7.5: Eigenvalues of L^2

Consider the angular wavefunction $\psi(\theta,\phi)=L_+\,Y_{l,m}(\theta,\phi)$. We know that

$$\oint\psi^*(\theta,\phi)\,\psi(\theta,\phi)\,d\Omega\geq 0, \tag{7.5.1}$$

because $\psi^*\,\psi\equiv|\psi|^{\,2}$ is a positive-definite real quantity. Hence, making use of Equations ([e5.48]) and ([e8.14]), we find that

$$\oint(L_+\,Y_{l,m})^*\,(L_+\,Y_{l,m})\,d\Omega=\oint Y_{l,m}^{\,*}\,(L_+)^\dagger\,(L_+\,Y_{l,m})\,d\Omega=\oint Y_{l,m}^{\,*}\,L_-\,L_+\,Y_{l,m}\,d\Omega\geq 0.$$

It follows from Equations ([e8.17]), and ([e8.29])–([e8.31]) that

$$\begin{aligned}\oint Y_{l,m}^{\,*}\,(L^2-L_z^2-\hbar\,L_z)\,Y_{l,m}\,d\Omega &=\oint Y_{l,m}^{\,*}\,\hbar^{\,2}\,[l\,(l+1)-m\,(m+1)]\,Y_{l,m}\,d\Omega\\ &=\hbar^{\,2}\,[l\,(l+1)-m\,(m+1)]\,\oint Y_{l,m}^{\,*}\,Y_{l,m}\,d\Omega\\ &=\hbar^{\,2}\,[l\,(l+1)-m\,(m+1)]\geq 0.\end{aligned}$$

We, thus, obtain the constraint

$$l\,(l+1)\geq m\,(m+1). \tag{7.5.2}$$

Likewise, the inequality

$$\oint(L_-\,Y_{l,m})^*\,(L_-\,Y_{l,m})\,d\Omega=\oint Y_{l,m}^{\,*}\,L_+\,L_-\,Y_{l,m}\,d\Omega\geq 0 \tag{7.5.3}$$

leads to a second constraint:

$$l\,(l+1)\geq m\,(m-1). \tag{7.5.4}$$

Without loss of generality, we can assume that $l\geq 0$. This is reasonable, from a physical standpoint, because $l\,(l+1)\,\hbar^{\,2}$ is supposed to represent the magnitude squared of something, and should, therefore, only take non-negative values. If l is non-negative then the constraints ([e8.42]) and ([e8.44]) are equivalent to the following constraint:

$$-l\leq m\leq l. \tag{7.5.5}$$

We, thus, conclude that the quantum number m can only take a restricted range of integer values.

Now, if m can only take a restricted range of integer values then there must exist a lowest possible value that it can take. Let us call this special value m_-, and let Y_{l,m_-} be the corresponding eigenstate. Suppose we act on this eigenstate with the lowering operator L_-. According to Equation ([e8.32]), this will have the effect of converting the eigenstate into that of a state with a lower value of m. However, no such state exists. A non-existent state is represented in quantum mechanics by the null wavefunction, $\psi=0$. Thus, we must have

$$L_-\,Y_{l,m_-}=0. \tag{7.5.6}$$

From Equation ([e8.15]),

$$L^2=L_+\,L_-+L_z^2-\hbar\,L_z \tag{7.5.7}$$

Hence,

$$L^2\,Y_{l,m_-}=(L_+\,L_-+L_z^2-\hbar\,L_z)\,Y_{l,m_-}, \tag{7.5.8}$$

or

$$l\,(l+1)\,Y_{l,m_-}=m_-\,(m_--1)\,Y_{l,m_-}, \tag{7.5.9}$$

where use has been made of ([e8.29]), ([e8.30]), and ([e8.46]). It follows that

$$l\,(l+1)=m_-\,(m_--1). \tag{7.5.10}$$

Assuming that m_- is negative, the solution to the previous equation is

$$m_- = -l. \tag{7.5.11}$$

We can similarly show that the largest possible value of m is

$$m_+ = +l. \tag{7.5.12}$$

The previous two results imply that l is an integer, because m_- and m_+ are both constrained to be integers.

We can now formulate the rules that determine the allowed values of the quantum numbers l and m. The quantum number l takes the non-negative integer values $0, 1, 2, 3, \cdots$. Once l is given, the quantum number m can take any integer value in the range

$$-l,\ -l+1,\ \cdots\ 0,\ \cdots, l-1,\ l. \tag{7.5.13}$$

Thus, if $l = 0$ then m can only take the value 0, if $l = 1$ then m can take the values $-1, 0, +1$, if $l = 2$ then m can take the values $-2, -1, 0, +1, +2$, and so on.

Contributors and Attributions

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