11.9: Zeeman Effect

Consider a hydrogen atom placed in a uniform z-directed external magnetic field of magnitude $|\mathbf{B}|$. The modification to the Hamiltonian of the system is

$$H_1 = -\mu\cdot\mathbf{B}, \tag{11.9.1}$$

where

$$\mu = -\frac{e}{2\,m_e}\,(\mathbf{L}+2\,\mathbf{S}) \tag{11.9.2}$$

is the total electron magnetic moment, including both orbital and spin contributions. [See Equations ([e10.57])–([e10.59]).] Thus,

$$H_1 = \frac{e\,B}{2\,m_e}\,(L_z+2\,S_z). \tag{11.9.3}$$

Suppose that the applied magnetic field is much weaker than the atom's internal magnetic field, ([e12.124]). Because the magnitude of the internal field is about 25 tesla, this is a fairly reasonable assumption. In this situation, we can treat H_1 as a small perturbation acting on the simultaneous eigenstates of the unperturbed Hamiltonian and the fine structure Hamiltonian. Of course, these states are the simultaneous eigenstates of L^2, S^2, $J^{\,2}$, and J_z. (See the previous section.) Hence, from standard perturbation theory, the first-order energy-shift induced by a weak external magnetic field is

$$\begin{aligned}\Delta E_{l,1/2;j,m_j} &= \langle l,1/2;j,m_j|H_1|l,1/2;j,m_j\rangle\\ &= \frac{e\,B}{2\,m_e}\,\left(m_j\,\hbar+\langle l,1/2;j,m_j|S_z|l,1/2;j,m_j\rangle\right),\end{aligned}$$

because $J_z = L_z+S_z$. Now, according to Equations ([e11.47]) and ([e11.48]),

$$\psi^{(2)}_{j,m_j} = \left(\frac{j+m_j}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m_j-1/2,1/2}+\left(\frac{j-m_j}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m_j+1/2,-1/2} \tag{11.9.4}$$

when $j = l+1/2$, and

$$\psi^{(2)}_{j,m_j} = \left(\frac{j+1-m_j}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m_j-1/2,1/2}-\left(\frac{j+1+m_j}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m_j+1/2,-1/2} \tag{11.9.5}$$

when $j = l-1/2$. Here, the $\psi^{(1)}_{m,m_s}$ are the simultaneous eigenstates of L^2, S^2, L_z, and S_z, whereas the $\psi^{(2)}_{j,m_j}$ are the simultaneous eigenstates of L^2, S^2, $J^{\,2}$, and J_z. In particular,

$$S_z\,\psi^{(1)}_{m,\pm 1/2} = \pm\frac{\hbar}{2}\,\psi^{(1)}_{m,\pm 1/2}. \tag{11.9.6}$$

It follows from Equations ([e12.143])–([e12.145]), and the orthormality of the $\psi^{(1)}$, that

$$\langle l,1/2;j,m_j|S_z|l,1/2;j,m_j\rangle = \pm\frac{m_j\,\hbar}{2\,l+1} \tag{11.9.7}$$

when $j = l\pm 1/2$. Thus, the induced energy-shift when a hydrogen atom is placed in an external magnetic field—which is known as the *Zeeman effect* —becomes

$$\Delta E_{l,1/2;j,m_j} = \mu_B\,B\,m_j\left(1\pm\frac{1}{2\,l+1}\right) \tag{11.9.8}$$

where the $\pm$ signs correspond to $j = l\pm 1/2$. Here,

$$\mu_B = \frac{e\,\hbar}{2\,m_e} = 5.788\times 10^{-5}\ \mathrm{eV/T} \tag{11.9.9}$$

is known as the *Bohr magnetron*. Of course, the quantum number m_j takes values differing by unity in the range $-j$ to j. It, thus, follows from Equation ([e12.147]) that the Zeeman effect splits degenerate states characterized by $j = l+1/2$ into $2\,j+1$ equally spaced states of interstate spacing

$$\Delta E_{j=l+1/2} = \mu_B\,B\left(\frac{2\,l+2}{2\,l+1}\right). \tag{11.9.10}$$

Likewise, the Zeeman effect splits degenerate states characterized by $j=l-1/2$ into $2\,j+1$ equally spaced states of interstate spacing

$$\Delta E_{j=l-1/2} = \mu_B\,B\left(\frac{2\,l}{2\,l+1}\right). \tag{11.9.11}$$

In conclusion, in the presence of a weak external magnetic field, the two degenerate $1S_{1/2}$ states of the hydrogen atom are split by $2\,\mu_B\,B$. Likewise, the four degenerate $2S_{1/2}$ and $2P_{1/2}$ states are split by $(2/3)\,\mu_B\,B$, whereas the four degenerate $2P_{3/2}$ states are split by $(4/3)\,\mu_B\,B$. This is illustrated in Figure [fzee]. Note, finally, that because the $\psi^{(2)}_{l,m_j}$ are not simultaneous eigenstates of the unperturbed and perturbing Hamiltonians, Equations ([e12.149]) and ([e12.150]) can only be regarded as the expectation values of the magnetic-field induced energy-shifts. However, as long as the external magnetic field is much weaker than the internal magnetic field, these expectation values are almost identical to the actual measured values of the energy-shifts.

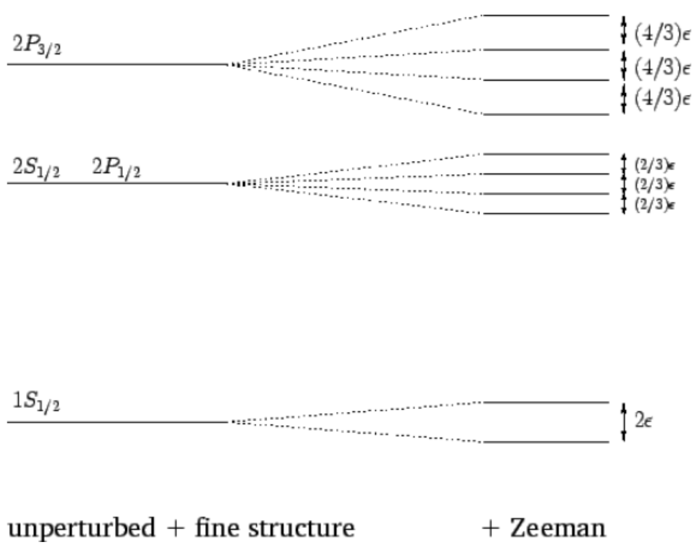

Figure 24: The Zeeman effect for the $n=1$ and 2 states of a hydrogen atom. Here, $\epsilon=\mu_B B$. Not to scale.

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)