7.4: Eigenvalues of Lz

It seems reasonable to attempt to write the eigenstate $Y_{l,m}(\theta,\phi)$ in the separable form

$$Y_{l,m}(\theta,\phi) = \Theta_{l,m}(\theta)\,\Phi_m(\phi). \tag{7.4.1}$$

We can satisfy the orthonormality constraint ([e8.31]) provided that

$$\int_0^\pi \Theta^{\,*}_{l',m'}(\theta)\,\Theta_{l,m}(\theta)\,\sin\theta\,d\theta = \delta_{ll'},$$
$$\int_0^{2\pi} \Phi^{\,*}_{m'}(\phi)\,\Phi_m(\phi)\,d\phi = \delta_{mm'}.$$

Note, from Equation ([e8.26]), that the differential operator which represents L_z only depends on the azimuthal angle ϕ, and is independent of the polar angle θ. It therefore follows from Equations ([e8.26]), ([e8.29]), and ([e8.34]) that

$$-{\rm i}\,\hbar\,\frac{d\Phi_m}{d\phi} = m\,\hbar\,\Phi_m. \tag{7.4.2}$$

The solution of this equation is

$$\Phi_m(\phi) \sim {\rm e}^{\,{\rm i}\,m\,\phi}. \tag{7.4.3}$$

Here, the symbol $\sim$ just means that we are neglecting multiplicative constants.

Our basic interpretation of a wavefunction as a quantity whose modulus squared represents the probability density of finding a particle at a particular point in space suggests that a physical wavefunction must be single-valued in space. Otherwise, the probability density at a given point would not, in general, have a unique value, which does not make physical sense. Hence, we demand that the wavefunction ([e8.38]) be single-valued: that is, $\Phi_m(\phi+2\,\pi)=\Phi_m(\phi)$ for all ϕ. This immediately implies that the quantity m is quantized. In fact, m can only take integer values. Thus, we conclude that the eigenvalues of L_z are also quantized, and take the values $m\,\hbar$, where m is an integer. [A more rigorous argument is that $\Phi_m(\phi)$ must be continuous in order to ensure that L_z is an Hermitian operator, because the proof of hermiticity involves an integration by parts in ϕ that has canceling contributions from $\phi=0$ and $\phi=2\pi$.]

Finally, we can easily normalize the eigenstate ([e8.38]) by making use of the orthonormality constraint ([e8.36]). We obtain

$$\Phi_m(\phi) = \frac{{\rm e}^{\,{\rm i}\,m\,\phi}}{\sqrt{2\pi}}. \tag{7.4.4}$$

This is the properly normalized eigenstate of L_z corresponding to the eigenvalue $m\,\hbar$.

Contributors and Attributions

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