6.3: Degenerate Electron Gases

Consider N electrons trapped in a cubic box of dimension a. Let us treat the electrons as essentially non-interacting particles. According to Section [snon], the total energy of a system consisting of many non-interacting particles is simply the sum of the single-particle energies of the individual particles. Furthermore, electrons are subject to the *Pauli exclusion principle* (see Section [siden]), because they are indistinguishable fermions. The exclusion principle states that no two electrons in our system can occupy the same single-particle energy level. Now, from the previous section, the single-particle energy levels for a particle in a box are characterized by the three quantum numbers, l_x, l_y, and l_z. Thus, we conclude that no two electrons in our system can have the same set of values of l_x, l_y, and l_z. It turns out that this is not quite true, because electrons possess an intrinsic angular momentum called spin. The spin states of an electron are governed by an additional quantum number, which can take one of two different values. (See Chapter [sspin].) Hence, when spin is taken into account, we conclude that a maximum of two electrons (with different spin quantum numbers) can occupy a single-particle energy level corresponding to a particular set of values of l_x, l_y, and l_z. Note, from Equations ([e7.38]) and ([e7.39]), that the associated particle energy is proportional to $l^{\,2} = l_x^2 + l_y^2 + l_z^2$.

Suppose that our electrons are cold: that is, they have comparatively little thermal energy. In this case, we would expect them to fill the lowest single-particle energy levels available to them. We can imagine the single-particle energy levels as existing in a sort of three-dimensional quantum number space whose Cartesian coordinates are l_x, l_y, and l_z. Thus, the energy levels are uniformly distributed in this space on a cubic lattice. Moreover, the distance between nearest neighbor energy levels is unity. This implies that the number of energy levels per unit volume is also unity. Finally, the energy of a given energy level is proportional to its distance, $l^{\,2} = l_x^2 + l_y^2 + l_z^2$, from the origin.

Because we expect cold electrons to occupy the lowest energy levels available to them, but only two electrons can occupy a given energy level, it follows that if the number of electrons, N, is very large then the filled energy levels will be approximately distributed in a sphere centered on the origin of quantum number space. The number of energy levels contained in a sphere of radius l is approximately equal to the volume of the sphere—because the number of energy levels per unit volume is unity. It turns out that this is not quite correct, because we have forgotten that the quantum numbers l_x, l_y, and l_z can only take positive values. Hence, the filled energy levels actually only occupy one octant of a sphere. The radius l_F of the octant of filled energy levels in quantum number space can be calculated by equating the number of energy levels it contains to the number of electrons, N. Thus, we can write

$$N = 2\times \frac{1}{8}\times \frac{4\,\pi}{3}\,l_F^{\,3}. \tag{6.3.1}$$

Here, the factor 2 is to take into account the two spin states of an electron, and the factor $1/8$ is to take account of the fact that l_x, l_y, and l_z can only take positive values. Thus,

$$l_F = \left(\frac{3\,N}{\pi}\right)^{1/3}. \tag{6.3.2}$$

According to Equation ([e7.38]), the energy of the most energetic electrons—which is known as the *Fermi energy* —is given by

$$E_F = \frac{l_F^{\,2}\,\pi^{\,2}\,\hbar^{\,2}}{2\,m_e\,a^{\,2}} = \frac{\pi^{\,2}\,\hbar^{\,2}}{2\,m\,a^{\,2}}\left(\frac{3\,N}{\pi}\right)^{2/3}, \tag{6.3.3}$$

where m_e is the electron mass. This can also be written as

$$E_F = \frac{\pi^{\,2}\,\hbar^{\,2}}{2\,m_e}\left(\frac{3\,n}{\pi}\right)^{2/3}, \tag{6.3.4}$$

where $n = N/a^{\,3}$ is the number of electrons per unit volume (in real space). Note that the Fermi energy only depends on the number density of the confined electrons.

The mean energy of the electrons is given by

$$\bar{E} = E_F \int_0^{l_F} l^{\,2}\,4\pi\,l^{\,2}\,dl \left/ \frac{4}{3}\,\pi\,l_F^{\,5} \right. = \frac{3}{5}\,E_F, \tag{6.3.5}$$

because $E \propto l^{2}$, and the energy levels are uniformly distributed in quantum number space within an octant of radius l_{F}. Now, according to classical physics, the mean thermal energy of the electrons is $(3/2)\,k_B\,T$, where T is the electron temperature, and k_B the Boltzmann constant . Thus, if $k_B\,T \ll E_F$ then our original assumption that the electrons are cold is valid. Note that, in this case, the electron energy is much larger than that predicted by classical physics—electrons in this state are termed *degenerate*. On the other hand, if $k_B\,T \gg E_F$ then the electrons are hot, and are essentially governed by classical physics—electrons in this state are termed *non-degenerate*.

The total energy of a degenerate electron gas is

$$E_{\rm total} = N\,\bar{E} = \frac{3}{5}\,N\,E_F. \tag{6.3.6}$$

Hence, the gas pressure takes the form

$$P = -\frac{\partial E_{\rm total}}{\partial V} = \frac{2}{5}\,n\,E_F, \tag{6.3.7}$$

because $E_F \propto a^{-2} = V^{-2/3}$. [See Equation ([e7.42]).] Now, the pressure predicted by classical physics is $P = n\,k_B\,T$. Thus, a degenerate electron gas has a much higher pressure than that which would be predicted by classical physics. This is an entirely quantum mechanical effect, and is due to the fact that identical fermions cannot get significantly closer together than a de Broglie wavelength without violating the Pauli exclusion principle. Note that, according to Equation ([e7.43]), the mean spacing between degenerate electrons is

$$d \sim n^{-1/3} \sim \frac{h}{\sqrt{m_e\,E}} \sim \frac{h}{p} \sim \lambda, \tag{6.3.8}$$

where λ is the de Broglie wavelength. Thus, an electron gas is non-degenerate when the mean spacing between the electrons is much greater than the de Broglie wavelength, and becomes degenerate as the mean spacing approaches the de Broglie wavelength.

In turns out that the conduction (i.e., free) electrons inside metals are highly degenerate (because the number of electrons per unit volume is very large, and $E_F \propto n^{2/3}$). Indeed, most metals are hard to compress as a direct consequence of the high degeneracy pressure of their conduction electrons. To be more exact, resistance to compression is usually measured in terms of a quantity known as the *bulk modulus*, which is defined

$$B = -V\,\frac{\partial P}{\partial V} \tag{6.3.9}$$

Now, for a fixed number of electrons, $P \propto V^{-5/3}$. [See Equations ([e7.42]) and ([e7.46]).] Hence,

$$B = \frac{5}{3}\,P = \frac{\pi^{3}\,\hbar^{2}}{9\,m}\left(\frac{3\,n}{\pi}\right)^{5/3}. \tag{6.3.10}$$

For example, the number density of free electrons in magnesium is $n \sim 8.6\times 10^{28}\,{\rm m^{-3}}$. This leads to the following estimate for the bulk modulus: $B \sim 6.4\times 10^{10}\,{\rm N\,m^{-2}}$. The actual bulk modulus is $B = 4.5\times 10^{10}\,{\rm N\,m^{-2}}$.

Contributors and Attributions

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