3.9: Measurement

Suppose that A is an Hermitian operator corresponding to some dynamical variable. By analogy with the discussion in Section [scoll], we expect that if a measurement of A yields the result a then the act of measurement will cause the wavefunction to collapse to a state in which a measurement of A is bound to give the result a. What sort of wavefunction, ψ, is such that a measurement of A is bound to yield a certain result, a? Well, expressing ψ as a linear combination of the eigenstates of A, we have

$$\psi=\sum_i c_i\,\psi_i, \tag{3.9.1}$$

where ψ_i is an eigenstate of A corresponding to the eigenvalue a_i. If a measurement of A is bound to yield the result a then

$$\langle A\rangle = a, \tag{3.9.2}$$

and

$$\sigma_A^{\,2} = \langle A^2\rangle - \langle A\rangle = 0. \tag{3.9.3}$$

Now, it is easily seen that

$$\langle A\rangle = \sum_i |c_i|^{\,2}\,a_i,$$
$$\langle A^2\rangle = \sum_i |c_i|^{\,2}\,a_i^{\,2}.$$

Thus, Equation ([e4.130]) gives

$$\sum_i a_i^{\,2}\,|c_i|^{\,2} - \left(\sum_i a_i\,|c_i|^{\,2}\right)^2 = 0. \tag{3.9.4}$$

Furthermore, the normalization condition yields

$$\sum_i |c_i|^{\,2} = 1. \tag{3.9.5}$$

For instance, suppose that there are only two eigenstates. The previous two equations then reduce to $|c_1|^{\,2}=x$, and $|c_2|^{\,2}=1-x$, where $0\leq x\leq 1$, and

$$(a_1-a_2)^2\,x\,(1-x)=0. \tag{3.9.6}$$

The only solutions are $x=0$ and $x=1$. This result can easily be generalized to the case where there are more than two eigenstates. It follows that a state associated with a definite value of A is one in which one of the $|c_i|^{\,2}$ is unity, and all of the others are zero. In other words, the only states associated with definite values of A are the eigenstates of A. It immediately follows that the result of a measurement of A must be one of the eigenvalues of A. Moreover, if a general wavefunction is expanded as a linear combination of the eigenstates of A, like in Equation ([e4.128]), then it is clear from Equation ([e4.131]), and the general definition of a mean, that the probability of a measurement of A yielding the eigenvalue a_i is simply $|c_i|^{\,2}$, where c_i is the coefficient in front of the ith eigenstate in the expansion. Note, from Equation ([e4.134]), that these probabilities are properly normalized: that is, the probability of a measurement of A resulting in any possible answer is unity. Finally, if a measurement of A results in the eigenvalue a_i then immediately after the measurement the system will be left in the eigenstate corresponding to a_i.

Consider two physical dynamical variables represented by the two Hermitian operators A and B. Under what circumstances is it possible to simultaneously measure these two variables (exactly)? Well, the possible results of measurements of A and B are the eigenvalues of A and B, respectively. Thus, to simultaneously measure A and B (exactly) there must exist states which are simultaneous eigenstates of A and B. In fact, in order for A and B to be simultaneously measurable under all circumstances, we need all of the eigenstates of A to also be eigenstates of B, and vice versa, so that all states associated with unique values of A are also associated with unique values of B, and vice versa.

Now, we have already seen, in Section 1.8, that if A and B do not commute (i.e., if $A\,B\neq B\,A$) then they cannot be simultaneously measured. This suggests that the condition for simultaneous measurement is that A and B should commute.

Suppose that this is the case, and that the ψ_i and a_i are the normalized eigenstates and eigenvalues of A, respectively. It follows that

$$(A\,B-B\,A)\,\psi_i = (A\,B-B\,a_i)\,\psi_i = (A-a_i)\,B\,\psi_i = 0, \tag{3.9.7}$$

or

$$A\,(B\,\psi_i) = a_i\,(B\,\psi_i). \tag{3.9.8}$$

Thus, $B\,\psi_i$ is an eigenstate of A corresponding to the eigenvalue a_i (though not necessarily a normalized one). In other words, $B\,\psi_i \propto \psi_i$, or

$$B\,\psi_i = b_i\,\psi_i, \tag{3.9.9}$$

where b_i is a constant of proportionality. Hence, ψ_i is an eigenstate of B, and, thus, a simultaneous eigenstate of A and B. We conclude that if A and B commute then they possess simultaneous eigenstates, and are thus simultaneously measurable (exactly).

Continuous Eigenvalues

In the previous two sections, it was tacitly assumed that we were dealing with operators possessing discrete eigenvalues and square-integrable eigenstates. Unfortunately, some operators—most notably, x and p—possess eigenvalues that lie in a continuous range and non-square-integrable eigenstates (in fact, these two properties go hand in hand). Let us, therefore, investigate the eigenstates and eigenvalues of the displacement and momentum operators.

Let $\psi_x(x,x')$ be the eigenstate of x corresponding to the eigenvalue x'. It follows that

$$x\,\psi_x(x,x') = x'\,\psi_x(x,x') \tag{3.9.10}$$

for all x. Consider the Dirac delta-function $\delta(x-x')$. We can write

$$x\,\delta(x-x') = x'\,\delta(x-x'), \tag{3.9.11}$$

because $\delta(x-x')$ is only non-zero infinitesimally close to $x=x'$. Evidently, $\psi_x(x,x')$ is proportional to $\delta(x-x')$. Let us make the constant of proportionality unity, so that

$$\psi_x(x,x') = \delta(x-x'). \tag{3.9.12}$$

It is easily demonstrated that

$$\int_{-\infty}^{\infty}\delta(x-x')\,\delta(x-x'')\,dx = \delta(x'-x''). \tag{3.9.13}$$

Hence, $\psi_x(x,x')$ satisfies the orthonormality condition

$$\int_{-\infty}^{\infty}\psi_x^\ast(x,x')\,\psi_x(x,x'')\,dx = \delta(x'-x''). \tag{3.9.14}$$

This condition is analogous to the orthonormality condition ([e3.125]) satisfied by square-integrable eigenstates. Now, by definition, $\delta(x-x')$ satisfies

$$\int_{-\infty}^{\infty}f(x)\,\delta(x-x')\,dx = f(x'), \tag{3.9.15}$$

where $f(x)$ is a general function. We can thus write

$$\psi(x) = \int_{-\infty}^{\infty}c(x')\,\psi_x(x,x')\,dx', \tag{3.9.16}$$

where $c(x') = \psi(x')$, or

$$c(x') = \int_{-\infty}^{\infty}\psi_x^\ast(x,x')\,\psi(x)\,dx. \tag{3.9.17}$$

In other words, we can expand a general wavefunction $\psi(x)$ as a linear combination of the eigenstates, $\psi_x(x,x')$, of the displacement operator. Equations ([e4.144]) and ([e4.145]) are analogous to Equations ([e3.123]) and ([e3.126]), respectively, for square-integrable eigenstates. Finally, by analogy with the results in Section 1.9, the probability density of a measurement of x yielding the value x' is $|c(x')|^2$, which is equivalent to the standard result $|\psi(x')|^2$. Moreover, these probabilities are properly normalized provided $\psi(x)$ is properly normalized [cf., Equation ([e3.127])]: that is,

$$\int_{-\infty}^{\infty} |c(x')|^2\, dx' = \int_{-\infty}^{\infty} |\psi(x')|^2\, dx' = 1. \tag{3.9.18}$$

Finally, if a measurement of x yields the value x' then the system is left in the corresponding displacement eigenstate, $\psi_x(x, x')$, immediately after the measurement. That is, the wavefunction collapses to a "spike-function", $\delta(x - x')$, as discussed in Section [scoll].

Now, an eigenstate of the momentum operator $p \equiv -\mathrm{i}\,\hbar\,\partial/\partial x$ corresponding to the eigenvalue p' satisfies

$$-\mathrm{i}\,\hbar\,\frac{\partial\psi_p(x,p')}{\partial x} = p'\,\psi_p(x,p'). \tag{3.9.19}$$

It is evident that

$$\psi_p(x,p') \propto \mathrm{e}^{+\mathrm{i}\,p'\,x/\hbar}. \tag{3.9.20}$$

We require $\psi_p(x,p')$ to satisfy an analogous orthonormality condition to Equation ([e4.143]): that is,

$$\int_{-\infty}^{\infty} \psi_p^*(x,p')\,\psi_p(x,p'')\,dx = \delta(p'-p''). \tag{3.9.21}$$

Thus, it follows from Equation ([e3.72]) that the constant of proportionality in Equation ([e4.148]) should be $(2\pi\,\hbar)^{-1/2}$: that is,

$$\psi_p(x,p') = \frac{\mathrm{e}^{+\mathrm{i}\,p'\,x/\hbar}}{(2\pi\,\hbar)^{1/2}}. \tag{3.9.22}$$

Furthermore, according to Equations ([e3.64]) and ([e3.65]),

$$\psi(x) = \int_{-\infty}^{\infty} c(p')\,\psi_p(x,p')\,dp', \tag{3.9.23}$$

where $c(p') = \phi(p')$ [see Equation ([e3.65])], or

$$c(p') = \int_{-\infty}^{\infty} \psi_p^*(x,p')\,\psi(x)\,dx. \tag{3.9.24}$$

In other words, we can expand a general wavefunction $\psi(x)$ as a linear combination of the eigenstates, $\psi_p(x,p')$, of the momentum operator. Equations ([e4.152]) and ([e4.153]) are again analogous to Equations ([e3.123]) and ([e3.126]), respectively, for square-integrable eigenstates. Likewise, the probability density of a measurement of p yielding the result p' is $|c(p')|^2$, which is equivalent to the standard result $|\phi(p')|^2$. The probabilities are also properly normalized provided $\psi(x)$ is properly normalized [cf., Equation ([e3.83])]: that is,

$$\int_{-\infty}^{\infty} |c(p')|^2\, dp' = \int_{-\infty}^{\infty} |\phi(p')|^2\, dp' = \int_{-\infty}^{\infty} |\psi(x')|^2\, dx' = 1. \tag{3.9.25}$$

Finally, if a mesurement of p yields the value p' then the system is left in the corresponding momentum eigenstate, $\psi_p(x,p')$, immediately after the measurement.

Contributors and Attributions

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