3.4: Ehrenfest's Theorem

A simple way to calculate the expectation value of momentum is to evaluate the time derivative of $\langle x\rangle$, and then multiply by the mass m: that is,

$$\langle p\rangle = m\,\frac{d\langle x\rangle}{dt} = m\,\frac{d}{dt}\int_{-\infty}^{\infty} x\,|\psi|^{\,2}\,dx = m\int_{-\infty}^{\infty} x\,\frac{\partial|\psi|^{\,2}}{\partial t}\,dx. \tag{3.4.1}$$

However, it is easily demonstrated that

$$\frac{\partial|\psi|^{\,2}}{\partial t} + \frac{\partial j}{\partial x} = 0 \tag{3.4.2}$$

[this is just the differential form of Equation ([epc])], where j is the probability current defined in Equation ([eprobc]). Thus,

$$\langle p\rangle = -m\int_{-\infty}^{\infty} x\,\frac{\partial j}{\partial x}\,dx = m\int_{-\infty}^{\infty} j\,dx, \tag{3.4.3}$$

where we have integrated by parts. It follows from Equation ([eprobc]) that

$$\langle p\rangle = -\frac{\mathrm{i}\,\hbar}{2}\int_{-\infty}^{\infty}\left(\psi^{*}\,\frac{\partial\psi}{\partial x} - \frac{\partial\psi^{*}}{\partial x}\,\psi\right)dx = -\mathrm{i}\,\hbar\int_{-\infty}^{\infty}\psi^{*}\,\frac{\partial\psi}{\partial x}\,dx, \tag{3.4.4}$$

where we have again integrated by parts. Hence, the expectation value of the momentum can be written

$$\langle p\rangle = m\,\frac{d\langle x\rangle}{dt} = -\mathrm{i}\,\hbar\int_{-\infty}^{\infty}\psi^{*}\,\frac{\partial\psi}{\partial x}\,dx. \tag{3.4.5}$$

It follows from the previous equation that

$$\frac{d\langle p\rangle}{dt} = -\mathrm{i}\,\hbar\int_{-\infty}^{\infty}\left(\frac{\partial\psi^{*}}{\partial t}\,\frac{\partial\psi}{\partial x} + \psi^{*}\,\frac{\partial^{\,2}\psi}{\partial t\,\partial x}\right)dx = \int_{-\infty}^{\infty}\left[\left(\mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t}\right)^{*}\frac{\partial\psi}{\partial x} + \frac{\partial\psi^{*}}{\partial x}\left(\mathrm{i}\,\hbar\,\frac{\partial\psi}{\partial t}\right)\right]dx,$$

where we have integrated by parts. Substituting from Schrödinger’s equation ([e3.1]), and simplifying, we obtain

$$\frac{d\langle p\rangle}{dt} = \int_{-\infty}^{\infty}\left[-\frac{\hbar^{\,2}}{2\,m}\,\frac{\partial}{\partial x}\!\left(\frac{\partial\psi^{*}}{\partial x}\,\frac{\partial\psi}{\partial x}\right) + V(x)\,\frac{\partial|\psi|^{\,2}}{\partial x}\right]dx = \int_{-\infty}^{\infty} V(x)\,\frac{\partial|\psi|^{\,2}}{\partial x}\,dx. \tag{3.4.6}$$

Integration by parts yields

$$\frac{d\langle p\rangle}{dt} = -\int_{-\infty}^{\infty}\frac{dV}{dx}\,|\psi|^{\,2}\,dx = -\left\langle\frac{dV}{dx}\right\rangle. \tag{3.4.7}$$

Hence, according to Equations ([e4.34x]) and ([e3.41]),

$$m\,\frac{d\langle x\rangle}{dt} = \langle p\rangle,$$
$$\frac{d\langle p\rangle}{dt} = -\left\langle\frac{dV}{dx}\right\rangle.$$

Evidently, the expectation values of displacement and momentum obey time evolution equations that are analogous to those of classical mechanics. This result is known as *Ehrenfest’s theorem* .

Suppose that the potential $V(x)$ is slowly varying. In this case, we can expand dV/dx as a Taylor series about $\langle x\rangle$. Keeping terms up to second order, we obtain

$$\frac{dV(x)}{dx} = \frac{dV(\langle x\rangle)}{d\langle x\rangle} + \frac{dV^{\,2}(\langle x\rangle)}{d\langle x\rangle^{\,2}}\,(x-\langle x\rangle) + \frac{1}{2}\,\frac{dV^{\,3}(\langle x\rangle)}{d\langle x\rangle^{3}}\,(x-\langle x\rangle)^{\,2}. \tag{3.4.8}$$

Substitution of the previous expansion into Equation ([e3.43]) yields

$$\frac{d\langle p\rangle}{dt} = -\frac{dV(\langle x\rangle)}{d\langle x\rangle} - \frac{\sigma_x^2}{2}\,\frac{dV^{\,3}(\langle x\rangle)}{d\langle x\rangle^{\,3}}, \tag{3.4.9}$$

because $\langle 1\rangle = 1$, and $\langle x-\langle x\rangle\rangle = 0$, and $\langle (x-\langle x\rangle)^{\,2}\rangle = \sigma_x^2$. The final term on the right-hand side of the previous equation can be neglected when the spatial extent of the particle wavefunction, σ_x, is much smaller than the variation length-scale of the potential. In this case, Equations ([e3.42]) and ([e3.43]) reduce to

$$m\,\frac{d\langle x\rangle}{dt} = \langle p\rangle,$$

$$\frac{d\langle p\rangle}{dt} = -\frac{dV(\langle x\rangle)}{d\langle x\rangle}.$$

These equations are exactly equivalent to the equations of classical mechanics, with $\langle x\rangle$ playing the role of the particle displacement. Of course, if the spatial extent of the wavefunction is negligible then a measurement of x is almost certain to yield a result that lies very close to $\langle x\rangle$. Hence, we conclude that quantum mechanics corresponds to classical mechanics in the limit that the spatial extent of the wavefunction (which is typically of order the de Boglie wavelength) is negligible. This is an important result, because we know that classical mechanics gives the correct answer in this limit.

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)
