3.3: Expectation Values (Averages) and Variances

We have seen that $|\psi(x,t)|^2$ is the probability density of a measurement of a particle's displacement yielding the value x at time t. Suppose that we make a large number of independent measurements of the displacement on an equally large number of identical quantum systems. In general, measurements made on different systems will yield different results. However, from the definition of probability (see Chapter [s2]), the mean of all these results is simply

$$\langle x\rangle = \int_{-\infty}^{\infty} x\,|\psi|^2\,dx. \tag{3.3.1}$$

Here, $\langle x\rangle$ is called the *expectation value* of x. (See Chapter [s2].) Similarly the expectation value of any function of x is

$$\langle f(x)\rangle = \int_{-\infty}^{\infty} f(x)\,|\psi|^2\,dx. \tag{3.3.2}$$

In general, the results of the various different measurements of x will be scattered around the expectation value, $\langle x\rangle$. The degree of scatter is parameterized by the quantity

$$\sigma_x^2 = \int_{-\infty}^{\infty} (x-\langle x\rangle)^2\,|\psi|^2\,dx \equiv \langle x^2\rangle - \langle x\rangle^2, \tag{3.3.3}$$

which is known as the *variance* of x. (See Chapter [s2].) The square-root of this quantity, σ_x, is called the *standard deviation* of x. (See Chapter [s2].) We generally expect the results of measurements of x to lie within a few standard deviations of the expectation value.

For instance, consider the normalized Gaussian wave-packet [see Equation ([eng])]

$$\psi(x) = \frac{\mathrm{e}^{\mathrm{i}\varphi}}{(2\pi\,\sigma^2)^{1/4}}\,\mathrm{e}^{-(x-x_0)^2/(4\,\sigma^2)}. \tag{3.3.4}$$

The expectation value of x associated with this wavefunction is

$$\langle x\rangle = \frac{1}{\sqrt{2\pi\,\sigma^2}}\int_{-\infty}^{\infty} x\,\mathrm{e}^{-(x-x_0)^2/(2\,\sigma^2)}\,dx. \tag{3.3.5}$$

Let $y=(x-x_0)/(\sqrt{2}\,\sigma)$. It follows that

$$\langle x\rangle = \frac{x_0}{\sqrt{\pi}}\int_{-\infty}^{\infty}\mathrm{e}^{-y^2}\,dy + \frac{\sqrt{2}\,\sigma}{\sqrt{\pi}}\int_{-\infty}^{\infty} y\,\mathrm{e}^{-y^2}\,dy. \tag{3.3.6}$$

However, the second integral on the right-hand side is zero, by symmetry. Hence, making use of Equation ([e3.8]), we obtain

$$\langle x\rangle = x_0. \tag{3.3.7}$$

Evidently, the expectation value of x for a Gaussian wave-packet is equal to the most likely value of x (i.e., the value of x that maximizes $|\psi|^2$).

The variance of x associated with the Gaussian wave-packet ([e3.24]) is

$$\sigma_x^2 = \frac{1}{\sqrt{2\pi\,\sigma^2}}\int_{-\infty}^{\infty}(x-x_0)^2\,\mathrm{e}^{-(x-x_0)^2/(2\,\sigma^2)}\,dx. \tag{3.3.8}$$

Let $y=(x-x_0)/(\sqrt{2}\,\sigma)$. It follows that

$$\sigma_x^2 = \frac{2\,\sigma^2}{\sqrt{\pi}}\int_{-\infty}^{\infty} y^2\,\mathrm{e}^{-y^2}\,dy. \tag{3.3.9}$$

However,

$$\int_{-\infty}^{\infty} y^2\,\mathrm{e}^{-y^2}\,dy = \frac{\sqrt{\pi}}{2}, \tag{3.3.10}$$

giving

$$\sigma_x^2 = \sigma^2. \tag{3.3.11}$$

This result is consistent with our earlier interpretation of σ as a measure of the spatial extent of the wave-packet. (See Section [s2.9].) It follows that we can rewrite the Gaussian wave-packet ([e3.24]) in the convenient form

$$\psi(x) = \frac{\mathrm{e}^{\,\mathrm{i}\,\varphi}}{(2\pi\,\sigma_x^2)^{1/4}}\,\mathrm{e}^{-(x-\langle x\rangle)^2/(4\,\sigma_x^2)}. \tag{3.3.12}$$

Contributors and Attributions

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