9.6: Conservation of Linear Momentum (Part 2)

Problem-Solving Strategy: Conservation of Momentum

Using conservation of momentum requires four basic steps. The first step is crucial:

1. Identify a closed system (total mass is constant, no net external force acts on the system).
2. Write down an expression representing the total momentum of the system before the "event" (explosion or collision).
3. Write down an expression representing the total momentum of the system after the "event."
4. Set these two expressions equal to each other, and solve this equation for the desired quantity

Example 9.6.1: Colliding Carts

Two carts in a physics lab roll on a level track, with negligible friction. These carts have small magnets at their ends, so that when they collide, they stick together (Figure 9.6.1). The first cart has a mass of 675 grams and is rolling at 0.75 m/s to the right; the second has a mass of 500 grams and is rolling at 1.33 m/s, also to the right. After the collision, what is the velocity of the two joined carts?

Figure 9.6.1: Two lab carts collide and stick together after the collision.

Strategy

We have a collision. We're given masses and initial velocities; we're asked for the final velocity. This all suggests using conservation of momentum as a method of solution. However, we can only use it if we have a closed system. So we need to be sure that the system we choose has no net external force on it, and that its mass is not changed by the collision.

Defining the system to be the two carts meets the requirements for a closed system: The combined mass of the two carts certainly doesn't change, and while the carts definitely exert forces on each other, those forces are internal to the system, so they do not change the momentum of the system as a whole. In the vertical direction, the weights of the carts are canceled by the normal forces on the carts from the track.

Solution

Conservation of momentum is

$$\vec{p}_f = \vec{p}_i.$$

Define the direction of their initial velocity vectors to be the +x-direction. The initial momentum is then

$$\vec{p}_i = m_1 v_1 \ \hat{i} + m_2 v_2 \ \hat{i}.$$

The final momentum of the now-linked carts is

$$\vec{p}_f = (m_1 + m_2)\vec{v}_f.$$

Equating:

$$\begin{aligned} (m_1 + m_2)\vec{v}_f &= m_1 v_1 \ \hat{i} + m_2 v_2 \ \hat{i} \\ \vec{v}_f &= \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right)\hat{i}. \end{aligned}$$

Substituting the given numbers:

$$\begin{aligned} \vec{v}_f &= \left[\frac{(0.675 \ kg)(0.75 \ m/s) + (0.5 \ kg)(1.33 \ m/s)}{1.175 \ kg}\right]\hat{i} \\ &= (0.997 \ m/s)\hat{i}. \end{aligned}$$

Significance

The principles that apply here to two laboratory carts apply identically to all objects of whatever type or size. Even for photons, the concepts of momentum and conservation of momentum are still crucially important even at that scale. (Since they are massless, the momentum of a photon is defined very differently from the momentum of ordinary objects. You will learn about this when you study quantum physics.)

? Exercise 9.6.1

Suppose the second, smaller cart had been initially moving to the left. What would the sign of the final velocity have been in this case?

✓ Example 9.6.2: A Bouncing Superball

A superball of mass 0.25 kg is dropped from rest from a height of h = 1.50 m above the floor. It bounces with no loss of energy and returns to its initial height (Figure 9.6.2).

a. What is the superball's change of momentum during its bounce on the floor?
b. What was Earth's change of momentum due to the ball colliding with the floor?
c. What was Earth's change of velocity as a result of this collision?

(This example shows that you have to be careful about defining your system.)

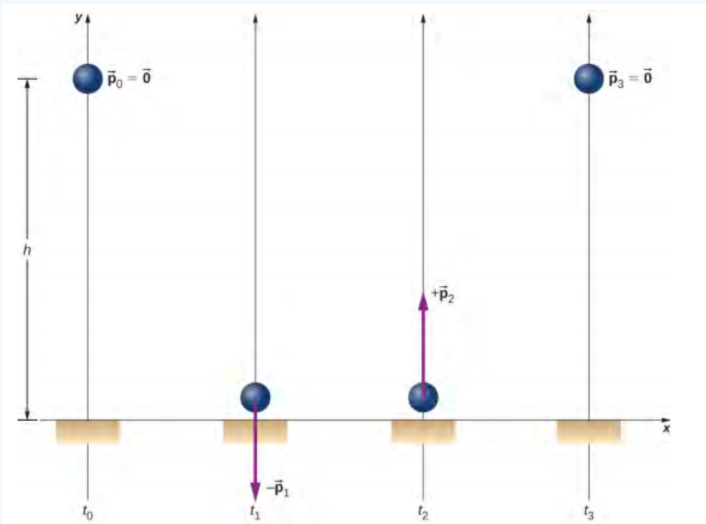

Figure 9.6.2: A superball is dropped to the floor (t_0), hits the floor (t_1), bounces (t_2), and returns to its initial height (t_3).

Strategy

Since we are asked only about the ball's change of momentum, we define our system to be the ball. But this is clearly not a closed system; gravity applies a downward force on the ball while it is falling, and the normal force from the floor applies a force during the bounce. Thus, we cannot use conservation of momentum as a strategy. Instead, we simply determine the ball's momentum just before it collides with the floor and just after, and calculate the difference. We have the ball's mass, so we need its velocities.

Solution

a. Since this is a one-dimensional problem, we use the scalar form of the equations. Let:
   - p_0 = the magnitude of the ball's momentum at time t_0, the moment it was released; since it was dropped from rest, this is zero.
   - p_1 = the magnitude of the ball's momentum at time t_1, the instant just before it hits the floor.
   - p_2 = the magnitude of the ball's momentum at time t_2, just after it loses contact with the floor after the bounce.

The ball's change of momentum is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= p_2\ \hat{j} - (-p_1\ \hat{j}) \\ &= (p_2 + p_1)\hat{j}.\end{aligned}$$

Its velocity just before it hits the floor can be determined from either conservation of energy or kinematics. We use kinematics here; you should re-solve it using conservation of energy and confirm you get the same result.

We want the velocity just before it hits the ground (at time t_1). We know its initial velocity $v_0 = 0$ (at time t_0), the height it falls, and its acceleration; we don't know the fall time. We could calculate that, but instead we use

$$\vec{v}_1 = -\hat{j}\sqrt{2gy} = -5.4\ m/s\ \hat{j}.$$

Thus the ball has a momentum of

$$\begin{aligned}\vec{p}_1 &= -(0.25\ kg)(-5.4\ m/s\ \hat{j}) \\ &= -(1.4\ kg \cdot m/s)\hat{j}.\end{aligned}$$

We don't have an easy way to calculate the momentum after the bounce. Instead, we reason from the symmetry of the situation.

Before the bounce, the ball starts with zero velocity and falls 1.50 m under the influence of gravity, achieving some amount of momentum just before it hits the ground. On the return trip (after the bounce), it starts with some amount of momentum, rises the same 1.50 m it fell, and ends with zero velocity. Thus, the motion after the bounce was the mirror image of the motion before the bounce. From this symmetry, it must be true that the ball's momentum after the bounce must be equal and opposite to its momentum before the bounce. (This is a subtle but crucial argument; make sure you understand it before you go on.) Therefore,

$$\vec{p}_2 = -\vec{p}_1 = +(1.4\ kg \cdot m/s)\hat{j}.$$

Thus, the ball's change of momentum during the bounce is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= (1.4\ kg \cdot m/s)\hat{j} - (-1.4\ kg \cdot m/s)\hat{j} \\ &= +(2.8\ kg \cdot m/s)\hat{j}.\end{aligned}$$

b. What was Earth's change of momentum due to the ball colliding with the floor? Your instinctive response may well have been either "zero; the Earth is just too massive for that tiny ball to have affected it" or possibly, "more than zero, but utterly negligible." But no—if we re-define our system to be the Superball + Earth, then this system is closed (neglecting the gravitational pulls of the Sun, the Moon, and the other planets in the solar system), and therefore the total change of momentum of this new system must be zero. Therefore, Earth's change of momentum is exactly the same magnitude:

$$\Delta\vec{p}_{Earth} = -2.8\ kg \cdot m/s\ \hat{j}. \tag{9.6.1}$$

c. What was Earth's change of velocity as a result of this collision? This is where your instinctive feeling is probably correct:

$$\begin{aligned}\Delta\vec{v}_{Earth} &= \frac{\Delta\vec{p}_{Earth}}{M_{Earth}} \\ &= -\frac{2.8\ kg \cdot m/s}{5.97 \times 10^{24}\ kg}\ \hat{j} \\ &= -(4.7 \times 10^{-25}\ m/s)\hat{j}.\end{aligned}$$

This change of Earth's velocity is utterly negligible

Significance

It is important to realize that the answer to part (c) is not a velocity; it is a change of velocity, which is a very different thing. Nevertheless, to give you a feel for just how small that change of velocity is, suppose you were moving with a velocity of 4.7 x 10^{-25} m/s. At this speed, it would take you about 7 million years to travel a distance equal to the diameter of a hydrogen atom.

Exercise 9.6.2

Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)? Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

Example 9.6.3: Ice hockey 1

Two hockey pucks of identical mass are on a flat, horizontal ice hockey rink. The red puck is motionless; the blue puck is moving at 2.5 m/s to the left (Figure 9.6.3). It collides with the motionless red puck. The pucks have a mass of 15 g. After the collision, the red puck is moving at 2.5 m/s, to the left. What is the final velocity of the blue puck?

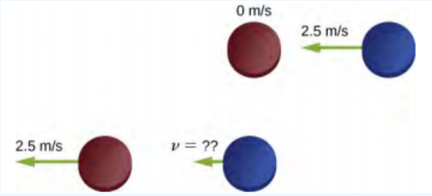

Figure 9.6.3: Two identical hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, we're told the masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy. Define the system to be the two pucks; there's no friction, so we have a closed system.

Before you look at the solution, what do you think the answer will be?

The blue puck final velocity will be:

a. zero
b. 2.5 m/s to the left
c. 2.5 m/s to the right
d. 1.25 m/s to the left
e. 1.25 m/s to the right
f. something else

Solution

Define the +x-direction to point to the right. Conservation of momentum then reads

$$\begin{aligned} \overrightarrow{p_f} &= \overrightarrow{p_i} \\ mv_{r_f}\ \hat{i} + mv_{b_f}\ \hat{i}\ &= mv_{r_i}\ \hat{i} - mv_{b_i}\ \hat{i}. \end{aligned}$$

Before the collision, the momentum of the system is entirely and only in the blue puck. Thus,

$$mv_{r_f}\ \hat{i} + mv_{b_f}\ \hat{i} = -mv_{b_i}\ \hat{i}$$

$$v_{r_f}\ \hat{i} + v_{b_f}\ \hat{i} = -v_{b_i}\ \hat{i}.$$

(Remember that the masses of the pucks are equal.) Substituting numbers:

$$\begin{aligned} -(2.5\ m/s)\hat{i} + \vec{v}_{b_f}\ &= -(2.5\ m/s)\hat{i} \\ \vec{v}_{b_f}\ &= 0. \end{aligned}$$

Significance

Evidently, the two pucks simply exchanged momentum. The blue puck transferred all of its momentum to the red puck. In fact, this is what happens in similar collision where $m_1 = m_2$.

? Exercise 9.6.3

Even if there were some friction on the ice, it is still possible to use conservation of momentum to solve this problem, but you would need to impose an additional condition on the problem. What is that additional condition?

✓ Philae

On November 12, 2014, the European Space Agency successfully landed a probe named **Philae** on Comet 67P/ Churyumov/Gerasimenko (Figure 9.6.4). During the landing, however, the probe actually landed three times, because it bounced twice. Let's calculate how much the comet's speed changed as a result of the first bounce.

Figure 9.6.4: An artist's rendering of Philae landing on a comet. (credit: modification of work by "DLR German Aerospace Center"/Flickr)

Let's define upward to be the +y-direction, perpendicular to the surface of the comet, and y = 0 to be at the surface of the comet. Here's what we know:

- The mass of Comet 67P: $M_c = 1.0 \times 10^{13}$ kg
- The acceleration due to the comet's gravity: $\vec{a} = -(5.0 \times 10^{-3}\ m/s^2)\ \hat{j}$
- **Philae**'s mass: $M_p = 96$ kg
- Initial touchdown speed: $\vec{v}_1 = -(1.0\ m/s)\ \hat{j}$
- Initial upward speed due to first bounce: $\vec{v}_2 = (0.38\ m/s)\ \hat{j}$
- Landing impact time: $\Delta t = 1.3$ s

Strategy

We're asked for how much the comet's speed changed, but we don't know much about the comet, beyond its mass and the acceleration its gravity causes. However, we are told that the **Philae** lander collides with (lands on) the comet, and bounces off of it. A collision suggests momentum as a strategy for solving this problem.

If we define a system that consists of both **Philae** and Comet 67/P, then there is no net external force on this system, and thus the momentum of this system is conserved. (We'll neglect the gravitational force of the sun.) Thus, if we calculate the change of momentum of the lander, we automatically have the change of momentum of the comet. Also, the comet's change of velocity is directly related to its change of momentum as a result of the lander "colliding" with it.

Solution

Let $\vec{p}_1$ be **Philae**'s momentum at the moment just before touchdown, and $\vec{p}_2$ be its momentum just after the first bounce. Then its momentum just before landing was

$$\vec{p}_1 = M_p\vec{v}_1 = (96\ kg)(-1.0\ m/s\ \hat{j}) = -(96\ kg \cdot m/s)\hat{j}$$

and just after was

$$\vec{p}_2 = M_p\vec{v}_2 = (96\ kg)(+0.38\ m/s\ \hat{j}) = (36.5\ kg \cdot m/s)\hat{j}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2\vec{p}_1 \\ &= (36.5\ kg\cdot m/s)\hat{j} - (-96.0\ kg\cdot m/s\ \hat{j}) \\ &= (133\ kg\cdot m/s)\hat{j}\end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the comet's momentum changed by exactly the negative of this:

$$\Delta\vec{p}_c = -\Delta\vec{p} = -(133\ kg\cdot m/s)\hat{j}.$$

Therefore, its change of velocity is

$$\Delta\vec{v}_c = \frac{\Delta\vec{p}_c}{M_c} = \frac{-(133\ kg\cdot m/s)\hat{j}}{1.0\times 10^{13}\ kg} = -(1.33\times 10^{-11}\ m/s)\hat{j}.$$

Significance

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is **not** zero.

? Exercise 9.6.4

The changes of momentum for **Philae** and for Comet 67/P were equal (in magnitude). Were the impulses experienced by **Philae** and the comet equal? How about the forces? How about the changes of kinetic energies?