4.E: One-Dimensional Potentials (Exercises)

1. Show that the wavefunction of a particle of mass m in an infinite one-dimensional square-well of width a returns to its original form after a quantum revival time $T = 4\,m\,a^{\,2}/\pi\,\hbar$.
2. A particle of mass m moves freely in one dimension between impenetrable walls located at $x = 0$ and a. Its initial wavefunction is

$$\psi(x,0) = \sqrt{2/a}\,\sin(3\pi\,x/a). \tag{4.E.1}$$

   What is the subsequent time evolution of the wavefunction? Suppose that the initial wavefunction is

$$\psi(x,0) = \sqrt{1/a}\,\sin(\pi\,x/a)\,[1+2\,\cos(\pi\,x/a)]. \tag{4.E.2}$$

   What now is the subsequent time evolution? Calculate the probability of finding the particle between 0 and $a/2$ as a function of time in each case.
3. A particle of mass m is in the ground-state of an infinite one-dimensional square-well of width a. Suddenly the well expands to twice its original size, as the right wall moves from a to $2a$, leaving the wavefunction momentarily undisturbed. The energy of the particle is now measured. What is the most probable result? What is the probability of obtaining this result? What is the next most probable result, and what is its probability of occurrence? What is the expectation value of the energy?
4. A stream of particles of mass m and energy $E > 0$ encounter a potential step of height $W(< E)$: that is, $V(x) = 0$ for $x < 0$ and $V(x) = W$ for $x > 0$ with the particles incident from $-\infty$. Show that the fraction reflected is

$$R = \left(\frac{k-q}{k+q}\right)^2, \tag{4.E.3}$$

   where $k^{\,2} = (2\,m/\hbar^{\,2})\,E$ and $q^{\,2} = (2\,m/\hbar^{\,2})\,(E-W)$.
5. A stream of particles of mass m and energy $E > 0$ encounter the delta-function potential $V(x) = -\alpha\,\delta(x)$, where $\alpha > 0$. Show that the fraction reflected is

$$R = \beta^{\,2}/(1+\beta^{\,2}), \tag{4.E.4}$$

   where $\beta = m\,\alpha/\hbar^{\,2}\,k$, and $k^{\,2} = (2\,m/\hbar^{\,2})\,E$. Does such a potential have a bound state? If so, what is its energy?
6. Two potential wells of width a are separated by a distance $L \gg a$. A particle of mass m and energy E is in one of the wells. Estimate the time required for the particle to tunnel to the other well.
7. Consider a particle trapped in the finite potential well whose potential is given by Equation ([e5.71]). Demonstrate that for a totally-symmetric state the ratio of the probability of finding the particle outside to the probability of finding the particle inside the well is

$$\frac{P_{\rm out}}{P_{\rm in}} = \frac{\cos^3 y}{\sin y\,(y+\sin y\,\cos y)}, \tag{4.E.5}$$

   where $(\lambda - y^{\,2})^{1/2} = y\,\tan y$, and $\lambda = V/E_0$. Hence, demonstrate that for a shallow well (i.e., $\lambda \ll 1$) $P_{\rm out} \simeq 1-2\,\lambda$, whereas for a deep well (i.e., $\lambda \gg 1$) $P_{\rm out} \simeq (\pi^{\,2}/4)/\lambda^{\,3/2}$ (assuming that the particle is in the ground state).[ex12.3]
8. Consider the half-infinite potential well

$$V(x) = \begin{cases} \infty & \text{\hspace{1cm}} \quad x \le 0 \\ -V_0 & \quad 0 < x < L \\ 0 & \quad x \ge L \end{cases}, \tag{4.E.6}$$

   where $V_0 > 0$. Demonstrate that the bound-states of a particle of mass m and energy $-V_0 < E < 0$ satisfy

$$\tan\!\left(\sqrt{2\,m\,(V_0+E)}\;L/\hbar\right) = -\sqrt{(V_0+E)/(-E)}. \tag{4.E.7}$$

9. Find the properly normalized first two excited energy eigenstates of the harmonic oscillator, as well as the expectation value of the potential energy in the nth energy eigenstate. Hint: Consider the raising and lowering operators $a_{\pm}$, defined in Equation ([e5.109]).

Contributors and Attributions

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