10.1: General Principles of Angular Momentum

The three fundamental orbital angular momentum operators, L_x, L_y, and L_z, obey the commutation relations ([e8.6])–([e8.8]), which can be written in the convenient vector form:

$$\mathbf{L}\times\mathbf{L} = \mathrm{i}\,\hbar\,\mathbf{L}. \tag{10.1.1}$$

Likewise, the three funcdamental spin angular momentum operators, S_x, S_y, and S_z, obey the commutation relations ([e10.1x])–([e10.2x]), which can also be written in vector form: that is,

$$\mathbf{S}\times\mathbf{S} = \mathrm{i}\,\hbar\,\mathbf{S}. \tag{10.1.2}$$

Because the orbital angular momentum operators are associated with the electron's motion through space, whereas the spin angular momentum operators are associated with its internal motion, and these two types of motion are completely unrelated (i.e., they correspond to different degrees of freedom—see Section [sfuncon]), it is reasonable to suppose that the two sets of operators commute with one another: that is,

$$[L_i, S_j] = 0, \tag{10.1.3}$$

where $i, j = 1, 2, 3$ corresponds to x, y, z, respectively.

Let us now consider the electron's total angular momentum vector

$$\mathbf{J} = \mathbf{L} + \mathbf{S}. \tag{10.1.4}$$

We have

$$\begin{aligned}\mathbf{J}\times\mathbf{J} &= (\mathbf{L}+\mathbf{S})\times(\mathbf{L}+\mathbf{S}) \\ &= \mathbf{L}\times\mathbf{L}+\mathbf{S}\times\mathbf{S}+\mathbf{L}\times\mathbf{S}+\mathbf{S}\times\mathbf{L} = \mathbf{L}\times\mathbf{L}+\mathbf{S}\times\mathbf{S} \\ &= \mathrm{i}\,\hbar\,\mathbf{L}+\mathrm{i}\,\hbar\,\mathbf{S} = \mathrm{i}\,\hbar\,\mathbf{J}.\end{aligned}$$

In other words,

$$\mathbf{J}\times\mathbf{J} = \mathrm{i}\,\hbar\,\mathbf{J}. \tag{10.1.5}$$

It is thus evident that the three funcdamental total angular momentum operators, J_x, J_y, and J_z, obey analogous commutation relations to the corresponding orbital and spin angular momentum operators. It therefore follows that the total angular momentum has similar properties to the orbital and spin angular momenta. For instance, it is only possible to simultaneously measure the magnitude squared of the total angular momentum vector,

$$J^{\,2} = J_x^{\,2}+J_y^{\,2}+J_z^{\,2}, \tag{10.1.6}$$

together with a single Cartesian component. By convention, we shall always choose to measure J_z. A simultaneous eigenstate of J_z and $J^{\,2}$ satisfies

$$\begin{aligned}J_z\,\psi_{j,m_j} &= m_j\,\hbar\,\psi_{j,m_j}, \\ J^{\,2}\,\psi_{j,m_j} &= j\,(j+1)\,\hbar^{\,2}\,\psi_{j,m_j},\end{aligned}$$

where the quantum number j can take positive integer, or half-integer, values, and the quantum number m_j is restricted to the following range of values:

$$-j, -j+1, \cdots, j-1, j. \tag{10.1.7}$$

Now,

$$J^{\,2} = (\mathbf{L}+\mathbf{S})\cdot(\mathbf{L}+\mathbf{S}) = L^2+S^{\,2}+2\,\mathbf{L}\cdot\mathbf{S}, \tag{10.1.8}$$

which can also be written as

$$J^{\,2} = L^2+S^{\,2}+2\,L_z\,S_z+L_+\,S_-+L_-\,S_+. \tag{10.1.9}$$

We know that the operator L^2 commutes with itself, with all of the Cartesian components of $\mathbf{L}$ (and, hence, with the raising and lowering operators $L_\pm$), and with all of the spin angular momentum operators. (See Section [s8.2].) It is therefore clear that

$$[J^{\,2}, L^2] = 0. \tag{10.1.10}$$

A similar argument allows us to also conclude that

$$[J^{\,2}, S^{\,2}] = 0. \tag{10.1.11}$$

Now, the operator L_z commutes with itself, with L^2, with all of the spin angular momentum operators, but not with the raising and lowering operators $L_\pm$. (See Section [s8.2].) It follows that

$$[J^{\,2}, L_z] \neq 0. \tag{10.1.12}$$

Likewise, we can also show that

$$[J^{\,2}, S_z] \neq 0. \tag{10.1.13}$$

Finally, we have

$$J_z = L_z + S_z, \tag{10.1.14}$$

where $[J_z, L_z] = [J_z, S_z] = 0$.

Recalling that only commuting operators correspond to physical quantities that can be simultaneously measured (see Section [smeas]), it follows, from the previous discussion, that there are two alternative sets of physical variables associated with angular momentum that we can measure simultaneously. The first set correspond to the operators L^2, $S^{\,2}$, L_z, S_z, and J_z. The second set correspond to the operators L^2, $S^{\,2}$, $J^{\,2}$, and J_z. In other words, we can always measure the magnitude squared of the orbital and spin angular momentum vectors, together with the z-component of the total angular momentum vector. In addition, we can either choose to measure the z-components of the orbital and spin angular momentum vectors, or the magnitude squared of the total angular momentum vector.

Let $\psi^{(1)}_{l,s;m,m_s}$ represent a simultaneous eigenstate of L^2, $S^{\,2}$, L_z, and S_z corresponding to the following eigenvalues:

$$\begin{aligned}
L^2\,\psi^{(1)}_{l,s;m,m_s} &= l\,(l+1)\,\hbar^{\,2}\,\psi^{(1)}_{l,s;m,m_s},\\
S^{\,2}\,\psi^{(1)}_{l,s;m,m_s} &= s\,(s+1)\,\hbar^{\,2}\,\psi^{(1)}_{l,s;m,m_s},\\
L_z\,\psi^{(1)}_{l,s;m,m_s} &= m\,\hbar\,\psi^{(1)}_{l,s;m,m_s},\\
S_z\,\psi^{(1)}_{l,s;m,m_s} &= m_s\,\hbar\,\psi^{(1)}_{l,s;m,m_s}.
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
J_z\,\psi^{(1)}_{l,s;m,m_s} &= (L_z+S_z)\,\psi^{(1)}_{l,s;m,m_s} = (m+m_s)\,\hbar\,\psi^{(1)}_{l,s;m,m_s}\\
&= m_j\,\hbar\,\psi^{(1)}_{l,s;m,m_s}.
\end{aligned}$$

Hence,

$$m_j = m + m_s. \tag{10.1.15}$$

In other words, the quantum numbers controlling the z-components of the various angular momentum vectors can simply be added algebraically.

Finally, let $\psi^{(2)}_{l,s;j,m_j}$ represent a simultaneous eigenstate of L^2, $S^{\,2}$, $J^{\,2}$, and J_z corresponding to the following eigenvalues:

$$\begin{aligned}
L^2\,\psi^{(2)}_{l,s;j,m_j} &= l\,(l+1)\,\hbar^{\,2}\,\psi^{(2)}_{l,s;j,m_j},\\
S^{\,2}\,\psi^{(2)}_{l,s;j,m_j} &= s\,(s+1)\,\hbar^{\,2}\,\psi^{(2)}_{l,s;j,m_j},\\
J^{\,2}\,\psi^{(2)}_{l,s;j,m_j} &= j\,(j+1)\,\hbar^{\,2}\,\psi^{(2)}_{l,s;j,m_j},\\
J_z\,\psi^{(2)}_{l,s;j,m_j} &= m_j\,\hbar\,\psi^{(2)}_{l,s;j,m_j}.
\end{aligned}$$

Contributors and Attributions

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