8.2: Expectation Values of x² and p² for the Harmonic Oscillator

As an example of all we have discussed let us look at the harmonic oscillator. Suppose we measure the average deviation from equilibrium for a harmonic oscillator in its ground state. This corresponds to measuring $\bar{x}$. Using

1. $$\phi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{8.2.1}$$

we find that

$$\langle x \rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} x \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx = 0. \tag{8.2.2}$$

Qn Why is it 0 ? Similarly, using $\hat{p} = -i\hbar\frac{d}{dx}$ and

$$\hat{p}\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) = im\ \omega x\left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \tag{8.2.3}$$

we find

$$\langle p \rangle = 0 \tag{8.2.4}$$

More challenging are the expectation values of x^2 and p^2. Let me look at the first one first:

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{\infty} \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} x^2 exp\left(-\frac{m\omega}{\hbar\pi}x^2\right) dx \\ &= \left(\frac{\hbar}{m\omega}\right) \pi^{-1/2} \int_{-\infty}^{\infty} exp\left(-y^2\right) dy \\ &= \left(\frac{\hbar}{m\omega}\right) \frac{1}{2} \end{aligned} \tag{8.2.5}$$

Now for $\hat{p}^2$,

$$\hat{p}^2 \exp\left(-\frac{m\omega}{2\hbar}x^2\right) = \left(-(m\omega x)^2 + \hbar m\omega\right) \exp\left(-\frac{m\omega}{2\hbar}x^2\right). \tag{8.2.6}$$

Thus,

$$\begin{aligned} &\left\langle \hat{p}^2 \right\rangle = \int_{-\infty}^{\infty} \left(\frac{m\omega}{\hbar\pi}\right)^{1/2} \left(-(m\omega x)^2 + \hbar m\omega\right) \exp\left(-\frac{m\omega}{\hbar}x^2\right) dx \\ &= \hbar m\omega\pi^{-1/2} \int_{-\infty}^{\infty} \left(1 - y^2\right) \exp\left(-y^2\right) dy \\ &= \hbar m\omega\frac{1}{2}. \end{aligned} \tag{8.2.7}$$

This is actually a form of the uncertainity relation, and shows that

$$\Delta x \Delta p \geq \frac{1}{2}\hbar! \tag{8.2.8}$$