11.1: Exercises

1. Consider the two-state system investigated in Section 1.3. Show that the most general expressions for the perturbed energy eigenvalues and eigenstates are

$$\begin{aligned} E_1' &= E_1 + e_{11} + \frac{|e_{12}|^{\,2}}{E_1 - E_2} + \mathcal{O}(\epsilon^{\,3}), \\ E_2' &= E_2 + e_{22} - \frac{|e_{12}|^{\,2}}{E_1 - E_2} + \mathcal{O}(\epsilon^{\,3}), \end{aligned}$$

and

$$\begin{aligned} \psi_1' &= \psi_1 + \frac{e_{12}^{\,*}}{E_1 - E_2}\,\psi_2 + \mathcal{O}(\epsilon^{\,2}), \\ \psi_2' &= \psi_2 - \frac{e_{12}}{E_1 - E_2}\,\psi_1 + \mathcal{O}(\epsilon^{\,2}), \end{aligned}$$

respectively. Here, $\epsilon = |e_{12}|/(E_1 - E_2) \ll 1$. You may assume that $|e_{11}|/(E_1 - E_2)$, $|e_{22}|/(E_1 - E_2) \sim \mathcal{O}(\epsilon)$.

2. Consider the two-state system investigated in Section 1.3. Show that if the unperturbed energy eigenstates are also eigenstates of the perturbing Hamiltonian then

$$\begin{aligned} E_1' &= E_1 + e_{11}, \\ E_2' &= E_2 + e_{22}, \end{aligned}$$

and

$$\begin{aligned} \psi_1' &= \psi_1 \\ \psi_2' &= \psi_2 \end{aligned}$$

to all orders in the perturbation expansion.

3. Consider the two-state system investigated in Section 1.3. Show that if the unperturbed energy eigenstates are degenerate, so that $E_1 = E_2 = E_{12}$, then the most general expressions for the perturbed energy eigenvalues and eigenstates are

$$E^{\pm} = E_{12} + e^{\pm}, \tag{11.1.1}$$

and

$$\psi^{\pm} = \langle 1|\psi^{\pm}\rangle\,\psi_1 + \langle 2|\psi^{\pm}\rangle\,\psi_2, \tag{11.1.2}$$

respectively, where

$$e^{\pm} = \frac{1}{2}\,(e_{11} + e_{22}) \pm \frac{1}{2}\left[(e_{11} - e_{22})^2 + 4\,|e_{12}|^{\,2}\right]^{\,1/2}, \tag{11.1.3}$$

and

$$\frac{\langle 1|\psi^{\pm}\rangle}{\langle 2|\psi^{\pm}\rangle} = -\left(\frac{e_{12}}{e_{11} - e^{\pm}}\right) = -\left(\frac{e_{22} - e^{\pm}}{e_{12}^{\,*}}\right). \tag{11.1.4}$$

Demonstrate that the $\psi^{\pm}$ are the simultaneous eigenstates of the unperturbed Hamiltonian, H_0, and the perturbed Hamiltonian, H_1, and that the $e^{\pm}$ are the corresponding eigenvalues of H_1.

4. Calculate the lowest-order energy-shift in the ground state of the one-dimensional harmonic oscillator when the perturbation

$$V = \lambda\,x^{\,4} \tag{11.1.5}$$

is added to

$$H = \frac{p_x^2}{2\,m} + \frac{1}{2}\,m\,\omega^{\,2}\,x^{\,2}. \tag{11.1.6}$$

Contributors and Attributions

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