2.2: Plane-Waves

As we have just seen, a wave of amplitude A, wavenumber k, angular frequency ω, and phase angle φ, propagating in the positive x-direction, is represented by the following wavefunction:

$$\psi(x,t) = A\,\cos(k\,x-\omega\,t+\varphi). \tag{2.2.1}$$

This type of wave is conventionally termed a *one-dimensional plane-wave*. It is one-dimensional because its associated wavefunction only depends on the single Cartesian coordinate, x. Furthermore, it is a plane-wave because the wave maxima, which are located at

$$k\,x-\omega\,t+\varphi = j\,2\pi, \tag{2.2.2}$$

where j is an integer, consist of a series of parallel planes, normal to the x-axis, that are equally spaced a distance $\lambda = 2\pi/k$ apart, and propagate along the positive x-axis at the velocity $v=\omega/k$. These conclusions follow because Equation (2.2.2) can be rewritten in the form

$$x = d, \tag{2.2.3}$$

where $d=(j-\varphi/2\pi)\,\lambda+v\,t$. Moreover, as is well known, Equation (2.2.3) is the equation of a plane, normal to the x-axis, whose distance of closest approach to the origin is d.

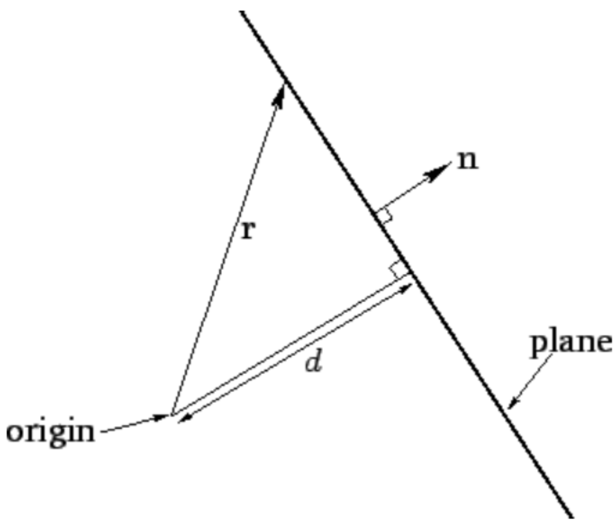

Figure 1: *The solution of* $\mathbf{n}\cdot\mathbf{r}=d$ *is a plane.*

The previous equation can also be written in the coordinate-free form

$$\mathbf{n}\cdot\mathbf{r} = d, \tag{2.2.4}$$

where $\mathbf{n}=(1,\,0,\,0)$ is a unit vector directed along the positive x-axis, and $\mathbf{r}=(x,\,y,\,z)$ represents the vector displacement of a general point from the origin. Because there is nothing special about the x-direction, it follows that if $\mathbf{n}$ is reinterpreted as a unit vector pointing in an arbitrary direction then Equation (2.2.4) can be reinterpreted as the general equation of a plane. As before, the plane is normal to $\mathbf{n}$, and its distance of closest approach to the origin is d. See Figure [f10.1]. This observation allows us to write the three-dimensional equivalent to the wavefunction (2.2.1) as

$$\psi(\mathbf{r},t) = A\,\cos(\mathbf{k}\cdot\mathbf{r}-\omega\,t+\varphi), \tag{2.2.5}$$

where the constant vector $\mathbf{k}=(k_x,\,k_y,\,k_z)=k\,\mathbf{n}$ is called the *wavevector*. The wave represented previously is conventionally termed a *three-dimensional plane-wave*. It is three-dimensional because its wavefunction, $\psi(\mathbf{r},t)$, depends on all three Cartesian coordinates. Moreover, it is a plane-wave because the wave maxima are located at

$$\mathbf{k}\cdot\mathbf{r}-\omega\,t+\varphi = j\,2\pi, \tag{2.2.6}$$

or

$$\mathbf{n}\cdot\mathbf{r} = (j - \varphi/2\pi)\,\lambda + v\,t, \tag{2.2.7}$$

where $\lambda = 2\pi/k$, and $v = \omega/k$. Note that the wavenumber, k, is the magnitude of the wavevector, $\mathbf{k}$: that is, $k \equiv |\mathbf{k}|$. It follows, by comparison with Equation (2.2.4), that the wave maxima consist of a series of parallel planes, normal to the wavevector, that are equally spaced a distance λ apart, and that propagate in the $\mathbf{k}$-direction at the velocity v. See Figure [f10.2]. Hence, the direction of the wavevector specifies the wave propagation direction, whereas its magnitude determines the wavenumber, k, and, thus, the wavelength, $\lambda = 2\pi/k$.

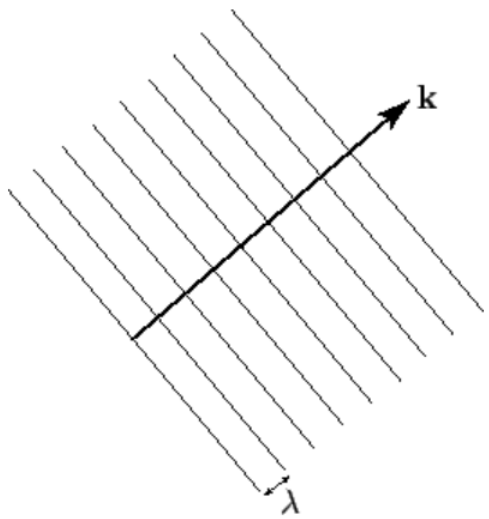

Figure 2: *Wave maxima associated with a three-dimensional plane wave.*

Contributors and Attributions

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