12.9: Radiation from Harmonic Oscillator

Consider an electron in a one-dimensional harmonic oscillator potential aligned along the x-axis. According to Section [sosc], the unperturbed energy eigenvalues of the system are

$$E_n = (n+1/2)\,\hbar\,\omega_0, \tag{12.9.1}$$

where ω_0 is the frequency of the corresponding classical oscillator. Here, the quantum number n takes the values $0, 1, 2, \cdots$. Let the $\psi_n(x)$ be the (real) properly normalized unperturbed eigenstates of the system.

Suppose that the electron is initially in an excited state: that is, $n > 0$. In principle, the electron can decay to a lower energy state via the spontaneous emission of a photon of the appropriate frequency. Let us investigate this effect. Now, according to Equation ([e3.115]), the system can only make a spontaneous transition from an energy state corresponding to the quantum number n to one corresponding to the quantum number n' if the associated electric dipole moment

$$(d_x)_{n,n'} = \langle n|e\,x|n'\rangle = e\int_{-\infty}^{\infty}\psi_n(x)\,x\,\psi_{n'}(x)\,dx \tag{12.9.2}$$

is non-zero [because $d_{if} \equiv (d_x)_{n,n'}^{\,2}$ for the case in hand]. However, according to Equation ([e5.xxx]),

$$\int_{-\infty}^{\infty}\psi_n\,x\,\psi_{n'}\,dx = \sqrt{\frac{\hbar}{2\,m_e\,\omega_0}}\left(\sqrt{n}\,\delta_{n,n'+1} + \sqrt{n'}\,\delta_{n,n'-1}\right). \tag{12.9.3}$$

Because we are dealing with emission, we must have $n > n'$. Hence, we obtain

$$(d_x)_{n,n'} = e\sqrt{\frac{\hbar\,n}{2\,m_e\,\omega_0}}\,\delta_{n,n'+1}. \tag{12.9.4}$$

It is clear that (in the electric dipole approximation) we can only have spontaneous emission between states whose quantum numbers differ by unity. Thus, the frequency of the photon emitted when the nth excited state decays is

$$\omega_{n,n-1} = \frac{E_n - E_{n-1}}{\hbar} = \omega_0. \tag{12.9.5}$$

Hence, we conclude that, no matter which state decays, the emitted photon always has the same frequency as the classical oscillator.

According to Equation ([e3.115]), the decay rate of the nth excited state is given by

$$w_n = \frac{\omega_{n,n-1}^{\,3}\,(d_x)_{n,n-1}^{\,2}}{3\pi\,\epsilon_0\,\hbar\,c^{\,3}}. \tag{12.9.6}$$

It follows that

$$w_n = \frac{n\,e^{\,2}\,\omega_0^{\,2}}{6\pi\,\epsilon_0\,m_e\,c^{\,3}}. \tag{12.9.7}$$

The mean radiated power is simply

$$P_n = \hbar\,\omega_0\,w_n = \frac{e^{\,2}\,\omega_0^{\,2}}{6\pi\,\epsilon_0\,m_e\,c^{\,3}}\,[E_n - (1/2)\,\hbar\,\omega_0]. \tag{12.9.8}$$

Classically, an electron in a one-dimensional oscillator potential radiates at the oscillation frequency ω_0 with the mean power

$$P = \frac{e^{\,2}\,\omega_0^{\,2}}{6\pi\,\epsilon_0\,m_e\,c^{\,3}}\,E, \tag{12.9.9}$$

where E is the oscillator energy. It can be seen that a quantum oscillator radiates in an almost exactly analogous manner to the equivalent classical oscillator. The only difference is the factor $(1/2)\,\hbar\,\omega_0$ in Equation ([e13.126])—this is needed to ensure that the ground-state of the quantum oscillator does not radiate.

Contributors and Attributions

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