# 14.5: Determination of Phase-Shifts

Let us now consider how the phase-shifts, $\delta_l$, in Equation ([e17.73]) can be evaluated. Consider a spherically symmetric potential, $V(r)$, that vanishes for $r > a$, where $a$ is termed the *range* of the potential. In the region $r > a$, the wavefunction $\psi(\mathbf{r})$ satisfies the free-space Schrödinger equation ([e17.54]). The most general solution that is consistent with no incoming spherical-waves is

$$\psi(\mathbf{r}) = \sqrt{n} \sum_{l=0,\infty} \mathrm{i}^{\,l}\,(2\,l+1)\,\mathcal{R}_l(r)\,P_l(\cos\theta), \tag{14.5.1}$$

where

$$\mathcal{R}_l(r) = \exp(\,\mathrm{i}\,\delta_l)\,\left[\cos\delta_l\,j_l(k\,r) - \sin\delta_l\,y_l(k\,r)\right]. \tag{14.5.2}$$

Note that $y_l(k\,r)$ functions are allowed to appear in the previous expression because its region of validity does not include the origin (where $V \neq 0$). The logarithmic derivative of the $l$th radial wavefunction, $\mathcal{R}_l(r)$, just outside the range of the potential is given by

$$\beta_{l+} = k\,a\left[\frac{\cos\delta_l\,j_l'(k\,a) - \sin\delta_l\,y_l'(k\,a)}{\cos\delta_l\,j_l(k\,a) - \sin\delta_l\,y_l(k\,a)}\right], \tag{14.5.3}$$

where $j_l'(x)$ denotes $dj_l(x)/dx$, et cetera. The previous equation can be inverted to give

$$\tan\delta_l = \frac{k\,a\,j_l'(k\,a) - \beta_{l+}\,j_l(k\,a)}{k\,a\,y_l'(k\,a) - \beta_{l+}\,y_l(k\,a)}. \tag{14.5.4}$$

Thus, the problem of determining the phase-shift, $\delta_l$, is equivalent to that of obtaining $\beta_{l+}$.

The most general solution to Schrödinger's equation inside the range of the potential ($r < a$) that does not depend on the azimuthal angle $\phi$ is

$$\psi(\mathbf{r}) = \sqrt{n} \sum_{l=0,\infty} \mathrm{i}^{\,l}\,(2\,l+1)\,\mathcal{R}_l(r)\,P_l(\cos\theta), \tag{14.5.5}$$

where

$$\mathcal{R}_l(r) = \frac{u_l(r)}{r}, \tag{14.5.6}$$

and

$$\frac{d^{\,2}u_l}{dr^{\,2}} + \left[k^{\,2} - \frac{l\,(l+1)}{r^{\,2}} - \frac{2\,m}{\hbar^{\,2}}\,V\right]u_l = 0. \tag{14.5.7}$$

The boundary condition

$$u_l(0) = 0 \tag{14.5.8}$$

ensures that the radial wavefunction is well behaved at the origin. We can launch a well-behaved solution of the previous equation from $r = 0$, integrate out to $r = a$, and form the logarithmic derivative

$$\beta_{l-} = \frac{1}{(u_l/r)}\left.\frac{d(u_l/r)}{dr}\right|_{r=a}. \tag{14.5.9}$$

Because $\psi(\mathbf{r})$ and its first derivatives are necessarily continuous for physically acceptable wavefunctions, it follows that

$$\beta_{l+} = \beta_{l-}. \tag{14.5.10}$$

The phase-shift, $\delta_l$, is then obtainable from Equation ([e17.82]).

## Contributors and Attributions

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