15.1: The Force Four-Vector

In classical mechanics, Newton's second law relates momenta and forces, through the time derivative of the momentum. In relativity, we'll therefore simply define the force four-vector as the derivative of the energy-momentum four-vector with respect to the proper time (which gives a four-vector, as you can check easily):

$$\overline{\boldsymbol{F}} = \frac{\mathrm{d}\overline{\boldsymbol{p}}}{\mathrm{d}\tau} = \gamma(v)\left(\frac{1}{c}\frac{\mathrm{d}E}{\mathrm{d}t}, F_x, F_y, F_z\right) \tag{15.1.1}$$

We define the components of the three-force[1] F as the ('regular' or 'coordinate') time derivative of the three momentum: $\boldsymbol{F} = \mathrm{d}\boldsymbol{p}/\mathrm{d}t$, so Newton's second law holds as long as you don't change your frame of reference. Likewise, Newton's third law holds, if you consider the three-force $\boldsymbol{F}$ in a fixed frame of reference. The zeroth term of $\overline{\boldsymbol{F}}$ contains the time derivative of the energy, which we defined as the power in Section 3.1: $P = \mathrm{d}E/\mathrm{d}t$, again within the context of a fixed frame of reference.

There is a classical result that involves the force that does translate to special relativity for arbitrary reference frames: the work-energy theorem. To see how that comes about, consider a Lorentz transform from a comoving system (or instantaneous inertial frame S') to an arbitrary inertial frame S. In S', $\gamma(u) = \gamma(0) = 1$, so the space components of the force four-vector are just the components of the force three-vector (and Newton's second law holds); moreover, in this frame,

$$\frac{\mathrm{d}E'}{\mathrm{d}t} = \frac{\mathrm{d}\left(m/\sqrt{1-(u')^2/c^2}\right)}{\mathrm{d}t} = 0 \tag{15.1.2}$$

because the derivative contains a factor u', which (by choice of frame) is zero. We thus have $\overline{\boldsymbol{F}}' = (0, F_x', F_y', F_z')$. The force is a four-vector, and therefore transforms according to Equation (13.7):

$$\overline{\boldsymbol{F}} = \begin{pmatrix} F_0 \\ F_1 \\ F_2 \\ F_3 \end{pmatrix} = \begin{pmatrix} \gamma(u) & \gamma(u)\frac{u}{c} & 0 & 0 \\ \gamma(u)\frac{u}{c} & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ F_x' \\ F_y' \\ F_z' \end{pmatrix} = \begin{pmatrix} \gamma(u)\frac{u}{c}F_x' \\ \gamma(u)F_x' \\ F_y' \\ F_z' \end{pmatrix} \tag{15.1.3}$$

so, comparing the components of $\overline{\boldsymbol{F}}$ in Equations 15.1.1 and 15.1.3, we get

$$\frac{\mathrm{d}E}{\mathrm{d}t} = uF_x', \quad F_x = F_x', \quad F_y = \frac{F_y'}{\gamma(u)}, \quad F_z = \frac{F_z'}{\gamma(u)} \tag{15.1.4}$$

The longitudinal force is thus the same in both frames, but the transversal force is not! Forces thus behave differently than you might naively expect under Lorentz transformations. Moreover, the transformation is *not* symmetric: we don't get $F_y' = F_y/\gamma(-u)$ (which would indeed directly contradict Equation 15.1.3. The reason why we've lost this symmetry is that for forces, there is a special frame: that of the particle (here S'), where Newton's second law holds. In all other frames, we have to transform the forces according to Equation 15.1.4.

There is a silver lining: the zero component in Equation 15.1.3 gives us that $\mathrm{d}E = uF_x'\mathrm{d}t = F_x'\mathrm{d}x = F_x\mathrm{d}x$, which integrated gives the work-energy theorem:

$$\Delta E = F\Delta x = \Delta W.$$

As long as we stay away from the forces, work and energy will behave as we've come to expect.

[1] Some authors use f to avoid confusion with the four-force $\overline{\boldsymbol{F}}$; others use $\boldsymbol{F}$ for the three-force and $\overline{\boldsymbol{K}}$ for the four-force.