2.6: Solving the Equations of Motion in Three Special Cases

In Section 2.3 we saw some examples of equations of motion originating from Newton's second law of motion. For the quite common case that the mass of our object of interest is constant, its trajectory will be given as the solution of a second-order ordinary differential equation, with time as our variable. In general, the force in Newton's second law may depend on time and position, as well as on the first derivative of the position, i.e., the velocity. In one dimension, we thus have

$$m\ddot{x} = F(x, \dot{x}, t) \tag{2.6.1}$$

Equation 2.6.1 can be hard to solve for complicated functions F. However, in each of the special cases that the force only depends on one of the three variables, we can write down a general solution - albeit as an integral over the force, which we may or may not be able to calculate explicitly.

Case 1: F=F(t)

If the force only depends on time, we can solve Equation (2.6.1) by direct integration. Using that $v = \dot{x}$ we have $m\dot{v} = F(t)$, which we integrate to find

$$\int_{t_0}^{t} F(t')\,\mathrm{d}t' = m \int_{v_0}^{v} \mathrm{d}v' = m\left[v(t) - v_0\right] \tag{2.6.2}$$

where at the initial time $t = t_0$ the object has velocity $v = v_0$. We can now find the position by integrating the velocity:

$$x(t) = \int_{t_0}^{t} v(t')dt' \tag{2.6.3}$$

Case 2: F=F(x)

If the force depends only on the position in space (as is the case for the harmonic oscillator), we cannot integrate over time, as to do so we would already need to know x(t). Instead, we invoke the chain rule to rewrite our differential equation as an equation in which the position is our variable. We have:

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}x}\frac{\mathrm{d}x}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}x} \tag{2.6.4}$$

and so our equation of motion becomes

$$mv\frac{\mathrm{d}v}{\mathrm{d}x} = F(x) \tag{2.6.5}$$

which we again solve by direct integration:

$$\int_{x_0}^{x} F(x')\,\mathrm{d}x' = m\int_{v_0}^{v} v'\,\mathrm{d}v' = \frac{m}{2}\left[v^2(x) - v_0^2\right] \tag{2.6.6}$$

To get x(t), we use the relation that $\frac{dx}{dt} = v(x)$. Separation of variables gives $\frac{dx}{v(x)} = dt$, which we can integrate to get

$$t - t_0 = \int_{x_0}^{x} \frac{1}{v(x')}\mathrm{d}x' \tag{2.6.7}$$

which gives us t(x). In principle we can invert this expression to give us x(t), although in practice this may not be easy.

Case 3: F=F(v)

If the force depends only on the velocity, there are two ways we can proceed. We can write the equation of motion as $m\frac{dv}{dt} = F(v)$ and use separation of variables to get:

$$t - t_0 = m\int_{\nu_0}^{v} \frac{1}{F(v')}\mathrm{d}v' \tag{2.6.8}$$

from which we can get $v(t)$ after inverting, and $x(t)$ after integrating $v(t)$ as in Equation (2.6.3). Alternatively, we could again rewrite our equation of motion as an equation in space instead of time, and arrive at:

$$x-x_0=m\int_{v_0}^{v}\frac{v'}{F(v')}\mathrm{d}v' \tag{2.6.9}$$

From Equation (2.6.9) we can get v(x) by inverting, and x(t) from Equation (2.6.7). Note that Equation (2.6.9) does not give us $x(t)$ directly, as x is the variable in that equation.

example 2.6.4: velocity of the harmonic oscillator

It may seem that what we've done so far in this section has hardly helped matters: the 'solutions' we found contain integrals and often need to be inverted to get our desired function $x(t)$ (or, depending on the problem we're studying, $v(t)$ or $v(x)$). To show you how these solutions may be useful, let's consider a specific example: a harmonic oscillator, consisting of a mass on a Hookean spring, with

$$F=F(x)=-kx.$$

Solution

We already wrote down the equation of motion (Equation 2.3.4) and its general solution (Equation 2.3.5). The general solution can be found through the substitution of exponentials, as we'll do in Section 8.1. However, we can also learn something useful from writing the equation of motion in the form (2.6.5). Its solution, formally given by equation (2.6.6), can be calculated explicitly for our force as

$$\frac{m}{2}\left[v^2(x)-v_0^2\right]=\int_{x_0}^{x}(-kx')\,\mathrm{d}x'=-\frac{k}{2}\left[x^2-x_0^2\right] \tag{2.6.10}$$

which gives

$$v(x)=\sqrt{v_0^2-\frac{k}{m}\left(x^2-x_0^2\right)}$$

for $v(x)$. Although $x(t)$ and $v(t)$ are more easily obtained from the solution given in Equation 2.3.5, that solution will not give you $v(x)$, and deriving it is tricky. Here we get it almost for free. Moreover, as you have probably noted, Equation 2.6.10 relates the kinetic to the potential energy of the harmonic oscillator - a special case of conservation of energy, which we'll discuss in the next section.