11.4: Rapidity and Repeated Lorentz Transformations

As stated at the end of section 11.2, the composition of two Lorentz transformations is again a Lorentz transformation, with a velocity boost given by the 'relativistic addition' equation (11.3.1) (you're asked to prove this in problem 11.1). You could of course repeat this process for successive transformations, but the repeated addition of velocities quickly leads to impractical expressions. You could also investigate whether the combination of a Lorentz transformation in the x direction and one in the y direction again gives a Lorentz transformation. The answer is, in general, no: it is the combination of a Lorentz transformation and a rotation. In some sense, we can also consider Lorentz transformations themselves as 'rotations' in (4-dimensional) spacetime. We'll discuss spacetime in more detail in the next two sections. Here, we'll work out a different way of writing the Lorentz transformations that shows their relation to rotations. As a bonus, it will allow us to easily calculate the speed of the n the Lorentz transformation (starting from rest, all in the positive x direction).

Let us again write the Lorentz transformation as a matrix. Using the $\gamma(u)$ factor and introducing $\beta(u)=u/c$, we have

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \gamma(u)\begin{pmatrix} 1 & \beta \\ \beta & 1 \end{pmatrix}\begin{pmatrix} x' \\ ct' \end{pmatrix}, \tag{11.4.1}$$

again beautifully illustrating the symmetry between time and space. We now define[1] the rapidity ϕ by

$$\tanh(\phi)=\beta(u)=\frac{u}{c}. \tag{11.4.2}$$

We then have

$$\gamma(u)=\frac{1}{\sqrt{1-\beta(u)^2}}=\frac{1}{\sqrt{1-\tanh^2\phi}}=\cosh\phi, \tag{11.4.3}$$

$$\gamma(u)\beta(u)=\frac{\beta(u)}{\sqrt{1-\beta(u)^2}}=\frac{\tanh\phi}{\sqrt{1-\tanh^2\phi}}=\sinh\phi. \tag{11.4.4}$$

Substituting these expressions back into the Lorentz transformations (11.4.1), we get

$$\begin{pmatrix} x \\ ct \end{pmatrix} = \begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}\begin{pmatrix} x' \\ ct' \end{pmatrix} \tag{11.4.5}$$

which closely resembles the expression for a rotation.

We can likewise rewrite the equation for velocity addition in terms of the rapidity. Suppose we want to add velocities u and v, let the resulting velocity be w, then:

$$\tanh(\phi_w)=\frac{w}{c}=\frac{u/c+v/c}{1+uv/c^2}=\frac{\beta_u+\beta_v}{1+\beta_u\beta_v}=\frac{\tanh(\phi_u)+\tanh(\phi_v)}{1+\tanh(\phi_u)\tanh(\phi_v)}=\tanh(\phi_u+\phi_v) \tag{11.4.6}$$

(in problem 11.9 you get to prove the last equality). We thus find a very simple addition rule for the rapidities:

$$\phi_w=\phi_u+\phi_v. \tag{11.4.7}$$

Suppose now that we have a (stationary) reference system S, a system S' that moves with speed u (and rapidity ϕ) with respect to S, a system S'' that moves with speed u with respect to S', and so on. By equation (11.4.7), the system $S^{(n)'}$ then moves with rapidity $\phi_n=n\phi$ with respect to S. To find the relative speed u_n at which $S^{(n)'}$ moves in S, we invert the definition of the rapidity, which gives us

$$\phi=\frac{1}{2}\log\left(\frac{1+\beta}{1-\beta}\right), \tag{11.4.8}$$

so u_n is given by

$$u_n=\frac{1-\left(\frac{1-u/c}{1+u/c}\right)^n}{1+\left(\frac{1-u/c}{1+u/c}\right)^n}c. \tag{11.4.9}$$

Note that equation (11.4.9) provides another proof of the statement that no inertial object can move at the speed of light: for arbitrary large values of n, u_n remains less than c, so no matter how many velocity boosts you give your massive particle, you can never make it move at the speed of light, let alone exceed that speed.

[1]If you're unfamiliar with the hypergeometric functions: they're defined like the trigonometric functions as combinations of powers of e, except that we drop the complex number i, so we have

$$\sinh(\phi)=\frac{1}{2}\left(e^{\phi}-e^{-\phi}\right), \quad \cosh(\phi)=\frac{1}{2}\left(e^{\phi}+e^{-\phi}\right), \quad \tanh(\phi)=\frac{\sinh(\phi)}{\cosh(\phi)}=\frac{e^{\phi}-e^{-\phi}}{e^{\phi}+e^{-\phi}} \tag{11.4.10}$$

Note that $d\sinh\phi/d\phi=\cosh\phi$ like the trigonometric counterpart, but $d\cosh\phi/d\phi=\sinh\phi$, so no minus sign in this case.
