14.2: Born Approximation

Equation ([e15.17]) is not particularly useful, as it stands, because the quantity $f(\mathbf{k},\mathbf{k}')$ depends on the, as yet, unknown wavefunction $\psi(\mathbf{r})$. [See Equation ([e5.12]).] Suppose, however, that the scattering is not particularly strong. In this case, it is reasonable to suppose that the total wavefunction, $\psi(\mathbf{r})$, does not differ substantially from the incident wavefunction, $\psi_0(\mathbf{r})$. Thus, we can obtain an expression for $f(\mathbf{k},\mathbf{k}')$ by making the substitution $\psi(\mathbf{r}) \rightarrow \psi_0(\mathbf{r}) = \sqrt{n}\,\exp(\,{\rm i}\,\mathbf{k}\cdot\mathbf{r})$ in Equation ([e5.12]). This procedure is called the *Born approximation* .

The Born approximation yields

$$f(\mathbf{k},\mathbf{k}') \simeq \frac{m}{2\pi\,\hbar^{\,2}}\int {\rm e}^{\,{\rm i}\,(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}'}\,V(\mathbf{r}')\,d^{\,3}\mathbf{r}'. \tag{14.2.1}$$

Thus, $f(\mathbf{k},\mathbf{k}')$ becomes proportional to the Fourier transform of the scattering potential $V(\mathbf{r})$ with respect to the wavevector $\mathbf{q} = \mathbf{k}-\mathbf{k}'$.

For a spherically symmetric potential,

$$f(\mathbf{k}',\mathbf{k}) \simeq -\frac{m}{2\pi\,\hbar^{\,2}}\iiint \exp(\,{\rm i}\,q\,r'\,\cos\theta')\,V(r')\,r'^{\,2}\,dr'\,\sin\theta'\,d\theta'\,d\phi', \tag{14.2.2}$$

giving

$$f(\mathbf{k}',\mathbf{k}) \simeq -\frac{2\,m}{\hbar^{\,2}\,q}\int_0^\infty r'\,V(r')\sin(q\,r')\,dr'. \tag{14.2.3}$$

Note that $f(\mathbf{k}',\mathbf{k})$ is just a function of q for a spherically symmetric potential. It is easily demonstrated that

$$q \equiv |\mathbf{k}-\mathbf{k}'| = 2\,k\,\sin(\theta/2), \tag{14.2.4}$$

where θ is the angle subtended between the vectors $\mathbf{k}$ and $\mathbf{k}'$. In other words, θ is the scattering angle. Recall that the vectors $\mathbf{k}$ and $\mathbf{k}'$ have the same length, via energy conservation.

Consider scattering by a *Yukawa potential* ,

$$V(r) = \frac{V_0\,\exp(-\mu\,r)}{\mu\,r}, \tag{14.2.5}$$

where V_0 is a constant, and $1/\mu$ measures the “range” of the potential. It follows from Equation ([e17.38]) that

$$f(\theta) = -\frac{2\,m\,V_0}{\hbar^{\,2}\,\mu}\,\frac{1}{q^{\,2}+\mu^{\,2}}, \tag{14.2.6}$$

because

$$\int_0^\infty \exp(-\mu\,r')\,\sin(q\,r')\,dr' = \frac{q}{q^{\,2}+\mu^{\,2}}. \tag{14.2.7}$$

Thus, in the Born approximation, the differential cross-section for scattering by a Yukawa potential is

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2\,m\,V_0}{\hbar^{\,2}\,\mu}\right)^2 \frac{1}{[2\,k^{\,2}\,(1-\cos\theta)+\mu^{\,2}]^{\,2}}, \tag{14.2.8}$$

given that

$$q^{\,2} = 4\,k^{\,2}\,\sin^2(\theta/2) = 2\,k^{\,2}\,(1-\cos\theta). \tag{14.2.9}$$

The Yukawa potential reduces to the familiar Coulomb potential as $\mu \rightarrow 0$, provided that $V_0/\mu \rightarrow Z\,Z'\,e^{\,2}/(4\pi\,\epsilon_0)$. In this limit, the Born differential cross-section becomes

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{2\,m\,Z\,Z'\,e^{\,2}}{4\pi\,\epsilon_0\,\hbar^{\,2}}\right)^2 \frac{1}{16\,k^{\,4}\,\sin^4(\theta/2)}. \tag{14.2.10}$$

Recall that $\hbar\,k$ is equivalent to $|\mathbf{p}|$, so the previous equation can be rewritten

$$\frac{d\sigma}{d\Omega} \simeq \left(\frac{Z\,Z'\,e^{2}}{16\pi\,\epsilon_0\,E}\right)^{2} \frac{1}{\sin^4(\theta/2)}, \tag{14.2.11}$$

where $E = p^{2}/2\,m$ is the kinetic energy of the incident particles. Of course, Equation ([e17.46]) is identical to the famous *Rutherford scattering cross-section* formula of classical physics .

The Born approximation is valid provided that $\psi(\mathbf{r})$ is not too different from $\psi_0(\mathbf{r})$ in the scattering region. It follows, from Equation ([e15.9]), that the condition for $\psi(\mathbf{r}) \simeq \psi_0(\mathbf{r})$ in the vicinity of $\mathbf{r} = \mathbf{0}$ is

$$\left|\frac{m}{2\pi\,\hbar^{2}} \int \frac{\exp(\,{\rm i}\,k\,r')}{r'}\,V(\mathbf{r}')\,d^{\,3}\mathbf{r}'\right| \ll 1. \tag{14.2.12}$$

Consider the special case of the Yukawa potential. At low energies, (i.e., $k \ll \mu$) we can replace $\exp(\,{\rm i}\,k\,r')$ by unity, giving

$$\frac{2\,m}{\hbar^{2}}\frac{|V_0|}{\mu^{2}} \ll 1 \tag{14.2.13}$$

as the condition for the validity of the Born approximation. The condition for the Yukawa potential to develop a bound state is

$$\frac{2\,m}{\hbar^{2}}\frac{|V_0|}{\mu^{2}} \geq 2.7, \tag{14.2.14}$$

where V_0 is negative . Thus, if the potential is strong enough to form a bound state then the Born approximation is likely to break down. In the high-k limit, Equation ([e17.47]) yields

$$\frac{2\,m}{\hbar^{2}}\frac{|V_0|}{\mu\,k} \ll 1. \tag{14.2.15}$$

This inequality becomes progressively easier to satisfy as k increases, implying that the Born approximation is more accurate at high incident particle energies.

Contributors and Attributions

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