

3.2: Normalization of the Wavefunction

Now, a probability is a real number lying between 0 and 1. An outcome of a measurement that has a probability 0 is an impossible outcome, whereas an outcome that has a probability 1 is a certain outcome. According to Equation ([e3.2]), the probability of a measurement of x yielding a result lying between $-\infty$ and $+\infty$ is

$$P_{x \in -\infty:\infty}(t) = \int_{-\infty}^{\infty} |\psi(x, t)|^2 dx. \quad (3.2.1)$$

However, a measurement of x must yield a value lying between $-\infty$ and $+\infty$, because the particle has to be located somewhere. It follows that $P_{x \in -\infty:\infty} = 1$, or

$$\int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 1, \quad (3.2.2)$$

which is generally known as the *normalization condition* for the wavefunction.

For example, suppose that we wish to normalize the wavefunction of a Gaussian wave-packet, centered on $x = x_0$, and of characteristic width σ (see Section [s2.9]): that is,

$$\psi(x) = \psi_0 e^{-(x-x_0)^2/(4\sigma^2)}. \quad (3.2.3)$$

In order to determine the normalization constant ψ_0 , we simply substitute Equation ([e3.5]) into Equation ([e3.4]) to obtain

$$|\psi_0|^2 \int_{-\infty}^{\infty} e^{-(x-x_0)^2/(2\sigma^2)} dx = 1. \quad (3.2.4)$$

Changing the variable of integration to $y = (x - x_0)/(\sqrt{2}\sigma)$, we get

$$|\psi_0|^2 \sqrt{2}\sigma \int_{-\infty}^{\infty} e^{-y^2} dy = 1. \quad (3.2.5)$$

However,

$$\int_{-\infty}^{\infty} e^{-y^2} dy = \sqrt{\pi}, \quad (3.2.6)$$

which implies that

$$|\psi_0|^2 = \frac{1}{(2\pi\sigma^2)^{1/2}}. \quad (3.2.7)$$

Hence, a general normalized Gaussian wavefunction takes the form

$$\psi(x) = \frac{e^{i\varphi}}{(2\pi\sigma^2)^{1/4}} e^{-(x-x_0)^2/(4\sigma^2)}, \quad (3.2.8)$$

where φ is an arbitrary real phase-angle.

It is important to demonstrate that if a wavefunction is initially normalized then it stays normalized as it evolves in time according to Schrödinger's equation. If this is not the case then the probability interpretation of the wavefunction is untenable, because it does not make sense for the probability that a measurement of x yields any possible outcome (which is, manifestly, unity) to change in time. Hence, we require that

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\psi(x, t)|^2 dx = 0, \quad (3.2.9)$$

for wavefunctions satisfying Schrödinger's equation. The previous equation gives

$$\frac{d}{dt} \int_{-\infty}^{\infty} \psi^* \psi dx = \int_{-\infty}^{\infty} \left(\frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t} \right) dx = 0. \quad (3.2.10)$$

Now, multiplying Schrödinger's equation by $\psi^*/(i\hbar)$, we obtain

$$\psi^* \frac{\partial \psi}{\partial t} = \frac{i \hbar}{2m} \psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{i}{\hbar} V |\psi|^2. \quad (3.2.11)$$

The complex conjugate of this expression yields

$$\psi \frac{\partial \psi^*}{\partial t} = -\frac{i \hbar}{2m} \psi \frac{\partial^2 \psi^*}{\partial x^2} + \frac{i}{\hbar} V |\psi|^2 \quad (3.2.12)$$

[because $(AB)^* = A^* B^*$, $A^{**} = A$, and $i^* = -i$].

Summing the previous two equations, we get

$$\frac{\partial \psi^*}{\partial t} \psi + \psi^* \frac{\partial \psi}{\partial t} = \frac{i \hbar}{2m} \left(\psi^* \frac{\partial^2 \psi}{\partial x^2} - \psi \frac{\partial^2 \psi^*}{\partial x^2} \right) = \frac{i \hbar}{2m} \frac{\partial}{\partial x} \left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right). \quad (3.2.13)$$

Equations (3.2.12) and (3.2.13) can be combined to produce

$$\frac{d}{dt} \int_{-\infty}^{\infty} |\psi|^2 dx = \frac{i \hbar}{2m} \left[\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x} \right]_{-\infty}^{\infty} = 0. \quad (3.2.14)$$

The previous equation is satisfied provided

$$|\psi| \rightarrow 0 \quad \text{as} \quad |x| \rightarrow \infty. \quad (3.2.15)$$

However, this is a necessary condition for the integral on the left-hand side of Equation (3.2.14) to converge. Hence, we conclude that all wavefunctions that are *square-integrable* [i.e., are such that the integral in Equation (3.2.14) converges] have the property that if the normalization condition (3.2.14) is satisfied at one instant in time then it is satisfied at all subsequent times.

It is also possible to demonstrate, via very similar analysis to that just described, that

$$\frac{dP_{x \in a:b}}{dt} + j(b, t) - j(a, t) = 0, \quad (3.2.16)$$

where $P_{x \in a:b}$ is defined in Equation (3.2.2), and

$$j(x, t) = \frac{i \hbar}{2m} \left(\psi \frac{\partial \psi^*}{\partial x} - \psi^* \frac{\partial \psi}{\partial x} \right) \quad (3.2.17)$$

is known as the *probability current*. Note that j is real. Equation (3.2.16) is a *probability conservation equation*. According to this equation, the probability of a measurement of x lying in the interval a to b evolves in time due to the difference between the flux of probability into the interval [i.e., $j(a, t)$], and that out of the interval [i.e., $j(b, t)$]. Here, we are interpreting $j(x, t)$ as the flux of probability in the $+x$ -direction at position x and time t .

Note, finally, that not all wavefunctions can be normalized according to the scheme set out in Equation (3.2.14). For instance, a plane-wave wavefunction

$$\psi(x, t) = \psi_0 e^{i(kx - \omega t)} \quad (3.2.18)$$

is not square-integrable, and, thus, cannot be normalized. For such wavefunctions, the best we can say is that

$$P_{x \in a:b}(t) \propto \int_a^b |\psi(x, t)|^2 dx. \quad (3.2.19)$$

In the following, all wavefunctions are assumed to be square-integrable and normalized, unless otherwise stated.

Contributors and Attributions

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