9.4: Normalisation and Hermitean conjugates

If you look at the expression $\int_{-\infty}^{\infty} f(y)^* \hat{a}^\dagger g(y) dy$ and use the explicit form $\hat{a}^\dagger = \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right)$, you may guess that we can use partial integration to get the operator acting on f,

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(y)^* \hat{a}^\dagger g(y) dy \\
&\quad = \int_{-\infty}^{\infty} f(y)^* \frac{1}{\sqrt{2}}\left(y - \frac{d}{dy}\right) g(y) dy \\
&\quad = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(y + \frac{d}{dy}\right) f(y)^* g(y) dy \\
&\quad = \int_{-\infty}^{\infty} [\hat{a} f(y)]^* g(y) dy
\end{aligned}$$

This is the first example of an operator that is clearly not Hermitean, but we see that $\hat{a}$ and $\hat{a}^\dagger$ are related by "Hermitean conjugation". We can actually use this to normalise the wave function! Let us look at

$$\begin{aligned}
O_n &= \int_{-\infty}^{\infty} \left[\left(\hat{a}^\dagger\right)^n e^{-y^2/2}\right]^* \left(\hat{a}^\dagger\right)^n e^{-y^2/2} dy \\
&= \int_{-\infty}^{\infty} \left[\hat{a}\left(\hat{a}^\dagger\right)^n e^{-y^2/2}\right]^* \left(\hat{a}^\dagger\right)^{n-1} e^{-y^2/2} dy
\end{aligned}$$

If we now use $\hat{a}\hat{a}^\dagger = \hat{a}^\dagger \hat{a} + \hat{1}$ repeatedly until the operator $\hat{a}$ acts on $u_0(y)$, we find

$$O_n = n O_{n-1} \tag{9.4.1}$$

Since $O_0 = \sqrt{\pi}$, we find that

$$u_n(y) = \frac{1}{\sqrt{n! \sqrt{\pi}}} \left(\hat{a}^+\right)^n e^{-y^2/2} \tag{9.4.2}$$

Question: Show that this agrees with the normalisation proposed in the previous study of the harmonic oscillator!

Question: Show that the states u_n for different n are orthogonal, using the techniques sketched above.