14.8: Resonances

There is a significant exception to the independence of the cross-section on energy mentioned previously. Suppose that the quantity $(2\,m\,|V_0|\,a^{\,2}/\hbar^{\,2})^{\,1/2}$ is slightly less than $\pi/2$. As the incident energy increases, $k'\,a$, which is given by Equation ([e17.112]), can reach the value $\pi/2$. In this case, $\tan(k'\,a)$ becomes infinite, so we can no longer assume that the right-hand side of Equation ([e17.107]) is small. In fact, it follows from Equation ([e17.107]) that if the value of the incident energy is such that $k'\,a=\pi/2$ then we also have $k\,a+\delta_0=\pi/2$, or $\delta_0\simeq\pi/2$ (because we are assuming that $k\,a\ll 1$). This implies that

$$\sigma_{\rm total} = \frac{4\pi}{k^{\,2}}\,\sin^2\delta_0 = 4\pi\,a^{\,2}\left(\frac{1}{k^{\,2}\,a^{\,2}}\right). \tag{14.8.1}$$

Note that the cross-section now depends on the energy. Furthermore, the magnitude of the cross-section is much larger than that given in Equation ([e17.111]) for $k'\,a\neq\pi/2$ (because $k\,a\ll 1$).

The origin of this rather strange behavior is quite simple. The condition

$$\sqrt{\frac{2\,m\,|V_0|\,a^{\,2}}{\hbar^{\,2}}} = \frac{\pi}{2} \tag{14.8.2}$$

is equivalent to the condition that a spherical well of depth V_0 possesses a bound state at zero energy. Thus, for a potential well that satisfies the previous equation, the energy of the scattering system is essentially the same as the energy of the bound state. In this situation, an incident particle would like to form a bound state in the potential well. However, the bound state is not stable, because the system has a small positive energy. Nevertheless, this sort of *resonance scattering* is best understood as the capture of an incident particle to form a metastable bound state, and the subsequent decay of the bound state and release of the particle. The cross-section for resonance scattering is generally much larger than that for non-resonance scattering.

We have seen that there is a resonant effect when the phase-shift of the S-wave takes the value $\pi/2$. There is nothing special about the $l=0$ partial wave, so it is reasonable to assume that there is a similar resonance when the phase-shift of the lth partial wave is $\pi/2$. Suppose that δ_l attains the value $\pi/2$ at the incident energy E_0, so that

$$\delta_l(E_0) = \frac{\pi}{2}. \tag{14.8.3}$$

Let us expand $\cot\delta_l$ in the vicinity of the resonant energy:

$$\begin{aligned}\cot\delta_l(E) &= \cot\delta_l(E_0) + \left(\frac{d\cot\delta_l}{dE}\right)_{E=E_0}(E-E_0)+\cdots\\ &= -\left(\frac{1}{\sin^2\delta_l}\,\frac{d\delta_l}{dE}\right)_{E=E_0}(E-E_0)+\cdots.\end{aligned}$$

Defining

$$\left(\frac{d\delta_l(E)}{dE}\right)_{E=E_0} = \frac{2}{\Gamma} \tag{14.8.4}$$

we obtain

$$\cot\delta_l(E) = -\frac{2}{\Gamma}(E-E_0)+\cdots \tag{14.8.5}$$

Recall, from Equation ([e17.75]), that the contribution of the lth partial wave to the scattering cross-section is

$$\sigma_l = \frac{4\pi}{k^{\,2}}\,(2\,l+1)\,\sin^2\delta_l = \frac{4\pi}{k^{\,2}}\,(2\,l+1)\,\frac{1}{1+\cot^2\delta_l}. \tag{14.8.6}$$

Thus,

$$\sigma_l \simeq \frac{4\pi}{k^{\,2}}\,(2\,l+1)\,\frac{\mathit{\Gamma}^{\,2}/4}{(E-E_0)^{\,2}+\mathit{\Gamma}^{\,2}/4}. \tag{14.8.7}$$

This is the famous *Breit-Wigner formula* . The variation of the partial cross-section σ_l with the incident energy has the form of a classical resonance curve. The quantity Γ is the width of the resonance (in energy). We can interpret the Breit-Wigner formula as describing the absorption of an incident particle to form a metastable state, of energy E_0, and lifetime $\tau = \hbar/\Gamma$.

Contributors and Attributions

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