4.1: Infinite Potential Well

Consider a particle of mass m and energy E moving in the following simple potential:

$$V(x) = \begin{cases} 0 & \text{for } 0 \leq x \leq a \\ \infty & \text{otherwise} \end{cases}. \tag{4.1.1}$$

It follows from Equation ([e5.2]) that if $d^{\,2}\psi/dx^{\,2}$ (and, hence, ψ) is to remain finite then ψ must go to zero in regions where the potential is infinite. Hence, $\psi = 0$ in the regions $x \leq 0$ and $x \geq a$. Evidently, the problem is equivalent to that of a particle trapped in a one-dimensional box of length a. The boundary conditions on ψ in the region $0 < x < a$ are

$$\psi(0) = \psi(a) = 0. \tag{4.1.2}$$

Furthermore, it follows from Equation ([e5.2]) that ψ satisfies

$$\frac{d^{\,2}\psi}{dx^{\,2}} = -k^{\,2}\,\psi \tag{4.1.3}$$

in this region, where

$$k^{\,2} = \frac{2\,m\,E}{\hbar^{\,2}}. \tag{4.1.4}$$

Here, we are assuming that $E > 0$. It is easily demonstrated that there are no solutions with $E < 0$ which are capable of satisfying the boundary conditions ([e5.4]).

The solution to Equation ([e5.5]), subject to the boundary conditions ([e5.4]), is

$$\psi_n(x) = A_n\,\sin(k_n\,x), \tag{4.1.5}$$

where the A_n are arbitrary (real) constants, and

$$k_n = \frac{n\,\pi}{a}, \tag{4.1.6}$$

for $n = 1, 2, 3, \cdots$. Now, it can be seen from Equations ([e5.6]) and ([e5.8]) that the energy E is only allowed to take certain discrete values: that is,

$$E_n = \frac{n^{\,2}\,\pi^{\,2}\,\hbar^{\,2}}{2\,m\,a^{\,2}}. \tag{4.1.7}$$

In other words, the eigenvalues of the energy operator are discrete. This is a general feature of bounded solutions: that is, solutions for which $|\psi| \rightarrow 0$ as $|x| \rightarrow \infty$. According to the discussion in Section [sstat], we expect the stationary eigenfunctions $\psi_n(x)$ to satisfy the orthonormality constraint

$$\int_0^a \psi_n(x)\,\psi_m(x)\,dx = \delta_{nm}. \tag{4.1.8}$$

It is easily demonstrated that this is the case, provided $A_n = \sqrt{2/a}$. Hence,

$$\psi_n(x) = \sqrt{\frac{2}{a}}\,\sin\!\left(n\,\pi\,\frac{x}{a}\right) \tag{4.1.9}$$

for $n = 1, 2, 3, \cdots$.

Finally, again from Section [sstat], the general time-dependent solution can be written as a linear superposition of stationary solutions:

$$\psi(x,t) = \sum_{n=0,\infty} c_n\,\psi_n(x)\,{\rm e}^{-{\rm i}\,E_n\,t/\hbar}, \tag{4.1.10}$$

where

$$c_n = \int_0^a \psi_n(x)\,\psi(x,0)\,dx. \tag{4.1.11}$$

Contributors and Attributions

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