5.6: Electromagnetic Waves

The Wave Equation

When Maxwell realized that his new addition to the theory meant that not only can changing magnetic fields induce electric fields (Faraday), but changing electric fields can also induce magnetic fields, it occurred to him that it might be possible for propagation to occur: A changing magnetic field creates a changing electric field, which creates a changing magnetic field, and so on.

It was not hard for a mathematician such as Maxwell to express this propagation mathematically. To see how it comes about, let's simplify our physical situation by considering a region free of charges. This results in a simplified set of Maxwell's equations:

$$\begin{array}{ll} \text{electric Gauss:} & \overrightarrow{\nabla} \cdot \overrightarrow{E} = 0 \\ \text{magnetic Gauss:} & \overrightarrow{\nabla} \cdot \overrightarrow{B} = 0 \\ \text{Faraday:} & \overrightarrow{\nabla} \times \overrightarrow{E} = -\dfrac{d}{dt}\overrightarrow{B} \\ \text{Maxwell:} & \overrightarrow{\nabla} \times \overrightarrow{B} = \mu_o\epsilon_o\dfrac{d}{dt}\overrightarrow{E} \end{array} \tag{5.6.1}$$

Let's start by taking a derivative of the equation of the Maxwell equation with respect to time:

$$\frac{d}{dt}\overrightarrow{\nabla} \times \overrightarrow{B} = \overrightarrow{\nabla} \times \frac{d}{dt}\overrightarrow{B} = \mu_o\epsilon_o\frac{d^2}{dt^2}\overrightarrow{E} \tag{5.6.2}$$

Now plug the equation of Faraday into the derivative of the magnetic field:

$$\overrightarrow{\nabla} \times \left(-\overrightarrow{\nabla} \times \overrightarrow{E}\right) = \mu_o\epsilon_o\frac{d^2}{dt^2}\overrightarrow{E} \tag{5.6.3}$$

Now we have an equation exclusively in terms of the electric field (electric field induces magnetic field which induces electric field again). The double curl looks quite daunting to simplify, but it turns out that there is a useful identity from vector calculus to save the day:

$$\overrightarrow{\nabla} \times \left(\overrightarrow{\nabla} \times \overrightarrow{E}\right) = \overrightarrow{\nabla}\left(\overrightarrow{\nabla} \cdot \overrightarrow{E}\right) - \nabla^2\overrightarrow{E} \tag{5.6.4}$$

Plugging the electric Gauss equation into this and then plugging this equation in for the double curl gives:

$$\nabla^2\overrightarrow{E} = \mu_o\epsilon_o\frac{d^2\overrightarrow{E}}{dt^2} \tag{5.6.5}$$

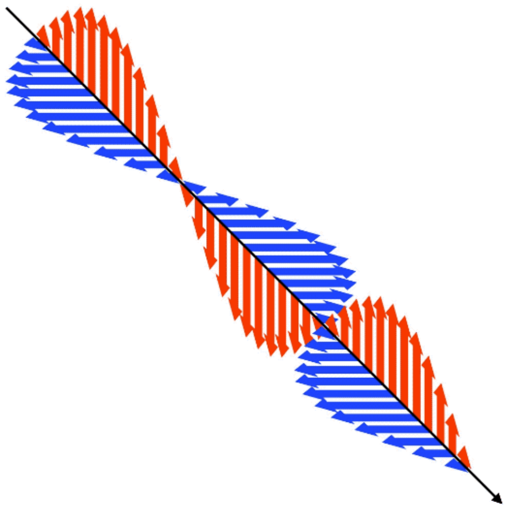
Perhaps you recognize this differential equation from Physics 9B? It is the wave equation – not surprising, really, given that a changing electric field seems to propagate another electric field (using the changing magnetic field as an intermediate step). Naturally Maxwell recognized the wave equation as well, and asked the most obvious question, "How fast is this wave?" Given that the velocity of a wave can be taken directly from the wave equation, this is not hard to calculate. The coefficient of the second time derivative term is the inverse of the square of the wave speed, so the speed of this wave is:

$$v = \frac{1}{\sqrt{\mu_o\epsilon_o}} = \frac{1}{\sqrt{\left(4\pi \times 10^{-7}\frac{Ns^2}{C^2}\right)\left(8.85 \times 10^{-12}\frac{C^2}{Nm^2}\right)}} = 3.0 \times 10^8\frac{m}{s} \tag{5.6.6}$$

Well of course Maxwell recognized this number immediately (as should you!) – it is the speed of light, c. Maxwell has shown that light is an electromagnetic phenomenon that exists because electric and magnetic fields can propagate by inducing each other.

If one begins the derivation above by taking a derivative of the Faraday equation with respect to time and follows the same steps, one finds that the very same wave equation applies to the magnetic field – both fields propagate together as a single light ("electromagnetic") wave.

Figure 5.6.1 – Electromagnetic Wave



EM Wave Properties

Let’s see what we can find out about these waves by looking at a specific example. Suppose we have a harmonic plane wave of electric field polarized in the x-z plane. Recall from 9B that this is expressed mathematically by:

$$\overrightarrow{E}(z,t)=\hat{i}\;E_o\cos\left(\frac{2\pi}{\lambda}z-\frac{2\pi}{T}t\right) \tag{5.6.7}$$

This represents a wave that propagates along the z direction, the "displacement" direction (polarization direction of the electric field vectors) along the x direction, has an amplitude of E_o, a wavelength of λ, and period of T. We have chosen the starting time such that the phase constant is zero.

Let's plug this field into Faraday's equation by taking its curl:

$$\overrightarrow{\nabla}\times\overrightarrow{E}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ E_x & E_y & E_z\end{vmatrix}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 0 & 0 & \frac{\partial}{\partial z}\\ E & 0 & 0\end{vmatrix}=\frac{\partial E}{\partial z}\left(+\hat{j}\right) \tag{5.6.8}$$

Performing the derivative, we get:

$$\overrightarrow{\nabla}\times\overrightarrow{E}=-\left(\frac{2\pi}{\lambda}\right)E_o\sin\left(\frac{2\pi}{\lambda}z-\frac{2\pi}{T}t\right)\;\hat{j} \tag{5.6.9}$$

Now that we know the curl of the electric field, we can plug the result into Faraday's law:

$$\overrightarrow{\nabla}\times\overrightarrow{E}=-\frac{d}{dt}\overrightarrow{B}=-\left(\frac{2\pi}{\lambda}\right)E_o\sin\left(\frac{2\pi}{\lambda}z-\frac{2\pi}{T}t\right)\;\hat{j} \tag{5.6.10}$$

We can now integrate to find the wave function for the magnetic field of this wave (for simplicity, we will assume that the electric and magnetic fields are in phase with each other, which will mean the arbitrary constant from the integral is just zero):

$$\overrightarrow{B}(z,t)=+\hat{j}\;\left(\frac{2\pi}{\lambda}\right)E_o\int\sin\left(\frac{2\pi}{\lambda}z-\frac{2\pi}{T}t\right)dt=+\hat{j}\;\left(\frac{T}{\lambda}\right)E_o\cos\left(\frac{2\pi}{\lambda}z-\frac{2\pi}{T}t\right) \tag{5.6.11}$$

We see that the magnetic field wave function has the same frequency and wavelength as the electric field wave function, and since the ratio $\frac{T}{\lambda}$ is just the inverse of the speed of the wave c, which means that the amplitudes of the electric and magnetic parts of the wave are related by:

$$B_o = \frac{E_o}{c} \tag{5.6.12}$$

We can also see a how the various directions are related. The velocity is in the $\hat{k}$ direction, the electric field in the $\hat{i}$ direction, and the magnetic field in the $\hat{j}$ direction – all three of these vectors are mutually orthogonal. In fact, the direction of the wave's velocity vector is the same direction as the vector $\overrightarrow{E} \times \overrightarrow{B}$.

Example 5.6.1

We know that electric and magnetic fields store energy in the space in which they exist. As a light wave passes through a region of space, the fluctuating fields cause the energy density in that space to fluctuate. Is more of the wave's energy a result of the electric field or the magnetic field? More specifically, compute the ratio of the maximum energy densities of the two fields within a single EM wave traveling through a vacuum.

Solution

The energy densities for electric and magnetic fields in a vacuum are given by:

$$U_E = \tfrac{1}{2}\epsilon_o E^2$$
$$U_B = \tfrac{1}{2\mu_o} B^2$$

The maximum energy densities come about when the fields equal their amplitudes, so taking the ratio of these energies gives:

$$\frac{U_E}{U_B} = \frac{\epsilon_o E^2}{\frac{1}{\mu_o} B^2} = \epsilon_o \mu_o \frac{E_o^2}{B_o^2}$$

Now plugging in Equation 5.6.6 and Equation 5.6.12, we get the simple result:

$$U_E = U_B$$

Both fields contribute equally to the energy density in the space through which the wave passes.