11.4: The hydrogen atom

For the hydrogen atom we have a Coulomb force exerted by the proton forcing the electron to orbit around it. Since the proton is 1837 heavier than the electron, we can ignore the reverse action. The potential is thus

$$V(r) = -\frac{e^2}{4\pi\epsilon_0 r} \tag{11.4.1}$$

If we substitute this in the Schrödinger equation for $u(r)$, we find

$$-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial r^2}u(r) - \frac{e^2}{4\pi\epsilon_0 r}u(r) = Eu(r). \tag{11.4.2}$$

The way to attack this problem is once again to combine physical quantities to set the scale of length, and see what emerges. From a dimensional analysis we find that the length scale is set by the Bohr radius a_0,

$$a_0 = \frac{4\pi\epsilon_0\hbar^2}{me^2} = 0.53 \times 10^{-10}\ \mathrm{m} \tag{11.4.3}$$

The scale of energy is set by these same parameters to be

$$\frac{e^2}{4\pi\epsilon_0 a_0} = 2\mathrm{Ry} \tag{11.4.4}$$

and one Ry (Rydberg) is 13.6 eV . Solutions can be found by a complicated argument similar to the one for the Harmonic oscillator, but (without proof) we have

$$E_n = -\frac{1}{2}\Box\frac{e^2}{4\pi\epsilon_0 a_0}\left[\frac{1}{n^2}\right. = -13.6\frac{1}{n^2}\mathrm{eV} \tag{11.4.5}$$

and

$$R_n = e^{-r(na_0)}\left(c_0 + c_1 r + \ldots + c_{n-1}r^{n-1}\right) \tag{11.4.6}$$

The explicit, and normalised, forms of a few of these states are

$$\begin{aligned} R_1(r) &= \tfrac{1}{\sqrt{4\pi}}2a_0^{-3/2}e^{-r/a_0}, \\ R_2(r) &= \tfrac{1}{\sqrt{4\pi}}2(2a_0)^{-32}\left[1 - \tfrac{r}{2a_0}\Box e^{-r(2a_0)}\right.. \end{aligned} \tag{11.4.7}$$

Remember these are normalised to

$$\int_0^\infty R_n(r)^* R_m(r)dr = \delta_{nm} \tag{11.4.8}$$

Notice that there are solution that do depend on θ and φ as well, and that we have not looked at such solutions here!