12.5: Harmonic Perturbation

Consider a (Hermitian) perturbation that oscillates sinusoidally in time. This is usually termed a *harmonic perturbation*. Such a perturbation takes the form

$$H_1(t) = V\,\exp(\,\mathrm{i}\,\omega\,t) + V^{\dagger}\,\exp(-\mathrm{i}\,\omega\,t), \tag{12.5.1}$$

where V is, in general, a function of position, momentum, and spin operators.

It follows from Equations ([e13.48]) and ([e13.51]) that, to first-order,

$$c_f(t) = -\frac{\mathrm{i}}{\hbar}\int_0^t \left[V_{fi}\,\exp(\,\mathrm{i}\,\omega\,t') + V_{fi}^{\dagger}\,\exp(-\mathrm{i}\,\omega\,t')\right]\exp(\,\mathrm{i}\,\omega_{fi}\,t')\,dt', \tag{12.5.2}$$

where

$$\begin{aligned} V_{fi} &= \langle f|V|i\rangle, \\ V_{fi}^{\dagger} &= \langle f|V^{\dagger}|i\rangle = \langle i|V|f\rangle^{*}. \end{aligned}$$

Integration with respect to t' yields

$$\begin{aligned} c_f(t) \;&= -\frac{\mathrm{i}\,t}{\hbar}\big(V_{fi}\,\exp[\,\mathrm{i}\,(\omega+\omega_{fi})\,t/2]\mathrm{sinc}\,[(\omega+\omega_{fi})\,t/2] \\ &\quad + V_{fi}^{\dagger}\,\exp[-\mathrm{i}\,(\omega-\omega_{fi})\,t/2]\mathrm{sinc}\,[(\omega-\omega_{fi})\,t/2]\big)\,, \end{aligned}$$

where

$$\mathrm{sinc}\,x \equiv \frac{\sin\,x}{x}. \tag{12.5.3}$$

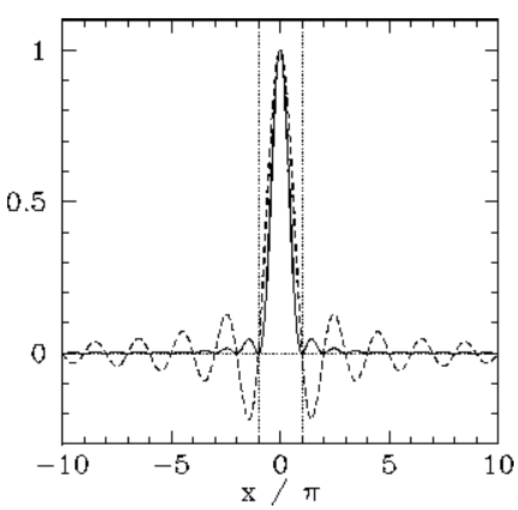

Figure 25: *The functions* $\mathrm{sinc}(x)$ *(dashed curve) and* $\mathrm{sinc}^2(x)$ *(solid curve). The vertical dotted lines denote the region* $|x| \leq \pi$

Now, the function $\mathrm{sinc}(x)$ takes its largest values when $|x| \lesssim \pi$, and is fairly negligible when $|x| \gg \pi$. (See Figure [fsinc].) Thus, the first and second terms on the right-hand side of Equation ([e13.55]) are only non-negligible when

$$|\omega+\omega_{fi}| \lesssim \frac{2\pi}{t} \tag{12.5.4}$$

and

$$|\omega-\omega_{fi}| \lesssim \frac{2\pi}{t} \tag{12.5.5}$$

respectively.

Clearly, as t increases, the ranges in ω over which these two terms are non-negligible gradually shrink in size. Eventually, when $t \gg 2\pi/|\omega_{fi}|$, these two ranges become strongly non-overlapping. Hence, in this limit, $P_{i\rightarrow f} = |c_f|^{\,2}$ yields

$$P_{i\rightarrow f}(t) = \frac{t^{\,2}}{\hbar^{\,2}}\left\{|V_{fi}|^{\,2}\,\mathrm{sinc}^2\left[(\omega+\omega_{fi})\,t/2\right] + |V_{fi}^\dagger|^{\,2}\,\mathrm{sinc}^2\left[(\omega-\omega_{fi})\,t/2\right]\right\}. \tag{12.5.6}$$

Now, the function $\mathrm{sinc}^2(x)$ is very strongly peaked at $x=0$, and is completely negligible for $|x| \gg \pi$. (See Figure [fsinc].) It follows that the previous expression exhibits a resonant response to the applied perturbation at the frequencies $\omega = \pm\omega_{fi}$. Moreover, the widths of these resonances decease linearly as time increases. At each of the resonances (i.e., at $\omega = \pm\omega_{fi}$), the transition probability $P_{i\rightarrow f}(t)$ varies as $t^{\,2}$ [because $\sinh(0) = 1$]. This behavior is entirely consistent with our earlier result ([e13.28]), for the two-state system, in the limit $\gamma\,t \ll 1$ (recall that our perturbative solution is only valid as long as $P_{i\rightarrow f} \ll 1$).

The resonance at $\omega = -\omega_{fi}$ corresponds to

$$E_f - E_i = -\hbar\,\omega. \tag{12.5.7}$$

This implies that the system loses energy $\hbar\,\omega$ to the perturbing field, while making a transition to a final state whose energy is less than the initial state by $\hbar\,\omega$. This process is known as *stimulated emission*. The resonance at $\omega = \omega_{fi}$ corresponds to

$$E_f - E_i = \hbar\,\omega. \tag{12.5.8}$$

This implies that the system gains energy $\hbar\,\omega$ from the perturbing field, while making a transition to a final state whose energy is greater than that of the initial state by $\hbar\,\omega$. This process is known as *absorption*.

Stimulated emission and absorption are mutually exclusive processes, because the first requires $\omega_{fi} < 0$, whereas the second requires $\omega_{fi} > 0$. Hence, we can write the transition probabilities for both processes separately. Thus, from Equation ([e13.49]), the transition probability for stimulated emission is

$$P_{i\rightarrow f}^{stm}(t) = \frac{t^{\,2}}{\hbar^{\,2}}\,|V_{if}^\dagger|^{\,2}\,\mathrm{sinc}^2\left[(\omega-\omega_{if})\,t/2\right], \tag{12.5.9}$$

where we have made use of the facts that $\omega_{if} = -\omega_{fi} > 0$, and $|V_{fi}|^{\,2} = |V_{if}^\dagger|^{\,2}$. Likewise, the transition probability for absorption is

$$P_{i\rightarrow f}^{abs}(t) = \frac{t^{\,2}}{\hbar^{\,2}}\,|V_{fi}^\dagger|^{\,2}\,\mathrm{sinc}^2\left[(\omega-\omega_{fi})\,t/2\right]. \tag{12.5.10}$$

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)

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