12.5.1: Hyperfine Structure

To this point, the nucleus has been assumed to interact with the electron only through its electric field. However, like the electron, the proton has spin angular momentum with $s=1/2$, and associated with this angular momentum is an intrinsic dipole moment

$$\mu_{\mathrm{p}}=\gamma_{p}\frac{e}{Mc}\mathbf{S}_{\mathrm{p}},\tag{12.5.1.1}$$

where M is the proton mass and γ_p is a numerical factor known experimentally to be $\gamma_p=2.7928$. Note that the proton dipole moment is weaker than the electron dipole moment by roughly a factor of $M/m\sim 2000$, and hence one expects the associated effects to be small, even in comparison to fine structure, so again treating the corrections as a perturbation is justified. The proton dipole moment will interact with both the spin dipole moment of the electron and the orbital dipole moment of the electron, and so there are two new contributions to the Hamiltonian, the nuclear spin-orbit interaction and the spin-spin interaction. The derivation for the nuclear spin-orbit Hamiltonian is the same as for the electron spin-orbit Hamiltonian, except that the calculation is done in the frame of the proton and hence there is no factor of $1/2$ from the Thomas precession. The nuclear spin-orbit Hamiltonian is

$$\Delta H_{pso}=\frac{\gamma_p e^2}{mMc^2r^3}\mathbf{L}\cdot\mathbf{S}_{\mathrm{p}}\tag{12.5.1.2}$$

The spin-spin Hamiltonian can be derived by considering the field produced by the proton spin dipole, which can be written

$$\mathbf{B}(\mathbf{r})=\frac{1}{r^3}\left[3\frac{(\mu_{\mathrm{p}}\cdot\mathbf{r})\,\mathbf{r}}{r^2}-\mu_{\mathrm{p}}\right]+\frac{8\pi}{3}\mu_{\mathrm{p}}\delta^3(\mathbf{r}).\tag{12.5.1.3}$$

The first term is just the usual field associated with a magnetic dipole, but the second term requires special explanation. Normally, when one considers a dipole field, it is implicit that one is interested in the field far from the dipole-that is, at distances far from the source compared to the size of the current loop producing the dipole. However, every field line outside the loop must return inside the loop, as shown in figure 6. If the size of the current loop goes to zero, then the field will be infinite at the origin, and this contribution is what is reflected by the second term in equation 77. The electron has additional energy

$$\Delta E_{ss}=-\mu_{\mathrm{e}}\cdot\mathbf{B}\tag{12.5.1.4}$$

due to the interaction of its spin dipole with this field, and hence the spinspin Hamiltonian is

$$\Delta H_{ss}=\frac{\gamma_p e^2}{mMc^2}\left\{\frac{1}{r^3}\left[3\left(\mathbf{S}_{\mathrm{p}}\cdot\hat{r}\right)\left(\mathbf{S}_{\mathrm{e}}\cdot\hat{r}\right)-\left(\mathbf{S}_{\mathrm{p}}\cdot\mathbf{S}_{\mathrm{e}}\right)\right]+\frac{8\pi}{3}\left(\mathbf{S}_{\mathrm{p}}\cdot\mathbf{S}_{\mathrm{e}}\right)\delta^3(\mathbf{r})\right\}.\tag{12.5.1.5}$$

The operator J_z does not commute with this Hamiltonian. However, one can define the total angular momentum

$$\mathbf{F}=\mathbf{L}+\mathbf{S}_{\mathrm{e}}+\mathbf{S}_{\mathrm{p}}=\mathbf{J}+\mathbf{S}_{\mathrm{p}}\tag{12.5.1.6}$$

The corresponding operators F^2 and F_z commute with the Hamiltonian, and they introduce new quantum numbers f and m_f through the relations

$$F^2\Psi=f(f+1)\hbar^2\Psi$$

$$F_z\Psi=m_f\hbar\Psi$$

The quantum number f has possible values $f=j+1/2, j-1/2$ since the proton is spin $1/2$, and hence every energy level associated with a particular set of quantum numbers n, l, and j will be split into two levels of slightly different energy, depending on the relative orientation of the proton magnetic dipole with the electron state.

Consider first the case $l=0$, since the hyperfine splitting of the hydrogen atom ground state is of the most interest. Since the electron has no orbital angular momentum, there is no nuclear spin-orbit effect. It can be shown that because the wavefunction has spherical symmetry, only the delta function term contributes from the spin-spin Hamiltonian. First order perturbation theory yields

$$\Delta E_{hf}=\frac{8\pi\gamma_p e^2}{3mMc^2}\left(\mathbf{S}_{\mathrm{p}}\cdot\mathbf{S}_{\mathrm{e}}\right)|\Psi(0)|^2\tag{12.5.1.7}$$

Like the Darwin term, this depends on the probability of finding the electron at the origin. The value of $\mathbf{S}_{\mathrm{p}}\cdot\mathbf{S}_{\mathrm{e}}$ can be found by squaring $\mathbf{F}$, which with $l=0$ gives

$$F^2 = S_e^2 + S_p^2 + 2\mathbf{S}_{\mathrm{e}} \cdot \mathbf{S}_{\mathrm{p}} \tag{12.5.1.8}$$

Hence

$$\mathbf{S}_{\mathrm{p}} \cdot \mathbf{S}_{\mathrm{e}} = \frac{\hbar^2}{2}\left[f(f+1) - s_p\left(s_p+1\right) - s_e\left(s_e+1\right)\right] = \frac{\hbar^2}{2}\left[f(f+1) - \frac{3}{2}\right], \tag{12.5.1.9}$$

where the last step includes the values $s_e = s_p = 1/2$. The hyperfine energy shift for $l=0$ is then

$$\Delta E_{hf} = \left(\frac{m}{M}\right) \alpha^4 mc^2 \frac{4\gamma_p}{3n^3}\left[f(f+1) - \frac{3}{2}\right]. \tag{12.5.1.10}$$

It is easy to see from this expression that the hyperfine splittings are smaller than fine structure by a factor of M/m. For the specific case of the ground state of the hydrogen atom $(n=1)$, the energy separation between the states of $f=1$ and $f=0$ is

$$\Delta E_{hf}(f=1) - \Delta E_{hf}(f=0) = 5.9 \times 10^{-6}\mathrm{eV} \tag{12.5.1.11}$$

The photon corresponding to the transition between these two states has frequency and wavelength

$$\begin{aligned} \nu &= 1420.4057517667(10)\mathrm{MHz} \\ \lambda &= 21.1 \text{ cm.} \end{aligned}$$

This is the source of the famous " 21 cm line," which is extremely useful to radio astronomers for tracking hydrogen in the interstellar medium of galaxies. The transition is exceedingly slow, but the huge amounts of interstellar hydrogen make it readily observable. It is too slow to be seen in a terrestrial laboratory by spontaneous emission, but the frequency can be measured to very high accuracy by using stimulated emission, and this frequency is in fact one of the best-known numbers in all of physics.

For $l \neq 0$, the δ term does not contribute but the other terms in the spin-spin Hamiltonian as well as the nuclear spin-orbit Hamiltonian do contribute. The calculation is much harder but yields

$$\Delta E_{hf} = \left(\frac{m}{M}\right) \alpha^4 mc^2 \frac{\gamma_p}{2n^3} \frac{\pm 1}{\left(f+\frac{1}{2}\right)\left(l+\frac{1}{2}\right)} \tag{12.5.1.12}$$

for $f = j \pm 1/2$.

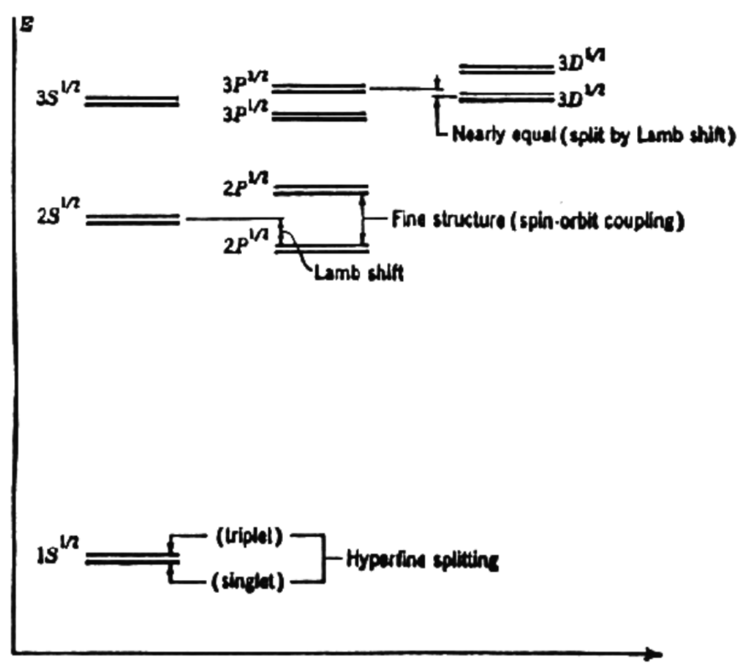

Figure 12.5.1.1: Some lowenergy states of the hydrogen atom including ne struc ture hyperne structure and the Lamb shift

Figure $12.5.1.1$ shows a revised version of the structure of the hydrogen atom, including the Lamb shift and hyperfine structure. Note that each hyperfine state still has a $2f+1$ degeneracy associated with the different possible values of m_f which correspond to different orientations of the total angular momentum with respect to the z-axis. For example, in the ground state, the higher-energy state $f=1$ is actually a triplet, consisting of three degenerate states, and the $f=0$ state is a singlet. This degeneracy can be broken by the presence of an external magnetic field.

Contributors and Attributions

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