9.3: Solution of the One-Dimensional Wave Equation

The one-dimensional wave Equation 9.2.6 has a surprisingly generic solution, due to the fact that it contains second derivatives in both space and time. As you can readily see by inspection, the function $q(x,t) = x - v_{\mathrm{w}}t$ is a solution, as is the same function with a plus instead of a minus sign. These functions represent waves traveling to the right (minus) or left (plus) at speed v_w. However, the shape of the wave does not matter - any function $F(q) = F(x - v_{\mathrm{w}}t)$ is a solution of 9.2.6, as is any function $G(x + v_{\mathrm{w}}t)$, and the general solution is the sum of these:

$$u(x,t) = F(x - v_{\mathrm{w}}t) + G(x + v_{\mathrm{w}}t) \tag{9.3.1}$$

To find a specific solution, we need to look at the initial conditions of the wave, i.e., the conditions at $t = 0$. Because the wave equation is second order in time, we need to specify both the initial displacement and the displacement's initial velocity, which can be functions of the position. For the most general case we write:

$$\begin{aligned} u(x,0) &= u_0(x) \\ \dot{u}(x,0) &= v_0(x) \end{aligned} \tag{9.3.2}$$

The resulting solution of the one-dimensional wave equation is known as *d'Alembert's equation*:

$$u(x,t) = \frac{1}{2}\left(u_0(x - v_{\mathrm{w}}t) + u_0(x + v_{\mathrm{w}}t)\right) + \frac{1}{2v_{\mathrm{w}}} \int_{x - v_{\mathrm{w}}t}^{x + v_{\mathrm{w}}t} v(y)\mathrm{d}y \tag{9.3.3}$$