6.3: Motion Under the Action of a Central Force

A *central force* is a force that points along the (positive or negative) radial direction $\hat{\boldsymbol{r}}$, and whose magnitude depends only on the distance r to the origin - so $\boldsymbol{F}(\boldsymbol{r}) = F(r)\hat{\boldsymbol{r}}$. Central forces can be defined in both two and three dimensions, with the three-dimensional concept of the radial distance (to the origin) and direction (direction of increasing r) completely analogous to the two-dimensional case. Two important examples of central forces are (general) Newtonian gravity (2.2.2) and the Coulomb force (2.2.4) between two charged objects. Although these forces are three-dimensional examples, discussing them here is appropriate, as the following theorem shows.

Theorem 6.3.1

The motion of a particle under the action of a central force takes place in a plane.

Proof

We first note that a central force can exert no torque on an object:

$$\boldsymbol{\tau} = \boldsymbol{r} \times \boldsymbol{F} = F(r)(\boldsymbol{r} \times \hat{\boldsymbol{r}}) = 0.$$

Consequently, under the action of a central force, angular momentum is conserved. Moreover, we have

$$\boldsymbol{r} \cdot \boldsymbol{L} = \boldsymbol{r} \cdot (\boldsymbol{r} \times \boldsymbol{p}) = 0$$

and

$$\boldsymbol{v} \cdot \boldsymbol{L} = \boldsymbol{v} \cdot (\boldsymbol{r} \times m\boldsymbol{v}) = 0.$$

Both the position vector $\boldsymbol{r}$ and the velocity vector $\boldsymbol{v}$ thus lie in the plane perpendicular to $\boldsymbol{L}$. As $\boldsymbol{L}$ is conserved $\boldsymbol{r}$ and $\boldsymbol{v}$ must be confined to the plane perpendicular to $\boldsymbol{L}$ and through the origin.

□

Applying the results of the previous section to the motion of a single particle under the action of a central force, we find (for the plane in which the particle moves):

$$F(r) = F_r = m\ddot{r} - mr\dot{\theta}^2 = m\ddot{r} - \frac{L^2}{mr^3} \tag{6.3.1}$$

where we used that for a single particle, the magnitude of the angular momentum is given by $L = mr^2\dot{\theta}$. Rewriting Equation 6.3.1 gives

$$m\ddot{r} = F(r) + \frac{L^2}{mr^3} = F(r) + F_{\mathrm{cf}} \tag{6.3.2}$$

where F_{cf} is known as the centrifugal force, as it tends to move our particle away from the origin. We can write the centrifugal force as the derivative of a potential:

$$F_{\mathrm{cf}} = -\frac{\mathrm{d}U_{\mathrm{cf}}}{\mathrm{d}r} = -\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{L^2}{2mr^2}\right) \tag{6.3.3}$$

Writing the original force as the derivative of a potential $U(r)$ as well, we can write down an equation for the total energy of the system:

$$E = K + U = \frac{1}{2}m\dot{r}^2 + U(r) + \frac{L^2}{2mr^2} \tag{6.3.4}$$

For both Newtonian gravity and the Coulomb force, the potential can be written as $U(r) = -\alpha/r$, where $\alpha = Gm_1m_2$ for gravity and $\alpha = -k_e q_1 q_2$ for Coulomb's law. We can then rewrite the energy equation as a differential equation for $r(t)$:

$$\frac{1}{2}m\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = E + \frac{\alpha}{r} - \frac{L^2}{2mr^2} \tag{6.3.5}$$

To describe the motion of the particle, rather than specifying $r(t)$ and $\theta(t)$, we would like to express r as a function of θ. We can rewrite Equation 6.3.5 to a differential equation for $r(\theta)$ by invoking the chain rule:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\frac{\mathrm{d}\theta}{\mathrm{d}t}\right)^2 = \left(\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2\left(\frac{L}{mr^2}\right)^2 \tag{6.3.6}$$

where we again used that $L = mr^2\dot{\theta}$. Equation 6.3.5 now becomes:

$$\left(\frac{1}{r^2}\frac{\mathrm{d}r}{\mathrm{d}\theta}\right)^2 = -\frac{1}{r^2} + \frac{2m\alpha}{L^2 r} + \frac{2mE}{L^2} \tag{6.3.7}$$

We can simplify Equation 6.3.7 further by introducing a new variable, $z = \frac{1}{r} - m\alpha/L^2$. We also introduce a dimensionless constant $\varepsilon = \sqrt{1 + 2EL^2/m\alpha^2}$ and an inverse length $q = m\alpha\varepsilon/L^2$. With these substitutions, our equation becomes:

$$\left(\frac{\mathrm{d}z}{\mathrm{d}\theta}\right)^2 = -z^2 + q^2 \tag{6.3.8}$$

We can solve Equation 6.3.8 by separation of variables:

$$\int \frac{1}{\sqrt{q^2 - z^2}}\mathrm{d}z = \int \mathrm{d}\theta \Rightarrow \arccos\left(\frac{z}{q}\right) = \theta - \theta_0 \tag{6.3.9}$$

Taking the reference angle θ_0 (our integration constant) to be zero, we find $z(\theta) = q\cos(\theta)$. Translating back to $r(\theta)$, we obtain a fairly simple solution:

$$r(\theta) = \frac{L^2}{m\alpha}\frac{1}{1 + \varepsilon\cos\theta} \tag{6.3.10}$$

What the solution 6.3.10 (the orbit of our particle under the action of the central force) actually looks like, depends on the value of our dimensionless variable ε, known as the eccentricity of the orbit. To find out which orbits we can get, we translate Equation 6.3.10 back to Cartesian coordinates, using $x = r\cos\theta$. Defining $k = L^2/m\alpha$, we get $k = r + \varepsilon r\cos\theta = r + \varepsilon x$, or $r = k - \varepsilon x$. Now using $r^2 = x^2 + y^2$, we get

$$x^2 + y^2 = (k - \varepsilon x)^2 = k^2 - 2\varepsilon kx + \varepsilon^2 x^2 \tag{6.3.11}$$

We can now distinguish four possibilities:

1. $\varepsilon = 0$: In this case, Equation 6.3.11 becomes $x^2 + y^2 = k^2$, so our orbit is a circle with the origin at its center.
2. $0 < \varepsilon < 1$: For this case, with some algebra, we can rewrite Equation 6.3.11 as

   $$((x - x_0)/a)^2 + (y/b)^2 = 1,$$

   where $a = k/\left(1 - \varepsilon^2\right)$, $x_0 = -\varepsilon a$, and $b = k/\sqrt{1 - \varepsilon^2}$. These orbits are ellipses, with the center of the ellipse at $(x_0, 0)$, semi-major axis a, semi-minor axis b, and focal length $f = \sqrt{a^2 - b^2} = k\varepsilon/\left(1 - \varepsilon^2\right) = -x_0$. One of the foci thus lies at the origin.
3. $\varepsilon = 1$: Equation 6.3.11 now becomes $y^2 = k^2 - 2kx$, which is the equation for a parabola (extending along the negative x-axis) with its 'top' (in this case, rightmost point) at $(k/2, 0)$ and focal length k/2, so the (single) focus is again located at the origin.
4. $\varepsilon > 1$: This case again requires some algebra to rewrite Equation 6.3.11 in a recognizable standard form: $((x - x_0)/a)^2 - (y/b)^2 = 1$, where $a = k/\left(\varepsilon^2 - 1\right)$, $x_0 = \varepsilon a$ and $b = k/\sqrt{\varepsilon^2 - 1}$. These orbits are hyperbola, crossing the x-axis at $(x_0, 0)$, and approaching asymptotes $y = \pm b((x/a) - \varepsilon)$, which meet at $(x_0 + a, 0)$. The focal length is now $f = \sqrt{a^2 + b^2} = k\varepsilon/\left(\varepsilon^2 - 1\right) = \varepsilon a = x_0 + a$, so the focus of the hyperbola is also located at the origin.

In mathematics, these four possible types of orbits are known as conic sections: the curves you can get by intersecting a cone with a plane. Specifically, when the central force is gravity, such as in the solar system (where the sun is so much heavier than everything else combined that to good approximation we can describe orbits as being determined by the sun's gravitational field alone), the four cases listed above are the only possible orbits bodies can have. The planets, dwarf planets, asteroids, and many minor objects in our solar system all follow elliptical orbits around the sun, albeit with different eccentricity[1] - Earth's is almost zero (0.017), but that of Mars is significantly higher (0.09), and of Pluto much higher still (0.25). Comets, on the other hand, typically parabolic or hyperbolic orbits. Spacecraft such as the Voyager and New Horizons probes are often put on trajectories past planets that are not their final destination, to pick up (or loose) speed through a gravitational assist (in which they take a little bit of momentum from the planet's orbit); those paths past planets are typically hyperbola. Getting a spacecraft to orbit another planet (i.e., in a bound, so elliptical) orbit is actually much harder, but again, the resulting orbit is described by the maths presented above.

[1] *See table B.4 for data on the orbits of all planets and a number of their moons.*