9.5: Spin Precession

According to classical physics, a small current loop possesses a *magnetic moment* of magnitude $\mu = I\,A$, where I is the current circulating around the loop, and A the area of the loop. The direction of the magnetic moment is conventionally taken to be normal to the plane of the loop, in the sense given by a standard right-hand circulation rule. Consider a small current loop consisting of an electron in uniform circular motion. It is easily demonstrated that the electron's orbital angular momentum $\mathbf{L}$ is related to the magnetic moment μ of the loop via

$$\mu = -\frac{e}{2\,m_e}\,\mathbf{L}, \tag{9.5.1}$$

where e is the magnitude of the electron charge, and m_e the electron mass.

The previous expression suggests that there may be a similar relationship between magnetic moment and spin angular momentum. We can write

$$\mu = -\frac{g\,e}{2\,m_e}\,\mathbf{S}, \tag{9.5.2}$$

where g is called the *gyromagnetic ratio*. Classically, we would expect $g = 1$. In fact,

$$g = 2\left(1 + \frac{\alpha}{2\pi} + \cdots\right) = 2.0023192, \tag{9.5.3}$$

here $\alpha = e^{\,2}/(2\,\epsilon_0\,h\,c) \simeq 1/137$ is the so-called *fine-structure constant*. The fact that the gyromagnetic ratio is (almost) twice that expected from classical physics is only explicable using relativistic quantum mechanics . Furthermore, the small corrections to the relativistic result $g = 2$ come from quantum field theory .

The energy of a classical magnetic moment μ in a uniform magnetic field $\mathbf{B}$ is

$$H = -\mu\cdot\mathbf{B}. \tag{9.5.4}$$

Assuming that the previous expression also holds good in quantum mechanics, the Hamiltonian of an electron in a z-directed magnetic field of magnitude B takes the form

$$H = \Omega\,S_z, \tag{9.5.5}$$

where

$$\Omega = \frac{g\,e\,B}{2\,m_e}. \tag{9.5.6}$$

Here, for the sake of simplicity, we are neglecting the electron's translational degrees of freedom.

Schrödinger's equation can be written [see Equation ([etimed])]

$$\mathrm{i}\,\hbar\,\frac{\partial\chi}{\partial t} = H\,\chi, \tag{9.5.7}$$

where the spin state of the electron is characterized by the spinor χ. Adopting the Pauli representation, we obtain

$$\chi = \begin{pmatrix} c_+(t) \\ c_-(t) \end{pmatrix}, \tag{9.5.8}$$

where $|c_+|^{\,2} + |c_-|^{\,2} = 1$. Here, $|c_+|^{\,2}$ is the probability of observing the spin-up state, and $|c_-|^{\,2}$ the probability of observing the spin-down state. It follows from Equations ([e10.46]), ([e10.53]), ([e10.60]), ([e10.62]), and ([e10.63]) that

$$\mathrm{i}\,\hbar\begin{pmatrix} \dot{c}_+ \\ \dot{c}_- \end{pmatrix} = \frac{\Omega\,\hbar}{2}\begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}, \tag{9.5.9}$$

where $\dot{} \equiv d/dt$. Hence,

$$\dot{c}_\pm = \mp \mathrm{i}\,\frac{\Omega}{2}\,c_\pm \tag{9.5.10}$$

Let

$$c_+(0) = \cos(\alpha/2),$$
$$c_-(0) = \sin(\alpha/2).$$

The significance of the angle α will become apparent presently. Solving Equation ([e10.65]), subject to the initial conditions ([e10.66]) and ([e10.67]), we obtain

$$c_+(t) = \cos(\alpha/2)\,\exp(-\mathrm{i}\,\Omega\,t/2),$$
$$c_-(t) = \sin(\alpha/2)\,\exp(+\mathrm{i}\,\Omega\,t/2).$$

We can most easily visualize the effect of the time dependence in the previous expressions for $c_\pm$ by calculating the expectation values of the three Cartesian components of the electron's spin angular momentum. By analogy with Equation ([e3.55]), the expectation value of a general spin operator A is simply

$$\langle A\rangle = \chi^\dagger\,A\,\chi. \tag{9.5.11}$$

Hence, the expectation value of S_z is

$$\langle S_z\rangle = \frac{\hbar}{2}(c_+^\ast, c_-^\ast)\begin{pmatrix}1, & 0\\ 0, & -1\end{pmatrix}\begin{pmatrix}c_+\\ c_-\end{pmatrix}, \tag{9.5.12}$$

which reduces to

$$\langle S_z\rangle = \frac{\hbar}{2}\cos\alpha \tag{9.5.13}$$

with the help of Equations ([e10.68]) and ([e10.69]). Likewise, the expectation value of S_x is

$$\langle S_x\rangle = \frac{\hbar}{2}(c_+^\ast, c_-^\ast)\begin{pmatrix}0, & 1\\ 1, & 0\end{pmatrix}\begin{pmatrix}c_+\\ c_-\end{pmatrix}, \tag{9.5.14}$$

which reduces to

$$\langle S_x\rangle = \frac{\hbar}{2}\sin\alpha\,\cos(\Omega\,t). \tag{9.5.15}$$

Finally, the expectation value of S_y is

$$\langle S_y\rangle = \frac{\hbar}{2}\sin\alpha\,\sin(\Omega\,t). \tag{9.5.16}$$

According to Equations ([e10.72]), ([e10.74]), and ([e10.75]), the expectation value of the spin angular momentum vector subtends a constant angle α with the z-axis, and precesses about this axis at the frequency

$$\Omega \simeq \frac{e\,B}{m_e}. \tag{9.5.17}$$

This behavior is actually equivalent to that predicted by classical physics. Note, however, that a measurement of S_x, S_y, or S_z will always yield either $+\hbar/2$ or $-\hbar/2$. It is the relative probabilities of obtaining these two results that varies as the expectation value of a given component of the spin varies.

Contributors and Attributions

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