2.3: Equations of Motion

Now that we have set our axioms - Newton's laws of motion and the various force laws - we are ready to start combining them to get useful results, things that we did not put into the axioms in the first place but follow from them. The first thing we can do is write down equations of motion: an equation that describes the motion of a particle due to the action of a certain type of force. For example, suppose you take a rock of a certain mass m and let go of it at some height h above the ground, then what will happen? Once you've let go of the rock, there is only one force acting on the rock, namely Earth's gravity, and we are well within the regime where Equation 2.2.2 applies, so we know the force. We also know that this net force will result in a change of momentum (Equation 2.1.4), which, because the rock won't lose any mass in the process of falling, can be rewritten as Equation 2.1.5. By equating the forces we arrive at an equation of motion for the rock, which in this case is very simple:

$$m\boldsymbol{g} = m\ddot{\boldsymbol{x}} \tag{2.3.1}$$

We immediately see that the mass of the rock does not matter (Galilei was right! - though of course he was in our set of axioms, because we arrived at them by assuming he was right...). Less trivially, Equation (2.3.1) is a second-order differential equation for the motion of the rock, which means that in order to find the actual motion, we need two initial conditions - which in our present example are that the rock starts at height h and zero velocity.

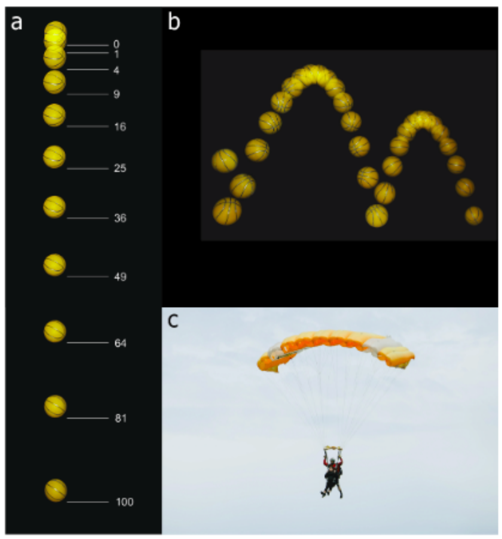

Figure 2.3.1: Dropping under the force of gravity. (a and b) A ball released from rest drops with a constant acceleration, resulting in a constantly increasing velocity. Images in (a) are taken every 0.05 s; distances are multiples of 12 mm. In (b), the trajectory of the ball resulting from repeated bounces is shown with intervals of 0.04 s [6], CC BY-SA 3.0. (c) Paragliders need to balance the force of gravity and that of drag to stop accelerating and fall at a continuous speed (known as their terminal velocity) [7], CC BY-SA 3.0.

Equation (2.3.1) is essentially one-dimensional - all motion occurs along the vertical line. Solving it is therefore straightforward - you simply integrate over time twice. The general solution is:

$$\boldsymbol{x}(t) = \boldsymbol{x}(0) + \boldsymbol{v}(0)t + \frac{1}{2}\boldsymbol{g}t^2 \tag{2.3.2}$$

which with our boundary conditions becomes

$$\boldsymbol{x}(t) = \left(h - \frac{1}{2}gt^2\right)\hat{\boldsymbol{z}} \tag{2.3.3}$$

where g is the magnitude of g (which points down, hence the minus sign). Of course Equation 2.3.3 breaks down when the rock hits the ground at $t = \sqrt{\frac{2h}{g}}$, which is easily understood because at that point gravity is no longer the only force acting on it.

We can also immediately write down the equation of motion for a mass on a spring (no gravity at present), in which the net force is given by Hooke's law. Equating that force to the net force in Newton's second law of motion gives:

$$-k\boldsymbol{x}(t) = m\ddot{\boldsymbol{x}}(t) \tag{2.3.4}$$

Of course, we find another second-order differential equation, so we again need the initial position and velocity to specify a solution. The general solution of Equation 2.3.4 is a combination of sines and cosines, with a frequency $\omega=\sqrt{\frac{k}{m}}$ (as we already know from the dimensional analysis in Section 1.2):

$$\boldsymbol{x}(t)=\boldsymbol{x}(0)\cos(\omega t)+\frac{\boldsymbol{v}(0)}{\omega}\sin(\omega t) \tag{2.3.5}$$

We'll study this case in more detail in Section 8.1. In general, the force in Newton's second law may depend on time and position, as well as on the first derivative of the position, i.e., the velocity. For the special case that it depends on only one of the three variables, we can write down the solution formally, in terms of an integral over the force. These formal solutions are given in Section 2.6. To see how they work in practice, let's consider a slightly more involved problem, that of a stone falling with drag.

Example 2.3.1: Falling Stone with Drag

Suppose we have a spherical stone of radius a that you drop from a height h at t=0. At what time, and with which velocity, will the stone hit the ground?

Solution

We already solved this problem in the simple case without drag above, but now let's include drag. There are then two forces acting on the stone: gravity (pointing down) with magnitude m_g, and drag (pointing in the direction opposite the motion, in this case up) with magnitude $6\pi\eta av=bv$, as given by Stokes' law (Equation 2.2.5). Our equation of motion is now given by (with x as the height of the particle, and the downward direction as positive):

$$m\ddot{x}=-b\dot{x}+mg \tag{2.3.6}$$

We see that our force does not depend on time or position, but only on velocity - so we have case 3 of Appendix 2.6. We could invoke either Equation (2.33) or (2.34) to write down a formal solution, but there is an easier way, which will allow us to evaluate the relevant integrals without difficulty. Since our equation of motion is linear, we know that the sum of two solutions is again a solution. One of the terms on the right hand side of Equation (2.19) is constant, which means that our equation is not homogeneous (we can rewrite it to $m\ddot{x}+b\dot{x}=mg$ to see this), so a useful thing to do is to split our solution in a homogeneous and a particular part. Rewriting our equation in terms of $v=\dot{x}$ instead of x, we get $m\dot{v}+bv=mg$, from which we can immediately get a particular solution: $v_{\mathrm{p}}=\frac{mg}{b}$, as the time derivative of this constant v_{p} vanishes. Subtracting v_{p},we are left with a homogeneous equation: $m\dot{v}_{\mathrm{h}}+bv_{\mathrm{h}}$, which we now solve by separation of variables. First we write˙ $\dot{v}_{\mathrm{h}}=\frac{\mathrm{d}v_{\mathrm{h}}}{\mathrm{d}t}$, then re-arrange so that all factors containing v_{h} are on one side and all factors containing t are on the other, which gives $-(\frac{m}{b})(\frac{1}{v_h})dv_h=dt$. We can now integrate to get:

$$-\frac{m}{b}\int_{\nu_0}^{\nu}\frac{1}{v'}\mathrm{d}v'=-\frac{m}{b}\log\left(\frac{v}{v_0}\right)=t-t_0 \tag{2.3.7}$$

which is an example of Equation (2.33). After rearranging and setting $t_0=0$:

$$v_{\mathrm{h}}(t)=v_0\exp\left(-\frac{b}{m}t\right) \tag{2.3.8}$$

Note that this homogeneous solution fits our intuition: if there is no extra force on the particle, the drag force will slow it down exponentially. Also note that we didn't set $v_0=0$, as the homogeneous solution does not equal the total solution. Instead v_0 is an integration constant that we'll need to set once we've written down the full solution, which is:

$$v(t)=v_{\mathrm{h}}(t)+v_{\mathrm{p}}(t)=v_0\exp\left(-\frac{b}{m}t\right)+\frac{mg}{b} \tag{2.3.9}$$

Now setting $v(0)=0$ gives $v_0=-\frac{mg}{b}$, so

$$v(t)=\frac{mg}{b}\left[1-\exp\left(-\frac{b}{m}t\right)\right] \tag{2.3.10}$$

To get x(t), we simply integrate v(t) over time, to get:

$$x(t)=\frac{mg}{b}\left[t+\frac{m}{b}\exp\left(-\frac{b}{m}t\right)\right] \tag{2.3.11}$$

We can find when the stone hits the ground by setting x(t)=h and solving for t; we can find how fast it is going at that point by substituting that value of t back into v(t).