7.5: A Few Solutions

The polynomial solutions occur for

$$\epsilon_n = \left(n + \frac{1}{2}\right). \tag{7.5.1}$$

The terminating solutions are the ones that contains only even coefficients for even n and odd coefficients for odd n. Let me construct a few, using the relation (7.16). For n even I start with $a_0 = 1, a_1 = 0$, and for n odd I start with $a_0 = 0, a_1 = 1$,

$$\begin{aligned} H_0(y) &= 1 \\ H_1(y) &= y \\ H_2(y) &= 1 - 2y^2, \\ H_3(y) &= y - \tfrac{2}{3}y^3. \end{aligned} \tag{7.5.2}$$

Question: Can you reproduce these results? What happens if I start with $a_0 = 0, a_1 = 1$ for, e.g., H_0 ?

In summary: The solutions of the Schrödinger equation occur for energies $\left(n + \frac{1}{2}\right)\hbar\omega$, an the wavefunctions are

$$\phi_n(x) \propto H_n \sqrt{\frac{m\omega}{\hbar}} x \exp\left(-\frac{m\omega}{\hbar}x^2\right) \tag{7.5.3}$$

(In analogy with matrix diagonalisation one often speaks of eigenvalues or eigenenergies for E, and eigenfunctions for ϕ.)
Once again it is relatively straightforward to show how to normalise these solutions. This can be done explicitly for the first few polynomials, and we can also show that

$$\int_{-\infty}^{\infty} \phi_{n_1}(x)\phi_{n_2}(x)dx = 0 \quad \text{if} \quad n_1 \neq n_2. \tag{7.5.4}$$

This defines the orthogonality of the wave functions. From a more formal theory of the polynomials $H_n(y)$ it can be shown that the normalised form of $\phi_n(x)$ is

$$\phi_n(x) = 2^{-m2}(n!)^{-12}\left[\frac{m\omega}{\hbar\pi}\right]^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) H_n\left[\frac{m\omega}{\hbar}x\right]. \tag{7.5.5}$$