5.1: Fundamental Concepts of Multi-Particle Systems

We have already seen that the instantaneous state of a system consisting of a single non-relativistic particle, whose position coordinate is x, is fully specified by a complex wavefunction $\psi(x,t)$. This wavefunction is interpreted as follows. The probability of finding the particle between x and $x+dx$ at time t is given by $|\psi(x,t)|^{\,2}\,dx$. This interpretation only makes sense if the wavefunction is normalized such that

$$\int_{-\infty}^{\infty}|\psi(x,t)|^{\,2}\,dx = 1 \tag{5.1.1}$$

at all times. The physical significance of this normalization requirement is that the probability of the particle being found anywhere on the x-axis must always be unity (which corresponds to certainty).

Consider a system containing N non-relativistic particles, labeled $i=1,N$, moving in one dimension. Let x_i and m_i be the position coordinate and mass, respectively, of the ith particle. By analogy with the single-particle case, the instantaneous state of a multi-particle system is specified by a complex wavefunction $\psi(x_1,x_2,\ldots,x_N,t)$. The probability of finding the first particle between x_1 and x_1+dx_1, the second particle between x_2 and x_2+dx_2, et cetera, at time t is given by $|\psi(x_1,x_2,\ldots,x_N,t)|^{\,2}\,dx_1\,dx_2\ldots dx_N$. It follows that the wavefunction must satisfy the normalization condition

$$\int|\psi(x_1,x_2,\ldots,x_N,t)|^{\,2}\,dx_1\,dx_2\ldots dx_N = 1 \tag{5.1.2}$$

at all times, where the integration is taken over all $x_1\,x_2\,\ldots x_N$ space.

In a single-particle system, position is represented by the algebraic operator x, whereas momentum is represented by the differential operator $-\mathrm{i}\,\hbar\,\partial/\partial x$. (See Section [s4.6].) By analogy, in a multi-particle system, the position of the ith particle is represented by the algebraic operator x_i, whereas the corresponding momentum is represented by the differential operator

$$p_i = -\mathrm{i}\,\hbar\,\frac{\partial}{\partial x_i}. \tag{5.1.3}$$

Because the x_i are independent variables (i.e., $\partial x_i/\partial x_j = \delta_{ij}$), we conclude that the various position and momentum operators satisfy the following commutation relations:

$$\begin{aligned}
[x_i,x_j] &= 0,\\
[p_i,p_j] &= 0,\\
[x_i,p_j] &= \mathrm{i}\,\hbar\,\delta_{ij}.
\end{aligned}$$

Now, we know, from Section [smeas], that two dynamical variables can only be (exactly) measured simultaneously if the operators that represent them in quantum mechanics commute with one another. Thus, it is clear, from the previous commutation relations, that the only restriction on measurement in a one-dimensional multi-particle system is that it is impossible to simultaneously measure the position and momentum of the same particle. Note, in particular, that a knowledge of the position or momentum of a given particle does not in any way preclude a similar knowledge for a different particle. The commutation relations ([xe6.4])–([xe6.6]) illustrate an important point in quantum mechanics: namely, that operators corresponding to different degrees of freedom of a dynamical system tend to commute with one another. In this case, the different degrees of freedom correspond to the different motions of the various particles making up the system.

Finally, if $H(x_1,x_2,\ldots,x_N,t)$ is the Hamiltonian of the system then the multi-particle wavefunction $\psi(x_1,x_2,\ldots,x_N,t)$ satisfies the usual time-dependent Schrödinger equation [see Equation ([etimed])]

$$\mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t} = H\,\psi. \tag{5.1.4}$$

Likewise, a multi-particle state of definite energy E (i.e., an eigenstate of the Hamiltonian with eigenvalue E) is written (see Section [sstat])

$$\psi(x_1,x_2,\ldots,x_N,t) = \psi_E(x_1,x_2,\ldots,x_N)\,\mathrm{e}^{-\mathrm{i}\,E\,t/\hbar}, \tag{5.1.5}$$

where the stationary wavefunction ψ_E satisfies the time-independent Schrödinger equation [see Equation ([etimei])]

$$H\,\psi_E = E\,\psi_E. \tag{5.1.6}$$

Here, H is assumed not to be an explicit function of t.

Contributors and Attributions

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