3.2: Lorentz Transformation

Transformations Between Inertial Frames

When we first studied relative motion in Physics 9HA, we wrote down a way of translating between the values measured in the two frames. This set of equations was called the Galilean transformation equations. As sensible as these are, they clearly are not correct in light of what we now know about relativity. Most notably, the Galilean transformation assumes a universal time variable that is common to all frames.

So now we seek a new set of transformation equations to relate the spacetime coordinates of frames in relative motion. We will start with a couple of simplifying assumptions. First, the two frames in question share a spatial origin at the moment in time we will call $t=t'=0$ – we will define "event A" as occurring at this spacetime point. The effect of doing this is that distances and time intervals between this event and a second event are now just the spacetime coordinates themselves. For example: $\Delta x=x-0=x$ and $\Delta t'=t'-0=t'$.

For our second assumption, we will continue to define the relative motion as the primed frame moving at a speed v in the $+x$-direction relative to the unprimed frame.

In order to get a set of equations that gives us a translation between the $(ct,\ x,\ y,\ z,)$ spacetime coordinates measured in one frame and the $(ct',\ x',\ y',\ z',)$ spacetime coordinates measured in the other, we begin by noting that with motion only along the x-axis, the y and z coordinates will remain unchanged. For example, we know that lengths along those directions do not contract, so we would not expect the coordinates to be related in any way other than $y'=y$ and $z'=z$. But what about the x and ct coordinates?

We start by assuming that the transformation is a *linear* one, not unlike the Galilean transformation (after all, the Galilean transformation *does* work for frames whose relative speed is low). This means that the primed values can be written as linear combinations of the unprimed values:

$$\begin{aligned} x' &= J\cdot x+K\cdot ct \\ ct' &= L\cdot x+M\cdot ct \end{aligned} \tag{3.2.1}$$

Our goal is to determine the unknown constants J, K, L, and M that work for relativity. Let's start by defining "event B" viewed by the primed observer. Let's say that this event occurs at this observer's time t', and takes place at the origin of the unprimed frame. Since the primed observer sees this frame moving in the $-x'$-direction for a time period of t' after starting at the origin, the primed observer sees this event occur at the position $x'=-vt'$. Plugging $x=0$ (the event occurs at the unprimed origin) into the first equation above and comparing gives us the constant K:

$$\left.\begin{aligned} x' &= -vt' \\ x' &= 0+K\cdot ct \end{aligned}\right\}\ \ K=-\left(\frac{v}{c}\right)\left(\frac{t'}{t}\right) \tag{3.2.2}$$

Events A and B both occur at the origin of the unprimed frame, so the time span between them is the proper time, and the frame is inertial, so it is the spacetime interval. Therefore the time measured between these events in the primed and unprimed frames are related according to the usual time dilation formula:

$$t'=\gamma_v t \tag{3.2.3}$$

Plugging this in above gives us the constant K:

$$K=-\left(\frac{v}{c}\right)\gamma_v \tag{3.2.4}$$

Using this same event B, we can obtain the constant M as well. Plugging in $x=0$ gives:

$$ct'=0+M\cdot ct \quad\Rightarrow\quad M=\frac{t'}{t}=\gamma_v \tag{3.2.5}$$

Recapping what we have so far:

$$\begin{aligned} x' &= J\cdot x-\left(\frac{v}{c}\right)\gamma_v ct \\ ct' &= L\cdot x+\gamma_v ct \end{aligned} \tag{3.2.6}$$

Now to determine the other two constants, define "event B" as occurring at the origin of the primed frame, $x'=0$. The unprimed observer will see this event occur at the position $x=vt$, which we can plug back in to get:

$$0=J\cdot vt-\gamma_v vt \quad \Rightarrow \quad J=\gamma_v \tag{3.2.7}$$

To find the final constant L requires noting that the time measured in the primed frame for event B is now the proper time, and a bit more algebra than was needed for the previous constants (which is omitted here):

$$\left.\begin{array}{l} ct'=L\cdot vt+\gamma_v ct \\ t=\gamma_v t' \end{array}\right\} \; L=-\left(\frac{v}{c}\right)\gamma_v \tag{3.2.8}$$

Putting everything together gives us the *Lorentz transformation equations*:

$$\begin{aligned} ct' &= \gamma_v\left[ct-\left(\frac{v}{c}\right)x\right] \\ x' &= \gamma_v\left[x-\left(\frac{v}{c}\right)ct\right] \\ y' &= y \\ z' &= z \end{aligned} \tag{3.2.9}$$

The symmetry between the x and t variable is apparent, and shows the important difference between relativity and galilean physics – time is not universal and unaffected by the position of an event. Notice that when the velocity is very small compared to the speed of light (as it is in our everyday experience), then letting $\frac{v}{c}\rightarrow 0$ changes the Lorentz transformation equations into the Galilean transformation equations.

Finally, it should be noted that these transformations can also be written in terms of *changes* in these variables from one event to another. In effect, this is hidden in the equations themselves, as event A simply has all the variables equal to zero.

These equations give the spacetime coordinates of an event in the primed frame given the spacetime coordinates of the same event in the unprimed frame. But what if we want to do the reverse – find the coordinates of the event in the unprimed frame from those in the primed frame? [*This is called the inverse of this transformation.*] It's actually quite easy to do – the only difference in perspectives between these two frames is the sign of the velocity. We get the inverse transformation by simply replacing the v everywhere in the equations with $-v$.

Example 3.2.1

We have said that the interval-squared $\Delta s^2=c^2\Delta t^2-\Delta x^2-\Delta y^2-\Delta z^2$ is an invariant, which means that it is the same in every inertial frame. Use the Lorentz transformation equations to show that this is true.

Solution

We want to show that $\Delta s'^2=\Delta s^2$, which makes this a pure plug-in. Clearly the y and z changes are equal in both frames, so we will ignore them and deal with just the t and x changes:

$$\begin{aligned} \Delta s'^2 &= c^2\Delta t'^2-\Delta x'^2 \\ &= \left(\gamma_v\left[c\Delta t-\left(\frac{v}{c}\right)\Delta x\right]\right)^2-\left(\gamma_v\left[\Delta x-\left(\frac{v}{c}\right)c\Delta t\right]\right)^2 \\ &= \gamma_v^2\left[\left(c^2\Delta t^2-\cancel{2v\Delta x\Delta t}+\frac{v^2}{c^2}\Delta x^2\right)-\left(\Delta x^2+\cancel{2v\Delta x\Delta t}+v^2\Delta t^2\right)\right] \\ &= \cancel{\gamma_v^2}\left[\cancel{\left(1-\frac{v^2}{c^2}\right)}\left(c^2\Delta t^2-\Delta x^2\right)\right] \\ &= \Delta s^2 \end{aligned}$$

Revisiting Previous Results

After all that struggle with thought experiments and spacetime diagrams, only now do we have a simple, powerful tool for achieving the same results. Time dilation is downright trivial. If (unprimed) Ann sees two events occur at the same place ($\Delta x=0$) separated by a time interval Δt, then the time span that (primed) Bob measures between these events is:

$$c\Delta t' = \gamma_v \left[c\Delta t - \left(\frac{v}{c}\right) \cancelto{0}{\Delta x}\right] \quad \Rightarrow \quad \Delta t' = \gamma_v \Delta t \tag{3.2.10}$$

We can also look at simultaneity. Events that are simultaneous in Ann's frame ($\Delta t = 0$) are not simultaneous in Bob's:

$$c\Delta t' = \gamma_v \left[c \cancelto{0}{\Delta t} - \left(\frac{v}{c}\right) \Delta x\right] \neq 0 \tag{3.2.11}$$

Looking at this expression, we also see that $\Delta t'$ is negative (i.e. $t_2 < t_1$) when $\Delta x'$ is positive (i.e. $x_2 > x_1$). This means that for the two events that Ann sees as simultaneous, Bob sees the event with the greater x-value as occurring first (note that we are still assuming that Bob is moving in the $+x$-direction relative to Ann). So is Ann flies by Bob in a spaceship where she sees lights on the front and rear of her ship flashing in sync, Bob sees the light on the rear of her ship flashing ahead of the light on the front.

Reproducing length contraction is a bit more difficult to obtain from the Lorentz transformation equations. the reason is that the length that is measured by one observer depends upon different events than the length measured by the other observer. That is, the length of an object in a given frame is the distance between events located at both ends of the object *that occur at the same time*, and as just noted, events simultaneous in one frame are not simultaneous in the other. Nevertheless, we can get the length contraction result with some care.

Two events that are simultaneous at both ends of an object according to Bob gives:

$$0 = c\Delta t' = \gamma_v \left[c\Delta t - \frac{v}{c}\Delta x\right] \quad \Rightarrow \quad c\Delta t = \frac{v}{c}\Delta x \tag{3.2.12}$$

Plugging this back into the transformation for the length measured by Bob gives the length contraction:

$$\Delta x' = \gamma_v \left[\Delta x - v\Delta t\right] = \gamma_v \left[\Delta x - v\left(\frac{v}{c}\Delta x\right)\right] = \gamma_v \Delta x \left[1 - \frac{v^2}{c^2}\right] = \frac{\Delta x}{\gamma_v} \tag{3.2.13}$$
