12.2: Two-State System

Consider a system in which the time-independent Hamiltonian possesses two eigenstates, denoted

$$H_0\,\psi_1 = E_1\,\psi_1,$$
$$H_0\,\psi_2 = E_2\,\psi_2.$$

Suppose, for the sake of simplicity, that the diagonal elements of the interaction Hamiltonian, H_1, are zero: that is,

$$\langle 1|H_1|1\rangle = \langle 2|H_1|2\rangle = 0. \tag{12.2.1}$$

The off-diagonal elements are assumed to oscillate sinusoidally at some frequency ω: that is,

$$\langle 1|H_1|2\rangle = \langle 2|H_1|1\rangle^* = \gamma\,\hbar\,\exp(\mathrm{i}\,\omega\,t), \tag{12.2.2}$$

where γ and ω are real. Note that it is only the off-diagonal matrix elements which give rise to the effect which we are interested in: namely, transitions between states 1 and 2.

For a two-state system, Equation ([e13.12]) reduces to

$$\mathrm{i}\,\frac{dc_1}{dt} = \gamma\,\exp[+\mathrm{i}\,(\omega-\omega_{21})\,t]\,c_2,$$
$$\mathrm{i}\,\frac{dc_2}{dt} = \gamma\,\exp[-\mathrm{i}\,(\omega-\omega_{21})\,t]\,c_1,$$

where $\omega_{21} = (E_2 - E_1)/\hbar$. The previous two equations can be combined to give a second-order differential equation for the time-variation of the amplitude c_2: that is,

$$\frac{d^{\,2}c_2}{dt^{\,2}} + \mathrm{i}\,(\omega-\omega_{21})\,\frac{dc_2}{dt} + \gamma^{\,2}\,c_2 = 0. \tag{12.2.3}$$

Once we have solved for c_2, we can use Equation ([e13.20]) to obtain the amplitude c_1. Let us search for a solution in which the system is certain to be in state 1 (and, thus, has no chance of being in state 2) at time $t=0$. Thus, our initial conditions are $c_1(0)=1$ and $c_2(0)=0$. It is easily demonstrated that the appropriate solutions to ([e13.21]) and ([e13.20]) are

$$c_2(t) = \left(\frac{-\mathrm{i}\gamma}{\Omega}\right)\exp\left[\frac{-\mathrm{i}\,(\omega-\omega_{21})\,t}{2}\right]\sin(\Omega t) \tag{12.2.4}$$

$$\begin{aligned} c_1(t) =& \exp\left[\frac{\mathrm{i}\,(\omega-\omega_{21})\,t}{2}\right]\cos(\Omega t) \\ &-\left[\frac{\mathrm{i}\,(\omega-\omega_{21})}{2\Omega}\right]\exp\left[\frac{\mathrm{i}\,(\omega-\omega_{21})\,t}{2}\right]\sin(\Omega t)\end{aligned} \tag{12.2.5}$$

where

$$\Omega = \sqrt{\gamma^2 + (\omega-\omega_{21})^{\,2}/4} \tag{12.2.6}$$

Now, the probability of finding the system in state 1 at time t is simply $P_1(t) = |c_1(t)|^{\,2}$. Likewise, the probability of finding the system in state 2 at time t is $P_2(t) = |c_2(t)|^{\,2}$. It follows that

$$P_1(t) = 1 - P_2(t),$$
$$P_2(t) = \left[\frac{\gamma^{\,2}}{\gamma^{\,2} + (\omega-\omega_{21})^{\,2}/4}\right]\sin^2(\Omega\,t).$$

This result is known as *Rabi's formula* .

Equation ([e13.25]) exhibits all the features of a classic resonance . At resonance, when the oscillation frequency of the perturbation, ω, matches the frequency ω_{21}, we find that

$$P_1(t) = \cos^2(\gamma\,t),$$
$$P_2(t) = \sin^2(\gamma\,t).$$

According to the previous result, the system starts off in state 1 at $t = 0$. After a time interval $\pi/(2\,\gamma)$ it is certain to be in state 2. After a further time interval $\pi/(2\,\gamma)$ it is certain to be in state 1 again, and so on. Thus, the system periodically flip-flops between states 1 and 2 under the influence of the time-dependent perturbation. This implies that the system alternatively absorbs and emits energy from the source of the perturbation.

The absorption-emission cycle also takes place away from the resonance, when $\omega \neq \omega_{21}$. However, the amplitude of the oscillation in the coefficient c_2 is reduced. This means that the maximum value of $P_2(t)$ is no longer unity, nor is the minimum of $P_1(t)$ zero. In fact, if we plot the maximum value of $P_2(t)$ as a function of the applied frequency, ω, then we obtain a resonance curve whose maximum (unity) lies at the resonance, and whose full-width half-maximum (in frequency) is $4\,\gamma$. Thus, if the applied frequency differs from the resonant frequency by substantially more than $2\,\gamma$ then the probability of the system jumping from state 1 to state 2 is always very small. In other words, the time-dependent perturbation is only effective at causing transitions between states 1 and 2 if its frequency of oscillation lies in the approximate range $\omega_{21} \pm 2\,\gamma$. Clearly, the weaker the perturbation (i.e., the smaller γ becomes), the narrower the resonance.

Contributors and Attributions

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