

## 9.4: Normalisation and Hermitean conjugates

If you look at the expression  $\int_{-\infty}^{\infty} f(y)^* \hat{a}^\dagger g(y) dy$  and use the explicit form  $\hat{a}^\dagger = \frac{1}{\sqrt{2}} \left( y - \frac{d}{dy} \right)$ , you may guess that we can use partial integration to get the operator acting on  $f$ ,

$$\begin{aligned} \int_{-\infty}^{\infty} f(y)^* \hat{a}^\dagger g(y) dy &= \int_{-\infty}^{\infty} f(y)^* \frac{1}{\sqrt{2}} \left( y - \frac{d}{dy} \right) g(y) dy \\ &= \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}} \left( y + \frac{d}{dy} \right) f(y)^* g(y) dy \\ &= \int_{-\infty}^{\infty} [\hat{a} f(y)]^* g(y) dy \end{aligned}$$

This is the first example of an operator that is clearly not Hermitean, but we see that  $\hat{a}$  and  $\hat{a}^\dagger$  are related by "Hermitean conjugation". We can actually use this to normalise the wave function! Let us look at

$$\begin{aligned} O_n &= \int_{-\infty}^{\infty} \left[ (\hat{a}^\dagger)^n e^{-y^2/2} \right]^* (\hat{a}^\dagger)^n e^{-y^2/2} dy \\ &= \int_{-\infty}^{\infty} \left[ \hat{a} (\hat{a}^\dagger)^n e^{-y^2/2} \right]^* (\hat{a}^\dagger)^{n-1} e^{-y^2/2} dy \end{aligned}$$

If we now use  $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger\hat{a} + \hat{1}$  repeatedly until the operator  $\hat{a}$  acts on  $u_0(y)$ , we find

$$O_n = n O_{n-1} \quad (9.4.1)$$

Since  $O_0 = \sqrt{\pi}$ , we find that

$$u_n(y) = \frac{1}{\sqrt{n! \sqrt{\pi}}} (\hat{a}^\dagger)^n e^{-y^2/2} \quad (9.4.2)$$

**Question:** Show that this agrees with the normalisation proposed in the previous study of the harmonic oscillator!

**Question:** Show that the states  $u_n$  for different  $n$  are orthogonal, using the techniques sketched above.

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