10.3.1 for the radius R, we get

$$\begin{aligned} R &= CF + FP \\ &= FP + FP \\ &= 2FP \\ &= 2f \end{aligned} \tag{11.10.3.3}$$

In other words, in the small-angle approximation, the focal length f of a concave spherical mirror is half of its radius of curvature, R:

$$f = \frac{R}{2}.$$

In this chapter, we assume that the **small-angle approximation** (also called the **paraxial approximation**) is always valid. In this approximation, all rays are paraxial rays, which means that they make a small angle with the optical axis and are at a distance much less than the radius of curvature from the optical axis. In this case, their angles θ of reflection are small angles, so

$$\sin\theta \approx \tan\theta \approx \theta. \tag{11.10.3.4}$$

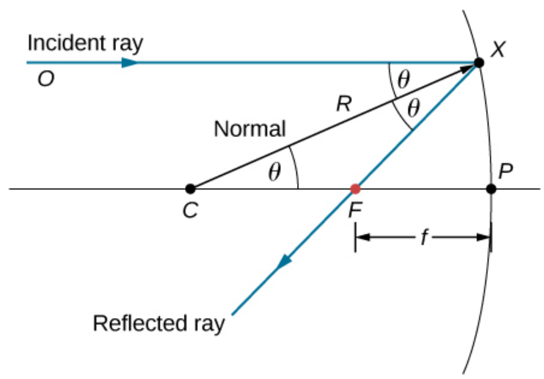


Figure 11.10.3.4: Reflection in a concave mirror. In the small-angle approximation, a ray that is parallel to the optical axis CP is reflected through the focal point F of the mirror.

Using Ray Tracing to Locate Images

To find the location of an image formed by a spherical mirror, we first use ray tracing, which is the technique of drawing rays and using the law of reflection to determine the reflected rays (later, for lenses, we use the law of refraction to determine refracted rays). Combined with some basic geometry, we can use ray tracing to find the focal point, the image location, and other information about how a mirror manipulates light. In fact, we already used ray tracing above to locate the focal point of spherical mirrors, or the image distance of flat mirrors. To locate the image of an object, you must locate at least two points of the image. Locating each point requires drawing at least two rays from a point on the object and constructing their reflected rays. The point at which the reflected rays intersect, either in real space or in virtual space, is where the corresponding point of the image is located. To make ray tracing easier, we concentrate on four "principal" rays whose reflections are easy to construct.

Figure $11.10.3.5$ shows a concave mirror and a convex mirror, each with an arrow-shaped object in front of it. These are the objects whose images we want to locate by ray tracing. To do so, we draw rays from point Q that is on the object but not on the optical axis. We choose to draw our ray from the tip of the object. Principal ray 1 goes from point Q and travels parallel to the optical axis. The reflection of this ray must pass through the focal point, as discussed above. Thus, for the concave mirror, the reflection of principal ray 1 goes through focal point F, as shown in Figure $11.10.3.5b$ For the convex mirror, the backward extension of the reflection of principal ray 1 goes through the focal point (i.e., a virtual focus). Principal ray 2 travels first on the line going through the focal point and then is reflected back along a line parallel to the optical axis. Principal ray 3 travels toward the center of curvature of the mirror, so it strikes the mirror at normal incidence and is reflected back along the line from which it came. Finally, principal ray 4 strikes the vertex of the mirror and is reflected symmetrically about the optical axis.

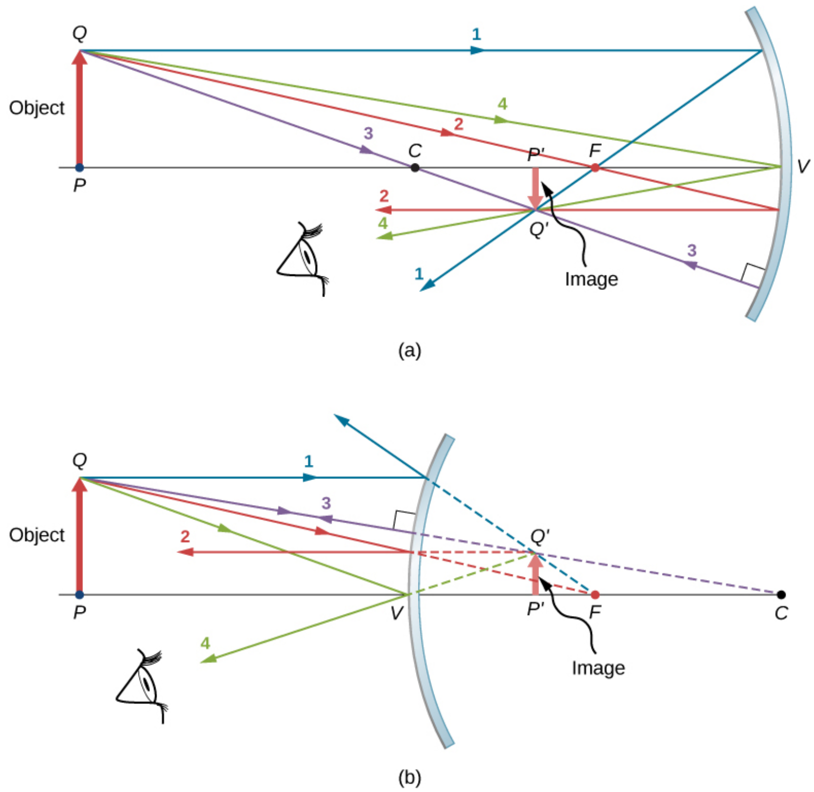


Figure 11.10.3.5: The four principal rays shown for both (a) a concave mirror and (b) a convex mirror. The image forms where the rays intersect (for real images) or where their backward extensions intersect (for virtual images).

The four principal rays intersect at point Q', which is where the image of point Q is located. To locate point Q', drawing any two of these principle rays would suffice. We are thus free to choose whichever of the principal rays we desire to locate the image. Drawing more than two principal rays is sometimes useful to verify that the ray tracing is correct.

To completely locate the extended image, we need to locate a second point in the image, so that we know how the image is oriented. To do this, we trace the principal rays from the base of the object. In this case, all four principal rays run along the optical axis, reflect from the mirror, and then run back along the optical axis. The difficulty is that, because these rays are collinear, we cannot determine a unique point where they intersect. All we know is that the base of the image is on the optical axis. However, because the mirror is symmetrical from top to bottom, it does not change the vertical orientation of the object. Thus, because the object is vertical, the image must be vertical. Therefore, the image of the base of the object is on the optical axis directly above the image of the tip, as drawn in the figure.

For the concave mirror, the extended image in this case forms between the focal point and the center of curvature of the mirror. It is inverted with respect to the object, is a real image, and is smaller than the object. Were we to move the object closer to or farther from the mirror, the characteristics of the image would change. For example, we show, as a later exercise, that an object placed between a concave mirror and its focal point leads to a virtual image that is upright and larger than the object. For the convex mirror, the extended image forms between the focal point and the mirror. It is upright with respect to the object, is a virtual image, and is smaller than the object.

Ray-Tracing Rules

Ray tracing is very useful for mirrors. The rules for ray tracing are summarized here for reference:

- A ray traveling parallel to the optical axis of a spherical mirror is reflected along a line that goes through the focal point of the mirror (ray 1 in Figure 11.10.3.5).
- A ray traveling along a line that goes through the focal point of a spherical mirror is reflected along a line parallel to the optical axis of the mirror (ray 2 in Figure 11.10.3.5).
- A ray traveling along a line that goes through the center of curvature of a spherical mirror is reflected back along the same line (ray 3 in Figure 11.10.3.5).

- A ray that strikes the vertex of a spherical mirror is reflected symmetrically about the optical axis of the mirror (ray 4 in Figure 11.10.3.5).

We use ray tracing to illustrate how images are formed by mirrors and to obtain numerical information about optical properties of the mirror. If we assume that a mirror is small compared with its radius of curvature, we can also use algebra and geometry to derive a mirror equation, which we do in the next section. Combining ray tracing with the mirror equation is a good way to analyze mirror systems.

Image Formation by Reflection—The Mirror Equation

For a plane mirror, we showed that the image formed has the same height and orientation as the object, and it is located at the same distance behind the mirror as the object is in front of the mirror. Although the situation is a bit more complicated for curved mirrors, using geometry leads to simple formulas relating the object and image distances to the focal lengths of concave and convex mirrors.

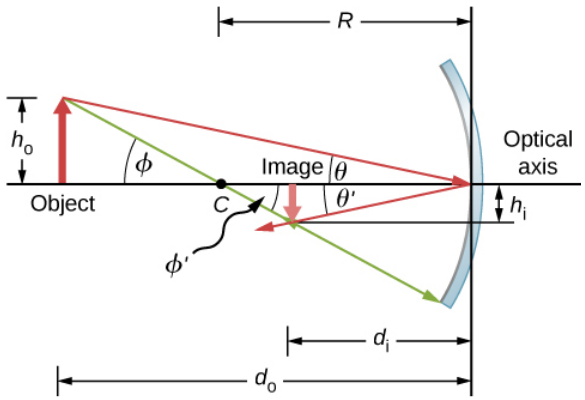


Figure 11.10.3.6: Image formed by a concave mirror.

Consider the object OP shown in Figure 11.10.3.6 The center of curvature of the mirror is labeled C and is a distance R from the vertex of the mirror, as marked in the figure. The object and image distances are labeled d_o and d_i, and the object and image heights are labeled h_o and h_i, respectively. Because the angles ϕ and ϕ' are alternate interior angles, we know that they have the same magnitude. However, they must differ in sign if we measure angles from the optical axis, so $\phi=-\phi'$. An analogous scenario holds for the angles θ and θ'. The law of reflection tells us that they have the same magnitude, but their signs must differ if we measure angles from the optical axis. Thus, $\theta=-\theta'$. Taking the tangent of the angles θ and θ', and using the property that $\tan(-\theta)=-\tan\theta$, gives us

$$\left.\begin{aligned}\tan\theta&=\frac{h_o}{d_o}\\ \tan\theta'=-\tan\theta&=\frac{h_i}{d_i}\end{aligned}\right\}=\frac{h_o}{d_o}=-\frac{h_i}{d_i} \tag{11.10.3.5}$$

or

$$-\frac{h_o}{h_i}=\frac{d_o}{d_i}. \tag{11.10.3.6}$$

Similarly, taking the tangent of ϕ and ϕ' gives

$$\left.\begin{aligned}\tan\phi&=\frac{h_o}{d_o-R}\\ \tan\phi'=-\tan\phi&=\frac{h_i}{R-d_i}\end{aligned}\right\}=\frac{h_o}{d_o-R}=-\frac{h_i}{R-d_i}$$

or

$$-\frac{h_o}{h_i}=\frac{d_o-R}{R-d_i}. \tag{11.10.3.7}$$

Combining Equation 11.10.3.5and 11.10.3.7gives

$$\frac{d_o}{d_i} = \frac{d_o - R}{R - d_i}.$$

After a little algebra, this becomes

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{2}{R}. \tag{11.10.3.8}$$

No approximation is required for this result, so it is exact. However, as discussed above, in the small-angle approximation, the focal length of a spherical mirror is one-half the radius of curvature of the mirror, or $f = R/2$. Inserting this into Equation 11.10.3.8 gives the **mirror equation**:

$$\underbrace{\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}}_{\text{mirror equation}}. \tag{11.10.3.9}$$

The mirror equation relates the image and object distances to the focal distance and is valid only in the small-angle approximation (Equation 11.10.3.2). Although it was derived for a concave mirror, it also holds for convex mirrors (proving this is left as an exercise). We can extend the mirror equation to the case of a plane mirror by noting that a plane mirror has an infinite radius of curvature. This means the focal point is at infinity, so the mirror equation simplifies to

$$d_o = -d_i$$

which is the same equation obtained earlier.

Notice that we have been very careful with the signs in deriving the mirror equation. For a plane mirror, the image distance has the opposite sign of the object distance. Also, the real image formed by the concave mirror in Figure $11.10.3.6$is on the opposite side of the optical axis with respect to the object. In this case, the image height should have the opposite sign of the object height. To keep track of the signs of the various quantities in the mirror equation, we now introduce a sign convention.

Sign convention for spherical mirrors

Using a consistent sign convention is very important in geometric optics. It assigns positive or negative values for the quantities that characterize an optical system. Understanding the sign convention allows you to describe an image without constructing a ray diagram. This text uses the following sign convention:

1. The focal length f is positive for concave mirrors and negative for convex mirrors.
2. The image distance d_i is positive for real images and negative for virtual images.

Notice that rule 1 means that the radius of curvature of a spherical mirror can be positive or negative. What does it mean to have a negative radius of curvature? This means simply that the radius of curvature for a convex mirror is defined to be negative.

Image Magnification

Let's use the sign convention to further interpret the derivation of the mirror equation. In deriving this equation, we found that the object and image heights are related by

$$-\frac{h_o}{h_i} = \frac{d_o}{d_i}. \tag{11.10.3.10}$$

See Equation 11.10.3.6 Both the object and the image formed by the mirror in Figure $11.10.3.6$are real, so the object and image distances are both positive. The highest point of the object is above the optical axis, so the object height is positive. The image, however, is below the optical axis, so the image height is negative. Thus, this sign convention is consistent with our derivation of the mirror equation.

Equation 11.10.3.10in fact describes the **linear magnification** (often simply called "**magnification**") of the image in terms of the object and image distances. We thus define the dimensionless **magnification** m as follows:

$$\underbrace{m = \frac{h_i}{h_o}}_{\text{linear magnification}} \quad . \tag{11.10.3.11}$$

If m is positive, the image is upright, and if m is negative, the image is inverted. If $|m| > 1$, the image is larger than the object, and if $|m| < 1$, the image is smaller than the object. With this definition of magnification, we get the following relation between the vertical and horizontal object and image distances:

$$m = \frac{h_i}{h_o} = -\frac{d_i}{d_o}.$$

This is a very useful relation because it lets you obtain the magnification of the image from the object and image distances, which you can obtain from the mirror equation.

Example 11.10.3.1: Solar Electric Generating System

One of the solar technologies used today for generating electricity involves a device (called a parabolic trough or concentrating collector) that concentrates sunlight onto a blackened pipe that contains a fluid. This heated fluid is pumped to a heat exchanger, where the thermal energy is transferred to another system that is used to generate steam and eventually generates electricity through a conventional steam cycle. Figure $11.10.3.7$shows such a working system in southern California. The real mirror is a parabolic cylinder with its focus located at the pipe; however, we can approximate the mirror as exactly one-quarter of a circular cylinder.

Figure 11.10.3.7: Parabolic trough collectors are used to generate electricity in southern California. (credit: “kjkolb”/Wikimedia Commons)

1. If we want the rays from the sun to focus at 40.0 cm from the mirror, what is the radius of the mirror?
2. What is the amount of sunlight concentrated onto the pipe, per meter of pipe length, assuming the insolation (incident solar radiation) is 900 W/m^2?
3. If the fluid-carrying pipe has a 2.00-cm diameter, what is the temperature increase of the fluid per meter of pipe over a period of 1 minute? Assume that all solar radiation incident on the reflector is absorbed by the pipe, and that the fluid is mineral oil.

Strategy

First identify the physical principles involved. Part (a) is related to the optics of spherical mirrors. Part (b) involves a little math, primarily geometry. Part (c) requires an understanding of heat and density.

Solution

a. The sun is the object, so the object distance is essentially infinity: $d_o = \infty$. The desired image distance is $d_i = 40.0\,cm$. We use the mirror equation (Equation $11.10.3.9$) to find the focal length of the mirror:

$$\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$$
$$f=\left(\frac{1}{d_o}+\frac{1}{d_i}\right)^{-1}$$
$$=\left(\frac{1}{\infty}+\frac{1}{40.0\,cm}\right)^{-1}$$
$$=40.0\,cm$$

Thus, the radius of the mirror is

$$R=2f=80.0\,cm.$$

b. The insolation is 900 W/m^2. You must find the cross-sectional area A of the concave mirror, since the power delivered is $900\,W/m^2\times A$. The mirror in this case is a quarter-section of a cylinder, so the area for a length L of the mirror is $A=\frac{1}{4}(2\pi R)L$. The area for a length of 1.00 m is then

$$A=\frac{\pi}{2}R(1.00m)$$
$$=\frac{(3.14)}{2}(0.800\,m)(1.00\,m)$$
$$=1.26\,m^2.$$

The insolation on the 1.00-m length of pipe is then

$$(9.00\times10^2\,\frac{W}{m^2})(1.26\,m^2)=1130\,W.$$

c. The increase in temperature is given by $Q=mc\Delta T$. The mass m of the mineral oil in the one-meter section of pipe is

$$m=\rho V=\rho\pi\left(\frac{d}{2}\right)^2(1.00\,m)$$
$$=(8.00\times10^2\,kg/m^3)(3.14)(0.0100\,m)^2(1.00\,m)$$
$$=0.251\,kg$$

Therefore, the increase in temperature in one minute is

$$\Delta T=\frac{Q}{mc}$$
$$=\frac{(1130\,W)(60.0\,s)}{(0.251\,kg)(1670\,J\cdot kg/{}^\circ C)}$$
$$=162^\circ$$

Significance

An array of such pipes in the California desert can provide a thermal output of 250 MW on a sunny day, with fluids reaching temperatures as high as 400°C. We are considering only one meter of pipe here and ignoring heat losses along the pipe.

Example 11.10.3.2: Image in a Convex Mirror

A keratometer is a device used to measure the curvature of the cornea of the eye, particularly for fitting contact lenses. Light is reflected from the cornea, which acts like a convex mirror, and the keratometer measures the magnification of the image. The smaller the magnification, the smaller the radius of curvature of the cornea. If the light source is 12 cm from the cornea and the image magnification is 0.032, what is the radius of curvature of the cornea?

Strategy

If you find the focal length of the convex mirror formed by the cornea, then you know its radius of curvature (it's twice the focal length). The object distance is d_o=12cm and the magnification is m=0.032. First find the image distance d_i and then solve for the focal length f.

Solution

Start with the equation for magnification (Equation 11.10.3.11) and solving for d_i and inserting the given values yields

$$d_i = -md_o = -(0.032)(12\,cm) = -0.384\,cm$$

where we retained an extra significant figure because this is an intermediate step in the calculation. Solve the mirror equation for the focal length f and insert the known values for the object and image distances. The result is

$$\begin{aligned} \frac{1}{d_o} + \frac{1}{d_i} &= \frac{1}{f} \\ f &= \left(\frac{1}{d_o} + \frac{1}{d_i}\right)^{-1} \\ &= \left(\frac{1}{12cm} + \frac{1}{-0.384cm}\right)^{-1} \\ &= -40.0\,cm \end{aligned}$$

The radius of curvature is twice the focal length, so

$$R = 2f = -0.80\,cm$$

Significance

The focal length is negative, so the focus is virtual, as expected for a concave mirror and a real object. The radius of curvature found here is reasonable for a cornea. The distance from cornea to retina in an adult eye is about 2.0 cm. In practice, corneas may not be spherical, which complicates the job of fitting contact lenses. Note that the image distance here is negative, consistent with the fact that the image is behind the mirror. Thus, the image is virtual because no rays actually pass through it. In the problems and exercises, you will show that, for a fixed object distance, a smaller radius of curvature corresponds to a smaller the magnification.

PROBLEM-SOLVING STRATEGY: SPHERICAL MIRRORS

- Step 1. First make sure that image formation by a spherical mirror is involved.
- Step 2. Determine whether ray tracing, the mirror equation, or both are required. A sketch is very useful even if ray tracing is not specifically required by the problem. Write symbols and known values on the sketch.
- Step 3. Identify exactly what needs to be determined in the problem (identify the unknowns).
- Step 4. Make a list of what is given or can be inferred from the problem as stated (identify the knowns).
- Step 5. If ray tracing is required, use the ray-tracing rules listed near the beginning of this section.
- Step 6. Most quantitative problems require using the mirror equation. Use the examples as guides for using the mirror equation.
- Step 7. Check to see whether the answer makes sense. Do the signs of object distance, image distance, and focal length correspond with what is expected from ray tracing? Is the sign of the magnification correct? Are the object and image distances reasonable?

Departure from the Small-Angle Approximation

The small-angle approximation (Equation 11.10.3.4) is a cornerstone of the above discussion of image formation by a spherical mirror. When this approximation is violated, then the image created by a spherical mirror becomes distorted. Such distortion is called aberration. Here we briefly discuss two specific types of aberrations: spherical aberration and coma.

Spherical aberration

Consider a broad beam of parallel rays impinging on a spherical mirror, as shown in Figure 11.10.3.8 The farther from the optical axis the rays strike, the worse the spherical mirror approximates a parabolic mirror. Thus, these rays are not focused at the same point as rays that are near the optical axis, as shown in the figure. Because of spherical aberration, the image of an extended object in a spherical mirror will be blurred. Spherical aberrations are characteristic of the mirrors and lenses that we consider in the following section of this chapter (more sophisticated mirrors and lenses are needed to eliminate spherical aberrations).

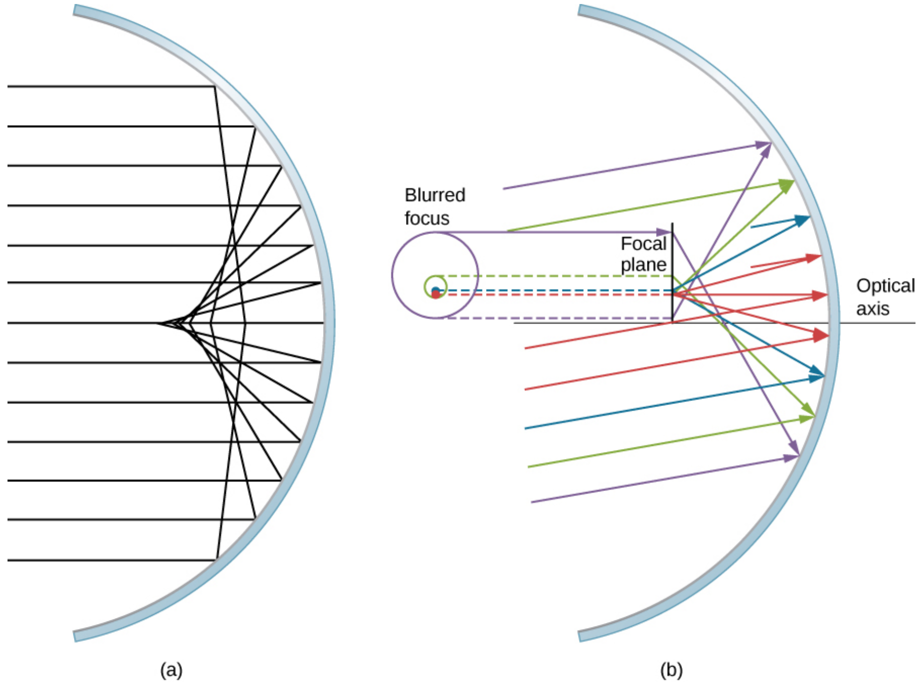


Figure 11.10.3.8: (a) With spherical aberration, the rays that are farther from the optical axis and the rays that are closer to the optical axis are focused at different points. Notice that the aberration gets worse for rays farther from the optical axis. (b) For comatic aberration, parallel rays that are not parallel to the optical axis are focused at different heights and at different focal lengths, so the image contains a "tail" like a comet (which is "coma" in Latin). Note that the colored rays are only to facilitate viewing; the colors do not indicate the color of the light.

Coma or Comatic Aberration

Coma is similar to spherical aberration, but arises when the incoming rays are not parallel to the optical axis, as shown in Figure $11.10.3.8b$ Recall that the small-angle approximation holds for spherical mirrors that are small compared to their radius. In this case, spherical mirrors are good approximations of parabolic mirrors. Parabolic mirrors focus all rays that are parallel to the optical axis at the focal point. However, parallel rays that are **not** parallel to the optical axis are focused at different heights and at different focal lengths, as show in Figure $11.10.3.8b$ Because a spherical mirror is symmetric about the optical axis, the various colored rays in this figure create circles of the corresponding color on the focal plane.

Although a spherical mirror is shown in Figure $11.10.3.8b$, comatic aberration occurs also for parabolic mirrors—it does not result from a breakdown in the small-angle approximation (Equation 11.10.3.4). Spherical aberration, however, occurs only for spherical mirrors and is a result of a breakdown in the small-angle approximation. We will discuss both coma and spherical aberration later in this chapter, in connection with telescopes.

11.10.4: Images Formed by Refraction

Learning Objectives

By the end of this section, you will be able to:

- Describe image formation by a single refracting surface
- Determine the location of an image and calculate its properties by using a ray diagram
- Determine the location of an image and calculate its properties by using the equation for a single refracting surface

When rays of light propagate from one medium to another, these rays undergo refraction, which is when light waves are bent at the interface between two media. The refracting surface can form an image in a similar fashion to a reflecting surface, except that the law of refraction (Snell's law) is at the heart of the process instead of the law of reflection.

Refraction at a Plane Interface—Apparent Depth

If you look at a straight rod partially submerged in water, it appears to bend at the surface. The reason behind this curious effect is that the image of the rod inside the water forms a little closer to the surface than the actual position of the rod, so it does not line up with the part of the rod that is above the water. The same phenomenon explains why a fish in water appears to be closer to the surface than it actually is.

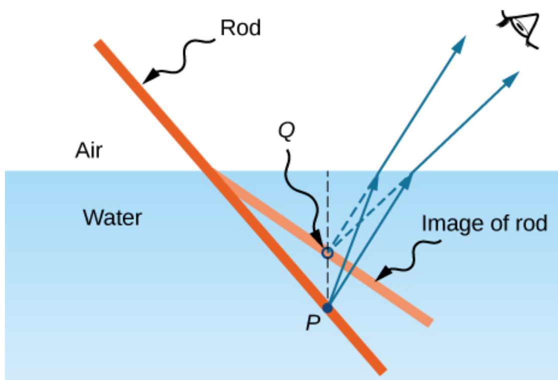


Figure 11.10.4.1: Bending of a rod at a water-air interface. Point P on the rod appears to be at point Q, which is where the image of point P forms due to refraction at the air-water interface.

To study image formation as a result of refraction, consider the following questions:

1. What happens to the rays of light when they enter or pass through a different medium?
2. Do the refracted rays originating from a single point meet at some point or diverge away from each other?

To be concrete, we consider a simple system consisting of two media separated by a plane interface (Figure $11.10.4.2$). The object is in one medium and the observer is in the other. For instance, when you look at a fish from above the water surface, the fish is in medium 1 (the water) with refractive index 1.33, and your eye is in medium 2 (the air) with refractive index 1.00, and the surface of the water is the interface. The depth that you "see" is the image height h_i and is called the **apparent depth**. The actual depth of the fish is the object height h_o.

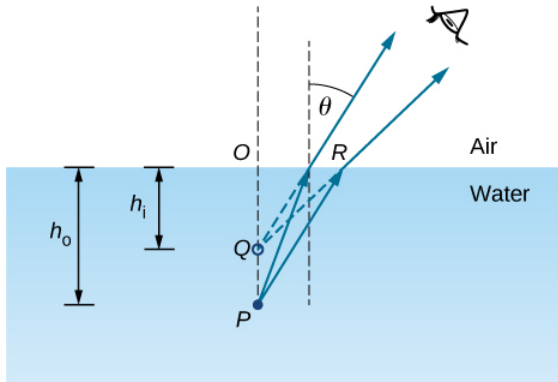


Figure 11.10.4.2: Apparent depth due to refraction. The real object at point P creates an image at point Q. The image is not at the same depth as the object, so the observer sees the image at an "apparent depth."

The apparent depth h_i depends on the angle at which you view the image. For a view from above (the so-called "normal" view), we can approximate the refraction angle θ to be small, and replace $\sin\theta$ in Snell's law by $\tan\theta$. With this approximation, you can use the triangles ΔOPR and ΔOQR to show that the apparent depth is given by

$$h_i = \left(\frac{n_2}{n_1}\right) h_o.$$

The derivation of this result is left as an exercise. Thus, a fish appears at 3/4 of the real depth when viewed from above.

Refraction at a Spherical Interface

Spherical shapes play an important role in optics primarily because high-quality spherical shapes are far easier to manufacture than other curved surfaces. To study refraction at a single spherical surface, we assume that the medium with the spherical surface at one end continues indefinitely (a "semi-infinite" medium).

Refraction at a Convex Surface

Consider a point source of light at point **P** in front of a convex surface made of glass (Figure 11.10.4.3). Let R be the radius of curvature, n_1 be the refractive index of the medium in which object point **P** is located, and n_2 be the refractive index of the medium with the spherical surface. We want to know what happens as a result of refraction at this interface.

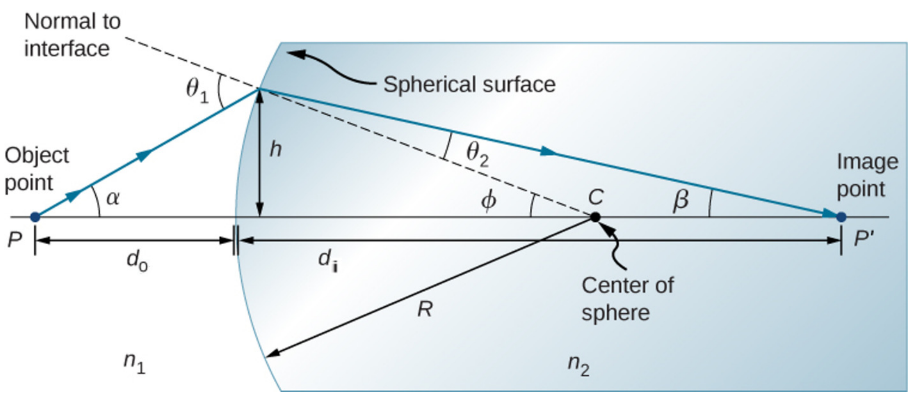


Figure 11.10.4.3: Refraction at a convex surface ($n_2 > n_1$).

Because of the symmetry involved, it is sufficient to examine rays in only one plane. The figure shows a ray of light that starts at the object point P, refracts at the interface, and goes through the image point P'. We derive a formula relating the object distance d_o, the image distance d_i, and the radius of curvature R.

Applying **Snell's law** to the ray emanating from point P gives

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Within the small-angle approximation

$$\sin\theta \approx \theta,$$

Snell's law then takes the form

$$n_1\theta_1 \approx n_2\theta_2. \tag{11.10.4.1}$$

From the geometry of Figure 11.10.4.3 we see that

$$\theta_1 = \alpha + \phi,$$

$$\theta_2 = \phi - \beta.$$

Inserting both expressions into Equation 11.10.4.1gives

$$n_1(\alpha+\phi) \approx n_2(\phi-\beta). \tag{11.10.4.2}$$

Using Figure 11.10.4.3 we calculate the tangent of the angles α, β, and ϕ:

- $\tan\alpha \approx \frac{h}{d_o}$
- $\tan\beta \approx \frac{h}{d_i}$
- $\tan\phi \approx \frac{h}{R}$

Again using the small-angle approximation, we find that $\tan\theta \approx \theta$, so the above relationships become

- $\alpha \approx \frac{h}{d_o}$
- $\beta \approx \frac{h}{d_i}$
- $\phi \approx \frac{h}{R}$.

Putting these angles into Equation 11.10.4.2gives

$$n_1\left(\frac{h}{d_o}+\frac{h}{R}\right) = n_2\left(\frac{h}{R}-\frac{h}{d_i}\right).$$

We can write this more conveniently as

$$\frac{n_1}{d_o}+\frac{n_2}{d_i} = \frac{n_2-n_1}{R}. \tag{11.10.4.3}$$

If the object is placed at a special point called the first focus, or the object focus F_1, then the image is formed at infinity, as shown in Figure 11.10.4.4a

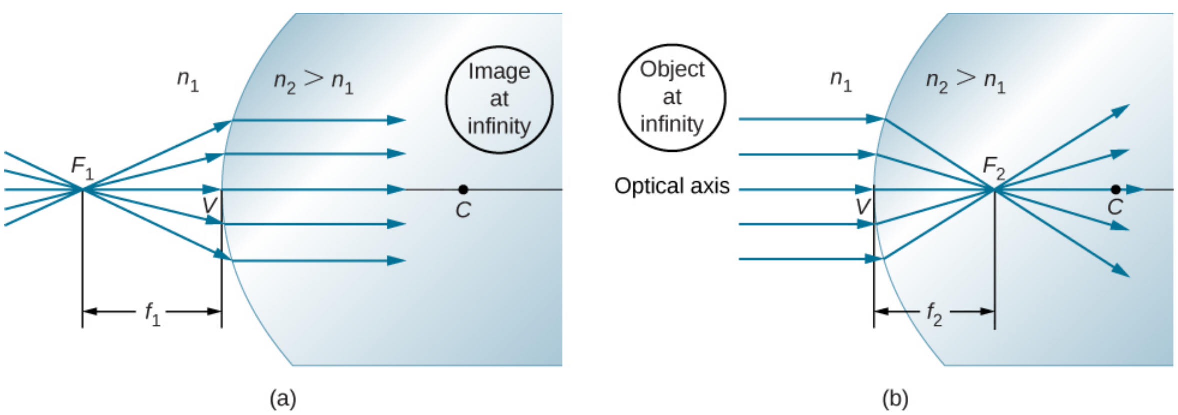


Figure 11.10.4.4: (a) First focus (called the "object focus") for refraction at a convex surface. (b) Second focus (called "image focus") for refraction at a convex surface.

We can find the location f_1 of the first focus F_1 by setting $d_i = \infty$ in Equation ???.

$$\frac{n_1}{f_1} + \frac{n_2}{\infty} = \frac{n_2 - n_1}{R} \tag{11.10.4.4}$$

$$f_1 = \frac{n_1 R}{n_2 - n_1} \tag{11.10.4.5}$$

Similarly, we can define a second focus or image focus F_2 where the image is formed for an object that is far away (Figure 11.10.4.4b). The location of the second focus F_2 is obtained from Equation ??? by setting $d_0 = \infty$:

$$\frac{n_1}{\infty} + \frac{n_2}{f_2} = \frac{n_2 - n_1}{R} \tag{11.10.4.6}$$

$$f_2 = \frac{n_2 R}{n_2 - n_1}. \tag{11.10.4.7}$$

Note that the object focus is at a different distance from the vertex than the image focus because $n_1 \neq n_2$.

Sign convention for single refracting surfaces

Although we derived this equation for refraction at a convex surface, the same expression holds for a concave surface, provided we use the following sign convention:

1. $R > 0$ if surface is convex toward object; otherwise, $R < 0$.
2. $d_i > 0$ if image is real and on opposite side from the object; otherwise, $d_i < 0$.

11.10.5: Thin Lenses

Learning Objectives

By the end of this section, you will be able to:

- Use ray diagrams to locate and describe the image formed by a lens
- Employ the thin-lens equation to describe and locate the image formed by a lens

Lenses are found in a huge array of optical instruments, ranging from a simple magnifying glass to a camera's zoom lens to the eye itself. In this section, we use the Snell's law to explore the properties of lenses and how they form images.

The word "lens" derives from the Latin word for a lentil bean, the shape of which is similar to a convex lens. However, not all lenses have the same shape. Figure $11.10.5.1$shows a variety of different lens shapes. The vocabulary used to describe lenses is the same as that used for spherical mirrors: The axis of symmetry of a lens is called the optical axis, where this axis intersects the lens surface is called the vertex of the lens, and so forth.

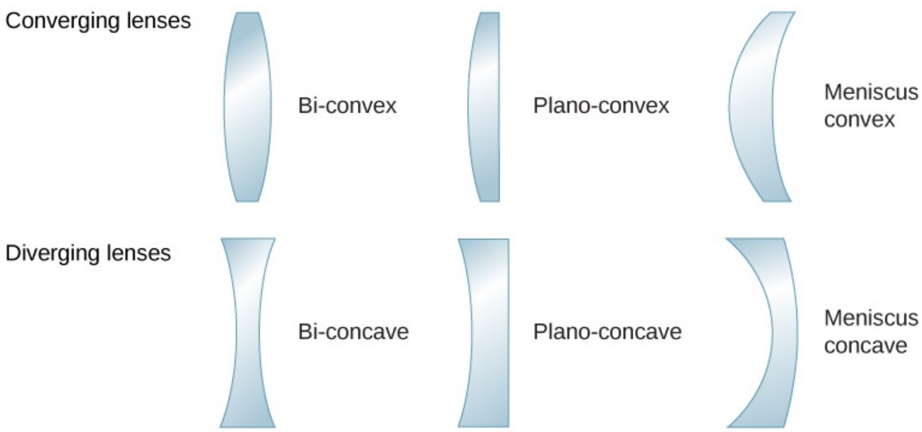


Figure 11.10.5.1: Various types of lenses: Note that a converging lens has a thicker "waist," whereas a diverging lens has a thinner waist.

A convex or converging lens is shaped so that all light rays that enter it parallel to its optical axis intersect (or focus) at a single point on the optical axis on the opposite side of the lens, as shown in Figure $11.10.5.1a$ Likewise, a concave or diverging lens is shaped so that all rays that enter it parallel to its optical axis diverge, as shown in part (b). To understand more precisely how a lens manipulates light, look closely at the top ray that goes through the converging lens in part (a). Because the index of refraction of the lens is greater than that of air, Snell's law tells us that the ray is bent toward the perpendicular to the interface as it enters the lens. Likewise, when the ray exits the lens, it is bent away from the perpendicular. The same reasoning applies to the diverging lenses, as shown in Figure $11.10.5.1b$ The overall effect is that light rays are bent toward the optical axis for a converging lens and away from the optical axis for diverging lenses. For a converging lens, the point at which the rays cross is the focal point **F** of the lens. For a diverging lens, the point from which the rays appear to originate is the (virtual) focal point. The distance from the center of the lens to its focal point is the focal length **f** of the lens.

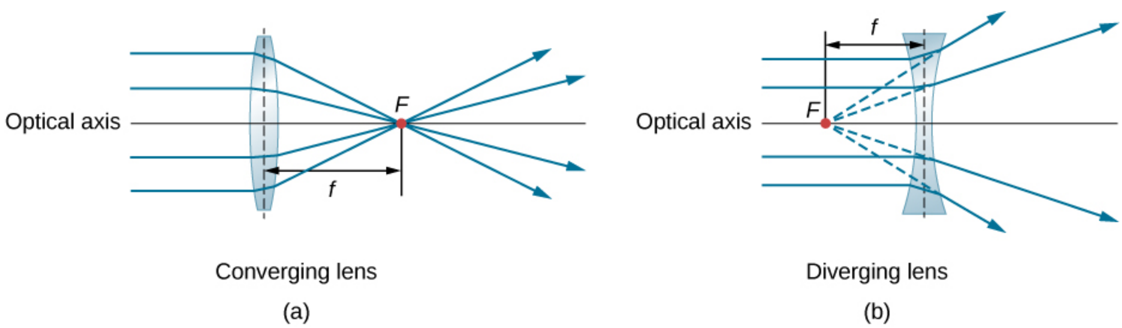


Figure 11.10.5.2: Rays of light entering (a) a converging lens and (b) a diverging lens, parallel to its axis, converge at its focal point F. The distance from the center of the lens to the focal point is the lens's focal length f. Note that the light rays are bent upon entering and exiting the lens, with the overall effect being to bend the rays toward the optical axis.

A lens is considered to be thin if its thickness **t** is much less than the radii of curvature of both surfaces, as shown in Figure 11.10.5.3 In this case, the rays may be considered to bend once at the center of the lens. For the case drawn in the figure, light ray 1 is parallel to the optical axis, so the outgoing ray is bent once at the center of the lens and goes through the focal point. Another important characteristic of thin lenses is that light rays that pass through the center of the lens are undeviated, as shown by light ray 2.

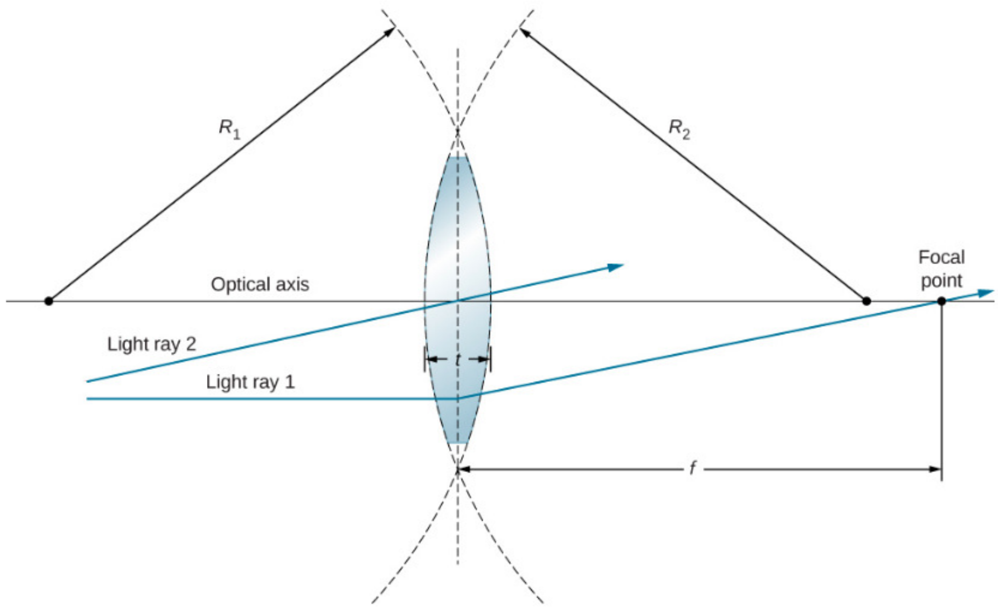


Figure 11.10.5.3: In the thin-lens approximation, the thickness t of the lens is much, much less than the radii R_1 and R_2 of curvature of the surfaces of the lens. Light rays are considered to bend at the center of the lens, such as light ray 1. Light ray 2 passes through the center of the lens and is undeviated in the thin-lens approximation.

As noted in the initial discussion of Snell's law, the paths of light rays are exactly reversible. This means that the direction of the arrows could be reversed for all of the rays in Figure 11.10.5.2 For example, if a point-light source is placed at the focal point of a convex lens, as shown in Figure 11.10.5.4, parallel light rays emerge from the other side.

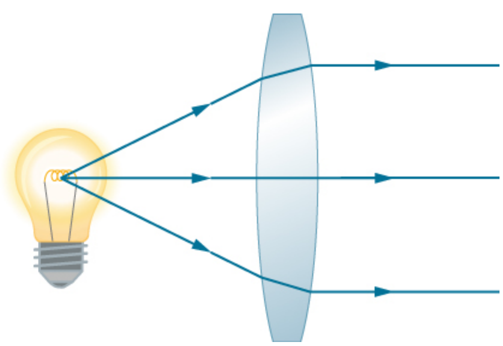


Figure 11.10.5.4 in converging and diverging lenses. This technique is used in lighthouses and sometimes in traffic lights to produce a directional beam of light from a source that emits light in all directions.

Ray Tracing and Thin Lenses

Ray tracing is the technique of determining or following (tracing) the paths taken by light rays. Ray tracing for thin lenses is very similar to the technique we used with spherical mirrors. As for mirrors, ray tracing can accurately describe the operation of a lens. The rules for ray tracing for thin lenses are similar to those of spherical mirrors:

1. A ray entering a converging lens parallel to the optical axis passes through the focal point on the other side of the lens (ray 1 in part (a) of Figure 11.10.5.4). A ray entering a diverging lens parallel to the optical axis exits along the line that passes through the focal point on the **same** side of the lens (ray 1 in part (b) of the figure).
2. A ray passing through the center of either a converging or a diverging lens is not deviated (ray 2 in parts (a) and (b)).
3. For a converging lens, a ray that passes through the focal point exits the lens parallel to the optical axis (ray 3 in part (a)). For a diverging lens, a ray that approaches along the line that passes through the focal point on the opposite side exits the lens parallel

to the axis (ray 3 in part (b)).

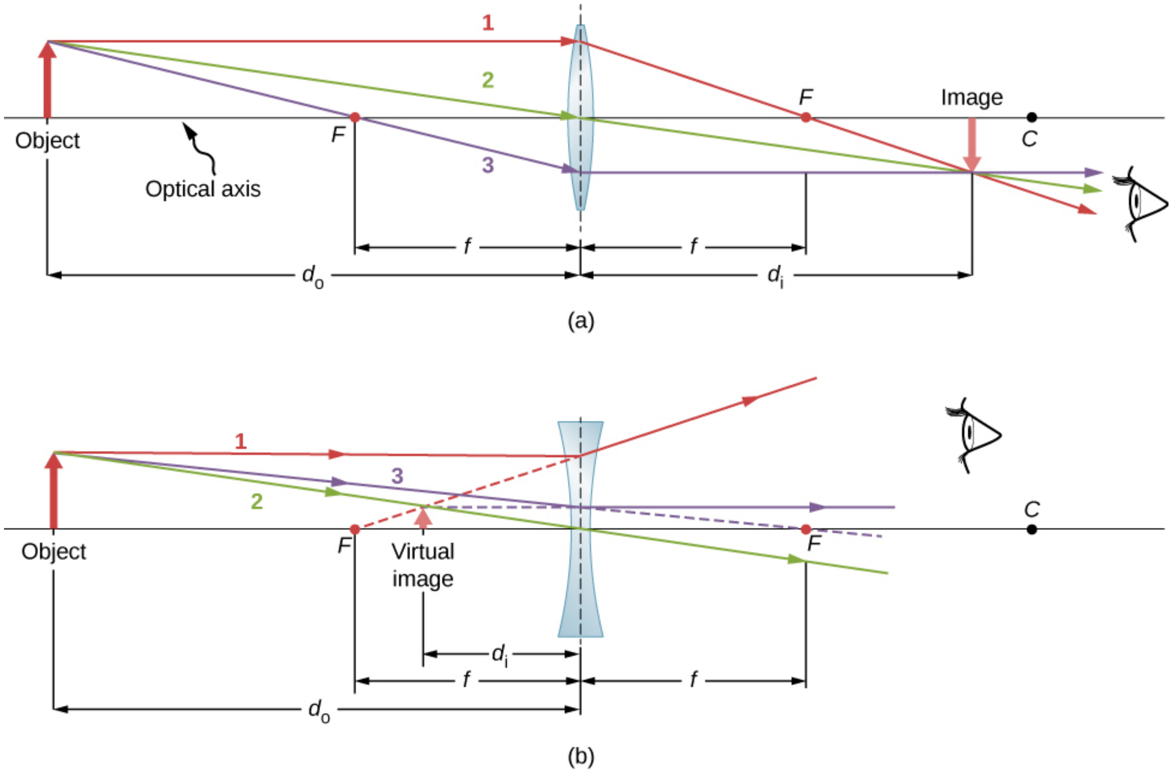


Figure 11.10.5.5: Thin lenses have the same focal lengths on either side. (a) Parallel light rays entering a converging lens from the right cross at its focal point on the left. (b) Parallel light rays entering a diverging lens from the right seem to come from the focal point on the right.

Thin lenses work quite well for monochromatic light (i.e., light of a single wavelength). However, for light that contains several wavelengths (e.g., white light), the lenses work less well. The problem is that, as we learned in the previous chapter, the index of refraction of a material depends on the wavelength of light. This phenomenon is responsible for many colorful effects, such as rainbows. Unfortunately, this phenomenon also leads to aberrations in images formed by lenses. In particular, because the focal distance of the lens depends on the index of refraction, it also depends on the wavelength of the incident light. This means that light of different wavelengths will focus at different points, resulting is so-called "**chromatic aberrations**." In particular, the edges of an image of a white object will become colored and blurred. Special lenses called **doublets** are capable of correcting chromatic aberrations. A doublet is formed by gluing together a converging lens and a diverging lens. The combined doublet lens produces significantly reduced chromatic aberrations.

Image Formation by Thin Lenses

We use ray tracing to investigate different types of images that can be created by a lens. In some circumstances, a lens forms a real image, such as when a movie projector casts an image onto a screen. In other cases, the image is a virtual image, which cannot be projected onto a screen. Where, for example, is the image formed by eyeglasses? We use ray tracing for thin lenses to illustrate how they form images, and then we develop equations to analyze quantitatively the properties of thin lenses.

Consider an object some distance away from a converging lens, as shown in Figure 11.10.5.6 To find the location and size of the image, we trace the paths of selected light rays originating from one point on the object, in this case, the tip of the arrow. The figure shows three rays from many rays that emanate from the tip of the arrow. These three rays can be traced by using the ray-tracing rules given above.

- Ray 1 enters the lens parallel to the optical axis and passes through the focal point on the opposite side (rule 1).
- Ray 2 passes through the center of the lens and is not deviated (rule 2).
- Ray 3 passes through the focal point on its way to the lens and exits the lens parallel to the optical axis (rule 3).

The three rays cross at a single point on the opposite side of the lens. Thus, the image of the tip of the arrow is located at this point. All rays that come from the tip of the arrow and enter the lens are refracted and cross at the point shown.

After locating the image of the tip of the arrow, we need another point of the image to orient the entire image of the arrow. We chose to locate the image base of the arrow, which is on the optical axis. As explained in the section on spherical mirrors, the base will be on the optical axis just above the image of the tip of the arrow (due to the top-bottom symmetry of the lens). Thus, the image spans the optical axis to the (negative) height shown. Rays from another point on the arrow, such as the middle of the arrow, cross at another common point, thus filling in the rest of the image.

Although three rays are traced in this figure, only two are necessary to locate a point of the image. It is best to trace rays for which there are simple ray-tracing rules.

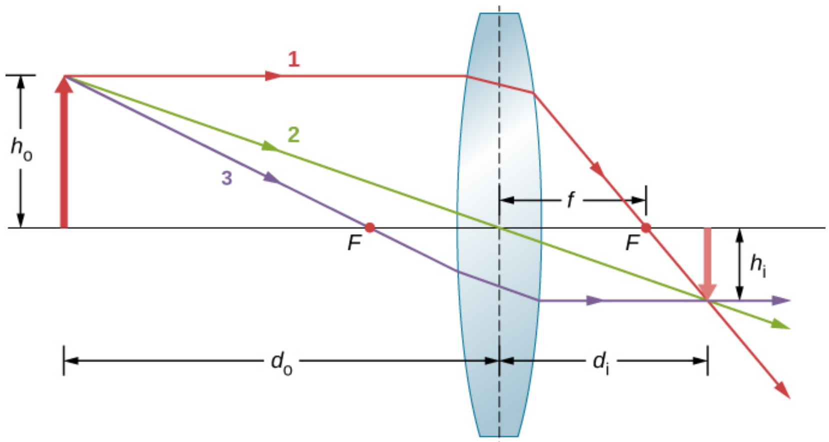


Figure 11.10.5.6: Ray tracing is used to locate the image formed by a lens. Rays originating from the same point on the object are traced—the three chosen rays each follow one of the rules for ray tracing, so that their paths are easy to determine. The image is located at the point where the rays cross. In this case, a real image—one that can be projected on a screen—is formed.

Several important distances appear in the figure. As for a mirror, we define d_o to be the object distance, or the distance of an object from the center of a lens. The image distance d_i is defined to be the distance of the image from the center of a lens. The height of the object and the height of the image are indicated by h_o and h_i, respectively. Images that appear upright relative to the object have positive heights, and those that are inverted have negative heights. By using the rules of ray tracing and making a scale drawing with paper and pencil, like that in Figure 11.10.5.6, we can accurately describe the location and size of an image. But the real benefit of ray tracing is in visualizing how images are formed in a variety of situations.

Oblique Parallel Rays and Focal Plane

We have seen that rays parallel to the optical axis are directed to the focal point of a converging lens. In the case of a diverging lens, they come out in a direction such that they appear to be coming from the focal point on the opposite side of the lens (i.e., the side from which parallel rays enter the lens). What happens to parallel rays that are not parallel to the optical axis (Figure 11.10.5.7)? In the case of a converging lens, these rays do not converge at the focal point. Instead, they come together on another point in the plane called the focal plane. The focal plane contains the focal point and is perpendicular to the optical axis. As shown in the figure, parallel rays focus where the ray through the center of the lens crosses the focal plane.

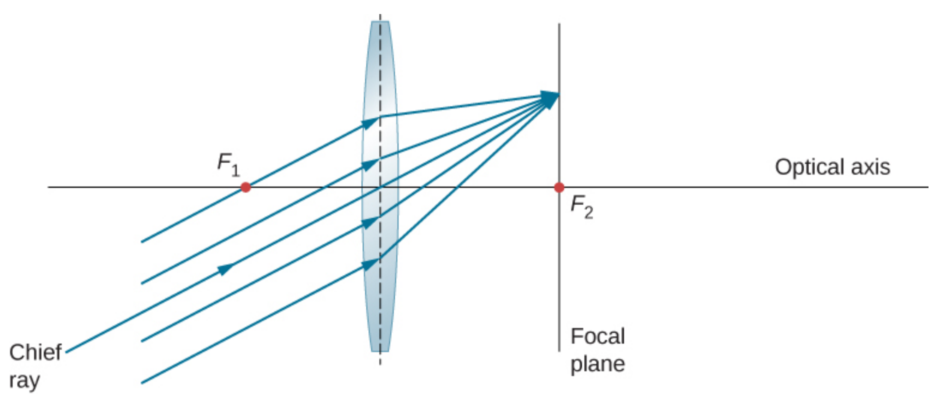


Figure 11.10.5.7: Parallel oblique rays focus on a point in a focal plane.

Thin-Lens Equation

Ray tracing allows us to get a qualitative picture of image formation. To obtain numeric information, we derive a pair of equations from a geometric analysis of ray tracing for thin lenses. These equations, called the thin-lens equation and the lens maker's equation, allow us to quantitatively analyze thin lenses.

Consider the thick bi-convex lens shown in Figure 11.10.5.8 The index of refraction of the surrounding medium is n_1 (if the lens is in air, then $n_1 = 1.00$) and that of the lens is n_2. The radii of curvatures of the two sides are R_1 and R_2. We wish to find a relation between the object distance d_o, the image distance d_i, and the parameters of the lens.

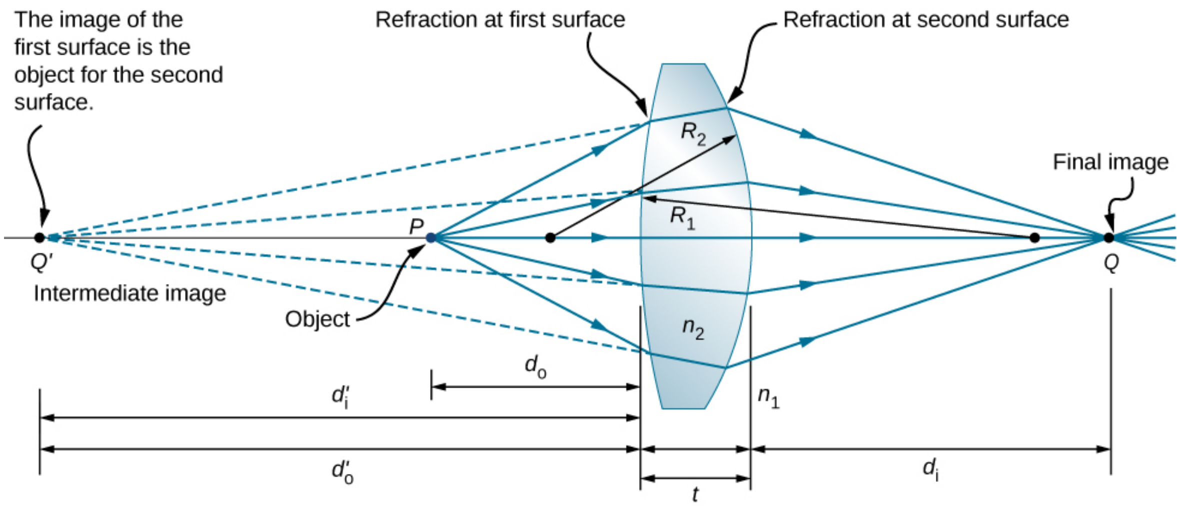


Figure 11.10.5.8 for deriving the lens maker's equation. Here, t is the thickness of lens, n_1 is the index of refraction of the exterior medium, and n_2 is the index of refraction of the lens. We take the limit of $t \to 0$ to obtain the formula for a thin lens.

To derive the thin-lens equation, we consider the image formed by the first refracting surface (i.e., left surface) and then use this image as the object for the second refracting surface. In the figure, the image from the first refracting surface is Q', which is formed by extending backwards the rays from inside the lens (these rays result from refraction at the first surface). This is shown by the dashed lines in the figure. Notice that this image is virtual because no rays actually pass through the point Q′. To find the image distance d'_i corresponding to the image Q′, we use Equation 2.4.3. In this case, the object distance is d_o, the image distance is (d_i\), and the radius of curvature is R_1. Inserting these into the relationship derived previous for refraction at curves surfaces gives

$$\frac{n_1}{d_o} + \frac{n_2}{d'_i} = \frac{n_2 - n_1}{R_1}. \tag{11.10.5.1}$$

The image is virtual and on the same side as the object, so $d_i'<0$ and $d_o>0$. The first surface is convex toward the object, so $R_1 > 0$.

To find the object distance for the object Q formed by refraction from the second interface, note that the role of the indices of refraction n_1 and n_2 are interchanged in Equation 2.4.3. In Figure 11.10.5.8 the rays originate in the medium with index n_2, whereas in Figure 2.4.3, the rays originate in the medium with index n_1. Thus, we must interchange n_1 and n_2 in Equation 2.4.3. In addition, by consulting again Figure 11.10.5.8 we see that the object distance is d'_o and the image distance is d_i. The radius of curvature is R_2 Inserting these quantities into Equation 2.4.3 gives

$$\frac{n_2}{d'_o} + \frac{n_1}{d_i} = \frac{n_1 - n_2}{R_2}. \tag{11.10.5.2}$$

The image is real and on the opposite side from the object, so $d_i > 0$ and $d_o{'} > 0$. The second surface is convex away from the object, so $R_2 < 0$. Equation 11.10.5.2can be simplified by noting that

$$d'_o = |d'_i| + t,$$

where we have taken the absolute value because d'_i is a negative number, whereas both d'_o and t are positive. We can dispense with the absolute value if we negate d'_i, which gives

$$d'_o = -d'_i + td.$$

Inserting this into Equation 11.10.5.2gives

$$\frac{n_2}{-d'_i+t}+\frac{n_1}{d_i}=\frac{n_1-n_2}{R_2}. \tag{11.10.5.3}$$

Summing Equations 11.10.5.2and 11.10.5.3gives

$$\frac{n_1}{d_o}+\frac{n_1}{d_i}+\frac{n_2}{d'_i}+\frac{n_2}{-d'_i+t}=(n_2-n_1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right). \tag{11.10.5.4}$$

In the thin-lens approximation, we assume that the lens is very thin compared to the first image distance, or $t \ll d'_i$ (or, equivalently, $t \ll R_1$ and $t \ll R_2$). In this case, the third and fourth terms on the left-hand side of Equation 11.10.5.4 cancel, leaving us with

$$\frac{n_1}{d_o}+\frac{n_1}{d_i}=(n_2-n_1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Dividing by n_1 gives us finally

$$\frac{1}{d_o}+\frac{1}{d_i}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right). \tag{11.10.5.5}$$

The left-hand side looks suspiciously like the mirror equation that we derived above for spherical mirrors. As done for spherical mirrors, we can use ray tracing and geometry to show that, for a thin lens,

$$\underbrace{\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}}_{\text{thin-lens equation}} \tag{11.10.5.6}$$

where f is the focal length of the thin lens (this derivation is left as an exercise). This is the *thin-lens equation*. The focal length of a thin lens is the same to the left and to the right of the lens. Combining Equations 11.10.5.6and 11.10.5.5gives

$$\underbrace{\frac{1}{f}=\left(\frac{n_2}{n_1}-1\right)\left(\frac{1}{R_1}-\frac{1}{R_2}\right)}_{\text{lens maker's equation}} \tag{11.10.5.7}$$

which is called the **lens maker's equation**. It shows that the focal length of a thin lens depends only of the radii of curvature and the index of refraction of the lens and that of the surrounding medium. For a lens in air, $n_1=1.0$ and $n_2 \equiv n$, so the lens maker's equation reduces to

$$\frac{1}{f}=(n-1)\left(\frac{1}{R_1}-\frac{1}{R_2}\right).$$

Sign conventions for lenses

To properly use the thin-lens equation, the following sign conventions must be obeyed:

- d_i is positive if the image is on the side opposite the object (i.e., real image); otherwise, d_i is negative (i.e., virtual image).
- f is positive for a converging lens and negative for a diverging lens.
- R is positive for a surface convex toward the object, and negative for a s urface concave toward object.

Magnification

By using a finite-size object on the optical axis and ray tracing, you can show that the magnification m of an image is

$$m \equiv \frac{h_i}{h_o}=-\frac{d_i}{d_o} \tag{11.10.5.8}$$

(where the three lines mean "is defined as"). This is exactly the same equation as we obtained for mirrors (see Equation 2.3.15). If $m>0$, then the image has the same vertical orientation as the object (called an "upright" image). If m<0, then the image has the opposite vertical orientation as the object (called an "**inverted**" image).

Using the Thin-Lens Equation

The thin-lens equation and the lens maker's equation are broadly applicable to situations involving thin lenses. We explore many features of image formation in the following examples.

Consider a thin converging lens. Where does the image form and what type of image is formed as the object approaches the lens from infinity? This may be seen by using the thin-lens equation for a given focal length to plot the image distance as a function of object distance. In other words, we plot

$$d_i = \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1}$$

for a given value of f. For $f = 1\,cm$, the result is shown in Figure $11.10.5.9a$

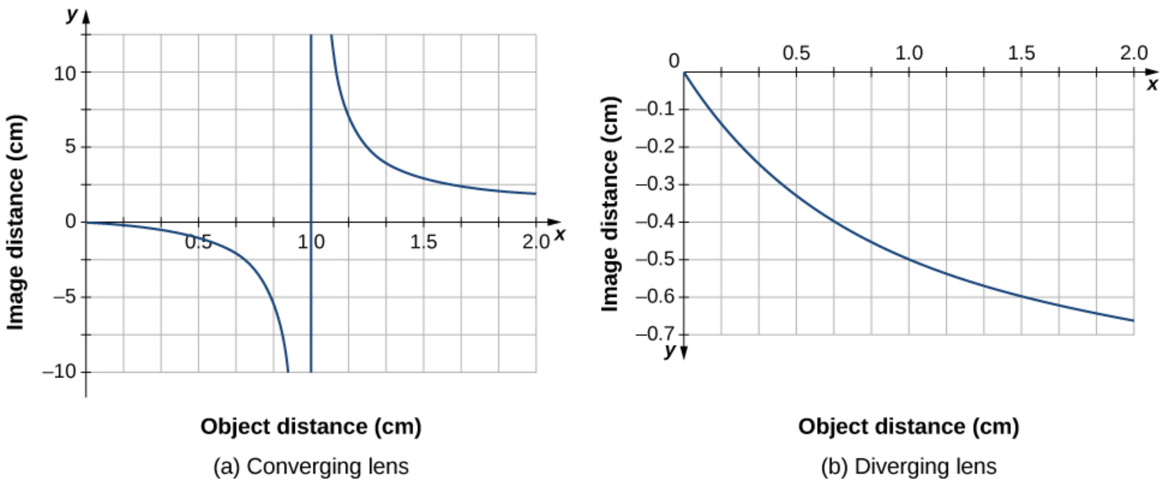


Figure 11.10.5.9: (a) Image distance for a thin converging lens with f=1.0 cm as a function of object distance. (b) Same thing but for a diverging lens with f=−1.0 cm.

An object much farther than the focal length **f** from the lens should produce an image near the focal plane, because the second term on the right-hand side of the equation above becomes negligible compared to the first term, so we have $d_i \approx f$. This can be seen in the plot of part (a) of the figure, which shows that the image distance approaches asymptotically the focal length of 1 cm for larger object distances. As the object approaches the focal plane, the image distance diverges to positive infinity. This is expected because an object at the focal plane produces parallel rays that form an image at infinity (i.e., very far from the lens). When the object is farther than the focal length from the lens, the image distance is positive, so the image is real, on the opposite side of the lens from the object, and inverted (because $m = -d_i/d_o$ via Equation 11.10.5.8). When the object is closer than the focal length from the lens, the image distance becomes negative, which means that the image is virtual, on the same side of the lens as the object, and upright.

For a thin diverging lens of focal length $f = -1.0\,cm$, a similar plot of image distance vs. object distance is shown in Figure $11.10.5.10b$ In this case, the image distance is negative for all positive object distances, which means that the image is virtual, on the same side of the lens as the object, and upright. These characteristics may also be seen by ray-tracing diagrams (Figure $11.10.5.10$).

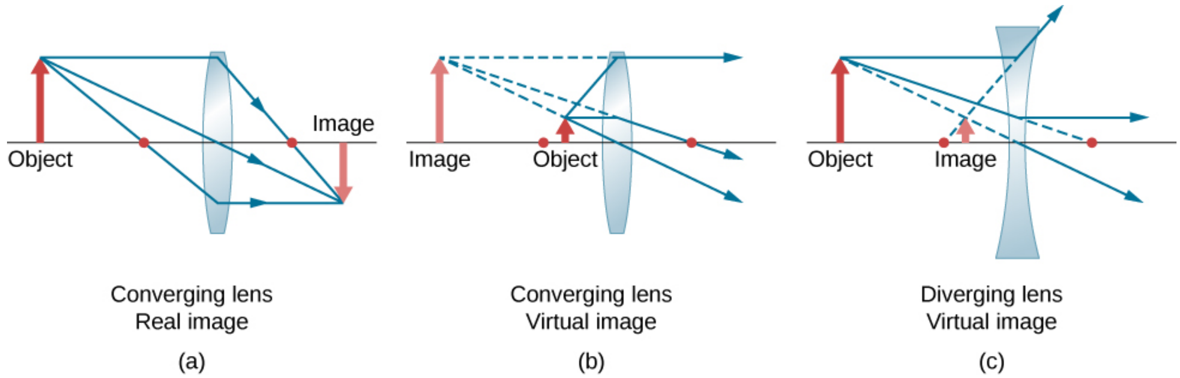


Figure 11.10.5.10: The red dots show the focal points of the lenses. (a) A real, inverted image formed from an object that is farther than the focal length from a converging lens. (b) A virtual, upright image formed from an object that is closer than a focal length from the lens. (c) A virtual, upright image formed from an object that is farther than a focal length from a diverging lens.

To see a concrete example of upright and inverted images, look at Figure 11.10.5.11, which shows images formed by converging lenses when the object (the person's face in this case) is place at different distances from the lens. In part (a) of the figure, the person's face is farther than one focal length from the lens, so the image is inverted. In part (b), the person's face is closer than one focal length from the lens, so the image is upright.

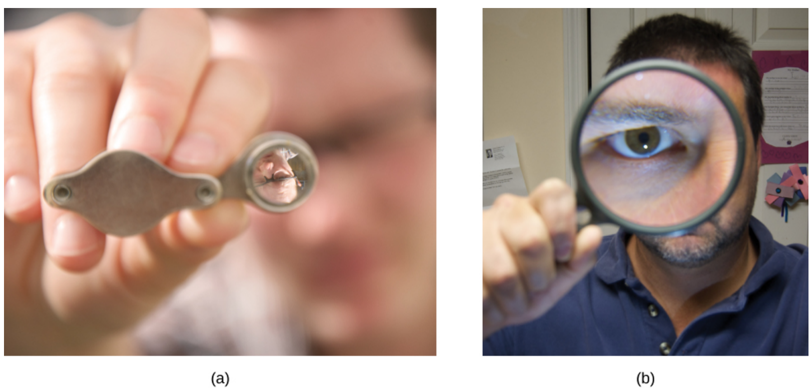


Figure 11.10.5.11: (a) When a converging lens is held farther than one focal length from the man's face, an inverted image is formed. Note that the image is in focus but the face is not, because the image is much closer to the camera taking this photograph than the face. (b) An upright image of the man's face is produced when a converging lens is held at less than one focal length from his face. (credit a: modification of work by "DaMongMan"/Flickr; credit b: modification of work by Casey Fleser)

Work through the following examples to better understand how thin lenses work.

PROBLEM-SOLVING STRATEGY: LENSES

- Step 1. Determine whether ray tracing, the thin-lens equation, or both would be useful. Even if ray tracing is not used, a careful sketch is always very useful. Write symbols and values on the sketch.
- Step 2. Identify what needs to be determined in the problem (identify the unknowns).
- Step 3. Make a list of what is given or can be inferred from the problem (identify the knowns).
- Step 4. If ray tracing is required, use the ray-tracing rules listed near the beginning of this section.
- Step 5. Most quantitative problems require the use of the thin-lens equation and/or the lens maker's equation. Solve these for the unknowns and insert the given quantities or use both together to find two unknowns.
- Step 7. Check to see if the answer is reasonable. Are the signs correct? Is the sketch or ray tracing consistent with the calculation?

Example 11.10.5.1: Using the Lens Maker's Equation

Find the radius of curvature of a biconcave lens symmetrically ground from a glass with index of refractive 1.55 so that its focal length in air is 20 cm (for a biconcave lens, both surfaces have the same radius of curvature).

Strategy

Use the thin-lens form of the lens maker's equation:

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

where $R_1 < 0$ and $R_2 > 0$. Since we are making a symmetric biconcave lens, we have $|R_1| = |R_2|$.

Solution

We can determine the radius R of curvature from

$$\frac{1}{f} = \left(\frac{n_2}{n_1} - 1\right)\left(\frac{-2}{R}\right).$$

Solving for R and inserting $f = -20\ cm$, $n_2 = 1.55$, and $n_1 = 1.00$ gives

$$\begin{aligned} R &= -2f\left(\frac{n_2}{n_1} - 1\right) \\ &= -2(-20\ cm)\left(\frac{1.55}{1.00} - 1\right) \\ &= 22\ cm. \end{aligned}$$

Example 11.10.5.2: Converging Lens and Different Object Distances

Find the location, orientation, and magnification of the image for an 3.0 cm high object at each of the following positions in front of a convex lens of focal length 10.0 cm. (a) $d_o = 50.0\ cm$, (b) $d_o = 5.00\ cm$, and (c) $d_o = 20.0\ cm$.

Strategy

We start with the thin-lens equation (Equation 11.10.5.6)

$$\frac{1}{d_i} + \frac{1}{d_o} = \frac{1}{f}.$$

Solve this for the image distance d_i and insert the given object distance and focal length.

Solution

a. For $d_o = 50\ cm$ and $f = +10\ cm$, this gives

$$\begin{aligned} d_i &= \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1} \\ &= \left(\frac{1}{10.0\ cm} - \frac{1}{50.0 cm}\right)^{-1} \\ &= 12.5\ cm \end{aligned}$$

The image is positive, so the image, is real, is on the opposite side of the lens from the object, and is 12.6 cm from the lens. To find the magnification and orientation of the image, use

$$\begin{aligned} m &= -\frac{d_i}{d_o} \\ &= -\frac{12.5\ cm}{50.0\ cm} \\ &= -0.250. \end{aligned}$$

The negative magnification means that the image is inverted. Since $|m| < 1$, the image is smaller than the object. The size of the image is given by

$$|h_i| = |m|h_o$$
$$= (0.250)(3.0\,cm)$$
$$= 0.75\,cm$$

b. For $d_o = 5.00\,cm$ and $f = +10.0\,cm$

$$d_i = \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1}$$
$$= \left(\frac{1}{10.0\,cm} - \frac{1}{5.00\,cm}\right)^{-1}$$
$$= -10.0\,cm$$

The image distance is negative, so the image is virtual, is on the same side of the lens as the object, and is 10 cm from the lens. The magnification and orientation of the image are found from

$$m = -\frac{d_i}{d_o}$$
$$= -\frac{-10.0\,cm}{5.00\,cm}$$
$$= +2.00.$$

The positive magnification means that the image is upright (i.e., it has the same orientation as the object). Since $|m| > 0$, the image is larger than the object. The size of the image is

$$|h_i| = |m|h_o$$
$$= (2.00)(3.0\,cm)$$
$$= 6.0\,cm.$$

c. For $d_o = 20\,cm$ and $f = +10cm$

$$d_i = \left(\frac{1}{f} - \frac{1}{d_o}\right)^{-1}$$
$$= \left(\frac{1}{10.0\,cm} - \frac{1}{20.0\,cm}\right)^{-1}$$
$$= 20.0\,cm$$

The image distance is positive, so the image is real, is on the opposite side of the lens from the object, and is 20.0 cm from the lens. The magnification is

$$m = -\frac{d_i}{d_o}$$
$$= -\frac{20.0\,cm}{20.0\,cm}$$
$$= -1.00.$$

The negative magnification means that the image is inverted. Since $|m| = 1$, the image is the same size as the object.

When solving problems in geometric optics, we often need to combine ray tracing and the lens equations. The following example demonstrates this approach.

Example 11.10.5.3: Choosing the Focal Length and Type of Lens

To project an image of a light bulb on a screen 1.50 m away, you need to choose what type of lens to use (converging or diverging) and its focal length (Figure $11.10.5.12$). The distance between the lens and the light bulb is fixed at 0.75 m. Also, what is the magnification and orientation of the image?

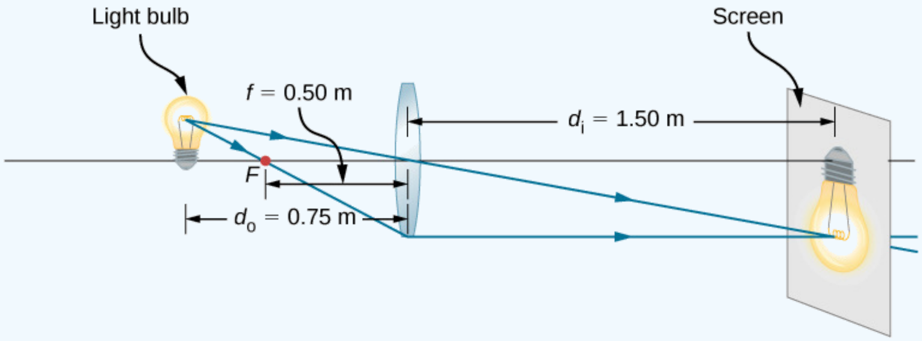


Figure 11.10.5.12: A light bulb placed 0.75 m from a lens having a 0.50-m focal length produces a real image on a screen, as discussed in the example. Ray tracing predicts the image location and size.

Strategy

The image must be real, so you choose to use a converging lens. The focal length can be found by using the thin-lens equation and solving for the focal length. The object distance is $d_o = 0.75\,m$ and the image distance is $d_i = 1.5\,m$.

Solution

Solve the thin lens for the focal length and insert the desired object and image distances:

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

$$\begin{aligned} f &= \left(\frac{1}{d_o} + \frac{1}{d_i}\right)^{-1} \\ &= \left(\frac{1}{0.75\,m} + \frac{1}{1.5\,m}\right)^{-1} \\ &= 0.50\,m \end{aligned}$$

The magnification is

$$\begin{aligned} m &= -\frac{d_i}{d_o} \\ &= -\frac{1.5\,m}{0.75\,m} \\ &= -2.0. \end{aligned}$$

Significance

The minus sign for the magnification means that the image is inverted. The focal length is positive, as expected for a converging lens. Ray tracing can be used to check the calculation (Figure $11.10.5.12$). As expected, the image is inverted, is real, and is larger than the object.

11.10.6: The Eye

Learning Objectives

By the end of this section, you will be able to:

- Understand the basic physics of how images are formed by the human eye
- Recognize several conditions of impaired vision as well as the optics principles for treating these conditions

Physics of the Eye

The eye is remarkable in how it forms images and in the richness of detail and color it can detect. However, our eyes often need some correction to reach what is called "normal" vision. Actually, normal vision should be called "ideal" vision because nearly one-half of the human population requires some sort of eyesight correction, so requiring glasses is by no means "abnormal." Image formation by our eyes and common vision correction can be analyzed with the optics discussed earlier in this chapter.

Figure $11.10.6.1$shows the basic anatomy of the eye. The cornea and lens form a system that, to a good approximation, acts as a single thin lens. For clear vision, a real image must be projected onto the light-sensitive retina, which lies a fixed distance from the lens. The flexible lens of the eye allows it to adjust the radius of curvature of the lens to produce an image on the retina for objects at different distances. The center of the image falls on the fovea, which has the greatest density of light receptors and the greatest acuity (sharpness) in the visual field. The variable opening (i.e., the pupil) of the eye, along with chemical adaptation, allows the eye to detect light intensities from the lowest observable to 10^{10}times greater (without damage). This is an incredible range of detection. Processing of visual nerve impulses begins with interconnections in the retina and continues in the brain. The optic nerve conveys the signals received by the eye to the brain.

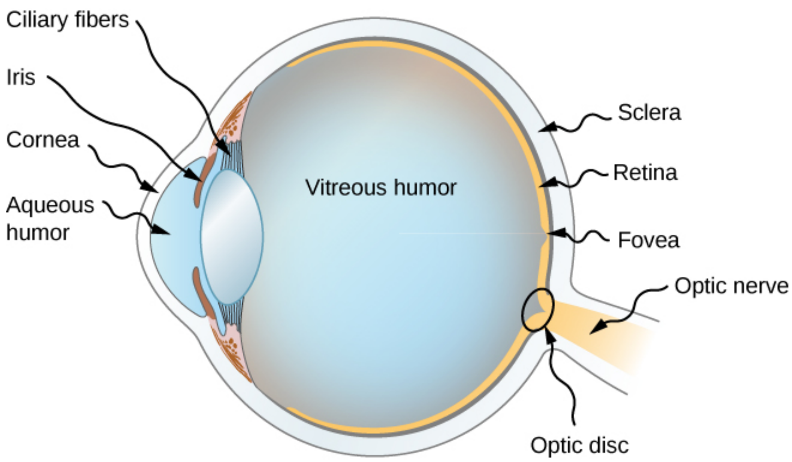


Figure 11.10.6.1: The cornea and lens of the eye act together to form a real image on the light-sensing retina, which has its densest concentration of receptors in the fovea and a blind spot over the optic nerve. The radius of curvature of the lens of an eye is adjustable to form an image on the retina for different object distances. Layers of tissues with varying indices of refraction in the lens are shown here. However, they have been omitted from other pictures for clarity.

The indices of refraction in the eye are crucial to its ability to form images. Table $11.10.6.1$lists the indices of refraction relevant to the eye. The biggest change in the index of refraction, which is where the light rays are most bent, occurs at the air-cornea interface rather than at the aqueous humor-lens interface. The ray diagram in Figure $11.10.6.2$shows image formation by the cornea and lens of the eye. The cornea, which is itself a converging lens with a focal length of approximately 2.3 cm, provides most of the focusing power of the eye. The lens, which is a converging lens with a focal length of about 6.4 cm, provides the finer focus needed to produce a clear image on the retina. The cornea and lens can be treated as a single thin lens, even though the light rays pass through several layers of material (such as cornea, aqueous humor, several layers in the lens, and vitreous humor), changing direction at each interface. The image formed is much like the one produced by a single convex lens (i.e., a real, inverted image). Although images formed in the eye are inverted, the brain inverts them once more to make them seem upright.

Table 11.10.6.1: Refractive Indices Relevant to the Eye*This is an average value. The actual index of refraction varies throughout the lens and is greatest in center of the lens.

Material	Index of Refraction

Material	Index of Refraction
Water	1.33
Air	1.0
Cornea	1.38
Aqueous humor	1.34
Lens	1.41*
Vitreous humor	1.34

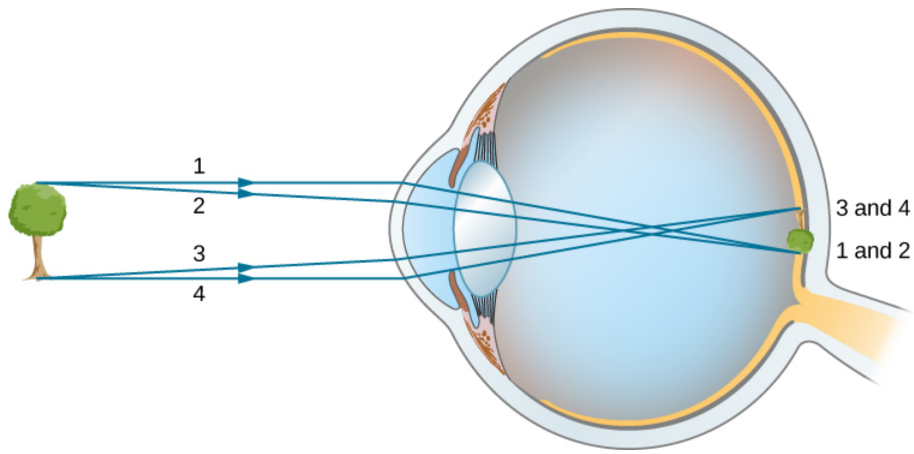


Figure 11.10.6.2: In the human eye, an image forms on the retina. Rays from the top and bottom of the object are traced to show how a real, inverted image is produced on the retina. The distance to the object is not to scale.

As noted, the image must fall precisely on the retina to produce clear vision—that is, the image distance d_i must equal the lens-to-retina distance. Because the lens-to-retina distance does not change, the image distance d_i must be the same for objects at all distances. The ciliary muscles adjust the shape of the eye lens for focusing on nearby or far objects. By changing the shape of the eye lens, the eye changes the focal length of the lens. This mechanism of the eye is called accommodation.

The nearest point an object can be placed so that the eye can form a clear image on the retina is called the near point of the eye. Similarly, the far point is the farthest distance at which an object is clearly visible. A person with normal vision can see objects clearly at distances ranging from 25 cm to essentially infinity. The near point increases with age, becoming several meters for some older people. In this text, we consider the near point to be 25 cm.

We can use the thin-lens equations to quantitatively examine image formation by the eye. First, we define the optical power of a lens as

$$P=\frac{1}{f}$$

with the focal length **f** given in meters. The units of optical power are called “diopters” (D). That is, 1D=1/m,or $1m^{-1}$. Optometrists prescribe common eyeglasses and contact lenses in units of diopters. With this definition of optical power, we can rewrite the thin-lens equations as

$$P=\frac{1}{d_o}+\frac{1}{d_i}.$$

Working with optical power is convenient because, for two or more lenses close together, the effective optical power of the lens system is approximately the sum of the optical power of the individual lenses:

$$P_{total}=P_{lens\,1}+P_{lens\,2}+P_{lens\,3}+\cdots \tag{11.10.6.1}$$

Example 11.10.6.1: Effective Focal Length of the Eye

The cornea and eye lens have focal lengths of 2.3 and 6.4 cm, respectively. Find the net focal length and optical power of the eye.

Strategy

The optical powers of the closely spaced lenses add, so $P_{eye} = P_{cornea} + P_{lens}$.

Solution

Writing the equation for power in terms of the focal lengths gives

$$\frac{1}{f_{eye}} = \frac{1}{f_{cornea}} + \frac{1}{f_{lens}} = \frac{1}{2.3cm} + \frac{1}{6.4cm}.$$

Hence, the focal length of the eye (cornea and lens together) is

$$f_{eye} = 1.69cm.$$

The optical power of the eye is

$$P_{eye} = \frac{1}{f_{eye}} = \frac{1}{0.0169m} = 59D.$$

For clear vision, the image distance d_i must equal the lens-to-retina distance. Normal vision is possible for objects at distances $d_o = 25\,cm$ to infinity. The following example shows how to calculate the image distance for an object placed at the near point of the eye.

Example 11.10.6.2: Image of an object placed at the near point

The net focal length of a particular human eye is 1.7 cm. An object is placed at the near point of the eye. How far behind the lens is a focused image formed?

Strategy

The near point is 25 cm from the eye, so the object distance is d_o=25 cm. We determine the image distance from the lens equation:

$$\frac{1}{d_i} = \frac{1}{f} - \frac{1}{d_o}.$$

Solution

$$d_i = (\frac{1}{f} - \frac{1}{d_o})^{-1}$$

$$= (\frac{1}{1.7cm} - \frac{1}{25cm})^{-1}$$

$$= 1.8cm$$

Therefore, the image is formed 1.8 cm behind the lens.

Significance

From the magnification formula, we find $m = -\frac{1.8cm}{25cm} = -0.073$. Since m<0, the image is inverted in orientation with respect to the object. From the absolute value of **m** we see that the image is much smaller than the object; in fact, it is only 7% of the size of the object.

Vision Correction

The need for some type of vision correction is very common. Typical vision defects are easy to understand with geometric optics, and some are simple to correct. Figure 11.10.6.3illustrates two common vision defects. Nearsightedness, or myopia, is the ability to see near objects, whereas distant objects are blurry. The eye over converges the nearly parallel rays from a distant object, and the rays cross in front of the retina. More divergent rays from a close object are converged on the retina for a clear image. The distance to the farthest object that can be seen clearly is called the far point of the eye (normally the far point is at infinity). Farsightedness, or hyperopia, is the ability to see far objects clearly, whereas near objects are blurry. A farsighted eye does not sufficiently converge the rays from a near object to make the rays meet on the retina.

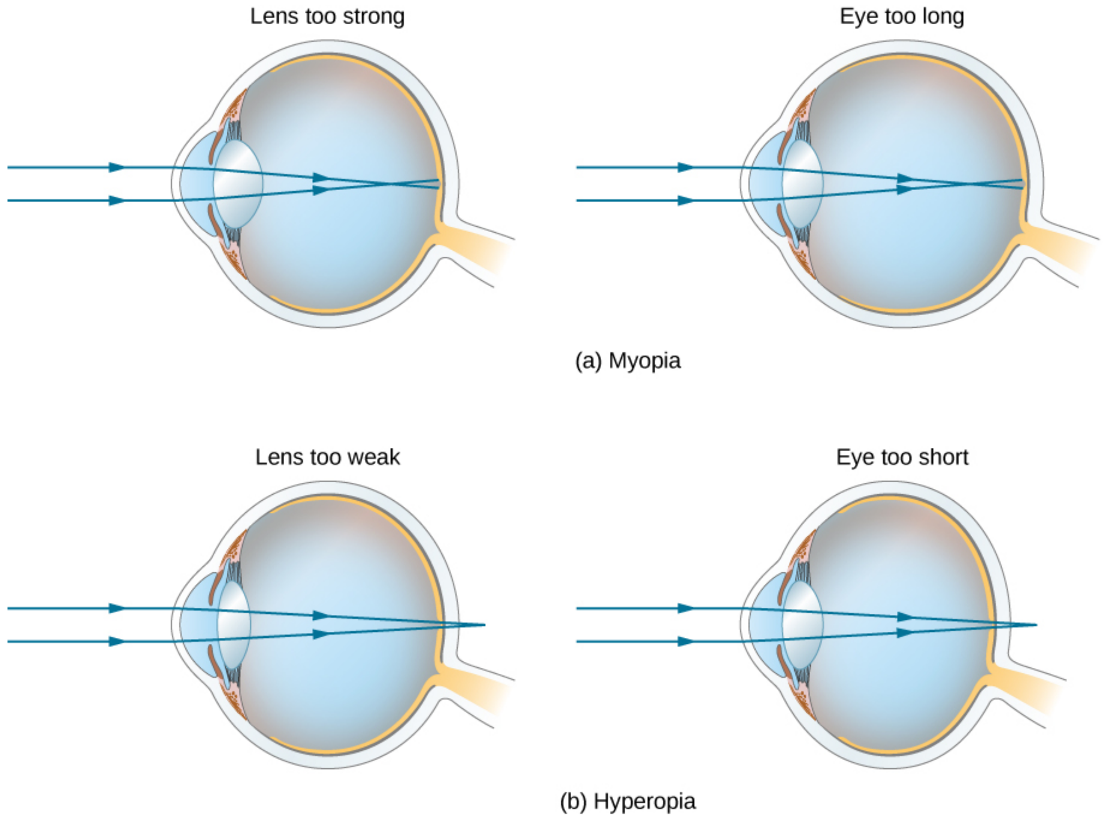


Figure 11.10.6.3: (a) The nearsighted (myopic) eye converges rays from a distant object in front of the retina, so they have diverged when they strike the retina, producing a blurry image. An eye lens that is too powerful can cause nearsightedness, or the eye may be too long. (b) The farsighted (hyperopic) eye is unable to converge the rays from a close object on the retina, producing blurry near-field vision. An eye lens with insufficient optical power or an eye that is too short can cause farsightedness.

Since the nearsighted eye over converges light rays, the correction for nearsightedness consists of placing a diverging eyeglass lens in front of the eye, as shown in Figure 11.10.6.4 This reduces the optical power of an eye that is too powerful (recall that the focal length of a diverging lens is negative, so its optical power is negative). Another way to understand this correction is that a diverging lens will cause the incoming rays to diverge more to compensate for the excessive convergence caused by the lens system of the eye. The image produced by the diverging eyeglass lens serves as the (optical) object for the eye, and because the eye cannot focus on objects beyond its far point, the diverging lens must form an image of distant (physical) objects at a point that is closer than the far point.

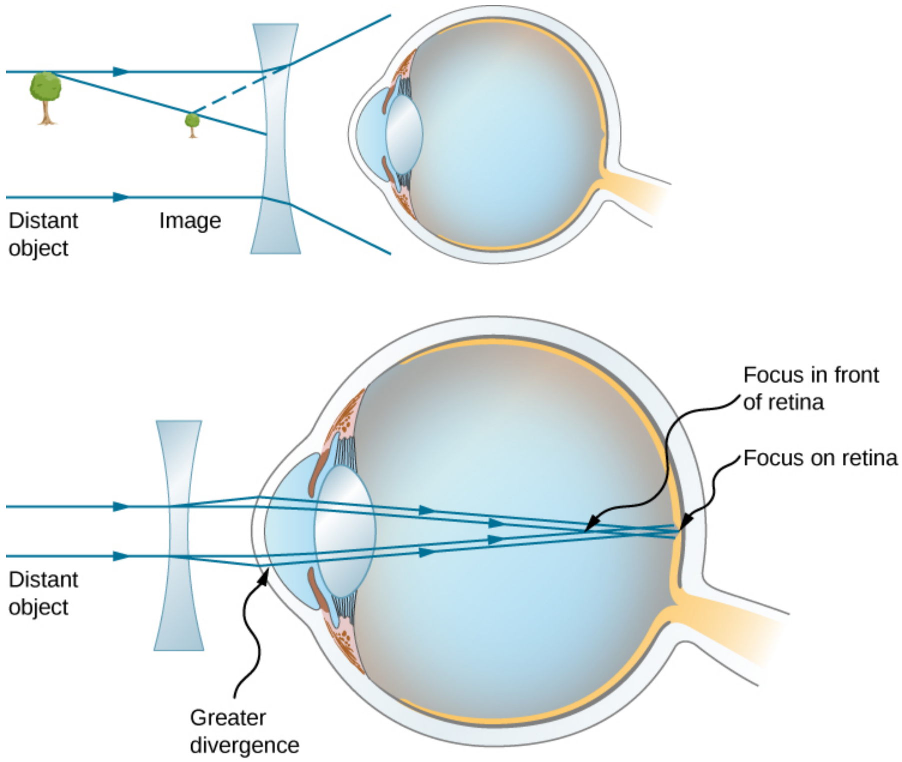


Figure 11.10.6.4: Correction of nearsightedness requires a diverging lens that compensates for over convergence by the eye. The diverging lens produces an image closer to the eye than the physical object. This image serves as the optical object for the eye, and the nearsighted person can see it clearly because it is closer than their far point.

✓ Example 11.10.6.3: Correcting Nearsightedness

What optical power of eyeglass lens is needed to correct the vision of a nearsighted person whose far point is 30.0 cm? Assume the corrective lens is fixed 1.50 cm away from the eye.

Strategy

You want this nearsighted person to be able to see distant objects clearly, which means that the eyeglass lens must produce an image 30.0 cm from the eye for an object at infinity. An image 30.0 cm from the eye will be 30.0 cm−1.50 cm=28.5 cm from the eyeglass lens. Therefore, we must have d_i=−28.5cm when d_o=∞. The image distance is negative because it is on the same side of the eyeglass lens as the object.

Solution

Since d_i and dodo are known, we can find the optical power of the eyeglass lens by using Equation ???:

$$P=\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{\infty}+\frac{1}{-0.285m}=-3.51D.$$

Significance

The negative optical power indicates a diverging (or concave) lens, as expected. If you examine eyeglasses for nearsighted people, you will find the lenses are thinnest in the center. Additionally, if you examine a prescription for eyeglasses for nearsighted people, you will find that the prescribed optical power is negative and given in units of diopters.

Correcting farsightedness consists simply of using the opposite type of lens as for nearsightedness (i.e., a converging lens), as shown in Figure 11.10.6.5

Such a lens will produce an image of physical objects that are closer than the near point at a distance that is between the near point and the far point, so that the person can see the image clearly. To determine the optical power needed for correction, you must therefore know the person’s near point, as explained in Example 11.10.6.4

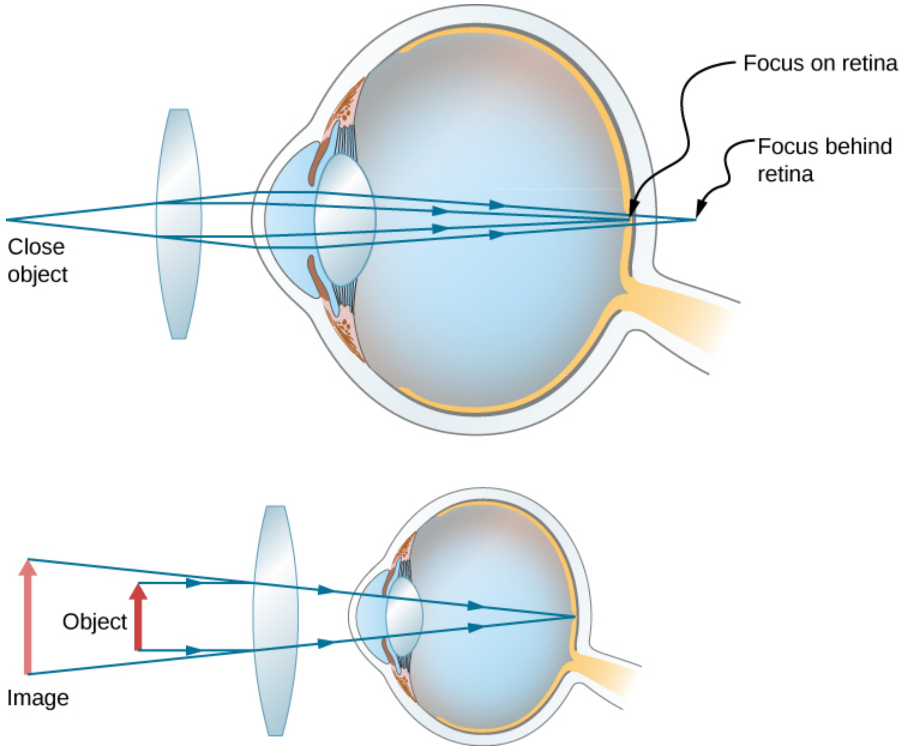


Figure 11.10.6.5: Correction of farsightedness uses a converging lens that compensates for the under convergence by the eye. The converging lens produces an image farther from the eye than the object, so that the farsighted person can see it clearly.

Example 11.10.6.4: Correcting Farsightedness

What optical power of eyeglass lens is needed to allow a farsighted person, whose near point is 1.00 m, to see an object clearly that is 25.0 cm from the eye? Assume the corrective lens is fixed 1.5 cm from the eye.

Strategy

When an object is 25.0 cm from the person's eyes, the eyeglass lens must produce an image 1.00 m away (the near point), so that the person can see it clearly. An image 1.00 m from the eye will be 100cm−1.5cm=98.5cm from the eyeglass lens because the eyeglass lens is 1.5 cm from the eye. Therefore, d_i=−98.5cm, where the minus sign indicates that the image is on the same side of the lens as the object. The object is 25.0cm−1.5cm=23.5cm from the eyeglass lens, so d_o=23.5cm.

Solution

Since d_i and dodo are known, we can find the optical power of the eyeglass lens by using Equation 11.10.6.1:

$$P=\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{0.285m}+\frac{1}{-0.985m}=+3.24D.$$

Significance

The positive optical power indicates a converging (convex) lens, as expected. If you examine eyeglasses of farsighted people, you will find the lenses to be thickest in the center. In addition, prescription eyeglasses for farsighted people have a prescribed optical power that is positive.

11.10.7: The Camera

Learning Objectives

By the end of this section, you will be able to:

- Describe the optics of a camera
- Characterize the image created by a camera

Cameras are very common in our everyday life. Between 1825 and 1827, French inventor Nicéphore Niépce successfully photographed an image created by a primitive camera. Since then, enormous progress has been achieved in the design of cameras and camera-based detectors.

Initially, photographs were recorded by using the light-sensitive reaction of silver-based compounds such as silver chloride or silver bromide. Silver-based photographic paper was in common use until the advent of digital photography in the 1980s, which is intimately connected to **charge-coupled device (CCD)** detectors. In a nutshell, a CCD is a semiconductor chip that records images as a matrix of tiny pixels, each pixel located in a "bin" in the surface. Each pixel is capable of detecting the intensity of light impinging on it. Color is brought into play by putting red-, blue-, and green-colored filters over the pixels, resulting in colored digital images (Figure 11.10.7.1). At its best resolution, one CCD pixel corresponds to one pixel of the image. To reduce the resolution and decrease the size of the file, we can "bin" several CCD pixels into one, resulting in a smaller but "pixelated" image.

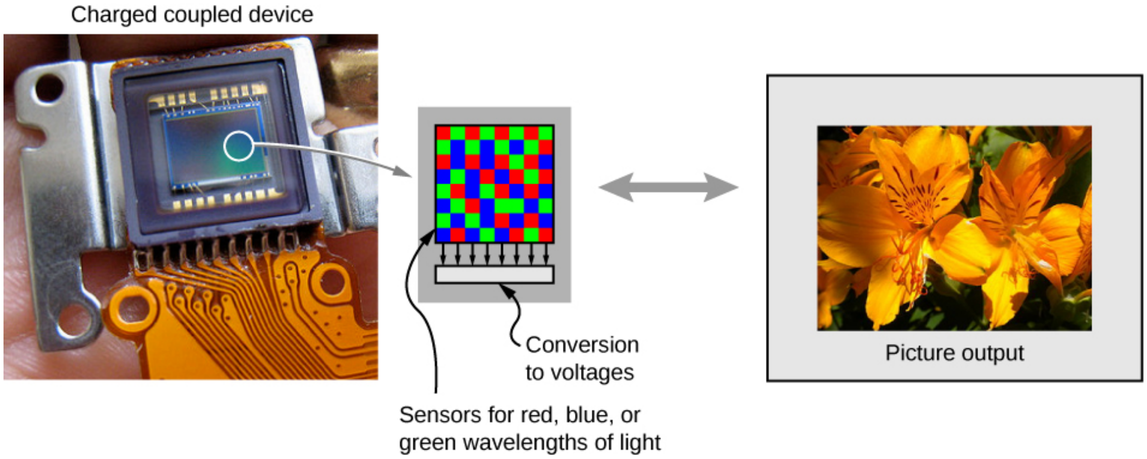


Figure 11.10.7.1: A charge-coupled device (CCD) converts light signals into electronic signals, enabling electronic processing and storage of visual images. This is the basis for electronic imaging in all digital cameras, from cell phones to movie cameras. (credit left: modification of work by Bruce Turner)

Clearly, electronics is a big part of a digital camera; however, the underlying physics is basic optics. As a matter of fact, the optics of a camera are pretty much the same as those of a single lens with an object distance that is significantly larger than the lens's focal distance (Figure 11.10.7.2).

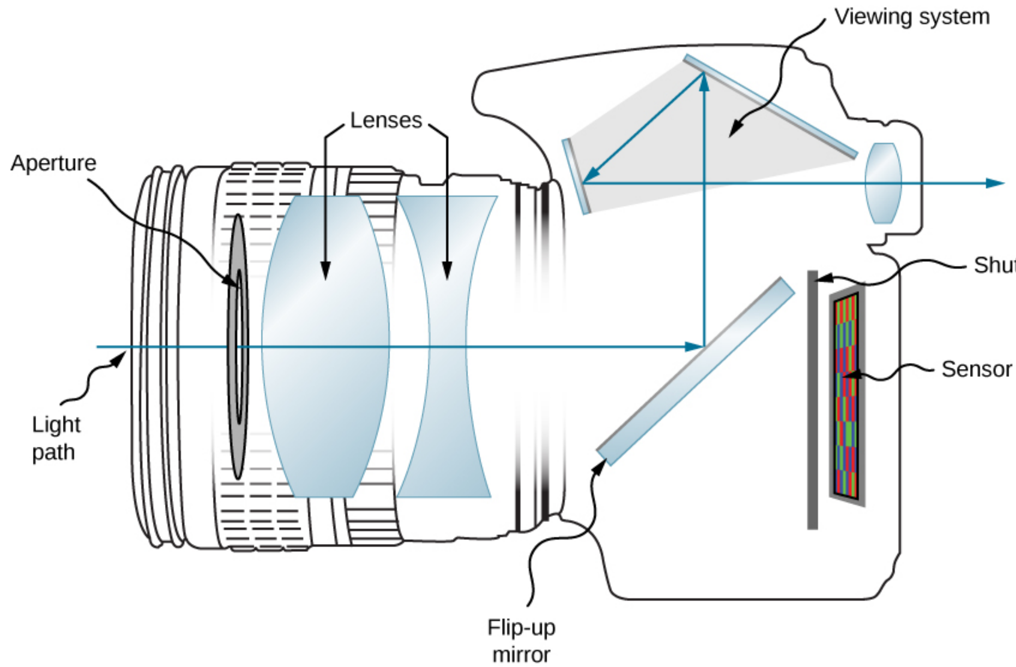


Figure 11.10.7.2: Modern digital cameras have several lenses to produce a clear image with minimal aberration and use red, blue, and green filters to produce a color image.

For instance, let us consider the camera in a smartphone. An average smartphone camera is equipped with a stationary wide-angle lens with a focal length of about 4–5 mm. (This focal length is about equal to the thickness of the phone.) The image created by the lens is focused on the CCD detector mounted at the opposite side of the phone. In a cell phone, the lens and the CCD cannot move relative to each other. So how do we make sure that both the images of a distant and a close object are in focus?

Recall that a human eye can accommodate for distant and close images by changing its focal distance. A cell phone camera cannot do that because the distance from the lens to the detector is fixed. Here is where the small focal distance becomes important. Let us assume we have a camera with a 5-mm focal distance. What is the image distance for a selfie? The object distance for a selfie (the length of the hand holding the phone) is about 50 cm. Using the thin-lens equation, we can write

$$\frac{1}{5mm}=\frac{1}{500mm}+\frac{1}{d_i}$$

We then obtain the image distance:

$$\frac{1}{d_i}=\frac{1}{5mm}-\frac{1}{500mm}$$

Note that the object distance is 100 times larger than the focal distance. We can clearly see that the 1/(500 mm) term is significantly smaller than 1/(5 mm), which means that the image distance is pretty much equal to the lens's focal length. An actual calculation gives us the image distance d_i=5.05mm. This value is extremely close to the lens's focal distance.

Now let us consider the case of a distant object. Let us say that we would like to take a picture of a person standing about 5 m from us. Using the thin-lens equation again, we obtain the image distance of 5.005 mm. The farther the object is from the lens, the closer the image distance is to the focal distance. At the limiting case of an infinitely distant object, we obtain the image distance exactly equal to the focal distance of the lens.

As you can see, the difference between the image distance for a selfie and the image distance for a distant object is just about 0.05 mm or 50 microns. Even a short object distance such as the length of your hand is two orders of magnitude larger than the lens's focal length, resulting in minute variations of the image distance. (The 50-micron difference is smaller than the thickness of an average sheet of paper.) Such a small difference can be easily accommodated by the same detector, positioned at the focal distance of the lens. Image analysis software can help improve image quality.

Conventional point-and-shoot cameras often use a movable lens to change the lens-to-image distance. Complex lenses of the more expensive mirror reflex cameras allow for superb quality photographic images. The optics of these camera lenses is beyond the scope of this textbook.

11.10.8: The Simple Magnifier

Learning Objectives

By the end of this section, you will be able to:

- Understand the optics of a simple magnifier
- Characterize the image created by a simple magnifier

The apparent size of an object perceived by the eye depends on the angle the object subtends from the eye. As shown in Figure $11.10.8.1$, the object at A subtends a larger angle from the eye than when it is position at point B. Thus, the object at A forms a larger image on the retina (see OA') than when it is positioned at B (see OB'). Thus, objects that subtend large angles from the eye appear larger because they form larger images on the retina.

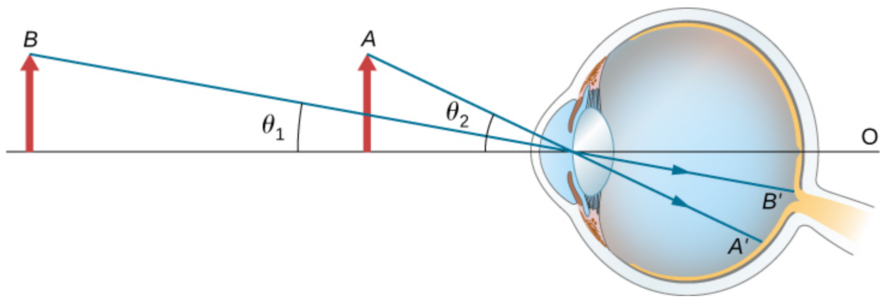


Figure 11.10.8.1: Size perceived by an eye is determined by the angle subtended by the object. An image formed on the retina by an object at A is larger than an image formed on the retina by the same object positioned at B (compared image heights OA' to OB').

We have seen that, when an object is placed within a focal length of a convex lens, its image is virtual, upright, and larger than the object (see part (b) of this Figure). Thus, when such an image produced by a convex lens serves as the object for the eye, as shown in Figure $11.10.8.2$, the image on the retina is enlarged, because the image produced by the lens subtends a larger angle in the eye than does the object. A convex lens used for this purpose is called a **magnifying glass** or a **simple magnifier**.

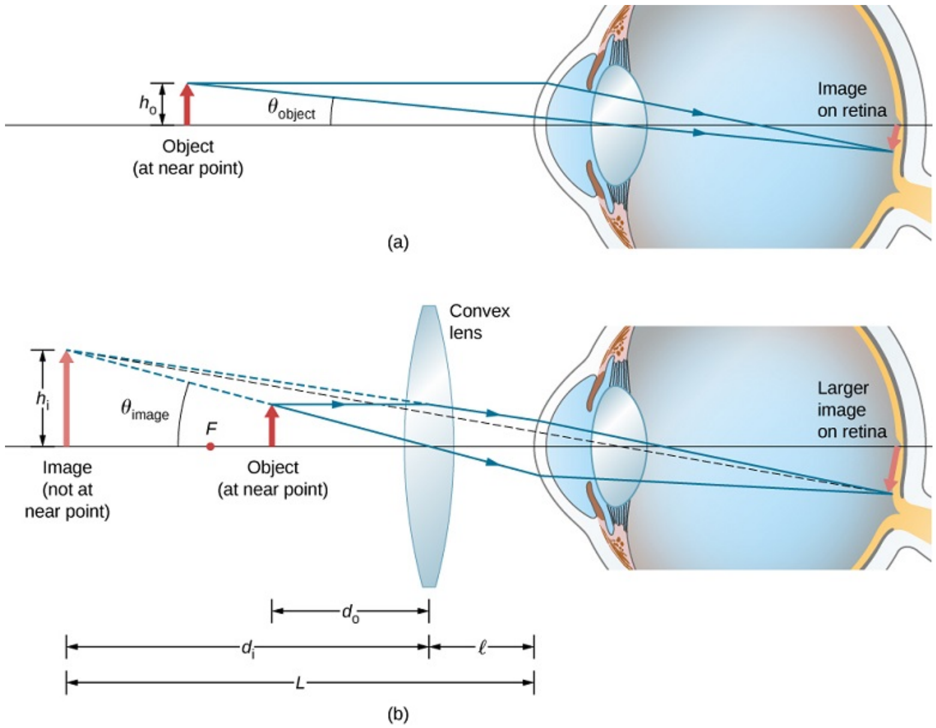


Figure 11.10.8.2: The simple magnifier is a convex lens used to produce an enlarged image of an object on the retina. (a) With no convex lens, the object subtends an angle θ_{object} from the eye. (b) With the convex lens in place, the image produced by the convex lens subtends an angle θ_{image} from the eye, with $\theta_{image} > \theta_{object}$. Thus, the image on the retina is larger with the convex lens in place.

To account for the magnification of a magnifying lens, we compare the angle subtended by the image (created by the lens) with the angle subtended by the object (viewed with no lens), as shown in Figure $11.10.8.1a$ We assume that the object is situated at the near point of the eye, because this is the object distance at which the unaided eye can form the largest image on the retina. We will compare the magnified images created by a lens with this maximum image size for the unaided eye. The magnification of an image when observed by the eye is the angular magnification M, which is defined by the ratio of the angle θ_{image} subtended by the image to the angle θ_{object} subtended by the object:

$$M = \frac{\theta_{image}}{\theta_{object}}.$$

Consider the situation shown in Figure $11.10.8.1b$ The magnifying lens is held a distance ℓ from the eye, and the image produced by the magnifier forms a distance L from the eye. We want to calculate the **angular magnification** for any arbitrary L and ℓ. In the small-angle approximation, the angular size θ_{image} of the image is h_i/L. The angular size θ_{object} of the object at the near point is $\theta_{object} = h_o/25\,cm$. The angular magnification is then

$$\underbrace{M = \frac{\theta_{image}}{\theta_{object}} = \frac{h_i(25cm)}{Lh_o}}_{\text{angular magnification}}. \tag{11.10.8.1}$$

Using the definition of linear magnification

$$m = -\frac{d_i}{d_o} = \frac{h_i}{h_o} \tag{11.10.8.2}$$

and the thin-lens equation

$$\frac{1}{d_o} + \frac{1}{d_i} = \frac{1}{f}$$

we arrive at the following expression for the angular magnification of a magnifying lens:

$$M = \left(-\frac{d_i}{d_o}\right)\left(\frac{25\;cm}{L}\right) \tag{11.10.8.3}$$

$$= -d_i\left(\frac{1}{f} - \frac{1}{d_i}\right)\left(\frac{25\;cm}{L}\right) \tag{11.10.8.4}$$

$$= \left(1 - \frac{d_i}{f}\right)\left(\frac{25\;cm}{L}\right) \tag{11.10.8.5}$$

From Figure 11.10.8.1b, we see that the absolute value of the image distance is $|d_i| = L - \ell$. Note that $d_i < 0$ because the image is virtual, so we can dispense with the absolute value by explicitly inserting the minus sign:

$$-d_i = L - \ell. \tag{11.10.8.6}$$

Inserting Equation 11.10.8.6 into Equation 11.10.8.5 gives us the final equation for the angular magnification of a magnifying lens:

$$M = \left(\frac{25\;cm}{L}\right)\left(1 + \frac{L-\ell}{f}\right). \tag{11.10.8.7}$$

Note that all the quantities in this equation have to be expressed in centimeters. Often, we want the image to be at the near-point distance (e.g., $L = 25\;cm$) to get maximum magnification, and we hold the magnifying lens close to the eye ($\ell = 0$). In this case, Equation 11.10.8.7 gives

$$M = 1 + \frac{25\;cm}{f} \tag{11.10.8.8}$$

which shows that the greatest magnification occurs for the lens with the shortest focal length. In addition, when the image is at the near-point distance and the lens is held close to the eye ($\ell = 0$), then $L = d_i = 25\;cm$ and Equation 11.10.8.7 becomes

$$M = \frac{h_i}{h_o} = m \tag{11.10.8.9}$$

where m is the linear magnification (Equation 11.10.8.2) previously derived for spherical mirrors and thin lenses. Another useful situation is when the image is at infinity ($L = \infty$). Equation 11.10.8.7 then takes the form

$$M(L = \infty) = \frac{25\;cm}{f}. \tag{11.10.8.10}$$

The resulting magnification is simply the ratio of the near-point distance to the focal length of the magnifying lens, so a lens with a shorter focal length gives a stronger magnification. Although this magnification is smaller by 1 than the magnification obtained with the image at the near point, it provides for the most comfortable viewing conditions, because the eye is relaxed when viewing a distant object.

By comparing Equations 11.10.8.8 and 11.10.8.10, we see that the range of angular magnification of a given converging lens is

$$\frac{25cm}{f} \le M \le 1 + \frac{25cm}{f}.$$

✓ Example 11.10.8.1: Magnifying a Diamond

A jeweler wishes to inspect a 3.0-mm-diameter diamond with a magnifier. The diamond is held at the jeweler's near point (25 cm), and the jeweler holds the magnifying lens close to his eye.

a. What should the focal length of the magnifying lens be to see a 15-mm-diameter image of the diamond?
b. What should the focal length of the magnifying lens be to obtain 10× magnification?

Strategy

We need to determine the requisite magnification of the magnifier. Because the jeweler holds the magnifying lens close to his eye, we can use Equation 11.10.8.8 to find the focal length of the magnifying lens.

Solution

a. The required linear magnification is the ratio of the desired image diameter to the diamond's actual diameter (Equation 11.10.8.10). Because the jeweler holds the magnifying lens close to his eye and the image forms at his near point, the linear magnification is the same as the angular magnification, so

$$\begin{aligned} M &= m = \frac{h_i}{h_o} \\ &= \frac{15\,mm}{3.0\,mm} \\ &= 5.0. \end{aligned}$$

The focal length **f** of the magnifying lens may be calculated by solving Equation 11.10.8.8for f, which gives

$$\begin{aligned} M &= 1 + \frac{25\,cm}{f} \\ f &= \frac{25\,cm}{M-1} \\ &= \frac{25\,cm}{5.0-1} \\ &= 6.3\,cm \end{aligned}$$

b. To get an image magnified by a factor of ten, we again solve Equation 11.10.8.8for f, but this time we use $M = 10$. The result is

$$\begin{aligned} f &= \frac{25\,cm}{M-1} \\ &= \frac{25\,cm}{10-1} \\ &= 2.8\,cm. \end{aligned}$$

Significance

Note that a greater magnification is achieved by using a lens with a smaller focal length. We thus need to use a lens with radii of curvature that are less than a few centimeters and hold it very close to our eye. This is not very convenient. A compound microscope, explored in the following section, can overcome this drawback.

11.10.9: Microscopes and Telescopes

Learning Objectives

By the end of this section, you will be able to:

- Explain the physics behind the operation of microscopes and telescopes
- Describe the image created by these instruments and calculate their magnifications

Microscopes and telescopes are major instruments that have contributed hugely to our current understanding of the micro- and macroscopic worlds. The invention of these devices led to numerous discoveries in disciplines such as physics, astronomy, and biology, to name a few. In this section, we explain the basic physics that make these instruments work.

Microscopes

Although the eye is marvelous in its ability to see objects large and small, it obviously is limited in the smallest details it can detect. The desire to see beyond what is possible with the naked eye led to the use of optical instruments. We have seen that a simple convex lens can create a magnified image, but it is hard to get large magnification with such a lens. A magnification greater than 5× is difficult without distorting the image. To get higher magnification, we can combine the simple magnifying glass with one or more additional lenses. In this section, we examine microscopes that enlarge the details that we cannot see with the naked eye.

Microscopes were first developed in the early 1600s by eyeglass makers in The Netherlands and Denmark. The simplest compound microscope is constructed from two convex lenses (Figure $11.10.9.1$). The objective lens is a convex lens of short focal length (i.e., high power) with typical magnification from 5× to 100×. The eyepiece, also referred to as the ocular, is a convex lens of longer focal length.

The purpose of a microscope is to create magnified images of small objects, and both lenses contribute to the final magnification. Also, the final enlarged image is produced sufficiently far from the observer to be easily viewed, since the eye cannot focus on objects or images that are too close (i.e., closer than the near point of the eye).

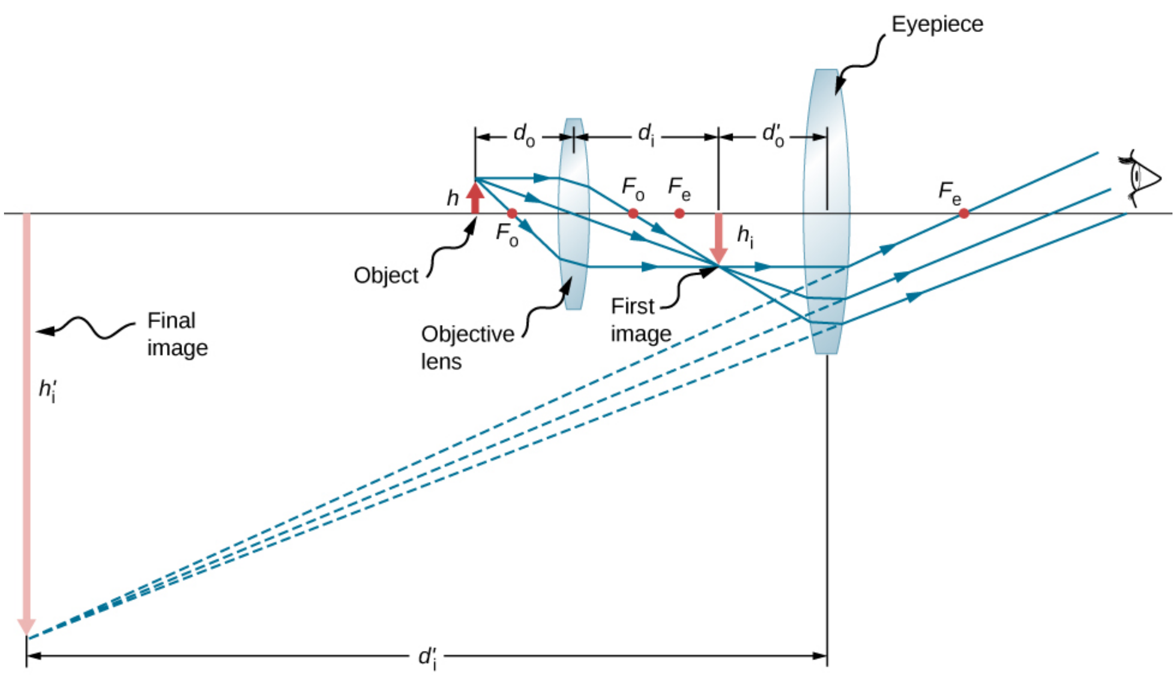


Figure 11.10.9.1: A compound microscope is composed of two lenses: an objective and an eyepiece. The objective forms the first image, which is larger than the object. This first image is inside the focal length of the eyepiece and serves as the object for the eyepiece. The eyepiece forms final image that is further magnified.

To see how the microscope in Figure $11.10.9.1$forms an image, consider its two lenses in succession. The object is just beyond the focal length f^{obj} of the objective lens, producing a real, inverted image that is larger than the object. This first image serves as the object for the second lens, or eyepiece. The eyepiece is positioned so that the first image is within its focal length f^{eye}, so that it can further magnify the image. In a sense, it acts as a magnifying glass that magnifies the intermediate image produced by the objective. The image produced by the eyepiece is a magnified virtual image. The final image remains inverted but is farther from the observer than the object, making it easy to view.

The eye views the virtual image created by the eyepiece, which serves as the object for the lens in the eye. The virtual image formed by the eyepiece is well outside the focal length of the eye, so the eye forms a real image on the retina.

The magnification of the microscope is the product of the linear magnification m^{obj} by the objective and the angular magnification M^{eye} by the eyepiece. These are given by

$$\underbrace{m^{obj} = -\frac{d_i^{obj}}{d_o^{obj}} \approx -\frac{d_i^{obj}}{f^{obj}}}_{\text{linear magnification by objective}}$$

$$\underbrace{M^{eye} = 1 + \frac{25cm}{f^{eye}}}_{\text{angular magnification by eyepiece}}$$

Here, f^{obj} and f^{eye} are the focal lengths of the objective and the eyepiece, respectively. We assume that the final image is formed at the near point of the eye, providing the largest magnification. Note that the angular magnification of the eyepiece is the same as obtained earlier for the simple magnifying glass. This should not be surprising, because the eyepiece is essentially a magnifying glass, and the same physics applies here. The **net magnification** M_{net} of the compound microscope is the product of the linear magnification of the objective and the angular magnification of the eyepiece:

$$M_{\text{net}} = m^{\text{obj}} M^{\text{eye}} = -\frac{d_{\text{i}}^{\text{obj}} \left(f^{\text{eye}} + 25\text{cm}\right)}{f^{\text{obj}} f^{\text{eye}}}. \tag{11.10.9.1}$$

Example 11.10.9.1: Microscope Magnification

Calculate the magnification of an object placed 6.20 mm from a compound microscope that has a 6.00 mm-focal length objective and a 50.0 mm-focal length eyepiece. The objective and eyepiece are separated by 23.0 cm.

Strategy

This situation is similar to that shown in Figure 11.10.9.1. To find the overall magnification, we must know the linear magnification of the objective and the angular magnification of the eyepiece. We can use Equation 11.10.9.1, but we need to use the thin-lens equation to find the image distance d_i^{obj} of the objective.

Solution

Solving the thin-lens equation for d_i^{obj} gives

$$\begin{aligned} d_i^{obj} &= \left(\frac{1}{f^{obj}} - \frac{1}{d_o^{obj}}\right)^{-1} \\ &= \left(\frac{1}{6.00\,mm} - \frac{1}{6.20mm}\right)^{-1} \\ &= 186\,mm \\ &= 18.6\,cm \end{aligned}$$

Inserting this result into Equation 11.10.9.1along with the known values

- $f^{obj} = 6.00\,mm = 0.600\,cm$
- $f^{eye} = 50.0mm = 5.00cm$

gives

$$\begin{aligned} M_{net} &= -\frac{d_i^{obj}(f^{eye} + 25\ cm)}{f^{obj} f^{eye}} \\ &= -\frac{(18.6\ cm)(5.00\ cm + 25\ cm)}{(0.600\ cm)(5.00\ cm)} \\ &= -186 \end{aligned}$$

Significance

Both the objective and the eyepiece contribute to the overall magnification, which is large and negative, consistent with Figure 11.10.9.1, where the image is seen to be large and inverted. In this case, the image is virtual and inverted, which **cannot** happen for a single element.

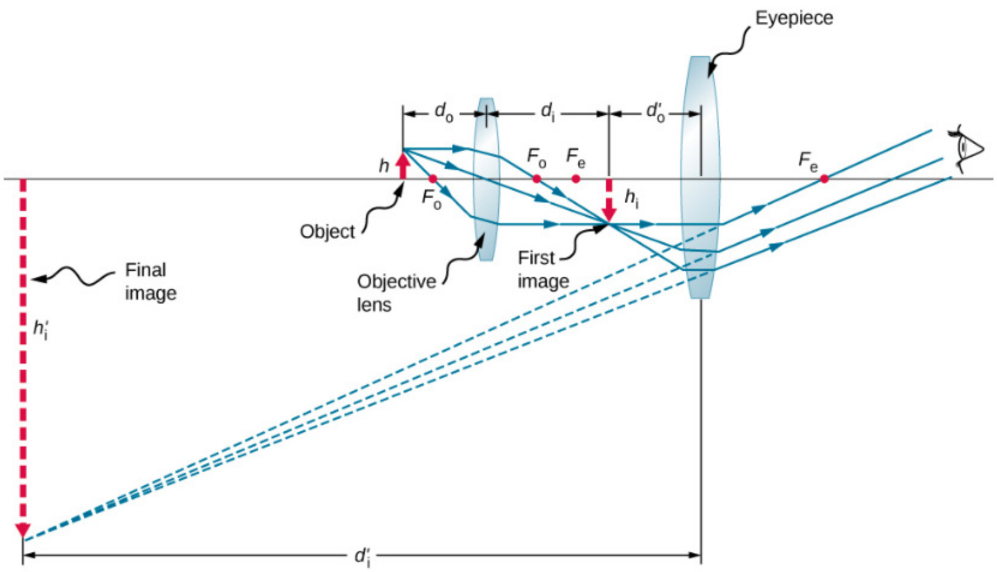


Figure 11.10.9.2: A compound microscope with the image created at infinity.

We now calculate the magnifying power of a microscope when the image is at infinity, as shown in Figure 11.10.9.2, because this makes for the most relaxed viewing. The magnifying power of the microscope is the product of linear magnification m^{obj} of the objective and the angular magnification M^{eye} of the eyepiece. We know that

$$m^{obj} = -\frac{d_i^{obj}}{d_o^{obj}}$$

and from the thin-lens equation we obtain

$$m^{\text{obj}} = -\frac{d_{\text{i}}^{\text{obj}}}{d_{\text{o}}^{\text{obj}}} = 1 - \frac{d_{\text{i}}^{\text{obj}}}{f^{\text{obj}}} = \frac{f^{\text{obj}} - d_{\text{i}}^{\text{obj}}}{f^{\text{obj}}}. \tag{11.10.9.2}$$

If the final image is at infinity, then the image created by the objective must be located at the focal point of the eyepiece. This may be seen by considering the thin-lens equation with $d_i = \infty$ or by recalling that rays that pass through the focal point exit the lens parallel to each other, which is equivalent to focusing at infinity. For many microscopes, the distance between the image-side focal point of the objective and the object-side focal point of the eyepiece is standardized at L = 16 cm. This distance is called the **tube length** of the microscope. From Figure 11.10.9.2, we see that

$$L = f^{obj} - d_i^{obj}.$$

Inserting this into Equation 11.10.9.2 gives

$$m^{obj} = \frac{L}{f^{obj}} = \frac{16cm}{f^{obj}}. \tag{11.10.9.3}$$

We now need to calculate the angular magnification of the eyepiece with the image at infinity. To do so, we take the ratio of the angle θ_{image} subtended by the image to the angle θ_{object} subtended by the object at the near point of the eye (this is the closest that the unaided eye can view the object, and thus this is the position where the object will form the largest image on the retina of the unaided eye). Using Figure $11.10.9.2$and working in the **small-angle approximation**, we have

$$\theta_{image} \approx \frac{h_i^{obj}}{f^{eye}}$$

and

$$\theta_{\text{object}} \approx \frac{h_i^{obj}}{25cm}$$

where h_i^{obj} is the height of the image formed by the objective, which is the object of the eyepiece. Thus, the angular magnification of the eyepiece is

$$M^{\text{eye}} = \frac{\theta_{\text{image}}}{\theta_{\text{object}}} = \frac{h_i^{\text{obj}}}{f^{\text{eye}}} \frac{25\text{cm}}{h_i^{\text{obj}}} = \frac{25\text{cm}}{f^{\text{eye}}}. \tag{11.10.9.4}$$

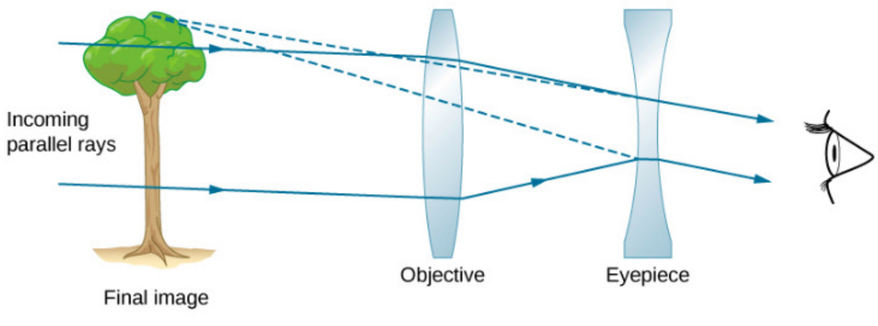
The net magnifying power of the compound microscope with the image at infinity is therefore

$$M_{net} = m^{obj} M^{eye} = -\frac{(16cm)(25cm)}{f^{obj} f^{eye}}. \tag{11.10.9.5}$$

The focal distances must be in centimeters. The minus sign indicates that the final image is inverted. Note that the only variables in the equation are the focal distances of the eyepiece and the objective, which makes this equation particularly useful.

Telescopes

Telescopes are meant for viewing distant objects and produce an image that is larger than the image produced in the unaided eye. Telescopes gather far more light than the eye, allowing dim objects to be observed with greater magnification and better resolution. Telescopes were invented around 1600, and Galileo was the first to use them to study the heavens, with monumental consequences. He observed the moons of Jupiter, the craters and mountains on the moon, the details of sunspots, and the fact that the Milky Way is composed of a vast number of individual stars.



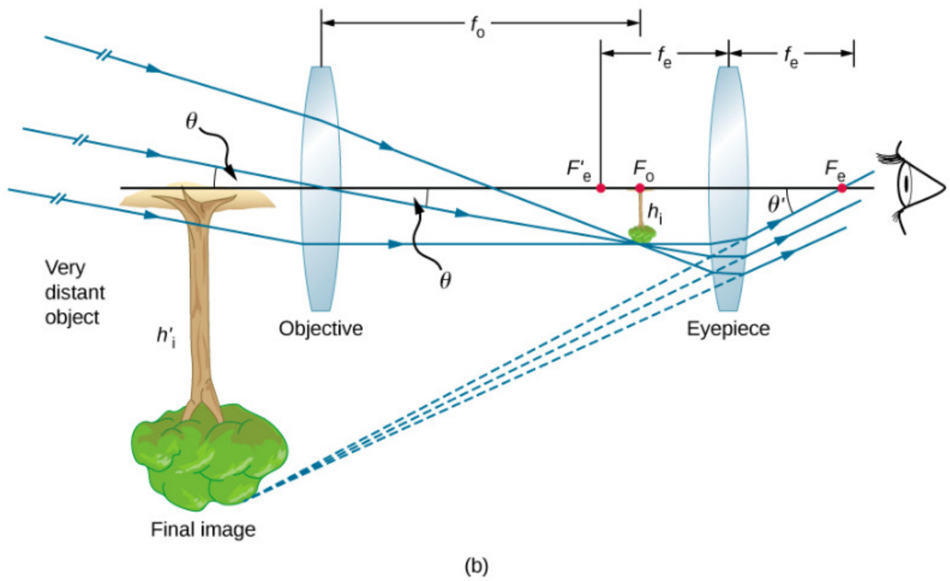


Figure 11.10.9.3: (a) Galileo made telescopes with a convex objective and a concave eyepiece. These produce an upright image and are used in spyglasses. (b) Most simple refracting telescopes have two convex lenses. The objective forms a real, inverted image at (or just within) the focal plane of the eyepiece. This image serves as the object for the eyepiece. The eyepiece forms a virtual, inverted image that is magnified.

Figure $11.10.9.3a$shows a refracting telescope made of two lenses. The first lens, called the **objective**, forms a real image within the focal length of the second lens, which is called the **eyepiece**. The image of the objective lens serves as the object for the eyepiece, which forms a magnified virtual image that is observed by the eye. This design is what Galileo used to observe the heavens.

Although the arrangement of the lenses in a refracting telescope looks similar to that in a microscope, there are important differences. In a telescope, the real object is far away and the intermediate image is smaller than the object. In a microscope, the real object is very close and the intermediate image is larger than the object. In both the telescope and the microscope, the eyepiece magnifies the intermediate image; in the telescope, however, this is the only magnification.

The most common two-lens telescope is shown in Figure $11.10.9.3b$ The object is so far from the telescope that it is essentially at infinity compared with the focal lengths of the lenses $d_o^{obj} \approx \infty$, so the incoming rays are essentially parallel and focus on the focal plane. Thus, the first image is produced at

$$d_i^{obj} = f^{obj}$$

as shown in the figure, and is not large compared with what you might see by looking directly at the object. However, the eyepiece of the telescope eyepiece (like the microscope eyepiece) allows you to get nearer than your near point to this first image and so magnifies it (because you are near to it, it subtends a larger angle from your eye and so forms a larger image on your retina). As for a simple magnifier, the angular magnification of a telescope is the ratio of the angle subtended by the image (θ_{image} in $11.10.9.3b$) to the angle subtended by the real object (θ_{object} in $11.10.9.3b$):

$$M = \frac{\theta_{image}}{\theta_{object}}. \tag{11.10.9.6}$$

To obtain an expression for the magnification that involves only the lens parameters, note that the focal plane of the objective lens lies very close to the focal plan of the eyepiece. If we assume that these planes are superposed, we have the situation shown in Figure 11.10.9.4

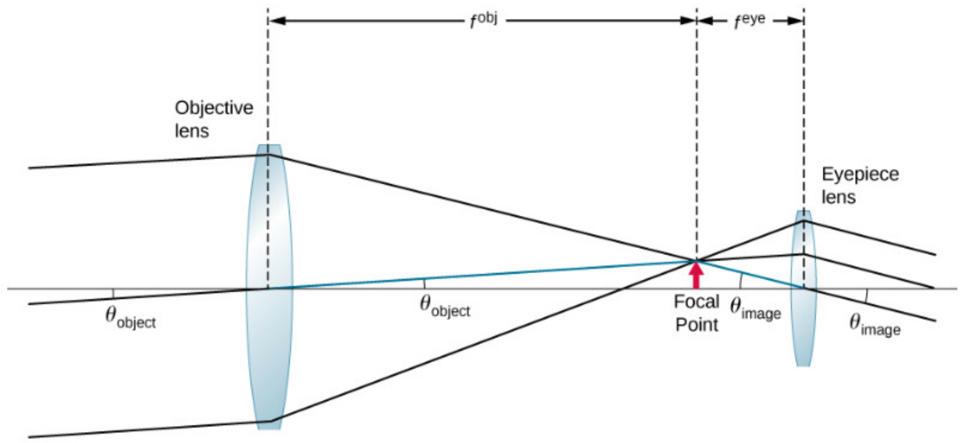


Figure 11.10.9.4: The focal plane of the objective lens of a telescope is very near to the focal plane of the eyepiece. The angle θ_{image} subtended by the image viewed through the eyepiece is larger than the angle θ_{object} subtended by the object when viewed with the unaided eye.

We further assume that the angles θ_{object} and θ_{image} are small, so that the small-angel approximation holds ($\tan\theta \approx \theta$). If the image formed at the focal plane has height h then

$$\theta_{\text{object}} \approx \tan\theta_{\text{object}} = \frac{h}{f^{\text{obj}}}$$
$$\theta_{\text{image}} \approx \tan\theta_{\text{image}} = \frac{-h}{f^{eye}}$$

where the minus sign is introduced because the height is negative if we measure both angles in the counterclockwise direction. Inserting these expressions into Equation 11.10.9.6gives

$$M = \frac{-h_{\text{i}}}{f^{\text{eye}}}\frac{f^{\text{obj}}}{h_{\text{i}}} = -\frac{f^{\text{obj}}}{f^{\text{eye}}}. \tag{11.10.9.7}$$

Thus, to obtain the greatest angular magnification, it is best to have an objective with a long focal length and an eyepiece with a short focal length. The greater the angular magnification M, the larger an object will appear when viewed through a telescope, making more details visible. Limits to observable details are imposed by many factors, including lens quality and atmospheric disturbance. Typical eyepieces have focal lengths of 2.5 cm or 1.25 cm. If the objective of the telescope has a focal length of 1 meter, then these eyepieces result in magnifications of 40× and 80×, respectively. Thus, the angular magnifications make the image appear 40 times or 80 times closer than the real object.

The minus sign in the magnification indicates the image is inverted, which is unimportant for observing the stars but is a real problem for other applications, such as telescopes on ships or telescopic gun sights. If an upright image is needed, Galileo's arrangement in $11.10.9.3a$can be used. But a more common arrangement is to use a third convex lens as an eyepiece, increasing the distance between the first two and inverting the image once again, as seen in Figure $11.10.9.5$

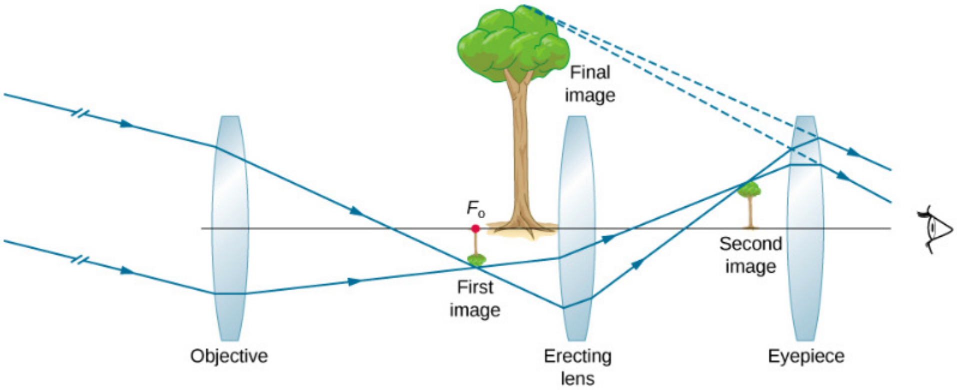


Figure 11.10.9.5: This arrangement of three lenses in a telescope produces an upright final image. The first two lenses are far enough apart that the second lens inverts the image of the first. The third lens acts as a magnifier and keeps the image upright and in a location that is easy to view.

The largest refracting telescope in the world is the 40-inch diameter Yerkes telescope located at Lake Geneva, Wisconsin (Figure 11.10.9.6), and operated by the University of Chicago.

It is very difficult and expensive to build large refracting telescopes. You need large defect-free lenses, which in itself is a technically demanding task. A refracting telescope basically looks like a tube with a support structure to rotate it in different directions. A refracting telescope suffers from several problems. The aberration of lenses causes the image to be blurred. Also, as the lenses become thicker for larger lenses, more light is absorbed, making faint stars more difficult to observe. Large lenses are also very heavy and deform under their own weight. Some of these problems with refracting telescopes are addressed by avoiding refraction for collecting light and instead using a curved mirror in its place, as devised by Isaac Newton. These telescopes are called reflecting telescopes.

Figure 11.10.9.6: In 1897, the Yerkes Observatory in Wisconsin (USA) built a large refracting telescope with an objective lens that is 40 inches in diameter and has a tube length of 62 feet. (credit: Yerkes Observatory, University of Chicago)

Reflecting Telescopes

Isaac Newton designed the first reflecting telescope around 1670 to solve the problem of chromatic aberration that happens in all refracting telescopes. In chromatic aberration, light of different colors refracts by slightly different amounts in the lens. As a result, a rainbow appears around the image and the image appears blurred. In the reflecting telescope, light rays from a distant source fall upon the surface of a concave mirror fixed at the bottom end of the tube. The use of a mirror instead of a lens eliminates chromatic aberration. The concave mirror focuses the rays on its focal plane. The design problem is how to observe the focused image. Newton used a design in which the focused light from the concave mirror was reflected to one side of the tube into an eyepiece (Figure 11.10.9.7a). This arrangement is common in many amateur telescopes and is called the Newtonian design.

Some telescopes reflect the light back toward the middle of the concave mirror using a convex mirror. In this arrangement, the light-gathering concave mirror has a hole in the middle (11.10.9.7b). The light then is incident on an eyepiece lens. This arrangement of the objective and eyepiece is called the **Cassegrain design**. Most big telescopes, including the Hubble space telescope, are of this design. Other arrangements are also possible. In some telescopes, a light detector is placed right at the spot where light is focused by the curved mirror.

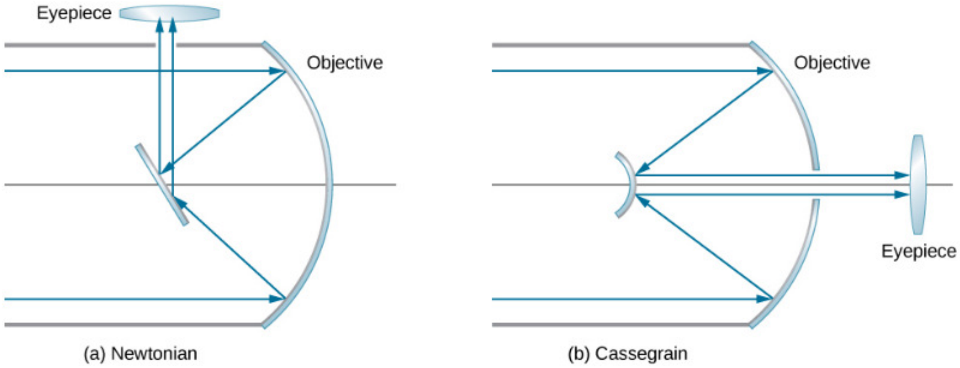


Figure 11.10.9.7: Reflecting telescopes: (a) In the Newtonian design, the eyepiece is located at the side of the telescope; (b) in the Cassegrain design, the eyepiece is located past a hole in the primary mirror.

Most astronomical research telescopes are now of the reflecting type. One of the earliest large telescopes of this kind is the Hale 200-inch (or 5-meter) telescope built on Mount Palomar in southern California, which has a 200 inch-diameter mirror. One of the largest telescopes in the world is the 10-meter Keck telescope at the Keck Observatory on the summit of the dormant Mauna Kea volcano in Hawaii. The Keck Observatory operates two 10-meter telescopes. Each is not a single mirror, but is instead made up of 36 hexagonal mirrors. Furthermore, the two telescopes on the Keck can work together, which increases their power to an effective 85-meter mirror. The Hubble telescope (Figure 11.10.9.8) is another large reflecting telescope with a 2.4 meter-diameter primary mirror. The Hubble was put into orbit around Earth in 1990.

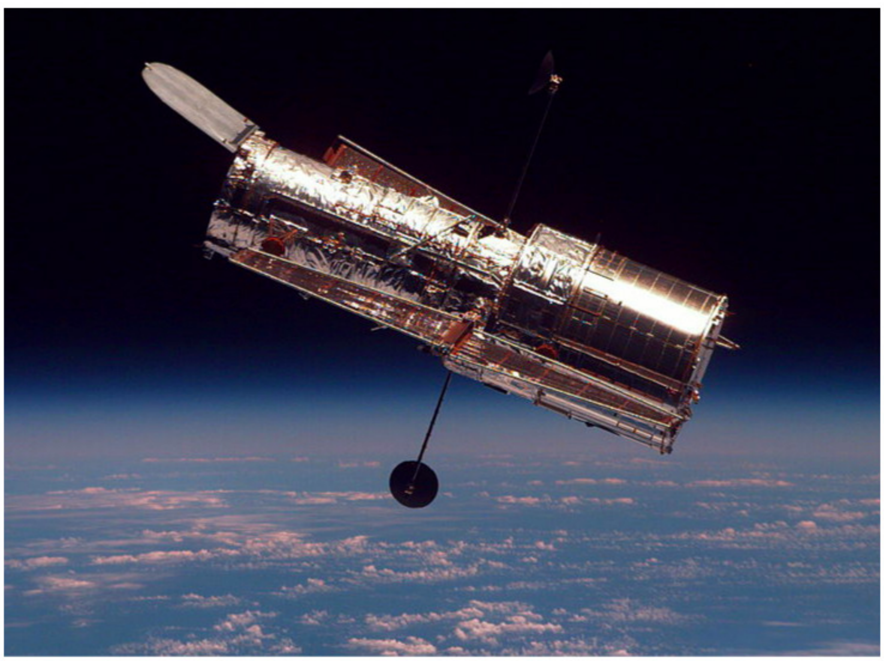


Figure 11.10.9.8: The Hubble space telescope as seen from the Space Shuttle Discovery. (credit: modification of work by NASA)

The angular magnification M of a reflecting telescope is also given by Equation 11.10.9.3. For a spherical mirror, the focal length is half the radius of curvature, so making a large objective mirror not only helps the telescope collect more light, but also increases the magnification of the image.

11.10.E: Geometric Optics and Image Formation (Exercises)

Conceptual Questions

2.1 Images Formed by Plane Mirrors

1. What are the differences between real and virtual images? How can you tell (by looking) whether an image formed by a single lens or mirror is real or virtual?

2. Can you see a virtual image? Explain your response.

3. Can you photograph a virtual image?

4. Can you project a virtual image onto a screen?

5. Is it necessary to project a real image onto a screen to see it?

6. Devise an arrangement of mirrors allowing you to see the back of your head. What is the minimum number of mirrors needed for this task?

7. If you wish to see your entire body in a flat mirror (from head to toe), how tall should the mirror be? Does its size depend upon your distance away from the mirror? Provide a sketch.

2.2 Spherical Mirrors

8. At what distance is an image always located: at d_o, d_i, or f ?

9. Under what circumstances will an image be located at the focal point of a spherical lens or mirror?

10. What is meant by a negative magnification? What is meant by a magnification whose absolute value is less than one?

11. Can an image be larger than the object even though its magnification is negative? Explain.

2.3 Images Formed by Refraction

12. Derive the formula for the apparent depth of a fish in a fish tank using Snell's law.

13. Use a ruler and a protractor to find the image by refraction in the following cases. Assume an air-glass interface. Use a refractive index of 1 for air and of 1.5 for glass. (Hint: Use Snell's law at the interface.)

(a) A point object located on the axis of a concave interface located at a point within the focal length from the vertex.

(b) A point object located on the axis of a concave interface located at a point farther than the focal length from the vertex.

(c) A point object located on the axis of a convex interface located at a point within the focal length from the vertex.

(d) A point object located on the axis of a convex interface located at a point farther than the focal length from the vertex.

(e) Repeat (a)–(d) for a point object off the axis.

2.4 Thin Lenses

14. You can argue that a flat piece of glass, such as in a window, is like a lens with an infinite focal length. If so, where does it form an image? That is, how are d_i and d_o related?

15. When you focus a camera, you adjust the distance of the lens from the film. If the camera lens acts like a thin lens, why can it not be a fixed distance from the film for both near and distant objects?

16. A thin lens has two focal points, one on either side of the lens at equal distances from its center, and should behave the same for light entering from either side. Look backward and forward through a pair of eyeglasses and comment on whether they are thin lenses.

17. Will the focal length of a lens change when it is submerged in water? Explain.

2.5 The Eye

18. If the lens of a person's eye is removed because of cataracts (as has been done since ancient times), why would you expect an eyeglass lens of about 16 D to be prescribed?

19. When laser light is shone into a relaxed normal-vision eye to repair a tear by spot-welding the retina to the back of the eye, the rays entering the eye must be parallel. Why?

20. Why is your vision so blurry when you open your eyes while swimming under water? How does a face mask enable clear vision?

21. It has become common to replace the cataract-clouded lens of the eye with an internal lens. This intraocular lens can be chosen so that the person has perfect distant vision. Will the person be able to read without glasses? If the person was nearsighted, is the power of the intraocular lens greater or less than the removed lens?

22. If the cornea is to be reshaped (this can be done surgically or with contact lenses) to correct myopia, should its curvature be made greater or smaller? Explain.

2.8 Microscopes and Telescopes

23. Geometric optics describes the interaction of light with macroscopic objects. Why, then, is it correct to use geometric optics to analyze a microscope's image?

24. The image produced by the microscope in Figure 2.38 cannot be projected. Could extra lenses or mirrors project it? Explain.

25. If you want your microscope or telescope to project a real image onto a screen, how would you change the placement of the eyepiece relative to the objective?

Problems

2.1 Images Formed by Plane Mirrors

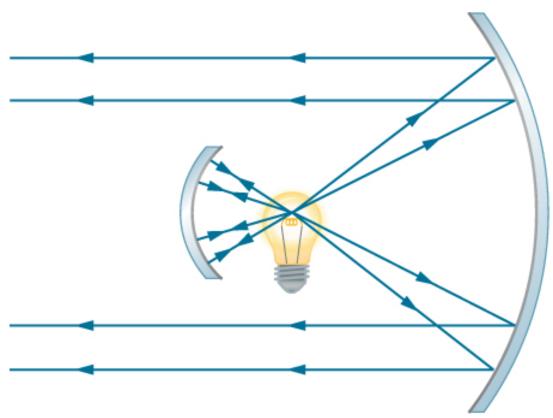
26. Consider a pair of flat mirrors that are positioned so that they form an angle of 120°. An object is placed on the bisector between the mirrors. Construct a ray diagram as in Figure 2.4 to show how many images are formed.

27. Consider a pair of flat mirrors that are positioned so that they form an angle of 60°. An object is placed on the bisector between the mirrors. Construct a ray diagram as in Figure 2.4 to show how many images are formed.

28. By using more than one flat mirror, construct a ray diagram showing how to create an inverted image.

2.2 Spherical Mirrors

29. The following figure shows a light bulb between two spherical mirrors. One mirror produces a beam of light with parallel rays; the other keeps light from escaping without being put into the beam. Where is the filament of the light in relation to the focal point or radius of curvature of each mirror?



A light bulb is shown in the centre, with a small concave mirror to its left and a bigger one to its right. The light rays originating from the bulb that hit the smaller mirror are reflected back to the bulb. Light rays from the bulb that hit the bigger mirror are reflected. These reflected rays are parallel and travel towards the left.

30. Why are diverging mirrors often used for rearview mirrors in vehicles? What is the main disadvantage of using such a mirror compared with a flat one?

31. Some telephoto cameras use a mirror rather than a lens. What radius of curvature mirror is needed to replace a 800 mm-focal length telephoto lens?

32. Calculate the focal length of a mirror formed by the shiny back of a spoon that has a 3.00 cm radius of curvature.

33. Electric room heaters use a concave mirror to reflect infrared (IR) radiation from hot coils. Note that IR radiation follows the same law of reflection as visible light. Given that the mirror has a radius of curvature of 50.0 cm and produces an image of the coils 3.00 m away from the mirror, where are the coils?

34. Find the magnification of the heater element in the previous problem. Note that its large magnitude helps spread out the reflected energy.

35. What is the focal length of a makeup mirror that produces a magnification of 1.50 when a person's face is 12.0 cm away? Explicitly show how you follow the steps in the Example 2.2.

36. A shopper standing 3.00 m from a convex security mirror sees his image with a magnification of 0.250.

(a) Where is his image?

(b) What is the focal length of the mirror?

(c) What is its radius of curvature?

37. An object 1.50 cm high is held 3.00 cm from a person's cornea, and its reflected image is measured to be 0.167 cm high.

(a) What is the magnification?

(b) Where is the image?

(c) Find the radius of curvature of the convex mirror formed by the cornea. (Note that this technique is used by optometrists to measure the curvature of the cornea for contact lens fitting. The instrument used is called a keratometer, or curve measurer.)

38. Ray tracing for a flat mirror shows that the image is located a distance behind the mirror equal to the distance of the object from the mirror. This is stated as $d_i = -d_o$, since this is a negative image distance (it is a virtual image). What is the focal length of a flat mirror?

39. Show that, for a flat mirror, $h_i = h_o$, given that the image is the same distance behind the mirror as the distance of the object from the mirror.

40. Use the law of reflection to prove that the focal length of a mirror is half its radius of curvature. That is, prove that $f = R/2$. Note this is true for a spherical mirror only if its diameter is small compared with its radius of curvature.

41. Referring to the electric room heater considered in problem 5, calculate the intensity of IR radiation in W/m^2 projected by the concave mirror on a person 3.00 m away. Assume that the heating element radiates 1500 W and has an area of $100cm^2$, and that half of the radiated power is reflected and focused by the mirror.

42. Two mirrors are inclined at an angle of 60° and an object is placed at a point that is equidistant from the two mirrors. Use a protractor to draw rays accurately and locate all images. You may have to draw several figures so that that rays for different images do not clutter your drawing.

43. Two parallel mirrors are facing each other and are separated by a distance of 3 cm. A point object is placed between the mirrors 1 cm from one of the mirrors. Find the coordinates of all the images.

2.3 Images Formed by Refraction

44. An object is located in air 30 cm from the vertex of a concave surface made of glass with a radius of curvature 10 cm. Where does the image by refraction form and what is its magnification? Use $n_{air} = 1$ and $n_{glass} = 1.5$.

45. An object is located in air 30 cm from the vertex of a convex surface made of glass with a radius of curvature 80 cm. Where does the image by refraction form and what is its magnification?

46. An object is located in water 15 cm from the vertex of a concave surface made of glass with a radius of curvature 10 cm. Where does the image by refraction form and what is its magnification? Use $n_{water} = 4/3$ and $n_{glass} = 1.5$.

47. An object is located in water 30 cm from the vertex of a convex surface made of Plexiglas with a radius of curvature of 80 cm. Where does the image form by refraction and what is its magnification? $n_{water} = 4/3$ and $n_{Plexiglas} = 1.65$.

48. An object is located in air 5 cm from the vertex of a concave surface made of glass with a radius of curvature 20 cm. Where does the image form by refraction and what is its magnification? Use $n_{air} = 1$ and $n_{glass} = 1.5$.

49. Derive the spherical interface equation for refraction at a concave surface. (Hint: Follow the derivation in the text for the convex surface.)

2.4 Thin Lenses

50. How far from the lens must the film in a camera be, if the lens has a 35.0-mm focal length and is being used to photograph a flower 75.0 cm away? Explicitly show how you follow the steps in the Figure 2.27.

51. A certain slide projector has a 100 mm-focal length lens.

(a) How far away is the screen if a slide is placed 103 mm from the lens and produces a sharp image?

(b) If the slide is 24.0 by 36.0 mm, what are the dimensions of the image? Explicitly show how you follow the steps in the Figure 2.27.

52. A doctor examines a mole with a 15.0-cm focal length magnifying glass held 13.5 cm from the mole.

(a) Where is the image?

(b) What is its magnification?

(c) How big is the image of a 5.00 mm diameter mole?

53. A camera with a 50.0-mm focal length lens is being used to photograph a person standing 3.00 m away.

(a) How far from the lens must the film be?

(b) If the film is 36.0 mm high, what fraction of a 1.75-m-tall person will fit on it?

(c) Discuss how reasonable this seems, based on your experience in taking or posing for photographs.

54. A camera lens used for taking close-up photographs has a focal length of 22.0 mm. The farthest it can be placed from the film is 33.0 mm.

(a) What is the closest object that can be photographed?

(b) What is the magnification of this closest object?

55. Suppose your 50.0 mm-focal length camera lens is 51.0 mm away from the film in the camera.

(a) How far away is an object that is in focus?

(b) What is the height of the object if its image is 2.00 cm high?

56. What is the focal length of a magnifying glass that produces a magnification of 3.00 when held 5.00 cm from an object, such as a rare coin?

57. The magnification of a book held 7.50 cm from a 10.0 cm-focal length lens is 4.00.

(a) Find the magnification for the book when it is held 8.50 cm from the magnifier.

(b) Repeat for the book held 9.50 cm from the magnifier.

(c) Comment on how magnification changes as the object distance increases as in these two calculations.

58. Suppose a 200 mm-focal length telephoto lens is being used to photograph mountains 10.0 km away.

(a) Where is the image?

(b) What is the height of the image of a 1000 m high cliff on one of the mountains?

59. A camera with a 100 mm-focal length lens is used to photograph the sun. What is the height of the image of the sun on the film, given the sun is $1.40 \times 10^6 km$ in diameter and is $1.50 \times 10^8 km$ away?

60. Use the thin-lens equation to show that the magnification for a thin lens is determined by its focal length and the object distance and is given by $m = f/(f - d_o)$.

61. An object of height 3.0 cm is placed 5.0 cm in front of a converging lens of focal length 20 cm and observed from the other side. Where and how large is the image?

62. An object of height 3.0 cm is placed at 5.0 cm in front of a diverging lens of focal length 20 cm and observed from the other side. Where and how large is the image?

63. An object of height 3.0 cm is placed at 25 cm in front of a diverging lens of focal length 20 cm. Behind the diverging lens, there is a converging lens of focal length 20 cm. The distance between the lenses is 5.0 cm. Find the location and size of the final image.

64. Two convex lenses of focal lengths 20 cm and 10 cm are placed 30 cm apart, with the lens with the longer focal length on the right. An object of height 2.0 cm is placed midway between them and observed through each lens from the left and from the right. Describe what you will see, such as where the image(s) will appear, whether they will be upright or inverted and their magnifications.

2.5 The Eye

Unless otherwise stated, the lens-to-retina distance is 2.00 cm.

65. What is the power of the eye when viewing an object 50.0 cm away?

66. Calculate the power of the eye when viewing an object 3.00 m away.

67. The print in many books averages 3.50 mm in height. How high is the image of the print on the retina when the book is held 30.0 cm from the eye?

68. Suppose a certain person's visual acuity is such that he can see objects clearly that form an image $4.00 \mu m$ high on his retina. What is the maximum distance at which he can read the 75.0-cm-high letters on the side of an airplane?

69. People who do very detailed work close up, such as jewelers, often can see objects clearly at much closer distance than the normal 25 cm.

(a) What is the power of the eyes of a woman who can see an object clearly at a distance of only 8.00 cm?

(b) What is the image size of a 1.00-mm object, such as lettering inside a ring, held at this distance?

(c) What would the size of the image be if the object were held at the normal 25.0 cm distance?

70. What is the far point of a person whose eyes have a relaxed power of 50.5 D?

71. What is the near point of a person whose eyes have an accommodated power of 53.5 D?

72. (a) A laser reshaping the cornea of a myopic patient reduces the power of his eye by 9.00 D, with a ± 5.0 uncertainty in the final correction. What is the range of diopters for eyeglass lenses that this person might need after this procedure?

(b) Was the person nearsighted or farsighted before the procedure? How do you know?

73. The power for normal close vision is 54.0 D. In a vision-correction procedure, the power of a patient's eye is increased by 3.00 D. Assuming that this produces normal close vision, what was the patient's near point before the procedure?

74. For normal distant vision, the eye has a power of 50.0 D. What was the previous far point of a patient who had laser vision correction that reduced the power of her eye by 7.00 D, producing normal distant vision?

75. The power for normal distant vision is 50.0 D. A severely myopic patient has a far point of 5.00 cm. By how many diopters should the power of his eye be reduced in laser vision correction to obtain normal distant vision for him?

76. A student's eyes, while reading the blackboard, have a power of 51.0 D. How far is the board from his eyes?

77. The power of a physician's eyes is 53.0 D while examining a patient. How far from her eyes is the object that is being examined?

78. The normal power for distant vision is 50.0 D. A young woman with normal distant vision has a 10.0% ability to accommodate (that is, increase) the power of her eyes. What is the closest object she can see clearly?

79. The far point of a myopic administrator is 50.0 cm.

(a) What is the relaxed power of his eyes?

(b) If he has the normal 8.00% ability to accommodate, what is the closest object he can see clearly?

80. A very myopic man has a far point of 20.0 cm. What power contact lens (when on the eye) will correct his distant vision?

81. Repeat the previous problem for eyeglasses held 1.50 cm from the eyes.

82. A myopic person sees that her contact lens prescription is –4.00 D. What is her far point?

83. Repeat the previous problem for glasses that are 1.75 cm from the eyes.

84. The contact lens prescription for a mildly farsighted person is 0.750 D, and the person has a near point of 29.0 cm. What is the power of the tear layer between the cornea and the lens if the correction is ideal, taking the tear layer into account?

2.7 The Simple Magnifier

85. If the image formed on the retina subtends an angle of 30° and the object subtends an angle of 5°, what is the magnification of the image?

86. What is the magnification of a magnifying lens with a focal length of 10 cm if it is held 3.0 cm from the eye and the object is 12 cm from the eye?

87. How far should you hold a 2.1 cm-focal length magnifying glass from an object to obtain a magnification of $10\times$? Assume you place your eye 5.0 cm from the magnifying glass.

88. You hold a 5.0 cm-focal length magnifying glass as close as possible to your eye. If you have a normal near point, what is the magnification?

89. You view a mountain with a magnifying glass of focal length $f = 10 cm$. What is the magnification?

90. You view an object by holding a 2.5 cm-focal length magnifying glass 10 cm away from it. How far from your eye should you hold the magnifying glass to obtain a magnification of $10\times$?

91. A magnifying glass forms an image 10 cm on the opposite side of the lens from the object, which is 10 cm away. What is the magnification of this lens for a person with a normal near point if their eye 12 cm from the object?

92. An object viewed with the naked eye subtends a 2° angle. If you view the object through a $10\times$ magnifying glass, what angle is subtended by the image formed on your retina?

93. For a normal, relaxed eye, a magnifying glass produces an angular magnification of 4.0. What is the largest magnification possible with this magnifying glass?

94. What range of magnification is possible with a 7.0 cm-focal length converging lens?

95. A magnifying glass produces an angular magnification of 4.5 when used by a young person with a near point of 18 cm. What is the maximum angular magnification obtained by an older person with a near point of 45 cm?

2.8 Microscopes and Telescopes

96. A microscope with an overall magnification of 800 has an objective that magnifies by 200.

(a) What is the angular magnification of the eyepiece?

(b) If there are two other objectives that can be used, having magnifications of 100 and 400, what other total magnifications are possible?

97. (a) What magnification is produced by a 0.150 cm-focal length microscope objective that is 0.155 cm from the object being viewed?

(b) What is the overall magnification if an $8\times$ eyepiece (one that produces an angular magnification of 8.00) is used?

98. Where does an object need to be placed relative to a microscope for its 0.50 cm-focal length objective to produce a magnification of -400?

99. An amoeba is 0.305 cm away from the 0.300 cm-focal length objective lens of a microscope.

(a) Where is the image formed by the objective lens?

(b) What is this image's magnification?

(c) An eyepiece with a 2.00-cm focal length is placed 20.0 cm from the objective. Where is the final image?

(d) What angular magnification is produced by the eyepiece?

(e) What is the overall magnification? (See Figure 2.39.)

100. Unreasonable Results Your friends show you an image through a microscope. They tell you that the microscope has an objective with a 0.500-cm focal length and an eyepiece with a 5.00-cm focal length. The resulting overall magnification is 250,000. Are these viable values for a microscope?

Unless otherwise stated, the lens-to-retina distance is 2.00 cm.

101. What is the angular magnification of a telescope that has a 100 cm-focal length objective and a 2.50 cm-focal length eyepiece?

102. Find the distance between the objective and eyepiece lenses in the telescope in the above problem needed to produce a final image very far from the observer, where vision is most relaxed. Note that a telescope is normally used to view very distant objects.

103. A large reflecting telescope has an objective mirror with a 10.0-m radius of curvature. What angular magnification does it produce when a 3.00 m-focal length eyepiece is used?

104. A small telescope has a concave mirror with a 2.00-m radius of curvature for its objective. Its eyepiece is a 4.00 cm-focal length lens.

(a) What is the telescope’s angular magnification?

(b) What angle is subtended by a 25,000 km-diameter sunspot?

(c) What is the angle of its telescopic image?

105. A 7.5×7.5× binocular produces an angular magnification of −7.50, acting like a telescope. (Mirrors are used to make the image upright.) If the binoculars have objective lenses with a 75.0-cm focal length, what is the focal length of the eyepiece lenses?

106. Construct Your Own Problem Consider a telescope of the type used by Galileo, having a convex objective and a concave eyepiece as illustrated in part (a) of Figure 2.40. Construct a problem in which you calculate the location and size of the image produced. Among the things to be considered are the focal lengths of the lenses and their relative placements as well as the size and location of the object. Verify that the angular magnification is greater than one. That is, the angle subtended at the eye by the image is greater than the angle subtended by the object.

107. Trace rays to find which way the given ray will emerge after refraction through the thin lens in the following figure. Assume thin-lens approximation. (**Hint:** Pick a point P on the given ray in each case. Treat that point as an object. Now, find its image Q. Use the rule: All rays on the other side of the lens will either go through Q or appear to be coming from Q.)

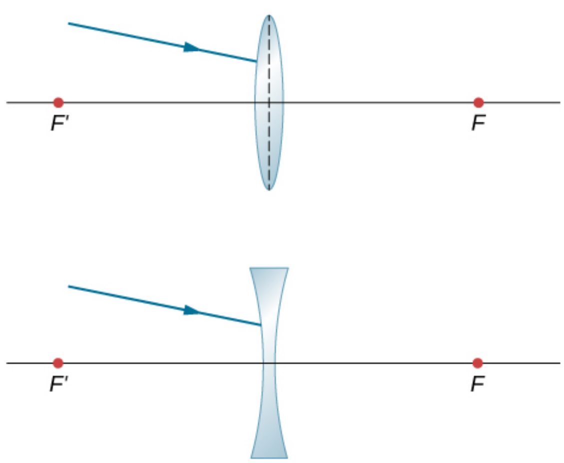


Figure a shows a ray not parallel to the optical axis striking a bi-convex lens. Figure a shows a ray not parallel to the optical axis striking a bi-concave lens.

108. Copy and draw rays to find the final image in the following diagram. (Hint: Find the intermediate image through lens alone. Use the intermediate image as the object for the mirror and work with the mirror alone to find the final image.)

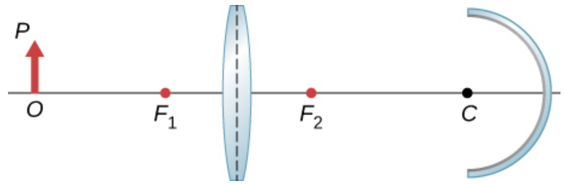


Figure shows from left to right: an object with base O on the optical axis and tip P, a bi-convex lens and a concave mirror with center of curvature C. The focal point of the bi-convex on the object side is labeled F subscript 1 and that on the mirror side is labeled F subscript 2.

109. A concave mirror of radius of curvature 10 cm is placed 30 cm from a thin convex lens of focal length 15 cm. Find the location and magnification of a small bulb sitting 50 cm from the lens by using the algebraic method.

110. An object of height 3 cm is placed at 25 cm in front of a converging lens of focal length 20 cm. Behind the lens there is a concave mirror of focal length 20 cm. The distance between the lens and the mirror is 5 cm. Find the location, orientation and size of the final image.

111. An object of height 3 cm is placed at a distance of 25 cm in front of a converging lens of focal length 20 cm, to be referred to as the first lens. Behind the lens there is another converging lens of focal length 20 cm placed 10 cm from the first lens. There is a concave mirror of focal length 15 cm placed 50 cm from the second lens. Find the location, orientation, and size of the final image.

112. An object of height 2 cm is placed at 50 cm in front of a diverging lens of focal length 40 cm. Behind the lens, there is a convex mirror of focal length 15 cm placed 30 cm from the converging lens. Find the location, orientation, and size of the final image.

113. Two concave mirrors are placed facing each other. One of them has a small hole in the middle. A penny is placed on the bottom mirror (see the following figure). When you look from the side, a real image of the penny is observed above the hole. Explain how that could happen.

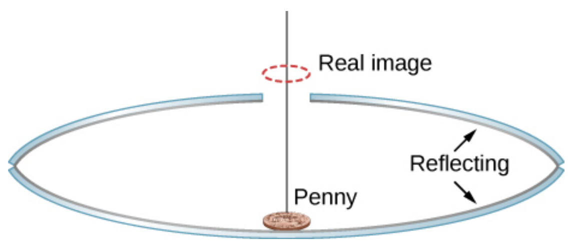


Figure shows the side view of two concave mirrors placed one on top of the other, facing each other. The top one has a small hole in the middle. A penny is placed on the bottom mirror. An image of the penny is shown above the top mirror, just above the hole.

114. A lamp of height 5 cm is placed 40 cm in front of a converging lens of focal length 20 cm. There is a plane mirror 15 cm behind the lens. Where would you find the image when you look in the mirror?

115. Parallel rays from a faraway source strike a converging lens of focal length 20 cm at an angle of 15 degrees with the horizontal direction. Find the vertical position of the real image observed on a screen in the focal plane.

116. Parallel rays from a faraway source strike a diverging lens of focal length 20 cm at an angle of 10 degrees with the horizontal direction. As you look through the lens, where in the vertical plane the image would appear?

117. A light bulb is placed 10 cm from a plane mirror, which faces a convex mirror of radius of curvature 8 cm. The plane mirror is located at a distance of 30 cm from the vertex of the convex mirror. Find the location of two images in the convex mirror. Are there other images? If so, where are they located?

118. A point source of light is 50 cm in front of a converging lens of focal length 30 cm. A concave mirror with a focal length of 20 cm is placed 25 cm behind the lens. Where does the final image form, and what are its orientation and magnification?

119. Copy and trace to find how a horizontal ray from S comes out after the lens. Use $n_{glass} = 1.5$ for the prism material.

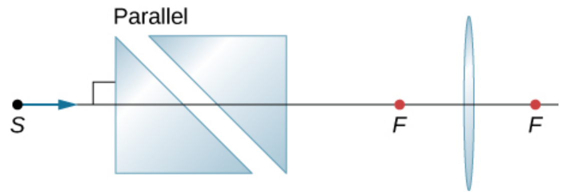


Figure shows two prisms with their bases parallel to each other at an angle of 45 degrees to the horizontal. To the right of this is a bi-convex lens. A ray along the optical axis enters this set up from the left.

120. Copy and trace how a horizontal ray from S comes out after the lens. Use $n = 1.55$ for the glass.

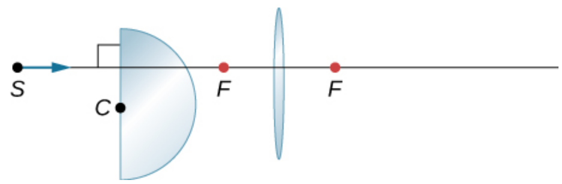


Figure shows the cross section of a hemisphere to the left and that of a bi-convex lens to the right. A ray along the optical axis enters this setup from the left.

121. Copy and draw rays to figure out the final image.

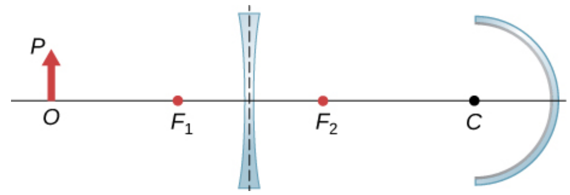


Figure shows from left to right: an object with base O on the axis and tip P. A bi-concave lens with focal point F_1 and F_2 on the left and right respectively and a concave mirror with centre of curvature C.

122. By ray tracing or by calculation, find the place inside the glass where rays from S converge as a result of refraction through the lens and the convex air-glass interface. Use a ruler to estimate the radius of curvature.

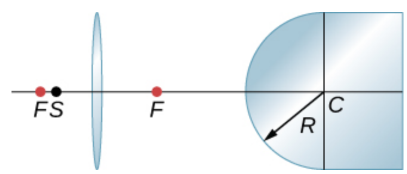


Figure shows a bi-convex lens on the left and a glass with a convex surface on the right. The lens has focal points F on both sides. The center of curvature of convex glass is C and its radius of curvature is R. Point S is between the lens and its focal point on the left.

123. A diverging lens has a focal length of 20 cm. What is the power of the lens in diopters?

124. Two lenses of focal lengths of f_1 and f_2 are glued together with transparent material of negligible thickness. Show that the total power of the two lenses simply add.

125. What will be the angular magnification of a convex lens with the focal length 2.5 cm?

126. What will be the formula for the angular magnification of a convex lens of focal length f if the eye is very close to the lens and the near point is located a distance D from the eye?

Additional Problems

127. Use a ruler and a protractor to draw rays to find images in the following cases.

(a) A point object located on the axis of a concave mirror located at a point within the focal length from the vertex.

(b) A point object located on the axis of a concave mirror located at a point farther than the focal length from the vertex.

(c) A point object located on the axis of a convex mirror located at a point within the focal length from the vertex.

(d) A point object located on the axis of a convex mirror located at a point farther than the focal length from the vertex.

(e) Repeat (a)–(d) for a point object off the axis.

128. Where should a 3 cm tall object be placed in front of a concave mirror of radius 20 cm so that its image is real and 2 cm tall?

129. A 3 cm tall object is placed 5 cm in front of a convex mirror of radius of curvature 20 cm. Where is the image formed? How tall is the image? What is the orientation of the image?

130. You are looking for a mirror so that you can see a four-fold magnified virtual image of an object when the object is placed 5 cm from the vertex of the mirror. What kind of mirror you will need? What should be the radius of curvature of the mirror?

131. Derive the following equation for a convex mirror: $\frac{1}{VO}-\frac{1}{VI}=-\frac{1}{VF}$, where **VO** is the distance to the object **O** from vertex **V**, **VI** the distance to the image **I** from **V**, and **VF** is the distance to the focal point **F** from **V**. (Hint: use two sets of similar triangles.)

132. (a) Draw rays to form the image of a vertical object on the optical axis and farther than the focal point from a converging lens.

(b) Use plane geometry in your figure and prove that the magnification m is given by $m=\frac{h_i}{h_o}=-\frac{d_i}{d_o}$.

133. Use another ray-tracing diagram for the same situation as given in the previous problem to derive the thin-lens equation, $\frac{1}{d_o}+\frac{1}{d_i}=\frac{1}{f}$.

134. You photograph a 2.0-m-tall person with a camera that has a 5.0 cm-focal length lens. The image on the film must be no more than 2.0 cm high.

(a) What is the closest distance the person can stand to the lens?

(b) For this distance, what should be the distance from the lens to the film?

135. Find the focal length of a thin plano-convex lens. The front surface of this lens is flat, and the rear surface has a radius of curvature of $R_2=-35cm$. Assume that the index of refraction of the lens is 1.5.

136. Find the focal length of a meniscus lens with $R_1=20cm$ and $R_2=15cm$. Assume that the index of refraction of the lens is 1.5.

137. A nearsighted man cannot see objects clearly beyond 20 cm from his eyes. How close must he stand to a mirror in order to see what he is doing when he shaves?

138. A mother sees that her child’s contact lens prescription is 0.750 D. What is the child’s near point?

139. Repeat the previous problem for glasses that are 2.20 cm from the eyes.

140. The contact-lens prescription for a nearsighted person is −4.00 D and the person has a far point of 22.5 cm. What is the power of the tear layer between the cornea and the lens if the correction is ideal, taking the tear layer into account?

141. Unreasonable Results A boy has a near point of 50 cm and a far point of 500 cm. Will a −4.00 D lens correct his far point to infinity?

142. Find the angular magnification of an image by a magnifying glass of $f=5.0cm$ if the object is placed $d_o=4.0cm$ from the lens and the lens is close to the eye.

143. Let objective and eyepiece of a compound microscope have focal lengths of 2.5 cm and 10 cm, respectively and be separated by 12 cm. A $70-\mu m$ object is placed 6.0 cm from the objective. How large is the virtual image formed by the objective-eyepiece system?

144. Draw rays to scale to locate the image at the retina if the eye lens has a focal length 2.5 cm and the near point is 24 cm. (Hint: Place an object at the near point.)

145. The objective and the eyepiece of a microscope have the focal lengths 3 cm and 10 cm respectively. Decide about the distance between the objective and the eyepiece if we need a $10\times$ magnification from the objective/eyepiece compound system.

146. A far-sighted person has a near point of 100 cm. How far in front or behind the retina does the image of an object placed 25 cm from the eye form? Use the cornea to retina distance of 2.5 cm.

147. A near-sighted person has afar point of 80 cm.

(a) What kind of corrective lens the person will need if the lens is to be placed 1.5 cm from the eye?

(b) What would be the power of the contact lens needed? Assume distance to contact lens from the eye to be zero.

148. In a reflecting telescope the objective is a concave mirror of radius of curvature 2 m and an eyepiece is a convex lens of focal length 5 cm. Find the apparent size of a 25-m tree at a distance of 10 km that you would perceive when looking through the telescope.

149. Two stars that are $10^9 km$ apart are viewed by a telescope and found to be separated by an angle of 10^{-5} radians. If the eyepiece of the telescope has a focal length of 1.5 cm and the objective has a focal length of 3 meters, how far away are the stars from the observer?

150. What is the angular size of the Moon if viewed from a binocular that has a focal length of 1.2 cm for the eyepiece and a focal length of 8 cm for the objective? Use the radius of the moon $1.74 \times 10^6 m$ and the distance of the moon from the observer to be $3.8 \times 10^8 m$.

151. An unknown planet at a distance of $10^{12} m$ from Earth is observed by a telescope that has a focal length of the eyepiece of 1 cm and a focal length of the objective of 1 m. If the far away planet is seen to subtend an angle of 10^{-5} radian at the eyepiece, what is the size of the planet?

CHAPTER OVERVIEW

12: Interference

The most certain indication of a wave is interference. This wave characteristic is most prominent when the wave interacts with an object that is not large compared with the wavelength. Interference is observed for water waves, sound waves, light waves, and, in fact, all types of waves.

12.1: Prelude to Interference

12.2: Young's Double-Slit Interference

12.3: Mathematics of Interference

12.4: Multiple-Slit Interference

12.5: Interference in Thin Films

12.E: Interference (Exercises)

12.1: Prelude to Interference

The most certain indication of a wave is interference. This wave characteristic is most prominent when the wave interacts with an object that is not large compared with the wavelength. Interference is observed for water waves, sound waves, light waves, and, in fact, all types of waves.

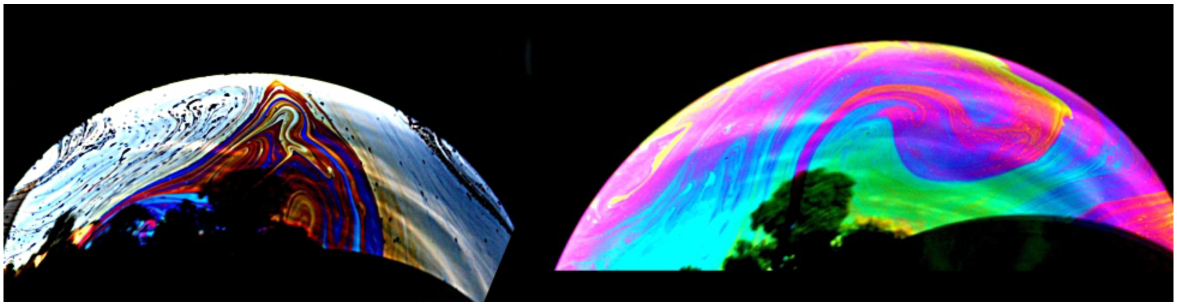


Figure 12.1.1: Soap bubbles are blown from clear fluid into very thin films. The colors we see are not due to any pigmentation but are the result of light interference, which enhances specific wavelengths for a given thickness of the film.

If you have ever looked at the reds, blues, and greens in a sunlit soap bubble and wondered how straw-colored soapy water could produce them, you have hit upon one of the many phenomena that can only be explained by the wave character of light (Figure 12.1.1). The same is true for the colors seen in an oil slick or in the light reflected from a DVD disc. These and other interesting phenomena cannot be explained fully by geometric optics. In these cases, light interacts with objects and exhibits wave characteristics. The branch of optics that considers the behavior of light when it exhibits wave characteristics is called wave optics (sometimes called physical optics). It is the topic of this chapter.

12.2: Young's Double-Slit Interference

Learning Objectives

By the end of this section, you will be able to:

- Explain the phenomenon of interference
- Define constructive and destructive interference for a double slit

The Dutch physicist Christiaan **Huygens** (1629–1695) thought that light was a wave, but Isaac Newton did not. Newton thought that there were other explanations for color, and for the interference and diffraction effects that were observable at the time. Owing to Newton's tremendous reputation, his view generally prevailed; the fact that Huygens's principle worked was not considered direct evidence proving that light is a wave. The acceptance of the wave character of light came many years later in 1801, when the English physicist and physician Thomas **Young** (1773–1829) demonstrated optical interference with his now-classic double-slit experiment.

If there were not one but two sources of waves, the waves could be made to interfere, as in the case of waves on water (Figure $12.2.1$). If light is an electromagnetic wave, it must therefore exhibit interference effects under appropriate circumstances. In Young's experiment, sunlight was passed through a pinhole on a board. The emerging beam fell on two pinholes on a second board. The light emanating from the two pinholes then fell on a screen where a pattern of bright and dark spots was observed. This pattern, called fringes, can only be explained through interference, a wave phenomenon.

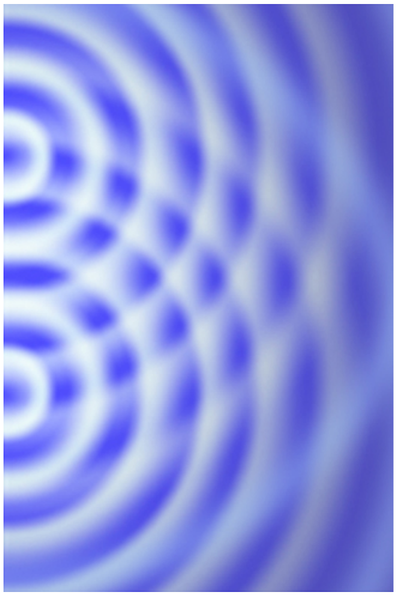


Figure 12.2.1: Photograph of an interference pattern produced by circular water waves in a ripple tank. Two thin plungers are vibrated up and down in phase at the surface of the water. Circular water waves are produced by and emanate from each plunger.

We can analyze double-slit interference with the help of Figure $12.2.2$, which depicts an apparatus analogous to Young's. Light from a monochromatic source falls on a slit S_0. The light emanating from S_0 is incident on two other slits S_1 and S_2 that are equidistant from S_0. A pattern of **interference fringes** on the screen is then produced by the light emanating from S_1 and S_2. All slits are assumed to be so narrow that they can be considered secondary point sources for Huygens' wavelets (The Nature of Light). Slits S_1 and S_2 are a distance **d** apart ($d \leq 1\,mm$), and the distance between the screen and the slits is D(≈1m), which is much greater than **d.**

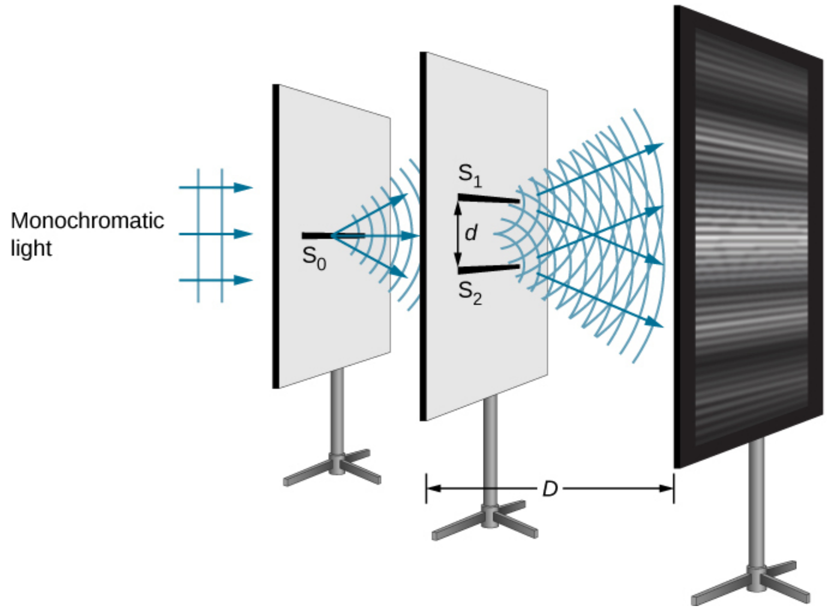


Figure 12.2.2: The double-slit interference experiment using monochromatic light and narrow slits. Fringes produced by interfering Huygens wavelets from slits S_1 and S_2 are observed on the screen.

Since S_0 is assumed to be a point source of monochromatic light, the secondary Huygens wavelets leaving S_1 and S_2 always maintain a constant phase difference (zero in this case because S_1 and S_2 are equidistant from S_0) and have the same frequency. The sources S_1 and S_2 are then said to be coherent. By coherent waves, we mean the waves are in phase or have a definite phase relationship. The term incoherent means the waves have random phase relationships, which would be the case if S_1 and S_2 were illuminated by two independent light sources, rather than a single source S_0. Two independent light sources (which may be two separate areas within the same lamp or the Sun) would generally not emit their light in unison, that is, not coherently. Also, because S_1 and S_2 are the same distance from S_0, the amplitudes of the two Huygens wavelets are equal.

Young used sunlight, where each wavelength forms its own pattern, making the effect more difficult to see. In the following discussion, we illustrate the double-slit experiment with monochromatic light (single λ) to clarify the effect. Figure $12.2.3$ shows the pure constructive and destructive interference of two waves having the same wavelength and amplitude.

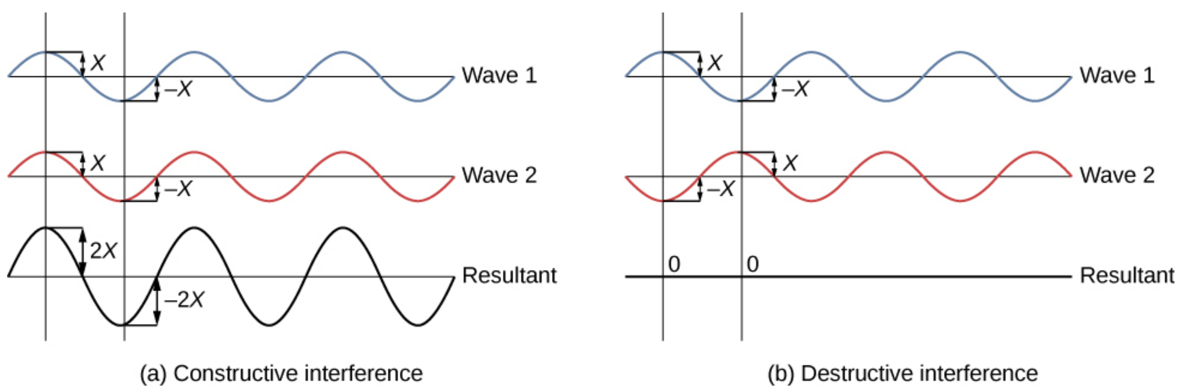


Figure 12.2.3: The amplitudes of waves add. (a) Pure constructive interference is obtained when identical waves are in phase. (b) Pure destructive interference occurs when identical waves are exactly out of phase, or shifted by half a wavelength.

When light passes through narrow slits, the slits act as sources of coherent waves and light spreads out as semicircular waves, as shown in Figure $12.2.1a$. Pure **constructive interference** occurs where the waves are crest to crest or trough to trough. Pure **destructive interference** occurs where they are crest to trough. The light must fall on a screen and be scattered into our eyes for us to see the pattern. An analogous pattern for water waves is shown in Figure $12.2.1$. Note that regions of constructive and destructive interference move out from the slits at well-defined angles to the original beam. These angles depend on wavelength and the distance between the slits, as we shall see below.

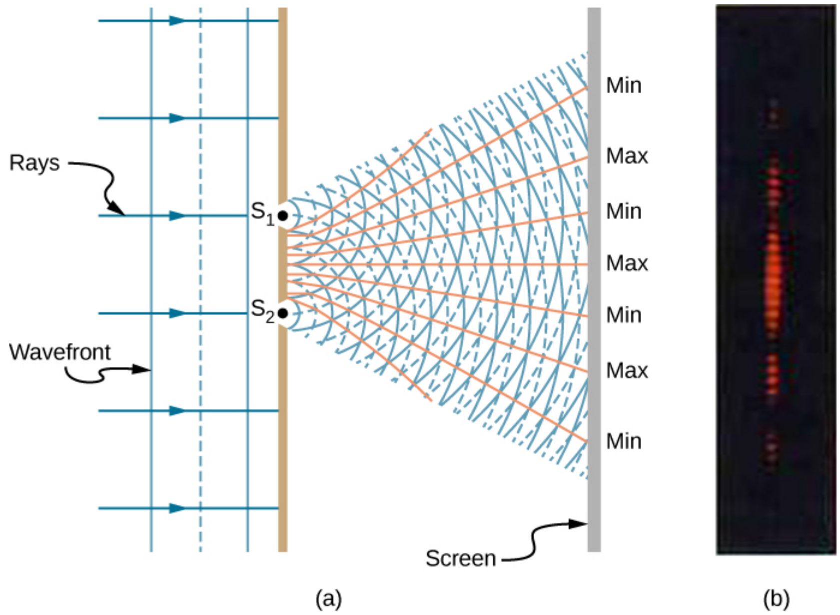


Figure 12.2.4: Double slits produce two coherent sources of waves that interfere. (a) Light spreads out (diffracts) from each slit, because the slits are narrow. These waves overlap and interfere constructively (bright lines) and destructively (dark regions). We can only see this if the light falls onto a screen and is scattered into our eyes. (b) When light that has passed through double slits falls on a screen, we see a pattern such as this.

To understand the double-slit interference pattern, consider how two waves travel from the slits to the screen (Figure $12.2.5$). Each slit is a different distance from a given point on the screen. Thus, different numbers of wavelengths fit into each path. Waves start out from the slits in phase (crest to crest), but they may end up out of phase (crest to trough) at the screen if the paths differ in length by half a wavelength, interfering destructively. If the paths differ by a whole wavelength, then the waves arrive in phase (crest to crest) at the screen, interfering constructively. More generally, if the path length difference Δl between the two waves is any half-integral number of wavelengths [(1 / 2)λ, (3 / 2)λ, (5 / 2)λ, etc.], then destructive interference occurs. Similarly, if the path length difference is any integral number of wavelengths (λ, 2λ, 3λ, etc.), then constructive interference occurs. These conditions can be expressed as equations:

$$\underbrace{\Delta l = m\lambda}_{\text{constructive interference}}$$

for $m = 0, \pm 1, \pm 2, \pm 3\ldots$

$$\underbrace{\Delta l = \left(m + \frac{1}{2}\right)\lambda}_{\text{destructuve interference}}$$

for $m = 0, \pm 1, \pm 2, \pm 3\ldots$

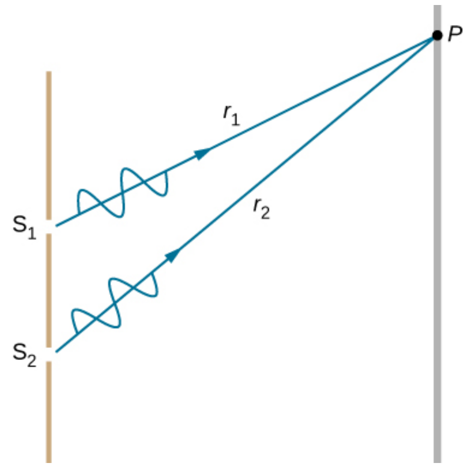


Figure 12.2.5: Waves follow different paths from the slits to a common point P on a screen. Destructive interference occurs where one path is a half wavelength longer than the other—the waves start in phase but arrive out of phase. Constructive interference occurs where one path is a whole wavelength longer than the other—the waves start out and arrive in phase.

12.3: Mathematics of Interference

Learning Objectives

By the end of this section, you will be able to:

- Determine the angles for bright and dark fringes for double slit interference
- Calculate the positions of bright fringes on a screen

Figure $12.3.1a$ shows how to determine the path length difference Δl for waves traveling from two slits to a common point on a screen. If the screen is a large distance away compared with the distance between the slits, then the angle θ between the path and a line from the slits to the screen ($12.3.1b$) is nearly the same for each path. In other words, r_1 and r_2 are essentially parallel. The lengths of r_1 and r_2 differ by Δl, as indicated by the two dashed lines in the $12.3.1$.

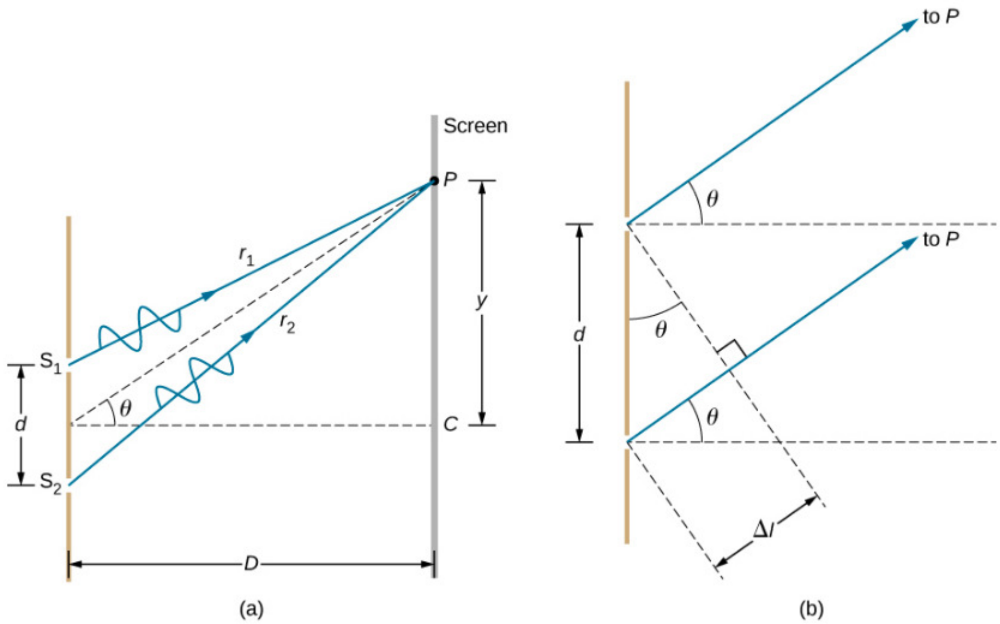


Figure 12.3.1: (a) To reach P, the light waves from S_1 and S_2 must travel different distances. (b) The path difference between the two rays is Δl.

Simple trigonometry shows

$$\Delta l = d \sin \theta \tag{12.3.1}$$

where **d** is the distance between the slits. Combining this with the interference equations discussed previously, we obtain constructive interference for a double slit when the path length difference is an **integral multiple** of the wavelength, or

$$\underbrace{d \sin \theta = m\lambda}_{\text{constructive interference}} \tag{12.3.2}$$

and

$$\underbrace{d \sin \theta = \left(m + \frac{1}{2}\right) \lambda}_{\text{destructive interference}} \tag{12.3.3}$$

where

- $m = 0, \pm 1, \pm 2, \pm 3 \ldots$,
- λ is the wavelength of the light,
- d is the distance between slits, and
- θ is the angle from the original direction of the beam as discussed above.

We call m the **order of the interference**. For example, $m = 4$ is fourth-order interference.

Equations 12.3.2 and 12.3.3 for double-slit interference imply that a series of bright and dark lines are formed. For vertical slits, the light spreads out horizontally on either side of the incident beam into a pattern called interference fringes (Figure 12.3.2). The closer the slits are, the more the bright fringes spread apart. We can see this by examining the Equation 12.3.2. For fixed λ and m, the smaller d is, the larger θ must be, since $\sin\theta = m\lambda/d$. This is consistent with our contention that wave effects are most noticeable when the object the wave encounters (here, slits a distance **d** apart) is small. Small d gives large θ, hence, a large effect.

Referring back to Figure 12.3.1a, θ is typically small enough that

$$\sin\theta \approx \tan\theta \approx y_m/D$$

where y_m is the distance from the central maximum to the **m**-th bright fringe and **D** is the distance between the slit and the screen. Equation 12.3.1 may then be written as

$$d\frac{y_m}{D} = m\lambda$$

or

$$y_m = \frac{m\lambda D}{d}.$$

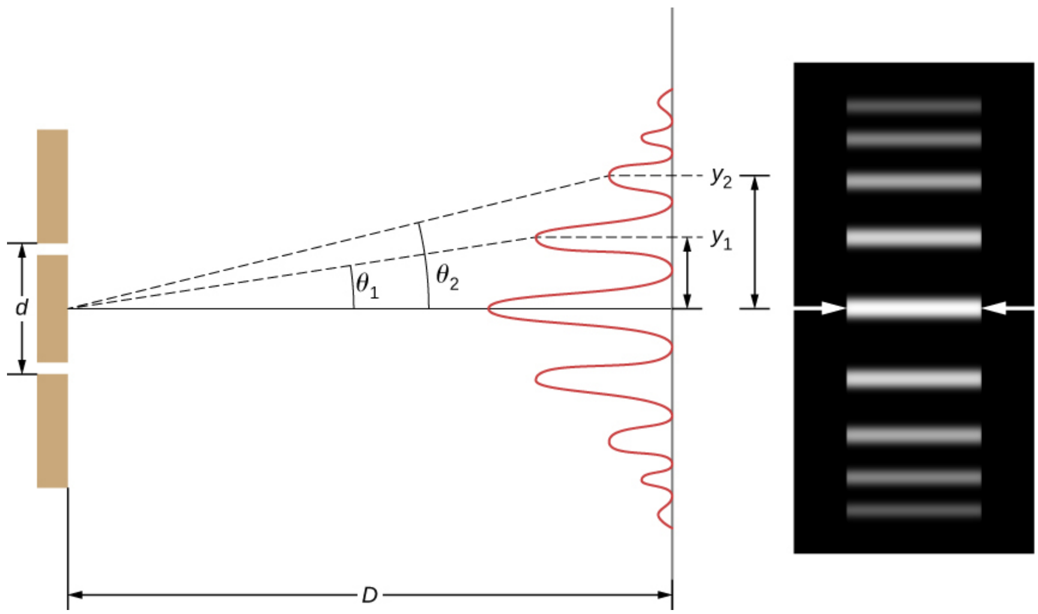


Figure 12.3.2: The interference pattern for a double slit has an intensity that falls off with angle. The image shows multiple bright and dark lines, or fringes, formed by light passing through a double slit.

Example 12.3.1: Finding a Wavelength from an Interference Pattern

Suppose you pass light from a He-Ne laser through two slits separated by 0.0100 mm and find that the third bright line on a screen is formed at an angle of 10.95° relative to the incident beam. What is the wavelength of the light?

Strategy

The phenomenon is two-slit interference as illustrated in Figure 12.3.2 and the third bright line is due to third-order constructive interference, which means that $m = 3$. We are given $d = 0.0100\,mm$ and $\theta = 10.95^o$. The wavelength can thus be found using Equation 12.3.2 for constructive interference.

Solution

Solving Equation 12.3.2 for the wavelength λ gives

$$\lambda = \frac{d\sin\theta}{m}.$$

Substituting known values yields

$$\begin{aligned}\lambda &= \frac{(0.0100\,mm)(\sin 10.95^o)}{3} \\ &= 6.33\times 10^{-4} mm \\ &= 633\,nm.\end{aligned}$$

Significance

To three digits, this is the wavelength of light emitted by the common He-Ne laser. Not by coincidence, this red color is similar to that emitted by neon lights. More important, however, is the fact that interference patterns can be used to measure wavelength. Young did this for visible wavelengths. This analytical technique is still widely used to measure electromagnetic spectra. For a given order, the angle for constructive interference increases with λ, so that spectra (measurements of intensity versus wavelength) can be obtained.

✔ Example 12.3.2: Calculating the Highest Order Possible

Interference patterns do not have an infinite number of lines, since there is a limit to how big **m** can be. What is the highest-order constructive interference possible with the system described in the preceding example?

Strategy

Equation 12.3.2 describes constructive interference from two slits. For fixed values of d and λ, the larger m is, the larger $\sin\theta$ is. However, the maximum value that $\sin\theta$ can have is 1, for an angle of 90°. (Larger angles imply that light goes backward and does not reach the screen at all.) Let us find what value of m corresponds to this maximum diffraction angle.

Solution

Solving the equation $d\,\sin\,\theta = m\lambda$ for **m** gives

$$m = \frac{d\,\sin\,\theta}{\lambda}.$$

Taking $\sin\,\theta = 1$ and substituting the values of d and λ from the preceding example gives

$$m = \frac{(0.0100\,mm)(1)}{633\,nm} \approx 15.8.$$

Therefore, the largest integer m can be is 15, or $m = 15$.

Significance

The number of fringes depends on the wavelength and slit separation. The number of fringes is very large for large slit separations. However, recall (see The Propagation of Light) that wave interference is only prominent when the wave interacts with objects that are not large compared to the wavelength. Therefore, if the slit separation and the sizes of the slits become much greater than the wavelength, the intensity pattern of light on the screen changes, so there are simply two bright lines cast by the slits, as expected, when light behaves like rays. We also note that the fringes get fainter farther away from the center. Consequently, not all 15 fringes may be observable.

? Exercise 12.3.1

In the system used in the preceding examples, at what angles are the first and the second bright fringes formed?

Answer

3.63^o and 7.27^o, respectively

12.4: Multiple-Slit Interference

Learning Objectives

By the end of this section, you will be able to:

- Describe the locations and intensities of secondary maxima for multiple-slit interference

Analyzing the interference of light passing through two slits lays out the theoretical framework of interference and gives us a historical insight into Thomas Young's experiments. However, much of the modern-day application of slit interference uses not just two slits but many, approaching infinity for practical purposes. The key optical element is called a **diffraction grating**, an important tool in optical analysis, which we discuss in detail in chapter on Diffraction. Here, we start the analysis of multiple-slit interference by taking the results from our analysis of the double slit (N=2) and extending it to configurations with three, four, and much larger numbers of slits.

Figure $12.4.1$ shows the simplest case of multiple-slit interference, with three slits, or N=3. The spacing between slits is **d**, and the path length difference between adjacent slits is d sin θ, same as the case for the double slit. What is new is that the path length difference for the first and the third slits is 2d sin θ. The condition for constructive interference is the same as for the double slit, that is

$$d \sin \theta = m\lambda$$

When this condition is met, 2d sin θ is automatically a multiple of λ, so all three rays combine constructively, and the bright fringes that occur here are called principal maxima. But what happens when the path length difference between adjacent slits is only λ/2? We can think of the first and second rays as interfering destructively, but the third ray remains unaltered. Instead of obtaining a dark fringe, or a minimum, as we did for the double slit, we see a secondary maximum with intensity lower than the principal maxima.

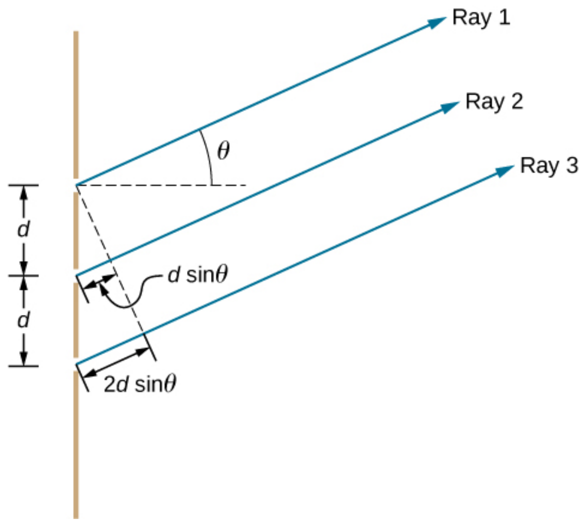


Figure $12.4.1$: Interference with three slits. Different pairs of emerging rays can combine constructively or destructively at the same time, leading to secondary maxima.

In general, for **N** slits, these secondary maxima occur whenever an unpaired ray is present that does not go away due to destructive interference. This occurs at (N−2) evenly spaced positions between the principal maxima. The amplitude of the electromagnetic wave is correspondingly diminished to 1/N of the wave at the principal maxima, and the light intensity, being proportional to the square of the wave amplitude, is diminished to $1/N^2$ of the intensity compared to the principal maxima. As Figure $12.4.2$ Interference fringe patterns for two, three and four slits. As the number of slits increases, more secondary maxima appear, but the principal maxima shows, a dark fringe is located between every maximum (principal or secondary). As **N** grows larger and the number of bright and dark fringes increase, the widths of the maxima become narrower due to the closely located neighboring dark fringes. Because the total amount of light energy remains unaltered, narrower maxima require that each maximum reaches a correspondingly higher intensity.

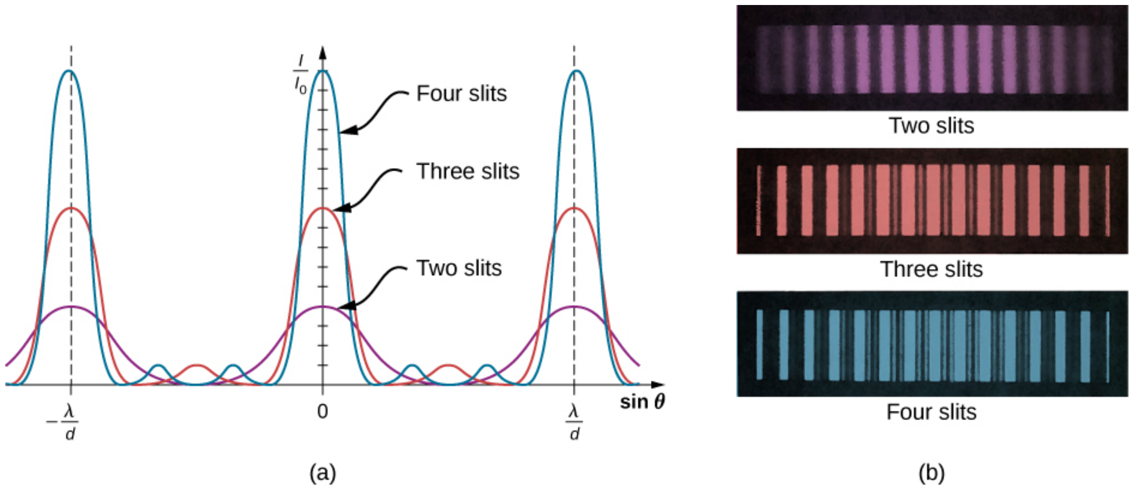


Figure 12.4.2: Interference fringe patterns for two, three and four slits. As the number of slits increases, more secondary maxima appear, but the principal maxima become brighter and narrower. (a) Graph and (b) photographs of fringe patterns.

12.5: Interference in Thin Films

Learning Objectives

By the end of this section, you will be able to:

- Describe the phase changes that occur upon reflection
- Describe fringes established by reflected rays of a common source
- Explain the appearance of colors in thin films

The bright colors seen in an oil slick floating on water or in a sunlit soap bubble are caused by interference. The brightest colors are those that interfere constructively. This interference is between light reflected from different surfaces of a thin film; thus, the effect is known as **thin-film interference.**

As we noted before, interference effects are most prominent when light interacts with something having a size similar to its wavelength. A thin film is one having a thickness t smaller than a few times the wavelength of light, λ. Since color is associated indirectly with λ and because all interference depends in some way on the ratio of λ to the size of the object involved, we should expect to see different colors for different thicknesses of a film, as in Figure $12.5.1$.

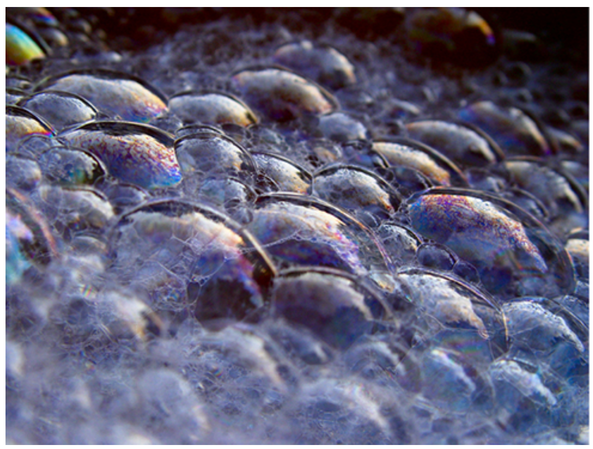


Figure 12.5.1: These soap bubbles exhibit brilliant colors when exposed to sunlight. (credit: Scott Robinson)

What causes thin-film interference? Figure $12.5.2$ shows how light reflected from the top and bottom surfaces of a film can interfere. Incident light is only partially reflected from the top surface of the film (ray 1). The remainder enters the film and is itself partially reflected from the bottom surface. Part of the light reflected from the bottom surface can emerge from the top of the film (ray 2) and interfere with light reflected from the top (ray 1). The ray that enters the film travels a greater distance, so it may be in or out of phase with the ray reflected from the top. However, consider for a moment, again, the bubbles in Figure $12.5.1$. The bubbles are darkest where they are thinnest. Furthermore, if you observe a soap bubble carefully, you will note it gets dark at the point where it breaks. For very thin films, the difference in path lengths of rays 1 and 2 in Figure $12.5.2$ is negligible, so why should they interfere destructively and not constructively? The answer is that a phase change can occur upon reflection, as discussed next.

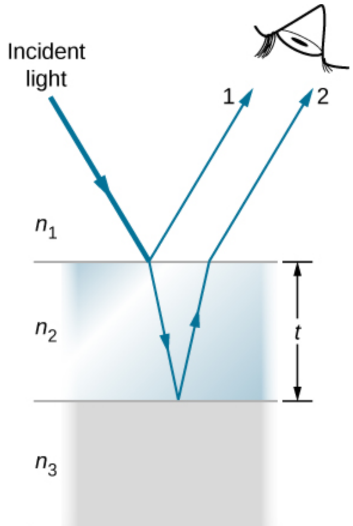


Figure 12.5.2: Light striking a thin film is partially reflected (ray 1) and partially refracted at the top surface. The refracted ray is partially reflected at the bottom surface and emerges as ray 2. These rays interfere in a way that depends on the thickness of the film and the indices of refraction of the various media.

Changes in Phase due to Reflection

We saw earlier (Waves) that reflection of mechanical waves can involve a 180° phase change. For example, a traveling wave on a string is inverted (i.e., a 180° phase change) upon reflection at a boundary to which a heavier string is tied. However, if the second string is lighter (or more precisely, of a lower linear density), no inversion occurs. Light waves produce the same effect, but the deciding parameter for light is the index of refraction. Light waves undergo a 180° or π radians phase change upon reflection at an interface beyond which is a medium of higher index of refraction. No phase change takes place when reflecting from a medium of lower refractive index (Figure $12.5.3$). Because of the periodic nature of waves, this phase change or inversion is equivalent to $\pm\lambda/2$ in distance travelled, or path length. Both the path length and refractive indices are important factors in thin-film interference.

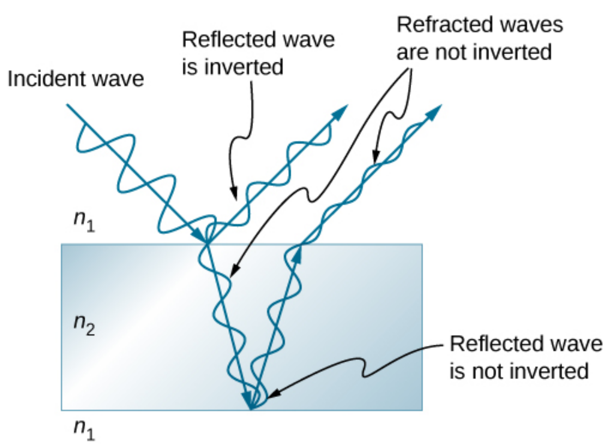


Figure 12.5.3: Reflection at an interface for light traveling from a medium with index of refraction n_1 to a medium with index of refraction n_2, $n_1 < n_2$, causes the phase of the wave to change by π radians.

If the film in Figure $12.5.3$ is a **soap bubble** (essentially water with air on both sides), then a phase shift of $\lambda/2$ occurs for ray 1 but not for ray 2. Thus, when the film is very thin and the path length difference between the two rays is negligible, they are exactly out of phase, and destructive interference occurs at all wavelengths. Thus, the soap bubble is dark here. The thickness of the film relative to the wavelength of light is the other crucial factor in thin-film interference. Ray 2 in Figure $12.5.3$ travels a greater distance than ray 1. For light incident perpendicular to the surface, ray 2 travels a distance approximately $2t$ farther than ray 1. When this distance is an integral or half-integral multiple of the wavelength in the medium ($\lambda_n = \lambda/n$, where λ is the wavelength in vacuum and n is the index of refraction), constructive or destructive interference occurs, depending also on whether there is a phase change in either ray.

✓ Example 12.5.1: Calculating the Thickness of a Nonreflective Lens Coating

Sophisticated cameras use a series of several lenses. Light can reflect from the surfaces of these various lenses and degrade image clarity. To limit these reflections, lenses are coated with a thin layer of magnesium fluoride, which causes destructive thin-film interference. What is the thinnest this film can be, if its index of refraction is 1.38 and it is designed to limit the reflection of 550-nm light, normally the most intense visible wavelength? Assume the index of refraction of the glass is 1.52.

Strategy

Refer to Figure $12.5.2$ and use $n_1 = 1.00$ for air, $n_2 = 1.38$, and $n_3 = 1.52$. Both ray 1 and ray 2 have a λ/2 shift upon reflection. Thus, to obtain destructive interference, ray 2 needs to travel a half wavelength farther than ray 1. For rays incident perpendicularly, the path length difference is 2**t**.

Solution

To obtain destructive interference here,

$$2t = \frac{\lambda_{n2}}{2}$$

where λ_{n2} is the wavelength in the film and is given by $\lambda_{n2} = \lambda / n_2$. Thus,

$$2t = \frac{\lambda / n_2}{2}.$$

Solving for **t** and entering known values yields

$$t = \frac{\lambda / n_2}{4} = \frac{(500\, nm)/1.38}{4} = 99.6\, nm.$$

Significance

Films such as the one in this example are most effective in producing destructive interference when the thinnest layer is used, since light over a broader range of incident angles is reduced in intensity. These films are called **nonreflective coatings**; this is only an approximately correct description, though, since other wavelengths are only partially cancelled. Nonreflective coatings are also used in car windows and sunglasses.

Combining Path Length Difference with Phase Change

Thin-film interference is most constructive or most destructive when the path length difference for the two rays is an integral or half-integral wavelength. That is, for rays incident perpendicularly,

$$2t = \lambda_n,\ 2\lambda_n,\ 3\lambda_n, \ldots\ or\ 2t = \lambda_n/2,\ 3\lambda_n/2,\ 5\lambda_n/2, \ldots$$

To know whether interference is constructive or destructive, you must also determine if there is a phase change upon reflection. Thin-film interference thus depends on film thickness, the wavelength of light, and the refractive indices. For white light incident on a film that varies in thickness, you can observe rainbow colors of constructive interference for various wavelengths as the thickness varies.

✓ Example 12.5.2: Soap Bubbles

a. What are the three smallest thicknesses of a soap bubble that produce constructive interference for red light with a wavelength of 650 nm? The index of refraction of soap is taken to be the same as that of water.
b. What three smallest thicknesses give destructive interference?

Strategy

Use Figure $12.5.3$ to visualize the bubble, which acts as a thin film between two layers of air. Thus $n_1 = n_3 = 1.00$ for air, and $n_2 = 1.333$ for soap (equivalent to water). There is a λ/2 shift for ray 1 reflected from the top surface of the bubble and no shift for ray 2 reflected from the bottom surface. To get constructive interference, then, the path length difference (2**t**) must be a half-integral multiple of the wavelength—the first three being $\lambda_n/2$, $3\lambda_n/2$, and $5\lambda_n/2$. To get destructive interference, the path length difference must be an integral multiple of the wavelength—the first three being 0, λ_n, and $2\lambda_n$.

Solution

a. Constructive interference occurs here when

$$2t_c = \frac{\lambda_n}{2}, \frac{3\lambda_n}{2}, \frac{5\lambda_n}{2}, \ldots$$

Thus, the smallest constructive thickness t_c is

$$t_c = \frac{\lambda_n}{4} = \frac{\lambda/n}{4} = \frac{(650\,nm)/1.333}{4} = 122\,nm.$$

The next thickness that gives constructive interference is $t'_c = 3\lambda_n/4$, so that

$$t'_c = 366\,nm.$$

Finally, the third thickness producing constructive interference is $t'_c = 5\lambda_n/4$, so that

$$t'_c = 610\,nm.$$

b. For destructive interference, the path length difference here is an integral multiple of the wavelength. The first occurs for zero thickness, since there is a phase change at the top surface, that is,

$$t_d = 0,$$

the very thin (or negligibly thin) case discussed above. The first non-zero thickness producing destructive interference is

$$2t'_d = \lambda_n.$$

Substituting known values gives

$$t'_d = \frac{\lambda}{2} = \frac{\lambda/n}{2} = \frac{(650\,nm)/1.333}{2} = 244\,nm.$$

Finally, the third destructive thickness is $2t''_d = 2\lambda_n$, so that

$$t''_d = \lambda_n = \frac{\lambda}{n} = \frac{650\,nm}{1.333} = 488\,nm.$$

Significance

If the bubble were illuminated with pure red light, we would see bright and dark bands at very uniform increases in thickness. First would be a dark band at 0 thickness, then bright at 122 nm thickness, then dark at 244 nm, bright at 366 nm, dark at 488 nm, and bright at 610 nm. If the bubble varied smoothly in thickness, like a smooth wedge, then the bands would be evenly spaced.

? Exercise 12.5.2

Going further with Example 12.5.2, what are the next two thicknesses of soap bubble that would lead to

a. constructive interference, and
b. destructive interference?

Answer a

853 nm and 1097 nm

Answer b

731 nm and 975 nm

Another example of thin-film interference can be seen when microscope slides are separated (see Figure 12.5.4). The slides are very flat, so that the wedge of air between them increases in thickness very uniformly. A phase change occurs at the second surface but not the first, so a dark band forms where the slides touch. The rainbow colors of constructive interference repeat, going from violet to red again and again as the distance between the slides increases. As the layer of air increases, the bands become more difficult to see, because slight changes in incident angle have greater effects on path length differences. If monochromatic light instead of white light is used, then bright and dark bands are obtained rather than repeating rainbow colors.

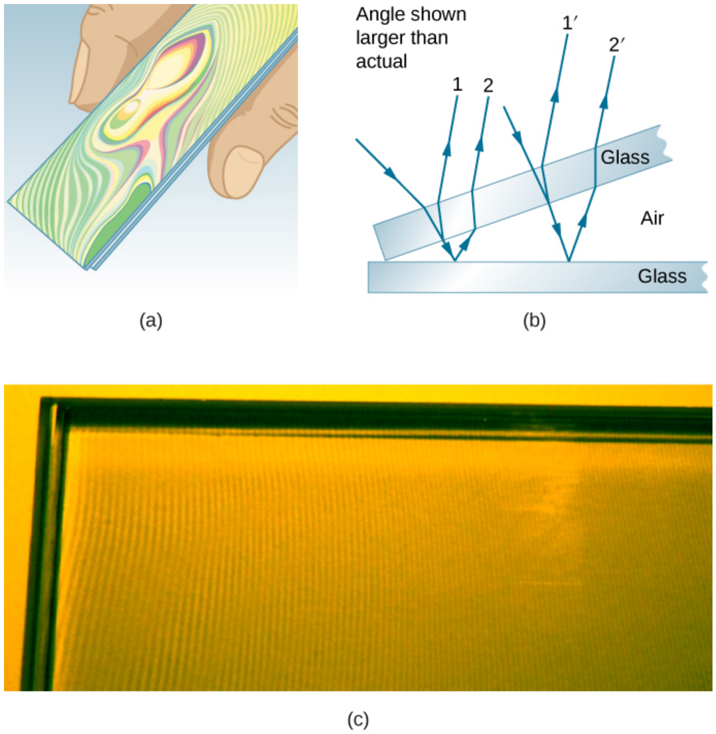


Figure 12.5.4: (a) The rainbow-color bands are produced by thin-film interference in the air between the two glass slides. (b) Schematic of the paths taken by rays in the wedge of air between the slides. (c) If the air wedge is illuminated with monochromatic light, bright and dark bands are obtained rather than repeating rainbow colors.

An important application of thin-film interference is found in the manufacturing of optical instruments. A lens or mirror can be compared with a master as it is being ground, allowing it to be shaped to an accuracy of less than a wavelength over its entire surface. Figure 12.5.5 illustrates the phenomenon called Newton's rings, which occurs when the plane surfaces of two lenses are placed together. (The circular bands are called Newton's rings because Isaac Newton described them and their use in detail. Newton did not discover them; Robert Hooke did, and Newton did not believe they were due to the wave character of light.) Each successive ring of a given color indicates an increase of only half a wavelength in the distance between the lens and the blank, so that great precision can be obtained. Once the lens is perfect, no rings appear.

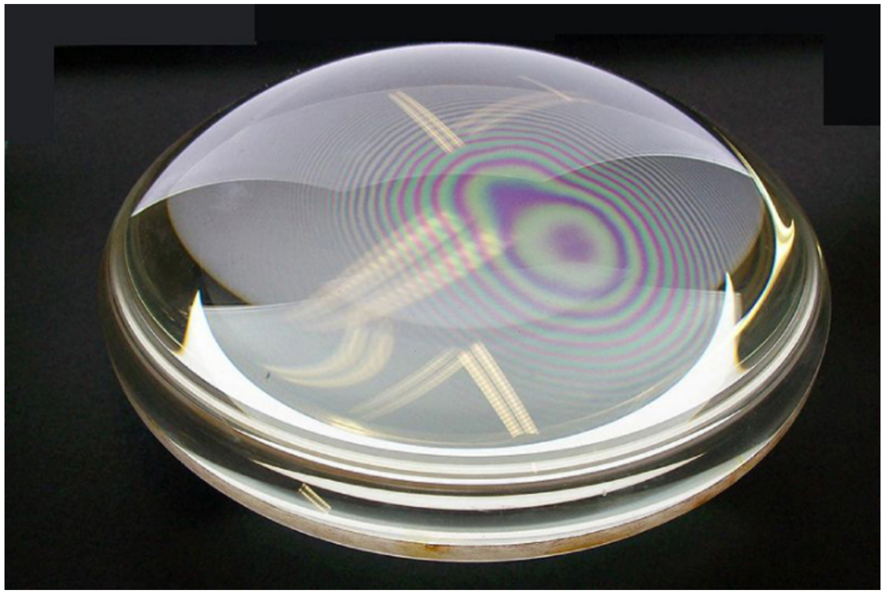


Figure 12.5.5:“Newton’s rings” interference fringes are produced when two plano-convex lenses are placed together with their plane surfaces in contact. The rings are created by interference between the light reflected off the two surfaces as a result of a slight gap between them, indicating that these surfaces are not precisely plane but are slightly convex. (credit: Ulf Seifert)

Thin-film interference has many other applications, both in nature and in manufacturing. The wings of certain moths and butterflies have nearly iridescent colors due to thin-film interference. In addition to pigmentation, the wing’s color is affected greatly by constructive interference of certain wavelengths reflected from its film-coated surface. Some car manufacturers offer special paint jobs that use thin-film interference to produce colors that change with angle. This expensive option is based on variation of thin-film path length differences with angle. Security features on credit cards, banknotes, driving licenses, and similar items prone to forgery use thin-film interference, diffraction gratings, or holograms. As early as 1998, Australia led the way with dollar bills printed on polymer with a diffraction grating security feature, making the currency difficult to forge. Other countries, such as Canada, New Zealand, and Taiwan, are using similar technologies, while US currency includes a thin-film interference effect.

12.E: Interference (Exercises)

Conceptual Questions

3.1 Young's Double-Slit Interference

1. Young’s double-slit experiment breaks a single light beam into two sources. Would the same pattern be obtained for two independent sources of light, such as the headlights of a distant car? Explain.

2. Is it possible to create a experimental setup in which there is only destructive interference? Explain.

3. Why won’t two small sodium lamps, held close together, produce an interference pattern on a distant screen? What if the sodium lamps were replaced by two laser pointers held close together?

3.2 Mathematics of Interference

4. Suppose you use the same double slit to perform Young’s double-slit experiment in air and then repeat the experiment in water. Do the angles to the same parts of the interference pattern get larger or smaller? Does the color of the light change? Explain.

5. Why is monochromatic light used in the double slit experiment? What would happen if white light were used?

3.4 Interference in Thin Films

6. What effect does increasing the wedge angle have on the spacing of interference fringes? If the wedge angle is too large, fringes are not observed. Why?

7. How is the difference in paths taken by two originally in-phase light waves related to whether they interfere constructively or destructively? How can this be affected by reflection? By refraction?

8. Is there a phase change in the light reflected from either surface of a contact lens floating on a person’s tear layer? The index of refraction of the lens is about 1.5, and its top surface is dry.

9. In placing a sample on a microscope slide, a glass cover is placed over a water drop on the glass slide. Light incident from above can reflect from the top and bottom of the glass cover and from the glass slide below the water drop. At which surfaces will there be a phase change in the reflected light?

10. Answer the above question if the fluid between the two pieces of crown glass is carbon disulfide.

11. While contemplating the food value of a slice of ham, you notice a rainbow of color reflected from its moist surface. Explain its origin.

12. An inventor notices that a soap bubble is dark at its thinnest and realizes that destructive interference is taking place for all wavelengths. How could she use this knowledge to make a nonreflective coating for lenses that is effective at all wavelengths? That is, what limits would there be on the index of refraction and thickness of the coating? How might this be impractical?

13. A nonreflective coating like the one described in Example 3.3 works ideally for a single wavelength and for perpendicular incidence. What happens for other wavelengths and other incident directions? Be specific.

14. Why is it much more difficult to see interference fringes for light reflected from a thick piece of glass than from a thin film? Would it be easier if monochromatic light were used?

3.5 The Michelson Interferometer

15. Describe how a Michelson interferometer can be used to measure the index of refraction of a gas (including air).

Problems

3.2 Mathematics of Interference

16. At what angle is the first-order maximum for 450-nm wavelength blue light falling on double slits separated by 0.0500 mm?

17. Calculate the angle for the third-order maximum of 580-nm wavelength yellow light falling on double slits separated by 0.100 mm.

18. What is the separation between two slits for which 610-nm orange light has its first maximum at an angle of 30.0°?

19. Find the distance between two slits that produces the first minimum for 410-nm violet light at an angle of 45.0°.

20. Calculate the wavelength of light that has its third minimum at an angle of 30.0° when falling on double slits separated by $3.00 \mu m$. Explicitly show how you follow the steps from the Problem-Solving Strategy: Wave Optics, located at the end of the chapter.

21. What is the wavelength of light falling on double slits separated by $2.00 \mu m$ if the third-order maximum is at an angle of 60.0°?

22. At what angle is the fourth-order maximum for the situation in the preceding problem?

23. What is the highest-order maximum for 400-nm light falling on double slits separated by $25.0 \mu m$?

24. Find the largest wavelength of light falling on double slits separated by $1.20 \mu m$ for which there is a first-order maximum. Is this in the visible part of the spectrum?

25. What is the smallest separation between two slits that will produce a second-order maximum for 720-nm red light?

26. (a) What is the smallest separation between two slits that will produce a second-order maximum for any visible light?

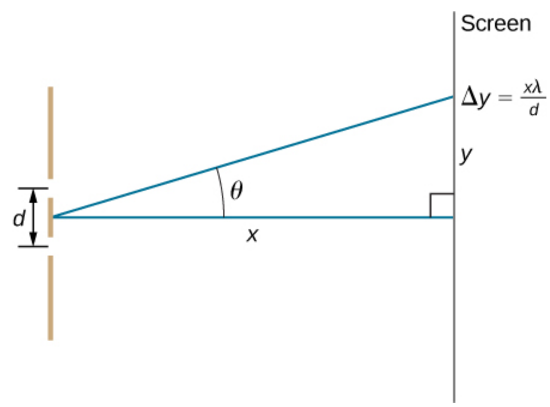
(b) For all visible light?

27. (a) If the first-order maximum for monochromatic light falling on a double slit is at an angle of 10.0°, at what angle is the second-order maximum?

(b) What is the angle of the first minimum?

(c) What is the highest-order maximum possible here?

28. Shown below is a double slit located a distance **x** from a screen, with the distance from the center of the screen given by y. When the distance **d** between the slits is relatively large, numerous bright spots appear, called fringes. Show that, for small angles (where $sin\theta \approx \theta$, with θ in radians), the distance between fringes is given by $\Delta y = x\lambda / d$



Picture shows a double slit located a distance x from a screen, with the distance from the center of the screen given by y. Distance between the slits is d.

29. Using the result of the preceding problem,

(a) calculate the distance between fringes for 633-nm light falling on double slits separated by 0.0800 mm, located 3.00 m from a screen.

(b) What would be the distance between fringes if the entire apparatus were submersed in water, whose index of refraction is 1.33?

30. Using the result of the problem two problems prior, find the wavelength of light that produces fringes 7.50 mm apart on a screen 2.00 m from double slits separated by 0.120 mm.

31. In a double-slit experiment, the fifth maximum is 2.8 cm from the central maximum on a screen that is 1.5 m away from the slits. If the slits are 0.15 mm apart, what is the wavelength of the light being used?

32. The source in Young's experiment emits at two wavelengths. On the viewing screen, the fourth maximum for one wavelength is located at the same spot as the fifth maximum for the other wavelength. What is the ratio of the two wavelengths?

33. If 500-nm and 650-nm light illuminates two slits that are separated by 0.50 mm, how far apart are the second-order maxima for these two wavelengths on a screen 2.0 m away?

34. Red light of wavelength of 700 nm falls on a double slit separated by 400 nm.

(a) At what angle is the first-order maximum in the diffraction pattern?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

3.3 Multiple-Slit Interference

35. Ten narrow slits are equally spaced 0.25 mm apart and illuminated with yellow light of wavelength 580 nm. (a) What are the angular positions of the third and fourth principal maxima? (b) What is the separation of these maxima on a screen 2.0 m from the slits?

36. The width of bright fringes can be calculated as the separation between the two adjacent dark fringes on either side. Find the angular widths of the third- and fourth-order bright fringes from the preceding problem.

37. For a three-slit interference pattern, find the ratio of the peak intensities of a secondary maximum to a principal maximum.

38. What is the angular width of the central fringe of the interference pattern of

(a) 20 slits separated by $d = 2.0 \times 10^{-3} mm$?

(b) 50 slits with the same separation? Assume that $\lambda = 600 nm$.

3.4 Interference in Thin Films

39. A soap bubble is 100 nm thick and illuminated by white light incident perpendicular to its surface. What wavelength and color of visible light is most constructively reflected, assuming the same index of refraction as water?

40. An oil slick on water is 120 nm thick and illuminated by white light incident perpendicular to its surface. What color does the oil appear (what is the most constructively reflected wavelength), given its index of refraction is 1.40?

41. Calculate the minimum thickness of an oil slick on water that appears blue when illuminated by white light perpendicular to its surface. Take the blue wavelength to be 470 nm and the index of refraction of oil to be 1.40.

42. Find the minimum thickness of a soap bubble that appears red when illuminated by white light perpendicular to its surface. Take the wavelength to be 680 nm, and assume the same index of refraction as water.

43. A film of soapy water ($n = 1.33$) on top of a plastic cutting board has a thickness of 233 nm. What color is most strongly reflected if it is illuminated perpendicular to its surface?

44. What are the three smallest non-zero thicknesses of soapy water ($n = 1.33$) on Plexiglas if it appears green (constructively reflecting 520-nm light) when illuminated perpendicularly by white light?

45. Suppose you have a lens system that is to be used primarily for 700-nm red light. What is the second thinnest coating of fluorite (magnesium fluoride) that would be nonreflective for this wavelength?

46. (a) As a soap bubble thins it becomes dark, because the path length difference becomes small compared with the wavelength of light and there is a phase shift at the top surface. If it becomes dark when the path length difference is less than one-fourth the wavelength, what is the thickest the bubble can be and appear dark at all visible wavelengths? Assume the same index of refraction as water.

(b) Discuss the fragility of the film considering the thickness found.

47. To save money on making military aircraft invisible to radar, an inventor decides to coat them with a nonreflective material having an index of refraction of 1.20, which is between that of air and the surface of the plane. This, he reasons, should be much cheaper than designing Stealth bombers.

(a) What thickness should the coating be to inhibit the reflection of 4.00-cm wavelength radar?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

3.5 The Michelson Interferometer

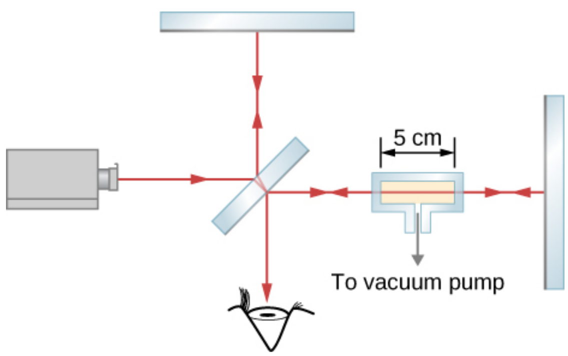
48. A Michelson interferometer has two equal arms. A mercury light of wavelength 546 nm is used for the interferometer and stable fringes are found. One of the arms is moved by $1.5 \mu m$. How many fringes will cross the observing field?

49. What is the distance moved by the traveling mirror of a Michelson interferometer that corresponds to 1500 fringes passing by a point of the observation screen? Assume that the interferometer is illuminated with a 606 nm spectral line of krypton-86.

50. When the traveling mirror of a Michelson interferometer is moved $2.40 \times 10^{-5} m$, 90 fringes pass by a point on the observation screen. What is the wavelength of the light used?

51. In a Michelson interferometer, light of wavelength 632.8 nm from a He-Ne laser is used. When one of the mirrors is moved by a distance D, 8 fringes move past the field of view. What is the value of the distance D?

52. A chamber 5.0 cm long with flat, parallel windows at the ends is placed in one arm of a Michelson interferometer (see below). The light used has a wavelength of 500 nm in a vacuum. While all the air is being pumped out of the chamber, 29 fringes pass by a point on the observation screen. What is the refractive index of the air?



Picture shows a schematics of a set-up utilized to measure the refractive index of a gas. The glass chamber with a gas is placed in the Michelson interferometer between the half-silvered mirror M and mirror M1. The space inside the container is 5 cm wide.

Additional Problems

53. For 600-nm wavelength light and a slit separation of 0.12 mm, what are the angular positions of the first and third maxima in the double slit interference pattern?

54. If the light source in the preceding problem is changed, the angular position of the third maximum is found to be 0.57°. What is the wavelength of light being used now?

55. Red light ($\lambda = 710.nm$) illuminates double slits separated by a distance $d = 0.150mm$. The screen and the slits are 3.00 m apart.

(a) Find the distance on the screen between the central maximum and the third maximum.

(b) What is the distance between the second and the fourth maxima?

56. Two sources as in phase and emit waves with $\lambda = 0.42m$. Determine whether constructive or destructive interference occurs at points whose distances from the two sources are

(a) 0.84 and 0.42 m,

(b) 0.21 and 0.42 m,

(c) 1.26 and 0.42 m,

(d) 1.87 and 1.45 m,

(e) 0.63 and 0.84 m and

(f) 1.47 and 1.26 m.

57. Two slits $4.0 \times 10^{-6} m$ apart are illuminated by light of wavelength 600 nm. What is the highest order fringe in the interference pattern?

58. Suppose that the highest order fringe that can be observed is the eighth in a double-slit experiment where 550-nm wavelength light is used. What is the minimum separation of the slits?

59. The interference pattern of a He-Ne laser light ($\lambda = 632.9 nm$) passing through two slits 0.031 mm apart is projected on a screen 10.0 m away. Determine the distance between the adjacent bright fringes.

60. Young's double-slit experiment is performed immersed in water ($n = 1.333$). The light source is a He-Ne laser, $\lambda = 632.9 nm$ in vacuum.

(a) What is the wavelength of this light in water?

(b) What is the angle for the third order maximum for two slits separated by 0.100 mm.

61. A double-slit experiment is to be set up so that the bright fringes appear 1.27 cm apart on a screen 2.13 m away from the two slits. The light source was wavelength 500 nm. What should be the separation between the two slits?

62. An effect analogous to two-slit interference can occur with sound waves, instead of light. In an open field, two speakers placed 1.30 m apart are powered by a single-function generator producing sine waves at 1200-Hz frequency. A student walks along a line 12.5 m away and parallel to the line between the speakers. She hears an alternating pattern of loud and quiet, due to constructive and destructive interference. What is (a) the wavelength of this sound and (b) the distance between the central maximum and the first maximum (loud) position along this line?

63. A hydrogen gas discharge lamp emits visible light at four wavelengths, $\lambda =$410, 434, 486, and 656 nm. (a) If light from this lamp falls on a N slits separated by 0.025 mm, how far from the central maximum are the third maxima when viewed on a screen 2.0 m from the slits? (b) By what distance are the second and third maxima separated for $l = 486 nm$?

64. Monochromatic light of frequency $5.5 \times 10^{14} Hz$ falls on 10 slits separated by 0.020 mm. What is the separation between the first and third maxima on a screen that is 2.0 m from the slits?

65. Eight slits equally separated by 0.149 mm is uniformly illuminated by a monochromatic light at $\lambda = 523 nm$. What is the width of the central principal maximum on a screen 2.35 m away?

66. Eight slits equally separated by 0.149 mm is uniformly illuminated by a monochromatic light at $\lambda = 523 nm$. What is the intensity of a secondary maxima compared to that of the principal maxima?

67. A transparent film of thickness 250 nm and index of refraction of 1.40 is surrounded by air. What wavelength in a beam of white light at near-normal incidence to the film undergoes destructive interference when reflected?

68. An intensity minimum is found for 450 nm light transmitted through a transparent film ($n = 1.20$) in air.

(a) What is minimum thickness of the film?

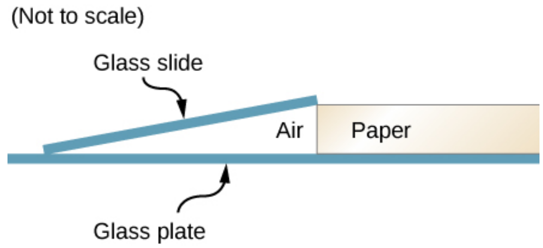
(b) If this wavelength is the longest for which the intensity minimum occurs, what are the next three lower values of λ for which this happens?

69. A thin film with $n = 1.32$ is surrounded by air. What is the minimum thickness of this film such that the reflection of normally incident light with $\lambda = 500 nm$ is minimized?

70. Repeat your calculation of the previous problem with the thin film placed on a flat glass ($n = 1.50$) surface.

71. After a minor oil spill, a think film of oil ($n = 1.40$) of thickness 450 nm floats on the water surface in a bay. (a) What predominant color is seen by a bird flying overhead? (b) What predominant color is seen by a seal swimming underwater?

72. A microscope slide 10 cm long is separated from a glass plate at one end by a sheet of paper. As shown below, the other end of the slide is in contact with the plate. he slide is illuminated from above by light from a sodium lamp ($\lambda = 589nm$), and 14 fringes per centimeter are seen along the slide. What is the thickness of the piece of paper? Picture shows a microscope slide that touches the glass plate at one end and is separated from it at another end by a sheet of paper.



73. Suppose that the setup of the preceding problem is immersed in an unknown liquid. If 18 fringes per centimeter are now seen along the slide, what is the index of refraction of the liquid?

74. A thin wedge filled with air is produced when two flat glass plates are placed on top of one another and a slip of paper is inserted between them at one edge. Interference fringes are observed when monochromatic light falling vertically on the plates are seen in reflection. Is the first fringe near the edge where the plates are in contact a bright fringe or a dark fringe? Explain.

75. Two identical pieces of rectangular plate glass are used to measure the thickness of a hair. The glass plates are in direct contact at one edge and a single hair is placed between them hear the opposite edge. When illuminated with a sodium lamp ($\lambda = 589nm$), the hair is seen between the 180th and 181st dark fringes. What are the lower and upper limits on the hair's diameter?

76. Two microscope slides made of glass are illuminated by monochromatic ($\lambda = 589nm$) light incident perpendicularly. The top slide touches the bottom slide at one end and rests on a thin copper wire at the other end, forming a wedge of air. The diameter of the copper wire is **29.45μm**. How many bright fringes are seen across these slides?

77. A good quality camera "lens" is actually a system of lenses, rather than a single lens, but a side effect is that a reflection from the surface of one lens can bounce around many times within the system, creating artifacts in the photograph. To counteract this problem, one of the lenses in such a system is coated with a thin layer of material ($n = 1.28$) on one side. The index of refraction of the lens glass is 1.68. What is the smallest thickness of the coating that reduces the reflection at 640 nm by destructive interference? (In other words, the coating's effect is to be optimized for $\lambda = 640nm$.)

78. Constructive interference is observed from directly above an oil slick for wavelengths (in air) 440 nm and 616 nm. The index of refraction of this oil is $n = 1.54$. What is the film's minimum possible thickness?

79. A soap bubble is blown outdoors. What colors (indicate by wavelengths) of the reflected sunlight are seen enhanced? The soap bubble has index of refraction 1.36 and thickness 380 nm.

80. A Michelson interferometer with a He-Ne laser light source ($\lambda = 632.8nm$) projects its interference pattern on a screen. If the movable mirror is caused to move by $8.54\mu m$, how many fringes will be observed shifting through a reference point on a screen?

81. An experimenter detects 251 fringes when the movable mirror in a Michelson interferometer is displaced. The light source used is a sodium lamp, wavelength 589 nm. By what distance did the movable mirror move?

82. A Michelson interferometer is used to measure the wavelength of light put through it. When the movable mirror is moved by exactly 0.100 mm, the number of fringes observed moving through is 316. What is the wavelength of the light?

83. A 5.08-cm-long rectangular glass chamber is inserted into one arm of a Michelson interferometer using a 633-nm light source. This chamber is initially filled with air ($n = 1.000293$) at standard atmospheric pressure but the air is gradually pumped out using a vacuum pump until a near perfect vacuum is achieved. How many fringes are observed moving by during the transition?

84. Into one arm of a Michelson interferometer, a plastic sheet of thickness $75\mu m$ is inserted, which causes a shift in the interference pattern by 86 fringes. The light source has wavelength of 610 nm in air. What is the index of refraction of this plastic?

85. The thickness of an aluminum foil is measured using a Michelson interferometer that has its movable mirror mounted on a micrometer. There is a difference of 27 fringes in the observed interference pattern when the micrometer clamps down on the foil compared to when the micrometer is empty. Calculate the thickness of the foil?

86. The movable mirror of a Michelson interferometer is attached to one end of a thin metal rod of length 23.3 mm. The other end of the rod is anchored so it does not move. As the temperature of the rod changes from $15^\circ C$ to $25C$, a change of 14 fringes is observed. The light source is a He Ne laser, $\lambda = 632.8 nm$. What is the change in length of the metal bar, and what is its thermal expansion coefficient?

87. In a thermally stabilized lab, a Michelson interferometer is used to monitor the temperature to ensure it stays constant. The movable mirror is mounted on the end of a 1.00-m-long aluminum rod, held fixed at the other end. The light source is a He Ne laser, $\lambda = 632.8 nm$. The resolution of this apparatus corresponds to the temperature difference when a change of just one fringe is observed. What is this temperature difference?

88. A 65-fringe shift results in a Michelson interferometer when a $42.0 - \mu m$ film made of an unknown material is placed in one arm. The light source has wavelength 632.9 nm. Identify the material using the indices of refraction found in Table 1.1.

Challenge Problems

89. Determine what happens to the double-slit interference pattern if one of the slits is covered with a thin, transparent film whose thickness is λ/[2(n−1)]λ/[2(n−1)], where λλ is the wavelength of the incident light and n is the index of refraction of the film.

90. Fifty-one narrow slits are equally spaced and separated by 0.10 mm. The slits are illuminated by blue light of wavelength 400 nm. What is angular position of the twenty-fifth secondary maximum? What is its peak intensity in comparison with that of the primary maximum?

91. A film of oil on water will appear dark when it is very thin, because the path length difference becomes small compared with the wavelength of light and there is a phase shift at the top surface. If it becomes dark when the path length difference is less than one-fourth the wavelength, what is the thickest the oil can be and appear dark at all visible wavelengths? Oil has an index of refraction of 1.40.

92. Figure 3.14 shows two glass slides illuminated by monochromatic light incident perpendicularly. The top slide touches the bottom slide at one end and rests on a 0.100-mm-diameter hair at the other end, forming a wedge of air. (a) How far apart are the dark bands, if the slides are 7.50 cm long and 589-nm light is used? (b) Is there any difference if the slides are made from crown or flint glass? Explain.

93. Figure 3.14 shows two 7.50-cm-long glass slides illuminated by pure 589-nm wavelength light incident perpendicularly. The top slide touches the bottom slide at one end and rests on some debris at the other end, forming a wedge of air. How thick is the debris, if the dark bands are 1.00 mm apart?

94. A soap bubble is 100 nm thick and illuminated by white light incident at a 45° angle to its surface. What wavelength and color of visible light is most constructively reflected, assuming the same index of refraction as water?

95. An oil slick on water is 120 nm thick and illuminated by white light incident at a 45° angle to its surface. What color does the oil appear (what is the most constructively reflected wavelength), given its index of refraction is 1.40?

CHAPTER OVERVIEW

13: Diffraction

In the preceding chapter, we implicitly regarded slits as objects with positions but no size. The widths of the slits were considered negligible. When the slits have finite widths, each point along the opening can be considered a point source of light—a foundation of Huygens's principle. Because real-world optical instruments must have finite apertures (otherwise, no light can enter), diffraction plays a major role in the way we interpret the output of these optical instruments. For example, diffraction places limits on our ability to resolve images or objects. This is a problem that we will study later in this chapter.

- 13.1: Prelude to Diffraction
- 13.2: Single-Slit Diffraction
- 13.3: Intensity in Single-Slit Diffraction
- 13.4: Double-Slit Diffraction
- 13.5: Diffraction Gratings
- 13.6: Circular Apertures and Resolution
- 13.7: X-Ray Diffraction
- 13.E: Diffraction (Exercises)

13.1: Prelude to Diffraction

Imagine passing a monochromatic light beam through a narrow opening—a slit just a little wider than the wavelength of the light. Instead of a simple shadow of the slit on the screen, you will see that an interference pattern appears, even though there is only one slit.

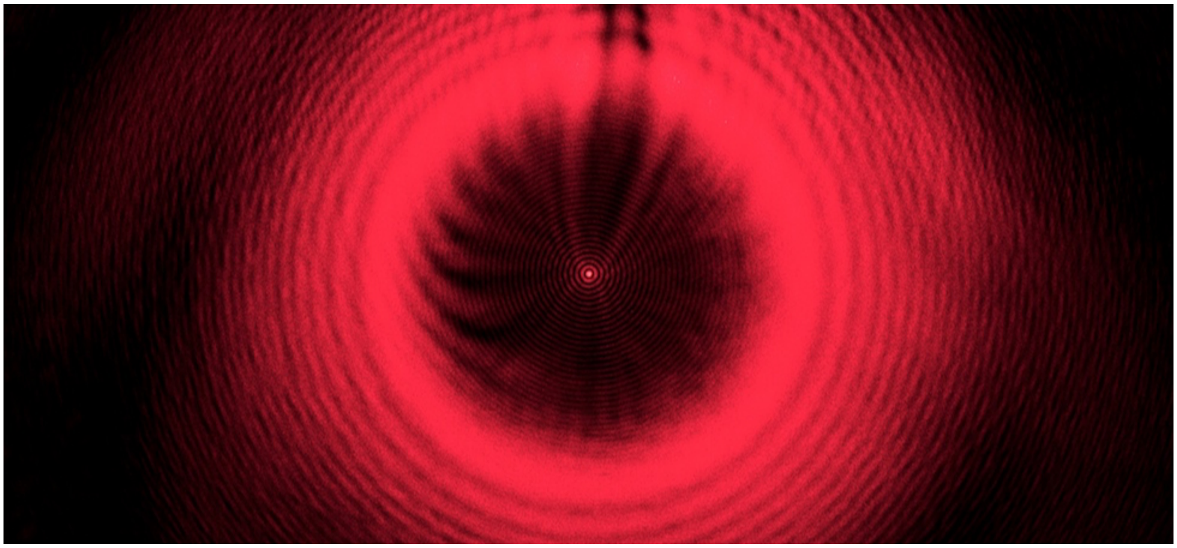


Figure 13.1.1: A steel ball bearing illuminated by a laser does not cast a sharp, circular shadow. Instead, a series of diffraction fringes and a central bright spot are observed. Known as Poisson's spot, the effect was first predicted by Augustin-Jean Fresnel (1788–1827) as a consequence of diffraction of light waves. Based on principles of ray optics, Siméon-Denis Poisson (1781–1840) argued against Fresnel's prediction. (credit: modification of work by Harvard Natural Science Lecture Demonstrations)

In the chapter on interference, we saw that you need two sources of waves for interference to occur. How can there be an interference pattern when we have only one slit? In The Nature of Light, we learned that, due to Huygens's principle, we can imagine a wave front as equivalent to infinitely many point sources of waves. Thus, a wave from a slit can behave not as one wave but as an infinite number of point sources. These waves can interfere with each other, resulting in an interference pattern without the presence of a second slit. This phenomenon is called diffraction.

Another way to view this is to recognize that a slit has a small but finite width. In the preceding chapter, we implicitly regarded slits as objects with positions but no size. The widths of the slits were considered negligible. When the slits have finite widths, each point along the opening can be considered a point source of light—a foundation of Huygens's principle. Because real-world optical instruments must have finite apertures (otherwise, no light can enter), diffraction plays a major role in the way we interpret the output of these optical instruments. For example, diffraction places limits on our ability to resolve images or objects. This is a problem that we will study later in this chapter.

13.2: Single-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Explain the phenomenon of diffraction and the conditions under which it is observed
- Describe diffraction through a single slit

After passing through a narrow aperture (opening), a wave propagating in a specific direction tends to spread out. For example, sound waves that enter a room through an open door can be heard even if the listener is in a part of the room where the geometry of ray propagation dictates that there should only be silence. Similarly, ocean waves passing through an opening in a breakwater can spread throughout the bay inside. (Figure $13.2.1$). The spreading and bending of sound and ocean waves are two examples of diffraction, which is the bending of a wave around the edges of an opening or an obstacle—a phenomenon exhibited by all types of waves.

Figure $13.2.1$: Because of the diffraction of waves, ocean waves entering through an opening in a breakwater can spread throughout the bay. (credit: modification of map data from Google Earth)

The diffraction of sound waves is apparent to us because wavelengths in the audible region are approximately the same size as the objects they encounter, a condition that must be satisfied if diffraction effects are to be observed easily. Since the wavelengths of visible light range from approximately 390 to 770 nm, most objects do not diffract light significantly. However, situations do occur in which apertures are small enough that the diffraction of light is observable. For example, if you place your middle and index fingers close together and look through the opening at a light bulb, you can see a rather clear diffraction pattern, consisting of light and dark lines running parallel to your fingers.

Diffraction through a Single Slit

Light passing through a single slit forms a diffraction pattern somewhat different from those formed by double slits or diffraction gratings, which we discussed in the chapter on interference. Figure $13.2.2$ shows a **single-slit diffraction pattern**. Note that the central maximum is larger than maxima on either side and that the intensity decreases rapidly on either side. In contrast, a diffraction grating produces evenly spaced lines that dim slowly on either side of the center.

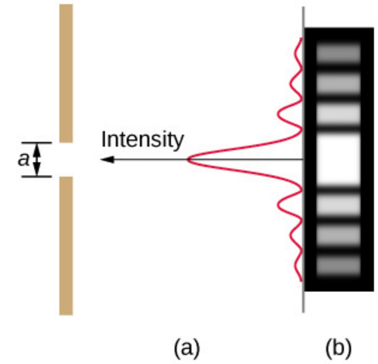


Figure 13.2.2: Single-slit diffraction pattern. (a) Monochromatic light passing through a single slit has a central maximum and many smaller and dimmer maxima on either side. The central maximum is six times higher than shown. (b) The diagram shows the bright central maximum, and the dimmer and thinner maxima on either side.

The analysis of single-slit diffraction is illustrated in Figure $13.2.2$. Here, the light arrives at the slit, illuminating it uniformly and is in phase across its width. We then consider light propagating onwards from different parts of the **same** slit. According to Huygens's principle, every part of the wave front in the slit emits wavelets, as we discussed in The Nature of Light. These are like rays that start out in phase and head in all directions. (Each ray is perpendicular to the wave front of a wavelet.) Assuming the screen is very far away compared with the size of the slit, rays heading toward a common destination are nearly parallel. When they travel straight ahead, as in part (a) of the figure, they remain in phase, and we observe a central maximum. However, when rays travel at an angle θ relative to the original direction of the beam, each ray travels a different distance to a common location, and they can arrive in or out of phase. In part (b), the ray from the bottom travels a distance of one wavelength λ farther than the ray from the top. Thus, a ray from the center travels a distance λ/2 less than the one at the bottom edge of the slit, arrives out of phase, and interferes destructively. A ray from slightly above the center and one from slightly above the bottom also cancel one another. In fact, each ray from the slit interferes destructively with another ray. In other words, a pair-wise cancellation of all rays results in a dark minimum in intensity at this angle. By symmetry, another minimum occurs at the same angle to the right of the incident direction (toward the bottom of the figure) of the light.

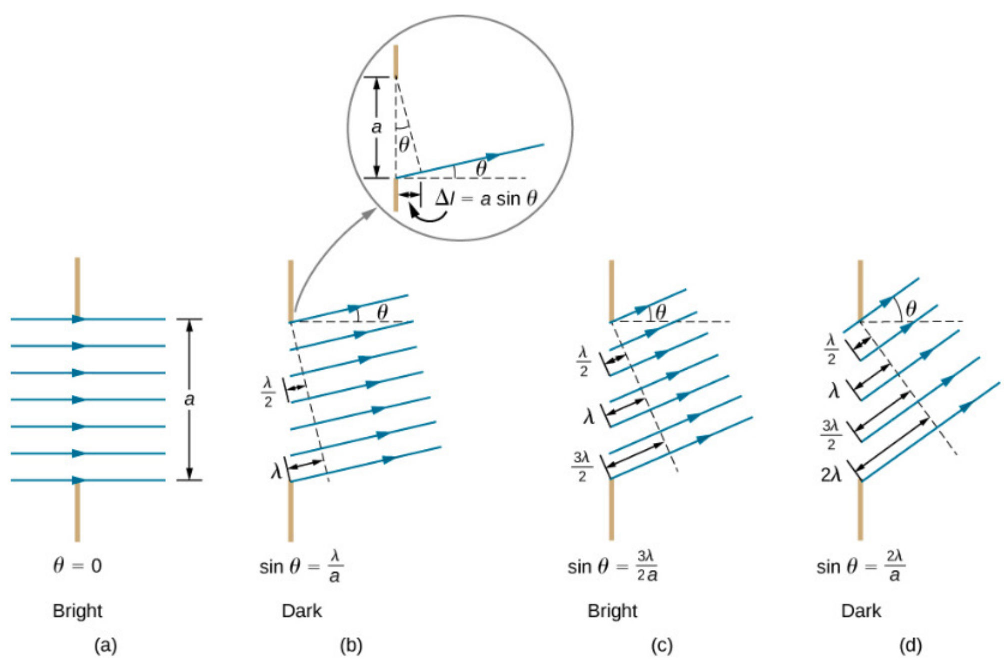


Figure $13.2.3$: Light passing through a single slit is diffracted in all directions and may interfere constructively or destructively, depending on the angle. The difference in path length for rays from either side of the slit is seen to be a sin θ.

At the larger angle shown in part (c), the path lengths differ by $3\lambda/2$ for rays from the top and bottom of the slit. One ray travels a distance λ different from the ray from the bottom and arrives in phase, interfering constructively. Two rays, each from slightly above those two, also add constructively. Most rays from the slit have another ray to interfere with constructively, and a maximum in intensity occurs at this angle. However, not all rays interfere constructively for this situation, so the maximum is not as intense as the central maximum. Finally, in part (d), the angle shown is large enough to produce a second minimum. As seen in the figure, the difference in path length for rays from either side of the slit is $a \sin\theta$, and we see that a destructive minimum is obtained when this distance is an integral multiple of the wavelength.

Thus, to obtain destructive interference for a single slit,

$$\underbrace{a \sin\theta = m\lambda}_{\text{destructive interference}}$$

where

- $m = \pm 1, \pm 2, \pm 3, \ldots$,
- a is the slit width,
- λ is the light's wavelength,
- θ is the angle relative to the original direction of the light, and
- m is the order of the minimum.

Figure $13.2.3$ shows a graph of intensity for single-slit interference, and it is apparent that the maxima on either side of the central maximum are much less intense and not as wide. This effect is explored in Double-Slit Diffraction.

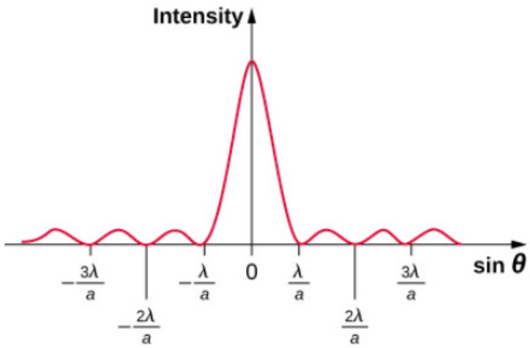


Figure 13.2.3: A graph of single-slit diffraction intensity showing the central maximum to be wider and much more intense than those to the sides. In fact, the central maximum is six times higher than shown here.

Example $13.2.1$: Calculating Single-Slit Diffraction

Visible light of wavelength 550 nm falls on a single slit and produces its second diffraction minimum at an angle of 45.0° relative to the incident direction of the light, as in Figure $13.2.5$.

a. What is the width of the slit?
b. At what angle is the first minimum produced?

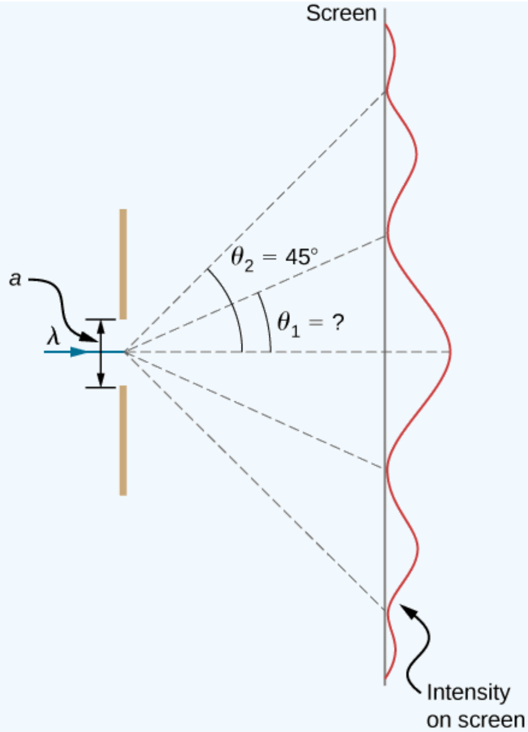


Figure 13.2.5: In this example, we analyze a graph of the single-slit diffraction pattern.

Strategy

From the given information, and assuming the screen is far away from the slit, we can use the equation $a \sin\theta = m\lambda$ first to find a, and again to find the angle for the first minimum θ_1.

Solution

a. We are given that $\lambda = 550\,nm$, $m = 2$, and $\theta_2 = 45.0^\circ$. Solving the equation $a\ \sin\ \theta = m\lambda$ for a and substituting known values gives

$$a = \frac{m\lambda}{\sin\ \theta_2} = \frac{2(550\,nm)}{\sin\ 45.0^\circ} = \frac{1100\times 10^{-9}m}{0.707} = 1.56\times 10^{-6}m.$$

b. Solving the equation $a\ \sin\ \theta = m\lambda$ for $\sin\ \theta_1$ and substituting the known values gives

$$\sin\ \theta_1 = \frac{m\lambda}{a} = \frac{1(550\times 10^{-9}m)}{1.56\times 10^{-6}m}.$$

Thus the angle θ_1 is

$$\theta_1 = \sin^{-1} 0.354 = 20.7^\circ.$$

Significance

We see that the slit is narrow (it is only a few times greater than the wavelength of light). This is consistent with the fact that light must interact with an object comparable in size to its wavelength in order to exhibit significant wave effects such as this single-slit diffraction pattern. We also see that the central maximum extends 20.7° on either side of the original beam, for a width of about 41°. The angle between the first and second minima is only about 24°(45.0°−20.7°). Thus, the second maximum is only about half as wide as the central maximum.

? Exercise 13.2.1

Suppose the slit width in Example 13.2.1 is increased to $1.8\times 10^{-6}m$. What are the new angular positions for the first, second, and third minima? Would a fourth minimum exist?

Answer

17.8^o, 37.7^o, 66.4^o; no

13.3: Intensity in Single-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Calculate the intensity relative to the central maximum of the single-slit diffraction peaks
- Calculate the intensity relative to the central maximum of an arbitrary point on the screen

To calculate the intensity of the diffraction pattern, we follow the phasor method used for calculations with ac circuits in Alternating-Current Circuits. If we consider that there are N Huygens sources across the slit shown previously, with each source separated by a distance a/N from its adjacent neighbors, the path difference between waves from adjacent sources reaching the arbitrary point P on the screen is $(a/N)\sin\theta$. This distance is equivalent to a phase difference of $(2\pi a/\lambda N)\sin\theta$. The phasor diagram for the waves arriving at the point whose angular position is θ is shown in Figure $13.3.1$. The amplitude of the phasor for each Huygens wavelet is ΔE_0, the amplitude of the resultant phasor is E, and the phase difference between the wavelets from the first and the last sources is

$$\phi = \left(\frac{2\pi}{\lambda}\right) a \sin\theta.$$

With $N \to \infty$, the phasor diagram approaches a circular arc of length $N\Delta E_0$ and radius r. Since the length of the arc is $N\Delta E_0$ for any ϕ, the radius r of the arc must decrease as ϕ increases (or equivalently, as the phasors form tighter spirals).

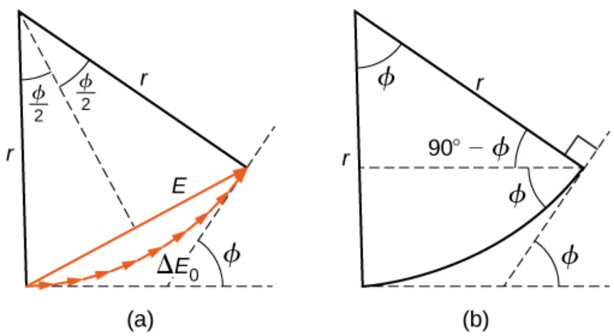


Figure 13.3.1: (a) Phasor diagram corresponding to the angular position θθ in the single-slit diffraction pattern. The phase difference between the wavelets from the first and last sources is $\phi = (2\pi/\lambda)a\,sin\,\theta$. (b) The geometry of the phasor diagram.

The phasor diagram for ϕ = 0 (the center of the diffraction pattern) is shown in Figure $13.3.1a$ using N=30. In this case, the phasors are laid end to end in a straight line of length $N\Delta E_0$, the radius **r** goes to infinity, and the resultant has its maximum value $E = N\Delta E_0$. The intensity of the light can be obtained using the relation $I = \frac{1}{2}c\epsilon_0 E^2$ from Electromagnetic Waves. The intensity of the maximum is then

$$I_0 = \frac{1}{2}c\epsilon_0(N\Delta E_0)^2 = \frac{1}{2\mu_0 c}(N\Delta E_0)^2,$$

where $\epsilon_0 = 1/\mu_0 c^2$. The phasor diagrams for the first two zeros of the diffraction pattern are shown in Figure $13.3.1b$ and Figure $13.3.1d$. In both cases, the phasors add to zero, after rotating through $\phi = 2\pi$ rad for m = 1 and 4π rad for m = 2.

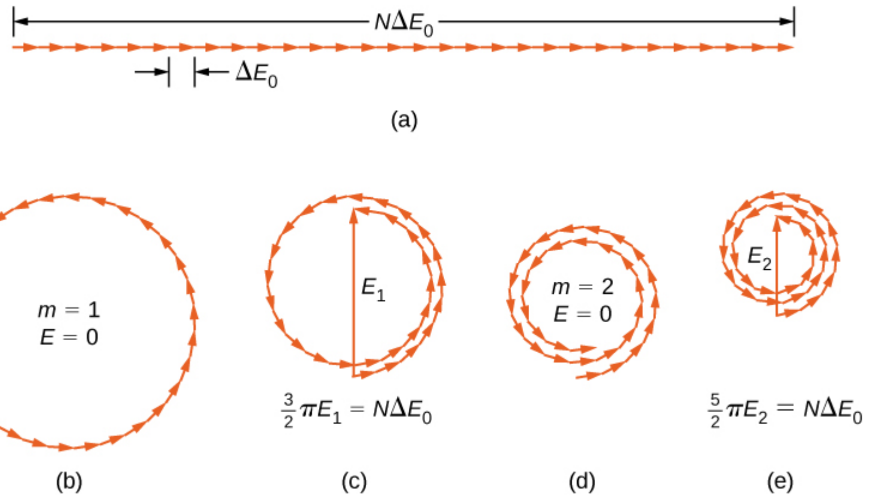


Figure 13.3.2: Phasor diagrams (with 30 phasors) for various points on the single-slit diffraction pattern. Multiple rotations around a given circle have been separated slightly so that the phasors can be seen. (a) Central maximum, (b) first minimum, (c) first maximum beyond central maximum, (d) second minimum, and (e) second maximum beyond central maximum.

The next two maxima beyond the central maxima are represented by the phasor diagrams of parts (c) and (e). In part (c), the phasors have rotated through $\phi = 3\pi$ rad and have formed a resultant phasor of magnitude E_1. The length of the arc formed by the phasors is $N\Delta E_0$. Since this corresponds to 1.5 rotations around a circle of diameter E_1, we have

$$\frac{3}{2}\pi E_1 = N\Delta E_0,$$

so

$$E_1 = \frac{2N\Delta E_0}{3\pi}$$

and

$$I_1 = \frac{1}{2\mu_0 c}E_1^2 = \frac{4(N\Delta E_0)^2}{(9\pi^2)(2\mu_0 c)} = 0.045 I_0,$$

where

$$I_0 = \frac{(N\Delta E_0)^2}{2\mu_0 c}.$$

In part (e), the phasors have rotated through $\phi = 5\pi$ rad, corresponding to 2.5 rotations around a circle of diameter E_2 and arc length $N\Delta E_0$. This results in $I_2 = 0.016 I_0$. The proof is left as an exercise for the student (Exercise 4.119).

These two maxima actually correspond to values of ϕ slightly less than 3π rad and 5π rad. Since the total length of the arc of the phasor diagram is always $N\Delta E_0$, the radius of the arc decreases as ϕ increases. As a result, E_1 and E_2 turn out to be slightly larger for arcs that have not quite curled through 3π rad and 5π rad, respectively. The exact values of ϕ for the maxima are investigated in Exercise 4.120. In solving that problem, you will find that they are less than, but very close to, $\phi = 3\pi,\ 5\pi,\ 7\pi, \ldots$ rad.

To calculate the intensity at an arbitrary point P on the screen, we return to the phasor diagram of Figure $13.3.1$. Since the arc subtends an angle ϕ at the center of the circle,

$$N\Delta E_0 = r\phi \tag{13.3.1}$$

and

$$\sin\left(\frac{\phi}{2}\right) = \frac{E}{2r}. \tag{13.3.2}$$

where E is the amplitude of the resultant field. Solving the Equation 13.3.2 for E and then substituting r from Equation 13.3.1, we find

$$E = 2r \sin \frac{\phi}{2}$$
$$= 2\frac{N\Delta E_0}{\phi} \sin \frac{\phi}{2}.$$

Now defining

$$\beta = \frac{\phi}{2} = \frac{\pi a \sin \theta}{\lambda} \tag{13.3.3}$$

we obtain

$$E = N\Delta E_0 \frac{\sin \beta}{\beta} \tag{13.3.4}$$

Equation 13.3.4 relates the amplitude of the resultant field at any point in the diffraction pattern to the amplitude $N\Delta E_0$ at the central maximum. The intensity is proportional to the square of the amplitude, so

$$I = I_0 \left(\frac{\sin \beta}{\beta}\right)^2 \tag{13.3.5}$$

where $I_0 = (N\delta E_0)^2/2\mu_0 c$ is the intensity at the center of the pattern.

For the central maximum, ϕ = 0, β is also zero and we see from l'Hôpital's rule that $\lim_{\beta\to 0}(sin\,\beta/\beta) = 1$, so that $lim_{\phi\to 0} I = I_0$. For the next maximum, $\phi = 3\pi$ rad, we have $\beta = 3\pi/2$ rad and when substituted into Equation 13.3.5, it yields

$$I_1 = I_0 \left(\frac{\sin 3\pi/2}{3\pi/2}\right)^2 = 0.045 I_0,$$

in agreement with what we found earlier in this section using the diameters and circumferences of phasor diagrams. Substituting $\phi = 5\pi$ rad into Equation 13.3.5 yields a similar result for I_2.

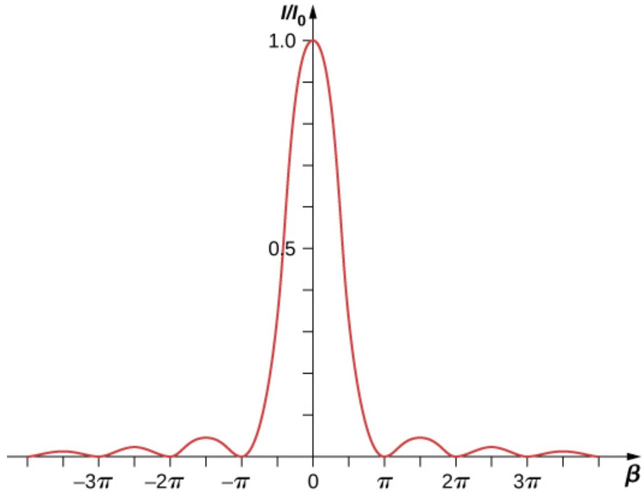
A plot of Equation 13.3.5 is shown in Figure $13.3.3$ and directly below it is a photograph of an actual diffraction pattern. Notice that the central peak is much brighter than the others, and that the zeros of the pattern are located at those points where $sin\,\beta = 0$, which occurs when $\beta = m\pi$ rad. This corresponds to

$$\frac{\pi a \sin\theta}{\lambda} = m\pi,$$

or

$$a \sin \theta = m\lambda,$$

which we derived for the destructive interference ro a single slit previously.



(b)

Figure 13.3.3: (a) The calculated intensity distribution of a single-slit diffraction pattern. (b) The actual diffraction pattern.

Example 13.3.1: Intensity in Single-Slit Diffraction

Light of wavelength 550 nm passes through a slit of width 2.00 μm and produces a diffraction pattern similar to that shown in Figure 13.3.3a.

a. Find the locations of the first two minima in terms of the angle from the central maximum.
b. Determine the intensity relative to the central maximum at a point halfway between these two minima.

Strategy

The minima are given by Equation 4.2.1, $a\,sin\,\theta = m\lambda$. The first two minima are for m = 1 and m = 2. Equation 13.3.5 and Equation 13.3.3 can be used to determine the intensity once the angle has been worked out.

Solution

1. Solving Equation 4.2.1 for θ gives us $\theta_m = \sin^{-1}(m\lambda/a)$, so that

$$\theta_1 = \sin^{-1}\left(\frac{(+1)(550\times10^{-9}m)}{2.00\times10^{-6}m}\right) = +16.0^\circ$$

and

$$\theta_2 = \sin^{-1}\left(\frac{(+2)(550\times10^{-9}m)}{2.00\times10^{-6}m}\right) = +33.4^\circ.$$

2. The halfway point between θ_1 and θ_2 is

$$\theta = (\theta_1+\theta_2)/2 = (16.0^\circ + 33.4^\circ)/2 = 24.7^\circ.$$

Equation 13.3.3 gives

$$\beta = \frac{\pi a\,sin\,\theta}{\lambda} = \frac{\pi(2.00\times10^{-6}m)\,\sin(24.7^\circ)}{(550\times10^{-9}m)} = 1.52\pi\;or\;4.77\;rad.$$

From Equation 13.3.5, we can calculate

$$\frac{I}{I_0} = \left(\frac{\sin \beta}{\beta}\right)^2 = \left(\frac{\sin(4.77)}{4.77}\right)^2 = \left(\frac{-0.9985}{4.77}\right)^2 = 0.044.$$

Significance

This position, halfway between two minima, is very close to the location of the maximum, expected near $\beta = 3\pi/2$, or 1.5π.

? Exercise 13.3.1

For the experiment in Example 13.3.1, at what angle from the center is the third maximum and what is its intensity relative to the central maximum?

Answer

74.3^o, $0.0083I_0$

If the slit width a is varied, the intensity distribution changes, as illustrated in Figure 13.3.4. The central peak is distributed over the region from $sin\,\theta = -\lambda/a$ to $sin\,\theta = +\lambda/a$. For small θ, this corresponds to an angular width $\Delta\theta \approx 2\lambda/a$. Hence, an increase in the slit width results in a decrease in the **width of the central peak**. For a slit with a ≫ λ, the central peak is very sharp, whereas if a ≈ λ, it becomes quite broad.

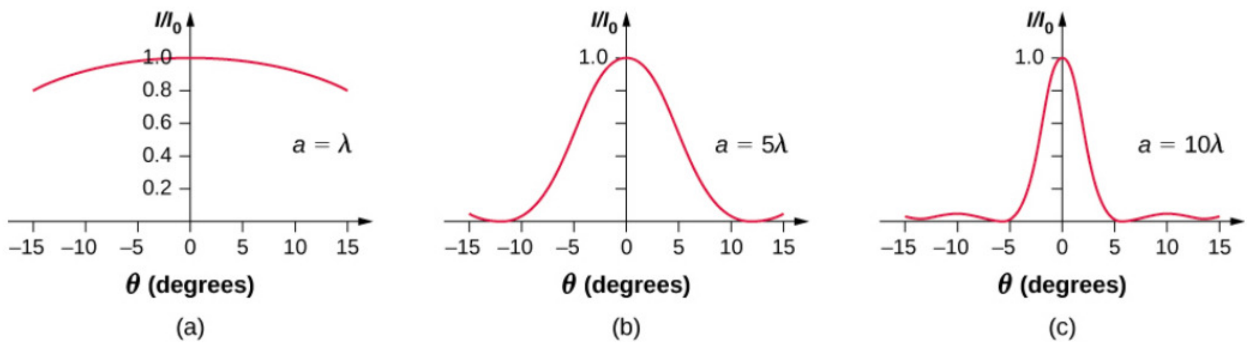


Figure 13.3.4: Single-slit diffraction patterns for various slit widths. As the slit width **a** increases from **a**=λ to 5λ and then to 10λ, the width of the central peak decreases as the angles for the first minima decrease as predicted by Equation 4.2.1.

Diffraction Simulation

A diffraction experiment in optics can require a lot of preparation but this simulation by Andrew Duffy offers not only a quick set up but also the ability to change the slit width instantly. Run the simulation and select “Single slit.” You can adjust the slit width and see the effect on the diffraction pattern on a screen and as a graph.

13.4: Double-Slit Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Describe the combined effect of interference and diffraction with two slits, each with finite width
- Determine the relative intensities of interference fringes within a diffraction pattern
- Identify missing orders, if any

When we studied interference in Young's double-slit experiment, we ignored the diffraction effect in each slit. We assumed that the slits were so narrow that on the screen you saw only the interference of light from just two point sources. If the slit is smaller than the wavelength, then Figure 4.3.4a shows that there is just a spreading of light and no peaks or troughs on the screen. Therefore, it was reasonable to leave out the diffraction effect in that chapter. However, if you make the slit wider, Figure 4.3.4b and (c) show that you cannot ignore diffraction. In this section, we study the complications to the double-slit experiment that arise when you also need to take into account the diffraction effect of each slit.

To calculate the diffraction pattern for two (or any number of) slits, we need to generalize the method we just used for a single slit. That is, across each slit, we place a uniform distribution of point sources that radiate Huygens wavelets, and then we sum the wavelets from all the slits. This gives the intensity at any point on the screen. Although the details of that calculation can be complicated, the final result is quite simple:

Two-Slit Diffraction Pattern

The diffraction pattern of two slits of width a that are separated by a distance **d** is the interference pattern of two point sources separated by **d** multiplied by the diffraction pattern of a slit of width a.

In other words, the **locations** of the interference fringes are given by the equation

$$d\,\sin\,\theta = m\lambda$$

the same as when we considered the slits to be point sources, but the **intensities** of the fringes are now reduced by diffraction effects, according to Equation 4.3.11. [Note that in the chapter on interference, we wrote $d\,sin\,\theta = m\lambda$ and used the integer m to refer to interference fringes. Equation 4.2.1 also uses m, but this time to refer to diffraction minima. If both equations are used simultaneously, it is good practice to use a different variable (such as n) for one of these integers in order to keep them distinct.]

Interference and diffraction effects operate simultaneously and generally produce minima at different angles. This gives rise to a complicated pattern on the screen, in which some of the maxima of interference from the two slits are missing if the maximum of the interference is in the same direction as the minimum of the diffraction. We refer to such a missing peak as a **missing order**. One example of a diffraction pattern on the screen is shown in Figure $13.4.1$. The solid line with multiple peaks of various heights is the intensity observed on the screen. It is a product of the interference pattern of waves from separate slits and the diffraction of waves from within one slit.

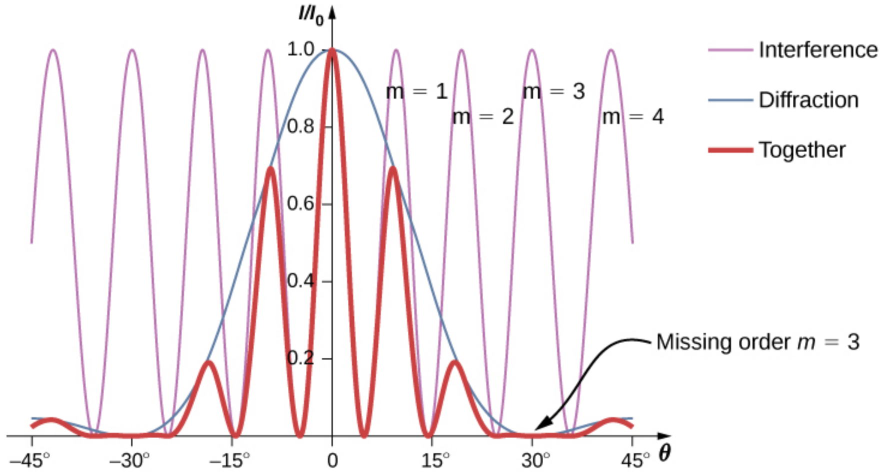


Figure 13.4.1: Diffraction from a double slit. The purple line with peaks of the same height are from the interference of the waves from two slits; the blue line with one big hump in the middle is the diffraction of waves from within one slit; and the thick red line is the product of the two, which is the pattern observed on the screen. The plot shows the expected result for a slit width $a = 2\lambda$ and slit separation $d = 6\lambda$. The maximum of $m = \pm 3$ order for the interference is missing because the minimum of the diffraction occurs in the same direction.

Example 13.4.1: Intensity of the Fringes

Figure 13.4.1 shows that the intensity of the fringe for m=3 is zero, but what about the other fringes? Calculate the intensity for the fringe at m=1 relative to I_0, the intensity of the central peak.

Strategy

Determine the angle for the double-slit interference fringe, using the equation from Interference, then determine the relative intensity in that direction due to diffraction by using Equation 4.3.11.

Solution

From the chapter on interference, we know that the bright interference fringes occur at $d\, sin\, \theta = m\lambda$, or

$$\sin\,\theta = \frac{m\lambda}{d}.$$

From Equation 4.3.11,

$$I = I_0 \left(\frac{\sin\,\beta}{\beta}\right)^2$$

where

$$\beta = \frac{\phi}{2} = \frac{\pi a \sin\,\theta}{\lambda}.$$

Substituting from above,

$$\beta = \frac{\pi a \sin\,\theta}{\lambda} = \frac{\pi a}{\lambda} \cdot \frac{m\lambda}{d} = \frac{m\pi a}{d}.$$

For $a = 2\lambda,\ d = 6\lambda$, and $m = 1$,

$$\beta = \frac{(1)\pi(2\lambda)}{(6\lambda)} = \frac{\pi}{3}.$$

Then, the intensity is

$$I = I_0 \left(\frac{\sin\,\beta}{\beta}\right)^2 = I_0 \left(\frac{sin\,(\pi/3)}{\pi/3}\right)^2 = 0.684 I_0.$$

Significance

Note that this approach is relatively straightforward and gives a result that is almost exactly the same as the more complicated analysis using phasors to work out the intensity values of the double-slit interference (thin line in Figure 13.4.1). The phasor approach accounts for the downward slope in the diffraction intensity (blue line) so that the peak **near** m=1 occurs at a value of θ ever so slightly smaller than we have shown here.

Example 13.4.2: Two-Slit Diffraction

Suppose that in Young's experiment, slits of width 0.020 mm are separated by 0.20 mm. If the slits are illuminated by monochromatic light of wavelength 500 nm, how many bright fringes are observed in the central peak of the diffraction pattern?

Solution

From Equation 4.2.1, the angular position of the first diffraction minimum is $\theta \approx sin\,\theta = \frac{\lambda}{a} = \frac{5.0 \times 10^{-7} m}{2.0 \times 10^{-5} m} = 2.5 \times 10^{-2} rad$.

Using $d \sin\,\theta = m\lambda$ for $\theta = 2.5 \times 10^{-2} rad$, we find

$$m = \frac{d\,sin\,\theta}{\lambda} = \frac{(0.20\,mm)(2.5 \times 10^{-2} rad)}{(5.0 \times 10^{-7} m)} = 10,$$

which is the maximum interference order that fits inside the central peak. We note that $m = \pm 10$ are missing orders as θ matches exactly. Accordingly, we observe bright fringes for

m = −9, −8, −7, −6, −5, −4, −3, −2, −1, 0, +1, +2, +3, +4, +5, +6, +7, +8, and +9

for a total of 19 bright fringes.

Exercise 13.4.1

For the experiment in Example 13.4.2, show that m=20 is also a missing order.

Solution

From $d\ \sin\,\theta = m\lambda$, the interference maximum occurs at 2.87^o for $m = 20$. From Equation 4.2.1, this is also the angle for the second diffraction minimum. (**Note:** Both equations use the index **m** but they refer to separate phenomena.)

Explore the effects of double-slit diffraction. In this simulation written by Fu-Kwun Hwang, select N=2 using the slider and see what happens when you control the slit width, slit separation and the wavelength. Can you make an order go "missing?"

13.5: Diffraction Gratings

Learning Objectives

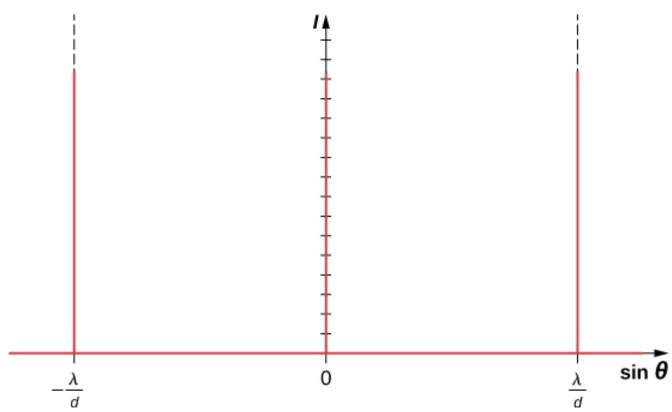
By the end of this section, you will be able to:

- Discuss the pattern obtained from diffraction gratings
- Explain diffraction grating effects

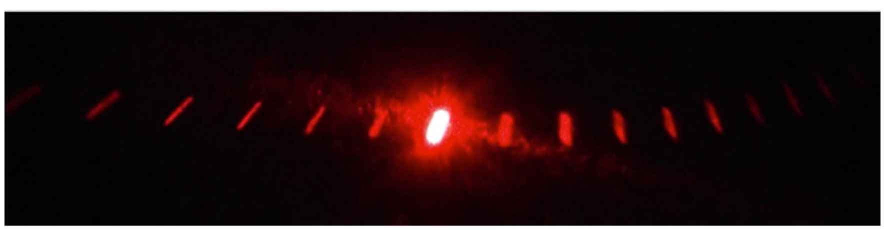
Analyzing the interference of light passing through two slits lays out the theoretical framework of interference and gives us a historical insight into Thomas Young’s experiments. However, most modern-day applications of slit interference use not just two slits but many, approaching infinity for practical purposes. The key optical element is called a diffraction grating, an important tool in optical analysis.

Diffraction Gratings: An Infinite Number of Slits

The analysis of multi-slit interference in Interference allows us to consider what happens when the number of slits **N** approaches infinity. Recall that $N-2$ secondary maxima appear between the principal maxima. We can see there will be an infinite number of secondary maxima that appear, and an infinite number of dark fringes between them. This makes the spacing between the fringes, and therefore the width of the maxima, infinitesimally small. Furthermore, because the intensity of the secondary maxima is proportional to $1/N^2$, it approaches zero so that the secondary maxima are no longer seen. What remains are only the principal maxima, now very bright and very narrow (Figure $13.5.1$).



(a)



(b)

Figure $13.5.1$: (a) Intensity of light transmitted through a large number of slits. When N approaches infinity, only the principal maxima remain as very bright and very narrow lines. (b) A laser beam passed through a diffraction grating. (credit b: modification of work by Sebastian Stapelberg)

In reality, the number of slits is not infinite, but it can be very large—large enough to produce the equivalent effect. A prime example is an optical element called a diffraction grating. A diffraction grating can be manufactured by carving glass with a sharp tool in a large number of precisely positioned parallel lines, with untouched regions acting like slits (Figure $13.5.2$). This type of grating can be photographically mass produced rather cheaply. Because there can be over 1000 lines per millimeter across the grating, when a section as small as a few millimeters is illuminated by an incoming ray, the number of illuminated slits is effectively infinite, providing for very sharp principal maxima.

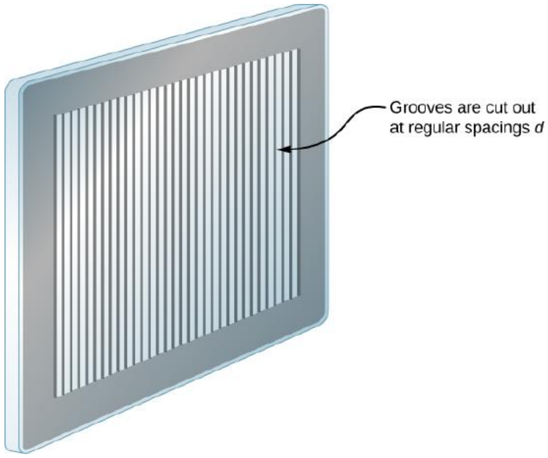


Figure 13.5.2: A diffraction grating can be manufactured by carving glass with a sharp tool in a large number of precisely positioned parallel lines.

Diffraction gratings work both for transmission of light, as in Figure 13.5.3, and for reflection of light, as on butterfly wings and the Australian opal in Figure 13.5.4a. Natural diffraction gratings also occur in the feathers of certain birds such as the hummingbird. Tiny, finger-like structures in regular patterns act as reflection gratings, producing constructive interference that gives the feathers colors not solely due to their pigmentation. This is called **iridescence.**

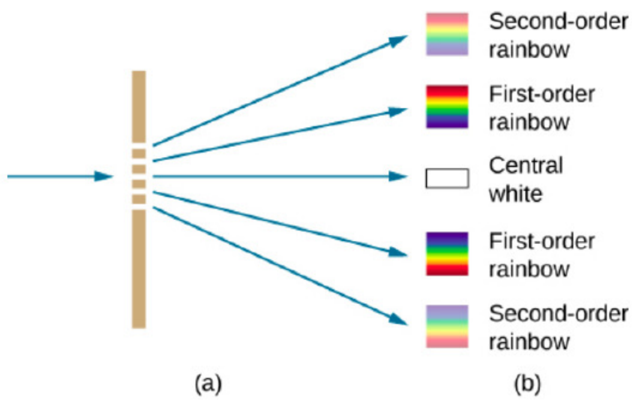


Figure 13.5.3: (a) Light passing through a diffraction grating is diffracted in a pattern similar to a double slit, with bright regions at various angles. (b) The pattern obtained for white light incident on a grating. The central maximum is white, and the higher-order maxima disperse white light into a rainbow of colors.

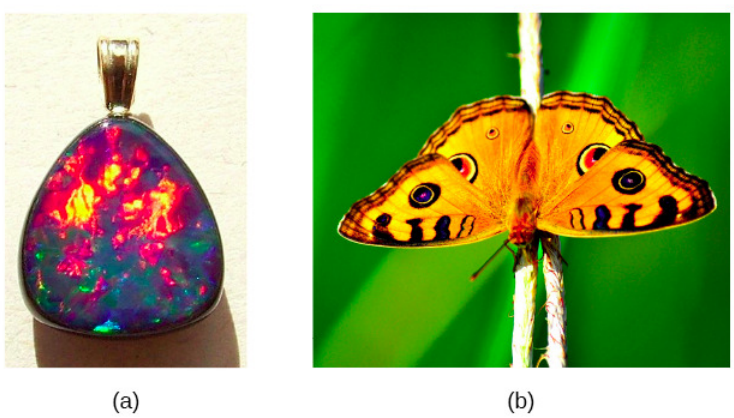


Figure 13.5.4: (a) This Australian opal and (b) butterfly wings have rows of reflectors that act like reflection gratings, reflecting different colors at different angles. (credit a: modification of work by "Opals-On-Black"/Flickr; credit b: modification of work by “whologwhy”/Flickr)

Applications of Diffraction Gratings

Where are diffraction gratings used in applications? Diffraction gratings are commonly used for **spectroscopic dispersion** and analysis of light. What makes them particularly useful is the fact that they form a sharper pattern than double slits do. That is, their bright fringes are narrower and brighter while their dark regions are darker. Diffraction gratings are key components of monochromators used, for example, in optical imaging of particular wavelengths from biological or medical samples. A diffraction grating can be chosen to specifically analyze a wavelength emitted by molecules in diseased cells in a biopsy sample or to help excite strategic molecules in the sample with a selected wavelength of light. Another vital use is in optical fiber technologies where fibers are designed to provide optimum performance at specific wavelengths. A range of diffraction gratings are available for selecting wavelengths for such use.

Example $13.5.1$: Calculating Typical Diffraction Grating Effects

Diffraction gratings with 10,000 lines per centimeter are readily available. Suppose you have one, and you send a beam of white light through it to a screen 2.00 m away.

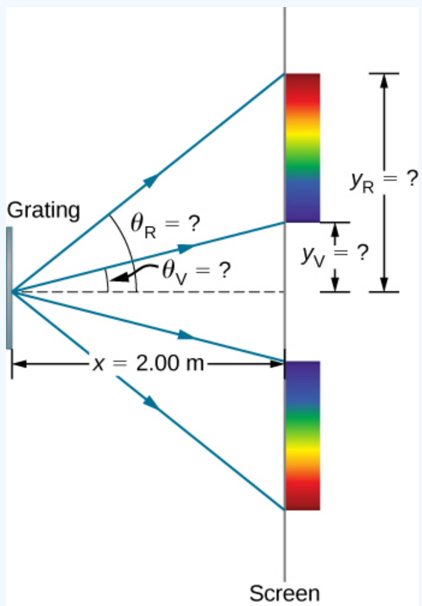
a. Find the angles for the first-order diffraction of the shortest and longest wavelengths of visible light (380 and 760 nm, respectively).
b. What is the distance between the ends of the rainbow of visible light produced on the screen for first-order interference? (Figure $13.5.5$).
c. 

Figure 13.5.5: (a) The diffraction grating considered in this example produces a rainbow of colors on a screen a distance $x = 2.00\,m$ from the grating. The distances along the screen are measured perpendicular to the x-direction. In other words, the rainbow pattern extends out of the page.

(b) In a bird's-eye view, the rainbow pattern can be seen on a table where the equipment is placed.

Strategy

Once a value for the diffraction grating's slit spacing d has been determined, the angles for the sharp lines can be found using the equation

$$d \sin \theta = m\lambda$$

for $m = 0, \pm 1, \pm 2, \ldots$.

Since there are 10,000 lines per centimeter, each line is separated by 1/10,000 of a centimeter. Once we know the angles, we an find the distances along the screen by using simple trigonometry.

Solution

1. The distance between slits is $d = (1\,cm)/10,000 = 1.00 \times 10^{-4}\,cm$ or $1.00 \times 10^{-6}m$. Let us call the two angles θ_V for violet (380 nm) and θ_R for red (760 nm). Solving the equation $d\,\sin\theta_V = m\lambda$ for sin θ_V,

$$\sin\,\theta_V = \frac{m\lambda_V}{d},$$

where m = 1 for the first-order and $\lambda_V = 380\,nm = 3.80 \times 10^{-7}m$. Substituting these values gives

$$\sin\,\theta_V = \frac{3.80 \times 10^{-7}m}{1.00 \times 10^{-6}m} = 0.380.$$

Thus the angle θ_V is

$$\theta_V = \sin^{-1} 0.380 = 22.33^\circ.$$

Similarly,

$$\sin\,\theta_R = \frac{7.60 \times 10^{-7}m}{1.00 \times 10^{-6}m} = 0.760.$$

Thus the angle θ_R is

$$\theta_R = \sin^{-1} 0.760 = 49.46^\circ.$$

Notice that in both equations, we reported the results of these intermediate calculations to four significant figures to use with the calculation in part (b).

2. The distances on the screen are labeled y_V and y_R in Figure $13.5.5$. Notice that $tan\,\theta = y/x$. We can solve for y_V and y_R. That is,

$$y_V = x\,\tan\,\theta_V = (2.00\,m)(\tan\,22.33^\circ) = 0.815\,m$$

and

$$y_R = x\,\tan\,\theta_R = (2.00\,m)(\tan\,49.46^\circ) = 2.338\,m.$$

The distance between them is therefore

$$y_R - y_V = 1.523\,m$$

Significance

The large distance between the red and violet ends of the rainbow produced from the white light indicates the potential this diffraction grating has as a spectroscopic tool. The more it can spread out the wavelengths (greater dispersion), the more detail can be seen in a spectrum. This depends on the quality of the diffraction grating—it must be very precisely made in addition to having closely spaced lines.

? Exercise 13.5.1

If the line spacing of a diffraction grating d is not precisely known, we can use a light source with a well-determined wavelength to measure it. Suppose the first-order constructive fringe of the H_β emission line of hydrogen ($\lambda = 656.3\,nm$) is measured at 11.36° using a spectrometer with a diffraction grating. What is the line spacing of this grating?

Answer

$3.332 \times 10^{-6}m$ or 300 lines per millimeter

Take the same simulation we used for double-slit diffraction and try increasing the number of slits from $N = 2$ to $N = 3, 4, 5, \ldots$. The primary peaks become sharper, and the secondary peaks become less and less pronounced. By the time you reach the maximum number of $N = 20$, the system is behaving much like a diffraction grating.

13.6: Circular Apertures and Resolution

Learning Objectives

By the end of this section, you will be able to:

- Describe the diffraction limit on resolution
- Describe the diffraction limit on beam propagation

Light diffracts as it moves through space, bending around obstacles, interfering constructively and destructively. This can be used as a spectroscopic tool—a diffraction grating disperses light according to wavelength, for example, and is used to produce spectra—but diffraction also limits the detail we can obtain in images.

Figure $13.6.1a$ shows the effect of passing light through a small circular **aperture**. Instead of a bright spot with sharp edges, we obtain a spot with a fuzzy edge surrounded by circles of light. This pattern is caused by diffraction, similar to that produced by a single slit. Light from different parts of the circular aperture interferes constructively and destructively. The effect is most noticeable when the aperture is small, but the effect is there for large apertures as well.

Figure 13.6.1: (a) Monochromatic light passed through a small circular aperture produces this diffraction pattern. (b) Two point-light sources that are close to one another produce overlapping images because of diffraction. (c) If the sources are closer together, they cannot be distinguished or resolved.

How does diffraction affect the detail that can be observed when light passes through an aperture? Figure $13.6.1b$ shows the diffraction pattern produced by two point-light sources that are close to one another. The pattern is similar to that for a single point source, and it is still possible to tell that there are two light sources rather than one. If they are closer together, as in Figure $13.6.1c$, we cannot distinguish them, thus limiting the detail or **resolution** we can obtain. This limit is an inescapable consequence of the wave nature of light.

Diffraction limits the resolution in many situations. The acuity of our vision is limited because light passes through the pupil, which is the circular aperture of the eye. Be aware that the diffraction-like spreading of light is due to the limited diameter of a light beam, not the interaction with an aperture. Thus, light passing through a lens with a diameter D shows this effect and spreads, blurring the image, just as light passing through an aperture of diameter D does. Thus, diffraction limits the resolution of any system having a lens or mirror. Telescopes are also limited by diffraction, because of the finite diameter D of the primary mirror.

Just what is the limit? To answer that question, consider the diffraction pattern for a circular aperture, which has a central maximum that is wider and brighter than the maxima surrounding it (similar to a slit) (Figure $13.6.1a$). It can be shown that, for a circular aperture of diameter D, the first minimum in the diffraction pattern occurs at $\theta = 1.22\lambda/D$ (providing the aperture is large compared with the wavelength of light, which is the case for most optical instruments). The accepted criterion for determining the diffraction limit to resolution based on this angle is known as the **Rayleigh criterion**, which was developed by Lord Rayleigh in the nineteenth century.

Rayleigh Criterion

The diffraction limit to resolution states that two images are just resolvable when the center of the diffraction pattern of one is directly over the first minimum of the diffraction pattern of the other (Figure $13.6.1b$).

The first minimum is at an angle of $\theta = 1.22\lambda/D$, so that two point objects are just resolvable if they are separated by the angle

$$\theta = 1.22 \frac{\lambda}{D} \tag{13.6.1}$$

where λ is the wavelength of light (or other electromagnetic radiation) and D is the diameter of the aperture, lens, mirror, etc., with which the two objects are observed. In this expression, θ has units of radians. This angle is also commonly known as the **diffraction limit**.

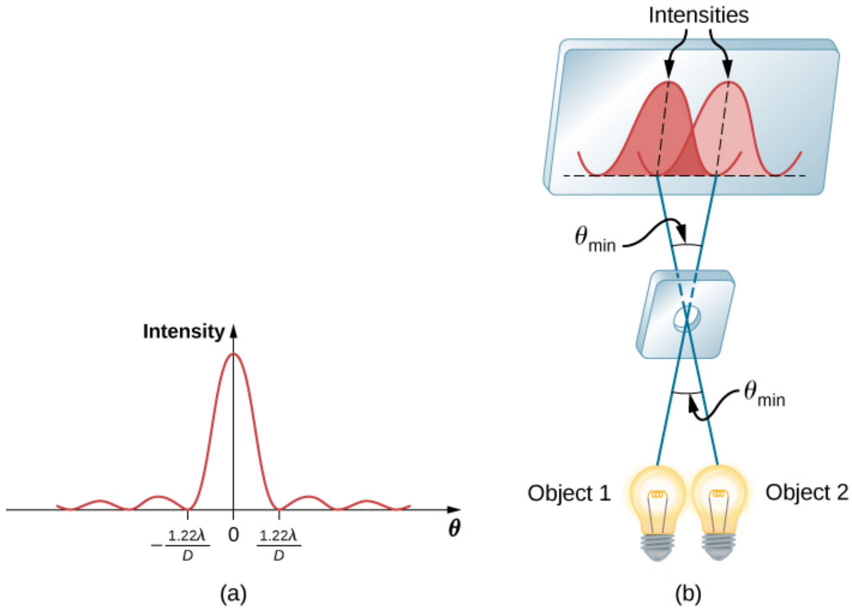


Figure 13.6.2: (a) Graph of intensity of the diffraction pattern for a circular aperture. Note that, similar to a single slit, the central maximum is wider and brighter than those to the sides. (b) Two point objects produce overlapping diffraction patterns. Shown here is the Rayleigh criterion for being just resolvable. The central maximum of one pattern lies on the first minimum of the other.

All attempts to observe the size and shape of objects are limited by the wavelength of the probe. Even the small wavelength of light prohibits exact precision. When extremely small wavelength probes are used, as with an electron microscope, the system is disturbed, still limiting our knowledge. Heisenberg's uncertainty principle asserts that this limit is fundamental and inescapable, as we shall see in the chapter on quantum mechanics.

Example 13.6.1: Calculating Diffraction Limits of the Hubble Space Telescope

The primary mirror of the orbiting Hubble Space Telescope has a diameter of 2.40 m. Being in orbit, this telescope avoids the degrading effects of atmospheric distortion on its resolution. (a) What is the angle between two just-resolvable point light sources (perhaps two stars)? Assume an average light wavelength of 550 nm. (b) If these two stars are at a distance of 2 million light-years, which is the distance of the Andromeda Galaxy, how close together can they be and still be resolved? (A light-year, or ly, is the distance light travels in 1 year.)

Strategy

The Rayleigh criterion stated in Equation 13.6.1, $\theta = 1.22\lambda/D$, gives the smallest possible angle θ between point sources, or the best obtainable resolution. Once this angle is known, we can calculate the distance between the stars, since we are given how far away they are.

Solution

1. The Rayleigh criterion for the minimum resolvable angle is

$$\theta = 1.22 \frac{\lambda}{D}.$$

Entering known values gives

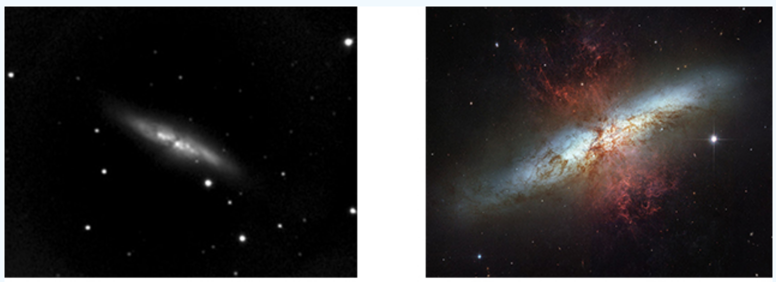
$$\theta = 1.22 \frac{550 \times 10^{-9} m}{2.40\, m} = 2.80 \times 10^{-7} rad.$$

2. The distance **s** between two objects a distance **r** away and separated by an angle θ is $s = r\theta$. Substituting known values gives

$$s = (2.0 \times 10^{6}\, ly)(2.80 \times 10^{-7}\, rad) = 0.56\, ly.$$

Significance

The angle found in part (a) is extraordinarily small (less than 1/50,000 of a degree), because the primary mirror is so large compared with the wavelength of light. As noticed, diffraction effects are most noticeable when light interacts with objects having sizes on the order of the wavelength of light. However, the effect is still there, and there is a diffraction limit to what is observable. The actual resolution of the Hubble Telescope is not quite as good as that found here. As with all instruments, there are other effects, such as nonuniformities in mirrors or aberrations in lenses that further limit resolution. However, Figure $13.6.3$ gives an indication of the extent of the detail observable with the Hubble because of its size and quality, and especially because it is above Earth's atmosphere.



(a) (b)

Figure 13.6.3: These two photographs of the M82 Galaxy give an idea of the observable detail using (a) a ground-based telescope and (b) the Hubble Space Telescope. (credit a: modification of work by "Ricnun"/Wikimedia Commons)

The answer in part (b) indicates that two stars separated by about half a light-year can be resolved. The average distance between stars in a galaxy is on the order of five light-years in the outer parts and about one light-year near the galactic center. Therefore, the Hubble can resolve most of the individual stars in Andromeda Galaxy, even though it lies at such a huge distance that its light takes 2 million years to reach us. Figure $13.6.4$ shows another mirror used to observe radio waves from outer space.

Figure 13.6.4: A 305-m-diameter paraboloid at Arecibo in Puerto Rico is lined with reflective material, making it into a radio telescope. It is the largest curved focusing dish in the world. Although D for Arecibo is much larger than for the Hubble Telescope, it detects radiation of a much longer wavelength and its diffraction limit is significantly poorer than Hubble's. The Arecibo telescope is still very useful, because important information is carried by radio waves that is not carried by visible light. (credit: Jeff Hitchcock)

? Exercise 13.6.1

What is the angular resolution of the Arecibo telescope shown in Figure $13.6.4$ when operated at 21-cm wavelength? How does it compare to the resolution of the Hubble Telescope?

Answer

$8.4 \times 10^{-4}\, rad$, 3000 times broader than the Hubble Telescope

Diffraction is not only a problem for optical instruments but also for the electromagnetic radiation itself. Any beam of light having a finite diameter D and a wavelength λ exhibits diffraction spreading. The beam spreads out with an angle θ given by Equation 13.6.1, $\theta = 1.22\lambda/D$. Take, for example, a laser beam made of rays as parallel as possible (angles between rays as close to θ = 0° as possible) instead spreads out at an angle $\theta = 1.22\lambda/D$, where D is the diameter of the beam and λ is its wavelength. This spreading is impossible to observe for a flashlight because its beam is not very parallel to start with. However, for long-distance transmission of laser beams or microwave signals, diffraction spreading can be significant (Figure 13.6.5). To avoid this, we can increase **D**. This is done for laser light sent to the moon to measure its distance from Earth. The laser beam is expanded through a telescope to make D much larger and θ smaller.

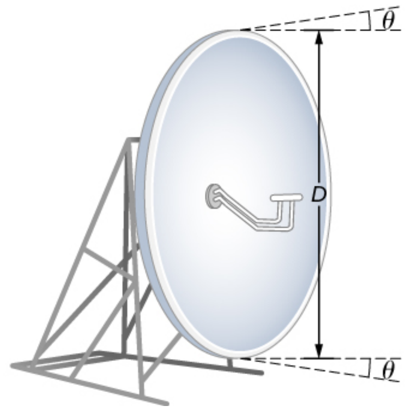


Figure 13.6.5: The beam produced by this microwave transmission antenna spreads out at a minimum angle $\theta = 1.22\lambda/D$ due to diffraction. It is impossible to produce a near-parallel beam because the beam has a limited diameter.

In most biology laboratories, resolution is an issue when the use of the microscope is introduced. The smaller the distance **x** by which two objects can be separated and still be seen as distinct, the greater the resolution. The resolving power of a lens is defined as that distance **x**. An expression for resolving power is obtained from the Rayleigh criterion. Figure $13.6.6a$ shows two point objects separated by a distance **x**. According to the Rayleigh criterion, resolution is possible when the minimum angular separation is

$$\theta = 1.22\frac{\lambda}{D} = \frac{x}{d},$$

where D is the distance between the specimen and the objective lens, and we have used the small angle approximation (i.e., we have assumed that **x** is much smaller than **d**), so that $tan\,\theta \approx sin\,\theta$. Therefore, the resolving power is

$$x = 1.22\frac{\lambda d}{D}.$$

Another way to look at this is by the concept of numerical aperture (**NA**), which is a measure of the maximum acceptance angle at which a lens will take light and still contain it within the lens. Figure $13.6.1b$shows a lens and an object at point **P**. The **NA** here is a measure of the ability of the lens to gather light and resolve fine detail. The angle subtended by the lens at its focus is defined to be $\theta = 2\alpha$. From the figure and again using the small angle approximation, we can write

$$sin\,\alpha = \frac{D/2}{d} = \frac{D}{2d}.$$

The **NA** for a lens is $NA = n\,sin\,\alpha$, where **n** is the index of refraction of the medium between the objective lens and the object at point **P**. From this definition for **NA**, we can see that

$$x = 1.22\frac{\lambda d}{D} = 1.22\frac{\lambda}{2\,sin\,\alpha} = 0.61\frac{\lambda n}{NA}.$$

In a microscope, **NA** is important because it relates to the resolving power of a lens. A lens with a large **NA** is able to resolve finer details. Lenses with larger **NA** are also able to collect more light and so give a brighter image. Another way to describe this situation is that the larger the **NA**, the larger the cone of light that can be brought into the lens, so more of the diffraction modes are collected. Thus the microscope has more information to form a clear image, and its resolving power is higher.

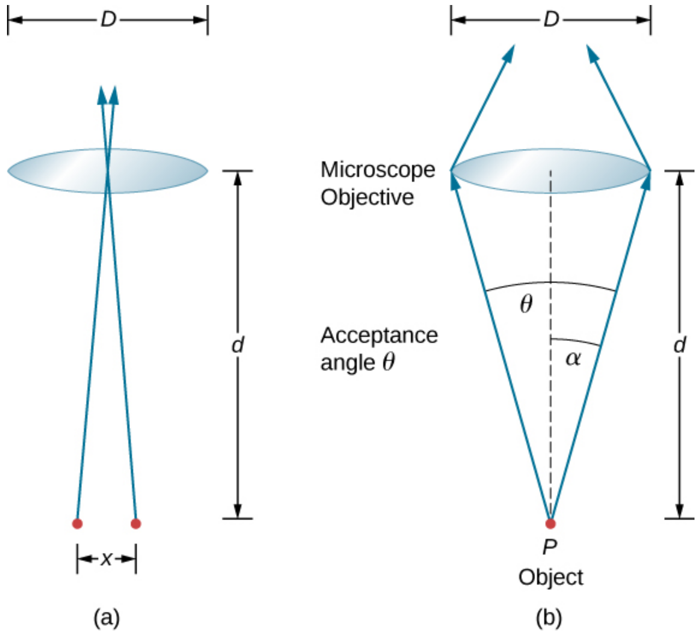


Figure 13.6.6: (a) Two points separated by a distance x and positioned a distance D away from the objective. (b) Terms and symbols used in discussion of resolving power for a lens and an object at point P (credit a: modification of work by "Infopro"/Wikimedia Commons).

One of the consequences of diffraction is that the focal point of a beam has a finite width and intensity distribution. Imagine focusing when only considering geometric optics, as in Figure $13.6.7a$. The focal point is regarded as an infinitely small point with a huge intensity and the capacity to incinerate most samples, irrespective of the **NA** of the objective lens—an unphysical oversimplification. For wave optics, due to diffraction, we take into account the phenomenon in which the focal point spreads to become a focal spot (Figure $13.6.7b$) with the size of the spot decreasing with increasing **NA**. Consequently, the intensity in the focal spot increases with increasing **NA**. The higher the **NA**, the greater the chances of photodegrading the specimen. However, the spot never becomes a true point.

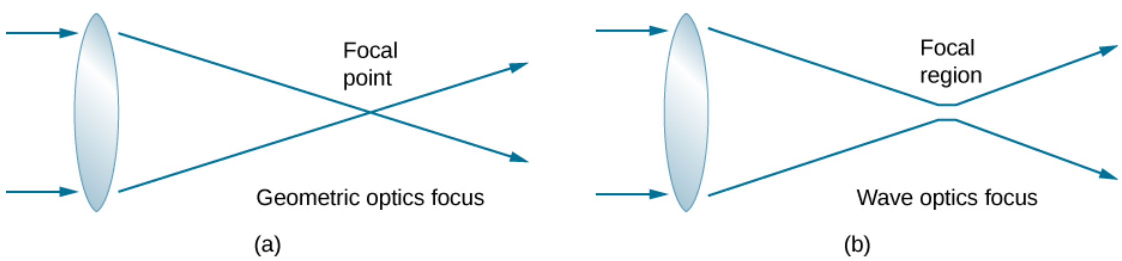


Figure 13.6.7:(a) In geometric optics, the focus is modelled as a point, but it is not physically possible to produce such a point because it implies infinite intensity. (b) In wave optics, the focus is an extended region.

In a different type of microscope, molecules within a specimen are made to emit light through a mechanism called fluorescence. By controlling the molecules emitting light, it has become possible to construct images with resolution much finer than the Rayleigh criterion, thus circumventing the diffraction limit. The development of super-resolved fluorescence microscopy led to the 2014 Nobel Prize in Chemistry.

Optical Resolution Simulation

In this Optical Resolution Model, two diffraction patterns for light through two circular apertures are shown side by side in this simulation by Fu-Kwun Hwang. Watch the patterns merge as you decrease the aperture diameters.

13.7: X-Ray Diffraction

Learning Objectives

By the end of this section, you will be able to:

- Describe interference and diffraction effects exhibited by X-rays in interaction with atomic-scale structures

Since X-ray photons are very energetic, they have relatively short wavelengths, on the order of 10^{-8} m to 10^{-12} m. Thus, typical X-ray photons act like rays when they encounter macroscopic objects, like teeth, and produce sharp shadows. However, since atoms are on the order of 0.1 nm in size, X-rays can be used to detect the location, shape, and size of atoms and molecules. The process is called **X-ray diffraction**, and it involves the interference of X-rays to produce patterns that can be analyzed for information about the structures that scattered the X-rays.

Perhaps the most famous example of X-ray diffraction is the discovery of the double-helical structure of **DNA** in 1953 by an international team of scientists working at England's Cavendish Laboratory—American James Watson, Englishman Francis Crick, and New Zealand-born Maurice Wilkins. Using X-ray diffraction data produced by Rosalind Franklin, they were the first to model the double-helix structure of DNA that is so crucial to life. For this work, Watson, Crick, and Wilkins were awarded the 1962 Nobel Prize in Physiology or Medicine. (There is some debate and controversy over the issue that Rosalind Franklin was not included in the prize, although she died in 1958, before the prize was awarded.)

Figure $13.7.1$ shows a diffraction pattern produced by the scattering of X-rays from a crystal. This process is known as X-ray crystallography because of the information it can yield about crystal structure, and it was the type of data Rosalind Franklin supplied to Watson and Crick for DNA. Not only do X-rays confirm the size and shape of atoms, they give information about the atomic arrangements in materials. For example, more recent research in high-temperature superconductors involves complex materials whose lattice arrangements are crucial to obtaining a superconducting material. These can be studied using X-ray crystallography.

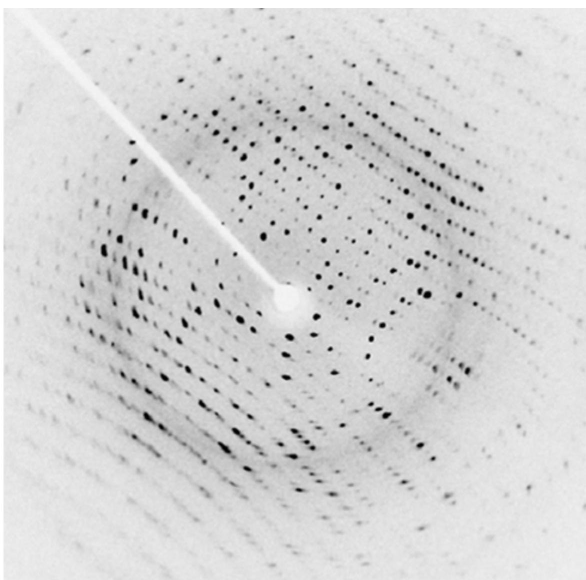


Figure 13.7.1: X-ray diffraction from the crystal of a protein (hen egg lysozyme) produced this interference pattern. Analysis of the pattern yields information about the structure of the protein. (credit: "Del45"/Wikimedia Commons)

Historically, the scattering of X-rays from crystals was used to prove that X-rays are energetic electromagnetic (EM) waves. This was suspected from the time of the discovery of X-rays in 1895, but it was not until 1912 that the German Max **von Laue** (1879–1960) convinced two of his colleagues to scatter X-rays from crystals. If a diffraction pattern is obtained, he reasoned, then the X-rays must be waves, and their wavelength could be determined. (The spacing of atoms in various crystals was reasonably well known at the time, based on good values for Avogadro's number.) The experiments were convincing, and the 1914 Nobel Prize in Physics was given to von Laue for his suggestion leading to the proof that X-rays are EM waves. In 1915, the unique father-and-son team of Sir William Henry **Bragg** and his son Sir William Lawrence Bragg were awarded a joint Nobel Prize for inventing the X-ray spectrometer and the then-new science of X-ray analysis.

In ways reminiscent of thin-film interference, we consider two plane waves at X-ray wavelengths, each one reflecting off a different plane of atoms within a crystal's lattice, as shown in Figure $13.7.2$ From the geometry, the difference in path lengths is $2d \sin \theta$. Constructive interference results when this distance is an integer multiple of the wavelength. This condition is captured by the **Bragg equation**,

$$m\lambda = 2d \sin \theta, \tag{13.7.1}$$

for $m = 1, 2, 3, \ldots$.

where m is a positive integer and d is the spacing between the planes. Following the Law of Reflection, both the incident and reflected waves are described by the same angle, θ, but unlike the general practice in geometric optics, θ is measured with respect to the surface itself, rather than the normal.

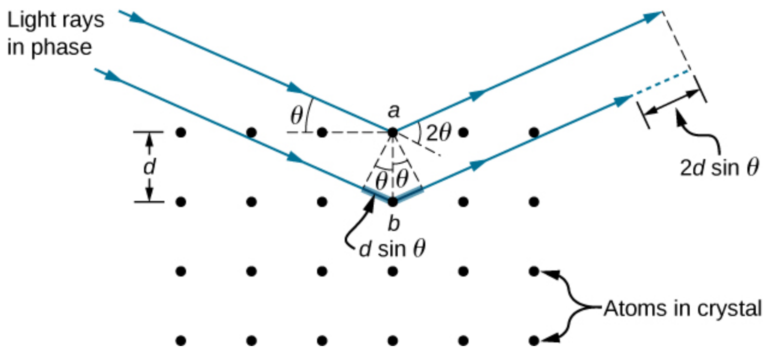


Figure 13.7.2: X-ray diffraction with a crystal. Two incident waves reflect off two planes of a crystal. The difference in path lengths is indicated by the dashed line.

Example 13.7.1: X-Ray Diffraction with Salt Crystals

Common table salt is composed mainly of $NaCl$ crystals. In a $NaCl$ crystal, there is a family of planes 0.252 nm apart. If the first-order maximum is observed at an incidence angle of 18.1°, what is the wavelength of the X-ray scattering from this crystal?

Strategy:

Use the Bragg equation, Equation 13.7.1, to solve for θ.

Solution

For first-order, $m = 1$, and the plane spacing d is known. Solving the Bragg equation for wavelength yields

$$\begin{aligned} \lambda &= \frac{2d \sin \theta}{m} \\ &= \frac{2(0.252 \times 10^{-9} m) \sin (18.1^o)}{1} \\ &= 1.57 \times 10^{-10} m, \; or \; 0.157 \, nm \end{aligned}$$

Significance

The determined wavelength fits within the X-ray region of the electromagnetic spectrum. Once again, the wave nature of light makes itself prominent when the wavelength ($\lambda = 0.157 \, nm$) is comparable to the size of the physical structures ($d = 0.252 \, nm$) it interacts with.

Exercise 13.7.1

For the experiment described in Example $13.7.1$, what are the two other angles where interference maxima may be observed? What limits the number of maxima?

Answer

38.4^o and 68.8^o; Between $\theta = 0^o \rightarrow 90^o$, orders 1, 2, and 3, are all that exist.

Although Figure $13.7.2$ depicts a crystal as a two-dimensional array of scattering centers for simplicity, real crystals are structures in three dimensions. Scattering can occur simultaneously from different families of planes at different orientations and spacing patterns known as called **Bragg planes**, as shown in Figure $13.7.3$. The resulting interference pattern can be quite complex.

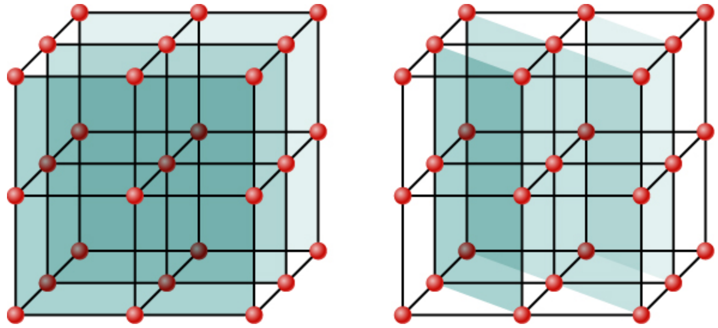


Figure $13.7.3$: Because of the regularity that makes a crystal structure, one crystal can have many families of planes within its geometry, each one giving rise to X-ray diffraction.

13.E: Diffraction (Exercises)

Conceptual Questions

4.1 Single-Slit Diffraction

1. As the width of the slit producing a single-slit diffraction pattern is reduced, how will the diffraction pattern produced change?

2. Compare interference and diffraction.

3. If you and a friend are on opposite sides of a hill, you can communicate with walkie-talkies but not with flashlights. Explain.

4. What happens to the diffraction pattern of a single slit when the entire optical apparatus is immersed in water?

5. In our study of diffraction by a single slit, we assume that the length of the slit is much larger than the width. What happens to the diffraction pattern if these two dimensions were comparable?

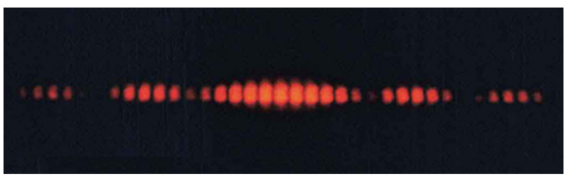
6. A rectangular slit is twice as wide as it is high. Is the central diffraction peak wider in the vertical direction or in the horizontal direction?

4.2 Intensity in Single-Slit Diffraction

7. In Equation 4.4, the parameter ββ looks like an angle but is not an angle that you can measure with a protractor in the physical world. Explain what ββ represents.

4.3 Double-Slit Diffraction

8. Shown below is the central part of the interference pattern for a pure wavelength of red light projected onto a double slit. The pattern is actually a combination of single- and double-slit interference. Note that the bright spots are evenly spaced. Is this a double- or single-slit characteristic? Note that some of the bright spots are dim on either side of the center. Is this a single- or double-slit characteristic? Which is smaller, the slit width or the separation between slits? Explain your responses. Figure is an image showing red interference pattern on a black background. The central part has brighter lines. The lines are cut off at the top and bottom, seemingly enclosed between two sinusoidal waves of opposite phase.



4.5 Circular Apertures and Resolution

9. Is higher resolution obtained in a microscope with red or blue light? Explain your answer.

10. The resolving power of refracting telescope increases with the size of its objective lens. What other advantage is gained with a larger lens?

11. The distance between atoms in a molecule is about $10^{-8} cm$. Can visible light be used to “see” molecules?

12. A beam of light always spreads out. Why can a beam not be created with parallel rays to prevent spreading? Why can lenses, mirrors, or apertures not be used to correct the spreading?

4.6 X-Ray Diffraction

13. Crystal lattices can be examined with X-rays but not UV. Why?

4.7 Holography

14. How can you tell that a hologram is a true three-dimensional image and that those in three-dimensional movies are not?

15. If a hologram is recorded using monochromatic light at one wavelength but its image is viewed at another wavelength, say 10 shorter, what will you see? What if it is viewed using light of exactly half the original wavelength?

16. What image will one see if a hologram is recorded using monochromatic light but its image is viewed in white light? Explain.

Problems

4.1 Single-Slit Diffraction

17. (a) At what angle is the first minimum for 550-nm light falling on a single slit of width $1.00 \mu m$?

(b) Will there be a second minimum?

18. (a) Calculate the angle at which a $2.00 - \mu m$-wide slit produces its first minimum for 410-nm violet light.

(b) Where is the first minimum for 700-nm red light?

19. (a) How wide is a single slit that produces its first minimum for 633-nm light at an angle of 28.0°?

(b) At what angle will the second minimum be?

20. (a) What is the width of a single slit that produces its first minimum at 60.0° for 600-nm light?

(b) Find the wavelength of light that has its first minimum at 62.0°.

21. Find the wavelength of light that has its third minimum at an angle of 48.6° when it falls on a single slit of width $3.00 \mu m$.

22. (a) Sodium vapor light averaging 589 nm in wavelength falls on a single slit of width $7.50 \mu m$. At what angle does it produces its second minimum?

(b) What is the highest-order minimum produced?

23. Consider a single-slit diffraction pattern for $\lambda = 589 nm$, projected on a screen that is 1.00 m from a slit of width 0.25 mm. How far from the center of the pattern are the centers of the first and second dark fringes?

24. (a) Find the angle between the first minima for the two sodium vapor lines, which have wavelengths of 589.1 and 589.6 nm, when they fall upon a single slit of width $2.00 \mu m$.

(b) What is the distance between these minima if the diffraction pattern falls on a screen 1.00 m from the slit?

(c) Discuss the ease or difficulty of measuring such a distance.

25. (a) What is the minimum width of a single slit (in multiples of λ) that will produce a first minimum for a wavelength λ?

(b) What is its minimum width if it produces 50 minima?

(c) 1000 minima?

26. (a) If a single slit produces a first minimum at 14.5°, at what angle is the second-order minimum?

(b) What is the angle of the third-order minimum?

(c) Is there a fourth-order minimum?

(d) Use your answers to illustrate how the angular width of the central maximum is about twice the angular width of the next maximum (which is the angle between the first and second minima).

27. If the separation between the first and the second minima of a single-slit diffraction pattern is 6.0 mm, what is the distance between the screen and the slit? The light wavelength is 500 nm and the slit width is 0.16 mm.

28. A water break at the entrance to a harbor consists of a rock barrier with a 50.0-m-wide opening. Ocean waves of 20.0-m wavelength approach the opening straight on. At what angles to the incident direction are the boats inside the harbor most protected against wave action?

29. An aircraft maintenance technician walks past a tall hangar door that acts like a single slit for sound entering the hangar. Outside the door, on a line perpendicular to the opening in the door, a jet engine makes a 600-Hz sound. At what angle with the door will the technician observe the first minimum in sound intensity if the vertical opening is 0.800 m wide and the speed of sound is 340 m/s?

4.2 Intensity in Single-Slit Diffraction

30. A single slit of width $3.0 \mu m$ is illuminated by a sodium yellow light of wavelength 589 nm. Find the intensity at a 15° angle to the axis in terms of the intensity of the central maximum.

31. A single slit of width 0.1 mm is illuminated by a mercury light of wavelength 576 nm. Find the intensity at a 10°10° angle to the axis in terms of the intensity of the central maximum.

32. The width of the central peak in a single-slit diffraction pattern is 5.0 mm. The wavelength of the light is 600 nm, and the screen is 2.0 m from the slit. (a) What is the width of the slit? (b) Determine the ratio of the intensity at 4.5 mm from the center of the pattern to the intensity at the center.

33. Consider the single-slit diffraction pattern for $\lambda = 600 nm$, $a = 0.025 m$, and $x = 2.0 m$. Find the intensity in terms of I_o at $\theta = 0.5^\circ$, 1.0°, 1.5°, 3.0°, and 10.0°.

4.3 Double-Slit Diffraction

34. Two slits of width $2 \mu m$, each in an opaque material, are separated by a center-to-center distance of $6 \mu m$. A monochromatic light of wavelength 450 nm is incident on the double-slit. One finds a combined interference and diffraction pattern on the screen.

(a) How many peaks of the interference will be observed in the central maximum of the diffraction pattern?

(b) How many peaks of the interference will be observed if the slit width is doubled while keeping the distance between the slits same?

(c) How many peaks of interference will be observed if the slits are separated by twice the distance, that is, $12 \mu m$, while keeping the widths of the slits same?

(d) What will happen in (a) if instead of 450-nm light another light of wavelength 680 nm is used?

(e) What is the value of the ratio of the intensity of the central peak to the intensity of the next bright peak in (a)?

(f) Does this ratio depend on the wavelength of the light?

(g) Does this ratio depend on the width or separation of the slits?

35. A double slit produces a diffraction pattern that is a combination of single- and double-slit interference. Find the ratio of the width of the slits to the separation between them, if the first minimum of the single-slit pattern falls on the fifth maximum of the double-slit pattern. (This will greatly reduce the intensity of the fifth maximum.)

36. For a double-slit configuration where the slit separation is four times the slit width, how many interference fringes lie in the central peak of the diffraction pattern?

37. Light of wavelength 500 nm falls normally on 50 slits that are $2.5 \times 10^{-3} mm$ wide and spaced $5.0 \times 10^{-3} mm$ apart. How many interference fringes lie in the central peak of the diffraction pattern?

38. A monochromatic light of wavelength 589 nm incident on a double slit with slit width $2.5 \mu m$ and unknown separation results in a diffraction pattern containing nine interference peaks inside the central maximum. Find the separation of the slits.

39. When a monochromatic light of wavelength 430 nm incident on a double slit of slit separation $5 \mu m$, there are 11 interference fringes in its central maximum. How many interference fringes will be in the central maximum of a light of the same wavelength and slit widths, but a new slit separation of $4 \mu m$?

40. Determine the intensities of two interference peaks other than the central peak in the central maximum of the diffraction, if possible, when a light of wavelength 628 nm is incident on a double slit of width 500 nm and separation 1500 nm. Use the intensity of the central spot to be $1 mW/cm^2$.

4.4 Diffraction Gratings

41. A diffraction grating has 2000 lines per centimeter. At what angle will the first-order maximum be for 520-nm-wavelength green light?

42. Find the angle for the third-order maximum for 580-nm-wavelength yellow light falling on a difraction grating having 1500 lines per centimeter.

43. How many lines per centimeter are there on a diffraction grating that gives a first-order maximum for 470-nm blue light at an angle of 25.0° ?

44. What is the distance between lines on a diffraction grating that produces a second-order maximum for 760-nm red light at an angle of 60.0°?

45. Calculate the wavelength of light that has its second-order maximum at 45.0° when falling on a diffraction grating that has 5000 lines per centimeter.

46. An electric current through hydrogen gas produces several distinct wavelengths of visible light. What are the wavelengths of the hydrogen spectrum, if they form first-order maxima at angles $24.2^\circ, 25.7^\circ, 29.1^\circ$,and 41.0° when projected on a diffraction grating having 10,000 lines per centimeter?

47. (a) What do the four angles in the preceding problem become if a 5000-line per centimeter diffraction grating is used?

(b) Using this grating, what would the angles be for the second-order maxima?

(c) Discuss the relationship between integral reductions in lines per centimeter and the new angles of various order maxima.

48. What is the spacing between structures in a feather that acts as a reflection grating, giving that they produce a first-order maximum for 525-nm light at a 30.0° angle?

49. An opal such as that shown in Figure 4.15 acts like a reflection grating with rows separated by about $8 \mu m$. If the opal is illuminated normally,

(a) at what angle will red light be seen and

(b) at what angle will blue light be seen?

50. At what angle does a diffraction grating produce a second-order maximum for light having a first-order maximum at 20.0°?

51. (a) Find the maximum number of lines per centimeter a diffraction grating can have and produce a maximum for the smallest wavelength of visible light.

(b) Would such a grating be useful for ultraviolet spectra?

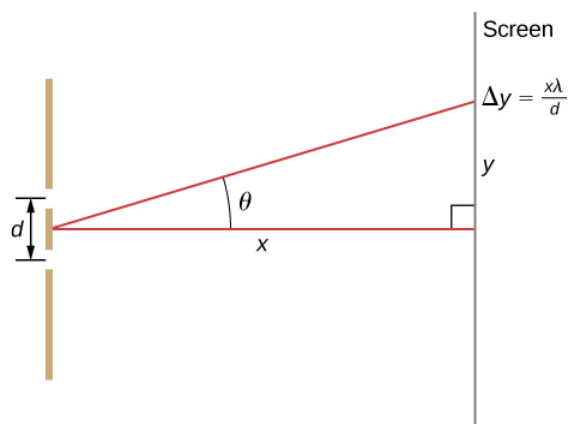
(c) For infrared spectra?

52. (a) Show that a 30,000 line per centimeter grating will not produce a maximum for visible light.

(b) What is the longest wavelength for which it does produce a first-order maximum?

(c) What is the greatest number of line per centimeter a diffraction grating can have and produce a complete second-order spectrum for visible light?

53. The analysis shown below also applies to diffraction gratings with lines separated by a distance **d**. What is the distance between fringes produced by a diffraction grating having 125 lines per centimeter for 600-nm light, if the screen is 1.50 m away? (**Hint**: The distance between adjacent fringes is $\Delta y = x\lambda/d$, assuming the slit separation **d** is comparable to λ.)



54. The 305-m-diameter Arecibo radio telescope pictured in Figure 4.20 detects radio waves with a 4.00-cm average wavelength.

(a) What is the angle between two just-resolvable point sources for this telescope?

(b) How close together could these point sources be at the 2 million light-year distance of the Andromeda Galaxy?

55. Assuming the angular resolution found for the Hubble Telescope in Example 4.6, what is the smallest detail that could be observed on the moon?

56. Diffraction spreading for a flashlight is insignificant compared with other limitations in its optics, such as spherical aberrations in its mirror. To show this, calculate the minimum angular spreading of a flashlight beam that is originally 5.00 cm in diameter with an average wavelength of 600 nm.

57. (a) What is the minimum angular spread of a 633-nm wavelength He-Ne laser beam that is originally 1.00 mm in diameter? (b) If this laser is aimed at a mountain cliff 15.0 km away, how big will the illuminated spot be? (c) How big a spot would be illuminated on the moon, neglecting atmospheric effects? (This might be done to hit a corner reflector to measure the round-trip time and, hence, distance.)

58. A telescope can be used to enlarge the diameter of a laser beam and limit diffraction spreading. The laser beam is sent through the telescope in opposite the normal direction and can then be projected onto a satellite or the moon. (a) If this is done with the Mount Wilson telescope, producing a 2.54-m-diameter beam of 633-nm light, what is the minimum angular spread of the beam? (b) Neglecting atmospheric effects, what is the size of the spot this beam would make on the moon, assuming a lunar distance of $3.84 \times 10^8 m.$?

59. The limit to the eye's acuity is actually related to diffraction by the pupil.

(a) What is the angle between two just-resolvable points of light for a 3.00-mm-diameter pupil, assuming an average wavelength of 550 nm?

(b) Take your result to be the practical limit for the eye. What is the greatest possible distance a car can be from you if you can resolve its two headlights, given they are 1.30 m apart?

(c) What is the distance between two just-resolvable points held at an arm's length (0.800 m) from your eye?

(d) How does your answer to (c) compare to details you normally observe in everyday circumstances?

60. What is the minimum diameter mirror on a telescope that would allow you to see details as small as 5.00 km on the moon some 384,000 km away? Assume an average wavelength of 550 nm for the light received.

61. Find the radius of a star's image on the retina of an eye if its pupil is open to 0.65 cm and the distance from the pupil to the retina is 2.8 cm. Assume $\lambda = 550 nm$.

62. (a) The dwarf planet Pluto and its moon, Charon, are separated by 19,600 km. Neglecting atmospheric effects, should the 5.08-m-diameter Palomar Mountain telescope be able to resolve these bodies when they are $4.50 \times 10^9 km$ from Earth? Assume an average wavelength of 550 nm.

(b) In actuality, it is just barely possible to discern that Pluto and Charon are separate bodies using a ground-based telescope. What are the reasons for this?

63. A spy satellite orbits Earth at a height of 180 km. What is the minimum diameter of the objective lens in a telescope that must be used to resolve columns of troops marching 2.0 m apart? Assume λ=550nm.λ=550nm.

64. What is the minimum angular separation of two stars that are just-resolvable by the 8.1-m Gemini South telescope, if atmospheric effects do not limit resolution? Use 550 nm for the wavelength of the light from the stars.

65. The headlights of a car are 1.3 m apart. What is the maximum distance at which the eye can resolve these two headlights? Take the pupil diameter to be 0.40 cm.

66. When dots are placed on a page from a laser printer, they must be close enough so that you do not see the individual dots of ink. To do this, the separation of the dots must be less than Raleigh's criterion. Take the pupil of the eye to be 3.0 mm and the distance from the paper to the eye of 35 cm; find the minimum separation of two dots such that they cannot be resolved. How many dots per inch (dpi) does this correspond to?

67. Suppose you are looking down at a highway from a jetliner flying at an altitude of 6.0 km. How far apart must two cars be if you are able to distinguish them? Assume that $\lambda = 550 nm$ and that the diameter of your pupils is 4.0 mm.

68. Can an astronaut orbiting Earth in a satellite at a distance of 180 km from the surface distinguish two skyscrapers that are 20 m apart? Assume that the pupils of the astronaut's eyes have a diameter of 5.0 mm and that most of the light is centered around 500 nm.

69. The characters of a stadium scoreboard are formed with closely spaced lightbulbs that radiate primarily yellow light. (Use $\lambda = 600 nm$.) How closely must the bulbs be spaced so that an observer 80 m away sees a display of continuous lines rather than the individual bulbs? Assume that the pupil of the observer's eye has a diameter of 5.0 mm.

70. If a microscope can accept light from objects at angles as large as $\alpha = 70^\circ$, what is the smallest structure that can be resolved when illuminated with light of wavelength 500 nm and

(a) the specimen is in air?

(b) When the specimen is immersed in oil, with index of refraction of 1.52?

71. A camera uses a lens with aperture 2.0 cm. What is the angular resolution of a photograph taken at 700 nm wavelength? Can it resolve the millimeter markings of a ruler placed 35 m away?

4.6 X-Ray Diffraction

72. X-rays of wavelength 0.103 nm reflects off a crystal and a second-order maximum is recorded at a Bragg angle of 25.5°. What is the spacing between the scattering planes in this crystal?

73. A first-order Bragg reflection maximum is observed when a monochromatic X-ray falls on a crystal at a 32.3° angle to a reflecting plane. What is the wavelength of this X-ray?

74. An X-ray scattering experiment is performed on a crystal whose atoms form planes separated by 0.440 nm. Using an X-ray source of wavelength 0.548 nm, what is the angle (with respect to the planes in question) at which the experimenter needs to illuminate the crystal in order to observe a first-order maximum?

75. The structure of the NaCl crystal forms reflecting planes 0.541 nm apart. What is the smallest angle, measured from these planes, at which X-ray diffraction can be observed, if X-rays of wavelength 0.085 nm are used?

76. On a certain crystal, a first-order X-ray diffraction maximum is observed at an angle of 27.1° relative to its surface, using an X-ray source of unknown wavelength. Additionally, when illuminated with a different, this time of known wavelength 0.137 nm, a second-order maximum is detected at 37.3°. Determine (a) the spacing between the reflecting planes, and (b) the unknown wavelength.

77. Calcite crystals contain scattering planes separated by 0.30 nm. What is the angular separation between first and second-order diffraction maxima when X-rays of 0.130 nm wavelength are used?

78. The first-order Bragg angle for a certain crystal is 12.1°. What is the second-order angle?

Additional Problems

79. White light falls on two narrow slits separated by 0.40 mm. The interference pattern is observed on a screen 3.0 m away. (a) What is the separation between the first maxima for red light ($\lambda = 700 nm$) and violet light ($\lambda = 400 nm$)? (b) At what point nearest the central maximum will a maximum for yellow light ($\lambda = 600 nm$) coincide with a maximum for violet light? Identify the order for each maximum.

80. Microwaves of wavelength 10.0 mm fall normally on a metal plate that contains a slit 25 mm wide.

(a) Where are the first minima of the diffraction pattern?

(b) Would there be minima if the wavelength were 30.0 mm?

81. Quasars, or **quasi-stellar radio sources**, are astronomical objects discovered in 1960. They are distant but strong emitters of radio waves with angular size so small, they were originally unresolved, the same as stars. The quasar 3C405 is actually two discrete radio sources that subtend an angle of 82 arcsec. If this object is studied using radio emissions at a frequency of 410 MHz, what is the minimum diameter of a radio telescope that can resolve the two sources?

82. Two slits each of width 1800 nm and separated by the center-to-center distance of 1200 nm are illuminated by plane waves from a krypton ion laser-emitting at wavelength 461.9 nm. Find the number of interference peaks in the central diffraction peak.

83. A microwave of an unknown wavelength is incident on a single slit of width 6 cm. The angular width of the central peak is found to be 25°. Find the wavelength.

84. Red light (wavelength 632.8 nm in air) from a Helium-Neon laser is incident on a single slit of width 0.05 mm. The entire apparatus is immersed in water of refractive index 1.333. Determine the angular width of the central peak.

85. A light ray of wavelength 461.9 nm emerges from a 2-mm circular aperture of a krypton ion laser. Due to diffraction, the beam expands as it moves out. How large is the central bright spot at

(a) 1 m,

(b) 1 km,

(c) 1000 km, and

(d) at the surface of the moon at a distance of 400,000 km from Earth.

86. How far apart must two objects be on the moon to be distinguishable by eye if only the diffraction effects of the eye's pupil limit the resolution? Assume 550 nm for the wavelength of light, the pupil diameter 5.0 mm, and 400,000 km for the distance to the moon.

87. How far apart must two objects be on the moon to be resolvable by the 8.1-m-diameter Gemini North telescope at Mauna Kea, Hawaii, if only the diffraction effects of the telescope aperture limit the resolution? Assume 550 nm for the wavelength of light and 400,000 km for the distance to the moon.

88. A spy satellite is reputed to be able to resolve objects 10. cm apart while operating 197 km above the surface of Earth. What is the diameter of the aperture of the telescope if the resolution is only limited by the diffraction effects? Use 550 nm for light.

89. Monochromatic light of wavelength 530 nm passes through a horizontal single slit of width $1.5 \mu m$ in an opaque plate. A screen of dimensions $2.0 m \times 2.0 m$ is 1.2 m away from the slit.

(a) Which way is the diffraction pattern spread out on the screen?

(b) What are the angles of the minima with respect to the center?

(c) What are the angles of the maxima?

(d) How wide is the central bright fringe on the screen?

(e) How wide is the next bright fringe on the screen?

90. A monochromatic light of unknown wavelength is incident on a slit of width $20 \mu m.$ A diffraction pattern is seen at a screen 2.5 m away where the central maximum is spread over a distance of 10.0 cm. Find the wavelength.

91. A source of light having two wavelengths 550 nm and 600 nm of equal intensity is incident on a slit of width $1.8 \mu m.$ Find the separation of the $m = 1$ bright spots of the two wavelengths on a screen 30.0 cm away.

92. A single slit of width 2100 nm is illuminated normally by a wave of wavelength 632.8 nm. Find the phase difference between waves from the top and one third from the bottom of the slit to a point on a screen at a horizontal distance of 2.0 m and vertical distance of 10.0 cm from the center.

93. A single slit of width $3.0 \mu m$ is illuminated by a sodium yellow light of wavelength 589 nm. Find the intensity at a 15° angle to the axis in terms of the intensity of the central maximum.

94. A single slit of width 0.10 mm is illuminated by a mercury lamp of wavelength 576 nm. Find the intensity at a 10° angle to the axis in terms of the intensity of the central maximum.

95. A diffraction grating produces a second maximum that is 89.7 cm from the central maximum on a screen 2.0 m away. If the grating has 600 lines per centimeter, what is the wavelength of the light that produces the diffraction pattern?

96. A grating with 4000 lines per centimeter is used to diffract light that contains all wavelengths between 400 and 650 nm. How wide is the first-order spectrum on a screen 3.0 m from the grating?

97. A diffraction grating with 2000 lines per centimeter is used to measure the wavelengths emitted by a hydrogen gas discharge tube. (a) At what angles will you find the maxima of the two first-order blue lines of wavelengths 410 and 434 nm? (b) The maxima of two other first-order lines are found at $\theta_1 = 0.097$rad and $\theta_2 = 0.132$rad. What are the wavelengths of these lines?

98. For white light ($400nm < \lambda < 700nm$) falling normally on a diffraction grating, show that the second and third-order spectra overlap no matter what the grating constant d is.

99. How many complete orders of the visible spectrum ($400nm < \lambda < 700nm$) can be produced with a diffraction grating that contains 5000 lines per centimeter?

100. Two lamps producing light of wavelength 589 nm are fixed 1.0 m apart on a wooden plank. What is the maximum distance an observer can be and still resolve the lamps as two separate sources of light, if the resolution is affected solely by the diffraction of light entering the eye? Assume light enters the eye through a pupil of diameter 4.5 mm.

101. On a bright clear day, you are at the top of a mountain and looking at a city 12 km away. There are two tall towers 20.0 m apart in the city. Can your eye resolve the two towers if the diameter of the pupil is 4.0 mm? If not, what should be the minimum magnification power of the telescope needed to resolve the two towers? In your calculations use 550 nm for the wavelength of the light.

102. Radio telescopes are telescopes used for the detection of radio emission from space. Because radio waves have much longer wavelengths than visible light, the diameter of a radio telescope must be very large to provide good resolution. For example, the radio telescope in Penticton, BC in Canada, has a diameter of 26 m and can be operated at frequencies as high as 6.6 GHz.

(a) What is the wavelength corresponding to this frequency?

(b) What is the angular separation of two radio sources that can be resolved by this telescope?

(c) Compare the telescope's resolution with the angular size of the moon.

Figure:4.30 (credit: Jason Nishiyama)

103. Calculate the wavelength of light that produces its first minimum at an angle of 36.9° when falling on a single slit of width $1.00\mu m$.

104. (a) Find the angle of the third diffraction minimum for 633-nm light falling on a slit of width $20.0\mu m$.

(b) What slit width would place this minimum at 85.0°?

105. As an example of diffraction by apertures of everyday dimensions, consider a doorway of width 1.0 m.

(a) What is the angular position of the first minimum in the diffraction pattern of 600-nm light?

(b) Repeat this calculation for a musical note of frequency 440 Hz (A above middle C). Take the speed of sound to be 343 m/s.

106. What are the angular positions of the first and second minima in a diffraction pattern produced by a slit of width 0.20 mm that is illuminated by 400 nm light? What is the angular width of the central peak?

107. How far would you place a screen from the slit of the previous problem so that the second minimum is a distance of 2.5 mm from the center of the diffraction pattern?

108. How narrow is a slit that produces a diffraction pattern on a screen 1.8 m away whose central peak is 1.0 m wide? Assume $\lambda = 589 nm$.

109. Suppose that the central peak of a single-slit diffraction pattern is so wide that the first minima can be assumed to occur at angular positions of ±90°±90°. For this case, what is the ratio of the slit width to the wavelength of the light?

110. The central diffraction peak of the double-slit interference pattern contains exactly nine fringes. What is the ratio of the slit separation to the slit width?

111. Determine the intensities of three interference peaks other than the central peak in the central maximum of the diffraction, if possible, when a light of wavelength 500 nm is incident normally on a double slit of width 1000 nm and separation 1500 nm. Use the intensity of the central spot to be $1 mW/cm^2$.

112. The yellow light from a sodium vapor lamp seems to be of pure wavelength, but it produces two first-order maxima at 36.093° and 36.129° when projected on a 10,000 line per centimeter diffraction grating. What are the two wavelengths to an accuracy of 0.1 nm?

113. Structures on a bird feather act like a reflection grating having 8000 lines per centimeter. What is the angle of the first-order maximum for 600-nm light?

114. If a diffraction grating produces a first-order maximum for the shortest wavelength of visible light at 30.0°, at what angle will the first-order maximum be for the largest wavelength of visible light?

115. (a) What visible wavelength has its fourth-order maximum at an angle of 25.0° when projected on a 25,000-line per centimeter diffraction grating?

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

116. Consider a spectrometer based on a diffraction grating. Construct a problem in which you calculate the distance between two wavelengths of electromagnetic radiation in your spectrometer. Among the things to be considered are the wavelengths you wish to be able to distinguish, the number of lines per meter on the diffraction grating, and the distance from the grating to the screen or detector. Discuss the practicality of the device in terms of being able to discern between wavelengths of interest.

117. An amateur astronomer wants to build a telescope with a diffraction limit that will allow him to see if there are people on the moons of Jupiter.

(a) What diameter mirror is needed to be able to see 1.00-m detail on a Jovian moon at a distance of $7.50 \times 10^8 km$ from Earth? The wavelength of light averages 600 nm.

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

Challenge Problems

118. Blue light of wavelength 450 nm falls on a slit of width 0.25 mm. A converging lens of focal length 20 cm is placed behind the slit and focuses the diffraction pattern on a screen.

(a) How far is the screen from the lens?

(b) What is the distance between the first and the third minima of the diffraction pattern?

119. (a) Assume that the maxima are halfway between the minima of a single-slit diffraction pattern. The use the diameter and circumference of the phasor diagram, as described in Intensity in Single-Slit Diffraction, to determine the intensities of the third and fourth maxima in terms of the intensity of the central maximum.

(b) Do the same calculation, using Equation 4.4.

120. (a) By differentiating Equation 4.4, show that the higher-order maxima of the single-slit diffraction pattern occur at values of β that satisfy $tan\beta = \beta$.

(b) Plot $y = tan\beta$ and $y = \beta$ versus β and find the intersections of these two curves. What information do they give you about the locations of the maxima?

(c) Convince yourself that these points do not appear exactly at $\beta = (n + \frac{1}{2})\pi$, where $n = 0, 1, 2, \ldots$, but are quite close to these values.

121. What is the maximum number of lines per centimeter a diffraction grating can have and produce a complete first-order spectrum for visible light?

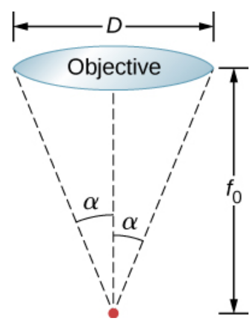
122. Show that a diffraction grating cannot produce a second-order maximum for a given wavelength of light unless the first-order maximum is at an angle less than 30.0°.

123. A He-Ne laser beam is reflected from the surface of a CD onto a wall. The brightest spot is the reflected beam at an angle equal to the angle of incidence. However, fringes are also observed. If the wall is 1.50 m from the CD, and the first fringe is 0.600 m from the central maximum, what is the spacing of grooves on the CD?

124. Objects viewed through a microscope are placed very close to the focal point of the objective lens. Show that the minimum separation x of two objects resolvable through the microscope is given by

$$x = \frac{1.22\lambda f_0}{D},$$

where f_0 is the focal length and **D** is the diameter of the objective lens as shown below.



CHAPTER OVERVIEW

14: Relativity

The theory of relativity led to a profound change in the way we perceive space and time. The “common sense” rules that we use to relate space and time measurements in the Newtonian worldview differ seriously from the correct rules at speeds near the speed of light. Unlike Newtonian mechanics, which describes the motion of particles, or Maxwell's equations, which specify how the electromagnetic field behaves, special relativity is not restricted to a particular type of phenomenon. Instead, its rules on space and time affect all fundamental physical theories.

14.1: Prelude to Relativity
14.2: Invariance of Physical Laws
14.3: Relativity of Simultaneity
14.4: Time Dilation
14.5: Length Contraction
14.6: The Lorentz Transformation
14.7: Relativistic Velocity Transformation
14.8: Doppler Effect for Light
14.9: Relativistic Momentum
14.10: Relativistic Energy
14.E: Relativity (Exercises)

14.1: Prelude to Relativity

The special theory of relativity was proposed in 1905 by **Albert Einstein** (1879–1955). It describes how time, space, and physical phenomena appear in different frames of reference that are moving at constant velocity with respect to each other. This differs from Einstein's later work on general relativity, which deals with any frame of reference, including accelerated frames.

Figure 14.1.1: Special relativity explains how time passes slightly differently on Earth and within the rapidly moving global positioning satellite (GPS). GPS units in vehicles could not find their correct location on Earth without taking this correction into account. (credit: USAF)

The theory of relativity led to a profound change in the way we perceive space and time. The "common sense" rules that we use to relate space and time measurements in the Newtonian worldview differ seriously from the correct rules at speeds near the speed of light. For example, the special theory of relativity tells us that measurements of length and time intervals are not the same in reference frames moving relative to one another. A particle might be observed to have a lifetime of $1.0 \times 10^{-8} s$ in one reference frame, but a lifetime of $2.0 \times 10^{-8} s$ in another; and an object might be measured to be 2.0 m long in one frame and 3.0 m long in another frame. These effects are usually significant only at speeds comparable to the speed of light, but even at the much lower speeds of the global positioning satellite, which requires extremely accurate time measurements to function, the different lengths of the same distance in different frames of reference are significant enough that they need to be taken into account.

Unlike **Newtonian mechanics**, which describes the motion of particles, or Maxwell's equations, which specify how the electromagnetic field behaves, special relativity is not restricted to a particular type of phenomenon. Instead, its rules on space and time affect all fundamental physical theories.

The modifications of Newtonian mechanics in special relativity do not invalidate classical Newtonian mechanics or require its replacement. Instead, the equations of relativistic mechanics differ meaningfully from those of classical Newtonian mechanics only for objects moving at relativistic speeds (i.e., speeds less than, but comparable to, the speed of light). In the macroscopic world that you encounter in your daily life, the relativistic equations reduce to classical equations, and the predictions of classical Newtonian mechanics agree closely enough with experimental results to disregard relativistic corrections.

14.2: Invariance of Physical Laws

Learning Objectives

By the end of this section, you will be able to:

- Describe the theoretical and experimental issues that Einstein's theory of special relativity addressed.
- State the two postulates of the special theory of relativity.

Suppose you calculate the hypotenuse of a right triangle given the base angles and adjacent sides. Whether you calculate the hypotenuse from one of the sides and the cosine of the base angle, or from the Pythagorean theorem, the results should agree. Predictions based on different principles of physics must also agree, whether we consider them principles of mechanics or principles of electromagnetism.

Albert Einstein pondered a disagreement between predictions based on electromagnetism and on assumptions made in classical mechanics. Specifically, suppose an observer measures the velocity of a light pulse in the observer's own **rest frame**; that is, in the frame of reference in which the observer is at rest. According to the assumptions long considered obvious in classical mechanics, if an observer measures a velocity $\vec{v}$ in one frame of reference, and that frame of reference is moving with velocity $\vec{u}$ past a second reference frame, an observer in the second frame measures the original velocity as

$$\vec{v}' = \vec{v} + \vec{u}.$$

This sum of velocities is often referred to as *Galilean relativity*. If this principle is correct, the pulse of light that the observer measures as traveling with speed **c** travels at speed **c** + **u** measured in the frame of the second observer. If we reasonably assume that the laws of electrodynamics are the same in both frames of reference, then the predicted speed of light (in vacuum) in both frames should be

$$c = 1/\sqrt{\epsilon_0 \mu_0}.$$

Each observer should measure the same speed of the light pulse with respect to that observer's own rest frame. To reconcile difficulties of this kind, Einstein constructed his special theory of relativity, which introduced radical new ideas about time and space that have since been confirmed experimentally.

Inertial Frames

All velocities are measured relative to some frame of reference. For example, a car's motion is measured relative to its starting position on the road it travels on; a projectile's motion is measured relative to the surface from which it is launched; and a planet's orbital motion is measured relative to the star it orbits. The frames of reference in which mechanics takes the simplest form are those that are not accelerating. Newton's first law, the law of inertia, holds exactly in such a frame.

Definition: Inertial Reference Frame

An **inertial frame of reference** is a reference frame in which a body at rest remains at rest and a body in motion moves at a constant speed in a straight line unless acted upon by an outside force.

For example, to a passenger inside a plane flying at constant speed and constant altitude, physics seems to work exactly the same as when the passenger is standing on the surface of Earth. When the plane is taking off, however, matters are somewhat more complicated. In this case, the passenger at rest inside the plane concludes that a net force **F** on an object is not equal to the product of mass and acceleration, **ma**. Instead, **F** is equal to **ma** plus a fictitious force. This situation is not as simple as in an inertial frame. Special relativity handles accelerating frames as a constant and velocities as relative to the observer. General relativity treats both velocity and acceleration as relative to the observer, thus making the use of curved space-time.

Einstein's First Postulate

Not only are the principles of classical mechanics simplest in inertial frames, but they are the same in all inertial frames. Einstein based the first postulate of his theory on the idea that this is true for all the laws of physics, not merely those in mechanics.

> **FIRST POSTULATE OF SPECIAL RELATIVITY**
>
> The laws of physics are the same in all *inertial frames of reference*.

This postulate denies the existence of a special or preferred inertial frame. The laws of nature do not give us a way to endow any one inertial frame with special properties. For example, we cannot identify any inertial frame as being in a state of "absolute rest." We can only determine the relative motion of one frame with respect to another.

There is, however, more to this postulate than meets the eye. The laws of physics include only those that satisfy this postulate. We will see that the definitions of energy and momentum must be altered to fit this postulate. Another outcome of this postulate is the famous equation $E = mc^2$, which relates energy to mass.

Einstein's Second Postulate

The second postulate upon which Einstein based his theory of special relativity deals with the **speed of light**. Late in the nineteenth century, the major tenets of classical physics were well established. Two of the most important were the laws of electromagnetism and Newton's laws. Investigations such as Young's double-slit experiment in the early 1800s had convincingly demonstrated that light is a wave. Maxwell's equations of electromagnetism implied that electromagnetic waves travel at $c = 3.00 \times 10^8\, m/s$ in a vacuum, but they do not specify the frame of reference in which light has this speed. Many types of waves were known, and all travelled in some medium. Scientists therefore assumed that some medium carried the light, even in a vacuum, and that light travels at a speed **c** relative to that medium (often called "the aether").

Starting in the mid-1880s, the American physicist A.A. Michelson, later aided by E.W. Morley, made a series of direct measurements of the speed of light. They intended to deduce from their data the speed v at which Earth was moving through the mysterious medium for light waves. The speed of light measured on Earth should have been $c+v$ when Earth's motion was opposite to the medium's flow at speed u past the Earth, and $c-v$ when Earth was moving in the same direction as the medium (Figure $14.2.1$). The results of their measurements were startling.

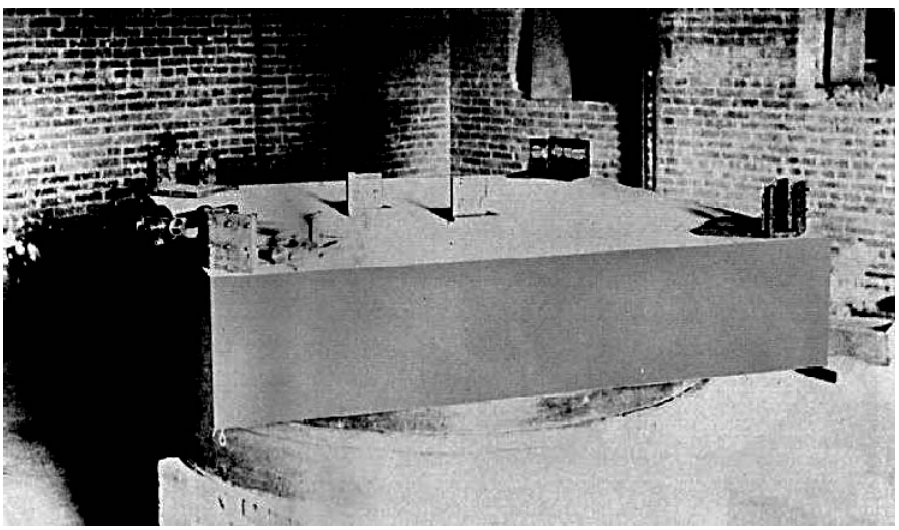


Figure 14.2.1: Michelson and Morley's interferometric setup, mounted on a stone slab that floats in an annular trough of mercury.

The eventual conclusion derived from this result is that light, unlike mechanical waves such as sound, does not need a medium to carry it. Furthermore, the Michelson-Morley results implied that the speed of light **c** is independent of the motion of the source relative to the observer. That is, everyone observes light to move at speed **c** regardless of how they move relative to the light source or to one another. For several years, many scientists tried unsuccessfully to explain these results within the framework of Newton's laws.

> **MICHELSON-MORLEY EXPERIMENT**
>
> The Michelson-Morley experiment demonstrated that the speed of light in a vacuum is independent of the motion of Earth about the Sun.

In addition, there was a contradiction between the principles of electromagnetism and the assumption made in Newton's laws about relative velocity. Classically, the velocity of an object in one frame of reference and the velocity of that object in a second frame of reference relative to the first should combine like simple vectors to give the velocity seen in the second frame. If that were correct, then two observers moving at different speeds would see light traveling at different speeds. Imagine what a light wave would look like to a person traveling along with it (in vacuum) at a speed c. If such a motion were possible, then the wave would be stationary relative to the observer. It would have electric and magnetic fields whose strengths varied with position but were constant in time. This is not allowed by Maxwell's equations. So either Maxwell's equations are different in different inertial frames, or an object with mass cannot travel at speed c. Einstein concluded that the latter is true: An object with mass cannot travel at speed c. Maxwell's equations are correct, but Newton's addition of velocities is not correct for light.

Not until 1905, when Einstein published his first paper on special relativity, was the currently accepted conclusion reached. Based mostly on his analysis that the laws of electricity and magnetism would not allow another speed for light, and only slightly aware of the Michelson-Morley experiment, Einstein detailed his second postulate of special relativity.

SECOND POSTULATE OF SPECIAL RELATIVITY

Light travels in a vacuum with the same speed c in any direction in all inertial frames.

In other words, the speed of light has the same definite speed for any observer, regardless of the relative motion of the source. This deceptively simple and counterintuitive postulate, along with the first postulate, leave all else open for change. Among the changes are the loss of agreement on the time between events, the variation of distance with speed, and the realization that matter and energy can be converted into one another. We describe these concepts in the following sections.

Exercise 14.2.1

Explain how special relativity differs from general relativity.

Answer

Special relativity applies only to objects moving at constant velocity, whereas general relativity applies to objects that undergo acceleration

14.3: Relativity of Simultaneity

Learning Objectives

By the end of this section, you will be able to:

- Show from Einstein's postulates that two events measured as simultaneous in one inertial frame are not necessarily simultaneous in all inertial frames.
- Describe how simultaneity is a relative concept for observers in different inertial frames in relative motion.

Do time intervals depend on who observes them? Intuitively, it seems that the time for a process, such as the elapsed time for a foot race (Figure 14.3.1), should be the same for all observers. In everyday experiences, disagreements over elapsed time have to do with the accuracy of measuring time. No one would be likely to argue that the actual time interval was different for the moving runner and for the stationary clock displayed. Carefully considering just how time is measured, however, shows that elapsed time does depends on the relative motion of an observer with respect to the process being measured.

Figure 14.3.1: Elapsed time for a foot race is the same for all observers, but at relativistic speeds, elapsed time depends on the motion of the observer relative to the location where the process being timed occurs. (credit: "Jason Edward Scott Bain"/Flickr)

Consider how we measure elapsed time. If we use a stopwatch, for example, how do we know when to start and stop the watch? One method is to use the arrival of light from the event. For example, if you're in a moving car and observe the light arriving from a traffic signal change from green to red, you know it's time to step on the brake pedal. The timing is more accurate if some sort of electronic detection is used, avoiding human reaction times and other complications.

Now suppose two observers use this method to measure the time interval between two flashes of light from flash lamps that are a distance apart (Figure 14.3.2). An observer **A** is seated midway on a rail car with two flash lamps at opposite sides equidistant from her. A pulse of light is emitted from each flash lamp and moves toward observer **A**, shown in frame (a) of the figure. The rail car is moving rapidly in the direction indicated by the velocity vector in the diagram. An observer **B** standing on the platform is facing the rail car as it passes and observes both flashes of light reach him simultaneously, as shown in frame (c). He measures the distances from where he saw the pulses originate, finds them equal, and concludes that the pulses were emitted simultaneously.

However, because of Observer **A**'s motion, the pulse from the right of the railcar, from the direction the car is moving, reaches her before the pulse from the left, as shown in frame (b). She also measures the distances from within her frame of reference, finds them equal, and concludes that the pulses were not emitted simultaneously.

The two observers reach conflicting conclusions about whether the two events at well-separated locations were simultaneous. Both frames of reference are valid, and both conclusions are valid. Whether two events at separate locations are simultaneous depends on the motion of the observer relative to the locations of the events.

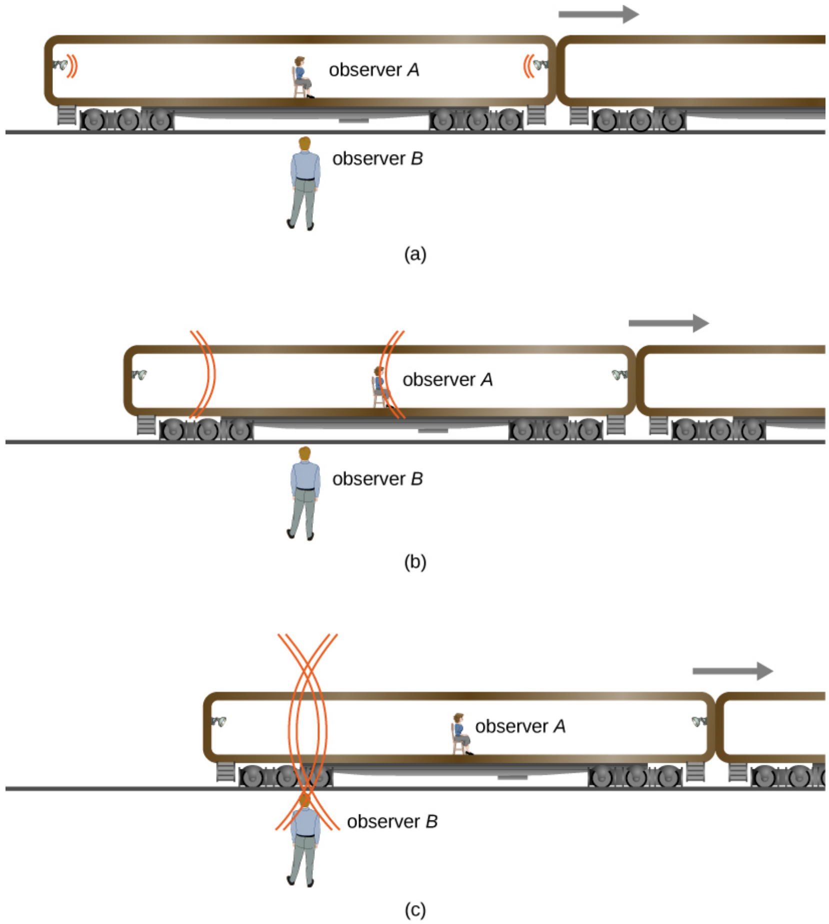


Figure 14.3.2: (a) Two pulses of light are emitted simultaneously relative to observer B. (c) The pulses reach observer B's position simultaneously. (b) Because of A's motion, she sees the pulse from the right first and concludes the bulbs did not flash simultaneously. Both conclusions are correct.

Here, the relative velocity between observers affects whether two events a distance apart are observed to be simultaneous. **Simultaneity is not absolute**. We might have guessed (incorrectly) that if light is emitted simultaneously, then two observers halfway between the sources would see the flashes simultaneously. But careful analysis shows this cannot be the case if the speed of light is the same in all inertial frames.

This type of **thought experiment** (in German, "Gedankenexperiment") shows that seemingly obvious conclusions must be changed to agree with the postulates of relativity. The validity of thought experiments can only be determined by actual observation, and careful experiments have repeatedly confirmed Einstein's theory of relativity.

14.4: Time Dilation

Learning Objectives

By the end of this section, you will be able to:

- Explain how time intervals can be measured differently in different reference frames.
- Describe how to distinguish a proper time interval from a dilated time interval.
- Describe the significance of the muon experiment.
- Explain why the twin paradox is not a contradiction.
- Calculate time dilation given the speed of an object in a given frame.

The analysis of simultaneity shows that Einstein's postulates imply an important effect: Time intervals have different values when measured in different inertial frames. Suppose, for example, an astronaut measures the time it takes for a pulse of light to travel a distance perpendicular to the direction of his ship's motion (relative to an earthbound observer), bounce off a mirror, and return (Figure $14.4.1a$). How does the elapsed time that the astronaut measures in the spacecraft compare with the elapsed time that an earthbound observer measures by observing what is happening in the spacecraft?

Examining this question leads to a profound result. The elapsed time for a process depends on which observer is measuring it. In this case, the time measured by the astronaut (within the spaceship where the astronaut is at rest) is smaller than the time measured by the earthbound observer (to whom the astronaut is moving). The time elapsed for the same process is different for the observers, because the distance the light pulse travels in the astronaut's frame is smaller than in the earthbound frame, as seen in Figure $14.4.1b$. Light travels at the same speed in each frame, so it takes more time to travel the greater distance in the earthbound frame.

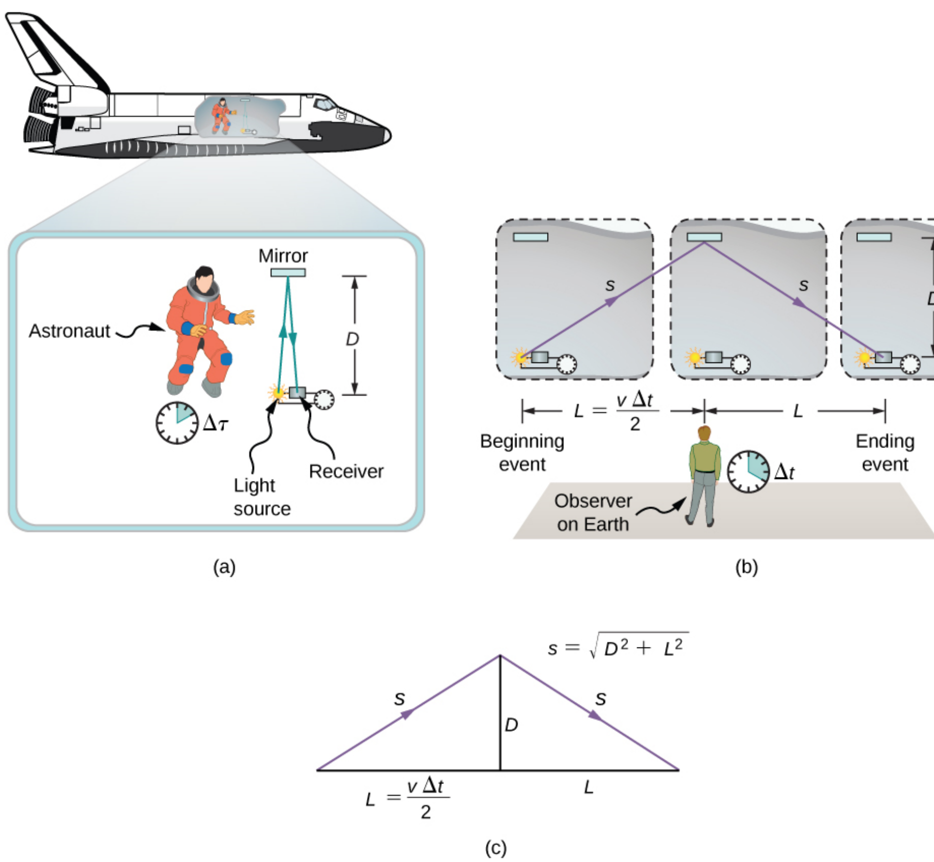


Figure 14.4.1: (a) An astronaut measures the time $\Delta\tau$ for light to travel distance 2D in the astronaut's frame. (b) A NASA scientist on Earth sees the light follow the longer path 2s and take a longer time Δt. (c) These triangles are used to find the relationship between the two distances D and s.

Definition: Time Dilation

Time dilation is the lengthening of the time interval between two events for an observer in an inertial frame that is moving with respect to the rest frame of the events (in which the events occur at the same location).

To quantitatively compare the time measurements in the two inertial frames, we can relate the distances in Figure $14.4.1b$ to each other, then express each distance in terms of the time of travel (respectively either Δt or $\Delta \tau$) of the pulse in the corresponding reference frame. The resulting equation can then be solved for Δt in terms of $\Delta \tau$.

The lengths D and L in Figure $14.4.1c$ are the sides of a right triangle with hypotenuse s. From the Pythagorean theorem,

$$s^2 = D^2 + L^2.$$

The lengths $2s$ and $2L$ are, respectively, the distances that the pulse of light and the spacecraft travel in time Δt in the earthbound observer's frame. The length D is the distance that the light pulse travels in time $\Delta \tau$ in the astronaut's frame. This gives us three equations:

$$2s = c\Delta t$$

$$2L = v\Delta t;$$

$$2D = c\Delta \tau.$$

Note that we used Einstein's second postulate by taking the speed of light to be **c** in both inertial frames. We substitute these results into the previous expression from the Pythagorean theorem:

$$\begin{aligned} s^2 &= D^2 + L^2 \\ \left(c\frac{\Delta t}{2}\right)^2 &= \left(c\frac{\Delta \tau}{2}\right)^2 + \left(v\frac{\Delta t}{2}\right)^2 \end{aligned}$$

Then we rearrange to obtain

$$(c\Delta t)^2 - (v\Delta t)^2 = (c\Delta \tau)^2.$$

Finally, solving for Δt in terms of $\Delta \tau$ gives us

$$\Delta t = \frac{\Delta \tau}{\sqrt{1-(v/c)^2}}.$$

This is equivalent to

$$\Delta t = \gamma \Delta \tau, \tag{14.4.1}$$

where γ is the relativistic factor (often called the **Lorentz factor**) given by

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and v and c are the speeds of the moving observer and light, respectively.

Note the asymmetry between the two measurements. Only one of them is a measurement of the time interval between two events—the emission and arrival of the light pulse—at the same position. It is a measurement of the time interval in the rest frame of a single clock. The measurement in the earthbound frame involves comparing the time interval between two events that occur at different locations. The time interval between events that occur at a single location has a separate name to distinguish it from the time measured by the earthbound observer, and we use the separate symbol $\Delta \tau$ to refer to it throughout this chapter.

Definition: Proper Time

The **proper time** interval $\Delta \tau$ between two events is the time interval measured by an observer for whom both events occur at the same location.

The equation relating δt and $\Delta\tau$ is truly remarkable. First, as stated earlier, elapsed time is not the same for different observers moving relative to one another, even though both are in inertial frames. A proper time interval $\Delta\tau$ for an observer who, like the astronaut, is moving with the apparatus, is smaller than the time interval for other observers. It is the smallest possible measured time between two events. The earthbound observer sees time intervals within the moving system as dilated (i.e., lengthened) relative to how the observer moving relative to Earth sees them within the moving system. Alternatively, according to the earthbound observer, less time passes between events within the moving frame. Note that the shortest elapsed time between events is in the inertial frame in which the observer sees the events (e.g., the emission and arrival of the light signal) occur at the same point.

This time effect is real and is not caused by inaccurate clocks or improper measurements. Time-interval measurements of the same event differ for observers in relative motion. The dilation of time is an intrinsic property of time itself. All clocks moving relative to an observer, including biological clocks, such as a person's heartbeat, or aging, are observed to run more slowly compared with a clock that is stationary relative to the observer.

Note that if the relative velocity is much less than the speed of light (v << c), then v^2/c^2 is extremely small, and the elapsed times Δt and $\Delta\tau$ are nearly equal. At low velocities, physics based on modern relativity approaches classical physics—everyday experiences involve very small relativistic effects. However, for speeds near the speed of light, v^2/c^2 is close to one, so $\sqrt{1-v^2/c^2}$ is very small and Δt becomes significantly larger than $\Delta\tau$.

Half-Life of a Muon

There is considerable experimental evidence that the equation $\Delta t=\gamma\Delta\tau$ is correct. One example is found in cosmic ray particles that continuously rain down on Earth from deep space. Some collisions of these particles with nuclei in the upper atmosphere result in short-lived particles called **muons**. The half-life (amount of time for half of a material to decay) of a muon is 1.52 μs when it is at rest relative to the observer who measures the half-life. This is the proper time interval $\Delta\tau$. This short time allows very few muons to reach Earth's surface and be detected if Newtonian assumptions about time and space were correct. However, muons produced by cosmic ray particles have a range of velocities, with some moving near the speed of light. It has been found that the muon's half-life as measured by an earthbound observer (Δt) varies with velocity exactly as predicted by the equation $\Delta t=\gamma\Delta\tau$. The faster the muon moves, the longer it lives. We on Earth see the muon last much longer than its half-life predicts within its own rest frame. As viewed from our frame, the muon decays more slowly than it does when at rest relative to us. A far larger fraction of muons reach the ground as a result.

Before we present the first example of solving a problem in relativity, we state a strategy you can use as a guideline for these calculations.

PROBLEM-SOLVING STRATEGY: RELATIVITY

1. Make a list of what is given or can be inferred from the problem as stated (identify the knowns). Look in particular for information on relative velocity **v**.
2. Identify exactly what needs to be determined in the problem (identify the unknowns).
3. Make certain you understand the conceptual aspects of the problem before making any calculations (express the answer as an equation). Decide, for example, which observer sees time dilated or length contracted before working with the equations or using them to carry out the calculation. If you have thought about who sees what, who is moving with the event being observed, who sees proper time, and so on, you will find it much easier to determine if your calculation is reasonable.
4. Determine the primary type of calculation to be done to find the unknowns identified above (do the calculation). You will find the section summary helpful in determining whether a length contraction, relativistic kinetic energy, or some other concept is involved.

Note **that you should not round off during the calculation**. As noted in the text, you must often perform your calculations to many digits to see the desired effect. You may round off at the very end of the problem solution, but do not use a rounded number in a subsequent calculation. Also, check the answer to see if it is reasonable: Does it make sense? This may be more difficult for relativity, which has few everyday examples to provide experience with what is reasonable. But you can look for velocities greater than **c** or relativistic effects that are in the wrong direction (such as a time contraction where a dilation was expected).

Example $14.4.1A$: Time Dilation in a High-Speed Vehicle

The Hypersonic Technology Vehicle 2 (HTV-2) is an experimental rocket vehicle capable of traveling at 21,000 km/h (5830 m/s). If an electronic clock in the HTV-2 measures a time interval of exactly 1-s duration, what would observers on Earth measure the time interval to be?

Strategy

Apply the time dilation formula to relate the proper time interval of the signal in HTV-2 to the time interval measured on the ground.

Solution

1. Identify the knowns: $\Delta\tau = 1\ s;\ v = 5830 m/s$.
2. Identify the unknown: Δt.
3. Express the answer as an equation:

$$\Delta t = \gamma\Delta\tau = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}}.$$

4. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \frac{1\ s}{\sqrt{1-\left(\frac{5830\ m/s}{3.00\times10^8 m/s}\right)^2}} \\ &= 1.000000000189\ s \\ &= 1\ s + 1.89\times10^{-10} s.\end{aligned}$$

Significance

The very high speed of the HTV-2 is still only 10^{-5} times the speed of light. Relativistic effects for the HTV-2 are negligible for almost all purposes, but are not zero.

What Speeds are Relativistic?

How fast must a vehicle travel for 1 second of time measured on a passenger’s watch in the vehicle to differ by 1% for an observer measuring it from the ground outside?

Strategy

Use the time dilation formula to find **v/c** for the given ratio of times.

Solution

1. Identify the known:

$$\frac{\Delta\tau}{\Delta t} = \frac{1}{1.01}.$$

2. Identify the unknown: **v/c**.
3. Express the answer as an equation:

$$\begin{aligned} \Delta t &= \gamma \Delta \tau \\ &= \frac{1}{\sqrt{1 - v^2/c^2}} \Delta \tau \\ \frac{\Delta \tau}{\Delta t} &= \sqrt{1 - v^2/c^2} \\ \left(\frac{\Delta \tau}{\Delta t}\right)^2 &= 1 - \frac{v^2}{c^2} \\ \frac{v}{c} &= \sqrt{1 - (\Delta \tau / \Delta t)^2}. \end{aligned}$$

4. Do the calculation:

$$\frac{v}{c} = \sqrt{1 - (1/1.01)^2} = 0.14.$$

Significance

The result shows that an object must travel at very roughly 10% of the speed of light for its motion to produce significant relativistic time dilation effects.

Calculating Δt for a Relativistic Event

Suppose a cosmic ray colliding with a nucleus in Earth's upper atmosphere produces a muon that has a velocity $v = 0.950c$. The muon then travels at constant velocity and lives 2.20 μs as measured in the muon's frame of reference. (You can imagine this as the muon's internal clock.) How long does the muon live as measured by an earthbound observer (Figure $14.4.2$)?

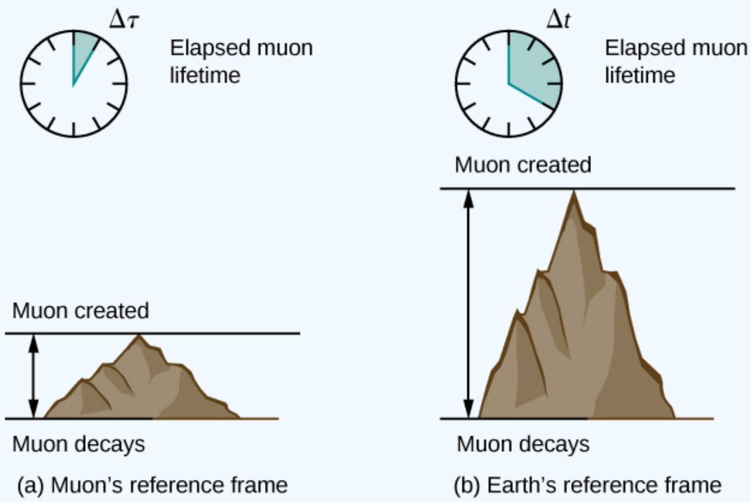


Figure 14.4.2: A muon in Earth's atmosphere lives longer as measured by an earthbound observer than as measured by the muon's internal clock.

As we will discuss later, in the muon's reference frame, it travels a shorter distance than measured in Earth's reference frame.

Strategy

A clock moving with the muon measures the proper time of its decay process, so the time we are given is $\Delta \tau = 2.20 \mu s$. The earthbound observer measures Δt as given by the equation $\Delta t = \gamma \Delta \tau$. Because the velocity is given, we can calculate the time in Earth's frame of reference.

Solution

1. Identify the knowns: $v = 0.950c$; $\delta \tau = 2.20 \mu s$.
2. Identify the unknown: Δt.
3. Express the answer as an equation. Use:

$$\Delta t = \gamma \Delta \tau.$$

with

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}.$$

4. Do the calculation. Use the expression for γ to determine Δt from $\Delta\tau$:

$$\begin{aligned}\Delta t &= \gamma\Delta\tau. \\ &= \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\delta\tau \\ &= \frac{2.20\mu s}{\sqrt{1-(0.950)^2}} \\ &= 7.05\ \mu s.\end{aligned}$$

Remember to keep extra significant figures until the final answer.

Significance

One implication of this example is that because $\gamma = 3.20$ at 95.0% of the speed of light ($v = 0.950c$), the relativistic effects are significant. The two time intervals differ by a factor of 3.20, when classically they would be the same. Something moving at 0.950**c** is said to be highly relativistic.

✓ Example $14.4.1B$: Relativistic Television

A non-flat screen, older-style television display (Figure $14.4.3$) works by accelerating electrons over a short distance to relativistic speed, and then using electromagnetic fields to control where the electron beam strikes a fluorescent layer at the front of the tube. Suppose the electrons travel at $6.00\times 10^7 m/s$ through a distance of 0.200m0.200m from the start of the beam to the screen.

a. What is the time of travel of an electron in the rest frame of the television set?
b. What is the electron's time of travel in its own rest frame?

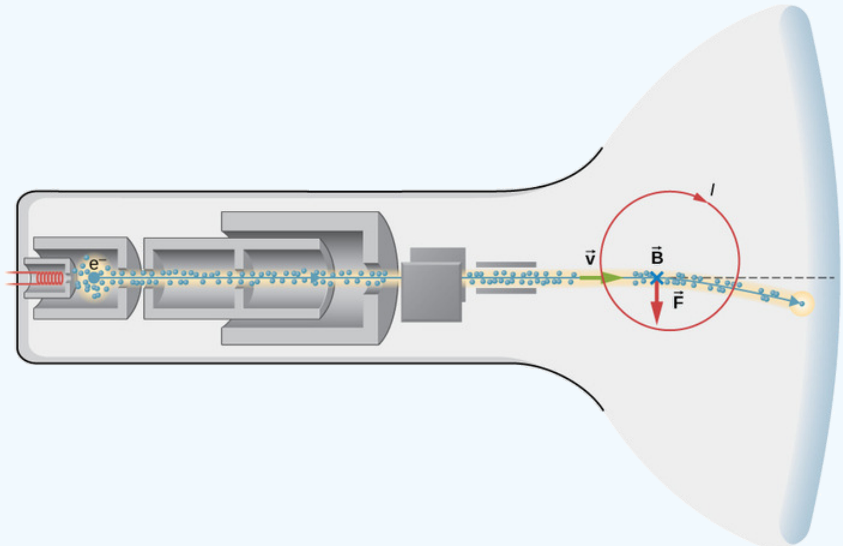


Figure 14.4.3: The electron beam in a cathode ray tube television display.

Strategy for (a)

(a) Calculate the time from $vt = d$. Even though the speed is relativistic, the calculation is entirely in one frame of reference, and relativity is therefore not involved.

Solution

1. Identify the knowns:

$$v = 6.00 \times 10^{7} m/s \, d = 0.200 \, m.$$

2. Identify the unknown: the time of travel Δt.
3. Express the answer as an equation:

$$\Delta t = \frac{d}{v}.$$

4. Do the calculation:

$$\begin{aligned} t &= \frac{0.200 \, m}{6.00 \times 10^{7} \; m/s} \\ &= 3.33 \times 10^{-9} \; s. \end{aligned}$$

Significance

The time of travel is extremely short, as expected. Because the calculation is entirely within a single frame of reference, relativity is not involved, even though the electron speed is close to **c**.

Strategy for (b)

(b) In the frame of reference of the electron, the vacuum tube is moving and the electron is stationary. The electron-emitting cathode leaves the electron and the front of the vacuum tube strikes the electron with the electron at the same location. Therefore we use the time dilation formula to relate the proper time in the electron rest frame to the time in the television frame.

Solution

1. Identify the knowns (from part a):

$$\Delta t = 3.33 \times 10^{-9} \; s; \; v = 6.00 \times 10^{7} \; m/s; \; d = 0.200 \, m.$$

2. Identify the unknown: τ.
3. Express the answer as an equation:

$$\Delta t = \gamma \Delta \tau = \frac{\Delta \tau}{\sqrt{1 - v^2/c^2}}.$$

4. Do the calculation:

$$\begin{aligned} \Delta \tau &= (3.33 \times 10^{-9} s) \sqrt{1 - \left(\frac{6.00 \times 10^{7} m/s}{3.00 \times 10^{8} m/s} \right)^2} \\ &= 3.26 \times 10^{-9} s. \end{aligned}$$

Significance

The time of travel is shorter in the electron frame of reference. Because the problem requires finding the time interval measured in different reference frames for the same process, relativity is involved. If we had tried to calculate the time in the electron rest frame by simply dividing the 0.200 m by the speed, the result would be slightly incorrect because of the relativistic speed of the electron.

? Exercise 14.4.1

What is γ if $v = 0.650c$?

Answer

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}=\frac{1}{\sqrt{1-\frac{(0.650c)}{c^2}}}=1.32$$

The Twin Paradox

An intriguing consequence of time dilation is that a space traveler moving at a high velocity relative to Earth would age less than the astronaut's earthbound twin. This is often known as the **twin paradox**. Imagine the astronaut moving at such a velocity that $\gamma=30.0$, as in Figure $14.4.4$ A trip that takes 2.00 years in her frame would take 60.0 years in the earthbound twin's frame. Suppose the astronaut travels 1.00 year to another star system, briefly explores the area, and then travels 1.00 year back. An astronaut who was 40 years old at the start of the trip would be would be 42 when the spaceship returns. Everything on Earth, however, would have aged 60.0 years. The earthbound twin, if still alive, would be 100 years old.

The situation would seem different to the astronaut in Figure $14.4.4$ Because motion is relative, the spaceship would seem to be stationary and Earth would appear to move. (This is the sensation you have when flying in a jet.) Looking out the window of the spaceship, the astronaut would see time slow down on Earth by a factor of $\gamma=30.0$. Seen from the spaceship, the earthbound sibling will have aged only 2/30, or 0.07, of a year, whereas the astronaut would have aged 2.00 years.

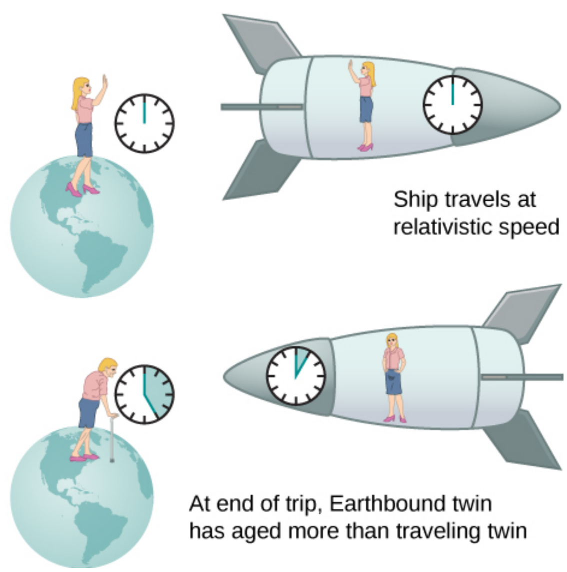


Figure 14.4.4: The twin paradox consists of the conflicting conclusions about which twin ages more as a result of a long space journey at relativistic speed.

The paradox here is that the two twins cannot both be correct. As with all paradoxes, conflicting conclusions come from a false premise. In fact, the astronaut's motion is significantly different from that of the earthbound twin. The astronaut accelerates to a high velocity and then decelerates to view the star system. To return to Earth, she again accelerates and decelerates. The spacecraft is not in a single inertial frame to which the time dilation formula can be directly applied. That is, the astronaut twin changes inertial references. The earthbound twin does not experience these accelerations and remains in the same inertial frame. Thus, the situation is not symmetric, and it is incorrect to claim that the astronaut observes the same effects as her twin. The lack of symmetry between the twins will be still more evident when we analyze the journey later in this chapter in terms of the path the astronaut follows through four-dimensional space-time.

In 1971, American physicists Joseph Hafele and Richard Keating verified time dilation at low relative velocities by flying extremely accurate atomic clocks around the world on commercial aircraft. They measured elapsed time to an accuracy of a few nanoseconds and compared it with the time measured by clocks left behind. Hafele and Keating's results were within experimental uncertainties of the predictions of relativity. Both special and general relativity had to be taken into account, because gravity and accelerations were involved as well as relative motion.

? Exercise 14.4.2A

a. A particle travels at $1.90 \times 10^8\ m/s$ and lives $2.1 \times 10^8\ s$ when at rest relative to an observer. How long does the particle live as viewed in the laboratory?

Answer

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{2.10 \times 10^{-8} s}{\sqrt{1-\frac{(1.90\times 10^8\ m/s)^2}{(3.00\times 10^8\ m/s)^2}}} = 2.71 \times 10^{-8}\ s.$$

? Exercise 14.4.2B

Spacecraft A and B pass in opposite directions at a relative speed of $4.00 \times 10^7\ m/s$. An internal clock in spacecraft A causes it to emit a radio signal for 1.00 s. The computer in spacecraft B corrects for the beginning and end of the signal having traveled different distances, to calculate the time interval during which ship A was emitting the signal. What is the time interval that the computer in spacecraft B calculates?

Answer

Only the relative speed of the two spacecraft matters because there is no absolute motion through space. The signal is emitted from a fixed location in the frame of reference of **A**, so the proper time interval of its emission is $\tau = 1.00\ s$. The duration of the signal measured from frame of reference **B** is then

$$\Delta t = \frac{\Delta\tau}{\sqrt{1-\frac{v^2}{c^2}}} = \frac{1.00\ s}{\sqrt{1-\frac{(4.00\times 10^7\ m/s)^2}{(3.00\times 10^8\ m/s)^2}}} = 1.01\ s.$$

14.5: Length Contraction

Learning Objectives

By the end of this section, you will be able to:

- Explain how simultaneity and length contraction are related.
- Describe the relation between length contraction and time dilation and use it to derive the length-contraction equation.

The length of the train car in Figure $14.5.1$ is the same for all the passengers. All of them would agree on the simultaneous location of the two ends of the car and obtain the same result for the distance between them. But simultaneous events in one inertial frame need not be simultaneous in another. If the train could travel at relativistic speeds, an observer on the ground would see the simultaneous locations of the two endpoints of the car at a different distance apart than observers inside the car. Measured distances need not be the same for different observers when relativistic speeds are involved.

Figure 14.5.1: People might describe distances differently, but at relativistic speeds, the distances really are different. (credit: "russavia"/Flickr).

Proper Length

Two observers passing each other always see the same value of their relative speed. Even though time dilation implies that the train passenger and the observer standing alongside the tracks measure different times for the train to pass, they still agree that relative speed, which is distance divided by elapsed time, is the same. If an observer on the ground and one on the train measure a different time for the length of the train to pass the ground observer, agreeing on their relative speed means they must also see different distances traveled.

The muon discussed previously illustrates this concept (Figure $14.5.2$). To an observer on Earth, the muon travels at 0.950**c** for 7.05 μs from the time it is produced until it decays. Therefore, it travels a distance relative to Earth of:

$$\begin{aligned} L_0 &= v\Delta t \\ &= (0.950)(3.00 \times 10^8\, m/s)(7.05 \times 10^{-6}\, s) \\ &= 2.01\, km. \end{aligned}$$

In the muon frame, the lifetime of the muon is 2.20 μs. In this frame of reference, the Earth, air, and ground have only enough time to travel:

$$\begin{aligned} L &= v\Delta r \\ &= (0.950)(3.00 \times 10^8\, m/s)(2.20 \times 10^{-6}\, s) \\ &= 0.627\, km. \end{aligned}$$

The distance between the same two events (production and decay of a muon) depends on who measures it and how they are moving relative to it.

Definition: Proper Length

Proper length L_0 is the distance between two points measured by an observer who is at rest relative to both of the points.

The earthbound observer measures the proper length L_0 because the points at which the muon is produced and decays are stationary relative to Earth. To the muon, Earth, air, and clouds are moving, so the distance **L** it sees is not the proper length.

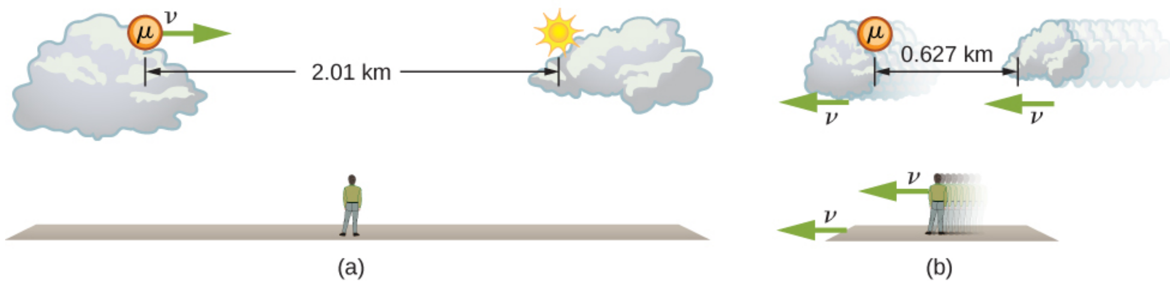


Figure 14.5.2: (a) The earthbound observer sees the muon travel 2.01 km. (b) The same path has length 0.627 km seen from the muon's frame of reference. The Earth, air, and clouds are moving relative to the muon in its frame, and have smaller lengths along the direction of travel.

Length Contraction

To relate distances measured by different observers, note that the velocity relative to the earthbound observer in our muon example is given by

$$v = \frac{L_0}{\Delta t}.$$

The time relative to the earthbound observer is Δt, because the object being timed is moving relative to this observer. The velocity relative to the moving observer is given by

$$v = \frac{L}{\Delta \tau}.$$

The moving observer travels with the muon and therefore observes the proper time $\Delta\tau$. The two velocities are identical; thus,

$$\frac{L_0}{\Delta t} = \frac{L}{\Delta \tau}. \tag{14.5.1}$$

We know that $\Delta t = \gamma \Delta \tau$ and substituting this into Equation 14.5.1 gives

$$L = \frac{L_0}{\gamma}.$$

Substituting for γ gives an equation relating the distances measured by different observers.

Definition: Lenght Contraction

Length contraction is the decrease in the measured length of an object from its proper length when measured in a reference frame that is moving with respect to the object:

$$L = L_0 \sqrt{1 - \frac{v^2}{c^2}} \tag{14.5.2}$$

where L_0 is the length of the object in its rest frame, and L is the length in the frame moving with velocity v.

If we measure the length of anything moving relative to our frame, we find its length **L** to be smaller than the proper length L_0 that would be measured if the object were stationary. For example, in the muon's rest frame, the distance Earth moves between where the muon was produced and where it decayed is shorter than the distance traveled as seen from the Earth's frame. Those points are fixed relative to Earth but are moving relative to the muon. Clouds and other objects are also contracted along the direction of motion as seen from muon's rest frame.

Thus, two observers measure different distances along their direction of relative motion, depending on which one is measuring distances between objects at rest.

But what about distances measured in a direction perpendicular to the relative motion? Imagine two observers moving along their **x**-axes and passing each other while holding meter sticks vertically in the **y**-direction. Figure $14.5.3$ shows two meter sticks M and M' that are at rest in the reference frames of two boys S and S', respectively. A small paintbrush is attached to the top (the 100-cm mark) of stick M'. Suppose that S' is moving to the right at a very high speed **v** relative to S, and the sticks are oriented so that they are perpendicular, or transverse, to their relative velocity vector. The sticks are held so that as they pass each other, their lower ends (the 0-cm marks) coincide. Assume that when S looks at his stick M afterwards, he finds a line painted on it, just below the top of the stick. Because the brush is attached to the top of the other boy's stick M', S can only conclude that stick M' is less than 1.0 m long.

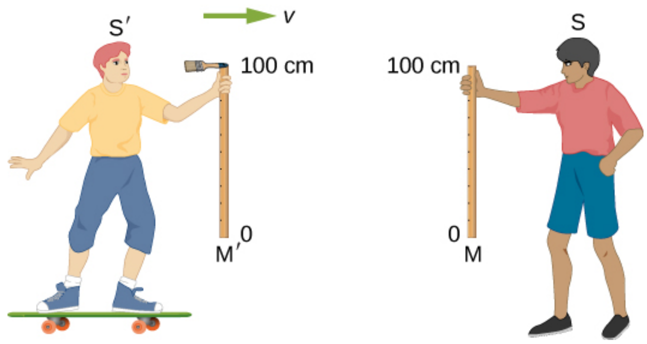


Figure $14.5.3$: Meter sticks M and M'M′ are stationary in the reference frames of observers S and S', respectively. As the sticks pass, a small brush attached to the 100-cm mark of M' paints a line on M.

Now when the boys approach each other, S', like S, sees a meter stick moving toward him with speed **v**. Because their situations are symmetric, each boy must make the same measurement of the stick in the other frame. So, if S measures stick M' to be less than 1.0 m long, S' must measure stick M to be also less than 1.0 m long, and S' must see his paintbrush pass over the top of stick M and not paint a line on it. In other words, after the same event, one boy sees a painted line on a stick, while the other does not see such a line on that same stick!

Einstein's first postulate requires that the laws of physics (as, for example, applied to painting) predict that S and S', who are both in inertial frames, make the same observations; that is, S and S' must either both see a line painted on stick M, or both not see that line. We are therefore forced to conclude our original assumption that S saw a line painted below the top of his stick was wrong! Instead, S finds the line painted right at the 100-cm mark on M. Then both boys will agree that a line is painted on M, and they will also agree that both sticks are exactly 1 m long. We conclude then that measurements of a transverse *length must be the same* **in different inertial frames**.

✔ Example $14.5.4$: Calculating Length Contraction

Suppose an astronaut, such as the twin in the twin paradox discussion, travels so fast that $\gamma = 30.00$. (a) The astronaut travels from Earth to the nearest star system, Alpha Centauri, 4.300 light years (ly) away as measured by an earthbound observer. How far apart are Earth and Alpha Centauri as measured by the astronaut? (b) In terms of **c**, what is the astronaut's velocity relative to Earth? You may neglect the motion of Earth relative to the sun (Figure $14.5.4$).

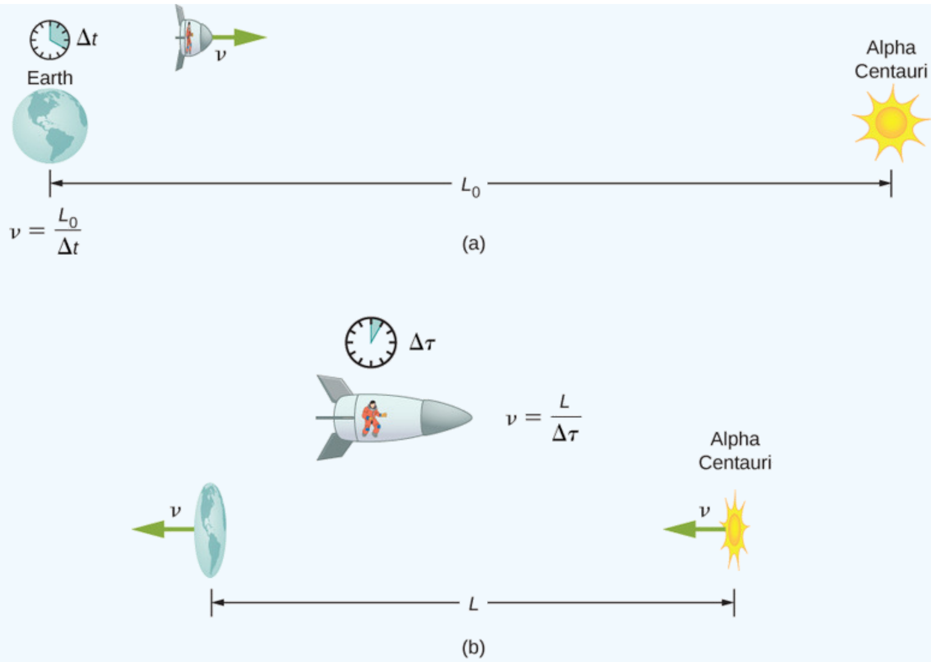


Figure 14.5.4: (a) The earthbound observer measures the proper distance between Earth and Alpha Centauri. (b) The astronaut observes a length contraction because Earth and Alpha Centauri move relative to her ship. She can travel this shorter distance in a smaller time (her proper time) without exceeding the speed of light.

Strategy

First, note that a light year (ly) is a convenient unit of distance on an astronomical scale—it is the distance light travels in a year. For part (a), the 4.300-ly distance between Alpha Centauri and Earth is the proper distance L_0, because it is measured by an earthbound observer to whom both stars are (approximately) stationary. To the astronaut, Earth and Alpha Centauri are moving past at the same velocity, so the distance between them is the contracted length **L**. In part (b), we are given γ, so we can find v by rearranging the definition of γ to express v in terms of c.

Solution for (a)

For part (a):

1. Identify the knowns: $L_0 = 4.300\, ly; \gamma = 30.00$.
2. Identify the unknown: **L**.
3. Express the answer as an equation: $L = \dfrac{L_0}{\gamma}$.
4. Do the calculation:

$$\begin{aligned} L &= \frac{L_0}{\gamma} \\ &= \frac{4.300\, ly}{30.00} \\ &= 0.1433\, ly. \end{aligned}$$

Solution for (b)

For part (b):

1. Identify the known: $\gamma = 30.00$.
2. Identify the unknown: **v** in terms of **c**.
3. Express the answer as an equation. Start with:

$$\gamma = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Then solve for the unknown **v/c** by first squaring both sides and then rearranging:

$$\gamma^2 = \frac{1}{1-\frac{v^2}{c^2}}$$

$$\frac{v^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\frac{v}{c} = \sqrt{1-\frac{1}{\gamma^2}}.$$

4. Do the calculation:

$$\begin{aligned}\frac{v}{c} &= \sqrt{1-\frac{1}{\gamma^2}} \\ &= \sqrt{1-\frac{1}{(30.00)^2}} \\ &= 0.99944\end{aligned}$$

or

$$v = 0.9994\,c.$$

Significance: Remember not to round off calculations until the final answer, or you could get erroneous results. This is especially true for special relativity calculations, where the differences might only be revealed after several decimal places. The relativistic effect is large here ($\gamma = 30.00$), and we see that **v** is approaching (not equaling) the speed of light. Because the distance as measured by the astronaut is so much smaller, the astronaut can travel it in much less time in her frame.

People traveling at extremely high velocities could cover very large distances (thousands or even millions of light years) and age only a few years on the way. However, like emigrants in past centuries who left their home, these people would leave the Earth they know forever. Even if they returned, thousands to millions of years would have passed on Earth, obliterating most of what now exists. There is also a more serious practical obstacle to traveling at such velocities; immensely greater energies would be needed to achieve such high velocities than classical physics predicts can be attained. This will be discussed later in the chapter.

Why don't we notice length contraction in everyday life? The distance to the grocery store does not seem to depend on whether we are moving or not. Examining Equation 14.5.2, we see that at low velocities ($v \ll c$), the lengths are nearly equal, which is the classical expectation. However, length contraction is real, if not commonly experienced. For example, a charged particle such as an electron traveling at relativistic velocity has electric field lines that are compressed along the direction of motion as seen by a stationary observer (Figure 14.5.5). As the electron passes a detector, such as a coil of wire, its field interacts much more briefly, an effect observed at particle accelerators such as the 3-km-long Stanford Linear Accelerator (SLAC). In fact, to an electron traveling down the beam pipe at SLAC, the accelerator and Earth are all moving by and are length contracted. The relativistic effect is so great that the accelerator is only 0.5 m long to the electron. It is actually easier to get the electron beam down the pipe, because the beam does not have to be as precisely aimed to get down a short pipe as it would to get down a pipe 3 km long. This, again, is an experimental verification of the special theory of relativity.

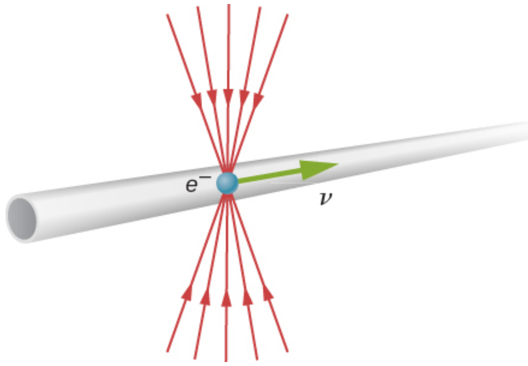


Figure 14.5.5: The electric field lines of a high-velocity charged particle are compressed along the direction of motion by length contraction, producing an observably different signal as the particle goes through a coil.

Exercise 14.5.1

A particle is traveling through Earth's atmosphere at a speed of $0.750c$. To an earthbound observer, the distance it travels is 2.50 km. How far does the particle travel as viewed from the particle's reference frame?

Answer

$$L=L_0\sqrt{1-\frac{v^2}{c^2}}=(2.50\,km)\sqrt{1-\frac{(0.750c)^2}{c^2}}=1.65\,km$$

14.6: The Lorentz Transformation

Learning Objectives

- Describe the Galilean transformation of classical mechanics, relating the position, time, velocities, and accelerations measured in different inertial frames
- Derive the corresponding Lorentz transformation equations, which, in contrast to the Galilean transformation, are consistent with special relativity
- Explain the Lorentz transformation and many of the features of relativity in terms of four-dimensional space-time

We have used the postulates of relativity to examine, in particular examples, how observers in different frames of reference measure different values for lengths and the time intervals. We can gain further insight into how the postulates of relativity change the Newtonian view of time and space by examining the transformation equations that give the space and time coordinates of events in one inertial reference frame in terms of those in another. We first examine how position and time coordinates transform between inertial frames according to the view in Newtonian physics. Then we examine how this has to be changed to agree with the postulates of relativity. Finally, we examine the resulting Lorentz transformation equations and some of their consequences in terms of four-dimensional space-time diagrams, to support the view that the consequences of special relativity result from the properties of time and space itself, rather than electromagnetism.

The Galilean Transformation Equations

An event is specified by its location and time (x, y, z, t) relative to one particular inertial frame of reference S. As an example, (x, y, z, t) could denote the position of a particle at time t, and we could be looking at these positions for many different times to follow the motion of the particle. Suppose a second frame of reference S' moves with velocity v with respect to the first. For simplicity, assume this relative velocity is along the x-axis. The relation between the time and coordinates in the two frames of reference is then

$$x = x' + vt \tag{14.6.1}$$

$$y = y' \tag{14.6.2}$$

$$x = z'. \tag{14.6.3}$$

Implicit in these equations is the assumption that time measurements made by observers in both S and S' are the same. That is,

$$t = t' \tag{14.6.4}$$

Equations 14.6.1-14.6.4are known collectively as the **Galilean transformation**.

We can obtain the Galilean velocity and acceleration transformation equations by differentiating these equations with respect to time. We use u for the velocity of a particle throughout this chapter to distinguish it from v, the relative velocity of two reference frames. Note that, for the Galilean transformation, the increment of time used in differentiating to calculate the particle velocity is the same in both frames, $dt = dt'$. Differentiation yields

$$u_x = u'_x + v,\ u_y = u'_y,\ u_z = u'_z$$

and

$$a_x = a'_x,\ a_y = a'_y,\ a_z = a'_z.$$

We denote the velocity of the particle by u rather than v to avoid confusion with the velocity v of one frame of reference with respect to the other. Velocities in each frame differ by the velocity that one frame has as seen from the other frame. Observers in both frames of reference measure the same value of the acceleration. Because the mass is unchanged by the transformation, and distances between points are uncharged, observers in both frames see the same forces $F = ma$ acting between objects and the same form of Newton’s second and third laws in all inertial frames. The laws of mechanics are consistent with the first postulate of relativity.

The Lorentz Transformation Equations

The Galilean transformation nevertheless violates Einstein's postulates, because the velocity equations state that a pulse of light moving with speed c along the **x**-axis would travel at speed $c-v$ in the other inertial frame. Specifically, the spherical pulse has radius $r=ct$ at time t in the unprimed frame, and also has radius $r'=ct'$ at time t' in the primed frame. Expressing these relations in Cartesian coordinates gives

$$x^2+y^2+z^2-c^2t^2 =0 \tag{14.6.5}$$

$$x'^2+y'^2+z'^2-c^2t'^2 =0. \tag{14.6.6}$$

The left-hand sides Equations 14.6.5 and 14.6.6 can be set equal because both are zero. Because $y=y'$ and $z=z'$, we obtain

$$x^2-c^2t^2=x'^2-c^2t'^2.$$

This cannot be satisfied for nonzero relative velocity v of the two frames if we assume the Galilean transformation results in $t=t'$ with $x=x'+vt'$.

To find the correct set of transformation equations, assume the two coordinate systems S and S' in Figure 14.6.1. First suppose that an event occurs at $(x', 0, 0, t')$ in S' and at $(x, 0, 0, t)$ in S, as depicted in Figure 14.6.1.

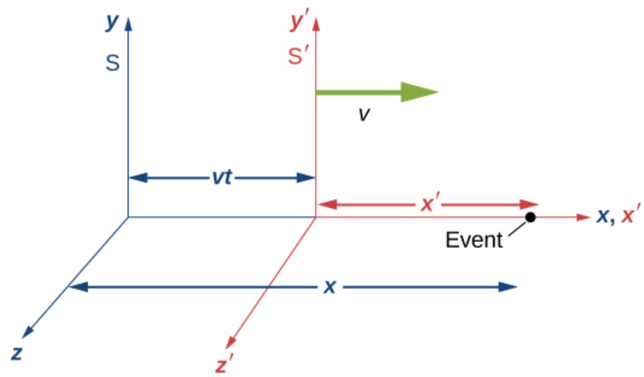


Figure 14.6.1: An event occurs at $(x,0,0,t)$ in S and at $(x',0,0,t')$ in S'. The Lorentz transformation equations relate events in the two systems.

Suppose that at the instant that the origins of the coordinate systems in **S** and S' coincide, a flash bulb emits a spherically spreading pulse of light starting from the origin. At time **t**, an observer in **S** finds the origin of S' to be at $x=vt$. With the help of a friend in **S**, the S' observer also measures the distance from the event to the origin of S' and finds it to be $x'\sqrt{1-v^2/c^2}$. This follows because we have already shown the postulates of relativity to imply length contraction. Thus the position of the event in **S** is

$$x=vt+x'\sqrt{1-v^2/c^2}$$

and

$$x'=\frac{x-vt}{\sqrt{1-v^2/c^2}}. \tag{14.6.7}$$

The postulates of relativity imply that the equation relating distance and time of the spherical wave front:

$$x^2+y^2+z^2-c^2t^2=0$$

must apply both in terms of primed and unprimed coordinates, which was shown above to lead to Equation:

$$x^2-c^2t^2=x'^2-c^2t'^2.$$

We combine this with Equation 14.6.7 that relates x and x' to obtain the relation between t and t':

$$t'=\frac{t-vx/c^2}{\sqrt{1-v^2/c^2}}.$$

The equations relating the time and position of the events as seen in S are then

$$t = \frac{t' + vx'/c^2}{\sqrt{1 - v^2/c^2}}.$$

$$x = \frac{x' + vt'}{\sqrt{1 - v^2/c^2}}.$$

$$y = y'$$

$$z = z'.$$

This set of equations, relating the position and time in the two inertial frames, is known as the *Lorentz transformation*. They are named in honor of H.A. Lorentz (1853–1928), who first proposed them. Interestingly, he justified the transformation on what was eventually discovered to be a fallacious hypothesis. The correct theoretical basis is Einstein's special theory of relativity.

The reverse transformation expresses the variables in S in terms of those in S'. Simply interchanging the primed and unprimed variables and substituting gives:

$$t' = \frac{t - vx/c^2}{\sqrt{1 - v^2/c^2}}$$

$$x' = \frac{x - vt}{\sqrt{1 - v^2/c^2}}$$

$$y' = y$$

$$z' = z.$$

Example 14.6.1: Using Lorentz Transformation for Time

Spacecraft S' is on its way to Alpha Centauri when Spacecraft **S** passes it at relative speed $c/2$. The captain of S' sends a radio signal that lasts 1.2 s according to that ship's clock. Use the Lorentz transformation to find the time interval of the signal measured by the communications officer of spaceship **S**.

Solution

1. Identify the known: $\Delta t' = t'_2 - t'_1 = 1.2s$; $\Delta x' = x'_2 - x'_1 = 0$.
2. Identify the unknown: $\Delta t = t_2 - t_1$.
3. Express the answer as an equation. The time signal starts as (x', t'_1) and stops at (x', t'_1). Note that the x' coordinate of both events is the same because the clock is at rest in S'. Write the first Lorentz transformation equation in terms of $\Delta t = t_2 - t_1$, $\Delta x = x_2 - x_1$, and similarly for the primed coordinates, as:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

Because the position of the clock in S' is fixed, $\Delta x' = 0$, and the time interval Δt becomes:

$$\Delta t = \frac{\Delta t'}{\sqrt{1 - \frac{v^2}{c^2}}}.$$

4. Do the calculation.
With $\Delta t' = 1.2\ s$ this gives:

$$\Delta t = \frac{1.2\ s}{\sqrt{1 - \left(\frac{1}{2}\right)^2}}$$
$$= 1.4\ s.$$

Note that the Lorentz transformation reproduces the time dilation equation.

Example 14.6.2: Using the Lorentz Transformation for Length

A surveyor measures a street to be $L = 100\,m$ long in Earth frame S. Use the Lorentz transformation to obtain an expression for its length measured from a spaceship S', moving by at speed $0.20c$, assuming the x coordinates of the two frames coincide at time $t = 0$.

Solution

1. Identify the known: $L = 100\,m$; $v = 0.20c$; $\Delta\tau = 0$.
2. Identify the unknown: L'.
3. Express the answer as an equation. The surveyor in frame S has measured the two ends of the stick simultaneously, and found them at rest at x_2 and x_1 a distance $L = x_2 - x_1 = 100\,m$ apart. The spaceship crew measures the simultaneous location of the ends of the sticks in their frame. To relate the lengths recorded by observers in S' and S, respectively, write the second of the four Lorentz transformation equations as:

$$\begin{aligned} x_2 - x_1 &= \frac{x'_2 - vt}{\sqrt{1 - v^2/c^2}} - \frac{x'_1 - vt}{\sqrt{1 - v^2/c^2}} \\ &= \frac{x'_2 - x'_1}{\sqrt{1 - v^2/c^2}}. \end{aligned}$$

4. Do the calculation. Because $x_2 - x_1 = 100\,m$, the length of the moving stick is equal to:

$$\begin{aligned} L' &= x'_2 - x'_1 \\ &= (L)\sqrt{1 - v^2/c^2} \\ &= (100\,m)\sqrt{1 - (0.20)^2} \\ &= 98.0\,m. \end{aligned}$$

Note that the Lorentz transformation gave the length contraction equation for the street.

Example 14.6.3: Lorentz Transformation and Simultaneity

The observer shown in Figure $14.6.2$ standing by the railroad tracks sees the two bulbs flash simultaneously at both ends of the 26 m long passenger car when the middle of the car passes him at a speed of $c/2$. Find the separation in time between when the bulbs flashed as seen by the train passenger seated in the middle of the car.

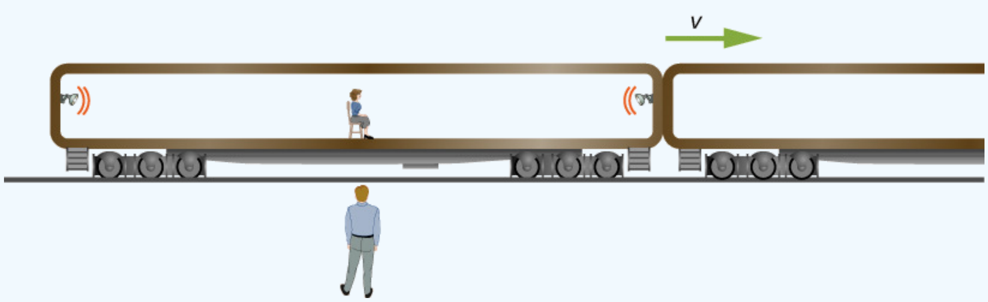


Figure 14.6.2: An person watching a train go by observes two bulbs flash simultaneously at opposite ends of a passenger car. There is another passenger inside of the car observing the same flashes but from a different perspective.

Solution

1. Identify the known: $\Delta t = 0$.

   Note that the spatial separation of the two events is between the two lamps, not the distance of the lamp to the passenger.

2. Identify the unknown: $\Delta t' = t'_2 - t'_1$. .

   Again, note that the time interval is between the flashes of the lamps, not between arrival times for reaching the passenger.

3. Express the answer as an equation:

$$\Delta t = \frac{\Delta t' + v\Delta x'/c^2}{\sqrt{1 - v^2/c^2}}.$$

4. Do the calculation:

$$0 = \frac{\Delta t' + \frac{c}{2}(26\,m)/c^2}{\sqrt{1 - v^2/c^2}}$$

$$\begin{aligned} \Delta t' &= -\frac{26\,m/s}{2c} = -\frac{26\,m/s}{2(3.00 \times 10^8\,m/s)} \\ &= -4.33 \times 10^{-8}\,s. \end{aligned}$$

Significance

The sign indicates that the event with the larger x'_2 namely, the flash from the right, is seen to occur first in the S' frame, as found earlier for this example, so that $t_2 < t_1$.

Space-Time

Relativistic phenomena can be analyzed in terms of events in a four-dimensional **space-time**. When phenomena such as the twin paradox, time dilation, length contraction, and the dependence of simultaneity on relative motion are viewed in this way, they are seen to be characteristic of the nature of space and time, rather than specific aspects of electromagnetism.

In three-dimensional space, positions are specified by three coordinates on a set of Cartesian axes, and the displacement of one point from another is given by:

$$(\Delta x,\, \Delta y,\, \Delta z) = (x_2 - x_1,\, y_2 - y - 1,\, z_2 - z_1).$$

The distance Δr between the points is

$$\Delta r^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2.$$

The distance Δr is invariant under a rotation of axes. If a new set of Cartesian axes rotated around the origin relative to the original axes are used, each point in space will have new coordinates in terms of the new axes, but the distance $\Delta r'$ given by

$$\Delta r'^2 = (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2.$$

That has the same value that Δr^2 had. Something similar happens with the Lorentz transformation in space-time.

Define the separation between two events, each given by a set of **x**, **y**, **z**¸ and **ct** along a four-dimensional Cartesian system of axes in space-time, as

$$(\Delta x,\, \Delta y,\, \Delta z,\, c\Delta t) = (x_2 - x_1,\, y_2 - y_1,\, z_2 - z_1,\, c(t_2 - t_1)).$$

Also define the space-time interval Δs between the two events as

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

If the two events have the same value of **ct** in the frame of reference considered, Δs would correspond to the distance Δr between points in space.

The path of a particle through space-time consists of the events (**x**, **y**, **z**¸ **ct**) specifying a location at each time of its motion. The path through space-time is called the **world line** of the particle. The world line of a particle that remains at rest at the same location is a straight line that is parallel to the time axis. If the particle moves at constant velocity parallel to the **x**-axis, its world line would be a sloped line $x = vt$, corresponding to a simple displacement vs. time graph. If the particle accelerates, its world line is curved. The increment of **s** along the world line of the particle is given in differential form as

$$ds^2 = (dx)^2 + (dy)^2 + (dz)^2 - c^2(dt)^2.$$

Just as the distance Δr is invariant under rotation of the space axes, the space-time interval:

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

is invariant under the Lorentz transformation. This follows from the postulates of relativity, and can be seen also by substitution of the previous Lorentz transformation equations into the expression for the space-time interval:

$$\begin{aligned}\Delta s^2 &= (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2. \\ &= \left(\frac{\Delta x' + v\Delta t'}{\sqrt{1-v^2/c^2}}\right)^2 + (\Delta y')^2 + (\Delta z')^2 - \left(c\frac{\Delta t' + \dfrac{v\Delta x'}{c^2}}{\sqrt{1-v^2/c^2}}\right)^2 \\ &= (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 - (c\Delta t')^2 \\ &= \Delta s'^2.\end{aligned}$$

In addition, the Lorentz transformation changes the coordinates of an event in time and space similarly to how a three-dimensional rotation changes old coordinates into new coordinates:

Lorentz transformation (x, t coordinates):	Axis–rotation around z - a axis (x, t coordinates):
$x' = (\gamma)x + (-\beta\gamma)ct$	$x' = (\cos\theta)x + (\sin\theta)y$
$ct' = (-\beta\gamma)x + (\gamma)ct$	$y' = (-\sin\theta)x + (\cos\theta)y$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$ and $\beta = v/c$.

Lorentz transformations can be regarded as generalizations of spatial rotations to space-time. However, there are some differences between a three-dimensional axis rotation and a Lorentz transformation involving the time axis, because of differences in how the metric, or rule for measuring the displacements Δr and Δs, differ. Although Δr is invariant under spatial rotations and Δs is invariant also under Lorentz transformation, the Lorentz transformation involving the time axis does not preserve some features, such as the axes remaining perpendicular or the length scale along each axis remaining the same.

Note that the quantity Δs^2 can have either sign, depending on the coordinates of the space-time events involved. For pairs of events that give it a negative sign, it is useful to define $c^2\Delta\tau^2$ as $-\Delta s^2$. The significance of $c^2\Delta\tau$ as just defined follows by noting that in a frame of reference where the two events occur at the same location, we have $\Delta x = \Delta y = \Delta z = 0$ and therefore (from the equation for $\Delta s^2 = -c^2\Delta\tau^2$):

$$c^2\Delta\tau^2 = -\Delta s^2 = (c^2\Delta t)^2.$$

Therefore $c^2\Delta\tau$ is the time interval $c^2\Delta t$ in the frame of reference where both events occur at the same location. It is the same interval of proper time discussed earlier. It also follows from the relation between Δs and that $c^2\Delta\tau$ that because Δs is Lorentz invariant, the proper time is also Lorentz invariant. All observers in all inertial frames agree on the proper time intervals between the same two events.

? Exercise 14.6.1

Show that if a time increment dt elapses for an observer who sees the particle moving with velocity v, it corresponds to a proper time particle increment for the particle of $d\tau = \gamma dt$.

Answer

Start with the definition of the proper time increment:

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - (dx^2 + dx^2 + dx^2)/c^2}.$$

where (dx, dy, dx, cdt) are measured in the inertial frame of an observer who does not necessarily see that particle at rest. This therefore becomes

$$d\tau = \sqrt{-(ds)^2/c^2} = \sqrt{dt^2 - [(dx)^2 + (dy)^2 + (dz)^2]/c^2}$$

$$dt\sqrt{1-\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2+\left(\frac{dz}{dt}\right)^2\right]/c^2}$$

$$dt\sqrt{1-v^2/c^2}$$

$$dt=\gamma d\tau.$$

The Light Cone

We can deal with the difficulty of visualizing and sketching graphs in four dimensions by imagining the three spatial coordinates to be represented collectively by a horizontal axis, and the vertical axis to be the **ct**-axis. Starting with a particular event in space-time as the origin of the space-time graph shown, the world line of a particle that remains at rest at the initial location of the event at the origin then is the time axis. Any plane through the time axis parallel to the spatial axes contains all the events that are simultaneous with each other and with the intersection of the plane and the time axis, as seen in the rest frame of the event at the origin.

It is useful to picture a **light cone** on the graph, formed by the world lines of all light beams passing through the origin event **A**, as shown in Figure $14.6.3$. The light cone, according to the postulates of relativity, has sides at an angle of 45° if the time axis is measured in units of **ct**, and, according to the postulates of relativity, the light cone remains the same in all inertial frames. Because the event **A** is arbitrary, every point in the space-time diagram has a light cone associated with it.

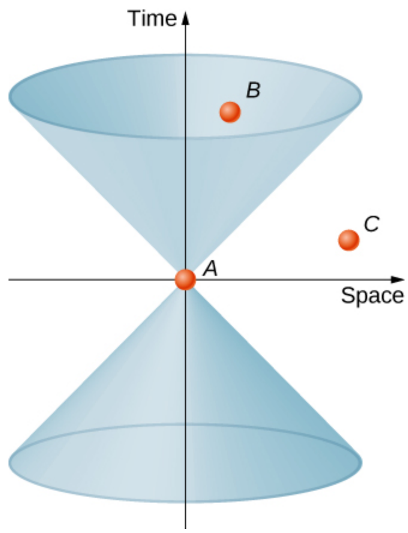


Figure 14.6.3: The light cone consists of all the world lines followed by light from the event A at the vertex of the cone.

Consider now the world line of a particle through space-time. Any world line outside of the cone, such as one passing from **A** through **C**, would involve speeds greater than **c**, and would therefore not be possible. Events such as **C** that lie outside the light cone are said to have a space-like separation from event **A**. They are characterized by:

$$\Delta s^2_{AC}=(x_A-x_C)^2+(y_A-y_C)^2+(z_A-z_C)^2-(c\Delta t)^2>0.$$

An event like **B** that lies in the upper cone is reachable without exceeding the speed of light in vacuum, and is characterized by

$$\Delta s^2_{AB}=(x_A-x_B)^2+(y_A-y_B)^2+(z_A-z_B)^2-(c\Delta t)^2<0.$$

The event is said to have a time-like separation from **A**. Time-like events that fall into the upper half of the light cone occur at greater values of **t** than the time of the event **A** at the vertex and are in the future relative to **A**. Events that have time-like separation from A and fall in the lower half of the light cone are in the past, and can affect the event at the origin. The region outside the light cone is labeled as neither past nor future, but rather as “elsewhere.”

For any event that has a space-like separation from the event at the origin, it is possible to choose a time axis that will make the two events occur at the same time, so that the two events are simultaneous in some frame of reference. Therefore, which of the events with space-like separation comes before the other in time also depends on the frame of reference of the observer. Since space-like separations can be traversed only by exceeding the speed of light; this violation of which event can cause the other provides another argument for why particles cannot travel faster than the speed of light, as well as potential material for science fiction about time travel. Similarly for any event with time-like separation from the event at the origin, a frame of reference can be found that will make the events occur at the same location. Because the relations

$$\Delta s^2_{AC} = (x_A - x_C)^2 + (y_A - x_C)^2 + (z_A - z_C)^2 - (c\Delta t)^2 > 0.$$

and

$$\Delta s^2_{AB} = (x_A - x_B)^2 + (y_A - y_B)^2 + (z_A - z_B)^2 - (c\Delta t)^2 < 0.$$

are Lorentz invariant, whether two events are time-like and can be made to occur at the same place or space-like and can be made to occur at the same time is the same for all observers. All observers in different inertial frames of reference agree on whether two events have a time-like or space-like separation.

The twin paradox seen in space-time

The **twin paradox** discussed earlier involves an astronaut twin traveling at near light speed to a distant star system, and returning to Earth. Because of time dilation, the space twin is predicted to age much less than the earthbound twin. This seems paradoxical because we might have expected at first glance for the relative motion to be symmetrical and naively thought it possible to also argue that the earthbound twin should age less.

To analyze this in terms of a space-time diagram, assume that the origin of the axes used is fixed in Earth. The world line of the earthbound twin is then along the time axis.

The world line of the astronaut twin, who travels to the distant star and then returns, must deviate from a straight line path in order to allow a return trip. As seen in Figure 14.6.4, the circumstances of the two twins are not at all symmetrical. Their paths in space-time are of manifestly different length. Specifically, the world line of the earthbound twin has length $2c\Delta t$, which then gives the proper time that elapses for the earthbound twin as $2\Delta t$. The distance to the distant star system is $\Delta x = v\Delta t$. The proper time that elapses for the space twin is $2\Delta\tau$ where

$$c^2\Delta\tau^2 = -\Delta s^2 = (c\Delta t)^2 - (\Delta x)^2.$$

This is considerably shorter than the proper time for the earthbound twin by the ratio

$$\frac{c\Delta\tau}{c\Delta t} = \sqrt{\frac{(c\Delta t)^2 - (\Delta x)^2}{(c\Delta t)^2}} = \sqrt{\frac{(c\Delta t)^2 - (v\Delta t)^2}{(c\Delta t)^2}} = \sqrt{1 - \frac{v^2}{c^2}} = \frac{1}{\gamma}.$$

consistent with the time dilation formula. The twin paradox is therefore seen to be no paradox at all. The situation of the two twins is not symmetrical in the space-time diagram. The only surprise is perhaps that the seemingly longer path on the space-time diagram corresponds to the smaller proper time interval, because of how $\Delta\tau$ and Δs depend on Δx and Δt.

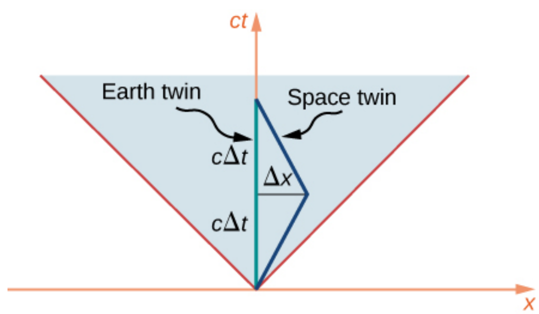


Figure 14.6.4. The space twin and the earthbound twin, in the twin paradox example, follow world lines of different length through space-time.

Lorentz Transformations in Space-time

We have already noted how the Lorentz transformation leaves

$$\Delta s^2 = (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 - (c\Delta t)^2.$$

unchanged and corresponds to a rotation of axes in the four-dimensional space-time. If the S and S' frames are in relative motion along their shared **x**-direction the space and time axes of S' are rotated by an angle αα as seen from S, in the way shown in shown in Figure 14.6.5, where:

$$tan\,\alpha = \frac{v}{c} = \beta.$$

This differs from a rotation in the usual three-dimension sense, insofar as the two space-time axes rotate toward each other symmetrically in a scissors-like way, as shown. The rotation of the time and space axes are both through the same angle. The mesh of dashed lines parallel to the two axes show how coordinates of an event would be read along the primed axes. This would be done by following a line parallel to the x' and one parallel to the t'-axis, as shown by the dashed lines. The length scale of both axes are changed by:

$$ct' = ct\sqrt{\frac{1+\beta^2}{1-\beta^2}};\; x' = x\sqrt{\frac{1+\beta^2}{1-\beta^2}}.$$

The line labeled "v = c" at 45° to the **x**-axis corresponds to the edge of the light cone, and is unaffected by the Lorentz transformation, in accordance with the second postulate of relativity. The "v = c" line, and the light cone it represents, are the same for both the **S** and S' frame of reference.

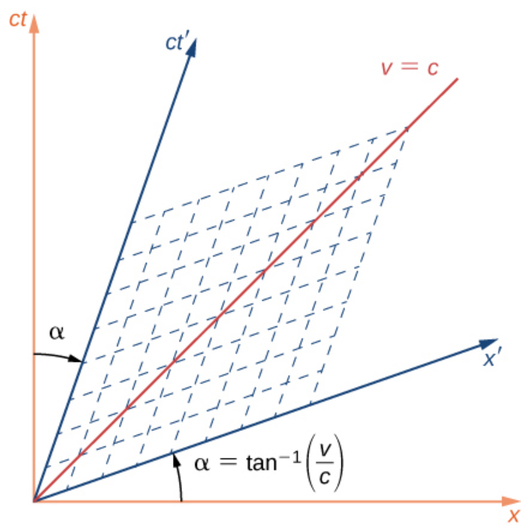


Figure 14.6.5: The Lorentz transformation results in new space and time axes rotated in a scissors-like way with respect to the original axes.

Simultaneity

Simultaneity of events at separated locations depends on the frame of reference used to describe them, as given by the scissors-like "rotation" to new time and space coordinates as described. If two events have the same t values in the unprimed frame of reference, they need not have the same values measured along the ct'-axis, and would then **not** be simultaneous in the primed frame.

As a specific example, consider the near-light-speed train in which flash lamps at the two ends of the car have flashed simultaneously in the frame of reference of an observer on the ground. The space-time graph is shown Figure 14.6.6. The flashes of the two lamps are represented by the dots labeled "Left flash lamp" and "Right flash lamp" that lie on the light cone in the past. The world line of both pulses travel along the edge of the light cone to arrive at the observer on the ground simultaneously. Their arrival is the event at the origin. They therefore had to be emitted simultaneously in the unprimed frame, as represented by the point labeled as t (both). But time is measured along the ct'-axis in the frame of reference of the observer seated in the middle of the train car. So in her frame of reference, the emission event of the bulbs labeled as t' (left) and t' (right) were not simultaneous.

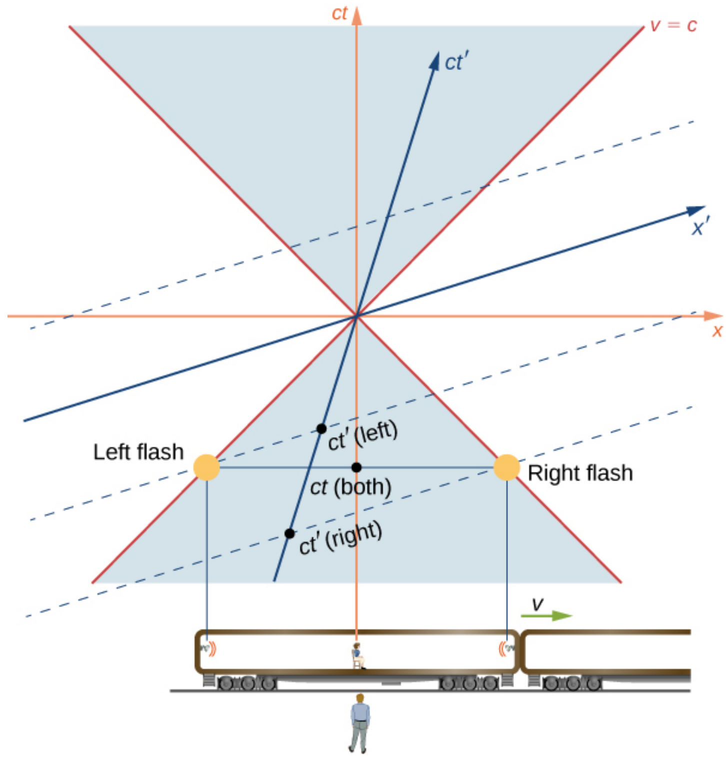


Figure 14.6.6: The train example revisited. The flashes occur at the same time t (both) along the time axis of the ground observer, but at different times, along the t't′ time axis of the passenger.

In terms of the space-time diagram, the two observers are merely using different time axes for the same events because they are in different inertial frames, and the conclusions of both observers are equally valid. As the analysis in terms of the space-time diagrams further suggests, the property of how simultaneity of events depends on the frame of reference results from the properties of space and time itself, rather than from anything specifically about electromagnetism.

14.7: Relativistic Velocity Transformation

Learning Objectives

By the end of this section, you will be able to:

- Derive the equations consistent with special relativity for transforming velocities in one inertial frame of reference into another.
- Apply the velocity transformation equations to objects moving at relativistic speeds.
- Examine how the combined velocities predicted by the relativistic transformation equations compare with those expected classically.

Remaining in place in a kayak in a fast-moving river takes effort. The river current pulls the kayak along. Trying to paddle against the flow can move the kayak upstream relative to the water, but that only accounts for part of its velocity relative to the shore. The kayak's motion is an example of how velocities in Newtonian mechanics combine by vector addition. The kayak's velocity is the vector sum of its velocity relative to the water and the water's velocity relative to the riverbank. However, the relativistic addition of velocities is quite different.

Velocity Transformations

Imagine a car traveling at night along a straight road, as in Figure $14.7.1$. The driver sees the light leaving the headlights at speed c within the car's frame of reference. If the Galilean transformation applied to light, then the light from the car's headlights would approach the pedestrian at a speed $u = v + c$, contrary to Einstein's postulates.

Figure 14.7.1: According to experimental results and the second postulate of relativity, light from the car's headlights moves away from the car at speed c and toward the observer on the sidewalk at speed c.

Both the distance traveled and the time of travel are different in the two frames of reference, and they must differ in a way that makes the speed of light the same in all inertial frames. The correct rules for transforming velocities from one frame to another can be obtained from the Lorentz transformation equations.

Relativistic Transformation of Velocity

Suppose an object **P** is moving at constant velocity $u = (u'_x, u'_y, u'_z)$ as measured in the S' frame. The S' frame is moving along its x'-axis at velocity v. In an increment of time dt', the particle is displaced by dx' along the x'-axis. Applying the Lorentz transformation equations gives the corresponding increments of time and displacement in the unprimed axes:

$$dt = \gamma(dt' + vdx'/c^2) \tag{14.7.1}$$

$$dx = \gamma(dx' + vdt') \tag{14.7.2}$$

$$dy = dy' \tag{14.7.3}$$

$$dz = dz'. \tag{14.7.4}$$

The velocity components of the particle seen in the unprimed coordinate system are then

$$\frac{dx}{dt} = \frac{\gamma(dx' + vdt')}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dx'}{dt'} + v}{1 + \frac{v}{c^2}\frac{dx'}{dt'}} \tag{14.7.5}$$

$$\frac{dy}{dt} = \frac{dy'}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dy'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \tag{14.7.6}$$

$$\frac{dz}{dt} = \frac{dz'}{\gamma(dt' + v\,dx'/c^2)} = \frac{\frac{dz'}{dt'}}{\gamma\left(1 + \frac{v}{c^2}\frac{dx'}{dt'}\right)} \tag{14.7.7}$$

We thus obtain the equations for the velocity components of the object as seen in frame S:

$$u_x = \left(\frac{u'_x + v}{1 + vu'_x/c^2}\right),\; u_y = \left(\frac{u'_y/\gamma}{1 + vu'_x/c^2}\right),\; u_z = \left(\frac{u'_z/\gamma}{1 + vu'_x/c^2}\right).$$

Compare this with how the Galilean transformation of classical mechanics says the velocities transform, by adding simply as vectors:

$$u_x = u'_x + u,\; u_y = u'_y,\; u_z = u'_z.$$

When the relative velocity of the frames is much smaller than the speed of light, that is, when $v \gg c,$ the special relativity velocity addition law reduces to the Galilean velocity law. When the speed v of S' relative to S is comparable to the speed of light, the relativistic velocity addition law gives a much smaller result than the classical (Galilean) velocity addition does.

Example 14.7.1: Velocity Transformation Equations for Light

Suppose a spaceship heading directly toward Earth at half the speed of light sends a signal to us on a laser-produced beam of light (Figure 14.7.2). Given that the light leaves the ship at speed c as observed from the ship, calculate the speed at which it approaches Earth.

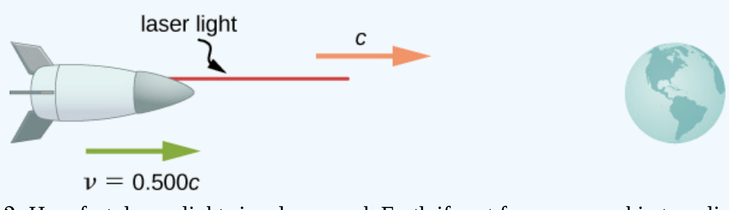


Figure 14.7.2: How fast does a light signal approach Earth if sent from a spaceship traveling at $0.500c$?

Strategy

Because the light and the spaceship are moving at relativistic speeds, we cannot use simple velocity addition. Instead, we determine the speed at which the light approaches Earth using relativistic velocity addition.

Solution

Identify the knowns: $v = 0.500c$; $u' = c$.

Identify the unknown: u.

Express the answer as an equation: $u = \dfrac{v+u'}{1+\dfrac{vu'}{c^2}}$.

Do the calculation:

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+c}{1+\frac{(0.500c)(c)}{c^2}} \\ &= \frac{(0.500+1)c}{\left(\frac{c^2+0.500c^2}{c^2}\right)} = c. \end{aligned}$$

Significance

Relativistic velocity addition gives the correct result. Light leaves the ship at speed c and approaches Earth at speed c. The speed of light is independent of the relative motion of source and observer, whether the observer is on the ship or earthbound.

Velocities cannot add to greater than the speed of light, provided that v is less than c and u' does not exceed c. The following example illustrates that relativistic velocity addition is not as symmetric as classical velocity addition.

Example 14.7.2: Relativistic Package Delivery

Suppose the spaceship in the previous example approaches Earth at half the speed of light and shoots a canister at a speed of $0.750c$(Figure $14.7.3$).

a. At what velocity does an earthbound observer see the canister if it is shot directly toward Earth?
b. If it is shot directly away from Earth?

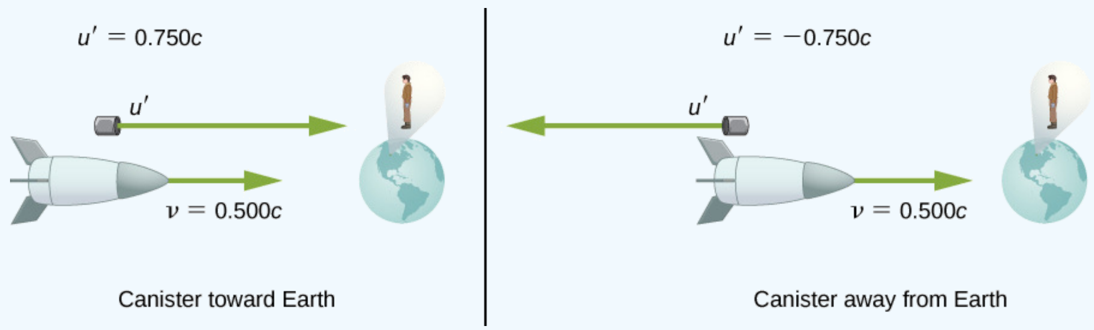


Figure 14.7.3: A canister is fired at 0.7500c toward Earth or away from Earth.

Strategy

Because the canister and the spaceship are moving at relativistic speeds, we must determine the speed of the canister by an earthbound observer using relativistic velocity addition instead of simple velocity addition.

Solution for (a)

1. Identify the knowns: $v=0.500c$; $u'=0.750c$.
2. Identify the unknown: u.
3. Express the answer as an equation: $u = \dfrac{v+u'}{1+\dfrac{vu'}{c^2}}$.
4. Do the calculation:

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+0.750c}{1+\frac{(0.500c)(0.750c)}{c^2}} \\ &= 0.909c. \end{aligned}$$

Solution for (b)

1. Identify the knowns: $v = 0.500c$; $u' = -0.750c$.
2. Identify the unknown: u.
3. Express the answer as an equation: $u = \frac{v+u'}{1+\frac{vu'}{c^2}}$.
4. Do the calculation:

$$\begin{aligned} u &= \frac{v+u'}{1+\frac{vu'}{c^2}} \\ &= \frac{0.500c+(-0.750c)}{1+\frac{(0.500c)(-0.750c)}{c^2}} \\ &= -0.400c. \end{aligned}$$

Significance

The minus sign indicates a velocity away from Earth (in the opposite direction from v), which means the canister is heading toward Earth in part (a) and away in part (b), as expected. But relativistic velocities do not add as simply as they do classically. In part (a), the canister does approach Earth faster, but at less than the vector sum of the velocities, which would give $1.250c$. In part (b), the canister moves away from Earth at a velocity of −0.400c, which is faster than the −0.250c expected classically. The differences in velocities are not even symmetric: In part (a), an observer on Earth sees the canister and the ship moving apart at a speed of 0.409**c**, and at a speed of 0.900**c** in part (b).

Exercise 14.7.1

Distances along a direction perpendicular to the relative motion of the two frames are the same in both frames. Why then are velocities perpendicular to the **x**-direction different in the two frames?

Answer

Although displacements perpendicular to the relative motion are the same in both frames of reference, the time interval between events differ, and differences in dt and dt' lead to different velocities seen from the two frames.

14.8: Doppler Effect for Light

Learning Objectives

By the end of this section, you will be able to:

- Explain the origin of the shift in frequency and wavelength of the observed wavelength when observer and source moved toward or away from each other
- Derive an expression for the relativistic Doppler shift
- Apply the Doppler shift equations to real-world examples

As discussed in the chapter on sound, if a source of sound and a listener are moving farther apart, the listener encounters fewer cycles of a wave in each second, and therefore lower frequency, than if their separation remains constant. For the same reason, the listener detects a higher frequency if the source and listener are getting closer. The resulting Doppler shift in detected frequency occurs for any form of wave. For sound waves, however, the equations for the Doppler shift differ markedly depending on whether it is the source, the observer, or the air, which is moving. Light requires no medium, and the Doppler shift for light traveling in vacuum depends only on the relative speed of the observer and source.

The Relativistic Doppler Effect

Suppose an observer in S sees light from a source in S' moving away at velocity v (Figure $14.8.1$). The wavelength of the light could be measured within S' — for example, by using a mirror to set up standing waves and measuring the distance between nodes. These distances are proper lengths with S' as their rest frame, and change by a factor $\sqrt{1-v^2/c^2}$ when measured in the observer's frame S, where the ruler measuring the wavelength in S' is seen as moving.

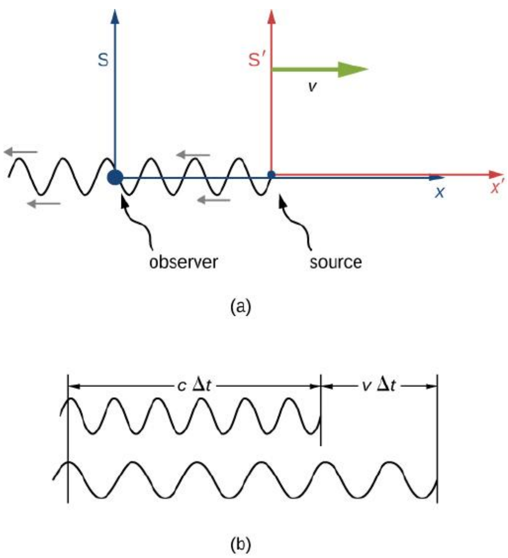


Figure 14.8.1: (a) When a light wave is emitted by a source fixed in the moving inertial frame S', the observer in S sees the wavelength measured in S'. to be shorter by a factor $\sqrt{1-v^2/c^2}$. (b) Because the observer sees the source moving away within S, the wave pattern reaching the observer in S is also stretched by the factor $(c\Delta t+v\Delta t)/(c\Delta t)=1+v/c$.

If the source were stationary in **S**, the observer would see a length cΔt of the wave pattern in time Δt. But because of the motion of S' relative to **S**, considered solely within **S**, the observer sees the wave pattern, and therefore the wavelength, stretched out by a factor of

$$\frac{c\Delta t_{period}+v\Delta t_{period}}{c\Delta t_{period}}=1+\frac{v}{c}$$

as illustrated in (b) of Figure $14.8.1$. The overall increase from both effects gives

$$\begin{aligned}\lambda_{obs} &= \lambda_{src}\left(1+\frac{v}{c}\right)\sqrt{\frac{1}{1-\frac{v^2}{c^2}}}\\ &= \lambda_{src}\left(1+\frac{v}{c}\right)\sqrt{\frac{1}{\left(1+\frac{v}{c}\right)\left(1-\frac{v}{c}\right)}}\\ &= \lambda_{src}\sqrt{\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}}\end{aligned}$$

where λ_{src} is the wavelength of the light seen by the source in S' and λ_{obs} is the wavelength that the observer detects within **S**.

Red Shifts and Blue Shifts

The observed wavelength λ_{obs} of electromagnetic radiation is longer (called a "red shift") than that emitted by the source when the source moves away from the observer. Similarly, the wavelength is shorter (called a "blue shift") when the source moves toward the observer. The amount of change is determined by

$$\lambda_{obs} = \lambda_s\sqrt{\frac{\left(1+\frac{v}{c}\right)}{\left(1-\frac{v}{c}\right)}}$$

where λ_s is the wavelength in the frame of reference of the source, and v is the relative velocity of the two frames S and S'. The velocity v is positive for motion away from an observer and negative for motion toward an observer. In terms of source frequency and observed frequency, this equation can be written as

$$f_{obs} = f_s\sqrt{\frac{\left(1-\frac{v}{c}\right)}{\left(1+\frac{v}{c}\right)}} \tag{14.8.1}$$

Notice that the signs are different from those of the wavelength equation.

Example 14.8.1: Calculating a Doppler Shift

Suppose a galaxy is moving away from Earth at a speed 0.825**c**. It emits radio waves with a wavelength of

0.525 m. What wavelength would we detect on Earth?

Strategy

Because the galaxy is moving at a relativistic speed, we must determine the Doppler shift of the radio waves using the relativistic Doppler shift instead of the classical Doppler shift.

Solution

1. Identify the knowns: $u = 0.825c$; $\lambda_s = 0.525\,m$.
2. Identify the unknown: λ_{obs}.
3. Express the answer as an equation:

$$\lambda_{obs} = \lambda_s\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}.$$

4. Do the calculation:

$$\begin{aligned}\lambda_{obs} &= \lambda_s\sqrt{\frac{1+\frac{v}{c}}{1-\frac{v}{c}}}\\ &= (0.525\,m)\sqrt{\frac{1+\frac{0.825c}{c}}{1-\frac{0.825c}{c}}}\\ &= 1.70\,m.\end{aligned}$$

Significance

Because the galaxy is moving away from Earth, we expect the wavelengths of radiation it emits to be redshifted. The wavelength we calculated is 1.70 m, which is redshifted from the original wavelength of 0.525 m. You will see in Particle Physics and Cosmology that detecting redshifted radiation led to present-day understanding of the origin and evolution of the universe.

Exercise 14.8.1

Suppose a space probe moves away from Earth at a speed 0.350**c**. It sends a radio-wave message back to Earth at a frequency of 1.50 GHz. At what frequency is the message received on Earth?

Solution

We can substitute the data directly into the equation for relativistic Doppler frequency (Equation 14.8.1):

$$\begin{aligned} f_{obs} &= f_s\sqrt{\frac{1-\frac{v}{c}}{1+\frac{v}{c}}} \\ &= (1.50\,GHz)\sqrt{\frac{1-\frac{0.350c}{c}}{1+\frac{0.350c}{c}}} \\ &= 1.04\,GHz. \end{aligned}$$

The relativistic Doppler effect has applications ranging from Doppler radar storm monitoring to providing information on the motion and distance of stars. We describe some of these applications in the exercises.

14.9: Relativistic Momentum

> **Learning Objectives**
>
> By the end of this section, you will be able to:
>
> - Define relativistic momentum in terms of mass and velocity
> - Show how relativistic momentum relates to classical momentum
> - Show how conservation of relativistic momentum limits objects with mass to speeds less than c

Momentum is a central concept in physics. The broadest form of Newton's second law is stated in terms of momentum. Momentum is conserved whenever the net external force on a system is zero. This makes momentum conservation a fundamental tool for analyzing collisions (Figure 14.9.1). Much of what we know about subatomic structure comes from the analysis of collisions of accelerator-produced relativistic particles, and momentum conservation plays a crucial role in this analysis.

Figure 14.9.1: Momentum is an important concept for these football players from the University of California at Berkeley and the University of California at Davis. A player with the same velocity but greater mass collides with greater impact because his momentum is greater. For objects moving at relativistic speeds, the effect is even greater.

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Does the law of conservation of momentum survive this requirement at high velocities? It can be shown that the momentum calculated as merely $\vec{p} = m\frac{d\vec{x}}{dt}$, even if it is conserved in one frame of reference, may not be conserved in another after applying the Lorentz transformation to the velocities. The correct equation for momentum can be shown, instead, to be the classical expression in terms of the increment dτ of proper time of the particle, observed in the particle's rest frame:

$$\begin{aligned} \vec{p} &= m\frac{d\vec{x}}{d\tau} = m\frac{d\vec{x}}{dt}\frac{dt}{d\tau} \\ &= m\frac{d\vec{x}}{dt}\frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{m\vec{u}}{\sqrt{1-u^2/c^2}} \\ &= \gamma m\vec{u}. \end{aligned}$$

Definition: Relativistic Momentum and Rest Mass

Relativistic momentum $\vec{p}$ is classical momentum multiplied by the relativistic factor γ:

$$\vec{p} = \gamma m \vec{u} \tag{14.9.1}$$

where m is the **rest mass** of the object, $\vec{u}$ is its velocity relative to an observer, and γ is the **relativistic factor**:

$$\gamma = \frac{1}{\sqrt{1 - \frac{u^2}{c^2}}}. \tag{14.9.2}$$

Note that we use u for velocity here to distinguish it from relative velocity v between observers. The factor γ that occurs here has the same form as the previous relativistic factor γ except that it is now in terms of the velocity of the particle u instead of the relative velocity **v** of two frames of reference.

With **p** expressed in this way, total momentump_{tot} is conserved whenever the net external force is zero, just as in classical physics. Again we see that the relativistic quantity becomes virtually the same as the classical quantity at low velocities, where u/c is small and γ is very nearly equal to 1. Relativistic momentum has the same intuitive role as classical momentum. It is greatest for large masses moving at high velocities, but because of the factor γ, relativistic momentum approaches infinity as u approaches c (Figure $14.9.2$). This is another indication that an object with mass cannot reach the speed of light. If it did, its momentum would become infinite—an unreasonable value.

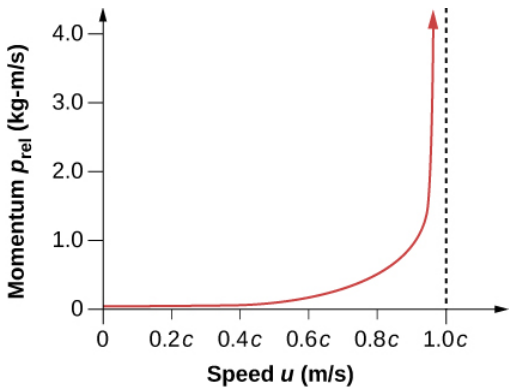


Figure 14.9.2: Relativistic momentum approaches infinity as the velocity of an object approaches the speed of light.

Mass vs. Rest mass

The relativistically correct definition of momentum (Equation 14.9.1) is sometimes taken to imply that mass varies with velocity: $m_{var} = \gamma m$, particularly in older textbooks. However, note that m is the mass of the object as measured by a person at rest relative to the object. Thus, m is defined to be the rest mass, which could be measured at rest, perhaps using gravity. When a mass is moving relative to an observer, the only way that its mass can be determined is through collisions or other means involving momentum. Because the mass of a moving object cannot be determined independently of momentum, the only meaningful mass is rest mass. Therefore, when we use the term "mass," assume it to be identical to "rest mass."

Relativistic momentum is defined in such a way that conservation of momentum holds in all inertial frames. Whenever the net external force on a system is zero, relativistic momentum is conserved, just as is the case for classical momentum. This has been verified in numerous experiments.

Exercise 14.9.1

What is the momentum of an electron traveling at a speed $0.985c$? The rest mass of the electron is $9.11 \times 10^{-31} kg$.

Answer

Substitute the data into Equation 14.9.1:

$$
\begin{aligned}
p &= \gamma m u \\
&= \frac{mu}{\sqrt{1-\frac{u^2}{c^2}}} \\
&= \frac{(9.11\times 10^{-31}\,kg)(0.985)(3.00\times 10^{8}\,m/s)}{\sqrt{1-\frac{(0.985c)^2}{c^2}}} \\
&= 1.56\times 10^{-21}\,kg-m/s.
\end{aligned}
$$

14.10: Relativistic Energy

Learning Objectives

By the end of this section, you will be able to:

- Explain how the work-energy theorem leads to an expression for the relativistic kinetic energy of an object
- Show how the relativistic energy relates to the classical kinetic energy, and sets a limit on the speed of any object with mass
- Describe how the total energy of a particle is related to its mass and velocity
- Explain how relativity relates to energy-mass equivalence, and some of the practical implications of energy-mass equivalence

The tokamak in Figure $14.10.1$ is a form of experimental fusion reactor, which can change mass to energy. Nuclear reactors are proof of the relationship between energy and matter.

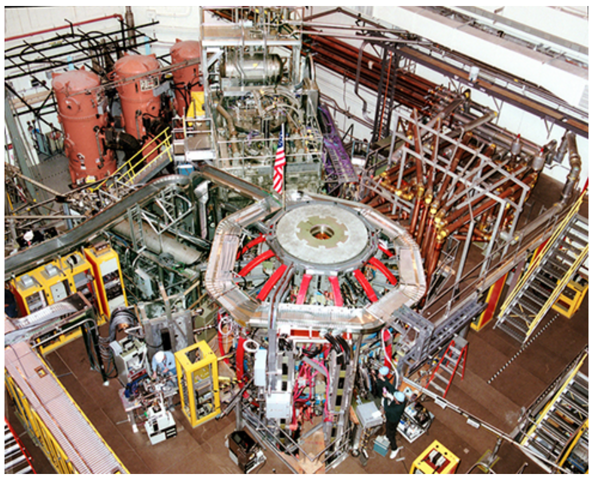


Figure 14.10.1: The National Spherical Torus Experiment (NSTX) is a fusion reactor in which hydrogen isotopes undergo fusion to produce helium. In this process, a relatively small mass of fuel is converted into a large amount of energy. (credit: Princeton Plasma Physics Laboratory)

Conservation of energy is one of the most important laws in physics. Not only does energy have many important forms, but each form can be converted to any other. We know that classically, the total amount of energy in a system remains constant. Relativistically, energy is still conserved, but energy-mass equivalence must now be taken into account, for example, in the reactions that occur within a nuclear reactor. Relativistic energy is intentionally defined so that it is conserved in all inertial frames, just as is the case for relativistic momentum. As a consequence, several fundamental quantities are related in ways not known in classical physics. All of these relationships have been verified by experimental results and have fundamental consequences. The altered definition of energy contains some of the most fundamental and spectacular new insights into nature in recent history.

Kinetic Energy and the Ultimate Speed Limit

The first postulate of relativity states that the laws of physics are the same in all inertial frames. Einstein showed that the law of conservation of energy of a particle is valid relativistically, but for energy expressed in terms of velocity and mass in a way consistent with relativity. Consider first the relativistic expression for the kinetic energy. We again use u for velocity to distinguish it from relative velocity v between observers. Classically, kinetic energy is related to mass and speed by the familiar expression

$$K=\frac{1}{2}mu^2.$$

The corresponding relativistic expression for kinetic energy can be obtained from the work-energy theorem. This theorem states that the net work on a system goes into kinetic energy. Specifically, if a force, expressed as

$$\vec{F}=\frac{d\vec{p}}{dt}=m\frac{d(\gamma\vec{u})}{dt}$$

accelerates a particle from rest to its final velocity, the work done on the particle should be equal to its final kinetic energy. In mathematical form, for one-dimensional motion:

$$\begin{aligned} K &= \int F dx = \int m \frac{d}{dt}(\gamma u) dx \\ &= m \int \frac{d(\gamma u)}{dt} \frac{dx}{dt} \\ &= m \int u \frac{d}{dt}\left(\frac{u}{\sqrt{1-(u/c)^2}}\right) dt. \end{aligned}$$

Integrate this by parts to obtain

$$\begin{aligned} K &= \frac{mu^2}{\sqrt{1-(u/c)^2}}\Bigg|_0^u - m\int \frac{u}{\sqrt{1-(u/c)^2}} \frac{du}{dt} dt \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - m\int \frac{u}{\sqrt{1-(u/c)^2}} du \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} - mc^2\left(\sqrt{1-(u/c)^2}\right)\Bigg|_0^u \\ &= \frac{mu^2}{\sqrt{1-(u/c)^2}} + \frac{mu^2}{\sqrt{1-(u/c)^2}} - mc^2 \\ &= mc^2\left[\frac{(u^2/c^2)+1-(u^2/c^2)}{\sqrt{1-(u/c)^2}}\right] - mc^2 \\ &= \frac{mc^2}{\sqrt{1-(u/c)^2}} - mc^2. \end{aligned}$$

Therefore, the **relativistic kinetic energy** of any particle of mass m is

$$K_{rel} = (\gamma - 1)mc^2. \tag{14.10.1}$$

When an object is motionless, its speed is $u = 0$ and

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = 1$$

so that $K_{rel} = 0$ at rest, as expected. However, the expression for relativistic kinetic energy (such as total energy and rest energy) does not look much like the classical $\frac{1}{2}mu^2$. To show that the expression for K_{rel} reduces to the classical expression for kinetic energy at low speeds, we use the binomial expansion to obtain an approximation for $(1+\varepsilon)^n$ valid for small ε:

$$(1+\varepsilon)^n = 1 + n\varepsilon + \frac{n(n-1)}{2!}\varepsilon^2 + \frac{n(n-1)(n-2)}{3!}\varepsilon^3 + \cdots \approx 1 + n\varepsilon$$

by neglecting the very small terms in $\varepsilon^2$and higher powers of ε. Choosing $\varepsilon = -u^2/c^2$ and $n = -\frac{1}{2}$ leads to the conclusion that γ at nonrelativistic speeds, where $\varepsilon = u/c$ is small, satisfies

$$\gamma = (1 - u^2/c^2)^{-1/2} \approx 1 + \frac{1}{2}\left(\frac{u^2}{c^2}\right).$$

A binomial expansion is a way of expressing an algebraic quantity as a sum of an infinite series of terms. In some cases, as in the limit of small speed here, most terms are very small. Thus, the expression derived here for γ is not exact, but it is a very accurate approximation. Therefore, at low speed:

$$\gamma - 1 \approx \frac{1}{2}\left(\frac{u^2}{c^2}\right).$$

Entering this into the expression for relativistic kinetic energy (Equation 14.10.1) gives

$$\begin{aligned} K_{rel} &\approx \left[\frac{1}{2}\left(\frac{u^2}{c^2}\right)\right] mc^2 \\ &\approx \frac{1}{2} mu^2 \\ &\approx K_{class}. \end{aligned}$$

That is, relativistic kinetic energy becomes the same as classical kinetic energy when $u \ll c$.

It is even more interesting to investigate what happens to kinetic energy when the speed of an object approaches the speed of light. We know that γ becomes infinite as u approaches c, so that K_{rel} also becomes infinite as the velocity approaches the speed of light (Figure 14.10.2). The increase in K_{rel} is far larger than in K_{class} as v approaches c. An infinite amount of work (and, hence, an infinite amount of energy input) is required to accelerate a mass to the speed of light.

> *No object with mass can attain the speed of light.*

The speed of light is the ultimate speed limit for any particle having mass. All of this is consistent with the fact that velocities less than **c** always add to less than c. Both the relativistic form for kinetic energy and the ultimate speed limit being c have been confirmed in detail in numerous experiments. No matter how much energy is put into accelerating a mass, its velocity can only approach—not reach—the speed of light.

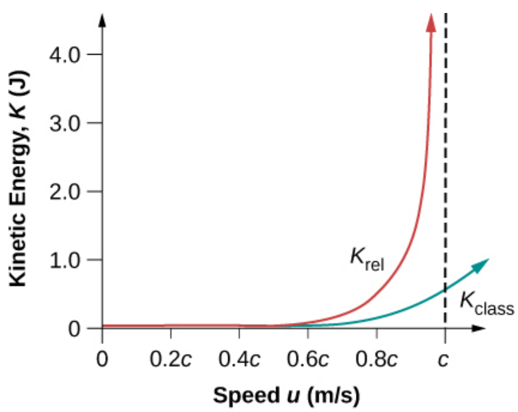


Figure 14.10.2:This graph of K_{rel} versus velocity shows how kinetic energy increases without bound as velocity approaches the speed of light. Also shown is K_{class}, the classical kinetic energy.

✔ Example 14.10.1: Comparing Kinetic Energy

An electron has a velocity $v = 0.990c$.

a. Calculate the kinetic energy in MeV of the electron.
b. Compare this with the classical value for kinetic energy at this velocity. (The mass of an electron is $9.11 \times 10^{-31} kg$.)

Strategy

The expression for relativistic kinetic energy is always correct, but for (a), it must be used because the velocity is highly relativistic (close to c). First, we calculate the relativistic factor γ, and then use it to determine the relativistic kinetic energy. For (b), we calculate the classical kinetic energy (which would be close to the relativistic value if v were less than a few percent of c) and see that it is not the same.

Solution for (a)

1. Identify the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31} kg$
2. Identify the unknown: K_{rel}.
3. Express the answer as an equation: $K_{rel} = (\gamma - 1)mc^2$ with $\gamma = \dfrac{1}{\sqrt{1 - u^2/c^2}}$.

4. Do the calculation. First calculate γ. Keep extra digits because this is an intermediate calculation:

$$\begin{aligned}\gamma &= \frac{1}{\sqrt{1-u^2/c^2}} \\ &= \frac{1}{\sqrt{1-\frac{(0.990c)^2}{c^2}}} \\ &= 7.0888.\end{aligned}$$

Now use this value to calculate the kinetic energy (Equatoin 14.10.1):

$$\begin{aligned}K_{rel} &= (\gamma - 1)mc^2 \\ &= (7.0888 - 1)(9.11 \times 10^{-31}\,kg)(3.00 \times 10^8\,m/s^2) \\ &= 4.9922 \times 10^{-13}\,J\end{aligned}$$

5. Convert units:

$$\begin{aligned}K_{rel} &= (4.9922 \times 10^{-13}\,J)\left(\frac{1\,MeV}{1.60 \times 10^{-13} J}\right) \\ &= 3.12\,MeV.\end{aligned}$$

Solution for (b)

1. List the knowns: $v = 0.990c$; $m = 9.11 \times 10^{-31} kg$.
2. List the unknown: K_{rel}
3. Express the answer as an equation:
4. Do the calculation:

$$\begin{aligned}K_{class} &= \frac{1}{2}mu^2 \\ &= \frac{1}{2}(9.11 \times 10^{-31} kg)(0.990)^2(3.00 \times 10^8\,m/s)^2 \\ &= 4.0179 \times 10^{-14} J.\end{aligned}$$

5. Convert units:

$$\begin{aligned}K_{class} &= 4.0179 \times 10^{-14} J\left(\frac{1\,MeV}{1.60 \times 10^{-13} J}\right) \\ &= 0.251\,MeV.\end{aligned}$$

Significance

As might be expected, because the velocity is 99.0% of the speed of light, the classical kinetic energy differs significantly from the correct relativistic value. Note also that the classical value is much smaller than the relativistic value. In fact, $K_{rel}/K_{class} = 12.4$ in this case. This illustrates how difficult it is to get a mass moving close to the speed of light. Much more energy is needed than predicted classically. Ever-increasing amounts of energy are needed to get the velocity of a mass a little closer to that of light. An energy of 3 MeV is a very small amount for an electron, and it can be achieved with present-day particle accelerators. SLAC, for example, can accelerate electrons to over $50 \times 10^9 eV = 50,000\,MeV$.

Is there any point in getting **v** a little closer to **c** than 99.0% or 99.9%? The answer is yes. We learn a great deal by doing this. The energy that goes into a high-velocity mass can be converted into any other form, including into entirely new particles. In the Large Hadron Collider in Figure 14.10.1, charged particles are accelerated before entering the ring-like structure. There, two beams of particles are accelerated to their final speed of about 99.7% the speed of light in opposite directions, and made to collide, producing totally new species of particles. Most of what we know about the substructure of matter and the collection of exotic short-lived particles in nature has been learned this way. Patterns in the characteristics of these previously unknown particles hint at a basic substructure for all matter. These particles and some of their characteristics will be discussed in a later chapter on particle physics.

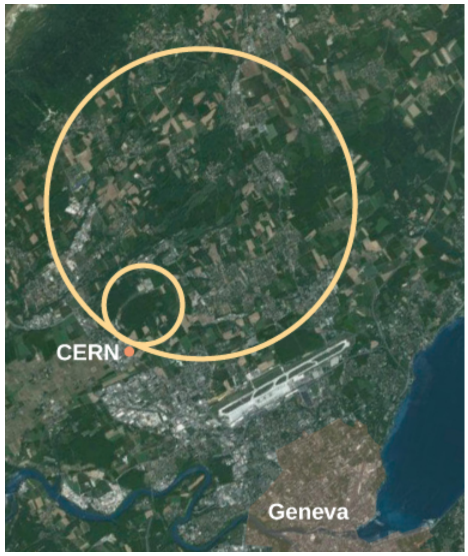


Figure 14.10.3: The European Organization for Nuclear Research (called CERN after its French name) operates the largest particle accelerator in the world, straddling the border between France and Switzerland.

Total Relativistic Energy

The expression for kinetic energy can be rearranged to:

$$\begin{aligned} E &= \frac{mc^2}{\sqrt{1-u^2/c^2}} \\ &= K + mc^2. \end{aligned}$$

Einstein argued in a separate article, also later published in 1905, that if the energy of a particle changes by ΔE, its mass changes by $\Delta m = \Delta E/C^2$. Abundant experimental evidence since then confirms that mc^2 corresponds to the energy that the particle of mass m has when at rest. For example, when a neutral pion of mass m at rest decays into two photons, the photons have zero mass but are observed to have total energy corresponding to mc^2 for the pion. Similarly, when a particle of mass m decays into two or more particles with smaller total mass, the observed kinetic energy imparted to the products of the decay corresponds to the decrease in mass. Thus, E is the total relativistic energy of the particle, and mc^2 is its rest energy.

Definition: Total Energy

Total energy (E) of a particle is

$$E = \gamma mc^2$$

where m is mass, c is the speed of light, $\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$, and u is the velocity of the mass relative to an observer.

Definition: Rest Enerey

Rest energy of an object is

$$E_0 = mc^2. \tag{14.10.2}$$

Equation 14.10.2is the correct form of Einstein's most famous equation, which for the first time showed that energy is related to the mass of an object at rest. For example, if energy is stored in the object, its rest mass increases. This also implies that mass can be destroyed to release energy. The implications of these first two equations regarding relativistic energy are so broad that they were not completely recognized for some years after Einstein published them in 1905, nor was the experimental proof that they are correct widely recognized at first. Einstein, it should be noted, did understand and describe the meanings and implications of his theory.

Example 14.10.2: Calculating Rest Energy

Calculate the rest energy of a 1.00-g mass.

Strategy

One gram is a small mass—less than one-half the mass of a penny. We can multiply this mass, in SI units, by the speed of light squared to find the equivalent rest energy.

Solution

1. Identify the knowns: $m = 1.00 \times 10^{-3} kg$; $c = 3.00 \times 10^{8} m/s$.
2. Identify the unknown: E_0.
3. Express the answer as an equation: $E_0 = mc^2$.
4. Do the calculation:

$$E_0 = mc^2 = (1.00 \times 10^{-3} kg)(3.00 \times 10^{8} m/s)^2 = 9.00 \times 10^{13} kg \cdot m^2/s^2.$$

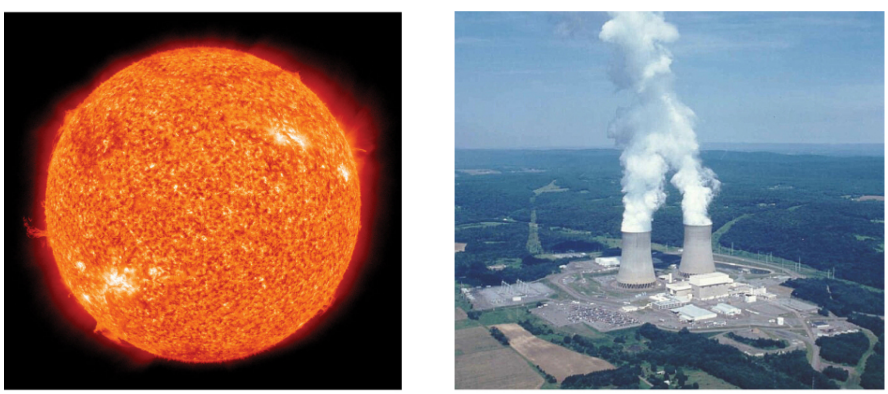
5. Convert units. Noting that $1\ kg \cdot m^2/s^2 = 1\ J$, we see the rest energy is:

$$E_0 = 9.00 \times 10^{13}\ J.$$

Significance

This is an enormous amount of energy for a 1.00-g mass. Rest energy is large because the speed of light **c** is a large number and c^2 is a very large number, so that mc^2 is huge for any macroscopic mass. The $9.00 \times 10^{13} J$ rest mass energy for 1.00 g is about twice the energy released by the Hiroshima atomic bomb and about 10,000 times the kinetic energy of a large aircraft carrier.

Today, the practical applications of *the conversion of mass into another form of energy*, such as in nuclear weapons and nuclear power plants, are well known. But examples also existed when Einstein first proposed the correct form of relativistic energy, and he did describe some of them. Nuclear radiation had been discovered in the previous decade, and it had been a mystery as to where its energy originated. The explanation was that, in some nuclear processes, a small amount of mass is destroyed and energy is released and carried by nuclear radiation. But the amount of mass destroyed is so small that it is difficult to detect that any is missing. Although Einstein proposed this as the source of energy in the radioactive salts then being studied, it was many years before there was broad recognition that mass could be and, in fact, commonly is, converted to energy (Figure 14.10.4).



(a) (b)

Figure 14.10.4: (a) The sun and (b) the Susquehanna Steam Electric Station both convert mass into energy—the sun via nuclear fusion, and the electric station via nuclear fission. (credit a: modification of work by NASA; credit b: modification of work by "ChNPP"/Wikimedia Commons)

Because of the relationship of rest energy to mass, we now consider mass to be a form of energy rather than something separate. There had not been even a hint of this prior to Einstein's work. Energy-mass equivalence is now known to be the source of the sun's energy, the energy of nuclear decay, and even one of the sources of energy keeping Earth's interior hot.

Stored Energy and Potential Energy

What happens to energy stored in an object at rest, such as the energy put into a battery by charging it, or the energy stored in a toy gun's compressed spring? The energy input becomes part of the total energy of the object and thus increases its rest mass. All stored and potential energy becomes mass in a system. In seeming contradiction, the principle of conservation of mass (meaning total mass is constant) was one of the great laws verified by nineteenth-century science. Why was it not noticed to be incorrect? The following example helps answer this question.

Example 14.10.3: Calculating Rest Mass

A car battery is rated to be able to move 600 ampere-hours $(A \cdot h)$ of charge at 12.0 V.

a. Calculate the increase in rest mass of such a battery when it is taken from being fully depleted to being fully charged, assuming none of the chemical reactants enter or leave the battery.
b. What percent increase is this, given that the battery's mass is 20.0 kg?

Strategy

In part (a), we first must find the energy stored as chemical energy E_{batt} in the battery, which equals the electrical energy the battery can provide. Because $E_{batt} = qV$, we have to calculate the charge q in $600\ A \cdot h$, which is the product of the current I and the time t. We then multiply the result by 12.0 V. We can then calculate the battery's increase in mass using $E_{batt} = (\Delta m)c^2$. Part (b) is a simple ratio converted into a percentage.

Solution for (a)

1. Identify the knowns:

$$I \cdot t = 600\ A \cdot h;\ V = 12.0\ V;\ c = 3.00 \times 10^8\ m/s.$$

2. Identify the unknown: Δm.
3. Express the answer as an equation:

$$\begin{aligned} E_{batt} &= (\Delta m)c^2 \\ \Delta m &= \frac{E_{batt}}{c^2} \\ &= \frac{qV}{c^2} \\ &= \frac{(It)V}{c^2}. \end{aligned}$$

4. Do the calculation:

$$\Delta m = \frac{(600\ A \cdot h)(12.0\ V)}{(3.00 \times 10^8)^2}.$$

5. Write amperes A as coulombs per second (C/s), and convert hours into seconds:

$$\begin{aligned} \Delta m &= \frac{(600\ C/s \cdot h)\left(\frac{3600\ s}{1\ h}\right)(12.0\ J/C)}{(3.00 \times 10^8\ m/s)^2} \\ &= 2.88 \times 10^{-10}\ kg. \end{aligned}$$

where we have used the conversion $1\ kg \cdot m^2/s^2 = 1\ J.$.

Solution for (b)

For part (b):

1. Identify the knowns: $\delta m = 2.88 \times 10^{-10} kg;\ m = 20.0\ kg.$
2. Identify the unknown: % change.
3. Express the answer as an equation:

$$\% \, increase = \frac{\delta m}{m} \times 100\%.$$

4. Do the calculation:

$$\begin{aligned}\% \, increase &= \frac{\Delta m}{m} \times 100\% \\ &= \frac{2.88 \times 10^{-10} \, kg}{20.0 \, kg} \times 100\% \\ &= 1.44 \times 10^{-9} \%\end{aligned}$$

Significance

Both the actual increase in mass and the percent increase are very small, because energy is divided by c^2, a very large number. We would have to be able to measure the mass of the battery to a precision of a billionth of a percent, or 1 part in 10^{11}, to notice this increase. It is no wonder that the mass variation is not readily observed. In fact, this change in mass is so small that we may question how anyone could verify that it is real. The answer is found in nuclear processes in which the percentage of mass destroyed is large enough to be measured accurately. The mass of the fuel of a nuclear reactor, for example, is measurably smaller when its energy has been used. In that case, stored energy has been released (converted mostly into thermal energy to power electric generators) and the rest mass has decreased. A decrease in mass also occurs from using the energy stored in a battery, except that the stored energy is much greater in nuclear processes, making the change in mass measurable in practice as well as in theory.

Relativistic Energy and Momentum

We know classically that kinetic energy and momentum are related to each other, because:

$$K_{class} = \frac{p^2}{2m} = \frac{(mu)^2}{2m} = \frac{1}{2}mu^2.$$

Relativistically, we can obtain a relationship between energy and momentum by algebraically manipulating their defining equations. This yields:

$$E^2 = (pc)^2 + (mc^2)^2, \tag{14.10.3}$$

where E is the relativistic total energy,

$$E = \frac{mc^2}{\sqrt{1-u^2/c^2}}$$

and p is the relativistic momentum. This relationship between relativistic energy and relativistic momentum is more complicated than the classical version, but we can gain some interesting new insights by examining it. First, total energy is related to momentum and rest mass. At rest, momentum is zero, and the equation gives the total energy to be the rest energy mc^2 (so this equation is consistent with the discussion of rest energy above). However, as the mass is accelerated, its momentum p increases, thus increasing the total energy. At sufficiently high velocities, the rest energy term $(mc^2)^2$ becomes negligible compared with the momentum term $(pc)^2$; thus, $E = pc$ at extremely relativistic velocities.

If we consider momentum p to be distinct from mass, we can determine the implications of the equation

$$E^2 = (pc)^2 + (mc^2)^2,$$

for a particle that has no mass. If we take m to be zero in this equation, then $E = pc$, $or p = E/c$. Massless particles have this momentum. There are several massless particles found in nature, including photons (which are packets of electromagnetic radiation). Another implication is that a massless particle must travel at speed **c** and only at speed **c**. It is beyond the scope of this text to examine the relationship in the equation $E^2 = (pc)^2 + (mc^2)^2$ in detail, but you can see that the relationship has important implications in special relativity.

Exercise 14.10.1

What is the kinetic energy of an electron if its speed is $0.992c$?

Answer

$$\begin{aligned} K_{rel} &= (\gamma - 1)mc^2 = \left(\frac{1}{\sqrt{1 - \frac{u^2}{c^2}}} - 1 \right) mc^2 \\ &= \left(\frac{1}{\sqrt{1 - \frac{(0.992c)^2}{c^2}}} - 1 \right) (9.11 \times 10^{-31}\, kg)(3.00 \times 10^8\, m/s)^2 \\ &= 5.67 \times 10^{-13}\, J \end{aligned}$$

14.E: Relativity (Exercises)

Conceptual Questions

5.1 Invariance of Physical Laws

1. Which of Einstein's postulates of special relativity includes a concept that does not fit with the ideas of classical physics? Explain.

2. Is Earth an inertial frame of reference? Is the sun? Justify your response.

3. When you are flying in a commercial jet, it may appear to you that the airplane is stationary and Earth is moving beneath you. Is this point of view valid? Discuss briefly.

5.3 Time Dilation

4. (a) Does motion affect the rate of a clock as measured by an observer moving with it?

(b) Does motion affect how an observer moving relative to a clock measures its rate?

5. To whom does the elapsed time for a process seem to be longer, an observer moving relative to the process or an observer moving with the process? Which observer measures the interval of proper time?

6. (a) How could you travel far into the future of Earth without aging significantly?

(b) Could this method also allow you to travel into the past?

5.4 Length Contraction

7. To whom does an object seem greater in length, an observer moving with the object or an observer moving relative to the object? Which observer measures the object's proper length?

8. Relativistic effects such as time dilation and length contraction are present for cars and airplanes. Why do these effects seem strange to us?

9. Suppose an astronaut is moving relative to Earth at a significant fraction of the speed of light.

(a) Does he observe the rate of his clocks to have slowed?

(b) What change in the rate of earthbound clocks does he see?

(c) Does his ship seem to him to shorten?

(d) What about the distance between two stars that lie in the direction of his motion? (e) Do he and an earthbound observer agree on his velocity relative to Earth?

5.7 Doppler Effect for Light

10. Explain the meaning of the terms "red shift" and "blue shift" as they relate to the relativistic Doppler effect.

11. What happens to the relativistic Doppler effect when relative velocity is zero? Is this the expected result?

12. Is the relativistic Doppler effect consistent with the classical Doppler effect in the respect that λ_{obs} is larger for motion away?

13. All galaxies farther away than about 50×10^6 **ly** exhibit a red shift in their emitted light that is proportional to distance, with those farther and farther away having progressively greater red shifts. What does this imply, assuming that the only source of red shift is relative motion?

5.8 Relativistic Momentum

14. How does modern relativity modify the law of conservation of momentum?

15. Is it possible for an external force to be acting on a system and relativistic momentum to be conserved? Explain.

5.9 Relativistic Energy

16. How are the classical laws of conservation of energy and conservation of mass modified by modern relativity?

17. What happens to the mass of water in a pot when it cools, assuming no molecules escape or are added? Is this observable in practice? Explain.

18. Consider a thought experiment. You place an expanded balloon of air on weighing scales outside in the early morning. The balloon stays on the scales and you are able to measure changes in its mass. Does the mass of the balloon change as the day progresses? Discuss the difficulties in carrying out this experiment.

19. The mass of the fuel in a nuclear reactor decreases by an observable amount as it puts out energy. Is the same true for the coal and oxygen combined in a conventional power plant? If so, is this observable in practice for the coal and oxygen? Explain.

20. We know that the velocity of an object with mass has an upper limit of c. Is there an upper limit on its momentum? Its energy? Explain.

21. Given the fact that light travels at c , can it have mass? Explain.

22. If you use an Earth-based telescope to project a laser beam onto the moon, you can move the spot across the moon's surface at a velocity greater than the speed of light. Does this violate modern relativity? (Note that light is being sent from the Earth to the moon, not across the surface of the moon.)

Problems

5.3 Time Dilation

23. (a) What is γ if $v = 0.250c$?

(b) If $v = 0.500c$?

24. (a) What is γ if $v = 0.100c$?

(b) If $v = 0.900c$?

25. Particles called π-mesons are produced by accelerator beams. If these particles travel at $2.70 \times 10^8 m/s$ and live $2.60 \times 10^{-8} s$ when at rest relative to an observer, how long do they live as viewed in the laboratory?

26. Suppose a particle called a kaon is created by cosmic radiation striking the atmosphere. It moves by you at $0.980c$, and it lives $1.24 \times 10^{-8} s$ when at rest relative to an observer. How long does it live as you observe it?

27. A neutral π-meson is a particle that can be created by accelerator beams. If one such particle lives $1.40 \times 10^{-16} s$ as measured in the laboratory, and $0.840 \times 10^{-16} s$ when at rest relative to an observer, what is its velocity relative to the laboratory?

28. A neutron lives 900 s when at rest relative to an observer. How fast is the neutron moving relative to an observer who measures its life span to be 2065 s?

29. If relativistic effects are to be less than 1%, then γ must be less than 1.01. At what relative velocity is $\gamma = 1.01$?

30. If relativistic effects are to be less than 3%, then γ must be less than 1.03. At what relative velocity is $\gamma = 1.03$?

5.4 Length Contraction

31. A spaceship, 200 m long as seen on board, moves by the Earth at 0.970c. What is its length as measured by an earthbound observer?

32. How fast would a 6.0 m-long sports car have to be going past you in order for it to appear only 5.5 m long?

33. (a) How far does the muon in Example 5.1 travel according to the earthbound observer?

(b) How far does it travel as viewed by an observer moving with it? Base your calculation on its velocity relative to the Earth and the time it lives (proper time).

(c) Verify that these two distances are related through length contraction $\gamma = 3.20$.

34. (a) How long would the muon in Example 5.1 have lived as observed on Earth if its velocity was $0.0500c$?

(b) How far would it have traveled as observed on Earth?

(c) What distance is this in the muon's frame?

35. Unreasonable Results A spaceship is heading directly toward Earth at a velocity of 0.800c. The astronaut on board claims that he can send a canister toward the Earth at 1.20c relative to Earth.

(a) Calculate the velocity the canister must have relative to the spaceship.

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

5.5 The Lorentz Transformation

36. Describe the following physical occurrences as events, that is, in the form **(x, y, z, t)**:

(a) A postman rings a doorbell of a house precisely at noon.

(b) At the same time as the doorbell is rung, a slice of bread pops out of a toaster that is located 10 m from the door in the east direction from the door.

(c) Ten seconds later, an airplane arrives at the airport, which is 10 km from the door in the east direction and 2 km to the south.

37. Describe what happens to the angle $\alpha = tan(v/c)$, and therefore to the transformed axes in Figure 5.17, as the relative velocity **v** of the S and S' frames of reference approaches **c**.

38. Describe the shape of the world line on a space-time diagram of

(a) an object that remains at rest at a specific position along the **x**-axis;

(b) an object that moves at constant velocity **u** in the **x**-direction;

(c) an object that begins at rest and accelerates at a constant rate of in the positive **x**-direction.

39. A man standing still at a train station watches two boys throwing a baseball in a moving train. Suppose the train is moving east with a constant speed of 20 m/s and one of the boys throws the ball with a speed of 5 m/s with respect to himself toward the other boy, who is 5 m west from him. What is the velocity of the ball as observed by the man on the station?

40. When observed from the sun at a particular instant, Earth and Mars appear to move in opposite directions with speeds 108,000 km/h and 86,871 km/h, respectively. What is the speed of Mars at this instant when observed from Earth?

41. A man is running on a straight road perpendicular to a train track and away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

42. A man is running on a straight road that makes **30°** with the train track. The man is running in the direction on the road that is away from the track at a speed of 12 m/s. The train is moving with a speed of 30 m/s with respect to the track. What is the speed of the man with respect to a passenger sitting at rest in the train?

43. In a frame at rest with respect to the billiard table, a billiard ball of mass m moving with speed v strikes another billiard ball of mass m at rest. The first ball comes to rest after the collision while the second ball takes off with speed v in the original direction of the motion of the first ball. This shows that momentum is conserved in this frame.

(a) Now, describe the same collision from the perspective of a frame that is moving with speed v in the direction of the motion of the first ball.

(b) Is the momentum conserved in this frame?

44. In a frame at rest with respect to the billiard table, two billiard balls of same mass **m** are moving toward each other with the same speed **v**. After the collision, the two balls come to rest.

(a) Show that momentum is conserved in this frame.

(b) Now, describe the same collision from the perspective of a frame that is moving with speed **v** in the direction of the motion of the first ball.

(c) Is the momentum conserved in this frame?

45. In a frame S, two events are observed: event 1: a pion is created at rest at the origin and event 2: the pion disintegrates after time τ. Another observer in a frame S' is moving in the positive direction along the positive x-axis with a constant speed v and observes the same two events in his frame. The origins of the two frames coincide at $t = t' = 0$.

(a) Find the positions and timings of these two events in the frame S' (a) according to the Galilean transformation, and

(b) according to the Lorentz transformation.

5.6 Relativistic Velocity Transformation

46. If two spaceships are heading directly toward each other at 0.800**c**, at what speed must a canister be shot from the first ship to approach the other at 0.999**c** as seen by the second ship?

47. Two planets are on a collision course, heading directly toward each other at 0.250**c**. A spaceship sent from one planet approaches the second at 0.750**c** as seen by the second planet. What is the velocity of the ship relative to the first planet?

48. When a missile is shot from one spaceship toward another, it leaves the first at 0.950**c** and approaches the other at 0.750**c**. What is the relative velocity of the two ships?

49. What is the relative velocity of two spaceships if one fires a missile at the other at 0.750**c** and the other observes it to approach at 0.950**c**?

50. Prove that for any relative velocity **v** between two observers, a beam of light sent from one to the other will approach at speed **c** (provided that **v** is less than **c**, of course).

51. Show that for any relative velocity **v** between two observers, a beam of light projected by one directly away from the other will move away at the speed of light (provided that **v** is less than **c**, of course).

5.7 Doppler Effect for Light

52. A highway patrol officer uses a device that measures the speed of vehicles by bouncing radar off them and measuring the Doppler shift. The outgoing radar has a frequency of 100 GHz and the returning echo has a frequency 15.0 kHz higher. What is the velocity of the vehicle? Note that there are two Doppler shifts in echoes. Be certain not to round off until the end of the problem, because the effect is small.

5.8 Relativistic Momentum

53. Find the momentum of a helium nucleus having a mass of $6.68 \times 10^{-27} kg$ that is moving at 0.200**c**.

54. What is the momentum of an electron traveling at 0.980**c**?

55. (a) Find the momentum of a $1.00 \times 10^9 - kg$ asteroid heading towards Earth at 30.0 km/s.

(b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

56. (a) What is the momentum of a 2000-kg satellite orbiting at 4.00 km/s? (b) Find the ratio of this momentum to the classical momentum. (Hint: Use the approximation that $\gamma = 1 + (1/2)v^2/c^2$ at low velocities.)

57. What is the velocity of an electron that has a momentum of $3.04 \times 10^{-21} kg \cdot m/s$? Note that you must calculate the velocity to at least four digits to see the difference from **c**.

58. Find the velocity of a proton that has a momentum of $4.48 \times 10^{-19} kg \cdot m/s$.

5.9 Relativistic Energy

59. What is the rest energy of an electron, given its mass is $9.11 \times 10^{-31} kg$? Give your answer in joules and MeV.

60. Find the rest energy in joules and MeV of a proton, given its mass is $1.67 \times 10^{-27} kg$.

61. If the rest energies of a proton and a neutron (the two constituents of nuclei) are 938.3 and 939.6 MeV, respectively, what is the difference in their mass in kilograms?

62. The Big Bang that began the universe is estimated to have released $10^{68} J$ of energy. How many stars could half this energy create, assuming the average star's mass is $4.00 \times 10^{30} kg$?

63. A supernova explosion of a $2.00 \times 10^{31} kg$ star produces $1.00 \times 10^{44} J$ of energy.

(a) How many kilograms of mass are converted to energy in the explosion?

(b) What is the ratio $\Delta m/m$ of mass destroyed to the original mass of the star?

64. (a) Using data from Potential Energy of a System, calculate the mass converted to energy by the fission of 1.00 kg of uranium.

(b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$?

65. (a) Using data from Potential Energy of a System, calculate the amount of mass converted to energy by the fusion of 1.00 kg of hydrogen.

(b) What is the ratio of mass destroyed to the original mass, $\Delta m/m$?

(c) How does this compare with $\Delta m/m$ for the fission of 1.00 kg of uranium?

66. There is approximately $10^{34} J$ of energy available from fusion of hydrogen in the world's oceans.

(a) If $10^{33} J$ of this energy were utilized, what would be the decrease in mass of the oceans?

(b) How great a volume of water does this correspond to?

(c) Comment on whether this is a significant fraction of the total mass of the oceans.

67. A muon has a rest mass energy of 105.7 MeV, and it decays into an electron and a massless particle.

(a) If all the lost mass is converted into the electron's kinetic energy, find γ for the electron.

(b) What is the electron's velocity?

68. A π-meson is a particle that decays into a muon and a massless particle. The π-meson has a rest mass energy of 139.6 MeV, and the muon has a rest mass energy of 105.7 MeV. Suppose the π-meson is at rest and all of the missing mass goes into the muon's kinetic energy. How fast will the muon move?

69. (a) Calculate the relativistic kinetic energy of a 1000-kg car moving at 30.0 m/s if the speed of light were only 45.0 m/s.

(b) Find the ratio of the relativistic kinetic energy to classical.

70. Alpha decay is nuclear decay in which a helium nucleus is emitted. If the helium nucleus has a mass of $6.80 \times 10^{-27} kg$ and is given 5.00 MeV of kinetic energy, what is its velocity?

71. (a) Beta decay is nuclear decay in which an electron is emitted. If the electron is given 0.750 MeV of kinetic energy, what is its velocity?

(b) Comment on how the high velocity is consistent with the kinetic energy as it compares to the rest mass energy of the electron.

Additional Problems

72. (a) At what relative velocity is $\gamma = 1.50$?

(b) At what relative velocity is $\gamma = 100$?γ=100?

73. (a) At what relative velocity is $\gamma = 2.00$?

(b) At what relative velocity is $\gamma = 10.0$?

74. Unreasonable Results (a) Find the value of γ required for the following situation. An earthbound observer measures 23.9 h to have passed while signals from a high-velocity space probe indicate that 24.0 h have passed on board.

(b) What is unreasonable about this result?

(c) Which assumptions are unreasonable or inconsistent?

75. (a) How long does it take the astronaut in Example 5.5 to travel 4.30 ly at $0.99944c$(as measured by the earthbound observer)?

(b) How long does it take according to the astronaut?

(c) Verify that these two times are related through time dilation with $\gamma = 30.00$ as given.

76. (a) How fast would an athlete need to be running for a 100-m race to look 100 yd long?

(b) Is the answer consistent with the fact that relativistic effects are difficult to observe in ordinary circumstances? Explain.

77. (a) Find the value of γ for the following situation. An astronaut measures the length of his spaceship to be 100 m, while an earthbound observer measures it to be 25.0 m.

(b) What is the speed of the spaceship relative to Earth?

78. A clock in a spaceship runs one-tenth the rate at which an identical clock on Earth runs. What is the speed of the spaceship?

79. An astronaut has a heartbeat rate of 66 beats per minute as measured during his physical exam on Earth. The heartbeat rate of the astronaut is measured when he is in a spaceship traveling at 0.5c with respect to Earth by an observer (A) in the ship and by an observer (B) on Earth.

(a) Describe an experimental method by which observer B on Earth will be able to determine the heartbeat rate of the astronaut when the astronaut is in the spaceship.

(b) What will be the heartbeat rate(s) of the astronaut reported by observers A and B?

80. A spaceship (A) is moving at speed **c/2** with respect to another spaceship (B). Observers in A and B set their clocks so that the event at **(x, y, z, t)** of turning on a laser in spaceship B has coordinates (0, 0, 0, 0) in A and also (0, 0, 0, 0) in B. An observer at the origin of B turns on the laser at $t = 0$ and turns it off at $t = \tau$ in his time. What is the time duration between on and off as seen by an observer in A?

81. Same two observers as in the preceding exercise, but now we look at two events occurring in spaceship A. A photon arrives at the origin of A at its time $t = 0$ and another photon arrives at $(x = 1.00m, 0, 0)$ at $t = 0$ in the frame of ship A.

(a) Find the coordinates and times of the two events as seen by an observer in frame B.

(b) In which frame are the two events simultaneous and in which frame are they are not simultaneous?

82. Same two observers as in the preceding exercises. A rod of length 1 m is laid out on the **x**-axis in the frame of B from origin to $(x = 1.00m, 0, 0)$. What is the length of the rod observed by an observer in the frame of spaceship A?

83. An observer at origin of inertial frame S sees a flashbulb go off at $x = 150km, y = 15.0km$, and $z = 1.00km$ at time $t = 4.5 \times 10^{-4} s$. At what time and position in the S' system did the flash occur, if S' is moving along shared **x**-direction with S at a velocity $v = 0.6c$?

84. An observer sees two events $1.5 \times 10^{-8} s$ apart at a separation of 800 m. How fast must a second observer be moving relative to the first to see the two events occur simultaneously?

85. An observer standing by the railroad tracks sees two bolts of lightning strike the ends of a 500-m-long train simultaneously at the instant the middle of the train passes him at 50 m/s. Use the Lorentz transformation to find the time between the lightning strikes as measured by a passenger seated in the middle of the train.

86. Two astronomical events are observed from Earth to occur at a time of 1 s apart and a distance separation of $1.5 \times 10^9 m$ from each other.

(a) Determine whether separation of the two events is space like or time like.

(b) State what this implies about whether it is consistent with special relativity for one event to have caused the other?

87. Two astronomical events are observed from Earth to occur at a time of 0.30 s apart and a distance separation of $2.0 \times 10^9 m$ from each other. How fast must a spacecraft travel from the site of one event toward the other to make the events occur at the same time when measured in the frame of reference of the spacecraft?

88. A spacecraft starts from being at rest at the origin and accelerates at a constant rate **g**, as seen from Earth, taken to be an inertial frame, until it reaches a speed of **c/2**.

(a) Show that the increment of proper time is related to the elapsed time in Earth's frame by:

$$d\tau = \sqrt{1 - v2/c^2}\, dt$$

.

(b) Find an expression for the elapsed time to reach speed **c/2** as seen in Earth's frame.

(c) Use the relationship in (a) to obtain a similar expression for the elapsed proper time to reach c/2 as seen in the spacecraft, and determine the ratio of the time seen from Earth with that on the spacecraft to reach the final speed.

89. (a) All but the closest galaxies are receding from our own Milky Way Galaxy. If a galaxy $12.0 \times 10^9 ly$ away is receding from us at 0.900c, at what velocity relative to us must we send an exploratory probe to approach the other galaxy at 0.990c as measured from that galaxy?

(b) How long will it take the probe to reach the other galaxy as measured from Earth? You may assume that the velocity of the other galaxy remains constant.

(c) How long will it then take for a radio signal to be beamed back? (All of this is possible in principle, but not practical.)

90. Suppose a spaceship heading straight toward the Earth at 0.750**c** can shoot a canister at 0.500**c** relative to the ship.

(a) What is the velocity of the canister relative to Earth, if it is shot directly at Earth?

(b) If it is shot directly away from Earth?

91. Repeat the preceding problem with the ship heading directly away from Earth.

92. If a spaceship is approaching the Earth at 0.100**c** and a message capsule is sent toward it at 0.100**c** relative to Earth, what is the speed of the capsule relative to the ship?

93. (a) Suppose the speed of light were only 3000 m/s. A jet fighter moving toward a target on the ground at 800 m/s shoots bullets, each having a muzzle velocity of 1000 m/s. What are the bullets' velocity relative to the target?

(b) If the speed of light was this small, would you observe relativistic effects in everyday life? Discuss.

94. If a galaxy moving away from the Earth has a speed of 1000 km/s and emits 656 nm light characteristic of hydrogen (the most common element in the universe).

(a) What wavelength would we observe on Earth?

(b) What type of electromagnetic radiation is this? (c) Why is the speed of Earth in its orbit negligible here?

95. A space probe speeding towards the nearest star moves at $0.250c$and sends radio information at a broadcast frequency of 1.00 GHz. What frequency is received on Earth?

96. Near the center of our galaxy, hydrogen gas is moving directly away from us in its orbit about a black hole. We receive 1900 nm electromagnetic radiation and know that it was 1875 nm when emitted by the hydrogen gas. What is the speed of the gas?

97. (a) Calculate the speed of a $1.00 - \mu g$ particle of dust that has the same momentum as a proton moving at 0.999**c**.

(b) What does the small speed tell us about the mass of a proton compared to even a tiny amount of macroscopic matter?

98. (a) Calculate γ for a proton that has a momentum of $1.00 kg \cdot m/s$.

(b) What is its speed? Such protons form a rare component of cosmic radiation with uncertain origins.

99. Show that the relativistic form of Newton's second law is

(a) $F = m\dfrac{du}{dt}\dfrac{1}{(1-u^2/c^2)^{3/2}}$;

(b) Find the force needed to accelerate a mass of 1 kg by 1 m/s^2 when it is traveling at a velocity of **c/2**.

100. A positron is an antimatter version of the electron, having exactly the same mass. When a positron and an electron meet, they annihilate, converting all of their mass into energy.

(a) Find the energy released, assuming negligible kinetic energy before the annihilation.

(b) If this energy is given to a proton in the form of kinetic energy, what is its velocity?

(c) If this energy is given to another electron in the form of kinetic energy, what is its velocity?

101. What is the kinetic energy in MeV of a π-meson that lives $1.40 \times 10^{-16} s$ as measured in the laboratory, and $0.840 \times 10^{-16} s$ when at rest relative to an observer, given that its rest energy is 135 MeV?

102. Find the kinetic energy in MeV of a neutron with a measured life span of 2065 s, given its rest energy is 939.6 MeV, and rest life span is 900s.

103. (a) Show that $(pc)^2/(mc^2)^2 = \gamma^2 - 1$. This means that at large velocities $pc >> mc^2$.

(b) Is $E \approx pc$ when $\gamma = 30.0$, as for the astronaut discussed in the twin paradox?

104. One cosmic ray neutron has a velocity of $0.250c$relative to the Earth.

(a) What is the neutron's total energy in MeV?

(b) Find its momentum.

(c) Is $E \approx pc$ in this situation? Discuss in terms of the equation given in part (a) of the previous problem.

105. What is γ for a proton having a mass energy of 938.3 MeV accelerated through an effective potential of 1.0 TV (teravolt)?

106. (a) What is the effective accelerating potential for electrons at the Stanford Linear Accelerator, if $\gamma = 1.00 \times 10^5$ for them?

(b) What is their total energy (nearly the same as kinetic in this case) in GeV?

107. (a) Using data from Potential Energy of a System, find the mass destroyed when the energy in a barrel of crude oil is released.

(b) Given these barrels contain 200 liters and assuming the density of crude oil is $750 kg/m^3$, what is the ratio of mass destroyed to original mass, $\Delta m/m$?

108. (a) Calculate the energy released by the destruction of 1.00 kg of mass.

(b) How many kilograms could be lifted to a 10.0 km height by this amount of energy?

109. A Van de Graaff accelerator utilizes a 50.0 MV potential difference to accelerate charged particles such as protons.

(a) What is the velocity of a proton accelerated by such a potential?

(b) An electron?

110. Suppose you use an average of $500 kW \cdot h$ of electric energy per month in your home.

(a) How long would 1.00 g of mass converted to electric energy with an efficiency of 38.0% last you?

(b) How many homes could be supplied at the $500 kW \cdot h$ per month rate for one year by the energy from the described mass conversion?

111. (a) A nuclear power plant converts energy from nuclear fission into electricity with an efficiency of 35.0%. How much mass is destroyed in one year to produce a continuous 1000 MW of electric power?

(b) Do you think it would be possible to observe this mass loss if the total mass of the fuel is $10^4 kg$?

112. Nuclear-powered rockets were researched for some years before safety concerns became paramount.

(a) What fraction of a rocket's mass would have to be destroyed to get it into a low Earth orbit, neglecting the decrease in gravity? (Assume an orbital altitude of 250 km, and calculate both the kinetic energy (classical) and the gravitational potential energy needed.)

(b) If the ship has a mass of $1.00 \times 10^5 kg$ (100 tons), what total yield nuclear explosion in tons of TNT is needed?

113. The sun produces energy at a rate of 3.85×10^{26} W by the fusion of hydrogen. About **0.7%** of each kilogram of hydrogen goes into the energy generated by the Sun.

(a) How many kilograms of hydrogen undergo fusion each second?

(b) If the sun is 90.0% hydrogen and half of this can undergo fusion before the sun changes character, how long could it produce energy at its current rate?

(c) How many kilograms of mass is the sun losing per second?

(d) What fraction of its mass will it have lost in the time found in part (b)?

114. Show that $E^2 - p^2c^2$ for a particle is invariant under Lorentz transformations.

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Glossary

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