11.6: Spherical Harmonics

The key issue about three-dimensional motion in a spherical potential is angular momentum. This is true classically as well as in quantum theories. The angular momentum in classical mechanics is defined as the vector (outer) product of r and p,

$$L = r \times p. \tag{11.6.1}$$

This has an easy quantum analog that can be written as

$$\hat{L} = \hat{r} \times \hat{p} \tag{11.6.2}$$

After exapnsion we find

$$\hat{L} = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}, z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}, x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right) \tag{11.6.3}$$

This operator has some very interesting properties:

$$[\hat{L}, \hat{r}] = 0. \tag{11.6.4}$$

Thus

$$[\hat{\boldsymbol{L}}, \hat{H}] = 0! \tag{11.6.5}$$

And even more surprising,

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$$\left[\hat{L}_x, \hat{L}_y\right] = i\hbar\hat{L}_z. \tag{11.6.6}$$

Thus the different components of L are not compatible (i.e., can't be determined at the same time). Since L commutes with H we can diagonalise one of the components of L at the same time as H. Actually, we diagonalsie $\hat{L}^2$, $\hat{L}_z$ and H at the same time!

The solutions to the equation

$$\hat{L}^2 Y_{LM}(\theta, \phi) = \hbar^2 L(L+1) Y_{LM}(\theta, \phi) \tag{11.6.7}$$

are called the spherical harmonics.

Question: check that $\hat{\boldsymbol{L}}^2$ is independent of $\boldsymbol{r}$!

The label M corresponds to the operator $\hat{L}_z$,

$$\hat{L}_z Y_{LM}(\theta, \phi) = \hbar M Y_{LM}(\theta, \phi). \tag{11.6.8}$$