# 10.3: Two Spin One-Half Particles

Consider a system consisting of two spin one-half particles. Suppose that the system does not possess any orbital angular momentum. Let $\mathbf{S}_1$ and $\mathbf{S}_2$ be the spin angular momentum operators of the first and second particles, respectively, and let

$$\mathbf{S}=\mathbf{S}_1+\mathbf{S}_2 \tag{10.3.1}$$

be the total spin angular momentum operator. By analogy with the previous analysis, we conclude that it is possible to simultaneously measure either $S_1^{\,2}$, $S_2^{\,2}$, $S^{\,2}$, and $S_z$, or $S_1^{\,2}$, $S_2^{\,2}$, $S_{1z}$, $S_{2z}$, and $S_z$. Let the quantum numbers associated with measurements of $S_1^{\,2}$, $S_{1z}$, $S_2^{\,2}$, $S_{2z}$, $S^{\,2}$, and $S_z$ be $s_1$, $m_{s_1}$, $s_2$, $m_{s_2}$, $s$, and $m_s$, respectively. In other words, if the spinor $\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}}$ is a simultaneous eigenstate of $S_1^{\,2}$, $S_2^{\,2}$, $S_{1z}$, and $S_{2z}$, then

$$\begin{aligned}
S_1^{\,2}\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}} &= s_1\,(s_1+1)\,\hbar^{\,2}\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}},\\
S_2^{\,2}\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}} &= s_2\,(s_2+1)\,\hbar^{\,2}\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}},\\
S_{1z}\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}} &= m_{s_1}\,\hbar\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}},\\
S_{2z}\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}} &= m_{s_2}\,\hbar\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}},\\
S_z\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}} &= m_s\,\hbar\,\chi^{(1)}_{s_1,s_2;m_{s_1},m_{s_2}}.
\end{aligned}$$

Likewise, if the spinor $\chi^{(2)}_{s_1,s_2;s,m_s}$ is a simultaneous eigenstate of $S_1^{\,2}$, $S_2^{\,2}$, $S^{\,2}$, and $S_z$, then

$$\begin{aligned}
S_1^{\,2}\,\chi^{(2)}_{s_1,s_2;s,m_s} &= s_1\,(s_1+1)\,\hbar^{\,2}\,\chi^{(2)}_{s_1,s_2;s,m_s},\\
S_2^{\,2}\,\chi^{(2)}_{s_1,s_2;s,m_s} &= s_2\,(s_2+1)\,\hbar^{\,2}\,\chi^{(2)}_{s_1,s_2;s,m_s},\\
S^{2}\,\chi^{(2)}_{s_1,s_2;s,m_s} &= s\,(s+1)\,\hbar^{\,2}\,\chi^{(2)}_{s_1,s_2;s,m_s},\\
S_z\,\chi^{(2)}_{s_1,s_2;s,m_s} &= m_s\,\hbar\,\chi^{(2)}_{s_1,s_2;s,m_s}.
\end{aligned}$$

Of course, because both particles have spin one-half, $s_1=s_2=1/2$, and $s_{1z}, s_{2z}=\pm 1/2$. Furthermore, by analogy with previous analysis,

$$m_s=m_{s_1}+m_{s_2}. \tag{10.3.2}$$

Now, we saw, in the previous section, that when spin $l$ is added to spin one-half then the possible values of the total angular momentum quantum number are $j=l\pm 1/2$. By analogy, when spin one-half is added to spin one-half then the possible values of the total spin quantum number are $s=1/2\pm 1/2$. In other words, when two spin one-half particles are combined, we either obtain a state with overall spin $s=1$, or a state with overall spin $s=0$. To be more exact, there are three possible $s=1$ states (corresponding to $m_s=-1$, 0, 1), and one possible $s=0$ state (corresponding to $m_s=0$). The three $s=1$ states are generally known as the *triplet* states, whereas the $s=0$ state is known as the *singlet* state.

Clebsch-Gordon coefficients for adding spin one-half to spin one-half. Only non-zero coefficients are shown.

| | $-1/2, -1/2$ | $-1/2, 1/2$ | $1/2, -1/2$ | $1/2, 1/2$ | $m_{s_1}, m_{s_2}$ |
|---|---|---|---|---|---|
| [0.5ex] $1, -1$ | 1 | | | | |
| [0.5ex] $1, 0$ | | $1/\sqrt{2}$ | $1/\sqrt{2}$ | | |
| [0.5ex] $0, 0$ | | $1/\sqrt{2}$ | $-1/\sqrt{2}$ | | |
| [0.5ex] $1, 1$ | | | | 1 | |
| $s, m_s$ | | | | | |

The Clebsch-Gordon coefficients for adding spin one-half to spin one-half can easily be inferred from Table [t2] (with $l=1/2$), and are listed in Table [t4]. It follows from this table that the three triplet states are:

$$\chi^{(2)}_{1,-1} = \chi^{(1)}_{-1/2,-1.2},$$
$$\chi^{(2)}_{1,0} = \frac{1}{\sqrt{2}}\left(\chi^{(1)}_{-1/2,1/2}+\chi^{(1)}_{1/2,-1/2}\right),$$
$$\chi^{(2)}_{1,1} = \chi^{(1)}_{1/2,1/2},$$

where $\chi^{(2)}_{s,m_s}$ is shorthand for $\chi^{(2)}_{s_1,s_2;s,m_s}$, et cetera. Likewise, the singlet state is written:

$$\chi^{(2)}_{0,0} = \frac{1}{\sqrt{2}}\left(\chi^{(1)}_{-1/2,1/2}-\chi^{(1)}_{1/2,-1/2}\right). \tag{10.3.3}$$

## Contributors and Attributions

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