1.2: Doppler Effect

Sound Source in Motion

All waves begin at some source, and have an effect some distance away at something we will call a "receiver." Sound waves in particular exhibit this source/receiver relationship very well: Something (a "source") vibrates, varying the air pressure in its vicinity in some fashion. This pressure variance then propagates as a sound wave to another place, where the varying pressure causes another object (the "receiver") to vibrate. We have already seen how the separation distance and the inverse-square law relates the vibration of the source to the vibration of the receiver in terms of their energies.

Let's consider the following three properties of a sound wave generated at a source and detected at a receiver:

- period (or frequency)

The period of the wave according to the source is simply the time span between the generation of each wave crest, and the frequency measured by the source is the inverse of this number. As each compression reaches the receiver, a compression occurs, so the time that elapses between crests "washing over" the receiver is the period of the wave as measured by the receiver, and the inverse of this period is the frequency measured by that receiver.

- wavelength

The spatial separation of the wave crests is the wavelength of the sound wave. This is not a measurement that "belongs" to either the source or receiver – a simple snapshot of the particles in the air can be used to measure this distance.

- wave velocity

The two values above – the period and wavelength – are related to each other according to the speed of the wave. That is, the source calculates the speed of the wave to be the ratio of the wavelength and the period measured by the source. The receiver calculates the wave speed as the ratio of the wavelength and the period that it measures.

The question we seek to answer here is, "How does motion of the source through the medium affect these quantities, if at all?" To answer this question, we'll start with the simple case where there is no movement through the medium, and create a model that simplifies comparisons of what happens at the sender and receiver. We'll assume that the sender is point source (depicted with a red dot in the figure below), and that the receiver is a point in space as well (depicted with a blue dot in the figure below). The radiating circles in the figure represent sound wave crests (regions of maximum density or pressure), propagating outward from the source. Every time a new crest is created at the source, the red dot flashes, and every time a wave crest is detected, the blue dot flashes. The time between red dot flashes is the period of the wave according to the source, and the time between blue dot flashes is the period of the wave according to the receiver.

Figure 1.2.1 – Source and Receiver Stationary

The source and receiver have been placed exactly three wavelengths apart, which has the effect of synchronizing their flashes, which makes it clear that the time between red dot flashes is the same as the time between blue dot flashes. Of course, the time interval between flashes would be the same even if the source and receiver were separated by a different distance and the flashes were not simultaneous. The point is that the source and receiver agree upon the period (frequency) of this wave. There is no reason that the source and receiver should ever disagree about the wavelength of the wave, as a snapshot can be taken at any moment and a meter stick can be used to measure this quantity. Also it is clear that the wave moves away from the source at the same speed that it moves toward the receiver. So both source and receiver agree upon measurements of all three of the quantities in the equation $v = \lambda f$.

Now let's suppose that the source is moving through the air toward the stationary receiver as the sound is emitted. After a wave crest leaves the source, it continues spherically outward from the point where the source was when it emitted the sound, but the source moves before it emits the next wave crest, which results in the spherical wave crests not not being concentric. The result is depicted in the figure below.

Figure 1.2.2 – Source Moving Toward Stationary Receiver

While the source is emitting a crest at the same regular time intervals as in the stationary case above (the time between red dot flashes is the same), the wave crests that reach the receiver are closer together. The speed of the wave crests according to the receiver is the same as before (the only difference is that they start at different places), so the time interval between blue dot flashes is shorter than before. It is clear that the blue dot is flashing twice as frequently as the red dot. So the source and receiver do *not* agree upon the period (frequency) of this wave!

The wavelength in this case is tricky – clearly it depends upon where you measure it. In front of the source it is much shorter than behind it. For the purposes of this discussion, just the wavelength along the line joining the source and receiver matters, and as before a snapshot can be taken, and the distance measured between wave crests is well-defined. So both the source and receiver agree upon the wavelength, but disagree upon the frequency, so what happens to the relation $v=\lambda f$? Clearly the source and receiver cannot agree on the speed of the wave. Of course they don't! The wave moves at a fixed velocity *through the medium*, and the receiver measures this speed because it is not moving through the medium. But the source *is* moving through the medium, which means it will measure a slower speed for the wave crests. Indeed, the source sees the crests moving away from it (in the direction specified) slower than the receiver sees the same crest coming toward it. So the source measures a slower wave speed and lower frequency for the wave than the receiver, and both measure the same wavelength, which allows both to satisfy the equation $v=\lambda f$.

Ultimately we would like a relationship between the frequencies measured by the source and receiver. To this end, let's make the following definitions:

$$\begin{array}{lll} f_s & \equiv & \text{the frequency of the wave measured by the source. Equals the inverse of the period of the wave, } T_s \\ f_r & \equiv & \text{the frequency of the wave measured by the receiver. Equals the inverse of the period of the wave, } T_r \\ v_s & \equiv & \text{the velocity of the source relative to the medium} \\ v & \equiv & \text{the velocity of the sound within the medium} \end{array}$$

We can compute the wavelength in front of the source using these quantities. A wave crest is emitted, and in the time it takes for a second one to be emitted (T_s), it travels a distance of vT_s. At this point, a second wave crest is emitted, but it is not emitted at the same position as the previous one. This one is emitted from a position that is closer to the receiver by the amount that the source has moved in the same time period: v_sT_s. The wavelength is the distance between these two crests:

$$\lambda = vT_s - v_sT_s = \left(v-v_s\right)T_s = \left(v-v_s\right)\frac{1}{f_s} \tag{1.2.1}$$

Now we can use the wavelength (which is the same for the receiver as the source) to relate the two frequencies:

$$\lambda=\frac{v}{f_r} \quad\Rightarrow\quad \frac{v}{f_r}=\left(v-v_s\right)\frac{1}{f_s} \quad\Rightarrow\quad f_r=\left(\frac{v}{v-v_s}\right)f_s \tag{1.2.2}$$

The fraction is greater than one, so this formula agrees with our observation that the frequency of the wave measured by the receiver is greater than the frequency sent by the source.

Suppose the source was moving away from the receiver. Then the wavelength is *increased*, which means that the time span between wave crests reaching the receiver is increased, and the receiver measures a lower frequency than the source. The amount that the wavelength is expanded is found the same way that the amount it was reduced in the previous case, and the effect is that the two terms in Equation 2.2.1 are added rather than subtracted. This changes the final answer such that the minus sign in the denominator becomes a plus sign. We can therefore summarize the relationship between the source and receiver frequencies (known as the *doppler effect*) for motion along a line as:

$$f_r = \left(\frac{v}{v \mp v_s}\right) f_s \tag{1.2.3}$$

The upper (–) sign refers to the source moving toward the receiver, and the lower (+) sign refers to it moving away from the receiver. If the source is not moving directly toward or away from the receiver, then things get only slightly more complicated. Note that the wavelength is different for every angle the velocity of the source makes with the line between the source and receiver. While this is not a particularly difficult extension to the doppler formula, it's more than we will need for our purposes, and we will examine it no further.

Example 1.2.1

A pedestrian standing on a corner hears the siren of a police car coming directly toward her. At this point in time, the car is $700\ m$ away and she hears a decibel level of $25\ dB$. The policeman in the car hears the siren at a frequency of $1000\ Hz$, while the pedestrian hears it at a frequency of $1100\ Hz$. Find the decibel level the pedestrian hears $22\ s$ later. Assume the police car maintains a constant speed and that sound exits the siren radially into 3-dimensions. The speed of sound in air is $344\ \frac{m}{s}$.

Solution

The loudness of the siren will of course increase as it gets closer to her. We can use the inverse-square law for intensity to determine how much louder it is when it is closer 22 seconds later, but to determine how much closer it is, we first have to figure out how fast it is going. We do this using the doppler effect. We are given the source's frequency and the listener's frequency, so we use those to compute the velocity of the source, noting that the listener is not moving, and the source is moving toward the listener:

$$f_r = \left(\frac{v}{v \mp v_s}\right) f_s \quad \Rightarrow \quad v_s = v\left(1 - \frac{f_s}{f_r}\right) = \left(344\frac{m}{s}\right)\left(1 - \frac{1000Hz}{1100Hz}\right) = \frac{344}{11}\frac{m}{s}$$

The distance traveled by the police car during the 22 seconds is therefore:

$$d = vt = \left(\frac{344}{11}\frac{m}{s}\right)(22s) = 688m$$

Subtracting this from the original distance tells us that the car is now a mere $12\ m$ away. We use this fact to determine the change in the intensity of the sound from the inverse-square law (Equation 1.3.14):

$$I_1 r_1^2 = I_2 r_2^2 \quad \Rightarrow \quad I_{close} = \frac{r_{far}^2}{r_{close}^2} I_{far} = \frac{(700m)^2}{(12m)^2} I_{far} = 3400\ I_{far}$$

Now we need to express this in the logarithmic scale for decibels. It is simplest to find the change in decibel level:

$$\beta_{close} - \beta_{far} = (10dB)\log\left(\frac{I_{close}}{I_o}\right) - (10dB)\log\left(\frac{I_{far}}{I_o}\right) = (10dB)\log\left(\frac{I_{close}}{I_{far}}\right) = (10dB)\log\ 3400 = 35dB \quad \Rightarrow \quad \beta_{close} = 25dB + 35dB = 60dB$$

Sound Receiver in Motion

Okay, now that we know what effect a moving source has on the frequency measured by a stationary receiver, we'll look at the opposite scenario – the effect on the frequency when the receiver is moving toward and away from the source. The overall effect is similar, in that moving toward the source increases the frequency and moving away from the source decreases the frequency, but the analysis is slightly different. With the source stationary, the wave crests are not squeezed closer together or stretched farther apart. The wavelength is simply determined from the source's frequency and the speed of sound in the medium:

$$\lambda = \frac{v}{f_s} \tag{1.2.4}$$

If the receiver is moving into the crests (toward the source), then the crests are moving toward the receiver at a relative speed of $v + v_r$, where now v_r is the velocity of the receiver. The time it takes between crests hitting the receiver (blue light flashes) is the distance traveled (one wavelength) divided by the relative speed, so:

$$T_r = \frac{\lambda}{v + v_r} \tag{1.2.5}$$

Putting these two equations together gives us the relationship between the frequencies:

$$\frac{1}{f_r} = T_r = \frac{v}{f_s\,(v + v_r)} \quad \Rightarrow \quad f_r = \left(\frac{v + v_r}{v}\right) f_s \tag{1.2.6}$$

The fraction is greater than one, so the frequency measured by the receiver is indeed higher than that of the source. As before, if the receiver is moving away, then there is a sign change from this case, giving:

$$f_r = \left(\frac{v \pm v_r}{v}\right) f_s \tag{1.2.7}$$

As before the upper sign indicates motion toward, and the lower sign motion away.

Example 1.2.2

Sound is emitted from a stationary source, and is detected by a stationary receiver. Naturally both measure the same frequency. The source now starts moving away from the receiver, and the frequency of the sound heard by the receiver is shifted lower by 25%. If the receiver had instead moved away from the source at the same speed, by what percentage would the frequency of the sound shift down?

Solution

We start by determining the speed we are talking about here. From the source-moving formula, if the receiver frequency is 25% lower, then it is 3/4 of the source frequency:

$$f_r=\left(\frac{v}{v+v_s}\right)f_s \quad\Rightarrow\quad \frac{v}{v+v_s}=\frac{3}{4} \quad\Rightarrow\quad v_s=\frac{1}{3}v$$

So the source is moving away at a speed of one third the speed of sound. Now we only need to compute the change in frequency when the receiver moves away from the stationary source at the same speed (i.e. plug in $\frac{1}{3}$ for v_r):

$$f_r=\left(\frac{v-v_r}{v}\right)f_s \quad\Rightarrow\quad \left(\frac{v-\frac{1}{3}v}{v}\right)=\frac{2}{3}$$

So if the source moves away at the same speed, then the frequency drops by one third (33%).

Combinations of Motions

The next natural question that arises is, "What if both the source and receiver are moving?" In this case, we can actually just "stack" these two results. When we calculated the effect of the receiver moving toward the source, we started with the wavelength of the sound. In that particular case, the wavelength resulted from a stationary source, but if it hadn't, the derivation would have been the same. So if the wavelength is compressed or stretched by the motion of the source, we use that wavelength, and get the answer from there. That is, instead of plugging Equation 2.2.4 (which expresses a stationary source) into Equation 2.2.5, we use Equation 2.2.1 (which expresses a moving source) instead:

$$\left.\begin{array}{l}\lambda=(v\mp v_s)\dfrac{1}{f_s}\\[2ex] \dfrac{1}{f_r}=\dfrac{\lambda}{v\pm v_r}\end{array}\right\} \quad\Rightarrow\quad f_r=\frac{v\pm v_r}{v\mp v_s}f_s \tag{1.2.8}$$

Once again the top sign handles motion toward the other, and the bottom sign handles motion away. There are a couple of checks we can do on this result. First, putting in $v_r=0$ or $v_s=0$ gives the same equations we found above for a stationary receiver and source, respectively. Second, we notice that if both objects are moving in the same direction at the same speed through the air, then one is moving toward while the other is moving away, and these speeds are equal, so the numerator equals the denominator, and both measure the same frequency.

Another interesting combination that comes up often is the *echo*. If a source is moving toward (say) a stationary wall, and the sound sent by the source is reflected off the wall and heard by the source (which now has become a receiver), how does the emitted frequency compare with the received frequency? The important concept to understand here is that when a wave strikes a new medium (in this case, sound going from air into a solid wall), the property of the wave that is maintained is the frequency. This is because if we measure the time between successive wave crests hitting the new medium, the same time elapses between wave crests emitted by that new medium. This applies to both reflection and transmission. So whatever frequency of sound is received by the wall then becomes the frequency of the sound transmitted by the wall in the echo.

So the sound received by the original sender after an echo undergoes *two* successive doppler effects. The wave that strikes the wall is a different frequency from the what was sent by the moving source. Then upon reflection, that new frequency is transmitted (with the wall treated as a new stationary source) and measured by the original sender (which is now a moving receiver).

Example 1.2.3

An automated flying drone comes equipped with an ultrasonic sensing device. This device emits sound pulses with a frequency of $100\ kHz$ to probe its surroundings by detecting echoes of those pulses from nearby objects. The drone flies along the x-axis in the $+x$ direction, when it detects an echo from a UFO that is directly in front of it that is also moving along the x-axis. The onboard computer for the drone immediately logs the following data:

airspeed of the drone:	$24.0\frac{m}{s}$
frequency of the echoed sound pulse:	$109kHz$

a. *Are the drone and the UFO moving toward or away from each other as this data is being recorded? Explain.*
b. *Find the direction in which the UFO is moving through the air ($+x$ or $-x$).*
c. *Find the speed of the UFO through the air.*

Solution

a. The frequency of the wave is increased in the course of the round-trip. We know that if the drone and UFO were moving at the same speed in the same direction, then the UFO would "hear" the same frequency as the drone emits, would then echo back that same frequency, and again since they *are moving the same speed and direction, the drone would "hear" the same frequency echoed by the UFO, which means there would be no doppler shift at all. Clearly then if the UFO slows down (or reverses direction) or the drone speeds up so that they are moving toward each other, the effect will be to doppler shift the detected echo to a higher frequency than the emitted sound pulse. So the drone and UFO are moving toward each other.*

b. There are a couple ways to do this. We will look at one way here, and the second way will be given in the answer to part (c). Suppose the UFO is stationary. The doppler effect for the echoed sound would then be found in the usual 2-step manner: UFO hears a doppler-shifted sound, reflects that frequency back, and the drone hears that sound doppler-shifted again. The first shift is due to a moving source, and the second shift is due to a moving listener, so:

$$\left.\begin{array}{ll} f_{echo}=\left(\dfrac{v}{v-v_{drone}}\right)f_{emitted} & [\text{moving drone sends sound}] \\ f_{received}=\left(\dfrac{v+v_{drone}}{v}\right)f_{echo} & [\text{moving drone receives echoed sound}] \end{array}\right\} \Rightarrow \quad f_{received}=\left(\frac{v+v_{drone}}{v-v_{drone}}\right)f_{emitted}$$
$$=\left(\frac{344\frac{m}{s}+24\frac{m}{s}}{344\frac{m}{s}-24\frac{m}{s}}\right)(100kHz)=115Hz$$

So if the UFO were stationary, then the frequency shift would be $15\ kHz$*, and this is more than the* $9\ kHz$*. For the frequency shift to be reduced from the case of when the UFO is stationary, the UFO must be moving away from the drone, which is in the* $+x$ *direction.*

c. We do this in two steps. The first incorporates the doppler shifted sound heard by the UFO, then for the echo that frequency becomes the source, and it is doppler shifted again when it is heard by the drone's detector. Unlike part (b), the doppler shifts are due to motion of both the source and the listener each time. Since we know the UFO and drone are getting closer, we'll assume that the UFO is moving in the –x direction (toward the drone):

$$\left.\begin{array}{l} f_{echo}=\left(\dfrac{v+v_{ufo}}{v-v_{drone}}\right)f_{emitted} \\ f_{received}=\left(\dfrac{v+v_{drone}}{v-v_{ufo}}\right)f_{echo} \end{array}\right\} \Rightarrow \quad f_{received}=\left(\frac{v+v_{drone}}{v-v_{drone}}\right)\left(\frac{v+v_{ufo}}{v-v_{ufo}}\right)f_{emitted}$$

We are given the emitted and received frequencies, as well as the drone speed, so we can solve for the UFO speed. The result of the algebra is:

$$v_{ufo}=-0.0268\ v=-9.2\frac{m}{s} \tag{1.2.9}$$

Notice that the sign came out negative. At the beginning we assumed that the UFO was moving toward the drone (in the $-x$ *direction). The negative sign indicates that the UFO is in fact moving in the opposite direction, in agreement with what we determined in part (b).*

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