3.7: Heisenberg's Uncertainty Principle

Consider a real-space Hermitian operator, $O(x)$. A straightforward generalization of Equation ([e3.55a]) yields

$$\int_{-\infty}^{\infty}\psi_1^\ast\,(O\,\psi_2)\,dx = \int_{-\infty}^{\infty}(O\,\psi_1)^\ast\,\psi_2\,dx, \tag{3.7.1}$$

where $\psi_1(x)$ and $\psi_2(x)$ are general functions.

Let $f = (A-\langle A\rangle)\,\psi$, where $A(x)$ is an Hermitian operator, and $\psi(x)$ a general wavefunction. We have

$$\int_{-\infty}^{\infty}|f|^{\,2}\,dx = \int_{-\infty}^{\infty}f^\ast\,f\,dx = \int_{-\infty}^{\infty}[(A-\langle A\rangle)\,\psi]^{\,\ast}\,[(A-\langle A\rangle)\,\psi]\,dx. \tag{3.7.2}$$

Making use of Equation ([e3.84]), we obtain

$$\int_{-\infty}^{\infty}|f|^{\,2}\,dx = \int_{-\infty}^{\infty}\psi^\ast\,(A-\langle A\rangle)^{\,2}\,\psi\,dx = \sigma_A^{\,2}, \tag{3.7.3}$$

where $\sigma_A^{\,2}$ is the variance of A. [See Equation ([e3.24a]).] q4 Similarly, if $g=(B-\langle B\rangle)\,\psi$, where B is a second Hermitian operator, then

$$\int_{-\infty}^{\infty}|g|^{\,2}\,dx = \sigma_B^{\,2}, \tag{3.7.4}$$

Now, there is a standard result in mathematics, known as the *Schwartz inequality* , which states that

$$\left|\int_a^b f^\ast(x)\,g(x)\,dx\right|^{\,2} \leq \int_a^b |f(x)|^{\,2}\,dx\,\int_a^b |g(x)|^{\,2}\,dx, \tag{3.7.5}$$

where f and g are two general functions. Furthermore, if z is a complex number then

$$|z|^{\,2} = [\mathrm{Re}(z)]^{\,2} + [\mathrm{Im}(z)]^{\,2} \geq [\mathrm{Im}(z)]^{\,2} = \left[\frac{1}{2\,\mathrm{i}}\,(z-z^\ast)\right]^{\,2}. \tag{3.7.6}$$

Hence, if $z=\int_{-\infty}^{\infty}f^\ast\,g\,dx$ then Equations ([e3.86])–([e3.89]) yield

$$\sigma_A^{\,2}\,\sigma_B^{\,2} \geq \left[\frac{1}{2\,\mathrm{i}}\,(z-z^\ast)\right]^{\,2}. \tag{3.7.7}$$

However,

$$z = \int_{-\infty}^{\infty}[(A-\langle A\rangle)\,\psi]^{\,\ast}\,[(B-\langle B\rangle)\,\psi]\,dx = \int_{-\infty}^{\infty}\psi^\ast\,(A-\langle A\rangle)\,(B-\langle B\rangle)\,\psi\,dx, \tag{3.7.8}$$

where use has been made of Equation ([e3.84]). The previous equation reduces to

$$z = \int_{-\infty}^{\infty}\psi^\ast\,A\,B\,\psi\,dx - \langle A\rangle\,\langle B\rangle. \tag{3.7.9}$$

Furthermore, it is easily demonstrated that

$$z^\ast = \int_{-\infty}^{\infty}\psi^\ast\,B\,A\,\psi\,dx - \langle A\rangle\,\langle B\rangle. \tag{3.7.10}$$

Hence, Equation ([e3.90]) gives

$$\sigma_A^{\,2}\,\sigma_B^{\,2} \geq \left(\frac{1}{2\,\mathrm{i}}\langle[A,B]\rangle\right)^{\,2}, \tag{3.7.11}$$

where

$$[A,B] \equiv A\,B - B\,A. \tag{3.7.12}$$

Equation ([e3.94]) is the general form of *Heisenberg's uncertainty principle* in quantum mechanics. It states that if two dynamical variables are represented by the two Hermitian operators A and B, and these operators do not commute (i.e., $A\,B \neq B\,A$), then it is impossible to simultaneously (exactly) measure the two variables. Instead, the product of the variances in the measurements is always greater than some critical value, which depends on the extent to which the two operators do not commute.

For instance, displacement and momentum are represented (in real-space) by the operators x and $p \equiv -\mathrm{i}\,\hbar\,\partial/\partial x$, respectively. Now, it is easily demonstrated that

$$[x, p] = \mathrm{i}\,\hbar. \tag{3.7.13}$$

Thus,

$$\sigma_x\,\sigma_p \geq \frac{\hbar}{2}, \tag{3.7.14}$$

which can be recognized as the standard displacement-momentum uncertainty principle (see Section [sun]). It turns out that the minimum uncertainty (i.e., $\sigma_x\,\sigma_p = \hbar/2$) is only achieved by Gaussian wave-packets (see Section [s2.9]): that is,

$$\psi(x) = \frac{\mathrm{e}^{+ip_0 x/\hbar}}{\left(2\pi\sigma_x^2\right)^{1/4}}\,\mathrm{e}^{-(x-x_0)^2/4\sigma_x^2} \tag{3.7.15}$$

$$\phi(p) = \frac{\mathrm{e}^{-\mathrm{i}p x_0/\hbar}}{\left(2\pi\sigma_p^2\right)^{1/4}}\,\mathrm{e}^{-(p-p_0)^2/4\sigma_p^2} \tag{3.7.16}$$

where $\phi(p)$ is the momentum-space equivalent of $\psi(x)$.

Energy and time are represented by the operators $H \equiv \mathrm{i}\,\hbar\,\partial/\partial t$ and t, respectively. These operators do not commute, indicating that energy and time cannot be measured simultaneously. In fact,

$$[H, t] = \mathrm{i}\,\hbar, \tag{3.7.17}$$

so

$$\sigma_E\,\sigma_t \geq \frac{\hbar}{2}. \tag{3.7.18}$$

This can be written, somewhat less exactly, as

$\Delta E \Delta t \gtrsim \hbar$ are the uncertainties in energy and time, respectively. The previous expression is generally known as the *energy-time uncertainty principle*.

For instance, suppose that a particle passes some fixed point on the x-axis. Because the particle is, in reality, an extended wave-packet, it takes a certain amount of time, Δt, for the particle to pass. Thus, there is an uncertainty, Δt, in the arrival time of the particle. Moreover, because $E = \hbar\,\omega$, the only wavefunctions that have unique energies are those with unique frequencies: that is, plane-waves. Because a wave-packet of finite extent is made up of a combination of plane-waves of different wavenumbers, and, hence, different frequencies, there will be an uncertainty ΔE in the particle's energy that is proportional to the range of frequencies of the plane-waves making up the wave-packet. The more compact the wave-packet (and, hence, the smaller Δt), the larger the range of frequencies of the constituent plane-waves (and, hence, the large ΔE), and vice versa.

To be more exact, if $\psi(t)$ is the wavefunction measured at the fixed point as a function of time then we can write

$$\psi(t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \chi(E)\mathrm{e}^{-iEt/\hbar}\,dE \tag{3.7.19}$$

In other words, we can express $\psi(t)$ as a linear combination of plane-waves of definite energy E. Here, $\chi(E)$ is the complex amplitude of plane-waves of energy E in this combination.

By Fourier's theorem, we also have

$$\chi(E) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} \psi(t)\mathrm{e}^{+iEt/\hbar}\,dt \tag{3.7.20}$$

For instance, if $\psi(t)$ is a Gaussian then it is easily shown that $\chi(E)$ is also a Gaussian: that is,

$$\psi(t)=\frac{\mathrm{e}^{-iE_0t/\hbar}}{\left(2\pi\sigma_t^2\right)^{1/4}}\mathrm{e}^{-(t-t_0)^2/4\sigma_t^2} \tag{3.7.21}$$

$$\chi(E)=\frac{\mathrm{e}^{+iEt_0/\hbar}}{\left(2\pi\sigma_E^2\right)^{1/4}}\mathrm{e}^{-(E-E_0)^2/4\sigma_E^2} \tag{3.7.22}$$

where $\sigma_E\,\sigma_t=\hbar/2$. As before, Gaussian wave-packets satisfy the minimum uncertainty principle $\sigma_E\,\sigma_t=\hbar/2$. Conversely, non-Gaussian wave-packets are characterized by $\sigma_E\,\sigma_t>\hbar/2$.

Contributors and Attributions

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