5.3: Two-Particle Systems

Consider a system consisting of two particles, mass m_1 and m_2, interacting via a potential $V(x_1-x_2)$ that only depends on the relative positions of the particles. According to Equations ([ex3]) and ([ex10]), the Hamiltonian of the system is written

$$H(x_1,x_2) = -\frac{\hbar^{\,2}}{2\,m_1}\,\frac{\partial^{\,2}}{\partial x_1^{\,2}} - \frac{\hbar^{\,2}}{2\,m_2}\,\frac{\partial^{\,2}}{\partial x_2^{\,2}} + V(x_1-x_2). \tag{5.3.1}$$

Let

$$x' = x_1 - x_2 \tag{5.3.2}$$

be the particles' relative position coordinate, and

$$X = \frac{m_1\,x_1 + m_2\,x_2}{m_1+m_2} \tag{5.3.3}$$

the coordinate of the center of mass. It is easily demonstrated that

$$\begin{aligned} \frac{\partial}{\partial x_1} &= \frac{m_1}{m_1+m_2}\,\frac{\partial}{\partial X} + \frac{\partial}{\partial x'}, \\ \frac{\partial}{\partial x_2} &= \frac{m_2}{m_1+m_2}\,\frac{\partial}{\partial X} - \frac{\partial}{\partial x'}. \end{aligned}$$

Hence, when expressed in terms of the new variables, x' and X, the Hamiltonian becomes

$$H(x',X) = -\frac{\hbar^{\,2}}{2\,M}\,\frac{\partial^{\,2}}{\partial X^{\,2}} - \frac{\hbar^{\,2}}{2\,\mu}\,\frac{\partial^{\,2}}{\partial x'^{\,2}} + V(x'), \tag{5.3.4}$$

where

$$M = m_1 + m_2 \tag{5.3.5}$$

is the total mass of the system, and

$$\mu = \frac{m_1\,m_2}{m_1+m_2} \tag{5.3.6}$$

the so-called *reduced mass* . Note that the total momentum of the system can be written

$$P = -{\rm i}\,\hbar\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) = -{\rm i}\,\hbar\,\frac{\partial}{\partial X}. \tag{5.3.7}$$

The fact that the Hamiltonian ([ex6.24]) is separable when expressed in terms of the new coordinates [i.e., $H(x',X) = H_{x'}(x') + H_X(X)$] suggests, by analogy with the analysis in the previous section, that the wavefunction can be factorized: that is,

$$\psi(x_1,x_2,t) = \psi_{x'}(x',t)\,\psi_X(X,t). \tag{5.3.8}$$

Hence, the time-dependent Schrödinger equation ([ex7]) also factorizes to give

$${\rm i}\,\hbar\,\frac{\partial\psi_{x'}}{\partial t} = -\frac{\hbar^{\,2}}{2\,\mu}\,\frac{\partial^{\,2}\psi_{x'}}{\partial x'^{\,2}} + V(x')\,\psi_{x'}, \tag{5.3.9}$$

and

$${\rm i}\,\hbar\,\frac{\partial\psi_X}{\partial t} = -\frac{\hbar^{\,2}}{2\,M}\,\frac{\partial^{\,2}\psi_X}{\partial X^{\,2}}. \tag{5.3.10}$$

The previous equation can be solved to give

$$\psi_X(X,t) = \psi_0\,{\rm e}^{\,{\rm i}\,(P'\,X/\hbar - E'\,t/\hbar)}, \tag{5.3.11}$$

where ψ_0, P', and $E' = P'^{\,2}/2\,M$ are constants. It is clear, from Equations ([exa]), ([exb]), and ([ex33]), that the total momentum of the system takes the constant value P'. In other words, momentum is conserved.

Suppose that we work in the *centre of mass frame* of the system, which is characterized by $P' = 0$. It follows that $\psi_X = \psi_0$. In this case, we can write the wavefunction of the system in the form $\psi(x_1, x_2, t) = \psi_{x'}(x', t)\,\psi_0 \equiv \psi(x_1 - x_2, t)$, where

$$\mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t} = -\frac{\hbar^{\,2}}{2\,\mu}\,\frac{\partial^{\,2}\psi}{\partial x^{\,2}} + V(x)\,\psi. \tag{5.3.12}$$

In other words, in the center of mass frame, two particles of mass m_1 and m_2, moving in the potential $V(x_1 - x_2)$, are equivalent to a single particle of mass μ, moving in the potential $V(x)$, where $x = x_1 - x_2$. This is a familiar result from classical dynamics .

Contributors and Attributions

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