7.3: Eigenstates of Angular Momentum

Let us find the simultaneous eigenstates of the angular momentum operators L_z and L^2. Because both of these operators can be represented as purely angular differential operators, it stands to reason that their eigenstates only depend on the angular coordinates θ and ϕ. Thus, we can write

$$\begin{aligned} L_z\,Y_{l,m}(\theta,\phi) &= m\,\hbar\,Y_{l,m}(\theta,\phi), \\ L^2\,Y_{l,m}(\theta,\phi) &= l\,(l+1)\,\hbar^{\,2}\,Y_{l,m}(\theta,\phi). \end{aligned}$$

Here, the $Y_{l,m}(\theta,\phi)$ are the eigenstates in question, whereas the dimensionless quantities m and l parameterize the eigenvalues of L_z and L^2, which are $m\,\hbar$ and $l\,(l+1)\,\hbar^{\,2}$, respectively. Of course, we expect the $Y_{l,m}$ to be both mutually orthogonal and properly normalized (see Section [seig]), so that

$$\oint Y_{l',m'}^{\,*}(\theta,\phi)\,Y_{l,m}(\theta,\phi)\,d\Omega = \delta_{ll'}\,\delta_{mm'}, \tag{7.3.1}$$

where $d\Omega = \sin\theta\,d\theta\,d\phi$ is an element of solid angle, and the integral is over all solid angle.

Now,

$$\begin{aligned} L_z\,(L_+\,Y_{l,m}) &= (L_+\,L_z + [L_z, L_+])\,Y_{l,m} = (L_+\,L_z + \hbar\,L_+)\,Y_{l,m} \\ &= (m+1)\,\hbar\,(L_+\,Y_{l,m}), \end{aligned}$$

where use has been made of Equation ([e8.19]). We, thus, conclude that when the operator L_+ operates on an eigenstate of L_z corresponding to the eigenvalue $m\,\hbar$ it converts it to an eigenstate corresponding to the eigenvalue $(m+1)\,\hbar$. Hence, L_+ is known as the *raising operator* (for L_z). It is also easily demonstrated that

$$L_z\,(L_-\,Y_{l,m}) = (m-1)\,\hbar\,(L_-\,Y_{l,m}). \tag{7.3.2}$$

In other words, when L_- operates on an eigenstate of L_z corresponding to the eigenvalue $m\,\hbar$ it converts it to an eigenstate corresponding to the eigenvalue $(m-1)\,\hbar$. Hence, L_- is known as the *lowering operator* (for L_z).

Writing

$$\begin{aligned} L_+\,Y_{l,m} &= c_{l,m}^{+}\,Y_{l,m+1}, \\ L_-\,Y_{l,m} &= c_{l,m}^{-}\,Y_{l,m-1}, \end{aligned}$$

we obtain

$$L_-\,L_+\,Y_{l,m} = c_{l,m}^{+}\,c_{l,m+1}^{-}\,Y_{l,m} = [l\,(l+1) - m\,(m+1)]\,\hbar^2\,Y_{l,m}, \tag{7.3.3}$$

where use has been made of Equation ([e8.17]). Likewise,

$$L_+\,L_-\,Y_{l,m} = c_{l,m-1}^{+}\,c_{l,m}^{-}\,Y_{l,m} = [l\,(l+1) - m\,(m-1)]\,\hbar^{\,2}\,Y_{l,m}, \tag{7.3.4}$$

where use has been made of Equation ([e8.15]). It follows that

$$\begin{aligned} c_{l,m}^{+}\,c_{l,m+1}^{-} &= [l\,(l+1) - m\,(m+1)]\,\hbar^{\,2}, \\ c_{l,m-1}^{+}\,c_{l,m}^{-} &= [l\,(l+1) - m\,(m-1)]\,\hbar^{\,2}. \end{aligned}$$

These equations are satisfied when

$$c_{l,m}^{\pm} = [l\,(l+l) - m\,(m\pm 1)]^{1/2}\,\hbar. \tag{7.3.5}$$

Hence, we can write

$$\begin{aligned} L_+\,Y_{l,m} &= [l\,(l+1) - m\,(m+1)]^{1/2}\,\hbar\,Y_{l,m+1}, \\ L_-\,Y_{l,m} &= [l\,(l+1) - m\,(m-1)]^{1/2}\,\hbar\,Y_{l,m-1}. \end{aligned}$$

Contributors and Attributions

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