12.1: The Uncertainty Principle

Before discussing specifics about the structure of the hydrogen atom, it is interesting to note what information about the hydrogen atom can be derived just from the Heisenberg uncertainty principle. A familiar form of the uncertainty principle looks like the following:

$$\Delta x \Delta p_x \sim \hbar \tag{12.1.1}$$

where Δx and Δp_x are the uncertainty in the x-component of the position and momentum of a particle, respectively. Consider an electron in a classical circular orbit in the xy-plane. It is then reasonable to write $\Delta x \sim r$, where r is the radius of the orbit. Assuming a state of minimum uncertainty, Δp_x is then known from the uncertainty principle, and it should be roughly equal

to the magnitude of the momentum for the circular orbit being considered. That is,

$$p \sim \Delta p_x \sim \frac{\hbar}{r}. \tag{12.1.2}$$

Classically, the energy is simply [1]

$$E = \frac{p^2}{2m} - \frac{e^2}{r} = \frac{\hbar^2}{2mr^2} - \frac{e^2}{r}, \tag{12.1.3}$$

where m is the electron mass and e the electron charge. The last step results from the substitution of p from equation 2 . The value of r is unknown, but one would expect it to have a value that minimizes the energy, as Nature likes to do. Differentiating equation 3 with respect to r and setting equal to zero gives

$$\frac{dE}{dr} = -\frac{\hbar^2}{mr^3} + \frac{e^2}{r^2} = 0. \tag{12.1.4}$$

This yields

$$r = \frac{\hbar^2}{me^2} \equiv a_0 = 0.529\text{Å}, \tag{12.1.5}$$

where a_0 is the Bohr radius. Substituting into equation 3 gives

$$E = -13.6\text{eV} \tag{12.1.6}$$

The Bohr radius is exactly the radius of the circular orbit in the ground state of the electron in Bohr theory, and it holds up as representative of the extent of the orbit in Schrödinger theory. The energy -13.6 eV is the known ground state energy of the hydrogen atom. So, starting with only a very rough view of the structure of the atom and the uncertainty principle, one can make some reasonable assumptions and derive two extremely important fundamental results - the "size" if the hydrogen atom in its ground state and its ionization energy. Of course, to get precisely the right results one needs to make the right assumptions, and so this calculation is certainly not rigorously accurate. It merely illustrates the relation of the fundamental physical structure of the hydrogen atom to the uncertainty principle. The fact that these results were derived assuming minimum uncertainty leads to a rather important conclusion-the hydrogen atom in its ground state is essentially in a state of minimum uncertainty. This explains why the electron in its ground state cannot radiate, as one expects classically, and get drawn in towards the nucleus - to do so would violate the uncertainty principle. If the electron were confined closer to the nucleus, so that Δx were much smaller, then Δp_x would be much larger and so it would not be possible to consider the electron as necessarily bound to the nucleus.

1. To achieve consistency and avoid confusion all equations are written in Gaussian units

Contributors and Attributions

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