3.8: Eigenstates and Eigenvalues

Consider a general real-space operator, $A(x)$. When this operator acts on a general wavefunction $\psi(x)$ the result is usually a wavefunction with a completely different shape. However, there are certain special wavefunctions which are such that when A acts on them the result is just a multiple of the original wavefunction. These special wavefunctions are called *eigenstates*, and the multiples are called *eigenvalues*. Thus, if

$$A\,\psi_a(x) = a\,\psi_a(x), \tag{3.8.1}$$

where a is a complex number, then ψ_a is called an eigenstate of A corresponding to the eigenvalue a.

Suppose that A is an Hermitian operator corresponding to some physical dynamical variable. Consider a particle whose wavefunction is ψ_a. The expectation of value A in this state is simply [see Equation ([e3.55])]

$$\langle A\rangle = \int_{-\infty}^{\infty}\psi_a^\ast\,A\,\psi_a\,dx = a\int_{-\infty}^{\infty}\psi_a^\ast\,\psi_a\,dx = a, \tag{3.8.2}$$

where use has been made of Equation ([e3.107]) and the normalization condition ([e3.4]). Moreover,

$$\langle A^2\rangle = \int_{-\infty}^{\infty}\psi_a^\ast\,A^2\,\psi_a\,dx = a\int_{-\infty}^{\infty}\psi_a^\ast\,A\,\psi_a\,dx = a^{\,2}\int_{-\infty}^{\infty}\psi_a^\ast\,\psi_a\,dx = a^{\,2}, \tag{3.8.3}$$

so the variance of A is [cf., Equation ([e3.24a])]

$$\sigma_A^{\,2} = \langle A^2\rangle - \langle A\rangle^2 = a^{\,2} - a^{\,2} = 0. \tag{3.8.4}$$

The fact that the variance is zero implies that every measurement of A is bound to yield the same result: namely, a. Thus, the eigenstate ψ_a is a state that is associated with a unique value of the dynamical variable corresponding to A. This unique value is simply the associated eigenvalue.

It is easily demonstrated that the eigenvalues of an Hermitian operator are all real. Recall [from Equation ([e3.84])] that an Hermitian operator satisfies

$$\int_{-\infty}^{\infty}\psi_1^\ast\,(A\,\psi_2)\,dx = \int_{-\infty}^{\infty}(A\,\psi_1)^\ast\,\psi_2\,dx. \tag{3.8.5}$$

Hence, if $\psi_1 = \psi_2 = \psi_a$ then

$$\int_{-\infty}^{\infty}\psi_a^\ast\,(A\,\psi_a)\,dx = \int_{-\infty}^{\infty}(A\,\psi_a)^\ast\,\psi_a\,dx, \tag{3.8.6}$$

which reduces to [see Equation ([e3.107])]

$$a = a^\ast, \tag{3.8.7}$$

assuming that ψ_a is properly normalized.

Two wavefunctions, $\psi_1(x)$ and $\psi_2(x)$, are said to be *orthogonal* if

$$\int_{-\infty}^{\infty}\psi_1^\ast\,\psi_2\,dx = 0. \tag{3.8.8}$$

Consider two eigenstates of A, ψ_a and $\psi_{a'}$, which correspond to the two different eigenvalues a and a', respectively. Thus,

$$A\,\psi_a = a\,\psi_a,$$
$$A\,\psi_{a'} = a'\,\psi_{a'}.$$

Multiplying the complex conjugate of the first equation by $\psi_{a'}$, and the second equation by $\psi_a^\ast$, and then integrating over all x, we obtain

$$\int_{-\infty}^{\infty}(A\,\psi_a)^*\,\psi_{a'}\,dx = a\int_{-\infty}^{\infty}\psi_a^*\,\psi_{a'}\,dx,$$
$$\int_{-\infty}^{\infty}\psi_a^*\,(A\,\psi_{a'})\,dx = a'\int_{-\infty}^{\infty}\psi_a^*\,\psi_{a'}\,dx.$$

However, from Equation ([e3.111]), the left-hand sides of the previous two equations are equal. Hence, we can write

$$(a-a')\int_{-\infty}^{\infty}\psi_a^*\,\psi_{a'}\,dx = 0. \tag{3.8.9}$$

By assumption, $a \neq a'$, yielding

$$\int_{-\infty}^{\infty}\psi_a^*\,\psi_{a'}\,dx = 0. \tag{3.8.10}$$

In other words, eigenstates of an Hermitian operator corresponding to different eigenvalues are automatically orthogonal.

Consider two eigenstates of A, ψ_a and ψ_a', that correspond to the same eigenvalue, a. Such eigenstates are termed *degenerate*. The previous proof of the orthogonality of different eigenstates fails for degenerate eigenstates. Note, however, that any linear combination of ψ_a and ψ_a' is also an eigenstate of A corresponding to the eigenvalue a. Thus, even if ψ_a and ψ_a' are not orthogonal, we can always choose two linear combinations of these eigenstates that are orthogonal. For instance, if ψ_a and ψ_a' are properly normalized, and

$$\int_{-\infty}^{\infty}\psi_a^*\,\psi_a'\,dx = c, \tag{3.8.11}$$

then it is easily demonstrated that

$$\psi_a'' = \frac{|c|}{\sqrt{1-|c|^2}}\left(\psi_a - c^{-1}\,\psi_a'\right) \tag{3.8.12}$$

is a properly normalized eigenstate of A, corresponding to the eigenvalue a, that is orthogonal to ψ_a. It is straightforward to generalize the previous argument to three or more degenerate eigenstates. Hence, we conclude that the eigenstates of an Hermitian operator are, or can be chosen to be, mutually orthogonal.

It is also possible to demonstrate that the eigenstates of an Hermitian operator form a complete set : that is, any general wavefunction can be written as a linear combination of these eigenstates. However, the proof is quite difficult, and we shall not attempt it here.

In summary, given an Hermitian operator A, any general wavefunction, $\psi(x)$, can be written

$$\psi = \sum_i c_i\,\psi_i, \tag{3.8.13}$$

where the c_i are complex weights, and the ψ_i are the properly normalized (and mutually orthogonal) eigenstates of A: that is,

$$A\,\psi_i = a_i\,\psi_i, \tag{3.8.14}$$

where a_i is the eigenvalue corresponding to the eigenstate ψ_i, and

$$\int_{-\infty}^{\infty}\psi_i^*\,\psi_j\,dx = \delta_{ij}. \tag{3.8.15}$$

Here, δ_{ij} is called the *Kronecker delta-function* , and takes the value unity when its two indices are equal, and zero otherwise.

It follows from Equations ([e3.123]) and ([e3.125]) that

$$c_i = \int_{-\infty}^{\infty}\psi_i^*\,\psi\,dx. \tag{3.8.16}$$

Thus, the expansion coefficients in Equation ([e3.123]) are easily determined, given the wavefunction ψ and the eigenstates ψ_i. Moreover, if ψ is a properly normalized wavefunction then Equations ([e3.123]) and ([e3.125]) yield

$$\sum_i |c_i|^2 = 1. \tag{3.8.17}$$

Contributors and Attributions

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