14.3: Totally Inelastic Collision

In a totally inelastic collision, particles stick together. A possible example is the absorption of a photon by a massive particle, resulting in an increase in its mass, as well as possibly a change in its momentum. Let's consider, as an example, a particle of mass m that is initially at rest, and absorbs an incoming photon with energy E_γ. There are now three ways to calculate the energy and momentum of the particle after this collision.

Method 1

We have conservation of both (total) energy and momentum. Before the collision, the massive particle has energy $E_\mathrm{i} = mc^2$ (as it is standing still), and the total energy of the system $E_\gamma + mc^2$, which must be conserved. The total energy of the particle after the collision is $E_\mathrm{f} = \gamma(v) m_\mathrm{f} c^2$, where both the velocity v and the mass m_f are unknown. The total momentum before the collision is E_γ / c, as the particle is initially standing still (and thus has momentum zero), while after the collision it is $\gamma(v) m_\mathrm{f} v$. We thus have:

$$E_\gamma + mc^2 = \gamma(v) m_\mathrm{f} c^2 \tag{14.3.1}$$

$$E_\gamma = \gamma(v) m_\mathrm{f} v c \tag{14.3.2}$$

We thus have two equations with two unknowns (v and m_f). If we divide Equation 14.3.2 by 14.3.1, we get an expression for the final velocity v, which we can substitute back in either equation to solve for m_f (and potentially use to calculate the momentum after the collision). This is not pretty though, as we'll have complicated factors due to the presence of $\gamma(v)$.

Method 2

The four-momentum of the system is conserved during the collision. We have $\overline{\boldsymbol{p}}_\gamma$ for the photon, $\overline{\boldsymbol{p}}_1$ for the massive particle before the collision, and $\overline{\boldsymbol{p}}_\mathrm{f}$ for that particle after the collision, given by the following equations:

$$\overline{\boldsymbol{p}}_\gamma = \left(\frac{E_\gamma}{c}, \frac{E_\gamma}{c}, 0, 0\right) \tag{14.3.3}$$

$$\overline{\boldsymbol{p}}_1 = (mc, 0, 0, 0) \tag{14.3.4}$$

$$\overline{\boldsymbol{p}}_\mathrm{f} = \left(\frac{E_\mathrm{f}}{c}, p_\mathrm{f}, 0, 0\right) \tag{14.3.5}$$

From $\overline{\boldsymbol{p}}_\gamma + \overline{\boldsymbol{p}}_1 = \overline{\boldsymbol{p}}_\mathrm{f}$ we can read off two equations:

$$E_\gamma + mc^2 = E_\mathrm{f} \tag{14.3.6}$$

$$E_\gamma / c = p_\mathrm{f} \tag{14.3.7}$$

which immediately give us the final energy and momentum in terms of the initial ones. We can now find the final mass through Einstein's equation (13.16):

$$m_\mathrm{f}^2 c^4 = E_\mathrm{f}^2 - p_\mathrm{f}^2 c^2 = \left(E_\gamma + mc^2\right)^2 - E_\gamma^2 \tag{14.3.8}$$

$$= \left(E_\gamma + mc^2\right) mc^2 \tag{14.3.9}$$

This approach circumvents the use of the $\gamma(v)$ factor because we only use energy and momentum, not (classical) velocity. If we now want the velocity, we could still calculate it from the combination of m_f and either E_f or p_f, but since it was the mass and momentum we were after, there's no need to do so.

Method 3

Since the total energy-momentum four-vector is conserved in the collision, so must be its length (or the square of the length), which is trivial to calculate (remember that $\overline{\boldsymbol{p}} \cdot \overline{\boldsymbol{p}} = m^2 c^2$). We can often exploit this fact to make the maths much simpler. To see how this works, let's consider the full four-vector equation for our example: $\overline{\boldsymbol{p}}_\gamma + \overline{\boldsymbol{p}}_\mathrm{i} = \overline{\boldsymbol{p}}_\mathrm{f}$, so

$$\left(\overline{\boldsymbol{p}}_\gamma + \overline{\boldsymbol{p}}_\mathrm{i}\right) \cdot \left(\overline{\boldsymbol{p}}_\gamma + \overline{\boldsymbol{p}}_\mathrm{i}\right) = \overline{\boldsymbol{p}}_\mathrm{f} \cdot \overline{\boldsymbol{p}}_\mathrm{f} \tag{14.3.10}$$

$$\overline{\boldsymbol{p}}_\gamma \cdot \overline{\boldsymbol{p}}_\gamma + \overline{\boldsymbol{p}}_\mathrm{i} \cdot \overline{\boldsymbol{p}}_\mathrm{i} + 2 \overline{\boldsymbol{p}}_\gamma \cdot \overline{\boldsymbol{p}}_\mathrm{i} = \overline{\boldsymbol{p}}_\mathrm{f} \cdot \overline{\boldsymbol{p}}_\mathrm{f} \tag{14.3.11}$$

$$0 + m^2c^2 + 2E_\gamma m = m_{\mathrm{f}}^2 c^2 \tag{14.3.12}$$

which immediately gives us m_{f}. If we also want E_{f} or p_{f}, we can again use Equations 14.3.10and 14.3.11for the components, but if we only wanted the final mass, we're done in one step.

Note that although method 3 usually is the easiest route to your answer, it is not always - and it is a good idea to at least be aware of the other options.