13.1: The Position Four-Vector

As we've seen in the previous section, we can define a 'length' ¢s that is invariant under Lorentz transformations as seen by Equation 12.2.2 (repeated again below):

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta \boldsymbol{x}) \cdot (\Delta \boldsymbol{x}) = (c\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \tag{13.1.1}$$

As is clear from the definition of Δs, to get invariant quantities, we should not think of space as measured in three dimensions, but of *spacetime*, measured in four dimensions. This four-dimensional world of special relativity is called *Minkowski space*, and its vectors have four components: one for time and three for space. Conventionally, we add the time component as the zeroth component of the vector. To distinguish between 'ordinary', three-dimensional vectors (which are represented in bold) and four-vectors, we'll put a line on top of the latter. The *position four-vector* is then given by:

$$\overline{\boldsymbol{x}} = (x_0, x_1, x_2, x_3) = (ct, x, y, z) \tag{13.1.2}$$

We would like to be able to determine the length of the position four-vector by taking the inner product of the vector with itself. However, the regular inner product is not going to work, because instead of $(ct)^2 + x^2 + y^2 + z^2$, the quantity that is independent of the reference frame is$(ct)^2 - x^2 - y^2 - z^2$. We, therefore, define the inner product of two four-vectors $\overline{\boldsymbol{a}}$ and $\overline{\boldsymbol{b}}$ as

$$\overline{\boldsymbol{a}} \cdot \overline{\boldsymbol{b}} = a_0 b_0 - a_1 b_1 - a_2 b_2 - a_3 b_3 \tag{13.1.3}$$

As we've already seen, the magnitude of the position four-vectors, as determined by its inner product with itself (Equation (12.2.2)) is independent of the inertial reference frame you use to measure it in. In problem 13.3 you'll show that consequently, the value of the inner product of any two four-vectors is reference frame independent.

It may seem that we're back to normal - we've added a dimension and introduced a new inner product, but with those, we should be able to do calculations just as easily as in ordinary 3D space. The last part is true, but the new inner product is actually different from the regular one in one very important respect: the value in equation (12.2.2) can be zero or even negative for nonzero four-vectors $\overline{\boldsymbol{x}}$! To see what's going on, we return to the spacetime diagram, in particular, figure 12.1.1c. Suppose we have a particle traveling in the x-direction (taking $y = z = 0$ for convenience). What speed does it need for the length of its four-vector to vanish? For that to happen we need $ct = x$, or $v = x/t = c$, so $\overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}}$ becomes zero for something traveling at the speed of light. Likewise, $\overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}}$ is positive for a particle traveling slower than light, and negative for a particle traveling faster than light (which is of course impossible, since such a particle would need to first reach the speed of light, which as we've seen can never be done). However, we can consider the four-vector $\overline{\boldsymbol{x}}$ between any two points in spacetime, and from the sign of $\overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}}$ tell whether they can be connected through regular (slower than light) travel, by a light beam, or not at all. The first we call *timelike*, the second *lightlike*, and the third *spacelike*:

$$\begin{array}{ll} \overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}} > 0 & \text{timelike} \\ \overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}} = 0 & \text{lightlike} \\ \overline{\boldsymbol{x}} \cdot \overline{\boldsymbol{x}} < 0 & \text{spacelike} \end{array} \tag{13.1.4}$$

Two events which are connected by a spacelike four-vector cannot influence each other: there is no way to send a signal between them, and therefore there is no way to transfer information.