14.5: Totally Elastic Collision - Compton Scattering

As a final example of a collision in special relativity, we consider the totally elastic case: a collision in which the total momentum, total kinetic energy, and the mass of all particles are conserved. An example of such a collision is *Compton scattering*: the collision between a photon and an electron, resulting in a transfer of energy from one to the other, visible in a change of wavelength of the photon. For our example, we'll take the electron to be initially stationary, and the photon to be coming in along the x-axis; after the collision, both particles have nonzero velocities in both the x and y directions (see Figure 14.4.1).

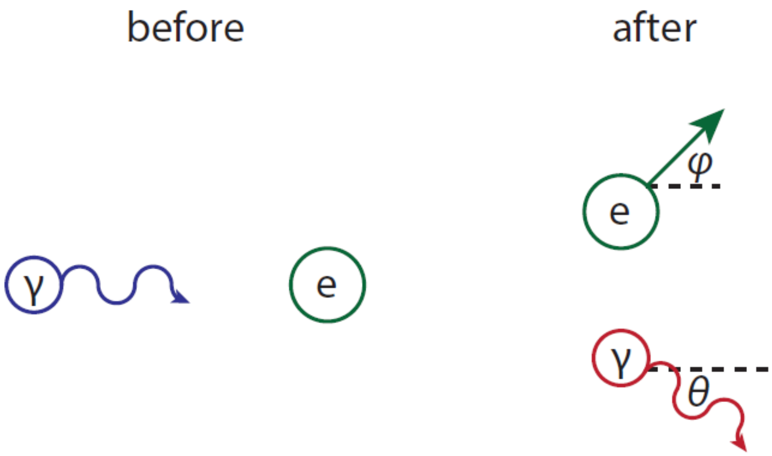

Figure 14.5.1: Compton scattering between a photon and an electron, resulting in a transfer of energy of the photon to the electron, measurable as a change in the photon's wavelength.

The four-momenta of the electron and photon before and after the collision are given by:

$$\overline{\boldsymbol{p}}_{\mathrm{e,i}}=\begin{pmatrix} m_{\mathrm{e}}c \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\boldsymbol{p}}_{\gamma,\mathrm{i}}=\frac{E_{\mathrm{i}}}{c}\begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \overline{\boldsymbol{p}}_{\mathrm{e,f}}=\begin{pmatrix} E_{\mathrm{e,f}}/c \\ p_{\mathrm{e,f}}\cos\phi \\ p_{\mathrm{e,f}}\sin\phi \\ 0 \end{pmatrix}, \quad \overline{\boldsymbol{p}}_{\gamma,\mathrm{f}}=\frac{E_{\mathrm{f}}}{c}\begin{pmatrix} 1 \\ \cos\theta \\ -\sin\theta \\ 0 \end{pmatrix} \tag{14.5.1}$$

We can now solve for the energy E_{f} of the outgoing photon (and thus its wavelength) in terms of that of the incoming photon (E_{i}) and the scattering angle θ. There are again (at least) two ways to do this. One is to compare the components of the initial and final energy-momentum four-vector term by term. The other is to again use the fact that we know about the length of the four-vector to immediately eliminate the scattering angle ϕ of the electron. To do so, we first rewrite the conservation of energy-momentum equation, $\overline{\boldsymbol{p}}_{\mathrm{e,i}}+\overline{\boldsymbol{p}}_{\gamma,\mathrm{i}}=\overline{\boldsymbol{p}}_{\mathrm{e,f}}+\overline{\boldsymbol{p}}_{\gamma,\mathrm{f}}$ to isolate the term of the outgoing electron, and then take the square, to get:

$$\left(\overline{\boldsymbol{p}}_{\mathrm{e,i}}+\overline{\boldsymbol{p}}_{\gamma,\mathrm{i}}-\overline{\boldsymbol{p}}_{\gamma,\mathrm{f}}\right)^{2}=\overline{\boldsymbol{p}}_{\mathrm{e,f}}^{2} \tag{14.5.2}$$

$$\overline{\boldsymbol{p}}_{\mathrm{e,i}}^{2}+\overline{\boldsymbol{p}}_{\gamma,\mathrm{i}}^{2}+\overline{\boldsymbol{p}}_{\gamma,\mathrm{f}}^{2}+\overline{\boldsymbol{p}}_{\mathrm{e,i}}\cdot\overline{\boldsymbol{p}}_{\gamma,\mathrm{i}}^{2}-2\overline{\boldsymbol{p}}_{\mathrm{e,i}}\cdot\overline{\boldsymbol{p}}_{\gamma,\mathrm{f}}-2\overline{\boldsymbol{p}}_{\gamma,\mathrm{i}}\cdot\overline{\boldsymbol{p}}_{\gamma,\mathrm{f}}=\overline{\boldsymbol{p}}_{\mathrm{ef}}^{2} \tag{14.5.3}$$

$$m_{e}^{2}c^{2}+0+0+2m_{\mathrm{e}}E_{\mathrm{i}}-2m_{\mathrm{e}}E_{\mathrm{f}}-2\frac{E_{\mathrm{i}}E_{\mathrm{f}}}{c^{2}}(1-\cos\theta)=m_{e}^{2}c^{2} \tag{14.5.4}$$

from which we can solve for E_{f}. Rewriting to wavelengths (through$E=hf=hc/\lambda$), we get

$$\lambda_{\mathrm{f}}=\lambda_{\mathrm{i}}+\frac{h}{m_{\mathrm{e}}c}(1-\cos\theta) \tag{14.5.5}$$