8.1: Oscillatory Motion

Harmonic Oscillator

We've already encountered two examples of oscillatory motion - the rotational motion of Chapter 5, and the mass-on-a-spring system in Section 2.3 (see Figure 1.1.1). The latter is the quintessential oscillator of physics, known as the *harmonic oscillator*. Recapping briefly, we get its equation of motion by considering a mass m that is being pulled on by a massless ideal spring of spring constant k. Equating the resulting spring force (Hooke's law) to the net force in Newton's second law of motion, we get:

$$m\ddot{x} = -kx \tag{8.1.1}$$

The harmonic oscillator is characterized by its *natural frequency* ω_0:

$$\omega_0 = \sqrt{\frac{k}{m}} \tag{8.1.2}$$

as follows readily by dimensional arguments (or, of course, by solving the differential equation). Because Equation 8.1.1 is second-order, its solution has two unknowns; moreover, as it has to be minus its own derivative we readily see that it should be a linear combination of sines and cosines (for a formal derivation, see Appendix A.3.2). We can write the solution in two different ways:

$$x(t) = x(0)\cos(\omega_0 t) + \frac{v(0)}{\omega_0}\sin(\omega_0 t) \tag{8.1.3}$$

$$= A\cos(\omega_0 t + \phi) \tag{8.1.4}$$

where the phase ϕ is given by $\tan\phi = -\frac{1}{\omega}\frac{v(0)}{x(0)}$ and the amplitude A by $A = \frac{x(0)}{\cos\phi}$. Unsurprisingly, as they are both simple periodic motions, there is a direct relationship between a harmonic oscillator with natural frequency ω_0, and a point on a disk rotating with uniform angular velocity ω_0 in the xy-plane - the motion of the harmonic oscillator is that of the disk projected on the x (or y) axis.

Torsional Oscillator

A torsional oscillator is the rotational analog of a harmonic oscillator - imagine a disk with moment of inertia I suspended by a massless, frictionless rope that has a torsional constant κ, i.e., the force to twist the rope is given by $F = -\kappa\theta$, with θ the twist angle. By invoking the rotational analog of Newton's second law of motion, Equation 5.4.1, we readily find for the equation of motion of the torsional oscillator:

$$I\ddot{\theta} = -\kappa\theta \tag{8.1.5}$$

so the torsional oscillator indeed is the exact rotational analog of the harmonic oscillator, and has a natural frequency of $\omega_0 = \sqrt{\frac{\kappa}{I}}$

Christiaan Huygens

Christiaan Huygens (1629-1695) was a Dutch physicist and astronomer, and one of the major figures in the scientific revolution. Huygens invented the pendulum clock in 1656, which revolutionized timekeeping and remained the most accurate clock for 300 years. Huygens was also the first to cast the laws of physics in mathematical form, writing down an early (quadratic) version of Newton's second law of motion, the equation for the centripetal force (Eq. 5.2.1), and the correct form of the laws of elastic collisions (Section 4.7). Observing two pendulum clocks on the same wall, Huygens observed that they synchronized (see Section 8.4). Huygens' study of optics led him to formulate the wave theory of light, which can correctly predict light diffraction. In astronomy, he discovered the first feature on the surface of Mars, the largest moon of Saturn (Titan), and that the previously observed 'shape changes' of Saturn were due to the presence of its rings. The Huygens probe that landed on Titan in 2005 was very appropriately named in his honor.

Figure 8.1.1: 1671 portrait of Huygens by Caspar Netscher [23].

Pendulum

A pendulum is an object that is suspended on a horizontal peg through any point x_{P} but its center of mass x_{CM} (it won't oscillate if you pin it at the center of mass). If the center of mass of the pendulum is pulled sideways, gravity will exert a torque around the position of the peg, pulling the pendulum back down. If the distance between x_{P} and x_{CM} is L, and the line connecting them makes an angle θ with the vertical through x_{P}, then the torque exerted by gravity around x_{P} equals $-mgL\sin\theta$, where as usual m is the mass of the pendulum. Now again invoking Equation 5.4.1, we can write for the equation of motion of the pendulum (with I its moment of inertia about x_{P}):

$$I\ddot{\theta} = -mgL\sin\theta \tag{8.1.6}$$

Unfortunately we can't solve Equation 8.1.6. For small angles however, we can Taylor-expand the sine, and write $\sin\theta \approx \theta$, which takes us back to the harmonic oscillator equation. From that we find that for this pendulum (called the physical pendulum), the natural frequency is $\omega_0 = \sqrt{\frac{mgL}{I}}$. For the special case that the pendulum consists of a mass m suspended on a massless rope of length L (the simple pendulum), we have $I = mL^2$ and thus $\omega_0 = \sqrt{g}L$.

Oscillations in a Potential Energy Landscape

The potential energy associated with a mass on a spring has a very simple form: $U_{\mathrm{s}}(x) = \frac{1}{2}kx^2$ (see Equation 3.3.7). The potential energy landscape of a harmonic oscillator thus has the shape of a parabola. Now that's a shape that we encounter very often: the shape of pretty much every landscape about a minimum closely resembles a parabola[1]. To see why this is the case, simply Taylor-expand the potential energy about a minimum at x_0: because the function has a minimum at $x_0, U'(x_0) = 0$, and the Taylor expansion gives

$$U(x) = U(x_0) + \frac{1}{2}U''(x_0)\, x^2 + \mathscr{O}\left(x^3\right) \tag{8.1.7}$$

Around a minimum in the potential energy, any potential energy thus resembles that of a harmonic oscillator. Any particle placed in such a potential energy landscape close to a minimum (i.e., a particle on which a force acts close to the point where the force vanishes) will therefore tend to oscillate. By comparing Equation 8.1.7 with the potential energy of the harmonic oscillator, we can immediately read off that the resulting oscillatory motion is identical to that of a harmonic oscillator with spring constant $k = U''(x_0)$. A particle released close to a minimum of the potential energy will thus oscillate with a frequency $\omega = \sqrt{U''(x_0)/m}$.

[1] *The only exception being functions of the form x^{2n} for n > 1.*