7.1: Linear and Angular Velocity

We related the linear and angular velocities of a rotating object in two dimensions in Section 5.1. There, we also already stated the relation between the linear velocity vector and rotation vector in three dimensions (Equation 5.1.5):

$$\boldsymbol{v}=\boldsymbol{\omega}\times\boldsymbol{r} \tag{7.1.1}$$

It is not hard to see that this expression indeed simplifies to the scalar relationship $v=\omega r$ for rotations in a plane, with the right sign for the linear velocity. That's hardly a proof though, so let's put this on some more solid footing. Suppose $\boldsymbol{r}$ makes an angle ϕ with $\boldsymbol{\omega}$. Suppose also that it changes by $\mathrm{d}\boldsymbol{r}$ in a time interval $\mathrm{d}t$, then if we have pure rotation, $\mathrm{d}\boldsymbol{r}$ is perpendicular to both $\boldsymbol{r}$ and $\boldsymbol{\omega}$, and its magnitude is given by $|d\boldsymbol{r}|=\omega r\sin\phi\mathrm{d}t=|\boldsymbol{\omega}\times\boldsymbol{r}|\mathrm{d}t$, where ω and r are the lengths of their respective vectors. Finally, as seen from the top (i.e., looking down the vector $\boldsymbol{\omega}$), the rotation should be counter-clockwise (by definition of the direction of $\boldsymbol{\omega}$), which corresponds with the direction of $\boldsymbol{\omega}\times\boldsymbol{r}$. We thus find that both the magnitude and direction of $\boldsymbol{v}=\mathrm{d}\boldsymbol{r}/\mathrm{d}t$ indeed equal $\boldsymbol{\omega}\times\boldsymbol{r}$, and Equation 5.1.5 holds.