12.11: 2P-1S Transitions in Hydrogen

Let us calculate the rate of spontaneous emission between the first excited state (i.e., $n=2$) and the ground-state (i.e., $n'=1$) of a hydrogen atom. Now, the ground-state is characterized by $l'=m'=0$. Hence, in order to satisfy the selection rules ([e13.133]) and ([e13.134]), the excited state must have the quantum numbers $l=1$ and $m=0,\ \pm 1$. Thus, we are dealing with a spontaneous transition from a $2P$ to a $1S$ state. Note, incidentally, that a spontaneous transition from a $2S$ to a $1S$ state is forbidden by our selection rules.

According to Section [s10.4], the wavefunction of a hydrogen atom takes the form

$$\psi_{n,l,m}(r,\theta,\phi) = R_{n,l}(r)\,Y_{l,m}(\theta,\phi), \tag{12.11.1}$$

where the radial functions $R_{n,l}$ are given in Section [s10.4], and the spherical harmonics $Y_{l,m}$ are given in Section [sharm]. Some straightforward, but tedious, integration reveals that

$$\begin{aligned}\langle 1,0,0|x|2,1,\pm 1\rangle &= \pm\frac{2^{\,7}}{3^{\,5}}\,a_0,\\ \langle 1,0,0|y|2,1,\pm 1\rangle &= {\rm i}\,\frac{2^{\,7}}{3^{\,5}}\,a_0,\\ \langle 1,0,0|z|2,1,0\rangle &= \sqrt{2}\,\frac{2^{\,7}}{3^{\,5}}\,a_0,\end{aligned}$$

where a_0 is the Bohr radius specified in Equation ([e9.57]). All of the other possible $2P\rightarrow 1S$ matrix elements are zero because of the selection rules. It follows from Equation ([e13.128]) that the modulus squared of the dipole moment for the $2P\rightarrow 1S$ transition takes the same value

$$d^{\,2} = \frac{2^{\,15}}{3^{\,10}}\,(e\,a_0)^{\,2} \tag{12.11.2}$$

for $m=0$, 1, or -1. Clearly, the transition rate is independent of the quantum number m. It turns out that this is a general result.

Now, the energy of the eigenstate of the hydrogen atom characterized by the quantum numbers n, l, m is $E=E_0/n^{\,2}$, where the ground-state energy E_0 is specified in Equation ([e9.56]). Hence, the energy of the photon emitted during a $2P\rightarrow 1S$ transition is

$$\hbar\,\omega = E_0/4 - E_0 = -\frac{3}{4}\,E_0 = 10.2\,{\rm eV}. \tag{12.11.3}$$

This corresponds to a wavelength of 1.215×10^{-7} m.

Finally, according to Equation ([e3.115]), the $2P\rightarrow 1S$ transition rate is written

$$w_{2P\rightarrow 1S} = \frac{\omega^{\,3}\,d^{\,2}}{3\pi\,\epsilon_0\,\hbar\,c^{\,3}}, \tag{12.11.4}$$

which reduces to

$$w_{2P\rightarrow 1S} = \left(\frac{2}{3}\right)^8\alpha^{\,5}\,\frac{m_e\,c^{\,2}}{\hbar} = 6.27\times 10^8\,{\rm s}^{-1} \tag{12.11.5}$$

with the aid of Equations ([e13.139]) and ([e13.140]). Here, $\alpha=1/137$ is the fine-structure constant. Hence, the mean life-time of a hydrogen $2P$ state is

$$\tau_{2P} = (w_{2P\rightarrow 1S})^{-1} = 1.6\,{\rm ns}. \tag{12.11.6}$$

Incidentally, because the $2P$ state only has a finite life-time, it follows from the energy-time uncertainty relation that the energy of this state is uncertain by an amount

$$\Delta E_{2P}\sim\frac{\hbar}{\tau_{2P}}\sim 4\times 10^{-7}\,{\rm eV}. \tag{12.11.7}$$

This uncertainty gives rise to a finite width of the spectral line associated with the $2P\rightarrow 1S$ transition. This natural line-width is of order

$$\frac{\Delta\lambda}{\lambda} \sim \frac{\Delta E_{2P}}{\hbar\omega} \sim 4 \times 10^{-8} \tag{12.11.8}$$

Contributors and Attributions

- Richard Fitzpatrick (Professor of Physics, The University of Texas at Austin)
