2.4: Classical Light-Waves

Consider a classical, monochromatic, linearly-polarized, plane light-wave, propagating through a vacuum in the x-direction. It is convenient to characterize a light-wave (which is, of course, a type of electromagnetic wave) by specifying its associated electric field. Suppose that the wave is polarized such that this electric field oscillates in the y-direction. (According to standard electromagnetic theory, the magnetic field oscillates in the z-direction, in phase with the electric field, with an amplitude which is that of the electric field divided by the velocity of light in vacuum.) Now, the electric field can be conveniently represented in terms of a *complex wavefunction*:

$$\psi(x,t) = \bar{\psi}\,\mathrm{e}^{\,\mathrm{i}\,(k\,x-\omega\,t)}. \tag{2.4.1}$$

Here, $\mathrm{i}=\sqrt{-1}$, k and ω are real parameters, and $\bar{\psi}$ is a complex wave amplitude. By convention, the physical electric field is the real part of the previous expression. Suppose that

$$\bar{\psi} = |\bar{\psi}|\,\mathrm{e}^{\,\mathrm{i}\,\varphi}, \tag{2.4.2}$$

where φ is real. It follows that the physical electric field takes the form

$$E_y(x,t) = \mathrm{Re}[\psi(x,t)] = |\bar{\psi}|\,\cos(k\,x-\omega\,t+\varphi), \tag{2.4.3}$$

where $|\bar{\psi}|$ is the amplitude of the electric oscillation, k the wavenumber, ω the angular frequency, and φ the phase angle. In addition, $\lambda = 2\pi/k$ is the wavelength, and $\nu=\omega/2\pi$ the frequency (in hertz).

According to standard electromagnetic theory , the frequency and wavelength of light-waves are related according to the well-known expression

$$c = \nu\,\lambda, \tag{2.4.4}$$

or, equivalently,

$$\omega = k\,c, \tag{2.4.5}$$

where $c = 3\times 10^8\ \mathrm{m/s}$ is the velocity of light in vacuum. Equations (2.4.3) and (2.4.5) yield

$$E_y(x,t) = |\bar{\psi}|\,\cos(k\,[x-(\omega/k)\,t]+\varphi) = |\bar{\psi}|\,\cos(k\,[x-c\,t]+\varphi). \tag{2.4.6}$$

Note that E_y depends on x and t only via the combination $x-c\,t$. It follows that the wave maxima and minima satisfy

$$x - c\,t = \text{constant}. \tag{2.4.7}$$

Thus, the wave maxima and minima propagate in the x-direction at the fixed velocity

$$\frac{dx}{dt} = c. \tag{2.4.8}$$

An expression, such as Equation (2.4.5), that determines the wave angular frequency as a function of the wavenumber, is generally termed a *dispersion relation*. As we have already seen, and as is apparent from Equation (2.4.6), the maxima and minima of a plane-wave propagate at the characteristic velocity

$$v_p = \frac{\omega}{k}, \tag{2.4.9}$$

which is known as the *phase-velocity*. Hence, the dispersion relation (2.4.5) is effectively saying that the phase-velocity of a plane light-wave, propagating through a vacuum, always takes the fixed value c, irrespective of its wavelength or frequency.

From standard electromagnetic theory , the *energy density* (i.e., the energy per unit volume) of a plane light-wave is

$$U = \frac{E_y^{\,2}}{\epsilon_0}, \tag{2.4.10}$$

where $\epsilon_0 = 8.85\times 10^{-12}\ \mathrm{F/m}$ is the *electrical permittivity of free space*. Hence, it follows from Equations (2.4.1) and (2.4.3) that

$$U \propto |\psi|^{\,2}. \tag{2.4.11}$$

Furthermore, a light-wave possesses linear momentum, as well as energy. This momentum is directed along the wave's direction of propagation, and is of density

$$G = \frac{U}{c}. \tag{2.4.12}$$

Contributors and Attributions

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