## 7.6: Spherical Harmonics

The simultaneous eigenstates, $Y_{l,m}(\theta,\phi)$, of $L^2$ and $L_z$ are known as the *spherical harmonics* . Let us investigate their functional form.

We know that

$$L_+\,Y_{l,l}(\theta,\phi) = 0, \tag{7.6.1}$$

because there is no state for which $m$ has a larger value than $+l$. Writing

$$Y_{l,l}(\theta,\phi) = \Theta_{l,l}(\theta)\,{\rm e}^{\,{\rm i}\,l\,\phi} \tag{7.6.2}$$

[see Equations ([e8.34]) and ([e8.38])], and making use of Equation ([e8.28]), we obtain

$$\hbar\,{\rm e}^{\,{\rm i}\,\phi}\left(\frac{\partial}{\partial\theta}+{\rm i}\,\cot\theta\,\frac{\partial}{\partial\phi}\right)\Theta_{l,l}(\theta)\,{\rm e}^{\,{\rm i}\,l\,\phi}=0. \tag{7.6.3}$$

This equation yields

$$\frac{d\Theta_{l,l}}{d\theta}-l\,\cot\theta\,\Theta_{l,l}=0. \tag{7.6.4}$$

which can easily be solved to give

$$\Theta_{l,l}\sim(\sin\theta)^{\,l}. \tag{7.6.5}$$

Hence, we conclude that

$$Y_{l,l}(\theta,\phi)\sim(\sin\theta)^{\,l}\,{\rm e}^{\,{\rm i}\,l\,\phi}. \tag{7.6.6}$$

Likewise, it is easy to demonstrate that

$$Y_{l,-l}(\theta,\phi)\sim(\sin\theta)^{\,l}\,{\rm e}^{-{\rm i}\,l\,\phi}. \tag{7.6.7}$$

Once we know $Y_{l,l}$, we can obtain $Y_{l,l-1}$ by operating on $Y_{l,l}$ with the lowering operator $L_-$. Thus,

$$Y_{l,l-1}\sim L_-\,Y_{l,l}\sim{\rm e}^{-{\rm i}\,\phi}\left(-\frac{\partial}{\partial\theta}+{\rm i}\,\cot\theta\,\frac{\partial}{\partial\phi}\right)(\sin\theta)^{\,l}\,{\rm e}^{\,{\rm i}\,l\,\phi}, \tag{7.6.8}$$

where use has been made of Equation ([e8.28]). The previous equation yields

$$Y_{l,l-1}\sim{\rm e}^{\,{\rm i}\,(l-1)\,\phi}\left(\frac{d}{d\theta}+l\,\cot\theta\right)(\sin\theta)^{\,l}. \tag{7.6.9}$$

Now,

$$\left(\frac{d}{d\theta}+l\,\cot\theta\right)f(\theta)\equiv\frac{1}{(\sin\theta)^{\,l}}\,\frac{d}{d\theta}\left[(\sin\theta)^{\,l}\,f(\theta)\right], \tag{7.6.10}$$

where $f(\theta)$ is a general function. Hence, we can write

$$Y_{l,l-1}(\theta,\phi)\sim\frac{{\rm e}^{\,{\rm i}\,(l-1)\,\phi}}{(\sin\theta)^{\,l-1}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)(\sin\theta)^{2\,l}. \tag{7.6.11}$$

ikewise, we can show that

$$Y_{l,-l+1}(\theta,\phi)\sim L_+\,Y_{l,-l}\sim\frac{{\rm e}^{-{\rm i}\,(l-1)\,\phi}}{(\sin\theta)^{\,l-1}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)(\sin\theta)^{2\,l}. \tag{7.6.12}$$

We can now obtain $Y_{l,l-2}$ by operating on $Y_{l,l-1}$ with the lowering operator. We get

$$Y_{l,l-2}\sim L_-\,Y_{l,l-1}\sim{\rm e}^{-{\rm i}\,\phi}\left(-\frac{\partial}{\partial\theta}+{\rm i}\,\cot\theta\,\frac{\partial}{\partial\phi}\right)\frac{{\rm e}^{\,{\rm i}\,(l-1)\,\phi}}{(\sin\theta)^{\,l-1}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)(\sin\theta)^{2\,l}, \tag{7.6.13}$$

which reduces to

$$Y_{l,l-2} \sim \mathrm{e}^{-\mathrm{i}\,(l-2)\,\phi}\left[\frac{d}{d\theta}+(l-1)\,\cot\theta\right]\frac{1}{(\sin\theta)^{\,l-1}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)(\sin\theta)^{2\,l}. \tag{7.6.14}$$

Finally, making use of Equation ([e8.64]), we obtain

$$Y_{l,l-2}(\theta,\phi) \sim \frac{\mathrm{e}^{\,\mathrm{i}\,(l-2)\,\phi}}{(\sin\theta)^{\,l-2}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)^{2}(\sin\theta)^{2\,l}. \tag{7.6.15}$$

Likewise, we can show that

$$Y_{l,-l+2}(\theta,\phi) \sim L_+\,Y_{l,-l+1} \sim \frac{\mathrm{e}^{-\mathrm{i}\,(l-2)\,\phi}}{(\sin\theta)^{\,l-2}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)^{2}(\sin\theta)^{2\,l}. \tag{7.6.16}$$

A comparison of Equations ([e8.59]), ([e8.64a]), and ([e8.68]) reveals the general functional form of the spherical harmonics:

$$Y_{l,m}(\theta,\phi) \sim \frac{\mathrm{e}^{\,\mathrm{i}\,m\,\phi}}{(\sin\theta)^{\,m}}\left(\frac{1}{\sin\theta}\,\frac{d}{d\theta}\right)^{l-m}(\sin\theta)^{2\,l}. \tag{7.6.17}$$

Here, $m$ is assumed to be non-negative. Making the substitution $u=\cos\theta$, we can also write

$$Y_{l,m}(u,\phi) \sim \mathrm{e}^{\,\mathrm{i}\,m\,\phi}\,(1-u^{\,2})^{-m/2}\left(\frac{d}{du}\right)^{l-m}(1-u^{\,2})^{\,l}. \tag{7.6.18}$$

Finally, it is clear from Equations ([e8.60]), ([e8.65]), and ([e8.69]) that

$$Y_{l,-m} \sim Y_{l,m}^{\,*}. \tag{7.6.19}$$

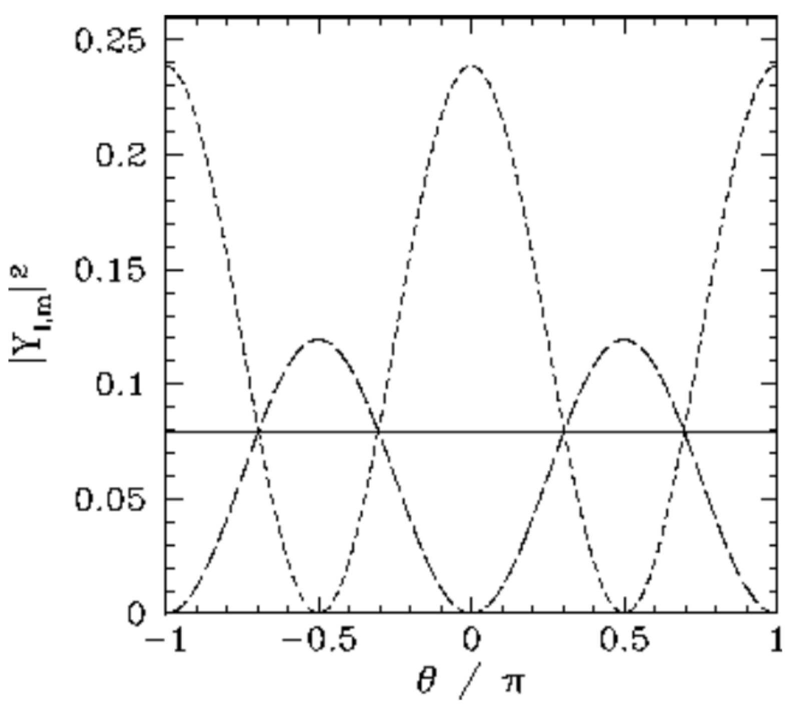


Figure 18: *The* $|Y_{l,m}(\theta,\phi)|^2$ *plotted as a functions of* $\theta$. *The solid, short-dashed, and long-dashed curves correspond to* $l,m=0,0,$ and $1,0,$ and $1,\pm 1$, *respectively.*

We now need to normalize our spherical harmonic functions so as to ensure that

$$\oint |Y_{l,m}(\theta,\phi)|^{\,2}\,d\Omega = 1. \tag{7.6.20}$$

After a great deal of tedious analysis, the normalized spherical harmonic functions are found to take the form

$$Y_{l,m}(\theta,\phi) = (-1)^m \left[\frac{2\,l+1}{4\pi}\,\frac{(l-m)!}{(l+m)!}\right]^{1/2} P_{l,m}(\cos\theta)\,\mathrm{e}^{\,\mathrm{i}\,m\,\phi} \tag{7.6.21}$$

for $m \geq 0$, where the $P_{l,m}$ are known as *associated Legendre polynomials* , and are written

$$P_{l,m}(u) = (-1)^{l+m}\,\frac{(l+m)!}{(l-m)!}\,\frac{(1-u^{\,2})^{-m/2}}{2^l\,l!}\left(\frac{d}{du}\right)^{l-m}(1-u^{\,2})^{\,l} \tag{7.6.22}$$

for $m \geq 0$. Alternatively,

$$P_{l,m}(u) = (-1)^{l}\,\frac{(1-u^{\,2})^{m/2}}{2^l\,l!}\left(\frac{d}{du}\right)^{l+m}(1-u^{\,2})^{\,l}, \tag{7.6.23}$$

for $m \geq 0$. The spherical harmonics characterized by $m<0$ can be calculated from those characterized by $m>0$ via the identity

$$Y_{l,-m} = (-1)^m\,Y_{l,m}^{\,\ast}. \tag{7.6.24}$$

The spherical harmonics are orthonormal: that is,

$$\oint Y_{l',m'}^{\,\ast}\,Y_{l,m}\,d\Omega = \delta_{ll'}\,\delta_{mm'}, \tag{7.6.25}$$

and also form a complete set. In other words, any well-behaved function of $\theta$ and $\phi$ can be represented as a superposition of spherical harmonics. Finally, and most importantly, the spherical harmonics are the simultaneous eigenstates of $L_z$ and $L^2$ corresponding to the eigenvalues $m\,\hbar$ and $l\,(l+1)\,\hbar^{\,2}$, respectively.

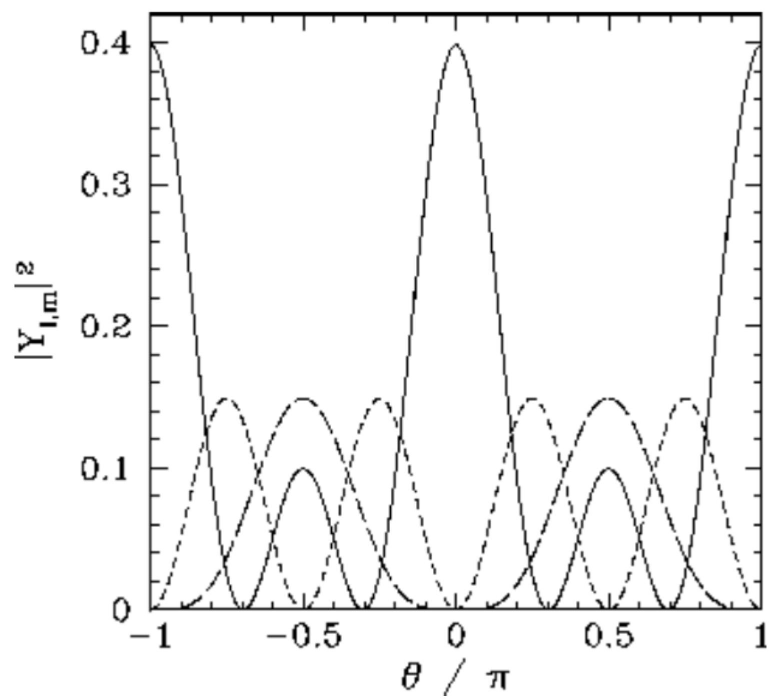


Figure 19: *The* $|Y_{l,m}(\theta,\phi)|^2$ *plotted as a functions of* $\theta$. *The solid, short-dashed, and long-dashed curves correspond to* $l,m=2,0,$ and $2,\pm 1,$ and $2,\pm 2$ *respectively.*

All of the $l=0$, $l=1$, and $l=2$ spherical harmonics are listed below:

$$
\begin{aligned}
Y_{0,0} &= \frac{1}{\sqrt{4\pi}},\\
Y_{1,0} &= \sqrt{\frac{3}{4\pi}}\,\cos\theta,\\
Y_{1,\pm 1} &= \mp\sqrt{\frac{3}{8\pi}}\,\sin\theta\,{\rm e}^{\pm{\rm i}\,\phi},\\
Y_{2,0} &= \sqrt{\frac{5}{16\pi}}\,(3\,\cos^2\theta-1),\\
Y_{2,\pm 1} &= \mp\sqrt{\frac{15}{8\pi}}\,\sin\theta\,\cos\theta\,{\rm e}^{\pm{\rm i}\,\phi},\\
Y_{2,\pm 2} &= \sqrt{\frac{15}{32\pi}}\,\sin^2\theta\,{\rm e}^{\pm 2\,{\rm i}\,\phi}.
\end{aligned}
$$

The $\theta$ variation of these functions is illustrated in Figures [ylm1] and [ylm2].

## Contributors and Attributions

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