9.1: Spin Operators

Because spin is a type of angular momentum, it is reasonable to suppose that it possesses similar properties to orbital angular momentum. Thus, by analogy with Section [s8.2], we would expect to be able to define three operators—S_x, S_y, and S_z—that represent the three Cartesian components of spin angular momentum. Moreover, it is plausible that these operators possess analogous commutation relations to the three corresponding orbital angular momentum operators, L_x, L_y, and L_z. [See Equations ([e8.6])–([e8.8]).] In other words,

$$\begin{aligned}
[S_x, S_y] &= \mathrm{i}\,\hbar\,S_z,\\
[S_y, S_z] &= \mathrm{i}\,\hbar\,S_x,\\
[S_z, S_x] &= \mathrm{i}\,\hbar\,S_y.
\end{aligned}$$

We can represent the magnitude squared of the spin angular momentum vector by the operator

$$S^2 = S_x^{\,2} + S_y^{\,2} + S_z^{\,2}. \tag{9.1.1}$$

By analogy with the analysis in Section [s8.2], it is easily demonstrated that

$$[S^2, S_x] = [S^2, S_y] = [S^2, S_z] = 0. \tag{9.1.2}$$

We thus conclude (see Section [smeas]) that we can simultaneously measure the magnitude squared of the spin angular momentum vector, together with, at most, one Cartesian component. By convention, we shall always choose to measure the z-component, S_z.

By analogy with Equation ([e8.13]), we can define raising and lowering operators for spin angular momentum:

$$S_\pm = S_x \pm \mathrm{i}\,S_y. \tag{9.1.3}$$

If S_x, S_y, and S_z are Hermitian operators, as must be the case if they are to represent physical quantities, then $S_\pm$ are the Hermitian conjugates of one another: that is,

$$(S_\pm)^\dagger = S_\mp. \tag{9.1.4}$$

Finally, by analogy with Section [s8.2], it is easily demonstrated that

$$\begin{aligned}
S_+\,S_- &= S^2 - S_z^{\,2} + \hbar\,S_z,\\
S_-\,S_+ &= S^2 - S_z^2 - \hbar\,S_z,\\
[S_+, S_z] &= -\hbar\,S_+,\\
[S_-, S_z] &= +\hbar\,S_-.
\end{aligned}$$

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)