14.4: Radioactive Decay and the Center-of-Momentum Frame

Radioactive decay is the process by which unstable particles with high mass; fall apart into more stable particles with lower mass. Although the process itself is quantum mechanical in nature, the dynamics of radioactive decay are described by special relativity and are essentially identical to those of an inelastic collision in reverse. Decay may occur spontaneously (as a random process), but can also be stimulated, by the absorption of a (typically small, e.g. a photon or electron) particle by the unstable one - a process used in nuclear reactors. Because the absorbed particle also carries energy, in stimulated decay the masses of the resulting particles can add up to something more than the rest mass of the original particle. An important question in nuclear physics is what the *threshold energy* of a given reaction is, i.e., the minimum energy the incoming particle must have for the process to be possible. This is not simply the differences in the mass-energy of the original and the resulting particles, as in the collision process momentum must also be conserved. To illustrate how to approach such a problem, let's again consider a concrete example: the threshold energy for the reaction in which a proton ($m_{\mathrm{p}} = 938\mathrm{MeV}/c^2$), initially at rest, absorbs a photon, and then emits a neutral pion ($m_\pi = 135\mathrm{MeV}/c^2$), see Figure 14.3.1 below.

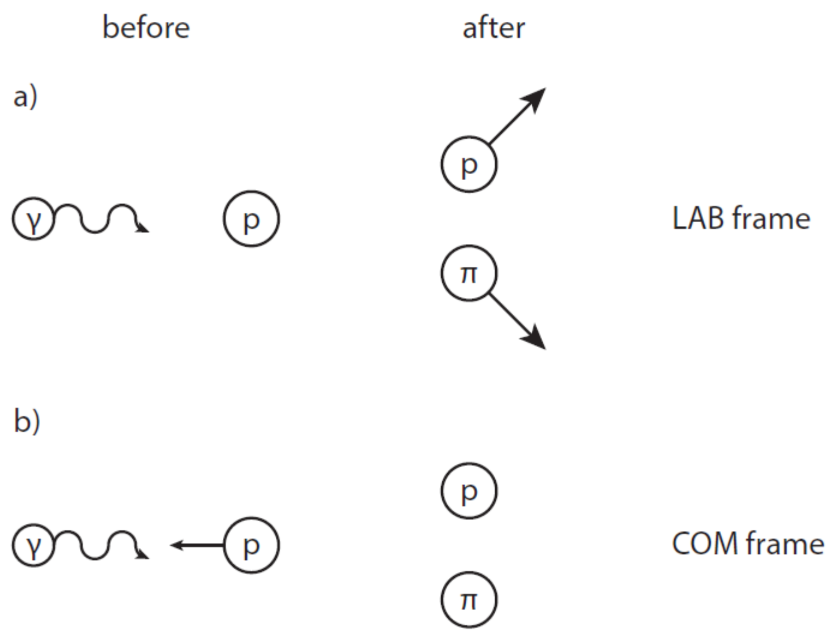

Figure 14.4.1: Example of stimulated radioactive decay: a proton, initially at rest, absorbs a photon and then emits a neutral pion. The reaction is shown in the lab frame in (a), and in the center-of-momentum frame in (b).

Figuring out what the minimum required energy is in the lab frame is not easy, as you have to account for the kinetic energy of the particles after the reaction. There is however a system in which the reaction products are standing still: the *center-of-momentum* frame, the relativistic analog of the center-of-mass frame of classical mechanics[1]. The center-of-momentum frame is defined as the frame in which the total momentum of all particles adds up to zero. In our specific example, before the collision, only the photon carries a momentum, equal to its energy E_γ divided by the speed of light. In general, the total momentum in the system can be a three-vector, equal to $\boldsymbol{p}_{\mathrm{T}} = \sum_i \boldsymbol{p}_i$, while the total energy is given by $E_{\mathrm{T}} = \sum_i E_i$. If we choose our coordinates such that the x-direction coincides with that of $\boldsymbol{p}_{\mathrm{T}}$, the energy-momentum four-vector of the entire system becomes $\overline{\boldsymbol{p}}_{\mathrm{T}} = (E_{\mathrm{T}}/c, p_{\mathrm{T}}, 0, 0)$, where $p_{\mathrm{T}} = |\boldsymbol{p}_{\mathrm{T}}|$. If we go to any different inertial frame S' moving with velocity v in the positive x direction, the components of the energy-momentum four-vector are given by the Lorentz transform of $\boldsymbol{p}_{\mathrm{T}}$:

$$\overline{\boldsymbol{p}}_{\mathrm{T}}' = \gamma(v)\left(\frac{E_{\mathrm{T}}}{c} - \frac{v}{c}p_{\mathrm{T}}, p_{\mathrm{T}} - \frac{v}{c}\frac{E_{\mathrm{T}}}{c}, 0, 0\right) \tag{14.4.1}$$

so we end up in a frame in which the total momentum is zero if we pick

$$v_{\mathrm{COM}} = \frac{c^2 p_{\mathrm{T}}}{E_{\mathrm{T}}} \tag{14.4.2}$$

for our velocity. In particular, we see that we can always make this transformation, and that the center-of-momentum frame is an inertial frame.

Back to our example: why do we care? The answer is almost tautological: if the total momentum is zero before the collision, it is also zero afterward - and so in the COM frame, the particles can all be standing still (see Figure 14.3.1b). That certainly corresponds to the lowest possible kinetic energy of the system, so the energy of the incoming photon is all converted to mass - and that must thus be the threshold energy we're looking for. Interestingly, to answer our original question, we don't even need to calculate what the actual velocity of the COM frame is, just the fact that it exists is sufficient. In the COM frame, we have, by conservation of four-momentum:

$$\overline{\boldsymbol{p}}_\gamma' + \overline{\boldsymbol{p}}_{\mathrm{p,i}}' = \overline{\boldsymbol{p}}_p, f' + \overline{\boldsymbol{p}}_\pi' \tag{14.4.3}$$

and therefore also

$$\left(\overline{\boldsymbol{p}}_\gamma' + \overline{\boldsymbol{p}}_{\mathrm{p,i}}'\right)^2 = \left(\overline{\boldsymbol{p}}_p, f' + \overline{\boldsymbol{p}}_\pi'\right)^2 = (m_{\mathrm{p}}c + m_\pi c)^2 \tag{14.4.4}$$

where the last equality follows from the fact that the reactants are standing still. Now the left-hand-side of equation (/ref{14.3.4}) is the length of a four-vector, and we've proven that these lengths are invariant under Lorentz transformations - so it's value is equal to that of $\left(\overline{\boldsymbol{p}}_\gamma + \overline{\boldsymbol{p}}_{\mathrm{p,i}}\right)^2$ in the lab frame. In that frame, we have $\overline{\boldsymbol{p}}_\gamma = (E_\gamma/c)(1,1,0,0)$ and $\overline{\boldsymbol{p}}_{\mathrm{p,i}} = (m_{\mathrm{p}}c, 0, 0, 0)$, so we end up with an easy equation form E_γ:

$$(m_{\mathrm{p}} + m_\pi)^2 c^2 = \overline{\boldsymbol{p}}_\gamma^2 + \overline{\boldsymbol{p}}_{\mathrm{p,i}}^2 + 2\overline{\boldsymbol{p}}_\gamma^2 \cdot \overline{\boldsymbol{p}}_{\mathrm{p,i}} = 0 + m_{\mathrm{p}}^2 c^2 + 2E_\gamma m_{\mathrm{p}} \tag{14.4.5}$$

or

$$E_\gamma = \frac{m_\pi^2 + 2m_\pi m_{\mathrm{p}}}{2m_{\mathrm{p}}} c^2 = 145\mathrm{MeV} \tag{14.4.6}$$

In this example, we thus need at least 10MeV of energy more than the mass of the particle we've created.

Note that in finding the threshold energy in the example, we again heavily relied on the four-vector properties of $\overline{\boldsymbol{p}}$ - not only it's length (like in the third method of section 14.2), but also the invariance of that length under Lorentz transformations. Using these properties results in easy equations to solve, while if you'd ignore them, you'd probably get stuck trying to figure out what the kinetic energy of the reaction products is.

[1] As our system includes a photon, a center-of-mass frame doesn't make sense here, as the photon has no mass - but it has nonzero momentum, so we can make a transformation to a system in which the total momentum vanishes.