6.3: Work Done by a Variable Force

Work Done by a Variable Force

Integration is used to calculate the work done by a variable force.

learning objectives

- Describe approaches used to calculate work done by a variable force

Using Integration to Calculate the Work Done by Variable Forces

A force is said to do work when it acts on a body so that there is a displacement of the point of application in the direction of the force. Thus, a force does work when it results in movement.

The work done by a constant force of magnitude F on a point that moves a displacement Δx in the direction of the force is simply the product

$$W = F \cdot \Delta x \tag{6.3.1}$$

In the case of a variable force, integration is necessary to calculate the work done. For example, let's consider work done by a spring. According to the Hooke's law the restoring force (or spring force) of a perfectly elastic spring is proportional to its extension (or compression), but opposite to the direction of extension (or compression). So the spring force acting upon an object attached to a horizontal spring is given by:

$$Fs = -kx \tag{6.3.2}$$

that is proportional to its displacement (extension or compression) in the x direction from the spring's equilibrium position, but its direction is opposite to the x direction. For a variable force, one must add all the infinitesimally small contributions to the work done during infinitesimally small time intervals dt (or equivalently, in infinitely small length intervals $dx=v_x dt$). In other words, an integral must be evaluated:

$$W_s = \int_0^t Fs \cdot vdt = \int_0^t -kxv_x dt = \int_{x_o}^{x} -kxdx = -\frac{1}{2}k\Delta x^2 \tag{6.3.3}$$

This is the work done by a spring exerting a variable force on a mass moving from position x_o to x (from time 0 to time t). The work done is positive if the applied force is in the same direction as the direction of motion; so the work done by the object on spring from time 0 to time t, is:

$$W_a = \int_0^t Fa \cdot vdt = \int_0^t -F_s \cdot vdt = \frac{1}{2}k\Delta x^2 \tag{6.3.4}$$

in this relation F_a is the force acted upon spring by the object. F_a and F_s are in fact action- reaction pairs; and W_a is equal to the elastic potential energy stored in spring.

Using Integration to Calculate the Work Done by Constant Forces

The same integration approach can be also applied to the work done by a constant force. This suggests that *integrating* the product of force and distance is the general way of determining the work done by a force on a moving body.

Consider the situation of a gas sealed in a piston, the study of which is important in Thermodynamics. In this case, the Pressure (Pressure =Force/Area) is constant and can be taken out of the integral:

$$W = \int_a^b PdV = P\int_a^b dV = P\Delta V \tag{6.3.5}$$

Another example is the work done by gravity (a constant force) on a free-falling object (we assign the y-axis to vertical motion, in this case):

$$W = \int_{t_1}^{t_2} F \cdot vdt = \int_{t_1}^{t_2} mgv_y dt = mg\int_{y_1}^{y_2} dy = mg\Delta y \tag{6.3.6}$$

Notice that the result is *the same* as we would have obtained by simply evaluating the product of force and distance.

Units Used for Work

The SI unit of work is the joule (J), which is defined as the work done by a force of one newton moving an object through a distance of one meter.

Non-SI units of work include the erg, the foot-pound, the foot-pound, the kilowatt hour, the liter-atmosphere, and the horsepower-hour.

Key Points

- The work done by a constant force of magnitude F on a point that moves a displacement d in the direction of the force is the product: $W = Fd$.
- Integration approach can be used both to calculate work done by a variable force and work done by a constant force.
- The SI unit of work is the joule; non- SI units of work include the erg, the foot-pound, the foot-poundal, the kilowatt hour, the litre-atmosphere, and the horsepower-hour.

Key Terms

- **work**: A measure of energy expended in moving an object; most commonly, force times displacement. No work is done if the object does not move.
- **force**: A physical quantity that denotes ability to push, pull, twist or accelerate a body, which is measured in a unit dimensioned in mass × distance/time^2 (ML/T^2): SI: newton (N); CGS: dyne (dyn)