3.9: Appendix- Some Exponential Operator Algebra

Suppose that the commutator of two operators A,B

$$[A,B]=c, \tag{3.9.1}$$

where c commutes with A and B, usually it's just a number, for instance 1 or $i\hbar$.

Then

$$[A,e^{\lambda B}] = \left[A,1+\lambda B+\left(\frac{\lambda^2}{2!}\right)B^2+\left(\frac{\lambda^3}{3!}\right)B^3+\ldots\right] \tag{3.9.2}$$

$$=\lambda c+\left(\frac{\lambda^2}{2!}\right)2Bc+\left(\frac{\lambda^3}{3!}\right)3B^2c+\ldots \tag{3.9.3}$$

$$=\lambda ce^{\lambda B}. \tag{3.9.4}$$

That is to say, the commutator of A with $e^{\lambda B}$ is proportional to $e^{\lambda B}$ itself. That is reminiscent of the simple harmonic oscillator commutation relation $[H,a^\dagger]=\hbar\omega a^\dagger$ which led directly to the ladder of eigenvalues of H separated by $\hbar\omega$. Will there be a similar "ladder" of eigenstates of A in general?

Assuming A (which is a general operator) has an eigenstate $|a\rangle$ with eigenvalue a,

$$A|a\rangle=a|a\rangle. \tag{3.9.5}$$

Applying $[A,e^{\lambda B}]=\lambda ce^{\lambda B}$ to the eigenstate $|a\rangle$:

$$Ae^{\lambda B}|a\rangle=e^{\lambda B}A|a\rangle+\lambda ce^{\lambda B}|a\rangle=(a+\lambda c)|a\rangle. \tag{3.9.6}$$

Therefore, unless it is identically zero, $e^{\lambda B}|a\rangle$ is *also* an eigenstate of A, with eigenvalue $a+\lambda c$. We conclude that instead of a *ladder* of eigenstates, we can apparently generate a whole *continuum* of eigenstates, since λ can be set arbitrarily!

To find more operator identities, premultiply $[A,e^{\lambda B}]=\lambda ce^{\lambda B}$ by $e^{-\lambda B}$ to find:

$$\begin{aligned} e^{-\lambda B}Ae^{\lambda B} &= A+\lambda[A,B] \\ &= A+\lambda c. \end{aligned}$$

This identity is *only* true for operators A,B whose commutator c is a number. (Well, c *could* be an operator, provided it still commutes with both A and B).

Our next task is to establish the following very handy identity, which is also only true if $[A,B]$ commutes with A and B:

$$e^{A+B}=e^Ae^Be-\frac{1}{2}[A,B]. \tag{3.9.7}$$

The proof is as follows:

Proof

Take $f(x)=e^{Ax}e^{Bx}$,

$$\begin{aligned} \frac{df}{dx} &= Ae^{Ax}e^{Bx}+e^{Ax}e^{Bx}B \\ &= f(x)(e^{-Bx}Ae^{Bx}+B) \\ &= f(x)(A+x[A,B]+B). \end{aligned}$$

It is easy to check that the solution to this first-order differential equation equal to one at $x=0$ is

$$f(x)=e^{x(A+B)}e^{\frac{1}{2}x^2[A,B]}$$

so taking $x=1$ gives the required identity,

$$e^{A+B}=e^Ae^Be^{-\frac{1}{2}[A,B]}.$$

□

It also follows that $e^B e^A = e^A e^B e^{-[A,B]}$ provided—as always—that $[A, B]$ commutes with A and B.

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