5.1: Rotation Basics

So far, we've been looking at motion that is easily described in Cartesian coordinates, often moving along straight lines. Such kind of motion happens a lot, but there is a second common class as well: rotational motion. It won't come as a surprise that to describe rotational motion, polar coordinates (or their 3D counterparts the cylindrical and spherical coordinates) are much handier than Cartesian ones[1]. For example, if we consider the case of a disk rotating with a uniform velocity around its center, the easiest way to do so is to specify over how many degrees (or radians) a point on the boundary advances per second. Compare this to linear motion - that is specified by how many meters you advance in the linear direction per second, which is the speed (with dimension L/T). The change of the angle per second gives you the angular speed!, where a counterclockwise rotation is taken to be in the positive direction. The angular speed has dimension 1/T, so it is a frequency. It is measured in degrees per second or radians per second. If the angle at a point in time is denoted by $\theta(t)$, then obviously $\omega = \dot{\theta}$, just like $v = \dot{x}$ in linear motion.

In three dimensions, ω becomes a vector, where the magnitude is still the rotational speed, and the direction gives you the direction of the rotation, by means of a right-hand rule: rotation is in the plane perpendicular to!, and in the direction the fingers of your right hand point if your thumb points along ω (this gives ω in the positive $\hat{z}$ direction for rotational motion in the xy plane).Going back to 2D for the moment, let's call the angular position $\theta(t)$, then

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \dot{\theta} \tag{5.1.1}$$

If we want to know the positionrin Cartesian coordinates, we can simply use the normal conversion from polar to Cartesian coordinates, and write

$$\boldsymbol{r}(t) = r\cos(\omega t)\hat{\boldsymbol{x}} + r\sin(\omega t)\hat{\boldsymbol{y}} = r\hat{\boldsymbol{r}} \tag{5.1.2}$$

where r is the distance to the origin. Note that r points in the direction of the polar unit vector $\hat{r}$. Equation 5.1.2 gives us an interpretation of ω as a frequency: if we consider an object undergoing uniform rotation (i.e., constant radius and constant velocity), in its x and y-directions it oscillates with frequency ω. As long as our motion remains purely rotational, the radial distance r does not change, and we can find the linear velocity by taking the time derivative of 5.1.2:

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) = -\omega r\sin(\omega t)\hat{x} + \omega r\cos(\omega t)\hat{y} = \omega r\hat{\boldsymbol{\theta}} \tag{5.1.3}$$

so in particular we have $v = \omega r$. Note that both v and ω denote instantaneous speeds, and Equation 5.1.2 only holds when ω is constant. However, the relation $v = \omega r$ always holds. To see that this is true, express θ in radians, $\theta = \frac{s}{r}$, where s is the distance traveled along the rotation direction. Then

$$\omega = \frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{1}{r}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{v}{r} \tag{5.1.4}$$

In three dimensions, we find

$$v = \omega \times r \tag{5.1.5}$$

where r points from the rotation axis to the rotating point.

Unlike in linear motion, in rotational motion there is always acceleration, even if the rotational velocity ω is constant. This acceleration originates in the fact that the direction of the (linear) velocity always changes as points revolve around the center, even if its magnitude, the net linear speed, is constant. In that special case, taking another derivative gives us the linear acceleration, which points towards the center of rotation:

$$\boldsymbol{a}(t) = \ddot{\boldsymbol{r}}(t) = -\omega^2 r\cos(\omega t)\hat{x} + \omega^2 r\sin(\omega t)\hat{y} = -\omega^2 r\hat{\boldsymbol{r}} \tag{5.1.6}$$

In Section 5.2 below we will use Equation 5.1.6 in combination with Newton's second law of motion to calculate the net centripetal force required to maintain rotation at a constant rate. Of course the angular velocity ω need not be constant at all. If it is not, we can define an angular acceleration by taking its time derivative:

$$\alpha = \frac{\mathrm{d}\omega}{\mathrm{d}t} = \ddot{\theta} \tag{5.1.7}$$

or in three dimensions, where ω is a vector:

$$\boldsymbol{\alpha}=\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t} \tag{5.1.8}$$

Note that when $\boldsymbol{\alpha}$ is parallel to $\boldsymbol{\omega}$, it simply represents a change in the rotation rate (i.e., a speeding up/slowing down of the rotation), but when it is not, it also represents a change of the plane of rotation. In both two and three dimensions, a change in rotation rate causes the linear acceleration to have a component in the tangential direction in addition to the radial acceleration (5.1.6). The tangential component of the acceleration is given by the derivative of the linear velocity:

$$\boldsymbol{a}_{\mathrm{t}}=\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t}=r\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}t}=r\boldsymbol{\alpha} \tag{5.1.9}$$

In two dimensions, $\boldsymbol{a}_{\mathrm{t}}$ points along the $\pm\hat{\boldsymbol{\theta}}$ direction.

Naturally, there are even more complicated possibilities - the radius of the rotational motion can change as well. We'll look at that case in more detail in Chapter 6, but first we consider 'pure' rotations, where the distance to the rotation axis is fixed.

[1] *If you need a refresher on polar coordinates, or are unfamiliar with polar basis vectors, check out appendix A.2.*