4.8: Elastic Collisions in the COM Frame

Equations (4.7.7) and (4.7.8) give the final velocities of two particles after a totally elastic collision. We did the calculation in the lab frame, i.e., from the point of view of a stationary observer. We could of course just as well have done the calculation in the center-of-mass (COM) frame of Section 4.3. Within that frame, as we'll see below, the relation between the initial and final velocities in an elastic collision is much simpler than in the lab frame. We will use Equation (4.3.1) to calculate velocities in the COM frame. For notational simplicity, we'll work in one dimension, and use an overbar to indicate velocities in the COM frame, so we get

$$v_i = v_{\text{cm}} + \bar{v}_i \tag{4.8.1}$$

for each particle i. The velocity of the center of mass is simply the time derivative of its position. For two particles, it is given by

$$v_{\text{cm}} = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \tag{4.8.2}$$

The velocities of the two particles in the COM frame is then

$$\bar{v}_1 = v_1 - v_{\text{cm}} = \frac{m_2}{m_1+m_2}(v_1 - v_2) = \frac{m_2}{m_1+m_2} v_{\text{rel}} \tag{4.8.3}$$

$$\bar{v}_2 = v_2 - v_{\text{cm}} = \frac{m_1}{m_1+m_2}(v_2 - v_1) = -\frac{m_1}{m_1+m_2} v_{\text{rel}} \tag{4.8.4}$$

where $v_{\text{rel}} = v_1 - v_2 = \bar{v}_1 - \bar{v}_2$ is the *relative velocity* of the two particles[3]. Note that it does not matter whether we calculate the relative velocity in the lab or COM frame. Equations (4.8.3) and (4.8.4) have a nice symmetry in their velocity components, but not in their mass components. The symmetry is more complete if instead of velocities, we consider momenta in the COM frame, where $\bar{p} = m\bar{v}$:

$$\bar{p}_1 = m_1 \bar{v}_1 = \frac{m_1 m_2}{m_1+m_2} v_{\text{rel}} = \mu v_{\text{rel}} \tag{4.8.5}$$

$$\bar{p}_2 = m_2 \bar{v}_2 = -\frac{m_1 m_2}{m_1+m_2} v_{\text{rel}} = -\mu v_{\text{rel}} \tag{4.8.6}$$

where we introduced a new variable μ, the *reduced mass*

$$\mu = \frac{m_1 m_2}{m_1+m_2} \tag{4.8.7}$$

Clearly the total momentum in the center of mass frame is zero[4] (as it should be), both before and after a collision, and is thus conserved. To find out what happens with the relative velocity in an elastic collision, we invoke conservation of kinetic energy, which we calculate using $K = \frac{1}{2} m v^2 = \frac{p^2}{2m}$:

$$\frac{\bar{p}_{1,\text{i}}^2}{2m_1} + \frac{\bar{p}_{2,\text{i}}^2}{2m_2} = \frac{\bar{p}_{1,\text{f}}^2}{2m_1} + \frac{\bar{p}_{2,\text{f}}^2}{2m_2} \tag{4.8.8}$$

$$\frac{\mu^2 \bar{v}_{\text{rel, i}}^2}{2m_1} + \frac{\mu^2 \bar{v}_{\text{rel, i}}^2}{2m_2} = \frac{\mu^2 \bar{v}_{\text{rel, f}}^2}{2m_1} + \frac{\mu^2 \bar{v}_{\text{rel, f}}^2}{2m_2} \tag{4.8.9}$$

$$\bar{v}_{\text{rel, i}}^2 = \bar{v}_{\text{rel, f}}^2 \tag{4.8.10}$$

We find that either $\bar{v}_{\text{rel,f}} = \bar{v}_{\text{rel,i}}$, in which case there would be no collision (as nothing changes), or $\bar{v}_{\text{rel,f}} = -\bar{v}_{\text{rel,i}}$, which means that in an elastic collision in the COM frame, the velocities (and momenta) of the colliding particles reverse. We get:

$$\bar{v}_{1,\text{f}} = -\bar{v}_{1,\text{i}} = -\frac{m_2}{m_1+m_2}(v_{1,\text{i}} - v_{2,\text{i}}) = -\frac{m_2}{m_1+m_2} v_{\text{rel,i}} \tag{4.8.11}$$

$$\bar{v}_{2,\text{f}} = -\bar{v}_{2,\text{i}} = -\frac{m_1}{m_1+m_2}(v_{2,\text{i}} - v_{1,\text{i}}) = \frac{m_1}{m_1+m_2} v_{\text{rel,i}} \tag{4.8.12}$$

We can of course transform these expressions back to the lab frame by adding the center of mass velocity (4.8.2), which gives Equations (4.7.7) and (4.7.8), the same as our calculation in the lab frame.

[3] *Equations (4.8.3) and (4.8.4) hold in multiple dimensions as well.*

[4] *Since the total momentum in the COM frame is zero, the frame is sometimes also referred to as the zero-momentum frame.*