1.E: Introduction to Classical Mechanics (Exercises)

1.1 **Harmonic oscillator revisited** Suppose you have a small object of mass m, which you attach to a spring of spring constant k (which itself is fixed to a wall at its other end, figure 1.1). Above, we derived an expression for the frequency of oscillation of the mass. We also argued that it should be the same for both a horizontally-positioned and a vertically-positioned oscillator, i.e., that the frequency is independent of the gravitational acceleration g.

a. Show that the frequency of oscillation is also independent of its amplitude A (the maximum distance from the equilibrium position the oscillating mass reaches).
b. Use dimensional analysis to derive an expression for the maximum velocity of the mass during the oscillation, as a function of m, k, and A.

1.2 In physics, we assume that quantities like the speed of light (c) and Newton's gravitational constant (G) have the same value throughout the universe, and are therefore known as physical constants. A third such constant from quantum mechanics is Planck's constant ($\hbar$, an h with a bar). In high-energy physics, people deal with processes that occur at very small length scales, so our regular SI-units like meters and seconds are not very useful. Instead, we can combine the fundamental physical constants into different basis values.

a. Combine c, G and $\hbar$ into a quantity that has the dimensions of length.
b. Calculate the numerical value of this length in SI units (this is known as the Planck length). You can find the numerical values of the physical constants in appendix B
c. Similarly, combine c, G and $\hbar$ into a quantity that has the dimensions of energy (indeed, known as the Planck energy) and calculate its numerical value.

1.3 **Reynolds numbers** Physicists often use dimensionless quantities to compare the magnitude of two physical quantities. Such numbers have two major advantages over quantities with numbers. First, as dimensionless quantities carry no units, it does not matter which unit system you use, you'll always get the same value. Second, by comparing quantities, the concepts 'big' and 'small' are well-defined, unlike for quantities with a dimension (for example, a distance may be small on human scales, but very big for a bacterium). Perhaps the best known example of a dimensionless quantity is the Reynolds number in fluid mechanics, which compares the relative magnitude of inertial and drag forces acting on a moving object:

$$Re = \frac{\text{inertial forces}}{\text{drag forces}} = \frac{\rho v L}{\eta} \tag{1.E.1}$$

where ρ is the density of the fluid (either a liquid or a gas), v the speed of the object, L its size, and η the viscosity of the fluid. Typical values of the viscosity are 1.0 $mPa \cdot s$ for water, 50 $mPa \cdot s$ for ketchup, and 1.0 $\mu Pa \cdot s$ for air.

a. Estimate the typical Reynolds number for a duck when flying and when swimming (you may assume that the swimming happens entirely submerged). NB: This will require you looking up or making educated guesses about some properties of these birds in motion. In either case, is the inertial or the drag force dominant?
b. Estimate the typical Reynolds number for a swimming bacterium. Again indicate which force is dominant.
c. Oil tankers that want to make port in Rotterdam already put their engines in reverse halfway across the North sea. Explain why they have to do so.
d. Express the Reynolds number for the flow of water through a (circular) pipe as a function of the radius R of the pipe, the volumetric flow rate (i.e., volume per second that flows through the pipe) Q,and the kinematic viscosity $v \equiv \frac{\eta}{\rho}$.
e. For low Reynolds number, fluids will typically exhibit so-called laminar flow, in which the fluid particles all follow paths that nicely align (this is the transparent flow of water from a tap at low flux). For higher Reynolds number, the flow becomes turbulent, with many eddies and vortices (the white-looking flow of water from the tap you observe when increasing the flow rate). The maximum Reynolds number for which the flow is typically laminar is experimentally measured to be about 2300. Estimate the flow velocity and volumetric flow rate of water from a tap with a 1.0 cm diameter in the case that the flow is just laminar.

1.4 The escape velocity of a planet is defined as the minimal initial velocity an object must have to escape its gravitational pull completely (and thus go fast enough to defy the rule that 'what goes up must come down').

a. From Newton's universal law of gravitation (equation 2.9), determine the dimension of the gravitational constant G.
b. Use dimensional analysis to show that for a planet of mass M and radius R, the escape velocity scales as $v \sim \sqrt{\frac{MG}{R}}$.

c. A more detailed calculation shows that in fact we have $v_{\text{escape}} = \sqrt{\frac{2GM}{R}}$. Express this value of the escape velocity in terms of the (mass) density ρ of the planet, instead of its mass M.

d. The average density of the moon is about $\frac{6}{10}^{\text{th}}$ that of the Earth, and the Moon's radius is about $\frac{11}{40}$ times that of the Earth. From these numbers and your answer at (c), calculate the ratio of the escape velocities of the Moon and the Earth, and explain why the Apollo astronauts needed a huge rocket to get to the Moon, and only a tiny one to get back.