9.3: Eigenstates of Sz and S²

Because the operators S_z and S^2 commute, they must possess simultaneous eigenstates. (See Section [smeas].) Let these eigenstates take the form [see Equations ([e8.29]) and ([e8.30])]:

$$\begin{aligned} S_z\,\chi_{s,m_s} &= m_s\,\hbar\,\chi_{s,m_s},\\ S^2\,\chi_{s,m_s} &= s\,(s+1)\,\hbar^{\,2}\,\chi_{s,m_s}. \end{aligned}$$

Now, it is easily demonstrated, from the commutation relations ([e10.9]) and ([e10.10]), that

$$S_z\,(S_+\,\chi_{s,m_s}) = (m_s+1)\,\hbar\,(S_+\,\chi_{s,m_s}), \tag{9.3.1}$$

and

$$S_z\,(S_-\,\chi_{s,m_s}) = (m_s-1)\,\hbar\,(S_-\,\chi_{s,m_s}). \tag{9.3.2}$$

Thus, S_+ and S_- are indeed the raising and lowering operators, respectively, for spin angular momentum. (See Section [seian].) The eigenstates of S_z and S^2 are assumed to be orthonormal: that is,

$$\chi^\dagger_{s,m_s}\,\chi_{s',m_s'} = \delta_{ss'}\,\delta_{m_s m_s'}. \tag{9.3.3}$$

Consider the wavefunction $\chi = S_+\,\chi_{s,m_s}$. Because we know, from Equation ([e10.11]), that $\chi^\dagger\,\chi\geq 0$, it follows that

$$(S_+\,\chi_{s,m_s})^\dagger\,(S_+\,\chi_{s,m_s}) = \chi^\dagger_{s,m_s}\,S_+^\dagger\,S_+\,\chi_{s,m_s} = \chi^\dagger_{s,m_s}\,S_-\,S_+\,\chi_{s,m_s}\geq 0, \tag{9.3.4}$$

where use has been made of Equation ([e10.7]). Equations ([e10.8]), ([e10.16]), ([e10.17]), and ([e10.20]) yield

$$s\,(s+1)\geq m_s\,(m_s+1). \tag{9.3.5}$$

Likewise, if $\chi = S_-\,\chi_{s,m_s}$ then we obtain

$$s\,(s+1)\geq m_s\,(m_s-1). \tag{9.3.6}$$

Assuming that $s\geq 0$, the previous two inequalities imply that

$$-s\leq m_s\leq s. \tag{9.3.7}$$

Hence, at fixed s, there is both a maximum and a minimum possible value that m_s can take.

Let $m_{s\,\rm min}$ be the minimum possible value of m_s. It follows that (see Section [slsq])

$$S_-\,\chi_{s,m_{s\,\rm min}} = 0. \tag{9.3.8}$$

Now, from Equation ([e10.7a]),

$$S^2 = S_+\,S_- + S_z^2 - \hbar\,S_z. \tag{9.3.9}$$

Hence,

$$S^2\,\chi_{s,m_{s\,\rm min}} = (S_+\,S_- + S_z^2 - \hbar\,S_z)\,\chi_{s,m_{s\,\rm min}}, \tag{9.3.10}$$

giving

$$s\,(s+1) = m_{s\,\rm min}\,(m_{s\,\rm min}-1). \tag{9.3.11}$$

Assuming that $m_{s\,\rm min}<0$, this equation yields

$$m_{s\,\rm min} = -s. \tag{9.3.12}$$

Likewise, it is easily demonstrated that

$$m_{s\,\rm max} = +s. \tag{9.3.13}$$

Moreover,

$$S_-\,\chi_{s,-s} = S_+\,\chi_{s,s} = 0. \tag{9.3.14}$$

Now, the raising operator S_+, acting upon $\chi_{s,-s}$, converts it into some multiple of $\chi_{s,-s+1}$. Employing the raising operator a second time, we obtain a multiple of $\chi_{s,-s+2}$. However, this process cannot continue indefinitely, because there is a maximum possible value of m_s. Indeed, after acting upon $\chi_{s,-s}$ a sufficient number of times with the raising operator S_+, we must obtain a multiple of $\chi_{s,s}$, so that employing the raising operator one more time leads to the null state. [See Equation ([e10.31]).] If this is not the case then we will inevitably obtain eigenstates of S_z corresponding to $m_s > s$, which we have already demonstrated is impossible.

It follows, from the previous argument, that

$$m_{s\,\mathrm{max}} - m_{s\,\mathrm{min}} = 2\,s = k, \tag{9.3.15}$$

where k is a positive integer. Hence, the quantum number s can either take positive integer or positive half-integer values. Up to now, our analysis has been very similar to that which we used earlier to investigate orbital angular momentum. (See Section [sorb].) Recall, that for orbital angular momentum the quantum number m, which is analogous to m_s, is restricted to take integer values. (See Section [slz].) This implies that the quantum number l, which is analogous to s, is also restricted to take integer values. However, the origin of these restrictions is the representation of the orbital angular momentum operators as differential operators in real space. (See Section [s8.3].) There is no equivalent representation of the corresponding spin angular momentum operators. Hence, we conclude that there is no reason why the quantum number s cannot take half-integer, as well as integer, values.

In 1940, Wolfgang Pauli proved the so-called *spin-statistics theorem* using relativistic quantum mechanics . According to this theorem, all fermions possess half-integer spin (i.e., a half-integer value of s), whereas all bosons possess integer spin (i.e., an integer value of s). In fact, all presently known fermions, including electrons and protons, possess *spin one-half*. In other words, electrons and protons are characterized by $s = 1/2$ and $m_s = \pm 1/2$.

Contributors and Attributions

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