3.10: Stationary States

An eigenstate of the energy operator $H \equiv \mathrm{i}\,\hbar\,\partial/\partial t$ corresponding to the eigenvalue E_i satisfies

$$\mathrm{i}\,\hbar\,\frac{\partial\psi_E(x,t,E_i)}{\partial t} = E_i\,\psi_E(x,t,E_i). \tag{3.10.1}$$

It is evident that this equation can be solved by writing

$$\psi_E(x,t,E_i) = \psi_i(x)\,\mathrm{e}^{-\mathrm{i}\,E_i\,t/\hbar}, \tag{3.10.2}$$

where $\psi_i(x)$ is a properly normalized stationary (i.e., non-time-varying) wavefunction. The wavefunction $\psi_E(x,t,E_i)$ corresponds to a so-called *stationary state*, because the probability density $|\psi_E|^{\,2}$ is non-time-varying. Note that a stationary state is associated with a unique value for the energy. Substitution of the previous expression into Schrödinger's equation ([e3.1]) yields the equation satisfied by the stationary wavefunction:

$$\frac{\hbar^{\,2}}{2\,m}\,\frac{d^{\,2}\psi_i}{dx^{\,2}} = [V(x)-E_i]\,\psi_i. \tag{3.10.3}$$

This is known as the *time-independent Schrödinger equation*. More generally, this equation takes the form

$$H\,\psi_i = E_i\,\psi_i, \tag{3.10.4}$$

where H is assumed not to be an explicit function of t. Of course, the ψ_i satisfy the usual orthonormality condition:

$$\int_{-\infty}^{\infty}\psi_i^\ast\,\psi_j\,dx = \delta_{ij}. \tag{3.10.5}$$

Moreover, we can express a general wavefunction as a linear combination of energy eigenstates:

$$\psi(x,t) = \sum_i c_i\,\psi_i(x)\,\mathrm{e}^{-\mathrm{i}\,E_i\,t/\hbar}, \tag{3.10.6}$$

where

$$c_i = \int_{-\infty}^{\infty}\psi_i^\ast(x)\,\psi(x,0)\,dx. \tag{3.10.7}$$

Here, $|c_i|^{\,2}$ is the probability that a measurement of the energy will yield the eigenvalue E_i. Furthermore, immediately after such a measurement, the system is left in the corresponding energy eigenstate. The generalization of the previous results to the case where H has continuous eigenvalues is straightforward.

If a dynamical variable is represented by some Hermitian operator A that commutes with H (so that it has simultaneous eigenstates with H), and contains no specific time dependence, then it is evident from Equations ([e4.157]) and ([e4.158]) that the expectation value and variance of A are time independent. In this sense, the dynamical variable in question is a constant of the motion.

Contributors and Attributions

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