14.6: Hard-Sphere Scattering

Let us test out this scheme using a particularly simple example. Consider scattering by a *hard sphere*, for which the potential is infinite for $r < a$, and zero for $r > a$. It follows that $\psi(\mathbf{r})$ is zero in the region $r < a$, which implies that $u_l = 0$ for all l. Thus,

$$\beta_{l-} = \beta_{l+} = \infty, \tag{14.6.1}$$

for all l. Equation ([e17.82]) thus gives

$$\tan\delta_l = \frac{j_l(k\,a)}{y_l(k\,a)}. \tag{14.6.2}$$

Consider the $l = 0$ partial wave, which is usually referred to as the S-wave. Equation ([e17.90]) yields

$$\tan\delta_0 = \frac{\sin(k\,a)/k\,a}{-\cos(k\,a)/ka} = -\tan(k\,a), \tag{14.6.3}$$

where use has been made of Equations ([e17.58a]) and ([e17.58b]). It follows that

$$\delta_0 = -k\,a. \tag{14.6.4}$$

The S-wave radial wave function is [see Equation ([e17.80])]

$$\begin{aligned}\mathcal{R}_0(r) &= \exp(-\mathrm{i}\,k\,a)\,\frac{[\cos(k\,a)\,\sin(k\,r) - \sin(k\,a)\,\cos(k\,r)]}{k\,r}\\ &= \exp(-\mathrm{i}\,k\,a)\,\frac{\sin[k\,(r-a)]}{k\,r}.\end{aligned}$$

The corresponding radial wavefunction for the incident wave takes the form [see Equation ([e15.49])]

$$\tilde{\mathcal{R}}_0(r) = \frac{\sin(k\,r)}{k\,r}. \tag{14.6.5}$$

Thus, the actual $l = 0$ radial wavefunction is similar to the incident $l = 0$ wavefunction, except that it is phase-shifted by $k\,a$.

Let us examine the low- and high-energy asymptotic limits of $\tan\delta_l$. Low energy implies that $k\,a \ll 1$. In this regime, the spherical Bessel functions reduce to:

$$\begin{aligned}j_l(k\,r) &\simeq \frac{(k\,r)^l}{(2\,l+1)!!},\\ y_l(k\,r) &\simeq -\frac{(2\,l-1)!!}{(k\,r)^{l+1}},\end{aligned}$$

where $n!! = n\,(n-2)\,(n-4)\cdots 1$. It follows that

$$\tan\delta_l = \frac{-(k\,a)^{2\,l+1}}{(2\,l+1)\,[(2\,l-1)!!]^{\,2}}. \tag{14.6.6}$$

It is clear that we can neglect δ_l, with $l > 0$, with respect to δ_0. In other words, at low energy, only S-wave scattering (i.e., spherically symmetric scattering) is important. It follows from Equations ([e15.17]), ([e17.73]), and ([e17.92]) that

$$\frac{d\sigma}{d\Omega} = \frac{\sin^2 k\,a}{k^{\,2}} \simeq a^{\,2} \tag{14.6.7}$$

for $k\,a \ll 1$. Note that the total cross-section

$$\sigma_{\text{total}} = \int \frac{d\sigma}{d\Omega}\,d\Omega = 4\pi\,a^{\,2} \tag{14.6.8}$$

is four times the geometric cross-section $\pi\,a^{\,2}$ (i.e., the cross-section for classical particles bouncing off a hard sphere of radius a). However, low energy scattering implies relatively long wavelengths, so we would not expect to obtain the classical result in this limit.

Consider the high-energy limit $k\,a \gg 1$. At high energies, all partial waves up to $l_{\rm max} = k\,a$ contribute significantly to the scattering cross-section. It follows from Equation ([e17.75]) that

$$\sigma_{\rm total} \simeq \frac{4\pi}{k^{\,2}} \sum_{l=0,l_{\rm max}} (2\,l+1)\,\sin^2\delta_l. \tag{14.6.9}$$

With so many l values contributing, it is legitimate to replace $\sin^2\delta_l$ by its average value $1/2$. Thus,

$$\sigma_{\rm total} \simeq \sum_{l=0,k\,a} \frac{2\pi}{k^{\,2}}\,(2\,l+1) \simeq 2\pi\,a^{\,2}. \tag{14.6.10}$$

This is twice the classical result, which is somewhat surprising, because we might expect to obtain the classical result in the short-wavelength limit. For hard-sphere scattering, incident waves with impact parameters less than a must be deflected. However, in order to produce a “shadow” behind the sphere, there must also be some scattering in the forward direction in order to produce destructive interference with the incident plane-wave. (Recall the optical theorem.) In fact, the interference is not completely destructive, and the shadow has a bright spot (the so-called “Poisson spot”) in the forward direction. The effective cross-section associated with this bright spot is $\pi\,a^{\,2}$ which, when combined with the cross-section for classical reflection, $\pi\,a^{\,2}$, gives the actual cross-section of $2\pi\,a^{\,2}$.

Contributors and Attributions

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