9.3: Eigenfunctions of H through ladder operations

If we start with the ground state we would expect that we can't go any lower,

$$\hat{a} u_0(y) = 0. \tag{9.3.1}$$

This can of course be checked explicitly,

$$\begin{aligned} \hat{a} e^{-y^2/2} &= \frac{1}{\sqrt{2}}\left(\hat{y} + \frac{d}{dy}\right) e^{-y^2/2} \\ &= \frac{1}{\sqrt{2}}\left(y e^{-y^2/2} - y e^{-y^2/2}\right) \\ &= 0 \end{aligned}$$

Quiz Can you show that $\boldsymbol{\epsilon_0 = 1/2}$ using the operators $\hat{\boldsymbol{a}}$?

Once we know that $\epsilon_0 = 1/2$, repeated application of

$$(a^\dagger a^+ 12) a^\dagger u(y) = (\epsilon + 1) a^\dagger u(y) \tag{9.3.2}$$

from the previous page shows that $\epsilon_n = n + 1/2$, which we know to be correct from our previous treatment.

Actually, once we know the ground state, we can now easily determine all the Hermite polynomials up to a normalisation constant:

$$\begin{aligned} u_1(y) &\propto a^\dagger e^{-y^2/2} \\ &= \frac{1}{\sqrt{2}}\left(\hat{y} - \frac{d}{dy}\right) e^{-y^2/2} \\ &= \frac{1}{\sqrt{2}}\left(y e^{-y^2/2} + y e^{-y^2/2}\right) \\ &= \sqrt{2} y e^{-y^2/2} \end{aligned}$$

Indeed $H_1(y) \propto y$.

From math books we can learn that the standard definition of the Hermite polynomials corresponds to

$$H_n(y) e^{-y^2/2} = (\sqrt{2})^n \left(\hat{a}^\dagger\right)^n e^{-y^2/2} \tag{9.3.3}$$

We thus learn $H_1(y) = 2y$ and $H_2(y) = \left(4y^2 - 2\right)$.

Question: Prove this last relation.