12.4: Perturbation Expansion

Let us recall the analysis of Section 1.2. The ψ_n are the stationary orthonormal eigenstates of the time-independent unperturbed Hamiltonian, H_0. Thus, $H_0\,\psi_n = E_n\,\psi_n$, where the E_n are the unperturbed energy levels, and $\langle n|m\rangle = \delta_{nm}$. Now, in the presence of a small time-dependent perturbation to the Hamiltonian, $H_1(t)$, the wavefunction of the system takes the form

$$\psi(t) = \sum_n c_n(t)\,\exp(-{\rm i}\,\omega_n\,t)\,\psi_n, \tag{12.4.1}$$

where $\omega_n = E_n/\hbar$. The amplitudes $c_n(t)$ satisfy

$$ {\rm i}\,\hbar\,\frac{dc_n}{dt} = \sum_m H_{nm}\,\exp(\,{\rm i}\,\omega_{nm}\,t)\,c_m, \tag{12.4.2}$$

where $H_{nm}(t) = \langle n|H_1(t)|m\rangle$ and $\omega_{nm} = (E_n-E_m)/\hbar$. Finally, the probability of finding the system in the nth eigenstate at time t is simply

$$P_n(t) = |c_n(t)|^{\,2} \tag{12.4.3}$$

(assuming that, initially, $\sum_n |c_n|^{\,2} = 1$).

Suppose that at $t=0$ the system is in some initial energy eigenstate labeled i. Equation ([e13.42]) is, thus, subject to the initial condition

$$c_n(0) = \delta_{ni}. \tag{12.4.4}$$

Let us attempt a perturbative solution of Equation ([e13.42]) using the ratio of H_1 to H_0 (or H_{nm} to $\hbar\,\omega_{nm}$, to be more exact) as our expansion parameter. Now, according to Equation ([e13.42]), the c_n are constant in time in the absence of the perturbation. Hence, the zeroth-order solution is simply

$$c_n^{(0)}(t) = \delta_{ni}. \tag{12.4.5}$$

The first-order solution is obtained, via iteration, by substituting the zeroth-order solution into the right-hand side of Equation ([e13.42]). Thus, we obtain

$$ {\rm i}\,\hbar\,\frac{dc_n^{(1)}}{dt} = \sum_m H_{nm}\,\exp(\,{\rm i}\,\omega_{nm}\,t)\,c_m^{(0)} = H_{ni}\,\exp(\,{\rm i}\,\omega_{ni}\,t), \tag{12.4.6}$$

subject to the boundary condition $c_n^{(1)}(0) = 0$. The solution to the previous equation is

$$c_n^{(1)} = -\frac{\rm i}{\hbar}\int_0^t H_{ni}(t')\,\exp(\,{\rm i}\,\omega_{ni}\,t')\,dt'. \tag{12.4.7}$$

It follows that, up to first-order in our perturbation expansion,

$$c_n(t) = \delta_{ni} - \frac{\rm i}{\hbar}\int_0^t H_{ni}(t')\,\exp(\,{\rm i}\,\omega_{ni}\,t')\,dt'. \tag{12.4.8}$$

Hence, the probability of finding the system in some final energy eigenstate labeled f at time t, given that it is definitely in a different initial energy eigenstate labeled i at time $t=0$, is

$$P_{i\rightarrow f}(t) = |c_f(t)|^{\,2} = \left|-\frac{\rm i}{\hbar}\int_0^t H_{fi}(t')\,\exp(\,{\rm i}\,\omega_{fi}\,t')\,dt'\right|^{\,2}. \tag{12.4.9}$$

Note, finally, that our perturbative solution is clearly only valid provided

$$P_{i\rightarrow f}(t) \ll 1. \tag{12.4.10}$$

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)