4.3: WKB Approximation

Consider a particle of mass m and energy $E > 0$ moving through some slowly varying potential $V(x)$. The particle's wavefunction satisfies

$$\frac{d^{2}\psi(x)}{dx^{2}} = -k^{2}(x)\,\psi(x), \tag{4.3.1}$$

where

$$k^{2}(x) = \frac{2\,m\,[E - V(x)]}{\hbar^{2}}. \tag{4.3.2}$$

Let us try a solution to Equation ([e5.38]) of the form

$$\psi(x) = \psi_0\,\exp\left(\int_0^x \mathrm{i}\,k(x')\,dx'\right), \tag{4.3.3}$$

where ψ_0 is a complex constant. Note that this solution represents a particle propagating in the positive x-direction [because the full wavefunction is multiplied by $\exp(-\mathrm{i}\,\omega\,t)$, where $\omega = E/\hbar > 0$] with the continuously varying wavenumber $k(x)$. It follows that

$$\frac{d\psi(x)}{dx} = \mathrm{i}\,k(x)\,\psi(x), \tag{4.3.4}$$

and

$$\frac{d^{2}\psi(x)}{dx^{2}} = \mathrm{i}\,k'(x)\,\psi(x) - k^{2}(x)\,\psi(x), \tag{4.3.5}$$

where $k' \equiv dk/dx$. A comparison of Equations ([e5.38]) and ([e5.42]) reveals that Equation ([e5.40]) represents an approximate solution to Equation ([e5.38]) provided that the first term on its right-hand side is negligible compared to the second. This yields the validity criterion $|k'| \ll k^{2}$, or

$$\frac{k}{|k'|} \gg k^{-1}. \tag{4.3.6}$$

In other words, the variation length-scale of $k(x)$, which is approximately the same as the variation length-scale of $V(x)$, must be much greater than the particle's de Broglie wavelength (which is of order k^{-1}). Let us suppose that this is the case. Incidentally, the approximation involved in dropping the first term on the right-hand side of Equation ([e5.42]) is generally known as the *WKB approximation*, after G. Wentzel, H.A. Kramers, and L. Brillouin . Similarly, Equation ([e5.40]) is termed a WKB solution.

According to the WKB solution ([e5.40]), the probability density remains constant: that is,

$$|\psi(x)|^{2} = |\psi_0|^{2}, \tag{4.3.7}$$

as long as the particle moves through a region in which $E > V(x)$, and $k(x)$ is consequently real (i.e., an allowed region according to classical physics). Suppose, however, that the particle encounters a potential barrier (i.e., a region from which the particle is excluded according to classical physics). By definition, $E < V(x)$ inside such a barrier, and $k(x)$ is consequently imaginary. Let the barrier extend from $x = x_1$ to x_2, where $0 < x_1 < x_2$. The WKB solution inside the barrier is written

$$\psi(x) = \psi_1\,\exp\left(-\int_{x_1}^x |k(x')|\,dx'\right), \tag{4.3.8}$$

where

$$\psi_1 = \psi_0\,\exp\left(\int_0^{x_1} \mathrm{i}\,k(x')\,dx'\right). \tag{4.3.9}$$

Here, we have neglected the unphysical exponentially growing solution.

According to the WKB solution ([e5.45]), the probability density decays exponentially inside the barrier: that is,

$$|\psi(x)|^2 = |\psi_1|^2 \exp\left(-2\int_{x_1}^{x} |k(x')|\,dx'\right), \tag{4.3.10}$$

where $|\psi_1|^2$ is the probability density at the left-hand side of the barrier (i.e., $x = x_1$). It follows that the probability density at the right-hand side of the barrier (i.e., $x = x_2$) is

$$|\psi_2|^2 = |\psi_1|^2 \exp\left(-2\int_{x_1}^{x_2} |k(x')|\,dx'\right). \tag{4.3.11}$$

Note that $|\psi_2|^2 < |\psi_1|^2$. Of course, in the region to the right of the barrier (i.e., $x > x_2$), the probability density takes the constant value $|\psi_2|^2$.

We can interpret the ratio of the probability densities to the right and to the left of the potential barrier as the probability, $|T|^2$, that a particle incident from the left will tunnel through the barrier and emerge on the other side: that is,

$$|T|^2 = \frac{|\psi_2|^2}{|\psi_1|^2} = \exp\left(-2\int_{x_1}^{x_2} |k(x')|\,dx'\right). \tag{4.3.12}$$

(See Section 1.3.) It is easily demonstrated that the probability of a particle incident from the right tunneling through the barrier is the same.

Note that the criterion ([e5.43]) for the validity of the WKB approximation implies that the previous transmission probability is very small. Hence, the WKB approximation only applies to situations in which there is very little chance of a particle tunneling through the potential barrier in question. Unfortunately, the validity criterion ([e5.43]) breaks down completely at the edges of the barrier (i.e., at $x = x_1$ and x_2), because $k(x) = 0$ at these points. However, it can be demonstrated that the contribution of those regions, around $x = x_1$ and x_2, in which the WKB approximation breaks down, to the integral in Equation ([e5.49]) is fairly negligible . Hence, the previous expression for the tunneling probability is a reasonable approximation provided that the incident particle's de Broglie wavelength is much smaller than the spatial extent of the potential barrier.

Contributors and Attributions

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