7.3: Behaviour for large |y|

Before solving the equation we are going to see how the solutions behave at large $|y|$ (and also large $|x|$, since these variable are proportional!). For $|y|$ very large, whatever the value of ϵ, $\epsilon \ll y^2$, and thus we have to solve

$$\frac{d^2u}{dy^2} = y^2 u(y) \tag{7.3.1}$$

This has two type of solutions, one proportional to $e^{y^2/2}$ and one to $e^{-y^2/2}$. We reject the first one as being not normalisable.

Question: Check that these are the solutions. Why doesn't it matter that they don't exactly solve the equations?

Substitute $u(y) = H(y)e^{-y^2/2}$. We find

$$\frac{d^2u}{dy^2} = \left[H''(y) - 2yH'(y) + y^2H(y)]e^{-y^2/2}\right]. \tag{7.3.2}$$

so we can obtain a differential equation for $H(y)$ in the form

$$H''(y) - 2yH'(y) + (2\epsilon - 1)H(y) = 0. \tag{7.3.3}$$

This equation will be solved by a substitution and infinite series (Taylor series!), and showing that it will have to terminates somewhere, i.e., $H(y)$ is a polynomial!