13.4: Relativistic Energy

The last three (or 'spatial') components of the momentum four-vector give us the regular components of the momentum, times the factor $\gamma(v)$. What about the zeroth (or 'temporal') component? To interpret it, we expand $\gamma(v)$, and find:

$$\begin{aligned} cp_0 &= \gamma(v) mc^2 && (13.4.1) \\ &= \left[1 + \frac{1}{2}\left(\frac{v}{c}\right)^2 + \frac{3}{8}\left(\frac{v}{c}\right)^4 + \ldots\right] mc^2 && (13.4.2) \\ &= mc^2 + \frac{1}{2} mv^2 + \ldots && (13.4.3) \end{aligned}$$

The second term in this expansion should be familiar: it's the kinetic energy of the particle. The third and higher terms are corrections to the classical kinetic energy - just like the higher-order terms in the spatial components are corrections to the classical momenta. The first term, however, is new: an extra energy contribution due to the mass of the particle. The whole term can now be interpreted as the *relativistic energy* of the particle:

$$\begin{aligned} E &= \gamma(v) mc^2 && (13.4.4) \\ &= mc^2 + K && (13.4.5) \end{aligned}$$

We can now write the zeroth component of the momentum four-vector as $p_0 = E/c$. Based on this interpretation, the four-vector is sometimes referred to as the **energy-momentum four-vector**.

A very useful relation can now easily be derived by calculating the length of the energy-momentum four-vector in two ways. On the one hand, it's given by (leaving out the square root for convenience)

$$\overline{\boldsymbol{p}} \cdot \overline{\boldsymbol{p}} = m^2 \overline{\boldsymbol{v}} \cdot \overline{\boldsymbol{v}} = m^2 c^2 \tag{13.4.6}$$

while on the other hand, we could also simply expand in the components of $\overline{\boldsymbol{p}}$ itself to get:

$$\overline{\boldsymbol{p}} \cdot \overline{\boldsymbol{p}} = \left(\frac{E}{c}\right)^2 - \boldsymbol{p} \cdot \boldsymbol{p} \tag{13.4.7}$$

where $\boldsymbol{p}$ is again the spatial part of $\overline{\boldsymbol{p}}$. Combining Equations 13.4.6 and 13.4.7, we get:

$$E^2 = m^2 c^4 + p^2 c^2 \tag{13.4.8}$$

where $p^2 = \boldsymbol{p} \cdot \boldsymbol{p}$. Equation 13.4.8 is the general form of Einstein's famous formula $E = mc^2$, to which it reduces for stationary particles (i.e. when $v = p = 0$).