

3.6: Momentum Representation

Fourier's theorem (see Section [s2.9]), applied to one-dimensional wavefunctions, yields

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \bar{\psi}(k, t) e^{+i k x} dk,$$

$$\bar{\psi}(k, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x, t) e^{-i k x} dx,$$

where k represents wavenumber. However, $p = \hbar k$. Hence, we can also write

$$\psi(x, t) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \phi(p, t) e^{+i p x / \hbar} dp,$$

$$\phi(p, t) = \frac{1}{\sqrt{2\pi \hbar}} \int_{-\infty}^{\infty} \psi(x, t) e^{-i p x / \hbar} dx,$$

where $\phi(p, t) = \bar{\psi}(k, t) / \sqrt{\hbar}$ is the momentum-space equivalent to the real-space wavefunction $\psi(x, t)$.

At this stage, it is convenient to introduce a useful function called the *Dirac delta-function*. This function, denoted $\delta(x)$, was first devised by Paul Dirac, and has the following rather unusual properties: $\delta(x)$ is zero for $x \neq 0$, and is infinite at $x = 0$. However, the singularity at $x = 0$ is such that

$$\int_{-\infty}^{\infty} \delta(x) dx = 1. \quad (3.6.1)$$

The delta-function is an example of what is known as a *generalized function*: that is, its value is not well defined at all x , but its integral is well defined. Consider the integral

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx. \quad (3.6.2)$$

Because $\delta(x)$ is only non-zero infinitesimally close to $x = 0$, we can safely replace $f(x)$ by $f(0)$ in the previous integral (assuming $f(x)$ is well behaved at $x = 0$), to give

$$\int_{-\infty}^{\infty} f(x) \delta(x) dx = f(0) \int_{-\infty}^{\infty} \delta(x) dx = f(0), \quad (3.6.3)$$

where use has been made of Equation ([e3.64a]). A simple generalization of this result yields

$$\int_{-\infty}^{\infty} f(x) \delta(x - x_0) dx = f(x_0), \quad (3.6.4)$$

which can also be thought of as an alternative definition of a delta-function.

Suppose that $\psi(x) = \delta(x - x_0)$. It follows from Equations ([e3.65]) and ([e3.69]) that

$$\phi(p) = \frac{e^{-i p x_0 / \hbar}}{\sqrt{2\pi \hbar}} \quad (3.6.5)$$

Hence, Equation ([e3.64]) yields the important result

$$\delta(x - x_0) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{+i p (x - x_0) / \hbar} dp. \quad (3.6.6)$$

Similarly,

$$\delta(p - p_0) = \frac{1}{2\pi \hbar} \int_{-\infty}^{\infty} e^{+i (p - p_0) x / \hbar} dx. \quad (3.6.7)$$

It turns out that we can just as easily formulate quantum mechanics using the momentum-space wavefunction, $\phi(p, t)$, as the real-space wavefunction, $\psi(x, t)$. The former scheme is known as the *momentum representation* of quantum mechanics. In the momentum representation, wavefunctions are the Fourier transforms of the equivalent real-space wavefunctions, and dynamical

variables are represented by different operators. Furthermore, by analogy with Equation ([e3.55]), the expectation value of some operator $O(p)$ takes the form

$$\langle O \rangle = \int_{-\infty}^{\infty} \phi^*(p, t) O(p) \phi(p, t) dp. \quad (3.6.8)$$

Consider momentum. We can write

$$\begin{aligned} \langle p \rangle &= \int_{-\infty}^{\infty} \psi^*(x, t) \left(-i\hbar \frac{\partial}{\partial x} \right) \psi(x, t) dx \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(p', t) \phi(p, t) p e^{+i(p-p')x/\hbar} dx dp dp', \end{aligned}$$

where use has been made of Equation ([e3.64]). However, it follows from Equation ([e3.72]) that

$$\langle p \rangle = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(p', t) \phi(p, t) p \delta(p-p') dp dp'. \quad (3.6.9)$$

Hence, using Equation ([e3.69]), we obtain

$$\langle p \rangle = \int_{-\infty}^{\infty} \phi^*(p, t) p \phi(p, t) dp = \int_{-\infty}^{\infty} p |\phi|^2 dp. \quad (3.6.10)$$

Evidently, momentum is represented by the operator p in the momentum representation. The previous expression also strongly suggests [by comparison with Equation ([e3.22])] that $|\phi(p, t)|^2$ can be interpreted as the probability density of a measurement of momentum yielding the value p at time t . It follows that $\phi(p, t)$ must satisfy an analogous normalization condition to Equation ([e3.4]): that is,

$$\int_{-\infty}^{\infty} |\phi(p, t)|^2 dp = 1. \quad (3.6.11)$$

Consider displacement. We can write

$$\begin{aligned} \langle x \rangle &= \int_{-\infty}^{\infty} \psi^*(x, t) x \psi(x, t) dx \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(p', t) \phi(p, t) \left(-i\hbar \frac{\partial}{\partial p} \right) e^{+i(p-p')x/\hbar} dx dp dp'. \end{aligned}$$

Integration by parts yields

$$\langle x \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(p', t) e^{+i(p-p')x/\hbar} \left(i\hbar \frac{\partial}{\partial p} \right) \phi(p, t) dx dp dp'. \quad (3.6.12)$$

Hence, making use of Equations ([e3.72]) and ([e3.69]), we obtain

$$\langle x \rangle = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \phi^*(p) \left(i\hbar \frac{\partial}{\partial p} \right) \phi(p) dp. \quad (3.6.13)$$

Evidently, displacement is represented by the operator

$$x \equiv i\hbar \frac{\partial}{\partial p} \quad (3.6.14)$$

in the momentum representation.

Finally, let us consider the normalization of the momentum-space wavefunction $\phi(p, t)$. We have

$$\int_{-\infty}^{\infty} \psi^*(x, t) \psi(x, t) dx = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi^*(p', t) \phi(p, t) e^{+i(p-p')x/\hbar} dx dp dp'. \quad (3.6.15)$$

Thus, it follows from Equations ([e3.69]) and ([e3.72]) that

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = \int_{-\infty}^{\infty} |\phi(p, t)|^2 dp. \quad (3.6.16)$$

Hence, if $\psi(x, t)$ is properly normalized [see Equation ([\[e3.4\]](#))] then $\phi(p, t)$, as defined in Equation ([\[e3.65\]](#)), is also properly normalized [see Equation ([\[enormp\]](#))].

The existence of the momentum representation illustrates an important point. Namely, there are many different, but entirely equivalent, ways of mathematically formulating quantum mechanics. For instance, it is also possible to represent wavefunctions as row and column vectors, and dynamical variables as matrices that act upon these vectors.

Contributors and Attributions

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