12.E: Time-Dependent Perturbation Theory (Exercises)

1. Consider the two-state system examined in Section 1.3.[ex8.1] Suppose that

$$\begin{aligned}\langle 1|\, H_1\,|1\rangle &= e_{11},\\ \langle 2|\, H_1\,|2\rangle &= e_{22},\\ \langle 1|\, H_1\,|2\rangle = \langle 2|\, H_1\,|1\rangle^\ast &= \frac{1}{2}\,\gamma\,\hbar\,\exp(\,{\rm i}\,\omega\,t),\end{aligned}$$

where e_{11}, e_{22}, γ, and ω are real. Show that

$$\begin{aligned}{\rm i}\,\frac{d\hat{c}_1}{dt} &= \frac{\gamma}{2}\,\exp[+{\rm i}\,(\omega-\hat{\omega}_{21})\,t]\,\hat{c}_2,\\ {\rm i}\,\frac{d\hat{c}_2}{dt} &= \frac{\gamma}{2}\,\exp[-{\rm i}\,(\omega-\hat{\omega}_{21})\,t]\,\hat{c}_1,\end{aligned}$$

where $\hat{c}_1 = c_1\,\exp(\,{\rm i}\,e_{11}\,t/\hbar)$, $\hat{c}_2 = c_2\,\exp(\,{\rm i}\,e_{22}\,t/\hbar)$, and

$$\hat{\omega}_{21} = \frac{E_2+e_{22}-E_1-e_{11}}{\hbar}. \tag{12.E.1}$$

Hence, deduce that if the system is definitely in state 1 at time $t=0$ then the probability of finding it in state 2 at some subsequent time, t, is

$$P_2(t) = \frac{\gamma^{\,2}}{\gamma^{\,2}+(\omega-\hat{\omega}_{21})^{\,2}}\,\sin^2\!\left(\left[\gamma^{\,2}+(\omega-\hat{\omega}_{21})^{\,2}\right]^{\,1/2}\frac{t}{2}\right). \tag{12.E.2}$$

2. Consider an atomic nucleus of spin-s and gyromagnetic ratio g placed in the magnetic field

$$\mathbf{B} = B_0\,\mathbf{e}_z + B_1\,[\cos(\omega\,t)\,\mathbf{e}_x - \sin(\omega\,t)\,\mathbf{e}_y]\,, \tag{12.E.3}$$

where $B_1 \ll B_0$. Let $\chi_{s,m}$ be a properly normalized simultaneous eigenstate of $S^{\,2}$ and S_z, where $\mathbf{S}$ is the nuclear spin. Thus, $S^{\,2}\,\chi_{s,m} = s\,(s+1)\,\hbar^{\,2}\,\chi_{s,m}$ and $S_z\,\chi_{s,m} = m\,\hbar\,\chi_{s,m}$, where $-s\le m\le s$. Furthermore, the instantaneous nuclear spin state is written

$$\chi = \sum_{m=-s,s} c_m(r)\,\chi_{s,m}, \tag{12.E.4}$$

where $\sum_{m=-s,s}|c_m|^{\,2} = 1$.

1. Demonstrate that

$$\begin{aligned}\frac{dc_m}{dt} = \frac{{\rm i}\,\gamma}{2}\Big(&[s\,(s+1)-m\,(m-1)]^{1/2}\,{\rm e}^{\,{\rm i}\,(\omega-\omega_0)\,t}\,c_{m-1}\\ &+[s\,(s+1)-m\,(m+1)]^{1/2}\,{\rm e}^{-{\rm i}\,(\omega-\omega_0)\,t}\,c_{m+1}\Big)\end{aligned}$$

for $-s\le m\le s$, where $\omega_0 = g\,\mu_N\,B_0/\hbar$, $\gamma = g\,\mu_N\,B_1/\hbar$, and $\mu_N = e\,\hbar/(2\,m_p)$.

2. Consider the case $s=1/2$. Demonstrate that if $\omega=\omega_0$ and $c_{1/2}(0)=1$ then

$$c_{1/2}(t) = \cos(\gamma\,t/2), \qquad c_{-1/2}(t) = {\rm i}\,\sin(\gamma\,t/2).$$

3. Consider the case $s=1$. Demonstrate that if $\omega=\omega_0$ and $c_1(0)=1$ then

$$\begin{aligned}c_1(t) &= \cos^2(\gamma\,t/2),\\ c_0(t) &= {\rm i}\sqrt{2}\,\cos(\gamma\,t/2)\,\sin(\gamma\,t/2),\\ c_{-1}(t) &= -\sin^2(\gamma\,t/2).\end{aligned}$$

4. Consider the case $s=3/2$. Demonstrate that if $\omega=\omega_0$ and $c_{3/2}(0)=1$ then

$$
\begin{aligned}
c_{3/2}(t) &= \cos^3(\gamma\,t/2),\\
c_{1/2}(t) &= \mathrm{i}\sqrt{3}\,\cos(\gamma\,t/2)\,\sin^2(\gamma\,t/2),\\
c_{-1/2}(t) &= -\sqrt{3}\,\cos^2(\gamma\,t/2)\,\sin(\gamma\,t/2),\\
c_{-3/2}(t) &= -\mathrm{i}\,\sin^3(\gamma\,t/2).
\end{aligned}
$$

3. Demonstrate that a spontaneous transition between two atomic states of zero orbital angular momentum is absolutely forbidden. (Actually, a spontaneous transition between two zero orbital angular momentum states is possible via the simultaneous emission of two photons, but takes place at a very slow rate .)

Contributors and Attributions

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