10.2: Angular Momentum in Hydrogen Atom

In a hydrogen atom, the wavefunction of an electron in a simultaneous eigenstate of L^2 and L_z has an angular dependence specified by the spherical harmonic $Y_{l,m}(\theta,\phi)$. (See Section [sharm].) If the electron is also in an eigenstate of $S^{\,2}$ and S_z then the quantum numbers s and m_s take the values $1/2$ and $\pm 1/2$, respectively, and the internal state of the electron is specified by the spinors $\chi_\pm$. (See Section [spauli].) Hence, the simultaneous eigenstates of L^2, $S^{\,2}$, L_z, and S_z can be written in the separable form

$$\psi^{(1)}_{l,1/2;m,\pm 1/2} = Y_{l,m}\,\chi_\pm. \tag{10.2.1}$$

Here, it is understood that orbital angular momentum operators act on the spherical harmonic functions, $Y_{l,m}$, whereas spin angular momentum operators act on the spinors, $\chi_\pm$.

Because the eigenstates $\psi^{(1)}_{l,1/2;m,\pm 1/2}$ are (presumably) orthonormal, and form a complete set, we can express the eigenstates $\psi^{(2)}_{l,1/2;j,m_j}$ as linear combinations of them. For instance,

$$\psi^{(2)}_{l,1/2;j,m+1/2} = \alpha\,\psi^{(1)}_{l,1/2;m,1/2} + \beta\,\psi^{(1)}_{l,1/2;m+1,-1/2}, \tag{10.2.2}$$

where α and β are, as yet, unknown coefficients. Note that the number of $\psi^{(1)}$ states that can appear on the right-hand side of the previous expression is limited to two by the constraint that $m_j = m + m_s$ [see Equation ([e11.23])], and the fact that m_s can only take the values $\pm 1/2$. Assuming that the $\psi^{(2)}$ eigenstates are properly normalized, we have

$$\alpha^{\,2} + \beta^{\,2} = 1. \tag{10.2.3}$$

Now, it follows from Equation ([e11.26]) that

$$J^{\,2}\,\psi^{(2)}_{l,1/2;j,m+1/2} = j\,(j+1)\,\hbar^{\,2}\,\psi^{(2)}_{l,1/2;j,m+1/2}, \tag{10.2.4}$$

where [see Equation ([e11.12])]

$$J^{\,2} = L^2 + S^{\,2} + 2\,L_z\,S_z + L_+\,S_- + L_-\,S_+. \tag{10.2.5}$$

Moreover, according to Equations ([e11.28]) and ([e11.29]), we can write

$$\psi^{(2)}_{l,1/2;j,m+1/2} = \alpha\,Y_{l,m}\,\chi_+ + \beta\,Y_{l,m+1}\,\chi_-. \tag{10.2.6}$$

Recall, from Equations ([eraise]) and ([elow]), that

$$\begin{aligned} L_+\,Y_{l,m} &= [l\,(l+1) - m\,(m+1)]^{1/2}\,\hbar\,Y_{l,m+1}, \\ L_-\,Y_{l,m} &= [l\,(l+1) - m\,(m-1)]^{1/2}\,\hbar\,Y_{l,m-1}. \end{aligned}$$

By analogy, when the spin raising and lowering operators, $S_\pm$, act on a general spinor, χ_{s,m_s}, we obtain

$$\begin{aligned} S_+\,\chi_{s,m_s} &= [s\,(s+1) - m_s\,(m_s+1)]^{1/2}\,\hbar\,\chi_{s,m_s+1}, \\ S_-\,\chi_{s,m_s} &= [s\,(s+1) - m_s\,(m_s-1)]^{1/2}\,\hbar\,\chi_{s,m_s-1}. \end{aligned}$$

For the special case of spin one-half spinors (i.e., $s = 1/2, m_s = \pm 1/2$), the previous expressions reduce to

$$S_+\,\chi_+ = S_-\,\chi_- = 0, \tag{10.2.7}$$

and

$$S_\pm\,\chi_\mp = \hbar\,\chi_\pm. \tag{10.2.8}$$

It follows from Equations ([e11.32]) and ([e11.34])–([e11.39]) that

$$\begin{aligned} J^{\,2}\,Y_{l,m}\,\chi_+ &= [l\,(l+1) + 3/4 + m]\,\hbar^{\,2}\,Y_{l,m}\,\chi_+ \\ &\quad + [l\,(l+1) - m\,(m+1)]^{1/2}\,\hbar^{\,2}\,Y_{l,m+1}\,\chi_-, \end{aligned}$$

and

$$\begin{aligned}J^{\,2}\,Y_{l,m+1}\,\chi_- &= [l\,(l+1)+3/4-m-1]\,\hbar^{\,2}\,Y_{l,m+1}\,\chi_- \\ &\quad+[l\,(l+1)-m\,(m+1)]^{1/2}\,\hbar^{\,2}\,Y_{l,m}\,\chi_+.\end{aligned}$$

Hence, Equations ([e11.31]) and ([e11.33]) yield

$$\begin{aligned}(x-m)\,\alpha-[l\,(l+1)-m\,(m+1)]^{1/2}\,\beta &= 0, \\ -[l\,(l+1)-m\,(m+1)]^{1/2}\,\alpha+(x+m+1)\,\beta &= 0,\end{aligned}$$

where

$$x = j\,(j+1)-l\,(l+1)-3/4. \tag{10.2.9}$$

Equations ([e11.42]) and ([e11.43]) can be solved to give

$$x\,(x+1) = l\,(l+1), \tag{10.2.10}$$

and

$$\frac{\alpha}{\beta} = \frac{[(l-m)\,(l+m+1)]^{1/2}}{x-m}. \tag{10.2.11}$$

It follows that $x = l$ or $x = -l-1$, which corresponds to $j = l+1/2$ or $j = l-1/2$, respectively. Once x is specified, Equations ([e11.30]) and ([e11.45]) can be solved for α and β. We obtain

$$\psi^{(2)}_{l+1/2,m+1/2} = \left(\frac{l+m+1}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m,1/2}+\left(\frac{l-m}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m+1,-1/2}, \tag{10.2.12}$$

and

$$\psi^{(2)}_{l-1/2,m+1/2} = \left(\frac{l-m}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m,1/2}-\left(\frac{l+m+1}{2\,l+1}\right)^{1/2}\psi^{(1)}_{m+1,-1/2}. \tag{10.2.13}$$

Here, we have neglected the common subscripts $l, 1/2$ for the sake of clarity: that is, $\psi^{(2)}_{l+1/2,m+1/2} \equiv \psi^{(2)}_{l,1/2;l+1/2,m+1/2}$, et cetera. The previous equations can easily be inverted to give the $\psi^{(1)}$ eigenstates in terms of the $\psi^{(2)}$ eigenstates:

$$\begin{aligned}\psi^{(1)}_{m,1/2} &= \left(\frac{l+m+1}{2\,l+1}\right)^{1/2}\psi^{(2)}_{l+1/2,m+1/2}+\left(\frac{l-m}{2\,l+1}\right)^{1/2}\psi^{(2)}_{l-1/2,m+1/2}, \\ \psi^{(1)}_{m+1,-1/2} &= \left(\frac{l-m}{2\,l+1}\right)^{1/2}\psi^{(2)}_{l+1/2,m+1/2}-\left(\frac{l+m+1}{2\,l+1}\right)^{1/2}\psi^{(2)}_{l-1/2,m+1/2}.\end{aligned}$$

The information contained in Equations ([e11.47])–([e11.50]) is neatly summarized in Table [t2]. For instance, Equation ([e11.47]) is obtained by reading the first row of this table, whereas Equation ([e11.50]) is obtained by reading the second column. The coefficients in this type of table are generally known as *Clebsch-Gordon coefficients* .

Clebsch-Gordon coefficients for adding spin one-half to spin l.

	$m, 1/2$	$m+1, -1/2$	m, m_s
[0.5ex] $l+1/2, m+1/2$	$\sqrt{(l+m+1)/(2\,l+1)}$	$\sqrt{(l-m)/(2\,l+1)}$	
[0.5ex] $l-1/2, m+1/2$	$\sqrt{(l-m)/(2\,l+1)}$	$-\sqrt{(l+m+1)/(2\,l+1)}$	
[0.5ex] j, m_j			

As an example, let us consider the $l = 1$ states of a hydrogen atom. The eigenstates of L^2, $S^{\,2}$, L_z, and S_z, are denoted $\psi^{(1)}_{m,m_s}$. Because m can take the values $-1, 0, 1$, whereas m_s can take the values $\pm 1/2$, there are clearly six such states: that is, $\psi^{(1)}_{1,\pm 1/2}$, $\psi^{(1)}_{0,\pm 1/2}$, and $\psi^{(1)}_{-1,\pm 1/2}$. The eigenstates of L^2, $S^{\,2}$, $J^{\,2}$, and J_z, are denoted $\psi^{(2)}_{j,m_j}$. Because $l = 1$ and $s = 1/2$ can be combined together to form either $j = 3/2$ or $j = 1/2$ (see previously), there are also six such states: that is, $\psi^{(2)}_{3/2,\pm 3/2}$, $\psi^{(2)}_{3/2,\pm 1/2}$, and $\psi^{(2)}_{1/2,\pm 1/2}$. According to Table [t2], the various different eigenstates are interrelated as follows:

$$\psi^{(2)}_{3/2,\pm 3/2} = \psi^{(1)}_{\pm 1,\pm 1/2},$$
$$\psi^{(2)}_{3/2,1/2} = \sqrt{\frac{2}{3}}\,\psi^{(1)}_{0,1/2} + \sqrt{\frac{1}{3}}\,\psi^{(1)}_{1,-1/2},$$
$$\psi^{(2)}_{1/2,1/2} = \sqrt{\frac{1}{3}}\,\psi^{(1)}_{0,1/2} - \sqrt{\frac{2}{3}}\,\psi^{(1)}_{1,-1/2},$$
$$\psi^{(2)}_{1/2,-1/2} = \sqrt{\frac{2}{3}}\,\psi^{(1)}_{-1,1/2} - \sqrt{\frac{1}{3}}\,\psi^{(1)}_{0,-1/2},$$
$$\psi^{(2)}_{3/2,-1/2} = \sqrt{\frac{1}{3}}\,\psi^{(1)}_{-1,1/2} + \sqrt{\frac{2}{3}}\,\psi^{(1)}_{0,-1/2},$$

and

$$\psi^{(1)}_{\pm 1,\pm 1/2} = \psi^{(2)}_{3/2,\pm 3/2},$$
$$\psi^{(1)}_{1,-1/2} = \sqrt{\frac{1}{3}}\,\psi^{(2)}_{3/2,1/2} - \sqrt{\frac{2}{3}}\,\psi^{(2)}_{1/2,1/2},$$
$$\psi^{(1)}_{0,1/2} = \sqrt{\frac{2}{3}}\,\psi^{(2)}_{3/2,1/2} + \sqrt{\frac{1}{3}}\,\psi^{(2)}_{1/2,1/2},$$
$$\psi^{(1)}_{0,-1/2} = \sqrt{\frac{2}{3}}\,\psi^{(2)}_{3/2,-1/2} - \sqrt{\frac{1}{3}}\,\psi^{(2)}_{1/2,-1/2},$$
$$\psi^{(1)}_{-1,1/2} = \sqrt{\frac{1}{3}}\,\psi^{(2)}_{3/2,-1/2} + \sqrt{\frac{2}{3}}\,\psi^{(2)}_{1/2,-1/2},$$

Thus, if we know that an electron in a hydrogen atom is in an $l=1$ state characterized by $m=0$ and $m_s=1/2$ [i.e., the state represented by $\psi^{(1)}_{0,1/2}$] then, according to Equation ([e11.57]), a measurement of the total angular momentum will yield $j=3/2$, $m_j=1/2$ with probability $2/3$, and $j=1/2$, $m_j=1/2$ with probability $1/3$. Suppose that we make such a measurement, and obtain the result $j=3/2$, $m_j=1/2$. As a result of the measurement, the electron is thrown into the corresponding eigenstate, $\psi^{(2)}_{3/2,1/2}$. It thus follows from Equation ([e11.52]) that a subsequent measurement of L_z and S_z will yield $m=0$, $m_s=1/2$ with probability $2/3$, and $m=1$, $m_s=-1/2$ with probability $1/3$.

Clebsch-Gordon coefficients for adding spin one-half to spin one. Only non-zero coefficients are shown.

	$-1,-1/2$	$-1,1/2$	$0,-1/2$	$0,1/2$	$1,-1/2$	$1,1/2$	m, m_s
[0.5ex] $3/2,-3/2$	1						
[0.5ex] $3/2,-1/2$		$\sqrt{1/3}$	$\sqrt{2/3}$				
[0.5ex] $1/2,-1/2$		$\sqrt{2/3}$	$-\sqrt{1/3}$				
[0.5ex] $3/2,1/2$				$\sqrt{2/3}$	$\sqrt{1/3}$		
[0.5ex] $1/2,1/2$				$\sqrt{1/3}$	$-\sqrt{2/3}$		
[0.5ex] $3/2,3/2$						1	
j, m_j							

The information contained in Equations ([ecgs])–([ecge]) is neatly summed up in Table [t3]. Note that each row and column of this table has unit norm, and also that the different rows and different columns are mutually orthogonal. Of course, this is because the

$\psi^{(1)}$ and $\psi^{(2)}$ eigenstates are orthonormal.

Contributors and Attributions

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