9.2: The operators Â and Â†.

In a previous chapter I have discussed a solution by a power series expansion. Here I shall look at a different technique, and define two operators $\hat{a}$ and $\hat{a}^{\dagger}$,

$$\hat{a}=\frac{1}{\sqrt{2}}\left(y+\frac{d}{dy}\right), \quad \hat{a}^{\dagger}=\frac{1}{\sqrt{2}}\left(y-\frac{d}{dy}\right). \tag{9.2.1}$$

Since

$$\frac{d}{dy}(y f(y))=y \frac{d}{dy} f(y)+f(y) \tag{9.2.2}$$

or in operator notation

$$\frac{d}{dy} \hat{y}=\hat{y} \frac{d}{dy}+\hat{1} \tag{9.2.3}$$

(the last term is usually written as just 1) we find

$$\begin{aligned} \hat{a} \hat{a}^{\dagger} &=\tfrac{1}{2}\left(\hat{y}^{2}-\tfrac{d^{2}}{dy^{2}}+\hat{1}\right) \\ \hat{a}^{\dagger} \hat{a} &=\tfrac{1}{2}\left(\hat{y}^{2}-\tfrac{d^{2}}{dy^{2}}-\hat{1}\right) \end{aligned} \tag{9.2.4}$$

If we define the commutator

$$[\hat{f}, \hat{g}]=\hat{f} \hat{g}-\hat{g} \hat{f} \tag{9.2.5}$$

we have

$$\left[\hat{a}, \hat{a}^{\dagger}\right]=\hat{1} \tag{9.2.6}$$

Now we see that we can replace the eigenvalue problem for the scaled Hamiltonian by either of

$$\begin{aligned} \left(\hat{a}^{\dagger} \hat{a}+\frac{1}{2}\right) u(y) &=\epsilon u(y) \\ \left(\hat{a} \hat{a}^{\dagger}-\frac{1}{2}\right) u(y) &=\epsilon u(y) \end{aligned}$$

By multiplying the first of these equations by $\hat{a}$ we get

$$\left(\hat{a} \hat{a}^{\dagger} \hat{a}+\frac{1}{2} \hat{a}\right) u(y)=\epsilon \hat{a} u(y). \tag{9.2.7}$$

If we just rearrange some brackets, we find

$$\left(\hat{a} \hat{a}^{\dagger}+\frac{1}{2}\right) \hat{a} u(y)=\epsilon \hat{a} u(y). \tag{9.2.8}$$

If we now use

$$\hat{a} \hat{a}^{+}=\hat{a}^{\dagger} \hat{a}-\hat{1}, \tag{9.2.9}$$

we see that

$$\left(\hat{a}^{\dagger} \hat{a}+\frac{1}{2}\right) \hat{a} u(y)=(\epsilon-1) \hat{a} u(y). \tag{9.2.10}$$

Question: Show that

$$\left(\hat{a}^{\dagger} \hat{a}+\frac{1}{2}\right) \hat{a}^{\dagger} u(y)=(\epsilon+1) \hat{a}^{\dagger} u(y). \tag{9.2.11}$$

We thus conclude that (we use the notation $u_n(y)$ for the eigenfunction corresponding to the eigenvalue ϵ_n)

$$\begin{aligned}\hat{a}u_n(y) &\propto u_{n-1}(y),\\ \hat{a}^\dagger u_n(y) &\propto u_{n+1}(y).\end{aligned}$$

So using $\hat{a}$ we can go down in eigenvalues, using $a^\dagger$ we can go up. This leads to the name lowering and raising operators (guess which is which?).

We also see from $9.2.12$ that the eigenvalues differ by integers only!