15.3: Relativistic Waves

We've seen that in special relativity, space and time are intimately coupled. There is a classical phenomenon for which this is also the case: the waves we discussed in Chapter 9. In Section 9.1 we introduced the sinusoidal wave, described by Equation 9.1.1:

$$u(x,t) = A\cos(k\cdot x - \omega t) \tag{15.3.1}$$

In Equation 15.3.1 we made the wave a function of all three space coordinates, introducing a *wave vector* $\boldsymbol{k}$ rather than just the wave number k of equation (9.1.1). The magnitude of the wave vector is simply that of the wave number, $|\boldsymbol{k}| = k = 2\pi/\lambda$, while its direction represents the direction the (traveling) wave is moving in. When written like Equation 15.3.1, you may guess that there exists a wave four-vector combining the temporal and spatial properties of the wave, and you would be correct. If we define

$$\overline{\boldsymbol{k}} = (\omega/c, \boldsymbol{k}) \tag{15.3.2}$$

then the argument of the cosine in equation (15.11), i.e., the phase $\phi(\boldsymbol{x},t)$ of the wave at the given point in space and time, is given by

$$\phi(\boldsymbol{x},t) \equiv \boldsymbol{k}\cdot\boldsymbol{x} - \omega t = -(\overline{\boldsymbol{k}}\cdot\overline{\boldsymbol{x}}) \tag{15.3.3}$$

We've already shown that dot products of two four-vectors are invariant under Lorentz transformations; as ϕ is a scalar (and thus invariant) and $\overline{\boldsymbol{x}}$ a four-vector, it follows that $\overline{\boldsymbol{k}}$ is indeed also a four-vector.

The main application of relativistic waves is light itself - in its occurrence as a wave. The wave four-vector of a light beam traveling in the positive x-direction is given by

$$\overline{\boldsymbol{k}} = (k, k, 0, 0) \tag{15.3.4}$$

where we used that for light, $\omega = ck$ (Equation 9.1.2). Unsurprisingly, this looks exactly like Equation 14.1.3 for the four-momentum of a photon - especially because that the energy of a photon is E $E = hc/\lambda = hck/2\pi$. Up to a physical constant, the wave and momentum four-vectors of light are thus identical:

$$\overline{\boldsymbol{p}}_{\text{photon}} = \frac{h}{2\pi}\overline{\boldsymbol{k}}_{\text{photon}} = \hbar\overline{\boldsymbol{k}}_{\text{photon}} \tag{15.3.5}$$

The combination $h/2\pi$ occurs so often that it got its own symbol, $\hbar$ ('h-bar'). Note that Equation 15.3.5 holds for light only.

You might expect that there is little more to say about light. After all, the light postulate ensures that the speed of light will be the same for all observers. Yet, different observers can observe the same ray of light (or the same photon) differently: although its speed is invariant, its frequency (and thus its color, as well as its momentum) is not! To see what happens, let us start with a stationary light source emitting rays in the positive x-direction in some system S, so the wave four-vector is given by Equation 15.3.4. We now Lorentz transform to a system S' moving with speed u in the x-direction. The wave four-vector as measured by an observer in S' is simply the Lorentz transform of Equation 15.3.4:

$$\overline{\boldsymbol{k}}' = \begin{pmatrix} \gamma(u) & -\gamma(u)\frac{u}{c} & 0 & 0 \\ -\gamma(u)\frac{u}{c} & \gamma(u) & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} k \\ k \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \gamma(u)k\left(1-\frac{u}{c}\right) \\ \gamma(u)k\left(1-\frac{u}{c}\right) \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} k' \\ k' \\ 0 \\ 0 \end{pmatrix} \tag{15.3.6}$$

We find that the moving observer still sees the light moving in the positive x-direction with speed c, but with a different wave number k', and thus a different frequency $\omega' = ck'$, given by

$$\omega' = \gamma(u)\left(1-\frac{u}{c}\right)\omega = \sqrt{\frac{1-u/c}{1+u/c}}\,\omega \tag{15.3.7}$$

Equation 15.3.7 gives the relativistic Doppler effect: a shift in observed frequency due to the motion of the observer, just as we found for sound in Section 9.7. In fact, Equation 15.3.7 reduces to equation (9.7.2) for small velocities $u << c$. In addition to the 'sound effect' where we account for the stretching or compression of the waves due to the motion of the observer, the relativistic Doppler effect also accounts for the time dilation between the two observers (it can also be derived by combining these two effects, as is done in many textbooks, see Problem 15.2.b). Unlike for sound, there is also a transverse relativistic Doppler effect (entirely due to the time dilation), for which we can find the expression by replacing the ray traveling in the positive x-direction in Equation 15.3.6 with one traveling in the positive y-direction.