5.2: Non-interacting Particles

In general, we expect the Hamiltonian of a multi-particle system to take the form

$$H(x_1, x_2, \ldots, x_N, t) = \sum_{i=1,N} \frac{p_i^2}{2\,m_i} + V(x_1, x_2, \ldots, x_N, t). \tag{5.2.1}$$

Here, the first term on the right-hand side represents the total kinetic energy of the system, whereas the potential V specifies the nature of the interaction between the various particles making up the system, as well as the interaction of the particles with any external forces.

Suppose that the particles do not interact with one another. This implies that each particle moves in a common potential: that is,

$$V(x_1, x_2, \ldots, x_N, t) = \sum_{i=1,N} V(x_i, t). \tag{5.2.2}$$

Hence, we can write

$$H(x_1, x_2, \ldots, x_N, t) = \sum_{i=1,N} H_i(x_i, t), \tag{5.2.3}$$

where

$$H_i = \frac{p_i^2}{2\,m_i} + V(x_i, t). \tag{5.2.4}$$

In other words, for the case of non-interacting particles, the multi-particle Hamiltonian of the system can be written as the sum of N independent single-particle Hamiltonians. Here, H_i represents the energy of the ith particle, and is completely unaffected by the energies of the other particles. Furthermore, given that the various particles that make up the system are non-interacting, we expect their instantaneous positions to be completely uncorrelated with one another. This immediately implies that the multi-particle wavefunction $\psi(x_1, x_2, \ldots x_N, t)$ can be written as the product of N independent single-particle wavefunctions: that is,

$$\psi(x_1, x_2, \ldots, x_N, t) = \psi_1(x_1, t)\,\psi_2(x_2, t)\ldots\psi_N(x_N, t). \tag{5.2.5}$$

Here, $|\psi_i(x_i, t)|^2\,dx_i$ is the probability of finding the ith particle between x_i and $x_i + dx_i$ at time t. This probability is completely unaffected by the positions of the other particles. It is evident that $\psi_i(x_i, t)$ must satisfy the normalization constraint

$$\int_{-\infty}^{\infty} |\psi_i(x_i, t)|^2\,dx_i = 1. \tag{5.2.6}$$

If this is the case then the normalization constraint ([en1]) for the multi-particle wavefunction is automatically satisfied. Equation ([ex13]) illustrates an important point in quantum mechanics: namely, that we can generally write the total wavefunction of a many degree of freedom system as a product of different wavefunctions corresponding to each degree of freedom.

According to Equations ([ex11]) and ([ex13]), the time-dependent Schrödinger equation ([ex7]) for a system of N non-interacting particles factorizes into N independent equations of the form

$$\mathrm{i}\,\hbar\,\frac{\partial \psi_i}{\partial t} = H_i\,\psi_i. \tag{5.2.7}$$

Assuming that $V(x, t) \equiv V(x)$, the time-independent Schrödinger equation ([ex9]) also factorizes to give

$$H_i\,\psi_{E_i} = E_i\,\psi_{E_i}, \tag{5.2.8}$$

where $\psi_i(x_i, t) = \psi_{E_i}(x_i)\,\exp(-\mathrm{i}\,E_i\,t/\hbar)$, and E_i is the energy of the ith particle. Hence, a multi-particle state of definite energy E has a wavefunction of the form

$$\psi(x_1, x_2, \ldots, x_n, t) = \psi_E(x_1, x_2, \ldots, x_N)\,\mathrm{e}^{-\mathrm{i}\,E\,t/\hbar}, \tag{5.2.9}$$

where

$$\psi_E(x_1, x_2, \ldots, x_N) = \psi_{E_1}(x_1)\,\psi_{E_2}(x_2)\ldots\psi_{E_N}(x_N), \tag{5.2.10}$$

and

$$E = \sum_{i=1,N} E_i. \tag{5.2.11}$$

Clearly, for the case of non-interacting particles, the energy of the whole system is simply the sum of the energies of the component particles.

Contributors and Attributions

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