9.2: Spin Space

We now have to discuss the wavefunctions upon which the previously introduced spin operators act. Unlike regular wavefunctions, spin wavefunctions do not exist in real space. Likewise, the spin angular momentum operators cannot be represented as differential operators in real space. Instead, we need to think of spin wavefunctions as existing in an abstract (complex) vector space. The different members of this space correspond to the different internal configurations of the particle under investigation. Note that only the directions of our vectors have any physical significance (just as only the shape of a regular wavefunction has any physical significance). Thus, if the vector χ corresponds to a particular internal state then $c\,\chi$ corresponds to the same state, where c is a complex number. Now, we expect the internal states of our particle to be superposable, because the superposibility of states is one of the fundamental assumptions of quantum mechanics . It follows that the vectors making up our vector space must also be superposable. Thus, if χ_1 and χ_2 are two vectors corresponding to two different internal states then $c_1\,\chi_1+c_2\,\chi_2$ is another vector corresponding to the state obtained by superposing c_1 times state 1 with c_2 times state 2 (where c_1 and c_2 are complex numbers). Finally, the dimensionality of our vector space is simply the number of linearly independent vectors required to span it (i.e., the number of linearly independent internal states of the particle under investigation).

We now need to define the length of our vectors. We can do this by introducing a second, or *dual*, vector space whose elements are in one to one correspondence with the elements of our first space . Let the element of the second space that corresponds to the element χ of the first space be called $\chi^\dagger$. Moreover, the element of the second space that corresponds to $c\,\chi$ is $c^\ast\,\chi^\dagger$. We shall assume that it is possible to combine χ and $\chi^\dagger$ in a multiplicative fashion to generate a real positive-definite number that we shall interpret as the length, or *norm*, of χ. Let us denote this number $\chi^\dagger\,\chi$. Thus, we have

$$\chi^\dagger\,\chi\geq 0 \tag{9.2.1}$$

for all χ. We shall also assume that it is possible to combine unlike states in an analogous multiplicative fashion to produce complex numbers. The product of two unlike states χ and χ' is denoted $\chi^\dagger\,\chi'$. Two states χ and χ' are said to be mutually orthogonal, or independent, if $\chi^\dagger\,\chi'=0$.

Now, when a general spin operator, A, operates on a general spin-state, χ, it converts it into a different spin-state that we shall denote $A\,\chi$. The dual of this state is $(A\,\chi)^\dagger\equiv\chi^\dagger\,A^\dagger$, where $A^\dagger$ is the Hermitian conjugate of A (this is the definition of an Hermitian conjugate in spin space). An eigenstate of A corresponding to the eigenvalue a satisfies

$$A\,\chi_a = a\,\chi_a. \tag{9.2.2}$$

As before, if A corresponds to a physical variable then a measurement of A will result in one of its eigenvalues. (See Section [smeas].) In order to ensure that these eigenvalues are all real, A must be Hermitian: that is, $A^\dagger=A$. (See Section [seig].) We expect the χ_a to be mutually orthogonal. We can also normalize them such that they all have unit length. In other words,

$$\chi_a^\dagger\,\chi_{a'}=\delta_{aa'}. \tag{9.2.3}$$

Finally, a general spin state can be written as a superposition of the normalized eigenstates of A: that is,

$$\chi=\sum_a c_a\,\chi_a. \tag{9.2.4}$$

A measurement of χ will then yield the result a with probability $|c_a|^2$.

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)