8.1: Key Postulates

1. Every single system has a wave function $\psi(x,t)$.
2. Every observable is represented by a Hermitean operator $\widehat{O}$.
3. The expectation value (average outcome of a measurement) is given by $\int \phi(x)^* \widehat{O}\phi(x)dx$
4. The outcome of an individual experiment can be any of the eigenvalues of $\widehat{O}$.

Let me take each of these in turn.

Wavefunction

The detailed statement is that: for every physical system there exists a wave function, a function of the parameters of the system (coordinates and such) and time, from which the outcome of any experiment can be predicted.

In these lectures I will not touch on systems that depend on other parameters than coordinates, but examples are known, such as the spin of an electron, which can be up or down, and is not like a coordinate at all.

Observables

In classical mechanics "observables" (the technical term for anything that can be measured) are represented by numbers. Think e.g., of $x, y, z, p_x, p_y, p_z, E$, In quantum mechanics "observables" are often quantised, they cannot take on all possible values: how to represent such quantities?

We have already seen that energy and momentum are represented by operators,

$$\hat{p} = -i\hbar\nabla = -i\hbar\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right). \tag{8.1.1}$$

and

$$\widehat{H} = -\frac{\hbar^2 d^2}{2m dx^2} + V(x) \tag{8.1.2}$$

Let me look at the Hamiltonian, the energy operator. We know that its normalisable solutions (eigenvalues) are discrete.

$$\widehat{H}\phi_n(x) = E_n\phi_n(x). \tag{8.1.3}$$

The numbers E_n are called the eigenvalues, and the functions $\phi_n(x)$ the eigenfunctions of the operator $\overset{\square}{H}$. Our postulate says that the only possible outcomes of any experiment where we measure energy are the values E_n !

Hermitean operators

Hermitean operators are those where the outcome of any measurement is always real, as they should be (complex position?). This means that both its eigenvalues are real, and that the average outcome of any experiment is real. The mathematical definition of a Hermitean operator can be given as

\int_{\text {all space }} \psi_1^*(x) \widehat{ O } \psi _ { 2 } (x) d x=\int_{\text {all space }} \widehat{O} \psi_1(x)\right]^* \psi_2(x) d x.

Quiz show that $\widehat{x}$ and $\hat{p}$ (in 1 dimension) are Hermitean.

Eigenvalues of Hermitean operators

Eigenvalues and eigen vectors of Hermitean operators are defined as for matrices, i.e., where there is a matrix-vector product we get an operator acting on a function, and the eigenvalue/function equation becomes

$$\widehat{O}f(x) = o_n f(x), \tag{8.1.4}$$

where o_n is a number (the "eigenvalue") and $f(x)$ is the "eigenfunction".

A list of important properties of the eigenvalue-eigenfunction pairs for Hermitean operators are:

1. The eigenvalues of an Hermitean operator are all real.
2. The eigenfunctions for different eigenvalues are orthogonal.
3. The set of all eigenfunction is complete.

- Ad 1. Let $\phi_n(x)$ be an eigenfunction of $\widehat{O}$. Use

$$o_n = \int dx \phi_n(x)^* \widehat{O} \phi_n(x) = \int dx \widehat{O} \phi_n(x)^* \phi_n(x) = o_n^* \tag{8.1.5}$$

- Ad 2. Let $\phi_n(x)$ and $\phi_m(x)$ be eigenfunctions of $\widehat{O}$. Use

This leads to

$$(o_n - o_m) \int dx \phi_n(x)^* \phi_n(x) = 0, \tag{8.1.6}$$

and if $o_n \neq o_m \int dx \phi_n(x)^* \phi_n(x) = 0$, which is the definition of two orthogonal functions.

- Ad 3. This is more complex, and no proof will be given. It means that any function can be written as a sum of eigenfunctions of $\widehat{O}$,

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \tag{8.1.7}$$

- Ad 3. This is more complex, and no proof will be given. It means that any function can be written as a sum of eigenfunctions of O,

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x) \tag{8.1.8}$$

(A good example of such a sum is the Fourier series.)

Outcome of a single experiment

The outcome of a measurement of any quantity can only be the set of natural values of such a quantity. These are just the eigenvalues of $\widehat{O}$

$$\widehat{O} f_n(x) = o_n f_n(x) \tag{8.1.9}$$

Is this immediately obvious from the formalism? The short answer is no, but suppose we measure the value of the obervable for a wave function known to be an eigenstate. The outcome of a measurement better be this eigenvalue and nothing else. This leads us to surmise that this rule holds for any wave function, and we get the answer we are looking for. This also agrees with the experimentally observed quantisation of observables such as energy.

Eigenfunctions of $\widehat{x}$

The operator $\widehat{x}$ multiplies with x. Solving the equation

$$\widehat{x}^{\square} \phi(x) = x_0 \phi(x) \tag{8.1.10}$$

we find that the solution must be exactly localised at $x = x_0$. The function that does that is called a Dirac δ function $\delta(x - x_0)$. This is defined through integration,

$$\int_{-\infty}^{\infty} \delta(x - x_0) g(x) dx = g(x_0) \tag{8.1.11}$$

and is not normalisable,

$$\int \delta(x - x_0)^2 dx = \infty \tag{8.1.12}$$

Eigenfunctions of $\hat{p}$

The operator $\hat{p}$ is $-i\hbar\frac{\partial}{\partial x}$. Solving the equation

$$-i\hbar\frac{d}{dx}\phi(x) = p_0\phi(x) \tag{8.1.13}$$

we get

$$\frac{d}{dx}\phi(x) = ip_0\hbar\phi(x) \tag{8.1.14}$$

with solution

$$\phi(x) = e^{ip_0xh} \tag{8.1.15}$$

a "plane wave". As we have seen before these states aren't normalised either!