9.1: Lorentz Transformations of Energy and Momentum

As you may know, like we can combine position and time in one four-vector $x=(\vec{x}, ct)$, we can also combine energy and momentum in a single four-vector, $p=(\vec{p}, E/c)$. From the Lorentz transformation property of time and position, for a change of velocity along the x-axis from a coordinate system at rest to one that is moving with velocity $\vec{v}=(v_x,0,0)$ we have

$$x' = \gamma(v)(x - v/ct), \tag{9.1.1}$$

$$t' = \gamma(t - xvx/c^2), \tag{9.1.2}$$

we can derive that energy and momentum behave in the same way,

$$p'_x = \gamma(v)(p_x - Ev/c^2) \tag{9.1.3}$$

$$= m u_x \gamma(|u|),$$

$$E' = \gamma(v)(E - vp_x) \tag{9.1.4}$$

$$= \gamma(|u|) m_0 c^2. \tag{9.1.5}$$

To understand the context of these equations remember the definition of γ

$$\gamma(v) = \frac{1}{\sqrt{1-\beta^2}}$$

and

$$\beta = \frac{v}{c}.$$

In Equation 9.1.5, we have also re-expressed the momentum energy in terms of a velocity $\vec{u}$. This is measured relative to the rest system of a particle, the system where the three-momentum $\vec{p}=0$.

Now all these exercises would be interesting mathematics but rather futile if there was no further information. We know however that the full four-momentum is conserved, i.e., if we have two particles coming into a collision and two coming out, the sum of four-momenta before and after is equal,

$$\begin{aligned} E_1^{\text{in}} + E_2^{\text{in}} &= E_1^{\text{out}} + E_2^{\text{out}}, \\ \vec{p}_1^{\,\text{in}} + \vec{p}_2^{\,\text{in}} &= \vec{p}_1^{\,\text{out}} + \vec{p}_2^{\,\text{out}}. \end{aligned}$$