5.4: Moment of Inertia

Suppose we have a mass m at the end of a massless stick of length r, rotating around the other end of the stick. If we want to increase the rotation rate, we need to apply a tangential acceleration at

$$\boldsymbol{a}_{\mathrm{t}}=r\boldsymbol{\alpha}$$

for which by Newton's second law of motion we need a force

$$\boldsymbol{F}=m\boldsymbol{a}_{\mathrm{t}}=\mathrm{mr}\,\boldsymbol{\alpha}.$$

This force in turn generates a torque of magnitude

$$\tau=r\cdot F=mr^{2}\alpha. \tag{5.4.1}$$

The last equality is reminiscent of Newton's second law of motion, but with force replaced by torque, acceleration by angular acceleration, and mass by the quantity mr^2. In analog with mass representing the inertia of a body undergoing linear acceleration, we'll identify this quantity as the inertia of a body undergoing rotational acceleration, which we'll call the moment of inertia and denote by I:

$$\boldsymbol{\tau}=I\boldsymbol{\alpha} \tag{5.4.2}$$

Equation 5.4.2 is the rotational analog of Newton's second law of motion. By extending our previous example, we can find the moment of inertia of an arbitrary collection of particles of masses m_α and distances to the rotation axis r_α (where α runs over all particles), and write:

$$I=\sum_{\alpha}m_{\alpha}r_{\alpha}^{2} \tag{5.4.3}$$

which like the center of mass in Section 4.1 easily generalizes to continuous objects as[2]

$$I=\int_{V}(\boldsymbol{r}\cdot\boldsymbol{r})\rho\mathrm{d}V=\int_{V}\rho r^{2}\mathrm{d}V \tag{5.4.4}$$

Note that it matters where we choose the rotation axis. For example, the moment of inertia of a rod of length L and mass m around an axis through its center perpendicular to the rod is $\frac{1}{12}mL^2$, whereas the moment of inertia around an axis perpendicular to the rod but located at one of its ends is $\frac{1}{3}mL^2$. Also, moments of inertia are different for hollow and solid objects - a hollow sphere of mass m and radius R has $\frac{2}{3}mR^2$ whereas a solid sphere has $\frac{2}{5}mR^2$, and for hollow and solid cylinders (or hoops and disks) around the long axis through the center we find mR^2 and $\frac{1}{4}mR^2$ respectively. These and some other examples are listed in Table 5.1. Below we'll relate the moment of inertia to the kinetic energy of a moving-and-rolling object, but first we present two handy theorems that will help in calculating them.

Table 5.1: Moments of inertia for some common objects, all with total mass M and length L / radius R

Object	Rotation Axis	Moment of Inertia
Stick	Center, perpendicular to stick	$\frac{1}{12}ML^2$
Stick	End, perpendicular to stick	$\frac{1}{3}ML^2$
Cylinder, hollow	Center, parallel to axis	MR^2
Cylinder, solid	Center, parallel to axis	$\frac{1}{2}MR^2$
Sphere, hollow	Any axis through center	$\frac{2}{3}MR^2$
Sphere, solid	Any axis through center	$\frac{2}{5}MR^2$
Planar object, size $a\times b$	Axis through center, in plane, parallel to side with length a	$\frac{1}{12}Mb^2$
Planar object, size $a\times b$	Axis through center, perpendicular to plane	$\frac{1}{12}M(a^2+b^2)$

Theorem 5.1: Parallel axis theorem

If the moment of inertia of a rigid body about an axis through its center of mass is given by $I_c m$, then the moment of inertia around an axis parallel to the original axis and separated from it by a distance d is given by

$$I = I_{\mathrm{cm}} + md^2 \tag{5.4.5}$$

where m is the object's mass.

Proof

Choose coordinates such that the center of mass is at the origin, and the original axis coincides with the $\hat{z}$ axis. Denote the position of the point in the xy plane through which the new axis passes by d, and the distance from that point for any other point in space by r_d, such that $r = d + r_d$. Now calculate the moment of inertia about the new axis through d:

$$\begin{aligned} I &= \int_V (\boldsymbol{r}_d \cdot \boldsymbol{r}_d)\,\rho \mathrm{d}V &(5.4.6)\\ &= \int_V (\mathbf{r}\cdot\mathbf{r} + \mathbf{d}\cdot\mathbf{d} - 2\mathbf{d}\cdot\mathbf{r})\rho \mathrm{d}V &(5.4.7)\\ &= I_{\mathrm{cm}} + md^2 - 2\mathbf{d}\cdot\int_V \mathbf{r}\rho\mathrm{d}V &(5.4.8)\end{aligned}$$

Here $d^2 = \boldsymbol{d}\cdot\boldsymbol{d}$. The last integral in the last line of 5.4.8 is equal to the position of the center of mass, which we chose to be at the origin, so the last term vanishes, and we arrive at 5.4.5. Note that the first two lines of Table 5.1 (moments of inertia of a stick) satisfy the perpendicular-axis theorem.

Theorem 5.2: Perpendicular axis theorem

If a rigid object lies entirely in a plane, and the moments of inertia around two perpendicular axes x and y in that plane are I_x and I_y, respectively, then the moment of inertia around the axis z perpendicular to the plane and passing through the intersection point, is given by

$$I_z = I_x + I_y \tag{5.4.9}$$

Proof

We simply calculate the moment of inertia around the z-axis (where A is the area of the object, and σ the mass per unit area):

$$I_z = \int_A (x^2 + y^2)\,\sigma\mathrm{d}A = \int_A x^2\sigma\mathrm{d}A + \int_A y^2\sigma\mathrm{d}A = I_y + I_x \tag{5.4.10}$$

Note that the last two lines of Table 5.1 (moments of inertia of a thin planar rectangle) satisfy the parallel axis theorem.

[2] *Like the one- and two-dimensional analogs of the center of mass of a continuous object (4.1.3), there are one- and two-dimensional analogs of 5.4.4, which you get by replacing ρ with λ or σ and dV by dx or dA, respectively.*