6.2: Interpreting Matter Waves

de Broglie Wavelength

We have described matter waves in terms of the momentum and energy of the particle, but it is still a wave, and as such possesses wave properties like wavelength and frequency. We can extract these quantities directly from the wave function by comparing it to a wave function of a general harmonic wave. We'll start with the matter wave's wavelength, known as the *de Broglie wavelength*:

$$\frac{2\pi}{\lambda}=\frac{p}{\hbar}\quad\Rightarrow\quad\lambda=\frac{h}{p}\tag{6.2.1}$$

This is the wavelength we need to use in (for example) a double slit calculation to predict interference patterns. Notice that unlike the photon, this wavelength depends upon the mass of the particle and its speed.

An Application

Obviously observations in science are highly-dependent upon light – telescopes collect light from outer space, and microscopes collect light from very small dimensions, for example. The microscope in particular runs into a limitation that comes from the wave nature of light. In the discussion in the previous section of trying to watch electrons as they go from the double slit to the screen, we said that the ability of the light to resolve the location of the electron depends on the wavelength of the light – the shorter the wavelength, the finer the granularity of the resolution.

So what if we use electrons for imaging, rather than light? They behave like waves like light does, but they have other features that light doesn't have, such as our ability to alter their speed and direction with electric and magnetic fields (so for example, we can focus them with magnetic fields to achieve the same effect as focusing light with a lens). But the real kicker is their resolving power thanks to their short de Broglie wavelengths. Such a device is known as an electron microscope. Let's do the math to see why these work so well...

Imagine accelerating a stream of electrons through a voltage of say 1000 volts (CRT televisions accelerated electrons with voltages over 10 times this great, and the best electron microscopes are much higher still). We can calculate the wavelength (and therefore the resolving power) of the matter waves thus created:

$$\begin{array}{c} KE=q_e\Delta V=\left(1.6\times10^{-19}C\right)\left(1000V\right)=1.6\times10^{-16}J \\ \dfrac{p^2}{2m_e}=KE\quad\Rightarrow\quad p=\sqrt{2\left(9.11\times10^{-31}kg\right)\left(1.6\times10^{-19}C\right)}=1.71\times10^{-23}\dfrac{kg\cdot m}{s} \\ \lambda=\dfrac{h}{p}=\dfrac{6.63\times10^{-34}J\cdot s}{1.71\times10^{-23}\frac{kg\cdot m}{s}}=3.88\times10^{-11}m \end{array}$$

This is on the atomic scale! If we wanted to use light to probe the same dimensions, we would need to use some serious X-rays (not the garden-variety dentist chair kind), and the energy per photon to achieve this wavelength would be:

$$E_{photon}=hf=\frac{hc}{\lambda}=\frac{\left(6.63\times10^{-34}J\cdot s\right)\left(3.0\times10^8\frac{m}{s}\right)}{3.88\times10^{-11}m}=5.13\times10^{-15}J$$

This is more than 30 times greater than the $1.6\times10^{-16}J$ of energy required by an electron, for the same result!

Probability Density

After associating the wave property of wavelength with a physical property (momentum), and the wave property of frequency with the physical property of energy, we turn our attention to wave amplitude. With light waves, we know that the amplitude-squared is a measure of intensity. When we first discussed wave-particle duality, we decided that in a particle context, intensity at a given position is a measure of the number of particles arriving at that position per second. With our one-photon-per-hour double-slit experiment, we concluded that this particle arrival rate is really a measure of the *probability* of a single particle arriving at that position.

Of course, the probability of arriving at a specific exact position is zero, since the sum of all the probabilities has to equal 1, and there is an infinite number of positions available. If we define a small range of positions dx, then we can reasonably talk about the probability of landing in that range. The intensity is proportional to the square of the wave function, so we have:

$$\text{probability of particle landing in tiny range of positions from } x \text{ to } x+dx: \quad P(x \leftrightarrow x+dx) = [\psi(x)]^2 dx \tag{6.2.2}$$

We are taking one liberty here, in that we are ignoring the time dependence of this wave function. The point is that we are waiting "long enough" for the electrons to get to the screen after passing through the double slit, and interpreting the distribution of the dots on the screen as a probability distribution – we are not (yet) considering the time evolution of the wave function as it makes the trip.

Closer examination of the equation above reveals that it can't quite be right. The reason is that our wave function is *complex-valued*, and complex numbers do not necessarily have squares that are positive, and all probabilities must be positive! We therefore make a small adjustment to the "square the amplitude" prescription for intensity in this case: We take the square of the *magnitude*:

$$P(x \leftrightarrow x+dx) = |\psi(x)|^2 dx \tag{6.2.3}$$

The magnitude-squared of a complex number is the product of that number with its complex conjugate. The complex conjugate of a complex number is found by changing the sign of the imaginary part of the number:

$$Z = a+bi \quad \Rightarrow \quad Z^* = a-bi \quad \Rightarrow \quad |Z|^2 = Z^* Z = (a-bi)(a+bi) = a^2-(bi)^2 = a^2+b^2 \tag{6.2.4}$$

The probability of the particle landing in a finite range (say between x_1 and x_2) is simply the sum of all the probabilities of landing in all the tiny ranges between those two points:

$$\text{probability of particle landing in range from } x_1 \text{ to } x_2 = P(x_1 \leftrightarrow x_2) = \int_{x_1}^{x_2} \psi^*(x)\,\psi(x)\,dx \tag{6.2.5}$$

The quantity $|\psi(x)|^2 = \psi^*(x)\,\psi(x)$ is called the *probability density*, for obvious reasons – integrating it over a range of positions gives the probability of the particle landing in that range.

Some Matter Wave Definitions

It's helpful to define some quantities that will help us manage the bookkeeping of rather complicated-looking wave functions (also so that we can understand what we read elsewhere!). There is no new physics here, just new language.

$$\text{angular frequency: } \omega \equiv \frac{2\pi}{T} = \frac{E}{\hbar} \tag{6.2.6}$$

$$\text{wave number: } k \equiv \frac{2\pi}{\lambda} = \frac{p}{\hbar} \tag{6.2.7}$$

These definitions make the expression of the wave function for a free particle (also called a *plane wave*, as it only moves in one direction, and regions of fixed phase form planes perpendicular to that direction):

$$\psi_{free}(x,t) = \psi_o e^{i(kx-\omega t)} \tag{6.2.8}$$

[*Note: By "free", we mean that it is not under the influence of any forces. We will later see how to deal with these situations – the wave equation (and therefore the wave functions that come from it) is different.*]

Alert

We have to be careful about associating matter waves too closely to "standard waves." For example, the speed of a standard wave is simply the wavelength divided by the period. This is known as the "phase velocity" of the matter wave, and it should not be confused with the speed of the particle (when it is observed as a particle). Indeed:

$$v_{phase} = \frac{\lambda}{T} = \frac{E}{p} = \frac{\frac{1}{2}mv^2_{particle}}{mv_{particle}} = \frac{1}{2}v_{particle}$$

For light, the phase and particle velocities do come out to be equal, but this is not so for massive particles. The way to think of it is this: The phase velocity is the rate at which the probability wave travels, but if we were to actually watch the particle (by reflecting light off it as it moves), it travels at a different rate. The phase velocity is not important in that it is not measurable – but be careful not to confuse these two quantities.