6.E: General Planar Motion (Exercises)

6.1 A particularly useful orbit for satellites is the *geosynchronous* one: the orbit in which the satellite rotates around the Earth in exactly one day, so with respect to the ground, it is always in the same position. Find the altitude (i.e., distance above the Earth's surface) for a circular geosynchronous orbit.

6.2 Kepler's laws apply to the case that an object with relatively small mass m orbits an object with large mass M, which we assume stays fixed. Technically, this is incorrect: both objects rotate about their common center of mass. Fortunately, we can still use the expressions derived in this section, with a small modification. To see how this works, we write down the equations of motion for the two objects, due to the force they exert on each other:

$$\ddot{x}_1 = -\frac{1}{m}\boldsymbol{F}(\boldsymbol{r}), \quad \ddot{\boldsymbol{x}}_2 = \frac{1}{M}\boldsymbol{F}(\boldsymbol{r}) \tag{6.E.1}$$

where $\boldsymbol{x}_1$ is the position of the object with mass m, $\boldsymbol{x}_2$ that of the object with mass M, and $\boldsymbol{r} = \boldsymbol{x}_2 - \boldsymbol{x}_1$ their separation. We denote the position of the center of mass of the system by $\boldsymbol{R}$.

a. As there is no external force acting on the system, the total momentum is conserved and therefore the center of mass cannot accelerate. Argue that this implies that

$$(m+M)\ddot{R} = 0 \tag{6.E.2}$$

and combine Equations 6.E.1 and 6.E.2 into an expression for $\boldsymbol{R}$ in terms of $\boldsymbol{x}_1$, $\boldsymbol{x}_2$, and the masses of the two objects.

b. From Equation 6.E.1, also derive an equation of motion for the separation $\boldsymbol{r}$ between the two objects. The equations you found in (a) and (b) together are equivalent to the equations of motion in 6.E.1, but only one is a differential equation, and they are uncoupled: we don't need to know the position of the center of mass to find the separation, and vice versa.

c. Show that you can re-write the equation of motion for the separation between the two objects as $\boldsymbol{F}(\boldsymbol{r}) = \mu\ddot{\boldsymbol{r}}$, where μ is the reduced mass that we also encountered when studying collisions in the center of mass frame, Equation 4.8.7, given by

$$\mu = \frac{mM}{m+M} \tag{6.E.3}$$

Note that solving the final equation for the separation $\boldsymbol{r}$ is entirely equivalent to solving the equation of motion of a single particle under the action of a central force, with the modification that the mass of the particle is replaced by the reduced mass. For the case that $m << M$, the reduced mass is approximately equal to m.

d. Calculate the reduced mass of the Earth-Moon two body problem. Can we state that the Moon revolves around the Earth?

e. Nowhere in the derivations in this problem did we assume that $m << M$. The same rules apply to any two objects. Consider the opposite limit: two objects (these might for instance be binary stars) of equal mass M that rotate around their common center of mass. Show that for this case, for circular orbits the orbital period is given by

$$T^2 = \frac{2\pi^2 d^3}{GM} \tag{6.E.4}$$

where d is the distance between the two objects.

6.3 A student with mass 65.0 kg stands at the center of a simple merry-go-round that consists of a large disk of radius 1.5 m and mass 25 kg and is making a full rotation every 2.0 s. The student walks out to a distance of 0.50 m from the center.

a. Find the rotational frequency of the merry-go-round with the student at this point.
b. What are the forces acting on the student at this point?