14.7: Low-Energy Scattering

In general, at low energies (i.e., when $1/k$ is much larger than the range of the potential), partial waves with $l>0$ make a negligible contribution to the scattering cross-section. It follows that, at these energies, with a finite range potential, only S-wave scattering is important.

As a specific example, let us consider scattering by a finite potential well, characterized by $V=V_0$ for $r<a$, and $V=0$ for $r\geq a$. Here, V_0 is a constant. The potential is repulsive for $V_0>0$, and attractive for $V_0<0$. The outside wavefunction is given by [see Equation ([e17.80])]

$$\begin{aligned}\mathcal{R}_0(r) &= \exp(\,\mathrm{i}\,\delta_0)\,\left[\cos\delta_0\,j_0(k\,r)-\sin\delta_0\,y_0(k\,r)\right]\\ &= \frac{\exp(\,\mathrm{i}\,\delta_0)\,\sin(k\,r+\delta_0)}{k\,r},\end{aligned}$$

where use has been made of Equations ([e17.58a]) and ([e17.58b]). The inside wavefunction follows from Equation ([e17.85]). We obtain

$$\mathcal{R}_0(r)=B\,\frac{\sin(k'\,r)}{r},\tag{14.7.1}$$

where use has been made of the boundary condition ([e17.86]). Here, B is a constant, and

$$E-V_0=\frac{\hbar^{\,2}\,k'^{\,2}}{2\,m}.\tag{14.7.2}$$

Note that Equation ([e17.103]) only applies when $E>V_0$. For $E<V_0$, we have

$$\mathcal{R}_0(r)=B\,\frac{\sinh(\kappa\,r)}{r},\tag{14.7.3}$$

where

$$V_0-E=\frac{\hbar^{\,2}\,\kappa^{\,2}}{2\,m}.\tag{14.7.4}$$

Matching $\mathcal{R}_0(r)$, and its radial derivative, at $r=a$ yields

$$\tan(k\,a+\delta_0)=\frac{k}{k'}\,\tan(k'\,a)\tag{14.7.5}$$

for $E>V_0$, and

$$\tan(k\,a+\delta_0)=\frac{k}{\kappa}\,\tanh(\kappa\,a)\tag{14.7.6}$$

for $E<V_0$.

Consider an attractive potential, for which $E>V_0$. Suppose that $|V_0|\gg E$ (i.e., the depth of the potential well is much larger than the energy of the incident particles), so that $k'\gg k$. We can see from Equation ([e17.107]) that, unless $\tan(k'\,a)$ becomes extremely large, the right-hand side is much less that unity, so replacing the tangent of a small quantity with the quantity itself, we obtain

$$k\,a+\delta_0\simeq\frac{k}{k'}\,\tan(k'\,a).\tag{14.7.7}$$

This yields

$$\delta_0\simeq k\,a\left[\frac{\tan(k'\,a)}{k'\,a}-1\right].\tag{14.7.8}$$

According to Equation ([e17.99]), the scattering cross-section is given by

$$\sigma_{\rm total}\simeq\frac{4\pi}{k^{\,2}}\,\sin^2\delta_0=4\pi\,a^{\,2}\left[\frac{\tan(k'a)}{k'\,a}-1\right]^{\,2}.\tag{14.7.9}$$

Now,

$$k' a = \sqrt{k^2 a^2 + \frac{2 m |V_0| a^2}{\hbar^2}}, \tag{14.7.10}$$

so for sufficiently small values of $k a$,

$$k' a \simeq \sqrt{\frac{2 m |V_0| a^2}{\hbar^2}}. \tag{14.7.11}$$

It follows that the total (S-wave) scattering cross-section is independent of the energy of the incident particles (provided that this energy is sufficiently small).

Note that there are values of $k' a$ (e.g., $k' a \simeq 4.49$) at which $\delta_0 \to \pi$, and the scattering cross-section ([e17.111]) vanishes, despite the very strong attraction of the potential. In reality, the cross-section is not exactly zero, because of contributions from $l > 0$ partial waves. But, at low incident energies, these contributions are small. It follows that there are certain values of V_0 and k that give rise to almost perfect transmission of the incident wave. This is called the *Ramsauer-Townsend effect*, and has been observed experimentally .

Contributors and Attributions

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