3.1: Work

How much work do you need to do to move a box? Well, that depends on two things: how heavy the box is,and how far you have to move it. Multiply the two, and you've got a good measure of how much work will be required. Of course, work can be done in other contexts as well - pulling a spring from equilibrium, or cycling against the wind. In each case, there's a force and a displacement. To be fair, we will only count the part of the force that is in the direction of the displacement (when cycling, you don't do work due to the fact that there'sa gravitational force pulling you down, since you don't move vertically; you do work because there's a drag force due to your moving through the air). We define work as the product of the component of the force in the direction of the displacement, times the displacement itself. We calculate this component by projecting the force vector on the displacement vector, using the dot product (see Appendix A.1 for an introduction in to vector math):

$$W=\boldsymbol{F}\cdot\boldsymbol{x} \tag{3.1.1}$$

Note that work is a scalar quantity - it has a magnitude but no direction. Work is measured in Joules (J), with one Joule being equal to one Newton times one meter.

Of course the force acting on our object need not be constant everywhere. Take for example the extension of a spring: the further you pull, the larger the force gets, as given by Hooke's law (2.2.1). To calculate the work done when extending the spring, we chop up the path (here a straight line) into many small pieces. For each piece, we approximate the force by the average value on that piece, then multiply with the length of the piece and sum. In the limit that we have infinitely many pieces, this approximation becomes exact, and the sum becomes an integral: for one dimension, we thus have:

$$W=\int_{x_1}^{x_2} F(x)\mathrm{d}x \tag{3.1.2}$$

Likewise, the path along which we move need not be a straight line. If the path consists of multiple straight segments, on each of which the force is constant, we can calculate the total work by adding the work done on the different segments. Taking the limit to infinitely many infinitesimally small segments dr, on each of which the force is given by the value F(r), the sum again becomes an integral:

$$W=\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} \boldsymbol{F}(\boldsymbol{r})\cdot\mathrm{d}\boldsymbol{r} \tag{3.1.3}$$

Equation (3.1.3) is the most general version of the definition of work; it simplifies to (3.1.2) for movement along a straight line, and to (3.1.1) if both the path is straight and the force constant[1].

In general, the work done depends on the path taken - for example, it's more work to take a detour when biking from home to work, assuming the air drag is the same everywhere. However, in many important cases the work done in getting from one point to another depends on the endpoints only. Forces for which this is true are called conservative forces. As we'll see below, the force exerted by a spring and that exerted by gravity are both conservative.

Sometimes we will not be interested in how much work is done in generating a certain displacement, but over a certain amount of time - for instance, a generator generates work by getting something to move, like a wheel or a valve, but we don't typically care about those details, we want to know how much work we can expect to get out of the generator, i.e., how much power it has. Power is defined as the amount of work per unit time, or

$$P=\frac{\mathrm{d}W}{\mathrm{d}t} \tag{3.1.4}$$

Power is measured in Joules per second, or Watts (W). To find out how much work is done by an engine that has a certain power output, we need to integrate that output over time:

$$W=\int P\mathrm{d}t \tag{3.1.5}$$

[1] *If you feel intimidated by the vector form of Equation (3.1.3), it may help to rewrite it in terms of the magnitudes of the force F(r) and the (infinitesimal) displacement dr, and the angle* θ *between them. In terms of* $F=|\boldsymbol{F}|$, $dr=|d\boldsymbol{r}|$ *and* θ, *we have* $\boldsymbol{F}\cdot d\boldsymbol{r}=F\cos\theta dr$ *, an expression you may have seen before for a force not pointing in the same direction as the displacement. If*

we now make the force and displacements functions of the position r, then so become the magnitude of the force and the angle, so we can also write Equation (3.1.3) as

$$W=\int_{\boldsymbol{r}_1}^{\boldsymbol{r}_2} F(\boldsymbol{r}) \cos \theta(\boldsymbol{r}) \cdot \mathrm{d} \boldsymbol{r} \tag{3.1.6}$$