

3.3: Potential Energy

We already encountered conservative forces in Section 3.1. The work done by a conservative force is (by definition) path-independent; that means that in particular the work done when moving along any closed path² must be zero:

$$\oint F \cdot dr = 0 \quad (3.3.1)$$

For a conservative force, we can thus define a potential energy difference between points 1 and 2 as the work necessary to move an object from point 1 to point 2:

$$\Delta U_{12} = - \int_{r_1}^{r_2} F \cdot dr \quad (3.3.2)$$

Note the minus sign in the definition - this is a choice of course, and you'll see below why we made this choice. Note also that the potential energy is defined only between two points. Often we will choose a convenient reference point and calculate the potential energy at any other point with respect to that point. The reference point is typically either the origin or infinity, if the force happens to be zero at either of these. Let's suppose we have set such a point, and know the potential energy difference with that point at any other point in space- this defines a (scalar) function $U(r)$. If we now want to know the force acting on a particle at r , all we need to do is take the derivative of $U(r)$ - that is the **gradient** in three dimensions (which simplifies to the ordinary derivative in one dimension):

$$F(r) = -\nabla U(r) \quad (3.3.3)$$

Equation 3.3.3 is extremely useful, as it gives us a means to calculate the force, which is a vector quantity, from the potential energy function, which is a scalar quantity - and therefore much simpler to work with. For instance, since energies are scalars, they can simply be added, as we'll do in the next section, whereas for forces you need to do vector addition. Equation 3.3.3 also reflects that we are free to choose a reference point for the potential energy, since the force does not change if we add a constant to the potential energy.

Gravitational Potential Energy

We saw in Section 2.2.2 that for low altitudes, the gravitational force is given by $F_g = mg$, where g is a vector of constant magnitude $g \approx 9.81 \frac{m}{s^2}$ and always points down. Therefore, the gravitational force does no work when you move horizontally, and if you first move up and then the same amount down again, it doesn't do any net work either, as the two contributions exactly cancel. F_g is therefore an example of a conservative force, and we can define and calculate the gravitational potential energy U_g between a point at height 0 (our reference point) and one at height h :

$$U_g(h) = - \int_{z=0}^{z=h} m(-g)dz = mgh \quad (3.3.4)$$

Note that by choosing a minus sign in the definition of the potential energy, we end up with a positive value of the energy here.

What about larger distances, i.e., Newton's law of gravity, Equation 2.2.3? Well, there the distances are measured radially, so any movement perpendicular to the radial direction doesn't matter, and if you move out and back in again, the net work done is zero, so by the same reasoning as before we again have a conservative force. This force vanishes at infinity, so it makes sense to set that as a reference point - though notice that that will make our potential energy always negative in this case:

$$U_G(r) = -\frac{GMm}{r} \quad (3.3.5)$$

where r is the distance between m and M , and M sits at the origin. Of course we can also calculate gravitational potential differences between two distances r_1 and r_2 from M :

$$\Delta U_G(r_1, r_2) = GMm \left(\frac{1}{r_1} - \frac{1}{r_2} \right) \quad (3.3.6)$$

📌 Emmy Noether

Emmy Noether (1882-1935) was a German mathematician, who made key contribution both to the development of abstract algebra and to ideas in theoretical physics. In physics, she uncovered a deep connection between symmetry and conservation laws (now known as Noether's theorem, considered by many as the most important theorem for the development of modern physics): for every continuous symmetry of a system, there exists a conserved quantity. A continuous symmetry is one that leaves a system invariant for an arbitrarily large given transformation; for example, the rotation of a circle under any angle. Applications of Noether's theorem include conservation of energy (corresponding to invariance under time translation, i.e., it doesn't matter where you set $t=0$, Section 3.4), conservation of momentum (invariance under space translation, i.e., it doesn't matter where you put the origin, Section 4.2) and conservation of angular momentum (invariance under space rotation, i.e., it doesn't matter in which direction you choose your x-axis, Section 5.7). Similar conservation laws are found in special and general relativity, quantum mechanics, and quantum field theory. Unfortunately, even in the early 20th century, women were still excluded from most academic positions. Noether therefore initially worked for free at the university of Erlangen, getting a paid position in Göttingen in 1915 at the invitation of Hilbert and Klein, who had both been convinced by the quality of her work. Her fame grew through the 1910s and 1920s, gaining worldwide recognition. Due to her Jewish descent, she was dismissed from her academic position by the Nazi government in 1933, and moved to the United States, where she died two years later at age 53. Various institutes and scholarship programs, mostly in Germany, are now named in her honor.



Figure 3.3.1: Emmy Noether [13].

Spring Potential Energy

Like the gravitational force, the Hookean spring force (Equation 2.2.1) also depends on displacement alone, and by the same reasoning is conservative (notice the pattern?). Calculating its associated potential energy is straight-forward, and taking the equilibrium position of the spring as the reference point, we find:

$$U_s(x) = \frac{1}{2}kx^2 \quad (3.3.7)$$

The minus sign in Hooke's Law gives us a positive spring potential energy. Note that x stands for displacement here; as we only consider one-dimensional springs the 1D-version is sufficient.

General Conservative Forces

In the case of the gravitational and spring force it was easy to reason that they had to be conservative. It is also easy to see that the friction force is not conservative: if you take a longer path, you need to do more net work against friction, which you can moreover never recover as mechanical energy. For more complicated systems, especially in three dimensions, it may not be so easy to see whether a force is conservative. Fortunately, there is an easy test you can perform: if the **curl** of a force is zero everywhere, it will be a conservative force, or expressed mathematically:

$$\nabla \times \mathbf{F} = 0 \quad \Leftrightarrow \quad \oint \mathbf{F} \cdot d\mathbf{r} = 0 \quad \Leftrightarrow \quad \mathbf{F} = -\nabla U \quad (3.3.8)$$

It is straightforward to show that if a force is conservative, its curl must vanish: a conservative force can be written as the gradient of some scalar function $U(\mathbf{x})$, and $\nabla \times \nabla U(\mathbf{x}) = 0$ for any function $U(\mathbf{x})$, as you can easily check for yourself. The proof the other way around is more complicated, and can be found in advanced mechanics textbooks.

² The integral sign with the circle in Equation (3.3.1) represents an integral over a closed path.

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