5.8: Motion in Two and Three Dimensions (Summary)

Key Terms

acceleration vector	instantaneous acceleration found by taking the derivative of the velocity function with respect to time in unit vector notation
angular frequency	ω, rate of change of an angle with which an object that is moving on a circular path
centripetal acceleration	component of acceleration of an object moving in a circle that is directed radially inward toward the center of the circle
displacement vector	vector from the initial position to a final position on a trajectory of a particle
position vector	vector from the origin of a chosen coordinate system to the position of a particle in two- or threedimensional space
projectile motion	motion of an object subject only to the acceleration of gravity
range	maximum horizontal distance a projectile travels
reference frame	coordinate system in which the position, velocity, and acceleration of an object at rest or moving is measured
relative velocity	velocity of an object as observed from a particular reference frame, or the velocity of one reference frame with respect to another reference frame
tangential accleration	magnitude of which is the time rate of change of speed. Its direction is tangent to the circle.
time of flight	elapsed time a projectile is in the air
total accleration	vector sum of centripetal and tangential accelerations
trajectory	path of a projectile through the air
velocity vector	vector that gives the instantaneous speed and direction of a particle; tangent to the trajectory

Key Equations

Position vector	$\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k} \quad (5.8.1)$
Displacement vector	$\Delta\vec{r}=\vec{r}(t_2)-\vec{r}(t_1) \quad (5.8.2)$
Velocity vector	$\vec{v}(t)=\lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}=\frac{d\vec{r}}{dt} \quad (5.8.3)$
Velocity in terms of components	$\vec{v}(t)=v_x(t)\hat{i}+v_y(t)\hat{j}+v_z(t)\hat{k} \quad (5.8.4)$
Velocity components	$v_x(t)=\frac{dx(t)}{dt} v_y(t)=\frac{dy(t)}{dt} v_z(t)=\frac{dz(t)}{dt} \quad (5.8.5)$
Average velocity	$\vec{v}_{avg}=\frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1} \quad (5.8.6)$

Instantaneous acceleration	$$\vec{a}(t)=\lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}=\frac{d\vec{v}}{dt} \quad (5.8.7)$$
Instantaneous acceleration, component form	$$\vec{a}(t)=\frac{dv_x(t)}{dt}\hat{i}+\frac{dv_y(t)}{dt}\hat{j}+\frac{dv_z(t)}{dt}\hat{k} \quad (5.8.8)$$
Instantaneous acceleration as second derivatives of position	$$\vec{a}(t)=\frac{d^2x(t)}{dt^2}\hat{i}+\frac{d^2y(t)}{dt^2}+\frac{d^2z(t)}{dt^2}\hat{k} \quad (5.8.9)$$
Time of flight	$$T_{tof}=\frac{2(v_0\sin\theta)}{g} \quad (5.8.10)$$
Trajectory	$$y=(\tan\theta_0)x-\left[\frac{g}{2(v_0\cos\theta_0)^2}\right]x^2 \quad (5.8.11)$$
Range	$$R=\frac{v_0^2\sin 2\theta_0}{g} \quad (5.8.12)$$
Centripetal accleration	$$a_C=\frac{v^2}{r} \quad (5.8.13)$$
Position vector, uniform circular motion	$$\vec{r}(t)=A\cos\omega t\hat{i}+A\sin\omega t\hat{j} \quad (5.8.14)$$
Velocity vector, uniform circular motion	$$\vec{v}(t)=\frac{d\vec{r}(t)}{dt}=-A\omega\sin\omega t\hat{i}+A\omega\cos\omega t\hat{j} \quad (5.8.15)$$
Acceleration vector, uniform circular motion	$$\vec{a}(t)=\frac{d\vec{v}(t)}{dt}=-A\omega^2\cos\omega t\hat{i}-A\omega^2\sin\omega t\hat{j} \quad (5.8.16)$$
Tangential acceleration	$$a_T=\frac{d\lvert\vec{v}\rvert}{dt} \quad (5.8.17)$$
Total acceleration	$$\vec{a}=\vec{a}_C+\vec{a}_T \quad (5.8.18)$$
Position vector in frame S is the position vector in frame S′ plus the vector from the origin of S to the origin of S′	$$\vec{r}_{PS}=\vec{r}_{PS'}+\vec{r}_{S'S} \quad (5.8.19)$$
Relative velocity equation connecting two reference frames	$$\vec{v}_{PS}=\vec{v}_{PS'}+\vec{v}_{S'S} \quad (5.8.20)$$
Relative velocity equation connecting more than two reference frames	$$\vec{v}_{PC}=\vec{v}_{PA}+\vec{v}_{AB}+\vec{v}_{BC} \quad (5.8.21)$$
Relative acceleration equation	$$\vec{a}_{PS}=\vec{a}_{PS'}+\vec{a}_{S'S} \quad (5.8.22)$$

Summary

4.1 Displacement and Velocity Vectors

- The position function $\vec{r}(t)$ gives the position as a function of time of a particle moving in two or three dimensions. Graphically, it is a vector from the origin of a chosen coordinate system to the point where the particle is located at a specific time.

- The displacement vector $\Delta\vec{r}$ gives the shortest distance between any two points on the trajectory of a particle in two or three dimensions.
- Instantaneous velocity gives the speed and direction of a particle at a specific time on its trajectory in two or three dimensions, and is a vector in two and three dimensions.
- The velocity vector is tangent to the trajectory of the particle.
- Displacement $\vec{r}$(t) can be written as a vector sum of the one-dimensional displacements $\vec{x}$(t), $\vec{y}$(t), $\vec{z}$(t) along the x, y, and z directions.
- Velocity $\vec{v}$(t) can be written as a vector sum of the one-dimensional velocities v_x(t), v_y(t), v_z(t) along the x, y, and z directions.
- Motion in any given direction is independent of motion in a perpendicular direction.

4.2 Acceleration Vector

- In two and three dimensions, the acceleration vector can have an arbitrary direction and does not necessarily point along a given component of the velocity.
- The instantaneous acceleration is produced by a change in velocity taken over a very short (infinitesimal) time period. Instantaneous acceleration is a vector in two or three dimensions. It is found by taking the derivative of the velocity function with respect to time.
- In three dimensions, acceleration $\vec{a}$(t) can be written as a vector sum of the one-dimensional accelerations a_x(t), a_y(t), and a_z (t) along the x-, y-, and z-axes.
- The kinematic equations for constant acceleration can be written as the vector sum of the constant acceleration equations in the x, y, and z directions.

4.3 Projectile Motion

- Projectile motion is the motion of an object subject only to the acceleration of gravity, where the acceleration is constant, as near the surface of Earth.
- To solve projectile motion problems, we analyze the motion of the projectile in the horizontal and vertical directions using the one-dimensional kinematic equations for x and y.
- The time of flight of a projectile launched with initial vertical velocity v_{0y} on an even surface is given by

$$T_{tof} = \frac{2(v_0 \sin\theta)}{g} \tag{5.8.23}$$

  This equation is valid only when the projectile lands at the same elevation from which it was launched.
- The maximum horizontal distance traveled by a projectile is called the range. Again, the equation for range is valid only when the projectile lands at the same elevation from which it was launched.

4.4 Uniform Circular Motion

- Uniform circular motion is motion in a circle at constant speed.
- Centripetal acceleration $\vec{a}_C$ is the acceleration a particle must have to follow a circular path. Centripetal acceleration always points toward the center of rotation and has magnitude $a_C = \frac{v^2}{r}$.
- Nonuniform circular motion occurs when there is tangential acceleration of an object executing circular motion such that the speed of the object is changing. This acceleration is called tangential acceleration $\vec{a}_T$. The magnitude of tangential acceleration is the time rate of change of the magnitude of the velocity. The tangential acceleration vector is tangential to the circle, whereas the centripetal acceleration vector points radially inward toward the center of the circle. The total acceleration is the vector sum of tangential and centripetal accelerations.
- An object executing uniform circular motion can be described with equations of motion. The position vector of the object is $\vec{r}$(t) = A cos ωt $\hat{i}$ + A sin ωt $\hat{j}$, where A is the magnitude $|\vec{r}(t)|$, which is also the radius of the circle, and ω is the angular frequency.

4.5 Relative Motion in One and Two Dimensions

- When analyzing motion of an object, the reference frame in terms of position, velocity, and acceleration needs to be specified.
- Relative velocity is the velocity of an object as observed from a particular reference frame, and it varies with the choice of reference frame.
- If S and S′ are two reference frames moving relative to each other at a constant velocity, then the velocity of an object relative to S is equal to its velocity relative to S′ plus the velocity of S′ relative to S.

- If two reference frames are moving relative to each other at a constant velocity, then the accelerations of an object as observed in both reference frames are equal.

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