7.4: Taylor Series Solution

Let us substitute a Taylor series for $H(y)$,

$$H(y)=\sum_{p=0}^{\infty} a_p y^p. \tag{7.4.1}$$

This leads to

$$\begin{aligned} H'(y) &= \sum_{p=0}^{\infty} p a_p y^{p-1} = \sum_{q=0}^{\infty} (q+1) a_{q+1} y^q \\ H''(y) &= \sum_{p=0}^{\infty} p(p-1) a_p y^{p-2} = \sum_{r=0}^{\infty} (r+1)(r+2) a_{r+2} y^r \end{aligned}$$

How to deal with equations involving polynomials.

If I ask you when is $a+by+cy^2=0$ for all y, I hope you will answer when $a=b=c=0$. In other words a polynomial is zero when all its coefficients are zero. In the same vein two polynomials are equal when all their coefficients are equal. So what happens for infinite polynomials? They are zero when all coefficients are zero, and they are equal when all coefficients are equal.

So lets deal with the equation, and collect terms of the same order in y.

$$\begin{aligned} y^0: &\quad 2a_2+(2\epsilon-1)a_0=0 \\ y^1: &\quad 6a_3-2a_1+(2\epsilon-1)a_1=0 \\ y^s: &\quad (s+1)(s+2)a_{s+2}-(2s+1-2\epsilon)a_s=0 \end{aligned} \tag{7.4.2}$$

These equations can be used to determine a_{s+2} if we know a_s. The only thing we do not want of our solutions is that they diverge at infinity. Notice that if there is an integer such that

$$2\epsilon=2n+1, \tag{7.4.3}$$

that $a_{n+2}=0$, and $a_{n+4}=0$, etc. These solutions are normalisable, and will be investigated later. If the series does not terminates, we just look at the behaviour of the coefficients for large s, using the following

Theorem 7.4.1

The behaviour of the coefficients a_s of a Taylor series $u(y)=\sum_s a_s y^s$ for large index s describes the behaviour of the function $u(y)$ for large value of y.

Now for large s,

$$a_{s+2}=\frac{2}{s}a_s, \tag{7.4.4}$$

which behaves the same as the Taylor coefficients of e^{y^2}:

$$e^{y^2}=\sum_{s\text{ even}} b_s y^s=\sum_{s\text{ even}} \frac{1}{\frac{s}{2}!} y^s, \tag{7.4.5}$$

and we find

$$b_{s+2}=\frac{2}{s+2}b_s, \tag{7.4.6}$$

which for large s is the same as the relation for a_s. Now $e^{y^2}e^{-y^2/2}=e^{y^2/2}$, and this diverges....