9.4: Pauli Representation

Let us denote the two independent spin eigenstates of an electron as

$$\chi_{\pm} \equiv \chi_{1/2,\pm 1/2}. \tag{9.4.1}$$

It thus follows, from Equations ([e10.16]) and ([e10.17]), that

$$S_z\,\chi_{\pm} = \pm\frac{1}{2}\,\hbar\,\chi_{\pm},$$
$$S^2\,\chi_{\pm} = \frac{3}{4}\,\hbar^{\,2}\,\chi_{\pm}.$$

Note that χ_+ corresponds to an electron whose spin angular momentum vector has a positive component along the z-axis. Loosely speaking, we could say that the spin vector points in the $+z$-direction (or its spin is “up”). Likewise, χ_- corresponds to an electron whose spin points in the $-z$-direction (or whose spin is “down”). These two eigenstates satisfy the orthonormality requirements

$$\chi_+^{\dagger}\,\chi_+ = \chi_-^{\dagger}\,\chi_- = 1, \tag{9.4.2}$$

and

$$\chi_+^{\dagger}\,\chi_- = 0. \tag{9.4.3}$$

A general spin state can be represented as a linear combination of χ_+ and χ_-: that is,

$$\chi = c_+\,\chi_+ + c_-\,\chi_-. \tag{9.4.4}$$

It is thus evident that electron spin space is two-dimensional.

Up to now, we have discussed spin space in rather abstract terms. In the following, we shall describe a particular representation of electron spin space due to Pauli . This so-called *Pauli representation* allows us to visualize spin space, and also facilitates calculations involving spin.

Let us attempt to represent a general spin state as a complex column vector in some two-dimensional space: that is,

$$\chi \equiv \begin{pmatrix} c_+ \\ c_- \end{pmatrix}. \tag{9.4.5}$$

The corresponding dual vector is represented as a row vector: that is,

$$\chi^{\dagger} \equiv (c_+^{\ast}, c_-^{\ast}). \tag{9.4.6}$$

Furthermore, the product $\chi^{\dagger}\,\chi$ is obtained according to the ordinary rules of matrix multiplication: that is,

$$\chi^{\dagger}\,\chi = (c_+^{\ast}, c_-^{\ast})\begin{pmatrix} c_+ \\ c_- \end{pmatrix} = c_+^{\ast}\,c_+ + c_-^{\ast}\,c_- = |c_+|^{\,2} + |c_-|^{\,2} \geq 0. \tag{9.4.7}$$

Likewise, the product $\chi^{\dagger}\,\chi'$ of two different spin states is also obtained from the rules of matrix multiplication: that is,

$$\chi^{\dagger}\,\chi' = (c_+^{\ast}, c_-^{\ast})\begin{pmatrix} c_+' \\ c_-' \end{pmatrix} = c_+^{\ast}\,c_+' + c_-^{\ast}\,c_-'. \tag{9.4.8}$$

Note that this particular representation of spin space is in complete accordance with the discussion in Section 1.3. For obvious reasons, a vector used to represent a spin state is generally known as *spinor*.

A general spin operator A is represented as a 2×2 matrix which operates on a spinor: that is,

$$A\,\chi \equiv \begin{pmatrix} A_{11}, & A_{12} \\ A_{21}, & A_{22} \end{pmatrix}\begin{pmatrix} c_+ \\ c_- \end{pmatrix}. \tag{9.4.9}$$

As is easily demonstrated, the Hermitian conjugate of A is represented by the transposed complex conjugate of the matrix used to represent A: that is,

$$A^\dagger \equiv \begin{pmatrix} A_{11}^*, & A_{21}^* \\ A_{12}^*, & A_{22}^* \end{pmatrix}. \tag{9.4.10}$$

Let us represent the spin eigenstates χ_+ and χ_- as

$$\chi_+ \equiv \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{9.4.11}$$

and

$$\chi_- \equiv \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{9.4.12}$$

respectively. Note that these forms automatically satisfy the orthonormality constraints ([e10.35]) and ([e10.36]). It is convenient to write the spin operators S_i (where $i = 1, 2, 3$ corresponds to x, y, z) as

$$S_i = \frac{\hbar}{2}\,\sigma_i. \tag{9.4.13}$$

Here, the σ_i are dimensionless 2×2 matrices. According to Equations ([e10.1x])–([e10.2x]), the σ_i satisfy the commutation relations

$$\begin{aligned} &= 2\,\mathrm{i}\,\sigma_z, \\ [\sigma_y, \sigma_z] &= 2\,\mathrm{i}\,\sigma_x, \\ [\sigma_z, \sigma_x] &= 2\,\mathrm{i}\,\sigma_y. \end{aligned}$$

Furthermore, Equation ([e10.34]) yields

$$\sigma_z\,\chi_\pm = \pm\chi_\pm. \tag{9.4.14}$$

It is easily demonstrated, from the previous expressions, that the σ_i are represented by the following matrices:

$$\begin{aligned} \sigma_x &\equiv \begin{pmatrix} 0, & 1 \\ 1, & 0 \end{pmatrix}, \\ \sigma_y &\equiv \begin{pmatrix} 0, & -\mathrm{i} \\ \mathrm{i}, & 0 \end{pmatrix}, \\ \sigma_z &\equiv \begin{pmatrix} 1, & 0 \\ 0, & -1 \end{pmatrix}. \end{aligned}$$

Incidentally, these matrices are generally known as the *Pauli matrices*.

Finally, a general spinor takes the form

$$\chi = c_+\,\chi_+ + c_-\,\chi_- = \begin{pmatrix} c_+ \\ c_- \end{pmatrix}. \tag{9.4.15}$$

If the spinor is properly normalized then

$$\chi^\dagger\,\chi = |c_+|^{\,2} + |c_-|^{\,2} = 1. \tag{9.4.16}$$

In this case, we can interpret $|c_+|^{\,2}$ as the probability that an observation of S_z will yield the result $+\hbar/2$, and $|c_-|^{\,2}$ as the probability that an observation of S_z will yield the result $-\hbar/2$.

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)