6.4: Kepler's Laws

The fact that the planets move in elliptical orbits was first discovered by Kepler, based on observational data alone (he didn't have the benefit, as we do, of living after Newton and thus knowing about Newton's law of gravity). Kepler summarized his observational facts in three laws, which we can, with the benefit of hindsight, prove to be corollaries of Newton's laws.

Theorem 6.4.1:Kepler's first law

The planets move in elliptical orbits, with the sun at one of the foci.

Proof

This is case two of the general result given by Equations 6.3.10 and 6.3.11.

Theorem 6.4.2: Kepler's second law

A line segment joining a planet and the sun sweeps out equal areas during equal intervals of time.

Proof

This law is nothing but a special case of conservation of angular momentum. Consider a small piece of the orbit, in which the planet moves a distance $\mathrm{d}x$. The lines connecting the initial and final points of this piece of orbit with the sun make an angle $\mathrm{d}\theta$. If the initial distance from the planet to the sun was r, and the final distance $r+\mathrm{d}r$, we have, to first order, $\mathrm{d}x=r\mathrm{d}\theta$. The infinitesimal area the planet has swiped out is then given by (area of a triangle): dA = 1 2 r dx = 1 2 r 2 dθ. If we want to know how much area was swept out over an amount of time, we need to know the time derivative of A, which is thus given by $\mathrm{d}A=\frac{1}{2}r\mathrm{d}x=\frac{1}{2}r^2\mathrm{d}\theta$. Now using that the angular momentum of the planet is given by $L=mr^2\dot{\theta}$, we find

$$\frac{\mathrm{d}A}{\mathrm{d}t}=\frac{r^2}{2}\frac{\mathrm{d}\theta}{\mathrm{d}t}=\frac{L}{2m} \tag{6.4.1}$$

which is constant if L is conserved.

Theorem 6.4.3: Kepler's third law

The square of the period T of an orbit is proportional to the cube of its semi-major axis length a:

$$T^2=\frac{4\pi^2}{GM_\odot}a^3 \tag{6.4.2}$$

where $M_\odot$ is the mass of the sun.

proof

We integrate Equation 6.4.1 over the period of a whole orbit, which gives $A=\frac{LT}{2m}$. By Kepler's first law, the orbit is an ellipse, so its area equals $A=\pi ab$, with a and b the ellipse's semi-major and semi-minor axes. The two axes are related by $b=a\sqrt{1-\varepsilon^2}$, with ε again the eccentricity of the ellipse. Making these substitutions and squaring the resulting relation, we get:

$$\pi^2a^4=\frac{L^2}{m\left(1-\varepsilon^2\right)}\frac{T^2}{4m} \tag{6.4.3}$$

Using $k=\frac{L^2}{m\alpha}$, like in Equation 6.3.11, and the observation that for an elliptical orbit $\frac{k}{(1-\varepsilon^2)}=a$, we get $\frac{L^2}{m(1-\varepsilon 2)}=\alpha a$. Now for orbits in the solar system, $\alpha=GM_\odot m$, so we arrive at Equation 6.4.2.

Johannes Kepler

Johannes Kepler (1571-1630) was a German astronomer and mathematician who made major contributions to understanding the motion of the planets. Copernicus had published his heliocentric (rather than geocentric) view of the universe posthumously in 1543, but the two systems were still heavily debated in Kepler's time. Having been convinced that Copernicus was right, Kepler set out to construct a geometric description of the solar system. He initially tried to do so using polyhedra and Platonic solids, but found that these could not accurately describe the data. In 1600, Kepler met with astronomer Tycho Brahe, who had made meticulous observations of the known planets, and, having been convinced of Kepler's skills in mathematics, shared his data with him. After Tycho's death in 1601, Kepler succeeded him as imperial mathematician in Prague, where he developed his laws over the next decade. Unfortunately, Kepler's Calvinist views got him in trouble frequently with both the Catholic and the Lutheran church, which led to his excummunication, but he managed to avoid further persecution by moving frequently, and he always could continue his scientific work. The Kepler spacecraft and mission, launched in 2009 to hunt for extrasolar terrestial planets, is named in his honor.

Figure 6.4.1: Portrait of Johannes Kepler (1610) [21].