3.11: Exercises

1. Monochromatic light with a wavelength of $6000\,\text{Å}$ passes through a fast shutter that opens for 10^{-9} sec. What is the subsequent spread in wavelengths of the no longer monochromatic light?
2. Calculate $\langle x\rangle$, $\langle x^{\,2}\rangle$, and σ_x, as well as $\langle p\rangle$, $\langle p^{\,2}\rangle$, and σ_p, for the normalized wavefunction
$$\psi(x)=\sqrt{\frac{2\,a^{\,3}}{\pi}}\,\frac{1}{x^{\,2}+a^{\,2}}. \tag{3.11.1}$$
Use these to find $\sigma_x\,\sigma_p$. Note that $\int_{-\infty}^{\infty}dx/(x^{\,2}+a^{\,2})=\pi/a$.
3. Classically, if a particle is not observed then the probability of finding it in a one-dimensional box of length L, which extends from $x=0$ to $x=L$, is a constant $1/L$ per unit length. Show that the classical expectation value of x is $L/2$, the expectation value of $x^{\,2}$ is $L^2/3$, and the standard deviation of x is $L/\sqrt{12}$.
4. Demonstrate that if a particle in a one-dimensional stationary state is bound then the expectation value of its momentum must be zero.
5. Suppose that $V(x)$ is complex. Obtain an expression for $\partial P(x,t)/\partial t$ and $d/dt\int P(x,t)\,dx$ from Schrödinger's equation. What does this tell us about a complex $V(x)$?
6. $\psi_1(x)$ and $\psi_2(x)$ are normalized eigenfunctions corresponding to the same eigenvalue. If
$$\int_{-\infty}^{\infty}\psi_1^\ast\,\psi_2\,dx=c, \tag{3.11.2}$$
where c is real, find normalized linear combinations of ψ_1 and ψ_2 that are orthogonal to (a) ψ_1, (b) $\psi_1+\psi_2$.
7. Demonstrate that $p=-{\rm i}\,\hbar\,\partial/\partial x$ is an Hermitian operator. Find the Hermitian conjugate of $a=x+{\rm i}\,p$.
8. An operator A, corresponding to a physical quantity α, has two normalized eigenfunctions $\psi_1(x)$ and $\psi_2(x)$, with eigenvalues a_1 and a_2. An operator B, corresponding to another physical quantity β, has normalized eigenfunctions $\phi_1(x)$ and $\phi_2(x)$, with eigenvalues b_1 and b_2. The eigenfunctions are related via
$$\begin{aligned}\psi_1&=(2\,\phi_1+3\,\phi_2)\,/\sqrt{13},\\ \psi_2&=(3\,\phi_1-2\,\phi_2)\,/\sqrt{13}.\end{aligned}$$
α is measured and the value a_1 is obtained. If β is then measured and then α again, show that the probability of obtaining a_1 a second time is $97/169$.
9. Demonstrate that an operator that commutes with the Hamiltonian, and contains no explicit time dependence, has an expectation value that is constant in time.
10. For a certain system, the operator corresponding to the physical quantity A does not commute with the Hamiltonian. It has eigenvalues a_1 and a_2, corresponding to properly normalized eigenfunctions
$$\begin{aligned}\phi_1&=(u_1+u_2)\,/\sqrt{2},\\ \phi_2&=(u_1-u_2)\,/\sqrt{2},\end{aligned}$$
where u_1 and u_2 are properly normalized eigenfunctions of the Hamiltonian with eigenvalues E_1 and E_2. If the system is in the state $\psi=\phi_1$ at time $t=0$, show that the expectation value of A at time t is
$$\langle A\rangle=\left(\frac{a_1+a_2}{2}\right)+\left(\frac{a_1-a_2}{2}\right)\cos\left(\frac{[E_1-E_2]\,t}{\hbar}\right). \tag{3.11.3}$$

Contributors and Attributions

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