2.10: Wave-Packets

The previous discussion suggests that the wavefunction of a massive particle of momentum p and energy E, moving in the positive x-direction, can be written

$$\psi(x,t) = \bar{\psi}\,{\rm e}^{\,{\rm i}\,(k\,x-\omega\,t)}, \tag{2.10.1}$$

where $k=p/\hbar>0$ and $\omega=E/\hbar>0$. Here, ω and k are linked via the dispersion relation ([e2.38]). Expression ([e2.41]) represents a plane-wave whose maxima and minima propagate in the positive x-direction with the phase-velocity $v_p=\omega/k$. As we have seen, this phase-velocity is only half of the classical velocity of a massive particle.

From before, the most reasonable physical interpretation of the wavefunction is that $|\psi(x,t)|^{\,2}$ is proportional to the probability density of finding the particle at position x at time t. However, the modulus squared of the wavefunction ([e2.41]) is $|\bar{\psi}|^{\,2}$, which depends on neither x nor t. In other words, this wavefunction represents a particle that is equally likely to be found anywhere on the x-axis at all times. Hence, the fact that the maxima and minima of the wavefunction propagate at a phase-velocity that does not correspond to the classical particle velocity does not have any real physical consequences.

How can we write the wavefunction of a particle that is localized in x: that is, a particle that is more likely to be found at some positions on the x-axis than at others? It turns out that we can achieve this goal by forming a linear combination of plane-waves of different wavenumbers: in other words,

$$\psi(x,t) = \int_{-\infty}^{\infty}\bar{\psi}(k)\,{\rm e}^{\,{\rm i}\,(k\,x-\omega\,t)}\,dk. \tag{2.10.2}$$

Here, $\bar{\psi}(k)$ represents the complex amplitude of plane-waves of wavenumber k in this combination. In writing the previous expression, we are relying on the assumption that particle waves are superposable: that is, that it is always possible to add two valid wave solutions to form a third valid wave solution. The ultimate justification for this assumption is that particle waves satisfy a differential wave equation that is linear in ψ. As we shall see, in Section 1.15, this is indeed the case. Incidentally, a plane-wave that varies as $\exp[\,{\rm i}\,(k\,x-\omega\,t)]$ and has a negative k (but positive ω) propagates in the negative x-direction at the phase-velocity $\omega/|k|$. Hence, the superposition ([e2.42]) includes both forward and backward propagating waves.

There is a useful mathematical theorem, known as *Fourier's theorem* , which states that if

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\bar{f}\,(k)\,{\rm e}^{\,{\rm i}\,k\,x}\,dk, \tag{2.10.3}$$

then

$$\bar{f}\,(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(x)\,{\rm e}^{-{\rm i}\,k\,x}\,dx. \tag{2.10.4}$$

Here, $\bar{f}\,(k)$ is known as the *Fourier transform* of the function $f(x)$. We can use Fourier's theorem to find the k-space function $\bar{\psi}(k)$ that generates any given x-space wavefunction $\psi(x)$ at a given time.

For instance, suppose that at $t=0$ the wavefunction of our particle takes the form

$$\psi(x,0)\propto \exp\left[{\rm i}\,k_0\,x - \frac{(x-x_0)^{\,2}}{4\,(\varDelta x)^{\,2}}\right]. \tag{2.10.5}$$

Thus, the initial probability density of the particle is written

$$|\psi(x,0)|^{\,2}\propto \exp\left[-\frac{(x-x_0)^{\,2}}{2\,(\varDelta x)^{\,2}}\right]. \tag{2.10.6}$$

This particular probability distribution is called a *Gaussian* distribution, and is plotted in Figure [f4]. It can be seen that a measurement of the particle's position is most likely to yield the value x_0, and very unlikely to yield a value which differs from x_0 by more than $3\,\varDelta x$. Thus, Equation ([e2.45]) is the wavefunction of a particle that is initially localized around $x=x_0$ in some region whose width is of order $\varDelta x$. This type of wavefunction is known as a *wave-packet*.

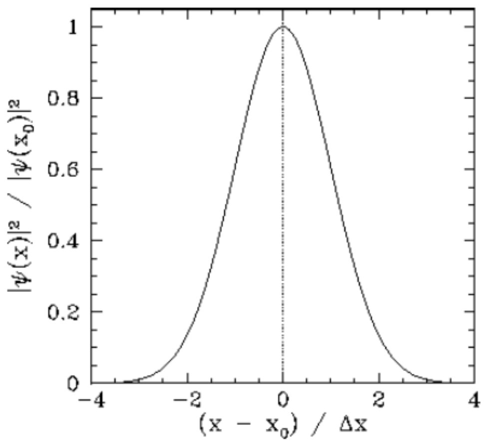

Figure 7: *A Gaussian probability distribution in x-space.*

According to Equation ([e2.42]),

$$\psi(x,0)=\int_{-\infty}^{\infty}\bar{\psi}(k)\,\mathrm{e}^{\,\mathrm{i}\,k\,x}\,dk. \tag{2.10.7}$$

Hence, we can employ Fourier's theorem to invert this expression to give

$$\bar{\psi}(k)\propto\int_{-\infty}^{\infty}\psi(x,0)\,\mathrm{e}^{-\mathrm{i}\,k\,x}\,dx. \tag{2.10.8}$$

Making use of Equation ([e2.45]), we obtain

$$\bar{\psi}(k)\propto\mathrm{e}^{-\mathrm{i}\,(k-k_0)\,x_0}\int_{-\infty}^{\infty}\exp\left[-\mathrm{i}\,(k-k_0)\,(x-x_0)-\frac{(x-x_0)^{\,2}}{4\,(\varDelta x)^{\,2}}\right]dx. \tag{2.10.9}$$

Changing the variable of integration to $y=(x-x_0)/(2\,\varDelta x)$, this reduces to

$$\bar{\psi}(k)\propto\mathrm{e}^{-\mathrm{i}\,k\,x_0}\int_{-\infty}^{\infty}\exp\left(-\mathrm{i}\,\beta\,y-y^{\,2}\right)dy, \tag{2.10.10}$$

where $\beta=2\,(k-k_0)\,\varDelta x$. The previous equation can be rearranged to give

$$\bar{\psi}(k)\propto\mathrm{e}^{-\mathrm{i}\,k\,x_0-\beta^{\,2}/4}\int_{-\infty}^{\infty}\mathrm{e}^{-(y-y_0)^{\,2}}\,dy, \tag{2.10.11}$$

where $y_0=-\mathrm{i}\,\beta/2$. The integral now just reduces to a number, as can easily be seen by making the change of variable $z=y-y_0$. Hence, we obtain

$$\bar{\psi}(k)\propto\exp\left[-\mathrm{i}\,k\,x_0-\frac{(k-k_0)^{\,2}}{4\,(\varDelta k)^{\,2}}\right], \tag{2.10.12}$$

where

$$\varDelta k=\frac{1}{2\,\varDelta x}. \tag{2.10.13}$$

If $|\psi(x)|^{\,2}$ is proportional to the probability density of a measurement of the particle's position yielding the value x then it stands to reason that $|\bar{\psi}(k)|^{\,2}$ is proportional to the probability density of a measurement of the particle's wavenumber yielding the value k. (Recall that $p=\hbar\,k$, so a measurement of the particle's wavenumber, k, is equivalent to a measurement of the particle's momentum, p). According to Equation ([e2.51]),

$$|\bar{\psi}(k)|^{\,2}\propto\exp\left[-\frac{(k-k_0)^{\,2}}{2\,(\varDelta k)^{\,2}}\right]. \tag{2.10.14}$$

Note that this probability distribution is a Gaussian in k-space. [See Equation ([e2.46]) and Figure [f4].] Hence, a measurement of k is most likely to yield the value k_0, and very unlikely to yield a value which differs from k_0 by more than $3\,\Delta k$. Incidentally, a Gaussian is the only simple mathematical function in x-space that has the same form as its Fourier transform in k-space.

We have just seen that a Gaussian probability distribution of characteristic width Δx in x-space [see Equation ([e2.46])] transforms to a Gaussian probability distribution of characteristic width Δk in k-space [see Equation ([e2.53])], where

$$\Delta x\,\Delta k = \frac{1}{2}. \tag{2.10.15}$$

This illustrates an important property of wave-packets. Namely, if we wish to construct a packet that is very localized in x-space (i.e., if Δx is small) then we need to combine plane-waves with a very wide range of different k-values (i.e., Δk will be large). Conversely, if we only combine plane-waves whose wavenumbers differ by a small amount (i.e., if Δk is small) then the resulting wave-packet will be very extended in x-space (i.e., Δx will be large).

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)