4.3: Reference Frames

Center of Mass Frame

The center of mass need not be fixed in space, so it can have a nonzero velocity, which of course is simply given by $v_{cm} = \dot{r}_{cm}$. For each of the particles in a multi-particle system, we can decompose its velocity by writing it as the sum of the center of mass velocity and a velocity relative to the center of mass:

$$v_\alpha = v_{\text{cm}} + v_{\alpha,\text{rel}} \tag{4.3.1}$$

In many applications, the information is in the velocity component relative to the center of mass. After all, conservation of momentum implies that for a system with no external forces acting on it, the center of mass velocity cannot change, even if all the individual momenta do change (as happens in collisions). Therefore, it is often convenient to analyze your system in a frame that moves with the center of mass, known (unsurprisingly), as the center of mass frame. In this frame, the center of mass velocity is identically zero, and again because of conservation of momentum, all other velocities in this frame must sum to zero. The 'real-world' frame with nonzero center of mass velocity is referred to as the lab frame.

Galilean Transformations and Inertial Frames

As Equation (4.3.1) shows, if you know a particle's velocity in the center of mass frame, you can easily calculate the velocity in the lab frame by adding the velocity of the center of mass. Going the other way, to calculate the velocity in the center of mass frame, you subtract v_{cm} from the velocity in the lab frame. Moreover, if the center of mass moves at constant velocity, we can also easily relate positions in both frames. If we denote coordinates in the lab frame by r and those in the center of mass frame by r', we readily obtain:

$$\boldsymbol{r} = \boldsymbol{r}' + \boldsymbol{v}_{\text{cm}} t \tag{4.3.2}$$

$$\boldsymbol{v} = \boldsymbol{v}' + \boldsymbol{v}_{\text{cm}} \tag{4.3.3}$$

Equation (4.3.2) is an example of a Galilean transformation between frames of reference (here the lab frame and the center of mass frame). It actually holds for any pair of reference frames that move with constant velocity with respect to each other. Such frames of reference are known as inertial frames if Newton's first law of motion holds in them; by Newton's second law, if one of the frames is an inertial frame, then the one obtained from it by a Galilean transformation (i.e., one moving at constant velocity with respect to the first frame) is also an inertial frame. The reason for this is that a constant velocity plays no role in Newton's second law, as it relates the derivative of the velocity (i.e., the acceleration) to a force. Consequently, not only is Newton's first law of motion valid in both inertial frames - all laws of physics are the same in two such frames. This fact is known as the principle of relativity. It does not apply to, for example, frames that rotate with respect to each other, as we'll see in Chapter 7. Moreover, although the principle of relativity is universally valid (it is in fact one of the two basic assumptions behind Einstein's theory of relativity), the Galilean transformations are not.They break down at velocities that approach the speed of light, as we'll explore in detail in Part II.

Kinetic Energy of a Collection of Particles

We've established above that both the total momentum and energy are conserved in closed systems, but the components can of course change. Momentum can be transferred from one particle to another, and so can (kinetic) energy; moreover kinetic energy can be generated from potential energy. Unfortunately, unlike momentum, the kinetic energy of a collection of particles does not equal that of the center of mass - this is because kinetic energy depends quadratically rather than linearly on the velocity. The total kinetic energy does of course equal the sum of the individual particles' kinetic energies. Moreover, here too the decomposition (4.3.1) is useful:

$$K = \sum_\alpha \frac{1}{2} m_\alpha \left(v_{\text{cm}} + v_{\alpha,\text{rel}}\right) \cdot \left(v_{\text{cm}} + v_{\alpha,\text{rel}}\right) \tag{4.3.4}$$

$$= \sum_\alpha \frac{1}{2} m_\alpha v_{\text{cm}}^2 + \sum_\alpha m_\alpha v_{\text{cm}} \cdot v_{\alpha,\text{rel}} + \sum_\alpha \frac{1}{2} m_\alpha v_{\alpha,\text{rel}}^2 \tag{4.3.5}$$

$$= \quad K_{\text{cm}} + K_{\text{int}} \tag{4.3.6}$$

Because the center of mass velocity is the same for all particles, it can be taken out of the sum in equation (4.3.5). Therefore, the first term equals $\frac{1}{2} M v_{\text{cm}}^2 = K_{\text{cm}}$, and in the second term we end up with the sum overall velocities relative to the center of mass - which is zero. We find that the total kinetic energy of a collection of particles equals the kinetic energy of the center of mass plus the total internal kinetic energy - which can change in both collisions and when potential energy gets converted into kinetic energy (or vice versa).