9.9: Center of Mass (Part 1)

Learning Objectives

- Explain the meaning and usefulness of the concept of center of mass
- Calculate the center of mass of a given system
- Apply the center of mass concept in two and three dimensions
- Calculate the velocity and acceleration of the center of mass

We have been avoiding an important issue up to now: When we say that an object moves (more correctly, accelerates) in a way that obeys Newton's second law, we have been ignoring the fact that all objects are actually made of many constituent particles. A car has an engine, steering wheel, seats, passengers; a football is leather and rubber surrounding air; a brick is made of atoms. There are many different types of particles, and they are generally not distributed uniformly in the object. How do we include these facts into our calculations?

Then too, an extended object might change shape as it moves, such as a water balloon or a cat falling (Figure $9.9.1$). This implies that the constituent particles are applying internal forces on each other, in addition to the external force that is acting on the object as a whole. We want to be able to handle this, as well.

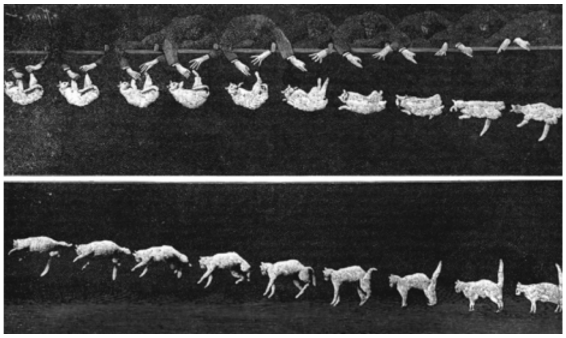


Figure $9.9.1$: As the cat falls, its body performs complicated motions so it can land on its feet, but one point in the system moves with the simple uniform acceleration of gravity.

The problem before us, then, is to determine what part of an extended object is obeying Newton's second law when an external force is applied and to determine how the motion of the object as a whole is affected by both the internal and external forces.

Be warned: To treat this new situation correctly, we must be rigorous and completely general. We won't make any assumptions about the nature of the object, or of its constituent particles, or either the internal or external forces. Thus, the arguments will be complex.

Internal and External Forces

Suppose we have an extended object of mass M, made of N interacting particles. Let's label their masses as m_j, where j = 1, 2, 3, …, N. Note that

$$M=\sum_{j=1}^{N} m_j. \tag{9.9.1}$$

If we apply some net **external force** $\vec{F}_{ext}$ on the object, every particle experiences some "share" or some fraction of that external force. Let:

$\vec{f}_j^{\,ext}$ = the fraction of the external force that the jth particle experiences

Notice that these fractions of the total force are not necessarily equal; indeed, they virtually never are. (They **can** be, but they usually aren't.) In general, therefore,

$$\vec{f}_1^{\,ext} \neq \vec{f}_2^{\,ext} \neq \cdots \neq \vec{f}_N^{\,ext}. \tag{9.9.2}$$

Next, we assume that each of the particles making up our object can interact (apply forces on) every other particle of the object. We won't try to guess what kind of forces they are; but since these forces are the result of particles of the object acting on other particles of the same object, we refer to them as **internal forces** $\vec{f}_j^{\,int}$; thus:

$\vec{f}_j^{\,int}$ = the net internal force that the jth particle experiences from all the other particles that make up the object.

Now, the **net** force, internal plus external, on the jth particle is the vector sum of these:

$$\vec{f}_j = \vec{f}_j^{\,int} + \vec{f}_j^{\,ext}. \tag{9.9.3}$$

where again, this is for all N particles; j = 1, 2, 3, … , N. As a result of this fractional force, the momentum of each particle gets changed:

$$\begin{aligned} \vec{f}_j &= \frac{d\vec{p}_j}{dt} \\ \vec{f}_j^{\,int} + \vec{f}_j^{\,ext} &= \frac{d\vec{p}_j}{dt}. \end{aligned}$$

The net force $\vec{F}$ on the object is the vector sum of these forces:

$$\begin{aligned} \vec{F}_{net} &= \sum_{j=1}^{N} (\vec{f}_j^{\,int} + \vec{f}_j^{\,ext}) \\ &= \sum_{j=1}^{N} \vec{f}_j^{\,int} + \sum_{j=1}^{N} \vec{f}_j^{\,ext}. \end{aligned}$$

This net force changes the momentum of the object as a whole, and the net change of momentum of the object must be the vector sum of all the individual changes of momentum of all of the particles:

$$\vec{F}_{net} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.9.4}$$

Combining Equation ??? and Equation 9.9.4 gives

$$\sum_{j=1}^{N} \vec{f}_j^{\,int} + \sum_{j=1}^{N} \vec{f}_j^{\,ext} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.9.5}$$

Let's now think about these summations. First consider the internal forces term; remember that each $\vec{f}_j^{\,int}$ is the force on the jth particle from the other particles in the object. But by Newton's third law, for every one of these forces, there must be another force that has the same magnitude, but the opposite sign (points in the opposite direction). These forces do not cancel; however, that's not what we're doing in the summation. Rather, we're simply **mathematically adding up** all the internal force vectors. That is, in general, the internal forces for any individual part of the object won't cancel, but when all the internal forces are added up, the internal forces must cancel in pairs. It follows, therefore, that the sum of all the internal forces must be zero:

$$\sum_{j=1}^{N} \vec{f}_j^{\,ext} = 0. \tag{9.9.6}$$

(This argument is subtle, but crucial; take plenty of time to completely understand it.)

For the external forces, this summation is simply the total external force that was applied to the whole object:

$$\sum_{j=1}^{N} \vec{f}_j^{\,ext} = \vec{F}_{ext}. \tag{9.9.7}$$

As a result,

$$\vec{F}_{ext} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.9.8}$$

This is an important result. Equation 9.9.8 tells us that the total change of momentum of the entire object (all N particles) is due only to the external forces; the internal forces do not change the momentum of the object as a whole. This is why you can't lift yourself in the air by standing in a basket and pulling up on the handles: For the system of you + basket, your upward pulling force is an internal force.

Force and Momentum

Remember that our actual goal is to determine the equation of motion for the entire object (the entire system of particles). To that end, let's define:

$\vec{p}_{CM}$ = the total momentum of the system of N particles (the reason for the subscript will become clear shortly)

Then we have

$$\vec{p}_{CM} = \equiv \sum_{j=1}^{N} \vec{p}_j. \tag{9.9.9}$$

and therefore Equation 9.9.8 can be written simply as

$$\vec{F} = \frac{d\vec{p}_{CM}}{dt}. \tag{9.9.10}$$

Since this change of momentum is caused by only the net external force, we have dropped the "ext" subscript. This is Newton's second law, but now for the entire extended object. If this feels a bit anticlimactic, remember what is hiding inside it: $\vec{p}_{CM}$ is the vector sum of the momentum of (in principle) hundreds of thousands of billions of billions of particles (6.02 x 10^{23}), all caused by one simple net external force—a force that you can calculate.

Center of Mass

Our next task is to determine what part of the extended object, if any, is obeying Equation 9.9.10.

It's tempting to take the next step; does the following equation mean anything?

$$\vec{F} = M\vec{a} \tag{9.9.11}$$

If it **does** mean something (acceleration of what, exactly?), then we could write

$$M\vec{a} = \frac{d\vec{p}_{CM}}{dt} \tag{9.9.12}$$

and thus

$$M\vec{a} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt} = \frac{d}{dt} \sum_{j=1}^{N} \vec{p}_j. \tag{9.9.13}$$

which follows because the derivative of a sum is equal to the sum of the derivatives.

Now, $\vec{p}_j$ is the momentum of the jth particle. Defining the positions of the constituent particles (relative to some coordinate system) as $\vec{r}_j$ = (x_j, y_j, z_j), we thus have

$$\vec{p}_j = m_j \vec{v}_j = m_j \frac{d\vec{r}_j}{dt}. \tag{9.9.14}$$

Substituting back, we obtain

$$\begin{aligned} M\vec{a} &= \frac{d}{dt}\sum_{j=1}^{N} m_j \frac{d\vec{r}_j}{dt} \\ &= \frac{d^2}{dt^2}\sum_{j=1}^{N} m_j \vec{r}_j. \end{aligned}$$

Dividing both sides by M (the total mass of the extended object) gives us

$$\vec{a} = \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j\right). \tag{9.9.15}$$

Thus, the point in the object that traces out the trajectory dictated by the applied force in Equation 9.9.11 is inside the parentheses in Equation 9.9.15.

Looking at this calculation, notice that (inside the parentheses) we are calculating the product of each particle's mass with its position, adding all N of these up, and dividing this sum by the total mass of particles we summed. This is reminiscent of an average; inspired by this, we'll (loosely) interpret it to be the weighted average position of the mass of the extended object. It's actually called the **center of mass** of the object. Notice that the position of the center of mass has units of meters; that suggests a definition:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j. \tag{9.9.16}$$

So, the point that obeys Equation 9.9.10 (and therefore Equation 9.9.11 as well) is the center of mass of the object, which is located at the position vector $\vec{r}_{CM}$.

It may surprise you to learn that there does not have to be any actual mass at the center of mass of an object. For example, a hollow steel sphere with a vacuum inside it is spherically symmetrical (meaning its mass is uniformly distributed about the center of the sphere); all of the sphere's mass is out on its surface, with no mass inside. But it can be shown that the center of mass of the sphere is at its geometric center, which seems reasonable. Thus, there is no mass at the position of the center of mass of the sphere. (Another example is a doughnut.) The procedure to find the center of mass is illustrated in Figure 9.9.2

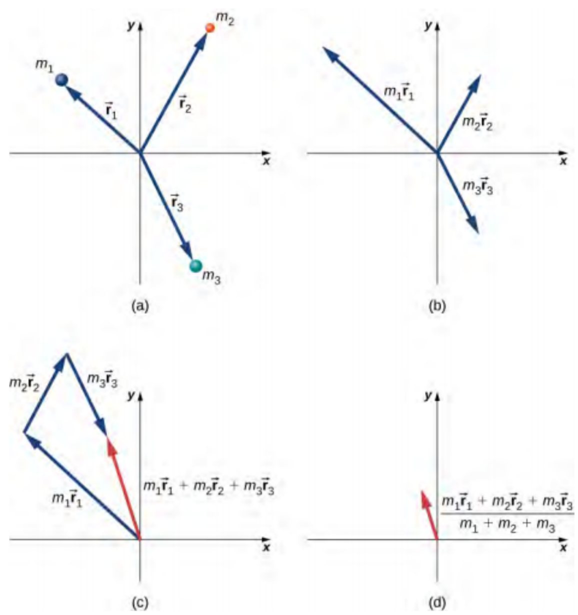


Figure 9.9.2: Finding the center of mass of a system of three different particles. (a) Position vectors are created for each object. (b) The position vectors are multiplied by the mass of the corresponding object. (c) The scaled vectors from part (b) are added together. (d) The final vector is divided by the total mass. This vector points to the center of mass of the system. Note that no mass is actually present at the center of mass of this system.

Since $\vec{r}_j = x_j\hat{i} + y_j\hat{j} + z_j\hat{k}$, it follows that:

$$r_{CM,x} = \frac{1}{m}\sum_{j=1}^{N} m_j x_j \tag{9.9.17}$$

$$r_{CM,y} = \frac{1}{m}\sum_{j=1}^{N} m_j y_j \tag{9.9.18}$$

$$r_{CM,z} = \frac{1}{m}\sum_{j=1}^{N} m_j z_j \tag{9.9.19}$$

and thus

$$\vec{r}_{CM} = r_{CM,x}\hat{i} + r_{CM,y}\hat{j} + r_{CM,z}\hat{k} \tag{9.9.20}$$

$$r_{CM} = |\vec{r}_{CM}| = (r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z})^{1/2}. \tag{9.9.21}$$

Therefore, you can calculate the components of the center of mass vector individually.

Finally, to complete the kinematics, the instantaneous velocity of the center of mass is calculated exactly as you might suspect:

$$\vec{v}_{CM} = \frac{d}{dt}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j\right) = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{v}_j \tag{9.9.22}$$

and this, like the position, has x-, y-, and z-components.

To calculate the center of mass in actual situations, we recommend the following procedure:

? Problem-Solving Strategy: Calculating the Center of Mass

The center of mass of an object is a position vector. Thus, to calculate it, do these steps:

1. Define your coordinate system. Typically, the origin is placed at the location of one of the particles. This is not required, however.
2. Determine the x, y, z-coordinates of each particle that makes up the object.
3. Determine the mass of each particle, and sum them to obtain the total mass of the object. Note that the mass of the object at the origin must be included in the total mass.
4. Calculate the x-, y-, and z-components of the center of mass vector, using Equation 9.9.17, Equation 9.9.18, and Equation 9.9.19.
5. If required, use the Pythagorean theorem to determine its magnitude.

Here are two examples that will give you a feel for what the center of mass is.

✓ Example 9.16: Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use Equation 9.9.16 with just N = 2 objects. We use a subscript "e" to refer to Earth, and subscript "m" to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, Equation 9.9.16 becomes

$$R = \frac{m_c r_c + m_m r_m}{m_c + m_m}. \tag{9.9.23}$$

From Appendix D,

$$m_c = 5.97 \times 10^{24}\ kg \tag{9.9.24}$$

$$m_m = 7.36 \times 10^{22}\ kg \tag{9.9.25}$$

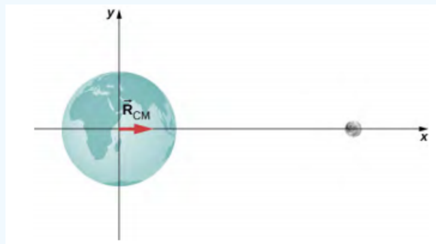
$$r_m = 3.82 \times 10^{5}\ m. \tag{9.9.26}$$

We defined the center of Earth as the origin, so $r_e = 0$ m. Inserting these into the equation for R gives

$$\begin{aligned} R &= \frac{(5.97 \times 10^{24}\ kg)(0\ m) + (7.36 \times 10^{22}\ kg)(3.82 \times 10^{8}\ m)}{(5.98 \times 10^{24}\ kg) + (7.36 \times 10^{22}\ kg)} \\ &= 4.64 \times 10^{6}\ m. \end{aligned}$$

Significance

The radius of Earth is 6.37×10^{6} m, so the center of mass of the Earth-moon system is $(6.37 - 4.64) \times 10^{6}$ m = 1.73×10^{6} m = 1730 km (roughly 1080 miles) **below** the surface of Earth. The location of the center of mass is shown (not to scale).



? Exercise 9.11

Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

✔ Example 9.17: Center of Mass of a Salt Crystal

Figure 9.9.3 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the unit cell) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36×10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates ($r_{CM,x}$, $r_{CM,y}$, $r_{CM,z}$), or by r_{CM} and two angles.

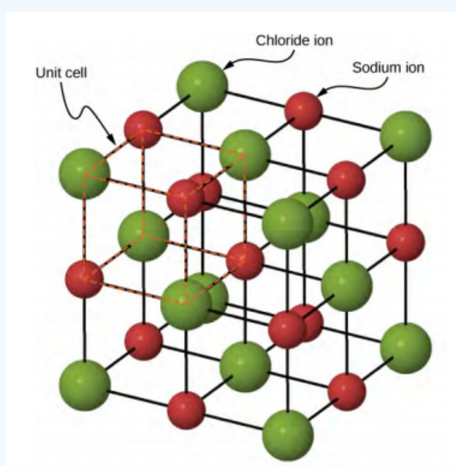


Figure 9.9.3: A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply Equation 9.9.17, Equation 9.9.18, and Equation 9.9.19 (along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. Figure 9.9.4 shows the coordinate system.

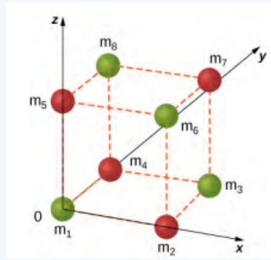


Figure 9.9.4: A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so N = 8:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{8} m_j \vec{r}_j. \tag{9.9.27}$$

The mass of each of the chloride ions is

$$35.453u \times \frac{1.660\times 10^{-27}\ kg}{u} = 5.885\times 10^{-26}\ kg \tag{9.9.28}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885\times 10^{-26}\ kg. \tag{9.9.29}$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816\times 10^{-26}\ kg. \tag{9.9.30}$$

The total mass of the unit cell is therefore

$$M = (4)(5.885\times 10^{-26}\ kg) + (4)(3.816\times 10^{-26}\ kg) = 3.880\times 10^{-25}\ kg. \tag{9.9.31}$$

From the geometry, the locations are

$$\begin{aligned}
\vec{r}_1 &= 0 \\
\vec{r}_2 &= (2.36\times 10^{-10}\ m)\hat{i} \\
\vec{r}_3 &= r_{3x}\hat{i} + r_{3y}\hat{j} = (2.36\times 10^{-10}\ m)\hat{i} + (2.36\times 10^{-10}\ m)\hat{j} \\
\vec{r}_4 &= (2.36\times 10^{-10}\ m)\hat{j} \\
\vec{r}_5 &= (2.36\times 10^{-10}\ m)\hat{k} \\
\vec{r}_6 &= r_{6x}\hat{i} + r_{6z}\hat{k} = (2.36\times 10^{-10}\ m)\hat{i} + (2.36\times 10^{-10}\ m)\hat{k} \\
\vec{r}_7 &= r_{7x}\hat{i} + r_{7y}\hat{j} + r_{7z}\hat{k} = (2.36\times 10^{-10}\ m)\hat{i} + (2.36\times 10^{-10}\ m)\hat{j} + (2.36\times 10^{-10}\ m)\hat{k} \\
\vec{r}_8 &= r_{8y}\hat{j} + r_{8z}\hat{k} = (2.36\times 10^{-10}\ m)\hat{j} + (2.36\times 10^{-10}\ m)\hat{k}.
\end{aligned}$$

Substituting:

https://phys.libretexts.org/@go/page/18220

$$\begin{aligned}
|\vec{r}_{CM,x}| &= \sqrt{r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z}} \\
&= \frac{1}{M}\sum_{j=1}^{8} m_j (r_x)_j \\
&= \frac{1}{M}(m_1 r_{1x} + m_2 r_{2x} + m_3 r_{3x} + m_4 r_{4x} + m_5 r_{5x} + m_6 r_{6x} + m_7 r_{7x} + m_8 r_{8x}) \\
&= \frac{1}{3.8804 \times 10^{-25}\ kg}\Big[(5.885 \times 10^{-26}\ kg)(0\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) \\
&+ (5.885 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0 + 0 \\
&+ (3.816 \times 10^{-26}\ kg)(2.36 \times 10^{-10}\ m) + 0\Big] \\
&= 1.18 \times 10^{-10}\ m.
\end{aligned}$$

Similar calculations give $r_{CM,y} = r_{CM,z} = 1.18 \times 10^{-10}$ m (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned}
r_{CM} &= \sqrt{r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z}} \\
&= (1.18 \times 10^{-10}\ m)\sqrt{3} \\
&= 2.044 \times 10^{-10}\ m.
\end{aligned}$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{r}_{CM}$. Let ϕ be the angle in the x,y-plane, measured from the +x-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1}\left(\frac{r_{CM,y}}{r_{CM,x}}\right) = 45^o. \tag{9.9.32}$$

Let θ be the angle in the y,z-plane, measured downward from the +z-axis; this is (not surprisingly):

$$\theta = \tan^{-1}\left(\frac{R_z}{R_y}\right) = 45^o. \tag{9.9.33}$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry

? Exercise 9.12

Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a **huge** number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. 1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in Equation 9.9.16to be zero. However, you must still include the mass of the object at your origin in your calculation of M, the total mass Equation 9.9.1. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.

This page titled 9.9: Center of Mass (Part 1) is shared under a CC BY license and was authored, remixed, and/or curated by OpenStax.

- **9.9: Center of Mass (Part 1)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/university-physics-volume-1.