7.1: Angular Momenum Operators

In classical mechanics, the vector angular momentum, $\mathbf{L}$, of a particle of position vector $\mathbf{r}$ and linear momentum $\mathbf{p}$ is defined as

$$\mathbf{L}=\mathbf{r}\times\mathbf{p}. \tag{7.1.1}$$

It follows that

$$\begin{aligned} L_x &= y\,p_z - z\,p_y,\\ L_y &= z\,p_x - x\,p_z,\\ L_z &= x\,p_y - y\,p_x. \end{aligned}$$

Let us, first of all, consider whether it is possible to use the previous expressions as the definitions of the operators corresponding to the components of angular momentum in quantum mechanics, assuming that the x_i and p_i (where $x_1\equiv x$, $p_1\equiv p_x$, $x_2\equiv y$, etc.) correspond to the appropriate quantum mechanical position and momentum operators. The first point to note is that expressions ([e8.1])–([e8.3]) are unambiguous with respect to the order of the terms in multiplicative factors, because the various position and momentum operators appearing in them all commute with one another. [See Equations ([commxp]).] Moreover, given that the x_i and the p_i are Hermitian operators, it is easily seen that the L_i are also Hermitian. This is important, because only Hermitian operators can represent physical variables in quantum mechanics. (See Section [s4.6].) We, thus, conclude that Equations ([e8.1])–([e8.3]) are plausible definitions for the quantum mechanical operators that represent the components of angular momentum.

Let us now derive the commutation relations for the L_i. For instance,

$$\begin{aligned} &= [y\,p_z - z\,p_y, z\,p_x - x\,p_z] = y\,p_x\,[p_z, z] + x\,p_y\,[z, p_z]\\ &= {\rm i}\,\hbar\,(x\,p_y - y\,p_x) = {\rm i}\,\hbar\,L_z, \end{aligned}$$

where use has been made of the definitions of the L_i [see Equations ([e8.1])–([e8.3])], and commutation relations ([commxx])–([commxp]) for the x_i and p_i. There are two similar commutation relations: one for L_y and L_z, and one for L_z and L_x. Collecting all of these commutation relations together, we obtain

$$\begin{aligned} [L_x, L_y] &= {\rm i}\,\hbar\,L_z,\\ [L_y, L_z] &= {\rm i}\,\hbar\,L_x,\\ [L_z, L_x] &= {\rm i}\,\hbar\,L_y. \end{aligned}$$

By analogy with classical mechanics, the operator L^2, that represents the magnitude squared of the angular momentum vector, is defined

$$L^2 = L_x^{\,2} + L_y^{\,2} + L_z^{\,2}. \tag{7.1.2}$$

Now, it is easily demonstrated that if A and B are two general operators then

$$[A^2, B] = A\,[A,B] + [A,B]\,A. \tag{7.1.3}$$

Hence,

$$\begin{aligned} &= [L_y^{\,2}, L_x] + [L_z^{\,2}, L_x]\\ &= L_y\,[L_y, L_x] + [L_y, L_x]\,L_y + L_z\,[L_z, L_x] + [L_z, L_x]\,L_z\\ &= {\rm i}\,\hbar\,(-L_y\,L_z - L_z\,L_y + L_z\,L_y + L_y\,L_z) = 0, \end{aligned}$$

where use has been made of Equations ([e8.6])–([e8.8]). In other words, L^2 commutes with L_x. Likewise, it is easily demonstrated that L^2 also commutes with L_y, and with L_z. Thus,

$$[L^2, L_x] = [L^2, L_y] = [L^2, L_z] = 0. \tag{7.1.4}$$

Recall, from Section [smeas], that in order for two physical quantities to be (exactly) measured simultaneously, the operators that represent them in quantum mechanics must commute with one another. Hence, the commutation relations ([e8.6])–([e8.8]) and ([e8.12]) imply that we can only simultaneously measure the magnitude squared of the angular momentum vector, L^2, together with, at most, one of its Cartesian components. By convention, we shall always choose to measure the z-component, L_z.

Finally, it is helpful to define the operators

$$L_{\pm} = L_x \pm \mathrm{i}\, L_y. \tag{7.1.5}$$

Note that L_+ and L_- are not Hermitian operators, but are the Hermitian conjugates of one another (see Section [s4.6]): that is,

$$(L_{\pm})^{\dagger} = L_{\mp}, \tag{7.1.6}$$

Moreover, it is easily seen that

$$\begin{aligned} L_+\, L_- &= (L_x + \mathrm{i}\, L_y)\,(L_x - \mathrm{i}\, L_y) = L_x^{\,2} + L_y^{\,2} - \mathrm{i}\,[L_x, L_y] = L_x^{\,2} + L_y^{\,2} + \hbar\, L_z \\ &= L^2 - L_z^{\,2} + \hbar\, L_z. \end{aligned}$$

Likewise,

$$L_-\, L_+ = L^2 - L_z^{\,2} - \hbar\, L_z, \tag{7.1.7}$$

giving

$$[L_+, L_-] = 2\,\hbar\, L_z. \tag{7.1.8}$$

We also have

$$[L_+, L_z] = [L_x, L_z] + \mathrm{i}\,[L_y, L_z] = -\mathrm{i}\,\hbar\, L_y - \hbar\, L_x = -\hbar\, L_+, \tag{7.1.9}$$

and, similarly,

$$[L_-, L_z] = \hbar\, L_-. \tag{7.1.10}$$

Contributors and Attributions

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