7.2: Representation of Angular Momentum

Now, we saw earlier, in Section [s7.2], that the operators, p_i, which represent the Cartesian components of linear momentum in quantum mechanics, can be represented as the spatial differential operators $-\mathrm{i}\,\hbar\,\partial/\partial x_i$. Let us now investigate whether angular momentum operators can similarly be represented as spatial differential operators.

It is most convenient to perform our investigation using conventional spherical polar coordinates: that is, r, θ, and ϕ. These are defined with respect to our usual Cartesian coordinates as follows:

$$\begin{aligned} x &= r\,\sin\theta\,\cos\phi, \\ y &= r\,\sin\theta\,\sin\phi, \\ z &= r\,\cos\theta. \end{aligned}$$

We deduce, after some tedious analysis, that

$$\begin{aligned} \frac{\partial}{\partial x} &= \sin\theta\,\cos\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\,\cos\phi}{r}\,\frac{\partial}{\partial\theta} - \frac{\sin\phi}{r\,\sin\theta}\,\frac{\partial}{\partial\phi}, \\ \frac{\partial}{\partial y} &= \sin\theta\,\sin\phi\,\frac{\partial}{\partial r} + \frac{\cos\theta\,\sin\phi}{r}\,\frac{\partial}{\partial\theta} + \frac{\cos\phi}{r\,\sin\theta}\,\frac{\partial}{\partial\phi}, \\ \frac{\partial}{\partial z} &= \cos\theta\,\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\,\frac{\partial}{\partial\theta}. \end{aligned}$$

Making use of the definitions ([e8.1]–[e8.3]), ([e8.9]), and ([e8.13]), the fundamental representation ([e6.12]–[e6.14]) of the p_i operators as spatial differential operators, Equations ([e8.21])–([e8zz]), and a great deal of tedious analysis, we finally obtain

$$\begin{aligned} L_x &= -\mathrm{i}\,\hbar\left(-\sin\phi\,\frac{\partial}{\partial\theta} - \cos\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right), \\ L_y &= -\mathrm{i}\,\hbar\left(\cos\phi\,\frac{\partial}{\partial\theta} - \sin\phi\,\cot\theta\,\frac{\partial}{\partial\phi}\right), \\ L_z &= -\mathrm{i}\,\hbar\,\frac{\partial}{\partial\phi}, \end{aligned}$$

as well as

$$L^2 = -\hbar^{\,2}\left[\frac{1}{\sin\theta}\,\frac{\partial}{\partial\theta}\left(\sin\theta\,\frac{\partial}{\partial\theta}\right) + \frac{1}{\sin^2\theta}\,\frac{\partial^{\,2}}{\partial\phi^{\,2}}\right], \tag{7.2.1}$$

and

$$L_\pm = \hbar\,\mathrm{e}^{\pm\mathrm{i}\,\phi}\left(\pm\frac{\partial}{\partial\theta} + \mathrm{i}\,\cot\theta\,\frac{\partial}{\partial\phi}\right). \tag{7.2.2}$$

We, thus, conclude that all of our angular momentum operators can be represented as differential operators involving the angular spherical coordinates, θ and ϕ, but not involving the radial coordinate, r.

Contributors and Attributions

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