10.5: Wave packets (states of minimal uncertainty)

One of the questions of some physical interest is "how can we create a qunatum-mechanical state that behaves as much as a classical particle as possible?" From the uncertainty principle,

$$\Delta x \Delta p \geq \frac{1}{2}\hbar \tag{10.5.1}$$

this must be a state where Δx and Δp are both as small as possible. Such a state is known as a "wavepacket". We shall see below (and by using a computer demo) that its behavior depends on the Hamiltonian governing the system that we are studying!

Let us start with the uncertainty in x. A state with width $\Delta x = \sigma$ should probably be a Gaussian, of the form

$$\psi(x,t) = \exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)A(x). \tag{10.5.2}$$

In order for ψ to be normalised, we need to require

$$|A(x)|^2 = \sqrt{\frac{1}{\sigma^2\pi}}. \tag{10.5.3}$$

Actually, I shall show below that with

$$A(x) = \sqrt[4]{\frac{1}{\sigma^2\pi}}e^{ip_0x/\hbar} \tag{10.5.4}$$

we have

$$\langle\hat{x}\rangle = x_0, \quad \langle\hat{p}\rangle = p_0, \quad \Delta x = \sigma, \quad \Delta p = \hbar/\sigma \tag{10.5.5}$$

The algebra behind this is relatively straightforward, but I shall just assume the first two, and only do the last two in all gory details.

$$\hat{p}\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)A(x) = \left(p_0 + i\hbar\frac{(x-x_0)}{\sigma^2}\right)\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)A(x) \tag{10.5.6}$$

Thus

$$\langle p\rangle = \sqrt{\frac{1}{\sigma^2\pi}}\int_{-\infty}^{\infty}\left(p_0 + i\hbar\frac{(x-x_0)}{\sigma^2}\right)\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right) = p_0. \tag{10.5.7}$$

Let $\hat{p}$ act twice,

$$\hat{p}^2\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)A(x) = \left(p_0^2 + 2i\hbar p_0\frac{(x-x_0)}{\sigma^2} - \hbar^2\left[\frac{(x-x_0)^2}{\sigma^4} - \frac{1}{\sigma^2}\right]\right)\times$$
$$\exp\left(-\frac{(x-x_0)^2}{2\sigma^2}\right)A(x).$$

Doing all the integrals we conclude that

$$\langle p^2\rangle = p_0^2 + \frac{\hbar^2}{2\sigma^2} \tag{10.5.8}$$

Thus, finally,

$$\Delta p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \frac{\hbar}{\sigma} \tag{10.5.9}$$

This is just the initial state, which clearly has minimal uncertainty. We shall now investigate how the state evolves in time by usin a numerical simulation. What we need to do is to decompose our state of minimal uncertainty in a sum over eigenstates of the Hamiltonian which describes our system!
