Joliet Junior College

JJC Physics 201 Engineering Physics I

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TABLE OF CONTENTS

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CHAPTER OVERVIEW

1: The Basics of Physics

Topic hierarchy

1.1: The Basics of Physics

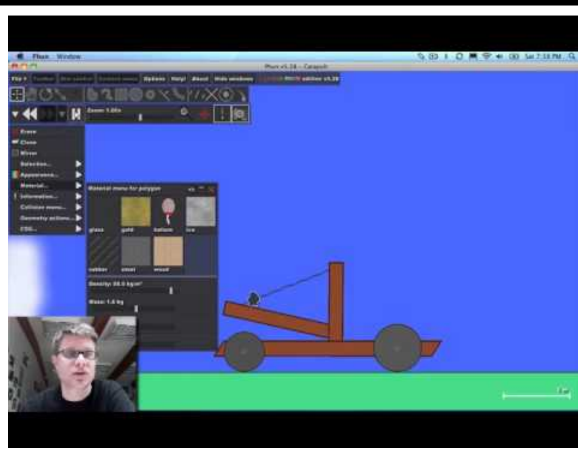
Introduction: Physics and Matter

Physics is a study of how the universe behaves.

learning objectives

- Apply physics to describe the function of daily life

Physics is a natural science that involves the study of matter and its motion through space and time, along with related concepts such as energy and force. More broadly, it is the study of nature in an attempt to understand how the universe behaves.

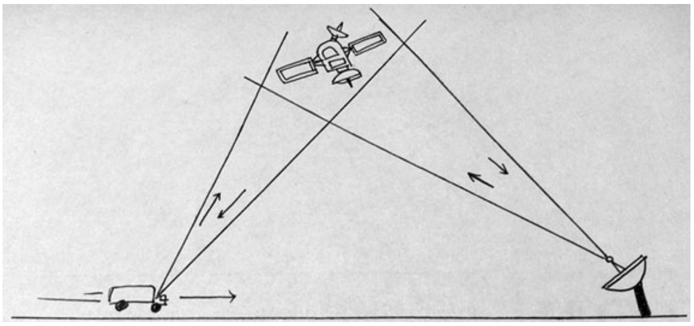


What is Physics?: Mr. Andersen explains the importance of physics as a science. History and virtual examples are used to give the discipline context.

Physics uses the scientific method to help uncover the basic principles governing light and matter, and to discover the implications of those laws. It assumes that there are rules by which the universe functions, and that those laws can be at least partially understood by humans. It is also commonly believed that those laws could be used to predict everything about the universe's future if complete information was available about the present state of all light and matter.

Matter is generally considered to be anything that has mass and volume. Many concepts integral to the study of classical physics involve theories and laws that explain matter and its motion. The law of conservation of mass, for example, states that mass cannot be created or destroyed. Further experiments and calculations in physics, therefore, take this law into account when formulating hypotheses to try to explain natural phenomena.

Physics aims to describe the function of everything around us, from the movement of tiny charged particles to the motion of people, cars, and spaceships. In fact, almost everything around you can be described quite accurately by the laws of physics. Consider a smart phone; physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and circuit layout when building the smart phone. Next, consider a GPS system; physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS device in a vehicle, it utilizes these physics equations to determine the travel time from one location to another. The study of physics is capable of making significant contributions through advances in new technologies that arise from theoretical breakthroughs.



Global Positioning System: GPS calculates the speed of an object, the distance over which it travels, and the time it takes to travel that distance using equations based on the laws of physics.

Physics and Other Fields

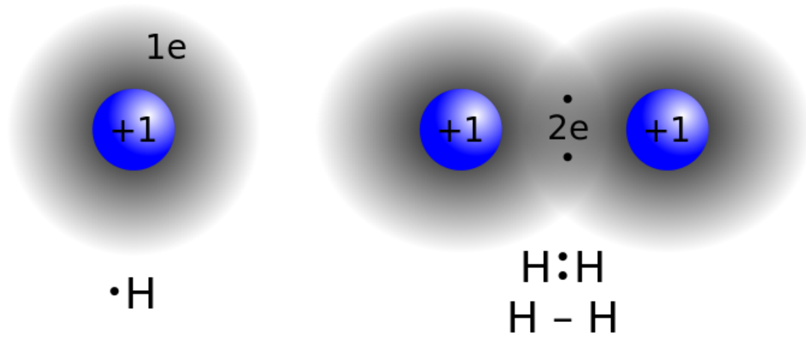
Physics is the foundation of many disciplines and contributes directly to chemistry, astronomy, engineering, and most scientific fields.

learning objectives

- Explain why the study of physics is integral to the study of other sciences

Physics and Other Disciplines

Physics is the foundation of many important disciplines and contributes directly to others. Chemistry deals with the interactions of atoms and molecules, so it is rooted in atomic and molecular physics. Most branches of engineering are applied physics. In architecture, physics is at the heart of structural stability and is involved in acoustics, heating, lighting, and the cooling of buildings. Parts of geology rely heavily on physics, such as the radioactive dating of rocks, earthquake analysis, and heat transfer in the Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.



Physics in Chemistry: The study of matter and electricity in physics is fundamental towards the understanding of concepts in chemistry, such as the covalent bond.

Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cell walls and cell membranes. On the macroscopic level, it can explain the heat, work, and power associated with the human body. Physics is involved in medical diagnostics, such as X-rays, magnetic resonance imaging (MRI), and ultrasonic blood flow measurements. Medical therapy sometimes directly involves physics: cancer radiotherapy uses ionizing radiation, for instance. Physics can also explain sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers can transmit information.

The boundary between physics and the other sciences is not always clear. For instance, chemists study atoms and molecules, which are what matter is built from, and there are some scientists who would be equally willing to call themselves physical chemists or chemical physicists. It might seem that the distinction between physics and biology would be clearer, since physics seems to deal with inanimate objects. In fact, almost all physicists would agree that the basic laws of physics that apply to molecules in a test tube work equally well for the combination of molecules that constitutes a bacterium. What differentiates physics from biology is that many of the scientific theories that describe living things ultimately result from the fundamental laws of physics, but cannot be rigorously derived from physical principles.

It is not necessary to formally study all applications of physics. What is most useful is the knowledge of the basic laws of physics and skill in the analytical methods for applying them. The study of physics can also improve your problem-solving skills. Furthermore, physics has retained the most basic aspects of science, so it is used by all of the sciences. The study of physics makes other sciences easier to understand.

Models, Theories, and Laws

The terms *model*, *theory*, and *law* have exact meanings in relation to their usage in the study of physics.

learning objectives

- Define the terms model, theory, and law

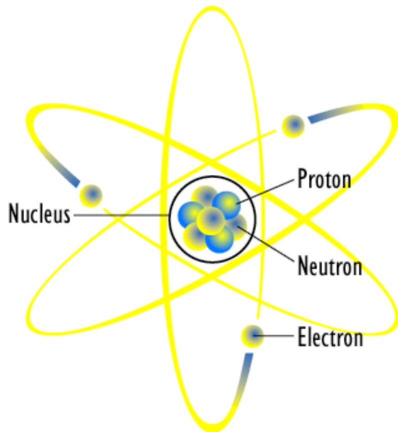
Definition of Terms: Model, Theory, Law

In colloquial usage, the terms *model*, *theory*, and *law* are often used interchangeably or have different interpretations than they do in the sciences. In relation to the study of physics, however, each term has its own specific meaning.

The *laws of nature* are concise descriptions of the universe around us. They are not explanations, but human statements of the underlying rules that all natural processes follow. They are intrinsic to the universe; humans did not create them and we cannot change them. We can only discover and understand them. The cornerstone of discovering natural laws is observation; science must describe the universe as it is, not as we may imagine it to be. Laws can never be known with absolute certainty, because it is impossible to perform experiments to establish and confirm a law in every possible scenario without exception. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable experiment contradicts a well-established law, then the law must be modified or overthrown completely.

Models

A *model* is a representation of something that is often too difficult (or impossible) to display directly. While a model's design is justified using experimental information, it is only accurate under limited situations. An example is the commonly used "planetary model" of the atom, in which electrons are pictured as orbiting the nucleus, analogous to the way planets orbit the Sun. We cannot observe electron orbits directly, but the mental image helps explain the observations we can make, such as the emission of light from hot gases. Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation, or they can be used to represent a situation in the form of a computer simulation.



Planetary Model of an Atom: The planetary model of the atom in which electrons are pictured as orbiting the nucleus, analogous to the way planets orbit the Sun

Theories

A *theory* is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers. *Some theories include models to help visualize phenomena, whereas others do not*. Newton's theory of gravity, for example, does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, makes use of a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what our instruments tell us about the behavior of gases.

Laws

A law uses concise language to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the designation law is reserved for a concise and very general statement that describes phenomena in nature, such as the law that energy is conserved during any process, or Newton's second law of motion, which relates force, mass, and acceleration by the simple equation $F = ma$. A theory, in contrast, is a less concise statement of observed phenomena. For example, the Theory of Evolution and the Theory of Relativity cannot be expressed concisely enough to be considered a law. The biggest difference between a law and a theory is that a law is much more complex and dynamic, and a theory is more explanatory. A law describes a single observable point of fact, whereas a theory explains an entire group of related phenomena. And, whereas a law is a postulate that forms the foundation of the scientific method, a theory is the end result of that process.

Key Points

- Physics is a natural science that involves the study of matter and its motion through space and time, along with related concepts such as energy and force.
- Matter is generally considered to be anything that has mass and volume.
- Scientific laws and theories express the general truths of nature and the body of knowledge they encompass. These laws of nature are rules that all natural processes appear to follow.
- Many scientific disciplines, such as biophysics, are hybrids of physics and other sciences.
- The study of physics encompasses all forms of matter and its motion in space and time.

- The application of physics is fundamental towards significant contributions in new technologies that arise from theoretical breakthroughs.
- Concepts in physics cannot be proven, they can only be supported or disproven through observation and experimentation.
- A model is an evidence-based representation of something that is either too difficult or impossible to display directly.
- A theory is an explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers.
- A law uses concise language, often expressed as a mathematical equation, to describe a generalized pattern in nature that is supported by scientific evidence and repeated experiments.

Key Terms

- **matter**: The basic structural component of the universe. Matter usually has mass and volume.
- **scientific method**: A method of discovering knowledge about the natural world based in making falsifiable predictions (hypotheses), testing them empirically, and developing peer-reviewed theories that best explain the known data.
- **application**: the act of putting something into operation
- **Model**: A representation of something difficult or impossible to display directly
- **Law**: A concise description, usually in the form of a mathematical equation, used to describe a pattern in nature
- **theory**: An explanation for patterns in nature that is supported by scientific evidence and verified multiple times by various groups of researchers

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1.2: Units

Length

Length is a physical measurement of distance that is fundamentally measured in the SI unit of a meter.

learning objectives

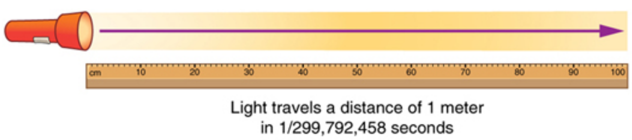
- Distinguish SI and customary units of length

Length can be defined as a measurement of the physical quantity of distance. Many qualitative observations fundamental to physics are commonly described using the measurement of length. The distance between objects, the rate at which objects are traveling, and how much force an object exerts are all dependent on length as a variable. In order to describe length in a standardized and quantitative manner, an accepted unit of measurement must be utilized.

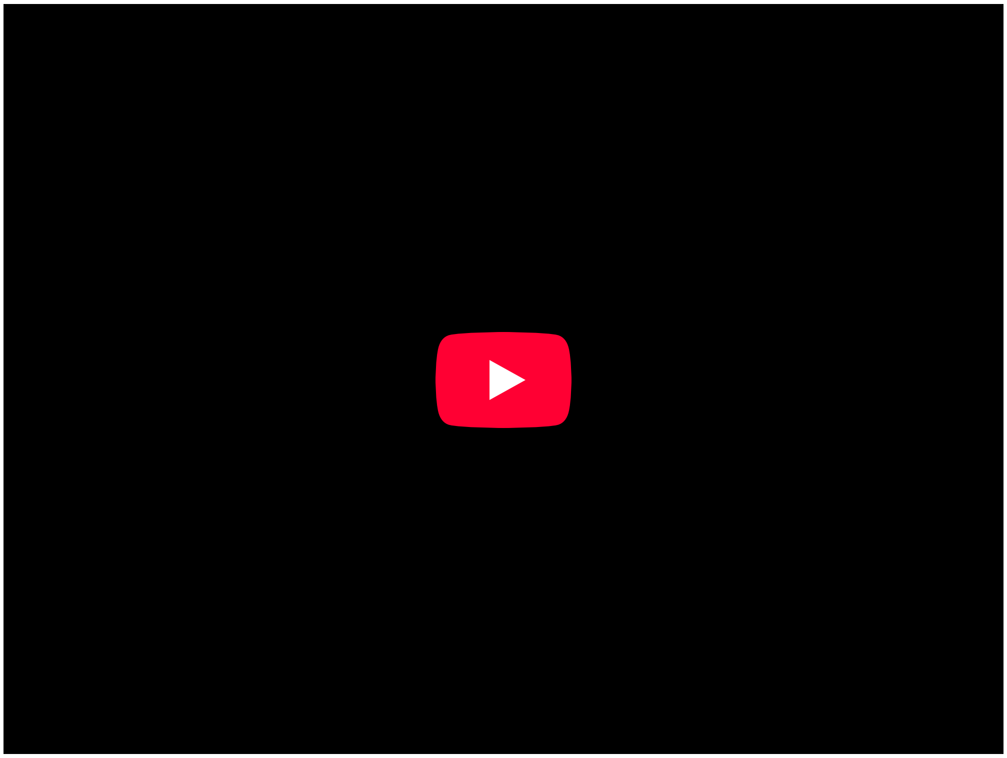
Many different units of length are used around the world. In the United States, the U.S. customary units operationally describe length in terms of the basic unit of an inch. Varying lengths are thus described in relation to the inch, such as a foot equaling 12 inches, a yard equaling three feet, and a mile equaling 1,760 yards.

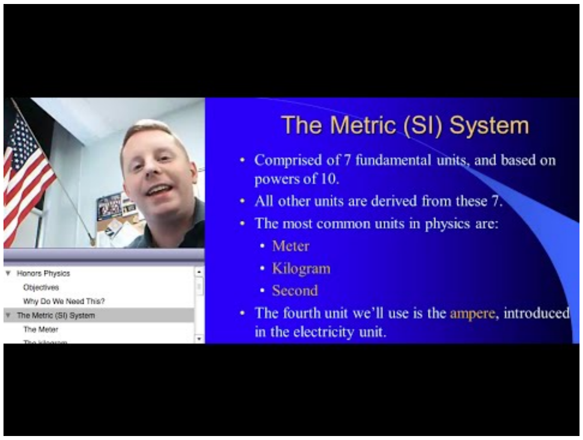
Though regional use of different measurement units is not generally problematic, it can raise issues of compatibility and understanding when working abroad or collaboratively with international partners. As such, a standard unit of measurement that is internationally accepted is needed. The basic unit of length as identified by the International System of Units (SI) is the meter. The meter is expressed more specifically in terms of speed of light.

One meter is defined as the distance that light travels in a vacuum in $\frac{1}{299,792,458}$ of a second. All lengths are measured in terms related to the meter, where its multiples are devised around the convenience of the number 10. For example, a centimeter is equal to $\frac{1}{100}$ of a meter (or 10^{-2} meters), and a kilometer is equal to 1,000 meters (or 10^3 meters).



Meter Defined by Speed of Light: The meter is defined to be the distance that light travels in $\frac{1}{299,792,458}$ of a second in a vacuum. Distance traveled is speed multiplied by time.





Metric System – Length: A brief introduction to the metric system and unit conversions.

Mass

Mass is the quantity of matter that an object contains, as measured by its resistance to acceleration.

learning objectives

- Explain the difference between mass and weight

Mass

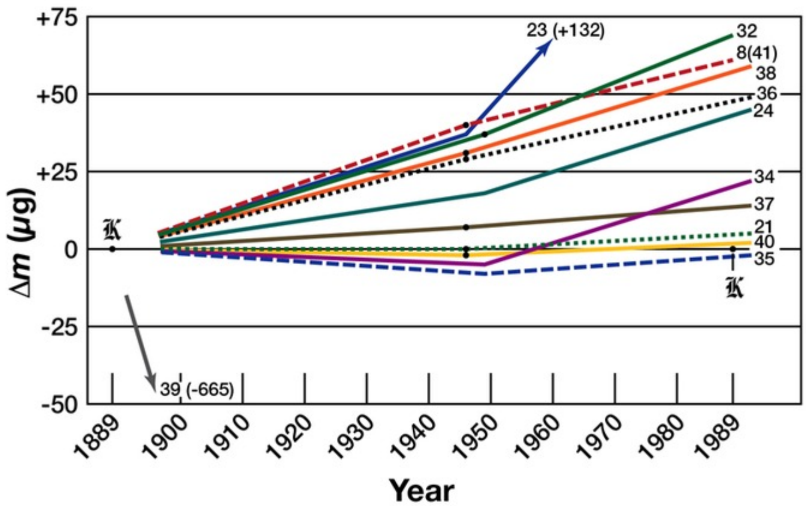
Mass, specifically inertial mass, is a quantitative measure of an object's resistance to acceleration. It is an intrinsic property of an object and does not change because of the environment. The SI unit of mass is the kilogram (kg).

The kilogram is defined as being equal to the mass of the International Prototype Kilogram (IPK), which is almost exactly equal to the mass of one liter of water. Until the year 2019, it was also the only SI unit that is directly defined by an artifact, rather than a fundamental physical property that can be reproduced in different laboratories. Four of the seven base units in the SI system are defined relative to the kilogram, so the stability of this measurement is crucial for accurate and consistent measurements.

In 2005, the International Committee for Weights and Measures (CIPM) recommended that the kilogram be redefined in terms of a fundamental constant of nature, due to evidence that the International Prototype Kilogram will vary in mass over time. At its 2011 meeting, the General Conference on Weights and Measures (CGPM) agreed that the kilogram should be redefined in terms of the Planck constant. The conference deferred a final decision until its next meeting in 2014.

Opportunity

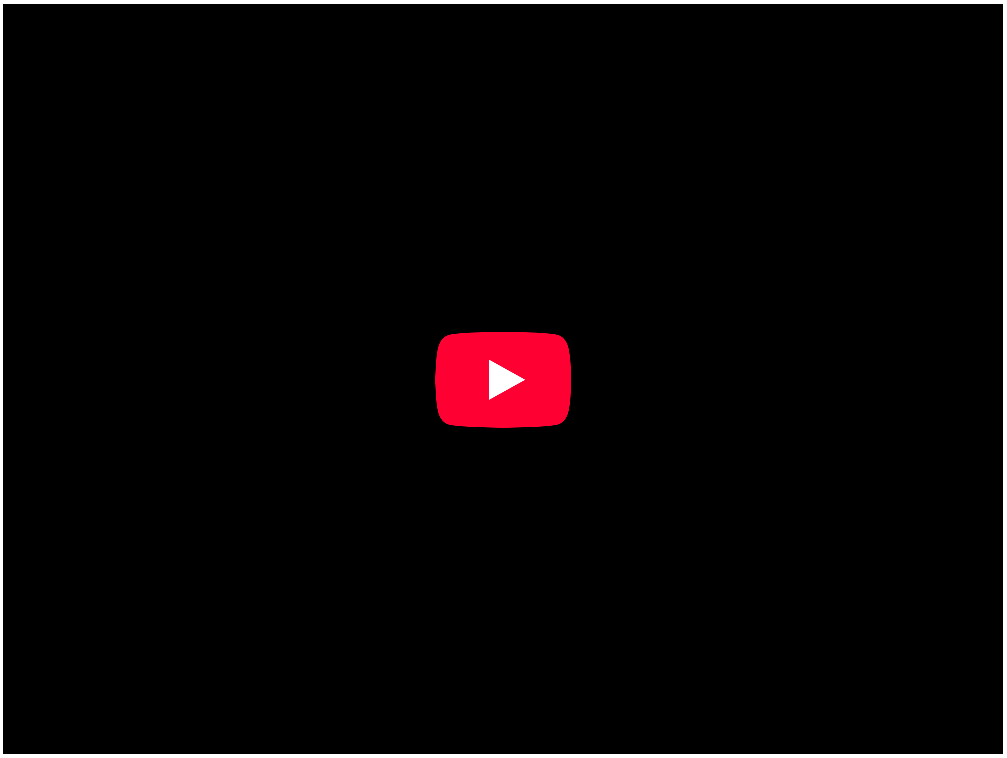
- Explain the story of how the kilogram came to be redefined in 2019.
- Update the graphic below

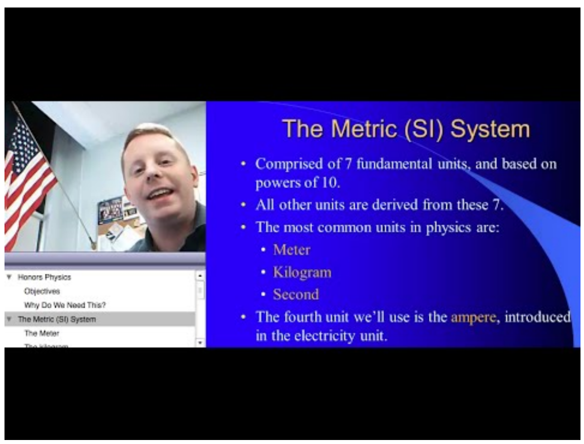


Prototype Mass Drifts: A graph of the relative change in mass of selected kilogram prototypes.

Mass and Weight

In everyday usage, the mass of an object in kilograms is often referred to as its weight. This value, though given in kilograms, is actually the non-SI unit of measure known as the kilogram- force. In scientific terms, 'weight' refers to the gravitational force acting on a given body. This measurement changes depending on the gravitational pull of the opposing body. For example, a person's weight on the Earth is different than a person's weight on the moon because of the differences in the gravitational pull of each body. In contrast, the mass of an object is an intrinsic property and remains the same regardless of gravitational fields. Accordingly, astronauts in microgravity must exert 10 times more force to accelerate a 10-kg object at the same rate as a 1-kg object, even though the differences in weight are imperceptible.





Metric System – Mass: A brief introduction to the metric system and unit conversions.

Time

Time is the fundamental physical quantity of duration and is measured by the SI Unit known as the second.

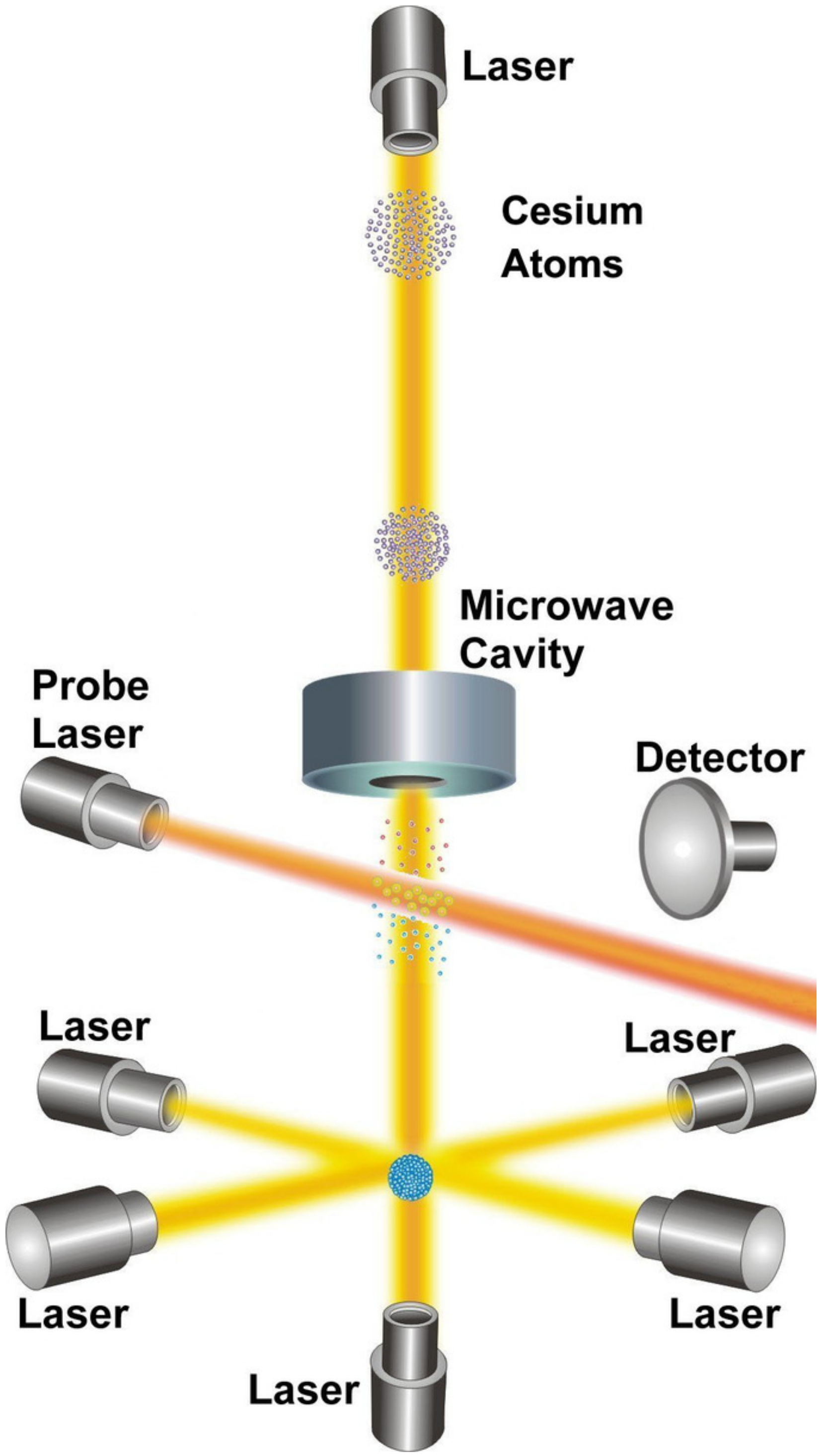
learning objectives

- Relate time with other physical quantities

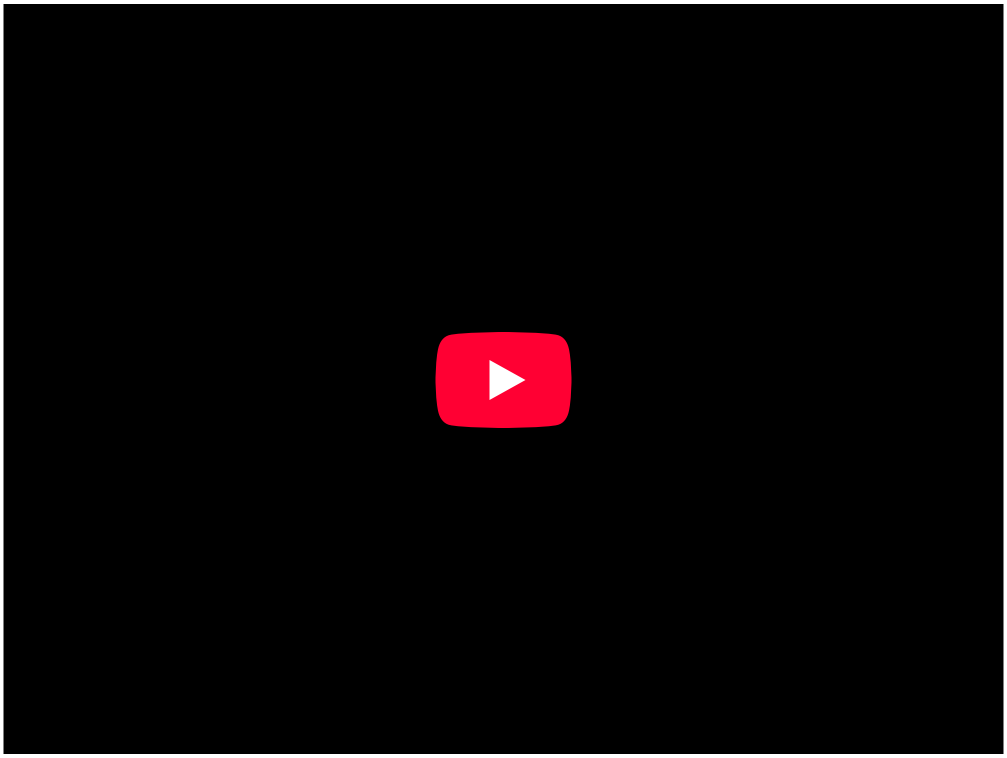
Time is one of the seven fundamental physical quantities in the International System (SI) of Units. Time is used to define other quantities, such as velocity or acceleration, and as such, it is important that it be standardized and quantified precisely. An operational definition of time is highly useful in the conduct of both advanced experiments and everyday affairs of life.

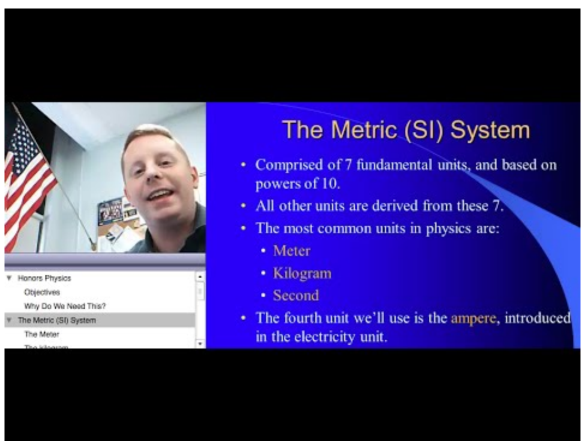
Historically, temporal measurement was a prime motivation in navigation and astronomy. Periodic events and motion have long served as standards for units of time. For example, the movement of the sun across the sky, the phases of the moon, the swing of a pendulum, and the beat of a heart have all been used as a standard for time keeping. These events and standards, however, are highly dynamic in nature and cannot reliably be utilized for accurate quantitative measures. Between 1000 and 1960 the second was defined as $\frac{1}{86,400}$ of a mean solar day. This definition changed between 1960 and 1967 and was defined in terms of the period of the Earth’s orbit around the Sun in 1900. Today, the SI Unit of the second is defined in terms of radiation emitted by cesium atoms.

The second is now operationally defined as “the duration of 9,192,631,770 periods of the radiation corresponding to the transition between the two hyperfine levels of the ground state of the cesium 133 atom.” It follows that the hyperfine splitting in the ground state of the cesium 133 atom is exactly 9,192,631,770 hertz. In other words, cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. The second is the time required for 9,192,631,770 of these vibrations to occur.



NIST-F1 Cesium Clock: NIST-F1 is referred to as a fountain clock because it uses a fountain-like movement of atoms to obtain its improved reckoning of time.





Metric System – Time: A brief introduction to the metric system and unit conversions.

Prefixes and Other Systems of Units

SI prefixes precede a basic unit of measure to indicate a multiple or fraction of the unit.

learning objectives

- Apply prefixes to units and distinguish between SI and customary units

Prefixes

A metric prefix, or SI prefix, is a unit prefix that precedes a basic unit of measure to indicate a multiple or fraction of the unit. Each prefix has a unique symbol that is prepended to the unit symbol. The prefix kilo-, for example, may be added to gram to indicate multiplication by one thousand; one kilogram is equal to one thousand grams (1 kg = 1000 g). The prefix centi-, likewise, may be added to meter to indicate division by one hundred; one centimeter is equal to one hundredth of a meter (1 cm = 0.01 m). Prefixes in varying multiples of 10 are a feature of all forms of the metric system, with many dating back to the system's introduction in the

1790s. Today, the prefixes are standardized for use in the International System of Units (SI) by the International Bureau of Weights and Measures.There are twenty prefixes officially specified by SI.

Metric prefixes

Prefix	Symbol	1000^m	10^n	Decimal	Short scale	Long scale
yotta	Y	1000^8	10^{24}	1000000000000000000000000	septillion	quadrillion
zetta	Z	1000^7	10^{21}	1000000000000000000000	sextillion	trilliard
exa	E	1000^6	10^{18}	1000000000000000000	quintillion	trillion
peta	P	1000^5	10^{15}	1000000000000000	quadrillion	billiard
tera	T	1000^4	10^{12}	1000000000000	trillion	billion
giga	G	1000^3	10^9	1000000000	billion	milliard
mega	M	1000^2	10^6	1000000	million	
kilo	k	1000^1	10^3	1000	thousand	
hecto	h	$1000^{2/3}$	10^2	100	hundred	
deca	da	$1000^{1/3}$	10^1	10	ten	
		1000^0	10^0	1	one	
deci	d	$1000^{-1/3}$	10^{-1}	0.1	tenth	
centi	c	$1000^{-2/3}$	10^{-2}	0.01	hundredth	
milli	m	1000^{-1}	10^{-3}	0.001	thousandth	
micro	µ	1000^{-2}	10^{-6}	0.000001	millionth	
nano	n	1000^{-3}	10^{-9}	0.000000001	billionth	milliardth
pico	p	1000^{-4}	10^{-12}	0.000000000001	trillionth	billionth
femto	f	1000^{-5}	10^{-15}	0.000000000000001	quadrillionth	billiardth
atto	a	1000^{-6}	10^{-18}	0.000000000000000001	quintillionth	trillionth
zepto	z	1000^{-7}	10^{-21}	0.000000000000000000001	sextillionth	trilliardth
yocto	y	1000^{-8}	10^{-24}	0.000000000000000000000001	septillionth	quadrillionth

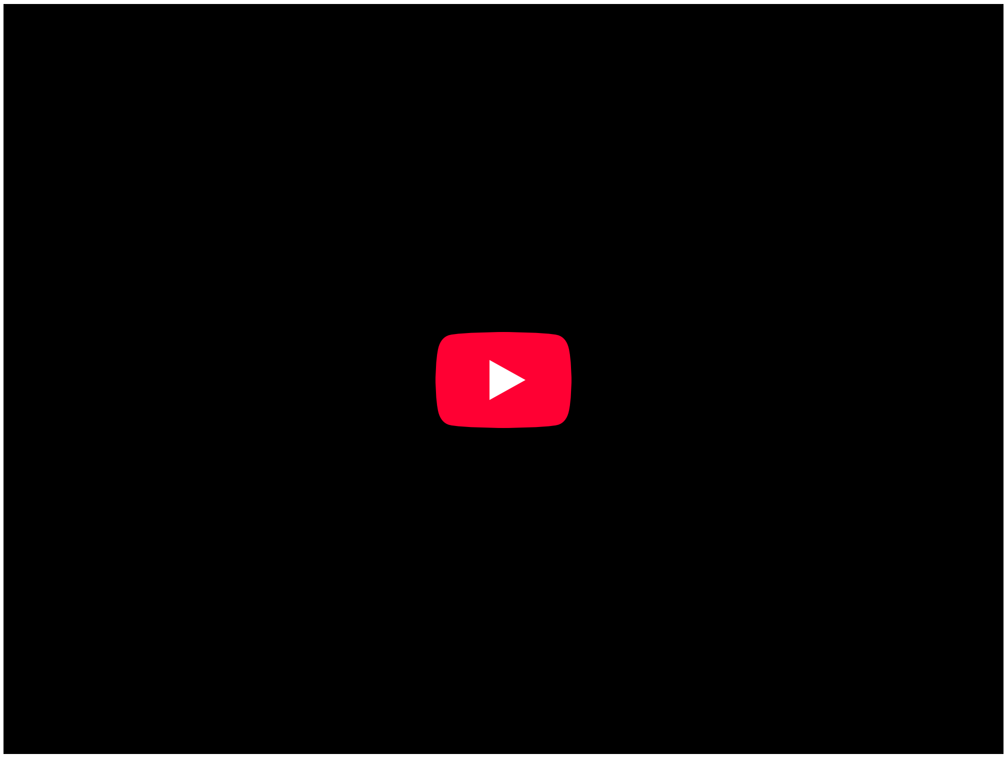
SI Unit Prefixes: The twenty prefixes officially specified by the International System of Units

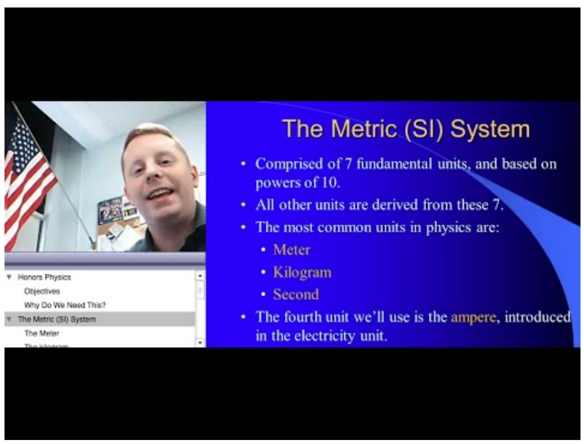
It is important to note that the kilogram is the only SI unit with a prefix as part of its name and symbol. Because multiple prefixes may not be used, in the case of the kilogram the prefix names are used with the unit name “gram” and the prefix symbols are used with the unit symbol “g. ” With this exception, any SI prefix may be used with any SI unit, including the degree Celsius and its symbol °C.

Other Systems of Units

The SI Unit system, or the metric system, is used by the majority of countries in the world, and is the standard system agreed upon by scientists and mathematicians. Colloquially, however, other systems of units are used in many countries. The United States, for example, teaches and uses the *United States customary units*. This system of units was developed from the English, or Imperial, unit standards of the United Kingdom.The United States customary units define measurements using different standards than those used in SI Units. The system for measuring length using the United States customary system is based on the inch, foot, yard, and mile. Likewise, units of area are measured in terms of square feet, and units of capacity and volume are measured in terms of cubic inches, cubic feet, or cubic yards. Units of mass are commonly defined in terms of ounces and pounds, rather than the SI unit of kilograms.Other commonly used units from the United States customary system include the fluid volume units of the teaspoon, tablespoon, fluid ounce, US cup, pint, quart, and gallon, as well as the degrees Fahrenheit used to measure temperature.

Some units that are widely used are not a part of the International System of Units and are considered Non-SI Units. These units, though not officially part of SI Units, are generally accepted for use in conjunction with SI units. These can include the minute, hour, and day used in temporal measurements, the liter for volumetric measurements, and the degree, minute, and second used to measure angles.





Metric System – Prefixes: A brief introduction to the metric system and unit conversions.

Converting Units

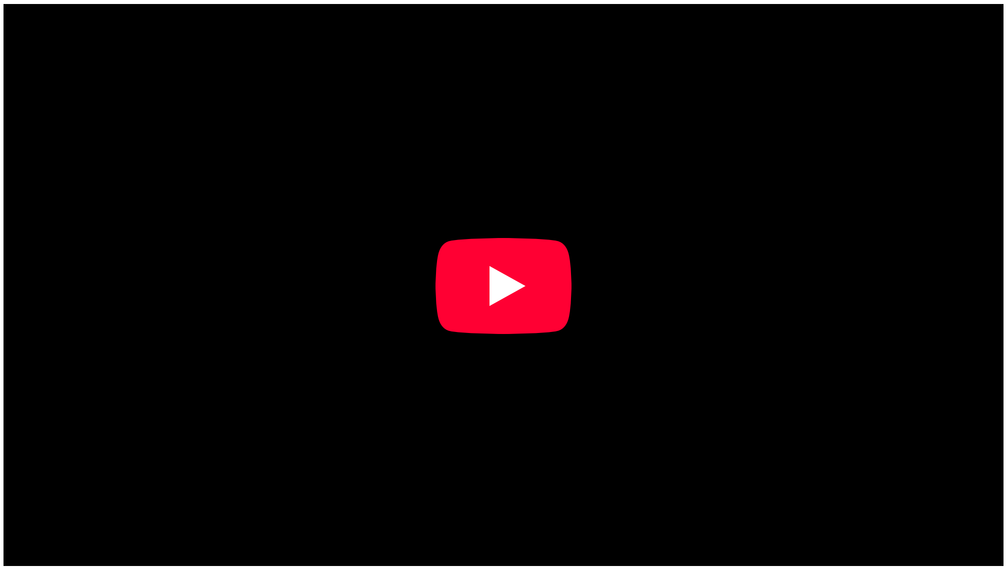
Converting between units can be done through the use of conversion factors or specific conversion formulas.

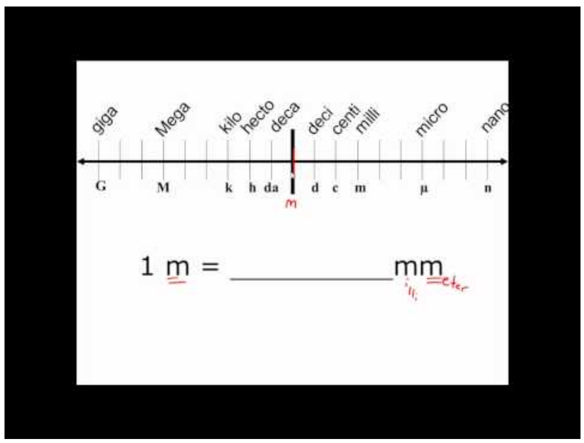
learning objectives

- Apply factor-label method for converting units

Translating Systems of Measurement

It is often necessary to convert from one type of unit to another. Conversion of units is the conversion of different units of measurement for the same quantity, typically using conversion factors. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters; if you're cooking in the US in a standard kitchen with standard tools, you will need to convert those measurements to cups. Or, perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you will need to convert units of feet to miles. This is a bit like translating a substitution code, using a formula that helps you understand what one measure means in terms of another system.





Unit Conversion in the Metric System: EASY Unit Conversion in the Metric System – This simple extra help video tutorial explains the metric system and how to make simple metric conversions.

Conversion Methods

There are several ways to approach doing conversions. One commonly used method is known as the Factor-label method for converting units, or the “railroad method. ”

The factor-label method is the sequential application of conversion factors expressed as fractions and arranged so that any dimensional unit appearing in both the numerator and denominator of any of the fractions can be cancelled out until only the desired set of dimensional units is obtained. For example, 10 miles per hour can be converted to meters per second by using a sequence of conversion factors.

Each conversion factor is equivalent to the value of one. For example, starting with 1 mile = 1609 meters and dividing both sides of the equation by 1 mile yields $\frac{1\text{ mile}}{1\text{ mile}} = \frac{1609\text{ meters}}{1\text{ mile}}$, which when simplified yields $1 = \frac{1609\text{ meters}}{1\text{ mile}}$. Physically crossing out the units that cancel each other out will also help visualize what’s left over.

$$1\ \cancel{\text{year}} \times \frac{365\ \cancel{\text{days}}}{1\ \cancel{\text{year}}} \times \frac{24\ \cancel{\text{hours}}}{1\ \cancel{\text{day}}} \times \frac{60\ \cancel{\text{min}}}{1\ \cancel{\text{hour}}} \times \frac{60\text{ s}}{1\ \cancel{\text{min}}} = \\ = 3.15 \times 10^7\text{ s}$$

Converting 1 year into seconds using the Factor-Label Method: Physically crossing out units that cancel out helps visualize the “leftover” unit(s).

So, when the units mile and hour are cancelled out and the arithmetic is done, 10 miles per hour converts to 4.47 meters per second.

A limitation of the factor-label method is that it can only convert between units that have a constant ratio that can be multiplied, or a multiplication factor. This method cannot be used between units that have a displacement, or difference factor. An example is the conversion between degrees Celsius and kelvins, or between Celsius and Fahrenheit. For these, it is best to use the specific conversion formulas.

For example, if you are planning a trip abroad in Spain and the weather forecast predicts the weather to be mostly cloudy and 16°C, you may want to convert the temperature into °F, a unit that you are more comfortable interpreting. In order to do this, you would need to know the conversion formula from Celsius to Fahrenheit. This formula is: $[^\circ\mathrm{F}] = [^\circ\mathrm{C}] \times \frac{9}{5} + 32$.

$$[^\circ\mathrm{F}] = [^\circ\mathrm{C}] \times \frac{9}{5} + 32 \tag{1.2.1}$$

$$[^\circ\mathrm{F}] = 28.8 + 32 \tag{1.2.2}$$

$$[^\circ\mathrm{F}] = 60.8 + 32 \tag{1.2.3}$$

So you would then know that 16°C is equivalent to 60.8°F and be able to pack the right type of clothing to be comfortable.

Key Points

- The SI unit for length is the meter.
- One meter is defined as the distance that light travels in a vacuum in $\frac{1}{299,792,458}$ of a second.
- Derivatives of measurement units related to the meter are devised around the convenience of the number 10.
- The kilogram is the only SI unit directly defined by the artifact itself.
- Mass is a property that does not depend on gravitational fields, unlike weight.
- One kilogram is defined as the mass of the International Prototype Kilogram (IPK), a platinum-iridium alloy cylinder.
- One kilogram is almost exactly equal to the mass of one liter of water.
- Time is a physical quantity of duration.
- The SI Unit for time is the second.
- The second is operationally defined in terms of radiation emitted by cesium atoms.
- The twenty standardized prefixes for use in the International System of Units are derived from multiples of 10.
- The kilogram is the only SI unit with a prefix as part of its name and symbol; as such, SI unit prefixes are prepended to the unit gram.
- The United States customary units define measurements based on the English, or Imperial, unit standards.
- Conversion of units is the conversion between different units of measurement for the same quantity, typically through multiplicative conversion factors.
- The factor-label method is the sequential application of conversion factors expressed as fractions in which units appearing in both the numerator and denominator can be cancelled out, leaving only the desired set of units.
- For conversions that have a difference factor, specific conversion formulas should be used.

Key Terms

- **Length**: How far apart objects are physically.
- **acceleration**: the rate at which the velocity of a body changes with time
- **inertia**: the tendency of an object to resist any change in its motion
- **Radiation**: the emission of energy as electromagnetic waves or as moving or oscillating subatomic particles.
- **prefix**: That which is prefixed; especially one or more letters or syllables added to the beginning of a word to modify its meaning; as, pre- in prefix, con- in conjure.
- **conversion**: a change between different units of measurement for the same quantity.

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1.3: Significant Figures and Order of Magnitude

Scientific Notation

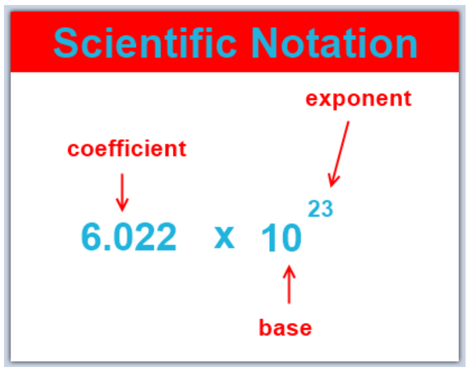
Scientific notation is a way of writing numbers that are too big or too small in a convenient and standard form.

learning objectives

- Convert properly between standard and scientific notation and identify appropriate situations to use it

Scientific Notation: A Matter of Convenience

Scientific notation is a way of writing numbers that are too big or too small in a convenient and standard form. Scientific notation has a number of useful properties and is commonly used in calculators and by scientists, mathematicians and engineers. In scientific notation all numbers are written in the form of $a \cdot 10^b$ (a multiplied by ten raised to the power of b), where the exponent b) is an integer, and the coefficient (a is any real number.



Scientific Notation: There are three parts to writing a number in scientific notation: the coefficient, the base, and the exponent.

Most of the interesting phenomena in our universe are not on the human scale. It would take about 1,000,000,000,000,000,000,000 bacteria to equal the mass of a human body. Thomas Young's discovery that light was a wave preceded the use of scientific notation, and he was obliged to write that the time required for one vibration of the wave was "$\frac{1}{500}$ of a millionth of a millionth of a second"; an inconvenient way of expressing the point. Scientific notation is a less awkward and wordy way to write very large and very small numbers such as these.

A Simple System

Scientific notation means writing a number in terms of a product of something from 1 to 10 and something else that is a power of ten.

For instance, $32 = 3.2 \cdot 10^1$

$320 = 3.2 \cdot 10^2$

$3200 = 3.2 \cdot 10^3$, and so forth…

Each number is ten times bigger than the previous one. Since 10^1 is ten times smaller than 10^2, it makes sense to use the notation 10^0 to stand for one, the number that is in turn ten times smaller than 10^1. Continuing on, we can write 10^{-1} to stand for 0.1, the number ten times smaller than 10^0. Negative exponents are used for small numbers:

$3.2 = 3.2 \cdot 10^0$

$0.32 = 3.2 \cdot 10^{-1}$

$0.032 = 3.2 \cdot 10^{-2}$

Scientific notation displayed calculators can take other shortened forms that mean the same thing. For example, $3.2 \cdot 10^6$(written notation) is the same as $3.2\mathrm{E}+6$ (notation on some calculators) and 3.2^6 (notation on some other calculators).

Round-off Error

A round-off error is the difference between the calculated approximation of a number and its exact mathematical value.

learning objectives

- Explain the impact round-off errors may have on calculations, and how to reduce this impact

Round-off Error

A round-off error, also called a rounding error, is the difference between the calculated approximation of a number and its exact mathematical value. Numerical analysis specifically tries to estimate this error when using approximation equations, algorithms, or both, especially when using finitely many digits to represent real numbers. When a sequence of calculations subject to rounding errors is made, errors may accumulate, sometimes dominating the calculation.

Calculations rarely lead to whole numbers. As such, values are expressed in the form of a decimal with infinite digits. The more digits that are used, the more accurate the calculations will be upon completion. Using a slew of digits in multiple calculations, however, is often unfeasible if calculating by hand and can lead to much more human error when keeping track of so many digits. To make calculations much easier, the results are often 'rounded off' to the nearest few decimal places.

For example, the equation for finding the area of a circle is $\mathrm{A} = \pi \mathrm{r}^2$. The number π(pi) has infinitely many digits, but can be truncated to a rounded representation of as 3.14159265359. However, for the convenience of performing calculations by hand, this number is typically rounded even further, to the nearest two decimal places, giving just 3.14. Though this technically decreases the accuracy of the calculations, the value derived is typically 'close enough' for most estimation purposes.

However, when doing a series of calculations, numbers are rounded off at each subsequent step. This leads to an accumulation of errors, and if profound enough, can misrepresent calculated values and lead to miscalculations and mistakes.

The following is an example of round-off error:

$$\sqrt{4.58^2 + 3.28^2} = \sqrt{21.0 + 10.8} = 5.64$$

Rounding these numbers off to one decimal place or to the nearest whole number would change the answer to 5.7 and 6, respectively. The more rounding off that is done, the more errors are introduced.

Order of Magnitude Calculations

An order of magnitude is the class of scale of any amount in which each class contains values of a fixed ratio to the class preceding it.

learning objectives

- Choose when it is appropriate to perform an order-of-magnitude calculation

Orders of Magnitude

An order of magnitude is the class of scale of any amount in which each class contains values of a fixed ratio to the class preceding it. In its most common usage, the amount scaled is 10, and the scale is the exponent applied to this amount (therefore, to be an order of magnitude greater is to be 10 times, or 10 to the power of 1, greater). Such differences in order of magnitude can be measured on the logarithmic scale in "decades," or factors of ten. It is common among scientists and technologists to say that a parameter whose value is not accurately known or is known only within a range is "on the order of" some value. The order of magnitude of a physical quantity is its magnitude in powers of ten when the physical quantity is expressed in powers of ten with one digit to the left of the decimal.

Orders of magnitude are generally used to make very approximate comparisons and reflect very large differences. If two numbers differ by one order of magnitude, one is about ten times larger than the other. If they differ by two orders of magnitude, they differ by a factor of about 100. Two numbers of the same order of magnitude have roughly the same scale — the larger value is less than ten times the smaller value.

It is important in the field of science that estimates be at least in the right ballpark. In many situations, it is often sufficient for an estimate to be within an order of magnitude of the value in question. Although making order-of-magnitude estimates seems simple and natural to experienced scientists, it may be completely unfamiliar to the less experienced.

Example 1.3.1:

Some of the mental steps of estimating in orders of magnitude are illustrated in answering the following example question: Roughly what percentage of the price of a tomato comes from the cost of transporting it in a truck?

Guessing the Number of Jelly Beans: Can you guess how many jelly beans are in the jar? If you try to guess directly, you will almost certainly underestimate. The right way to do it is to estimate the linear dimensions and then estimate the volume indirectly.

Incorrect solution: Let's say the trucker needs to make a profit on the trip. Taking into account her benifits, the cost of gas, and maintenance and payments on the truck, let's say the total cost is more like 2000. You might guess about 5000 tomatoes would fit in the back of the truck, so the extra cost per tomato is 40 cents. That means the cost of transporting one tomato is comparable to the cost of the tomato itself.

The problem here is that the human brain is not very good at estimating area or volume — it turns out the estimate of 5000 tomatoes fitting in the truck is way off. (This is why people have a hard time in volume-estimation contests, such as the one shown below.) When estimating area or volume, you are much better off estimating linear dimensions and computing the volume from there.

So, here's a better solution: As before, let's say the cost of the trip is $2000. The dimensions of the bin are probably 4m by 2m by 1m, for a volume of $8 \mathrm{~m}^3$. Since our goal is just an order-of-magnitude estimate, let's round that volume off to the nearest power of ten: $10 \mathrm{~m}^3$. The shape of a tomato doesn't follow linear dimensions, but since this is just an estimate, let's pretend that a tomato is an 0.1m by 0.1m by 0.1m cube, with a volume of $1 \cdot 10^{-3} \mathrm{~m}^3$. We can find the total number of tomatoes by dividing the volume of the bin by the volume of one tomato: $\frac{10^3 \mathrm{~m}^3}{10^{-3} \mathrm{~m}^3}=10^6$ tomatoes. The transportation cost per tomato is $\frac{\$2000}{10^6 \text{ tomatoes}}=\$0.002$ per tomato. That means that transportation really doesn't contribute very much to the cost of a tomato. Approximating the shape of a tomato as a cube is an example of another general strategy for making order-of-magnitude estimates.

Key Points

- Scientific notation means writing a number in terms of a product of something from 1 to 10 and something else that is a power of 10.
- In scientific notation all numbers are written in the form of $a \cdot 10^b$ (a times ten raised to the power of b).
- Each consecutive exponent number is ten times bigger than the previous one; negative exponents are used for small numbers.
- When a sequence of calculations subject to rounding error is made, these errors can accumulate and lead to the misrepresentation of calculated values.
- Increasing the number of digits allowed in a representation reduces the magnitude of possible round-off errors, but may not always be feasible, especially when doing manual calculations.
- The degree to which numbers are rounded off is relative to the purpose of calculations and the actual value.
- Orders of magnitude are generally used to make very approximate comparisons and reflect very large differences.
- In the field of science, it is often sufficient for an estimate to be within an order of magnitude of the value in question.

- When estimating area or volume, you are much better off estimating linear dimensions and computing volume from those linear dimensions.

Key Terms

- **exponent**: The power to which a number, symbol or expression is to be raised. For example, the 3 in x^3.
- **Scientific notation**: A method of writing, or of displaying real numbers as a decimal number between 1 and 10 followed by an integer power of 10
- **approximation**: An imprecise solution or result that is adequate for a defined purpose.
- **Order of Magnitude**: The class of scale or magnitude of any amount, where each class contains values of a fixed ratio (most often 10) to the class preceding it. For example, something that is 2 orders of magnitude larger is 100 times larger; something that is 3 orders of magnitude larger is 1000 times larger; and something that is 6 orders of magnitude larger is one million times larger, because $10^2 = 100, 10^3 = 1000, \text{ and } 10^6 = \text{one million}$

1.4: Solving Physics Problems

Dimensional Analysis

Any physical quantity can be expressed as a product of a combination of the basic physical dimensions.

learning objectives

- Calculate the conversion from one kind of dimension to another

Dimensions

The dimension of a physical quantity indicates how it relates to one of the seven basic quantities. These fundamental quantities are:

- [M] Mass
- [L] Length
- [T] Time
- [A] Current
- [K] Temperature
- [mol] Amount of a Substance
- [cd] Luminous Intensity

As you can see, the symbol is enclosed in a pair of square brackets. This is often used to represent the dimension of individual basic quantity. An example of the use of basic dimensions is speed, which has a dimension of 1 in length and -1 in time; $\frac{[L]}{[T]} = [LT^{-1}]$. Any physical quantity can be expressed as a product of a combination of the basic physical dimensions.

Dimensional Analysis

Dimensional analysis is the practice of checking relations between physical quantities by identifying their dimensions. The dimension of any physical quantity is the combination of the basic physical dimensions that compose it. Dimensional analysis is based on the fact that physical law must be independent of the units used to measure the physical variables. It can be used to check the plausibility of derived equations, computations and hypotheses.

Derived Dimensions

The dimensions of derived quantities may include few or all dimensions in individual basic quantities. In order to understand the technique to write dimensions of a derived quantity, we consider the case of force. Force is defined as:

$$F = m \cdot a \tag{1.4.1}$$

$$F = [M][a] \tag{1.4.2}$$

The dimension of acceleration, represented as [a], is itself a derived quantity being the ratio of velocity and time. In turn, velocity is also a derived quantity, being ratio of length and time.

$$F = [M][a] = [M][vT^{-1}] \tag{1.4.3}$$

$$F = [M][LT^{-1}T^{-1}] = [MLT^{-2}] \tag{1.4.4}$$

Dimensional Conversion

In practice, one might need to convert from one kind of dimension to another. For common conversions, you might already know how to convert off the top of your head. But for less common ones, it is helpful to know how to find the conversion factor:

$$Q = n_1 u_1 = n_2 u_2 \tag{1.4.5}$$

where n represents the amount per u dimensions. You can then use ratios to figure out the conversion:

$$n_2 = \frac{u_2}{u_1} \cdot n_1 \tag{1.4.6}$$

Trigonometry

Trigonometry is central to the use of free body diagrams, which help visually represent difficult physics problems.

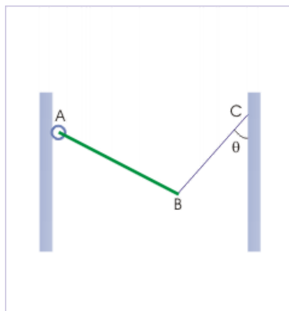
learning objectives

- Explain why trigonometry is useful in determining horizontal and vertical components of forces

Trigonometry and Solving Physics Problems

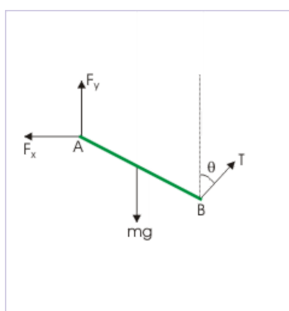
In physics, most problems are solved much more easily when a free body diagram is used. Free body diagrams use geometry and vectors to visually represent the problem. Trigonometry is also used in determining the horizontal and vertical components of forces and objects. Free body diagrams are very helpful in visually identifying which components are unknown and where the moments are applied. They can help analyze a problem, whether it is static or dynamic.

When people draw free body diagrams, often not everything is perfectly parallel and perpendicular. Sometimes people need to analyze the horizontal and vertical components of forces and object orientation. When the force or object is not acting parallel to the *x* or *y* axis, people can employ basic trigonometry to use the simplest components of the action to analyze it. Basically, everything should be considered in terms of *x* and *y*, which sometimes takes some manipulation.



Free Body Diagram: The rod is hinged from a wall and is held with the help of a string.

A rod 'AB' is hinged at 'A' from a wall and is held still with the help of a string, as shown in. This exercise involves drawing the free body diagram. To make the problem easier, the force *F* will be expressed in terms of its horizontal and vertical components. Removing all other elements from the image helps produce the finished free body diagram.



Free Body Diagram: The free body diagram as a finished product

Given the finished free body diagram, people can use their knowledge of trigonometry and the laws of sine and cosine to mathematically and numerical represent the horizontal and vertical components:

General Problem-Solving Tricks

Free body diagrams use geometry and vectors to visually represent the problem.

learning objectives

- Construct a free-body diagram for a physical scenario

In physics, most problems are solved much more easily when a free body diagram is used. This uses geometry and vectors to visually represent to problem, and trigonometry is also used in determining horizontal and vertical components of forces and objects.

Purpose: Free body diagrams are very helpful in visually identifying which components are unknown, where the moments are applied, and help analyze a problem, whether static or dynamic.

How to Make A Free Body Diagram

To draw a free body diagram, do not worry about drawing it to scale, this will just be what you use to help yourself identify the problems. First you want to model the body, in one of three ways:

- As a particle. This model may be used when any turning effects are zero or have zero interest even though the body itself may be extended. The body may be represented by a small symbolic blob and the diagram reduces to a set of concurrent arrows. A force on a particle is a *bound* vector.
- *rigid extended*. Stresses and strains are of no interest but turning effects are. A force arrow should lie along the line of force, but where along the line is irrelevant. A force on an extended rigid body is a*sliding* vector.
- *non-rigid extended*. The *point of application* of a force becomes crucial and has to be indicated on the diagram. A force on a non-rigid body is a *bound* vector. Some engineers use the tail of the arrow to indicate the point of application. Others use the tip.

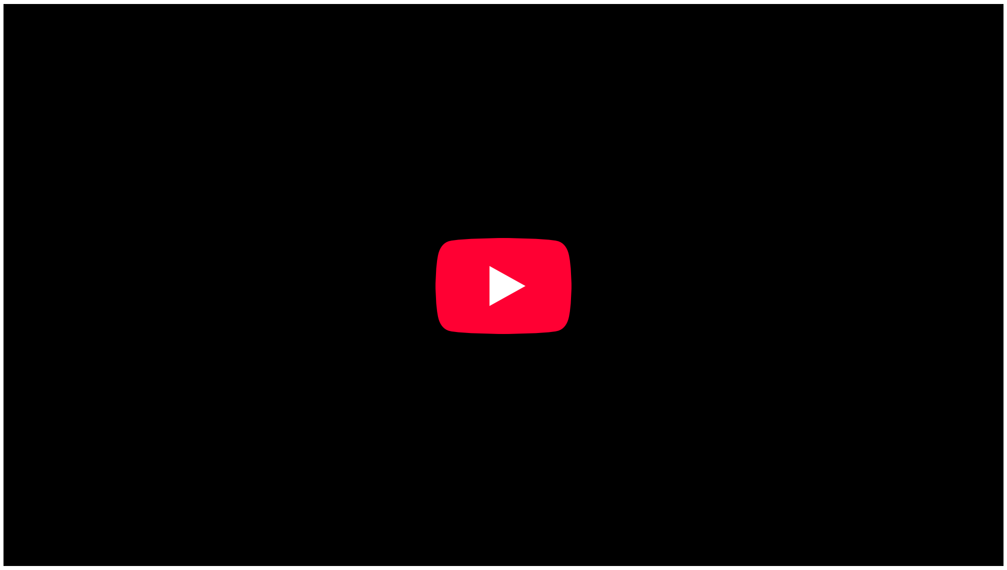
Do's and Don'ts

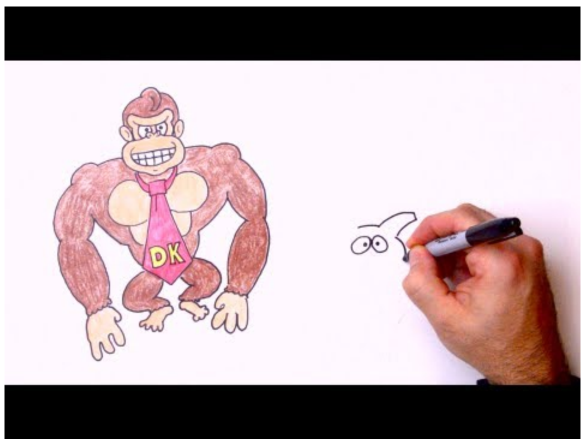
What to include: Since a free body diagram represents the body itself and the external forces on it. So you will want to include the following things in the diagram:

- The body: This is usually sketched in a schematic way depending on the body – particle/extended, rigid/non-rigid – and on what questions are to be answered. Thus if rotation of the body and torque is in consideration, an indication of size and shape of the body is needed.
- The external forces: These are indicated by labelled arrows. In a fully solved problem, a force arrow is capable of indicating the direction, the magnitude the point of application. These forces can be friction, gravity, normal force, drag, tension, etc…

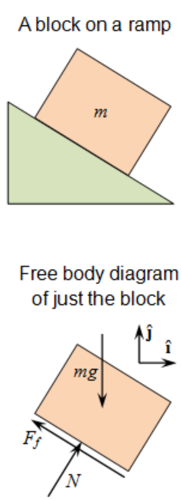
Do not include:

- Do not show bodies other than the body of interest.
- Do not show forces exerted by the body.
- Internal forces acting on various parts of the body by other parts of the body.
- Any velocity or acceleration is left out.





How To Solve Any Physics Problem: Learn five simple steps in five minutes! In this episode we cover the most effective problem-solving method I’ve encountered and call upon some fuzzy friends to help us remember the steps.



Free Body Diagram: Use this figure to work through the example problem.

Key Points

- Dimensional analysis is the practice of checking relations amount physical quantities by identifying their dimensions.
- It is common to be faced with a problem that uses different dimensions to express the same basic quantity. The following equation can be used to find the conversion factor between the two derived dimensions: $\mathrm{n_2} = \frac{\mathrm{u_2}}{\mathrm{u_1}} \times \mathrm{n_1}$.
- Dimensional analysis can also be used as a simple check to computations, theories and hypotheses.
- It is important to identify the problem and the unknowns and draw them in a free body diagram.
- The laws of cosine and sine can be used to determine the vertical and horizontal components of the different elements of the diagram.
- Free body diagrams use geometry and vectors to visually represent physics problems.
- A free body diagram lets you visually isolate the problem you are trying to solve, and simplify it into simple geometry and trigonometry.
- When drawing these diagrams, it is helpful to only draw the body it self, and the forces acting on it.
- Drawing other objects and internal forces can condense the diagram and cause it to be less helpful.

Key Terms

- **dimension**: A measure of spatial extent in a particular direction, such as height, width or breadth, or depth.
- **trigonometry**: The branch of mathematics that deals with the relationships between the sides and the angles of triangles and the calculations based on them, particularly the trigonometric functions.
- **static**: Fixed in place; having no motion.
- **dynamic**: Changing; active; in motion.

- How To Solve Any Physics Problem. **Located at**: **http://www.youtube.com/watch?v=YocWuzi4JhY**. **License**: ***Public Domain: No Known Copyright***. **License Terms**: Standard YouTube license

1.5: Prelude to Units and Measurement

Galaxies are as immense as atoms are small, yet the same laws of physics describe both, along with all the rest of nature—an indication of the underlying unity in the universe. The laws of physics are surprisingly few, implying an underlying simplicity to nature's apparent complexity. In this text, you learn about the laws of physics. Galaxies and atoms may seem far removed from your daily life, but as you begin to explore this broad-ranging subject, you may soon come to realize that physics plays a much larger role in your life than you first thought, no matter your life goals or career choice.

Figure 1.5.1: This image might be showing any number of things. It might be a whirlpool in a tank of water or perhaps a collage of paint and shiny beads done for art class. Without knowing the size of the object in units we all recognize, such as meters or inches, it is difficult to know what we're looking at. In fact, this image shows the Whirlpool Galaxy (and its companion galaxy), which is about 60,000 light-years in diameter (about 6×1017km across). (credit: S. Beckwith (STScI) Hubble Heritage Team, (STScI/AURA), ESA, NASA)

1.6: The Scope and Scale of Physics

Learning Objectives

- Describe the scope of physics.
- Calculate the order of magnitude of a quantity.
- Compare measurable length, mass, and timescales quantitatively.
- Describe the relationships among models, theories, and laws.

Physics is devoted to the understanding of all natural phenomena. In physics, we try to understand physical phenomena at all scales—from the world of subatomic particles to the entire universe. Despite the breadth of the subject, the various subfields of physics share a common core. The same basic training in physics will prepare you to work in any area of physics and the related areas of science and engineering. In this section, we investigate the scope of physics; the scales of length, mass, and time over which the laws of physics have been shown to be applicable; and the process by which science in general, and physics in particular, operates.

The Scope of Physics

Take another look at the thumbnail image. The Whirlpool Galaxy contains billions of individual stars as well as huge clouds of gas and dust. Its companion galaxy is also visible to the right. This pair of galaxies lies a staggering billion trillion miles (1.4×10^{21} mi) from our own galaxy (which is called the **Milky Way**). The stars and planets that make up the Whirlpool Galaxy might seem to be the furthest thing from most people's everyday lives, but the Whirlpool is a great starting point to think about the forces that hold the universe together. The forces that cause the Whirlpool Galaxy to act as it does are thought to be the same forces we contend with here on Earth, whether we are planning to send a rocket into space or simply planning to raise the walls for a new home. The gravity that causes the stars of the Whirlpool Galaxy to rotate and revolve is thought to be the same as what causes water to flow over hydroelectric dams here on Earth. When you look up at the stars, realize the forces out there are the same as the ones here on Earth. Through a study of physics, you may gain a greater understanding of the interconnectedness of everything we can see and know in this universe.

Think, now, about all the technological devices you use on a regular basis. Computers, smartphones, global positioning systems (GPSs), MP3 players, and satellite radio might come to mind. Then, think about the most exciting modern technologies you have heard about in the news, such as trains that levitate above tracks, "invisibility cloaks" that bend light around them, and microscopic robots that fight cancer cells in our bodies. All these groundbreaking advances, commonplace or unbelievable, rely on the principles of physics. Aside from playing a significant role in technology, professionals such as engineers, pilots, physicians, physical therapists, electricians, and computer programmers apply physics concepts in their daily work. For example, a pilot must understand how wind forces affect a flight path; a physical therapist must understand how the muscles in the body experience forces as they move and bend. As you will learn in this text, the principles of physics are propelling new, exciting technologies, and these principles are applied in a wide range of careers.

The underlying order of nature makes science in general, and physics in particular, interesting and enjoyable to study. For example, what do a bag of chips and a car battery have in common? Both contain energy that can be converted to other forms. The law of conservation of energy (which says that energy can change form but is never lost) ties together such topics as food calories, batteries, heat, light, and watch springs. Understanding this law makes it easier to learn about the various forms energy takes and how they relate to one another. Apparently unrelated topics are connected through broadly applicable physical laws, permitting an understanding beyond just the memorization of lists of facts.

Science consists of theories and laws that are the general truths of nature, as well as the body of knowledge they encompass. Scientists are continuously trying to expand this body of knowledge and to perfect the expression of the laws that describe it. **Physics**, which comes from the Greek **phúsis**, meaning "nature," is concerned with describing the interactions of energy, matter, space, and time to uncover the fundamental mechanisms that underlie every phenomenon. This concern for describing the basic phenomena in nature essentially defines the **scope of physics**.

Physics aims to understand the world around us at the most basic level. It emphasizes the use of a small number of quantitative laws to do this, which can be useful to other fields pushing the performance boundaries of existing technologies. Consider a smartphone (Figure 1.6.1). Physics describes how electricity interacts with the various circuits inside the device. This knowledge helps engineers select the appropriate materials and circuit layout when building a smartphone. Knowledge of the physics underlying these devices is required to shrink their size or increase their processing speed. Or, think about a GPS. Physics describes the relationship between the speed of an object, the distance over which it travels, and the time it takes to travel that distance. When you use a GPS in a vehicle, it relies on physics equations to determine the travel time from one location to another.

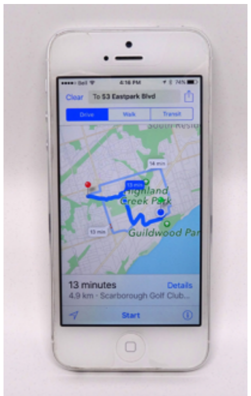


Figure 1.6.1: The Apple iPhone is a common smartphone with a GPS function. Physics describes the way that electricity flows through the circuits of this device. Engineers use their knowledge of physics to construct an iPhone with features that consumers will enjoy. One specific feature of an iPhone is the GPS function. A GPS uses physics equations to determine the drive time between two locations on a map.

Knowledge of physics is useful in everyday situations as well as in nonscientific professions. It can help you understand how microwave ovens work, why metals should not be put into them, and why they might affect pacemakers. Physics allows you to understand the hazards of radiation and to evaluate these hazards rationally and more easily. Physics also explains the reason why a black car radiator helps remove heat in a car engine, and it explains why a white roof helps keep the inside of a house cool. Similarly, the operation of a car's ignition system as well as the transmission of electrical signals throughout our body's nervous system are much easier to understand when you think about them in terms of basic physics.

Physics is a key element of many important disciplines and contributes directly to others. Chemistry, for example—since it deals with the interactions of atoms and molecules—has close ties to atomic and molecular physics. Most branches of engineering are concerned with designing new technologies, processes, or structures within the constraints set by the laws of physics. In architecture, physics is at the heart of structural stability and is involved in the acoustics, heating, lighting, and cooling of buildings. Parts of geology rely heavily on physics, such as radioactive dating of rocks, earthquake analysis, and heat transfer within Earth. Some disciplines, such as biophysics and geophysics, are hybrids of physics and other disciplines.

Physics has many applications in the biological sciences. On the microscopic level, it helps describe the properties of cells and their environments. On the macroscopic level, it explains the heat, work, and power associated with the human body and its various organ systems. Physics is involved in medical diagnostics, such as radiographs, magnetic resonance imaging, and ultrasonic blood flow measurements. Medical therapy sometimes involves physics directly; for example, cancer radiotherapy uses ionizing radiation. Physics also explains sensory phenomena, such as how musical instruments make sound, how the eye detects color, and how lasers transmit information.

It is not necessary to study all applications of physics formally. What is most useful is knowing the basic laws of physics and developing skills in the analytical methods for applying them. The study of physics also can improve your problem-solving skills. Furthermore, physics retains the most basic aspects of science, so it is used by all the sciences, and the study of physics makes other sciences easier to understand.

The Scale of Physics

From the discussion so far, it should be clear that to accomplish your goals in any of the various fields within the natural sciences and engineering, a thorough grounding in the laws of physics is necessary. The reason for this is simply that the laws of physics govern everything in the observable universe at all measurable scales of length, mass, and time. Now, that is easy enough to say, but to come to grips with what it really means, we need to get a little bit quantitative. So, before surveying the various scales that physics allows us to explore, let's first look at the concept of "order of magnitude," which we use to come to terms with the vast ranges of length, mass, and time that we consider in this text (Figure 1.6.2).

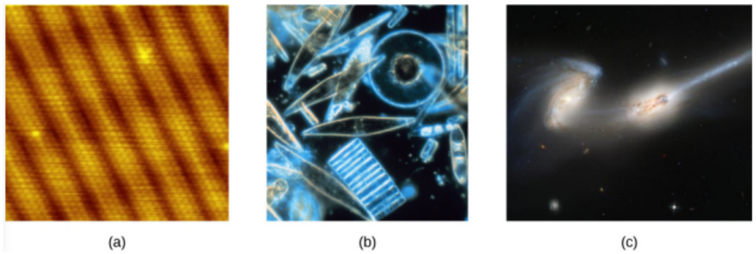


Figure 1.6.2: (a) Using a scanning tunneling microscope, scientists can see the individual atoms (diameters around 10^{-10} m) that compose this sheet of gold. (b) Tiny phytoplankton swim among crystals of ice in the Antarctic Sea. They range from a few micrometers (1 μm is 10^{-6} m) to as much as 2 mm (1 mm is 10^{-2} m) in length. (c) These two colliding galaxies, known as NGC 4676A (right) and NGC 4676B (left), are nicknamed "The Mice" because of the tail of gas emanating from each one. They are located 300 million light-years from Earth in the constellation Coma Berenices. Eventually, these two galaxies will merge into one. (credit a: modification of work by Erwinrossen; credit b: modification of work by Prof. Gordon T. Taylor, Stony Brook University; NOAA Corps Collections; credit c: modification of work by NASA, H. Ford (JHU), G. Illingworth (UCSC/LO), M. Clampin (STScI), G. Hartig (STScI), the ACS Science Team, and ESA)

Order of Magnitude

The **order of magnitude** of a number is the power of 10 that most closely approximates it. Thus, the order of magnitude refers to the scale (or size) of a value. Each power of 10 represents a different order of magnitude. For example, 10^1, 10^2, 10^3, and so forth, are all different orders of magnitude, as are $10^0 = 1$, 10^{-1}, 10^{-2}, and 10^{-3}. To find the order of magnitude of a number, take the base-10 logarithm of the number and round it to the nearest integer, then the order of magnitude of the number is simply the resulting power of 10. For example, the order of magnitude of 800 is 10^3 because $\log_{10} 800 \approx 2.903$, which rounds to 3. Similarly, the order of magnitude of 450 is 10^3 because $\log_{10} 450 \approx 2.653$, which rounds to 3 as well. Thus, we say the numbers 800 and 450 are of the same order of magnitude: 10^3. However, the order of magnitude of 250 is 10^2 because $\log_{10} 250 \approx 2.397$, which rounds to 2.

An equivalent but quicker way to find the order of magnitude of a number is first to write it in scientific notation and then check to see whether the first factor is greater than or less than $\sqrt{10} = 10^{0.5} \approx 3$. The idea is that $\sqrt{10} = 10^{0.5}$ is halfway between $1 = 10^0$ and $10 = 10^1$ on a log base-10 scale. Thus, if the first factor is less than $\sqrt{10}$, then we round it down to 1 and the order of magnitude is simply whatever power of 10 is required to write the number in scientific notation. On the other hand, if the first factor is greater than $\sqrt{10}$, then we round it up to 10 and the order of magnitude is one power of 10 higher than the power needed to write the number in scientific notation. For example, the number 800 can be written in scientific notation as 8 x 10^2. Because 8 is bigger than $\sqrt{10} \approx 3$, we say the order of magnitude of 800 is $10^{2+1} = 10^3$. The number 450 can be written as 4.5 x 10^2, so its order of magnitude is also 10^3 because 4.5 is greater than 3. However, 250 written in scientific notation is 2.5 x 10^2 and 2.5 is less than 3, so its order of magnitude is 10^2.

The order of magnitude of a number is designed to be a ballpark estimate for the scale (or size) of its value. It is simply a way of rounding numbers consistently to the nearest power of 10. This makes doing rough mental math with very big and very small numbers easier. For example, the diameter of a hydrogen atom is on the order of 10^{-10} m, whereas the diameter of the Sun is on the order of 10^9 m, so it would take roughly $10^9/10^{-10} = 10^{19}$ hydrogen atoms to stretch across the diameter of the Sun. This is much easier to do in your head than using the more precise values of 1.06 x 10^{-10} m for a hydrogen atom diameter and 1.39 x 10^9 m for the Sun's diameter, to find that it would take 1.31 x 10^{19} hydrogen atoms to stretch across the Sun's diameter. In addition to being easier, the rough estimate is also nearly as informative as the precise calculation.

Known Ranges of Length, Mass, and Time

The vastness of the universe and the breadth over which physics applies are illustrated by the wide range of examples of known lengths, masses, and times (given as orders of magnitude) in Figure 1.6.3. Examining this table will give you a feeling for the range of possible topics in physics and numerical values. A good way to appreciate the vastness of the ranges of values in Figure 1.6.3 is to try to answer some simple comparative questions, such as the following:

? Exercise 1.6.1

a. How many hydrogen atoms does it take to stretch across the diameter of the Sun?
b. How many protons are there in a bacterium?
c. How many floating-point operations can a supercomputer do in 1 day?

Answer a

10^9 m/10^{-10} m = 10^{19} hydrogen atoms

Answer b

10^{-15} kg/10^{-27} kg = 10^{12} protons

Answer c

10^5 s/10^{-17} s = 10^{22} floating-point operations

In studying Figure 1.6.3, take some time to come up with similar questions that interest you and then try answering them. Doing this can breathe some life into almost any table of numbers.

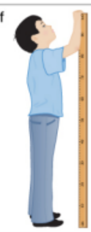
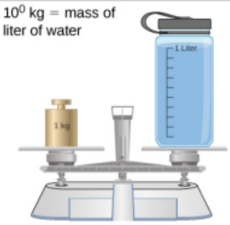
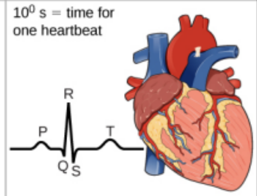
Length in Meters (m)	Masses in Kilograms (kg)	Time in Seconds (s)
10^{-15} m = diameter of proton	10^{-30} kg = mass of electron	10^{-22} s = mean lifetime of very unstable nucleus
10^{-14} m = diameter of large nucleus	10^{-27} kg = mass of proton	10^{-17} s = time for single floating-point operation in a supercomputer
10^{-10} m = diameter of hydrogen atom	10^{-15} kg = mass of bacterium	10^{-15} s = time for one oscillation of visible light
10^{-7} m = diameter of typical virus	10^{-5} kg = mass of mosquito	10^{-13} s = time for one vibration of an atom in a solid
10^{-2} m = pinky fingernail width	10^{-2} kg = mass of hummingbird	10^{-3} s = duration of a nerve impulse
10^{0} m = height of 4 year old child 	10^{0} kg = mass of liter of water 	10^{0} s = time for one heartbeat 
10^{2} m = length of football field	10^{2} kg = mass of person	10^{5} s = one day
10^{7} m = diameter of Earth	10^{19} kg = mass of atmosphere	10^{7} s = one year
10^{13} m = diameter of solar system	10^{22} kg = mass of Moon	10^{9} s = human lifetime
10^{16} m = distance light travels in a year (one light-year)	10^{25} kg = mass of Earth	10^{11} s = recorded human history
10^{21} m = Milky Way diameter	10^{30} kg = mass of Sun	10^{17} s = age of Earth
10^{26} m = distance to edge of observable universe	10^{53} kg = upper limit on mass of known universe	10^{18} s = age of the universe

Figure 1.6.3: This table shows the orders of magnitude of length, mass, and time.

Building Models

How did we come to know the laws governing natural phenomena? What we refer to as the laws of nature are concise descriptions of the universe around us. They are human statements of the underlying laws or rules that all natural processes follow. Such laws are intrinsic to the universe; humans did not create them and cannot change them. We can only discover and understand them. Their discovery is a very human endeavor, with all the elements of mystery, imagination, struggle, triumph, and disappointment inherent in any creative effort (Figure 1.5). The cornerstone of discovering natural laws is observation; scientists must describe the universe as it is, not as we imagine it to be.

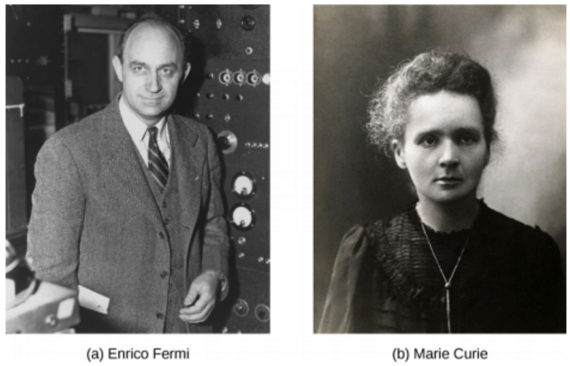


Figure 1.6.4: (a) Enrico Fermi (1901–1954) was born in Italy. On accepting the Nobel Prize in Stockholm in 1938 for his work on artificial radioactivity produced by neutrons, he took his family to America rather than return home to the government in power at the time. He became an American citizen and was a leading participant in the Manhattan Project. (b) Marie Curie (1867–1934) sacrificed monetary assets to help finance her early research and damaged her physical well-being with radiation exposure. She is the only person to win Nobel prizes in both physics and chemistry. One of her daughters also won a Nobel Prize. (credit a: United States Department of Energy)

A **model** is a representation of something that is often too difficult (or impossible) to display directly. Although a model is justified by experimental tests, it is only accurate in describing certain aspects of a physical system. An example is the Bohr model of single-electron atoms, in which the electron is pictured as orbiting the nucleus, analogous to the way planets orbit the Sun (Figure 1.6.5). We cannot observe electron orbits directly, but the mental image helps explain some of the observations we can make, such as the emission of light from hot gases (atomic spectra). However, other observations show that the picture in the Bohr model is not really what atoms look like. The model is “wrong,” but is still useful for some purposes. Physicists use models for a variety of purposes. For example, models can help physicists analyze a scenario and perform a calculation or models can be used to represent a situation in the form of a computer simulation. Ultimately, however, the results of these calculations and simulations need to be double-checked by other means—namely, observation and experimentation.

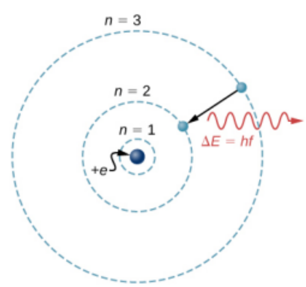


Figure 1.6.5: What is a model? The Bohr model of a single-electron atom shows the electron orbiting the nucleus in one of several possible circular orbits. Like all models, it captures some, but not all, aspects of the physical system.

The word **theory** means something different to scientists than what is often meant when the word is used in everyday conversation. In particular, to a scientist a theory is not the same as a “guess” or an “idea” or even a “hypothesis.” The phrase “it’s just a theory” seems meaningless and silly to scientists because science is founded on the notion of theories. To a scientist, a **theory** is a testable explanation for patterns in nature supported by scientific evidence and verified multiple times by various groups of researchers. Some theories include models to help visualize phenomena whereas others do not. Newton’s theory of gravity, for example, does not require a model or mental image, because we can observe the objects directly with our own senses. The kinetic theory of gases, on the other hand, is a model in which a gas is viewed as being composed of atoms and molecules. Atoms and molecules are too small to be observed directly with our senses—thus, we picture them mentally to understand what the instruments tell us about the behavior of gases. Although models are meant only to describe certain aspects of a physical system accurately, a theory should describe all aspects of any system that falls within its domain of applicability. In particular, any experimentally testable implication of a theory should be verified. If an experiment ever shows an implication of a theory to be false, then the theory is either thrown out or modified suitably (for example, by limiting its domain of applicability).

A **law** uses concise language to describe a generalized pattern in nature supported by scientific evidence and repeated experiments. Often, a law can be expressed in the form of a single mathematical equation. Laws and theories are similar in that they are both scientific statements that result from a tested hypothesis and are supported by scientific evidence. However, the designation **law** is usually reserved for a concise and very general statement that describes phenomena in nature, such as the law that energy is conserved during any process, or Newton's second law of motion, which relates force (F), mass (m), and acceleration (a) by the simple equation **F = ma**. A theory, in contrast, is a less concise statement of observed behavior. For example, the theory of evolution and the theory of relativity cannot be expressed concisely enough to be considered laws. The biggest difference between a law and a theory is that a theory is much more complex and dynamic. A law describes a single action whereas a theory explains an entire group of related phenomena. Less broadly applicable statements are usually called principles (such as Pascal's principle, which is applicable only in fluids), but the distinction between laws and principles often is not made carefully.

The models, theories, and laws we devise sometimes imply the existence of objects or phenomena that are as yet unobserved. These predictions are remarkable triumphs and tributes to the power of science. It is the underlying order in the universe that enables scientists to make such spectacular predictions. However, if experimentation does not verify our predictions, then the theory or law is wrong, no matter how elegant or convenient it is. Laws can never be known with absolute certainty because it is impossible to perform every imaginable experiment to confirm a law for every possible scenario. Physicists operate under the assumption that all scientific laws and theories are valid until a counterexample is observed. If a good-quality, verifiable experiment contradicts a well-established law or theory, then the law or theory must be modified or overthrown completely. The study of science in general, and physics in particular, is an adventure much like the exploration of an uncharted ocean. Discoveries are made; models, theories, and laws are formulated; and the beauty of the physical universe is made more sublime for the insights gained.

1.7: Units and Standards

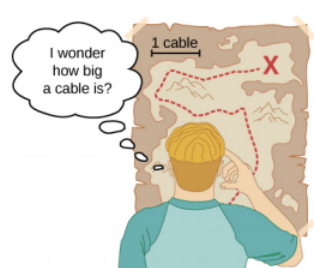
Learning Objectives

- Describe how SI base units are defined.
- Describe how derived units are created from base units.
- Express quantities given in SI units using metric prefixes.

As we saw previously, the range of objects and phenomena studied in physics is immense. From the incredibly short lifetime of a nucleus to the age of Earth, from the tiny sizes of subnuclear particles to the vast distance to the edges of the known universe, from the force exerted by a jumping flea to the force between Earth and the Sun, there are enough factors of 10 to challenge the imagination of even the most experienced scientist. Giving numerical values for physical quantities and equations for physical principles allows us to understand nature much more deeply than qualitative descriptions alone. To comprehend these vast ranges, we must also have accepted units in which to express them. We shall find that even in the potentially mundane discussion of meters, kilograms, and seconds, a profound simplicity of nature appears: all physical quantities can be expressed as combinations of only seven base physical quantities.

We define a **physical quantity** either by specifying how it is measured or by stating how it is calculated from other measurements. For example, we might define distance and time by specifying methods for measuring them, such as using a meter stick and a stopwatch. Then, we could define average speed by stating that it is calculated as the total distance traveled divided by time of travel.

Measurements of physical quantities are expressed in terms of **units**, which are standardized values. For example, the length of a race, which is a physical quantity, can be expressed in units of meters (for sprinters) or kilometers (for distance runners). Without standardized units, it would be extremely difficult for scientists to express and compare measured values in a meaningful way (Figure 1.7.1).



Distances given in unknown units are maddeningly useless.

Figure 1.7.1: Distances given in unknown units are maddeningly useless.

Two major systems of units are used in the world: **SI units** (for the French **Système International d'Unités**), also known as the **metric system**, and **English units** (also known as the **customary** or **imperial system**). English units were historically used in nations once ruled by the British Empire and are still widely used in the United States. English units may also be referred to as the **foot–pound–second** (fps) system, as opposed to the **centimeter–gram–second** (cgs) system. You may also encounter the term **SAE units**, named after the Society of Automotive Engineers. Products such as fasteners and automotive tools (for example, wrenches) that are measured in inches rather than metric units are referred to as **SAE fasteners** or **SAE wrenches**.

Virtually every other country in the world (except the United States) now uses SI units as the standard. The metric system is also the standard system agreed on by scientists and mathematicians.

SI Units: Base and Derived Units

In any system of units, the units for some physical quantities must be defined through a measurement process. These are called the **base quantities** for that system and their units are the system's **base units**. All other physical quantities can then be expressed as algebraic combinations of the base quantities. Each of these physical quantities is then known as a **derived quantity** and each unit is called a **derived unit**. The choice of base quantities is somewhat arbitrary, as long as they are independent of each other and all other quantities can be derived from them. Typically, the goal is to choose physical quantities that can be measured accurately to a high precision as the base quantities. The reason for this is simple. Since the derived units can be expressed as algebraic combinations of the base units, they can only be as accurate and precise as the base units from which they are derived.

Based on such considerations, the International Standards Organization recommends using seven base quantities, which form the International System of Quantities (ISQ). These are the base quantities used to define the SI base units. Table 1.7.1 lists these seven ISQ base quantities and the corresponding SI base units.

Table 1.7.1: ISQ Base Quantities and Their SI Units

ISQ Base Quantity	SI Base Unit
Length	meter (m)
Mass	kilogram (kg)
Time	second (s)
Electrical Current	ampere (A)
Thermodynamic Temperature	kelvin (K)
Amount of Substance	mole (mol)
Luminous Intensity	candela (cd)

You are probably already familiar with some derived quantities that can be formed from the base quantities in Table 1.7.1. For example, the geometric concept of area is always calculated as the product of two lengths. Thus, area is a derived quantity that can be expressed in terms of SI base units using square meters ($m \times m = m^2$). Similarly, volume is a derived quantity that can be expressed in cubic meters (m^3). Speed is length per time; so in terms of SI base units, we could measure it in meters per second (m/s). Volume mass density (or just density) is mass per volume, which is expressed in terms of SI base units such as kilograms per cubic meter (kg/m^3). Angles can also be thought of as derived quantities because they can be defined as the ratio of the arc length subtended by two radii of a circle to the radius of the circle. This is how the radian is defined. Depending on your background and interests, you may be able to come up with other derived quantities, such as the mass flow rate (kg/s) or volume flow rate (m^3/s) of a fluid, electric charge ($A \cdot s$), mass flux density [$kg/(m^2 \cdot s)$], and so on. We will see many more examples throughout this text. For now, the point is that every physical quantity can be derived from the seven base quantities in Table 1.7.1, and the units of every physical quantity can be derived from the seven SI base units.

For the most part, we use SI units in this text. Non-SI units are used in a few applications in which they are in very common use, such as the measurement of temperature in degrees Celsius (°C), the measurement of fluid volume in liters (L), and the measurement of energies of elementary particles in electron-volts (eV). Whenever non-SI units are discussed, they are tied to SI units through conversions. For example, 1 L is 10^{-3} m^3.

Check out a comprehensive source of information on SI units at the National Institute of Standards and Technology (NIST) Reference on Constants, Units, and Uncertainty.

Units of Time, Length, and Mass: The Second, Meter, and Kilogram

The initial chapters in this textmap are concerned with mechanics, fluids, and waves. In these subjects all pertinent physical quantities can be expressed in terms of the base units of length, mass, and time. Therefore, we now turn to a discussion of these three base units, leaving discussion of the others until they are needed later.

The Second

The SI unit for time, the **second** (abbreviated s), has a long history. For many years it was defined as 1/86,400 of a mean solar day. More recently, a new standard was adopted to gain greater accuracy and to define the second in terms of a nonvarying or constant physical phenomenon (because the solar day is getting longer as a result of the very gradual slowing of Earth's rotation). Cesium atoms can be made to vibrate in a very steady way, and these vibrations can be readily observed and counted. In 1967, the second was redefined as the time required for 9,192,631,770 of these vibrations to occur (Figure 1.7.2). Note that this may seem like more precision than you would ever need, but it isn't—GPSs rely on the precision of atomic clocks to be able to give you turn-by-turn directions on the surface of Earth, far from the satellites broadcasting their location.

Figure 1.7.2: An atomic clock such as this one uses the vibrations of cesium atoms to keep time to a precision of better than a microsecond per year. The fundamental unit of time, the second, is based on such clocks. This image looks down from the top of an atomic fountain nearly 30 feet tall. (credit: Steve Jurvetson)

The Meter

The SI unit for length is the **meter** (abbreviated m); its definition has also changed over time to become more precise. The meter was first defined in 1791 as 1/10,000,000 of the distance from the equator to the North Pole. This measurement was improved in 1889 by redefining the meter to be the distance between two engraved lines on a platinum–iridium bar now kept near Paris. By 1960, it had become possible to define the meter even more accurately in terms of the wavelength of light, so it was again redefined as 1,650,763.73 wavelengths of orange light emitted by krypton atoms. In 1983, the meter was given its current definition (in part for greater accuracy) as the distance light travels in a vacuum in 1/299,792,458 of a second (Figure 1.7.3). This change came after knowing the speed of light to be exactly 299,792,458 m/s. The length of the meter will change if the speed of light is someday measured with greater accuracy.

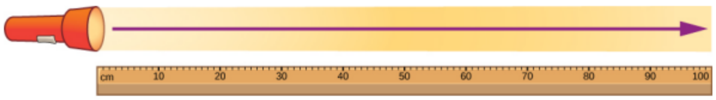


Figure 1.7.3: The meter is defined to be the distance light travels in 1/299,792,458 of a second in a vacuum. Distance traveled is speed multiplied by time.

The Kilogram

The SI unit for mass is the **kilogram** (abbreviated kg); From 1795–2018 it was defined to be the mass of a platinum–iridium cylinder kept with the old meter standard at the International Bureau of Weights and Measures near Paris. However, this cylinder has lost roughly 50 micrograms since it was created. Because this is the standard, this has shifted how we defined a kilogram. Therefore, a new definition was adopted in May 2019 based on the Planck constant and other constants which will never change in value. We will study Planck's constant in quantum mechanics, which is an area of physics that describes how the smallest pieces of the universe work. The kilogram is measured on a Kibble balance (see 1.7.4). When a weight is placed on a Kibble balance, an electrical current is produced that is proportional to Planck's constant. Since Planck's constant is defined, the exact current measurements in the balance define the kilogram.

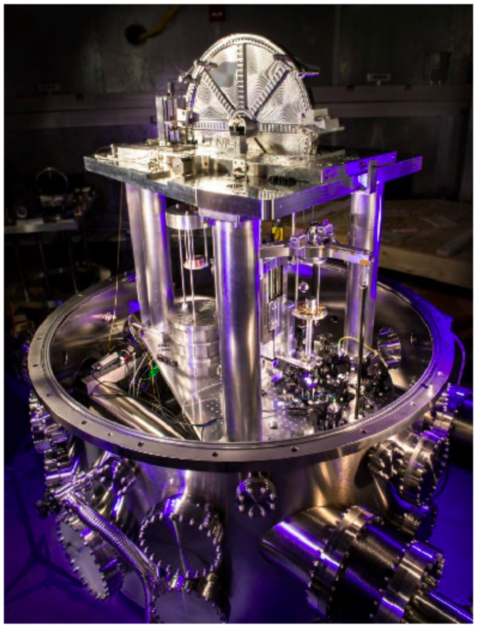


Figure 1.7.4: Redefining the SI unit of mass. The U.S. National Institute of Standards and Technology's Kibble balance is a machine that balances the weight of a test mass with the resulting electrical current needed for a force to balance it.

Metric Prefixes

SI units are part of the **metric system**, which is convenient for scientific and engineering calculations because the units are categorized by factors of 10. Table 1.7.1 lists the metric prefixes and symbols used to denote various factors of 10 in SI units. For example, a centimeter is one-hundredth of a meter (in symbols, 1 cm = 10^{-2} m) and a kilometer is a thousand meters (1 km = 10^3 m). Similarly, a megagram is a million grams (1 Mg = 10^6 g), a nanosecond is a billionth of a second (1 ns = 10^{-9} s), and a terameter is a trillion meters (1 Tm = 10^{12} m).

Table 1.7.2: Metric Prefixes for Powers of 10 and Their Symbols

Prefix	Symbol	Meaning	Prefix	Symbol	Meaning
yotta-	Y	10^{24}	yocto-	Y	10^{-24}
zetta-	Z	10^{21}	zepto-	Z	10^{-21}
exa-	E	10^{18}	atto-	E	10^{-18}
peta-	P	10^{15}	femto-	P	10^{-15}
tera-	T	10^{12}	pico-	T	10^{-12}
giga-	G	10^{9}	nano-	G	10^{-9}
mega-	M	10^{6}	micro-	M	10^{-6}
kilo-	k	10^{3}	milli-	k	10^{-3}
hecto-	h	10^{2}	centi-	h	10^{-2}
deka-	da	10^{1}	deci-	da	10^{-1}

The only rule when using metric prefixes is that you cannot "double them up." For example, if you have measurements in petameters (1 Pm = 10^{15} m), it is not proper to talk about megagigameters, although 10^6 x 10^9 = 10^{15}. In practice, the only time this becomes a bit confusing is when discussing masses. As we have seen, the base SI unit of mass is the kilogram (kg), but metric prefixes need to be applied to the gram (g), because we are not allowed to "double-up" prefixes. Thus, a thousand kilograms (10^3 kg) is written as a megagram (1 Mg) since

$$10^3\ kg = 10^3 \times 10^3\ g = 10^6\ g = 1\ Mg. \tag{1.7.1}$$

Incidentally, 10^3 kg is also called a **metric ton**, abbreviated t. This is one of the units outside the SI system considered acceptable for use with SI units.

As we see in the next section, metric systems have the advantage that conversions of units involve only powers of 10. There are 100 cm in 1 m, 1000 m in 1 km, and so on. In nonmetric systems, such as the English system of units, the relationships are not as simple—there are 12 in in 1 ft, 5280 ft in 1 mi, and so on.

Another advantage of metric systems is that the same unit can be used over extremely large ranges of values simply by scaling it with an appropriate metric prefix. The prefix is chosen by the order of magnitude of physical quantities commonly found in the task at hand. For example, distances in meters are suitable in construction, whereas distances in kilometers are appropriate for air travel, and nanometers are convenient in optical design. With the metric system there is no need to invent new units for particular applications. Instead, we rescale the units with which we are already familiar.

✔ Example 1.7.1: Using Metric Prefixes

Restate the mass 1.93 x 10^{13} kg using a metric prefix such that the resulting numerical value is bigger than one but less than 1000.

Strategy

Since we are not allowed to "double-up" prefixes, we first need to restate the mass in grams by replacing the prefix symbol k with a factor of 10^3 (Table $1.7.2$). Then, we should see which two prefixes in Table $1.7.2$ are closest to the resulting power of 10 when the number is written in scientific notation. We use whichever of these two prefixes gives us a number between one and 1000.

Solution

Replacing the k in kilogram with a factor of 10^3, we find that

$$1.93 \times 10^{13}\ kg = 1.93 \times 10^{13} \times 10^3\ g = 1.93 \times 10^{16}\ g.$$

From Table $1.7.2$, we see that 10^{16} is between "peta-" (10^{15}) and "exa-" (10^{18}). If we use the "peta-" prefix, then we find that 1.93×10^{16} g = 1.93×10^1 Pg, since 16 = 1 + 15. Alternatively, if we use the "exa-" prefix we find that 1.93 x 10^{16} g = 1.93 x 10^{-2} Eg, since 16 = −2 + 18. Because the problem asks for the numerical value between one and 1000, we use the "peta-" prefix and the answer is 19.3 Pg.

Significance

It is easy to make silly arithmetic errors when switching from one prefix to another, so it is always a good idea to check that our final answer matches the number we started with. An easy way to do this is to put both numbers in scientific notation and count powers of 10, including the ones hidden in prefixes. If we did not make a mistake, the powers of 10 should match up. In this problem, we started with 1.93 x 10^{13} kg, so we have 13 + 3 = 16 powers of 10. Our final answer in scientific notation is 1.93 x 10^1 Pg, so we have 1 + 15 = 16 powers of 10. So, everything checks out.

If this mass arose from a calculation, we would also want to check to determine whether a mass this large makes any sense in the context of the problem. For this, Figure 1.4 might be helpful.

? Exercises 1.7.1

Restate 4.79 x 10^5 kg using a metric prefix such that the resulting number is bigger than one but less than 1000.

Answer

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LibreTexts

1.8: Unit Conversion

Learning Objectives

- Use conversion factors to express the value of a given quantity in different units.

It is often necessary to convert from one unit to another. For example, if you are reading a European cookbook, some quantities may be expressed in units of liters and you need to convert them to cups. Or perhaps you are reading walking directions from one location to another and you are interested in how many miles you will be walking. In this case, you may need to convert units of feet or meters to miles.

Let's consider a simple example of how to convert units. Suppose we want to convert 80 m to kilometers. The first thing to do is to list the units you have and the units to which you want to convert. In this case, we have units in meters and we want to convert to kilometers. Next, we need to determine a conversion factor relating meters to kilometers. A **conversion factor** is a ratio that expresses how many of one unit are equal to another unit. For example, there are 12 in. in 1 ft, 1609 m in 1 mi, 100 cm in 1 m, 60 s in 1 min, and so on. Refer to Appendix B for a more complete list of conversion factors. In this case, we know that there are 1000 m in 1 km. Now we can set up our unit conversion. We write the units we have and then multiply them by the conversion factor so the units cancel out, as shown:

$$80\ \cancel{m} \times \frac{1\ km}{1000\ \cancel{m}} = 0.080\ km. \tag{1.8.1}$$

Note that the unwanted meter unit cancels, leaving only the desired kilometer unit. You can use this method to convert between any type of unit. Now, the conversion of 80 m to kilometers is simply the use of a metric prefix, as we saw in the preceding section, so we can get the same answer just as easily by noting that

$$80\ m = 8.0 \times 10^1\ m = 8.0 \times 10^{-2}\ km = 0.080\ km, \tag{1.8.2}$$

since "kilo-" means 10^3 and 1 = −2 + 3. However, using conversion factors is handy when converting between units that are not metric or when converting between derived units, as the following examples illustrate.

Example 1.8.1: Converting Nonmetric Units to Metric

The distance from the university to home is 10 mi and it usually takes 20 min to drive this distance. Calculate the average speed in meters per second (m/s). (**Note**: Average speed is distance traveled divided by time of travel.)

Strategy

First we calculate the average speed using the given units, then we can get the average speed into the desired units by picking the correct conversion factors and multiplying by them. The correct conversion factors are those that cancel the unwanted units and leave the desired units in their place. In this case, we want to convert miles to meters, so we need to know the fact that there are 1609 m in 1 mi. We also want to convert minutes to seconds, so we use the conversion of 60 s in 1 min.

Solution

1. Calculate average speed. Average speed is distance traveled divided by time of travel. (Take this definition as a given for now. Average speed and other motion concepts are covered in later chapters.) In equation form,

$$Average\ speed = \frac{Distance}{Time}.$$

2. Substitute the given values for distance and time:

$$Average\ speed = \frac{10\ mi}{20\ min} = 0.50\ \frac{mi}{min}.$$

3. Convert miles per minute to meters per second by multiplying by the conversion factor that cancels miles and leave meters, and also by the conversion factor that cancels minutes and leave seconds:

$$0.50 \frac{\cancel{mile}}{\cancel{min}} \times \frac{1609\ m}{1\ \ \cancel{mile}} \times \frac{1\ \cancel{min}}{60\ s} = \frac{(0.50)(1609)}{60}\ m/s = 13\ m/s.$$

Significance

Check the answer in the following ways:

1. Be sure the units in the unit conversion cancel correctly. If the unit conversion factor was written upside down, the units do not cancel correctly in the equation. We see the “miles” in the numerator in 0.50 mi/min cancels the “mile” in the denominator in the first conversion factor. Also, the “min” in the denominator in 0.50 mi/min cancels the “min” in the numerator in the second conversion factor.
2. Check that the units of the final answer are the desired units. The problem asked us to solve for average speed in units of meters per second and, after the cancelations, the only units left are a meter (m) in the numerator and a second (s) in the denominator, so we have indeed obtained these units.

Exercise 1.8.1

Light travels about 9 Pm in a year. Given that a year is about 3 x 10^7 s, what is the speed of light in meters per second?

Answer

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Example 1.8.2: Converting between Metric Units

The density of iron is 7.86 g/cm^3 under standard conditions. Convert this to kg/m^3.

Strategy

We need to convert grams to kilograms and cubic centimeters to cubic meters. The conversion factors we need are 1 kg = 10^3 g and 1 cm = 10^{-2}m. However, we are dealing with cubic centimeters (cm^3 = cm x cm x cm), so we have to use the second conversion factor three times (that is, we need to cube it). The idea is still to multiply by the conversion factors in such a way that they cancel the units we want to get rid of and introduce the units we want to keep.

Solution

$$7.86 \frac{\cancel{g}}{\cancel{cm^3}} \times \frac{kg}{10^3\ \ \cancel{g}} \times \left(\frac{\cancel{cm}}{10^{-2}\ m}\right)^3 = \frac{7.86}{(10^3)(10^{-6})}\ kg/m^3 = 7.86 \times 10^3\ kg/m^3$$

Significance

Remember, it’s always important to check the answer.

1. Be sure to cancel the units in the unit conversion correctly. We see that the gram (“g”) in the numerator in 7.86 g/cm^3 cancels the “g” in the denominator in the first conversion factor. Also, the three factors of “cm” in the denominator in 7.86 g/cm^3 cancel with the three factors of “cm” in the numerator that we get by cubing the second conversion factor.
2. Check that the units of the final answer are the desired units. The problem asked for us to convert to kilograms per cubic meter. After the cancelations just described, we see the only units we have left are “kg” in the numerator and three factors of “m” in the denominator (that is, one factor of “m” cubed, or “m^3”). Therefore, the units on the final answer are correct.

Exercise 1.8.2

We know from Figure 1.4 that the diameter of Earth is on the order of 10^7 m, so the order of magnitude of its surface area is 10^{14} m^2. What is that in square kilometers (that is, km^2)? (Try doing this both by converting 10^7 m to km and then squaring it and then by converting 10^{14} m^2 directly to square kilometers. You should get the same answer both ways.)

Answer

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Unit conversions may not seem very interesting, but not doing them can be costly. One famous example of this situation was seen with the **Mars Climate Orbiter**. This probe was launched by NASA on December 11, 1998. On September 23, 1999, while attempting to guide the probe into its planned orbit around Mars, NASA lost contact with it. Subsequent investigations showed a piece of software called SM_FORCES (or “small forces”) was recording thruster performance data in the English units of pound-seconds (lb • s). However, other pieces of software that used these values for course corrections expected them to be recorded in the SI units of newton-seconds (N • s), as dictated in the software interface protocols. This error caused the probe to follow a very different trajectory from what NASA thought it was following, which most likely caused the probe either to burn up in the Martian atmosphere or to shoot out into space. This failure to pay attention to unit conversions cost hundreds of millions of dollars, not to mention all the time invested by the scientists and engineers who worked on the project.

? Exercise 1.8.3

Given that 1 lb (pound) is 4.45 N, were the numbers being output by SM_FORCES too big or too small?

1.9: Dimensional Analysis

Learning Objectives

- Find the dimensions of a mathematical expression involving physical quantities.
- Determine whether an equation involving physical quantities is dimensionally consistent.

The **dimension** of any physical quantity expresses its dependence on the base quantities as a product of symbols (or powers of symbols) representing the base quantities. Table 1.9.1 lists the base quantities and the symbols used for their dimension. For example, a measurement of length is said to have dimension L or L^1, a measurement of mass has dimension M or M^1, and a measurement of time has dimension T or T^1. Like units, dimensions obey the rules of algebra. Thus, area is the product of two lengths and so has dimension L^2, or length squared. Similarly, volume is the product of three lengths and has dimension L^3, or length cubed. Speed has dimension length over time, L/T or LT^{-1}. Volumetric mass density has dimension M/L^3 or ML^{-3}, or mass over length cubed. In general, the dimension of any physical quantity can be written as

$$L^aM^bT^cI^d\Theta^eN^fJ^g \tag{1.9.1}$$

for some powers a, b, c, d, e, f, and g. We can write the dimensions of a length in this form with a = 1 and the remaining six powers all set equal to zero:

$$L^1 = L^1M^0T^0I^0\Theta^0N^0J^0. \tag{1.9.2}$$

Any quantity with a dimension that can be written so that all seven powers are zero (that is, its dimension is $L^0M^0T^0I^0\Theta^0N^0J^0$) is called **dimensionless** (or sometimes "of dimension 1," because anything raised to the zero power is one). Physicists often call dimensionless quantities **pure numbers**.

Table 1.9.1: Base Quantities and Their Dimensions

Base Quantity	Symbol for Dimension
Length	L
Mass	M
Time	T
Current	I
Thermodynamic Temperature	Θ
Amount of Substance	N
Luminous Intensity	J

Physicists often use square brackets around the symbol for a physical quantity to represent the dimensions of that quantity. For example, if r is the radius of a cylinder and h is its height, then we write [r] = L and [h] = L to indicate the dimensions of the radius and height are both those of length, or L. Similarly, if we use the symbol A for the surface area of a cylinder and V for its volume, then [A] = L^2 and [V] = L^3. If we use the symbol m for the mass of the cylinder and ρ for the density of the material from which the cylinder is made, then [m] = M and [ρ] = ML^{-3}.

The importance of the concept of dimension arises from the fact that any mathematical equation relating physical quantities must be **dimensionally consistent**, which means the equation must obey the following rules:

- Every term in an expression must have the same dimensions; it does not make sense to add or subtract quantities of differing dimension (think of the old saying: "You can't add apples and oranges"). In particular, the expressions on each side of the equality in an equation must have the same dimensions.
- The arguments of any of the standard mathematical functions such as trigonometric functions (such as sine and cosine), logarithms, or exponential functions that appear in the equation must be dimensionless. These functions require pure numbers as inputs and give pure numbers as outputs.

If either of these rules is violated, an equation is not dimensionally consistent and cannot possibly be a correct statement of physical law. This simple fact can be used to check for typos or algebra mistakes, to help remember the various laws of physics, and even to suggest the form that new laws of physics might take. This last use of dimensions is beyond the scope of this text, but is something you will undoubtedly learn later in your academic career.

Example 1.9.1: Using Dimensions to Remember an Equation

Suppose we need the formula for the area of a circle for some computation. Like many people who learned geometry too long ago to recall with any certainty, two expressions may pop into our mind when we think of circles: πr^2 and $2\pi r$. One expression is the circumference of a circle of radius r and the other is its area. But which is which?

Strategy

One natural strategy is to look it up, but this could take time to find information from a reputable source. Besides, even if we think the source is reputable, we shouldn't trust everything we read. It is nice to have a way to double-check just by thinking about it. Also, we might be in a situation in which we cannot look things up (such as during a test). Thus, the strategy is to find the dimensions of both expressions by making use of the fact that dimensions follow the rules of algebra. If either expression does not have the same dimensions as area, then it cannot possibly be the correct equation for the area of a circle.

Solution

We know the dimension of area is L^2. Now, the dimension of the expression πr^2 is

$$[\pi r^2] = [\pi]\cdot[r]^2 = 1\cdot L^2 = L^2, \tag{1.9.3}$$

since the constant π is a pure number and the radius r is a length. Therefore, πr^2 has the dimension of area. Similarly, the dimension of the expression $2\pi r$ is

$$[2\pi r] = [2]\cdot[\pi]\cdot[r] = 1\cdot 1\cdot L = L, \tag{1.9.4}$$

since the constants 2 and π are both dimensionless and the radius r is a length. We see that $2\pi r$ has the dimension of length, which means it cannot possibly be an area.

We rule out $2\pi r$ because it is not dimensionally consistent with being an area. We see that πr^2 is dimensionally consistent with being an area, so if we have to choose between these two expressions, πr^2 is the one to choose.

Significance

This may seem like kind of a silly example, but the ideas are very general. As long as we know the dimensions of the individual physical quantities that appear in an equation, we can check to see whether the equation is dimensionally consistent. On the other hand, knowing that true equations are dimensionally consistent, we can match expressions from our imperfect memories to the quantities for which they might be expressions. Doing this will not help us remember dimensionless factors that appear in the equations (for example, if you had accidentally conflated the two expressions from the example into $2\pi r^2$, then dimensional analysis is no help), but it does help us remember the correct basic form of equations.

Exercise 1.9.1

Suppose we want the formula for the volume of a sphere. The two expressions commonly mentioned in elementary discussions of spheres are $4\pi r^2$ and $\frac{4}{3}\pi r^3$. One is the volume of a sphere of radius r and the other is its surface area. Which one is the volume?

Answer

Add texts here. Do not delete this text first.

Example 1.9.2: Checking Equations for Dimensional Consistency

Consider the physical quantities s, v, a, and t with dimensions [s] = L, [v] = LT^{-1}, [a] = LT^{-2}, and [t] = T. Determine whether each of the following equations is dimensionally consistent:

a. s = vt + 0.5at^2;
b. s = vt^2 + 0.5at; and
c. v = sin ($\frac{at^2}{s}$).

Strategy

By the definition of dimensional consistency, we need to check that each term in a given equation has the same dimensions as the other terms in that equation and that the arguments of any standard mathematical functions are dimensionless.

Solution

a. There are no trigonometric, logarithmic, or exponential functions to worry about in this equation, so we need only look at the dimensions of each term appearing in the equation. There are three terms, one in the left expression and two in the expression on the right, so we look at each in turn:

$$[s] = L \tag{1.9.5}$$

$$[vt] = [v]\cdot[t] = LT^{-1}\cdot T = LT^0 = L \tag{1.9.6}$$

$$[0.5at^2] = [a]\cdot[t]^2 = LT^{-2}\cdot T^2 = LT^0 = L. \tag{1.9.7}$$

b. Again, there are no trigonometric, exponential, or logarithmic functions, so we only need to look at the dimensions of each of the three terms appearing in the equation:

$$[s] = L \tag{1.9.8}$$

$$[vt^2] = [v]\cdot[t]^2 = LT^{-1}\cdot T^2 = LT \tag{1.9.9}$$

$$[at] = [a]\cdot[t] = LT^{-2}\cdot T = LT^{-1}. \tag{1.9.10}$$

None of the three terms has the same dimension as any other, so this is about as far from being dimensionally consistent as you can get. The technical term for an equation like this is **nonsense**.

c. This equation has a trigonometric function in it, so first we should check that the argument of the sine function is dimensionless:

$$\left[\frac{at^2}{s}\right] = \frac{[a]\cdot[t]^2}{[s]} = \frac{LT^{-2}\cdot T^2}{L} = \frac{L}{L} = 1. \tag{1.9.11}$$

The argument is dimensionless. So far, so good. Now we need to check the dimensions of each of the two terms (that is, the left expression and the right expression) in the equation:

$$[v] = LT^{-1} \tag{1.9.12}$$

$$\left[sin\left(\frac{at^2}{s}\right)\right] = 1. \tag{1.9.13}$$

The two terms have different dimensions—meaning, the equation is not dimensionally consistent. This equation is another example of “nonsense.”

Significance

If we are trusting people, these types of dimensional checks might seem unnecessary. But, rest assured, any textbook on a quantitative subject such as physics (including this one) almost certainly contains some equations with typos. Checking equations routinely by dimensional analysis save us the embarrassment of using an incorrect equation. Also, checking the dimensions of an equation we obtain through algebraic manipulation is a great way to make sure we did not make a mistake (or to spot a mistake, if we made one).

? Exercise 1.9.2

Is the equation v = at dimensionally consistent?

Answer

Add texts here. Do not delete this text first.

One further point that needs to be mentioned is the effect of the operations of calculus on dimensions. We have seen that dimensions obey the rules of algebra, just like units, but what happens when we take the derivative of one physical quantity with respect to another or integrate a physical quantity over another? The derivative of a function is just the slope of the line tangent to its graph and slopes are ratios, so for physical quantities v and t, we have that the dimension of the derivative of v with respect to t is just the ratio of the dimension of v over that of t:

$$\left[\frac{dv}{dt}\right] = \frac{[v]}{[t]}. \tag{1.9.14}$$

Similarly, since integrals are just sums of products, the dimension of the integral of v with respect to t is simply the dimension of v times the dimension of t:

$$\left[\int v dt\right] = [v] \cdot [t]. \tag{1.9.15}$$

By the same reasoning, analogous rules hold for the units of physical quantities derived from other quantities by integration or differentiation.

1.10: Estimates and Fermi Calculations

Learning Objectives

- Estimate the values of physical quantities.

On many occasions, physicists, other scientists, and engineers need to make estimates for a particular quantity. Other terms sometimes used are **guesstimates**, **order-of-magnitude approximations**, **back-of-the-envelope calculations**, or **Fermi calculations**. (The physicist Enrico Fermi mentioned earlier was famous for his ability to estimate various kinds of data with surprising precision.) Will that piece of equipment fit in the back of the car or do we need to rent a truck? How long will this download take? About how large a current will there be in this circuit when it is turned on? How many houses could a proposed power plant actually power if it is built? Note that estimating does not mean guessing a number or a formula at random. Rather, **estimation** means using prior experience and sound physical reasoning to arrive at a rough idea of a quantity's value. Because the process of determining a reliable approximation usually involves the identification of correct physical principles and a good guess about the relevant variables, estimating is very useful in developing physical intuition. Estimates also allow us perform "sanity checks" on calculations or policy proposals by helping us rule out certain scenarios or unrealistic numbers. They allow us to challenge others (as well as ourselves) in our efforts to learn truths about the world.

Many estimates are based on formulas in which the input quantities are known only to a limited precision. As you develop physics problem-solving skills (which are applicable to a wide variety of fields), you also will develop skills at estimating. You develop these skills by thinking more quantitatively and by being willing to take risks. As with any skill, experience helps. Familiarity with dimensions (see Table 1.5.1) and units (see Table 1.3.1 and Table 1.3.2), and the scales of base quantities (see Figure 1.2.3) also helps.

To make some progress in estimating, you need to have some definite ideas about how variables may be related. The following strategies may help you in practicing the art of estimation:

- **Get big lengths from smaller lengths**. When estimating lengths, remember that anything can be a ruler. Thus, imagine breaking a big thing into smaller things, estimate the length of one of the smaller things, and multiply to get the length of the big thing. For example, to estimate the height of a building, first count how many floors it has. Then, estimate how big a single floor is by imagining how many people would have to stand on each other's shoulders to reach the ceiling. Last, estimate the height of a person. The product of these three estimates is your estimate of the height of the building. It helps to have memorized a few length scales relevant to the sorts of problems you find yourself solving. For example, knowing some of the length scales in Figure 1.2.3 might come in handy. Sometimes it also helps to do this in reverse—that is, to estimate the length of a small thing, imagine a bunch of them making up a bigger thing. For example, to estimate the thickness of a sheet of paper, estimate the thickness of a stack of paper and then divide by the number of pages in the stack. These same strategies of breaking big things into smaller things or aggregating smaller things into a bigger thing can sometimes be used to estimate other physical quantities, such as masses and times.
- **Get areas and volumes from lengths**. When dealing with an area or a volume of a complex object, introduce a simple model of the object such as a sphere or a box. Then, estimate the linear dimensions (such as the radius of the sphere or the length, width, and height of the box) first, and use your estimates to obtain the volume or area from standard geometric formulas. If you happen to have an estimate of an object's area or volume, you can also do the reverse; that is, use standard geometric formulas to get an estimate of its linear dimensions.
- **Get masses from volumes and densities**. When estimating masses of objects, it can help first to estimate its volume and then to estimate its mass from a rough estimate of its average density (recall, density has dimension mass over length cubed, so mass is density times volume). For this, it helps to remember that the density of air is around 1 kg/m^3, the density of water is 10^3 kg/m^3, and the densest everyday solids max out at around 10^4 kg/m^3. Asking yourself whether an object floats or sinks in either air or water gets you a ballpark estimate of its density. You can also do this the other way around; if you have an estimate of an object's mass and its density, you can use them to get an estimate of its volume.
- **If all else fails, bound it**. For physical quantities for which you do not have a lot of intuition, sometimes the best you can do is think something like: Well, it must be bigger than this and smaller than that. For example, suppose you need to estimate the mass of a moose. Maybe you have a lot of experience with moose and know their average mass offhand. If so, great. But for most people, the best they can do is to think something like: It must be bigger than a person (of order 10^2 kg) and less than a car

(of order 10^3 kg). If you need a single number for a subsequent calculation, you can take the geometric mean of the upper and lower bound—that is, you multiply them together and then take the square root. For the moose mass example, this would be

$$\left(10^2 \times 10^3\right)^{0.5} = 10^{2.5} = 10^{0.5} \times 10^2 \approx 3 \times 10^2 \ kg. \tag{1.10.1}$$

The tighter the bounds, the better. Also, no rules are unbreakable when it comes to estimation. If you think the value of the quantity is likely to be closer to the upper bound than the lower bound, then you may want to bump up your estimate from the geometric mean by an order or two of magnitude.

- **One "sig. fig." is fine**. There is no need to go beyond one significant figure when doing calculations to obtain an estimate. In most cases, the order of magnitude is good enough. The goal is just to get in the ballpark figure, so keep the arithmetic as simple as possible.
- **Ask yourself: Does this make any sense?** Last, check to see whether your answer is reasonable. How does it compare with the values of other quantities with the same dimensions that you already know or can look up easily? If you get some wacky answer (for example, if you estimate the mass of the Atlantic Ocean to be bigger than the mass of Earth, or some time span to be longer than the age of the universe), first check to see whether your units are correct. Then, check for arithmetic errors. Then, rethink the logic you used to arrive at your answer. If everything checks out, you may have just proved that some slick new idea is actually bogus.

Example 1.10.1: Mass of Earth's Oceans

Estimate the total mass of the oceans on Earth.

Strategy

We know the density of water is about 10^3 kg/m^3, so we start with the advice to "get masses from densities and volumes." Thus, we need to estimate the volume of the planet's oceans. Using the advice to "get areas and volumes from lengths," we can estimate the volume of the oceans as surface area times average depth, or V = AD. We know the diameter of Earth from Figure 1.4 and we know that most of Earth's surface is covered in water, so we can estimate the surface area of the oceans as being roughly equal to the surface area of the planet. By following the advice to "get areas and volumes from lengths" again, we can approximate Earth as a sphere and use the formula for the surface area of a sphere of diameter d—that is, A = πd^2, to estimate the surface area of the oceans. Now we just need to estimate the average depth of the oceans. For this, we use the advice: "If all else fails, bound it." We happen to know the deepest points in the ocean are around 10 km and that it is not uncommon for the ocean to be deeper than 1 km, so we take the average depth to be around $(10^3 \text{ x } 10^4)^{0.5} \approx 3 \text{ x } 10^3$m. Now we just need to put it all together, heeding the advice that "one 'sig. fig.' is fine."

Solution

We estimate the surface area of Earth (and hence the surface area of Earth's oceans) to be roughly

$$A = \pi d^2 = \pi \left(10^7 \ m\right)^2 \approx 3 \times 10^{14} \ m^2. \tag{1.10.2}$$

Next, using our average depth estimate of D = $3 \text{ x } 10^3$m, which was obtained by bounding, we estimate the volume of Earth's oceans to be

$$V = AD = \left(3 \times 10^{14} \ m^2\right) \left(3 \times 10^3 \ m\right) = 9 \times 10^{17} m^3. \tag{1.10.3}$$

Last, we estimate the mass of the world's oceans to be

$$M = \rho V = \left(10^3 \ kg/m^3\right) \left(9 \times 10^{17} \ m^3\right) = 9 \times 10^{20} \ kg. \tag{1.10.4}$$

Thus, we estimate that the order of magnitude of the mass of the planet's oceans is 10^{21} kg.

Significance

To verify our answer to the best of our ability, we first need to answer the question: Does this make any sense? From Figure 1.4, we see the mass of Earth's atmosphere is on the order of 10^{19} kg and the mass of Earth is on the order of 10^{25} kg. It is reassuring that our estimate of 10^{21} kg for the mass of Earth's oceans falls somewhere between these two. So, yes, it does seem to make sense. It just so happens that we did a search on the Web for "mass of oceans" and the top search results all said 1.4 x 10^{21} kg, which is the same order of magnitude as our estimate. Now, rather than having to trust blindly whoever first put that number up on a website (most of the other sites probably just copied it from them, after all), we can have a little more confidence in it.

Exercise 1.10.1

Figure 1.4 says the mass of the atmosphere is 10^{19} kg. Assuming the density of the atmosphere is 1 kg/m^3, estimate the height of Earth's atmosphere. Do you think your answer is an underestimate or an overestimate? Explain why.

How many piano tuners are there in New York City? How many leaves are on that tree? If you are studying photosynthesis or thinking of writing a smartphone app for piano tuners, then the answers to these questions might be of great interest to you. Otherwise, you probably couldn't care less what the answers are. However, these are exactly the sorts of estimation problems that people in various tech industries have been asking potential employees to evaluate their quantitative reasoning skills. If building physical intuition and evaluating quantitative claims do not seem like sufficient reasons for you to practice estimation problems, how about the fact that being good at them just might land you a high-paying job?

Phet Simulation: Estimation

For practice estimating relative lengths, areas, and volumes, check out this PhET simulation, titled "Estimation."

1.11: Significant Figures

Learning Objectives

- Determine the correct number of significant figures for the result of a computation.
- Describe the relationship between the concepts of accuracy, precision, uncertainty, and discrepancy.
- Calculate the percent uncertainty of a measurement, given its value and its uncertainty.
- Determine the uncertainty of the result of a computation involving quantities with given uncertainties.

Figure 1.11.1 shows two instruments used to measure the mass of an object. The digital scale has mostly replaced the double-pan balance in physics labs because it gives more accurate and precise measurements. But what exactly do we mean by **accurate and precise**? Aren't they the same thing? In this section we examine in detail the process of making and reporting a measurement.

(a) (b)

Figure 1.11.1: (a) A double-pan mechanical balance is used to compare different masses. Usually an object with unknown mass is placed in one pan and objects of known mass are placed in the other pan. When the bar that connects the two pans is horizontal, then the masses in both pans are equal. The "known masses" are typically metal cylinders of standard mass such as 1 g, 10 g, and 100 g. (b) Many mechanical balances, such as double-pan balances, have been replaced by digital scales, which can typically measure the mass of an object more precisely. A mechanical balance may read only the mass of an object to the nearest tenth of a gram, but many digital scales can measure the mass of an object up to the nearest thousandth of a gram. (credit a: modification of work by Serge Melki; credit b: modification of work by Karel Jakubec)

Accuracy and Precision of a Measurement

Science is based on observation and experiment—that is, on measurements. **Accuracy** is how close a measurement is to the accepted reference value for that measurement. For example, let's say we want to measure the length of standard printer paper. The packaging in which we purchased the paper states that it is 11.0 in. long. We then measure the length of the paper three times and obtain the following measurements: 11.1 in., 11.2 in., and 10.9 in. These measurements are quite accurate because they are very close to the reference value of 11.0 in. In contrast, if we had obtained a measurement of 12 in., our measurement would not be very accurate. Notice that the concept of accuracy requires that an accepted reference value be given.

The **precision** of measurements refers to how close the agreement is between repeated independent measurements (which are repeated under the same conditions). Consider the example of the paper measurements. The precision of the measurements refers to the spread of the measured values. One way to analyze the precision of the measurements is to determine the range, or difference, between the lowest and the highest measured values. In this case, the lowest value was 10.9 in. and the highest value was 11.2 in. Thus, the measured values deviated from each other by, at most, 0.3 in. These measurements were relatively precise because they did not vary too much in value. However, if the measured values had been 10.9 in., 11.1 in., and 11.9 in., then the measurements would not be very precise because there would be significant variation from one measurement to another. Notice that the concept of precision depends only on the actual measurements acquired and does not depend on an accepted reference value.

The measurements in the paper example are both accurate and precise, but in some cases, measurements are accurate but not precise, or they are precise but not accurate. Let's consider an example of a GPS attempting to locate the position of a restaurant in a city. Think of the restaurant location as existing at the center of a bull's-eye target and think of each GPS attempt to locate the restaurant as a black dot. In Figure $1.11.1a$, we see the GPS measurements are spread out far apart from each other, but they are all relatively close to the actual location of the restaurant at the center of the target. This indicates a low-precision, high-accuracy measuring system. However, in Figure $1.11.1b$, the GPS measurements are concentrated quite closely to one another, but they are far away from the target location. This indicates a high-precision, low-accuracy measuring system.

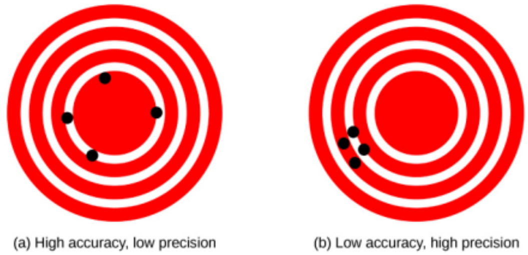

Figure $1.11.2$: A GPS attempts to locate a restaurant at the center of the bull's-eye. The black dots represent each attempt to pinpoint the location of the restaurant. (a) The dots are spread out quite far apart from one another, indicating low precision, but they are each rather close to the actual location of the restaurant, indicating high accuracy. (b) The dots are concentrated rather closely to one another, indicating high precision, but they are rather far away from the actual location of the restaurant, indicating low accuracy. (credit a and credit b: modification of works by Dark Evil)

Accuracy, Precision, Uncertainty, and Discrepancy

The precision of a measuring system is related to the **uncertainty** in the measurements whereas the accuracy is related to the **discrepancy** from the accepted reference value. Uncertainty is a quantitative measure of how much your measured values deviate from one another. There are many different methods of calculating uncertainty, each of which is appropriate to different situations. Some examples include taking the range (that is, the biggest less the smallest) or finding the standard deviation of the measurements. Discrepancy (or "measurement error") is the difference between the measured value and a given standard or expected value. If the measurements are not very precise, then the uncertainty of the values is high. If the measurements are not very accurate, then the discrepancy of the values is high.

Recall our example of measuring paper length; we obtained measurements of 11.1 in., 11.2 in., and 10.9 in., and the accepted value was 11.0 in. We might average the three measurements to say our best guess is 11.1 in.; in this case, our discrepancy is 11.1 – 11.0 = 0.1 in., which provides a quantitative measure of accuracy. We might calculate the uncertainty in our best guess by using the range of our measured values: 0.3 in. Then we would say the length of the paper is 11.1 in. plus or minus 0.3 in. The uncertainty in a measurement, A, is often denoted as δA (read "delta A"), so the measurement result would be recorded as A ± δA. Returning to our paper example, the measured length of the paper could be expressed as 11.1 ± 0.3 in. Since the discrepancy of 0.1 in. is less than the uncertainty of 0.3 in., we might say the measured value agrees with the accepted reference value to within experimental uncertainty.

Some factors that contribute to uncertainty in a measurement include the following:

- Limitations of the measuring device
- The skill of the person taking the measurement
- Irregularities in the object being measured
- Any other factors that affect the outcome (highly dependent on the situation)

In our example, such factors contributing to the uncertainty could be the smallest division on the ruler is 1/16 in., the person using the ruler has bad eyesight, the ruler is worn down on one end, or one side of the paper is slightly longer than the other. At any rate, the uncertainty in a measurement must be calculated to quantify its precision. If a reference value is known, it makes sense to calculate the discrepancy as well to quantify its accuracy.

Percent uncertainty

Another method of expressing uncertainty is as a percent of the measured value. If a measurement A is expressed with uncertainty δA, the percent uncertainty is defined as

$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% \tag{1.11.1}$$

✔ Example 1.11.1: Calculating Percent Uncertainty: A Bag of Apples

A grocery store sells 5-lb bags of apples. Let's say we purchase four bags during the course of a month and weigh the bags each time. We obtain the following measurements:

- Week 1 weight: 4.8 lb
- Week 2 weight: 5.3 lb
- Week 3 weight: 4.9 lb
- Week 4 weight: 5.4 lb

We then determine the average weight of the 5-lb bag of apples is 5.1 ± 0.3 lb. What is the percent uncertainty of the bag's weight?

Strategy

First, observe that the average value of the bag's weight, A, is 5.1 lb. The uncertainty in this value, δA, is 0.3 lb. We can use the following equation to determine the percent uncertainty of the weight:

$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% \tag{1.11.2}$$

Solution

Substitute the values into the equation:

$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% = \frac{0.3\ lb}{5.1\ lb} \times 100\% = 5.9\% \approx 6\% \tag{1.11.3}$$

Significance

We can conclude the average weight of a bag of apples from this store is 5.1 lb ± 6%. Notice the percent uncertainty is dimensionless because the units of weight in δA = 0.3 lb canceled those inn A = 5.1 lb when we took the ratio.

? Exercises 1.11.1

A high school track coach has just purchased a new stopwatch. The stopwatch manual states the stopwatch has an uncertainty of ±0.05 s. Runners on the track coach's team regularly clock 100-m sprints of 11.49 s to 15.01 s. At the school's last track meet, the first-place sprinter came in at 12.04 s and the second-place sprinter came in at 12.07 s. Will the coach's new stopwatch be helpful in timing the sprint team? Why or why not?

Uncertainties in Calculations

Uncertainty exists in anything calculated from measured quantities. For example, the area of a floor calculated from measurements of its length and width has an uncertainty because the length and width have uncertainties. How big is the uncertainty in something you calculate by multiplication or division? If the measurements going into the calculation have small uncertainties (a few percent or less), then the **method of adding percents** can be used for multiplication or division. This method states **the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation**. For example, if a floor has a length of 4.00 m and a width of 3.00 m, with uncertainties of 2% and 1%, respectively, then the area of the floor is 12.0 m^2 and has an uncertainty of 3%. (Expressed as an area, this is 0.36 m^2 [12.0 m^2 x 0.03], which we round to 0.4 m^2 since the area of the floor is given to a tenth of a square meter.)

Precision of Measuring Tools and Significant Figures

An important factor in the precision of measurements involves the precision of the measuring tool. In general, a precise measuring tool is one that can measure values in very small increments. For example, a standard ruler can measure length to the nearest millimeter whereas a caliper can measure length to the nearest 0.01 mm. The caliper is a more precise measuring tool because it can measure extremely small differences in length. The more precise the measuring tool, the more precise the measurements.

When we express measured values, we can only list as many digits as we measured initially with our measuring tool. For example, if we use a standard ruler to measure the length of a stick, we may measure it to be 36.7 cm. We can't express this value as 36.71 cm because our measuring tool is not precise enough to measure a hundredth of a centimeter. It should be noted that the last digit in a measured value has been estimated in some way by the person performing the measurement. For example, the person measuring the length of a stick with a ruler notices the stick length seems to be somewhere in between 36.6 cm and 36.7 cm, and he or she must estimate the value of the last digit. Using the method of **significant figures**, the rule is that **the last digit written down in a measurement is the first digit with some uncertainty**. To determine the number of significant digits in a value, start with the first measured value at the left and count the number of digits through the last digit written on the right. For example, the measured value 36.7 cm has three digits, or three significant figures. Significant figures indicate the precision of the measuring tool used to measure a value.

Zeros

Special consideration is given to zeros when counting significant figures. The zeros in 0.053 are not significant because they are placeholders that locate the decimal point. There are two significant figures in 0.053. The zeros in 10.053 are not placeholders; they are significant. This number has five significant figures. The zeros in 1300 may or may not be significant, depending on the style of writing numbers. They could mean the number is known to the last digit or they could be placeholders. So 1300 could have two, three, or four significant figures. To avoid this ambiguity, we should write 1300 in scientific notation as 1.3 x 10^3, 1.30 x 10^3, or 1.300 x 10^3, depending on whether it has two, three, or four significant figures. **Zeros are significant except when they serve only as placeholders**.

Significant Figures in Calculations

When combining measurements with different degrees of precision, **the number of significant digits in the final answer can be no greater than the number of significant digits in the least-precise measured value**. There are two different rules, one for multiplication and division and the other for addition and subtraction.

1. **For multiplication and division, the result should have the same number of significant figures as the quantity with the least number of significant figures entering into the calculation.** For example, the area of a circle can be calculated from its radius using A = πr^2. Let's see how many significant figures the area has if the radius has only two—say, r = 1.2 m. Using a calculator with an eight-digit output, we would calculate
$$A=\pi r^2=(3.1415927...)\times(1.2\ m)^2=4.5238934\ m^2. \tag{1.11.4}$$
But because the radius has only two significant figures, it limits the calculated quantity to two significant figures, or
$$A=4.5\ m^2. \tag{1.11.5}$$
although π is good to at least eight digits.
2. **For addition and subtraction, the answer can contain no more decimal places than the least-precise measurement**. Suppose we buy 7.56 kg of potatoes in a grocery store as measured with a scale with precision 0.01 kg, then we drop off 6.052 kg of potatoes at your laboratory as measured by a scale with precision 0.001 kg. Then, we go home and add 13.7 kg of potatoes as measured by a bathroom scale with precision 0.1 kg. How many kilograms of potatoes do we now have and how many significant figures are appropriate in the answer? The mass is found by simple addition and subtraction:
$$\begin{array}{l} \ \ 7.56\ kg \\ -6.052\ kg \\ \underline{+13.7\ kg \qquad\qquad} \\ 15.208\ kg=15.2\ kg. \end{array}$$
Next, we identify the least-precise measurement: 13.7 kg. This measurement is expressed to the 0.1 decimal place, so our final answer must also be expressed to the 0.1 decimal place. Thus, the answer is rounded to the tenths place, giving us 15.2 kg.

Significant Figures in This Text

In this text, most numbers are assumed to have three significant figures. Furthermore, consistent numbers of significant figures are used in all worked examples. An answer given to three digits is based on input good to at least three digits, for example. If the input has fewer significant figures, the answer will also have fewer significant figures. Care is also taken that the number of significant figures is reasonable for the situation posed. In some topics, particularly in optics, more accurate numbers are needed and we use more than three significant figures. Finally, if a number is exact, such as the two in the formula for the circumference of a circle, C = $2\pi r$, it does not affect the number of significant figures in a calculation. Likewise, conversion factors such as 100 cm/1 m are considered exact and do not affect the number of significant figures in a calculation.

1.12: Solving Problems in Physics

Learning Objectives

- Describe the process for developing a problem-solving strategy.
- Explain how to find the numerical solution to a problem.
- Summarize the process for assessing the significance of the numerical solution to a problem.

Problem-solving skills are clearly essential to success in a quantitative course in physics. More important, the ability to apply broad physical principles—usually represented by equations—to specific situations is a very powerful form of knowledge. It is much more powerful than memorizing a list of facts. Analytical skills and problem-solving abilities can be applied to new situations whereas a list of facts cannot be made long enough to contain every possible circumstance. Such analytical skills are useful both for solving problems in this text and for applying physics in everyday life.

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Figure 1.12.1: Problem-solving skills are essential to your success in physics. (credit: “scui3asteveo”/Flickr)

As you are probably well aware, a certain amount of creativity and insight is required to solve problems. No rigid procedure works every time. Creativity and insight grow with experience. With practice, the basics of problem solving become almost automatic. One way to get practice is to work out the text’s examples for yourself as you read. Another is to work as many end-of-section problems as possible, starting with the easiest to build confidence and then progressing to the more difficult. After you become involved in physics, you will see it all around you, and you can begin to apply it to situations you encounter outside the classroom, just as is done in many of the applications in this text.

Although there is no simple step-by-step method that works for every problem, the following three-stage process facilitates problem solving and makes it more meaningful. The three stages are strategy, solution, and significance. This process is used in examples throughout the book. Here, we look at each stage of the process in turn.

Strategy

Strategy is the beginning stage of solving a problem. The idea is to figure out exactly what the problem is and then develop a strategy for solving it. Some general advice for this stage is as follows:

- **Examine the situation to determine which physical principles are involved**. It often helps to **draw a simple sketch** at the outset. You often need to decide which direction is positive and note that on your sketch. When you have identified the physical principles, it is much easier to find and apply the equations representing those principles. Although finding the correct equation is essential, keep in mind that equations represent physical principles, laws of nature, and relationships among physical quantities. Without a conceptual understanding of a problem, a numerical solution is meaningless.
- **Make a list of what is given or can be inferred from the problem as stated (identify the “knowns”)**. Many problems are stated very succinctly and require some inspection to determine what is known. Drawing a sketch be very useful at this point as well. Formally identifying the knowns is of particular importance in applying physics to real-world situations. For example, the word stopped means the velocity is zero at that instant. Also, we can often take initial time and position as zero by the appropriate choice of coordinate system.

- **Identify exactly what needs to be determined in the problem (identify the unknowns).** In complex problems, especially, it is not always obvious what needs to be found or in what sequence. Making a list can help identify the unknowns.
- **Determine which physical principles can help you solve the problem**. Since physical principles tend to be expressed in the form of mathematical equations, a list of knowns and unknowns can help here. It is easiest if you can find equations that contain only one unknown—that is, all the other variables are known—so you can solve for the unknown easily. If the equation contains more than one unknown, then additional equations are needed to solve the problem. In some problems, several unknowns must be determined to get at the one needed most. In such problems it is especially important to keep physical principles in mind to avoid going astray in a sea of equations. You may have to use two (or more) different equations to get the final answer.

Solution

The solution stage is when you do the math. **Substitute the knowns (along with their units) into the appropriate equation and obtain numerical solutions complete with units**. That is, do the algebra, calculus, geometry, or arithmetic necessary to find the unknown from the knowns, being sure to carry the units through the calculations. This step is clearly important because it produces the numerical answer, along with its units. Notice, however, that this stage is only one-third of the overall problem-solving process.

Significance

After having done the math in the solution stage of problem solving, it is tempting to think you are done. But, always remember that physics is not math. Rather, in doing physics, we use mathematics as a tool to help us understand nature. So, after you obtain a numerical answer, you should always assess its significance:

- **Check your units**. If the units of the answer are incorrect, then an error has been made and you should go back over your previous steps to find it. One way to find the mistake is to check all the equations you derived for dimensional consistency. However, be warned that correct units do not guarantee the numerical part of the answer is also correct.
- **Check the answer to see whether it is reasonable. Does it make sense?** This step is extremely important: –the goal of physics is to describe nature accurately. To determine whether the answer is reasonable, check both its magnitude and its sign, in addition to its units. The magnitude should be consistent with a rough estimate of what it should be. It should also compare reasonably with magnitudes of other quantities of the same type. The sign usually tells you about direction and should be consistent with your prior expectations. Your judgment will improve as you solve more physics problems, and it will become possible for you to make finer judgments regarding whether nature is described adequately by the answer to a problem. This step brings the problem back to its conceptual meaning. If you can judge whether the answer is reasonable, you have a deeper understanding of physics than just being able to solve a problem mechanically.
- **Check to see whether the answer tells you something interesting. What does it mean?** This is the flip side of the question: Does it make sense? Ultimately, physics is about understanding nature, and we solve physics problems to learn a little something about how nature operates. Therefore, assuming the answer does make sense, you should always take a moment to see if it tells you something about the world that you find interesting. Even if the answer to this particular problem is not very interesting to you, what about the method you used to solve it? Could the method be adapted to answer a question that you do find interesting? In many ways, it is in answering questions such as these that science progresses.

1.13: Units and Measurement (Answers)

Check your Understanding

1.1. 4.79×10^2 Mg

1.2. $3 \times 10^8 m/s$

1.3. $10^8 km^2$

1.4. The numbers were too small, by a factor of 4.45.

1.5. $4\pi r^3/3$

1.6. yes

1.7. $3 \times 10^4 m$ or 30 km. It is probably an underestimate because the density of the atmosphere decreases with altitude. (In fact, 30 km does not even get us out of the stratosphere.)

1.8. No, the coach's new stopwatch will not be helpful. The uncertainty in the stopwatch is too great to differentiate between the sprint times effectively.

Conceptual Questions

1. Physics is the science concerned with describing the interactions of energy, matter, space, and time to uncover the fundamental mechanisms that underlie every phenomenon.

3. No, neither of these two theories is more valid than the other. Experimentation is the ultimate decider. If experimental evidence does not suggest one theory over the other, then both are equally valid. A given physicist might prefer one theory over another on the grounds that one seems more simple, more natural, or more beautiful than the other, but that physicist would quickly acknowledge that he or she cannot say the other theory is invalid. Rather, he or she would be honest about the fact that more experimental evidence is needed to determine which theory is a better description of nature.

5. Probably not. As the saying goes, "Extraordinary claims require extraordinary evidence."

7. Conversions between units require factors of 10 only, which simplifies calculations. Also, the same basic units can be scaled up or down using metric prefixes to sizes appropriate for the problem at hand.

9. a. Base units are defined by a particular process of measuring a base quantity whereas derived units are defined as algebraic combinations of base units.

b. A base quantity is chosen by convention and practical considerations. Derived quantities are expressed as algebraic combinations of base quantities.

c. A base unit is a standard for expressing the measurement of a base quantity within a particular system of units. So, a measurement of a base quantity could be expressed in terms of a base unit in any system of units using the same base quantities. For example, length is a base quantity in both SI and the English system, but the meter is a base unit in the SI system only.

11. a. Uncertainty is a quantitative measure of precision. b. Discrepancy is a quantitative measure of accuracy.

13. Check to make sure it makes sense and assess its significance.

Problems

15. a. 10^3;

b. 10^5;

c. 10^2;

d. 10^{15};

e. 10^2;

f. 10^{57}

17. 10^2 generations

19. 10^{11} atoms

21. 10^3 nerve impulses/s

23. 10^{26} floating-point operations per human lifetime

25. a. 957 ks;

b. 4.5 cs or 45 ms;

c. 550 ns;

d. 31.6 Ms

27. a. 75.9 Mm;

b. 7.4 mm;

c. 88 pm;

d. 16.3 Tm

29. a. 3.8 cg or 38 mg;

b. 230 Eg;

c. 24 ng;

d. 8 Eg

e. 4.2 g

31. a. 27.8 m/s;

b. 62 mi/h

33. a. 3.6 km/h;

b. 2.2 mi/h

35. $1.05 \times 10^5 ft^2$

37. 8.847 km

39. a. $1.3 \times 10^{-9} m;$

b. 40 km/My

41. $10^6 Mg/\mu L$

43. 62.4 lbm/ft^3

45. 0.017 rad

47. 1 light-nanosecond

49. $3.6 \times 10^{-4} m^3$

51. a. Yes, both terms have dimension L^2T^{-2}

b. No.

c. Yes, both terms have dimension LT^{-1}

d. Yes, both terms have dimension LT^{-2}

53. a. $[v] = LT^{-1}$;

b. $[a] = LT^{-2}$;

c. $[\int vdt] = L$;

d. $[\int adt] = LT^{-1}$;

e. $[\frac{da}{dt}] = LT^{-3}$

55. a. L;

b. L;

c. $L^0 = 1$ (that is, it is dimensionless)

57. 10^{28} atoms

59. 10^{51} molecules

61. 10^{16} solar systems

63. a. Volume = $10^{27} m^3$, diameter is 10^9 m.;

b. 10^{11} m

65. a. A reasonable estimate might be one operation per second for a total of 10^ in a lifetime.;

b. about $(10^9)(10^{-17} s) = 10^{-8} s$, or about 10 ns

67. 2 kg

69. 4%

71. 67 mL

73. a. The number 99 has 2 significant figures; 100. has 3 significant figures.

b. 1.00%;

c. percent uncertainties

75. a. 2%;

b. 1 mm Hg

77. 7.557 cm^2

79. a. 37.2 lb; because the number of bags is an exact value, it is not considered in the significant figures;

b. 1.4 N; because the value 55 kg has only two significant figures, the final value must also contain two significant figures

Additional Problems

81. a. $[s_0] = L$ and units are meters (m);

b. $[v_0] = LT^{-1}$ and units are meters per second (m/s);

c. $[a_0] = LT^{-2}$ and units are meters per second squared (m/s^2);

d. $[j_0] = LT^{-3}$ and units are meters per second cubed (m/s^3);

e. $[S_0] = LT^{-4}$ and units are m/s^4;

f. $[c] = LT^{-5}$ and units are m/s^5.

83. a. 0.059%;

b. 0.01%;

c. 4.681 m/s;

d. 0.07%,

0.003 m/s

85. a. 0.02%;

b. 1×10^4 lbm

87. a. 143.6 cm^3;

b. 0.2 cm^3 or 0.14%

Challenge Problems

89. Since each term in the power series involves the argument raised to a different power, the only way that every term in the power series can have the same dimension is if the argument is dimensionless. To see this explicitly, suppose $[x]=L^aM^bT^c$. Then, $[x^n]=[x]^n=L^{an}M^{bn}T^{cn}$. If we want $[x]=[x^n]$, then an = a, bn = b, and cn = c for all n. The only way this can happen is if a = b = c = 0.

Contributors and Attributions

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1.14: Units and Measurement (Exercises)

Conceptual Questions

1.1 The Scope and Scale of Physics

1. What is physics?
2. Some have described physics as a “search for simplicity.” Explain why this might be an appropriate description.
3. If two different theories describe experimental observations equally well, can one be said to be more valid than the other (assuming both use accepted rules of logic)?
4. What determines the validity of a theory?
5. Certain criteria must be satisfied if a measurement or observation is to be believed. Will the criteria necessarily be as strict for an expected result as for an unexpected result?
6. Can the validity of a model be limited or must it be universally valid? How does this compare with the required validity of a theory or a law?

1.2 Units and Standards

7. Identify some advantages of metric units.
8. What are the SI base units of length, mass, and time?
9. What is the difference between a base unit and a derived unit? (b) What is the difference between a base quantity and a derived quantity? (c) What is the difference between a base quantity and a base unit?
10. For each of the following scenarios, refer to Figure 1.4 and Table 1.2 to determine which metric prefix on the meter is most appropriate for each of the following scenarios. (a) You want to tabulate the mean distance from the Sun for each planet in the solar system. (b) You want to compare the sizes of some common viruses to design a mechanical filter capable of blocking the pathogenic ones. (c) You want to list the diameters of all the elements on the periodic table. (d) You want to list the distances to all the stars that have now received any radio broadcasts sent from Earth 10 years ago.

1.6 Significant Figures

11. (a) What is the relationship between the precision and the uncertainty of a measurement? (b) What is the relationship between the accuracy and the discrepancy of a measurement?

1.7 Solving Problems in Physics

12. What information do you need to choose which equation or equations to use to solve a problem?
13. What should you do after obtaining a numerical answer when solving a problem?

Problems

1.1 The Scope and Scale of Physics

14. Find the order of magnitude of the following physical quantities.
    a. The mass of Earth’s atmosphere: 5.1×10^{18} kg;
    b. The mass of the Moon’s atmosphere: 25,000 kg;
    c. The mass of Earth’s hydrosphere: 1.4×10^{21} kg;
    d. The mass of Earth: 5.97×10^{24} kg;
    e. The mass of the Moon: 7.34×10^{22} kg;
    f. The Earth–Moon distance (semi-major axis): 3.84×10^{8} m;
    g. The mean Earth–Sun distance: 1.5×10^{11} m;
    h. The equatorial radius of Earth: 6.38×10^{6} m;
    i. The mass of an electron: 9.11×10^{-31} kg;
    j. The mass of a proton: 1.67×10^{-27} kg;
    k. The mass of the Sun: 1.99×10^{30} kg.
15. Use the orders of magnitude you found in the previous problem to answer the following questions to within an order of magnitude.
    a. How many electrons would it take to equal the mass of a proton?
    b. How many Earths would it take to equal the mass of the Sun?

c. How many Earth–Moon distances would it take to cover the distance from Earth to the Sun?
d. How many Moon atmospheres would it take to equal the mass of Earth’s atmosphere?
e. How many moons would it take to equal the mass of Earth?
f. How many protons would it take to equal the mass of the Sun?

For the remaining questions, you need to use Figure 1.4 to obtain the necessary orders of magnitude of lengths, masses, and times.

16. Roughly how many heartbeats are there in a lifetime?
17. A generation is about one-third of a lifetime. Approximately how many generations have passed since the year 0 AD?
18. Roughly how many times longer than the mean life of an extremely unstable atomic nucleus is the lifetime of a human?
19. Calculate the approximate number of atoms in a bacterium. Assume the average mass of an atom in the bacterium is 10 times the mass of a proton.
20. (a) Calculate the number of cells in a hummingbird assuming the mass of an average cell is 10 times the mass of a bacterium. (b) Making the same assumption, how many cells are there in a human?
21. Assuming one nerve impulse must end before another can begin, what is the maximum firing rate of a nerve in impulses per second?
22. About how many floating-point operations can a supercomputer perform each year?
23. Roughly how many floating-point operations can a supercomputer perform in a human lifetime?

1.2 Units and Standards

24. The following times are given using metric prefixes on the base SI unit of time: the second. Rewrite them in scientific notation without the prefix. For example, 47 Ts would be rewritten as 4.7×10^{13} s.
    a. 980 Ps;
    b. 980 fs;
    c. 17 ns;
    d. 577 μs.
25. The following times are given in seconds. Use metric prefixes to rewrite them so the numerical value is greater than one but less than 1000. For example, 7.9×10^{-2} s could be written as either 7.9 cs or 79 ms.
    a. 9.57×10^{5} s;
    b. 0.045 s;
    c. 5.5×10^{-7} s;
    d. 3.16×10^{7} s.
26. The following lengths are given using metric prefixes on the base SI unit of length: the meter. Rewrite them in scientific notation without the prefix. For example, 4.2 Pm would be rewritten as 4.2×10^{15} m.
    a. 89 Tm;
    b. 89 pm;
    c. 711 mm;
    d. 0.45 μm.
27. The following lengths are given in meters. Use metric prefixes to rewrite them so the numerical value is bigger than one but less than 1000. For example, 7.9×10^{-2} m could be written either as 7.9 cm or 79 mm.
    a. 7.59×10^{7} m;
    b. 0.0074 m;
    c. 8.8×10^{-11} m;
    d. 1.63×10^{13} m.
28. The following masses are written using metric prefixes on the gram. Rewrite them in scientific notation in terms of the SI base unit of mass: the kilogram. For example, 40 Mg would be written as 4×10^{4} kg.
    a. 23 mg;
    b. 320 Tg;
    c. 42 ng;
    d. 7 g;
    e. 9 Pg.

29. The following masses are given in kilograms. Use metric prefixes on the gram to rewrite them so the numerical value is bigger than one but less than 1000. For example, 7 × 10−4 kg could be written as 70 cg or 700 mg.
    a. 3.8 × 10−5 kg;
    b. 2.3 × 1017 kg;
    c. 2.4 × 10−11 kg;
    d. 8 × 1015 kg;
    e. 4.2 × 10−3 kg.

1.3 Unit Conversion

30. The volume of Earth is on the order of 10^{21} m^3. (a) What is this in cubic kilometers (km^3)? (b) What is it in cubic miles (mi^3)? (c) What is it in cubic centimeters (cm^3)?
31. The speed limit on some interstate highways is roughly 100 km/h. (a) What is this in meters per second? (b) How many miles per hour is this?
32. A car is traveling at a speed of 33 m/s. (a) What is its speed in kilometers per hour? (b) Is it exceeding the 90 km/ h speed limit?
33. In SI units, speeds are measured in meters per second (m/s). But, depending on where you live, you're probably more comfortable of thinking of speeds in terms of either kilometers per hour (km/h) or miles per hour (mi/h). In this problem, you will see that 1 m/s is roughly 4 km/h or 2 mi/h, which is handy to use when developing your physical intuition. More precisely, show that (a) 1.0 m/s = 3.6 km/h and (b) 1.0 m/s = 2.2 mi/h.
34. American football is played on a 100-yd-long field, excluding the end zones. How long is the field in meters? (Assume that 1 m = 3.281 ft.)
35. Soccer fields vary in size. A large soccer field is 115 m long and 85.0 m wide. What is its area in square feet? (Assume that 1 m = 3.281 ft.)
36. What is the height in meters of a person who is 6 ft 1.0 in. tall?
37. Mount Everest, at 29,028 ft, is the tallest mountain on Earth. What is its height in kilometers? (Assume that 1 m = 3.281 ft.)
38. The speed of sound is measured to be 342 m/s on a certain day. What is this measurement in kilometers per hour?
39. Tectonic plates are large segments of Earth's crust that move slowly. Suppose one such plate has an average speed of 4.0 cm/yr. (a) What distance does it move in 1.0 s at this speed? (b) What is its speed in kilometers per million years?
40. The average distance between Earth and the Sun is 1.5 × 10^{11} m. (a) Calculate the average speed of Earth in its orbit (assumed to be circular) in meters per second. (b) What is this speed in miles per hour?
41. The density of nuclear matter is about 10^{18} kg/m^3. Given that 1 mL is equal in volume to cm^3, what is the density of nuclear matter in megagrams per microliter (that is, Mg/μL)?
42. The density of aluminum is 2.7 g/cm^3. What is the density in kilograms per cubic meter?
43. A commonly used unit of mass in the English system is the pound-mass, abbreviated lbm, where 1 lbm = 0.454 kg. What is the density of water in pound-mass per cubic foot?
44. A furlong is 220 yd. A fortnight is 2 weeks. Convert a speed of one furlong per fortnight to millimeters per second.
45. It takes 2π radians (rad) to get around a circle, which is the same as 360°. How many radians are in 1°?
46. Light travels a distance of about 3 × 10^8 m/s. A light-minute is the distance light travels in 1 min. If the Sun is 1.5 × 10^{11} m from Earth, how far away is it in lightminutes?
47. A light-nanosecond is the distance light travels in 1 ns. Convert 1 ft to light-nanoseconds.
48. An electron has a mass of 9.11 × 10^{-31} kg. A proton has a mass of 1.67 × 10^{-27} kg. What is the mass of a proton in electron-masses?
49. A fluid ounce is about 30 mL. What is the volume of a 12 fl-oz can of soda pop in cubic meters?

1.4 Dimensional Analysis

50. A student is trying to remember some formulas from geometry. In what follows, assume A is area, V is volume, and all other variables are lengths. Determine which formulas are dimensionally consistent. (a) V = $\pi r^2 h$; (b) A = $2\pi r^2 + 2\pi r h$; (c) V = 0.5bh; (d) V = πd^2 ; (e) V = $\frac{\pi d^3}{6}$
51. Consider the physical quantities s, v, a, and t with dimensions [s] = L, [v] = LT^{-1}, [a] = LT^{-2}, and [t] = T. Determine whether each of the following equations is dimensionally consistent. (a) v^2 = 2as; (b) s = vt^2 + 0.5at^2; (c) v = s/t; (d) a = v/t.
52. Consider the physical quantities m, s, v, a, and t with dimensions [m] = M, [s] = L, [v] = LT^{-1}, [a] = LT^{-2}, and [t] = T. Assuming each of the following equations is dimensionally consistent, find the dimension of the quantity on the left-hand side of the equation: (a) F = ma; (b) K = 0.5mv^2 ; (c) p = mv; (d) W = mas; (e) L = mvr.

53. Suppose quantity s is a length and quantity t is a time. Suppose the quantities v and a are defined by v = ds/dt and a = dv/dt. (a) What is the dimension of v? (b) What is the dimension of the quantity a? What are the dimensions of (c) $\int$vdt, (d) $\int$adt, and (e) da/dt?
54. Suppose [V] = L3 , [ρ] = ML^{-3}, and [t] = T. (a) What is the dimension of $\int \rho dV$? (b) What is the dimension of dV/dt? (c) What is the dimension of ρ(dV/dt)?
55. The arc length formula says the length s of arc subtended by angle Θ in a circle of radius r is given by the equation s = rΘ. What are the dimensions of (a) s, (b) r, and (c) Θ?

1.5 Estimates and Fermi Calculations

56. Assuming the human body is made primarily of water, estimate the volume of a person.
57. Assuming the human body is primarily made of water, estimate the number of molecules in it. (Note that water has a molecular mass of 18 g/mol and there are roughly 10^{24} atoms in a mole.)
58. Estimate the mass of air in a classroom.
59. Estimate the number of molecules that make up Earth, assuming an average molecular mass of 30 g/mol. (Note there are on the order of 10^{24} objects per mole.)
60. Estimate the surface area of a person.
61. Roughly how many solar systems would it take to tile the disk of the Milky Way?
62. (a) Estimate the density of the Moon. (b) Estimate the diameter of the Moon. (c) Given that the Moon subtends at an angle of about half a degree in the sky, estimate its distance from Earth.
63. The average density of the Sun is on the order 10^3 kg/ m^3. (a) Estimate the diameter of the Sun. (b) Given that the Sun subtends at an angle of about half a degree in the sky, estimate its distance from Earth. 64. Estimate the mass of a virus.
64. A floating-point operation is a single arithmetic operation such as addition, subtraction, multiplication, or division. (a) Estimate the maximum number of floating-point operations a human being could possibly perform in a lifetime. (b) How long would it take a supercomputer to perform that many floating-point operations?

1.6 Significant Figures

66. Consider the equation 4000/400 = 10.0. Assuming the number of significant figures in the answer is correct, what can you say about the number of significant figures in 4000 and 400?
67. Suppose your bathroom scale reads your mass as 65 kg with a 3% uncertainty. What is the uncertainty in your mass (in kilograms)?
68. A good-quality measuring tape can be off by 0.50 cm over a distance of 20 m. What is its percent uncertainty?
69. An infant's pulse rate is measured to be 130 ± 5 beats/ min. What is the percent uncertainty in this measurement?
70. (a) Suppose that a person has an average heart rate of 72.0 beats/min. How many beats does he or she have in 2.0 years? (b) In 2.00 years? (c) In 2.000 years?
71. A can contains 375 mL of soda. How much is left after 308 mL is removed?
72. State how many significant figures are proper in the results of the following calculations: (a) (106.7)(98.2) / (46.210)(1.01); (b) $(18.7)^2$; (c) $(1.60 \times 10^{-19})(3712)$
73. (a) How many significant figures are in the numbers 99 and 100.? (b) If the uncertainty in each number is 1, what is the percent uncertainty in each? (c) Which is a more meaningful way to express the accuracy of these two numbers: significant figures or percent uncertainties?
74. (a) If your speedometer has an uncertainty of 2.0 km/h at a speed of 90 km/h, what is the percent uncertainty? (b) If it has the same percent uncertainty when it reads 60 km/ h, what is the range of speeds you could be going?
75. (a) A person's blood pressure is measured to be 120 ± 2 mm Hg. What is its percent uncertainty? (b) Assuming the same percent uncertainty, what is the uncertainty in a blood pressure measurement of 80 mm Hg?
76. A person measures his or her heart rate by counting the number of beats in 30 s. If 40 ± 1 beats are counted in 30.0 ± 0.5 s, what is the heart rate and its uncertainty in beats per minute?
77. What is the area of a circle 3.102 cm in diameter?
78. Determine the number of significant figures in the following measurements: (a) 0.0009, (b) 15,450.0, (c) 6×103 , (d) 87.990, and (e) 30.42.
79. Perform the following calculations and express your answer using the correct number of significant digits. (a) A woman has two bags weighing 13.5 lb and one bag with a weight of 10.2 lb. What is the total weight of the bags? (b) The force F on an object is equal to its mass m multiplied by its acceleration a. If a wagon with mass 55 kg accelerates at a rate of 0.0255 m/s^2, what is the force on the wagon? (The unit of force is called the newton and it is expressed with the symbol N.)

Additional Problems

80. Consider the equation y = mt +b, where the dimension of y is length and the dimension of t is time, and m and b are constants. What are the dimensions and SI units of (a) m and (b) b?
81. Consider the equation $s=s_0+v_0t+\frac{a_0t^2}{2}+\frac{j_0t^3}{6}+\frac{S_0t^4}{24}+\frac{ct^5}{120}$, where s is a length and t is a time. What are the dimensions and SI units of (a) s_0, (b) v_0, (c) a_0, (d) j_0, (e) S_0, and (f) c?
82. (a) A car speedometer has a 5% uncertainty. What is the range of possible speeds when it reads 90 km/h? (b) Convert this range to miles per hour. Note 1 km = 0.6214 mi.
83. A marathon runner completes a 42.188-km course in 2 h, 30 min, and 12 s. There is an uncertainty of 25 m in the distance traveled and an uncertainty of 1 s in the elapsed time. (a) Calculate the percent uncertainty in the distance. (b) Calculate the percent uncertainty in the elapsed time. (c) What is the average speed in meters per second? (d) What is the uncertainty in the average speed?
84. The sides of a small rectangular box are measured to be 1.80 ± 0.1 cm, 2.05 ± 0.02 cm, and 3.1 ± 0.1 cm long. Calculate its volume and uncertainty in cubic centimeters.
85. When nonmetric units were used in the United Kingdom, a unit of mass called the pound-mass (lbm) was used, where 1 lbm = 0.4539 kg. (a) If there is an uncertainty of 0.0001 kg in the pound-mass unit, what is its percent uncertainty? (b) Based on that percent uncertainty, what mass in pound-mass has an uncertainty of 1 kg when converted to kilograms?
86. The length and width of a rectangular room are measured to be 3.955 ± 0.005 m and 3.050 ± 0.005 m. Calculate the area of the room and its uncertainty in square meters.
87. A car engine moves a piston with a circular cross-section of 7.500 ± 0.002 cm in diameter a distance of 3.250 ± 0.001 cm to compress the gas in the cylinder. (a) By what amount is the gas decreased in volume in cubic centimeters? (b) Find the uncertainty in this volume.

Challenge Problems

88. The first atomic bomb was detonated on July 16, 1945, at the Trinity test site about 200 mi south of Los Alamos. In 1947, the U.S. government declassified a film reel of the explosion. From this film reel, British physicist G. I. Taylor was able to determine the rate at which the radius of the fireball from the blast grew. Using dimensional analysis, he was then able to deduce the amount of energy released in the explosion, which was a closely guarded secret at the time. Because of this, Taylor did not publish his results until 1950. This problem challenges you to recreate this famous calculation.
    a. Using keen physical insight developed from years of experience, Taylor decided the radius r of the fireball should depend only on time since the explosion, t, the density of the air, ρ, and the energy of the initial explosion, E. Thus, he made the educated guess that $r=kE^a\rho^bt^c$ for some dimensionless constant k and some unknown exponents a, b, and c. Given that [E] = ML^2T^{-2}, determine the values of the exponents necessary to make this equation dimensionally consistent. (Hint: Notice the equation implies that $k=rE^{-a}\rho^{-b}t^{-c}$ and that [k] = 1.)
    b. By analyzing data from high-energy conventional explosives, Taylor found the formula he derived seemed to be valid as long as the constant k had the value 1.03. From the film reel, he was able to determine many values of r and the corresponding values of t. For example, he found that after 25.0 ms, the fireball had a radius of 130.0 m. Use these values, along with an average air density of 1.25 kg/m^3, to calculate the initial energy release of the Trinity detonation in joules (J). (Hint: To get energy in joules, you need to make sure all the numbers you substitute in are expressed in terms of SI base units.) (c) The energy released in large explosions is often cited in units of "tons of TNT" (abbreviated "t TNT"), where 1 t TNT is about 4.2 GJ. Convert your answer to (b) into kilotons of TNT (that is, kt TNT). Compare your answer with the quick-and-dirty estimate of 10 kt TNT made by physicist Enrico Fermi shortly after witnessing the explosion from what was thought to be a safe distance. (Reportedly, Fermi made his estimate by dropping some shredded bits of paper right before the remnants of the shock wave hit him and looked to see how far they were carried by it.)
89. The purpose of this problem is to show the entire concept of dimensional consistency can be summarized by the old saying "You can't add apples and oranges." If you have studied power series expansions in a calculus course, you know the standard mathematical functions such as trigonometric functions, logarithms, and exponential functions can be expressed as infinite sums of the form $\sum_{n=0}^{\infty}a_nx^n=a_0+a_1x+a_2x^2+a^3x^3+\cdots$, where the a_n are dimensionless constants for all n = 0, 1, 2, ⋯ and x is the argument of the function. (If you have not studied power series in calculus yet, just trust us.) Use this fact to explain why the requirement that all terms in an equation have the same dimensions is sufficient as a definition of dimensional consistency. That is, it actually implies the arguments of standard mathematical functions must be dimensionless, so it is not

really necessary to make this latter condition a separate requirement of the definition of dimensional consistency as we have done in this section.

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1.15: Units and Measurement (Summary)

Key Terms

accuracy	the degree to which a measured value agrees with an accepted reference value for that measurement
base quantity	physical quantity chosen by convention and practical considerations such that all other physical quantities can be expressed as algebraic combinations of them
base unit	standard for expressing the measurement of a base quantity within a particular system of units; defined by a particular procedure used to measure the corresponding base quantity
conversion factor	a ratio that expresses how many of one unit are equal to another unit
derived quantity	physical quantity defined using algebraic combinations of base quantities
derived units	units that can be calculated using algebraic combinations of the fundamental units
dimension	expression of the dependence of a physical quantity on the base quantities as a product of powers of symbols representing the base quantities; in general, the dimension of a quantity has the form $L^a M^b T^c I^d \Theta^e N^f J^g$ for some powers a, b, c, d, e, f, and g
dimensionally consistent	equation in which every term has the same dimensions and the arguments of any mathematical functions appearing in the equation are dimensionless
dimensionless	quantity with a dimension of $L^0 M^0 T^0 I^0 \Theta^e N^0 J^0 = 1$; also called quantity of dimension 1 or a pure number
discrepancy	the difference between the measured value and a given standard or expected value
English units	system of measurement used in the United States; includes units of measure such as feet, gallons, and pounds
estimation	using prior experience and sound physical reasoning to arrive at a rough idea of a quantity's value; sometimes called an "order-of-magnitude approximation," a "guesstimate," a "back-of-the-envelope calculation", or a "Fermi calculation"
kilogram	SI unit for mass, abbreviated kg
law	description, using concise language or a mathematical formula, of a generalized pattern in nature supported by scientific evidence and repeated experiments
meter	SI unit for length, abbreviated m
method of adding percents	the percent uncertainty in a quantity calculated by multiplication or division is the sum of the percent uncertainties in the items used to make the calculation
metric system	system in which values can be calculated in factors of 10
model	representation of something often too difficult (or impossible) to display directly

order of magnitude	the size of a quantity as it relates to a power of 10
percent uncertainty	the ratio of the uncertainty of a measurement to the measured value, expressed as a percentage
physical quantity	characteristic or property of an object that can be measured or calculated from other measurements
physics	science concerned with describing the interactions of energy, matter, space, and time; especially interested in what fundamental mechanisms underlie every phenomenon
precision	the degree to which repeated measurements agree with each other
second	the SI unit for time, abbreviated s
SI units	the international system of units that scientists in most countries have agreed to use; includes units such as meters, liters, and grams
significant figures	used to express the precision of a measuring tool used to measure a value
theory	testable explanation for patterns in nature supported by scientific evidence and verified multiple times by various groups of researchers
uncertainty	a quantitative measure of how much measured values deviate from one another
units	standards used for expressing and comparing measurements

Key Equations

Percent uncertainty	$$Percent\ uncertainty = \frac{\delta A}{A} \times 100\% \tag{1.15.1}$$

Summary

1.1 The Scope and Scale of Physics

- Physics is about trying to find the simple laws that describe all natural phenomena.
- Physics operates on a vast range of scales of length, mass, and time. Scientists use the concept of the order of magnitude of a number to track which phenomena occur on which scales. They also use orders of magnitude to compare the various scales.
- Scientists attempt to describe the world by formulating models, theories, and laws

1.2 Units and Standards

- Systems of units are built up from a small number of base units, which are defined by accurate and precise measurements of conventionally chosen base quantities. Other units are then derived as algebraic combinations of the base units.
- Two commonly used systems of units are English units and SI units. All scientists and most of the other people in the world use SI, whereas nonscientists in the United States still tend to use English units.
- The SI base units of length, mass, and time are the meter (m), kilogram (kg), and second (s), respectively.
- SI units are a metric system of units, meaning values can be calculated by factors of 10. Metric prefixes may be used with metric units to scale the base units to sizes appropriate for almost any application.

1.3 Unit Conversion

- To convert a quantity from one unit to another, multiply by conversion factors in such a way that you cancel the units you want to get rid of and introduce the units you want to end up with.
- Be careful with areas and volumes. Units obey the rules of algebra so, for example, if a unit is squared we need two factors to cancel it.

1.4 Dimensional Analysis

- The dimension of a physical quantity is just an expression of the base quantities from which it is derived.
- All equations expressing physical laws or principles must be dimensionally consistent. This fact can be used as an aid in remembering physical laws, as a way to check whether claimed relationships between physical quantities are possible, and even to derive new physical laws.

1.5 Estimates and Fermi Calculations

- An estimate is a rough educated guess at the value of a physical quantity based on prior experience and sound physical reasoning. Some strategies that may help when making an estimate are as follows:
  - Get big lengths from smaller lengths.
  - Get areas and volumes from lengths.
  - Get masses from volumes and densities.
  - If all else fails, bound it. One "sig. fig." is fine.
  - Ask yourself: Does this make any sense?

1.6 Significant Figures

- Accuracy of a measured value refers to how close a measurement is to an accepted reference value. The discrepancy in a measurement is the amount by which the measurement result differs from this value.
- Precision of measured values refers to how close the agreement is between repeated measurements. The uncertainty of a measurement is a quantification of this.
- The precision of a measuring tool is related to the size of its measurement increments. The smaller the measurement increment, the more precise the tool.
- Significant figures express the precision of a measuring tool.
- When multiplying or dividing measured values, the final answer can contain only as many significant figures as the least-precise value.
- When adding or subtracting measured values, the final answer cannot contain more decimal places than the least-precise value.

1.7 Solving Problems in Physics

The three stages of the process for solving physics problems used in this textmap are as follows:

- **Strategy**: Determine which physical principles are involved and develop a strategy for using them to solve the problem.
- **Solution**: Do the math necessary to obtain a numerical solution complete with units.
- **Significance**: Check the solution to make sure it makes sense (correct units, reasonable magnitude and sign) and assess its significance.

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CHAPTER OVERVIEW

2: Kinematics

Topic hierarchy

2.1: Basics of Kinematics

Defining Kinematics

Kinematics is the study of the motion of points, objects, and groups of objects without considering the causes of its motion.

learning objectives

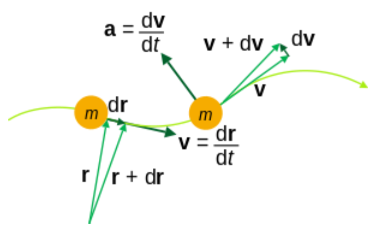
- Define kinematics

Kinematics is the branch of classical mechanics that describes the motion of points, objects and systems of groups of objects, without reference to the causes of motion (i.e., forces). The study of kinematics is often referred to as the "geometry of motion."

Objects are in motion all around us. Everything from a tennis match to a space-probe flyby of the planet Neptune involves motion. When you are resting, your heart moves blood through your veins. Even in inanimate objects there is continuous motion in the vibrations of atoms and molecules. Interesting questions about motion can arise: how long will it take for a space probe to travel to Mars? Where will a football land if thrown at a certain angle? An understanding of motion, however, is also key to understanding other concepts in physics. An understanding of acceleration, for example, is crucial to the study of force.

To describe motion, kinematics studies the trajectories of points, lines and other geometric objects, as well as their differential properties (such as velocity and acceleration). Kinematics is used in astrophysics to describe the motion of celestial bodies and systems; and in mechanical engineering, robotics and biomechanics to describe the motion of systems composed of joined parts (such as an engine, a robotic arm, or the skeleton of the human body).

A formal study of physics begins with kinematics. The word "kinematics" comes from a Greek word "kinesis" meaning motion, and is related to other English words such as "cinema" (movies) and "kinesiology" (the study of human motion). Kinematic analysis is the process of measuring the kinematic quantities used to describe motion. The study of kinematics can be abstracted into purely mathematical expressions, which can be used to calculate various aspects of motion such as velocity, acceleration, displacement, time, and trajectory.



Kinematics of a particle trajectory: Kinematic equations can be used to calculate the trajectory of particles or objects. The physical quantities relevant to the motion of a particle include: mass m, position r, velocity v, acceleration a.

Reference Frames and Displacement

In order to describe an object's motion, you need to specify its position relative to a convenient reference frame.

learning objectives

- Evaluate displacement within a frame of reference.

In order to describe the motion of an object, you must first describe its position — where it is at any particular time. More precisely, you need to specify its position relative to a convenient reference frame. Earth is often used as a reference frame, and we often describe the position of objects related to its position to or from Earth. Mathematically, the position of an object is generally represented by the variable x.

Frames of Reference

There are two choices you have to make in order to define a position variable x. You have to decide where to put $x = 0$ and which direction will be positive. This is referred to as choosing a coordinate system, or choosing a frame of reference. As long as you are consistent, any frame is equally valid. But you don't want to change coordinate systems in the middle of a calculation. Imagine sitting in a train in a station when suddenly you notice that the station is moving backward. Most people would say that they just failed to notice that the train was moving — it only *seemed* like the station was moving. But this shows that there is a *third* arbitrary choice that goes into choosing a coordinate system: valid frames of reference can differ from each other by moving relative to one another. It might seem strange to use a coordinate system moving relative to the earth — but, for instance, the frame of reference moving along with a train might be far more convenient for describing things happening inside the train. Frames of reference are particularly important when describing an object's displacement.

FRAMES OF REFERENCE by Professor Hume and Professor Donald Ivey of the University of Toronto

In this classic film, Professors Hume and Ivey cleverly illustrate reference frames and distinguish between fixed and moving frames of reference.

Frames of Reference (1960) Educational Film: Frames of Reference is a 1960 educational film by Physical Sciences Study Committee. The film was made to be shown in high school physics courses. In the film University of Toronto physics professors Patterson Hume and Donald Ivey explain the distinction between inertial and nonintertial frames of reference, while demonstrating these concepts through humorous camera tricks. For example, the film opens with Dr. Hume, who appears to be upside down, accusing Dr. Ivey of being upside down. Only when the pair flip a coin does it become obvious that Dr. Ivey — and the camera — are indeed inverted. The film's humor serves both to hold students' interest and to demonstrate the concepts being discussed. This PSSC film utilizes a fascinating set consisting of a rotating table and furniture occupying surprisingly unpredictable spots within the viewing area. The fine cinematography by Abraham Morochnik, and funny narration by University of Toronto professors Donald Ivey and Patterson Hume is a wonderful example of the fun a creative team of filmmakers can have with a subject that other, less imaginative types might find pedestrian. Producer: Richard Leacock Production Company: Educational Development Corp. Sponsor: Eric Prestamon

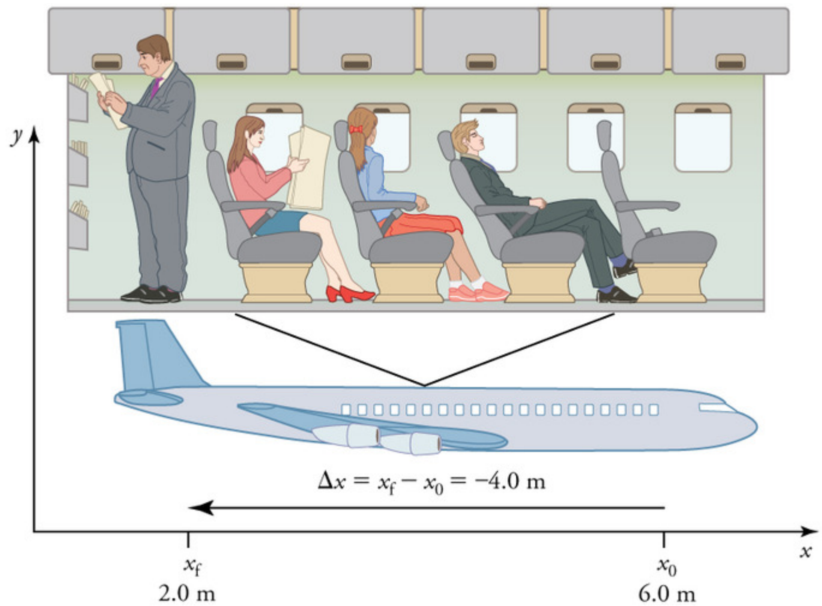
Displacement

Displacement is the change in position of an object relative to its reference frame. For example, if a car moves from a house to a grocery store, its displacement is the relative distance of the grocery store to the reference frame, or the house. The word "displacement" implies that an object has moved or has been displaced. Displacement is the change in position of an object and can be represented mathematically as follows:

$$\Delta \mathrm{x} = \mathrm{x_f} - \mathrm{x_0} \tag{2.1.1}$$

where Δx is displacement, x_f is the final position, and x_0 is the initial position.

shows the importance of using a frame of reference when describing the displacement of a passenger on an airplane.



Displacement in Terms of Frame of Reference: A passenger moves from his seat to the back of the plane. His location relative to the airplane is given by x. The -4.0m displacement of the passenger relative to the plane is represented by an arrow toward the rear of the plane. Notice that the arrow representing his displacement is twice as long as the arrow representing the displacement of the professor (he moves twice as far).

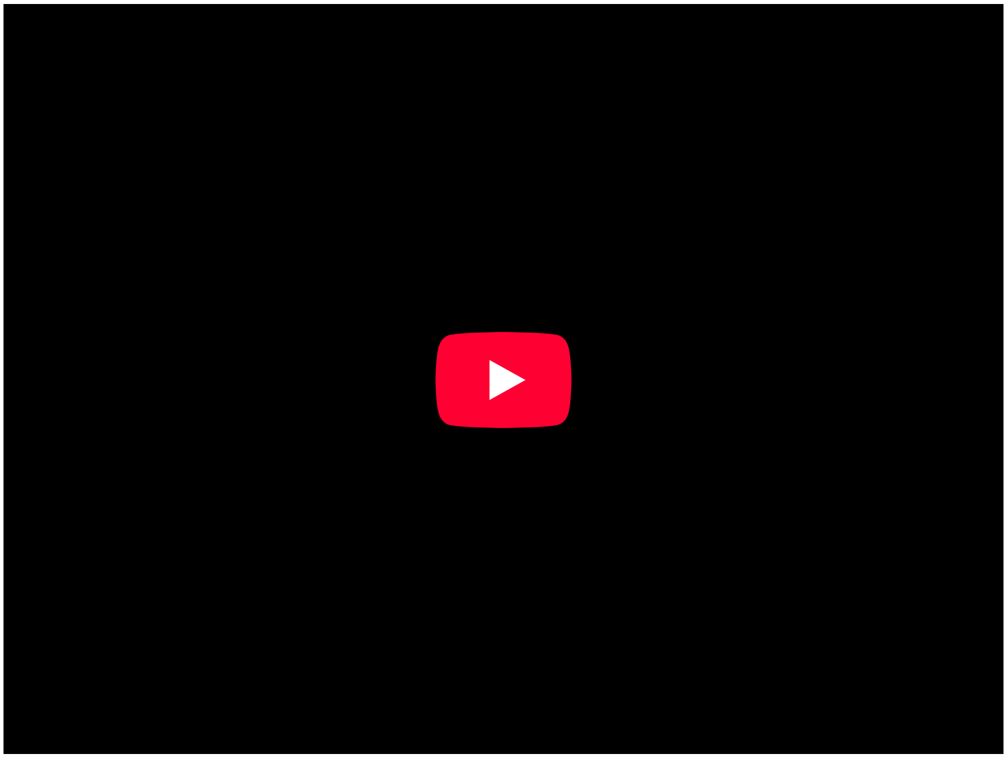
Introduction to Scalars and Vectors

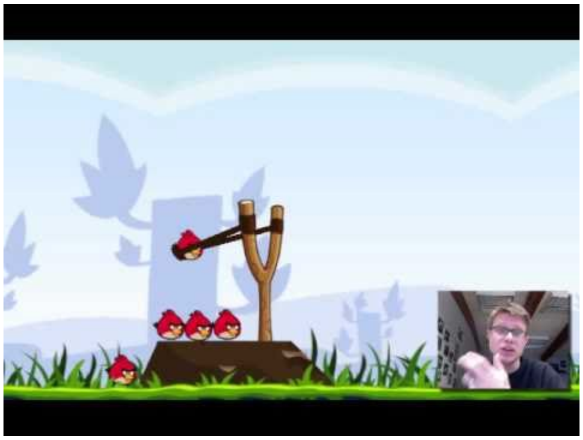
A vector is any quantity that has both magnitude and direction, whereas a scalar has only magnitude.

learning objectives

- Distinguish the difference between scalars and vectors

What is the difference between distance and displacement? Whereas displacement is defined by both direction and magnitude, distance is defined by magnitude alone. Displacement is an example of a vector quantity. Distance is an example of a scalar quantity. A vector is any quantity with both magnitude and direction. Other examples of vectors include a velocity of 90 km/h east and a force of 500 newtons straight down.





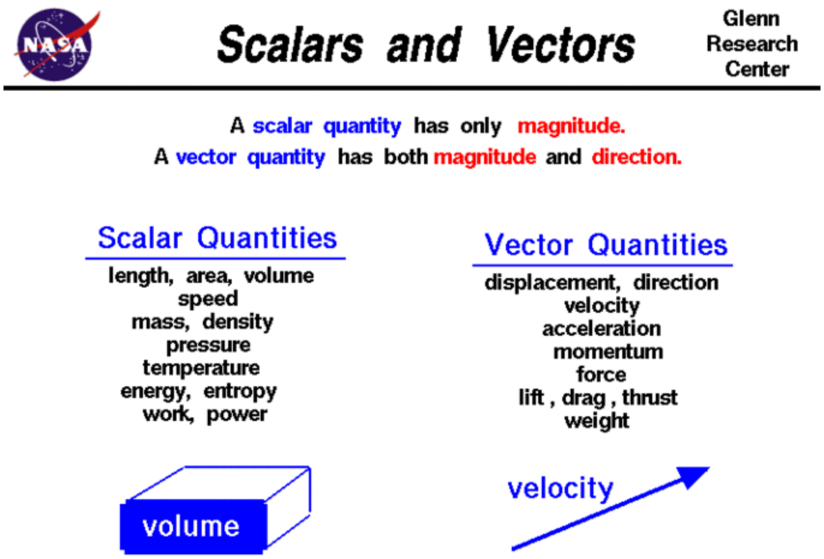
Scalars and Vectors: Mr. Andersen explains the differences between scalar and vectors quantities. He also uses a demonstration to show the importance of vectors and vector addition.

In mathematics, physics, and engineering, a vector is a geometric object that has a magnitude (or length) and direction and can be added to other vectors according to vector algebra. The direction of a vector in one-dimensional motion is given simply by a plus (+) or minus (−) sign. A vector is frequently represented by a line segment with a definite direction, or graphically as an arrow, connecting an initial point A with a terminal point B, as shown in.

$$\overrightarrow{AB}.$$

Vector representation: A vector is frequently represented by a line segment with a definite direction, or graphically as an arrow, connecting an initial point A with a terminal point B.

Some physical quantities, like distance, either have no direction or no specified direction. In physics, a scalar is a simple physical quantity that is not changed by coordinate system rotations or translations. It is any quantity that can be expressed by a single number and has a magnitude, but no direction. For example, a 20ºC temperature, the 250 kilocalories (250 Calories) of energy in a candy bar, a 90 km/h speed limit, a person's 1.8 m height, and a distance of 2.0 m are all scalars, or quantities with no specified direction. Note, however, that a scalar can be negative, such as a −20ºC temperature. In this case, the minus sign indicates a point on a scale rather than a direction. Scalars are never represented by arrows. (A comparison of scalars vs. vectors is shown in.)



Scalars vs. Vectors: A brief list of quantities that are either scalars or vectors.

Key Points

- To describe motion, kinematics studies the trajectories of points, lines and other geometric objects.
- The study of kinematics can be abstracted into purely mathematical expressions.
- Kinematic equations can be used to calculate various aspects of motion such as velocity, acceleration, displacement, and time.
- Choosing a frame of reference requires deciding where the object's initial position is and which direction will be considered positive.
- Valid frames of reference can differ from each other by moving relative to one another.
- Frames of reference are particularly important when describing an object's displacement.
- Displacement is the change in position of an object relative to its reference frame.
- A vector is any quantity that has magnitude and direction.
- A scalar is any quantity that has magnitude but no direction.
- Displacement and velocity are vectors, whereas distance and speed are scalars.

Key Terms

- **kinematics**: The branch of mechanics concerned with objects in motion, but not with the forces involved.
- **displacement**: A vector quantity that denotes distance with a directional component.
- **frame of reference**: A coordinate system or set of axes within which to measure the position, orientation, and other properties of objects in it.
- **scalar**: A quantity that has magnitude but not direction; compare vector.
- **vector**: A directed quantity, one with both magnitude and direction; the between two points.

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2.2: Speed and Velocity

Average Velocity: A Graphical Interpretation

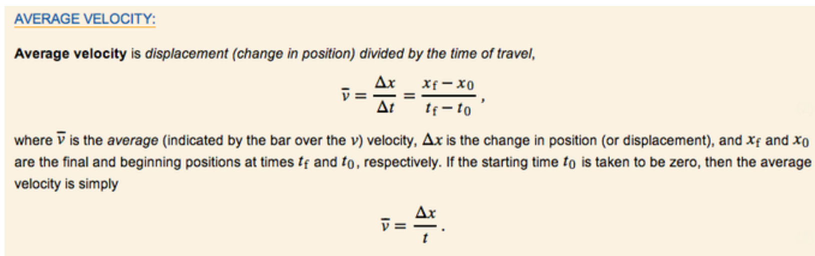
Average velocity is defined as the change in position (or displacement) over the time of travel.

learning objectives

- Contrast speed and velocity in physics

In everyday usage, the terms "speed" and "velocity" are used interchangeably. In physics, however, they are distinct quantities. Speed is a scalar quantity and has only magnitude. Velocity, on the other hand, is a vector quantity and so has both magnitude and direction. This distinction becomes more apparent when we calculate average speed and velocity.

Average speed is calculated as the distance traveled over the total time of travel. In contrast, average velocity is defined as the change in *position* (or displacement) over the total time of travel.



AVERAGE VELOCITY:

Average velocity is *displacement (change in position) divided by the time of travel,*

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_0}{t_f - t_0},$$

where $\bar{v}$ is the *average* (indicated by the bar over the v) velocity, Δx is the change in position (or displacement), and x_f and x_0 are the final and beginning positions at times t_f and t_0, respectively. If the starting time t_0 is taken to be zero, then the average velocity is simply

$$\bar{v} = \frac{\Delta x}{t}.$$

Average Velocity: The kinematic formula for calculating average velocity is the change in position over the time of travel.

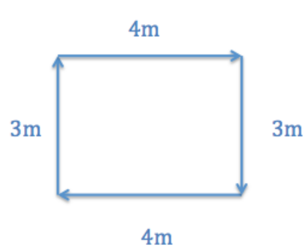
The SI unit for velocity is meters per second, or m/s, but many other units (such as km/h, mph, and cm/s) are commonly used. Suppose, for example, an airplane passenger took five seconds to move -4 m (the negative sign indicates that displacement is toward the back of the plane). His average velocity would be:

$$\mathrm{v} = \frac{\Delta \mathrm{x}}{\mathrm{t}} = \frac{-4\ \mathrm{m}}{5\ \mathrm{s}} = -0.8\,\frac{\mathrm{m}}{\mathrm{s}} \tag{2.2.1}$$

The minus sign indicates that the average velocity is also toward the rear of the plane.

The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point, however. For example, we cannot tell from average velocity whether the airplane passenger stops momentarily or backs up before he gets to the back of the plane. To get more details, we must consider smaller segments of the trip over smaller time intervals.

To illustrate the difference between average speed and average velocity, consider the following additional example. Imagine you are walking in a small rectangle. You walk three meters north, four meters east, three meters south, and another four meters west. The entire walk takes you 30 seconds. If you are calculating average speed, you would calculate the entire distance (3 + 4 + 3 + 4 = 14 meters) over the total time, 30 seconds. From this, you would get an average speed of 14/30 = 0.47 m/s. When calculating average velocity, however, you are looking at the displacement over time. Because you walked in a full rectangle and ended up exactly where you started, your displacement is 0 meters. Therefore, your average velocity, or displacement over time, would be 0 m/s.



Average Speed vs. Average Velocity: If you started walking from one corner and went all the way around the rectangle in 30 seconds, your average speed would be 0.47 m/s, but your average velocity would be 0 m/s.

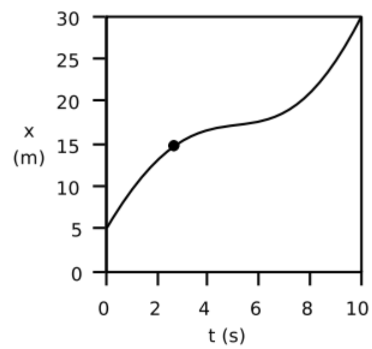
Instananeous Velocity: A Graphical Interpretation

Instantaneous velocity is the velocity of an object at a single point in time and space as calculated by the slope of the tangent line.

learning objectives

- Differentiate instantaneous velocity from other ways of determining velocity

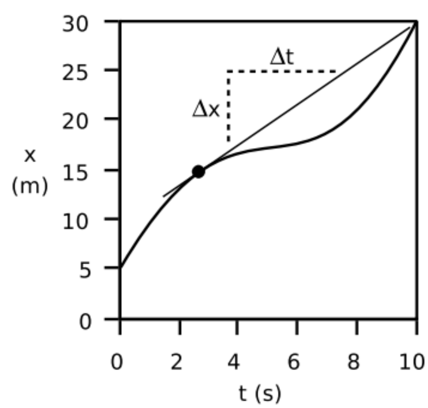
Typically, motion is not with constant velocity nor speed. While driving in a car, for example, we continuously speed up and slow down. A graphical representation of our motion in terms of distance vs. time, therefore, would be more variable or "curvy" rather than a straight line, indicating motion with a constant velocity as shown below. (We limit our discussion to one dimensional motion. It should be straightforward to generalize to three dimensional cases.)



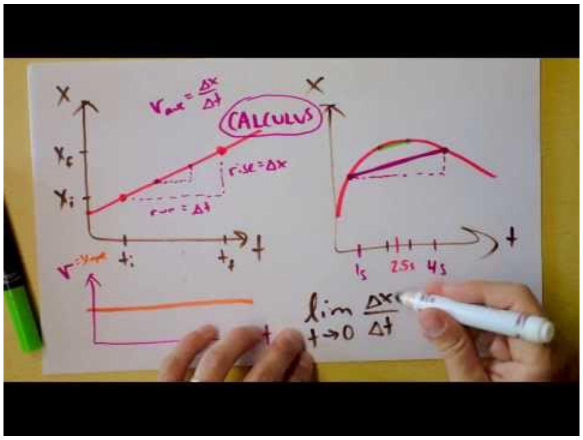
Motion with Changing Velocity: Motion is often observed with changing velocity. This would result in a curvy line when graphed with distance over time.

To calculate the speed of an object from a graph representing constant velocity, all that is needed is to find the slope of the line; this would indicate the change in distance over the change in time. However, changing velocity it is not as straightforward.

Since our velocity is constantly changing, we can estimate velocity in different ways. One way is to look at our instantaneous velocity, represented by one point on our curvy line of motion graphed with distance vs. time. In order to determine our velocity at any given moment, we must determine the slope at that point. To do this, we find a line that represents our velocity in that moment, shown graphically in. That line would be the line tangent to the curve at that point. If we extend this line, we can easily calculate the displacement of distance over time and determine our velocity at that given point. The velocity of an object at any given moment is the slope of the tangent line through the relevant point on its x vs. t graph.



Determining instantaneous velocity: The velocity at any given moment is defined as the slope of the tangent line through the relevant point on the graph



Instantaneous Velocity, Acceleration, Jerk, Slopes, Graphs vs. Time: This is how kinematics begins.

In calculus, finding the slope of curve $f(x)$ at $x = x_0$ $x=x_0$ is equivalent to finding the first derivative:

$$\frac{df(x)}{dx}|_{x=x_0}. \tag{2.2.2}$$

One interpretation of this definition is that the velocity shows how many meters the object would travel in one second if it continues moving at the same speed for at least one second.

Key Points

- Average velocity can be calculated by determining the total displacement divided by the total time of travel.
- The average velocity of an object does not tell us anything about what happens to it between the starting point and ending point.
- Average velocity is different from average speed in that it considers the direction of travel and the overall change in position.
- When velocity is constantly changing, we can estimate our velocity by looking at instantaneous velocity.
- Instantaneous velocity is calculated by determining the slope of the line tangent to the curve at the point of interest.
- Instantaneous velocity is similar to determining how many meters the object would travel in one second at a specific moment.

Key Terms

- **velocity**: A vector quantity that denotes the rate of change of position with respect to time, or a speed with a directional component.
- **instantaneous**: (As in velocity)—occurring, arising, or functioning without any delay; happening within an imperceptibly brief period of time.

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2.3: Acceleration

Graphical Interpretation

The graphical representation of acceleration over time can be derived through the graph of an object's position over time.

learning objectives

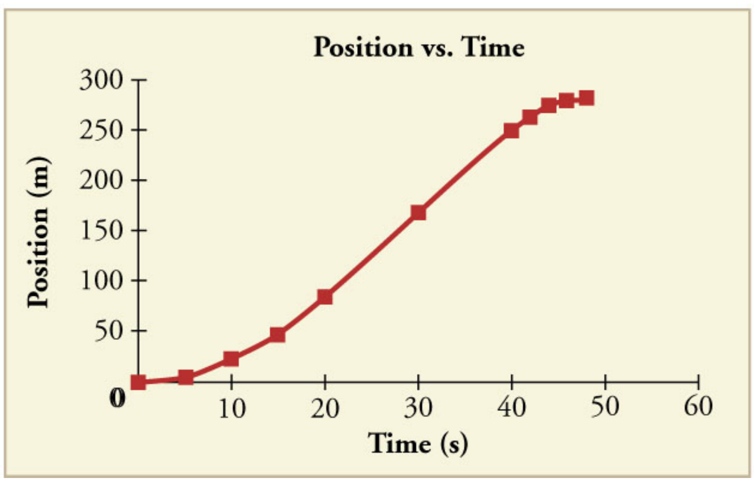
- Distinguish the difference between how to plot a velocity graph and how to plot an acceleration graph

In physics, acceleration is the rate at which the velocity of a body changes with time. It is a vector quantity with both magnitude and direction. Acceleration is accompanied by a force, as described by Newton's Second Law; the force, as a vector, is the product of the mass of the object being accelerated and the acceleration (vector), or $\mathrm{F}=\mathrm{ma}$. The SI unit of acceleration is the meter per second squared: $\frac{\mathrm{m}}{\mathrm{s}^2}$

Acceleration is a vector that points in the same direction as the change in velocity, though it may not always be in the direction of motion. For example, when an object slows down, or decelerating, its acceleration is in the opposite direction of its motion.

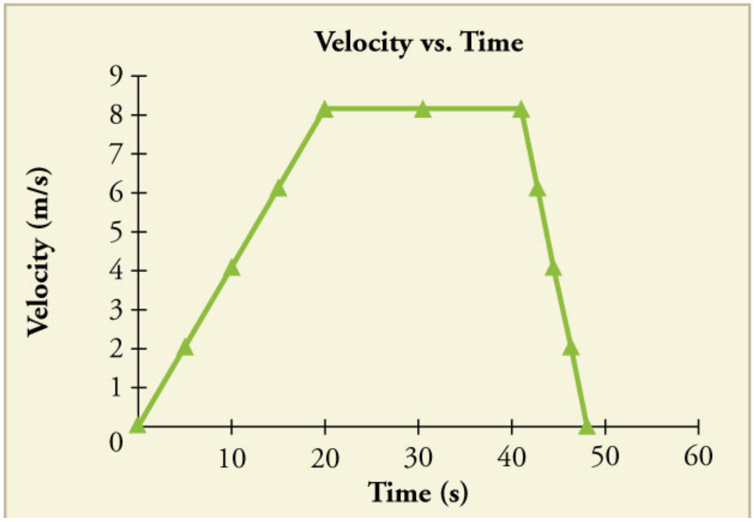
The motion of an object can be depicted graphically by plotting the position of an object over time. This distance-time graph can be used to create another graph that shows changes in velocity over time. Because acceleration is velocity in $\frac{\mathrm{m}}{\mathrm{s}}$ divided by time in s, we can further derive a graph of acceleration from a graph of an object's speed or position.

is a graph of an object's position over time. This graph is similar to the motion of a car. In the beginning, the object's position changes slowly as it gains speed. In the middle, the speed is constant and the position changes at a constant rate. As it slows down toward the end, the position changes more slowly. From this graph, we can derive a velocity vs time graph.



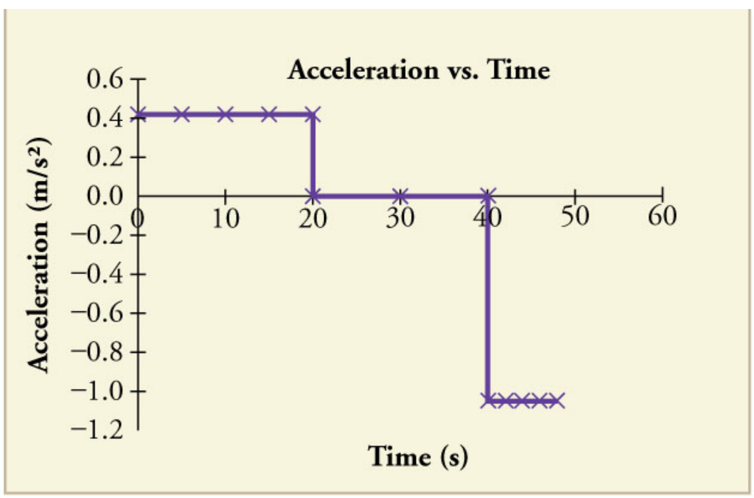
Position vs Time Graph: Notice that the object's position changes slowly at the beginning of the journey, then more and more quickly as it picks up speed. Its position then changes more slowly as it slows down at the end of the journey. In the middle of the journey, while the velocity remains constant, the position changes at a constant rate.

This shows the velocity of the object over time. The object's velocity increases in the beginning as it accelerates at the beginning, then remains constant in the middle before it slows down toward the end. Notice that this graph is a representation of the slope of the previous position vs time graph. From this graph, we can further derive an acceleration vs time graph.

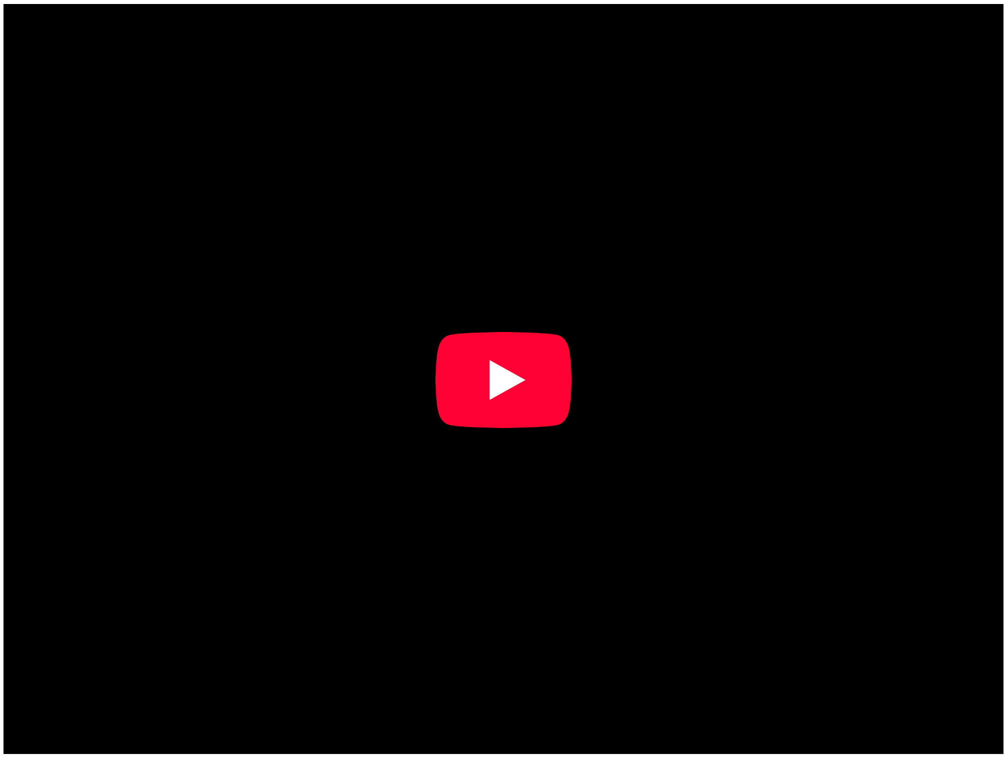


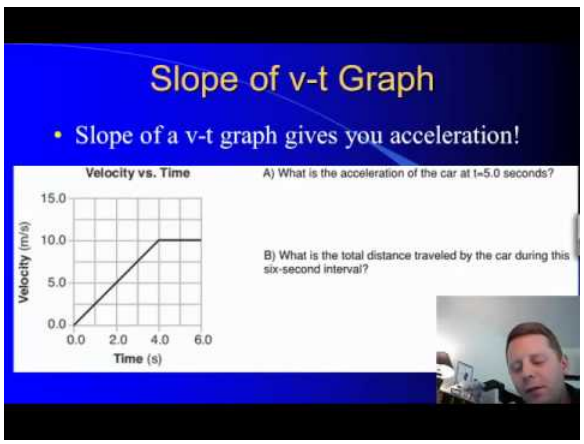
Velocity vs Time: The object's velocity increases as it accelerates at the beginning of the journey. It remains the same in the middle of the journey (where there is no acceleration). It decreases as the object decelerates at the end of the journey.

To do this, we would also plot the slope of the velocity vs time graph. In this graph, the acceleration is constant in the three different stages of motion. As we noted earlier, the object is increasing speed and changing positions slowly in the beginning. The acceleration graph shows that the object was increasing at a positive constant acceleration during this time. In the middle, when the object was changing position at a constant velocity, the acceleration was 0. This is because the object is no longer changing its velocity and is moving at a constant rate. Towards the end of the motion, the object slows down. This is depicted as a negative value on the acceleration graph. Note that in this example, the motion of the object is still forward (positive), but since it is decelerating, the acceleration is negative.



Acceleration vs Time Graph: The object has positive acceleration as it speeds up at the beginning of the journey. It has no acceleration as it travels at constant velocity in the middle of the journey. Its acceleration is negative as it slows down at the end of the journey.



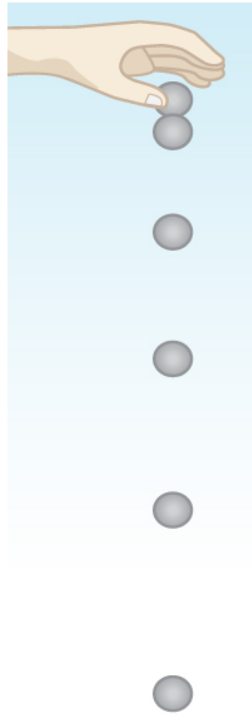


Graphing Motion: A brief introduction to particle diagrams and motion graphs.

Motion with Constant Acceleration

Constant acceleration occurs when an object's velocity changes by an equal amount in every equal time period.

> learning objectives
>
> - Describe how constant acceleration affects the motion of an object



One-Dimensional Motion: When you drop an object, it falls vertically toward the center of the earth due to the constant acceleration of gravity.

An object experiencing constant acceleration has a velocity that increases or decreases by an equal amount for any constant period of time. Acceleration can be derived easily from basic kinematic principles. It is defined as the first time derivative of velocity (so the second derivative of position with respect to time):

$$a=\frac{\partial v}{\partial t}=\frac{\partial^2 x}{\partial t^2} \tag{2.3.1}$$

Assuming acceleration to be constant does not seriously limit the situations we can study and does not degrade the accuracy of our treatment, because in a great number of situations, acceleration *is* constant. When it is not, we can either consider it in separate parts of constant acceleration or use an average acceleration over a period of time.

The motion of falling objects is a simple, one-dimensional type of projectile motion in which there is no horizontal movement. For example, if you held a rock out and dropped it, the rock would fall only vertically downward toward the earth. If you were to throw the rock instead of just dropping it, it would follow a more projectile-like pattern, similar to the one a kicked ball follows.

Projectile motion is the motion of an object thrown or projected into the air and is subject only to the acceleration of gravity. The object thrown is called a projectile, and the object's path is called its trajectory. In two-dimensional projectile motion, there is both a vertical and a horizontal component.

Due to the algebraic properties of constant acceleration, there are kinematic equations that relate displacement, initial velocity, final velocity, acceleration, and time. A summary of these equations is given below.

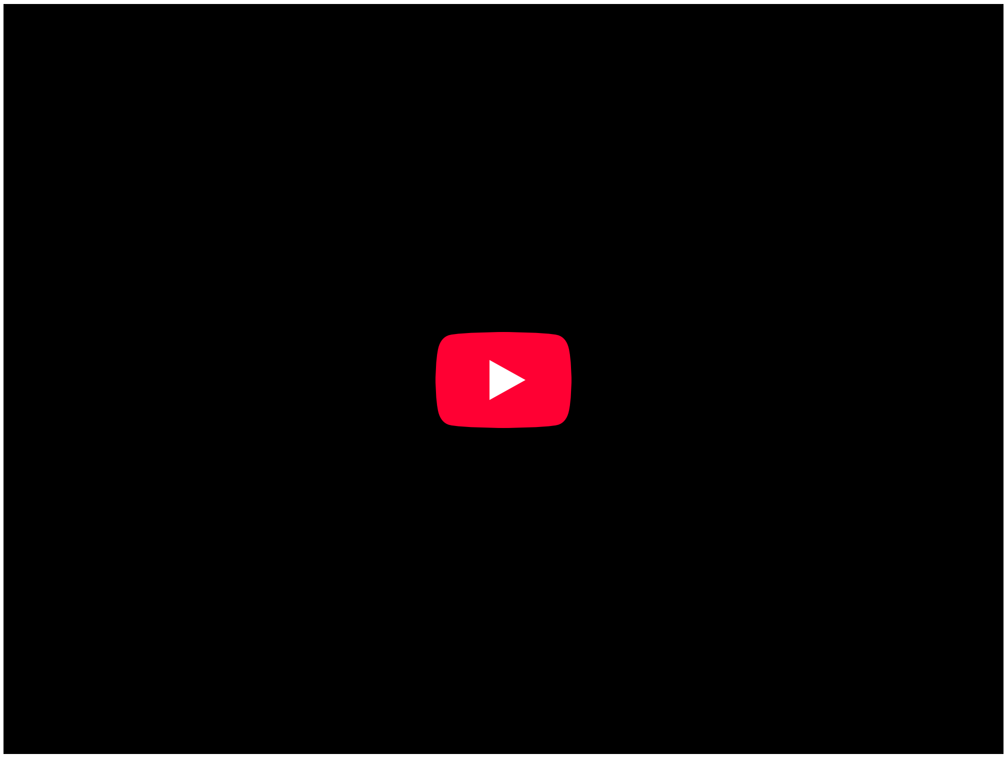
$$x=x_0+\bar{v}t \tag{2.3.2}$$

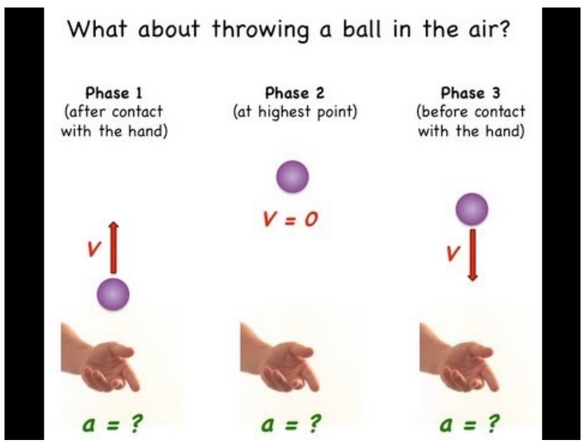
$$\bar{v}=\frac{v_0+v}{2} \tag{2.3.3}$$

$$v=v_0+at \tag{2.3.4}$$

$$x=x_0+v_0t+\frac{1}{2}at^2 \tag{2.3.5}$$

$$v^2=v_0^2+2a(x-x_0) \tag{2.3.6}$$





Constant Acceleration Explained with Vectors and Algebra: This video answers the question “what is acceleration? ".

Key Points

- Acceleration is the rate at which the velocity of a body changes with time.
- Acceleration is a vector that points in the same direction as the change in velocity, though it may not always be in the direction of motion.
- Because acceleration is velocity in m/s divided by time in s, we can derive a graph of acceleration from a graph of an object’s speed or position.
- Assuming acceleration to be constant does not seriously limit the situations we can study and does not degrade the accuracy of our treatment.
- Due to the algebraic properties of constant acceleration, there are kinematic equations that can be used to calculate displacement, velocity, acceleration, and time.
- Calculations with constant acceleration can be done in relation to one-dimensional motion as well as two-dimensional motion.

Key Terms

- **acceleration**: The amount by which a speed or velocity increases (and so a scalar quantity or a vector quantity).
- **velocity**: A vector quantity that denotes the rate of change of position with respect to time, or a speed with a directional component.
- **position**: A place or location.
- **kinematic**: of or relating to motion or kinematics

2.4: Problem-Solving for Basic Kinematics

Applications

There are four kinematic equations that describe the motion of objects without consideration of its causes.

learning objectives

- Choose which kinematics equation to use in problems in which the initial starting position is equal to zero

Kinematics is the branch of classical mechanics that describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without consideration of the causes of motion. There are four kinematic equations when the initial starting position is the origin, and the acceleration is constant:

1. $v = v_0 + at$
2. $d = \frac{1}{2}(v_0 + v)t$ or alternatively $v_{average} = \frac{d}{t}$
3. $d = v_0 t + (\frac{at^2}{2})$
4. $v^2 = v_0^2 + 2ad$

Notice that the four kinematic equations involve five kinematic variables: d, v, v_0, a and t. Each of these equations contains only four of the five variables and has a different one missing. This tells us that we need the values of three variables to obtain the value of the fourth and we need to choose the equation that contains the three known variables and one unknown variable for each specific situation.

Here the basic problem solving steps to use these equations:

Step one – Identify exactly what needs to be determined in the problem (identify the unknowns).

Step two – Find an equation or set of equations that can help you solve the problem.

Step three – Substitute the knowns along with their units into the appropriate equation, and obtain numerical solutions complete with units.

Step four – Check the answer to see if it is reasonable: Does it make sense?

Problem-solving skills are obviously essential to success in a quantitative course in physics. More importantly, the ability to apply broad physical principles, usually represented by equations, to specific situations is a very powerful form of knowledge. It is much more powerful than memorizing a list of facts. Analytical skills and problem-solving abilities can be applied to new situations, whereas a list of facts cannot be made long enough to contain every possible circumstance. Such analytical skills are useful both for solving problems in a physics class and for applying physics in everyday and professional life.

Motion Diagrams

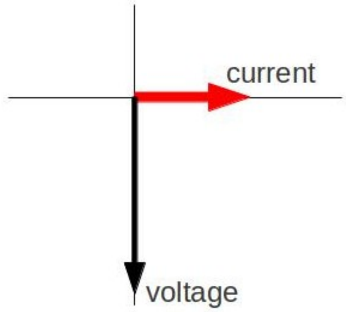
A motion diagram is a pictorial description of an object's motion and represents the position of an object at equally spaced time intervals.

learning objectives

- Construct a motion diagram

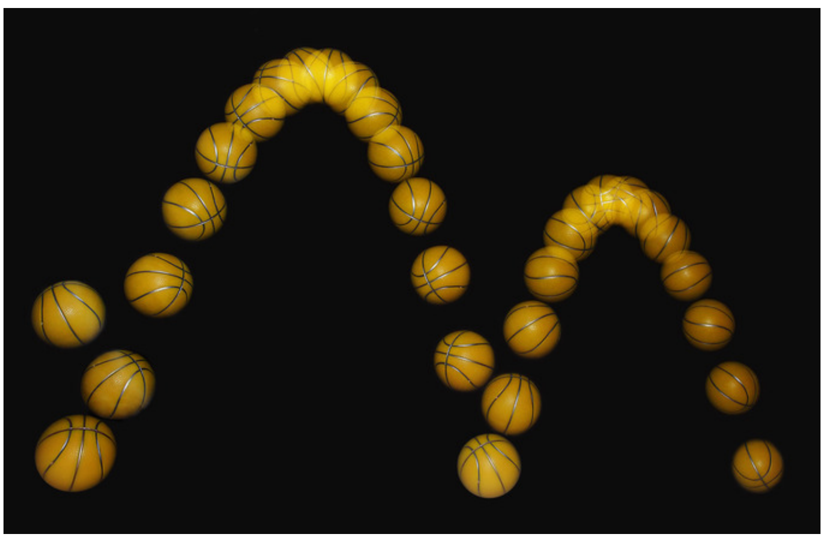
A motion diagram is a pictorial description of the motion of an object. It displays the object's location at various equally spaced times on the same diagram; shows an object's initial position and velocity; and presents several spots in the center of the diagram. These spots reveal whether or not the object has accelerated or decelerated. For simplicity, the object is represented by a simple shape, such as a filled circle, which contains information about an object's position at particular time instances. For this reason, a motion diagram is more information than a path diagram. It may also display the forces acting on the object at each time instance.

is a motion diagram of a simple trajectory. Imagine the object as a hockey puck sliding on ice. Notice that the puck covers the same distance per unit interval along the trajectory. We can conclude that the puck is moving at a constant velocity and, therefore, there is no acceleration or deceleration during the motion.



Puck Sliding on Ice: Motion diagram of a puck sliding on ice. The puck is moving at a constant velocity.

One major use of motion diagrams is the presentation of film through a series of frames taken by a camera; this is sometimes called stroboscopic technique (as seen in). Viewing an object on a motion diagram allows one to determine whether an object is speeding up or slowing down, or if it is at constant rest. As the frames are taken, we can assume that an object is at a constant rest if it occupies the same position over time. We can assume that an object is speeding up if there is a visible increase in the space between objects as time passes, and that it is slowing down if there is a visible decrease in the space between objects as time passes. The objects on the frame come very close together.



Bouncing Ball: A bouncing ball captured with a stroboscopic flash at 25 images per second.

Key Points

- The four kinematic equations involve five kinematic variables: d, v, v_0, a and t.
- Each equation contains only four of the five variables and has a different one missing.
- It is important to choose the equation that contains the three known variables and one unknown variable for each specific situation.
- Motion diagrams represent the motion of an object by displaying its location at various equally spaced times on the same diagram.
- Motion diagrams show an object's initial position and velocity and presents several spots in the center of the diagram. These spots reveal the object's state of motion.
- Motion diagrams contain information about an object's position at particular time instances and is therefore more informative than a path diagram.

Key Terms

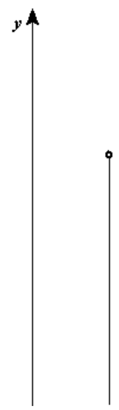
- **kinematics**: The branch of physics concerned with objects in motion.
- **stroboscopic**: Relating to an instrument used to make a cyclically moving object appear to be slow-moving, or stationary.
- **diagram**: A graph or chart.
- **motion**: A change of position with respect to time.

2.5: Free-Falling Objects

learning objectives

- Solve basic problems concerning free fall and distinguish it from other kinds of motion

The motion of falling objects is the simplest and most common example of motion with changing velocity. If a coin and a piece of paper are simultaneously dropped side by side, the paper takes much longer to hit the ground. However, if you crumple the paper into a compact ball and drop the items again, it will look like both the coin and the paper hit the floor simultaneously. This is because the amount of force acting on an object is a function of not only its mass, but also area. Free fall is the motion of a body where its weight is the only force acting on an object.



Free Fall: This clip shows an object in free fall.

Galileo also observed this phenomena and realized that it disagreed with the Aristotle principle that heavier items fall more quickly. Galileo then hypothesized that there is an upward force exerted by air in addition to the downward force of gravity. If air resistance and friction are negligible, then in a given location (because gravity changes with location), all objects fall toward the center of Earth with the *same constant acceleration, independent of their mass*, that constant acceleration is gravity. Air resistance opposes the motion of an object through the air, while friction opposes motion between objects and the medium through which they are traveling. The acceleration of free-falling objects is referred to as the acceleration due to gravity gg. As we said earlier, gravity varies depending on location and altitude on Earth (or any other planet), but the average acceleration due to gravity on Earth is 9.8 $\frac{m}{s^2}$. This value is also often expressed as a negative acceleration in mathematical calculations due to the downward direction of gravity.

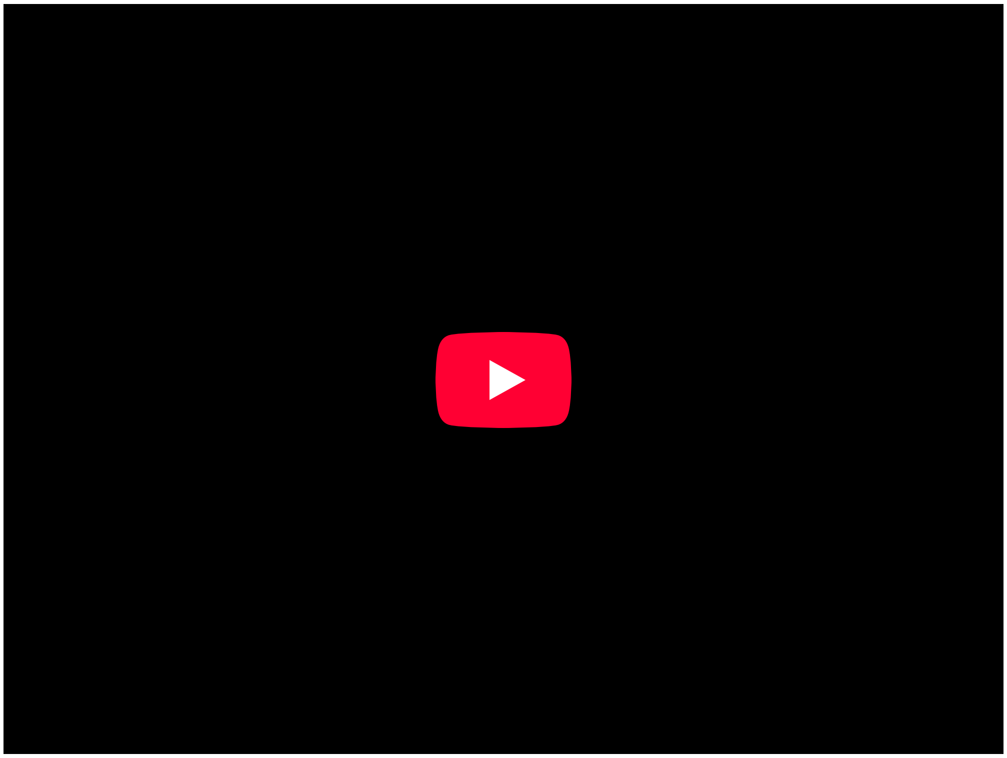
The best way to see the basic features of motion involving gravity is to start by considering straight up and down motion with no air resistance or friction. This means that if the object is dropped, we know the initial velocity is zero. Once the object is in motion, the object is in free-fall. Under these circumstances, the motion is one-dimensional and has constant acceleration, gg. The kinematic equations for objects experiencing free fall are:

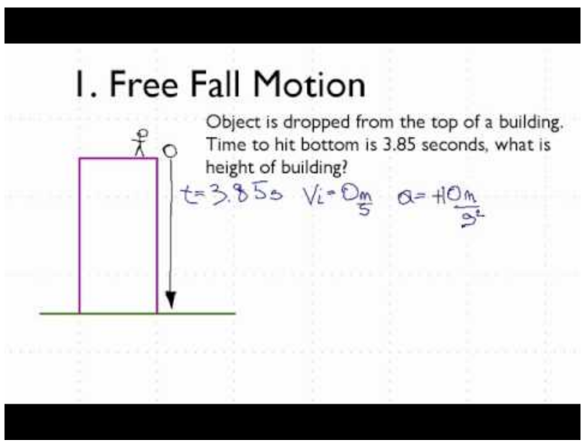
$$v = v_0 - gt \tag{2.5.1}$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{2.5.2}$$

$$v^2 = v_0^2 - 2g(y - y_0), \tag{2.5.3}$$

where $v = \text{velocity}, g = \text{gravity}, t = \text{time}, \text{ and } y = \text{vertical displacement}.$





Video 2.5.1: Free Fall Motion - Describes how to calculate the time for an object to fall if given the height and the height that an object fell if given the time to fall.

Example 2.5.1:

Some examples of objects that are in free fall include:

- A spacecraft in continuous orbit. The free fall would end once the propulsion devices turned on.
- An stone dropped down an empty well.
- An object, in projectile motion, on its descent.

Key Points

- The acceleration of free-falling objects is called the acceleration due to gravity, since objects are pulled towards the center of the earth.
- The acceleration due to gravity is constant on the surface of the Earth and has the value of 9.80 $\frac{m}{s^2}$.

Glossary

Acceleration

The amount by which a speed or velocity changes within a certain period of time (and so a scalar quantity or a vector quantity).

CHAPTER OVERVIEW

3: Vectors

Vectors are essential to physics and engineering. Many fundamental physical quantities are vectors, including displacement, velocity, force, and electric and magnetic vector fields. Scalar products of vectors define other fundamental scalar physical quantities, such as energy. Vector products of vectors define still other fundamental vector physical quantities, such as torque and angular momentum. In other words, vectors are a component part of physics in much the same way as sentences are a component part of literature. In introductory physics, vectors are Euclidean quantities that have geometric representations as arrows in one dimension (in a line), in two dimensions (in a plane), or in three dimensions (in space). They can be added, subtracted, or multiplied. In this chapter, we explore elements of vector algebra for applications in mechanics and in electricity and magnetism. Vector operations also have numerous generalizations in other branches of physics.

Thumbnail: Figure 2.1 - A signpost gives information about distances and directions to towns or to other locations relative to the location of the signpost. Distance is a scalar quantity. Knowing the distance alone is not enough to get to the town; we must also know the direction from the signpost to the town. The direction, together with the distance, is a vector quantity commonly called the displacement vector. A signpost, therefore, gives information about displacement vectors from the signpost to towns. (credit: modification of work by "studio tdes"/Flickr).

Contributors

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3.1: Prelude to Vectors

Vectors are essential to physics and engineering. Many fundamental physical quantities are vectors, including displacement, velocity, force, and electric and magnetic vector fields. Scalar products of vectors define other fundamental scalar physical quantities, such as energy. Vector products of vectors define still other fundamental vector physical quantities, such as torque and angular momentum. In other words, vectors are a component part of physics in much the same way as sentences are a component part of literature.

Figure 3.1.1: A signpost gives information about distances and directions to towns or to other locations relative to the location of the signpost. Distance is a scalar quantity. Knowing the distance alone is not enough to get to the town; we must also know the direction from the signpost to the town. The direction, together with the distance, is a vector quantity commonly called the displacement vector. A signpost, therefore, gives information about displacement vectors from the signpost to towns. (credit: modification of work by "studio tdes"/Flickr)

In introductory physics, vectors are Euclidean quantities that have geometric representations as arrows in one dimension (in a line), in two dimensions (in a plane), or in three dimensions (in space). They can be added, subtracted, or multiplied. In this chapter, we explore elements of vector algebra for applications in mechanics and in electricity and magnetism. Vector operations also have numerous generalizations in other branches of physics.

3.2: Scalars and Vectors (Part 1)

Learning Objectives

- Describe the difference between vector and scalar quantities.
- Identify the magnitude and direction of a vector.
- Explain the effect of multiplying a vector quantity by a scalar.
- Describe how one-dimensional vector quantities are added or subtracted.
- Explain the geometric construction for the addition or subtraction of vectors in a plane.
- Distinguish between a vector equation and a scalar equation.

Many familiar physical quantities can be specified completely by giving a single number and the appropriate unit. For example, “a class period lasts 50 min” or “the gas tank in my car holds 65 L” or “the distance between two posts is 100 m.” A physical quantity that can be specified completely in this manner is called a **scalar quantity**. Scalar is a synonym of “number.” Time, mass, distance, length, volume, temperature, and energy are examples of **scalar** quantities.

Scalar quantities that have the same physical units can be added or subtracted according to the usual rules of algebra for numbers. For example, a class ending 10 min earlier than 50 min lasts 50 min − 10 min = 40 min. Similarly, a 60-cal serving of corn followed by a 200-cal serving of donuts gives 60 cal + 200 cal = 260 cal of energy. When we multiply a scalar quantity by a number, we obtain the same scalar quantity but with a larger (or smaller) value. For example, if yesterday’s breakfast had 200 cal of energy and today’s breakfast has four times as much energy as it had yesterday, then today’s breakfast has 4(200 cal) = 800 cal of energy. Two scalar quantities can also be multiplied or divided by each other to form a derived scalar quantity. For example, if a train covers a distance of 100 km in 1.0 h, its speed is 100.0 km/1.0 h = 27.8 m/s, where the speed is a derived scalar quantity obtained by dividing distance by time.

Many physical quantities, however, cannot be described completely by just a single number of physical units. For example, when the U.S. Coast Guard dispatches a ship or a helicopter for a rescue mission, the rescue team must know not only the distance to the distress signal, but also the direction from which the signal is coming so they can get to its origin as quickly as possible. Physical quantities specified completely by giving a number of units (magnitude) and a direction are called **vector quantities**. Examples of vector quantities include displacement, velocity, position, force, and torque. In the language of mathematics, physical vector quantities are represented by mathematical objects called **vectors** (Figure $3.2.1$). We can add or subtract two vectors, and we can multiply a vector by a scalar or by another vector, but we cannot divide by a vector. The operation of division by a vector is not defined.

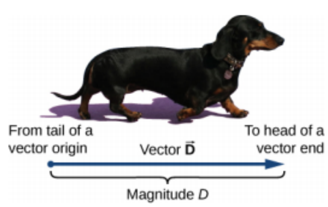

Figure $3.2.1$: We draw a vector from the initial point or origin (called the “tail” of a vector) to the end or terminal point (called the “head” of a vector), marked by an arrowhead. Magnitude is the length of a vector and is always a positive scalar quantity. (credit: modification of work by Cate Sevilla)

Let’s examine vector algebra using a graphical method to be aware of basic terms and to develop a qualitative understanding. In practice, however, when it comes to solving physics problems, we use analytical methods, which we’ll see in the next section. Analytical methods are more simple computationally and more accurate than graphical methods. From now on, to distinguish between a vector and a scalar quantity, we adopt the common convention that a letter in bold type with an arrow above it denotes a vector, and a letter without an arrow denotes a scalar. For example, a distance of 2.0 km, which is a scalar quantity, is denoted by d = 2.0 km, whereas a displacement of 2.0 km in some direction, which is a vector quantity, is denoted by $\vec{d}$.

Suppose you tell a friend on a camping trip that you have discovered a terrific fishing hole 6 km from your tent. It is unlikely your friend would be able to find the hole easily unless you also communicate the direction in which it can be found with respect to your campsite. You may say, for example, “Walk about 6 km northeast from my tent.” The key concept here is that you have to give not one but two pieces of information—namely, the distance or magnitude (6 km) **and** the direction (northeast).

Displacement is a general term used to describe a change in position, such as during a trip from the tent to the fishing hole. Displacement is an example of a vector quantity. If you walk from the tent (location A) to the hole (location B), as shown in Figure 3.2.2, the vector $\vec{D}$, representing your **displacement**, is drawn as the arrow that originates at point A and ends at point B. The arrowhead marks the end of the vector. The direction of the displacement vector $\vec{D}$ is the direction of the arrow. The length of the arrow represents the **magnitude** D of vector $\vec{D}$. Here, D = 6 km. Since the magnitude of a vector is its length, which is a positive number, the magnitude is also indicated by placing the absolute value notation around the symbol that denotes the vector; so, we can write equivalently that D ≡ $|\vec{D}|$. To solve a vector problem graphically, we need to draw the vector $\vec{D}$ to scale. For example, if we assume 1 unit of distance (1 km) is represented in the drawing by a line segment of length u = 2 cm, then the total displacement in this example is represented by a vector of length d = 6u = 6(2 cm) = 12 cm , as shown in Figure 3.2.3. Notice that here, to avoid confusion, we used D = 6 km to denote the magnitude of the actual displacement and d = 12 cm to denote the length of its representation in the drawing.

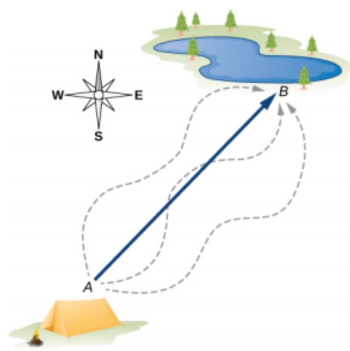

Figure 3.2.2: The displacement vector from point A (the initial position at the campsite) to point B (the final position at the fishing hole) is indicated by an arrow with origin at point A and end at point B. The displacement is the same for any of the actual paths (dashed curves) that may be taken between points A and B.

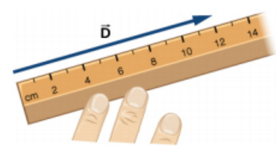

Figure 3.2.3: A displacement $\vec{D}$of magnitude 6 km is drawn to scale as a vector of length 12 cm when the length of 2 cm represents 1 unit of displacement (which in this case is 1 km).

Suppose your friend walks from the campsite at A to the fishing pond at B and then walks back: from the fishing pond at B to the campsite at A. The magnitude of the displacement vector $\vec{D}_{AB}$ from A to B is the same as the magnitude of the displacement vector $\vec{D}_{BA}$ from B to A (it equals 6 km in both cases), so we can write $\vec{D}_{AB} = \vec{D}_{BA}$. However, vector $\vec{D}_{AB}$ is not equal to vector $\vec{D}_{BA}$ because these two vectors have different directions: $\vec{D}_{AB} \neq \vec{D}_{BA}$. In Figure 2.3, vector $\vec{D}_{BA}$ would be represented by a vector with an origin at point B and an end at point A, indicating vector $\vec{D}_{BA}$ points to the southwest, which is exactly 180° opposite to the direction of vector $\vec{D}_{AB}$. We say that vector $\vec{D}_{BA}$ is **antiparallel** to vector $\vec{D}_{AB}$ and write $\vec{D}_{AB} = -\vec{D}_{BA}$, where the minus sign indicates the antiparallel direction.

Two vectors that have identical directions are said to be **parallel vectors**—meaning, they are **parallel** to each other. Two parallel vectors $\vec{A}$ and $\vec{B}$ are equal, denoted by $\vec{A} = \vec{B}$, if and only if they have equal magnitudes $|\vec{A}| = |\vec{B}|$. Two vectors with directions perpendicular to each other are said to be **orthogonal vectors**. These relations between vectors are illustrated in Figure 3.2.4.

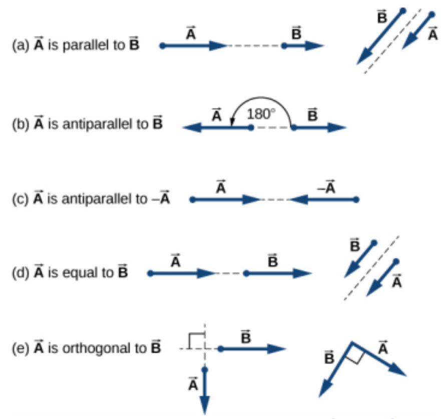

Figure 3.2.4: Various relations between two vectors $\vec{A}$ and $\vec{B}$. (a) $\vec{A} \neq \vec{B}$ because $A \neq B$. (b) $\vec{A} \neq \vec{B}$ because they are not parallel and $A \neq B$. (c) $\vec{A} \neq -\vec{A}$ because they have different directions (even though $|\vec{A}| = |-\vec{A}| = A$) . (d) $\vec{A} = \vec{B}$ because they are parallel and have identical magnitudes A = B. (e) $\vec{A} \neq \vec{B}$ because they have different directions (are not parallel); here, their directions differ by 90° —meaning, they are orthogonal.

Exercise 2.1

Two motorboats named **Alice** and **Bob** are moving on a lake. Given the information about their velocity vectors in each of the following situations, indicate whether their velocity vectors are equal or otherwise.

a. **Alice** moves north at 6 knots and **Bob** moves west at 6 knots.
b. **Alice** moves west at 6 knots and **Bob** moves west at 3 knots.
c. **Alice** moves northeast at 6 knots and **Bob** moves south at 3 knots.
d. **Alice** moves northeast at 6 knots and **Bob** moves southwest at 6 knots.
e. **Alice** moves northeast at 2 knots and **Bob** moves closer to the shore northeast at 2 knots.

Algebra of Vectors in One Dimension

Vectors can be multiplied by scalars, added to other vectors, or subtracted from other vectors. We can illustrate these vector concepts using an example of the fishing trip seen in Figure $3.2.5$.

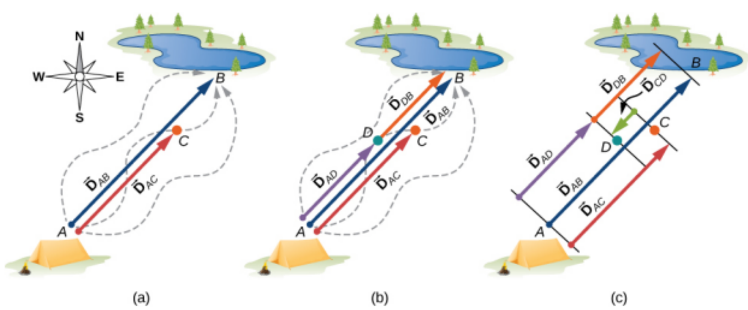

Figure $3.2.5$: Displacement vectors for a fishing trip. (a) Stopping to rest at point C while walking from camp (point A) to the pond (point B). (b) Going back for the dropped tackle box (point D). (c) Finishing up at the fishing pond.

Suppose your friend departs from point A (the campsite) and walks in the direction to point B (the fishing pond), but, along the way, stops to rest at some point C located three-quarters of the distance between A and B, beginning from point A (Figure $3.2.5a$). What is his displacement vector $\vec{D}_{AC}$ when he reaches point C? We know that if he walks all the way to B, his displacement vector relative to A is $\vec{D}_{AB}$, which has magnitude $D_{AB} = 6$ km and a direction of northeast. If he walks only a 0.75 fraction of the total distance, maintaining the northeasterly direction, at point C he must be $0.75\ D_{AB} = 4.5$ km away from the campsite at A. So, his displacement vector at the rest point C has magnitude $D_{AC} = 4.5\text{ km} = 0.75\ D_{AB}$ and is parallel to the displacement vector $\vec{D}_{AB}$. All of this can be stated succinctly in the form of the following **vector equation**:

$$\vec{D}_{AC} = 0.75\ \vec{D}_{AB}.$$

In a vector equation, both sides of the equation are vectors. The previous equation is an example of a vector multiplied by a positive scalar (number) α = 0.75. The result, $\vec{D}_{AC}$, of such a multiplication is a new vector with a direction parallel to the direction of the original vector $\vec{D}_{AB}$. In general, when a vector $\vec{D}_{A}$ is multiplied by a positive scalar α, the result is a new vector $\vec{D}_{B}$ that is parallel to $\vec{D}_{A}$:

$$\vec{B} = \alpha\vec{A} \tag{3.2.1}$$

The magnitude $|\vec{B}|$ of this new vector is obtained by multiplying the magnitude $|\vec{A}|$ of the original vector, as expressed by the **scalar equation**:

$$B = |\alpha|A. \tag{3.2.2}$$

In a scalar equation, both sides of the equation are numbers. Equation 3.2.2 is a scalar equation because the magnitudes of vectors are scalar quantities (and positive numbers). If the scalar α is **negative** in the vector equation Equation 3.2.1, then the magnitude $|\vec{B}|$ of the new vector is still given by Equation 3.2.2, but the direction of the new vector $\vec{B}$ is **antiparallel** to the direction of $\vec{A}$. These principles are illustrated in Figure $3.2.6a$ by two examples where the length of vector $\vec{A}$ is 1.5 units. When α = 2, the new vector $\vec{B}$ = 2$\vec{A}$ has length B = 2A = 3.0 units (twice as long as the original vector) and is parallel to the original vector. When α = −2, the new vector $\vec{C}$ = −2$\vec{A}$ has length C = |−2| A = 3.0 units (twice as long as the original vector) and is antiparallel to the original vector.

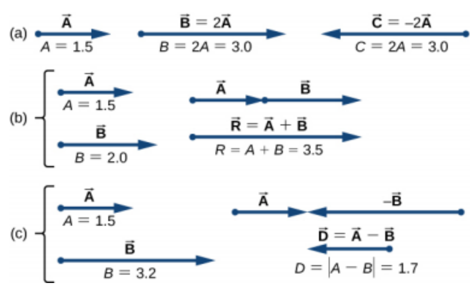

Figure $3.2.6$: Algebra of vectors in one dimension. (a) Multiplication by a scalar. (b) Addition of two vectors ($\vec{R}$ is called the resultant of vectors ($\vec{A}$ and ($\vec{B}$). (c) Subtraction of two vectors ($\vec{D}$ is the difference of vectors ($\vec{A}$ and $\vec{B}$).

Now suppose your fishing buddy departs from point A (the campsite), walking in the direction to point B (the fishing hole), but he realizes he lost his tackle box when he stopped to rest at point C (located three-quarters of the distance between A and B, beginning from point A). So, he turns back and retraces his steps in the direction toward the campsite and finds the box lying on the path at some point D only 1.2 km away from point C (see Figure $3.2.5b$). What is his displacement vector $\vec{D}_{AD}$ when he finds the box at point D? What is his displacement vector $\vec{D}_{DB}$ from point D to the hole? We have already established that at rest point C his displacement vector is $\vec{D}_{AC}$ = 0.75 $\vec{D}_{AB}$. Starting at point C, he walks southwest (toward the campsite), which means his new displacement vector $\vec{D}_{CD}$ from point C to point D is antiparallel to $\vec{D}_{AB}$. Its magnitude $|\vec{D}_{CD}|$ is D_{CD} = 1.2 km = 0.2 D_{AB}, so his second displacement vector is $\vec{D}_{CD}$ = −0.2 $\vec{D}_{AB}$. His total displacement $\vec{D}_{AD}$ relative to the campsite is the vector sum of the two displacement vectors: vector $\vec{D}_{AC}$ (from the campsite to the rest point) and vector $\vec{D}_{CD}$ (from the rest point to the point where he finds his box):

$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD}. \tag{3.2.3}$$

The vector sum of two (or more vectors is called the **resultant vector** or, for short, the **resultant**. When the vectors on the right-hand-side of Equation 3.2.3 are known, we can find the resultant $\vec{D}_{AD}$ as follows:

$$\vec{D}_{AD} = \vec{D}_{AC} + \vec{D}_{CD} = 0.75\ \vec{D}_{AB} - 0.2\ \vec{D}_{AB} = (0.75 - 0.2)\vec{D}_{AB} = 0.55\vec{D}_{AB}. \tag{3.2.4}$$

When your friend finally reaches the pond at B, his displacement vector $\vec{D}_{AB}$ from point A is the vector sum of his displacement vector $\vec{D}_{AD}$ from point A to point D and his displacement vector $\vec{D}_{DB}$ from point D to the fishing hole: $\vec{D}_{AB}$ = $\vec{D}_{AD}$ + $\vec{D}_{DB}$ (see Figure $3.2.5c$). This means his displacement vector $\vec{D}_{DB}$ is the difference of two vectors:

$$\vec{D}_{DB} = \vec{D}_{AB} - \vec{D}_{AD} = \vec{D}_{AB} + (-\vec{D}_{AD}). \tag{3.2.5}$$

Notice that a difference of two vectors is nothing more than a vector sum of two vectors because the second term in Equation 3.2.5 is vector $-\vec{D}_{AD}$ (which is antiparallel to $\vec{D}_{AD}$). When we substitute Equation 3.2.4 into Equation 3.2.5, we obtain the second displacement vector:

$$\vec{D}_{DB} = \vec{D}_{AB} - \vec{D}_{AD} = \vec{D}_{AB} - 0.55\ \vec{D}_{AB} = (1.0 - 0.55)\ \vec{D}_{AB} = 0.45\ \vec{D}_{AB}. \tag{3.2.6}$$

This result means your friend walked D_{DB} = 0.45 D_{AB} = 0.45(6.0 km) = 2.7 km from the point where he finds his tackle box to the fishing hole.

When vectors $\vec{A}$ and $\vec{B}$ lie along a line (that is, in one dimension), such as in the camping example, their resultant $\vec{R} = \vec{A} + \vec{B}$ and their difference $\vec{D} = \vec{A} - \vec{B}$ both lie along the same direction. We can illustrate the addition or subtraction of vectors by drawing the corresponding vectors to scale in one dimension, as shown in Figure 3.2.6.

To illustrate the resultant when $\vec{A}$ and $\vec{B}$ are two parallel vectors, we draw them along one line by placing the origin of one vector at the end of the other vector in head-to-tail fashion (see Figure (\PageIndex{6b}\)). The magnitude of this resultant is the sum of their magnitudes: R = A + B. The direction of the resultant is parallel to both vectors. When vector $\vec{A}$ is antiparallel to vector $\vec{B}$, we draw them along one line in either head-to-head fashion (Figure (\PageIndex{6c}\)) or tail-to-tail fashion. The magnitude of the vector difference, then, is the **absolute value** D = |A − B| of the difference of their magnitudes. The direction of the difference vector $\vec{D}$ is parallel to the direction of the longer vector.

In general, in one dimension—as well as in higher dimensions, such as in a plane or in space—we can add any number of vectors and we can do so in any order because the addition of vectors is **commutative**,

$$\vec{A} + \vec{B} = \vec{B} + \vec{A}. \tag{3.2.7}$$

and **associative,**

$$(\vec{A} + \vec{B}) + \vec{C} = \vec{A} + (\vec{B} + \vec{C}). \tag{3.2.8}$$

Moreover, multiplication by a scalar is **distributive**:

$$\alpha_1 \vec{A} + \alpha_2 \vec{A} = (\alpha_1 + \alpha_2)\vec{A}. \tag{3.2.9}$$

We used the distributive property in Equation 3.2.4 and Equation 3.2.6.

When adding many vectors in one dimension, it is convenient to use the concept of a **unit vector**. A unit vector, which is denoted by a letter symbol with a hat, such as $\hat{u}$, has a magnitude of one and does not have any physical unit so that $|\hat{u}| \equiv u = 1$. The only role of a unit vector is to specify direction. For example, instead of saying vector $\vec{D}_{AB}$ has a magnitude of 6.0 km and a direction of northeast, we can introduce a unit vector $\hat{u}$ that points to the northeast and say succinctly that $\vec{D}_{AB}$ = (6.0 km) $\hat{u}$. Then the southwesterly direction is simply given by the unit vector $-\hat{u}$. In this way, the displacement of 6.0 km in the southwesterly direction is expressed by the vector

$$\vec{D}_{BA} = (-6.0\ km)\ \hat{u}.$$

3.3: Scalars and Vectors (Part 2)

Example 3.3.1: A Ladybug Walker

A long measuring stick rests against a wall in a physics laboratory with its 200-cm end at the floor. A ladybug lands on the 100-cm mark and crawls randomly along the stick. It first walks 15 cm toward the floor, then it walks 56 cm toward the wall, then it walks 3 cm toward the floor again. Then, after a brief stop, it continues for 25 cm toward the floor and then, again, it crawls up 19 cm toward the wall before coming to a complete rest (Figure 3.3.1). Find the vector of its total displacement and its final resting position on the stick.

Strategy

If we choose the direction along the stick toward the floor as the direction of unit vector $\hat{u}$, then the direction toward the floor is $+\hat{u}$ and the direction toward the wall is $-\hat{u}$. The ladybug makes a total of five displacements:

$$\vec{D}_1 = (15\ cm)(+\hat{u}),$$
$$\vec{D}_2 = (56\ cm)(-\hat{u}),$$
$$\vec{D}_3 = (3\ cm)(+\hat{u}),$$
$$\vec{D}_4 = (25\ cm)(+\hat{u}),\ and$$
$$\vec{D}_5 = (19\ cm)(-\hat{u}).$$

The total displacement $\vec{D}$ is the resultant of all its displacement vectors.

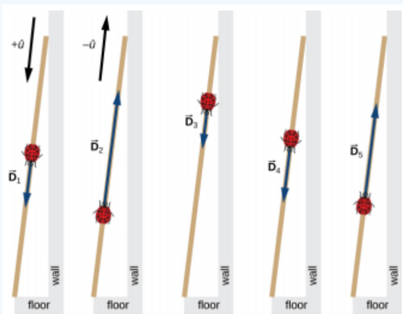

Figure 3.3.1: Five displacements of the ladybug. Note that in this schematic drawing, magnitudes of displacements are not drawn to scale. (credit: modification of work by "Persian Poet Gal"/Wikimedia Commons)

Solution

The resultant of all the displacement vectors is

$$\begin{aligned}\vec{D} &= \vec{D}_1 + \vec{D}_2 + \vec{D}_3 + \vec{D}_4 + \vec{D}_5 \\ &= (15\ cm)(+\hat{u}) + (56\ cm)(-\hat{u}) + (3\ cm)(+\hat{u}) + (25\ cm)(+\hat{u}) + (19\ cm)(-\hat{u}) \\ &= (15 - 56 + 3 + 25 - 19) cm\ \hat{u} \\ &= -32\ cm\ \hat{u}.\end{aligned}$$

In this calculation, we use the distributive law given by Equation 2.2.9. The result reads that the total displacement vector points away from the 100-cm mark (initial landing site) toward the end of the meter stick that touches the wall. The end that touches the wall is marked 0 cm, so the final position of the ladybug is at the (100 – 32) cm = 68-cm mark.

Exercise 2.2

A cave diver enters a long underwater tunnel. When her displacement with respect to the entry point is 20 m, she accidentally drops her camera, but she doesn't notice it missing until she is some 6 m farther into the tunnel. She swims back 10 m but cannot find the camera, so she decides to end the dive. How far from the entry point is she? Taking the positive direction out of the tunnel, what is her displacement vector relative to the entry point?

Algebra of Vectors in Two Dimensions

When vectors lie in a plane—that is, when they are in two dimensions—they can be multiplied by scalars, added to other vectors, or subtracted from other vectors in accordance with the general laws expressed by Equation 2.2.1, Equation 2..2.2, Equation 2.2.7, and Equation 2.2.8. However, the addition rule for two vectors in a plane becomes more complicated than the rule for vector addition in one dimension. We have to use the laws of geometry to construct resultant vectors, followed by trigonometry to find vector magnitudes and directions. This geometric approach is commonly used in navigation (Figure $3.3.2$). In this section, we need to have at hand two rulers, a triangle, a protractor, a pencil, and an eraser for drawing vectors to scale by geometric constructions.

Figure $3.3.2$: In navigation, the laws of geometry are used to draw resultant displacements on nautical maps.

For a geometric construction of the sum of two vectors in a plane, we follow the **parallelogram rule**. Suppose two vectors $\vec{A}$ and $\vec{B}$ are at the arbitrary positions shown in Figure $3.3.3$. Translate either one of them in parallel to the beginning of the other vector, so that after the translation, both vectors have their origins at the same point. Now, at the end of vector $\vec{A}$ we draw a line parallel to vector $\vec{B}$ and at the end of vector $\vec{B}$ we draw a line parallel to vector $\vec{A}$ (the dashed lines in Figure $3.3.3$). In this way, we obtain a parallelogram. From the origin of the two vectors we draw a diagonal that is the resultant $\vec{R}$ of the two vectors: $\vec{R} = \vec{A} + \vec{B}$ (Figure $3.3.3a$). The other diagonal of this parallelogram is the vector difference of the two vectors $\vec{D} = \vec{A} - \vec{B}$, as shown in Figure $3.3.3b$. Notice that the end of the difference vector is placed at the end of vector $\vec{A}$.

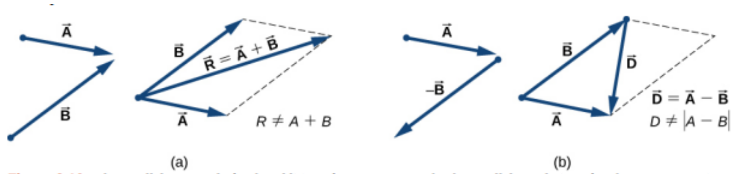

Figure $3.3.3$: The parallelogram rule for the addition of two vectors. Make the parallel translation of each vector to a point where their origins (marked by the dot) coincide and construct a parallelogram with two sides on the vectors and the other two sides (indicated by dashed lines) parallel to the vectors. (a) Draw the resultant vector $\vec{R}$ along the diagonal of the parallelogram from the common point to the opposite corner. Length R of the resultant vector is not equal to the sum of the magnitudes of the two vectors. (b) Draw the difference vector $\vec{D} = \vec{A} - \vec{B}$ along the diagonal connecting the ends of the vectors. Place the origin of vector $\vec{D}$ at the end of vector $\vec{B}$ and the end (arrowhead) of vector $\vec{D}$ at the end of vector $\vec{A}$. Length D of the difference vector is not equal to the difference of magnitudes of the two vectors.

It follows from the parallelogram rule that neither the magnitude of the resultant vector nor the magnitude of the difference vector can be expressed as a simple sum or difference of magnitudes A and B, because the length of a diagonal cannot be expressed as a simple sum of side lengths. When using a geometric construction to find magnitudes $|\vec{R}|$ and $|\vec{D}|$, we have to use trigonometry laws for triangles, which may lead to complicated algebra. There are two ways to circumvent this algebraic complexity. One way is to use the method of components, which we examine in the next section. The other way is to draw the vectors to scale, as is done in navigation, and read approximate vector lengths and angles (directions) from the graphs. In this section we examine the second approach.

If we need to add three or more vectors, we repeat the parallelogram rule for the pairs of vectors until we find the resultant of all of the resultants. For three vectors, for example, we first find the resultant of vector 1 and vector 2, and then we find the resultant of this resultant and vector 3. The order in which we select the pairs of vectors does not matter because the operation of vector addition is commutative and associative (see Equation 2.2.7 and Equation 2.2.8). Before we state a general rule that follows from repetitive applications of the parallelogram rule, let’s look at the following example.

Suppose you plan a vacation trip in Florida. Departing from Tallahassee, the state capital, you plan to visit your uncle Joe in Jacksonville, see your cousin Vinny in Daytona Beach, stop for a little fun in Orlando, see a circus performance in Tampa, and visit the University of Florida in Gainesville. Your route may be represented by five displacement vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, $\vec{D}$, and $\vec{E}$, which are indicated by the red vectors in Figure $3.3.4$. What is your total displacement when you reach Gainesville? The total displacement is the vector sum of all five displacement vectors, which may be found by using the parallelogram rule four times. Alternatively, recall that the displacement vector has its beginning at the initial position (Tallahassee) and its end at the final position (Gainesville), so the total displacement vector can be drawn directly as an arrow connecting Tallahassee with Gainesville (see the green vector in Figure $3.3.4$). When we use the parallelogram rule four times, the resultant $\vec{R}$ we obtain is exactly this green vector connecting Tallahassee with Gainesville: $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$.

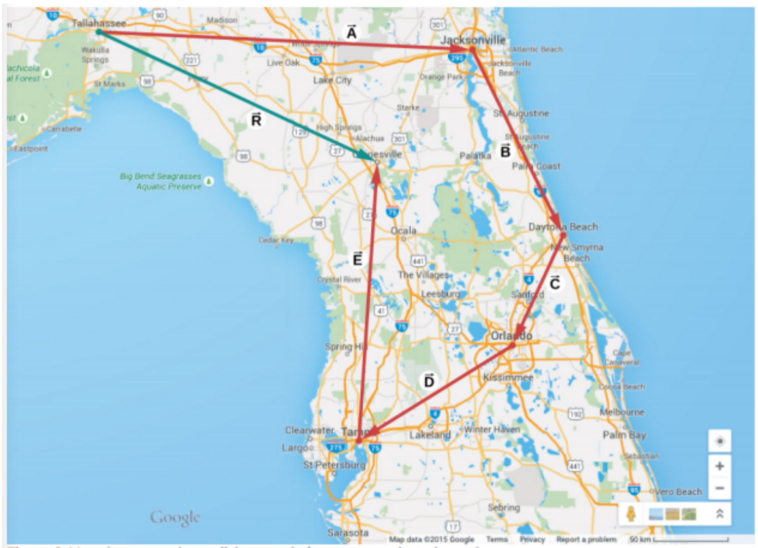

Figure $3.3.4$: When we use the parallelogram rule four times, we obtain the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} + \vec{E}$, which is the green vector connecting Tallahassee with Gainesville.

Drawing the resultant vector of many vectors can be generalized by using the following tail-to-head geometric construction. Suppose we want to draw the resultant vector $\vec{R}$ of four vectors $\vec{A}$, $\vec{B}$, $\vec{C}$, and $\vec{D}$ (Figure $3.3.5a$). We select any one of the vectors as the first vector and make a parallel translation of a second vector to a position where the origin (“tail”) of the second vector coincides with the end (“head”) of the first vector. Then, we select a third vector and make a parallel translation of the third vector to a position where the origin of the third vector coincides with the end of the second vector. We repeat this procedure until all the vectors are in a head-to-tail arrangement like the one shown in Figure $3.3.5$. We draw the resultant vector $\vec{R}$ by connecting the origin (“tail”) of the first vector with the end (“head”) of the last vector. The end of the resultant vector is at the end of the last vector. Because the addition of vectors is associative and commutative, we obtain the same resultant vector regardless of which vector we choose to be first, second, third, or fourth in this construction.

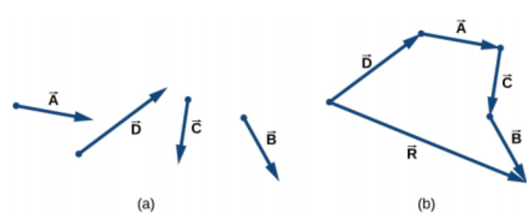

Figure $3.3.5$: Tail-to-head method for drawing the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D}$. (a) Four vectors of different magnitudes and directions. (b) Vectors in (a) are translated to new positions where the origin (“tail”) of one vector is at the end (“head”) of another vector. The resultant vector is drawn from the origin (“tail”) of the first vector to the end (“head”) of the last vector in this arrangement.

✓ Example 3.3.2: Geometric Construction of the Resultant

The three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in Figure 3.3.6 are specified by their magnitudes A = 10.0, B = 7.0, and C = 8.0, respectively, and by their respective direction angles with the horizontal direction α = 35°, β = −110°, and γ = 30°. The physical units of the magnitudes are centimeters. Choose a convenient scale and use a ruler and a protractor to find the following vector sums: (a) $\vec{R} = \vec{A} + \vec{B}$, (b) $\vec{D} = \vec{A} - \vec{B}$, and (c) $\vec{S} = \vec{A} - 3\vec{B} + \vec{C}$.

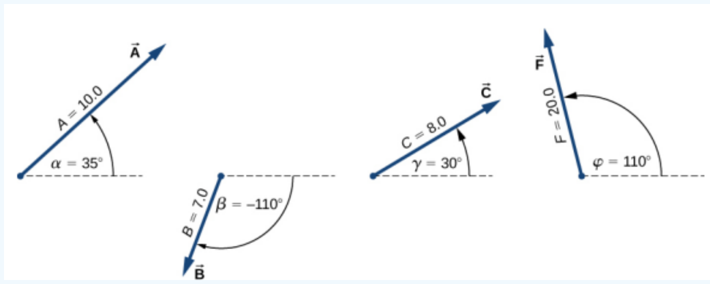

Figure 3.3.6: Vectors used in Example 3.3.2 and in the Exercise feature that follows.

Strategy

In geometric construction, to find a vector means to find its magnitude and its direction angle with the horizontal direction. The strategy is to draw to scale the vectors that appear on the right-hand side of the equation and construct the resultant vector. Then, use a ruler and a protractor to read the magnitude of the resultant and the direction angle. For parts (a) and (b) we use the parallelogram rule. For (c) we use the tail-to-head method.

Solution

For parts (a) and (b), we attach the origin of vector $\vec{B}$ to the origin of vector $\vec{A}$, as shown in Figure 3.3.7, and construct a parallelogram. The shorter diagonal of this parallelogram is the sum $\vec{A} + \vec{B}$. The longer of the diagonals is the difference $\vec{A} - \vec{B}$. We use a ruler to measure the lengths of the diagonals, and a protractor to measure the angles with the horizontal. For the resultant $\vec{R}$, we obtain R = 5.8 cm and $\theta_R \approx 0°$. For the difference $\vec{D}$, we obtain D = 16.2 cm and θ_D = 49.3°, which are shown in Figure 3.3.7.

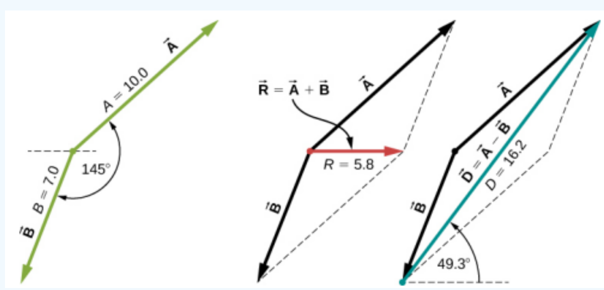

Figure 3.3.7: Using the parallelogram rule to solve (a) (finding the resultant, red) and (b) (finding the difference, blue).

For (c), we can start with vector $-3\vec{B}$ and draw the remaining vectors tail-to-head as shown in Figure 3.3.8. In vector addition, the order in which we draw the vectors is unimportant, but drawing the vectors to scale is very important. Next, we draw vector $\vec{S}$ from the origin of the first vector to the end of the last vector and place the arrowhead at the end of $\vec{S}$. We use a ruler to measure the length of $\vec{S}$, and find that its magnitude is S = 36.9 cm. We use a protractor and find that its direction angle is θ_S = 52.9°. This solution is shown in Figure 3.3.8.

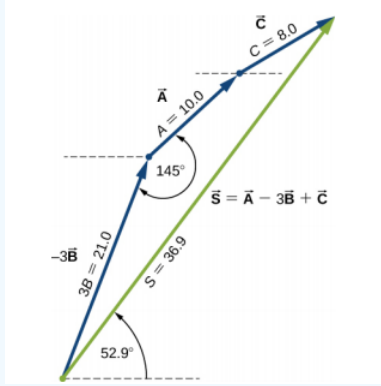

Figure 3.3.8: Using the tail-to-head method to solve (c) (finding vector $\vec{S}$, green).

Exercise 2.3

Using the three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ in Figure 3.3.6, choose a convenient scale, and use a ruler and a protractor to find vector $\vec{G}$ given by the vector equation $\vec{G} = \vec{A} + 2\vec{B} - \vec{F}$.

Simulation

Observe the addition of vectors in a plane by visiting this vector calculator and this PhET simulation.

3.4: Coordinate Systems and Components of a Vector (Part 1)

Learning Objectives

- Describe vectors in two and three dimensions in terms of their components, using unit vectors along the axes.
- Distinguish between the vector components of a vector and the scalar components of a vector.
- Explain how the magnitude of a vector is defined in terms of the components of a vector.
- Identify the direction angle of a vector in a plane.
- Explain the connection between polar coordinates and Cartesian coordinates in a plane.

Vectors are usually described in terms of their components in a coordinate system. Even in everyday life we naturally invoke the concept of orthogonal projections in a rectangular coordinate system. For example, if you ask someone for directions to a particular location, you will more likely be told to go 40 km east and 30 km north than 50 km in the direction 37° north of east.

In a rectangular (Cartesian) xy-coordinate system in a plane, a point in a plane is described by a pair of coordinates (x, y). In a similar fashion, a vector $\vec{A}$ in a plane is described by a pair of its vector coordinates. The x-coordinate of vector $\vec{A}$ is called its x-component and the y-coordinate of vector $\vec{A}$ is called its y-component. The vector x-component is a vector denoted by $\vec{A}_x$. The vector y-component is a vector denoted by $\vec{A}_y$. In the Cartesian system, the x and y **vector components** of a vector are the orthogonal projections of this vector onto the x- and y-axes, respectively. In this way, following the parallelogram rule for vector addition, each vector on a Cartesian plane can be expressed as the vector sum of its vector components:

$$\vec{A}=\vec{A}_x+\vec{A}_y. \tag{3.4.1}$$

As illustrated in Figure $3.4.1$, vector $\vec{A}$ is the diagonal of the rectangle where the x-component $\vec{A}_x$ is the side parallel to the x-axis and the y-component $\vec{A}_y$ is the side parallel to the y-axis. Vector component $\vec{A}_x$ is orthogonal to vector component $\vec{A}_y$.

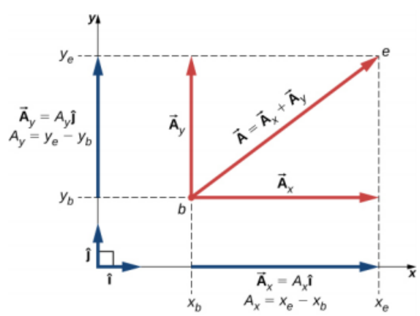

Figure $3.4.1$: Vector $\vec{A}$ in a plane in the Cartesian coordinate system is the vector sum of its vector x- and y-components. The x-vector component $\vec{A}_x$ is the orthogonal projection of vector $\vec{A}$ onto the x-axis. The y-vector component $\vec{A}_y$ is the orthogonal projection of vector $\vec{A}$ onto the y-axis. The numbers A_x and A_y that multiply the unit vectors are the scalar components of the vector.

It is customary to denote the positive direction on the x-axis by the unit vector $\hat{i}$ and the positive direction on the y-axis by the unit vector $\hat{j}$. Unit vectors of the axes, $\hat{i}$ and $\hat{j}$, define two orthogonal directions in the plane. As shown in Figure $3.4.1$, the x- and y-components of a vector can now be written in terms of the unit vectors of the axes:

$$\begin{cases} \vec{A}_x=A_x\hat{i} \\ \vec{A}_y=A_y\hat{j} \end{cases} \tag{3.4.2}$$

The vectors $\vec{A}_x$ and $\vec{A}_y$ defined by Equation 2.11 are the vector components of vector $\vec{A}$. The numbers A_x and A_y that define the vector components in Equation 3.4.2 are the **scalar components** of vector $\vec{A}$. Combining Equation 3.4.1 with Equation 3.4.2, we obtain **the component form of a vector**:

$$\vec{A}=A_x\hat{i}+A_y\hat{j}. \tag{3.4.3}$$

If we know the coordinates $b(x_b, y_b)$ of the origin point of a vector (where b stands for "beginning") and the coordinates e(x_e, y_e) of the end point of a vector (where e stands for "end"), we can obtain the scalar components of a vector simply by subtracting the origin point coordinates from the end point coordinates:

$$\begin{cases} A_x = x_e - x_b \\ A_y = y_e - y_b. \end{cases} \tag{3.4.4}$$

✔ Example 3.4.1: Displacement of a Mouse Pointer

A mouse pointer on the display monitor of a computer at its initial position is at point (6.0 cm, 1.6 cm) with respect to the lower left-side corner. If you move the pointer to an icon located at point (2.0 cm, 4.5 cm), what is the displacement vector of the pointer?

Strategy

The origin of the xy-coordinate system is the lower left-side corner of the computer monitor. Therefore, the unit vector $\hat{i}$ on the x-axis points horizontally to the right and the unit vector $\hat{j}$ on the y-axis points vertically upward. The origin of the displacement vector is located at point b(6.0, 1.6) and the end of the displacement vector is located at point e(2.0, 4.5). Substitute the coordinates of these points into Equation 3.4.4 to find the scalar components D_x and D_y of the displacement vector $\vec{D}$. Finally, substitute the coordinates into Equation 3.4.3 to write the displacement vector in the vector component form.

Solution

We identify x_b = 6.0, x_e = 2.0, y_b = 1.6, and y_e = 4.5, where the physical unit is 1 cm. The scalar x- and y-components of the displacement vector are

$$D_x = x_e - x_b = (2.0 - 6.0)\ cm = -4.0\ cm, \tag{3.4.5}$$

$$D_y = y_e - y_b = (4.5 - 1.6)\ cm = +2.9\ cm. \tag{3.4.6}$$

The vector component form of the displacement vector is

$$\vec{D} = D_x\ \hat{i} + D_y\ \hat{j} = (-4.0\ cm)\ \hat{i} + (2.9\ cm)\ \hat{j} = (-4.0\ \hat{i} + 2.9\ \hat{j})\ cm. \tag{3.4.7}$$

This solution is shown in Figure $3.4.2$.

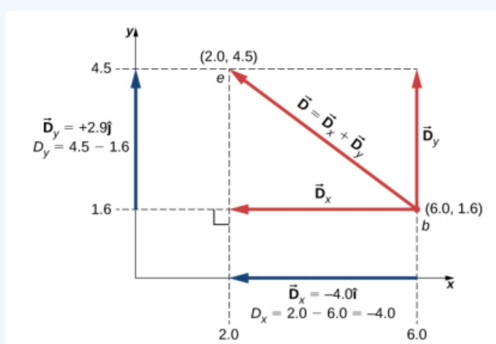

Figure $3.4.2$: The graph of the displacement vector. The vector points from the origin point at b to the end point at e.

Significance

Notice that the physical unit—here, 1 cm—can be placed either with each component immediately before the unit vector or globally for both components, as in Equation 3.4.7. Often, the latter way is more convenient because it is simpler.

The vector x-component $\vec{D}_x$ = −4.0 $\hat{i}$ = 4.0$(-\hat{i})$ of the displacement vector has the magnitude $|\vec{D}_x|$ = |− 4.0||$\hat{i}$| = 4.0 because the magnitude of the unit vector is |$\hat{i}$| = 1. Notice, too, that the direction of the x-component is $-\hat{i}$, which is antiparallel to the direction of the +x-axis; hence, the x-component vector $\vec{D}_x$ points to the left, as shown in Figure $3.4.2$. The scalar x-component of vector $\vec{D}$ is D_x = −4.0. Similarly, the vector y-component $\vec{D}_y = +2.9\hat{j}$ of the displacement vector has magnitude $|\vec{D}_y|$ = |2.9||$\hat{j}$| = 2.9 because the magnitude of the unit vector is |$\hat{j}$| = 1. The direction of the y-component is $+\hat{j}$, which is parallel to the direction of the +y-axis. Therefore, the y-component vector $\vec{D}_y$ points up, as seen in Figure 3.4.2. The scalar y-component of vector $\vec{D}$ is D_y = + 2.9. The displacement vector $\vec{D}$ is the resultant of its two vector components.

The vector component form of the displacement vector Equation 3.4.7 tells us that the mouse pointer has been moved on the monitor 4.0 cm to the left and 2.9 cm upward from its initial position.

? Exercise 2.4

A blue fly lands on a sheet of graph paper at a point located 10.0 cm to the right of its left edge and 8.0 cm above its bottom edge and walks slowly to a point located 5.0 cm from the left edge and 5.0 cm from the bottom edge. Choose the rectangular coordinate system with the origin at the lower left-side corner of the paper and find the displacement vector of the fly. Illustrate your solution by graphing.

When we know the scalar components A_x and A_y of a vector $\vec{A}$, we can find its magnitude A and its direction angle θ_A. The **direction angle**—or direction, for short—is the angle the vector forms with the positive direction on the x-axis. The angle θ_A is measured in the counterclockwise direction from the +x-axis to the vector (Figure 3.4.3). Because the lengths A, A_x, and A_y form a right triangle, they are related by the Pythagorean theorem:

$$A^2 = A_x^2 + A_y^2 \Leftrightarrow A = \sqrt{A_x^2 + A_y^2}. \tag{3.4.8}$$

This equation works even if the scalar components of a vector are negative. The direction angle θ_A of a vector is defined via the tangent function of angle θ_A in the triangle shown in Figure 3.4.3:

$$\tan\theta = \frac{A_y}{A_x} \Rightarrow \theta = \tan^{-1}\left(\frac{A_y}{A_x}\right). \tag{3.4.9}$$

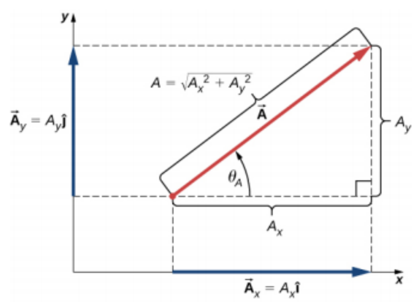

Figure 3.4.3: For vector $\vec{A}$, its magnitude A and its direction angle θ_A are related to the magnitudes of its scalar components because A, A_x, and A_y form a right triangle.

When the vector lies either in the first quadrant or in the fourth quadrant, where component A_x is positive (Figure 3.4.4), the angle θ in Equation 3.4.9) is identical to the direction angle θ_A. For vectors in the fourth quadrant, angle θ is negative, which means that for these vectors, direction angle θ_A is measured clockwise from the positive x-axis. Similarly, for vectors in the second quadrant, angle θ is negative. When the vector lies in either the second or third quadrant, where component A_x is negative, the direction angle is $\theta_A = \theta + 180°$ (Figure 3.4.4).

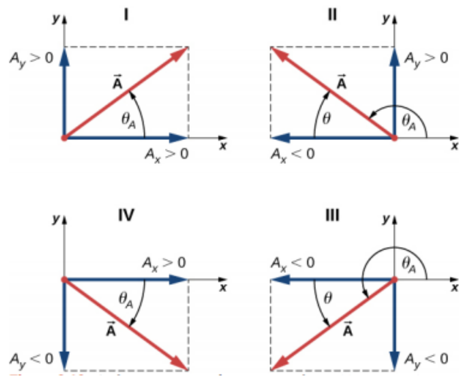

Figure 3.4.4: Scalar components of a vector may be positive or negative. Vectors in the first quadrant (I) have both scalar components positive and vectors in the third quadrant have both scalar components negative. For vectors in quadrants II and III, the direction angle of a vector is $\theta_A = \theta + 180°$.

Example 3.4.2: Magnitude and Direction of the Displacement Vector

You move a mouse pointer on the display monitor from its initial position at point (6.0 cm, 1.6 cm) to an icon located at point (2.0 cm, 4.5 cm). What is the magnitude and direction of the displacement vector of the pointer?

Strategy

In Example 3.4.1, we found the displacement vector $\vec{D}$ of the mouse pointer (see Equation 3.4.7). We identify its scalar components $D_x = -4.0$ cm and $D_y = +2.9$ cm and substitute into Equation 3.4.8 and Equation 3.4.9 to find the magnitude D and direction θ_D, respectively.

Solution

The magnitude of vector $\vec{D}$ is

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{(-4.0\ cm)^2 + (2.9\ cm)^2} = \sqrt{(4.0)^2 + (2.9)^2}\ cm = 4.9\ cm. \tag{3.4.10}$$

The direction angle is

$$\tan\theta = \frac{D_y}{D_x} = \frac{+2.9\ cm}{-4.0\ cm} = -0.725 \Rightarrow \theta = \tan^{-1}(-0.725) = -35.9^o. \tag{3.4.11}$$

Vector $\vec{D}$ lies in the second quadrant, so its direction angle is

$$\theta_D = \theta + 180^o = -35.9^o + 180^o = 144.1^o. \tag{3.4.12}$$

Exercise 2.5

If the displacement vector of a blue fly walking on a sheet of graph paper is $\vec{D} = (-5.00\ \hat{i} - 3.00\ \hat{j})$ cm, find its magnitude and direction.

In many applications, the magnitudes and directions of vector quantities are known and we need to find the resultant of many vectors. For example, imagine 400 cars moving on the Golden Gate Bridge in San Francisco in a strong wind. Each car gives the bridge a different push in various directions and we would like to know how big the resultant push can possibly be. We have already gained some experience with the geometric construction of vector sums, so we know the task of finding the resultant by drawing the vectors and measuring their lengths and angles may become intractable pretty quickly, leading to huge errors. Worries like this do not appear when we use analytical methods. The very first step in an analytical approach is to find vector components when the direction and magnitude of a vector are known.

Let us return to the right triangle in Figure 3.4.3. The quotient of the adjacent side A_x to the hypotenuse A is the cosine function of direction angle θ_A, $A_x/A = \cos\theta_A$, and the quotient of the opposite side A_y to the hypotenuse A is the sine function of θ_A, $A_y/A = \sin\theta_A$. When magnitude A and direction θ_A are known, we can solve these relations for the scalar components:

$$\begin{cases} A_x = A\cos\theta_A \\ A_y = A\sin\theta_A. \end{cases} \tag{3.4.13}$$

When calculating vector components with Equation 3.4.13, care must be taken with the angle. The direction angle θ_A of a vector is the angle measured **counterclockwise** from the positive direction on the x-axis to the vector. The clockwise measurement gives a negative angle.

Example 3.4.3: Components of Displacement Vectors

A rescue party for a missing child follows a search dog named Trooper. Trooper wanders a lot and makes many trial sniffs along many different paths. Trooper eventually finds the child and the story has a happy ending, but his displacements on various legs seem to be truly convoluted. On one of the legs he walks 200.0 m southeast, then he runs north some 300.0 m. On the third leg, he examines the scents carefully for 50.0 m in the direction 30° west of north. On the fourth leg, Trooper goes directly south for 80.0 m, picks up a fresh scent and turns 23° west of south for 150.0 m. Find the scalar components of Trooper's displacement vectors and his displacement vectors in vector component form for each leg.

Strategy

Let's adopt a rectangular coordinate system with the positive x-axis in the direction of geographic east, with the positive y-direction pointed to geographic north. Explicitly, the unit vector $\hat{i}$ of the x-axis points east and the unit vector $\hat{j}$ of the y-axis points north. Trooper makes five legs, so there are five displacement vectors. We start by identifying their magnitudes and direction angles, then we use Equation 3.4.13 to find the scalar components of the displacements and Equation 3.4.3 for the displacement vectors.

Solution

On the first leg, the displacement magnitude is L_1 = 200.0 m and the direction is southeast. For direction angle θ_1 we can take either 45° measured clockwise from the east direction or 45° + 270° measured counterclockwise from the east direction. With the first choice, θ_1 = −45°. With the second choice, θ_1 = + 315°. We can use either one of these two angles. The components are

$$L_{1x} = L_1\cos\theta_1 = (200.0\ m)\cos 315^o = 141.4\ m, \tag{3.4.14}$$

$$L_{1y} = L_1\sin\theta_1 = (200.0\ m)\sin 315^o = -141.4\ m, \tag{3.4.15}$$

The displacement vector of the first leg is

$$\vec{L}_1 = L_{1x}\ \hat{i} + L_{1y}\ \hat{j} = (14.4\ \hat{i} - 141.4\ \hat{j})\ m. \tag{3.4.16}$$

On the second leg of Trooper's wanderings, the magnitude of the displacement is L_2 = 300.0 m and the direction is north. The direction angle is θ_2 = + 90°. We obtain the following results:

$$L_{2x} = L_2\cos\theta_2 = (300.0\ m)\cos 90^o = 0.0, \tag{3.4.17}$$

$$L_{2y} = L_2\sin\theta_2 = (300.0\ m)\sin 90^o = 300.0\ m, \tag{3.4.18}$$

$$\vec{L}_2 = L_{2x}\ \hat{i} + L_{2y}\ \hat{j} = (300.0\ m)\ \hat{j}. \tag{3.4.19}$$

On the third leg, the displacement magnitude is L_3 = 50.0 m and the direction is 30° west of north. The direction angle measured counterclockwise from the eastern direction is θ_3 = 30° + 90° = + 120°. This gives the following answers:

$$L_{3x} = L_3\cos\theta_3 = (50.0\ m)\cos 120^o = -25.0\ m, \tag{3.4.20}$$

$$L_{3y} = L_3\sin\theta_3 = (50.0\ m)\sin 120^o = +43.3\ m, \tag{3.4.21}$$

$$\vec{L}_3 = L_{3x}\ \hat{i} + L_{3y}\ \hat{j} = (-25.0\ \hat{i} + 43.3\ \hat{j})\ m. \tag{3.4.22}$$

On the fourth leg of the excursion, the displacement magnitude is L_4 = 80.0 m and the direction is south. The direction angle can be taken as either θ_4 = −90° or \(\theta_{4} = + 270°. We obtain

$$L_{4x} = L_4\cos\theta_4 = (80.0\ m)\cos(-90^o) = 0, \tag{3.4.23}$$

$$L_{4y} = L_4\sin\theta_4 = (80.0\ m)\sin(-90^o) = -80.0\ m, \tag{3.4.24}$$

$$\vec{L}_4 = L_{4x}\ \hat{i} + L_{4y}\ \hat{j} = (-80.0\ m)\ \hat{j}. \tag{3.4.25}$$

On the last leg, the magnitude is L_5 = 150.0 m and the angle is θ_5 = −23° + 270° = + 247° (23° west of south), which gives

$$L_{5x} = L_5 \cos\theta_5 = (150.0\ m)\cos 247^o = -58.6\ m, \tag{3.4.26}$$

$$L_{5y} = L_5 \sin\theta_5 = (150.0\ m)\sin 247^o = -138.1\ m, \tag{3.4.27}$$

$$\vec{L}_5 = L_{5x}\ \hat{i} + L_{5y}\ \hat{j} = (-58.6\ \hat{i} - 138.1\ \hat{j})\ m. \tag{3.4.28}$$

? Exercise 2.6

If Trooper runs 20 m west before taking a rest, what is his displacement vector?

3.5: Coordinate Systems and Components of a Vector (Part 2)

Polar Coordinates

To describe locations of points or vectors in a plane, we need two orthogonal directions. In the Cartesian coordinate system these directions are given by unit vectors $\hat{i}$ and $\hat{j}$ along the x-axis and the y-axis, respectively. The Cartesian coordinate system is very convenient to use in describing displacements and velocities of objects and the forces acting on them. However, it becomes cumbersome when we need to describe the rotation of objects. When describing rotation, we usually work in the **polar coordinate system**.

In the polar coordinate system, the location of point P in a plane is given by two **polar coordinates** (Figure 3.5.1). The first polar coordinate is the **radial coordinate** r, which is the distance of point P from the origin. The second polar coordinate is an angle φ that the radial vector makes with some chosen direction, usually the positive x-direction. In polar coordinates, angles are measured in radians, or rads. The radial vector is attached at the origin and points away from the origin to point P. This radial direction is described by a unit radial vector $\hat{r}$. The second unit vector $\hat{t}$ is a vector orthogonal to the radial direction $\hat{r}$. The positive $+\hat{t}$ direction indicates how the angle φ changes in the counterclockwise direction. In this way, a point P that has coordinates (x, y) in the rectangular system can be described equivalently in the polar coordinate system by the two polar coordinates (r, φ). Equation 2.4.13 is valid for any vector, so we can use it to express the x- and y-coordinates of vector $\vec{r}$. In this way, we obtain the connection between the polar coordinates and rectangular coordinates of point P:

$$\begin{cases} x = r\cos\varphi \\ y = r\sin\varphi \end{cases}. \tag{3.5.1}$$

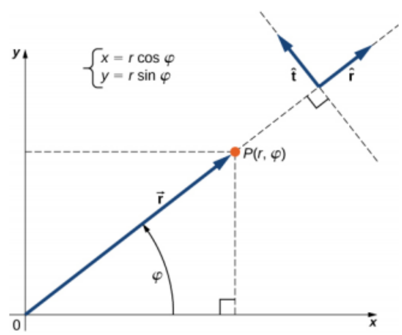

Figure 3.5.1: Using polar coordinates, the unit vector $\hat{r}$ defines the positive direction along the radius r (radial direction) and, orthogonal to it, the unit vector $\hat{t}$ defines the positive direction of rotation by the angle φ.

✔ Example 3.5.1: Polar Coordinates

A treasure hunter finds one silver coin at a location 20.0 m away from a dry well in the direction 20° north of east and finds one gold coin at a location 10.0 m away from the well in the direction 20° north of west. What are the polar and rectangular coordinates of these findings with respect to the well?

Strategy

The well marks the origin of the coordinate system and east is the +x-direction. We identify radial distances from the locations to the origin, which are r_S = 20.0 m (for the silver coin) and r_G = 10.0 m (for the gold coin). To find the angular coordinates, we convert 20° to radians: 20° = $\frac{\pi\, 20}{180}$ = $\frac{\pi}{9}$. We use Equation 3.5.1 to find the x- and y-coordinates of the coins.

Solution

The angular coordinate of the silver coin is $\varphi_S = \frac{\pi}{9}$, whereas the angular coordinate of the gold coin is $\varphi_G = \pi - \frac{\pi}{9} = \frac{8\pi}{9}$. Hence, the polar coordinates of the silver coin are (r_S, φ_S) = (20.0 m, $\frac{\pi}{9}$) and those of the gold coin are (r_G, φ_G) = (10.0 m, \frac{8 \pi}{9}\)). We substitute these coordinates into Equation 3.5.1 to obtain rectangular coordinates. For the gold coin, the coordinates are

$$\begin{cases} x_G = r_G\cos\varphi_G = (10.0\ m)\cos\frac{8\pi}{9} = -9.4\ m \\ y_G = r_G\sin\varphi_G = (10.0\ m)\sin\frac{8\pi}{9} = 3.4\ m \end{cases} \Rightarrow (x_G, y_G) = (-9.4\ m, 3.4\ m). \tag{3.5.2}$$

For the silver coin, the coordinates are

$$\left\{\begin{array}{l} x_S = r_S \cos\varphi_S = (20.0\ m)\cos\frac{\pi}{9} = 18.9\ m \\ y_S = r_S \sin\varphi_S = (20.0\ m)\sin\frac{\pi}{9} = 6.8\ m \end{array}\right. \Rightarrow (x_S, y_S) = (18.9\ m, 6.8\ m). \tag{3.5.3}$$

Vectors in Three Dimensions

To specify the location of a point in space, we need three coordinates (x, y, z), where coordinates x and y specify locations in a plane, and coordinate z gives a vertical positions above or below the plane. Three-dimensional space has three orthogonal directions, so we need not two but three unit vectors to define a three-dimensional coordinate system. In the Cartesian coordinate system, the first two unit vectors are the unit vector of the x-axis $\hat{i}$ and the unit vector of the y-axis $\hat{j}$. The third unit vector $\hat{k}$ is the direction of the z-axis (Figure $3.5.2$). The order in which the axes are labeled, which is the order in which the three unit vectors appear, is important because it defines the orientation of the coordinate system. The order x-y-z, which is equivalent to the order $\hat{i}$ - $\hat{j}$ - $\hat{k}$, defines the standard right-handed coordinate system (positive orientation).

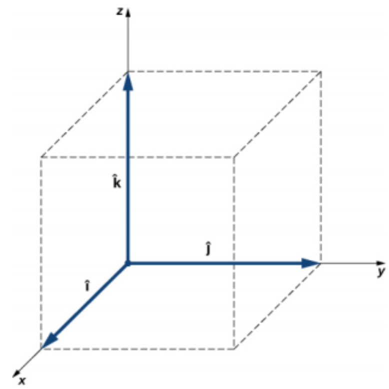

Figure $3.5.2$: Three unit vectors define a Cartesian system in three-dimensional space. The order in which these unit vectors appear defines the orientation of the coordinate system. The order shown here defines the right-handed orientation.

In three-dimensional space, vector $\vec{A}$ has three vector components: the x-component $\vec{A}_x$ = A_x $\hat{i}$, which is the part of vector $\vec{A}$ along the x-axis; the y-component $\vec{A}_y$ = A_y $\hat{j}$, which is the part of $\vec{A}$ along the y-axis; and the z-component $\vec{A}_z$ = A_z $\hat{k}$, which is the part of the vector along the z-axis. A vector in three-dimensional space is the vector sum of its three vector components (Figure $3.5.3$):

$$\vec{A} = A_x\hat{i} + A_y\hat{j} + A_z\hat{k}. \tag{3.5.4}$$

If we know the coordinates of its origin b(x_b, y_b, z_b) and of its end e(x_e y_e, z_e), its scalar components are obtained by taking their differences: A_x and A_y are given by

$$\left\{\begin{array}{l} A_x = x_e - x_b \\ A_y = y_e - y_b. \end{array}\right.$$

and the z-component is given by

$$A_z = z_e - z_b. \tag{3.5.5}$$

Magnitude A is obtained by generalizing Equation 2.4.8 to three dimensions:

$$A = \sqrt{A_x^2 + A_y^2 + A_z^2}\,. \tag{3.5.6}$$

This expression for the vector magnitude comes from applying the Pythagorean theorem twice. As seen in Figure $3.5.3$, the diagonal in the xy-plane has length $\sqrt{A_x^2 + A_y^2}$ and its square adds to the square $A_z{}^2$ to give A^2. Note that when the z-component is zero, the vector lies entirely in the xy-plane and its description is reduced to two dimensions.

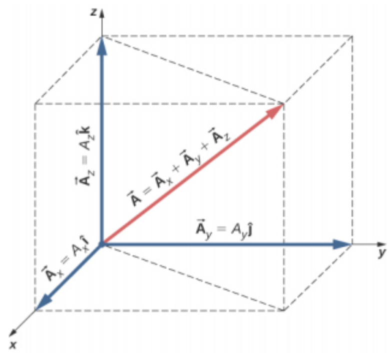

Figure 3.5.3: A vector in three-dimensional space is the vector sum of its three vector components.

✓ Example 3.5.2: Takeoff of a Drone

During a takeoff of IAI Heron (Figure 3.5.4), its position with respect to a control tower is 100 m above the ground, 300 m to the east, and 200 m to the north. One minute later, its position is 250 m above the ground, 1200 m to the east, and 2100 m to the north. What is the drone's displacement vector with respect to the control tower? What is the magnitude of its displacement vector?

Figure 3.5.4: The drone IAI Heron in flight. (credit: SSgt Reynaldo Ramon, USAF)

Strategy

We take the origin of the Cartesian coordinate system as the control tower. The direction of the +x-axis is given by unit vector $\hat{i}$ to the east, the direction of the +y-axis is given by unit vector $\hat{j}$ to the north, and the direction of the +z-axis is given by unit vector $\hat{k}$, which points up from the ground. The drone's first position is the origin (or, equivalently, the beginning) of the displacement vector and its second position is the end of the displacement vector.

Solution

We identify b(300.0 m, 200.0 m, 100.0 m) and e(480.0 m, 370.0 m, 250.0m), and use Equation 2.4.4 and Equation 3.5.5 to find the scalar components of the drone's displacement vector:

$$\begin{cases} D_x = x_e - x_b = 1200.0\ m - 300.0\ m = 900.0\ m, \\ D_y = y_e - y_b = 2100.0\ m - 200.0\ m = 1900.0\ m, \\ D_z = z_e - z_b = 250.0\ m - 100.0\ m = 150\ m. \end{cases} \tag{3.5.7}$$

We substitute these components into Equation 3.5.4 to find the displacement vector:

$$\vec{D} = D_x\ \hat{i} + D_y\ \hat{j} + D_z\ \hat{k} = 900.0\ \hat{i} + 1900.0\ \hat{j} + 150.0\ \hat{k} = (0.90\ \hat{i} + 1.90\ \hat{j} + 0.15\ \hat{k})\ km. \tag{3.5.8}$$

We substitute into Equation 3.5.6 to find the magnitude of the displacement:

$$D = \sqrt{D_x^2 + D_y^2 + D_z^2} = \sqrt{(0.90\ km)^2 + (1.90\ km)^2 + (0.15\ km)^2} = 4.44\ km. \tag{3.5.9}$$

? Exercise 2.7

If the average velocity vector of the drone in the displacement in Example 2.7 is $\vec{u}$ = (15.0 $\hat{i}$ + 31.7 $\hat{j}$ + 2.5 $\hat{k}$) m/s, what is the magnitude of the drone's velocity vector?

3.6: Algebra of Vectors

Learning Objectives

- Apply analytical methods of vector algebra to find resultant vectors and to solve vector equations for unknown vectors.
- Interpret physical situations in terms of vector expressions.

Vectors can be added together and multiplied by scalars. Vector addition is associative (Equation 2.2.8) and commutative (Equation 2.2.7), and vector multiplication by a sum of scalars is distributive (Equation 2.2.9). Also, scalar multiplication by a sum of vectors is distributive:

$$\alpha(\vec{A}+\vec{B}) = \alpha\vec{A}+\alpha B. \tag{3.6.1}$$

In this equation, α is any number (a scalar). For example, a vector antiparallel to vector $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ can be expressed simply by multiplying $\vec{A}$ by the scalar $\alpha = -1$:

$$-\vec{A} = A_x\hat{i} - A_y\hat{j} - A_z\hat{k}. \tag{3.6.2}$$

Example 3.6.1: Direction of Motion

In a Cartesian coordinate system where $\hat{i}$ denotes geographic east, $\hat{j}$ denotes geographic north, and $\hat{k}$ denotes altitude above sea level, a military convoy advances its position through unknown territory with velocity $\vec{v} = (4.0\,\hat{i} + 3.0\,\hat{j} + 0.1\,\hat{k})$ km/h. If the convoy had to retreat, in what geographic direction would it be moving?

Solution

The velocity vector has the third component $\vec{v}_z = (+\,0.1$ km/h$)\,\hat{k}$, which says the convoy is climbing at a rate of 100 m/h through mountainous terrain. At the same time, its velocity is 4.0 km/h to the east and 3.0 km/h to the north, so it moves on the ground in direction $\tan^{-1}(3/4) \approx 37°$ north of east. If the convoy had to retreat, its new velocity vector $\vec{u}$ would have to be antiparallel to $\vec{v}$ and be in the form $\vec{u} = -\alpha\vec{v}$, where α is a positive number. Thus, the velocity of the retreat would be $\vec{u} = \alpha(-4.0\,\hat{i} - 3.0\,\hat{j} - 0.1\,\hat{k})$ km/h. The negative sign of the third component indicates the convoy would be descending. The direction angle of the retreat velocity is $\tan^{-1}(-3\,\alpha - 4\,\alpha) \approx 37°$ south of west. Therefore, the convoy would be moving on the ground in direction 37° south of west while descending on its way back.

The generalization of the number zero to vector algebra is called the **null vector**, denoted by $\vec{0}$. All components of the null vector are zero, $\vec{0} = 0\,\hat{i} + 0\,\hat{j} + 0\,\hat{k}$, so the null vector has no length and no direction.

Two vectors $\vec{A}$ and $\vec{B}$ are **equal vectors** if and only if their difference is the null vector: $\vec{0} = \vec{A} - \vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) - (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) = (A_x - B_x)\,\hat{i} + (A_y - B_y)\,\hat{j} + (A_z - B_z)\,\hat{k}$. This vector equation means we must have simultaneously $A_x - B_x = 0$, $A_y - B_y = 0$, and $A_z - B_z = 0$. Hence, we can write $\vec{A} = \vec{B}$ if and only if the corresponding components of vectors $\vec{A}$ and $\vec{B}$ are equal:

$$\vec{A} = \vec{B} \Leftrightarrow \begin{cases} A_x = B_x \\ A_y = B_y \\ A_z = B_z \end{cases}. \tag{3.6.3}$$

Two vectors are equal when their corresponding scalar components are equal. Resolving vectors into their scalar components (i.e., finding their scalar components) and expressing them analytically in vector component form (given by Equation 2.5.4) allows us to use vector algebra to find sums or differences of many vectors **analytically** (i.e., without using graphical methods). For example, to find the resultant of two vectors $\vec{A}$ and $\vec{B}$, we simply add them component by component, as follows:

$$\vec{R} = \vec{A}+\vec{B} = (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) + (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) = (A_x + B_x)\,\hat{i} + (A_y + B_y)\,\hat{j} + (A_z + B_z)\,\hat{k}. \tag{3.6.4}$$

In this way, using Equation 3.6.3, scalar components of the resultant vector $\vec{R} = R_x\,\hat{i} + R_y\,\hat{j} + R_z\,\hat{k}$ are the sums of corresponding scalar components of vectors $\vec{A}$ and $\vec{B}$:

$$\begin{cases} R_x = A_x + B_x, \\ R_y = A_y + B_y, \\ R_z = A_z + B_z \end{cases}. \tag{3.6.5}$$

Analytical methods can be used to find components of a resultant of many vectors. For example, if we are to sum up N vectors $\vec{F}_1$, $\vec{F}_2$, $\vec{F}_3$, … , $\vec{F}_N$, where each vector is $\vec{F}_k$ = F_{kx} $\hat{i}$ + F_{ky} $\hat{j}$ + F_{kz} $\hat{k}$, the resultant vector $\vec{F}_R$ is

$$\vec{F}_R = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \ldots + \vec{F}_N = \sum_{k=1}^{N} \vec{F}_k = \sum_{k=1}^{N} \left(F_{kx}\hat{i} + F_{ky}\hat{j} + F_{kz}\hat{k}\right) = \left(\sum_{k=1}^{N} F_{kx}\right)\hat{i} + \left(\sum_{k=1}^{N} F_{ky}\right)\hat{j} + \left(\sum_{k=1}^{N} F_{kz}\right)\hat{k}. \tag{3.6.6}$$

Therefore, scalar components of the resultant vector are

$$\begin{cases} F_{Rx} = \sum_{k=1}^{N} F_{kx} = F_{1x} + F_{2x} + \ldots + F_{Nx} \\ F_{Ry} = \sum_{k=1}^{N} F_{ky} = F_{1y} + F_{2y} + \ldots + F_{Ny} \\ F_{Rz} = \sum_{k=1}^{N} F_{kz} = F_{1z} + F_{2z} + \ldots + F_{Nz}. \end{cases} \tag{3.6.7}$$

Having found the scalar components, we can write the resultant in vector component form:

$$\vec{F}_R = F_{Rx}\ \hat{i} + F_{Ry}\ \hat{j} + F_{Rz}\ \hat{k}. \tag{3.6.8}$$

Analytical methods for finding the resultant and, in general, for solving vector equations are very important in physics because many physical quantities are vectors. For example, we use this method in kinematics to find resultant displacement vectors and resultant velocity vectors, in mechanics to find resultant force vectors and the resultants of many derived vector quantities, and in electricity and magnetism to find resultant electric or magnetic vector fields.

In many physical situations, we often need to know the direction of a vector. For example, we may want to know the direction of a magnetic field vector at some point or the direction of motion of an object. We have already said direction is given by a unit vector, which is a dimensionless entity—that is, it has no physical units associated with it. When the vector in question lies along one of the axes in a Cartesian system of coordinates, the answer is simple, because then its unit vector of direction is either parallel or antiparallel to the direction of the unit vector of an axis. For example, the direction of vector $\vec{d}$ = −5 m $\hat{i}$ is unit vector $\hat{d}$ = − $\hat{i}$. The general rule of finding the unit vector $\hat{V}$ of direction for any vector $\vec{V}$ is to divide it by its magnitude V:

$$\hat{V} = \frac{\vec{V}}{V}. \tag{3.6.9}$$

We see from this expression that the unit vector of direction is indeed dimensionless because the numerator and the denominator in Equation 3.6.9 have the same physical unit. In this way, Equation 3.6.9 allows us to express the unit vector of direction in terms of unit vectors of the axes. Example 2.7.6 illustrates this principle.

Contributors

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3.7: Algebra of Vectors Examples

Example 3.7.1: Analytical Computation of a Resultant

Three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ in a plane (Figure 2.3.6) are specified by their magnitudes A = 10.0, B = 7.0, and C = 8.0, respectively, and by their respective direction angles with the horizontal direction α = 35°, β = −110°, and γ = 30°. The physical units of the magnitudes are centimeters. Resolve the vectors to their scalar components and find the following vector sums:

a. $\vec{R} = \vec{A} + \vec{B} + \vec{C}$,
b. $\vec{D} = \vec{A} - \vec{B}$, and
c. $\vec{S} = \vec{A} - 3\,\vec{B} + \vec{C}$.

Strategy

First, we use Equation 2.4.13 to find the scalar components of each vector and then we express each vector in its vector component form given by $\overrightarrow{\mathbf{A}} = A_x\hat{\mathbf{i}} + A_y\hat{\mathbf{j}}$. Then, we use analytical methods of vector algebra to find the resultants.

Solution

We resolve the given vectors to their scalar components:

$$\begin{cases} A_x = A\cos\alpha = (10.0\ cm)\cos 35^o = 8.19\ cm \\ A_y = A\sin\alpha = (10.0\ cm)\sin 35^o = 5.73\ cm \end{cases} \tag{3.7.1}$$

$$\begin{cases} B_x = B\cos\beta = (7.0\ cm)\cos(-110^o) = -2.39\ cm \\ B_y = B\sin\beta = (7.0\ cm)\sin(-110^o) = -6.58\ cm \end{cases} \tag{3.7.2}$$

$$\begin{cases} C_x = C\cos\gamma = (8.0\ cm)\cos(30^o) = 6.93\ cm \\ C_y = C\sin\gamma = (8.0\ cm)\sin(30^o) = 4.00\ cm \end{cases} \tag{3.7.3}$$

For (a) we may substitute directly into Equation 2.6.7 to find the scalar components of the resultant:

$$\begin{cases} R_x = A_x + B_x + C_x = 8.19\ cm - 2.39\ cm + 6.93\ cm = 12.73\ cm \\ R_y = A_y + B_y + C_y = 5.73\ cm - 6.58\ cm + 4.00\ cm = 3.15\ cm \end{cases} \tag{3.7.4}$$

Therefore, the resultant vector is $\vec{R} = R_x\hat{i} + R_y\hat{j} = (12.7\hat{i} + 3.1\hat{j})$ cm. For (b), we may want to write the vector difference as

$$\vec{D} = \vec{A} - \vec{B} = (A_x\hat{i} + A_y\hat{j}) - (B_x\hat{i} + B_y\hat{j}) = (A_x - B_x)\hat{i} + (A_y - B_y)\hat{j}. \tag{3.7.5}$$

Hence the difference vector is $\vec{D} = D_x\hat{i} + D_y\hat{j} = (10.6\hat{i} + 12.3\hat{j})$ cm.

For (c), we can write vector $\vec{S}$ in the following explicit form:

$$\vec{S} = \vec{A} - 3\vec{B} + \vec{C} = (A_x\hat{i} + A_y\hat{j}) - 3(B_x\hat{i} + B_y\hat{j}) + (C_x\hat{i} + C_y\hat{j}) = (A_x - 3B_x + C_x)\hat{i} + (A_y - 3B_y + C_y)\hat{j}. \tag{3.7.6}$$

Then, the scalar components of $\vec{S}$ are

$$\begin{cases} S_x = A_x - 3B_x + C_x = 8.19\ cm - 3(-2.39\ cm) + 6.93\ cm = 22.29\ cm \\ S_y = A_y - 3B_y + C_y = 5.73\ cm - 3(-6.58\ cm) + 4.00\ cm = 29.47\ cm \end{cases} \tag{3.7.7}$$

The vector is $\vec{S} = S_x\hat{i} + S_y\hat{j} = (22.3\hat{i} + 29.5\hat{j})$ cm.

Significance

Having found the vector components, we can illustrate the vectors by graphing or we can compute magnitudes and direction angles, as shown in Figure 3.7.1. Results for the magnitudes in (b) and (c) can be compared with results for the same problems obtained with the graphical method, shown in Figure 2.3.7 and Figure 2.3.8. Notice that the analytical method produces exact results and its accuracy is not limited by the resolution of a ruler or a protractor, as it was with the graphical method used in Example 2.3.2 for finding this same resultant.

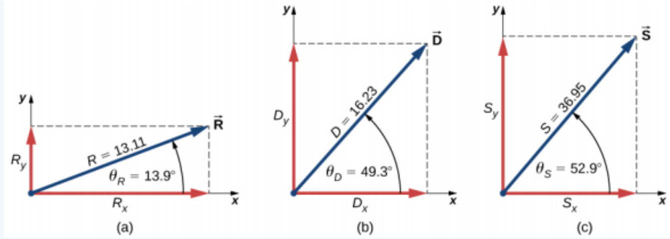

Figure 3.7.1: Graphical illustration of the solutions obtained analytically.

? Exercise 2.8

Three displacement vectors $\vec{A}$, $\vec{B}$, and $\vec{F}$ (Figure 2.3.6) are specified by their magnitudes A = 10.00, B = 7.00, and F = 20.00, respectively, and by their respective direction angles with the horizontal direction α = 35°, β = −110°, and φ = 110°. The physical units of the magnitudes are centimeters. Use the analytical method to find vector $\vec{F}$ = $\vec{A}$ + 2 $\vec{B}$ − $\vec{F}$. Verify that G = 28.15 cm and that θ_G = −68.65°.

✔ Example 3.7.2: The Tug-of-War Game

Four dogs named Astro, Balto, Clifford, and Dug play a tug-of-war game with a toy (Figure 3.7.2). Astro pulls on the toy in direction α = 55° south of east, Balto pulls in direction β = 60° east of north, and Clifford pulls in direction γ = 55° west of north. Astro pulls strongly with 160.0 units of force (N), which we abbreviate as A = 160.0 N. Balto pulls even stronger than Astro with a force of magnitude B = 200.0 N, and Clifford pulls with a force of magnitude C = 140.0 N. When Dug pulls on the toy in such a way that his force balances out the resultant of the other three forces, the toy does not move in any direction. With how big a force and in what direction must Dug pull on the toy for this to happen?

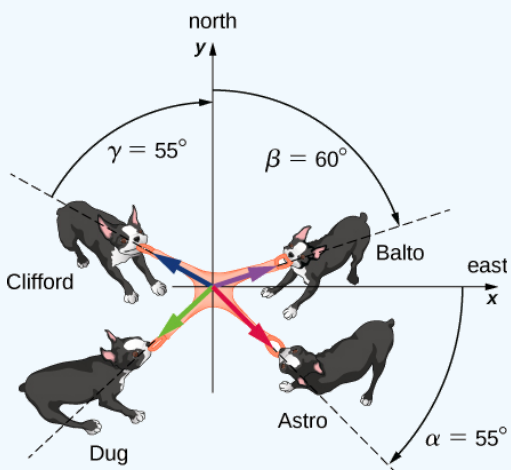

Figure 3.7.2: Four dogs play a tug-of-war game with a toy.

Strategy

We assume that east is the direction of the positive x-axis and north is the direction of the positive y-axis. As in Example 3.7.1, we have to resolve the three given forces — $\vec{A}$ (the pull from Astro), $\vec{B}$ (the pull from Balto), and $\vec{C}$ (the pull from Clifford)—into their scalar components and then find the scalar components of the resultant vector $\vec{R}$ = $\vec{A}$ + $\vec{B}$ + $\vec{C}$. When the pulling force $\vec{D}$ from Dug balances out this resultant, the sum of $\vec{D}$ and $\vec{R}$ must give the null vector $\vec{D}$ + $\vec{R}$ = $\vec{0}$. This means that $\vec{D}$ = $-\vec{R}$ so the pull from Dug must be antiparallel to $\vec{R}$.

Solution

The direction angles are θ_A = $-\alpha$ = −55°, θ_B = 90° − β = 30°, and θ_C = 90° + γ = 145°, and substituting them into Equation 2.4.13 gives the scalar components of the three given forces:

$$\begin{cases} A_x = A\cos\theta_A = (160.0\ N)\cos(-55^o) = +91.8\ N \\ A_y = A\sin\theta_A = (160.0\ N)\sin(-55^o) = -131.1\ N \end{cases} \tag{3.7.8}$$

$$\begin{cases} B_x = B\cos\theta_B = (200.0\ N)\cos 30^o = +173.2\ N \\ B_y = B\sin\theta_B = (200.0\ N)\sin 30^o = +100.0\ N \end{cases} \tag{3.7.9}$$

$$\begin{cases} C_x = C\cos\theta_C = (140.0\ N)\cos 145^o = -114.7\ N \\ C_y = C\sin\theta_C = (140.0\ N)\sin 145^o = +80.3\ N \end{cases} \tag{3.7.10}$$

Now we compute scalar components of the resultant vector $\vec{R} = \vec{A} + \vec{B} + \vec{C}$:

$$\begin{cases} R_x = A_x + B_x + C_x = +91.8\ N + 173.2\ N - 114.7\ N = +150.3\ N \\ R_y = A_y + B_y + C_y = -131.1\ N + 100.0\ N + 80.3\ N = +49.2\ N \end{cases} \tag{3.7.11}$$

The antiparallel vector to the resultant $\vec{R}$ is

$$\vec{D} = -\vec{R} = -R_x\hat{i} - R_y\hat{j} = (-150.3\hat{i} - 49.2\hat{j})N. \tag{3.7.12}$$

The magnitude of Dug's pulling force is

$$D = \sqrt{D_x^2 + D_y^2} = \sqrt{(-150.3)^2 + (-49.2)^2}N = 158.1\ N. \tag{3.7.13}$$

The direction of Dug's pulling force is

$$\theta = \tan^{-1}\left(\frac{D_y}{D_x}\right) = \tan^{-1}\left(\frac{-49.2\ N}{-150.3\ N}\right) = \tan^{-1}\left(\frac{49.2}{150.3}\right) = 18.1^o. \tag{3.7.14}$$

Dug pulls in the direction 18.1° south of west because both components are negative, which means the pull vector lies in the third quadrant (Figure 2.4.4).

? Exercise 2.9

Suppose that Balto in Example 3.7.2 leaves the game to attend to more important matters, but Astro, Clifford, and Dug continue playing. Astro and Clifford's pull on the toy does not change, but Dug runs around and bites on the toy in a different place. With how big a force and in what direction must Dug pull on the toy now to balance out the combined pulls from Clifford and Astro? Illustrate this situation by drawing a vector diagram indicating all forces involved.

✓ Example 3.7.3: Vector Algebra

Find the magnitude of the vector $\vec{C}$ that satisfies the equation $2\vec{A} - 6\vec{B} + 3\vec{C} = 2\hat{j}$, $\vec{A} = \hat{i} - 2\hat{k}$ and $\vec{B} = -\hat{j} + \frac{\hat{k}}{2}$.

Strategy

We first solve the given equation for the unknown vector $\vec{C}$. Then we substitute $\vec{A}$ and $\vec{B}$; group the terms along each of the three directions $\hat{i}$, $\hat{j}$, and $\hat{k}$; and identify the scalar components C_x, C_y, and C_z. Finally, we substitute into Equation 2.5.6 to find magnitude C.

Solution

$$\begin{aligned} 2\vec{A} - 6\vec{B} + 3\vec{C} &= 2\hat{j} \\ 3\vec{C} &= 2\hat{j} - 2\vec{A} + 6\vec{B} \\ \vec{C} &= \frac{2}{3}\hat{j} - \frac{2}{3}\vec{A} + 2\vec{B} \\ &= \frac{2}{3}\hat{j} - \frac{2}{3}(\hat{i} - 2\hat{k}) + 2\left(-\hat{j} + \frac{\hat{k}}{2}\right) \\ &= \frac{2}{3}\hat{j} - \frac{2}{3}\hat{i} + \frac{4}{3}\hat{k} - 2\hat{j} + \hat{k} \\ &= -\frac{2}{3}\hat{i} + \left(\frac{2}{3} - 2\right)\hat{j} + \left(\frac{4}{3} + 1\right)\hat{k} \\ &= -\frac{2}{3}\hat{i} - \frac{4}{3}\hat{j} + \frac{7}{3}\hat{k} \end{aligned}$$

The components are $C_x = -\frac{2}{3}$, $C_y = -\frac{4}{3}$, and $C_z = \frac{7}{3}$, and substituting into Equation 2.5.6 gives

$$C=\sqrt{C_x^2+C_y^2+C_z^2}=\sqrt{\left(-\frac{2}{3}\right)^2+\left(-\frac{4}{3}\right)^2+\left(\frac{7}{3}\right)^2}=\sqrt{\frac{23}{3}}. \tag{3.7.15}$$

✓ Example 3.7.4: Displacement of a Skier

Starting at a ski lodge, a cross-country skier goes 5.0 km north, then 3.0 km west, and finally 4.0 km southwest before taking a rest. Find his total displacement vector relative to the lodge when he is at the rest point. How far and in what direction must he ski from the rest point to return directly to the lodge?

Strategy

We assume a rectangular coordinate system with the origin at the ski lodge and with the unit vector $\hat{i}$ pointing east and the unit vector $\hat{j}$ pointing north. There are three displacements: $\vec{D}_1$, $\vec{D}_2$, and $\vec{D}_3$. We identify their magnitudes as D_1 = 5.0 km , D_2 = 3.0 km , and D_3 = 4.0 km . We identify their directions are the angles θ_1 = 90°, θ_2 = 180°, and θ_3 = 180° + 45° = 225°. We resolve each displacement vector to its scalar components and substitute the components into Equation 2.6.5 to obtain the scalar components of the resultant displacement $\vec{D}$ from the lodge to the rest point. On the way back from the rest point to the lodge, the displacement is $\vec{B}$ = − $\vec{D}$. Finally, we find the magnitude and direction of $\vec{B}$.

Solution

Scalar components of the displacement vectors are

$$\begin{cases} D_{1x}=D_1\cos\theta_1=(5.0\ km)\cos 90^o=0 \\ D_{1y}=D_1\sin\theta_1=(5.0\ km)\sin 90^o=5.0\ km \end{cases} \tag{3.7.16}$$

$$\begin{cases} D_{2x}=D_2\cos\theta_2=(3.0\ km)\cos 180^o=-3.0\ km \\ D_{2y}=D_2\sin\theta_2=(3.0\ km)\sin 180^o=0 \end{cases} \tag{3.7.17}$$

$$\begin{cases} D_{3x}=D_3\cos\theta_3=(4.0\ km)\cos 225^o=-2.8\ km \\ D_{3y}=D_3\sin\theta_3=(4.0\ km)\sin 225^o=-2.8\ km \end{cases} \tag{3.7.18}$$

Scalar components of the net displacement vector are

$$\begin{cases} D_x=D_{1x}+D_{2x}+D_{3x}=(0-3.0-2.8)km=-5.8\ km \\ D_y=D_{1y}+D_{2y}+D_{3y}=(5.0+0-2.8)km=+2.2\ km \end{cases} \tag{3.7.19}$$

Hence, the skier's net displacement vector is $\vec{D}$ = D_x $\hat{i}$ + D_y $\hat{j}$ = (−5.8 $\hat{i}$ + 2.2 $\hat{j}$)km . On the way back to the lodge, his displacement is $\vec{B}$ = − $\vec{D}$ = −(−5.8 $\hat{i}$+ 2.2 $\hat{j}$)km = (5.8 $\hat{i}$ − 2.2 $\hat{j}$)km. Its magnitude is B = $\sqrt{B_x^2+B_y^2}$ = $\sqrt{(5.8)^2+(-2.2)^2}$ km = 6.2 km and its direction angle is θ= $\tan^{-1}\left(\frac{-2.2}{5.8}\right)$ = −20.8°. Therefore, to return to the lodge, he must go 6.2 km in a direction about 21° south of east.

Significance

Notice that no figure is needed to solve this problem by the analytical method. Figures are required when using a graphical method; however, we can check if our solution makes sense by sketching it, which is a useful final step in solving any vector problem.

✓ Example 3.7.5: Displacement of a Jogger

A jogger runs up a flight of 200 identical steps to the top of a hill and then runs along the top of the hill 50.0 m before he stops at a drinking fountain (Figure 3.7.3). His displacement vector from point A at the bottom of the steps to point B at the fountain is $\vec{D}_{AB}$ = (−90.0 $\hat{i}$ + 30.0 $\hat{j}$)m. What is the height and width of each step in the flight? What is the actual distance the jogger covers? If he makes a loop and returns to point A, what is his net displacement vector?

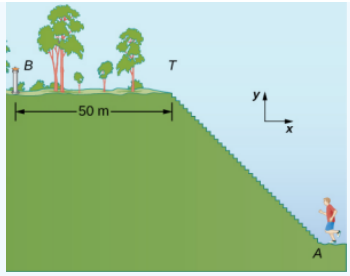

Figure 3.7.3: A jogger runs up a flight of steps.

Strategy

The displacement vector $\vec{D}_{AB}$ is the vector sum of the jogger's displacement vector $\vec{D}_{AT}$ along the stairs (from point A at the bottom of the stairs to point T at the top of the stairs) and his displacement vector $\vec{D}_{RB}$ on the top of the hill (from point T at the top of the stairs to the fountain at point B). We must find the horizontal and the vertical components of $\vec{D}_{TB}$. If each step has width w and height h, the horizontal component of $\vec{D}_{TB}$ must have a length of 200w and the vertical component must have a length of 200h. The actual distance the jogger covers is the sum of the distance he runs up the stairs and the distance of 50.0 m that he runs along the top of the hill.

Solution

In the coordinate system indicated in Figure 3.7.3, the jogger's displacement vector on the top of the hill is $\vec{D}_{RB}$ = (−50.0 m) $\hat{i}$. His net displacement vector is

$$\vec{D}_{AB} = \vec{D}_{AT} + \vec{D}_{TB}.$$

Therefore, his displacement vector $\vec{D}_{TB}$ along the stairs is

$$\begin{aligned}\vec{D}_{AT} &= \vec{D}_{AB} - \vec{D}_{TB} = (-90.0\hat{i} + 30.0\hat{j})m - (-50.0m)\hat{i}) = [(-90.050.0)hati + 30.0\hat{j})]m \\ &= (-40.0\hat{i} + 30.0\hat{j})m.\end{aligned}$$

Its scalar components are D_{ATx} = −40.0 m and D_{ATy} = 30.0 m. Therefore, we must have

$$200w = |-40.0|\text{m and } 200h = 30.0\text{ m}.$$

Hence, the step width is w = $\frac{40.0\ m}{200}$ = 0.2 m = 20 cm, and the step height is w = $\frac{30.0\ m}{200}$ = 0.15 m = 15 cm. The distance that the jogger covers along the stairs is

$$\vec{D}_{AT} = \sqrt{\vec{D}^2_{ATx} + \vec{D}^2_{ATy}} = \sqrt{(-40.0)^2 + (30.0)^2}m = 50.0\ m.$$

Thus, the actual distance he runs is D_{AT} + D_{TB} = 50.0 m + 50.0 m = 100.0 m. When he makes a loop and comes back from the fountain to his initial position at point A, the total distance he covers is twice this distance,or 200.0 m. However, his net displacement vector is zero, because when his final position is the same as his initial position, the scalar components of his net displacement vector are zero (Equation 2.4.4).

In many physical situations, we often need to know the direction of a vector. For example, we may want to know the direction of a magnetic field vector at some point or the direction of motion of an object. We have already said direction is given by a unit vector, which is a dimensionless entity—that is, it has no physical units associated with it. When the vector in question lies along one of the axes in a Cartesian system of coordinates, the answer is simple, because then its unit vector of direction is either parallel or antiparallel to the direction of the unit vector of an axis. For example, the direction of vector $\vec{d}$ = -5 m $\hat{i}$ is unit vector $\vec{d}$ = -$\hat{i}$. The general rule of finding the unit vector $\vec{V}$ of direction for any vector $\vec{V}$ is to divide it by its magnitude V:

$$\widehat{\mathbf{V}} = \frac{\overrightarrow{\mathbf{V}}}{V}. \tag{3.7.20}$$

We see from this expression that the unit vector of direction is indeed dimensionless because the numerator and the denominator in Equation 3.7.20 have the same physical unit. In this way, Equation 3.7.20 allows us to express the unit vector of direction in terms of unit vectors of the axes. The following example illustrates this principle.

Example 3.7.6: The Unit Vector of Direction

If the velocity vector of the military convoy in Example 2.6.1 is $\vec{v}$ = (4.000 $\hat{i}$ + 3.000 $\hat{j}$ + 0.100 $\hat{k}$)km/h, what is the unit vector of its direction of motion.

Strategy

The unit vector of the convoy's direction of motion is the unit vector $\hat{v}$ that is parallel to the velocity vector. The unit vector is obtained by dividing a vector by its magnitude, in accordance with Equation 3.7.20.

Solution

The magnitude of the vector $\vec{v}$ is

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{4.000^2 + 3.000^2 + 0.100^2}\, km/h = 5.001\ km/h.$$

To obtain the unit vector $\hat{v}$, divide $\vec{v}$ by its magnitude:

$$\begin{aligned} \hat{v} = \frac{\vec{v}}{v} &= \frac{(4.000\hat{i} + 3.00\hat{j} + 0.100\hat{k})km/h}{5.001\ km/h} \\ &= \frac{(4.000\hat{i} + 3.000\hat{j} + 0.1100\hat{k})}{5.001} \\ &= \frac{4.000}{5.001}\hat{i} + \frac{3.000}{5.001}\hat{j} + \frac{0.100}{5.001}\hat{k} \\ &= (79.98\hat{i} + 59.99\hat{j} + 2.00\hat{k}) \times 10^{-2}. \end{aligned}$$

Significance

Note that when using the analytical method with a calculator, it is advisable to carry out your calculations to at least three decimal places and then round off the final answer to the required number of significant figures, which is the way we performed calculations in this example. If you round off your partial answer too early, you risk your final answer having a huge numerical error, and it may be far off from the exact answer or from a value measured in an experiment.

Exercise 2.10

Verify that vector $\hat{v}$ obtained in Example $3.7.3$ is indeed a unit vector by computing its magnitude. If the convoy in Example 2.6.1 was moving across a desert flatland—that is, if the third component of its velocity was zero—what is the unit vector of its direction of motion? Which geographic direction does it represent?

3.8: Products of Vectors (Part 1)

Learning Objectives

- Explain the difference between the scalar product and the vector product of two vectors.
- Determine the scalar product of two vectors.
- Determine the vector product of two vectors.
- Describe how the products of vectors are used in physics.

A vector can be multiplied by another vector but may not be divided by another vector. There are two kinds of products of vectors used broadly in physics and engineering. One kind of multiplication is a **scalar multiplication of two vectors**. Taking a scalar product of two vectors results in a number (a scalar), as its name indicates. Scalar products are used to define work and energy relations. For example, the work that a force (a vector) performs on an object while causing its displacement (a vector) is defined as a scalar product of the force vector with the displacement vector. A quite different kind of multiplication is a **vector multiplication of vectors**. Taking a vector product of two vectors returns as a result a vector, as its name suggests. Vector products are used to define other derived vector quantities. For example, in describing rotations, a vector quantity called **torque** is defined as a vector product of an applied force (a vector) and its lever arm (a vector). It is important to distinguish between these two kinds of vector multiplications because the scalar product is a scalar quantity and a vector product is a vector quantity.

The Scalar Product of Two Vectors (the Dot Product)

Scalar multiplication of two vectors yields a scalar product.

Definition: Scalar Product (Dot Product)

The scalar product $\vec{A} \cdot \vec{B}$ of two vectors $\vec{A}$ and $\vec{B}$ is a number defined by the equation

$$\vec{A} \cdot \vec{B} = AB\cos\varphi, \tag{3.8.1}$$

where ϕ is the angle between the vectors (shown in Figure $3.8.1$). The scalar product is also called the **dot product** because of the dot notation that indicates it.

In the definition of the dot product, the direction of angle φ does not matter, and φ can be measured from either of the two vectors to the other because $\cos\varphi = \cos(-\varphi) = cos(2\pi - \varphi)$. The dot product is a negative number when 90° < φ ≤ 180° and is a positive number when 0° ≤ ϕ < 90°. Moreover, the dot product of two parallel vectors is $\vec{A} \cdot \vec{B}$ = AB cos 0° = AB, and the dot product of two antiparallel vectors is $\vec{A} \cdot \vec{B}$ = AB cos 180° = −AB. The scalar product of two orthogonal vectors vanishes: $\vec{A} \cdot \vec{B}$ = AB cos 90° = 0. The scalar product of a vector with itself is the square of its magnitude:

$$\vec{A}^2 \equiv \vec{A} \cdot \vec{A} = AA\cos 0^o = A^2 \tag{3.8.2}$$

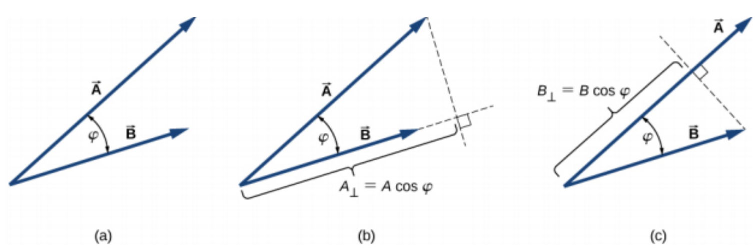

Figure $3.8.1$: The scalar product of two vectors. (a) The angle between the two vectors. (b) The orthogonal projection $A_\perp$ of vector $\vec{A}$ onto the direction of vector $\vec{B}$. (c) The orthogonal projection $B_\perp$ of vector $\vec{B}$ onto the direction of vector $\vec{A}$.

Example $3.8.1$: The Scalar Product

For the vectors shown in Figure 2.3.6, find the scalar product $\vec{A} \cdot \vec{F}$.

Strategy

From Figure 2.3.6, the magnitudes of vectors $\vec{A}$ and $\vec{B}$ are A = 10.0 and F = 20.0. Angle θ, between them, is the difference: $\theta=\varphi-\alpha$ = 110° − 35° = 75°. Substituting these values into Equation 3.8.1 gives the scalar product.

Solution

A straightforward calculation gives us

$$\vec{A}\cdot\vec{F}=AF\cos\theta=(10.0)(20.0)\cos 75^{o}=51.76. \tag{3.8.3}$$

? Exercise 2.11

For the vectors given in Figure 2.3.6, find the scalar products $\vec{A}\cdot\vec{B}$ and $\vec{B}\cdot\vec{C}$.

In the Cartesian coordinate system, scalar products of the unit vector of an axis with other unit vectors of axes always vanish because these unit vectors are orthogonal:

$$\hat{i}\ \cdot\ \hat{j}=|\hat{i}||\hat{j}|\cos 90^{o}=(1)(1)(0)=0, \tag{3.8.4}$$

$$\hat{i}\ \cdot\ \hat{k}=|\hat{i}||\hat{k}|\cos 90^{o}=(1)(1)(0)=0, \tag{3.8.5}$$

$$\hat{k}\cdot\ \hat{j}=|\hat{k}||\hat{j}|\cos 90^{o}=(1)(1)(0)=0. \tag{3.8.6}$$

In these equations, we use the fact that the magnitudes of all unit vectors are one: $|\hat{i}|=|\hat{j}|=|\hat{k}|$ = 1. For unit vectors of the axes, Equation 3.8.2 gives the following identities:

$$\hat{i}\ \cdot\ \hat{i}=i^2=\hat{j}\ \cdot\ \hat{j}=j^2=\hat{k}\ \cdot\ \hat{k}=1. \tag{3.8.7}$$

The scalar product $\vec{A}\cdot\vec{B}$ can also be interpreted as either the product of B with the projection $A_{\parallel}$ of vector $\vec{A}$ onto the direction of vector $\vec{B}$ (Figure 3.8.1(b)) or the product of A with the projection $B_{\parallel}$ of vector $\vec{B}$ onto the direction of vector $\vec{A}$ (Figure 3.8.1(c)):

$$\begin{aligned}\vec{A}\cdot\vec{B}&=AB\cos\varphi\\&=B(A\cos\varphi)=BA_{\parallel}\\&=A(B\cos\varphi)=AB_{\parallel}.\end{aligned}$$

For example, in the rectangular coordinate system in a plane, the scalar x-component of a vector is its dot product with the unit vector $\hat{i}$, and the scalar y-component of a vector is its dot product with the unit vector $\hat{j}$:

$$\begin{cases}\vec{A}\cdot\ \hat{i}=|\vec{A}||\hat{i}|\cos\theta_A=A\cos\theta_A=A\cos\theta_A=A_x\\ \vec{A}\cdot\ \hat{j}=|\vec{A}||\hat{j}|\cos(90^{o}-\theta_A)=A\sin\theta_A=A_y\end{cases} \tag{3.8.8}$$

Scalar multiplication of vectors is communtative,

$$\vec{A}\cdot\vec{B}=\vec{B}\cdot\vec{A}, \tag{3.8.9}$$

and obeys the distributive law:

$$\vec{A}\cdot(\vec{B}+\vec{C})=\vec{A}\cdot\vec{B}+\vec{A}\cdot\vec{C}. \tag{3.8.10}$$

We can use the commutative and distributive laws to derive various relations for vectors, such as expressing the dot product of two vectors in terms of their scalar components.

? Exercise 2.12

For vector $\vec{A}=A_x\ \hat{i}+A_y\ \hat{j}+A_z\ \hat{k}$ in a rectangular coordinate system, use Equation 3.8.4 through Equation 3.8.10 to show that $\vec{A}\cdot\hat{i}=A_x\vec{A}\cdot\ \hat{j}=A_y$ and $\vec{A}\cdot\ \hat{k}=A_z$.

When the vectors in Equation 3.8.1 are given in their vector component forms,

$$\vec{A}=A_x\ \hat{i}+A_y\ \hat{j}+A_z\ \hat{k}\ and \vec{B}=B_x\ \hat{i}+B_y\ \hat{j}+B_z\ \hat{k}, \tag{3.8.11}$$

we can compute their scalar product as follows:

$$\begin{aligned}\vec{A}\cdot\vec{B} &= (A_x\,\hat{i}+A_y\,\hat{j}+A_z\,\hat{k})\cdot(B_x\,\hat{i}+B_y\,\hat{j}+B_z\,\hat{k}) \\ &= A_xB_x\,\hat{i}\cdot\hat{i}+A_xB_y\,\hat{i}\cdot\hat{j}+A_xB_z\,\hat{i}\cdot\hat{k} \\ &+A_yB_x\,\hat{j}\cdot\hat{i}+A_yB_y\,\hat{j}\cdot\hat{j}+A_yB_z\,\hat{j}\cdot\hat{k} \\ &+A_zB_x\,\hat{k}\cdot\hat{i}+A_zB_y\,\hat{k}\cdot\hat{j}+A_zB_z\,\hat{k}\cdot\hat{k}.\end{aligned}$$

Since scalar products of two different unit vectors of axes give zero, and scalar products of unit vectors with themselves give one (see Equation 3.8.4 and Equation 3.8.7), there are only three nonzero terms in this expression. Thus, the scalar product simplifies to

$$\vec{A}\cdot\vec{B}=A_xB_x+A_yB_y+A_zB_z. \tag{3.8.12}$$

We can use Equation 3.8.12 for the scalar product in terms of scalar components of vectors to find the angle between two vectors. When we divide Equation 3.8.1 by AB, we obtain the equation for cos φ, into which we substitute Equation 3.8.12:

$$\cos\varphi=\frac{\vec{A}\cdot\vec{B}}{AB}=\frac{A_xB_x+A_yB_y+A_zB_z}{AB}. \tag{3.8.13}$$

Angle φ between vectors $\vec{A}$ and $\vec{B}$ is obtained by taking the inverse cosine of the expression in Equation 3.8.13.

Example 3.8.2

Three dogs are pulling on a stick in different directions, as shown in Figure 3.8.2. The first dog pulls with force $\vec{F}_1$ = (10.0 $\hat{i}$ − 20.4 $\hat{j}$ + 2.0 $\hat{k}$)N, the second dog pulls with force $\vec{F}_2$ = (−15.0 $\hat{i}$ − 6.2 $\hat{k}$)N , and the third dog pulls with force $\vec{F}_3$ = (5.0 $\hat{i}$ + 12.5 $\hat{j}$)N . What is the angle between forces $\vec{F}_1$ and $\vec{F}_2$?

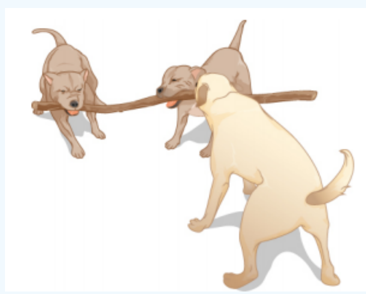

Figure 3.8.2: Three dogs are playing with a stick.

Strategy

The components of force vector $\vec{F}_1$ are F_{1x} = 10.0 N, F_{1y}= −20.4 N, and F_{1z} = 2.0 N, whereas those of force vector $\vec{F}_2$ are F_{2x} = −15.0 N, F_{2y} = 0.0 N, and F_{2z} = −6.2 N. Computing the scalar product of these vectors and their magnitudes, and substituting into Equation 3.8.13 gives the angle of interest.

Solution

The magnitudes of forces $\vec{F}_1$ and $\vec{F}_2$ are

$$F_1=\sqrt{F_{1x}^2+F_{1y}^2+F_{1z}^2}=\sqrt{10.0^2+20.4^2+2.0^2}\;N=22.8\;N \tag{3.8.14}$$

and

$$F_2=\sqrt{F_{2x}^2+F_{2y}^2+F_{2z}^2}=\sqrt{15.0^2+6.2^2}\;N=16.2\;N. \tag{3.8.15}$$

Substituting the scalar components into Equation 3.8.12 yields the scalar product

$$\begin{aligned}\vec{F}_1\cdot\vec{F}_2 &= F_{1x}F_{2x}+F_{1y}F_{2y}+F_{1z}F_{2z} \\ &=(10.0\;N)(-15.0\;N)+(-20.4\;N)(0.0\;N)+(2.0\;N)(-6.2\;N) \\ &=-162.4\;N^2.\end{aligned}$$

Finally, substituting everything into Equation 3.8.13 gives the angle

$$\cos\varphi = \frac{\vec{F}_1 \cdot \vec{F}_2}{F_1 F_2} = \frac{-162.4\ N^2}{(22.8\ N)(16.2\ N)} = -0.439 \Rightarrow \varphi = \cos^{-1}(-0.439) = 116.0^o. \tag{3.8.16}$$

Significance

Notice that when vectors are given in terms of the unit vectors of axes, we can find the angle between them without knowing the specifics about the geographic directions the unit vectors represent. Here, for example, the +x-direction might be to the east and the +y-direction might be to the north. But, the angle between the forces in the problem is the same if the +x-direction is to the west and the +y-direction is to the south.

? Exercise 2.13

Find the angle between forces $\vec{F}_1$ and $\vec{F}_3$ in Example 3.8.2.

✓ Example 3.8.3: The Work of a Force

When force $\vec{F}$ pulls on an object and when it causes its displacement $\vec{D}$, we say the force performs work. The amount of work the force does is the scalar product $\vec{F} \cdot \vec{D}$. If the stick in Example 3.8.2 moves momentarily and gets displaced by vector $\vec{D}$ = (−7.9 $\hat{j}$ − 4.2 $\hat{k}$)cm, how much work is done by the third dog in Example 3.8.2?

Strategy

We compute the scalar product of displacement vector $\vec{D}$ with force vector $\vec{F}_3$ = (5.0 $\hat{i}$ + 12.5 $\hat{j}$)N, which is the pull from the third dog. Let's use W_3 to denote the work done by force $\vec{F}_3$ on displacement $\vec{D}$.

Solution

Calculating the work is a straightforward application of the dot product:

$$\begin{aligned} W_3 &= \vec{F}_3 \cdot \vec{D} = F_{3x}D_x + F_{3y}D_y + F_{3z}D_z \\ &= (5.0\ N)(0.0\ cm) + (12.5\ N)(-7.9\ cm) + (0.0\ N)(-4.2\ cm) \\ &= -98.7\ N \cdot cm. \end{aligned}$$

Significance

The SI unit of work is called the joule (J) , where 1 J = 1 N · m. The unit cm · N can be written as 10^{-2} m · N = 10^{-2} J, so the answer can be expressed as W_3 = −0.9875 J ≈ −1.0 J.

? Exercise 2.14

How much work is done by the first dog and by the second dog in Example 3.8.2 on the displacement in Example 3.8.3?

Contributors and Attributions

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3.9: Products of Vectors (Part 2)

The Vector Products of Two Vectors (the Cross Product)

Vector multiplication of two vectors yields a vector product.

Vector Product (Cross Product)

The vector product of two vectors $\vec{A}$ and $\vec{B}$ is denoted by $\vec{A} \times \vec{B}$ and is often referred to as a cross product. The vector product is a vector that has its direction perpendicular to both vectors $\vec{A}$ and $\vec{B}$. In other words, vector $\vec{A} \times \vec{B}$ is perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$, as shown in Figure $3.9.1$. The magnitude of the vector product is defined as

$$|\vec{A} \times \vec{B}| = AB \sin \varphi, \tag{3.9.1}$$

where angle φ, between the two vectors, is measured from vector $\vec{A}$ (first vector in the product) to vector $\vec{B}$ (second vector in the product), as indicated in Figure $3.9.1$, and is between 0° and 180°.

According to Equation $3.9.1$, the vector product vanishes for pairs of vectors that are either parallel (φ = 0°) or antiparallel (φ = 180°) because sin 0° = sin 180° = 0.

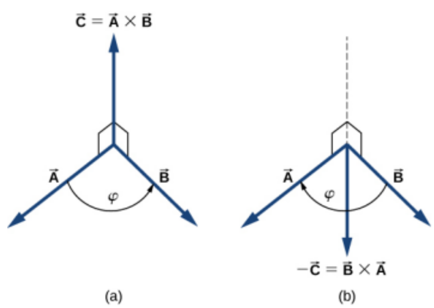

Figure $3.9.1$: The vector product of two vectors is drawn in three-dimensional space. (a) The vector product $\vec{A} \times \vec{B}$ is a vector perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$. Small squares drawn in perspective mark right angles between $\vec{A}$ and $\vec{C}$, and between $\vec{B}$ and $\vec{C}$ so that if $\vec{A}$ and $\vec{B}$ lie on the floor, vector $\vec{B}$ points vertically upward to the ceiling. (b) The vector product $\vec{B} \times \vec{A}$ is a vector antiparallel to vector $\vec{A} \times \vec{B}$.

On the line perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$ there are two alternative directions—either up or down, as shown in Figure $3.9.1$—and the direction of the vector product may be either one of them. In the standard right-handed orientation, where the angle between vectors is measured counterclockwise from the first vector, vector $\vec{A} \times \vec{B}$ points **upward**, as seen in Figure $3.9.1$(a). If we reverse the order of multiplication, so that now $\vec{B}$ comes first in the product, then vector $\vec{B} \times \vec{A}$ must point **downward**, as seen in Figure $3.9.1$(b). This means that vectors $\vec{A} \times \vec{B}$ and $\vec{B} \times \vec{A}$ are **antiparallel** to each other and that vector multiplication is **not** commutative but **anticommutative**. The **anticommutative property** means the vector product reverses the sign when the order of multiplication is reversed:

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}. \tag{3.9.2}$$

The **corkscrew right-hand rule** is a common mnemonic used to determine the direction of the vector product. As shown in Figure $3.9.2$, a corkscrew is placed in a direction perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$, and its handle is turned in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew.

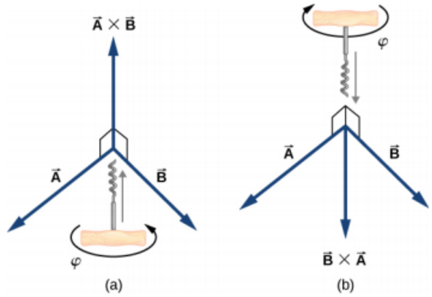

Figure 3.9.2: The corkscrew right-hand rule can be used to determine the direction of the cross product $\vec{A} \times \vec{B}$. Place a corkscrew in the direction perpendicular to the plane that contains vectors $\vec{A}$ and $\vec{B}$, and turn it in the direction from the first to the second vector in the product. The direction of the cross product is given by the progression of the corkscrew. (a) Upward movement means the cross-product vector points up. (b) Downward movement means the cross-product vector points downward.

Example 3.9.1: The Torque of a Force

The mechanical advantage that a familiar tool called a **wrench** provides (Figure 3.9.3) depends on magnitude F of the applied force, on its direction with respect to the wrench handle, and on how far from the nut this force is applied. The distance R from the nut to the point where force vector $\vec{F}$ is attached is called the **lever arm** and is represented by the radial vector $\vec{R}$. The physical vector quantity that makes the nut turn is called **torque** (denoted by $\vec{\tau}$), and it is the vector product of the lever arm with the force: $\vec{\tau} = \vec{R} \times \vec{F}$.

To loosen a rusty nut, a 20.00-N force is applied to the wrench handle at angle φ = 40° and at a distance of 0.25 m from the nut, as shown in Figure 3.9.3(a). Find the magnitude and direction of the torque applied to the nut. What would the magnitude and direction of the torque be if the force were applied at angle φ = 45°, as shown in Figure 3.9.3(b)? For what value of angle φ does the torque have the largest magnitude?

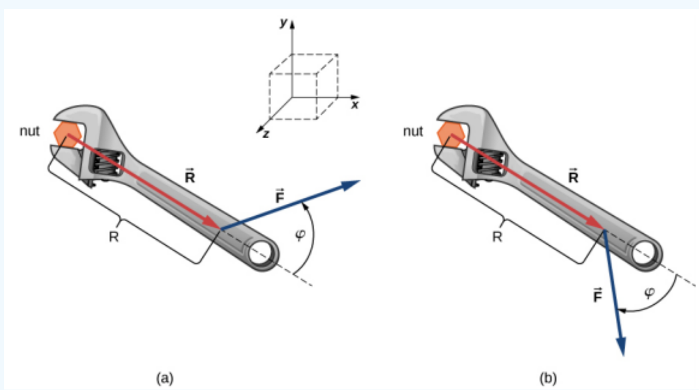

Figure 3.9.3: A wrench provides grip and mechanical advantage in applying torque to turn a nut. (a) Turn counterclockwise to loosen the nut. (b) Turn clockwise to tighten the nut.

Strategy

We adopt the frame of reference shown in Figure 3.9.3, where vectors $\vec{R}$ and $\vec{F}$ lie in the xy-plane and the origin is at the position of the nut. The radial direction along vector $\vec{R}$ (pointing away from the origin) is the reference direction for measuring the angle φ because $\vec{R}$ is the first vector in the vector product $\vec{\tau} = \vec{R} \times \vec{F}$. Vector $\vec{\tau}$ must lie along the z-axis because this is the axis that is perpendicular to the xy-plane, where both $\vec{R}$ and $\vec{F}$ lie. To compute the magnitude τ, we use Equation 3.9.1. To find the direction of $\vec{\tau}$, we use the corkscrew right-hand rule (Figure 3.9.2).

Solution

For the situation in (a), the corkscrew rule gives the direction of $\vec{R} \times \vec{F}$ in the positive direction of the z-axis. Physically, it means the torque vector $\vec{\tau}$ points out of the page, perpendicular to the wrench handle. We identify F = 20.00 N and R = 0.25 m, and compute the magnitude using Equation 3.9.1:

$$\tau = |\vec{R} \times \vec{F}| = RF \sin\varphi = (0.25\ m)(20.00\ N) \sin 40^o = 3.21\ N \cdot m. \tag{3.9.3}$$

For the situation in (b), the corkscrew rule gives the direction of $\vec{R} \times \vec{F}$ in the negative direction of the z-axis. Physically, it means the vector $\vec{\tau}$ points into the page, perpendicular to the wrench handle. The magnitude of this torque is

$$\tau = |\vec{R} \times \vec{F}| = RF \sin \varphi = (0.25\ m)(20.00\ N) \sin 45^{o} = 3.53\ N \cdot m. \tag{3.9.4}$$

The torque has the largest value when sin φ = 1, which happens when φ = 90°. Physically, it means the wrench is most effective—giving us the best mechanical advantage—when we apply the force perpendicular to the wrench handle. For the situation in this example, this best-torque value is τ_{best} = RF = (0.25 m)(20.00 N) = 5.00 N • m.

Significance

When solving mechanics problems, we often do not need to use the corkscrew rule at all, as we'll see now in the following equivalent solution. Notice that once we have identified that vector $\vec{R} \times \vec{F}$ lies along the z-axis, we can write this vector in terms of the unit vector $\hat{k}$ of the z-axis:

$$\vec{R} \times \vec{F} = RF \sin \varphi \hat{k}. \tag{3.9.5}$$

In this equation, the number that multiplies $\hat{k}$ is the scalar z-component of the vector $\vec{R} \times \vec{F}$. In the computation of this component, care must be taken that the angle φ is measured counterclockwise from $\vec{R}$ (first vector) to $\vec{F}$ (second vector) Following this principle for the angles, we obtain RF sin (+ 40°) = + 3.2 N • m for the situation in (a), and we obtain RF sin (−45°) = −3.5 N • m for the situation in (b). In the latter case, the angle is negative because the graph in Figure 3.9.3 indicates the angle is measured clockwise; but, the same result is obtained when this angle is measured counterclockwise because +(360° − 45°) = + 315° and sin (+ 315°) = sin (−45°). In this way, we obtain the solution without reference to the corkscrew rule. For the situation in (a), the solution is $\vec{R} \times \vec{F}$ = + 3.2 N • m $\hat{k}$; for the situation in (b), the solution is $\vec{R} \times \vec{F}$ = −3.5 N • m $\hat{k}$.

? Exercise 2.15

For the vectors given in Figure 2.3.6, find the vector products $\vec{A} \times \vec{B}$ and $\vec{C} \times \vec{F}$.

Similar to the dot product (Equation 2.8.10), the cross product has the following distributive property:

$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}. \tag{3.9.6}$$

The distributive property is applied frequently when vectors are expressed in their component forms, in terms of unit vectors of Cartesian axes. When we apply the definition of the cross product, Equation 3.9.1, to unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ that define the positive x-, y-, and z-directions in space, we find that

$$\hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = 0. \tag{3.9.7}$$

All other cross products of these three unit vectors must be vectors of unit magnitudes because $\hat{i}$, $\hat{j}$, and $\hat{k}$ are orthogonal. For example, for the pair $\hat{i}$ and $\hat{j}$, the magnitude is $|\hat{i} \times \hat{j}|$ = ij sin 90° = (1)(1)(1) = 1. The direction of the vector product $\hat{i} \times \hat{j}$ must be orthogonal to the xy-plane, which means it must be along the z-axis. The only unit vectors along the z-axis are $-\hat{k}$ or $+\hat{k}$. By the corkscrew rule, the direction of vector $\hat{i} \times \hat{j}$ must be parallel to the positive z-axis. Therefore, the result of the multiplication $\hat{i} \times \hat{j}$ is identical to + $\hat{k}$. We can repeat similar reasoning for the remaining pairs of unit vectors. The results of these multiplications are

$$\begin{cases} \hat{i} \times \hat{j} = +\hat{k}, \\ \hat{j} \times \hat{k} = +\hat{i}, \\ \hat{k} \times \hat{i} = +\hat{j}. \end{cases} \tag{3.9.8}$$

Notice that in Equation 3.9.8, the three unit vectors $\hat{i}$, $\hat{j}$, and $\hat{k}$ appear in the cyclic order shown in a diagram in Figure 3.9.4(a). The cyclic order means that in the product formula, $\hat{i}$ follows $\hat{k}$ and comes before $\hat{j}$, or $\hat{k}$ follows $\hat{j}$ and comes before $\hat{i}$, or $\hat{j}$ follows $\hat{i}$ and comes before $\hat{k}$. The cross product of two different unit vectors is always a third unit vector. When two unit vectors in the cross product appear in the cyclic order, the result of such a multiplication is the remaining unit vector, as illustrated in Figure 3.9.4(b). When unit vectors in the cross product appear in a different order, the result is a unit vector that is antiparallel to the remaining unit vector (i.e., the result is with the minus sign, as shown by the examples in Figure 3.9.4(c) and Figure 3.9.4(d).

In practice, when the task is to find cross products of vectors that are given in vector component form, this rule for the cross-multiplication of unit vectors is very useful.

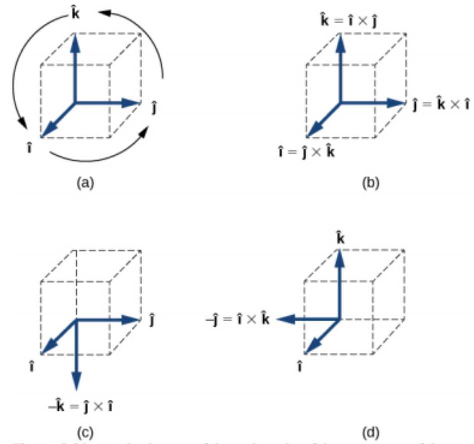

Figure 3.9.4: (a) The diagram of the cyclic order of the unit vectors of the axes. (b) The only cross products where the unit vectors appear in the cyclic order. These products have the positive sign. (c, d) Two examples of cross products where the unit vectors do not appear in the cyclic order. These products have the negative sign.

Suppose we want to find the cross product $\vec{A} \times \vec{B}$ for vectors $\vec{A} = A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}$ and $\vec{B} = B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}$. We can use the distributive property (Equation 3.9.6), the anticommutative property (Equation 3.9.2), and the results in Equation 3.9.7 and Equation 3.9.8 for unit vectors to perform the following algebra:

$$\begin{aligned}\vec{A} \times \vec{B} &= (A_x\,\hat{i} + A_y\,\hat{j} + A_z\,\hat{k}) \times (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) \\ &= A_x\,\hat{i} \times (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) + A_y\,\hat{j} \times (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) + A_z\,\hat{k} \times (B_x\,\hat{i} + B_y\,\hat{j} + B_z\,\hat{k}) \\ &= A_xB_x\,\hat{i} \times \hat{i} + A_xB_y\,\hat{i} \times \hat{j} + A_xB_z\,\hat{i} \times \hat{k} \\ &+ A_yB_x\,\hat{j} \times \hat{i} + A_yB_y\,\hat{j} \times \hat{j} + A_yB_z\,\hat{j} \times \hat{k} \\ &+ A_zB_x\,\hat{k} \times \hat{i} + A_zB_y\,\hat{k} \times \hat{j} + A_zB_z\,\hat{k} \times \hat{k} \\ &= A_xB_x(0) + A_xB_y(+\hat{k}) + A_xB_z(-\hat{j}) \\ &+ A_yB_x(-\hat{k}) + A_yB_y(0) + A_yB_z(+\hat{i}) \\ &+ A_zB_x(+\hat{j}) + A_zB_y(-\hat{i}) + A_zB_z(0).\end{aligned}$$

When performing algebraic operations involving the cross product, be very careful about keeping the correct order of multiplication because the cross product is anticommutative. The last two steps that we still have to do to complete our task are, first, grouping the terms that contain a common unit vector and, second, factoring. In this way we obtain the following very useful expression for the computation of the cross product:

$$\vec{C} = \vec{A} \times \vec{B} = (A_yB_z - A_zB_y)\,\hat{i} + (A_zB_x - A_xB_z)\,\hat{j} + (A_xB_y - A_yB_x)\,\hat{k}. \tag{3.9.9}$$

In this expression, the scalar components of the cross-product vector are

$$\begin{cases} C_x = A_yB_z - A_zB_y, \\ C_y = A_zB_x - A_xB_z, \\ C_z = A_xB_y - A_yB_x. \end{cases} \tag{3.9.10}$$

When finding the cross product, in practice, we can use either Equation 3.9.1 or Equation 3.9.9, depending on which one of them seems to be less complex computationally. They both lead to the same final result. One way to make sure if the final result is correct is to use them both.

✓ Example 3.9.2: A Particle in a Magnetic Field

When moving in a magnetic field, some particles may experience a magnetic force. Without going into details—a detailed study of magnetic phenomena comes in later chapters—let's acknowledge that the magnetic field $\vec{B}$ is a vector, the magnetic force $\vec{F}$ is a vector, and the velocity $\vec{u}$ of the particle is a vector. The magnetic force vector is proportional to the vector product of the velocity vector with the magnetic field vector, which we express as $\vec{F} = \zeta \vec{u} \times \vec{B}$. In this equation, a constant ζ takes care of the consistency in physical units, so we can omit physical units on vectors $\vec{u}$ and $\vec{B}$. In this example, let's assume the constant ζ is positive. A particle moving in space with velocity vector $\vec{u} = -5.0\,\hat{i} - 2.0\,\hat{j} + 3.5\,\hat{k}$ enters a region with a magnetic field and experiences a magnetic force. Find the magnetic force $\vec{F}$ on this particle at the entry point to the region where the magnetic field vector is (a) $\vec{B} = 7.2\,\hat{i} - \hat{j} - 2.4\,\hat{k}$ and (b) $\vec{B} = 4.5\,\hat{k}$. In each case, find magnitude F of the magnetic force and angle θ the force vector $\vec{F}$ makes with the given magnetic field vector $\vec{B}$.

Strategy

First, we want to find the vector product $\vec{u} \times \vec{B}$, because then we can determine the magnetic force using $\vec{F} = \zeta \vec{u} \times \vec{B}$. Magnitude F can be found either by using components, F = $\sqrt{F_x^2 + F_y^2 + F_z^2}$, or by computing the magnitude $|\vec{u} \times \vec{B}|$ directly using Equation 3.9.1. In the latter approach, we would have to find the angle between vectors $\vec{u}$ and $\vec{B}$. When we have $\vec{F}$, the general method for finding the direction angle θ involves the computation of the scalar product $\vec{F} \cdot \vec{B}$ and substitution into Equation 2.8.13. To compute the vector product we can either use Equation 3.9.9 or compute the product directly, whichever way is simpler.

Solution

The components of the velocity vector are $u_x = -5.0$, $u_y = -2.0$, and $u_z = 3.5$. (a) The components of the magnetic field vector are $B_x = 7.2$, $B_y = -1.0$, and $B_z = -2.4$. Substituting them into Equation 3.9.10 gives the scalar components of vector $\vec{F} = \zeta \vec{u} \times \vec{B}$:

$$\begin{cases} F_x = \zeta(u_y B_z - u_z B_y) = \zeta[(-2.0)(-2.4) - (3.5)(-1.0)] = 8.3\zeta \\ F_y = \zeta(u_z B_x - u_x B_z) = \zeta[(3.5)(7.2) - (-5.0)(-2.4)] = 13.2\zeta \\ F_z = \zeta(u_x B_y - u_y B_x) = \zeta[(-5.0)(-1.0) - (-2.0)(7.2)] = 19.4\zeta \end{cases} \tag{3.9.11}$$

Thus, the magnetic force is $\vec{F} = \zeta(8.3\,\hat{i} + 13.2\,\hat{j} + 19.4\,\hat{k})$ and its magnitude is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta\sqrt{(8.3)^2 + (13.2)^2 + (19.4)^2} = 24.9\zeta. \tag{3.9.12}$$

To compute angle θ, we may need to find the magnitude of the magnetic field vector

$$B = \sqrt{B_x^2 + B_y^2 + B_z^2} = \sqrt{(7.2)^2 + (-1.0)^2 + (-2.4)^2} = 7.6, \tag{3.9.13}$$

and the scalar product $\vec{F} \cdot \vec{B}$:

$$\vec{F} \cdot \vec{B} = F_x B_x + F_y B_y + F_z B_z = (8.3\zeta)(7.2) + (13.2\zeta)(-1.0) + (19.4\zeta)(-2.4) =. \tag{3.9.14}$$

Now, substituting into Equation 2.8.13 gives angle θ:

$$\cos\theta = \frac{\vec{F} \cdot \vec{B}}{FB} = \frac{0}{(18.2\zeta)(7.6)} = 0 \Rightarrow \theta = 90^o. \tag{3.9.15}$$

Hence, the magnetic force vector is perpendicular to the magnetic field vector. (We could have saved some time if we had computed the scalar product earlier.)

(b) Because vector $\vec{B} = 4.5\,\hat{k}$ has only one component, we can perform the algebra quickly and find the vector product directly:

$$\begin{aligned} \vec{F} &= \zeta \vec{u} \times \vec{B} = \zeta(-5.0\hat{i} - 2.0\hat{j} + 3.5\hat{k}) \times (4.5\hat{k}) \\ &= \zeta[(-5.0)(4.5)\hat{i} \times \hat{k} + (-2.0)(4.5)\hat{j} \times \hat{k} + (3.5)(4.5)\hat{k} \times \hat{k}] \\ &= \zeta[-22.5(-\hat{j}) - 9.0(+\hat{i}) + 0] = \zeta(-9.0\hat{i} + 22.5\hat{j}). \end{aligned}$$

The magnitude of the magnetic force is

$$F = \sqrt{F_x^2 + F_y^2 + F_z^2} = \zeta\sqrt{(-9.0)^2 + (22.5)^2 + (0.0)^2} = 24.2\zeta. \tag{3.9.16}$$

Because the scalar product is

$$\vec{F}\cdot\vec{B} = F_xB_x + F_yB_y + F_zB_z = (-9.0\zeta)(90) + (22.5\zeta)(0) + (0)(4.5) = 0, \tag{3.9.17}$$

the magnetic force vector $\vec{F}$ is perpendicular to the magnetic field vector $\vec{B}$.

Significance

Even without actually computing the scalar product, we can predict that the magnetic force vector must always be perpendicular to the magnetic field vector because of the way this vector is constructed. Namely, the magnetic force vector is the vector product $\vec{F} = \zeta\vec{u}\times\vec{B}$ and, by the definition of the vector product (see Figure 3.9.1), vector $\vec{F}$ must be perpendicular to both vectors $\vec{u}$ and $\vec{B}$.

? Exercise 2.16

Given two vectors $\vec{A} = -\hat{i} + \hat{j}$ and $\vec{B} = 3\,\hat{i} - \hat{j}$, find (a) $\vec{A}\times\vec{B}$, (b) $|\vec{A}\times(\vec{B}|$, (c) the angle between $\vec{A}$ and $\vec{B}$, and (d) the angle between $\vec{A}\times\vec{B}$ and vector $\vec{C} = \hat{i} + \hat{k}$.

In conclusion to this section, we want to stress that “dot product” and “cross product” are entirely different mathematical objects that have different meanings. The dot product is a scalar; the cross product is a vector. Later chapters use the terms **dot product** and **scalar product** interchangeably. Similarly, the terms **cross product** and **vector product** are used interchangeably.

Contributors

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3.10: Vectors (Answers)

Check Your Understanding

2.1. a. not equal because they are orthogonal;

b. not equal because they have different magnitudes;

c. not equal because they have different magnitudes and directions;

d. not equal because they are antiparallel;

e. equal.

2.2. 16 m; $\vec{D} = -16m\hat{u}$

2.3. G = 28.2 cm, $\theta_G = 291^\circ$

2.4. $\vec{D} = (-5.0\hat{i} - 3.0\hat{j})cm$; the fly moved 5.0 cm to the left and 3.0 cm down from its landing site.

2.5. 5.83 cm, 211°

2.6. $\vec{D} = (-20m)\hat{j}$

2.7. 35.1 m/s = 126.4 km/h

2.8. $\vec{G} = (10.25\hat{i} - 26.22\hat{j})cm$

2.9. **D =** 55.7 N; direction 65.7° north of east

2.10. $\hat{v} = 0.8\hat{i} + 0.6\hat{j}, 36.87^\circ$ north of east

2.11. $\vec{A} \cdot \vec{B} = -57.3, \vec{F} \cdot \vec{C} = 27.8$

2.13. 131.9°

2.14. $W_1 = 1.5J, W_2 = 0.3J$

2.15. $\vec{A} \times \vec{B} = -40.1\hat{k}$ or, equivalently, $\left|\vec{A} \times \vec{B}\right| = 40.1$, and the direction is into the page; $\vec{C} \times \vec{F} = +157.6\hat{k}$ or, equivalently, $\left|\vec{C} \times \vec{F}\right| = 157.6$, and the direction is out of the page.

2.16. a. $-2\hat{k}$,

b. 2,

c. 153.4°,

d. 135°

Conceptual Questions

1. scalar

3. answers may vary

5. parallel, sum of magnitudes, antiparallel, zero

7. no, yes

9. zero, yes

11. no

13. equal, equal, the same

15. a unit vector of the **x**-axis

17. They are equal.

19. yes

21. a. $C = \vec{A} \cdot \vec{B}$

b. $\vec{C} = \vec{A} \cdot \vec{B}$ or $\vec{C} = \vec{A} - \vec{B}$

c. $\vec{C} = \vec{A} \times \vec{B}$,

d. $\vec{C} = A\vec{B}$,

e. $\vec{C} + 2\vec{A} = \vec{B}$,

f. $\vec{C} = \vec{A} \times \vec{B}$,

g. left side is a scalar and right side is a vector,

h. $\vec{C} = 2\vec{A} \times \vec{B}$,

i. $\vec{C} = \vec{A}/B$,

j. $\vec{C} = \vec{A}/B$

23. They are orthogonal.

Problems

25. $\vec{h} = -49m\hat{u}$, 49 m

27. 30.8 m, 35.7° west of north

29. 134 km, 80°

31. 7.34 km, 63.5° south of east

33. 3.8 km east, 3.2 km north, 7.0 km

35. 14.3 km, 65°

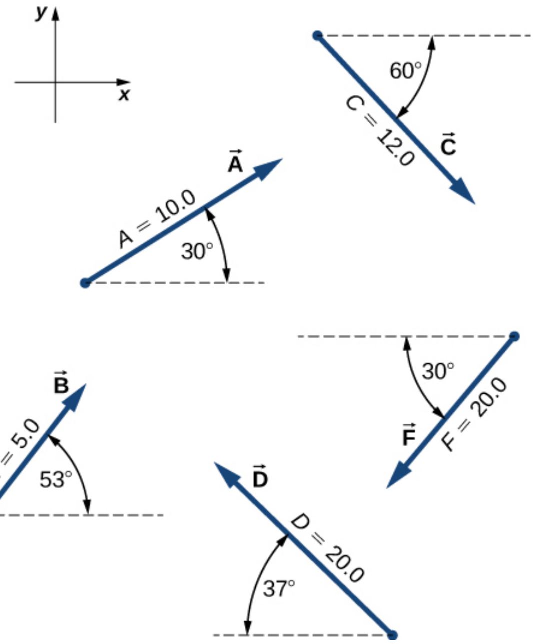
37. a. $\vec{A} = +8.66\hat{i} + 5.00\hat{j}$,

b. $\vec{B} = +3.01\hat{i} + 3.99\hat{j}$,

c. $\vec{C} = +6.00\hat{i} - 10.39\hat{j}$,

d. $\overrightarrow{D} = -15.97\hat{i} + 12.04\hat{j}$,

f. $\vec{F} = -17.32\hat{i} - 10.00\hat{j}$



39. a. 1.94 km, 7.24 km;

b. proof

41. 3.8 km east, 3.2 km north, 2.0 km, \(\vec{D}=(3.8\hat{i}+3.2\hat{j})km

43. $P_1(2.165m, 1.250m), P_2(-1.900m, 3.290m), 5.27m$

45. 8.60 m, $A(2\sqrt{5}m, 0.647\pi), B(3\sqrt{2}m, 0.75\pi)$

47. a. $\vec{A}+\vec{B}=-4\hat{i}-6\hat{j}$, $|\vec{A}+\vec{B}|=7.211, \theta=236.3^\circ$;

b. $\vec{A}-\vec{B}=-2\hat{i}+2\hat{j}, |\vec{A}-\vec{B}|=2\sqrt{2}, \theta=135^\circ$

49. a. $\vec{C}=(5.0\hat{i}-1.0\hat{j}-3.0\hat{k})m, C=5.92m$;

b. $\vec{D}=(4.0\hat{i}-11.0\hat{j}+15.0\hat{k})m, D=19.03m$.

51. $\vec{D}=(3.3\hat{i}-6.6\hat{j})km, \hat{i} is to the east, 7.34km, -63.5^\circ$

53. a.$\vec{R}=-1.35\hat{i}-22.04\hat{j}$,

b. $\vec{R}=-17.98\hat{i}+0.89\hat{j}$

55. $\vec{D}=(200\hat{i}+300\hat{j})yd$, D = 360.5 yd, 56.3° north of east; The numerical answers would stay the same but the physical unit would be meters. The physical meaning and distances would be about the same because 1 yd is comparable with 1 m.

57. $\vec{R}=-3\hat{i}-16\hat{j}$

59. $\overrightarrow{E}=E\hat{E}, E_x=+178.9V/m, E_y=-357.8V/m, E_z=0.0V/m, \theta_E=-tan^{-1}(2)$

61. a. $\vec{R}_B=(12.278\hat{i}+7.089\hat{j}+2.500\hat{k})km$, \($\vec{R}_D=(-0.262\hat{i}+3.000\hat{k})km$;

b. $\left|\vec{R}_B-\vec{R}_D\right|=14.414km$|R→B−R→D|=14.414km

63. a. 8.66,

b. 10.39,

c. 0.866,

d. 17.32

65. $\theta_i=64.12^\circ, \theta_j=150.79^\circ, \theta_k=77.39^\circ$

67. a. $-119.98\hat{k}$

b. $0\hat{k}$,

c. $+93.69\hat{k}$,

d. $-240.0\hat{k}$,

e. $+3.993\hat{k}$,

f. $-3.009\hat{k}$,

g. $+14.99\hat{k}$,

h. 0

69. a. 0,

b. 173,194,

c. +199,993$\hat{k}$

Additional Problems

71. a. 18.4 km and 26.2 km,

b. 31.5 km and 5.56 km

73. a. $(r, \varphi + \pi/2)$,

b. $(2r, \varphi + 2\pi)$(,

c. $(3r, -\varphi)$

75. $d_{PM} = 33.12nmi = 61.34km, d_{NP} = 35.47nmi = 65.69km$

77. proof

79. a. 10.00 m,

b. $5\pi m$,

c. 0

81. 22.2 km/h, 35.8°. south of west

83. 240.2 m, 2.2° south of west

85. $\vec{B} = -4.0\hat{i} + 3.0\hat{j}$ or $\vec{B} = 4.0\hat{i} - 3.0\hat{j}$

87. proof

Challenge Problems

89. $G_\perp = 2375\sqrt{17} \approx 9792$

91. proof

3.11: Vectors (Exercises)

Conceptual Questions

2.1 Scalars and Vectors

1. A weather forecast states the temperature is predicted to be −5 °C the following day. Is this temperature a vector or a scalar quantity? Explain.
2. Which of the following is a vector: a person's height, the altitude on Mt. Everest, the velocity of a fly, the age of Earth, the boiling point of water, the cost of a book, Earth's population, or the acceleration of gravity?
3. Give a specific example of a vector, stating its magnitude, units, and direction.
4. What do vectors and scalars have in common? How do they differ?
5. Suppose you add two vectors $\vec{A}$ and $\vec{B}$. What relative direction between them produces the resultant with the greatest magnitude? What is the maximum magnitude? What relative direction between them produces the resultant with the smallest magnitude? What is the minimum magnitude?
6. Is it possible to add a scalar quantity to a vector quantity?
7. Is it possible for two vectors of different magnitudes to add to zero? Is it possible for three vectors of different magnitudes to add to zero? Explain.
8. Does the odometer in an automobile indicate a scalar or a vector quantity?
9. When a 10,000-m runner competing on a 400-m track crosses the finish line, what is the runner's net displacement? Can this displacement be zero? Explain.
10. A vector has zero magnitude. Is it necessary to specify its direction? Explain.
11. Can a magnitude of a vector be negative?
12. Can the magnitude of a particle's displacement be greater that the distance traveled?
13. If two vectors are equal, what can you say about their components? What can you say about their magnitudes? What can you say about their directions?
14. If three vectors sum up to zero, what geometric condition do they satisfy?

2.2 Coordinate Systems and Components of a Vector

15. Give an example of a nonzero vector that has a component of zero.
16. Explain why a vector cannot have a component greater than its own magnitude.
17. If two vectors are equal, what can you say about their components?
18. If vectors $\vec{A}$ and $\vec{B}$ are orthogonal, what is the component of $\vec{B}$ along the direction of $\vec{A}$? What is the component of $\vec{A}$ along the direction of $\vec{B}$?
19. If one of the two components of a vector is not zero, can the magnitude of the other vector component of this vector be zero?
20. If two vectors have the same magnitude, do their components have to be the same?

2.4 Products of Vectors

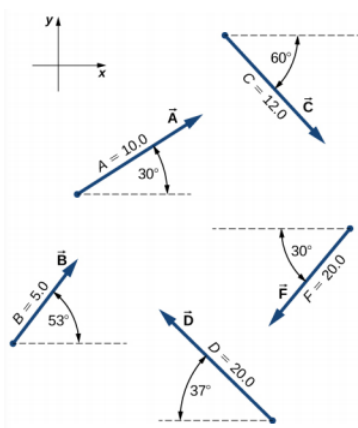
21. What is wrong with the following expressions? How can you correct them?
    a. $C = \vec{A}\vec{B}$,
    b. $\vec{C} = \vec{A}\vec{B}$,
    c. $C = \vec{A} \times \vec{B}$,
    d. $C = A\vec{B}$,
    e. $C + 2\vec{A} = B$,
    f. $\vec{C} = A \times \vec{B}$,
    g. $\vec{A} \cdot \vec{B} = \vec{A} \times \vec{B}$,
    h. $\vec{C} = 2\vec{A} \cdot \vec{B}$,
    i. $C = \vec{A}/\vec{B}$, and
    j. $C = \vec{A}/B$.
22. If the cross product of two vectors vanishes, what can you say about their directions?

23. If the dot product of two vectors vanishes, what can you say about their directions?
24. What is the dot product of a vector with the cross product that this vector has with another vector?

Problems

2.1 Scalars and Vectors

25. A scuba diver makes a slow descent into the depths of the ocean. His vertical position with respect to a boat on the surface changes several times. He makes the first stop 9.0 m from the boat but has a problem with equalizing the pressure, so he ascends 3.0 m and then continues descending for another 12.0 m to the second stop. From there, he ascends 4 m and then descends for 18.0 m, ascends again for 7 m and descends again for 24.0 m, where he makes a stop, waiting for his buddy. Assuming the positive direction up to the surface, express his net vertical displacement vector in terms of the unit vector. What is his distance to the boat?
26. In a tug-of-war game on one campus, 15 students pull on a rope at both ends in an effort to displace the central knot to one side or the other. Two students pull with force 196 N each to the right, four students pull with force 98 N each to the left, five students pull with force 62 N each to the left, three students pull with force 150 N each to the right, and one student pulls with force 250 N to the left. Assuming the positive direction to the right, express the net pull on the knot in terms of the unit vector. How big is the net pull on the knot? In what direction?
27. Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point and what is the compass direction of a line connecting your starting point to your final position? Use a graphical method.
28. For the vectors given in the following figure, use a graphical method to find the following resultants:
    a. $\vec{A}+\vec{B}$,
    b. $\vec{C}+\vec{B}$,
    c. $\vec{D}+\vec{F}$,
    d. $\vec{A}-\vec{B}$,
    e. $\vec{D}-\vec{F}$,
    f. $\vec{A}+2\vec{F}$,
    g. $\vec{A}-4\vec{D}+2\vec{F}$.



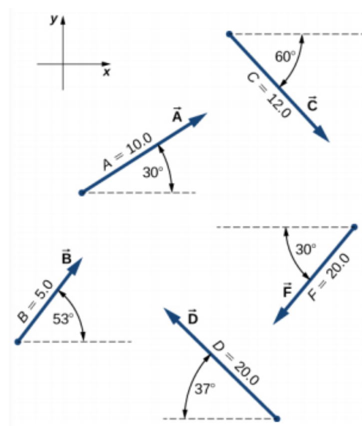
29. A delivery man starts at the post office, drives 40 km north, then 20 km west, then 60 km northeast, and finally 50 km north to stop for lunch. Use a graphical method to find his net displacement vector.
30. An adventurous dog strays from home, runs three blocks east, two blocks north, one block east, one block north, and two blocks west. Assuming that each block is about 100 m, how far from home and in what direction is the dog? Use a graphical method.
31. In an attempt to escape a desert island, a castaway builds a raft and sets out to sea. The wind shifts a great deal during the day and he is blown along the following directions: 2.50 km and 45.0° north of west, then 4.70 km and 60.0° south of east, then 1.30 km and 25.0° south of west, then 5.10 km straight east, then 1.70 km and 5.00° east of north, then 7.20 km and 55.0° south of west, and finally 2.80 km and 10.0° north of east. Use a graphical method to find the castaway's final position relative to the island.

32. A small plane flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east. Use a graphical method to find the total distance the plane covers from the starting point and the direction of the path to the final position.
33. A trapper walks a 5.0-km straight-line distance from his cabin to the lake, as shown in the following figure. Use a graphical method (the parallelogram rule) to determine the trapper's displacement directly to the east and displacement directly to the north that sum up to his resultant displacement vector. If the trapper walked only in directions east and north, zigzagging his way to the lake, how many kilometers would he have to walk to get to the lake?

34. A surveyor measures the distance across a river that flows straight north by the following method. Starting directly across from a tree on the opposite bank, the surveyor walks 100 m along the river to establish a baseline. She then sights across to the tree and reads that the angle from the baseline to the tree is 35°. How wide is the river?
35. A pedestrian walks 6.0 km east and then 13.0 km north. Use a graphical method to find the pedestrian's resultant displacement and geographic direction.
36. The magnitudes of two displacement vectors are A = 20 m and B = 6 m. What are the largest and the smallest values of the magnitude of the resultant $\vec{R} = \vec{A} + \vec{B}$?

2.2 Coordinate Systems and Components of a Vector

37. Assuming the +x-axis is horizontal and points to the right, resolve the vectors given in the following figure to their scalar components and express them in vector component form.



38. Suppose you walk 18.0 m straight west and then 25.0 m straight north. How far are you from your starting point? What is your displacement vector? What is the direction of your displacement? Assume the +x-axis is to the east.
39. You drive 7.50 km in a straight line in a direction 15° east of north. (a) Find the distances you would have to drive straight east and then straight north to arrive at the same point. (b) Show that you still arrive at the same point if the east and north legs are reversed in order. Assume the +xaxis is to the east.
40. A sledge is being pulled by two horses on a flat terrain. The net force on the sledge can be expressed in the Cartesian coordinate system as vector $\vec{F}$= (−2980.0 $\hat{i}$ + 8200.0 $\hat{j}$)N , where $\hat{i}$ and $\hat{j}$ denote directions to the east and north, respectively. Find the magnitude and direction of the pull.
41. A trapper walks a 5.0-km straight-line distance from her cabin to the lake, as shown in the following figure. Determine the east and north components of her displacement vector. How many more kilometers would she have to walk if she walked along the component displacements? What is her displacement vector?

42. The polar coordinates of a point are $\frac{4\pi}{3}$ and 5.50 m. What are its Cartesian coordinates?
43. Two points in a plane have polar coordinates P_1 (2.500 m, $\frac{\pi}{6}$) and P_2 (3.800 m, $\frac{2\pi}{3}$). Determine their Cartesian coordinates and the distance between them in the Cartesian coordinate system. Round the distance to a nearest centimeter.
44. A chameleon is resting quietly on a lanai screen, waiting for an insect to come by. Assume the origin of a Cartesian coordinate system at the lower left-hand corner of the screen and the horizontal direction to the right as the +x-direction. If its coordinates are (2.000 m, 1.000 m), (a) how far is it from the corner of the screen? (b) What is its location in polar coordinates?
45. Two points in the Cartesian plane are A(2.00 m, −4.00 m) and B(−3.00 m, 3.00 m). Find the distance between them and their polar coordinates.
46. A fly enters through an open window and zooms around the room. In a Cartesian coordinate system with three axes along three edges of the room, the fly changes its position from point b(4.0 m, 1.5 m, 2.5 m) to point e(1.0 m, 4.5 m, 0.5 m). Find the scalar components of the fly's displacement vector and express its displacement vector in vector component form. What is its magnitude?

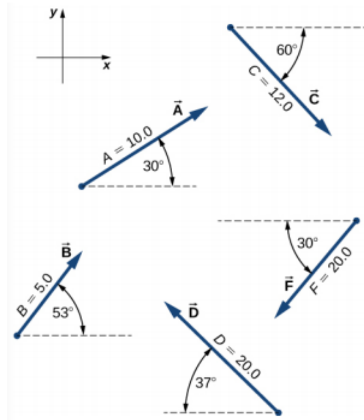
2.3 Algebra of Vectors

47. For vectors $\vec{B} = -\hat{i} - 4\hat{j}$ and $\vec{A} = -3\hat{i} - 2\hat{j}$, calculate (a) $\vec{A} + \vec{B}$ and its magnitude and direction angle, and (b) $\vec{A} - \vec{B}$ and its magnitude and direction angle.
48. A particle undergoes three consecutive displacements given by vectors $\vec{D}_1 = (3.0\,\hat{i} - 4.0\,\hat{j} - 2.0\,\hat{k})$mm, $\vec{D}_2 = (1.0\,\hat{i} - 7.0\,\hat{j} + 4.0\,\hat{i})$mm, and $\vec{D}_3 = (-7.0\,\hat{i} + 4.0\,\hat{j} + 1.0\,\hat{k})$mm. (a) Find the resultant displacement vector of the particle. (b) What is the magnitude of the resultant displacement? (c) If all displacements were along one line, how far would the particle travel?
49. Given two displacement vectors $\vec{A} = (3.00\,\hat{i} - 4.00\,\hat{j} + 4.00\,\hat{k})$m and $\vec{B} = (2.00\,\hat{i} + 3.00\,\hat{j} - 7.00\,\hat{k})$m, find the displacements and their magnitudes for (a) $\vec{C} = \vec{A} + \vec{B}$ and (b) $\vec{D} = 2\vec{A} - \vec{B}$.
50. A small plane flies 40.0 km in a direction 60° north of east and then flies 30.0 km in a direction 15° north of east. Use the analytical method to find the total distance the plane covers from the starting point, and the geographic direction of its displacement vector. What is its displacement vector?
51. . In an attempt to escape a desert island, a castaway builds a raft and sets out to sea. The wind shifts a great deal during the day, and she is blown along the following straight lines: 2.50 km and 45.0° north of west, then 4.70 km and 60.0° south of east, then 1.30 km and 25.0° south of west, then 5.10 km due east, then 1.70 km and 5.00° east of north, then 7.20 km and 55.0° south of west, and finally 2.80 km and 10.0° north of east. Use the analytical method to find the resultant vector of all her displacement vectors. What is its magnitude and direction?
52. Assuming the +x-axis is horizontal to the right for the vectors given in the following figure, use the analytical method to find the following resultants:
    a. $\vec{A} + \vec{B}$,
    b. $\vec{C} + \vec{B}$,
    c. $\vec{D} + \vec{F}$,
    d. $\vec{A} - \vec{B}$,
    e. $\vec{D} - \vec{F}$,
    f. $\vec{A} + 2\vec{F}$,
    g. $\vec{C} - 2\vec{B} + 3\vec{F}$, and
    h. $\vec{A} - 4\vec{D} + 2\vec{F}$.

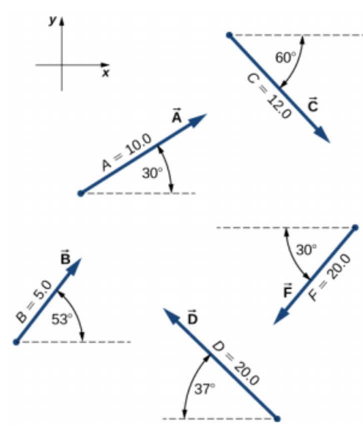
53. Given the vectors in the preceding figure, find vector $\vec{R}$ that solves equations (a) $\vec{D}+\vec{R}=\vec{F}$ and (b) $\vec{C}-2\vec{D}+5\vec{R}=3\vec{F}$. Assume the +x-axis is horizontal to the right.
54. A delivery man starts at the post office, drives 40 km north, then 20 km west, then 60 km northeast, and finally 50 km north to stop for lunch. Use the analytical method to determine the following: (a) Find his net displacement vector. (b) How far is the restaurant from the post office? (c) If he returns directly from the restaurant to the post office, what is his displacement vector on the return trip? (d) What is his compass heading on the return trip? Assume the +x-axis is to the east.
55. An adventurous dog strays from home, runs three blocks east, two blocks north, and one block east, one block north, and two blocks west. Assuming that each block is about a 100 yd, use the analytical method to find the dog's net displacement vector, its magnitude, and its direction. Assume the +x-axis is to the east. How would your answer be affected if each block was about 100 m?
56. If $\vec{D}$ = (6.00 $\hat{i}$ − 8.00 $\hat{j}$m, $\vec{B}$ = (−8.00 $\hat{i}$ + 3.00 $\hat{j}$)m , and $\vec{A}$ = (26.0 $\hat{i}$ + 19.0 $\hat{j}$)m, find the unknown constants a and b such that a $\vec{D}+b\vec{B}+\vec{A}=\vec{0}$.
57. Given the displacement vector $\vec{D}$ = (3 $\hat{i}$ − 4 $\hat{j}$)m, find the displacement vector $\vec{R}$ so that $\vec{D}$ + $\vec{R}$ = −4D $\hat{j}$.
58. Find the unit vector of direction for the following vector quantities: (a) Force $\vec{F}$ = (3.0 $\hat{i}$ − 2.0 $\hat{j}$)N, (b) displacement $\vec{D}$ = (−3.0 $\hat{i}$ − 4.0 $\hat{j}$)m, and (c) velocity $\vec{v}$ = (−5.00 $\hat{i}$ + 4.00 $\hat{j}$)m/s.
59. At one point in space, the direction of the electric field vector is given in the Cartesian system by the unit vector $\hat{E}=\frac{1}{\sqrt{5}}\hat{i}-\frac{2}{\sqrt{5}}\hat{j}$. If the magnitude of the electric field vector is E = 400.0 V/m, what are the scalar components E_x, E_y, and E_z of the electric field vector $\vec{E}$ at this point? What is the direction angle θ_E of the electric field vector at this point?
60. A barge is pulled by the two tugboats shown in the following figure. One tugboat pulls on the barge with a force of magnitude 4000 units of force at 15° above the line AB (see the figure and the other tugboat pulls on the barge with a force of magnitude 5000 units of force at 12° below the line AB. Resolve the pulling forces to their scalar components and find the components of the resultant force pulling on the barge. What is the magnitude of the resultant pull? What is its direction relative to the line AB?
61. In the control tower at a regional airport, an air traffic controller monitors two aircraft as their positions change with respect to the control tower. One plane is a cargo carrier Boeing 747 and the other plane is a Douglas DC-3. The Boeing is at an altitude of 2500 m, climbing at 10° above the horizontal, and moving 30° north of west. The DC-3 is at an altitude of 3000 m, climbing at 5° above the horizontal, and cruising directly west. (a) Find the position vectors of the planes relative to the control tower. (b) What is the distance between the planes at the moment the air traffic controller makes a note about their positions?

2.4 Products of Vectors

62. Assuming the +x-axis is horizontal to the right for the vectors in the following figure, find the following scalar products:
    a. $\vec{A}\cdot\vec{C}$,
    b. $\vec{A}\cdot\vec{F}$,
    c. $\vec{D}\cdot\vec{C}$,
    d. $\vec{A}\cdot(\vec{F}+2\vec{C})$,
    e. $\hat{i}\cdot\vec{B}$,
    f. $\hat{j}\cdot\vec{B}$,
    g. $(3\hat{i}-\hat{j})\cdot\vec{B}$ and
    h. $\hat{B}\cdot\vec{B}$.

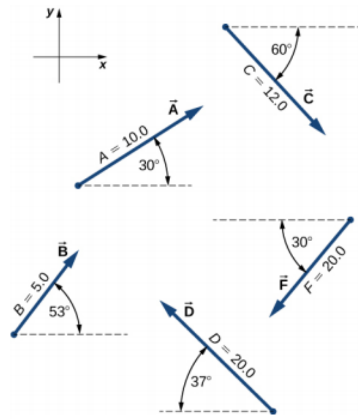


63. Assuming the +x-axis is horizontal to the right for the vectors in the preceding figure, find (a) the component of vector $\vec{A}$ along vector $\vec{C}$, (b) the component of vector $\vec{C}$ along vector $\vec{A}$, (c) the component of vector $\hat{i}$ along vector $\vec{F}$, and (d) the component of vector $\vec{F}$ along vector $\hat{i}$.
64. Find the angle between vectors for
    a. $\vec{D} = (-3.0\,\hat{i} - 4.0\,\hat{j})$m and $\vec{A} = (-3.0\,\hat{i} + 4.0\,\hat{j})$m and
    b. $\vec{D} = (2.0\,\hat{i} - 4.0\,\hat{j} + \hat{k})$m and $\vec{B} = (-2.0\,\hat{i} + 3.0\,\hat{j} + 2.0\,\hat{k})$m.
65. Find the angles that vector $\vec{D} = (2.0\,\hat{i} - 4.0\,\hat{j} + \hat{k})$m makes with the x-, y-, and z-axes.
66. Show that the force vector $\vec{D} = (2.0\,\hat{i} - 4.0\,\hat{j} + \hat{k})$N is orthogonal to the force vector $\vec{G} = (3.0\,\hat{i} + 4.0\,\hat{j} + 10.0\,\hat{k})$N.
67. Assuming the +x-axis is horizontal to the right for the vectors in the following figure, find the following vector products:
    a. $\vec{A} \times \vec{C}$,
    b. $\vec{A} \times \vec{F}$,
    c. $\vec{D} \times \vec{C}$
    d. $\vec{A} \times (\vec{F} + 2\vec{C})$,
    e. $\hat{i} \times \vec{B}$,
    f. $\hat{j} \times \vec{B}$,
    g. $(3\hat{i} - \hat{j}) \times \vec{B}$ and
    h. $\hat{B} \times \vec{B}$.



68. Find the cross product $\vec{A} \times \vec{C}$ for
    a. $\vec{A} = 2.0\,\hat{i} - 4.0\,\hat{j} + \hat{k}$ and $\vec{C} = 3.0\,\hat{i} + 4.0\,\hat{j} + 10.0\hat{k}$,
    b. $\vec{A} = 3.0\,\hat{i} + 4.0\,\hat{j} + 10.0\,\hat{k}$ and $\vec{C} = 2.0\,\hat{i} - 4.0\,\hat{j} + \hat{k}$,
    c. $\vec{A} = -3.0\,\hat{i} - 4.0\,\hat{j}$ and $\vec{C} = -3.0\,\hat{i} + 4.0\,\hat{j}$, and
    d. $\vec{C} = -2.0\,\hat{i} + 3.0\,\hat{j} + 2.0\,\hat{k}$ and $\vec{A} = -9.0\,\hat{j}$.

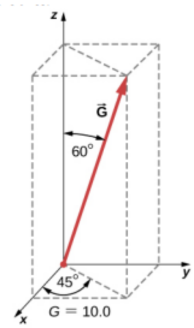
69. For the vectors in the following figure, find (a) $(\vec{A} \times \vec{F} \cdot \vec{D})$, (b) $(\vec{A} \times \vec{F}) \cdot (\vec{A} \times \vec{C})$, and (c) $(\vec{A} \cdot \vec{F})(\vec{D} \times \vec{B})$.

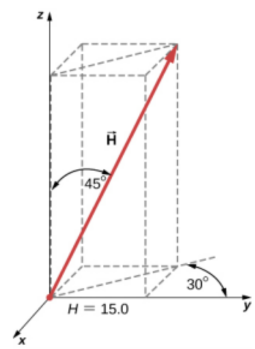


70. (a) If $\vec{A} \times \vec{F} = \vec{B} \times \vec{F}$, can we conclude $\vec{A} = \vec{B}$? (b) If $\vec{A} \cdot \vec{F} = \vec{B} \cdot \vec{F}$, can we conclude $\vec{A} = \vec{B}$? (c) If $F\vec{A} = \vec{B}F$, can we conclude $\vec{A} = \vec{B}$? Why or why not?

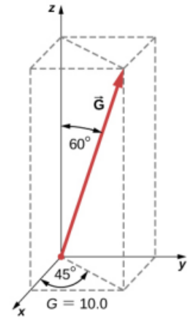
Additional Problems

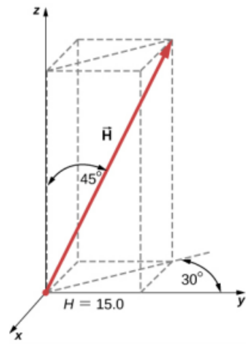
71. You fly 32.0 km in a straight line in still air in the direction 35.0° south of west. (a) Find the distances you would have to fly due south and then due west to arrive at the same point. (b) Find the distances you would have to fly first in a direction 45.0° south of west and then in a direction 45.0° west of north. Note these are the components of the displacement along a different set of axes—namely, the one rotated by 45° with respect to the axes in (a).
72. Rectangular coordinates of a point are given by (2, y) and its polar coordinates are given by (r, $\frac{\pi}{6}$). Find y and r.
73. If the polar coordinates of a point are (r,φ) and its rectangular coordinates are (x, y), determine the polar coordinates of the following points: (a) (−x, y), (b) (−2x, −2y), and (c) (3x, −3y).
74. Vectors $\vec{A}$ and $\vec{B}$ have identical magnitudes of 5.0 units. Find the angle between them if $\vec{A} + \vec{B} = 5\ 2\ \hat{j}$.
75. Starting at the island of Moi in an unknown archipelago, a fishing boat makes a round trip with two stops at the islands of Noi and Poi. It sails from Moi for 4.76 nautical miles (nmi) in a direction 37° north of east to Noi. From Noi, it sails 69° west of north to Poi. On its return leg from Poi, it sails 28° east of south. What distance does the boat sail between Noi and Poi? What distance does it sail between Moi and Poi? Express your answer both in nautical miles and in kilometers. Note: 1 nmi = 1852 m.
76. An air traffic controller notices two signals from two planes on the radar monitor. One plane is at altitude 800 m and in a 19.2-km horizontal distance to the tower in a direction 25° south of west. The second plane is at altitude 1100 m and its horizontal distance is 17.6 km and 20° south of west. What is the distance between these planes?
77. Show that when $\vec{A} + \vec{B} = \vec{C}$, then $C^2 = A^2 + B^2 + 2AB \cos \varphi$, where φ is the angle between vectors $\vec{A}$ and $\vec{B}$.
78. Four force vectors each have the same magnitude f. What is the largest magnitude the resultant force vector may have when these forces are added? What is the smallest magnitude of the resultant? Make a graph of both situations.
79. A skater glides along a circular path of radius 5.00 m in clockwise direction. When he coasts around one-half of the circle, starting from the west point, find (a) the magnitude of his displacement vector and (b) how far he actually skated. (c) What is the magnitude of his displacement vector when he skates all the way around the circle and comes back to the west point?
80. A stubborn dog is being walked on a leash by its owner. At one point, the dog encounters an interesting scent at some spot on the ground and wants to explore it in detail, but the owner gets impatient and pulls on the leash with force $\vec{F} = (98.0\ \hat{i} + 132.0\ \hat{j} + 32.0\ \hat{j})$N along the leash. (a) What is the magnitude of the pulling force? (b) What angle does the leash make with the vertical?
81. If the velocity vector of a polar bear is $\vec{u} = (-18.0\ \hat{i} - 13.0\ \hat{j})$km/h , how fast and in what geographic direction is it heading? Here, $\hat{i}$ and $\hat{j}$ are directions to geographic east and north, respectively.
82. Find the scalar components of three-dimensional vectors $\vec{G}$ and $\vec{H}$ in the following figure and write the vectors in vector component form in terms of the unit vectors of the axes.



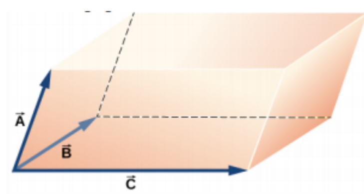


83. A diver explores a shallow reef off the coast of Belize. She initially swims 90.0 m north, makes a turn to the east and continues for 200.0 m, then follows a big grouper for 80.0 m in the direction 30° north of east. In the meantime, a local current displaces her by 150.0 m south. Assuming the current is no longer present, in what direction and how far should she now swim to come back to the point where she started?
84. A force vector $\vec{A}$ has x- and y-components, respectively, of −8.80 units of force and 15.00 units of force. The x- and y-components of force vector $\vec{B}$ are, respectively, 13.20 units of force and −6.60 units of force. Find the components of force vector $\vec{C}$ that satisfies the vector equation $\vec{A} - \vec{B} + 3\vec{C} = 0$.
85. Vectors $\vec{A}$ and $\vec{B}$ are two orthogonal vectors in the xy-plane and they have identical magnitudes. If $\vec{A} = 3.0\,\hat{i} + 4.0\,\hat{j}$, find $\vec{B}$.
86. For the three-dimensional vectors in the following figure, find (a) $\vec{G} \times \vec{H}$, (b) $|\vec{G} \times \vec{H}|$, and (c) $\vec{G} \cdot \vec{H}$.



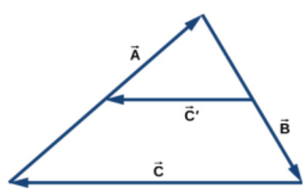


87. Show that $(\vec{B} \times \vec{C}) \cdot \vec{A}$ is the volume of the parallelepiped, with edges formed by the three vectors in the following figure.



Challenge Problems

88. Vector $\vec{B}$ is 5.0 cm long and vector $\vec{A}$ is 4.0 cm long. Find the angle between these two vectors when $|\vec{A} + \vec{B}|$ = 3.0 cm and $|\vec{A} - \vec{B}|$ = 3.0 cm.
89. What is the component of the force vector $\vec{G}$ = (3.0 $\hat{i}$ + 4.0 $\hat{j}$ + 10.0 $\hat{k}$)N along the force vector $\vec{H}$ = (1.0 $\hat{i}$ + 4.0 $\hat{j}$)N?
90. The following figure shows a triangle formed by the three vectors $\vec{A}$, $\vec{B}$ and $\vec{C}$. If vector $\vec{C}\,'$ is drawn between the midpoints of vectors $\vec{A}$ and $\vec{B}$, show that $\vec{C}\,' = \frac{\vec{C}}{2}$.



91. Distances between points in a plane do not change when a coordinate system is rotated. In other words, the magnitude of a vector is **invariant** under rotations of the coordinate system. Suppose a coordinate system S is rotated about its origin by angle φ to become a new coordinate system S′, as shown in the following figure. A point in a plane has coordinates (x, y) in S and coordinates (x′, y′) in S′.
    a. Show that, during the transformation of rotation, the coordinates in S′ are expressed in terms of the coordinates in S by the following relations:

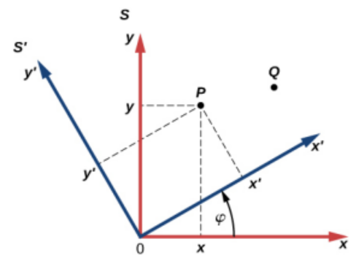
$$\begin{cases} x' = x\cos\varphi + y\sin\varphi \\ y' = -x\sin\varphi + y\cos\varphi \end{cases}. \tag{3.11.1}$$

b. Show that the distance of point P to the origin is invariant under rotations of the coordinate system. Here, you have to show that

$$\sqrt{x^2+y^2} = \sqrt{x'^2+y'^2}. \tag{3.11.2}$$

c. Show that the distance between points P and Q is invariant under rotations of the coordinate system. Here, you have to show that

$$\sqrt{(x_P-x_Q)^2+(y_P-y_Q)^2} = \sqrt{(x'_P-x'_Q)^2+(y'_P-y'_Q)^2}. \tag{3.11.3}$$



Contributors

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3.12: Vectors (Summary)

Key Terms

anticommutative property	change in the order of operation introduces the minus sign
antiparallel vectors	two vectors with directions that differ by 180°
associative	terms can be grouped in any fashion
commutative	operations can be performed in any order
component form of a vector	a vector written as the vector sum of its components in terms of unit vectors
corkscrew right-hand rule	a rule used to determine the direction of the vector product
cross product	the result of the vector multiplication of vectors is a vector called a cross product; also called a vector product
difference of two vectors	vector sum of the first vector with the vector antiparallel to the second
direction angle	in a plane, an angle between the positive direction of the x-axis and the vector, measured counterclockwise from the axis to the vector
displacement	change in position
distributive	multiplication can be distributed over terms in summation
dot product	the result of the scalar multiplication of two vectors is a scalar called a dot product; also called a scalar product
equal vectors	two vectors are equal if and only if all their corresponding components are equal; alternately, two parallel vectors of equal magnitudes
magnitude	length of a vector
null vector	a vector with all its components equal to zero
orthogonal vectors	two vectors with directions that differ by exactly 90°, synonymous with perpendicular vectors
parallel vectors	two vectors with exactly the same direction angles
parallelogram rule	geometric construction of the vector sum in a plane
polar coordinate system	an orthogonal coordinate system where location in a plane is given by polar coordinates
polar coordinates	a radial coordinate and an angle
radical coordinate	distance to the origin in a polar coordinate system
resultant vector	vector sum of two (or more) vectors
scalar	a number, synonymous with a scalar quantity in physics
scalar component	a number that multiplies a unit vector in a vector component of a vector
scalar equation	equation in which the left-hand and right-hand sides are numbers
scalar product	the result of the scalar multiplication of two vectors is a scalar called a scalar product; also called a dot product
scalar quantity	quantity that can be specified completely by a single number with an appropriate physical unit
tail-to-head geometric construction	geometric construction for drawing the resultant vector of many vectors
unit vector	vector of a unit magnitude that specifies direction; has no physical unit
unit vectors of the axes	unit vectors that define orthogonal directions in a plane or in space
vector	mathematical object with magnitude and direction
vector components	orthogonal components of a vector; a vector is the vector sum of its vector components
vector equation	equation in which the left-hand and right-hand sides are vectors
vector product	the result of the vector multiplication of vectors is a vector called a vector product; also called a cross product

vector quantity	physical quantity described by a mathematical vector—that is, by specifying both its magnitude and its direction; synonymous with a vector in physics
vector sum	resultant of the combination of two (or more) vectors

Key Equations

Multiplication by a scalar (vector equation)	$$\vec{B}=\alpha\vec{A} \tag{3.12.1}$$
Multiplication by a scalar (scalar equation for magnitudes)	$$B=\lvert\alpha\rvert A \tag{3.12.2}$$
Resultant of two vectors	$$\vec{D}_{AD}=\vec{D}_{AC}+\vec{D}_{CD} \tag{3.12.3}$$
Commutative law	$$\vec{A}+\vec{B}=\vec{B}+\vec{A} \tag{3.12.4}$$
Associative law	$$(\vec{A}+\vec{B})+\vec{C}=\vec{A}+(\vec{B}+\vec{C}) \tag{3.12.5}$$
Distributive law	$$\alpha_1\vec{A}+\alpha_2\vec{A}=(\alpha_1+\alpha_2)\vec{A} \tag{3.12.6}$$
The component form of a vector in two dimensions	$$\vec{A}=A_x\hat{i}+A_y\hat{j} \tag{3.12.7}$$
Scalar components of a vector in two dimensions	$$\begin{cases} A_x=x_e-x_b \\ A_y=y_e-y_b \end{cases} \tag{3.12.8}$$
Magnitude of a vector in a plane	$$A=\sqrt{A_x^2+A_y^2} \tag{3.12.9}$$
The direction angle of a vector in a plane	$$\theta_A=\tan^{-1}\left(\frac{A_y}{A_x}\right) \tag{3.12.10}$$
Scalar components of a vector in a plane	$$\begin{cases} A_x=A\cos\theta_A \\ A_y=A\sin\theta_A \end{cases} \tag{3.12.11}$$
Polar coordinates in a plane	$$\begin{cases} x=r\cos\varphi \\ y=r\sin\varphi \end{cases} \tag{3.12.12}$$
The component form of a vector in three dimensions	$$\vec{A}=A_x\hat{i}+A_y\hat{j}+A_z\hat{k} \tag{3.12.13}$$
The scalar z-component of a vector in three dimensions	$$A_z=z_e-z_b \tag{3.12.14}$$
Magnitude of a vector in three dimensions	$$A=\sqrt{A_x^2+A_y^2+A_z^2} \tag{3.12.15}$$
Distributive property	$$\alpha(\vec{A}+\vec{B})=\alpha\vec{A}+\alpha\vec{B} \tag{3.12.16}$$
Antiparallel vector to $\vec{A}$	$$-\vec{A}=A_x\hat{i}-A_y\hat{j}-A_z\hat{k} \tag{3.12.17}$$
Equal vectors	$$\vec{A}=\vec{B}\Leftrightarrow\begin{cases} A_x=B_x \\ A_y=B_y \\ A_z=B_z \end{cases} \tag{3.12.18}$$
Components of the resultant of N vectors	$$\begin{cases} F_{Rx}=\sum_{k=1}^{N}F_{kx}=F_{1x}+F_{2x}+\ldots+F_{Nx} \\ F_{Ry}=\sum_{k=1}^{N}F_{ky}=F_{1y}+F_{2y}+\ldots+F_{Ny} \\ F_{Rz}=\sum_{k=1}^{N}F_{kz}=F_{1z}+F_{2z}+\ldots+F_{Nz} \end{cases} \tag{3.12.19}$$

General unit vector	$$\hat{V} = \frac{\vec{V}}{V} \tag{3.12.20}$$
Definition of the scalar product	$$\vec{A} \cdot \vec{B} = AB\cos\varphi \tag{3.12.21}$$
Commutative property of the scalar product	$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} \tag{3.12.22}$$
Distributive property of the scalar product	$$\vec{A} \cdot (\vec{B} + \vec{C}) = \vec{A} \cdot \vec{B} + \vec{A} \cdot \vec{C} \tag{3.12.23}$$
Scalar product in terms of scalar components of vectors	$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z \tag{3.12.24}$$
Cosine of the angle between two vectors	$$\cos\varphi = \frac{\vec{A} \cdot \vec{B}}{AB} \tag{3.12.25}$$
Dot products of unit vectors	$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \tag{3.12.26}$$
Magnitude of the vector product (definition)	$$\lvert \vec{A} \times \vec{B} \rvert = AB\sin\varphi \tag{3.12.27}$$
Anticommutative property of the vector product	$$\lvert \vec{A} \times \vec{B} = -\vec{B} \times \vec{A} \tag{3.12.28}$$
Distributive property of the vector product	$$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C} \tag{3.12.29}$$
Cross products of unit vectors	$$\begin{cases} \hat{i} \times \hat{j} = +\hat{k}, \\ \hat{j} \times \hat{l} = +\hat{i}, \\ \hat{l} \times \hat{i} = +\hat{j}. \end{cases} \tag{3.12.30}$$
The cross product in terms of scalar components of vectors	$$\vec{A} \times \vec{B} = (A_y B_z - A_z B_y)\hat{i} + (A_z B_x - A_x B_z)\hat{j} + (A_x B_y - A_y B_x)\hat{k}$$

Summary

2.1 Scalars and Vectors

- A vector quantity is any quantity that has magnitude and direction, such as displacement or velocity.
- Geometrically, vectors are represented by arrows, with the end marked by an arrowhead. The length of the vector is its magnitude, which is a positive scalar. On a plane, the direction of a vector is given by the angle the vector makes with a reference direction, often an angle with the horizontal. The direction angle of a vector is a scalar.
- Two vectors are equal if and only if they have the same magnitudes and directions. Parallel vectors have the same direction angles but may have different magnitudes. Antiparallel vectors have direction angles that differ by 180°. Orthogonal vectors have direction angles that differ by 90°.
- When a vector is multiplied by a scalar, the result is another vector of a different length than the length of the original vector. Multiplication by a positive scalar does not change the original direction; only the magnitude is affected. Multiplication by a negative scalar reverses the original direction. The resulting vector is antiparallel to the original vector. Multiplication by a scalar is distributive. Vectors can be divided by nonzero scalars but cannot be divided by vectors.
- Two or more vectors can be added to form another vector. The vector sum is called the resultant vector. We can add vectors to vectors or scalars to scalars, but we cannot add scalars to vectors. Vector addition is commutative and associative.
- To construct a resultant vector of two vectors in a plane geometrically, we use the parallelogram rule. To construct a resultant vector of many vectors in a plane geometrically, we use the tail-to-head method.

2.2 Coordinate Systems and Components of a Vector

- Vectors are described in terms of their components in a coordinate system. In two dimensions (in a plane), vectors have two components. In three dimensions (in space), vectors have three components.
- A vector component of a vector is its part in an axis direction. The vector component is the product of the unit vector of an axis with its scalar component along this axis. A vector is the resultant of its vector components.
- Scalar components of a vector are differences of coordinates, where coordinates of the origin are subtracted from end point coordinates of a vector. In a rectangular system, the magnitude of a vector is the square root of the sum of the squares of its components.
- In a plane, the direction of a vector is given by an angle the vector has with the positive x-axis. This direction angle is measured counterclockwise. The scalar x-component of a vector can be expressed as the product of its magnitude with the cosine of its direction angle, and the scalar y-component can be expressed as the product of its magnitude with the sine of its direction angle.

- In a plane, there are two equivalent coordinate systems. The Cartesian coordinate system is defined by unit vectors $\hat{i}$ and $\hat{j}$ along the x-axis and the y-axis, respectively. The polar coordinate system is defined by the radial unit vector $\hat{r}$, which gives the direction from the origin, and a unit vector $\hat{t}$, which is perpendicular (orthogonal) to the radial direction.

2.3 Algebra of Vectors

- Analytical methods of vector algebra allow us to find resultants of sums or differences of vectors without having to draw them. Analytical methods of vector addition are exact, contrary to graphical methods, which are approximate.
- Analytical methods of vector algebra are used routinely in mechanics, electricity, and magnetism. They are important mathematical tools of physics.

2.4 Products of Vectors

- There are two kinds of multiplication for vectors. One kind of multiplication is the scalar product, also known as the dot product. The other kind of multiplication is the vector product, also known as the cross product. The scalar product of vectors is a number (scalar). The vector product of vectors is a vector.
- Both kinds of multiplication have the distributive property, but only the scalar product has the commutative property. The vector product has the anticommutative property, which means that when we change the order in which two vectors are multiplied, the result acquires a minus sign.
- The scalar product of two vectors is obtained by multiplying their magnitudes with the cosine of the angle between them. The scalar product of orthogonal vectors vanishes; the scalar product of antiparallel vectors is negative.
- The vector product of two vectors is a vector perpendicular to both of them. Its magnitude is obtained by multiplying their magnitudes by the sine of the angle between them. The direction of the vector product can be determined by the corkscrew right-hand rule. The vector product of two either parallel or antiparallel vectors vanishes. The magnitude of the vector product is largest for orthogonal vectors.
- The scalar product of vectors is used to find angles between vectors and in the definitions of derived scalar physical quantities such as work or energy.
- The cross product of vectors is used in definitions of derived vector physical quantities such as torque or magnetic force, and in describing rotations.

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CHAPTER OVERVIEW

4: Motion Along a Straight Line - with Vectors

Our universe is full of objects in motion. From the stars, planets, and galaxies; to the motion of people and animals; down to the microscopic scale of atoms and molecules—everything in our universe is in motion. We can describe motion using the two disciplines of kinematics and dynamics. We study dynamics, which is concerned with the causes of motion, in Newton's Laws of Motion; but, there is much to be learned about motion without referring to what causes it, and this is the study of kinematics. Kinematics involves describing motion through properties such as position, time, velocity, and acceleration.

A full treatment of **kinematics** considers motion in two and three dimensions. For now, we discuss motion in one dimension, which provides us with the tools necessary to study multidimensional motion. A good example of an object undergoing one-dimensional motion is the maglev (magnetic levitation) train depicted at the beginning of this chapter. As it travels, say, from Tokyo to Kyoto, it is at different positions along the track at various times in its journey, and therefore has displacements, or changes in position. It also has a variety of velocities along its path and it undergoes accelerations (changes in velocity). With the skills learned in this chapter we can calculate these quantities and average velocity. All these quantities can be described using kinematics, without knowing the train's mass or the forces involved.

Topic hierarchy

Thumbnail Figure 3.1 - A JR Central L0 series five-car maglev (magnetic levitation) train undergoing a test run on the Yamanashi Test Track. The maglev train's motion can be described using kinematics, the subject of this chapter. (credit: modification of work by "Maryland GovPics"/Flickr).

Contributors

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4.1: Prelude Motion Along a Straight Line

Our universe is full of objects in motion. From the stars, planets, and galaxies; to the motion of people and animals; down to the microscopic scale of atoms and molecules—everything in our universe is in motion. We can describe motion using the two disciplines of kinematics and dynamics. We study dynamics, which is concerned with the causes of motion, in Newton's Laws of Motion; but, there is much to be learned about motion without referring to what causes it, and this is the study of kinematics. Kinematics involves describing motion through properties such as position, time, velocity, and acceleration.

Figure 4.1.1: A JR Central L0 series five-car maglev (magnetic levitation) train undergoing a test run on the Yamanashi Test Track. The maglev train's motion can be described using kinematics, the subject of this chapter. (credit: modification of work by "Maryland GovPics"/Flickr)

A full treatment of **kinematics** considers motion in two and three dimensions. For now, we discuss motion in one dimension, which provides us with the tools necessary to study multidimensional motion. A good example of an object undergoing one-dimensional motion is the maglev (magnetic levitation) train depicted at the beginning of this chapter. As it travels, say, from Tokyo to Kyoto, it is at different positions along the track at various times in its journey, and therefore has displacements, or changes in position. It also has a variety of velocities along its path and it undergoes accelerations (changes in velocity). With the skills learned in this chapter we can calculate these quantities and average velocity. All these quantities can be described using kinematics, without knowing the train's mass or the forces involved.

4.2: Position, Displacement, and Average Velocity

> **Learning Objectives**
>
> - Define position, displacement, and distance traveled.
> - Calculate the total displacement given the position as a function of time.
> - Determine the total distance traveled.
> - Calculate the average velocity given the displacement and elapsed time.

When you're in motion, the basic questions to ask are: Where are you? Where are you going? How fast are you getting there? The answers to these questions require that you specify your position, your displacement, and your average velocity—the terms we define in this section.

Position

To describe the motion of an object, you must first be able to describe its **position** (x): **where it is at any particular time**. More precisely, we need to specify its position relative to a convenient frame of reference. A frame of reference is an arbitrary set of axes from which the position and motion of an object are described. Earth is often used as a frame of reference, and we often describe the position of an object as it relates to stationary objects on Earth. For example, a rocket launch could be described in terms of the position of the rocket with respect to Earth as a whole, whereas a cyclist's position could be described in terms of where she is in relation to the buildings she passes Figure $4.2.1$. In other cases, we use reference frames that are not stationary but are in motion relative to Earth. To describe the position of a person in an airplane, for example, we use the airplane, not Earth, as the reference frame. To describe the position of an object undergoing onedimensional motion, we often use the variable x. Later in the chapter, during the discussion of free fall, we use the variable y.

Figure $4.2.1$: These cyclists in Vietnam can be described by their position relative to buildings or a canal. Their motion can be described by their change in position, or displacement, in a frame of reference. (credit: Suzan Black)

Displacement

If an object moves relative to a frame of reference—for example, if a professor moves to the right relative to a whiteboard Figure $4.2.2$—then the object's position changes. This change in position is called **displacement**. The word displacement implies that an object has moved, or has been displaced. Although position is the numerical value of x along a straight line where an object might be located, displacement gives the change in position along this line. Since displacement indicates direction, it is a vector and can be either positive or negative, depending on the choice of positive direction. Also, an analysis of motion can have many displacements embedded in it. If right is positive and an object moves 2 m to the right, then 4 m to the left, the individual displacements are 2 m and −4 m, respectively.

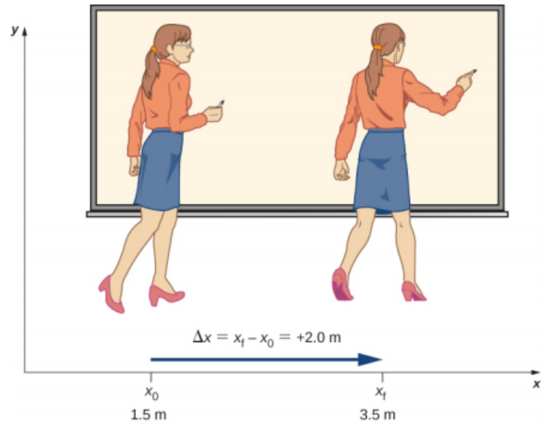


Figure 4.2.2: A professor paces left and right while lecturing. Her position relative to Earth is given by x. The +2.0-m displacement of the professor relative to Earth is represented by an arrow pointing to the right.

Displacement

Displacement Δx is the change in position of an object:

$$\Delta x = x_f - x_0, \tag{4.2.1}$$

where Δx is displacement, x_f is the final position, and x_0 is the initial position.

We use the uppercase Greek letter delta (Δ) to mean "change in" whatever quantity follows it; thus, Δx means **change in position** (final position less initial position). We always solve for displacement by subtracting initial position x_0 from final position x_f. Note that the SI unit for displacement is the meter, but sometimes we use kilometers or other units of length. Keep in mind that when units other than meters are used in a problem, you may need to convert them to meters to complete the calculation (see Appendix B).

Objects in motion can also have a series of displacements. In the previous example of the pacing professor, the individual displacements are 2 m and −4 m, giving a total displacement of −2 m. We define **total displacement** Δx_{Total}, as the **sum of the individual displacements**, and express this mathematically with the equation

$$\Delta x_{Total} = \sum \Delta x_i, \tag{4.2.2}$$

where δx_i are the individual displacements. In the earlier example,

$$\Delta x_1 = x_1 - x_0 = 2 - 0 = 2\ m. \tag{4.2.3}$$

Similarly,

$$\Delta x_2 = x_2 - x_1 = -2 - (2) = -4\ m. \tag{4.2.4}$$

Thus,

$$\Delta x_{total} = x_1 + x_2 = 2 - 4 = -2\ m. \tag{4.2.5}$$

The total displacement is 2 − 4 = −2 m to the left, or in the negative direction. It is also useful to calculate the magnitude of the displacement, or its size. The magnitude of the displacement is always positive. This is the absolute value of the displacement, because displacement is a vector and cannot have a negative value of magnitude. In our example, the magnitude of the total displacement is 2 m, whereas the magnitudes of the individual displacements are 2 m and 4 m.

The magnitude of the total displacement should not be confused with the distance traveled. Distance traveled x_{Total}, is the total length of the path traveled between two positions. In the previous problem, the **distance traveled** is the sum of the magnitudes of the individual displacements:

$$x_{total} = |x_1| + |x_2| = 2 + 4 = 6\ m. \tag{4.2.6}$$

Average Velocity

To calculate the other physical quantities in kinematics we must introduce the time variable. The time variable allows us not only to state where the object is (its position) during its motion, but also how fast it is moving. How fast an object is moving is given by the rate at which the position changes with time.

For each position x_i, we assign a particular time t_i. If the details of the motion at each instant are not important, the rate is usually expressed as the **average velocity** $\bar{v}$. This vector quantity is simply the total displacement between two points divided by the time taken to travel between them. The time taken to travel between two points is called the **elapsed time** Δt.

Average Velocity

If x_1 and x_2 are the positions of an object at times t_1 and t_2, respectively, then

$$Average\ velocity = \bar{v} = \frac{Displacement\ between\ two\ points}{Elapsed\ time\ between\ two\ points}$$

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}.$$

It is important to note that the average velocity is a vector and can be negative, depending on positions x_1 and x_2.

Example 3.1: Delivering Flyers

Jill sets out from her home to deliver flyers for her yard sale, traveling due east along her street lined with houses. At 0.5 km and 9 minutes later she runs out of flyers and has to retrace her steps back to her house to get more. This takes an additional 9 minutes. After picking up more flyers, she sets out again on the same path, continuing where she left off, and ends up 1.0 km from her house. This third leg of her trip takes 15 minutes. At this point she turns back toward her house, heading west. After 1.75 km and 25 minutes she stops to rest.

a. What is Jill’s total displacement to the point where she stops to rest?
b. What is the magnitude of the final displacement?
c. What is the average velocity during her entire trip?
d. What is the total distance traveled?
e. Make a graph of position versus time. A sketch of Jill’s movements is shown in Figure $4.2.3$.

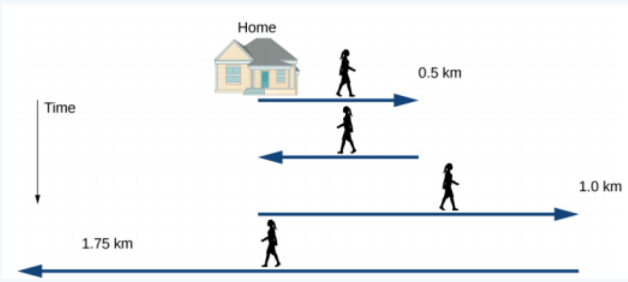


Figure $4.2.3$: Timeline of Jill’s movements.

Strategy

The problem contains data on the various legs of Jill’s trip, so it would be useful to make a table of the physical quantities. We are given position and time in the wording of the problem so we can calculate the displacements and the elapsed time. We take east to be the positive direction. From this information we can find the total displacement and average velocity. Jill’s home is the starting point x_0. The following table gives Jill’s time and position in the first two columns, and the displacements are calculated in the third column.

Time t_i (min)	Position x_i (km)	Displacement Δx_i (km)
$t_0 = 0$	$x_0 = 0$	$\Delta x_0 = 0$
$t_1 = 9$	$x_1 = 0.5$	$\Delta x_1 = x_1 - x_0 = 0.5$

$t_2 = 18$	$x_2 = 0$	$\Delta x2 = x_2 - x_1 = -0.5$
$t_3 = 33$	$x_3 = 1.0$	$\Delta x_3 = x_3 - x_2 = 1.0$
$t_4 = 58$	$x_4 = -0.75$	$\Delta x_4 = x_4 - x_3 = -1.75$

Solution

a. From the above table, the total displacement is

$$\sum \Delta x_i = 0.5 - 0.5 + 1.0 - 1.75\ km = -0.75\ km. \tag{4.2.7}$$

b. The magnitude of the total displacement is |−0.75| km = 0.75 km.

c.
$$Average\ velocity = \frac{Total\ displacement}{Elapsed\ time} = \bar{v} = \frac{-0.75\ km}{58\ min} = -0.013\ km/min \tag{4.2.8}$$

d. The total distance traveled (sum of magnitudes of individual displacements) is

$$x_{Total} = \sum |\Delta x_i| = 0.5 + 0.5 + 1.0 + 1.75\ km = 3.75\ km. \tag{4.2.9}$$

e. We can graph Jill's position versus time as a useful aid to see the motion; the graph is shown in Figure $4.2.4$.

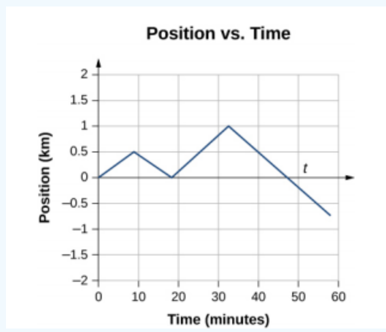


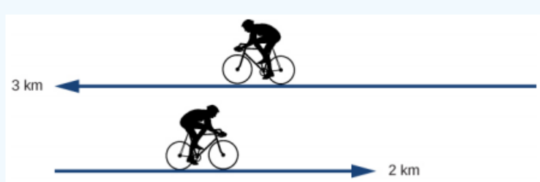
Figure $4.2.4$: This graph depicts Jill's position versus time. The average velocity is the slope of a line connecting the initial and final points.

Significance

Jill's total displacement is −0.75 km, which means at the end of her trip she ends up 0.75 km due west of her home. The average velocity means if someone was to walk due west at 0.013 km/min starting at the same time Jill left her home, they both would arrive at the final stopping point at the same time. Note that if Jill were to end her trip at her house, her total displacement would be zero, as well as her average velocity. The total distance traveled during the 58 minutes of elapsed time for her trip is 3.75 km.

? Exercise 3.1

A cyclist rides 3 km west and then turns around and rides 2 km east. (a) What is his displacement? (b) What is the distance traveled? (c) What is the magnitude of his displacement?



4.3: Instantaneous Velocity and Speed

Learning Objectives

- Explain the difference between average velocity and instantaneous velocity.
- Describe the difference between velocity and speed.
- Calculate the instantaneous velocity given the mathematical equation for the velocity.
- Calculate the speed given the instantaneous velocity.

We have now seen how to calculate the average velocity between two positions. However, since objects in the real world move continuously through space and time, we would like to find the velocity of an object at any single point. We can find the velocity of the object anywhere along its path by using some fundamental principles of calculus. This section gives us better insight into the physics of motion and will be useful in later chapters.

Instantaneous Velocity

The quantity that tells us how fast an object is moving anywhere along its path is the **instantaneous velocity**, usually called simply **velocity**. It is the average velocity between two points on the path in the limit that the time (and therefore the displacement) between the two points approaches zero. To illustrate this idea mathematically, we need to express position x as a continuous function of t denoted by x(t). The expression for the average velocity between two points using this notation is $\bar{v} = \frac{x(t_2)-x(t_1)}{t_2-t_1}$. To find the instantaneous velocity at any position, we let t_1 = t and t_2 = t + Δt. After inserting these expressions into the equation for the average velocity and taking the limit as Δt → 0, we find the expression for the instantaneous velocity:

$$v(t) = \lim_{\Delta t \to 0} \frac{x(t+\Delta t) - x(t)}{\Delta t} = \frac{dx(t)}{dt}. \tag{4.3.1}$$

Instantaneous Velocity

The instantaneous velocity of an object is the limit of the average velocity as the elapsed time approaches zero, or the derivative of x with respect to t:

$$v(t) = \frac{d}{dt}x(t). \tag{4.3.2}$$

Like average velocity, instantaneous velocity is a vector with dimension of length per time. The instantaneous velocity at a specific time point t_0 is the rate of change of the position function, which is the slope of the position function x(t) at t_0. Figure 4.3.1 shows how the average velocity $\bar{v} = \frac{\Delta x}{\Delta t}$ between two times approaches the instantaneous velocity at t_0. The instantaneous velocity is shown at time t_0, which happens to be at the maximum of the position function. The slope of the position graph is zero at this point, and thus the instantaneous velocity is zero. At other times, t_1, t_2, and so on, the instantaneous velocity is not zero because the slope of the position graph would be positive or negative. If the position function had a minimum, the slope of the position graph would also be zero, giving an instantaneous velocity of zero there as well. Thus, the zeros of the velocity function give the minimum and maximum of the position function.

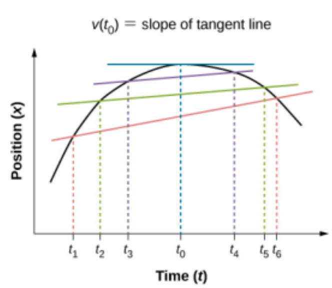


Figure 4.3.1: In a graph of position versus time, the instantaneous velocity is the slope of the tangent line at a given point. The average velocities $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_f - x_i}{t_f - t_i}$ between times Δt = t_6 − t_1, Δt = t_5 − t_2, and Δt = t_4 − t_3 are shown. When Δt → 0, the average velocity approaches the instantaneous velocity at t = t_0.

Example 3.2: Finding Velocity from a Position-Versus-Time Graph

Given the position-versus-time graph of Figure 4.3.2, find the velocity-versus-time graph.

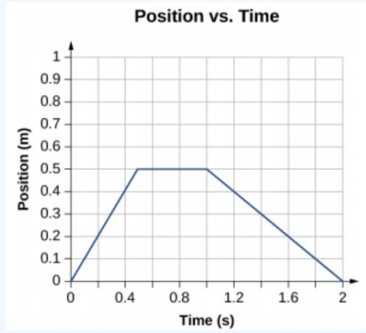


Figure 4.3.2: The object starts out in the positive direction, stops for a short time, and then reverses direction, heading back toward the origin. Notice that the object comes to rest instantaneously, which would require an infinite force. Thus, the graph is an approximation of motion in the real world. (The concept of force is discussed in Newton's Laws of Motion.)

Strategy

The graph contains three straight lines during three time intervals. We find the velocity during each time interval by taking the slope of the line using the grid.

Solution

Time interval 0 s to 0.5 s: $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{0.5\ m - 0.0\ m}{0.5\ s - 0.0\ s} = 1.0\ m/s$

Time interval 0.5 s to 1.0 s: $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{0.0\ m - 0.0\ m}{1.0\ s - 0.5\ s} = 0.0\ m/s$

Time interval 1.0 s to 2.0 s: $\bar{v} = \frac{\Delta x}{\Delta t} = \frac{0.0\ m - 0.5\ m}{2.0\ s - 1.0\ s} = -0.5\ m/s$

The graph of these values of velocity versus time is shown in Figure 4.3.3.

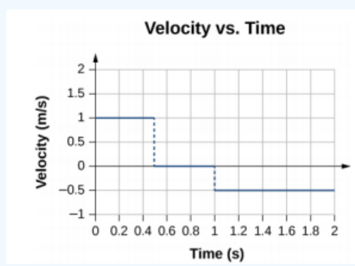


Figure 4.3.3: The velocity is positive for the first part of the trip, zero when the object is stopped, and negative when the object reverses direction.

Significance

During the time interval between 0 s and 0.5 s, the object's position is moving away from the origin and the position-versus-time curve has a positive slope. At any point along the curve during this time interval, we can find the instantaneous velocity by taking its slope, which is +1 m/s, as shown in Figure 4.3.3. In the subsequent time interval, between 0.5 s and 1.0 s, the position doesn't change and we see the slope is zero. From 1.0 s to 2.0 s, the object is moving back toward the origin and the slope is −0.5 m/s. The object has reversed direction and has a negative velocity.

Speed

In everyday language, most people use the terms speed and velocity interchangeably. In physics, however, they do not have the same meaning and are distinct concepts. One major difference is that speed has no direction; that is, speed is a scalar.

We can calculate the **average speed** by finding the total distance traveled divided by the elapsed time:

$$Average\ speed = \bar{s} = \frac{Total\ distance}{Elapsed\ time}. \tag{4.3.3}$$

Average speed is not necessarily the same as the magnitude of the average velocity, which is found by dividing the magnitude of the total displacement by the elapsed time. For example, if a trip starts and ends at the same location, the total displacement is zero, and therefore the average velocity is zero. The average speed, however, is not zero, because the total distance traveled is greater than zero. If we take a road trip of 300 km and need to be at our destination at a certain time, then we would be interested in our average speed.

However, we can calculate the **instantaneous speed** from the magnitude of the instantaneous velocity:

$$Instantaneous\ speed = |v(t)|. \tag{4.3.4}$$

If a particle is moving along the x-axis at +7.0 m/s and another particle is moving along the same axis at −7.0 m/s, they have different velocities, but both have the same speed of 7.0 m/s. Some typical speeds are shown in the following table.

Table 3.1 - Speeds of Various Objects

Speed	m/s	mi/h
Continental drift	10^{-7}	2×10^{-7}
Brisk walk	1.7	3.9
Cyclist	4.4	10
Sprint runner	12.2	27
Rural speed limit	24.6	56
Official land speed record	341.1	763
Speed of sound at sea level	343	768
Space shuttle on reetry	7800	17,500
Escape velocity of Earth*	11,200	25,000
Orbital speed of Earth around the Sun	29,783	66,623
Speed of light in a vacuum	299,792,458	670,616,629

*Escape velocity is the velocity at which an object must be launched so that it overcomes Earth’s gravity and is not pulled back toward Earth.

Calculating Instantaneous Velocity

When calculating instantaneous velocity, we need to specify the explicit form of the position function x(t). If each term in the x(t) equation has the form of At^n where A is a constant and n is an integer, this can be differentiated using the power rule to be:

$$\frac{d\left(At^n\right)}{dt} = Ant^{n-1}. \tag{4.3.5}$$

Note that if there are additional terms added together, this power rule of differentiation can be done multiple times and the solution is the sum of those terms. The following example illustrates the use of Equation 4.3.5.

✓ Example 3.3: Instantaneous Velocity Versus Average Velocity

The position of a particle is given by $x(t) = 3.0t + 0.5t^3$ m.

a. Using Equation 4.3.2 and Equation 4.3.5, find the instantaneous velocity at t = 2.0 s.
b. Calculate the average velocity between 1.0 s and 3.0 s.

Strategy

Equation 4.3.2 give the instantaneous velocity of the particle as the derivative of the position function. Looking at the form of the position function given, we see that it is a polynomial in t. Therefore, we can use Equation 4.3.5, the power rule from calculus, to find the solution. We use Equation 4.3.4 to calculate the average velocity of the particle.

Solution

a. v(t) = $\frac{dx(t)}{dt}$ = 3.0 + 1.5t^2 m/s. Substituting t = 2.0 s into this equation gives v(2.0 s) = [3.0 + 1.5(2.0)^2] m/s = 9.0 m/s.

b. To determine the average velocity of the particle between 1.0 s and 3.0 s, we calculate the values of x(1.0 s) and x(3.0 s):

$$x(1.0s) = \left[(3.0)(1.0) + 0.5(1.0)^3\right] m = 3.5\ m \tag{4.3.6}$$

$$x(3.0s) = \left[(3.0)(3.0) + 0.5(3.0)^3\right] m = 22.5\ m \tag{4.3.7}$$

Then the average velocity is

$$\bar{v} = \frac{x(3.0\ s) - x(1.0\ s)}{t(3.0\ s) - t(1.0\ s)} = \frac{22.5 - 3.5\ m}{3.0 - 1.0\ s} = 9.5\ m/s. \tag{4.3.8}$$

Significance

In the limit that the time interval used to calculate $\bar{v}$ goes to zero, the value obtained for $\bar{v}$ converges to the value of v.

Example 3.4: Instantaneous Velocity Versus Speed

Consider the motion of a particle in which the position is x(t) = 3.0t − 3t^2 m.

a. What is the instantaneous velocity at t = 0.25 s, t = 0.50 s, and t = 1.0 s?
b. What is the speed of the particle at these times?

Strategy

The instantaneous velocity is the derivative of the position function and the speed is the magnitude of the instantaneous velocity. We use Equation 4.3.2 and Equation 4.3.5 to solve for instantaneous velocity.

Solution

a. v(t) = $\frac{dx(t)}{dt}$ = 3.0 − 6.0t m/s
b. v(0.25 s) = 1.50 m/s, v(0.5 s) = 0 m/s, v(1.0 s) = −3.0 m/s
c. Speed = |v(t)| = 1.50 m/s, 0.0 m/s, and 3.0 m/s

Significance

The velocity of the particle gives us direction information, indicating the particle is moving to the left (west) or right (east). The speed gives the magnitude of the velocity. By graphing the position, velocity, and speed as functions of time, we can understand these concepts visually Figure 4.3.4. In (a), the graph shows the particle moving in the positive direction until t = 0.5 s, when it reverses direction. The reversal of direction can also be seen in (b) at 0.5 s where the velocity is zero and then turns negative. At 1.0 s it is back at the origin where it started. The particle's velocity at 1.0 s in (b) is negative, because it is traveling in the negative direction. But in (c), however, its speed is positive and remains positive throughout the travel time. We can also interpret velocity as the slope of the position-versus-time graph. The slope of x(t) is decreasing toward zero, becoming zero at 0.5 s and increasingly negative thereafter. This analysis of comparing the graphs of position, velocity, and speed helps catch errors in calculations. The graphs must be consistent with each other and help interpret the calculations.

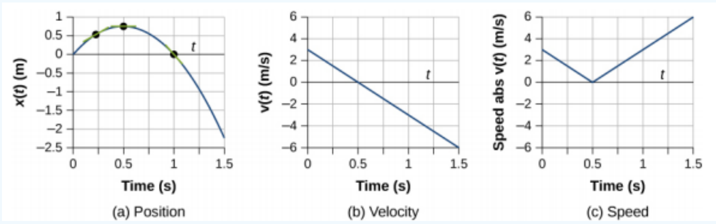


Figure 4.3.4: (a) Position: x(t) versus time. (b) Velocity: v(t) versus time. The slope of the position graph is the velocity. A rough comparison of the slopes of the tangent lines in (a) at 0.25 s, 0.5 s, and 1.0 s with the values for velocity at the corresponding times indicates they are the same values. (c) Speed: |v(t)| versus time. Speed is always a positive number.

? Exercise 3.2

The position of an object as a function of time is $x(t) = -3t^2$ m. (a) What is the velocity of the object as a function of time? (b) Is the velocity ever positive? (c) What are the velocity and speed at $t = 1.0$ s?

4.4: Average and Instantaneous Acceleration

Learning Objectives

- Calculate the average acceleration between two points in time.
- Calculate the instantaneous acceleration given the functional form of velocity.
- Explain the vector nature of instantaneous acceleration and velocity.
- Explain the difference between average acceleration and instantaneous acceleration.
- Find instantaneous acceleration at a specified time on a graph of velocity versus time.

The importance of understanding acceleration spans our day-to-day experience, as well as the vast reaches of outer space and the tiny world of subatomic physics. In everyday conversation, to **accelerate** means to speed up; applying the brake pedal causes a vehicle to slow down. We are familiar with the acceleration of our car, for example. The greater the acceleration, the greater the change in velocity over a given time. Acceleration is widely seen in experimental physics. In linear particle accelerator experiments, for example, subatomic particles are accelerated to very high velocities in collision experiments, which tell us information about the structure of the subatomic world as well as the origin of the universe. In space, cosmic rays are subatomic particles that have been accelerated to very high energies in supernovas (exploding massive stars) and active galactic nuclei. It is important to understand the processes that accelerate cosmic rays because these rays contain highly penetrating radiation that can damage electronics flown on spacecraft, for example.

Average Acceleration

The formal definition of acceleration is consistent with these notions just described, but is more inclusive.

Average Acceleration

Average acceleration is the rate at which velocity changes:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}, \tag{4.4.1}$$

where $\bar{a}$ is **average acceleration**, v is velocity, and t is time. (The bar over the a means **average** acceleration.)

Because acceleration is velocity in meters divided by time in seconds, the SI units for acceleration are often abbreviated m/s^2—that is, meters per second squared or meters per second per second. This literally means by how many meters per second the velocity changes every second. Recall that velocity is a vector—it has both magnitude and direction—which means that a change in velocity can be a change in magnitude (or speed), but it can also be a change in direction. For example, if a runner traveling at 10 km/h due east slows to a stop, reverses direction, continues her run at 10 km/h due west, her velocity has changed as a result of the change in direction, although the **magnitude** of the velocity is the same in both directions. Thus, acceleration occurs when velocity changes in magnitude (an increase or decrease in speed) or in direction, or both.

Acceleration as a Vector

Acceleration is a vector in the same direction as the **change** in velocity, Δv. Since velocity is a vector, it can change in magnitude or in direction, or both. Acceleration is, therefore, a change in speed or direction, or both.

Keep in mind that although acceleration is in the direction of the change in velocity, it is not always in the direction of motion. When an object slows down, its acceleration is opposite to the direction of its motion. Although this is commonly referred to as **deceleration** Figure 4.4.1, we say the train is accelerating in a direction opposite to its direction of motion.

Figure 4.4.1: A subway train in Sao Paulo, Brazil, decelerates as it comes into a station. It is accelerating in a direction opposite to its direction of motion. (credit: Yusuke Kawasaki)

The term **deceleration** can cause confusion in our analysis because it is not a vector and it does not point to a specific direction with respect to a coordinate system, so we do not use it. Acceleration is a vector, so we must choose the appropriate sign for it in our chosen coordinate system. In the case of the train in Figure 4.4.1, acceleration is in the **negative direction in the chosen coordinate system**, so we say the train is undergoing negative acceleration.

If an object in motion has a velocity in the positive direction with respect to a chosen origin and it acquires a constant negative acceleration, the object eventually comes to a rest and reverses direction. If we wait long enough, the object passes through the origin going in the opposite direction. This is illustrated in Figure 4.4.2.

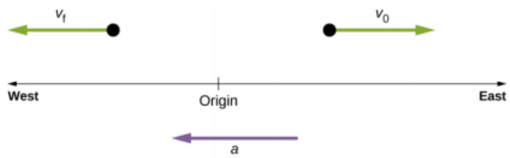


Figure 4.4.2: An object in motion with a velocity vector toward the east under negative acceleration comes to a rest and reverses direction. It passes the origin going in the opposite direction after a long enough time.

Example 3.5: Calculating Average Acceleration: A Racehorse Leaves the Gate

A racehorse coming out of the gate accelerates from rest to a velocity of 15.0 m/s due west in 1.80 s. What is its average acceleration?

Figure 4.4.3: Racehorses accelerating out of the gate. (credit: Jon Sullivan)

Strategy

First we draw a sketch and assign a coordinate system to the problem Figure 4.4.4. This is a simple problem, but it always helps to visualize it. Notice that we assign east as positive and west as negative. Thus, in this case, we have negative velocity.

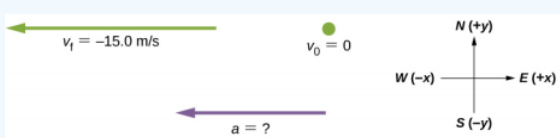


Figure 4.4.4: Identify the coordinate system, the given information, and what you want to determine.

We can solve this problem by identifying Δv and Δt from the given information, and then calculating the average acceleration directly from the equation $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$.

Solution

First, identify the knowns: $v_0 = 0$, $v_f = -15.0$ m/s (the negative sign indicates direction toward the west), $\Delta t = 1.80$ s. Second, find the change in velocity. Since the horse is going from zero to –15.0 m/s, its change in velocity equals its final velocity:

$$\Delta v = v_f - v_0 = v_f = -15.0\ m/s. \tag{4.4.2}$$

Last, substitute the known values (Δv and Δt) and solve for the unknown $\bar{a}$:

$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{-15.0\ m/s}{1.80\ s} = -8.33\ m/s^2. \tag{4.4.3}$$

Significance

The negative sign for acceleration indicates that acceleration is toward the west. An acceleration of 8.33 m/s^2 due west means the horse increases its velocity by 8.33 m/s due west each second; that is, 8.33 meters per second per second, which we write as 8.33 m/s^2. This is truly an average acceleration, because the ride is not smooth. We see later that an acceleration of this magnitude would require the rider to hang on with a force nearly equal to his weight.

? Exercise 3.3

Protons in a linear accelerator are accelerated from rest to 2.0 × 10^7 m/s in 10^{-4} s. What is the average acceleration of the protons?

Instantaneous Acceleration

Instantaneous acceleration a, or **acceleration at a specific instant in time**, is obtained using the same process discussed for instantaneous velocity. That is, we calculate the average velocity between two points in time separated by Δt and let Δt approach zero. The result is the derivative of the velocity function v(t), which is **instantaneous acceleration** and is expressed mathematically as

$$a(t) = \frac{d}{dt}v(t). \tag{4.4.4}$$

Thus, similar to velocity being the derivative of the position function, instantaneous acceleration is the derivative of the velocity function. We can show this graphically in the same way as instantaneous velocity. In Figure 4.4.5, instantaneous acceleration at time t_0 is the slope of the tangent line to the velocity-versus-time graph at time t_0. We see that average acceleration $\bar{a} = \frac{\Delta v}{\Delta t}$ approaches instantaneous acceleration as Δt approaches zero. Also in part (a) of the figure, we see that velocity has a maximum when its slope is zero. This time corresponds to the zero of the acceleration function. In part (b), instantaneous acceleration at the minimum velocity is shown, which is also zero, since the slope of the curve is zero there, too. Thus, for a given velocity function, the zeros of the acceleration function give either the minimum or the maximum velocity

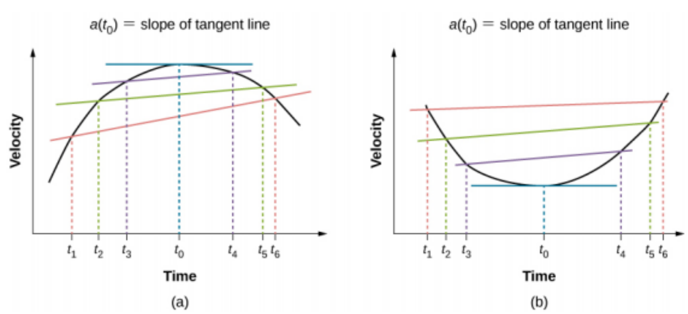


Figure 4.4.5: In a graph of velocity versus time, instantaneous acceleration is the slope of the tangent line. (a) Shown is average acceleration $\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - v_0}{t_f - t_0}$ between times $\Delta t = t_6 - t_1$, $\Delta t = t_5 - t_2$, and $\Delta t = t_4 - t_3$. When $\Delta t \to 0$, the average acceleration approaches instantaneous acceleration at time t_0. In view (a), instantaneous acceleration is shown for the point on the velocity curve at maximum velocity. At this point, instantaneous acceleration is the slope of the tangent line, which is zero. At any other time, the slope of the tangent line—and thus instantaneous acceleration—would not be zero. (b) Same as (a) but shown for instantaneous acceleration at minimum velocity.

To illustrate this concept, let's look at two examples. First, a simple example is shown using Figure 3.3.4(b), the velocityversus-time graph of Example 3.3, to find acceleration graphically. This graph is depicted in Figure 4.4.6(a), which is a straight line. The corresponding graph of acceleration versus time is found from the slope of velocity and is shown in Figure 4.4.6(b). In this example, the velocity function is a straight line with a constant slope, thus acceleration is a constant. In the next example, the velocity function is has a more complicated functional dependence on time.

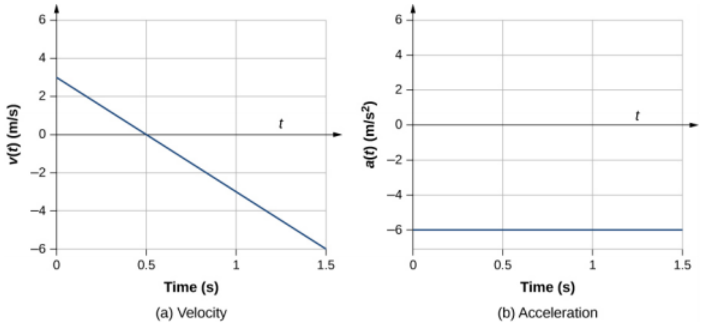


Figure 4.4.6: (a, b) The velocity-versus-time graph is linear and has a negative constant slope (a) that is equal to acceleration, shown in (b).

If we know the functional form of velocity, v(t), we can calculate instantaneous acceleration a(t) at any time point in the motion using Equation 4.4.4.

Example 3.6: Calculating Instantaneous Acceleration

A particle is in motion and is accelerating. The functional form of the velocity is $v(t) = 20t - 5t^2$ m/s.

a. Find the functional form of the acceleration.
b. Find the instantaneous velocity at t = 1, 2, 3, and 5 s.
c. Find the instantaneous acceleration at t = 1, 2, 3, and 5 s.
d. Interpret the results of (c) in terms of the directions of the acceleration and velocity vectors.

Strategy

We find the functional form of acceleration by taking the derivative of the velocity function. Then, we calculate the values of instantaneous velocity and acceleration from the given functions for each. For part (d), we need to compare the directions of velocity and acceleration at each time.

Solution

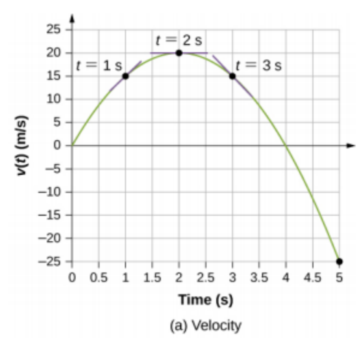
a. $a(t) = \frac{dv(t)}{dt}$dv(t) dt = 20 − 10t m/s$_2$
b. v(1 s) = 15 m/s, v(2 s) = 20 m/s, v(3 s) = 15 m/s, v(5 s) = −25 m/s
c. a(1 s) = 10m/s^2, a(2 s) = 0m/s^2, a(3 s) = −10m/s^2, a(5 s) = −30m/s^2
d. At t = 1 s, velocity v(1 s) = 15 m/s is positive and acceleration is positive, so both velocity and acceleration are in the same direction. The particle is moving faster.

At t = 2 s, velocity has increased to v(2 s) = 20 m/s , where it is maximum, which corresponds to the time when the acceleration is zero. We see that the maximum velocity occurs when the slope of the velocity function is zero, which is just the zero of the acceleration function.

At t = 3 s, velocity is v(3 s) = 15 m/s and acceleration is negative. The particle has reduced its velocity and the acceleration vector is negative. The particle is slowing down.

At t = 5 s, velocity is v(5 s) = −25 m/s and acceleration is increasingly negative. Between the times t = 3 s and t = 5 s the particle has decreased its velocity to zero and then become negative, thus reversing its direction. The particle is now speeding up again, but in the opposite direction.

We can see these results graphically in Figure 4.4.7.



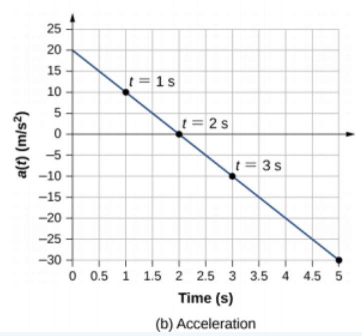


Figure 4.4.7: (a) Velocity versus time. Tangent lines are indicated at times 1, 2, and 3 s. The slopes of the tangents lines are the accelerations. At t = 3 s, velocity is positive. At t = 5 s, velocity is negative, indicating the particle has reversed direction. (b) Acceleration versus time. Comparing the values of accelerations given by the black dots with the corresponding slopes of the tangent lines (slopes of lines through black dots) in (a), we see they are identical.

Significance

By doing both a numerical and graphical analysis of velocity and acceleration of the particle, we can learn much about its motion. The numerical analysis complements the graphical analysis in giving a total view of the motion. The zero of the acceleration function corresponds to the maximum of the velocity in this example. Also in this example, when acceleration is positive and in the same direction as velocity, velocity increases. As acceleration tends toward zero, eventually becoming negative, the velocity reaches a maximum, after which it starts decreasing. If we wait long enough, velocity also becomes negative, indicating a reversal of direction. A real-world example of this type of motion is a car with a velocity that is increasing to a maximum, after which it starts slowing down, comes to a stop, then reverses direction.

? Exercise 3.4

An airplane lands on a runway traveling east. Describe its acceleration.

Getting a Feel for Acceleration

You are probably used to experiencing acceleration when you step into an elevator, or step on the gas pedal in your car. However, acceleration is happening to many other objects in our universe with which we don’t have direct contact. Table 3.2 presents the acceleration of various objects. We can see the magnitudes of the accelerations extend over many orders of magnitude.

Table 3.2 - Typical Values of Acceleration

(credit: Wikipedia: Orders of Magnitude (acceleration))

Acceleration	Value (m/s^2)
High-speed train	0.25
Elevator	2
Cheetah	5
Object in a free fall without air resistance near the surface of Earth	9.8

Acceleration	Value (m/s^2)
Space shuttle maximum during launch	29
Parachutist peak during normal opening of parachute	59
F16 aircraft pulling out of a dive	79
Explosive seat ejection from aircraft	147
Sprint missile	982
Fastest rocket sled peak acceleration	1540
Jumping flea	3200
Baseball struck by a bat	30,000
Closing jaws of a trap-jaw ant	1,000,000
Proton in the large Hadron collider	1.9×10^9

In this table, we see that typical accelerations vary widely with different objects and have nothing to do with object size or how massive it is. Acceleration can also vary widely with time during the motion of an object. A drag racer has a large acceleration just after its start, but then it tapers off as the vehicle reaches a constant velocity. Its average acceleration can be quite different from its instantaneous acceleration at a particular time during its motion. Figure 4.4.8 compares graphically average acceleration with instantaneous acceleration for two very different motions.

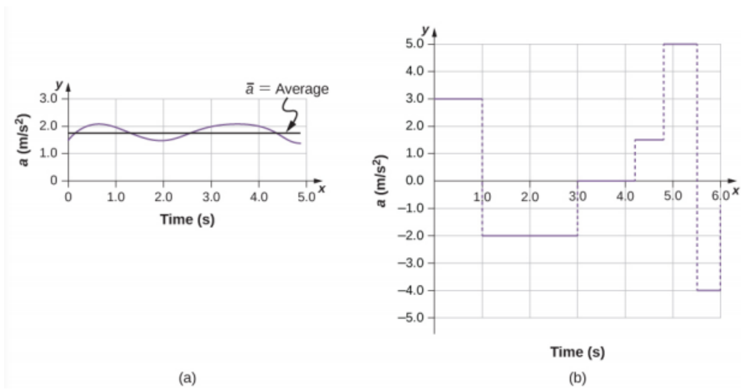


Figure 4.4.8: Graphs of instantaneous acceleration versus time for two different one-dimensional motions. (a) Acceleration varies only slightly and is always in the same direction, since it is positive. The average over the interval is nearly the same as the acceleration at any given time. (b) Acceleration varies greatly, perhaps representing a package on a post office conveyor belt that is accelerated forward and backward as it bumps along. It is necessary to consider small time intervals (such as from 0–1.0 s) with constant or nearly constant acceleration in such a situation.

Simulation

Learn about position, velocity, and acceleration graphs. Move the little man back and forth with a mouse and plot his motion. Set the position, velocity, or acceleration and let the simulation move the man for you. Visit this link to use the moving man simulation.

4.5: Motion with Constant Acceleration (Part 1)

Learning Objectives

- Identify which equations of motion are to be used to solve for unknowns.
- Use appropriate equations of motion to solve a two-body pursuit problem.

You might guess that the greater the acceleration of, say, a car moving away from a stop sign, the greater the car's displacement in a given time. But, we have not developed a specific equation that relates acceleration and displacement. In this section, we look at some convenient equations for kinematic relationships, starting from the definitions of displacement, velocity, and acceleration. We first investigate a single object in motion, called single-body motion. Then we investigate the motion of two objects, called **two-body pursuit problems**.

Notation

First, let us make some simplifications in notation. Taking the initial time to be zero, as if time is measured with a stopwatch, is a great simplification. Since elapsed time is $\Delta t = t_f - t_0$, taking $t_0 = 0$ means that $\Delta t = t_f$, the final time on the stopwatch. When initial time is taken to be zero, we use the subscript 0 to denote initial values of position and velocity. That is, x_0 **is the initial position** and v_0 **is the initial velocity**. We put no subscripts on the final values. That is, t **is the final time**, x **is the final position**, and v **is the final velocity**. This gives a simpler expression for elapsed time, $\Delta t = t$. It also simplifies the expression for x displacement, which is now $\Delta x = x - x_0$. Also, it simplifies the expression for change in velocity, which is now $\Delta v = v - v_0$. To summarize, using the simplified notation, with the initial time taken to be zero,

$$\Delta t = t \tag{4.5.1}$$

$$\Delta x = x - x_0 \tag{4.5.2}$$

$$\Delta v = v - v_0, \tag{4.5.3}$$

where the subscript 0 denotes an initial value and the absence of a subscript denotes a final value in whatever motion is under consideration.

We now make the important assumption that acceleration is constant. This assumption allows us to avoid using calculus to find instantaneous acceleration. Since acceleration is constant, the average and instantaneous accelerations are equal—that is,

$$\bar{a} = a = constant. \tag{4.5.4}$$

Thus, we can use the symbol a for acceleration at all times. Assuming acceleration to be constant does not seriously limit the situations we can study nor does it degrade the accuracy of our treatment. For one thing, acceleration is constant in a great number of situations. Furthermore, in many other situations we can describe motion accurately by assuming a constant acceleration equal to the average acceleration for that motion. Lastly, for motion during which acceleration changes drastically, such as a car accelerating to top speed and then braking to a stop, motion can be considered in separate parts, each of which has its own constant acceleration.

Displacement and Position from Velocity

To get our first two equations, we start with the definition of average velocity:

$$\bar{v} = \frac{\Delta x}{\Delta t}. \tag{4.5.5}$$

Substituting the simplified notation for Δx and Δt yields

$$\bar{v} = \frac{x - x_0}{t}. \tag{4.5.6}$$

Solving for x gives us

$$x = x_0 + \bar{v}t, \tag{4.5.7}$$

where the average velocity is

$$\bar{v} = \frac{v_0 + v}{2}. \tag{4.5.8}$$

The equation $\bar{v} = \frac{v_0+v}{2}$ reflects the fact that when acceleration is constant, v is just the simple average of the initial and final velocities. Figure 4.5.1 illustrates this concept graphically. In part (a) of the figure, acceleration is constant, with velocity increasing at a constant rate. The average velocity during the 1-h interval from 40 km/h to 80 km/h is 60 km/h:

$$\bar{v} = \frac{v_0 + v}{2} = \frac{40\ km/h + 80\ km/h}{2} = 60\ km/h. \tag{4.5.9}$$

In part (b), acceleration is not constant. During the 1-h interval, velocity is closer to 80 km/h than 40 km/h. Thus, the average velocity is greater than in part (a).

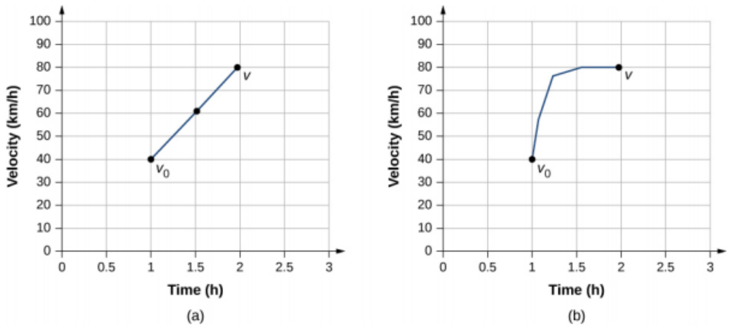


Figure 4.5.1: (a) Velocity-versus-time graph with constant acceleration showing the initial and final velocities v_0 and v. The average velocity is $\frac{1}{2}(v_0 + v)$ = 60 km/h. (b) Velocity-versus-time graph with an acceleration that changes with time. The average velocity is not given by $\frac{1}{2}(v_0 + v)$, but is greater than 60 km/h.

Solving for Final Velocity from Acceleration and Time

We can derive another useful equation by manipulating the definition of acceleration:

$$a = \frac{\Delta v}{\Delta t}. \tag{4.5.10}$$

Substituting the simplified notation for Δv and Δt gives us

$$a = \frac{v - v_0}{t}\ (constant\ a). \tag{4.5.11}$$

Solving for v yields

$$v = v_0 + at\ (constant\ a). \tag{4.5.12}$$

Example 3.7: Calculating Final Velocity

An airplane lands with an initial velocity of 70.0 m/s and then decelerates at 1.50 m/s^2 for 40.0 s. What is its final velocity?

Strategy

First, we identify the knowns: v_0 = 70 m/s, a = −1.50 m/s^2, t = 40 s.

Second, we identify the unknown; in this case, it is final velocity v_f.

Last, we determine which equation to use. To do this we figure out which kinematic equation gives the unknown in terms of the knowns. We calculate the final velocity using Equation 4.5.12, v = v_0 + at.

Solution

Substitute the known values and solve:

$$v = v_0 + at = 70.0\ m/s + (-1.50\ m/s^2)(40.0\ s) = 10.0\ m/s. \tag{4.5.13}$$

Figure 4.5.2 is a sketch that shows the acceleration and velocity vectors.

Figure 4.5.2: The airplane lands with an initial velocity of 70.0 m/s and slows to a final velocity of 10.0 m/s before heading for the terminal. Note the acceleration is negative because its direction is opposite to its velocity, which is positive.

Significance

The final velocity is much less than the initial velocity, as desired when slowing down, but is still positive (see figure). With jet engines, reverse thrust can be maintained long enough to stop the plane and start moving it backward, which is indicated by a negative final velocity, but is not the case here.

In addition to being useful in problem solving, the equation $v = v_0 + at$ gives us insight into the relationships among velocity, acceleration, and time. We can see, for example, that

- Final velocity depends on how large the acceleration is and how long it lasts
- If the acceleration is zero, then the final velocity equals the initial velocity ($v = v_0$), as expected (in other words, velocity is constant)
- If a is negative, then the final velocity is less than the initial velocity

All these observations fit our intuition. Note that it is always useful to examine basic equations in light of our intuition and experience to check that they do indeed describe nature accurately.

Solving for Final Position with Constant Acceleration

We can combine the previous equations to find a third equation that allows us to calculate the final position of an object experiencing constant acceleration. We start with

$$v = v_0 + at. \tag{4.5.14}$$

Adding v_0 to each side of this equation and dividing by 2 gives

$$\frac{v_0 + v}{2} = v_0 + \frac{1}{2}at. \tag{4.5.15}$$

Since $\frac{v_0+v}{2} = \bar{v}$ for constant acceleration, we have

$$\bar{v} = v_0 + \frac{1}{2}at. \tag{4.5.16}$$

Now we substitute this expression for $\bar{v}$ into the equation for displacement, $x = x_0 + \bar{v}t$, yielding

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \ (constant\ a). \tag{4.5.17}$$

✓ Example 3.8: Calculating Displacement of an Accelerating Object

Dragsters can achieve an average acceleration of 26.0 m/s^2. Suppose a dragster accelerates from rest at this rate for 5.56 s Figure 4.5.3. How far does it travel in this time?

Figure 4.5.3: U.S. Army Top Fuel pilot Tony "The Sarge" Schumacher begins a race with a controlled burnout. (credit: Lt. Col. William Thurmond. Photo Courtesy of U.S. Army.)

Strategy

First, let's draw a sketch Figure 4.5.4. We are asked to find displacement, which is x if we take x_0 to be zero. (Think about x_0 as the starting line of a race. It can be anywhere, but we call it zero and measure all other positions relative to it.) We can use the equation $x = x_0 + v_0 t + \frac{1}{2}at^2$ when we identify v_0, a, and t from the statement of the problem.

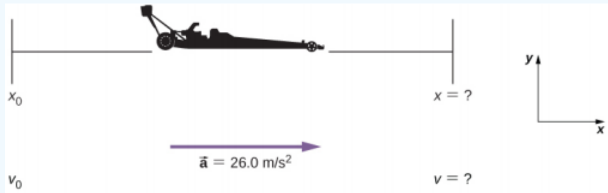


Figure 4.5.4: Sketch of an accelerating dragster.

Solution

First, we need to identify the knowns. Starting from rest means that $v_0 = 0$, a is given as 26.0 m/s^2 and t is given as 5.56 s.

Second, we substitute the known values into the equation to solve for the unknown:

$$x = x_0 + v_0 t + \frac{1}{2}at^2. \tag{4.5.18}$$

Since the initial position and velocity are both zero, this equation simplifies to

$$x = \frac{1}{2}at^2. \tag{4.5.19}$$

Substituting the identified values of a and t gives

$$x = \frac{1}{2}(26.0\ m/s^2)(5.56\ s)^2 = 402\ m. \tag{4.5.20}$$

Significance

If we convert 402 m to miles, we find that the distance covered is very close to one-quarter of a mile, the standard distance for drag racing. So, our answer is reasonable. This is an impressive displacement to cover in only 5.56 s, but top-notch dragsters can do a quarter mile in even less time than this. If the dragster were given an initial velocity, this would add another term to the distance equation. If the same acceleration and time are used in the equation, the distance covered would be much greater.

What else can we learn by examining the equation $x = x_0 + v_0 t + \frac{1}{2}at^2$? We can see the following relationships:

- Displacement depends on the square of the elapsed time when acceleration is not zero. In Example 3.8, the dragster covers only one-fourth of the total distance in the first half of the elapsed time.
- If acceleration is zero, then initial velocity equals average velocity ($v_0 = \bar{v}$) , and $x = x_0 + v_0 t + \frac{1}{2}at^2$ becomes $x = x_0 + v_0 t$.

Solving for Final Velocity from Distance and Acceleration

A fourth useful equation can be obtained from another algebraic manipulation of previous equations. If we solve $v = v_0 + at$ for t, we get

$$t = \frac{v - v_0}{a}. \tag{4.5.21}$$

Substituting this and $\bar{v} = \frac{v_0 + v}{2}$ into $x = x_0 + \bar{v}t$, we get

$$v^2 = v_0^2 + 2a(x - x_0)\ (constant\ a). \tag{4.5.22}$$

✓ Example 3.9: Calculating Final Velocity

Calculate the final velocity of the dragster in Example 3.8 without using information about time.

Strategy

The equation $v^2 = v_0^2 + 2a(x - x_0)$ is ideally suited to this task because it relates velocities, acceleration, and displacement, and no time information is required.

Solution

First, we identify the known values. We know that $v_0 = 0$, since the dragster starts from rest. We also know that $x - x_0 = 402$ m (this was the answer in Example 3.8). The average acceleration was given by $a = 26.0\ m/s^2$. Second, we substitute the knowns into the equation $v^2 = v_0^2 + 2a(x - x_0)$ and solve for v:

$$v^2 = 0 + 2(26.0\ m/s^2)(402\ m). \tag{4.5.23}$$

Thus,

$$v^2 = 2.09 \times 10^4\ m/s^2 \tag{4.5.24}$$

$$v = \sqrt{2.09 \times 10^4\ m^2/s^2} = 145\ m/s. \tag{4.5.25}$$

Significance

A velocity of 145 m/s is about 522 km/h, or about 324 mi/h, but even this breakneck speed is short of the record for the quarter mile. Also, note that a square root has two values; we took the positive value to indicate a velocity in the same direction as the acceleration.

An examination of the equation $v^2 = v_0^2 + 2a(x - x_0)$ can produce additional insights into the general relationships among physical quantities:

- The final velocity depends on how large the acceleration is and the distance over which it acts.
- For a fixed acceleration, a car that is going twice as fast doesn't simply stop in twice the distance. It takes much farther to stop. (This is why we have reduced speed zones near schools.)

4.6: Motion with Constant Acceleration (Part 2)

Putting Equations Together

In the following examples, we continue to explore one-dimensional motion, but in situations requiring slightly more algebraic manipulation. The examples also give insight into problem-solving techniques. The note that follows is provided for easy reference to the equations needed. Be aware that these equations are not independent. In many situations we have two unknowns and need two equations from the set to solve for the unknowns. We need as many equations as there are unknowns to solve a given situation.

Summary of Kinematic Equations (constant a)

$$x = x_0 + \bar{v}t \tag{4.6.1}$$

$$\bar{v} = \frac{v_0 + v}{2} \tag{4.6.2}$$

$$v = v_0 + at \tag{4.6.3}$$

$$x = x_0 + v_0 t + \frac{1}{2}at^2 \tag{4.6.4}$$

$$v^2 = v_0^2 + 2a(x - x_0) \tag{4.6.5}$$

Before we get into the examples, let's look at some of the equations more closely to see the behavior of acceleration at extreme values. Rearranging $v = v_0 + at$, we have

$$a = \frac{v - v_0}{t}. \tag{4.6.6}$$

From this we see that, for a finite time, if the difference between the initial and final velocities is small, the acceleration is small, approaching zero in the limit that the initial and final velocities are equal. On the contrary, in the limit t → 0 for a finite difference between the initial and final velocities, acceleration becomes infinite.

Similarly, rearranging $v^2 = v_0^2 + 2a(x - x_0)$, we can express acceleration in terms of velocities and displacement:

$$a = \frac{v^2 - v_0^2}{2(x - x_0)}. \tag{4.6.7}$$

Thus, for a finite difference between the initial and final velocities acceleration becomes infinite in the limit the displacement approaches zero. Acceleration approaches zero in the limit the difference in initial and final velocities approaches zero for a finite displacement.

Example 3.10: How Far Does a Car Go?

On dry concrete, a car can decelerate at a rate of 7.00 m/s^2, whereas on wet concrete it can decelerate at only 5.00 m/s^2. Find the distances necessary to stop a car moving at 30.0 m/s (about 110 km/h) on (a) dry concrete and (b) wet concrete. (c) Repeat both calculations and find the displacement from the point where the driver sees a traffic light turn red, taking into account his reaction time of 0.500 s to get his foot on the brake.

Strategy

First, we need to draw a sketch Figure 4.6.1. To determine which equations are best to use, we need to list all the known values and identify exactly what we need to solve for.

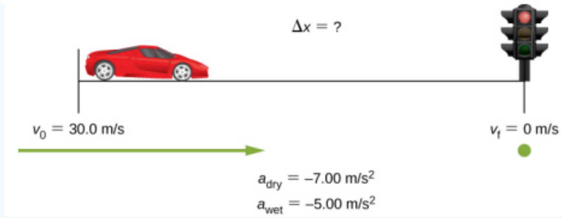


Figure 4.6.1: Sample sketch to visualize deceleration and stopping distance of a car.

Solution

a. First, we need to identify the knowns and what we want to solve for. We know that v_0 = 30.0 m/s, v = 0, and a = −7.00 m/s^2 (a is negative because it is in a direction opposite to velocity). We take x0 to be zero. We are looking for displacement Δx, or x − x_0. Second, we identify the equation that will help us solve the problem. The best equation to use is

$$v^2 = v_0^2 + 2a(x - x_0). \tag{4.6.8}$$

This equation is best because it includes only one unknown, x. We know the values of all the other variables in this equation. (Other equations would allow us to solve for x, but they require us to know the stopping time, t, which we do not know. We could use them, but it would entail additional calculations.) Third, we rearrange the equation to solve for x:

$$x - x_0 = \frac{v^2 - v_0^2}{2a} \tag{4.6.9}$$

and substitute the known values:

$$x - 0 = \frac{0^2 - (30.0\ m/s)^2}{2(-7.00\ m/s^2)}. \tag{4.6.10}$$

Thus,

$$x = 64.3\ m\ on\ dry\ concrete. \tag{4.6.11}$$

b. This part can be solved in exactly the same manner as (a). The only difference is that the acceleration is −5.00 m/s^2. The result is

$$x_{wet} = 90.0\ m\ on\ wet\ concrete. \tag{4.6.12}$$

c. When the driver reacts, the stopping distance is the same as it is in (a) and (b) for dry and wet concrete. So, to answer this question, we need to calculate how far the car travels during the reaction time, and then add that to the stopping time. It is reasonable to assume the velocity remains constant during the driver's reaction time. To do this, we, again, identify the knowns and what we want to solve for. We know that $\bar{v}$ = 30.0 m/s, $t_{reaction}$ = 0.500 s, and $a_{reaction}$ = 0. We take $x_{0\text{-}reaction}$ to be zero. We are looking for $x_{reaction}$. Second, as before, we identify the best equation to use. In this case, x = x_0 + $\bar{v}$t works well because the only unknown value is x, which is what we want to solve for. Third, we substitute the knowns to solve the equation:

$$x = 0 + (30.0\ m/s)(0.500\ s) = 15.0\ m. \tag{4.6.13}$$

This means the car travels 15.0 m while the driver reacts, making the total displacements in the two cases of dry and wet concrete 15.0 m greater than if he reacted instantly. Last, we then add the displacement during the reaction time to the displacement when braking (Figure 4.6.2),

$$x_{braking} + x_{reaction} = x_{total}, \tag{4.6.14}$$

and find (a) to be 64.3 m + 15.0 m = 79.3 m when dry and (b) to be 90.0 m + 15.0 m = 105 m when wet.

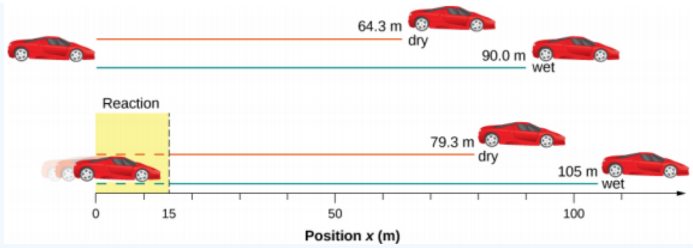


Figure 4.6.2: The distance necessary to stop a car varies greatly, depending on road conditions and driver reaction time. Shown here are the braking distances for dry and wet pavement, as calculated in this example, for a car traveling initially at 30.0 m/s. Also shown are the total distances traveled from the point when the driver first sees a light turn red, assuming a 0.500-s reaction time.

Significance

The displacements found in this example seem reasonable for stopping a fast-moving car. It should take longer to stop a car on wet pavement than dry. It is interesting that reaction time adds significantly to the displacements, but more important is the general approach to solving problems. We identify the knowns and the quantities to be determined, then find an appropriate equation. If there is more than one unknown, we need as many independent equations as there are unknowns to solve. There is often more than one way to solve a problem. The various parts of this example can, in fact, be solved by other methods, but the solutions presented here are the shortest.

Example 3.11: Calculating Time

Suppose a car merges into freeway traffic on a 200-m-long ramp. If its initial velocity is 10.0 m/s and it accelerates at 2.00 m/s^2, how long does it take the car to travel the 200 m up the ramp? (Such information might be useful to a traffic engineer.)

Strategy

First, we draw a sketch Figure 4.6.3. We are asked to solve for time t. As before, we identify the known quantities to choose a convenient physical relationship (that is, an equation with one unknown, t.)

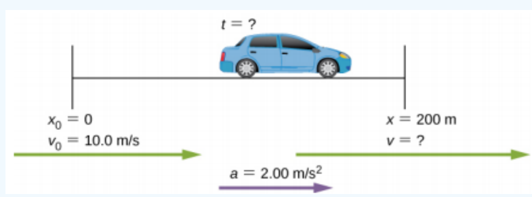


Figure 4.6.3: Sketch of a car accelerating on a freeway ramp.

Solution

Again, we identify the knowns and what we want to solve for. We know that $x_0 = 0$, $v_0 = 10$ m/s, $a = 2.00$ m/s^2, and $x = 200$ m.

We need to solve for t. The equation $x = x_0 + v_0 t + \frac{1}{2} a t^2$ works best because the only unknown in the equation is the variable t, for which we need to solve. From this insight we see that when we input the knowns into the equation, we end up with a quadratic equation.

We need to rearrange the equation to solve for t, then substituting the knowns into the equation:

$$200\ m = 0\ m + (10.0\ m/s)t + \frac{1}{2}(2.00\ m/s^2)t^2. \tag{4.6.15}$$

We then simplify the equation. The units of meters cancel because they are in each term. We can get the units of seconds to cancel by taking t = t s, where t is the magnitude of time and s is the unit. Doing so leaves

$$200 = 10t + t^2. \tag{4.6.16}$$

We then use the quadratic formula to solve for t,

$$t^2 + 10t - 200 = 0 \tag{4.6.17}$$

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}, \tag{4.6.18}$$

which yields two solutions: t = 10.0 and t = −20.0. A negative value for time is unreasonable, since it would mean the event happened 20 s before the motion began. We can discard that solution. Thus,

$$t = 10.0\ s. \tag{4.6.19}$$

Significance

Whenever an equation contains an unknown squared, there are two solutions. In some problems both solutions are meaningful; in others, only one solution is reasonable. The 10.0-s answer seems reasonable for a typical freeway on-ramp.

Exercise 3.5

A manned rocket accelerates at a rate of 20 m/s^2 during launch. How long does it take the rocket to reach a velocity of 400 m/s?

Example 3.12: Acceleration of a Spaceship

A spaceship has left Earth's orbit and is on its way to the Moon. It accelerates at 20 m/s^2 for 2 min and covers a distance of 1000 km. What are the initial and final velocities of the spaceship?

Strategy

We are asked to find the initial and final velocities of the spaceship. Looking at the kinematic equations, we see that one equation will not give the answer. We must use one kinematic equation to solve for one of the velocities and substitute it into another kinematic equation to get the second velocity. Thus, we solve two of the kinematic equations simultaneously.

Solution

First we solve for v_0 using $x = x_0 + v_0 t + \frac{1}{2}at^2 = \frac{1}{2}t^2$:

$$x - x_0 = v_0 t + \frac{1}{2}at^2 = \frac{1}{2}t^2 \tag{4.6.20}$$

$$1.0 \times 10^6\ m = v_0(120.0\ s) + \frac{1}{2}(20.0\ m/s^2)(120.0\ s)^2 \tag{4.6.21}$$

$$v_0 = 7133.3\ m/s. \tag{4.6.22}$$

Then we substitute v_0 into v = v_0 + at to solve for the final velocity:

$$v = v_0 + at = 7133.3\ m/s + (20.0\ m/s^2)(120.0\ s) = 9533.3\ m/s. \tag{4.6.23}$$

Significance

There are six variables in displacement, time, velocity, and acceleration that describe motion in one dimension. The initial conditions of a given problem can be many combinations of these variables. Because of this diversity, solutions may not be easy as simple substitutions into one of the equations. This example illustrates that solutions to kinematics may require solving two simultaneous kinematic equations.

With the basics of kinematics established, we can go on to many other interesting examples and applications. In the process of developing kinematics, we have also glimpsed a general approach to problem solving that produces both correct answers and insights into physical relationships. The next level of complexity in our kinematics problems involves the motion of two interrelated bodies, called **two-body pursuit problems**.

Two-Body Pursuit Problems

Up until this point we have looked at examples of motion involving a single body. Even for the problem with two cars and the stopping distances on wet and dry roads, we divided this problem into two separate problems to find the answers. In a **two-body pursuit problem**, the motions of the objects are coupled—meaning, the unknown we seek depends on the motion of both objects. To solve these problems we write the equations of motion for each object and then solve them simultaneously to find the unknown. This is illustrated in Figure 4.6.4.

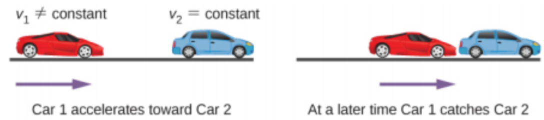


Figure 4.6.4: A two-body pursuit scenario where car 2 has a constant velocity and car 1 is behind with a constant acceleration. Car 1 catches up with car 2 at a later time.

The time and distance required for car 1 to catch car 2 depends on the initial distance car 1 is from car 2 as well as the velocities of both cars and the acceleration of car 1. The kinematic equations describing the motion of both cars must be solved to find these unknowns.

Consider the following example.

Example 3.13: Cheetah Catching a Gazelle

A cheetah waits in hiding behind a bush. The cheetah spots a gazelle running past at 10 m/s. At the instant the gazelle passes the cheetah, the cheetah accelerates from rest at 4 m/s^2 to catch the gazelle. (a) How long does it take the cheetah to catch the gazelle? (b) What is the displacement of the gazelle and cheetah?

Strategy

We use the set of equations for constant acceleration to solve this problem. Since there are two objects in motion, we have separate equations of motion describing each animal. But what links the equations is a common parameter that has the same value for each animal. If we look at the problem closely, it is clear the common parameter to each animal is their position x at a later time t. Since they both start at $x_0 = 0$, their displacements are the same at a later time t, when the cheetah catches up with the gazelle. If we pick the equation of motion that solves for the displacement for each animal, we can then set the equations equal to each other and solve for the unknown, which is time.

Solution

a. Equation for the gazelle: The gazelle has a constant velocity, which is its average velocity, since it is not accelerating. Therefore, we use Equation 3.5.7 with $x_0 = 0$:

$$x = x_0 + \bar{v}t = \bar{v}t. \tag{4.6.24}$$

Equation for the cheetah: The cheetah is accelerating from rest, so we use Equation 3.5.17 with $x_0 = 0$ and $v_0 = 0$:

$$x = x_0 + v_0 t + \frac{1}{2}at^2 = \frac{1}{2}at^2. \tag{4.6.25}$$

Now we have an equation of motion for each animal with a common parameter, which can be eliminated to find the solution. In this case, we solve for t:

$$x = \bar{v}t = \frac{1}{2}at^2 \tag{4.6.26}$$

$$t = \frac{2\bar{v}}{a}. \tag{4.6.27}$$

The gazelle has a constant velocity of 10 m/s, which is its average velocity. The acceleration of the cheetah is 4 m/s^2. Evaluating t, the time for the cheetah to reach the gazelle, we have

$$t = \frac{2\bar{v}}{a} = \frac{2(10)}{4} = 5\ s. \tag{4.6.28}$$

b. To get the displacement, we use either the equation of motion for the cheetah or the gazelle, since they should both give the same answer. Displacement of the cheetah:

$$x = \frac{1}{2}at^2 = \frac{1}{2}(4)(5)^2 - 50\ m. \tag{4.6.29}$$

Displacement of the gazelle:

$$x = \bar{v}t = 10(5) = 50\ m. \tag{4.6.30}$$

We see that both displacements are equal, as expected.

Significance

It is important to analyze the motion of each object and to use the appropriate kinematic equations to describe the individual motion. It is also important to have a good visual perspective of the two-body pursuit problem to see the common parameter that links the motion of both objects.

? Exercise 3.6

A bicycle has a constant velocity of 10 m/s. A person starts from rest and begins to run to catch up to the bicycle in 30 s when the bicycle is at the same position as the person. What is the acceleration of the person?

4.7: Free Fall

Learning Objectives

- Use the kinematic equations with the variables y and g to analyze free-fall motion.
- Describe how the values of the position, velocity, and acceleration change during a free fall.
- Solve for the position, velocity, and acceleration as functions of time when an object is in a free fall.

An interesting application of Equation 3.3.2 through Equation 3.5.22 is called **free fall**, which describes the motion of an object falling in a gravitational field, such as near the surface of Earth or other celestial objects of planetary size. Let's assume the body is falling in a straight line perpendicular to the surface, so its motion is one-dimensional. For example, we can estimate the depth of a vertical mine shaft by dropping a rock into it and listening for the rock to hit the bottom. But "falling," in the context of free fall, does not necessarily imply the body is moving from a greater height to a lesser height. If a ball is thrown upward, the equations of free fall apply equally to its ascent as well as its descent.

Gravity

The most remarkable and unexpected fact about falling objects is that if air resistance and friction are negligible, then in a given location all objects fall toward the center of Earth with the **same constant acceleration, independent of their mass**. This experimentally determined fact is unexpected because we are so accustomed to the effects of air resistance and friction that we expect light objects to fall slower than heavy ones. Until Galileo Galilei (1564–1642) proved otherwise, people believed that a heavier object has a greater acceleration in a free fall. We now know this is not the case. In the absence of air resistance, heavy objects arrive at the ground at the same time as lighter objects when dropped from the same height Figure 4.7.1.

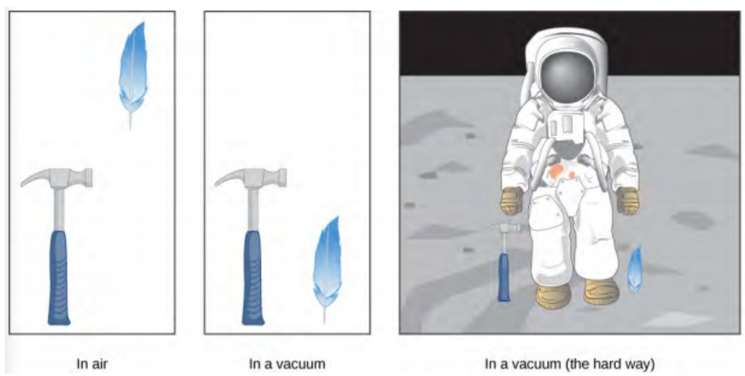


Figure 4.7.1: A hammer and a feather fall with the same constant acceleration if air resistance is negligible. This is a general characteristic of gravity not unique to Earth, as astronaut David R. Scott demonstrated in 1971 on the Moon, where the acceleration from gravity is only 1.67 m/s^2 and there is no atmosphere.

In the real world, air resistance can cause a lighter object to fall slower than a heavier object of the same size. A tennis ball reaches the ground after a baseball dropped at the same time. (It might be difficult to observe the difference if the height is not large.) Air resistance opposes the motion of an object through the air, and friction between objects—such as between clothes and a laundry chute or between a stone and a pool into which it is dropped—also opposes motion between them.

For the ideal situations of these first few chapters, an object **falling without air resistance or friction** is defined to be in **free fall**. The force of gravity causes objects to fall toward the center of Earth. The acceleration of free-falling objects is therefore called **acceleration due to gravity**. Acceleration due to gravity is constant, which means we can apply the kinematic equations to any falling object where air resistance and friction are negligible. This opens to us a broad class of interesting situations.

Acceleration due to gravity is so important that its magnitude is given its own symbol, g. It is constant at any given location on Earth and has the average value

$$g = 9.81\ m/s^2\ (or\ 32.2\ ft/s^2). \tag{4.7.1}$$

Although g varies from 9.78 m/s^2 to 9.83 m/s^2, depending on latitude, altitude, underlying geological formations, and local topography, let's use an average value of 9.8 m/s^2 rounded to two significant figures in this text unless specified otherwise. Neglecting these effects on the value of g as a result of position on Earth's surface, as well as effects resulting from Earth's rotation, we take the direction of acceleration due to gravity to be downward (toward the center of Earth). In fact, its direction **defines** what we call vertical. Note that whether acceleration a in the kinematic equations has the value +g or −g depends on how we define our coordinate system. If we define the upward direction as positive, then a = −g = −9.8 m/s^2, and if we define the downward direction as positive, then a = g = 9.8 m/s^2.

One-Dimensional Motion Involving Gravity

The best way to see the basic features of motion involving gravity is to start with the simplest situations and then progress toward more complex ones. So, we start by considering straight up-and-down motion with no air resistance or friction. These assumptions mean the velocity (if there is any) is vertical. If an object is dropped, we know the initial velocity is zero when in free fall. When the object has left contact with whatever held or threw it, the object is in free fall. When the object is thrown, it has the same initial speed in free fall as it did before it was released. When the object comes in contact with the ground or any other object, it is no longer in free fall and its acceleration of g is no longer valid. Under these circumstances, the motion is one-dimensional and has constant acceleration of magnitude g. We represent vertical displacement with the symbol y.

Kinematic Equations for Objects in Free Fall

We assume here that acceleration equals −g (with the positive direction upward).

$$v = v_0 - gt \tag{4.7.2}$$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{4.7.3}$$

$$v^2 = v_0^2 - 2g(y - y_0) \tag{4.7.4}$$

Problem-Solving Strategy: Free Fall

1. Decide on the sign of the acceleration of gravity. In Equation 4.7.2 through Equation 4.7.4, acceleration g is negative, which says the positive direction is upward and the negative direction is downward. In some problems, it may be useful to have acceleration g as positive, indicating the positive direction is downward.
2. Draw a sketch of the problem. This helps visualize the physics involved.
3. Record the knowns and unknowns from the problem description. This helps devise a strategy for selecting the appropriate equations to solve the problem.
4. Decide which of Equation 4.7.2 through Equation 4.7.4 are to be used to solve for the unknowns.

Example 3.14: Free Fall of a Ball

Figure 4.7.2 shows the positions of a ball, at 1-s intervals, with an initial velocity of 4.9 m/s downward, that is thrown from the top of a 98-m-high building. (a) How much time elapses before the ball reaches the ground? (b) What is the velocity when it arrives at the ground?

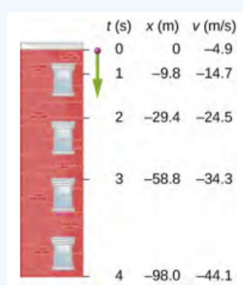


Figure 4.7.2: The positions and velocities at 1-s intervals of a ball thrown downward from a tall building at 4.9 m/s.

Strategy

Choose the origin at the top of the building with the positive direction upward and the negative direction downward. To find the time when the position is −98 m, we use Equation 4.7.3, with $y_0 = 0$, $v_0 = -4.9$ m/s, and g = 9.8 m/s^2.

Solution

a. Substitute the given values into the equation:

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{4.7.5}$$

$$-98.0\ m = 0 - (4.9\ m/s)t - \frac{1}{2}(9.8\ m/s^2)t^2. \tag{4.7.6}$$

This simplifies to

$$t^2 + t - 20 = 0. \tag{4.7.7}$$

This is a quadratic equation with roots t = −5.0 s and t = 4.0 s. The positive root is the one we are interested in, since time t = 0 is the time when the ball is released at the top of the building. (The time t = −5.0 s represents the fact that a ball thrown upward from the ground would have been in the air for 5.0 s when it passed by the top of the building moving downward at 4.9 m/s.)

b. Using Equation 4.7.2, we have

$$v = v_0 - gt = -4.9\ m/s - (9.8\ m/s^2)(4.0\ s) = -44.1\ m/s. \tag{4.7.8}$$

Significance

For situations when two roots are obtained from a quadratic equation in the time variable, we must look at the physical significance of both roots to determine which is correct. Since t = 0 corresponds to the time when the ball was released, the negative root would correspond to a time before the ball was released, which is not physically meaningful. When the ball hits the ground, its velocity is not immediately zero, but as soon as the ball interacts with the ground, its acceleration is not g and it accelerates with a different value over a short time to zero velocity. This problem shows how important it is to establish the correct coordinate system and to keep the signs of g in the kinematic equations consistent.

✔ Example 3.15: Vertical Motion of a Baseball

A batter hits a baseball straight upward at home plate and the ball is caught 5.0 s after it is struck Figure 4.7.3. (a) What is the initial velocity of the ball? (b) What is the maximum height the ball reaches? (c) How long does it take to reach the maximum height? (d) What is the acceleration at the top of its path? (e) What is the velocity of the ball when it is caught? Assume the ball is hit and caught at the same location.

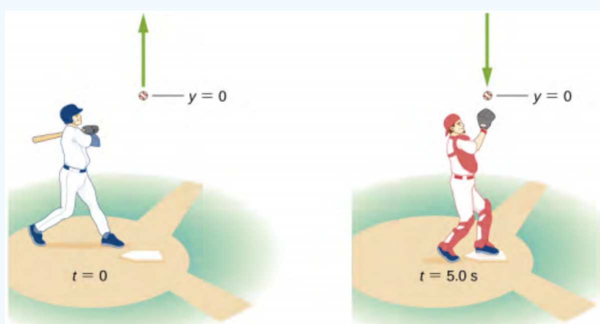


Figure 4.7.3: A baseball hit straight up is caught by the catcher 5.0 s later.

Strategy

Choose a coordinate system with a positive y-axis that is straight up and with an origin that is at the spot where the ball is hit and caught.

Solution

a. Equation 4.7.3 gives

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \tag{4.7.9}$$

$$0 = 0 + v_0(5.0\ s) - \frac{1}{2}(9.8\ m/s^2)(5.0\ s)^2. \tag{4.7.10}$$

which gives v_0 = 24.5 m/sec.

b. At the maximum height, v = 0. With v_0 = 24.5 m/s, Equation 4.7.4 gives

$$v^2 = v_0^2 - 2g(y - y_0) \tag{4.7.11}$$

$$0 = (24.5\ m/s^2) - 2(9.8\ m/s^2)(y - 0) \tag{4.7.12}$$

or

$$y = 30.6\ m. \tag{4.7.13}$$

c. To find the time when v = 0 , we use Equation 4.7.2:

$$v = v_0 - gt \tag{4.7.14}$$

$$0 = 24..5\ m/s - (9.8\ m/s^2)t. \tag{4.7.15}$$

This gives t = 2.5 s. Since the ball rises for 2.5 s, the time to fall is 2.5 s.

d. The acceleration is 9.8 m/s^2 everywhere, even when the velocity is zero at the top of the path. Although the velocity is zero at the top, it is changing at the rate of 9.8 m/s^2 downward.

e. The velocity at t = 5.0 s can be determined with Equation 4.7.2:

$$\begin{aligned} v &= v_0 - gt \\ &= 24.5\ m/s - 9.8\ m/s^2(5.0\ s) \\ &= -24.5\ m/s. \end{aligned}$$

Significance

The ball returns with the speed it had when it left. This is a general property of free fall for any initial velocity. We used a single equation to go from throw to catch, and did not have to break the motion into two segments, upward and downward. We are used to thinking of the effect of gravity is to create free fall downward toward Earth. It is important to understand, as illustrated in this example, that objects moving upward away from Earth are also in a state of free fall.

? Exercise 3.7

A chunk of ice breaks off a glacier and falls 30.0 m before it hits the water. Assuming it falls freely (there is no air resistance), how long does it take to hit the water? Which quantity increases faster, the speed of the ice chunk or its distance traveled?

✔ Example 3.16: Rocket Booster

A small rocket with a booster blasts off and heads straight upward. When at a height of 5.0 km and velocity of 200.0 m/s, it releases its booster. (a) What is the maximum height the booster attains? (b) What is the velocity of the booster at a height of 6.0 km? Neglect air resistance.

Figure 4.7.4: A rocket releases its booster at a given height and velocity. How high and how fast does the booster go?

Strategy

We need to select the coordinate system for the acceleration of gravity, which we take as negative downward. We are given the initial velocity of the booster and its height. We consider the point of release as the origin. We know the velocity is zero at the maximum position within the acceleration interval; thus, the velocity of the booster is zero at its maximum height, so we can use this information as well. From these observations, we use Equation 4.7.4, which gives us the maximum height of the booster. We also use Equation 4.7.4 to give the velocity at 6.0 km. The initial velocity of the booster is 200.0 m/s.

Solution

a. From Equation 4.7.4, $v^2 = v_0^2 - 2g(y - y_0)$. With v = 0 and y_0 = 0, we can solve for y:

$$y = \frac{v_0^2}{-2g} = \frac{(2.0 \times 10^2\ m/s)^2}{-2(9.8\ m/s^2)} = 2040.8\ m. \tag{4.7.16}$$

This solution gives the maximum height of the booster in our coordinate system, which has its origin at the point of release, so the maximum height of the booster is roughly 7.0 km.

b. An altitude of 6.0 km corresponds to y = 1.0 x 10^3 m in the coordinate system we are using. The other initial conditions are y_0 = 0, and v_0 = 200.0 m/s. We have, from Equation 4.7.4,

$$v^2 = (200.0\ m/s)^2 - 2(9.8\ m/s^2)(1.0 \times 10^3\ m) \Rightarrow v = \pm 142.8\ m/s. \tag{4.7.17}$$

Significance

We have both a positive and negative solution in (b). Since our coordinate system has the positive direction upward, the +142.8 m/s corresponds to a positive upward velocity at 6000 m during the upward leg of the trajectory of the booster. The value v = −142.8 m/s corresponds to the velocity at 6000 m on the downward leg. This example is also important in that an object is given an initial velocity at the origin of our coordinate system, but the origin is at an altitude above the surface of Earth, which must be taken into account when forming the solution.

Simulation

Visit this site to learn about graphing polynomials. The shape of the curve changes as the constants are adjusted. View the curves for the individual terms (for example, y = bx) to see how they add to generate the polynomial curve.

4.8: Finding Velocity and Displacement from Acceleration

Learning Objectives

- Derive the kinematic equations for constant acceleration using integral calculus.
- Use the integral formulation of the kinematic equations in analyzing motion.
- Find the functional form of velocity versus time given the acceleration function.
- Find the functional form of position versus time given the velocity function.

This section assumes you have enough background in calculus to be familiar with integration. In Instantaneous Velocity and Speed and Average and Instantaneous Acceleration we introduced the kinematic functions of velocity and acceleration using the derivative. By taking the derivative of the position function we found the velocity function, and likewise by taking the derivative of the velocity function we found the acceleration function. Using integral calculus, we can work backward and calculate the velocity function from the acceleration function, and the position function from the velocity function.

Kinematic Equations from Integral Calculus

Let's begin with a particle with an acceleration a(t) is a known function of time. Since the time derivative of the velocity function is acceleration,

$$\frac{d}{dt}v(t) = a(t), \tag{4.8.1}$$

we can take the indefinite integral of both sides, finding

$$\int \frac{d}{dt}v(t)dt = \int a(t)dt + C_1, \tag{4.8.2}$$

where C_1 is a constant of integration. Since $\int \frac{d}{dt}v(t)dt = v(t)$, the velocity is given by

$$v(t) = \int a(t)dt + C_1. \tag{4.8.3}$$

Similarly, the time derivative of the position function is the velocity function,

$$\frac{d}{dt}x(t) = v(t). \tag{4.8.4}$$

Thus, we can use the same mathematical manipulations we just used and find

$$x(t) = \int v(t)dt + C_2, \tag{4.8.5}$$

where C_2 is a second constant of integration.

We can derive the kinematic equations for a constant acceleration using these integrals. With a(t) = a, a constant, and doing the integration in Equation 4.8.3, we find

$$v(t) = \int a dt + C_1 = at + C_1. \tag{4.8.6}$$

If the initial velocity is $v(0) = v_0$, then

$$v_0 = 0 + C_1. \tag{4.8.7}$$

Then, $C_1 = v_0$ and

$$v(t) = v_0 + at, \tag{4.8.8}$$

which is Equation 3.5.12. Substituting this expression into Equation 4.8.5 gives

$$x(t) = \int (v_0 + at)dt + C_2. \tag{4.8.9}$$

Doing the integration, we find

$$x(t)=v_0t+\frac{1}{2}at^2+C_2. \tag{4.8.10}$$

If x(0) = x_0, we have

$$x_0=0+0+C_2. \tag{4.8.11}$$

so, C_2 = x_0. Substituting back into the equation for x(t), we finally have

$$x(t)=x_0+v_0t+\frac{1}{2}at^2. \tag{4.8.12}$$

which is Equation 3.5.17.

Example 3.17: Motion of a Motorboat

A motorboat is traveling at a constant velocity of 5.0 m/s when it starts to decelerate to arrive at the dock. Its acceleration is a(t) = $-\frac{1}{4}$ t m/s^2. (a) What is the velocity function of the motorboat? (b) At what time does the velocity reach zero? (c) What is the position function of the motorboat? (d) What is the displacement of the motorboat from the time it begins to decelerate to when the velocity is zero? (e) Graph the velocity and position functions.

Strategy

(a) To get the velocity function we must integrate and use initial conditions to find the constant of integration. (b) We set the velocity function equal to zero and solve for t. (c) Similarly, we must integrate to find the position function and use initial conditions to find the constant of integration. (d) Since the initial position is taken to be zero, we only have to evaluate the position function at t = 0 .

Solution

We take t = 0 to be the time when the boat starts to decelerate.

a. From the functional form of the acceleration we can solve Equation 4.8.3 to get v(t):

$$v(t)=\int a(t)dt+C_1=\int -\frac{1}{4}tdt+C_1=-\frac{1}{8}t^2+C_1. \tag{4.8.13}$$

At t = 0 we have v(0) = 5.0 m/s = 0 + C_1, so C_1 = 5.0 m/s or v(t) = 5.0 m/s − $\frac{1}{8}$ t^2.

b. v(t) = 0 = 5.0 m/s − $\frac{1}{8}$ t^2 (\Rightarrow\) t = 6.3 s

c. Solve Equation 4.8.5:

$$x(t)=\int v(t)dt+C_2=\int (5.0-\frac{1}{8}t^2)dt+C_2=5.0t-\frac{1}{24}t^3+C_2. \tag{4.8.14}$$

At t = 0, we set x(0) = 0 = x_0, since we are only interested in the displacement from when the boat starts to decelerate. We have

$$x(0)=0=C_2. \tag{4.8.15}$$

Therefore, the equation for the position is

$$x(t)=5.0t-\frac{1}{24}t^3. \tag{4.8.16}$$

d. Since the initial position is taken to be zero, we only have to evaluate x(t) when the velocity is zero. This occurs at t = 6.3 s. Therefore, the displacement is

$$x(6.3)=5.0(6.3)-\frac{1}{24}(6.3)^3=21.1\ m. \tag{4.8.17}$$

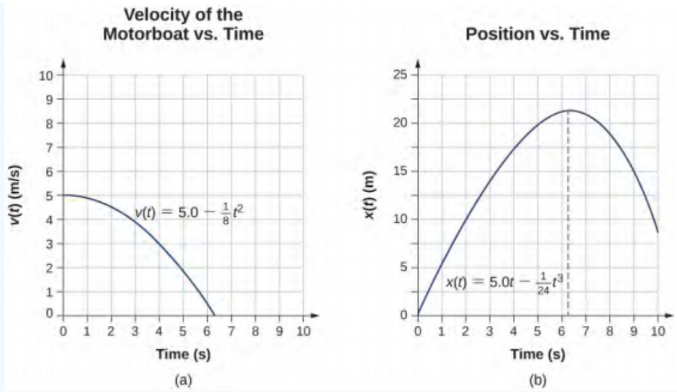


Figure 4.8.1: (a) Velocity of the motorboat as a function of time. The motorboat decreases its velocity to zero in 6.3 s. At times greater than this, velocity becomes negative—meaning, the boat is reversing direction. (b) Position of the motorboat as a function of time. At t = 6.3 s, the velocity is zero and the boat has stopped. At times greater than this, the velocity becomes negative—meaning, if the boat continues to move with the same acceleration, it reverses direction and heads back toward where it originated.

Significance

The acceleration function is linear in time so the integration involves simple polynomials. In Figure 4.8.1, we see that if we extend the solution beyond the point when the velocity is zero, the velocity becomes negative and the boat reverses direction. This tells us that solutions can give us information outside our immediate interest and we should be careful when interpreting them.

Exercise 3.8

A particle starts from rest and has an acceleration function $a(t) = \left(5 - \left(10\frac{1}{s}\right)t\right)\frac{m}{s^2}$. (a) What is the velocity function? (b) What is the position function? (c) When is the velocity zero?

4.9: Motion Along a Straight Line (Exercises)

Conceptual Questions

3.1 Position, Displacement, and Average Velocity

1. Give an example in which there are clear distinctions among distance traveled, displacement, and magnitude of displacement. Identify each quantity in your example specifically.
2. Under what circumstances does distance traveled equal magnitude of displacement? What is the only case in which magnitude of displacement and displacement are exactly the same?
3. Bacteria move back and forth using their flagella (structures that look like little tails). Speeds of up to 50 μm/s (50 x 10^{-6} m/s) have been observed. The total distance traveled by a bacterium is large for its size, whereas its displacement is small. Why is this?
4. Give an example of a device used to measure time and identify what change in that device indicates a change in time.
5. Does a car's odometer measure distance traveled or displacement?
6. During a given time interval the average velocity of an object is zero. What can you say conclude about its displacement over the time interval?

3.2 Instantaneous Velocity and Speed

7. There is a distinction between average speed and the magnitude of average velocity. Give an example that illustrates the difference between these two quantities.
8. Does the speedometer of a car measure speed or velocity?
9. If you divide the total distance traveled on a car trip (as determined by the odometer) by the elapsed time of the trip, are you calculating average speed or magnitude of average velocity? Under what circumstances are these two quantities the same?
10. How are instantaneous velocity and instantaneous speed related to one another? How do they differ?

3.3 Average and Instantaneous Acceleration

11. Is it possible for speed to be constant while acceleration is not zero?
12. Is it possible for velocity to be constant while acceleration is not zero? Explain.
13. Give an example in which velocity is zero yet acceleration is not.
14. If a subway train is moving to the left (has a negative velocity) and then comes to a stop, what is the direction of its acceleration? Is the acceleration positive or negative?
15. Plus and minus signs are used in one-dimensional motion to indicate direction. What is the sign of an acceleration that reduces the magnitude of a negative velocity? Of a positive velocity?

3.4 Motion with Constant Acceleration

16. When analyzing the motion of a single object, what is the required number of known physical variables that are needed to solve for the unknown quantities using the kinematic equations?
17. State two scenarios of the kinematics of single object where three known quantities require two kinematic equations to solve for the unknowns.

3.5 Free Fall

18. What is the acceleration of a rock thrown straight upward on the way up? At the top of its flight? On the way down? Assume there is no air resistance.
19. An object that is thrown straight up falls back to Earth. This is one-dimensional motion. (a) When is its velocity zero? (b) Does its velocity change direction? (c) Does the acceleration have the same sign on the way up as on the way down?
20. Suppose you throw a rock nearly straight up at a coconut in a palm tree and the rock just misses the coconut on the way up but hits the coconut on the way down. Neglecting air resistance and the slight horizontal variation in motion to account for the hit and miss of the coconut, how does the speed of the rock when it hits the coconut on the way down compare with what it would have been if it had hit the coconut on the way up? Is it more likely to dislodge the coconut on the way up or down? Explain.
21. The severity of a fall depends on your speed when you strike the ground. All factors but the acceleration from gravity being the same, how many times higher could a safe fall on the Moon than on Earth (gravitational acceleration on the

Moon is about one-sixth that of the Earth)?

22. How many times higher could an astronaut jump on the Moon than on Earth if her takeoff speed is the same in both locations (gravitational acceleration on the Moon is about on-sixth of that on Earth)?

3.6 Finding Velocity and Displacement from Acceleration

23. When given the acceleration function, what additional information is needed to find the velocity function and position function?

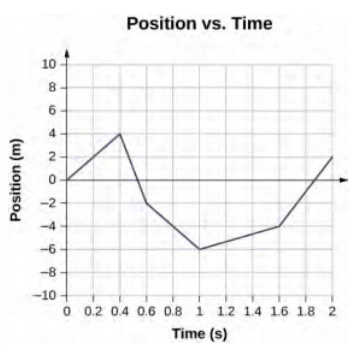
Problems

3.1 Position, Displacement, and Average Velocity

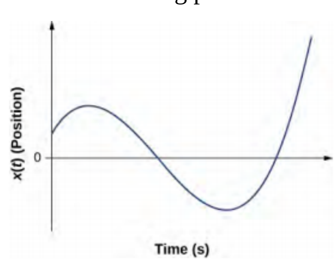
24. Consider a coordinate system in which the positive x axis is directed upward vertically. What are the positions of a particle (a) 5.0 m directly above the origin and (b) 2.0 m below the origin?
25. A car is 2.0 km west of a traffic light at t = 0 and 5.0 km east of the light at t = 6.0 min. Assume the origin of the coordinate system is the light and the positive x direction is eastward. (a) What are the car's position vectors at these two times? (b) What is the car's displacement between 0 min and 6.0 min?
26. The Shanghai maglev train connects Longyang Road to Pudong International Airport, a distance of 30 km. The journey takes 8 minutes on average. What is the maglev train's average velocity?
27. The position of a particle moving along the x-axis is given by $x(t) = 4.0 - 2.0t$ m. (a) At what time does the particle cross the origin? (b) What is the displacement of the particle between t = 3.0 s and t = 6.0 s?
28. A cyclist rides 8.0 km east for 20 minutes, then he turns and heads west for 8 minutes and 3.2 km. Finally, he rides east for 16 km, which takes 40 minutes. (a) What is the final displacement of the cyclist? (b) What is his average velocity?
29. On February 15, 2013, a superbolide meteor (brighter than the Sun) entered Earth's atmosphere over Chelyabinsk, Russia, and exploded at an altitude of 23.5 km. Eyewitnesses could feel the intense heat from the fireball, and the blast wave from the explosion blew out windows in buildings. The blast wave took approximately 2 minutes 30 seconds to reach ground level. (a) What was the average velocity of the blast wave? b) Compare this with the speed of sound, which is 343 m/s at sea level.

3.2 Instantaneous Velocity and Speed

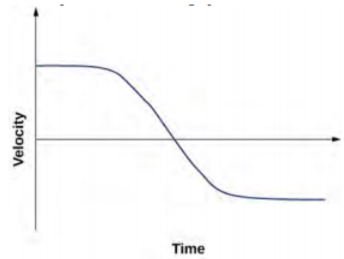
30. A woodchuck runs 20 m to the right in 5 s, then turns and runs 10 m to the left in 3 s. (a) What is the average velocity of the woodchuck? (b) What is its average speed?
31. Sketch the velocity-versus-time graph from the following position-versus-time graph.



32. Sketch the velocity-versus-time graph from the following position-versus-time graph.



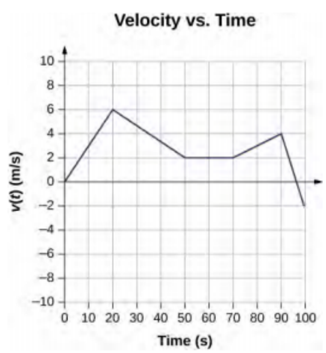
33. Given the following velocity-versus-time graph, sketch the position-versus-time graph.



34. An object has a position function x(t) = 5t m. (a) What is the velocity as a function of time? (b) Graph the position function and the velocity function. 35. A particle moves along the x-axis according to $x(t) = 10t - 2t^2$ m. (a) What is the instantaneous velocity at t = 2 s and t = 3 s? (b) What is the instantaneous speed at these times? (c) What is the average velocity between t = 2 s and t = 3 s?
35. **Unreasonable results**. A particle moves along the x-axis according to $x(t) = 3t^3 + 5t$. At what time is the velocity of the particle equal to zero? Is this reasonable?

3.3 Average and Instantaneous Acceleration

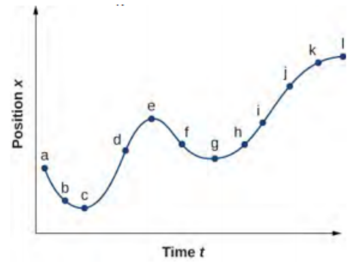
37. A cheetah can accelerate from rest to a speed of 30.0 m/s in 7.00 s. What is its acceleration?
38. Dr. John Paul Stapp was a U.S. Air Force officer who studied the effects of extreme acceleration on the human body. On December 10, 1954, Stapp rode a rocket sled, accelerating from rest to a top speed of 282 m/s (1015 km/h) in 5.00 s and was brought jarringly back to rest in only 1.40 s. Calculate his (a) acceleration in his direction of motion and (b) acceleration opposite to his direction of motion. Express each in multiples of g ($9.80\ m/s^2$) by taking its ratio to the acceleration of gravity.
39. Sketch the acceleration-versus-time graph from the following velocity-versus-time graph.



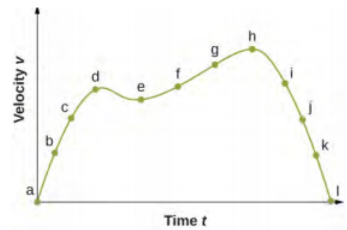
40. A commuter backs her car out of her garage with an acceleration of $1.40\ m/s^2$. (a) How long does it take her to reach a speed of 2.00 m/s? (b) If she then brakes to a stop in 0.800 s, what is her acceleration?
41. Assume an intercontinental ballistic missile goes from rest to a suborbital speed of 6.50 km/s in 60.0 s (the actual speed and time are classified). What is its average acceleration in meters per second and in multiples of g ($9.80\ m/s^2$)?
42. An airplane, starting from rest, moves down the runway at constant acceleration for 18 s and then takes off at a speed of 60 m/s. What is the average acceleration of the plane?

3.4 Motion with Constant Acceleration

43. A particle moves in a straight line at a constant velocity of 30 m/s. What is its displacement between t = 0 and t = 5.0 s?
44. A particle moves in a straight line with an initial velocity of 0 m/s and a constant acceleration of $30\ m/s^2$. If x = 0 at t = 0, what is the particle's position at t = 5 s?
45. A particle moves in a straight line with an initial velocity of 30 m/s and constant acceleration $30\ m/s^2$. (a) What is its displacement at t = 5 s? (b) What is its velocity at this same time?
46. (a) Sketch a graph of velocity versus time corresponding to the graph of displacement versus time given in the following figure. (b) Identify the time or times (t_a, t_b, t_c, etc.) at which the instantaneous velocity has the greatest positive value. (c) At which times is it zero? (d) At which times is it negative?



47. (a) Sketch a graph of acceleration versus time corresponding to the graph of velocity versus time given in the following figure. (b) Identify the time or times (ta , tb, tc , etc.) at which the acceleration has the greatest positive value. (c) At which times is it zero? (d) At which times is it negative?



48. A particle has a constant acceleration of 6.0 m/s^2. (a) If its initial velocity is 2.0 m/s, at what time is its displacement 5.0 m? (b) What is its velocity at that time?
49. At t = 10 s, a particle is moving from left to right with a speed of 5.0 m/s. At t = 20 s, the particle is moving right to left with a speed of 8.0 m/s. Assuming the particle's acceleration is constant, determine (a) its acceleration, (b) its initial velocity, and (c) the instant when its velocity is zero.
50. A well-thrown ball is caught in a well-padded mitt. If the acceleration of the ball is 2.10 x 10^4 m/s^2, and 1.85 ms (1 ms = 10^{-3} s) elapses from the time the ball first touches the mitt until it stops, what is the initial velocity of the ball?
51. A bullet in a gun is accelerated from the firing chamber to the end of the barrel at an average rate of 6.20 x 10^5 m/s^2 for 8.10 x 10^{-4} s. What is its muzzle velocity (that is, its final velocity)?
52. (a) A light-rail commuter train accelerates at a rate of 1.35 m/s^2. How long does it take to reach its top speed of 80.0 km/h, starting from rest? (b) The same train ordinarily decelerates at a rate of 1.65 m/s^2. How long does it take to come to a stop from its top speed? (c) In emergencies, the train can decelerate more rapidly, coming to rest from 80.0 km/h in 8.30 s. What is its emergency acceleration in meters per second squared?
53. While entering a freeway, a car accelerates from rest at a rate of 2.04 m/s^2 for 12.0 s. (a) Draw a sketch of the situation. (b) List the knowns in this problem. (c) How far does the car travel in those 12.0 s? To solve this part, first identify the unknown, then indicate how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, check your units, and discuss whether the answer is reasonable. (d) What is the car's final velocity? Solve for this unknown in the same manner as in (c), showing all steps explicitly.
54. **Unreasonable results** At the end of a race, a runner decelerates from a velocity of 9.00 m/s at a rate of 2.00 m/s^2. (a) How far does she travel in the next 5.00 s? (b) What is her final velocity? (c) Evaluate the result. Does it make sense?
55. Blood is accelerated from rest to 30.0 cm/s in a distance of 1.80 cm by the left ventricle of the heart. (a) Make a sketch of the situation. (b) List the knowns in this problem. (c) How long does the acceleration take? To solve this part, first identify the unknown, then discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking your units. (d) Is the answer reasonable when compared with the time for a heartbeat?
56. During a slap shot, a hockey player accelerates the puck from a velocity of 8.00 m/s to 40.0 m/s in the same direction. If this shot takes 3.33 x 10^{-2} s , what is the distance over which the puck accelerates?
57. A powerful motorcycle can accelerate from rest to 26.8 m/s (100 km/h) in only 3.90 s. (a) What is its average acceleration? (b) Assuming constant acceleration, how far does it travel in that time?
58. Freight trains can produce only relatively small accelerations. (a) What is the final velocity of a freight train that accelerates at a rate of 0.0500 m/s^2 for 8.00 min, starting with an initial velocity of 4.00 m/s? (b) If the train can slow down at a rate of 0.550 m/s^2, how long will it take to come to a stop from this velocity? (c) How far will it travel in each case?

59. A fireworks shell is accelerated from rest to a velocity of 65.0 m/s over a distance of 0.250 m. (a) Calculate the acceleration. (b) How long did the acceleration last?
60. A swan on a lake gets airborne by flapping its wings and running on top of the water. (a) If the swan must reach a velocity of 6.00 m/s to take off and it accelerates from rest at an average rate of $0.35\ m/s^2$, how far will it travel before becoming airborne? (b) How long does this take?
61. A woodpecker's brain is specially protected from large accelerations by tendon-like attachments inside the skull. While pecking on a tree, the woodpecker's head comes to a stop from an initial velocity of 0.600 m/s in a distance of only 2.00 mm. (a) Find the acceleration in meters per second squared and in multiples of g, where $g = 9.80\ m/s^2$. (b) Calculate the stopping time. (c) The tendons cradling the brain stretch, making its stopping distance 4.50 mm (greater than the head and, hence, less acceleration of the brain). What is the brain's acceleration, expressed in multiples of g?
62. An unwary football player collides with a padded goalpost while running at a velocity of 7.50 m/s and comes to a full stop after compressing the padding and his body 0.350 m. (a) What is his acceleration? (b) How long does the collision last?
63. A care package is dropped out of a cargo plane and lands in the forest. If we assume the care package speed on impact is 54 m/s (123 mph), then what is its acceleration? Assume the trees and snow stops it over a distance of 3.0 m.
64. An express train passes through a station. It enters with an initial velocity of 22.0 m/s and decelerates at a rate of 0.150 m/s^2 as it goes through. The station is 210.0 m long. (a) How fast is it going when the nose leaves the station? (b) How long is the nose of the train in the station? (c) If the train is 130 m long, what is the velocity of the end of the train as it leaves? (d) When does the end of the train leave the station?
65. **Unreasonable results** Dragsters can actually reach a top speed of 145.0 m/s in only 4.45 s. (a) Calculate the average acceleration for such a dragster. (b) Find the final velocity of this dragster starting from rest and accelerating at the rate found in (a) for 402.0 m (a quarter mile) without using any information on time. (c) Why is the final velocity greater than that used to find the average acceleration? (**Hint**: Consider whether the assumption of constant acceleration is valid for a dragster. If not, discuss whether the acceleration would be greater at the beginning or end of the run and what effect that would have on the final velocity.)

3.5 Free Fall

66. Calculate the displacement and velocity at times of (a) 0.500 s, (b) 1.00 s, (c) 1.50 s, and (d) 2.00 s for a ball thrown straight up with an initial velocity of 15.0 m/s. Take the point of release to be $y_0 = 0$.
67. Calculate the displacement and velocity at times of (a) 0.500 s, (b) 1.00 s, (c) 1.50 s, (d) 2.00 s, and (e) 2.50 s for a rock thrown straight down with an initial velocity of 14.0 m/s from the Verrazano Narrows Bridge in New York City. The roadway of this bridge is 70.0 m above the water.
68. A basketball referee tosses the ball straight up for the starting tip-off. At what velocity must a basketball player leave the ground to rise 1.25 m above the floor in an attempt to get the ball?
69. A rescue helicopter is hovering over a person whose boat has sunk. One of the rescuers throws a life preserver straight down to the victim with an initial velocity of 1.40 m/s and observes that it takes 1.8 s to reach the water. (a) List the knowns in this problem. (b) How high above the water was the preserver released? Note that the downdraft of the helicopter reduces the effects of air resistance on the falling life preserver, so that an acceleration equal to that of gravity is reasonable.
70. **Unreasonable results** A dolphin in an aquatic show jumps straight up out of the water at a velocity of 15.0 m/s. (a) List the knowns in this problem. (b) How high does his body rise above the water? To solve this part, first note that the final velocity is now a known, and identify its value. Then, identify the unknown and discuss how you chose the appropriate equation to solve for it. After choosing the equation, show your steps in solving for the unknown, checking units, and discuss whether the answer is reasonable. (c) How long a time is the dolphin in the air? Neglect any effects resulting from his size or orientation.
71. A diver bounces straight up from a diving board, avoiding the diving board on the way down, and falls feet first into a pool. She starts with a velocity of 4.00 m/s and her takeoff point is 1.80 m above the pool. (a) What is her highest point above the board? (b) How long a time are her feet in the air? (c) What is her velocity when her feet hit the water?
72. (a) Calculate the height of a cliff if it takes 2.35 s for a rock to hit the ground when it is thrown straight up from the cliff with an initial velocity of 8.00 m/s. (b) How long a time would it take to reach the ground if it is thrown straight down with the same speed?
73. A very strong, but inept, shot putter puts the shot straight up vertically with an initial velocity of 11.0 m/s. How long a time does he have to get out of the way if the shot was released at a height of 2.20 m and he is 1.80 m tall?

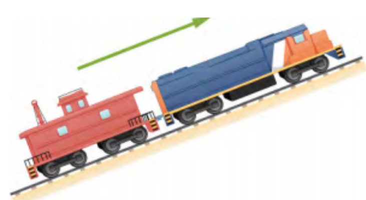
74. You throw a ball straight up with an initial velocity of 15.0 m/s. It passes a tree branch on the way up at a height of 7.0 m. How much additional time elapses before the ball passes the tree branch on the way back down?
75. A kangaroo can jump over an object 2.50 m high. (a) Considering just its vertical motion, calculate its vertical speed when it leaves the ground. (b) How long a time is it in the air?
76. Standing at the base of one of the cliffs of Mt. Arapiles in Victoria, Australia, a hiker hears a rock break loose from a height of 105.0 m. He can't see the rock right away, but then does, 1.50 s later. (a) How far above the hiker is the rock when he can see it? (b) How much time does he have to move before the rock hits his head?
77. There is a 250-m-high cliff at Half Dome in Yosemite National Park in California. Suppose a boulder breaks loose from the top of this cliff. (a) How fast will it be going when it strikes the ground? (b) Assuming a reaction time of 0.300 s, how long a time will a tourist at the bottom have to get out of the way after hearing the sound of the rock breaking loose (neglecting the height of the tourist, which would become negligible anyway if hit)? The speed of sound is 335.0 m/s on this day.

3.6 Finding Velocity and Displacement from Acceleration

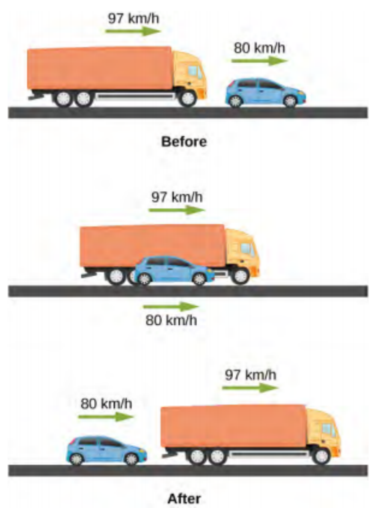
78. The acceleration of a particle varies with time according to the equation $a(t) = pt^2 - qt^3$. Initially, the velocity and position are zero. (a) What is the velocity as a function of time? (b) What is the position as a function of time?
79. Between $t = 0$ and $t = t_0$, a rocket moves straight upward with an acceleration given by $a(t) = A - Bt^{1/2}$, where A and B are constants. (a) If x is in meters and t is in seconds, what are the units of A and B? (b) If the rocket starts from rest, how does the velocity vary between $t = 0$ and $t = t_0$? (c) If its initial position is zero, what is the rocket's position as a function of time during this same time interval?
80. The velocity of a particle moving along the x-axis varies with time according to $v(t) = A + Bt^{-1}$, where A = 2 m/s, B = 0.25 m, and $1.0\ s \le t \le 8.0\ s$. Determine the acceleration and position of the particle at t = 2.0 s and t = 5.0 s. Assume that x(t = 1 s) = 0.
81. A particle at rest leaves the origin with its velocity increasing with time according to v(t) = 3.2t m/s. At 5.0 s, the particle's velocity starts decreasing according to [16.0 – 1.5(t – 5.0)] m/s. This decrease continues until t = 11.0 s, after which the particle's velocity remains constant at 7.0 m/s. (a) What is the acceleration of the particle as a function of time? (b) What is the position of the particle at t = 2.0 s, t = 7.0 s, and t = 12.0 s?

Additional Problems

82. Professional baseball player Nolan Ryan could pitch a baseball at approximately 160.0 km/h. At that average velocity, how long did it take a ball thrown by Ryan to reach home plate, which is 18.4 m from the pitcher's mound? Compare this with the average reaction time of a human to a visual stimulus, which is 0.25 s.
83. An airplane leaves Chicago and makes the 3000-km trip to Los Angeles in 5.0 h. A second plane leaves Chicago one-half hour later and arrives in Los Angeles at the same time. Compare the average velocities of the two planes. Ignore the curvature of Earth and the difference in altitude between the two cities.
84. **Unreasonable Results** A cyclist rides 16.0 km east, then 8.0 km west, then 8.0 km east, then 32.0 km west, and finally 11.2 km east. If his average velocity is 24 km/ h, how long did it take him to complete the trip? Is this a reasonable time?
85. An object has an acceleration of $+1.2\ cm/s^2$. At t = 4.0 s , its velocity is −3.4 cm/s. Determine the object's velocities at t = 1.0 s and t = 6.0 s.
86. A particle moves along the x-axis according to the equation $x(t) = 2.0 - 4.0t^2$ m. What are the velocity and acceleration at t = 2.0 s and t = 5.0 s?
87. A particle moving at constant acceleration has velocities of 2.0 m/s at t = 2.0 s and −7.6 m/s at t = 5.2 s. What is the acceleration of the particle?
88. A train is moving up a steep grade at constant velocity (see following figure) when its caboose breaks loose and starts rolling freely along the track. After 5.0 s, the caboose is 30 m behind the train. What is the acceleration of the caboose?



89. An electron is moving in a straight line with a velocity of 4.0×10^5 m/s. It enters a region 5.0 cm long where it undergoes an acceleration of 6.0×10^{12} m/s^2 along the same straight line. (a) What is the electron's velocity when it emerges from this region? b) How long does the electron take to cross the region?
90. An ambulance driver is rushing a patient to the hospital. While traveling at 72 km/h, she notices the traffic light at the upcoming intersections has turned amber. To reach the intersection before the light turns red, she must travel 50 m in 2.0 s. (a) What minimum acceleration must the ambulance have to reach the intersection before the light turns red? (b) What is the speed of the ambulance when it reaches the intersection?
91. A motorcycle that is slowing down uniformly covers 2.0 successive km in 80 s and 120 s, respectively. Calculate (a) the acceleration of the motorcycle and (b) its velocity at the beginning and end of the 2-km trip.
92. A cyclist travels from point A to point B in 10 min. During the first 2.0 min of her trip, she maintains a uniform acceleration of 0.090 m/s^2. She then travels at constant velocity for the next 5.0 min. Next, she decelerates at a constant rate so that she comes to a rest at point B 3.0 min later. (a) Sketch the velocity-versus-time graph for the trip. (b) What is the acceleration during the last 3 min? (c) How far does the cyclist travel?
93. Two trains are moving at 30 m/s in opposite directions on the same track. The engineers see simultaneously that they are on a collision course and apply the brakes when they are 1000 m apart. Assuming both trains have the same acceleration, what must this acceleration be if the trains are to stop just short of colliding?
94. A 10.0-m-long truck moving with a constant velocity of 97.0 km/h passes a 3.0-m-long car moving with a constant velocity of 80.0 km/h. How much time elapses between the moment the front of the truck is even with the back of the car and the moment the back of the truck is even with the front of the car?



95. A police car waits in hiding slightly off the highway. A speeding car is spotted by the police car doing 40 m/s. At the instant the speeding car passes the police car, the police car accelerates from rest at 4 m/s^2 to catch the speeding car. How long does it take the police car to catch the speeding car?
96. Pablo is running in a half marathon at a velocity of 3 m/s. Another runner, Jacob, is 50 meters behind Pablo with the same velocity. Jacob begins to accelerate at 0.05 m/s^2. (a) How long does it take Jacob to catch Pablo? (b) What is the distance covered by Jacob? (c) What is the final velocity of the Jacob?
97. **Unreasonable results** A runner approaches the finish line and is 75 m away; her average speed at this position is 8 m/s. She decelerates at this point at 0.5 m/s^2. How long does it take her to cross the finish line from 75 m away? Is this reasonable?
98. An airplane accelerates at 5.0 m/s^2 for 30.0 s. During this time, it covers a distance of 10.0 km. What are the initial and final velocities of the airplane?
99. Compare the distance traveled of an object that undergoes a change in velocity that is twice its initial velocity with an object that changes its velocity by four times its initial velocity over the same time period. The accelerations of both objects are constant.
100. An object is moving east with a constant velocity and is at position x_0 at time $t_0 = 0$. (a) With what acceleration must the object have for its total displacement to be zero at a later time t? (b) What is the physical interpretation of the solution in the case for $t \to \infty$?

101. A ball is thrown straight up. It passes a 2.00-m-high window 7.50 m off the ground on its path up and takes 1.30 s to go past the window. What was the ball's initial velocity?
102. A coin is dropped from a hot-air balloon that is 300 m above the ground and rising at 10.0 m/s upward. For the coin, find (a) the maximum height reached, (b) its position and velocity 4.00 s after being released, and (c) the time before it hits the ground.
103. A soft tennis ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.10 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 3.50 ms (3.50×10^{-3} s) (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?
104. **Unreasonable results**. A raindrop falls from a cloud 100 m above the ground. Neglect air resistance. What is the speed of the raindrop when it hits the ground? Is this a reasonable number?
105. Compare the time in the air of a basketball player who jumps 1.0 m vertically off the floor with that of a player who jumps 0.3 m vertically.
106. Suppose that a person takes 0.5 s to react and move his hand to catch an object he has dropped. (a) How far does the object fall on Earth, where $g = 9.8\ m/s^2$? (b) How far does the object fall on the Moon, where the acceleration due to gravity is 1/6 of that on Earth?
107. A hot-air balloon rises from ground level at a constant velocity of 3.0 m/s. One minute after liftoff, a sandbag is dropped accidentally from the balloon. Calculate (a) the time it takes for the sandbag to reach the ground and (b) the velocity of the sandbag when it hits the ground.
108. (a) A world record was set for the men's 100-m dash in the 2008 Olympic Games in Beijing by Usain Bolt of Jamaica. Bolt "coasted" across the finish line with a time of 9.69 s. If we assume that Bolt accelerated for 3.00 s to reach his maximum speed, and maintained that speed for the rest of the race, calculate his maximum speed and his acceleration. (b) During the same Olympics, Bolt also set the world record in the 200-m dash with a time of 19.30 s. Using the same assumptions as for the 100-m dash, what was his maximum speed for this race?
109. An object is dropped from a height of 75.0 m above ground level. (a) Determine the distance traveled during the first second. (b) Determine the final velocity at which the object hits the ground. (c) Determine the distance traveled during the last second of motion before hitting the ground.
110. A steel ball is dropped onto a hard floor from a height of 1.50 m and rebounds to a height of 1.45 m. (a) Calculate its velocity just before it strikes the floor. (b) Calculate its velocity just after it leaves the floor on its way back up. (c) Calculate its acceleration during contact with the floor if that contact lasts 0.0800 ms (8.00×10^{-5} s) (d) How much did the ball compress during its collision with the floor, assuming the floor is absolutely rigid?
111. An object is dropped from a roof of a building of height h. During the last second of its descent, it drops a distance h/3. Calculate the height of the building.

Challenge Problems

112. In a 100-m race, the winner is timed at 11.2 s. The second-place finisher's time is 11.6 s. How far is the second-place finisher behind the winner when she crosses the finish line? Assume the velocity of each runner is constant throughout the race.
113. The position of a particle moving along the x-axis varies with time according to $x(t) = 5.0t^2 - 4.0t^3$ m. Find (a) the velocity and acceleration of the particle as functions of time, (b) the velocity and acceleration at t = 2.0 s, (c) the time at which the position is a maximum, (d) the time at which the velocity is zero, and (e) the maximum position.
114. A cyclist sprints at the end of a race to clinch a victory. She has an initial velocity of 11.5 m/s and accelerates at a rate of $0.500\ m/s^2$ for 7.00 s. (a) What is her final velocity? (b) The cyclist continues at this velocity to the finish line. If she is 300 m from the finish line when she starts to accelerate, how much time did she save? (c) The second-place winner was 5.00 m ahead when the winner started to accelerate, but he was unable to accelerate, and traveled at 11.8 m/s until the finish line. What was the difference in finish time in seconds between the winner and runner-up? How far back was the runner-up when the winner crossed the finish line?
115. In 1967, New Zealander Burt Munro set the world record for an Indian motorcycle, on the Bonneville Salt Flats in Utah, of 295.38 km/h. The one-way course was 8.00 km long. Acceleration rates are often described by the time it takes to reach 96.0 km/h from rest. If this time was 4.00 s and Burt accelerated at this rate until he reached his maximum speed, how long did it take Burt to complete the course?

4.10: Motion Along a Straight Line (Summary)

Key Terms

acceleration due to gravity	acceleration of an object as a result of gravity
average acceleration	the rate of change in velocity; the change in velocity over time
average speed	the total distance traveled divided by elapsed time
average velocity	the displacement divided by the time over which displacement occurs
displacement	the change in position of an object
distance traveled	the total length of the path traveled between two positions
elapsed time	the difference between the ending time and the beginning time
free fall	the state of movement that results from gravitational force only
instantaneous acceleration	acceleration at a specific point in time
instantaneous speed	the absolute value of the instantaneous velocity
instantaneous velocity	the velocity at a specific instant or time point
kinematics	the description of motion through properties such as position, time, velocity, and acceleration
position	the location of an object at a particular time
total displacement	the sum of individual displacements over a given time period
two-body pursuit problem	a kinematics problem in which the unknowns are calculated by solving the kinematic equations simultaneously for two moving objects

Key Equations

Displacement	$$\Delta x = x_f - x_i \tag{4.10.1}$$
Total displacement	$$\Delta x_{Total} = \sum \Delta x_i \tag{4.10.2}$$
Average velocity	$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1} \tag{4.10.3}$$
Instantaneous velocity	$$v(t) = \frac{dx(t)}{dt} \tag{4.10.4}$$
Average speed	$$\bar{s} = \frac{Total\ distance}{Elapsed\ time} \tag{4.10.5}$$
Instantaneous speed	$$Instantaneous\ speed = \lvert v(t) \rvert \tag{4.10.6}$$
Average acceleration	$$\bar{a} = \frac{\Delta v}{\Delta t} = \frac{v_f - x_0}{t_f - t_0} \tag{4.10.7}$$

Instantaneous acceleration	$$a(t) = \frac{dv(t)}{dt} \tag{4.10.8}$$
Position from average velocity	$$x = x_0 + \bar{v}t \tag{4.10.9}$$
Average velocity	$$\bar{v} = \frac{v_0 + v}{2} \tag{4.10.10}$$
Velocity from acceleration	$$v = v_0 + at \ (constant\ a) \tag{4.10.11}$$
Position from velocity and acceleration	$$x = x_0 + v_0 t + \frac{1}{2}at^2 \ (constant\ a) \tag{4.10.12}$$
Velocity from distance	$$v^2 = v_0^2 + 2a(x - x_0) \ (constant\ a) \tag{4.10.13}$$
Velocity of free fall	$$v = v_0 - gt(positive\ upward) \tag{4.10.14}$$
Height of free fall	$$y = y_0 + v_0 t - \frac{1}{2}gt^2 \tag{4.10.15}$$
Velocity of free fall from height	$$v^2 = v_0^2 - 2g(y - y_0) \tag{4.10.16}$$
Velocity from acceleration	$$v(t) = \int a(t)dt + C_1 \tag{4.10.17}$$
Position from velocity	$$x(t) = \int v(t)dt + C_2 \tag{4.10.18}$$

Summary

3.1 Position, Displacement, and Average Velocity

- Kinematics is the description of motion without considering its causes. In this chapter, it is limited to motion along a straight line, called one-dimensional motion.
- Displacement is the change in position of an object. The SI unit for displacement is the meter. Displacement has direction as well as magnitude.
- Distance traveled is the total length of the path traveled between two positions.
- Time is measured in terms of change. The time between two position points x_1 and x_2 is $\Delta t = t_2 - t_1$. Elapsed time for an event is $\Delta t = t_f - t_0$, where t_f is the final time and t_0 is the initial time. The initial time is often taken to be zero.
- Average velocity $\bar{v}$ is defined as displacement divided by elapsed time. If x_1, t_1 and x_2, t_2 are two position time points, the average velocity between these points is

$$\bar{v} = \frac{\Delta x}{\Delta t} = \frac{x_2 - x_1}{t_2 - t_1}. \tag{4.10.19}$$

3.2 Instantaneous Velocity and Speed

- Instantaneous velocity is a continuous function of time and gives the velocity at any point in time during a particle's motion. We can calculate the instantaneous velocity at a specific time by taking the derivative of the position function, which gives us the functional form of instantaneous velocity v(t).
- Instantaneous velocity is a vector and can be negative.
- Instantaneous speed is found by taking the absolute value of instantaneous velocity, and it is always positive.
- Average speed is total distance traveled divided by elapsed time.

- The slope of a position-versus-time graph at a specific time gives instantaneous velocity at that time.

3.3 Average and Instantaneous Acceleration

- Acceleration is the rate at which velocity changes. Acceleration is a vector; it has both a magnitude and direction. The SI unit for acceleration is meters per second squared.
- Acceleration can be caused by a change in the magnitude or the direction of the velocity, or both.
- Instantaneous acceleration a(t) is a continuous function of time and gives the acceleration at any specific time during the motion. It is calculated from the derivative of the velocity function. Instantaneous acceleration is the slope of the velocity-versus-time graph.
- Negative acceleration (sometimes called deceleration) is acceleration in the negative direction in the chosen coordinate system.

3.4 Motion with Constant Acceleration

- When analyzing one-dimensional motion with constant acceleration, identify the known quantities and choose the appropriate equations to solve for the unknowns. Either one or two of the kinematic equations are needed to solve for the unknowns, depending on the known and unknown quantities.
- Two-body pursuit problems always require two equations to be solved simultaneously for the unknowns.

3.5 Free Fall

- An object in free fall experiences constant acceleration if air resistance is negligible.
- On Earth, all free-falling objects have an acceleration g due to gravity, which averages g = 9.81 m/s^2.
- For objects in free fall, the upward direction is normally taken as positive for displacement, velocity, and acceleration.

3.6 Finding Velocity and Displacement from Acceleration

- Integral calculus gives us a more complete formulation of kinematics.
- If acceleration a(t) is known, we can use integral calculus to derive expressions for velocity v(t) and position x(t).
- If acceleration is constant, the integral equations reduce to Equation 3.12 and Equation 3.13 for motion with constant acceleration.

Contributors and Attributions

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CHAPTER OVERVIEW

5: Two-Dimensional Kinematics

5.1: Prelude to Motion in Two and Three Dimensions

To give a complete description of kinematics, we must explore motion in two and three dimensions. After all, most objects in our universe do not move in straight lines; rather, they follow curved paths. From kicked footballs to the flight paths of birds to the orbital motions of celestial bodies and down to the flow of blood plasma in your veins, most motion follows curved trajectories.

Figure 5.1.1: The Red Arrows is the aerobatics display team of Britain's Royal Air Force. Based in Lincolnshire, England, they perform precision flying shows at high speeds, which requires accurate measurement of position, velocity, and acceleration in three dimensions. (credit: modification of work by Phil Long)

Fortunately, the treatment of motion in one dimension in the previous chapter has given us a foundation on which to build, as the concepts of position, displacement, velocity, and acceleration defined in one dimension can be expanded to two and three dimensions. Consider the Red Arrows, also known as the Royal Air Force Aerobatic team of the United Kingdom. Each jet follows a unique curved trajectory in three-dimensional airspace, as well as has a unique velocity and acceleration. Thus, to describe the motion of any of the jets accurately, we must assign to each jet a unique position vector in three dimensions as well as a unique velocity and acceleration vector. We can apply the same basic equations for displacement, velocity, and acceleration we derived in Motion Along a Straight Line to describe the motion of the jets in two and three dimensions, but with some modifications—in particular, the inclusion of vectors.

In this chapter we also explore two special types of motion in two dimensions: projectile motion and circular motion. Last, we conclude with a discussion of relative motion. In the chapter-opening picture, each jet has a relative motion with respect to any other jet in the group or to the people observing the air show on the ground.

5.2: Displacement and Velocity Vectors

LEARNING OBJECTIVES

- Calculate position vectors in a multidimensional displacement problem.
- Solve for the displacement in two or three dimensions.
- Calculate the velocity vector given the position vector as a function of time.
- Calculate the average velocity in multiple dimensions.

Displacement and velocity in two or three dimensions are straightforward extensions of the one-dimensional definitions. However, now they are vector quantities, so calculations with them have to follow the rules of vector algebra, not scalar algebra.

Displacement Vector

To describe motion in two and three dimensions, we must first establish a coordinate system and a convention for the axes. We generally use the coordinates x, y, and z to locate a particle at point $P(x, y, z)$ in three dimensions. If the particle is moving, the variables x, y, and z are functions of time (t):

$$x = x(t) \quad y = y(t) \quad z = z(t). \tag{5.2.1}$$

The position vector from the origin of the coordinate system to point P is $\vec{r}(t)$. In unit vector notation, introduced in Coordinate Systems and Components of a Vector, $\vec{r}$(t) is

$$\vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{j} + z(t)\ \hat{k}. \tag{5.2.2}$$

Figure $5.2.1$ shows the coordinate system and the vector to point P, where a particle could be located at a particular time t. Note the orientation of the x, y, and z axes. This orientation is called a right-handed coordinate system and it is used throughout the chapter.

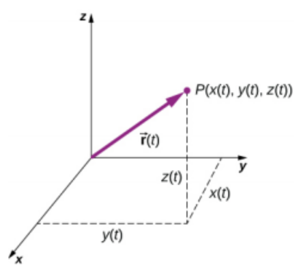

Figure 5.2.1: A three-dimensional coordinate system with a particle at position $P(x(t), y(t), z(t))$

With our definition of the position of a particle in three-dimensional space, we can formulate the three-dimensional displacement. Figure $5.2.3$ shows a particle at time t_1 located at P_1 with position vector $\vec{r}(t_1)$. At a later time t_2, the particle is located at P_2 with position vector $\vec{r}(t_2)$. The displacement vector $\Delta\vec{r}$ is found by subtracting $\vec{r}(t_1)$ from $\vec{r}(t_2)$:

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1). \tag{5.2.3}$$

Vector addition is discussed in Vectors. Note that this is the same operation we did in one dimension, but now the vectors are in three-dimensional space.

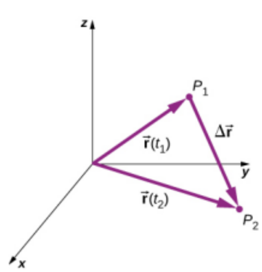

Figure 5.2.2: The displacement $\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1)$ is the vector from P_1 to P_2.

The following examples illustrate the concept of displacement in multiple dimensions

Example 4.1: Polar Orbiting Satellite

A satellite is in a circular polar orbit around Earth at an altitude of 400 km—meaning, it passes directly overhead at the North and South Poles. What is the magnitude and direction of the displacement vector from when it is directly over the North Pole to when it is at −45° latitude?

Strategy

We make a picture of the problem to visualize the solution graphically. This will aid in our understanding of the displacement. We then use unit vectors to solve for the displacement.

Solution

Figure $5.2.3$ shows the surface of Earth and a circle that represents the orbit of the satellite. Although satellites are moving in three-dimensional space, they follow trajectories of ellipses, which can be graphed in two dimensions. The position vectors are drawn from the center of Earth, which we take to be the origin of the coordinate system, with the y-axis as north and the x-axis as east. The vector between them is the displacement of the satellite. We take the radius of Earth as 6370 km, so the length of each position vector is 6770 km.

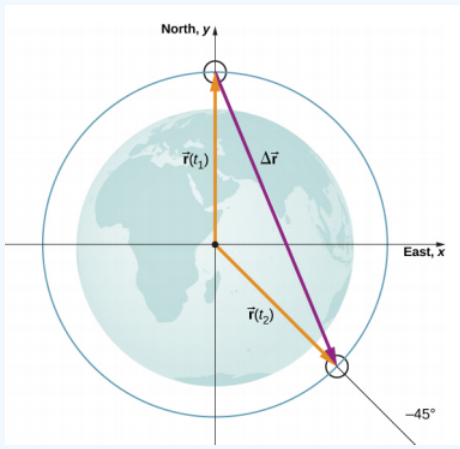

Figure $5.2.3$: Two position vectors are drawn from the center of Earth, which is the origin of the coordinate system, with the y-axis as north and the x-axis as east. The vector between them is the displacement of the satellite.

In unit vector notation, the position vectors are

$$\vec{r}(t_1) = 6770.\ km\ \hat{j}$$

$$\vec{r}(t_2) = 6770.\ km(\cos(-45^\circ))\ \hat{i} + 6770.\ km(\sin(-45^\circ))\ \hat{j}.$$

Evaluating the sine and cosine, we have

$$\vec{r}(t_1) = 6770.\hat{j}$$

$$\vec{r}(t_2) = 4787\ \hat{i} - 4787\ \hat{j}.$$

Now we can find $\Delta\vec{r}$, the displacement of the satellite:

$$\Delta\vec{r} = \vec{r}(t_2) - \vec{r}(t_1) = 4787\ \hat{i} - 11,557\ \hat{j}.$$

The magnitude of the displacement is

$$|\Delta\vec{r}| = \sqrt{(4787)^2 + (-11,557)^2} = 12,509\ km.$$

The angle the displacement makes with the x-axis is

$$\theta = \tan^{-1}\left(\frac{-11,557}{4787}\right) = -67.5^o.$$

Significance

Plotting the displacement gives information and meaning to the unit vector solution to the problem. When plotting the displacement, we need to include its components as well as its magnitude and the angle it makes with a chosen axis—in this case, the x-axis (Figure 5.2.4).

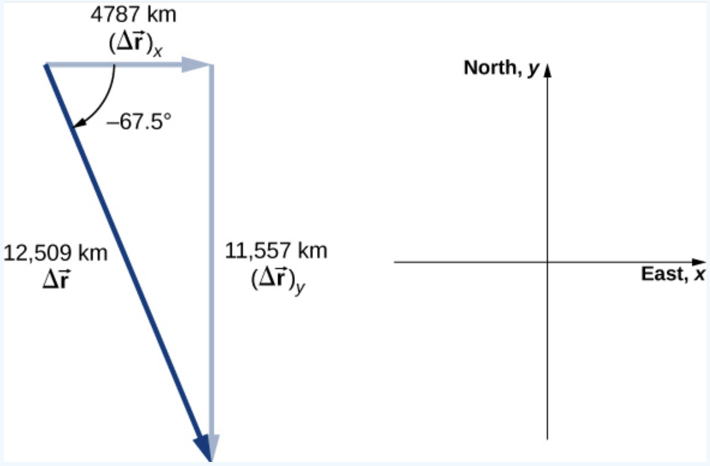

Figure 5.2.4: Displacement vector with components, angle, and magnitude.

Note that the satellite took a curved path along its circular orbit to get from its initial position to its final position in this example. It also could have traveled 4787 km east, then 11,557 km south to arrive at the same location. Both of these paths are longer than the length of the displacement vector. In fact, the displacement vector gives the shortest path between two points in one, two, or three dimensions.

Many applications in physics can have a series of displacements, as discussed in the previous chapter. The total displacement is the sum of the individual displacements, only this time, we need to be careful, because we are adding vectors. We illustrate this concept with an example of Brownian motion.

✓ Example 4.2: Brownian Motion

Brownian motion is a chaotic random motion of particles suspended in a fluid, resulting from collisions with the molecules of the fluid. This motion is three-dimensional. The displacements in numerical order of a particle undergoing Brownian motion could look like the following, in micrometers (Figure 5.2.5):

$$\Delta\vec{r}_1 = 2.0\ \hat{i} + \hat{j} + 3.0\hat{k} \tag{5.2.4}$$

$$\Delta\vec{r}_2 = -\hat{i} + 3.0\ \hat{k} \tag{5.2.5}$$

$$\Delta\vec{r}_3 = 4.0\ \hat{i} - 2.0\ \hat{j} + \hat{k} \tag{5.2.6}$$

$$\Delta\vec{r}_4 = -3.0\ \hat{i} + \hat{j} + 3.0\ \hat{k}. \tag{5.2.7}$$

What is the total displacement of the particle from the origin?

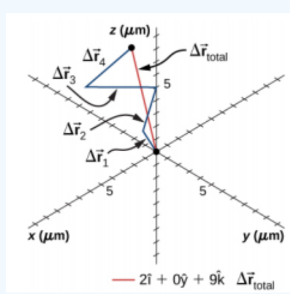

Figure 5.2.5: Trajectory of a particle undergoing random displacements of Brownian motion. The total displacement is shown in red.

Solution

We form the sum of the displacements and add them as vectors:

$$\begin{aligned}\Delta\vec{r}_{Total} &= \sum\Delta\vec{r}_i = \Delta\vec{r}_1+\Delta\vec{r}_2+\Delta\vec{r}_3+\Delta\vec{r}_4\\ &= (2.0-1.0+4.0-3.0)\ \hat{i}+(1.0+0-2.0+1.0)\ \hat{j}+(3.0+3.0+1.0+2.0)\ \hat{k}\\ &= 2.0\ \hat{i}+0\ \hat{j}+9.0\ \hat{k}\ \mu m.\end{aligned}$$

To complete the solution, we express the displacement as a magnitude and direction,

$$|\Delta\vec{r}_{Total}| = \sqrt{2.0^2+0^2+9.0^2} = 9.2\mu m, \quad \theta = \tan^{-1}\left(\frac{9}{2}\right) = 77^o, \tag{5.2.8}$$

with respect to the x-axis in the xz-plane.

Significance

From the figure we can see the magnitude of the total displacement is less than the sum of the magnitudes of the individual displacements.

Velocity Vector

In the previous chapter we found the instantaneous velocity by calculating the derivative of the position function with respect to time. We can do the same operation in two and three dimensions, but we use vectors. The instantaneous **velocity vector** is now

$$\vec{v}(t) = \lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t} = \frac{d\vec{r}}{dt}. \tag{5.2.9}$$

Let's look at the relative orientation of the position vector and velocity vector graphically. In Figure 5.2.6 we show the vectors $\vec{r}$(t) and $\vec{r}$(t + Δt), which give the position of a particle moving along a path represented by the gray line. As Δt goes to zero, the velocity vector, given by Equation 5.2.9, becomes tangent to the path of the particle at time t.

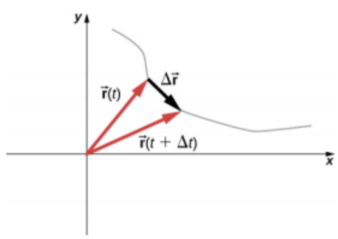

Figure 5.2.6: A particle moves along a path given by the gray line. In the limit as Δt approaches zero, the velocity vector becomes tangent to the path of the particle.

Equation 5.2.9 can also be written in terms of the components of $\vec{v}$(t). Since

$$\vec{r}(t) = x(t)\ \hat{i}+y(t)\ \hat{j}+z(t)\ \hat{k}, \tag{5.2.10}$$

we can write

$$\vec{v}(t) = v_x(t)\ \hat{i}+v_y(t)\ \hat{j}+v_z(t)\ \hat{k} \tag{5.2.11}$$

where

$$v_x(t) = \frac{dx(t)}{dt}, \quad v_y(t) = \frac{dy(t)}{dt}, \quad v_z(t) = \frac{dz(t)}{dt}. \tag{5.2.12}$$

If only the average velocity is of concern, we have the vector equivalent of the one-dimensional average velocity for two and three dimensions:

$$\vec{v}_{avg} = \frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1}. \tag{5.2.13}$$

✔ Example 4.3: Calculating the Velocity Vector

The position function of a particle is $\vec{r}$(t) = 2.0t^2 $\hat{i}$ + (2.0 + 3.0t) $\hat{j}$+ 5.0t $\hat{k}$ m. (a) What is the instantaneous velocity and speed at t = 2.0 s? (b) What is the average velocity between 1.0 s and 3.0 s?

Solution

Using Equation 5.2.11 and Equation 5.2.12, and taking the derivative of the position function with respect to time, we find

a.
$$v(t) = \frac{d\vec{r}(t)}{dt} = 4.0t\ \hat{i} + 3.0\ \hat{j} + 5.0\ \hat{k}\ m/s \tag{5.2.14}$$

$$\vec{v}(2.0\ s) = 8.0\ \hat{i} + 3.0\ \hat{j} + 5.0\ \hat{k}\ m/s \tag{5.2.15}$$

$$Speed\ |\vec{v}(2.0\ s)| = \sqrt{8^2 + 3^2 + 5^2} = 9.9\ m/s. \tag{5.2.16}$$

b. From Equation 5.2.13,

$$\begin{aligned}\vec{v}_{avg} &= \frac{\vec{r}(t_2) - \vec{r}(t_1)}{t_2 - t_1} = \frac{\vec{r}(3.0\ s) - \vec{r}(1.0\ s)}{3.0\ s - 1.0\ s} = \frac{(18\ \hat{i} + 11\ \hat{j} + 15\ \hat{k})m - (2\ \hat{i} + 5\ \hat{j} + 5\ \hat{k})m}{2.0\ s} \\ &= \frac{(16\ \hat{i} + 6\ \hat{j} + 10\ \hat{k})m}{2.0\ s} = 8.0\ \hat{i} + 3.0\ \hat{j} + 5.0\ \hat{k}\ m/s.\end{aligned}$$

Significance

We see the average velocity is the same as the instantaneous velocity at t = 2.0 s, as a result of the velocity function being linear. This need not be the case in general. In fact, most of the time, instantaneous and average velocities are not the same.

? Exercise 4.1

The position function of a particle is $\vec{r}(t) = 3.0t^3\ \hat{i} + 4.0\ \hat{j}$. (a) What is the instantaneous velocity at t = 3 s? (b) Is the average velocity between 2 s and 4 s equal to the instantaneous velocity at t = 3 s?

The Independence of Perpendicular Motions

When we look at the three-dimensional equations for position and velocity written in unit vector notation, Equation 5.2.2 and Equation 5.2.11, we see the components of these equations are separate and unique functions of time that do not depend on one another. Motion along the x direction has no part of its motion along the y and z directions, and similarly for the other two coordinate axes. Thus, the motion of an object in two or three dimensions can be divided into separate, independent motions along the perpendicular axes of the coordinate system in which the motion takes place.

To illustrate this concept with respect to displacement, consider a woman walking from point A to point B in a city with square blocks. The woman taking the path from A to B may walk east for so many blocks and then north (two perpendicular directions) for another set of blocks to arrive at B. How far she walks east is affected only by her motion eastward. Similarly, how far she walks north is affected only by her motion northward.

Independence of Motion

In the kinematic description of motion, we are able to treat the horizontal and vertical components of motion separately. In many cases, motion in the horizontal direction does not affect motion in the vertical direction, and vice versa.

An example illustrating the independence of vertical and horizontal motions is given by two baseballs. One baseball is dropped from rest. At the same instant, another is thrown horizontally from the same height and it follows a curved path. A stroboscope captures the positions of the balls at fixed time intervals as they fall (Figure 5.2.7).

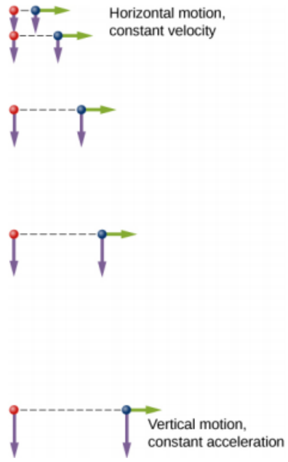

Figure 5.2.7: A diagram of the motions of two identical balls: one falls from rest and the other has an initial horizontal velocity. Each subsequent position is an equal time interval. Arrows represent the horizontal and vertical velocities at each position. The ball on the right has an initial horizontal velocity whereas the ball on the left has no horizontal velocity. Despite the difference in horizontal velocities, the vertical velocities and positions are identical for both balls, which shows the vertical and horizontal motions are independent.

It is remarkable that for each flash of the strobe, the vertical positions of the two balls are the same. This similarity implies vertical motion is independent of whether the ball is moving horizontally. (Assuming no air resistance, the vertical motion of a falling object is influenced by gravity only, not by any horizontal forces.) Careful examination of the ball thrown horizontally shows it travels the same horizontal distance between flashes. This is because there are no additional forces on the ball in the horizontal direction after it is thrown. This result means horizontal velocity is constant and is affected neither by vertical motion nor by gravity (which is vertical). Note this case is true for ideal conditions only. In the real world, air resistance affects the speed of the balls in both directions.

The two-dimensional curved path of the horizontally thrown ball is composed of two independent one-dimensional motions (horizontal and vertical). The key to analyzing such motion, called **projectile motion**, is to resolve it into motions along perpendicular directions. Resolving two-dimensional motion into perpendicular components is possible because the components are independent.

5.3: Acceleration Vector

Learning Objectives

- Calculate the acceleration vector given the velocity function in unit vector notation.
- Describe the motion of a particle with a constant acceleration in three dimensions.
- Use the one-dimensional motion equations along perpendicular axes to solve a problem in two or three dimensions with a constant acceleration.
- Express the acceleration in unit vector notation.

Instantaneous Acceleration

In addition to obtaining the displacement and velocity vectors of an object in motion, we often want to know its **acceleration vector** at any point in time along its trajectory. This acceleration vector is the instantaneous acceleration and it can be obtained from the derivative with respect to time of the velocity function, as we have seen in a previous chapter. The only difference in two or three dimensions is that these are now vector quantities. Taking the derivative with respect to time $\vec{v}$(t), we find

$$\vec{a}(t)=\lim_{t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}=\frac{d\vec{v}(t)}{dt}. \tag{5.3.1}$$

The acceleration in terms of components is

$$\vec{a}(t)=\frac{dv_x(t)}{dt}\,\hat{i}+\frac{dv_y(t)}{dt}\,\hat{j}+\frac{dv_z(t)}{dt}\,\hat{k}. \tag{5.3.2}$$

Also, since the velocity is the derivative of the position function, we can write the acceleration in terms of the second derivative of the position function:

$$\vec{a}(t)=\frac{d^2x(t)}{dt^2}\,\hat{i}+\frac{d^2y(t)}{dt^2}\,\hat{j}+\frac{d^2z(t)}{dt^2}\,\hat{k}. \tag{5.3.3}$$

Example 4.4: Finding an Acceleration Vector

A particle has a velocity of $\vec{v}(t)=5.0t\hat{i}+t^2\hat{j}-2.0t^3\hat{k}\,m/s$.

a. What is the acceleration function?
b. What is the acceleration vector at t = 2.0 s? Find its magnitude and direction.

Solution

a. We take the first derivative with respect to time of the velocity function to find the acceleration. The derivative is taken component by component:

$$\vec{a}(t)=5.0\;\hat{i}+2.0t\;\hat{j}-6.0t^2\;\hat{k}\;m/s^2.$$

b. Evaluating $\vec{a}(2.0\ s)=5.0\hat{i}+4.0\hat{j}-24.0\hat{k}\,m/s^2$ gives us the direction in unit vector notation. The magnitude of the acceleration is

$$|\vec{a}(2.20\ s)|=\sqrt{5.0^2+4.0^2+(-24.0)^2}=24.8\ m/s^2.$$

Significance

In this example we find that acceleration has a time dependence and is changing throughout the motion. Let’s consider a different velocity function for the particle.

Example 4.5: Finding a Particle Acceleration

A particle has a position function: $\vec{r}(t)=(10t-t^2)\hat{i}+5t\hat{j}+5t\hat{k}\,m$.

a. What is the velocity?
b. What is the acceleration?

c. Describe the motion from $t = 0\,s$.

Strategy

We can gain some insight into the problem by looking at the position function. It is linear in y and z, so we know the acceleration in these directions is zero when we take the second derivative. Also, note that the position in the x direction is zero for t = 0 s and t = 10 s.

Solution

a. Taking the derivative with respect to time of the position function, we find $\vec{v}(t) = (10 - 2t)\hat{i} + 5\hat{j} + 5\hat{k}\, m/s$. The velocity function is linear in time in the x direction and is constant in the y and z directions.
b. Taking the derivative of the velocity function, we find

$$\vec{a}(t) = -2\ \hat{i}\ m/s^2.$$

The acceleration vector is a constant in the negative x-direction.
c. The trajectory of the particle can be seen in Figure 5.3.1. Let's look in the y and z directions first. The particle's position increases steadily as a function of time with a constant velocity in these directions. In the x direction, however, the particle follows a path in positive x until t = 5 s, when it reverses direction. We know this from looking at the velocity function, which becomes zero at this time and negative thereafter. We also know this because the acceleration is negative and constant—meaning, the particle is decelerating, or accelerating in the negative direction. The particle's position reaches 25 m, where it then reverses direction and begins to accelerate in the negative x direction. The position reaches zero at t = 10 s.

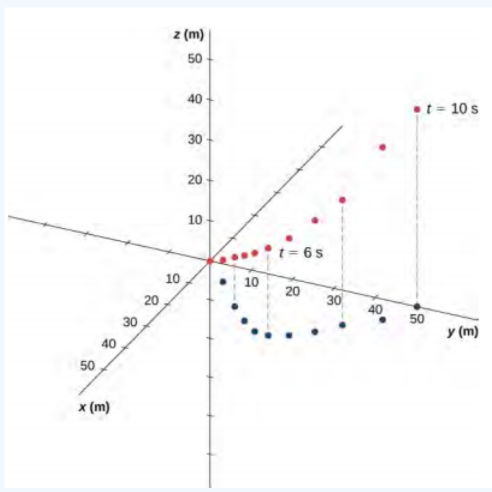

Figure 5.3.1: The particle starts at point (x, y, z) = (0, 0, 0) with position vector $\vec{r}$ = 0. The projection of the trajectory onto the xy-plane is shown. The values of y and z increase linearly as a function of time, whereas x has a turning point at t = 5 s and 25 m, when it reverses direction. At this point, the x component of the velocity becomes negative. At t = 10 s, the particle is back to 0 m in the x direction.

? Exercise 4.2

Suppose the acceleration function has the form $\vec{a}$(t) = a $\hat{i}$ + b $\hat{j}$ + c $\hat{k}$ m/s^2, where a, b, and c are constants. What can be said about the functional form of the velocity function?

Constant Acceleration

Multidimensional motion with constant acceleration can be treated the same way as shown in the previous chapter for one-dimensional motion. Earlier we showed that three-dimensional motion is equivalent to three one-dimensional motions, each along an axis perpendicular to the others. To develop the relevant equations in each direction, let's consider the two-dimensional problem of a particle moving in the xy plane with constant acceleration, ignoring the z-component for the moment. The acceleration vector is

$$\vec{a} = a_{0x}\ \hat{i} + a_{0y}\ \hat{j}. \tag{5.3.4}$$

Each component of the motion has a separate set of equations similar to Equation 3.10–Equation 3.14 of the previous chapter on one-dimensional motion. We show only the equations for position and velocity in the x- and y-directions. A similar set of kinematic equations could be written for motion in the z-direction:

$$x(t) = x_0 + (v_x)_{avg}t \tag{5.3.5}$$

$$v_x(t) = v_{0x} + a_x t \tag{5.3.6}$$

$$x(t) = x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \tag{5.3.7}$$

$$v_x^2(t) = v_{0x}^2 + 2a_x(x - x_0) \tag{5.3.8}$$

$$y(t) = y_0 + (v_y)_{avg}t \tag{5.3.9}$$

$$v_y(t) = v_{0y} + a_y t \tag{5.3.10}$$

$$y(t) = y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \tag{5.3.11}$$

$$v_y^2(t) = v_{0y}^2 + 2a_y(y - y_0). \tag{5.3.12}$$

Here the subscript 0 denotes the initial position or velocity. Equation 5.3.5 to 5.3.12 can be substituted into Equation 4.2 and Equation 4.5 without the z-component to obtain the position vector and velocity vector as a function of time in two dimensions:

$$\vec{r}(t) = x(t)\ \hat{i} + y(t)\ \hat{j} \tag{5.3.13}$$

and

$$\vec{v}(t) = v_x(t)\ \hat{i} + v_y(t)\ \hat{j}. \tag{5.3.14}$$

The following example illustrates a practical use of the kinematic equations in two dimensions.

Example 4.6: A Skier

Figure 5.3.2 shows a skier moving with an acceleration of 2.1 m/s^2 down a slope of 15° at t = 0. With the origin of the coordinate system at the front of the lodge, her initial position and velocity are

$$\vec{r}(0) = (7.50\ \hat{i} - 50.0\ \hat{j})m$$

and

$$\vec{v}(0) = (4.1\ \hat{i} - 1.1\ \hat{j})m/s$$

a. What are the x- and y-components of the skier's position and velocity as functions of time?
b. What are her position and velocity at t = 10.0 s?

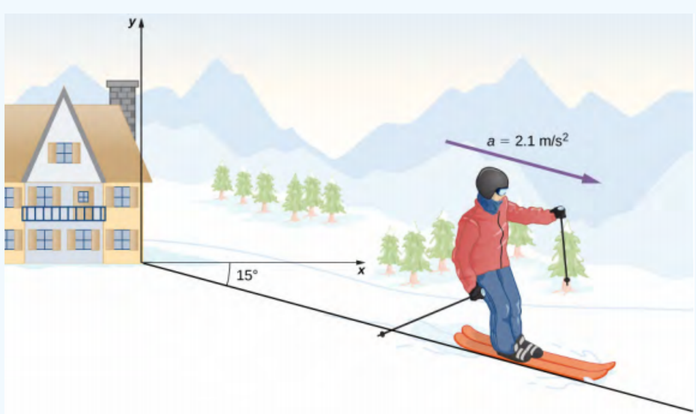

Figure 5.3.2: A skier has an acceleration of 2.1 m/s^2 down a slope of 15°. The origin of the coordinate system is at the ski lodge.

Strategy

Since we are evaluating the components of the motion equations in the x and y directions, we need to find the components of the acceleration and put them into the kinematic equations. The components of the acceleration are found by referring to the coordinate system in Figure 5.3.2. Then, by inserting the components of the initial position and velocity into the motion equations, we can solve for her position and velocity at a later time t.

Solution

a. The origin of the coordinate system is at the top of the hill with y-axis vertically upward and the x-axis horizontal. By looking at the trajectory of the skier, the x-component of the acceleration is positive and the y-component is negative. Since the angle is 15° down the slope, we find

$$a_x = (2.1\ m/s^2)\cos(15^o) = 2.0\ m/s^2 \tag{5.3.15}$$

$$a_y = (-2.1\ m/s^2)\sin(15^o) = -0.54\ m/s^2. \tag{5.3.16}$$

Inserting the initial position and velocity into Equations 5.3.6 and 5.3.7 for x, we have

$$x(t) = 75.0\ m + (4.1\ m/s)t + \frac{1}{2}(2.0\ m/s^2)t^2 \tag{5.3.17}$$

$$v_x(t) = 4.1\ m/s + (2.0\ m/s^2)t. \tag{5.3.18}$$

For y, we have

$$y(t) = -50.0.0\ m + (-1.1\ m/s)t + \frac{1}{2}(-0.54\ m/s^2)t^2 \tag{5.3.19}$$

$$v_y(t) = -1.1\ m/s + (-0.54\ m/s^2)t. \tag{5.3.20}$$

b. Now that we have the equations of motion for x and y as functions of time, we can evaluate them at t = 10.0 s:

$$x(10.0\ s) = 75.0\ m + (4.1\ m/s)(10.0\ s) + \frac{1}{2}(2.0\ m/s^2)(10.0\ s)^2 = 216.0\ m \tag{5.3.21}$$

$$v_x(10.0\ s) = 4.1\ m/s + (2.0\ m/s^2)(10.0\ s) = 24.1\ m/s \tag{5.3.22}$$

$$y(10.0) = -50.0.0\ m + (-1.1\ m/s)(10.0\ s) + \frac{1}{2}(-0.54\ m/s^2)(10.0\ s)^2 \tag{5.3.23}$$

$$v_y(10.0\ s) = -1.1\ m/s + (-0.54\ m/s^2)(10.0\ s). \tag{5.3.24}$$

The position and velocity at t = 10.0 s are, finally

$$\vec{r}(10.0\ s) = (216.0\ \hat{i} - 88.0\ \hat{j})m \tag{5.3.25}$$

$$\vec{v}(10.0\ s) = (24.1\ \hat{i} - 6.5\ \hat{j})m/s. \tag{5.3.26}$$

The magnitude of the velocity of the skier at 10.0 s is 25 m/s, which is 60 mi/h.

Significance

It is useful to know that, given the initial conditions of position, velocity, and acceleration of an object, we can find the position, velocity, and acceleration at any later time.

With Equations 5.3.1-5.3.3 we have completed the set of expressions for the position, velocity, and acceleration of an object moving in two or three dimensions. If the trajectories of the objects look something like the “Red Arrows” in the opening picture for the chapter, then the expressions for the position, velocity, and acceleration can be quite complicated. In the sections to follow we examine two special cases of motion in two and three dimensions by looking at projectile motion and circular motion.

Simulation

At this University of Colorado Boulder website, you can explore the position velocity and acceleration of a ladybug with an interactive simulation that allows you to change these parameters.

- **4.3: Acceleration Vector** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/university-physics-volume-1.

5.4: Projectile Motion

Learning Objectives

- Use one-dimensional motion in perpendicular directions to analyze projectile motion.
- Calculate the range, time of flight, and maximum height of a projectile that is launched and impacts a flat, horizontal surface.
- Find the time of flight and impact velocity of a projectile that lands at a different height from that of launch.
- Calculate the trajectory of a projectile.

Projectile motion is the motion of an object thrown or projected into the air, subject only to acceleration as a result of gravity. The applications of projectile motion in physics and engineering are numerous. Some examples include meteors as they enter Earth's atmosphere, fireworks, and the motion of any ball in sports. Such objects are called **projectiles** and their path is called a **trajectory**. The motion of falling objects as discussed in Motion Along a Straight Line is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In this section, we consider two-dimensional projectile motion, and our treatment neglects the effects of air resistance.

The most important fact to remember here is that **motions along perpendicular axes are independent** and thus can be analyzed separately. We discussed this fact in Displacement and Velocity Vectors, where we saw that vertical and horizontal motions are independent. The key to analyzing two-dimensional projectile motion is to break it into two motions: one along the horizontal axis and the other along the vertical. (This choice of axes is the most sensible because acceleration resulting from gravity is vertical; thus, there is no acceleration along the horizontal axis when air resistance is negligible.) As is customary, we call the horizontal axis the x-axis and the vertical axis the y-axis. It is not required that we use this choice of axes; it is simply convenient in the case of gravitational acceleration. In other cases we may choose a different set of axes. Figure $5.4.1$ illustrates the notation for displacement, where we define $\vec{s}$ to be the total displacement, and $\vec{x}$ and $\vec{y}$ are its component vectors along the horizontal and vertical axes, respectively. The magnitudes of these vectors are s, x, and y.

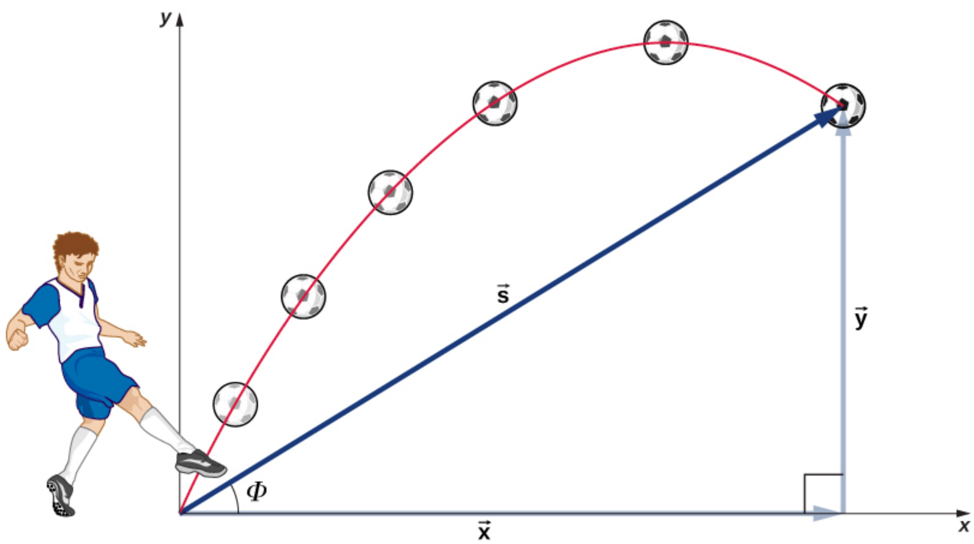

Figure $5.4.1$: The total displacement s of a soccer ball at a point along its path. The vector $\vec{s}$ has components $\vec{x}$ and $\vec{y}$ along the horizontal and vertical axes. Its magnitude is s and it makes an angle ϕ with the horizontal.

To describe projectile motion completely, we must include velocity and acceleration, as well as displacement. We must find their components along the x- and y-axes. Let's assume all forces except gravity (such as air resistance and friction, for example) are negligible. Defining the positive direction to be upward, the components of acceleration are then very simple:

$$a_y = -g = -9.8\ m/s^2 (-32\ ft/s^2). \tag{5.4.1}$$

Because gravity is vertical, $a_x = 0$. If $a_x = 0$, this means the initial velocity in the x direction is equal to the final velocity in the x direction, or $v_x = v_{0x}$. With these conditions on acceleration and velocity, we can write the kinematic Equation 4.11 through Equation 4.18 for motion in a uniform gravitational field, including the rest of the kinematic equations for a constant acceleration from Motion with Constant Acceleration. The kinematic equations for motion in a uniform gravitational field become kinematic equations with $a_y = -g$, $a_x = 0$:

Horizontal Motion

$$v_{0x} = v_x, \quad x = x_0 + v_x t \tag{5.4.2}$$

Vertical Motion

$$y = y_0 + \frac{1}{2}(v_{0y} + v_y)t \tag{5.4.3}$$

$$v_y = v_{0y} - gt \tag{5.4.4}$$

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2 \tag{5.4.5}$$

$$v_y^2 = v_{0y}^2 + 2g(y - y_0) \tag{5.4.6}$$

Using this set of equations, we can analyze projectile motion, keeping in mind some important points.

Problem-Solving Strategy: Projectile Motion

1. Resolve the motion into horizontal and vertical components along the x- and y-axes. The magnitudes of the components of displacement $\vec{s}$ along these axes are x and y. The magnitudes of the components of velocity $\vec{v}$ are $v_x = v\cos\theta$ and $v_y = v\sin\theta$, where v is the magnitude of the velocity and θ is its direction relative to the horizontal, as shown in Figure $5.4.2$.
2. Treat the motion as two independent one-dimensional motions: one horizontal and the other vertical. Use the kinematic equations for horizontal and vertical motion presented earlier.
3. Solve for the unknowns in the two separate motions: one horizontal and one vertical. Note that the only common variable between the motions is time t. The problem-solving procedures here are the same as those for one-dimensional kinematics and are illustrated in the following solved examples.
4. Recombine quantities in the horizontal and vertical directions to find the total displacement $\vec{s}$ and velocity $\vec{v}$. Solve for the magnitude and direction of the displacement and velocity using

$$s = \sqrt{x^2 + y^2}, \quad \phi = \tan^{-1}\left(\frac{y}{x}\right), \quad v = \sqrt{v_x^2 + v_y^2}. \tag{5.4.7}$$

where ϕ is the direction of the displacement $\vec{s}$.

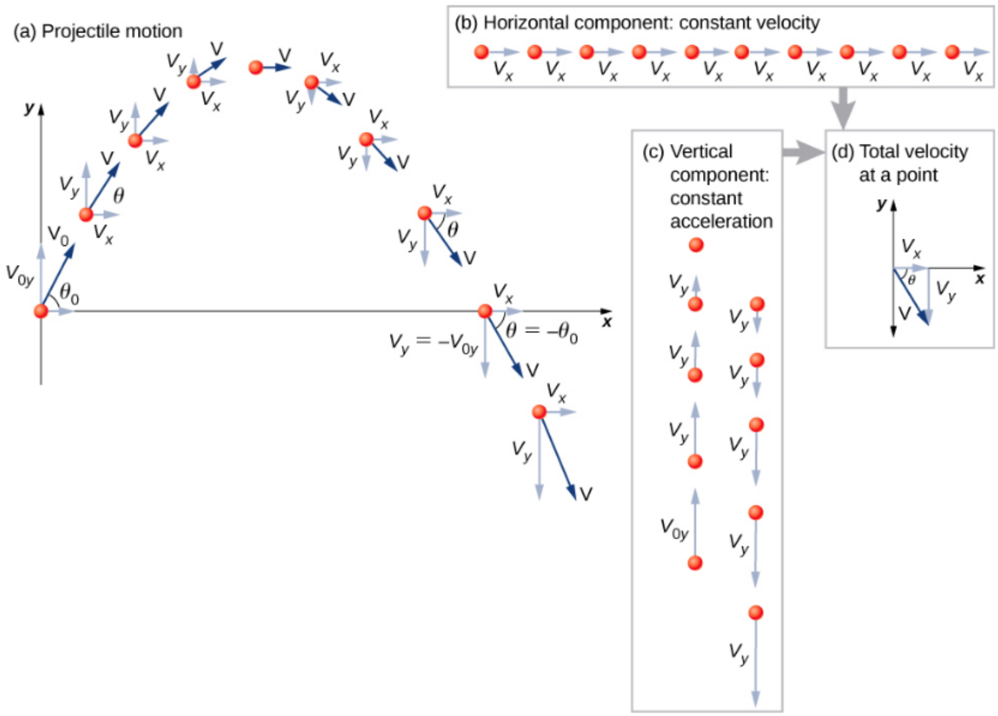

Figure 5.4.2: (a) We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. (b) The horizontal motion is simple, because $a_x = 0$ and v_x is a constant. (c) The velocity in the vertical direction begins to decrease as the object rises. At its highest point, the vertical velocity is zero. As the object falls toward Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity. (d) The x and y motions are recombined to give the total velocity at any given point on the trajectory.

✓ Example 4.7: A Fireworks Projectile Explodes high and away

During a fireworks display, a shell is shot into the air with an initial speed of 70.0 m/s at an angle of 75.0° above the horizontal, as illustrated in Figure 5.4.3. The fuse is timed to ignite the shell just as it reaches its highest point above the ground. (a) Calculate the height at which the shell explodes. (b) How much time passes between the launch of the shell and the explosion? (c) What is the horizontal displacement of the shell when it explodes? (d) What is the total displacement from the point of launch to the highest point?

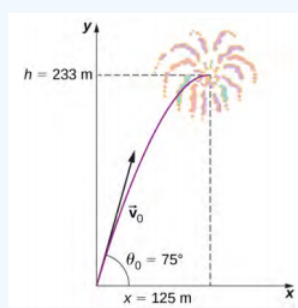

Figure 5.4.3: The trajectory of a fireworks shell. The fuse is set to explode the shell at the highest point in its trajectory, which is found to be at a height of 233 m and 125 m away horizontally.

Strategy

The motion can be broken into horizontal and vertical motions in which $a_x = 0$ and $a_y = -g$. We can then define x_0 and y_0 to be zero and solve for the desired quantities.

Solution

a. By “height” we mean the altitude or vertical position y above the starting point. The highest point in any trajectory, called the apex, is reached when $v_y = 0$. Since we know the initial and final velocities, as well as the initial position, we use the following equation to find y:

$$v_y^2 = v_{0y}^2 - 2g(y - y_0). \tag{5.4.8}$$

Because y_0 and v_y are both zero, the equation simplifies to

$$0 = v_{0y}^2 - 2gy. \tag{5.4.9}$$

Solving for y gives

$$y = \frac{v_{0y}^2}{2g}. \tag{5.4.10}$$

Now we must find v_{0y}, the component of the initial velocity in the y direction. It is given by $v_{0y} = v_0 \sin\theta_0$, where v_0 is the initial velocity of 70.0 m/s and $\theta_0 = 75°$ is the initial angle. Thus

$$v_{0y} = v_0 \sin\theta = (70.0\ m/s) \sin 75^o = 67.6\ m/s \tag{5.4.11}$$

and y is

$$y = \frac{(67.6\ m/s)^2}{2(9.80\ m/s^2)}. \tag{5.4.12}$$

Thus, we have

$$y = 233\ m. \tag{5.4.13}$$

Note that because up is positive, the initial vertical velocity is positive, as is the maximum height, but the acceleration resulting from gravity is negative. Note also that the maximum height depends only on the vertical component of the initial velocity, so that any projectile with a 67.6-m/s initial vertical component of velocity reaches a maximum height of 233 m (neglecting air resistance). The numbers in this example are reasonable for large fireworks displays, the shells of which do reach such heights before exploding. In practice, air resistance is not completely negligible, so the initial velocity would have to be somewhat larger than that given to reach the same height.

b. As in many physics problems, there is more than one way to solve for the time the projectile reaches its highest point. In this case, the easiest method is to use $v_y = v_{0y} - gt$. Because $v_y = 0$ at the apex, this equation reduces

$$0 = v_{0y} - gt \tag{5.4.14}$$

or

$$t = \frac{v_{0y}}{g} = \frac{67.6\ m/s}{9.80\ m/s^2} = 6.90\ s. \tag{5.4.15}$$

This time is also reasonable for large fireworks. If you are able to see the launch of fireworks, notice that several seconds pass before the shell explodes. Another way of finding the time is by using $y = y_0 + \frac{1}{2}(v_{0y} + v_y)t$. This is left for you as an exercise to complete.

c. Because air resistance is negligible, $a_x = 0$ and the horizontal velocity is constant, as discussed earlier. The horizontal displacement is the horizontal velocity multiplied by time as given by $x = x_0 + v_x t$, where x_0 is equal to zero. Thus,

$$x = v_x t, \tag{5.4.16}$$

where v_x is the x-component of the velocity, which is given by

$$v_x = v_0 \cos\theta = (70.0\ m/s) \cos 75^o = 18.1\ m/s. \tag{5.4.17}$$

Time t for both motions is the same, so x is

$$x = (18.1\ m/s)(6.90\ s) = 125\ m. \tag{5.4.18}$$

Horizontal motion is a constant velocity in the absence of air resistance. The horizontal displacement found here could be useful in keeping the fireworks fragments from falling on spectators. When the shell explodes, air resistance has a major effect, and many fragments land directly below.

d. The horizontal and vertical components of the displacement were just calculated, so all that is needed here is to find the magnitude and direction of the displacement at the highest point:

$$\vec{s} = 125\hat{i} + 233\hat{j} \tag{5.4.19}$$

$$|\vec{s}| = \sqrt{125^2 + 233^2} = 264\ m \tag{5.4.20}$$

$$\theta = \tan^{-1}\left(\frac{233}{125}\right) = 61.8^o. \tag{5.4.21}$$

Note that the angle for the displacement vector is less than the initial angle of launch. To see why this is, review Figure 5.4.1, which shows the curvature of the trajectory toward the ground level. When solving Example 4.7(a), the expression we found for y is valid for any projectile motion when air resistance is negligible. Call the maximum height y = h. Then,

$$h = \frac{v_{0y}^2}{2g}. \tag{5.4.22}$$

This equation defines the **maximum height of a projectile above its launch position** and it depends only on the vertical component of the initial velocity.

Exercise 4.3

A rock is thrown horizontally off a cliff 100.0 m high with a velocity of 15.0 m/s. (a) Define the origin of the coordinate system. (b) Which equation describes the horizontal motion? (c) Which equations describe the vertical motion? (d) What is the rock's velocity at the point of impact?

Example 4.8: Calculating projectile motion- Tennis Player

A tennis player wins a match at Arthur Ashe stadium and hits a ball into the stands at 30 m/s and at an angle 45° above the horizontal (Figure 5.4.4). On its way down, the ball is caught by a spectator 10 m above the point where the ball was hit. (a) Calculate the time it takes the tennis ball to reach the spectator. (b) What are the magnitude and direction of the ball's velocity at impact?

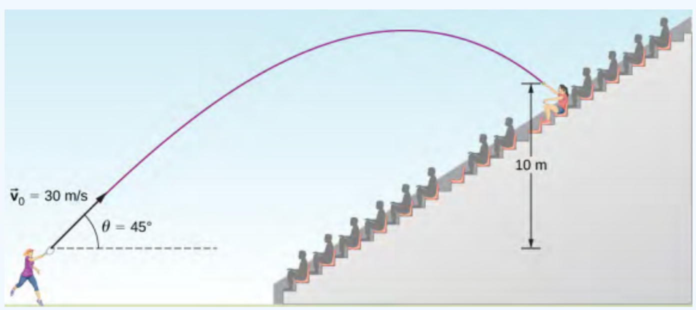

Figure 5.4.4: The trajectory of a tennis ball hit into the stands.

Strategy

Again, resolving this two-dimensional motion into two independent one-dimensional motions allows us to solve for the desired quantities. The time a projectile is in the air is governed by its vertical motion alone. Thus, we solve for t first. While the ball is rising and falling vertically, the horizontal motion continues at a constant velocity. This example asks for the final velocity. Thus, we recombine the vertical and horizontal results to obtain $\vec{v}$ at final time t, determined in the first part of the example.

Solution

a. While the ball is in the air, it rises and then falls to a final position 10.0 m higher than its starting altitude. We can find the time for this by using Equation 5.4.5:

$$y = y_0 + v_{0y}t - \frac{1}{2}gt^2. \tag{5.4.23}$$

If we take the initial position y_0 to be zero, then the final position is y = 10 m. The initial vertical velocity is the vertical component of the initial velocity:

$$v_{0y} = v_0 \sin\theta_0 = (30.0\ m/s)\sin 45^o = 21.2\ m/s. \tag{5.4.24}$$

Substituting into Equation 5.4.5 for y gives us

$$10.0\ m = (21.2\ m/s)t - (4.90\ m/s^2)t^2. \tag{5.4.25}$$

Rearranging terms gives a quadratic equation in t:

$$(4.90\ m/s^2)t^2 - (21.2\ m/s)t + 10.0\ m = 0. \tag{5.4.26}$$

Use of the quadratic formula yields t = 3.79 s and t = 0.54 s. Since the ball is at a height of 10 m at two times during its trajectory—once on the way up and once on the way down—we take the longer solution for the time it takes the ball to reach the spectator:

$$t = 3.79\ s. \tag{5.4.27}$$

The time for projectile motion is determined completely by the vertical motion. Thus, any projectile that has an initial vertical velocity of 21.2 m/s and lands 10.0 m above its starting altitude spends 3.79 s in the air.

b. We can find the final horizontal and vertical velocities v_x and v_y with the use of the result from (a). Then, we can combine them to find the magnitude of the total velocity vector $\vec{v}$ and the angle θ it makes with the horizontal. Since v_x is constant, we can solve for it at any horizontal location. We choose the starting point because we know both the initial velocity and the initial angle. Therefore,

$$v_x = v_0 \cos\theta_0 = (30\ m/s)\cos 45^o = 21.2\ m/s. \tag{5.4.28}$$

The final vertical velocity is given by Equation 5.4.4:

$$v_y = v_{0y} - gt. \tag{5.4.29}$$

Since v_{0y} was found in part (a) to be 21.2 m/s, we have

$$v_y = 21.2\ m/s - (9.8\ m/s^2)(3.79s) = -15.9\ m/s. \tag{5.4.30}$$

The magnitude of the final velocity $\vec{v}$ is

$$v = \sqrt{v_x^2 + v_y^2} = \sqrt{(21.2\ m/s)^2 + (-15.9\ m/s)^2} = 26.5\ m/s. \tag{5.4.31}$$

The direction θ_v is found using the inverse tangent:

$$\theta_v = \tan^{-1}\left(\frac{v_y}{v_x}\right) = \tan^{-1}\left(\frac{21.2}{-15.9}\right) = -53.1^o. \tag{5.4.32}$$

Significance

a. As mentioned earlier, the time for projectile motion is determined completely by the vertical motion. Thus, any projectile that has an initial vertical velocity of 21.2 m/s and lands 10.0 m above its starting altitude spends 3.79 s in the air.
b. The negative angle means the velocity is 53.1° below the horizontal at the point of impact. This result is consistent with the fact that the ball is impacting at a point on the other side of the apex of the trajectory and therefore has a negative y component of the velocity. The magnitude of the velocity is less than the magnitude of the initial velocity we expect since it is impacting 10.0 m above the launch elevation.

Time of Flight, Trajectory, and Range

Of interest are the time of flight, trajectory, and range for a projectile launched on a flat horizontal surface and impacting on the same surface. In this case, kinematic equations give useful expressions for these quantities, which are derived in the following sections.

Time of flight

We can solve for the time of flight of a projectile that is both launched and impacts on a flat horizontal surface by performing some manipulations of the kinematic equations. We note the position and displacement in y must be zero at launch and at impact on an even surface. Thus, we set the displacement in y equal to zero and find

$$y - y_0 = v_{0y}t - \frac{1}{2}gt^2 = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2 = 0. \tag{5.4.33}$$

Factoring, we have

$$t\left(v_0 \sin\theta_0 - \frac{gt}{2}\right) = 0. \tag{5.4.34}$$

Solving for t gives us

$$T_{tof} = \frac{2(v_0 \sin\theta_0)}{g}. \tag{5.4.35}$$

This is the **time of flight** for a projectile both launched and impacting on a flat horizontal surface. Equation 5.4.35 does not apply when the projectile lands at a different elevation than it was launched, as we saw in Example 4.8 of the tennis player hitting the ball into the stands. The other solution, t = 0, corresponds to the time at launch. The time of flight is linearly proportional to the initial velocity in the y direction and inversely proportional to g. Thus, on the Moon, where gravity is one-sixth that of Earth, a projectile launched with the same velocity as on Earth would be airborne six times as long.

Trajectory

The trajectory of a projectile can be found by eliminating the time variable t from the kinematic equations for arbitrary t and solving for y(x). We take $x_0 = y_0 = 0$ so the projectile is launched from the origin. The kinematic equation for x gives

$$x = v_{0x}t \Rightarrow t = \frac{x}{v_{0x}} = \frac{x}{v_0 \cos\theta_0}. \tag{5.4.36}$$

Substituting the expression for t into the equation for the position $y = (v_0 \sin\theta_0)t - \frac{1}{2}gt^2$ gives

$$y = (v_0 \sin\theta_0)\left(\frac{x}{v_0\cos\theta_0}\right) - \frac{1}{2}g\left(\frac{x}{v_0\cos\theta_0}\right)^2. \tag{5.4.37}$$

Rearranging terms, we have

$$y = (\tan\theta_0)x - \left[\frac{g}{2(v_0\cos\theta_0)^2}\right]x^2. \tag{5.4.38}$$

This trajectory equation is of the form $y = ax + bx^2$, which is an equation of a parabola with coefficients

$$a = \tan\theta_0, \quad b = -\frac{g}{2(v_0\cos\theta_0)^2}. \tag{5.4.39}$$

Range

From the trajectory equation we can also find the **range**, or the horizontal distance traveled by the projectile. Factoring Equation 5.4.38, we have

$$y = x\left[\tan\theta_0 - \frac{g}{2(v_0\cos\theta_0)^2}x\right]. \tag{5.4.40}$$

The position y is zero for both the launch point and the impact point, since we are again considering only a flat horizontal surface. Setting y = 0 in this equation gives solutions x = 0, corresponding to the launch point, and

$$x = \frac{2v_0^2 \sin\theta_0 \cos\theta_0}{g}, \tag{5.4.41}$$

corresponding to the impact point. Using the trigonometric identity $2\sin\theta\cos\theta = \sin 2\theta$ and setting x = R for range, we find

$$R = \frac{v_0^2 \sin 2\theta_0}{g}. \tag{5.4.42}$$

Note particularly that Equation 5.4.42 is valid only for launch and impact on a horizontal surface. We see the range is directly proportional to the square of the initial speed v_0 and $\sin^2\theta_0$, and it is inversely proportional to the acceleration of gravity. Thus, on the Moon, the range would be six times greater than on Earth for the same initial velocity. Furthermore, we see from the factor $\sin^2\theta_0$ that the range is maximum at 45°. These results are shown in Figure 5.4.5. In (a) we see that the greater the initial velocity, the greater the range. In (b), we see that the range is maximum at 45°. This is true only for conditions neglecting air resistance. If air resistance is considered, the maximum angle is somewhat smaller. It is interesting that the same range is found for two initial launch angles that sum to 90°. The projectile launched with the smaller angle has a lower apex than the higher angle, but they both have the same range.

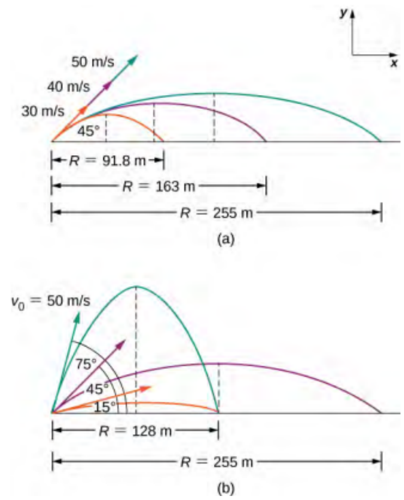

Figure 5.4.5: Trajectories of projectiles on level ground. (a) The greater the initial speed v_0, the greater the range for a given initial angle. (b) The effect of initial angle (\theta_{0}\) on the range of a projectile with a given initial speed. Note that the range is the same for initial angles of 15° and 75°, although the maximum heights of those paths are different.

Example 4.9: Comparing golf shots

A golfer finds himself in two different situations on different holes. On the second hole he is 120 m from the green and wants to hit the ball 90 m and let it run onto the green. He angles the shot low to the ground at 30° to the horizontal to let the ball roll after impact. On the fourth hole he is 90 m from the green and wants to let the ball drop with a minimum amount of rolling after impact. Here, he angles the shot at 70° to the horizontal to minimize rolling after impact. Both shots are hit and impacted on a level surface. (a) What is the initial speed of the ball at the second hole? (b) What is the initial speed of the ball at the fourth hole? (c) Write the trajectory equation for both cases. (d) Graph the trajectories.

Strategy

We see that the range equation has the initial speed and angle, so we can solve for the initial speed for both (a) and (b). When we have the initial speed, we can use this value to write the trajectory equation.

Solution

a.
$$R=\frac{v_0^2\sin 2\theta_0}{g}\Rightarrow v_0=\sqrt{\frac{Rg}{\sin 2\theta_0}}=\sqrt{\frac{(90.0\ m)(9.8\ m/s^2)}{\sin(2(30^o))}}=31.9\ m/s \tag{5.4.43}$$

b.
$$R=\frac{v_0^2\sin 2\theta_0}{g}\Rightarrow v_0=\sqrt{\frac{Rg}{\sin 2\theta_0}}=\sqrt{\frac{(90.0\ m)(9.8\ m/s^2)}{\sin(2(70^o))}}=37.0\ m/s \tag{5.4.44}$$

c.
$$y=x\left[\tan\theta_0-\frac{g}{2(v_0\cos\theta_0)^2}x\right] \tag{5.4.45}$$

Second hole:

$$y=x\left[\tan 30^o-\frac{9.8\ m/s^2}{2[(31.9\ m/s)(\cos 30^o)]^2}x\right]=0.58x-0.0064x^2 \tag{5.4.46}$$

Fourth hole:

$$y=x\left[\tan 70^o-\frac{9.8\ m/s^2}{2[(37.0\ m/s)(\cos 70^o)]^2}x\right]=2.75x-0.0306x^2 \tag{5.4.47}$$

d. Using a graphing utility, we can compare the two trajectories, which are shown in Figure 5.4.6.

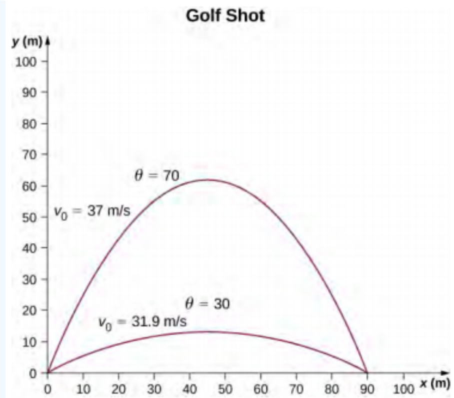

Figure 5.4.6: Two trajectories of a golf ball with a range of 90 m. The impact points of both are at the same level as the launch point.

Significance

The initial speed for the shot at 70° is greater than the initial speed of the shot at 30°. Note from Figure 5.4.6 that two projectiles launched at the same speed but at different angles have the same range if the launch angles add to 90°. The launch angles in this example add to give a number greater than 90°. Thus, the shot at 70° has to have a greater launch speed to reach 90 m, otherwise it would land at a shorter distance.

? Exercise 4.4

If the two golf shots in Example 4.9 were launched at the same speed, which shot would have the greatest range?

When we speak of the range of a projectile on level ground, we assume R is very small compared with the circumference of Earth. If, however, the range is large, Earth curves away below the projectile and the acceleration resulting from gravity changes direction along the path. The range is larger than predicted by the range equation given earlier because the projectile has farther to fall than it would on level ground, as shown in Figure 5.4.7, which is based on a drawing in Newton's **Principia**. If the initial speed is great enough, the projectile goes into orbit. Earth's surface drops 5 m every 8000 m. In 1 s an object falls 5 m without air resistance. Thus, if an object is given a horizontal velocity of 8000 m/s (or 18,000 mi/hr) near Earth's surface, it will go into orbit around the planet because the surface continuously falls away from the object. This is roughly the speed of the Space Shuttle in a low Earth orbit when it was operational, or any satellite in a low Earth orbit. These and other aspects of orbital motion, such as Earth's rotation, are covered in greater depth in Gravitation.

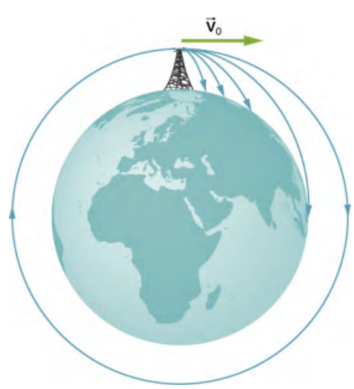

Figure 5.4.7: Projectile to satellite. In each case shown here, a projectile is launched from a very high tower to avoid air resistance. With increasing initial speed, the range increases and becomes longer than it would be on level ground because Earth curves away beneath its path. With a speed of 8000 m/s, orbit is achieved.

Simulation

At PhET Explorations: Projectile Motion, learn about projectile motion in terms of the launch angle and initial velocity.

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5.5: Uniform Circular Motion

Learning Objectives

- Solve for the centripetal acceleration of an object moving on a circular path.
- Use the equations of circular motion to find the position, velocity, and acceleration of a particle executing circular motion.
- Explain the differences between centripetal acceleration and tangential acceleration resulting from nonuniform circular motion.
- Evaluate centripetal and tangential acceleration in nonuniform circular motion, and find the total acceleration vector.

Uniform circular motion is a specific type of motion in which an object travels in a circle with a constant speed. For example, any point on a propeller spinning at a constant rate is executing uniform circular motion. Other examples are the second, minute, and hour hands of a watch. It is remarkable that points on these rotating objects are actually accelerating, although the rotation rate is a constant. To see this, we must analyze the motion in terms of vectors.

Centripetal Acceleration

In one-dimensional kinematics, objects with a constant speed have zero acceleration. However, in two- and three-dimensional kinematics, even if the speed is a constant, a particle can have acceleration if it moves along a curved trajectory such as a circle. In this case the velocity vector is changing, or $\frac{d\vec{v}}{dt} \neq 0$. This is shown in Figure $5.5.1$. As the particle moves counterclockwise in time Δt on the circular path, its position vector moves from $\vec{r}(t)$ to $\vec{r}(t+\Delta t)$. The velocity vector has constant magnitude and is tangent to the path as it changes from $\vec{v}$(t) to $\vec{v}\,(t+\Delta t)$, changing its direction only. Since the velocity vector $\vec{v}(t)$ is perpendicular to the position vector $\vec{r}$(t), the triangles formed by the position vectors and $\Delta\vec{r}$, and the velocity vectors and $\Delta\vec{v}$ are similar. Furthermore, since

$$|\vec{r}(t)| = |\vec{r}(t+\Delta t)|$$

and

$$|\vec{v}(t)| = |\vec{v}(t+\Delta t)|,$$

the two triangles are isosceles. From these facts we can make the assertion

$$\frac{\Delta v}{v} = \frac{\Delta r}{r} \tag{5.5.1}$$

or

$$\Delta v = \frac{v}{r}\Delta r. \tag{5.5.2}$$

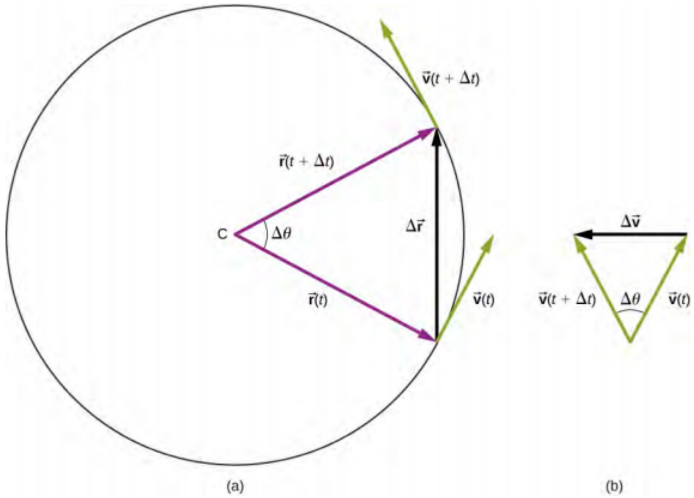

Figure 5.5.1: (a) A particle is moving in a circle at a constant speed, with position and velocity vectors at times t and $t+\Delta t$. (b) Velocity vectors forming a triangle. The two triangles in the figure are similar. The vector $\Delta\vec{v}$ points toward the center of the circle in the limit $\Delta t \rightarrow 0$.

We can find the magnitude of the acceleration from

$$a=\lim_{\Delta t \rightarrow 0}\left(\frac{\Delta v}{\Delta t}\right)=\frac{v}{r}\left(\lim_{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t}\right)=\frac{v^{2}}{r}. \tag{5.5.3}$$

The direction of the acceleration can also be found by noting that as Δt and therefore $\Delta\theta$ approach zero, the vector $\Delta\vec{v}$ approaches a direction perpendicular to $\vec{v}$. In the limit $\Delta t \rightarrow 0$, $\Delta\vec{v}$ is perpendicular to $\vec{v}$. Since $\vec{v}$ is tangent to the circle, the acceleration $\frac{d\vec{v}}{dt}$ points toward the center of the circle. Summarizing, a particle moving in a circle at a constant speed has an acceleration with magnitude

$$a_{c}=\frac{v^{2}}{r}. \tag{5.5.4}$$

The direction of the acceleration vector is toward the center of the circle (Figure 5.5.2). This is a radial acceleration and is called the **centripetal acceleration**, which is why we give it the subscript c. The word **centripetal** comes from the Latin words **centrum** (meaning "center") and **petere** (meaning to seek"), and thus takes the meaning "center seeking."

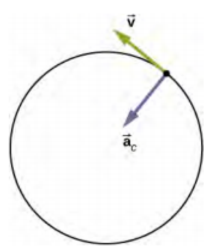

Figure 5.5.2: The centripetal acceleration vector points toward the center of the circular path of motion and is an acceleration in the radial direction. The velocity vector is also shown and is tangent to the circle.

Let's investigate some examples that illustrate the relative magnitudes of the velocity, radius, and centripetal acceleration.

Example 5.5.1: Creating an Acceleration of 1 g

A jet is flying at 134.1 m/s along a straight line and makes a turn along a circular path level with the ground. What does the radius of the circle have to be to produce a centripetal acceleration of 1 g on the pilot and jet toward the center of the circular trajectory?

Strategy

Given the speed of the jet, we can solve for the radius of the circle in the expression for the centripetal acceleration.

Solution

Set the centripetal acceleration equal to the acceleration of gravity: 9.8 m/s^2 = $\frac{v^2}{r}$.

Solving for the radius, we find

$$r = \frac{(134.1\ m/s)^2}{9.8\ m/s^2} = 1835\ m = 1.835\ km. \tag{5.5.5}$$

Significance

To create a greater acceleration than g on the pilot, the jet would either have to decrease the radius of its circular trajectory or increase its speed on its existing trajectory or both.

Exercise 4.5

A flywheel has a radius of 20.0 cm. What is the speed of a point on the edge of the flywheel if it experiences a centripetal acceleration of 900.0 cm/s^2?

Centripetal acceleration can have a wide range of values, depending on the speed and radius of curvature of the circular path. Typical centripetal accelerations are given in Table $5.5.1$.

Table $5.5.1$: Typical Centripetal Accelerations

Object	Centripetal Acceleration (m/s^2 or factors of g)
Earth around the Sun	5.93 x 10^{-3}
Moon around the Earth	2.73 x 10^{-3}
Satellite in geosynchronous orbit	0.233
Outer edge of a CD when playing	5.75
Jet in a barrel roll	(2-3 g)
Roller coaster	(5 g)
Electron orbiting a proton in a simple Bohr model of the atom	9.0 x 10^{22}

Equations of Motion for Uniform Circular Motion

A particle executing circular motion can be described by its position vector $\vec{r}(t)$. Figure $5.5.3$ shows a particle executing circular motion in a counterclockwise direction. As the particle moves on the circle, its position vector sweeps out the angle θ with the x-axis. Vector $\vec{r}(t)$ making an angle θ with the x-axis is shown with its components along the x- and y-axes. The magnitude of the position vector is $A = |\vec{r}(t)|$ and is also the radius of the circle, so that in terms of its components,

$$\vec{r}(t) = A \cos \omega \hat{i} + A \sin \omega t \hat{j}. \tag{5.5.6}$$

Here, ω is a constant called the **angular frequency** of the particle. The angular frequency has units of radians (rad) per second and is simply the number of radians of angular measure through which the particle passes per second. The angle θ that the position vector has at any particular time is ωt.

If T is the period of motion, or the time to complete one revolution ($2\pi\ rad$), then

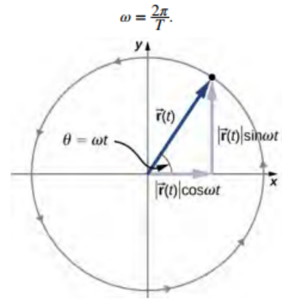

Figure 5.5.3: The position vector for a particle in circular motion with its components along the x- and y-axes. The particle moves counterclockwise. Angle θ is the angular frequency ω in radians per second multiplied by t.

Velocity and acceleration can be obtained from the position function by differentiation:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -A\omega\sin\omega t\,\hat{i} + A\omega\cos\omega t\,\hat{j}. \tag{5.5.7}$$

It can be shown from Figure 5.5.3 that the velocity vector is tangential to the circle at the location of the particle, with magnitude A ω. Similarly, the acceleration vector is found by differentiating the velocity:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -A\omega^2\cos\omega t\,\hat{i} - A\omega^2\sin\omega t\,\hat{j}. \tag{5.5.8}$$

From this equation we see that the acceleration vector has magnitude Aω^2 and is directed opposite the position vector, toward the origin, because $\vec{a}$(t) = $-\omega^2\vec{r}$(t).

Example 5.5.2: Circular Motion of a Proton

A proton has speed 5 x 10^6 m/s and is moving in a circle in the xy plane of radius r = 0.175 m. What is its position in the xy plane at time t = 2.0 x 10^{-7} s = 200 ns? At t = 0, the position of the proton is 0.175 m $\hat{i}$ and it circles counterclockwise. Sketch the trajectory.

Solution

From the given data, the proton has period and angular frequency:

$$T = \frac{2\pi r}{v} = \frac{2\pi(0.175\ m)}{5.0\times 10^6\ m/s} = 2.20\times 10^{-7}\ s$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2.20\times 10^{-7}\ s} = 2.856\times 10^7\ rad/s.$$

The position of the particle at t = 2.0 x 10^{-7} s with A = 0.175 m is

$$\begin{aligned}\vec{r}(2.0\times 10^{-7}\ s) &= A\cos\omega(2.0\times 10^{-7}\ s)\hat{i} + A\sin\omega(2.0\times 10^{-7}\ s)\hat{j}\ m \\ &= 0.175\cos(2.856\times 10^7\ rad/s)(2.0\times 10^{-7}\ s)\hat{i} + 0.175\sin(2.856\times 10^7\ rad/s)(2.0\times 10^{-7}\ s)\hat{j}\ m \\ &= 0.175\cos(5.712\ rad)\hat{i} + 0.175\sin(5.172\ rad)\hat{j}\ m \\ &= 0.147\hat{i} - 0.095\hat{j}\ m.\end{aligned}$$

From this result we see that the proton is located slightly below the x-axis. This is shown in Figure 5.5.4.

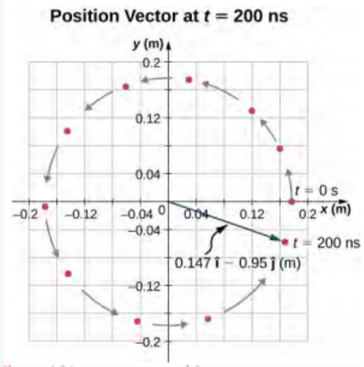

Figure 5.5.4: Position vector of the proton at $t = 2.0 \times 10^{-7}\, ms = 200\, ns$. The trajectory of the proton is shown. The angle through which the proton travels along the circle is 5.712 rad, which a little less than one complete revolution.

Significance

We picked the initial position of the particle to be on the x-axis. This was completely arbitrary. If a different starting position were given, we would have a different final position at t = 200 ns.

Nonuniform Circular Motion

Circular motion does not have to be at a constant speed. A particle can travel in a circle and speed up or slow down, showing an acceleration in the direction of the motion.

In uniform circular motion, the particle executing circular motion has a constant speed and the circle is at a fixed radius. If the speed of the particle is changing as well, then we introduce an additional acceleration in the direction tangential to the circle. Such accelerations occur at a point on a top that is changing its spin rate, or any accelerating rotor. In Displacement and Velocity Vectors we showed that centripetal acceleration is the time rate of change of the direction of the velocity vector. If the speed of the particle is changing, then it has a **tangential acceleration** that is the time rate of change of the magnitude of the velocity:

$$a_T = \frac{d|\vec{v}|}{dt}. \tag{5.5.9}$$

The direction of tangential acceleration is tangent to the circle whereas the direction of centripetal acceleration is radially inward toward the center of the circle. Thus, a particle in circular motion with a tangential acceleration has a **total acceleration** that is the vector sum of the centripetal and tangential accelerations:

$$\vec{a} = \vec{a}_c + \vec{a}_T. \tag{5.5.10}$$

The acceleration vectors are shown in Figure $5.5.5$. Note that the two acceleration vectors $\vec{a}_c$ and $\vec{a}_T$ are perpendicular to each other, with $\vec{a}_c$ in the radial direction and $\vec{a}_T$ in the tangential direction. The total acceleration $\vec{a}$ points at an angle between $\vec{a}_c$ and $\vec{a}_T$.

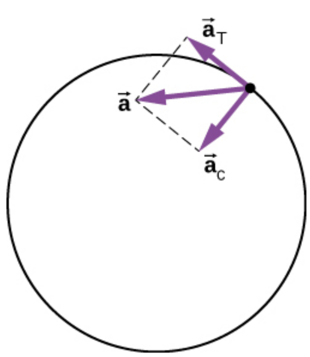

Figure 5.5.5: The centripetal acceleration points toward the center of the circle. The tangential acceleration is tangential to the circle at the particle's position. The total acceleration is the vector sum of the tangential and centripetal accelerations, which are perpendicular.

✓ Example 5.5.3: Total Acceleration during Circular Motion

A particle moves in a circle of radius r = 2.0 m. During the time interval from t = 1.5 s to t = 4.0 s its speed varies with time according to

$$v(t) = c_1 - \frac{c_2}{t^2}, c_1 = 4.0\ m/s, c_2 = 6.0\ m \cdot s. \tag{5.5.11}$$

What is the total acceleration of the particle at t = 2.0 s?

Strategy

We are given the speed of the particle and the radius of the circle, so we can calculate centripetal acceleration easily. The direction of the centripetal acceleration is toward the center of the circle. We find the magnitude of the tangential acceleration by taking the derivative with respect to time of |v(t)| using Equation 5.5.9 and evaluating it at t = 2.0 s. We use this and the magnitude of the centripetal acceleration to find the total acceleration.

Solution

Centripetal acceleration is

$$v(2.0\ s) = \left(4.0 - \frac{6.0}{(2.0)^2}\right) m/s = 2.5\ m/s$$

$$a_c = \frac{v^2}{r} = \frac{(2.5\ m/s)^2}{2.0\ m} = 3.1\ m/s^2$$

directed toward the center of the circle. Tangential acceleration is

$$a_T = \left|\frac{d\vec{v}}{dt}\right| = \frac{2c_2}{t^3} = \frac{12.0}{(2.0)^3} m/s^2 = 1.5\ m/s^2.$$

Total acceleration is

$$|\vec{a}| = \sqrt{3.1^2 + 1.5^2}\, m/s^2 = 3.44\ m/s^2 \tag{5.5.12}$$

and $\theta = \tan^{-1}\left(\frac{3.1}{1.5}\right) = 64°$ from the tangent to the circle. See Figure 5.5.6.

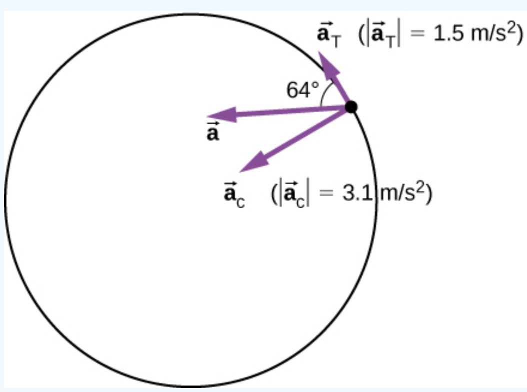

Figure 5.5.6: The tangential and centripetal acceleration vectors. The net acceleration $\vec{a}$ is the vector sum of the two accelerations.

Significance

The directions of centripetal and tangential accelerations can be described more conveniently in terms of a polar coordinate system, with unit vectors in the radial and tangential directions. This coordinate system, which is used for motion along curved paths, is discussed in detail later in the book.

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5.6: Relative Motion in One and Two Dimensions

Learning Objectives

- Explain the concept of reference frames.
- Write the position and velocity vector equations for relative motion.
- Draw the position and velocity vectors for relative motion.
- Analyze one-dimensional and two-dimensional relative motion problems using the position and velocity vector equations.

Motion does not happen in isolation. If you're riding in a train moving at 10 m/s east, this velocity is measured relative to the ground on which you're traveling. However, if another train passes you at 15 m/s east, your velocity relative to this other train is different from your velocity relative to the ground. Your velocity relative to the other train is 5 m/s west. To explore this idea further, we first need to establish some terminology.

Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of **reference frames**. When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth. If you say a person is sitting in a train moving at 10 m/s east, then you imply the person on the train is moving relative to the surface of Earth at this velocity, and Earth is the reference frame. We can expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes more complicated. In this case, the solar system is the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion.

Relative Motion in One Dimension

We introduce relative motion in one dimension first, because the velocity vectors simplify to having only two possible directions. Take the example of the person sitting in a train moving east. If we choose east as the positive direction and Earth as the reference frame, then we can write the velocity of the train with respect to the Earth as $\vec{v}_{TE}$ = 10 m/s $\hat{i}$ east, where the subscripts TE refer to train and Earth. Let's now say the person gets up out of /her seat and walks toward the back of the train at 2 m/s. This tells us she has a velocity relative to the reference frame of the train. Since the person is walking west, in the negative direction, we write her velocity with respect to the train as $\vec{v}_{PT}$ = −2 m/s $\hat{i}$. We can add the two velocity vectors to find the velocity of the person with respect to Earth. This relative velocity is written as

$$\vec{v}_{PE} = \vec{v}_{PT} + \vec{v}_{TE}. \tag{5.6.1}$$

Note the ordering of the subscripts for the various reference frames in Equation 5.6.1. The subscripts for the coupling reference frame, which is the train, appear consecutively in the right-hand side of the equation. Figure 5.6.1 shows the correct order of subscripts when forming the vector equation.

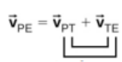


Figure 5.6.1: When constructing the vector equation, the subscripts for the coupling reference frame appear consecutively on the inside. The subscripts on the left-hand side of the equation are the same as the two outside subscripts on the right-hand side of the equation.

Adding the vectors, we find $\vec{v}_{PE}$ = 8 m/s $\hat{i}$, so the person is moving 8 m/s east with respect to Earth. Graphically, this is shown in Figure 5.6.2.

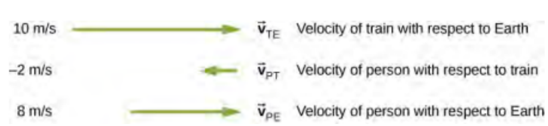


Figure 5.6.2: Velocity vectors of the train with respect to Earth, person with respect to the train, and person with respect to Earth.

Relative Velocity in Two Dimensions

We can now apply these concepts to describing motion in two dimensions. Consider a particle P and reference frames S and S′, as shown in Figure 5.6.3. The position of the origin of S′ as measured in S is $\vec{r}_{S'S}$, the position of P as measured in S′ is $\vec{r}_{PS'}$, and the position of P as measured in S is $\vec{r}_{PS}$.

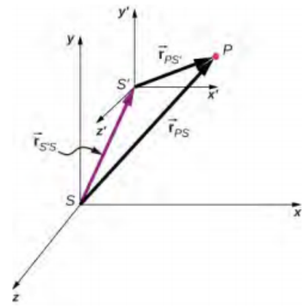


Figure $5.6.3$: The positions of particle P relative to frames S and S′ are $\vec{r}_{PS}$ and $\vec{r}_{PS'}$, respectively.

From Figure $5.6.3$ we see that

$$\vec{r}_{PS} = \vec{r}_{PS} + \vec{r}_{S'S}. \tag{5.6.2}$$

The relative velocities are the time derivatives of the position vectors. Therefore,

$$\vec{v}_{PS} = \vec{v}_{PS'} + \vec{v}_{S'S}. \tag{5.6.3}$$

The velocity of a particle relative to S is equal to its velocity relative to S′ plus the velocity of S′ relative to S.

We can extend Equation 5.6.3 to any number of reference frames. For particle P with velocities $\vec{v}_{PA}$, $\vec{v}_{PB}$, and $\vec{v}_{PC}$ in frames A, B, and C,

$$\vec{v}_{PC} = \vec{v}_{PA} + \vec{v}_{AB} + \vec{v}_{BC}. \tag{5.6.4}$$

We can also see how the accelerations are related as observed in two reference frames by differentiating Equation 5.6.3:

$$\vec{a}_{PS} = \vec{a}_{PS'} + \vec{a}_{S'S}. \tag{5.6.5}$$

We see that if the velocity of S′ relative to S is a constant, then $\vec{a}_{S'S}$ = 0 and

$$\vec{a}_{PS} = \vec{a}_{PS'}. \tag{5.6.6}$$

This says the acceleration of a particle is the same as measured by two observers moving at a constant velocity relative to each other.

Example 4.13: Motion of a Car Relative to a Truck

A truck is traveling south at a speed of 70 km/h toward an intersection. A car is traveling east toward the intersection at a speed of 80 km/h (Figure $5.6.4$). What is the velocity of the car relative to the truck?

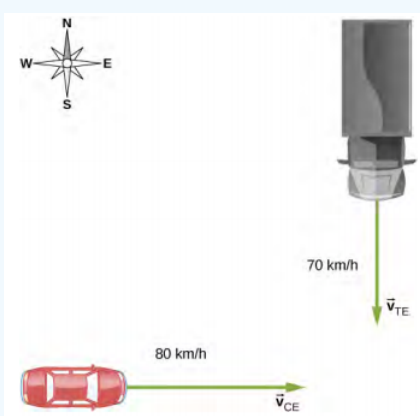


Figure $5.6.4$: A car travels east toward an intersection while a truck travels south toward the same intersection.

Strategy

First, we must establish the reference frame common to both vehicles, which is Earth. Then, we write the velocities of each with respect to the reference frame of Earth, which enables us to form a vector equation that links the car, the truck, and Earth to solve for the velocity of the car with respect to the truck.

Solution

The velocity of the car with respect to Earth is $\vec{v}_{CE}$ = 80 km/h $\hat{i}$. The velocity of the truck with respect to Earth is $\vec{v}_{TE}$ = −70 km/h $\hat{j}$. Using the velocity addition rule, the relative motion equation we are seeking is

$$\vec{v}_{CT} = \vec{v}_{CE} + \vec{v}_{ET}. \tag{5.6.7}$$

Here, $\vec{v}_{CT}$ is the velocity of the car with respect to the truck, and Earth is the connecting reference frame. Since we have the velocity of the truck with respect to Earth, the negative of this vector is the velocity of Earth with respect to the truck: $\vec{v}_{ET} = -\vec{v}_{TE}$. The vector diagram of this equation is shown in Figure 5.6.5.

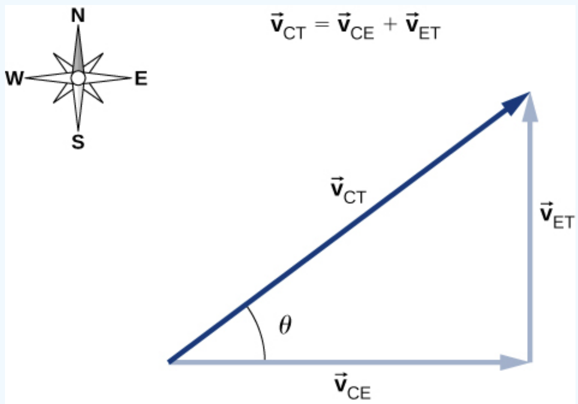


Figure 5.6.5: Vector diagram of the vector Equation 5.6.7.

We can now solve for the velocity of the car with respect to the truck:

$$|\vec{v}_{CT}| = \sqrt{(80.0\ km/h)^2 + (70.0\ km/h)^2} = 106.\ km/h$$

and

$$\theta = \tan^{-1}\left(\frac{70.0}{80.0}\right) = 41.2^o\ north\ of\ east.$$

Significance

Drawing a vector diagram showing the velocity vectors can help in understanding the relative velocity of the two objects.

Exercise 4.6

A boat heads north in still water at 4.5 m/s directly across a river that is running east at 3.0 m/s. What is the velocity of the boat with respect to Earth?

Example 4.14: Flying a Plane in a Wind

A pilot must fly his plane due north to reach his destination. The plane can fly at 300 km/h in still air. A wind is blowing out of the northeast at 90 km/h. (a) What is the speed of the plane relative to the ground? (b) In what direction must the pilot head her plane to fly due north?

Strategy

The pilot must point her plane somewhat east of north to compensate for the wind velocity. We need to construct a vector equation that contains the velocity of the plane with respect to the ground, the velocity of the plane with respect to the air, and the velocity of the air with respect to the ground. Since these last two quantities are known, we can solve for the velocity of the plane with respect to the ground. We can graph the vectors and use this diagram to evaluate the magnitude of the plane's velocity with respect to the ground. The diagram will also tell us the angle the plane's velocity makes with north with respect to the air, which is the direction the pilot must head her plane.

Solution

The vector equation is $\vec{v}_{PG} = \vec{v}_{PA} + \vec{v}_{AG}$, where P = plane, A = air, and G = ground. From the geometry in Figure 5.6.6, we can solve easily for the magnitude of the velocity of the plane with respect to the ground and the angle of the plane's heading, θ.

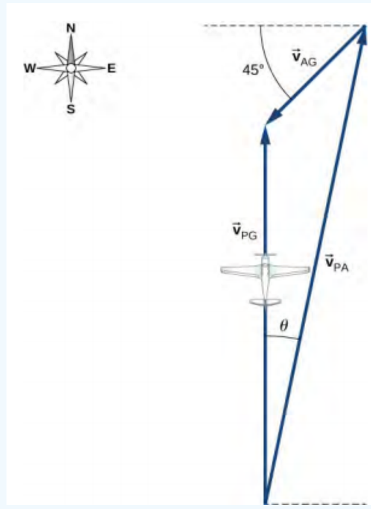


Figure 5.6.6: Vector diagram for Equation 5.6.2 showing the vectors $\vec{v}_{PA}$, $\vec{v}_{AG}$, and $\vec{v}_{PG}$.

a. Known quantities:

$$\left|\vec{v}_{PA}\right| = 300\ km/h \tag{5.6.8}$$

$$\left|\vec{v}_{AG}\right| = 90\ km/h \tag{5.6.9}$$

Substituting into the equation of motion, we obtain $\left|\vec{v}_{PG}\right|$ = 230 km/h.

b. The angle $\theta = \tan^{-1}\left(\frac{63.64}{300}\right)$ = 12° east of north.

5.7: Motion in Two and Three Dimensions (Exercises)

Conceptual Questions

4.1 Displacement and Velocity Vectors

1. What form does the trajectory of a particle have if the distance from any point A to point B is equal to the magnitude of the displacement from A to B?
2. Give an example of a trajectory in two or three dimensions caused by independent perpendicular motions.
3. If the instantaneous velocity is zero, what can be said about the slope of the position function?

4.2 Acceleration Vector

4. If the position function of a particle is a linear function of time, what can be said about its acceleration?
5. If an object has a constant x-component of the velocity and suddenly experiences an acceleration in the y direction, does the x-component of its velocity change?
6. If an object has a constant x-component of velocity and suddenly experiences an acceleration at an angle of 70° in the x direction, does the x-component of velocity change?

4.3 Projectile Motion

7. Answer the following questions for projectile motion on level ground assuming negligible air resistance, with the initial angle being neither 0° nor 90° : (a) Is the velocity ever zero? (b) When is the velocity a minimum? A maximum? (c) Can the velocity ever be the same as the initial velocity at a time other than at t = 0? (d) Can the speed ever be the same as the initial speed at a time other than at t = 0?
8. Answer the following questions for projectile motion on level ground assuming negligible air resistance, with the initial angle being neither 0° nor 90° : (a) Is the acceleration ever zero? (b) Is the vector $\vec{v}$ ever parallel or antiparallel to the vector $\vec{a}$? (c) Is the vector $\vec{v}$ ever perpendicular to the vector $\vec{a}$? If so, where is this located?
9. A dime is placed at the edge of a table so it hangs over slightly. A quarter is slid horizontally on the table surface perpendicular to the edge and hits the dime head on. Which coin hits the ground first?

4.4 Uniform Circular Motion

10. Can centripetal acceleration change the speed of a particle undergoing circular motion?
11. Can tangential acceleration change the speed of a particle undergoing circular motion?

4.5 Relative Motion in One and Two Dimensions

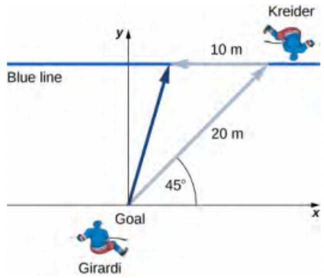
12. What frame or frames of reference do you use instinctively when driving a car? When flying in a commercial jet?
13. A basketball player dribbling down the court usually keeps his eyes fixed on the players around him. He is moving fast. Why doesn't he need to keep his eyes on the ball?
14. If someone is riding in the back of a pickup truck and throws a softball straight backward, is it possible for the ball to fall straight down as viewed by a person standing at the side of the road? Under what condition would this occur? How would the motion of the ball appear to the person who threw it?
15. The hat of a jogger running at constant velocity falls off the back of his head. Draw a sketch showing the path of the hat in the jogger's frame of reference. Draw its path as viewed by a stationary observer. Neglect air resistance.
16. A clod of dirt falls from the bed of a moving truck. It strikes the ground directly below the end of the truck. (a) What is the direction of its velocity relative to the truck just before it hits? (b) Is this the same as the direction of its velocity relative to ground just before it hits? Explain your answers.

Problems

4.1 Displacement and Velocity Vectors

17. The coordinates of a particle in a rectangular coordinate system are (1.0, –4.0, 6.0). What is the position vector of the particle?
18. The position of a particle changes from $\vec{r}_1 = (2.0\ \hat{i} + 3.0\ \hat{j})$cm to $\vec{r}_2 = (-4.0\ \hat{i} + 3.0\ \hat{j})$ cm. What is the particle's displacement?

19. The 18th hole at Pebble Beach Golf Course is a dogleg to the left of length 496.0 m. The fairway off the tee is taken to be the x direction. A golfer hits his tee shot a distance of 300.0 m, corresponding to a displacement $\Delta\vec{r}_1$ = 300.0 m $\hat{i}$, and hits his second shot 189.0 m with a displacement $\Delta\vec{r}_2$ = 172.0 m $\hat{i}$ + 80.3 m $\hat{j}$. What is the final displacement of the golf ball from the tee?
20. A bird flies straight northeast a distance of 95.0 km for 3.0 h. With the x-axis due east and the y-axis due north, what is the displacement in unit vector notation for the bird? What is the average velocity for the trip?
21. A cyclist rides 5.0 km due east, then 10.0 km 20° west of north. From this point she rides 8.0 km due west. What is the final displacement from where the cyclist started?
22. New York Rangers defenseman Daniel Girardi stands at the goal and passes a hockey puck 20 m and 45° from straight down the ice to left wing Chris Kreider waiting at the blue line. Kreider waits for Girardi to reach the blue line and passes the puck directly across the ice to him 10 m away. What is the final displacement of the puck? See the following figure.



23. The position of a particle is $\vec{r}(t) = 4.0t^2\,\hat{i} - 3.0\,\hat{j} + 2.0t^3\,\hat{k}$ m. (a) What is the velocity of the particle at 0 s and at 1.0 s? (b) What is the average velocity between 0 s and 1.0 s?
24. Clay Matthews, a linebacker for the Green Bay Packers, can reach a speed of 10.0 m/s. At the start of a play, Matthews runs downfield at 45° with respect to the 50-yard line and covers 8.0 m in 1 s. He then runs straight down the field at 90° with respect to the 50-yard line for 12 m, with an elapsed time of 1.2 s. (a) What is Matthews' final displacement from the start of the play? (b) What is his average velocity?
25. The F-35B Lighting II is a short-takeoff and vertical landing fighter jet. If it does a vertical takeoff to 20.00-m height above the ground and then follows a flight path angled at 30° with respect to the ground for 20.00 km, what is the final displacement?

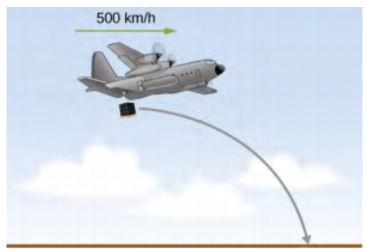
4.2 Acceleration Vector

26. The position of a particle is $\vec{r}$ (t) = (3.0$^2\,\hat{i}$ + 5.0 $\hat{j}$ − 6.0t $\hat{k}$) m. (a) Determine its velocity and acceleration as functions of time. (b) What are its velocity and acceleration at time t = 0?
27. A particle's acceleration is (4.0 $\hat{i}$ + 3.0 $\hat{j}$)m/s^2. At t = 0, its position and velocity are zero. (a) What are the particle's position and velocity as functions of time? (b) Find the equation of the path of the particle. Draw the x- and y-axes and sketch the trajectory of the particle.
28. A boat leaves the dock at t = 0 and heads out into a lake with an acceleration of 2.0 m/s^2 $\hat{i}$. A strong wind is pushing the boat, giving it an additional velocity of 2.0 m/s $\hat{i}$ + 1.0 m/s $\hat{j}$. (a) What is the velocity of the boat at t = 10 s? (b) What is the position of the boat at t = 10s? Draw a sketch of the boat's trajectory and position at t = 10 s, showing the x- and y-axes.
29. The position of a particle for t > 0 is given by $\vec{r}(t) = (3.0t^2\,\hat{i} - 7.0t^3\,\hat{j} - 5.0t^{-2}\,\hat{k})$ m. (a) What is the velocity as a function of time? (b) What is the acceleration as a function of time? (c) What is the particle's velocity at t = 2.0 s? (d) What is its speed at t = 1.0 s and t = 3.0 s? (e) What is the average velocity between t = 1.0 s and t = 2.0 s?
30. The acceleration of a particle is a constant. At t = 0 the velocity of the particle is (10 $\hat{i}$ + 20 $\hat{j}$)m/s. At t = 4 s the velocity is 10 $\hat{j}$ m/s. (a) What is the particle's acceleration? (b) How do the position and velocity vary with time? Assume the particle is initially at the origin.
31. A particle has a position function $\vec{r}(t) = \cos(1.0t)\,\hat{i} + \sin(1.0t)\,\hat{j} + t\,\hat{k}$, where the arguments of the cosine and sine functions are in radians. (a) What is the velocity vector? (b) What is the acceleration vector?
32. A Lockheed Martin F-35 II Lighting jet takes off from an aircraft carrier with a runway length of 90 m and a takeoff speed 70 m/s at the end of the runway. Jets are catapulted into airspace from the deck of an aircraft carrier with two sources of propulsion: the jet propulsion and the catapult. At the point of leaving the deck of the aircraft carrier, the F-

35’s acceleration decreases to a constant acceleration of 5.0 m/s^2 at 30° with respect to the horizontal. (a) What is the initial acceleration of the F-35 on the deck of the aircraft carrier to make it airborne? (b) Write the position and velocity of the F-35 in unit vector notation from the point it leaves the deck of the aircraft carrier. (c) At what altitude is the fighter 5.0 s after it leaves the deck of the aircraft carrier? (d) What is its velocity and speed at this time? (e) How far has it traveled horizontally?

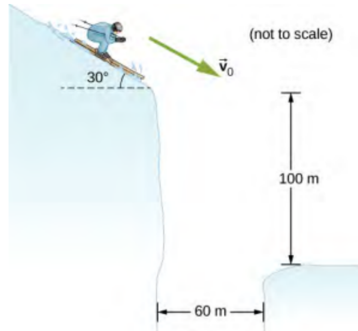
4.3 Projectile Motion

33. A bullet is shot horizontally from shoulder height (1.5 m) with and initial speed 200 m/s. (a) How much time elapses before the bullet hits the ground? (b) How far does the bullet travel horizontally?
34. A marble rolls off a tabletop 1.0 m high and hits the floor at a point 3.0 m away from the table’s edge in the horizontal direction. (a) How long is the marble in the air? (b) What is the speed of the marble when it leaves the table’s edge? (c) What is its speed when it hits the floor?
35. A dart is thrown horizontally at a speed of 10 m/ s at the bull’s-eye of a dartboard 2.4 m away, as in the following figure. (a) How far below the intended target does the dart hit? (b) What does your answer tell you about how proficient dart players throw their darts?
36. An airplane flying horizontally with a speed of 500 km/h at a height of 800 m drops a crate of supplies (see the following figure). If the parachute fails to open, how far in front of the release point does the crate hit the ground?

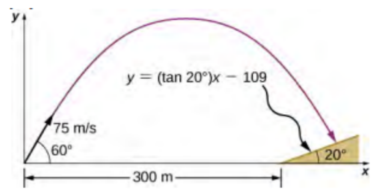


37. Suppose the airplane in the preceding problem fires a projectile horizontally in its direction of motion at a speed of 300 m/s relative to the plane. (a) How far in front of the release point does the projectile hit the ground? (b) What is its speed when it hits the ground?
38. A fastball pitcher can throw a baseball at a speed of 40 m/s (90 mi/h). (a) Assuming the pitcher can release the ball 16.7 m from home plate so the ball is moving horizontally, how long does it take the ball to reach home plate? (b) How far does the ball drop between the pitcher’s hand and home plate?
39. A projectile is launched at an angle of 30° and lands 20 s later at the same height as it was launched. (a) What is the initial speed of the projectile? (b) What is the maximum altitude? (c) What is the range? (d) Calculate the displacement from the point of launch to the position on its trajectory at 15 s.
40. A basketball player shoots toward a basket 6.1 m away and 3.0 m above the floor. If the ball is released 1.8 m above the floor at an angle of 60° above the horizontal, what must the initial speed be if it were to go through the basket?
41. At a particular instant, a hot air balloon is 100 m in the air and descending at a constant speed of 2.0 m/s. At this exact instant, a girl throws a ball horizontally, relative to herself, with an initial speed of 20 m/s. When she lands, where will she find the ball? Ignore air resistance.
42. A man on a motorcycle traveling at a uniform speed of 10 m/s throws an empty can straight upward relative to himself with an initial speed of 3.0 m/s. Find the equation of the trajectory as seen by a police officer on the side of the road. Assume the initial position of the can is the point where it is thrown. Ignore air resistance.
43. An athlete can jump a distance of 8.0 m in the broad jump. What is the maximum distance the athlete can jump on the Moon, where the gravitational acceleration is onesixth that of Earth?
44. The maximum horizontal distance a boy can throw a ball is 50 m. Assume he can throw with the same initial speed at all angles. How high does he throw the ball when he throws it straight upward?
45. A rock is thrown off a cliff at an angle of 53° with respect to the horizontal. The cliff is 100 m high. The initial speed of the rock is 30 m/s. (a) How high above the edge of the cliff does the rock rise? (b) How far has it moved horizontally when it is at maximum altitude? (c) How long after the release does it hit the ground? (d) What is the range of the rock? (e) What are the horizontal and vertical positions of the rock relative to the edge of the cliff at t = 2.0 s, t = 4.0 s, and t = 6.0 s?

46. Trying to escape his pursuers, a secret agent skis off a slope inclined at 30° below the horizontal at 60 km/h. To survive and land on the snow 100 m below, he must clear a gorge 60 m wide. Does he make it? Ignore air resistance.



47. A golfer on a fairway is 70 m away from the green, which sits below the level of the fairway by 20 m. If the golfer hits the ball at an angle of 40° with an initial speed of 20 m/s, how close to the green does she come?
48. A projectile is shot at a hill, the base of which is 300 m away. The projectile is shot at 60° above the horizontal with an initial speed of 75 m/s. The hill can be approximated by a plane sloped at 20° to the horizontal. Relative to the coordinate system shown in the following figure, the equation of this straight line is y = (tan 20°)x − 109. Where on the hill does the projectile land?



49. An astronaut on Mars kicks a soccer ball at an angle of 45° with an initial velocity of 15 m/s. If the acceleration of gravity on Mars is 3.7 m/s, (a) what is the range of the soccer kick on a flat surface? (b) What would be the range of the same kick on the Moon, where gravity is one-sixth that of Earth?
50. Mike Powell holds the record for the long jump of 8.95 m, established in 1991. If he left the ground at an angle of 15°, what was his initial speed?
51. MIT's robot cheetah can jump over obstacles 46 cm high and has speed of 12.0 km/h. (a) If the robot launches itself at an angle of 60° at this speed, what is its maximum height? (b) What would the launch angle have to be to reach a height of 46 cm?
52. Mt. Asama, Japan, is an active volcano. In 2009, an eruption threw solid volcanic rocks that landed 1 km horizontally from the crater. If the volcanic rocks were launched at an angle of 40° with respect to the horizontal and landed 900 m below the crater, (a) what would be their initial velocity and (b) what is their time of flight?
53. Drew Brees of the New Orleans Saints can throw a football 23.0 m/s (50 mph). If he angles the throw at 10° from the horizontal, what distance does it go if it is to be caught at the same elevation as it was thrown?
54. The Lunar Roving Vehicle used in NASA's late Apollo missions reached an unofficial lunar land speed of 5.0 m/s by astronaut Eugene Cernan. If the rover was moving at this speed on a flat lunar surface and hit a small bump that projected it off the surface at an angle of 20°, how long would it be "airborne" on the Moon?
55. A soccer goal is 2.44 m high. A player kicks the ball at a distance 10 m from the goal at an angle of 25°. What is the initial speed of the soccer ball?
56. Olympus Mons on Mars is the largest volcano in the solar system, at a height of 25 km and with a radius of 312 km. If you are standing on the summit, with what initial velocity would you have to fire a projectile from a cannon horizontally to clear the volcano and land on the surface of Mars? Note that Mars has an acceleration of gravity of 3.7 m/s^2.
57. In 1999, Robbie Knievel was the first to jump the Grand Canyon on a motorcycle. At a narrow part of the canyon (69.0 m wide) and traveling 35.8 m/s off the takeoff ramp, he reached the other side. What was his launch angle?
58. You throw a baseball at an initial speed of 15.0 m/s at an angle of 30° with respect to the horizontal. What would the ball's initial speed have to be at 30° on a planet that has twice the acceleration of gravity as Earth to achieve the same range? Consider launch and impact on a horizontal surface.

59. Aaron Rogers throws a football at 20.0 m/s to his wide receiver, who is running straight down the field at 9.4 m/s. If Aaron throws the football when the wide receiver is 10.0 m in front of him, what angle does Aaron have to launch the ball at so the receiver catches it 20.0 m in front of Aaron?

4.4 Uniform Circular Motion

60. A flywheel is rotating at 30 rev/s. What is the total angle, in radians, through which a point on the flywheel rotates in 40 s?
61. A particle travels in a circle of radius 10 m at a constant speed of 20 m/s. What is the magnitude of the acceleration?
62. Cam Newton of the Carolina Panthers throws a perfect football spiral at 8.0 rev/s. The radius of a pro football is 8.5 cm at the middle of the short side. What is the centripetal acceleration of the laces on the football?
63. A fairground ride spins its occupants inside a flying saucer-shaped container. If the horizontal circular path the riders follow has an 8.00-m radius, at how many revolutions per minute are the riders subjected to a centripetal acceleration equal to that of gravity?
64. A runner taking part in the 200-m dash must run around the end of a track that has a circular arc with a radius of curvature of 30.0 m. The runner starts the race at a constant speed. If she completes the 200-m dash in 23.2 s and runs at constant speed throughout the race, what is her centripetal acceleration as she runs the curved portion of the track?
65. What is the acceleration of Venus toward the Sun, assuming a circular orbit?
66. An experimental jet rocket travels around Earth along its equator just above its surface. At what speed must the jet travel if the magnitude of its acceleration is g?
67. A fan is rotating at a constant 360.0 rev/min. What is the magnitude of the acceleration of a point on one of its blades 10.0 cm from the axis of rotation?
68. A point located on the second hand of a large clock has a radial acceleration of 0.1 cm/s^2. How far is the point from the axis of rotation of the second hand?

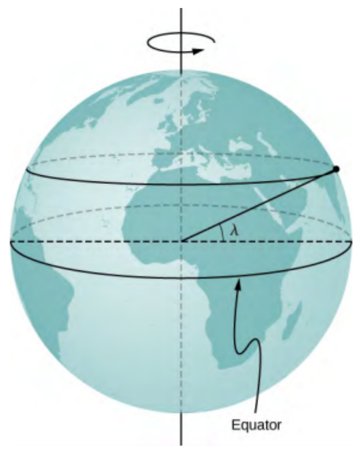
4.5 Relative Motion in One and Two Dimensions

69. The coordinate axes of the reference frame S′ remain parallel to those of S, as S′ moves away from S at a constant velocity $\vec{v}_{S'} = (4.0\,\hat{i} + 3.0\,\hat{j} + 5.0\,\hat{k})$ m/s. (a) If at time t = 0 the origins coincide, what is the position of the origin O′ in the S frame as a function of time? (b) How is particle position for $\vec{r}(t)$ and $\vec{r}\,'(t)$, as measured in S and S′, respectively, related? (c) What is the relationship between particle velocities $\vec{v}(t)$ and $\vec{v}\,'(t)$? (d) How are accelerations $\vec{a}(t)$ and $\vec{a}\,'(t)$ related?
70. The coordinate axes of the reference frame S′ remain parallel to those of S, as S′ moves away from S at a constant velocity $\vec{v}_{S'S} = (1.0\,\hat{i} + 2.0\,\hat{j} + 3.0\,\hat{k})t$ m/s. (a) If at time t = 0 the origins coincide, what is the position of origin O′ in the S frame as a function of time? (b) How is particle position for $\vec{r}(t)$ and $\vec{r}\,'(t)$, as measured in S and S′, respectively, related? (c) What is the relationship between particle velocities $\vec{v}(t)$ and $\vec{v}\,'(t)$? (d) How are accelerations $\vec{a}(t)$ and $\vec{a}\,'(t)$ related?
71. The velocity of a particle in reference frame A is $(2.0\,\hat{i} + 3.0\,\hat{j})$ m/s. The velocity of reference frame A with respect to reference frame B is $4.0\,\hat{k}$ m/s, and the velocity of reference frame B with respect to C is $2.0\,\hat{j}$ m/s. What is the velocity of the particle in reference frame C?
72. Raindrops fall vertically at 4.5 m/s relative to the earth. What does an observer in a car moving at 22.0 m/s in a straight line measure as the velocity of the raindrops?
73. A seagull can fly at a velocity of 9.00 m/s in still air. (a) If it takes the bird 20.0 min to travel 6.00 km straight into an oncoming wind, what is the velocity of the wind? (b) If the bird turns around and flies with the wind, how long will it take the bird to return 6.00 km?
74. A ship sets sail from Rotterdam, heading due north at 7.00 m/s relative to the water. The local ocean current is 1.50 m/s in a direction 40.0° north of east. What is the velocity of the ship relative to Earth?
75. A boat can be rowed at 8.0 km/h in still water. (a) How much time is required to row 1.5 km downstream in a river moving 3.0 km/h relative to the shore? (b) How much time is required for the return trip? (c) In what direction must the boat be aimed to row straight across the river? (d) Suppose the river is 0.8 km wide. What is the velocity of the boat with respect to Earth and how much time is required to get to the opposite shore? (e) Suppose, instead, the boat is aimed straight across the river. How much time is required to get across and how far downstream is the boat when it reaches the opposite shore?

76. A small plane flies at 200 km/h in still air. If the wind blows directly out of the west at 50 km/h, (a) in what direction must the pilot head her plane to move directly north across land and (b) how long does it take her to reach a point 300 km directly north of her starting point?
77. A cyclist traveling southeast along a road at 15 km/h feels a wind blowing from the southwest at 25 km/h. To a stationary observer, what are the speed and direction of the wind?
78. A river is moving east at 4 m/s. A boat starts from the dock heading 30° north of west at 7 m/s. If the river is 1800 m wide, (a) what is the velocity of the boat with respect to Earth and (b) how long does it take the boat to cross the river?

Additional Problems

79. A Formula One race car is traveling at 89.0 m/s along a straight track enters a turn on the race track with radius of curvature of 200.0 m. What centripetal acceleration must the car have to stay on the track?
80. A particle travels in a circular orbit of radius 10 m. Its speed is changing at a rate of 15.0 m/s^2 at an instant when its speed is 40.0 m/s. What is the magnitude of the acceleration of the particle?
81. The driver of a car moving at 90.0 km/h presses down on the brake as the car enters a circular curve of radius 150.0 m. If the speed of the car is decreasing at a rate of 9.0 km/h each second, what is the magnitude of the acceleration of the car at the instant its speed is 60.0 km/h?
82. A race car entering the curved part of the track at the Daytona 500 drops its speed from 85.0 m/s to 80.0 m/s in 2.0 s. If the radius of the curved part of the track is 316.0 m, calculate the total acceleration of the race car at the beginning and ending of reduction of speed.
83. An elephant is located on Earth's surface at a latitude λ. Calculate the centripetal acceleration of the elephant resulting from the rotation of Earth around its polar axis. Express your answer in terms of λ, the radius RE of Earth, and time T for one rotation of Earth. Compare your answer with g for λ = 40°.

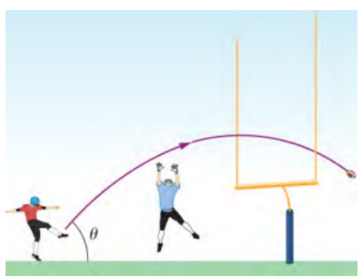


84. A proton in a synchrotron is moving in a circle of radius 1 km and increasing its speed by $v(t) = c_1 + c_2t^2$, where $c_1 = 2.0 \times 10^5$ m/s, $c_2 = 10^5$ m/s^3. (a) What is the proton's total acceleration at t = 5.0 s? (b) At what time does the expression for the velocity become unphysical?
85. A propeller blade at rest starts to rotate from t = 0 s to t = 5.0 s with a tangential acceleration of the tip of the blade at 3.00 m/s^2. The tip of the blade is 1.5 m from the axis of rotation. At t = 5.0 s, what is the total acceleration of the tip of the blade?
86. A particle is executing circular motion with a constant angular frequency of ω = 4.00 rad/s. If time t = 0 corresponds to the position of the particle being located at y = 0 m and x = 5 m, (a) what is the position of the particle at t = 10 s? (b) What is its velocity at this time? (c) What is its acceleration?
87. A particle's centripetal acceleration is a_C = 4.0 m/s^2 at t = 0 s where it is on the x-axis and moving counterclockwise in the xy plane. It is executing uniform circular motion about an axis at a distance of 5.0 m. What is its velocity at t = 10 s?
88. A rod 3.0 m in length is rotating at 2.0 rev/s about an axis at one end. Compare the centripetal accelerations at radii of (a) 1.0 m, (b) 2.0 m, and (c) 3.0 m.
89. A particle located initially at $(1.5\,\hat{j} + 4.0\,\hat{k})$m undergoes a displacement of $(2.5\,\hat{i} + 3.2\,\hat{j} - 1.2\,\hat{k})$ m. What is the final position of the particle?

90. The position of a particle is given by $\vec{r}(t) = (50\ m/s)t\ \hat{i} - (4.9\ m/s^2)t^2\ \hat{j}$. (a) What are the particle's velocity and acceleration as functions of time? (b) What are the initial conditions to produce the motion?
91. A spaceship is traveling at a constant velocity of $\vec{v}(t) = 250.0\ \hat{i}$ m/s when its rockets fire, giving it an acceleration of $\vec{a}(t) = (3.0\ \vec{i} + 4.0\ \hat{k})m/s^2$. What is its velocity 5 s after the rockets fire?
92. A crossbow is aimed horizontally at a target 40 m away. The arrow hits 30 cm below the spot at which it was aimed. What is the initial velocity of the arrow?
93. A long jumper can jump a distance of 8.0 m when he takes off at an angle of 45° with respect to the horizontal. Assuming he can jump with the same initial speed at all angles, how much distance does he lose by taking off at 30°?
94. On planet Arcon, the maximum horizontal range of a projectile launched at 10 m/s is 20 m. What is the acceleration of gravity on this planet?
95. A mountain biker encounters a jump on a race course that sends him into the air at 60° to the horizontal. If he lands at a horizontal distance of 45.0 m and 20 m below his launch point, what is his initial speed?
96. Which has the greater centripetal acceleration, a car with a speed of 15.0 m/s along a circular track of radius 100.0 m or a car with a speed of 12.0 m/s along a circular track of radius 75.0 m?
97. A geosynchronous satellite orbits Earth at a distance of 42,250.0 km and has a period of 1 day. What is the centripetal acceleration of the satellite?
98. Two speedboats are traveling at the same speed relative to the water in opposite directions in a moving river. An observer on the riverbank sees the boats moving at 4.0 m/s and 5.0 m/s. (a) What is the speed of the boats relative to the river? (b) How fast is the river moving relative to the shore?

Challenge Problems

99. World's Longest Par 3. The tee of the world's longest par 3 sits atop South Africa's Hanglip Mountain at 400.0 m above the green and can only be reached by helicopter. The horizontal distance to the green is 359.0 m. Neglect air resistance and answer the following questions. (a) If a golfer launches a shot that is 40° with respect to the horizontal, what initial velocity must she give the ball? (b) What is the time to reach the green?
100. When a field goal kicker kicks a football as hard as he can at 45° to the horizontal, the ball just clears the 3-m-high crossbar of the goalposts 45.7 m away. (a) What is the maximum speed the kicker can impart to the football? (b) In addition to clearing the crossbar, the football must be high enough in the air early during its flight to clear the reach of the onrushing defensive lineman. If the lineman is 4.6 m away and has a vertical reach of 2.5 m, can he block the 45.7-m field goal attempt? (c) What if the lineman is 1.0 m away?



101. A truck is traveling east at 80 km/h. At an intersection 32 km ahead, a car is traveling north at 50 km/h. (a) How long after this moment will the vehicles be closest to each other? (b) How far apart will they be at that point?

Contributors and Attributions

5.8: Motion in Two and Three Dimensions (Summary)

Key Terms

acceleration vector	instantaneous acceleration found by taking the derivative of the velocity function with respect to time in unit vector notation
angular frequency	ω, rate of change of an angle with which an object that is moving on a circular path
centripetal acceleration	component of acceleration of an object moving in a circle that is directed radially inward toward the center of the circle
displacement vector	vector from the initial position to a final position on a trajectory of a particle
position vector	vector from the origin of a chosen coordinate system to the position of a particle in two- or threedimensional space
projectile motion	motion of an object subject only to the acceleration of gravity
range	maximum horizontal distance a projectile travels
reference frame	coordinate system in which the position, velocity, and acceleration of an object at rest or moving is measured
relative velocity	velocity of an object as observed from a particular reference frame, or the velocity of one reference frame with respect to another reference frame
tangential accleration	magnitude of which is the time rate of change of speed. Its direction is tangent to the circle.
time of flight	elapsed time a projectile is in the air
total accleration	vector sum of centripetal and tangential accelerations
trajectory	path of a projectile through the air
velocity vector	vector that gives the instantaneous speed and direction of a particle; tangent to the trajectory

Key Equations

Position vector	$\vec{r}(t)=x(t)\hat{i}+y(t)\hat{j}+z(t)\hat{k} \quad (5.8.1)$
Displacement vector	$\Delta\vec{r}=\vec{r}(t_2)-\vec{r}(t_1) \quad (5.8.2)$
Velocity vector	$\vec{v}(t)=\lim_{\Delta t\to 0}\frac{\vec{r}(t+\Delta t)-\vec{r}(t)}{\Delta t}=\frac{d\vec{r}}{dt} \quad (5.8.3)$
Velocity in terms of components	$\vec{v}(t)=v_x(t)\hat{i}+v_y(t)\hat{j}+v_z(t)\hat{k} \quad (5.8.4)$
Velocity components	$v_x(t)=\frac{dx(t)}{dt} v_y(t)=\frac{dy(t)}{dt} v_z(t)=\frac{dz(t)}{dt} \quad (5.8.5)$
Average velocity	$\vec{v}_{avg}=\frac{\vec{r}(t_2)-\vec{r}(t_1)}{t_2-t_1} \quad (5.8.6)$

Instantaneous acceleration	$$\vec{a}(t)=\lim_{\Delta t\to 0}\frac{\vec{v}(t+\Delta t)-\vec{v}(t)}{\Delta t}=\frac{d\vec{v}}{dt} \quad (5.8.7)$$
Instantaneous acceleration, component form	$$\vec{a}(t)=\frac{dv_x(t)}{dt}\hat{i}+\frac{dv_y(t)}{dt}\hat{j}+\frac{dv_z(t)}{dt}\hat{k} \quad (5.8.8)$$
Instantaneous acceleration as second derivatives of position	$$\vec{a}(t)=\frac{d^2x(t)}{dt^2}\hat{i}+\frac{d^2y(t)}{dt^2}+\frac{d^2z(t)}{dt^2}\hat{k} \quad (5.8.9)$$
Time of flight	$$T_{tof}=\frac{2(v_0\sin\theta)}{g} \quad (5.8.10)$$
Trajectory	$$y=(\tan\theta_0)x-\left[\frac{g}{2(v_0\cos\theta_0)^2}\right]x^2 \quad (5.8.11)$$
Range	$$R=\frac{v_0^2\sin 2\theta_0}{g} \quad (5.8.12)$$
Centripetal accleration	$$a_C=\frac{v^2}{r} \quad (5.8.13)$$
Position vector, uniform circular motion	$$\vec{r}(t)=A\cos\omega t\,\hat{i}+A\sin\omega t\,\hat{j} \quad (5.8.14)$$
Velocity vector, uniform circular motion	$$\vec{v}(t)=\frac{d\vec{r}(t)}{dt}=-A\omega\sin\omega t\,\hat{i}+A\omega\cos\omega t\,\hat{j} \quad (5.8.15)$$
Acceleration vector, uniform circular motion	$$\vec{a}(t)=\frac{d\vec{v}(t)}{dt}=-A\omega^2\cos\omega t\,\hat{i}-A\omega^2\sin\omega t\,\hat{j} \quad (5.8.16)$$
Tangential acceleration	$$a_T=\frac{d\lvert\vec{v}\rvert}{dt} \quad (5.8.17)$$
Total acceleration	$$\vec{a}=\vec{a}_C+\vec{a}_T \quad (5.8.18)$$
Position vector in frame S is the position vector in frame S′ plus the vector from the origin of S to the origin of S′	$$\vec{r}_{PS}=\vec{r}_{PS'}+\vec{r}_{S'S} \quad (5.8.19)$$
Relative velocity equation connecting two reference frames	$$\vec{v}_{PS}=\vec{v}_{PS'}+\vec{v}_{S'S} \quad (5.8.20)$$
Relative velocity equation connecting more than two reference frames	$$\vec{v}_{PC}=\vec{v}_{PA}+\vec{v}_{AB}+\vec{v}_{BC} \quad (5.8.21)$$
Relative acceleration equation	$$\vec{a}_{PS}=\vec{a}_{PS'}+\vec{a}_{S'S} \quad (5.8.22)$$

Summary

4.1 Displacement and Velocity Vectors

- The position function $\vec{r}(t)$ gives the position as a function of time of a particle moving in two or three dimensions. Graphically, it is a vector from the origin of a chosen coordinate system to the point where the particle is located at a specific time.

- The displacement vector $\Delta\vec{r}$ gives the shortest distance between any two points on the trajectory of a particle in two or three dimensions.
- Instantaneous velocity gives the speed and direction of a particle at a specific time on its trajectory in two or three dimensions, and is a vector in two and three dimensions.
- The velocity vector is tangent to the trajectory of the particle.
- Displacement $\vec{r}$(t) can be written as a vector sum of the one-dimensional displacements $\vec{x}$(t), $\vec{y}$(t), $\vec{z}$(t) along the x, y, and z directions.
- Velocity $\vec{v}$(t) can be written as a vector sum of the one-dimensional velocities v_x(t), v_y(t), v_z(t) along the x, y, and z directions.
- Motion in any given direction is independent of motion in a perpendicular direction.

4.2 Acceleration Vector

- In two and three dimensions, the acceleration vector can have an arbitrary direction and does not necessarily point along a given component of the velocity.
- The instantaneous acceleration is produced by a change in velocity taken over a very short (infinitesimal) time period. Instantaneous acceleration is a vector in two or three dimensions. It is found by taking the derivative of the velocity function with respect to time.
- In three dimensions, acceleration $\vec{a}$(t) can be written as a vector sum of the one-dimensional accelerations a_x(t), a_y(t), and a_z (t) along the x-, y-, and z-axes.
- The kinematic equations for constant acceleration can be written as the vector sum of the constant acceleration equations in the x, y, and z directions.

4.3 Projectile Motion

- Projectile motion is the motion of an object subject only to the acceleration of gravity, where the acceleration is constant, as near the surface of Earth.
- To solve projectile motion problems, we analyze the motion of the projectile in the horizontal and vertical directions using the one-dimensional kinematic equations for x and y.
- The time of flight of a projectile launched with initial vertical velocity v_{0y} on an even surface is given by
$$T_{tof} = \frac{2(v_0 \sin\theta)}{g} \tag{5.8.23}$$
This equation is valid only when the projectile lands at the same elevation from which it was launched.
- The maximum horizontal distance traveled by a projectile is called the range. Again, the equation for range is valid only when the projectile lands at the same elevation from which it was launched.

4.4 Uniform Circular Motion

- Uniform circular motion is motion in a circle at constant speed.
- Centripetal acceleration $\vec{a}_C$ is the acceleration a particle must have to follow a circular path. Centripetal acceleration always points toward the center of rotation and has magnitude $a_C = \frac{v^2}{r}$.
- Nonuniform circular motion occurs when there is tangential acceleration of an object executing circular motion such that the speed of the object is changing. This acceleration is called tangential acceleration $\vec{a}_T$. The magnitude of tangential acceleration is the time rate of change of the magnitude of the velocity. The tangential acceleration vector is tangential to the circle, whereas the centripetal acceleration vector points radially inward toward the center of the circle. The total acceleration is the vector sum of tangential and centripetal accelerations.
- An object executing uniform circular motion can be described with equations of motion. The position vector of the object is $\vec{r}$(t) = A cos ωt $\hat{i}$ + A sin ωt $\hat{j}$, where A is the magnitude $|\vec{r}(t)|$, which is also the radius of the circle, and ω is the angular frequency.

4.5 Relative Motion in One and Two Dimensions

- When analyzing motion of an object, the reference frame in terms of position, velocity, and acceleration needs to be specified.
- Relative velocity is the velocity of an object as observed from a particular reference frame, and it varies with the choice of reference frame.
- If S and S′ are two reference frames moving relative to each other at a constant velocity, then the velocity of an object relative to S is equal to its velocity relative to S′ plus the velocity of S′ relative to S.

- If two reference frames are moving relative to each other at a constant velocity, then the accelerations of an object as observed in both reference frames are equal.

Contributors and Attributions

5.9: Motion in Two Dimensions

Constant Velocity

An object moving with constant velocity must have a constant speed in a constant direction.

learning objectives

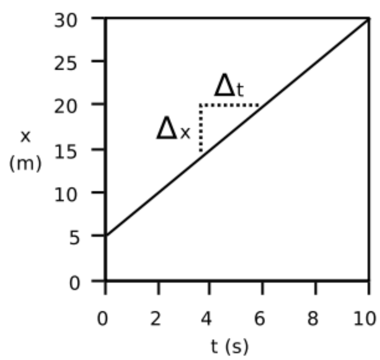
- Examine the terms for constant velocity and how they apply to acceleration

Motion with constant velocity is one of the simplest forms of motion. This type of motion occurs when an an object is moving (or sliding) in the presence of little or negligible friction, similar to that of a hockey puck sliding across the ice. To have a constant velocity, an object must have a constant speed in a constant direction. Constant direction constrains the object to motion to a straight path.

Newton's second law ($F = ma$) suggests that when a force is applied to an object, the object would experience acceleration. If the acceleration is 0, the object shouldn't have any external forces applied on it. Mathematically, this can be shown as the following:

$$a = \frac{dv}{dt} = 0 \Rightarrow v = \text{const.} \tag{5.9.1}$$

If an object is moving at constant velocity, the graph of distance vs. time (x vs. t) shows the same change in position over each interval of time. Therefore the motion of an object at constant velocity is represented by a straight line: $x = x_0 + vt$, where x_0 is the displacement when $t = 0$ (or at the y-axis intercept).



Motion with Constant Velocity: When an object is moving with constant velocity, it does not change direction nor speed and therefore is represented as a straight line when graphed as distance over time.

You can also obtain an object's velocity if you know its trace over time. Given a graph as in, we can calculate the velocity from the change in distance over the change in time. In graphical terms, the velocity can be interpreted as the slope of the line. The velocity can be positive or negative, and is indicated by the sign of our slope. This tells us in which direction the object moves.

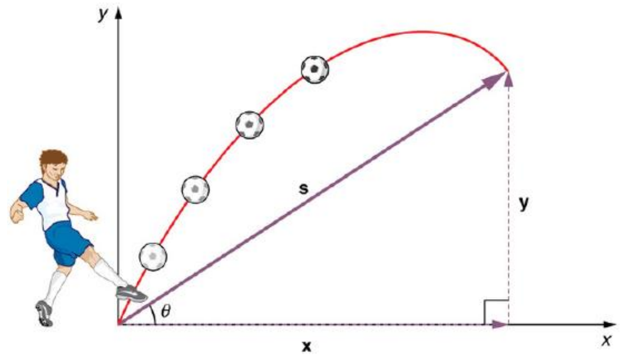
Constant Acceleration

Analyzing two-dimensional projectile motion is done by breaking it into two motions: along the horizontal and vertical axes.

learning objectives

- Analyze a two-dimensional projectile motion along horizontal and vertical axes

Projectile motion is the motion of an object thrown, or projected, into the air, subject only to the force of gravity. The object is called a projectile, and its path is called its trajectory. The motion of falling objects is a simple one-dimensional type of projectile motion in which there is no horizontal movement. In two-dimensional projectile motion, such as that of a football or other thrown object, there is both a vertical and a horizontal component to the motion.



Projectile Motion: Throwing a rock or kicking a ball generally produces a projectile pattern of motion that has both a vertical and a horizontal component.

The most important fact to remember is that motion along perpendicular axes are independent and thus can be analyzed separately. The key to analyzing two-dimensional projectile motion is to break it into two motions, one along the horizontal axis and the other along the vertical. To describe motion we must deal with velocity and acceleration, as well as with displacement.

We will assume all forces except for gravity (such as air resistance and friction, for example) are negligible. The components of acceleration are then very simple: $a_y = -g = -9.81\frac{m}{s^2}$ (we assume that the motion occurs at small enough heights near the surface of the earth so that the acceleration due to gravity is constant). Because the acceleration due to gravity is along the vertical direction *only*, $a_x = 0$. Thus, the kinematic equations describing the motion along the x and y directions respectively, can be used:

$$x = x_0 + v_x t \tag{5.9.2}$$

$$y = v_{0y} + a_y t \tag{5.9.3}$$

$$y = y_0 + v_{0y} t + \frac{1}{2} a_y t^2 \tag{5.9.4}$$

$$v_y^2 = v_{0y}^2 + 2a_y (y - y_0) \tag{5.9.5}$$

We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes. The horizontal motion is simple, because $a_x = 0$ and v_x is thus constant. The velocity in the vertical direction begins to decrease as an object rises; at its highest point, the vertical velocity is zero. As an object falls towards the Earth again, the vertical velocity increases again in magnitude but points in the opposite direction to the initial vertical velocity. The xx and yy motions can be recombined to give the total velocity at any given point on the trajectory.

Key Points

- Constant velocity means that the object in motion is moving in a straight line at a constant speed.
- This line can be represented algebraically as: $x = x_0 + vt$, where x_0 represents the position of the object at $t = 0$, and the slope of the line indicates the object's speed.
- The velocity can be positive or negative, and is indicated by the sign of our slope. This tells us in which direction the object moves.
- Constant acceleration in motion in two dimensions generally follows a projectile pattern.
- Projectile motion is the motion of an object thrown or projected into the air, subject to only the (vertical) acceleration due to gravity.
- We analyze two-dimensional projectile motion by breaking it into two independent one-dimensional motions along the vertical and horizontal axes.

Key Terms

- **constant velocity**: Motion that does not change in speed nor direction.
- **kinematic**: of or relating to motion or kinematics

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5.10: Vectors Revisited

Components of a Vector

Vectors are geometric representations of magnitude and direction and can be expressed as arrows in two or three dimensions.

learning objectives

- Contrast two-dimensional and three-dimensional vectors

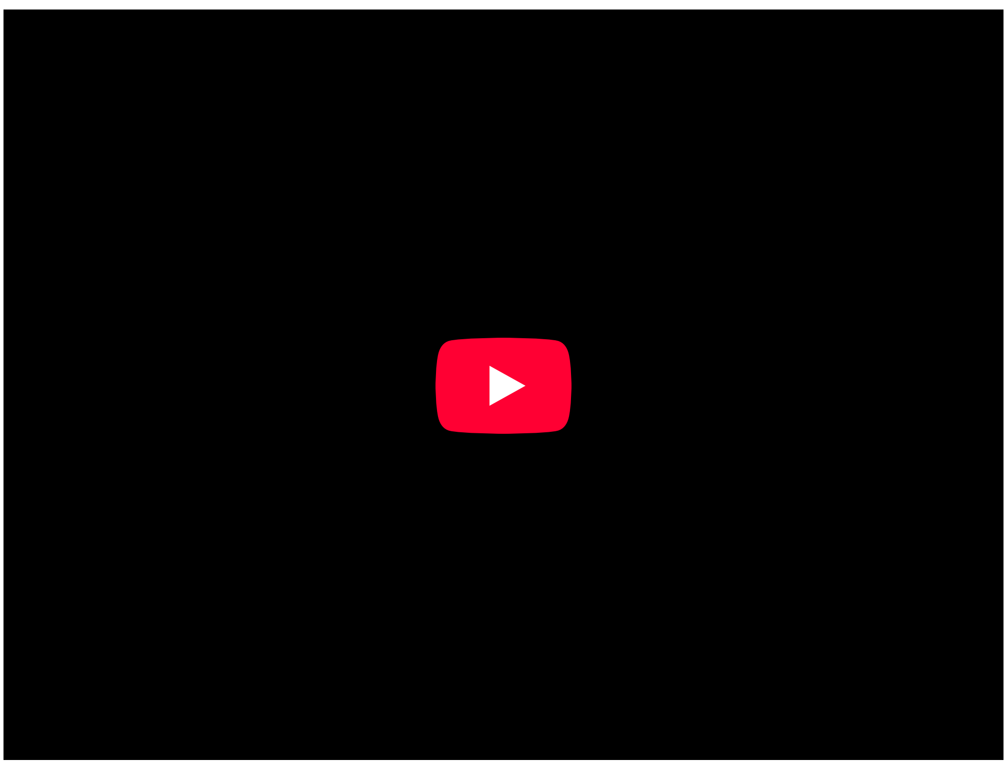
Vectors are geometric representations of magnitude and direction which are often represented by straight arrows, starting at one point on a coordinate axis and ending at a different point. All vectors have a length, called the magnitude, which represents some quality of interest so that the vector may be compared to another vector. Vectors, being arrows, also have a direction. This differentiates them from scalars, which are mere numbers without a direction.

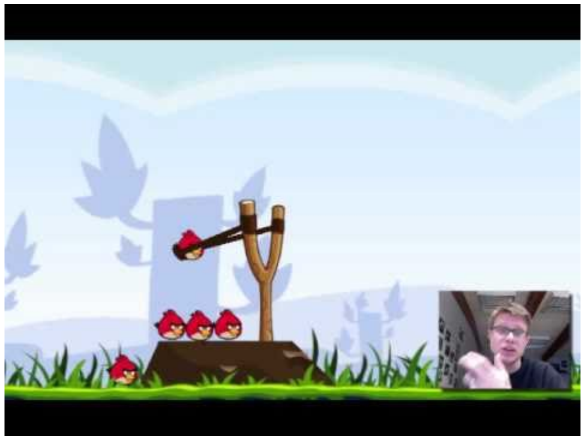
A vector is defined by its magnitude and its orientation with respect to a set of coordinates. It is often useful in analyzing vectors to break them into their component parts. For two-dimensional vectors, these components are horizontal and vertical. For three dimensional vectors, the magnitude component is the same, but the direction component is expressed in terms of xx, yy and zz.

Decomposing a Vector

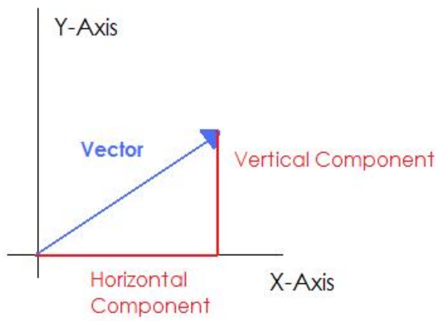
To visualize the process of decomposing a vector into its components, begin by drawing the vector from the origin of a set of coordinates. Next, draw a straight line from the origin along the x-axis until the line is even with the tip of the original vector. This is the horizontal component of the vector. To find the vertical component, draw a line straight up from the end of the horizontal vector until you reach the tip of the original vector. You should find you have a right triangle such that the original vector is the hypotenuse.

Decomposing a vector into horizontal and vertical components is a very useful technique in understanding physics problems. Whenever you see motion at an angle, you should think of it as moving horizontally and vertically at the same time. Simplifying vectors in this way can speed calculations and help to keep track of the motion of objects.





Scalars and Vectors: Mr. Andersen explains the differences between scalar and vectors quantities. He also uses a demonstration to show the importance of vectors and vector addition.



Components of a Vector: The original vector, defined relative to a set of axes. The horizontal component stretches from the start of the vector to its furthest x-coordinate. The vertical component stretches from the x-axis to the most vertical point on the vector. Together, the two components and the vector form a right triangle.

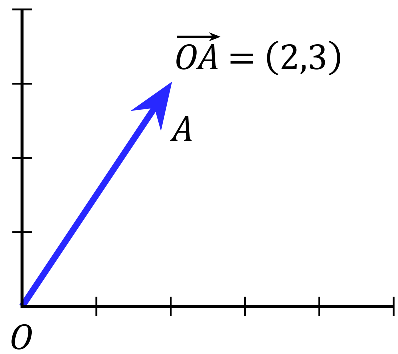
Scalars vs. Vectors

Scalars are physical quantities represented by a single number, and vectors are represented by both a number and a direction.

learning objectives

- Distinguish the difference between the quantities scalars and vectors represent

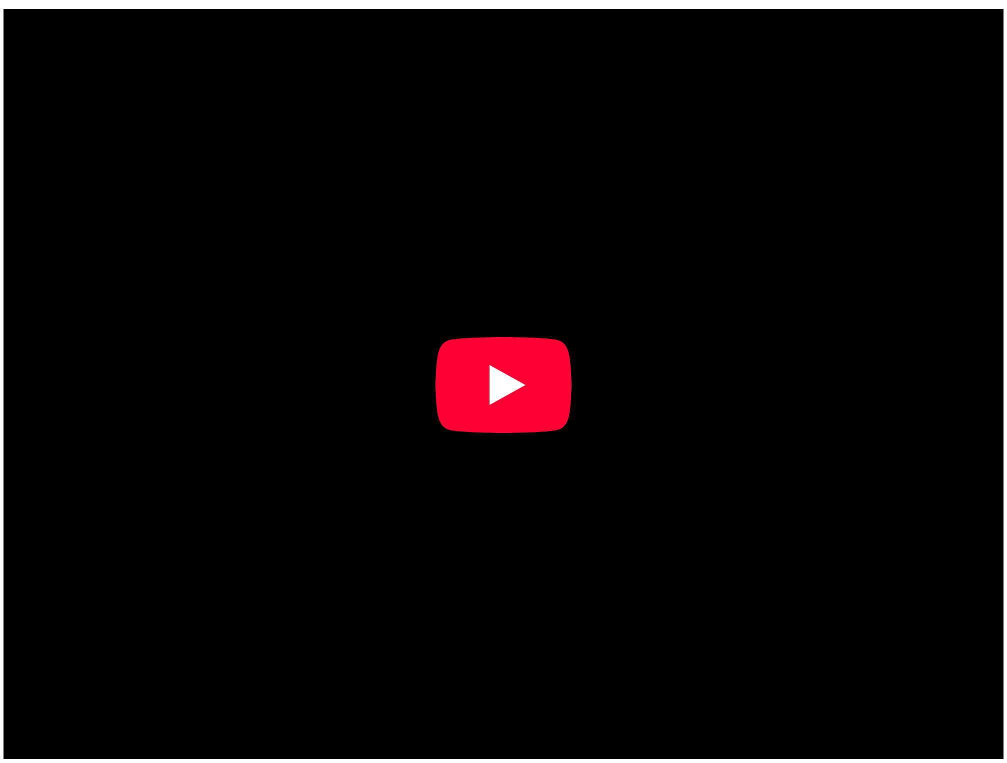
Physical quantities can usually be placed into two categories, vectors and scalars. These two categories are typified by what information they require. Vectors require two pieces of information: the magnitude and direction. In contrast, scalars require only the magnitude. Scalars can be thought of as numbers, whereas vectors must be thought of more like arrows pointing in a specific direction.

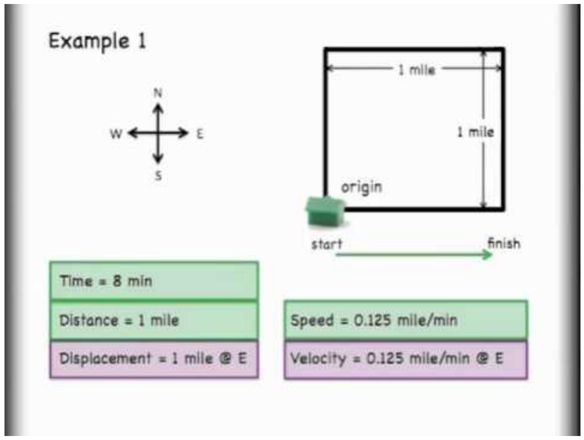


A Vector: An example of a vector. Vectors are usually represented by arrows with their length representing the magnitude and their direction represented by the direction the arrow points.

Vectors require both a magnitude and a direction. The magnitude of a vector is a number for comparing one vector to another. In the geometric interpretation of a vector the vector is represented by an arrow. The arrow has two parts that define it. The two parts are its length which represents the magnitude and its direction with respect to some set of coordinate axes. The greater the magnitude, the longer the arrow. Physical concepts such as displacement, velocity, and acceleration are all examples of quantities that can be represented by vectors. Each of these quantities has both a magnitude (how far or how fast) and a direction. In order to specify a direction, there must be something to which the direction is relative. Typically this reference point is a set of coordinate axes like the x-y plane.

Scalars differ from vectors in that they do not have a direction. Scalars are used primarily to represent physical quantities for which a direction does not make sense. Some examples of these are: mass, height, length, volume, and area. Talking about the direction of these quantities has no meaning and so they cannot be expressed as vectors.





The difference between Vectors and Scalars, Introduction and Basics: This video introduces the difference between scalars and vectors. Ideas about magnitude and direction are introduced and examples of both vectors and scalars are given.

Adding and Subtracting Vectors Graphically

Vectors may be added or subtracted graphically by laying them end to end on a set of axes.

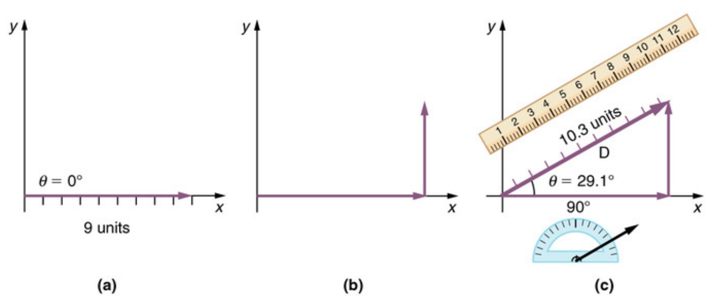
learning objectives

- Distinguish the difference between the quantities scalars and vectors represent

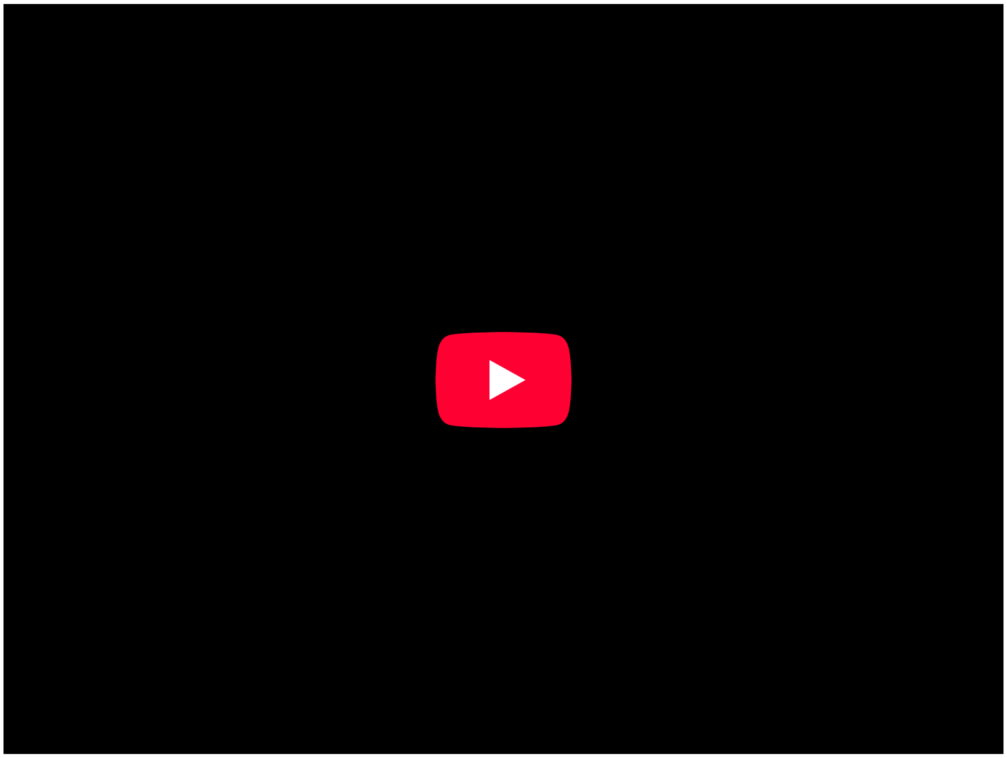
Adding and Subtracting Vectors

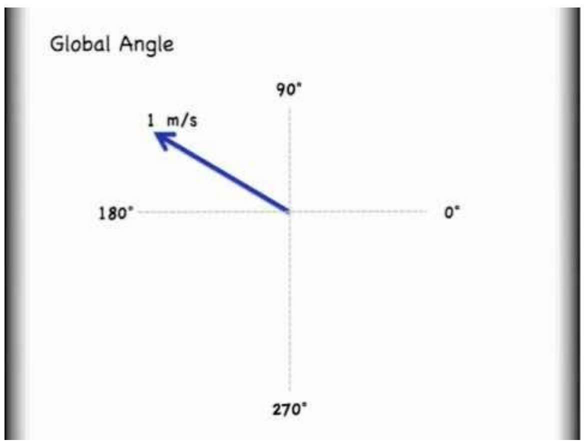
One of the ways in which representing physical quantities as vectors makes analysis easier is the ease with which vectors may be added to one another. Since vectors are graphical visualizations, addition and subtraction of vectors can be done graphically.

The graphical method of vector addition is also known as the head-to-tail method. To start, draw a set of coordinate axes. Next, draw out the first vector with its tail (base) at the origin of the coordinate axes. For vector addition it does not matter which vector you draw first since addition is commutative, but for subtraction ensure that the vector you draw first is the one you are subtracting *from*. The next step is to take the next vector and draw it such that its tail starts at the previous vector's head (the arrow side). Continue to place each vector at the head of the preceding one until all the vectors you wish to add are joined together. Finally, draw a straight line from the origin to the head of the final vector in the chain. This new line is the vector result of adding those vectors together.



Graphical Addition of Vectors: The head-to-tail method of vector addition requires that you lay out the first vector along a set of coordinate axes. Next, place the tail of the next vector on the head of the first one. Draw a new vector from the origin to the head of the last vector. This new vector is the sum of the original two.





Vector Addition Lesson 1 of 2: Head to Tail Addition Method: This video gets viewers started with vector addition and subtraction. The first lesson shows graphical addition while the second video takes a more mathematical approach and shows vector addition by components.

To subtract vectors the method is similar. Make sure that the first vector you draw is the one to be subtracted from. Then, to subtract a vector, proceed as if adding the *opposite* of that vector. In other words, flip the vector to be subtracted across the axes and then join it tail to head as if adding. To flip the vector, simply put its head where its tail was and its tail where its head was.

Adding and Subtracting Vectors Using Components

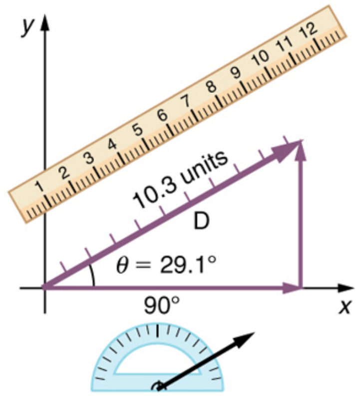
It is often simpler to add or subtract vectors by using their components.

learning objectives

- Demonstrate how to add and subtract vectors by components

Using Components to Add and Subtract Vectors

Another way of adding vectors is to add the components. Previously, we saw that vectors can be expressed in terms of their horizontal and vertical components. To add vectors, merely express both of them in terms of their horizontal and vertical components and then add the components together.

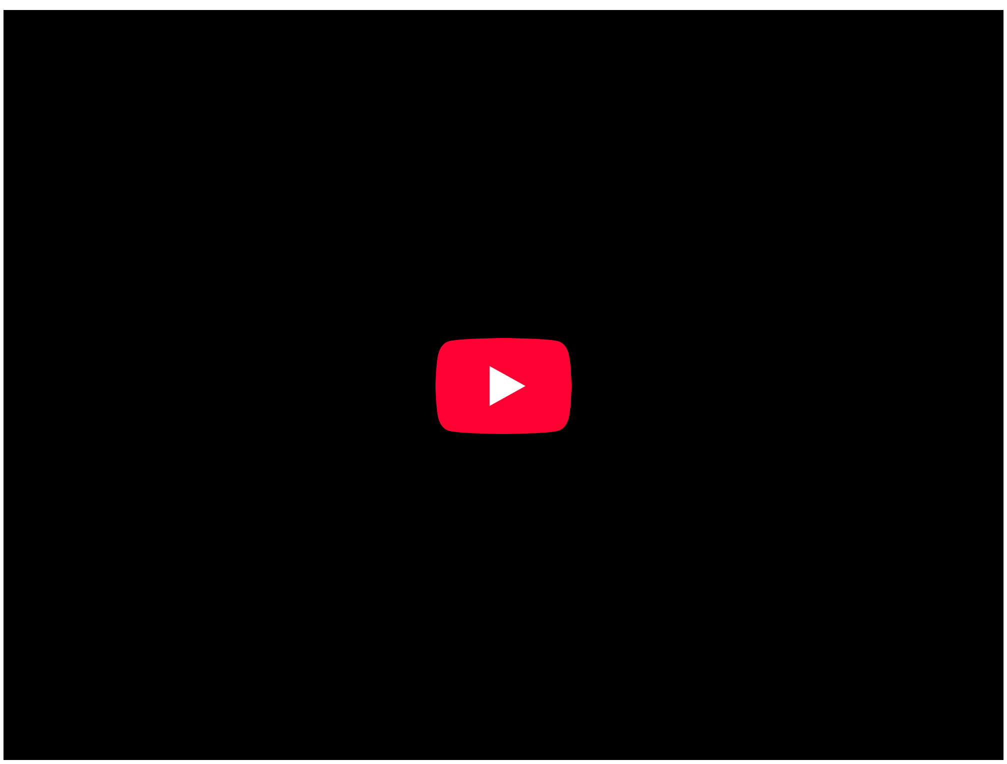


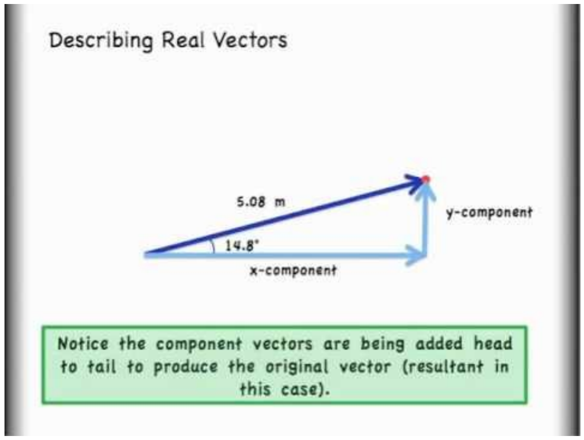
(c)

Vector with Horizontal and Vertical Components: The vector in this image has a magnitude of 10.3 units and a direction of 29.1 degrees above the x-axis. It can be decomposed into a horizontal part and a vertical part as shown.

For example, a vector with a length of 5 at a 36.9 degree angle to the horizontal axis will have a horizontal component of 4 units and a vertical component of 3 units. If we were to add this to another vector of the same magnitude and direction, we would get a vector twice as long at the same angle. This can be seen by adding the horizontal components of the two vectors $(4+4)$ and the two vertical components $(3+3)$. These additions give a new vector with a horizontal component of $8(4+4)$ and a vertical component of $6(3+3)$. To find the resultant vector, simply place the tail of the vertical component at the head (arrow side) of the horizontal component and then draw a line from the origin to the head of the vertical component. This new line is the resultant vector. It should be twice as long as the original, since both of its components are twice as large as they were previously.

To subtract vectors by components, simply subtract the two horizontal components from each other and do the same for the vertical components. Then draw the resultant vector as you did in the previous part.





Vector Addition Lesson 2 of 2: How to Add Vectors by Components: This video gets viewers started with vector addition using a mathematical approach and shows vector addition by components.

Multiplying Vectors by a Scalar

Multiplying a vector by a scalar changes the magnitude of the vector but not the direction.

learning objectives

- Summarize the interaction between vectors and scalars

Overview

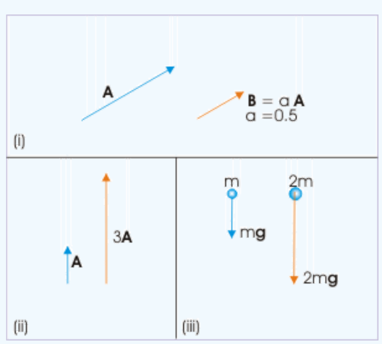
Although vectors and scalars represent different types of physical quantities, it is sometimes necessary for them to interact. While adding a scalar to a vector is impossible because of their different dimensions in space, it is possible to multiply a vector by a scalar. A scalar, however, cannot be multiplied by a vector.

To multiply a vector by a scalar, simply multiply the similar components, that is, the vector's magnitude by the scalar's magnitude. This will result in a new vector with the same direction but the product of the two magnitudes.

Example 5.10.1:

For example, if you have a vector A with a certain magnitude and direction, multiplying it by a scalar a with magnitude 0.5 will give a new vector with a magnitude of half the original. Similarly if you take the number 3 which is a pure and unit-less scalar and multiply it to a vector, you get a version of the original vector which is 3 times as long. As a more physical example take the gravitational force on an object. The force is a vector with its magnitude depending on the scalar known as mass and its direction being down. If the mass of the object is doubled, the force of gravity is doubled as well.

Multiplying vectors by scalars is very useful in physics. Most of the units used in vector quantities are intrinsically scalars multiplied by the vector. For example, the unit of meters per second used in velocity, which is a vector, is made up of two scalars, which are magnitudes: the scalar of length in meters and the scalar of time in seconds. In order to make this conversion from magnitudes to velocity, one must multiply the unit vector in a particular direction by these scalars.



Scalar Multiplication: (i) Multiplying the vector $\mathbf{A}$ by the scalar $\mathbf{a} = 0.5$ yields the vector $\mathbf{B}$ which is half as long. (ii) Multiplying the vector $\mathbf{A}$ by 3 triples its length. (iii) Doubling the mass (scalar) doubles the force (vector) of gravity.

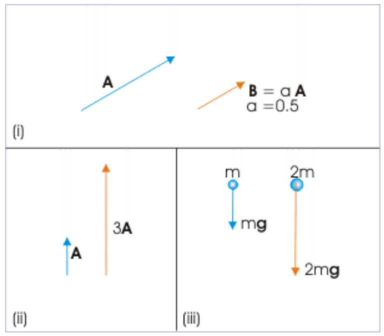
Unit Vectors and Multiplication by a Scalar

Multiplying a vector by a scalar is the same as multiplying its magnitude by a number.

learning objectives

- Predict the influence of multiplying a vector by a scalar

In addition to adding vectors, vectors can also be multiplied by constants known as scalars. Scalars are distinct from vectors in that they are represented by a magnitude but no direction. Examples of scalars include an object's mass, height, or volume.



Scalar Multiplication: (i) Multiplying the vector $\mathbf{A}$ by the scalar $\mathbf{a} = 0.5$ yields the vector $\mathbf{B}$ which is half as long. (ii) Multiplying the vector $\mathbf{A}$ by 3 triples its length. (iii) Doubling the mass (scalar) doubles the force (vector) of gravity.

When multiplying a vector by a scalar, the direction of the vector is unchanged and the magnitude is multiplied by the magnitude of the scalar. This results in a new vector arrow pointing in the same direction as the old one but with a longer or shorter length. You can also accomplish scalar multiplication through the use of a vector's components. Once you have the vector's components, multiply each of the components by the scalar to get the new components and thus the new vector.

A useful concept in the study of vectors and geometry is the concept of a unit vector. A unit vector is a vector with a length or magnitude of one. The unit vectors are different for different coordinates. In Cartesian coordinates the directions are x and y usually denoted $\hat{\mathrm{x}}$ and $\hat{\mathrm{y}}$. With the triangle above the letters referred to as a "hat". The unit vectors in Cartesian coordinates describe a circle known as the "unit circle" which has radius one. This can be seen by taking all the possible vectors of length one at all the possible angles in this coordinate system and placing them on the coordinates. If you were to draw a line around connecting all the heads of all the vectors together, you would get a circle of radius one.

Position, Displacement, Velocity, and Acceleration as Vectors

Position, displacement, velocity, and acceleration can all be shown vectors since they are defined in terms of a magnitude and a direction.

learning objectives

- Examine the applications of vectors in analyzing physical quantities

Use of Vectors

Vectors can be used to represent physical quantities. Most commonly in physics, vectors are used to represent displacement, velocity, and acceleration. Vectors are a combination of magnitude and direction, and are drawn as arrows. The length represents the magnitude and the direction of that quantity is the direction in which the vector is pointing. Because vectors are constructed this way, it is helpful to analyze physical quantities (with both size and direction) as vectors.

Applications

In physics, vectors are useful because they can visually represent position, displacement, velocity and acceleration. When drawing vectors, you often do not have enough space to draw them to the scale they are representing, so it is important to denote somewhere what scale they are being drawn at. For example, when drawing a vector that represents a magnitude of 100, one may draw a line that is 5 units long at a scale of $\frac{1}{20}$. When the inverse of the scale is multiplied by the drawn magnitude, it should equal the actual magnitude.

Position and Displacement

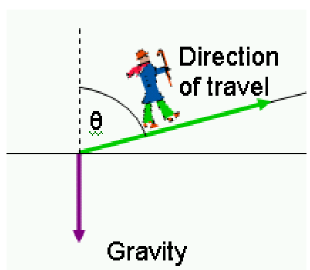
Displacement is defined as the distance, in any direction, of an object relative to the position of another object. Physicists use the concept of a position vector as a graphical tool to visualize displacements. A position vector expresses the position of an object from the origin of a coordinate system. A position vector can also be used to show the position of an object in relation to a reference point, secondary object or initial position (if analyzing how far the object has moved from its original location). The position vector is a straight line drawn from the arbitrary origin to the object. Once drawn, the vector has a length and a direction relative to the coordinate system used.

Velocity

Velocity is also defined in terms of a magnitude and direction. To say that something is gaining or losing velocity one must also say how much and in what direction. For example, an airplane flying at 200 $\frac{\text{km}}{\text{h}}$ to the northeast can be represented by an vector pointing in the northeast direction with a magnitude of 200 $\frac{\text{km}}{\text{h}}$. In drawing the vector, the magnitude is only important as a way to compare two vectors of the same units. So, if there were another airplane flying 100 $\frac{\text{km}}{\text{h}}$ to the southwest, the vector arrow should be half as long and pointing in the direction of southwest.

Acceleration

Acceleration, being the time rate of change of velocity, is composed of a magnitude and a direction, and is drawn with the same concept as a velocity vector. A value for acceleration would not be helpful in physics if the magnitude and direction of this acceleration was unknown, which is why these vectors are important. In a free body diagram, for example, of an object falling, it would be helpful to use an acceleration vector near the object to denote its acceleration towards the ground. If gravity is the only force acting on the object, this vector would be pointing downward with a magnitude of 9.81 $\frac{\text{m}}{\text{s}^2}$ of 32.2 $\frac{\text{ft}}{\text{s}^2}$.



Vector Diagram: Here is a man walking up a hill. His direction of travel is defined by the angle theta relative to the vertical axis and by the length of the arrow going up the hill. He is also being accelerated downward by gravity.

Key Points

- Vectors can be broken down into two components: magnitude and direction.
- By taking the vector to be analyzed as the hypotenuse, the horizontal and vertical components can be found by completing a right triangle. The bottom edge of the triangle is the horizontal component and the side opposite the angle is the vertical component.
- The angle that the vector makes with the horizontal can be used to calculate the length of the two components.
- Scalars are physical quantities represented by a single number and no direction.
- Vectors are physical quantities that require both magnitude and direction.
- Examples of scalars include height, mass, area, and volume. Examples of vectors include displacement, velocity, and acceleration.
- To add vectors, lay the first one on a set of axes with its tail at the origin. Place the next vector with its tail at the previous vector’s head. When there are no more vectors, draw a straight line from the origin to the head of the last vector. This line is the sum of the vectors.
- To subtract vectors, proceed as if adding the two vectors, but flip the vector to be subtracted across the axes and then join it tail to head as if adding.

- Adding or subtracting any number of vectors yields a resultant vector.
- Vectors can be decomposed into horizontal and vertical components.
- Once the vectors are decomposed into components, the components can be added.
- Adding the respective components of two vectors yields a vector which is the sum of the two vectors.
- A vector is a quantity with both magnitude and direction.
- A scalar is a quantity with only magnitude.
- Multiplying a vector by a scalar is equivalent to multiplying the vector's magnitude by the scalar. The vector lengthens or shrinks but does not change direction.
- A unit vector is a vector of magnitude (length) 1.
- A scalar is a physical quantity that can be represented by a single number. Unlike vectors, scalars do not have direction.
- Multiplying a vector by a scalar is the same as multiplying the vector's magnitude by the number represented by the scalar.
- Vectors are arrows consisting of a magnitude and a direction. They are used in physics to represent physical quantities that also have both magnitude and direction.
- Displacement is a physics term meaning the distance of an object from a reference point. Since the displacement contains two pieces of information: the distance from the reference point and the direction away from the point, it is well represented by a vector.
- Velocity is defined as the rate of change in time of the displacement. To know the velocity of an object one must know both how fast the displacement is changing and in what direction. Therefore it is also well represented by a vector.
- Acceleration, being the rate of change of velocity also requires both a magnitude and a direction relative to some coordinates.
- When drawing vectors, you often do not have enough space to draw them to the scale they are representing, so it is important to denote somewhere what scale they are being drawn at.

Key Terms

- **coordinates**: Numbers indicating a position with respect to some axis. Ex: x and y coordinates indicate position relative to xx and yy axes.
- **axis**: An imaginary line around which an object spins or is symmetrically arranged.
- **magnitude**: A number assigned to a vector indicating its length.
- **Coordinate axes**: A set of perpendicular lines which define coordinates relative to an origin. Example: x and y coordinate axes define horizontal and vertical position.
- **origin**: The center of a coordinate axis, defined as being the coordinate 0 in all axes.
- **Component**: A part of a vector. For example, horizontal and vertical components.
- **vector**: A directed quantity, one with both magnitude and direction; the between two points.
- **magnitude**: A number assigned to a vector indicating its length.
- **scalar**: A quantity that has magnitude but not direction; compare vector.
- **unit vector**: A vector of magnitude 1.
- **velocity**: The rate of change of displacement with respect to change in time.
- **displacement**: The length and direction of a straight line between two objects.
- **acceleration**: the rate at which the velocity of a body changes with time

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5.11: Projectile Motion Revisited

Basic Equations and Parabolic Path

Projectile motion is a form of motion where an object moves in parabolic path; the path that the object follows is called its trajectory.

learning objectives

- Assess the effect of angle and velocity on the trajectory of the projectile; derive maximum height using displacement

Projectile Motion

Projectile motion is a form of motion where an object moves in a bilaterally symmetrical, parabolic path. The path that the object follows is called its trajectory. Projectile motion only occurs when there is one force applied at the beginning on the trajectory, after which the only interference is from gravity. In a previous atom we discussed what the various components of an object in projectile motion are. In this atom we will discuss the basic equations that go along with them in the special case in which the projectile initial positions are null (i.e. $x_0 = 0$ and $y_0 = 0$).

Initial Velocity

The initial velocity can be expressed as x components and y components:

$$u_x = u \cdot \cos\theta \tag{5.11.1}$$

$$u_y = u \cdot \sin\theta \tag{5.11.2}$$

In this equation, u stands for initial velocity magnitude and θ refers to projectile angle.

Time of Flight

The time of flight of a projectile motion is the time from when the object is projected to the time it reaches the surface. As we discussed previously, T depends on the initial velocity magnitude and the angle of the projectile:

$$T = \frac{2 \cdot u_y}{g} \tag{5.11.3}$$

$$T = \frac{2 \cdot u \cdot \sin\theta}{g} \tag{5.11.4}$$

Acceleration

In projectile motion, there is no acceleration in the horizontal direction. The acceleration, a, in the vertical direction is just due to gravity, also known as free fall:

$$a_x = 0 \tag{5.11.5}$$

$$a_y = -g \tag{5.11.6}$$

Velocity

The horizontal velocity remains constant, but the vertical velocity varies linearly, because the acceleration is constant. At any time, t, the velocity is:

$$u_x = u \cdot \cos\theta \tag{5.11.7}$$

$$u_y = u \cdot \sin\theta - g \cdot t \tag{5.11.8}$$

You can also use the Pythagorean Theorem to find velocity:

$$u = \sqrt{u_x^2 + u_y^2} \tag{5.11.9}$$

Displacement

At time, t, the displacement components are:

$$x = u \cdot t \cdot \cos\theta \tag{5.11.10}$$

$$y = u \cdot t \cdot \sin\theta - \frac{1}{2} g t^2 \tag{5.11.11}$$

The equation for the magnitude of the displacement is $\Delta r = \sqrt{x^2 + y^2}$.

Parabolic Trajectory

We can use the displacement equations in the x and y direction to obtain an equation for the parabolic form of a projectile motion:

$$y = \tan\theta \cdot x - \frac{g}{2 \cdot u^2 \cdot \cos^2\theta} \cdot x^2 \tag{5.11.12}$$

Maximum Height

The maximum height is reached when $v_y = 0$. Using this we can rearrange the velocity equation to find the time it will take for the object to reach maximum height

$$t_h = \frac{u \cdot \sin\theta}{g} \tag{5.11.13}$$

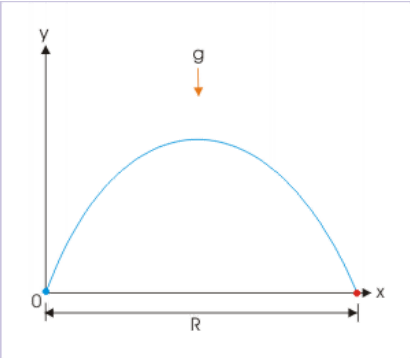
where t_h stands for the time it takes to reach maximum height. From the displacement equation we can find the maximum height

$$h = \frac{u^2 \cdot \sin^2\theta}{2 \cdot g} \tag{5.11.14}$$

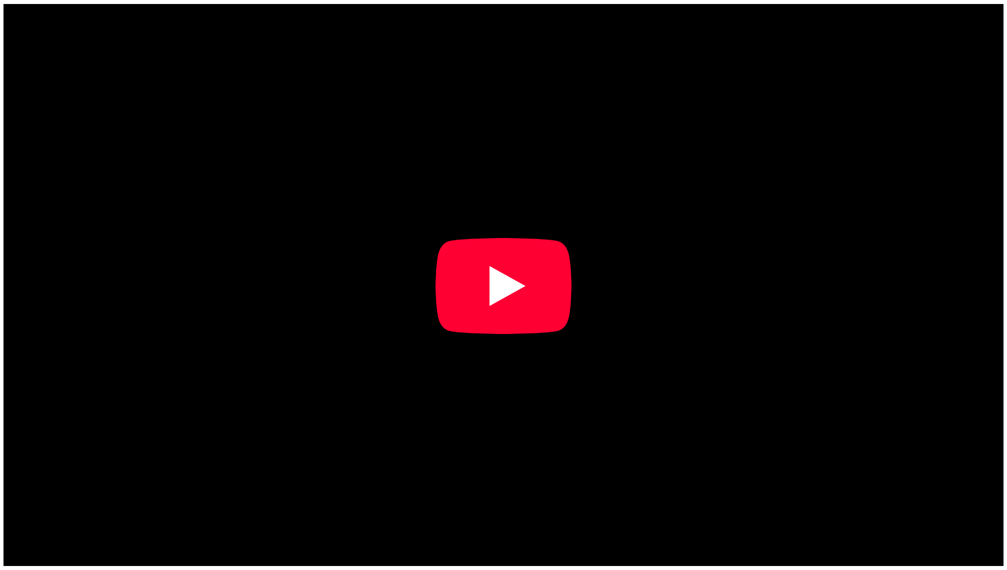
Range

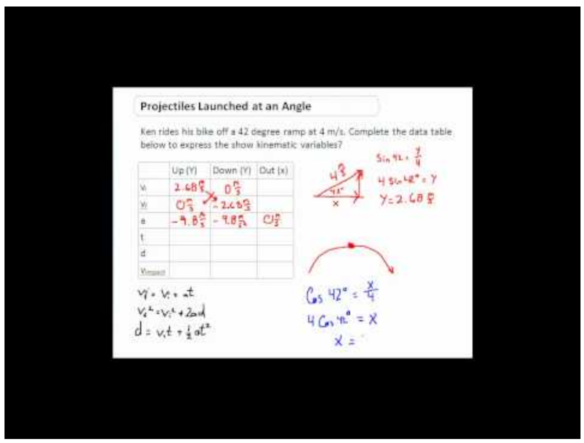
The range of the motion is fixed by the condition $y = 0$. Using this we can rearrange the parabolic motion equation to find the range of the motion:

$$R = \frac{u^2 \cdot \sin 2\theta}{g}. \tag{5.11.15}$$



Range of Trajectory: The range of a trajectory is shown in this figure.





Projectiles at an Angle: This video gives a clear and simple explanation of how to solve a problem on Projectiles Launched at an Angle. I try to go step by step through this difficult problem to layout how to solve it in a super clear way. 2D kinematic problems take time to solve, take notes on the order of how I solved it. Best wishes. Tune into my other videos for more help. Peace.

Solving Problems

In projectile motion, an object moves in parabolic path; the path the object follows is called its trajectory.

learning objectives

- Identify which components are essential in determining projectile motion of an object

We have previously discussed projectile motion and its key components and basic equations. Using that information, we can solve many problems involving projectile motion. Before we do this, let's review some of the key factors that will go into this problem-solving.

What is Projectile Motion?

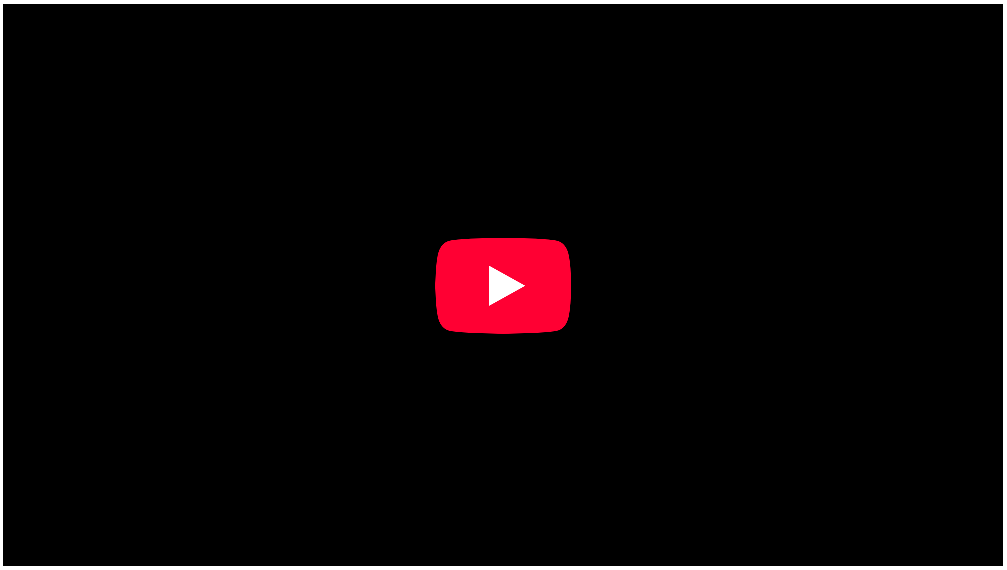
Projectile motion is when an object moves in a bilaterally symmetrical, parabolic path. The path that the object follows is called its trajectory. Projectile motion only occurs when there is one force applied at the beginning, after which the only influence on the trajectory is that of gravity.

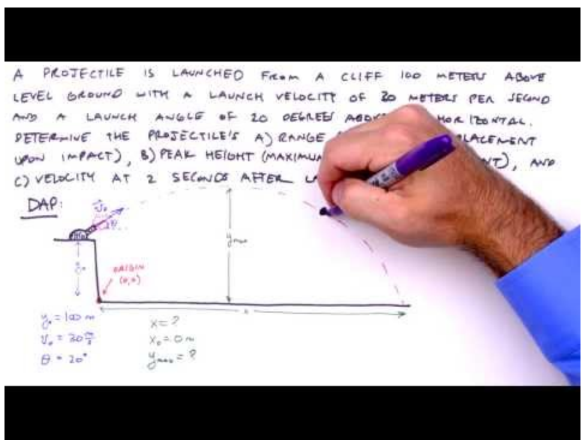
What are the Key Components of Projectile Motion?

The key components that we need to remember in order to solve projectile motion problems are:

- Initial launch angle, θ

- Initial velocity, u
- Time of flight, T
- Acceleration, a
- Horizontal velocity, v_x
- Vertical velocity, v_y
- Displacement, d
- Maximum height, H
- Range, R





How To Solve Any Projectile Motion Problem (The Toolbox Method): Introducing the "Toolbox" method of solving projectile motion problems! Here we use kinematic equations and modify with initial conditions to generate a "toolbox" of equations with which to solve a classic three-part projectile motion problem.

Now, let's look at two examples of problems involving projectile motion.

Example 5.11.1:

Example 1

Let's say you are given an object that needs to clear two posts of equal height separated by a specific distance. Refer to for this example. The projectile is thrown at $25\sqrt{2}$ m/s at an angle of 45°. If the object is to clear both posts, each with a height of 30m, find the minimum: (a) position of the launch on the ground in relation to the posts and (b) the separation between the posts. For simplicity's sake, use a gravity constant of 10. Problems of any type in physics are much easier to solve if you list the things that you know (the "givens").

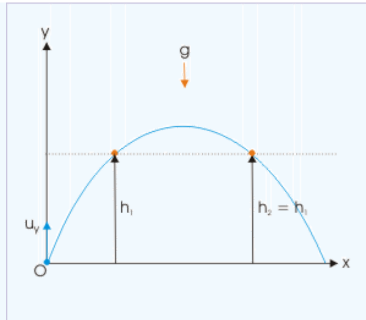


Diagram for Example 1: Use this figure as a reference to solve example 1. The problem is to make sure the object is able to clear both posts.

Solution: The first thing we need to do is figure out at what time tt the object reaches the specified height. Since the motion is in a parabolic shape, this will occur twice: once when traveling upward, and again when the object is traveling downward. For this we can use the equation of displacement in the vertical direction, $y - y_0$:

$$y - y_0 = (v_y \cdot t) - (\frac{1}{2} \cdot g \cdot t^2) \tag{5.11.16}$$

We substitute in the appropriate variables:

$$v_y = u \cdot \sin\theta = 25\sqrt{2}\frac{m}{s} \cdot \sin 45^\circ = 25\frac{m}{s} \tag{5.11.17}$$

Therefore:

$$30m = 25 \cdot t - \frac{1}{2} \cdot 10 \cdot t^2 \tag{5.11.18}$$

We can use the quadratic equation to find that the roots of this equation are 2s and 3s. This means that the projectile will reach 30m after 2s, on its way up, *and* after 3s, on its way down.

Example 2

An object is launched from the base of an incline, which is at an angle of 30°. If the launch angle is 60° *from the horizontal* and the launch speed is 10 m/s, what is the total flight time? The following information is given: $u = 10\frac{m}{s}; \theta = 60^\circ; g = 10\frac{m}{s^2}$.

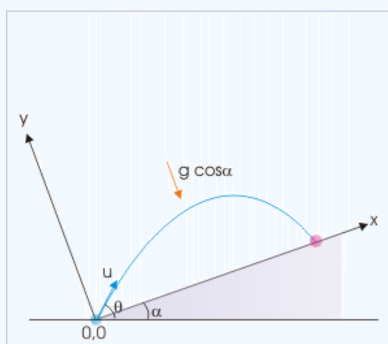


Diagram for Example 2: When dealing with an object in projectile motion on an incline, we first need to use the given information to reorient the coordinate system in order to have the object launch and fall on the same surface.

Solution: In order to account for the incline angle, we have to reorient the coordinate system so that the points of projection and return are on the same level. The angle of projection with respect to the x direction is $\theta - \alpha$, and the acceleration in the y direction is $g \cdot \cos\alpha$. We replace θ with $\theta - \alpha$ and g with $g \cdot \cos\alpha$:

$$T = \frac{2 \cdot u \cdot \sin(\theta)}{g} = \frac{2 \cdot u \cdot \sin(\theta - \alpha)}{g \cdot \cos(\alpha)} = \frac{2 \cdot 10 \cdot \sin(60 - 30)}{10 \cdot \cos(30)} = \frac{20 \cdot \sin(30)}{10 \cdot \cos(30)} \tag{5.11.19}$$

$$T = \frac{2}{\sqrt{3}} s \tag{5.11.20}$$

Zero Launch Angle

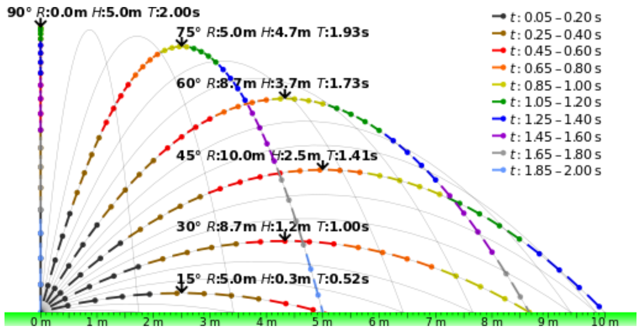
An object launched horizontally at a height H travels a range $\mathrm{v}_0 \sqrt{\frac{2\mathrm{H}}{\mathrm{g}}}$ during a time of flight $\mathrm{T} = \sqrt{\frac{2\mathrm{H}}{\mathrm{g}}}$.

learning objectives

- Explain the relationship between the range and the time of flight

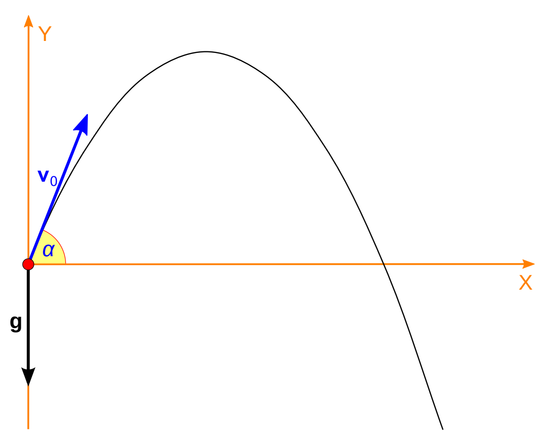
Projectile motion is a form of motion where an object moves in a parabolic path. The path followed by the object is called its trajectory. Projectile motion occurs when a force is applied at the beginning of the trajectory for the launch (after this the projectile is subject only to the gravity).

One of the key components of the projectile motion, and the trajectory it follows, is the initial launch angle. The angle at which the object is launched dictates the range, height, and time of flight the object will experience while in projectile motion. shows different paths for the same object being launched at the same initial velocity and different launch angles. As illustrated by the figure, the larger the initial launch angle and maximum height, the longer the flight time of the object.



Projectile Trajectories: The launch angle determines the range and maximum height that an object will experience after being launched.This image shows that path of the same object being launched at the same speed but different angles.

We have previously discussed the effects of different launch angles on range, height, and time of flight. However, what happens if there is no angle, and the object is just launched horizontally? It makes sense that the object should be launched at a certain height (H), otherwise it wouldn't travel very far before hitting the ground. Let's examine how an object launched horizontally at a height H travels. In our case is when α is 0.



Projectile motion: Projectile moving following a parabola.Initial launch angle is αα, and the velocity is v_0.

Duration of Flight

There is no vertical component in the initial velocity (v_0) because the object is launched horizontally. Since the object travels distance H in the vertical direction before it hits the ground, we can use the kinematic equation for the vertical motion:

$$(y - y_0) = -H = 0 \cdot T - \frac{1}{2} g T^2 \tag{5.11.21}$$

Here, T is the duration of the flight before the object its the ground. Therefore:

$$T = \sqrt{\frac{2H}{g}} \tag{5.11.22}$$

Range

In the horizontal direction, the object travels at a constant speed v_0 during the flight. Therefore, the range R (in the horizontal direction) is given as:

$$R = v_0 \cdot T = v_0 \sqrt{\frac{2H}{g}} \tag{5.11.23}$$

General Launch Angle

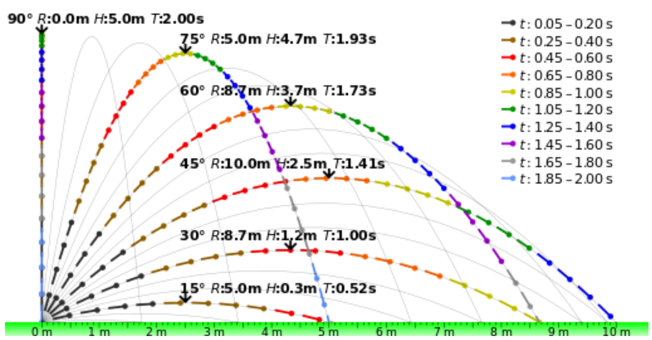
The initial launch angle (0-90 degrees) of an object in projectile motion dictates the range, height, and time of flight of that object.

learning objectives

- Choose the appropriate equation to find range, maximum height, and time of flight

Projectile motion is a form of motion where an object moves in a bilaterally symmetrical, parabolic path. The path that the object follows is called its trajectory. Projectile motion only occurs when there is one force applied at the beginning of the trajectory, after which the only interference is from gravity.

One of the key components of projectile motion and the trajectory that it follows is the initial launch angle. This angle can be anywhere from 0 to 90 degrees. The angle at which the object is launched dictates the range, height, and time of flight it will experience while in projectile motion. shows different paths for the same object launched at the same initial velocity at different launch angles. As you can see from the figure, the larger the initial launch angle, the closer the object comes to maximum height and the longer the flight time. The largest range will be experienced at a launch angle up to 45 degrees.



Launch Angle: The launch angle determines the range and maximum height that an object will experience after being launched. This image shows that path of the same object being launched at the same velocity but different angles.

The range, maximum height, and time of flight can be found if you know the initial launch angle and velocity, using the following equations:

$$R = \frac{v_i^2 \sin 2\theta_i}{g} \tag{5.11.24}$$

$$h = \frac{v_i^2 \sin^2 \theta_i}{2g} \tag{5.11.25}$$

$$T = \frac{2 v_i \sin \theta}{g} \tag{5.11.26}$$

Where R – Range, h – maximum height, T – time of flight, v_i – initial velocity, θ_i – initial launch angle, g – gravity.

Now that we understand how the launch angle plays a major role in many other components of the trajectory of an object in projectile motion, we can apply that knowledge to making an object land where we want it. If there is a certain distance, d, that you want your object to go and you know the initial velocity at which it will be launched, the initial launch angle required to get it that distance is called the angle of reach. It can be found using the following equation:

$$\theta=\frac{1}{2}\sin^{-1}(\frac{\mathrm{gd}}{\mathrm{v}^2}) \tag{5.11.27}$$

Key Points: Range, Symmetry, Maximum Height

Projectile motion is a form of motion where an object moves in parabolic path. The path that the object follows is called its trajectory.

learning objectives

- Construct a model of projectile motion by including time of flight, maximum height, and range

What is Projectile Motion ?

Projectile motion is a form of motion where an object moves in a bilaterally symmetrical, parabolic path. The path that the object follows is called its trajectory. Projectile motion only occurs when there is one force applied at the beginning on the trajectory, after which the only interference is from gravity. In this atom we are going to discuss what the various components of an object in projectile motion are, we will discuss the basic equations that go along with them in another atom, “Basic Equations and Parabolic Path”

Key Components of Projectile Motion:

Time of Flight, T:

The time of flight of a projectile motion is exactly what it sounds like. It is the time from when the object is projected to the time it reaches the surface. The time of flight depends on the initial velocity of the object and the angle of the projection, θθ. When the point of projection and point of return are on the same horizontal plane, the net vertical displacement of the object is zero.

Symmetry:

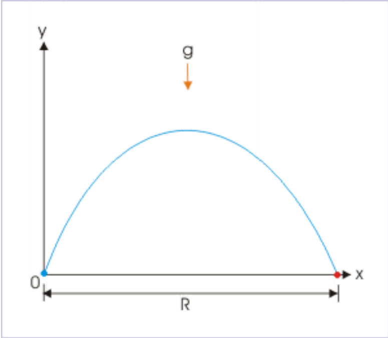
All projectile motion happens in a bilaterally symmetrical path, as long as the point of projection and return occur along the same horizontal surface. Bilateral symmetry means that the motion is symmetrical in the vertical plane. If you were to draw a straight vertical line from the maximum height of the trajectory, it would mirror itself along this line.

Maximum Height, H:

The maximum height of a object in a projectile trajectory occurs when the vertical component of velocity, vyvy, equals zero. As the projectile moves upwards it goes against gravity, and therefore the velocity begins to decelerate. Eventually the vertical velocity will reach zero, and the projectile is accelerated downward under gravity immediately. Once the projectile reaches its maximum height, it begins to accelerate downward. This is also the point where you would draw a vertical line of symmetry.

Range of the Projectile, R:

The range of the projectile is the displacement in the horizontal direction. There is no acceleration in this direction since gravity only acts vertically. shows the line of range. Like time of flight and maximum height, the range of the projectile is a function of initial speed.



Range: The range of a projectile motion, as seen in this image, is independent of the forces of gravity.

Key Points

- Objects that are projected from, and land on the same horizontal surface will have a vertically symmetrical path.
- The time it takes from an object to be projected and land is called the time of flight. This depends on the initial velocity of the projectile and the angle of projection.
- When the projectile reaches a vertical velocity of zero, this is the maximum height of the projectile and then gravity will take over and accelerate the object downward.
- The horizontal displacement of the projectile is called the range of the projectile, and depends on the initial velocity of the object.
- When solving problems involving projectile motion, we must remember all the key components of the motion and the basic equations that go along with them.
- Using that information, we can solve many different types of problems as long as we can analyze the information we are given and use the basic equations to figure it out.
- To clear two posts of equal height, and to figure out what the distance between these posts is, we need to remember that the trajectory is a parabolic shape and that there are two different times at which the object will reach the height of the posts.
- When dealing with an object in projectile motion on an incline, we first need to use the given information to reorientate the coordinate system in order to have the object launch and fall on the same surface.
- For the zero launch angle, there is no vertical component in the initial velocity.
- The duration of the flight before the object hits the ground is given as $\mathrm{T}=\sqrt{\frac{2\mathrm{H}}{\mathrm{g}}}$.
- In the horizontal direction, the object travels at a constant speed v_0 during the flight. The range R (in the horizontal direction) is given as: $\mathrm{R}=\mathrm{v}_0\cdot\mathrm{T}=\mathrm{v}_0\sqrt{\frac{2\mathrm{H}}{\mathrm{g}}}$.
- If the same object is launched at the same initial velocity, the height and time of flight will increase proportionally to the initial launch angle.
- An object launched into projectile motion will have an initial launch angle anywhere from 0 to 90 degrees.
- The range of an object, given the initial launch angle and initial velocity is found with: $\mathrm{R}=\frac{\mathrm{v}_\mathrm{i}^2\sin 2\theta_\mathrm{i}}{\mathrm{g}}$.
- The maximum height of an object, given the initial launch angle and initial velocity is found with: $\mathsf{h}=\frac{\mathrm{v}_\mathrm{i}^2\sin^2\theta_\mathrm{i}}{2\mathrm{g}}$.
- The time of flight of an object, given the initial launch angle and initial velocity is found with: $\mathrm{T}=\frac{2\mathrm{v}_\mathrm{i}\sin\theta}{\mathrm{g}}$.
- The angle of reach is the angle the object must be launched at in order to achieve a specific distance: $\theta=\frac{1}{2}\sin^{-1}(\frac{\mathrm{gd}}{\mathrm{v}^2})$.
- Objects that are projected from and land on the same horizontal surface will have a path symmetric about a vertical line through a point at the maximum height of the projectile.
- The time it takes from an object to be projected and land is called the time of flight. It depends on the initial velocity of the projectile and the angle of projection.
- The maximum height of the projectile is when the projectile reaches zero vertical velocity. From this point the vertical component of the velocity vector will point downwards.

- The horizontal displacement of the projectile is called the range of the projectile and depends on the initial velocity of the object.
- If an object is projected at the same initial speed, but two complementary angles of projection, the range of the projectile will be the same.

Key Terms

- **trajectory**: The path of a body as it travels through space.
- **symmetrical**: Exhibiting symmetry; having harmonious or proportionate arrangement of parts; having corresponding parts or relations.
- **reorientate**: to orientate anew; to cause to face a different direction
- **gravity**: Resultant force on Earth's surface, of the attraction by the Earth's masses, and the centrifugal pseudo-force caused by the Earth's rotation.
- **bilateral symmetry**: the property of being symmetrical about a vertical plane

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5.12: Multiple Velocities

Addition of Velocities

Relative velocities can be found by adding the velocity of the observed object to the velocity of the frame of reference it was measured in.

As learned in a previous atom, relative velocity is the velocity of an object as observed from a certain frame of reference.

demonstrates the concept of relative velocity. The girl is riding in a sled at 1.0 m/s, relative to an observer. When she throws the snowball forward at a speed of 1.5 m/s, relative to the sled, the velocity of the snowball to the observer is the sum of the velocity of the sled and the velocity of the snowball relative to the sled:

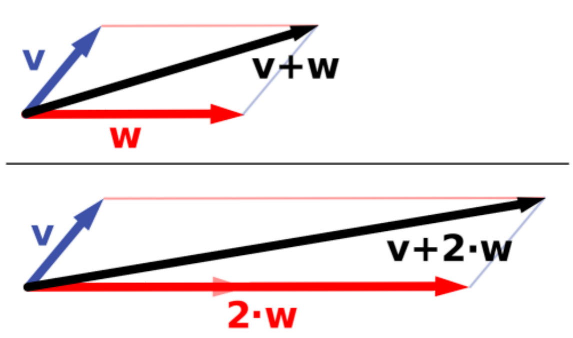
$$1.0 \mathrm{m/s} + 1.5 \mathrm{m/s} = 2.5 \mathrm{m/s} \tag{5.12.1}$$

If the girl were to throw the snowball behind her at the same speed, the velocity of the ball relative to the observer would be:

$$1.0 \mathrm{m/s} - 1.5 \mathrm{m/s} = -0.5 \mathrm{m/s} \tag{5.12.2}$$

The concept of relative velocity can also be demonstrated using the example of a boat in a river with a current. The boat is only trying to move forward, but since the river is in motion, it carries the boat sideways while it moves forward. The person on the boat is only observing the forward motion, while an observer on the shore will notice that the boat is moving sideways. In order to calculate the velocity that the object is moving relative to earth, it is helpful to remember that velocity is a vector. In order to analytically add these vectors, you need to remember the relationship between the magnitude and direction of the vector and its components on the x and y axis of the coordinate system:

- Magnitude: $\mathrm{v} = \sqrt{\mathrm{v_x^2} + \mathrm{v_y^2}}$
- Direction: $\theta = \tan^{-1}(\frac{\mathrm{v_y}}{\mathrm{v_x}})$
- x-component: $\mathrm{v_x} = \mathrm{v} \cos \theta$
- y-component: $\mathrm{v_y} = \mathrm{v} \sin \theta$



Vector Addition: Addition of velocities is simply the addition of vectors.

These components are shown above. The first two equations are for when the magnitude and direction are known, but you are looking for the components. The last two equations are for when the components are known, and you are looking for the magnitude and direction. The magnitude of the observed velocity from the shore is the square root sum of the squared velocity of the boat and the squared velocity of the river.

Relative Velocity

Relative velocity is the velocity of an object B measured with respect to the velocity of another object A, denoted as $\mathrm{v_{BA}}$.

Relative velocity is the velocity of an object B, in the rest frame of another object A. This is denoted as $\mathrm{v_{AB}}$, where A is the velocity in the rest frame of B.

Galileo observed the concept of relative velocity by using an example of a fly and a boat. He observed that while you are aboard the boat, if you see a fly, you can measure its velocity, u. You can then go back on land and measure the velocity of the boat, v. Is the velocity of the fly, u, the actual velocity of the fly? No, because what you measured was the velocity of the fly relative to the velocity of the boat. To obtain the velocity of the fly relative to the shore, s, you can use the vector sum as shown: $\mathrm{s} = \mathrm{u} + \mathrm{v}$

Examples of Relative Velocity

This concept is best explained using examples. Pretend you are sitting in a passenger train that is moving east. If you were to look out the window and see a man walking in the same direction, it would appear the the man is moving much slower than he actually is. Now imagine you are standing outside and observe the same man walking next to the train. It will appear the the man is walking much faster than it seemed when you were inside the train.

Now, imagine you are on a boat, and you see a man walking from one end of the deck to the other. The velocity that you observe the man walking in will be the same velocity that he would be walking in if you both were on land. Now imagine that you are on land and see the man on the moving boat, walking from one end of the deck to another. It will now appear that the man is walking much faster than it appeared when you were on the boat with him.

Why is this? The concept of relative velocity has to do with your frame of reference. When you were on the train, your frame of reference was moving in the same direction that the man was walking, so it appeared that he was walking slower. But once you were off the train, you were in a stationary frame of reference, so you were able to observe him moving at his actual speed. When you were on the boat, you were in a moving frame of reference, but so was the object you were observing, so you were able to observe the man walking at his actual velocity. Once you were back on land, you were in a stationary frame of reference, but the man was not, so the velocity you saw was his relative velocity.

Key Points

- In order to find the velocity of an object B that is moving on object A that is observed by an observer that is not moving, add the velocity of B and A together.
- Velocity is a vector quantity, so the relationships between the magnitude, direction, x- axis component and y-axis component are important.
- These vector components can be added analytically or graphically.
- In order to calculate the magnitude and direction, you must know the values of the axis components (either x and y, or x, y, and z) and to calculate the component values, you must know the magnitude and direction.
- Relative velocity is the velocity of an object in motion being observed from a frame of reference that is either also in motion or stationary.
- If the frame of reference is moving in the same direction as the object being observed, it will appear as though the object is moving slower than it actually is.
- If the object being observed is on a moving surface, the velocity observed from that surface will be less than the velocity observed from a stationary surface looking onto the moving surface.
- In Galileo’s example, the observed velocity of the fly, uu, measured in reference to the velocity of the boat, v. In order to find the velocity of the fly with respect to the shore, S, he had to add the velocity of the boat to the observed velocity of the fly: $s=u+v$.

Key Terms

- **relative**: Expressed in relation to another item, rather than in complete form.

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CHAPTER OVERVIEW

6: The Laws of Motion

Topic hierarchy

6.1: Introduction

Newton and His Laws

There are three laws of motion that describe the relationship between forces, mass, and acceleration.

learning objectives

- Apply three Newton's laws of motion to relate forces, mass, and acceleration

Newton's laws of motion describe the relationship between the forces acting on a body and its motion due to those forces. For example, if your car breaks down and you need to push it, you must exert a force with your hands on the car in order for it to move. The laws of motion will tell you how quickly the car will move from your pushing. There are three laws of motion:

First law: If an object experiences no net force, then its velocity is constant: the object is either at rest (if its velocity is zero), or it moves in a straight line with constant speed (if its velocity is nonzero). For example, if you don't push the car (no force), then it doesn't move.

Second law: The acceleration aa of a body is parallel and directly proportional to the net force F acting on the body, is in the direction of the net force, and is inversely proportional to the mass mm of the body:

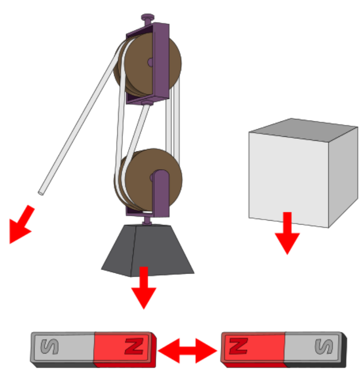
$$\mathrm{F}=\mathrm{m} \cdot \mathrm{a} \text{ or } \mathrm{a}=\frac{\mathrm{F}}{\mathrm{m}} \tag{6.1.1}$$

For example, if you push the car with a greater force it will accelerate more. But, if the car is more massive (mm is larger) then it won't accelerate as much from the same size force as a lighter car.

Third law: When a first body exerts a force F_1 on a second body, the second body simultaneously exerts a force $\mathrm{F}_2=-\mathrm{F}_1$ on the first body. This means that F_1 and F_2 are equal in magnitude and opposite in direction. For example, when you push a car, if it is exerting the same force on you that you are exerting on it, you might wonder why you don't move backwards? The answer is there are also forces from the ground on your feet pushing you forward. So, in fact, the car is pushing a force back on you that is of the same magnitude that you are using to push it forward.

In the figure below there are some practical examples illustrating the concept of force:

- Strain: by using a machine known as pulley you can easily raise or lower a massive body
- Gravitational Force: a massive body is attracted downward by the gravitational force practiced by the Earth
- Magnetic Force: two magnets repel each other when the same poles get closer



Examples of Force: Some situations in which forces are at play.

Key Points

- Acceleration of an object is proportional to the force on it.
- Force causes an object to move.
- Objects with more mass require more force to move.

Key Terms

- **force**: Any influence that causes an object to undergo a certain change, either concerning its movement, direction or geometrical construction.

6.2: Force and Mass

Force

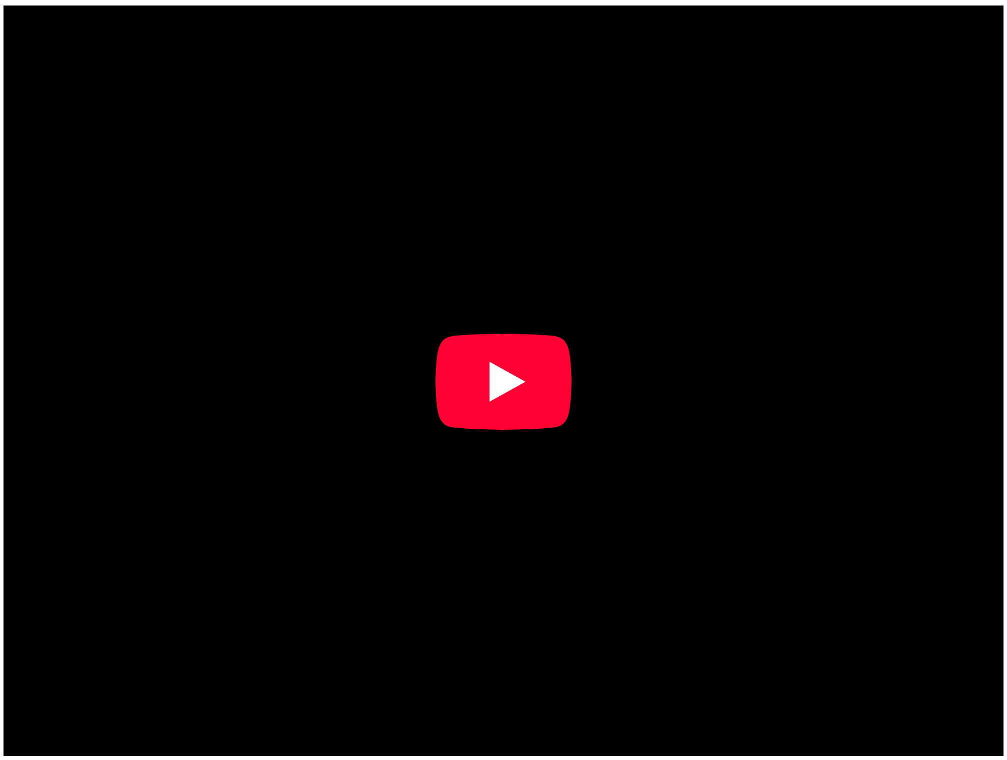
Force is any influence that causes an object to change, either concerning its movement, direction, or geometrical construction.

learning objectives

- Develop the relationship between mass and acceleration in determining force

Overview of Forces

In physics, a force is any influence that causes an object to undergo a certain change, either concerning its movement, direction, or geometrical construction. It is measured with the SI unit of Newtons. A force is that which can cause an object with mass to change its velocity, i.e., to accelerate, or which can cause a flexible object to deform. Force can also be described by intuitive concepts such as a push or pull. A force has both magnitude and direction, making it a vector quantity.



What is a force?: Describes what forces are and what they do.

Qualities of Force

The original form of Newton's second law states that the net force acting upon an object is equal to the rate at which its momentum changes. This law is further given to mean that the acceleration of an object is directly proportional to the net force acting on the object, is in the direction of the net force, and is inversely proportional to the mass of the object.

As we mentioned, force is a vector quantity. A vector is a one dimensional array with elements of both magnitude and direction. In a force vector, the mass, m, is the magnitude component and the acceleration, a, is the directional component. The equation for force is written:

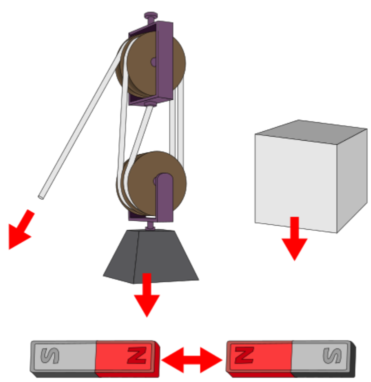
$$\mathrm{F} = \mathrm{m} \cdot \mathrm{a} \tag{6.2.1}$$

Related concepts to force include thrust, which increases the velocity of an object; drag, which decreases the velocity of an object; and torque which produces changes in rotational speed of an object. Forces which do not act uniformly on all parts of a body will also cause mechanical stresses, a technical term for influences which cause deformation of matter. While mechanical stress can remain embedded in a solid object, gradually deforming it, mechanical stress in a fluid determines changes in its pressure and volume.

Dynamics

Dynamics is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. Our intuitive definition of force — that is, a push or a pull —is a good place to start. We know that a push or pull has both magnitude and direction (therefore, it is a vector quantity) and can vary considerably in each regard.

shows a few examples of the "push-pull" nature of force. The top left example is that of a pulley system. The force that someone would have to pull down on the cable would have to equal and exceed the force made by the mass the object and the effects of gravity on those object in order for the system to move up. The top right example shows that any object resting on a surface will still exert force on that surface. The bottom example is that of two magnets being attracted to each other due to magnetic force.



Examples of Force: Some situations in which forces are at play.

Mass

Mass is a physical property of matter that depends on size and shape of matter, and is expressed as kilograms by the SI system.

learning objectives

- Justify the significance of understanding mass in physics

What is Mass?

All elements have physical properties whose values can help describe an elements physical state. Changes to these properties can describe elemental transformations. Physical properties do not change the chemical nature of matter. The physical property we are covering in this atom is called mass.

Mass is defined as a quantitative measure of an object's resistance to acceleration. The terms mass and weight are often interchanged, however it is incorrect to do so. Weight is a different property of matter that, while related to mass, is not mass, but rather the amount of gravitational force acting on a given body of matter. Mass is an intrinsic property that never changes.

Units of Mass

In order to measure something, a standard value must be established to use in relation to the object of measurement. This relation is called a unit. The International System of Units (SI) measures mass in kilograms, or kg. There are other units of mass, including the following (only the first two are accepted by the SI system):

- t – Tonne; 1t = 1000kg
- u – atomic mass unit; 1u ~= 1.66×10^{-27}kg
- sl – slug
- lb – pound

Concepts Using Mass

- Weight – see
- Newtons Second Law – mass has a central role in determining the behavior of bodies. Newtons Second Law relates force f, exerted in a body of mass m, to the body's acceleration a: $\mathrm{F} = \mathrm{ma}$
- Momentum – mass relates a body's momentum, p, to its linear velocity, v: $\mathrm{p} = \mathrm{mv}$
- Kinetic Energy – mass relates kinetic energy, K to velocity, v: $\mathrm{K} = \frac{1}{2}\mathrm{m}|\mathrm{v}^2|$

Key Points

- Force is stated as a vector quantity, meaning it has elements of both magnitude and direction. Mass and acceleration respectively.
- In layman's terms, force is a push or pull that can be defined in terms of various standards.
- Dynamics is the study of the force that causes objects and systems to move or deform.
- External forces are any outside forces that act on a body, and internal forces are any force acting within a body.
- Mass is defined as a quantitative measure of an object's resistance to acceleration.
- According to Newton's second law of motion, if a body of fixed mass m is subjected to a single force F, its acceleration a is given by F/m.
- Mass is central in many concepts of physics, including:weight, momentum, acceleration, and kinetic energy.
- According to Newton's second law of motion, if a body of fixed mass m is subjected to a single force F, its acceleration a is given by F/m.

Key Terms

- **force**: A force is any influence that causes an object to undergo a certain change, either concerning its movement, direction or geometrical construction.
- **velocity**: A vector quantity that denotes the rate of change of position with respect to time, or a speed with a directional component.
- **vector**: A directed quantity, one with both magnitude and direction; the between two points.
- **mass**: The quantity of matter which a body contains, irrespective of its bulk or volume. It is one of four fundamental properties of matter. It is measured in kilograms in the SI system of measurement.

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6.3: Newton's Laws

The First Law: Inertia

Newton's first law of motion describes inertia. According to this law, a body at rest tends to stay at rest, and a body in motion tends to stay in motion, unless acted on by a net external force.

learning objectives

- Define the First Law of Motion

History

Sir Isaac Newton was an English scientist who was interested in the motion of objects under various conditions. In 1687, he published a work called *Philosophiae Naturalis Principla Mathematica*, which described his three laws of motion. Newton used these laws to explain and explore the motion of physical objects and systems. These laws form the basis for mechanics. The laws describe the relationship between forces acting on a body and the motions experienced due to these forces. The three laws are as follows:

1. If an object experiences no net force, its velocity will remain constant. The object is either at rest and the velocity is zero or it moves in a straight line with a constant speed.
2. The acceleration of an object is parallel and directly proportional to the net force acting on the object, is in the direction of the net force, and is inversely proportional to the mass of the object.
3. When a first object exerts a force on a second object, the second object simultaneously exerts a force on the first object, meaning that the force of the first object and the force of the second object are equal in magnitude and opposite in direction.

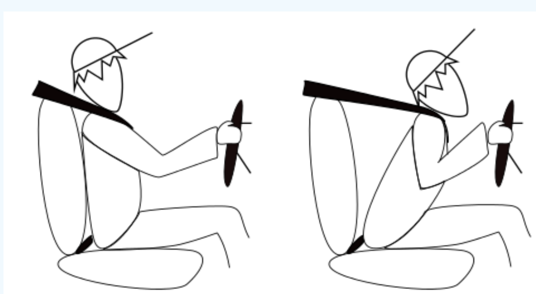
The First Law of Motion

You have most likely heard Newton's first law of motion before. If you haven't heard it in the form written above, you have probably heard that "a body in motion stays in motion, and a body at rest stays at rest." This means that an object that is in motion will not change its velocity unless an unbalanced force acts upon it. This is called uniform motion. It is easier to explain this concept through examples.

Example 6.3.1:

If you are ice skating, and you push yourself away from the side of the rink, according to Newton's first law you will continue all the way to the other side of the rink. But, this won't actually happen. Newton says that a body in motion will stay in motion until an outside force acts upon it. In this and most other real world cases, this outside force is friction. The friction between your ice skates and the ice is what causes you to slow down and eventually stop.

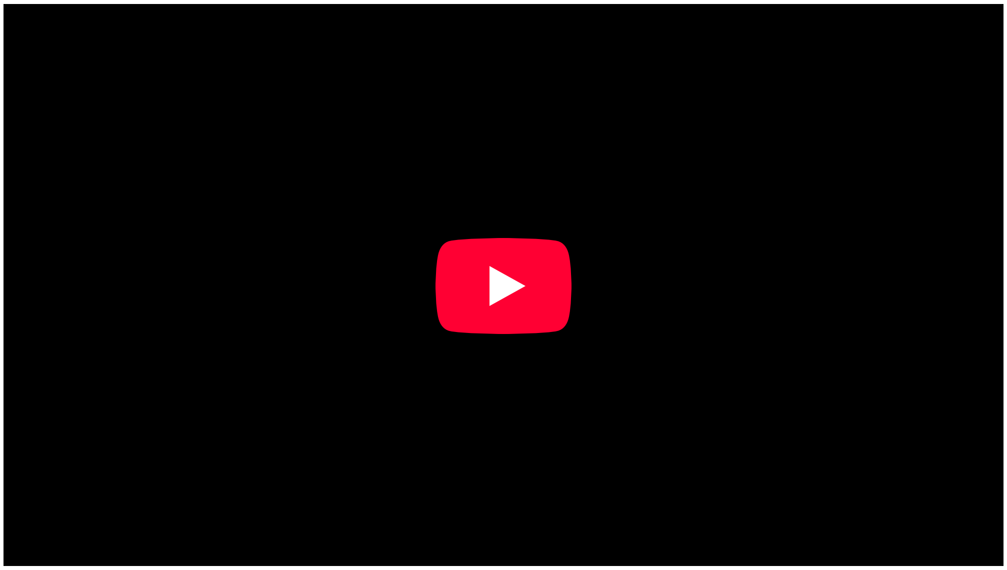
Let's look at another situation. Refer to for this example. Why do we wear seat belts? Obviously, they're there to protect us from injury in case of a car accident. If a car is traveling at 60 mph, the driver is also traveling at 60 mph. When the car suddenly stops, an external force is applied to the car that causes it to slow down. But there is no force acting on the driver, so the driver continues to travel at 60 mph. The seat belt is there to counteract this and act as that external force to slow the driver down along with the car, preventing them from being harmed.

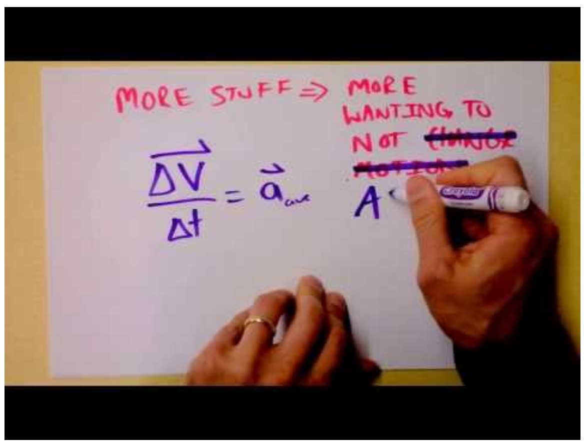


Newton's First Law: Newton's first law in effect on the driver of a car

Inertia

Sometimes this first law of motion is referred to as the law of inertia. Inertia is the property of a body to remain at rest or to remain in motion with constant velocity. Some objects have more inertia than others because the inertia of an object is equivalent to its mass. This is why it is more difficult to change the direction of a boulder than a baseball.





Doc Physics – Newton: Newton’s first law is hugely counterintuitive. You may have learned it in gradeschool, though. Let’s see it for the mind-blowing conclusion it really is.

The Second Law: Force and Acceleration

The second law states that the net force on an object is equal to the rate of change, or derivative, of its linear momentum.

learning objectives

- Define the Second Law of Motion

English scientist Sir Isaac Newton examined the motion of physical objects and systems under various conditions. In 1687, he published his three laws of motion in *Philosophiae Naturalis Principla Mathematica*. The laws form the basis for mechanics—they describe the relationship between forces acting on a body, and the motion experienced due to these forces. These three laws state:

1. If an object experiences no net force, its velocity will remain constant. The object is either at rest and the velocity is zero, or it moves in a straight line with a constant speed.
2. The acceleration of an object is parallel and directly proportional to the net force acting on the object, is in the direction of the net force and is inversely proportional to the mass of the object.

3. When a first object exerts a force on a second object, the second object simultaneously exerts a force on the first object, meaning that the force of the first object and the force of the second object are equal in magnitude and opposite in direction.

The first law of motion defines only the natural state of the motion of the body (i.e., when the net force is zero). It does not allow us to quantify the force and acceleration of a body. The acceleration is the rate of change in velocity; it is caused only by an external force acting on it. The second law of motion states that the net force on an object is equal to the rate of change of its linear momentum.

Linear Momentum

Linear momentum of an object is a vector quantity that has both magnitude and direction. It is the product of mass and velocity of a particle at a given time:

$$\mathrm{p}=\mathrm{mv} \tag{6.3.1}$$

where, $\mathrm{p}=\text{momentum}, \mathrm{m}=\text{mass}, \text{ and } \mathrm{v}=\text{velocity}$. From this equation, we see that objects with more mass will have more momentum.

The Second Law of Motion

Picture two balls of different mass, traveling in the same direction at the same velocity. If they both collide with a wall at the same time, the heavier ball will exert a larger force on the wall. This concept, illustrated below, explains Newton's second law, which emphasizes the importance of force and motion, over velocity alone. It states: the net force on an object is equal to the rate of change of its linear momentum. From calculus we know that the rate of change is the same as a derivative. When we the linear momentum of an object we get:

Force and Mass: This animation demonstrates the connection between force and mass.

$$\mathrm{F}=\frac{\mathrm{dp}}{\mathrm{dt}} \tag{6.3.2}$$

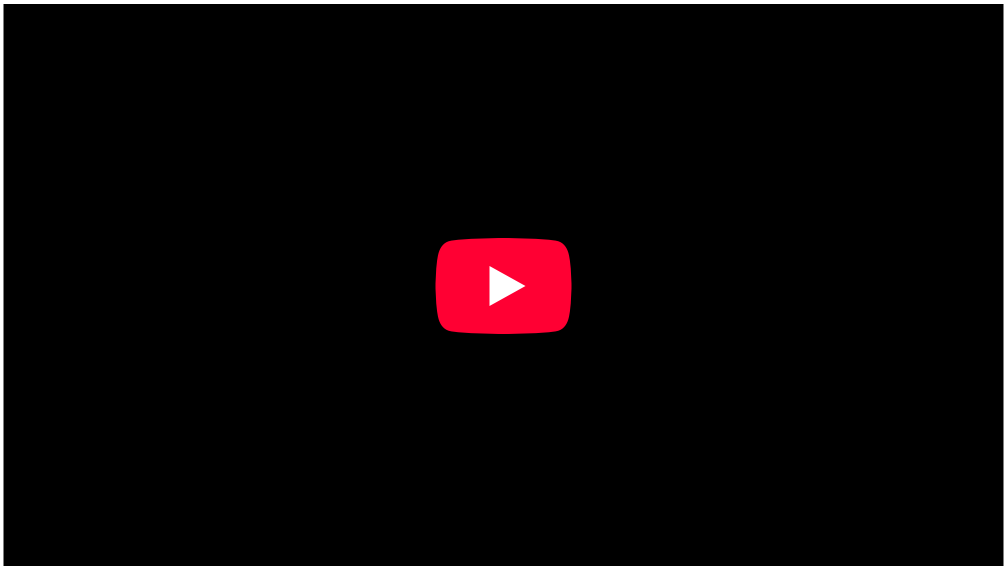
$$\mathrm{F}=\frac{\mathrm{d}(\mathrm{m}\cdot\mathrm{v})}{\mathrm{dt}} \tag{6.3.3}$$

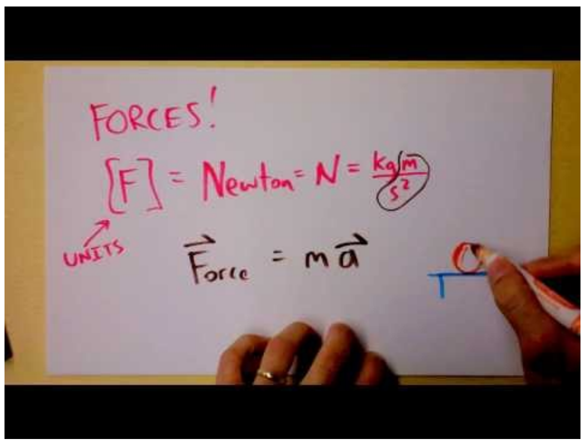
where, $\mathrm{F}=\text{Force}$ and $\mathrm{t}=\text{time}$. From this we can further simplify the equation:

$$\mathrm{F}=\mathrm{m}\frac{\mathrm{d}(\mathrm{v})}{\mathrm{dt}} \tag{6.3.4}$$

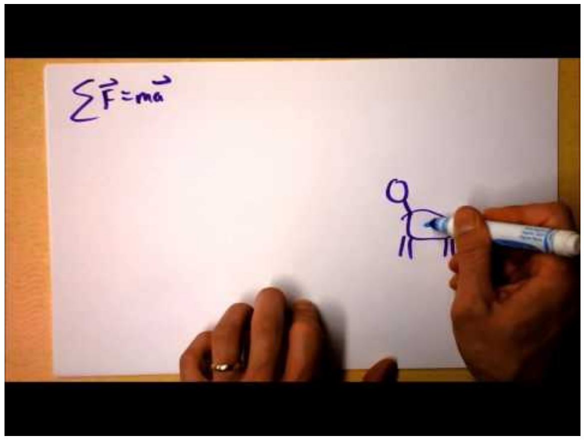
$$\mathrm{F}=\mathrm{m}\cdot\mathrm{a} \tag{6.3.5}$$

where, a=accelerationa=acceleration. As we stated earlier, acceleration is the rate of change of velocity, or velocity divided by time.





Newton’s Three Laws of Mechanics – Second Law – Part 1: Here we’ll see how many people can confuse your understanding of Newton’s 2nd law of motion through oversight, sloppy language, or cruel intentions.



Newton's Three Laws of Mechanics – Second Law – Part Two: Equilibrium is investigated and Newton's 1st law is seen as a special case of Newton's 2nd law!

The Third Law: Symmetry in Forces

The third law of motion states that for every action, there is an equal and opposite reaction.

learning objectives

- Define the Third Law of Motion

Sir Isaac Newton was a scientist from England who was interested in the motion of objects under various conditions. In 1687, he published a work called *Philosophiae Naturalis Principla Mathematica*, which contained his three laws of motion. Newton used these laws to explain and explore the motion of physical objects and systems. These laws form the bases for mechanics. The laws describe the relationship between forces acting on a body, and the motion is an experience due to these forces. Newton's three laws are:

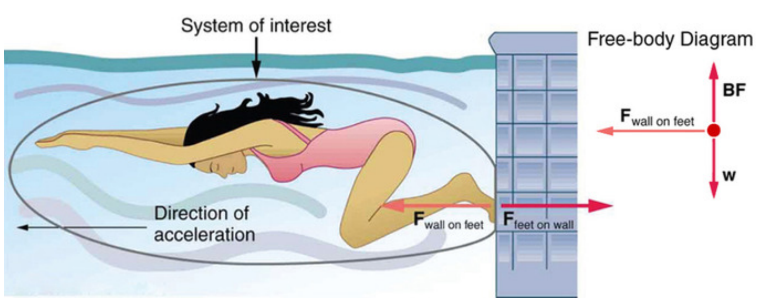
1. If an object experiences no net force, its velocity will remain constant. The object is either at rest and the velocity is zero or it moves in a straight line with a constant speed.
2. The acceleration of an object is parallel and directly proportional to the net force acting on the object, is in the direction of the net force and is inversely proportional to the mass of the object.
3. When a first object exerts a force on a second object, the second object simultaneously exerts a force on the first object, meaning that the force of the first object and the force of the second object are equal in magnitude and opposite in direction.

Newton's Third Law of Motion

Newton's third law basically states that for every action, there is an equal and opposite reaction. If object A exerts a force on object B, because of the law of symmetry, object B will exert a force on object A that is equal to the force acted on it:

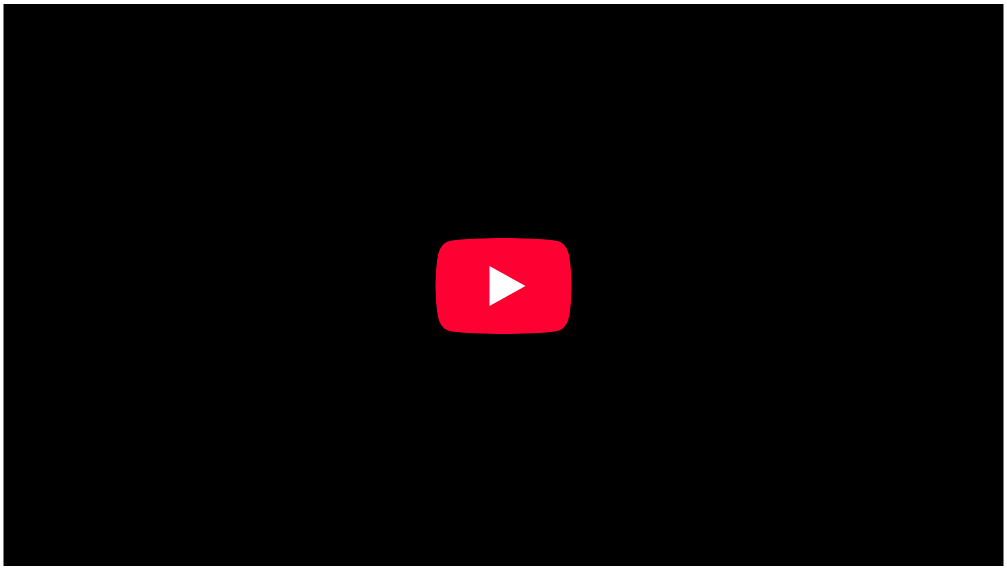
$$\mathrm{FA} = -\mathrm{FB} \tag{6.3.6}$$

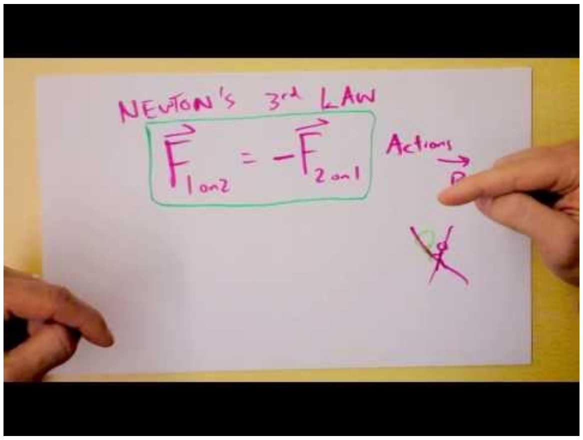
In this example, F_A is the action and F_B is the reaction. You have undoubtedly witnessed this law of motion. For example, take a swimmer who uses her feet to push off the wall in order to gain speed. The more force she exerts on the wall, the harder she pushes off. This is because the wall exerts the same force on her that she forces on it. She pushes the wall in the direction behind her, therefore the wall will exert a force on her that is in the direction in front of her and propel her forward.



Newton's Third Law of Motion: When a swimmer pushes off the wall, the swimmer is using the third law of motion.

Take as another example, the concept of thrust. When a rocket launches into outer space, it expels gas backward at a high velocity. The rocket exerts a large backward force on the gas, and the gas exerts and equal and opposite reaction force forward on the rocket, causing it to launch. This force is called thrust. Thrust is used in cars and planes as well.





Newton's Third Law: The most fundamental statement of basic physical reality is also the most often misunderstood. As your mom if she's clear on Newton's Third. Then ask her why things can move if every force has a paired opposite force all the time, forever.

Key Points

- Newton's three laws of physics are the basis for mechanics.
- The first law states that a body at rest will stay at rest until a net external force acts upon it and that a body in motion will remain in motion at a constant velocity until acted on by a net external force.
- Net external force is the sum of all of the forcing acting on an object.
- Just because there are forces acting on an object doesn't necessarily mean that there is a net external force; forces that are equal in magnitude but acting in opposite directions can cancel one another out.
- Friction is the force between an object in motion and the surface on which it moves. Friction is the external force that acts on objects and causes them to slow down when no other external force acts upon them.
- Inertia is the tendency of a body in motion to remain in motion. Inertia is dependent on mass, which is why it is harder to change the direction of a heavy body in motion than it is to change the direction of a lighter object in motion.
- Newton's three laws of motion explain the relationship between forces acting on an object and the motion they experience due to these forces. These laws act as the basis for mechanics.

- The second law explains the relationship between force and motion, as opposed to velocity and motion. It uses the concept of linear momentum to do this.
- Linear momentum p, is the product of mass m, and velocity v : $p = mv$.
- The second law states that the net force is equal to the derivative, or rate of change of its linear momentum.
- By simplifying this relationship and remembering that acceleration is the rate of change of velocity, we can see that the second law of motion is where the relationship between force and acceleration comes from.
- If an object A exerts a force on object B, object B exerts an equal and opposite force on object A.
- Newton's third law can be seen in many everyday circumstances. When you walk, the force you use to push off the ground backwards makes you move forward.
- Thrust is an application of the third law of motion. A helicopter uses thrust to push the air under the propeller down, and therefore lift off the ground.

Key Terms

- **inertia**: The property of a body that resists any change to its uniform motion; equivalent to its mass.
- **friction**: A force that resists the relative motion or tendency to such motion of two bodies in contact.
- **uniform motion**: Motion at a constant velocity (with zero acceleration). Note that an object in motion will not change its velocity unless an unbalanced force acts upon it.
- **net force**: The combination of all the forces that act on an object.
- **momentum**: (of a body in motion) the product of its mass and velocity.
- **acceleration**: The amount by which a speed or velocity increases (and so a scalar quantity or a vector quantity).
- **symmetry**: Exact correspondence on either side of a dividing line, plane, center or axis.
- **thrust**: The force generated by propulsion, as in a jet engine.

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6.4: Other Examples of Forces

Weight

Weight is taken as the force on an object due to gravity, and is different than the mass of an object.

learning objectives

- Infer what factors other than gravity will contribute to the apparent weight of an object

In physics, it is important to differentiate the weight of an object from its mass. The mass of an object is an intrinsic quantity, independent of the location of the object. On the other hand, the weight of an object is an extrinsic quantity. It is considered as the force on an object due to gravity. Since gravitational acceleration changes depending on the location in the universe, weight does as well.

Mathematically, the weight of an object (W) can be found by multiplying its mass (m) by the acceleration due to gravity (g): $\mathrm{W}=\mathrm{M} \cdot \mathrm{g}$. The strength of gravity varies very little over the surface of the Earth. In fact, the greatest percent difference in the value of the acceleration due to gravity on Earth is 0.5%.

For most calculations involving the weight of an object on Earth, it is sufficient to assume that $\mathrm{g}=9.8 \frac{\mathrm{m}}{\mathrm{s}^2}$.

The weight of an object has the same SI unit as force—the Newton ($1\mathrm{N}=1\mathrm{kg} \cdot \frac{\mathrm{m}}{\mathrm{s}^2}$).

In US customary units, the weight of an object can be expressed in pounds. Keep in mind that in US units the pound is either a unit of force or of mass. If one must find the weight (as opposed to the mass) of an object in US units, it can be calculated in terms of pounds of force.

It is important to note that the apparent weight of an object (i.e., the weight of an object determined by a scale) will vary if forces other than gravity are acting upon the object. For example, if you weigh a given mass underwater you will find a different result than if you weigh that mass in air. In this case, the weight of the object varies due to the force of buoyancy. While the mass is in the water it displaces fluid, resulting in an upward force upon it. This upward force affects the net force that the mass exerts on the scale, and thus alters its “apparent” weight.

Spring Scale: A spring scale measures weight by finding the extent to which a spring is compressed.This is proportional to the force that a mass exerts on the scale due to its weight.

Normal Forces

The normal force comes about when an object contacts a surface; the resulting force is always perpendicular to the surface of contact.

learning objectives

- Evaluate Newton's Second and Third Laws in determining the normal force on an object

Overview

The normal force, F_N, comes about when an object contacts a surface. According to Newton's third law, when one object exerts a force on a second object, the second object always exerts a force that is equal in magnitude and opposite in direction on the first object. This is the reason that the normal force exists.

A common situation in which a normal force exists is when a person stands on the ground. Because of Newton's third law, the ground exerts a force on the person that is equal in magnitude to the person's weight. In this simple case, the weight of the person and the opposing normal force are the only two forces considered on the person. The person remains still because the forces due to weight and the normal force create a net force of zero on the person.

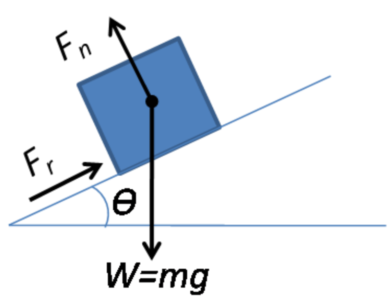
Forces on Inclined Planes

A more complex example of a situation in which a normal force exists is when a mass rests on an inclined plane. In this case, the normal force is not in the exact opposite direction as the force due to the weight of the mass. This is because the mass contacts the surface at an angle. By taking this angle into account, the magnitude of the normal force (F_N) can be found from:

$$F_N = mg\cos(\theta), \tag{6.4.1}$$

where:

- m is the mass under consideration,
- g is the acceleration due to gravity,
- and θ is the angle between the inclined surface and the horizontal.



Inclined Plane: A mass rests on an inclined plane that is at an angle θ to the horizontal. The following forces act on the mass: the weight of the mass ($m \cdot g$),the force due to friction (Fr),and the normal force (Fn).

Another interesting example involving normal forces is when a person stands in an elevator. When the elevator goes up, the normal force is actually greater than the force due to gravity. In this situation there are only two forces acting on the person. The first is the force of gravity on the person, which does not change. The second is the normal force. By summing the forces and setting them equal to m·am·a (utilizing Newton's second law), we find:

$$F_N - m \cdot g = m \cdot a \tag{6.4.2}$$

where:

- F_N is the normal force,
- $m \cdot g$ is the force due to gravity,
- m is the mass of the person,
- and a is the acceleration.

Since acceleration is positive, the normal force must actually be greater than the force due to gravity (the weight of the person).

Key Points

- Weight is taken to be the force on an object due to gravity.
- Weight and mass are not the same thing!
- The weight of a given mass will be different when the acceleration due to gravity is different.
- Apparent weight can change because of the effect of buoyancy.
- The strength of gravity is almost the same everywhere on the surface of the Earth.
- The normal force, F_N, comes about when an object contacts a surface.
- The normal force exists because for every force, there is always an equal and opposite force.
- The normal force is always perpendicular to the plane that the object contacts or rests on.

Key Terms

- **Gravitational acceleration**: Gravitational acceleration is the acceleration that an object undergoes due solely to gravity
- **perpendicular**: at or forming a right angle (to).
- **normal**: A line or vector that is perpendicular to another line, surface, or plane.

6.5: Problem-Solving

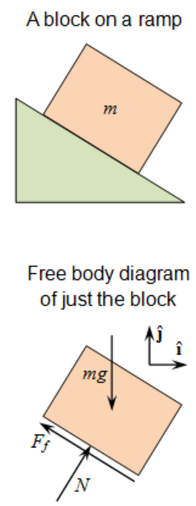
A General Approach

Basic problem-solving techniques can aid in the solution of problems involving motion (i.e., the laws of motion).

learning objectives

- Assess the laws of motion through practiced problem solving techniques

When dealing with the laws of motion, although knowledge of concepts and equations is important, understanding basic problem solving techniques can simplify the process of solving problems that may appear difficult. Your approach to problem solving can involve several key steps.



Free body diagram: An example of a drawing to help identify forces and directions.

First, gather all relevant information from the problem. Identify all quantities that are given (the *knowns*), then do the same for all quantities needed (the *unknowns*). Also, identify the physical principles involved (e.g., force, gravity, friction, etc.).

Next, a drawing may be helpful. Sometimes a drawing can even help determine the known and unknown quantities. It need not be a work of art, but it should be clear enough to illustrate proper dimension, (meaning one, two, or three dimensions). You can then use this drawing to determine which direction is positive and which is negative (making note of this on the drawing).

A next step is to use what is known to find the appropriate equation to find what is unknown. While it is easiest to find an equation that leaves only one unknown, sometimes this is not possible. In these situations, you can solve multiple equations to find the right answer. Remember that equations represent physical principles and relationships, so use the equations and drawings in tandem.

You may then substitute the knowns into the appropriate equations and find a numerical solution.

Check the answer to see if it is reasonable and makes sense. Your judgment will improve and fine tune as you solve more problems of this nature. This "judgement" step helps intuit the problem in terms of its conceptual meaning. If you can judge whether the answer is reasonable, you have a deeper understanding of physics than simply the mechanics of problem solving.

When solving problems, we tend to perform these steps in different order, as well as do several steps simultaneously. There is no rigid procedure that will work every time. Creativity and insight grow with experience. In time, the basics of problem solving can become relatively automatic.

Key Points

- Gathering all relevant information and identifying knowns and unknowns is an important first step.
- Always make a drawing to help identify directions of forces and to establish x, y, and z axes.
- Choose the correct equations, solve the problem, and check that the answer fits expectations numerically.

Key Terms

- **equation**: An assertion that two expressions are equal, expressed by writing the two expressions separated by an equal sign; from which one is to determine a particular quantity.

6.6: Vector Nature of Forces

Forces in Two Dimensions

Forces act in a particular direction and have sizes dependent upon how strong the push or pull is.

learning objectives

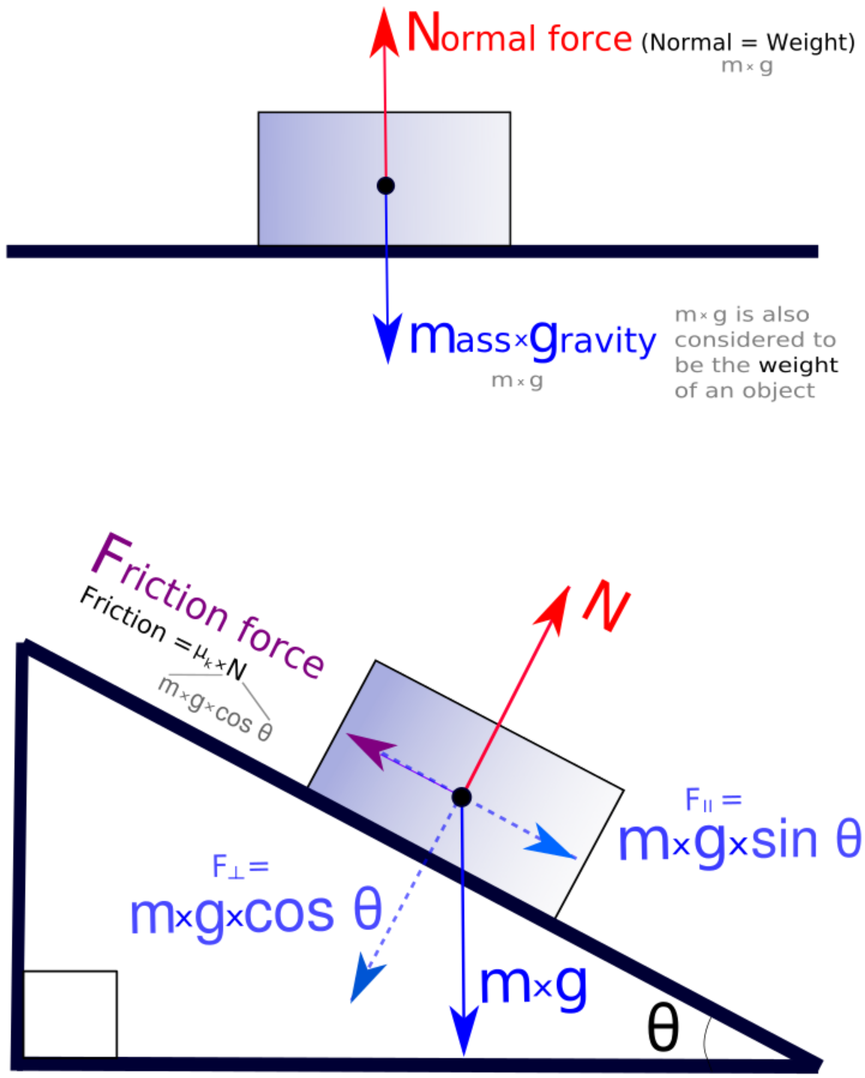
- Explain why forces are classified as “vector quantities”

Forces act in a particular direction and have sizes dependent upon how strong the push or pull is. Because of these characteristics, forces are classified as “vector quantities. ” This means that forces follow a different set of mathematical rules than physical quantities that do not have direction (denoted scalar quantities).

For example, when determining what happens when two forces act on the same object, it is necessary to know both the magnitude and the direction of both forces to calculate the result. If both of these pieces of information are not known for each force, the situation is ambiguous. For example, if you know that two people are pulling on the same rope with known magnitudes of force but you do not know which direction either person is pulling, it is impossible to determine what the acceleration of the rope will be. The two people could be pulling against each other as in tug of war or the two people could be pulling in the same direction. In this simple one-dimensional example, without knowing the direction of the forces it is impossible to decide whether the net force is the result of adding the two force magnitudes or subtracting one from the other. Associating forces with vectors avoids such problems.

When two forces act on a point particle, the resulting force or the resultant (also called the net force) can be determined by following the parallelogram rule of vector addition: the addition of two vectors represented by sides of a parallelogram gives an equivalent resultant vector which is equal in magnitude and direction to the transversal of the parallelogram. The magnitude of the resultant varies from the difference of the magnitudes of the two forces to their sum, depending on the angle between their lines of action.

Free-body diagrams can be used as a convenient way to keep track of forces acting on a system. Ideally, these diagrams are drawn with the angles and relative magnitudes of the force vectors preserved so that graphical vector addition can be done to determine the net force.



Forces as Vectors: Free-body diagrams of an object on a flat surface and an inclined plane. Forces are resolved and added together to determine their magnitudes and the net force.

Key Points

- When determining what happens when two forces act on the same object, it is necessary to know both the magnitude and the direction of both forces to calculate the result.
- When two forces act on a point particle, the resulting force or the resultant (also called the net force), can be determined by following the parallelogram rule of vector addition.
- Free-body diagrams can be used as a convenient way to keep track of forces acting on an object.

Key Terms

- **vector**: A directed quantity, one with both magnitude and direction; the between two points.
- **free-body diagram**: A free body diagram, also called a force diagram, is a pictorial representation often used by physicists and engineers to analyze the forces acting on a body of interest.
- **resultant**: A vector that is the vector sum of multiple vectors

6.7: Further Applications of Newton’s Laws

Applications of Newton’s Laws

Net force affects the motion, postion and/or shape of objects (some important and commonly used forces are friction, drag and deformation).

learning objectives

- Explain the effect of forces on an object’s motion and shape

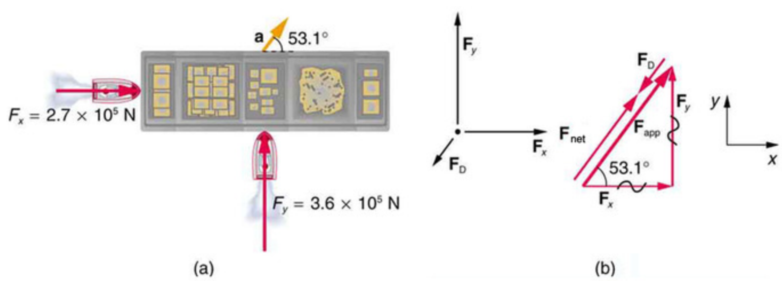
We know that a net force affects the motion, position and shape of an object. It is useful at this point to look at some particularly interesting and common forces that will provide further applications of Newton’s laws of motion. Specifically, we will discuss the forces of friction, air or liquid drag, and deformation.

Friction

Friction is a force that resists movement between two surfaces sliding against each other. When surfaces in contact move relative to each other, the friction between the two surfaces converts kinetic energy into heat. This property can have a dramatic effect, as seen in the use of friction created by rubbing pieces of wood together to start a fire. Friction is not itself a fundamental force, but arises from fundamental electromagnetic forces between the charged particles constituting the two contacting surfaces.

Drag

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either gas or liquid). You feel this drag force when you move your hand through water, or through the wind. Like friction, the force of drag is a force that resists motion. As we will discuss in later units, the drag force is proportional to the velocity of the object moving through it. We see an illustrated example of drag force in.



Drag Force on a Barge: (a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Since the applied forces are perpendicular, the x– and y-axes are in the same direction as F_x and F_y. The problem quickly becomes a one-dimensional problem along the direction of F_{app}, since friction is in the direction opposite to F_{app}.

Deformation

We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object’s shape. If a bulldozer pushes a car into a wall, the car will not move but it will noticeably change shape. The change in shape of an object due to the application of a force is a deformation. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed (that is, the deformation is elastic for small deformations). Second, the size of the deformation is proportional to the force.

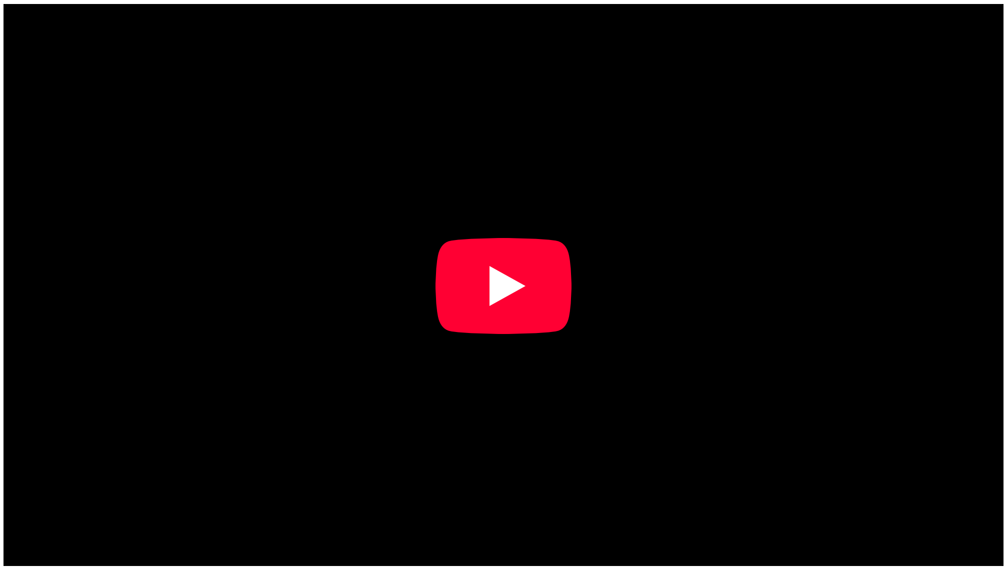
Friction: Kinetic

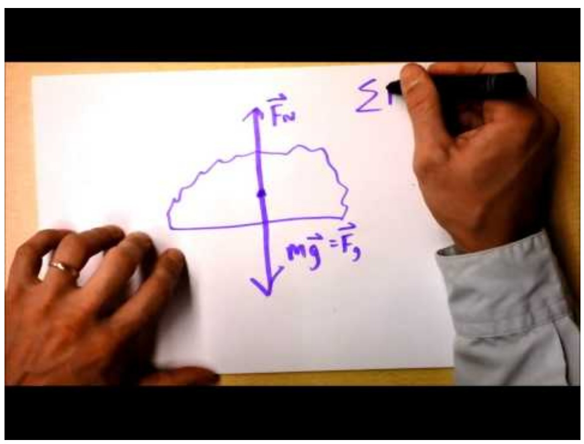
If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction.

learning objectives

- Explain the dynamics of energy for friction between two surfaces

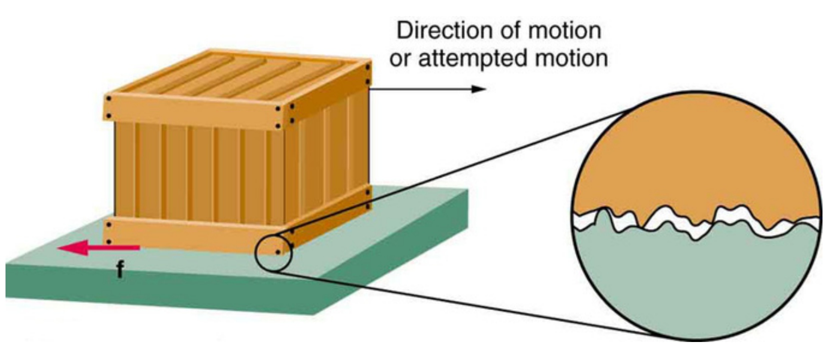
When surfaces in contact move relative to each other, the friction between the two surfaces converts kinetic energy into heat. This property can have dramatic consequences, as illustrated by the use of friction created by rubbing pieces of wood together to start a fire. Kinetic energy is converted to heat whenever motion with friction occurs, for example when a viscous fluid is stirred.





Kinetic Friction Introduction: Here, I’ll explain the microscopic justification of friction and what we can know about it. The coefficient of friction, too!

Kinetic (or dynamic) friction occurs when two objects are moving relative to each other and rub together; a sled on the ground would be a good example of kinetic friction.



Friction: Frictional forces always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. In order for the object to move, it must rise to where the peaks can skip along the bottom surface. Thus, a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. Such adhesive forces also depend on the substances the surfaces are made of, explaining, for example, why rubber-soled shoes slip less than those with leather soles.

The force of friction is what slows an object sliding over a surface. This force is what makes the brakes on cars work or causes resistance when you slide your hand across a surface. The force of friction can be represented by an equation: $\mathrm{F_{friction}} = \mu \mathrm{F_n}$. In this equation μ is something called the coefficient of friction. This is a unitless number that represents the strength of the friction of the object. A very “grippy” surface like rubber might have a high coefficient of friction, whereas a slippery surface like ice has a much lower coefficient. $\mathrm{F_n}$ is called the normal force and is the force of the surface pushing up on the object. In most cases on level ground, the normal force will be the equal and opposite of the object’s weight. In other words, it is the force that the surface must exert to keep the object from falling through.

The coefficient of kinetic friction is typically represented as μ_k and is usually less than the coefficient of static friction for the same materials.

Friction: Static

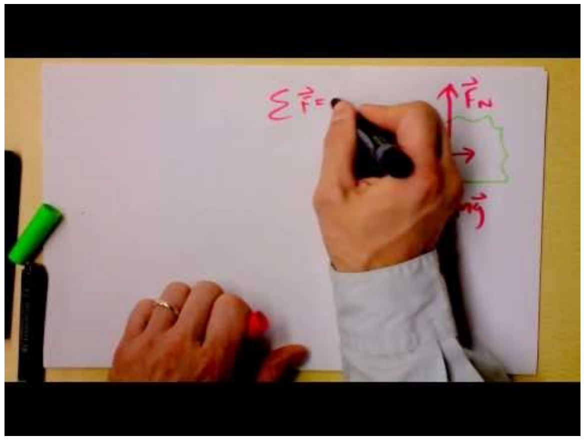
Static friction is a type of friction that occurs to resist motion when two objects are at rest against each other.

learning objectives

- Demonstrate the relationship of maximum force of static friction

Static Friction

Another type of frictional force is static friction, otherwise known as stiction. Like all friction, it acts to resist the motion of an object moving over a surface. Unlike kinetic friction, however, static friction acts to resist the start of motion.



Static Friction and some friction challenges: Here, I talk about sneaky ol' static friction.

Static friction is friction between two objects that are not moving relative to each other. This frictional force is what prevents a parked car from sliding down a hill, for example. Before an object at rest on a surface can move, it must overcome the force of static friction.

Static friction originates from multiple sources. For any given material on another material of the same composition, friction will be greater as the material surfaces become rougher (consider sandpaper) on the macroscopic level. Additionally, intermolecular forces can greatly influence friction when two materials are put into contact. When surface area is below the micrometer range, Van der Waals' forces, electrostatic interactions and hydrogen bonding can cause two materials to adhere to one another. A force is required to overcome these interactions and cause the surfaces to move across one another.

Like kinetic friction, the force of static friction is given by a coefficient multiplied by the normal force. The normal force is the force of the surface pushing up on the object, which is usually equal to the object's weight. The coefficient of static friction is usually greater than the coefficient of kinetic friction and is usually represented by μ_s.

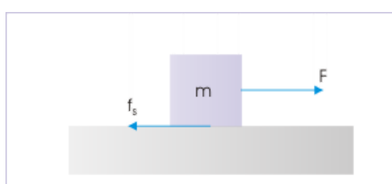
Putting these elements together gives the maximum force of static friction as:

$$\mathrm{F_s} = \mu_\mathrm{s}\mathrm{F_n} \tag{6.7.1}$$

In general, the force of static friction can be represented as:

$$\mathrm{F_s} \leq \mu_\mathrm{s}\mathrm{F_n} \tag{6.7.2}$$

As with all frictional forces, the force of friction can never exceed the force applied. Thus the force of static friction will vary between 0 and ${}_\mathrm{s}\mathrm{F_n}$ depending on the strength of the applied force. Any force smaller than ${}_\mathrm{s}\mathrm{F_n}$ attempting to slide one surface over the other is opposed by a frictional force of equal magnitude and opposite direction. Any force larger than that overcomes the force of static friction and causes sliding to occur. The instant sliding occurs, static friction is no longer applicable—the friction between the two surfaces is then called kinetic friction.



Static Friction: To move a block at rest on a surface, a force must be applied which is great enough to overcome the force of static friction.

Problem-Solving With Friction and Inclines

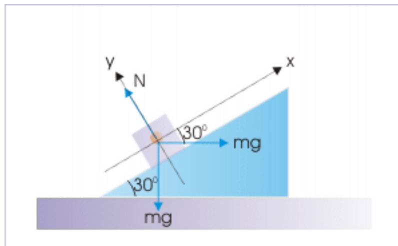
Combining motion on inclines with friction uses such concepts as equilibrium and contact force on an incline.

learning objectives

- Calculate the force of friction on an incline

Contact Force on an Incline

The incline plane has two contact or interface surfaces. One is the incline surface, where the block is placed and the other is the base of the incline, which is in contact with the surface underneath. The motion of the block, therefore, may depend on the motion of the incline itself.



Block and incline system: Forces on the block

When on an incline, calculating the force of friction is different than when the object is on a level surface. Recall that the force of friction depends on both the coefficient of friction and the normal force. $F_f = \mu F_n$ When on an incline with an angle θ, the normal force becomes $F_n = mg\cos(\theta)$

As always, the frictional force resists motion. If the block is being pushed up the incline the friction force points down the incline. If the block is being pulled down the incline, the friction force will hold the block up.

Equilibrium of Forces on an Incline

When not acted on by any other forces, only by gravity and friction, the frictional force will resist the tendency of the block to slide down the incline. If the frictional force is equal to the gravitational force the block will not slide down the incline. The block is said to be in equilibrium since the sum of the forces on it is 0.

Gravitational force down an incline is given by $mg\sin(\theta)$.

Where θ is the angle the incline makes with the horizontal. For the block to be in equilibrium, the maximum force of friction $F_f = \mu mg\cos(\theta)$ must be greater than or equal to $F_G = mg\sin(\theta)$ FG=mgsin(θ). If the maximum frictional force is greater than the force of gravity, the sum of the forces is still 0. The force of friction can never exceed the other forces acting on it. The frictional forces only act to counter motion.

Drag

The drag force is the resistive force felt by objects moving through fluids and is proportional to the square of the object's speed.

learning objectives

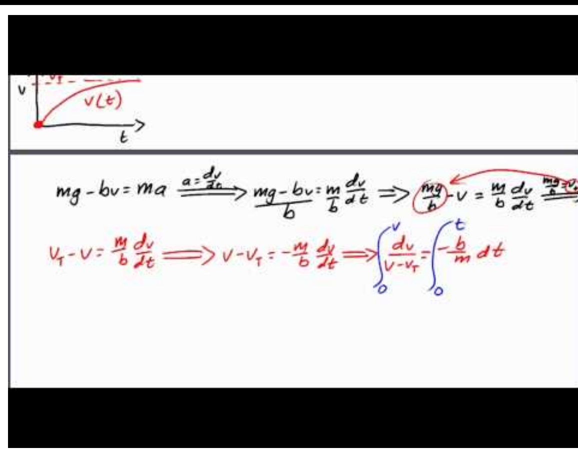
- Relate the magnitude of drag force to the speed of an object

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion.

Like friction, the drag force always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. Aerodynamic objects tend to have small surface areas and be designed to have low drag coefficients.

For most large objects such as bicyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force F_D is found to be proportional to the square of the speed of the object. We can write this relationship mathematically as $FD \propto v^2$. When taking into account other factors, this relationship becomes $F_D = \frac{\frac{1}{2C}}{\text{rhoAv}^2}$, where C is known as the drag coefficient, a unit-less number that represents the aerodynamic properties of the object, A is the cross-sectional area of the object which is facing the direction of motion, and ρ is the density of the fluid the object is moving through.

Aerodynamic Shape: From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed. They are shaped like a bullet with tapered fins. (credit: U.S. Army, via Wikimedia Commons)



Retarding and Drag Forces: A brief look at retarding (drag) forces in physics, for students in introductory physics classes that use calculus. This video walks through a single scenario of an object experiencing a drag force where the drag force is proportional to the object's velocity.

Stress and Strain

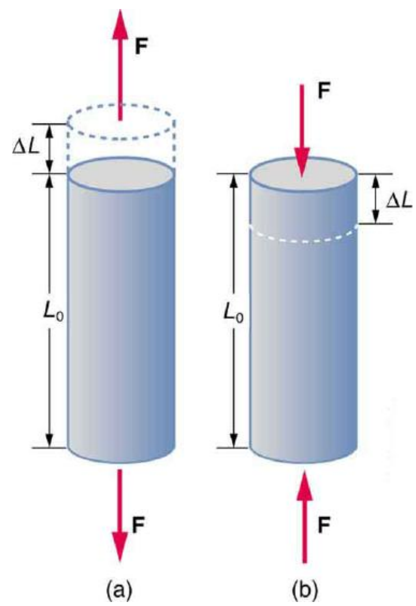
The ratio of force to area $\frac{F}{A}$ is called stress and the ratio of change in length to length $\frac{\Delta L}{L}$ is called the strain.

learning objectives

- Explain how forces affects the shape of an object

We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object’s shape. If a bulldozer pushes a car into a wall, the car will not move past the wall but it will noticeably change shape. A change in shape due to the application of a force is a deformation. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation is proportional to the force—that is, for small deformations, Hooke’s law is obeyed. In equation form, Hooke’s law is given by $F = k \cdot \Delta L$ where ΔL is the change in length and kk is a constant which depends on the material properties of the object.

Deformations come in several types: changes in length (tension and compression), sideways shear (stress), and changes in volume.



Tension/Compression: Tension: The rod is stretched a length ΔL when a force is applied parallel to its length. (b) Compression: The same rod is compressed by forces with the same magnitude in the opposite direction. For very small deformations and uniform materials, ΔL is approximately the same for the same magnitude of tension or compression. For larger deformations, the cross-sectional area changes as the rod is compressed or stretched.

The ratio of force to area $\frac{F}{A}$ is called stress and the ratio of change in length to length $\frac{\Delta L}{L}$ is called the strain.

Stress and strain are related to each other by a constant called Young’s Modulus or the elastic modulus which varies depending on the material. Using Young’s Modulus the relation between stress and strain is given by: $\text{stress} = Y \cdot \text{strain}$.

A material with a high elastic modulus is said to have high tensile strength. Such materials are very resistant to being stretched and require a large amount of force to deform a small amount.

Translational Equilibrium

An object is said to be in equilibrium when there is no external net force acting on it.

learning objectives

- Assess the role each type of equilibrium plays in mechanical devices

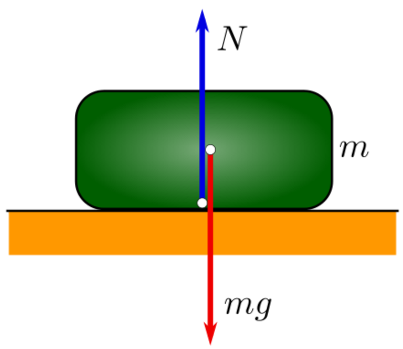
We are surrounded by great engineering architectures and mechanical devices, which are at rest in the frame of reference of Earth. A large part of engineering creations are static objects. Yet we also seek equilibrium of moving objects like that of floating ship, airplane cruising at high speed, and such other moving mechanical devices. In both cases – static or dynamic – net external forces and torques are zero.

A body is said to be in mechanical equilibrium when net external force is equal to zero and net external torque is also zero. Mathematically,

$$\Sigma \overrightarrow{\mathrm{F}}\mathrm{ext} = 0 \text{ and } \Sigma \overrightarrow{\tau}\mathrm{ext} = 0 \tag{6.7.3}$$

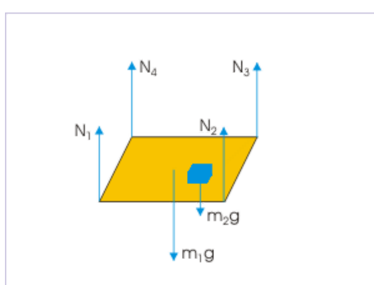
Since there is no net force on the object, the object does not accelerate. This implies two types of possible equilibrium. The first type, where all particles in the system are at rest and do have not velocity, is known as static equilibrium. In the second type, the object has a velocity, but since there are no net forces acting on it, the velocity remains constant. In the second case, the particle is said to be in dynamic equilibrium. Static or dynamic, these kinds of equilibrium can be categorized as translational equilibrium.

Examples of translational equilibrium are all around us. A book resting on a table is pushing down on the table with the force of its weight. The table, in turn, is pushing back on the book, keeping the book from falling through the table. Since neither the table nor the book are moving, this is an example of static equilibrium. The force of gravity on the book is perfectly counteracted by the force of the table pushing on it.



Forces Acting on an Object at Rest: A force diagram showing the forces acting on an object at rest on a surface. Notice that the amount of force that the table is pushing upward on the object (the N vector) is equal to the downward force of the object's weight (shown here as mg, as weight is equal to the object's mass multiplied by the acceleration due to gravity): because these forces are equal, the object is in a state of equilibrium (all the forces acting on it balance to zero).

An example of dynamic (or mechanical) equilibrium is an object sliding down a wedge. The force of gravity pulls the object down the wedge, but it is counteracted by the force of friction between the wedge and the object. If the force of friction is equal to the force of gravity, the object will proceed at a constant velocity.



Forces on a Table: These six forces are in equilibrium. The four forces of the table leg counteract the force of the table and the object pushing on them.

Connected Objects

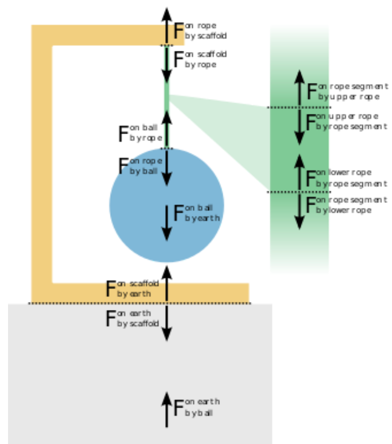
Forces can be transferred from one object to another through connections.

> **learning objectives**
>
> - Analyze the affect a rigid connection has on the movement of objects

The physics of connected objects is very similar to physics of simple objects. There are a variety of ways objects can be connected to each other, and a corresponding variety of mathematical ways to model such connections.

The simplest form of connection is a perfectly rigid connection. If two objects are connected by a perfectly rigid connector then they may be thought of as the same object. Perfectly rigid connectors cannot stretch nor deform, and transfer forces instantaneously from one side of the connection to the other. For example, given two blocks (both of mass 1 kg) connected by a perfectly rigid bar, if the first block is pulled with a force of 1 Newton, then both blocks will accelerate at the same time and the same acceleration. In this case the acceleration is $\frac{1}{2}\mathrm{m/s^2}$ —the same as if a force of mass 2 kg is exerted on one object. Thus it can be said that a perfectly rigid connection makes two objects into one large object. Of course, perfectly rigid connections do not exist in nature. Some deformation will always exist in any object as force travels along it. However, many materials are sufficiently rigid, so that using the perfectly rigid approximation is useful for simplicity's sake.

One can think of the force transferring through the connection by means of the "tension" force. Tension is the pulling force exerted by a string, chain, or similar connector on another object. If two objects are connected by a string, a force exerted on one is balanced by a tension force in the string which pulls on the other. Of course, if the tension force is greater than the rope can withstand, the rope will break.



Tension Forces: The forces involved in supporting a ball by a rope. Tension is the force of the rope on the scaffold, the force of the rope on the ball, and the balanced forces acting on and produced by segments of the rope.

Circular Motion

An object in circular motion undergoes acceleration due to centripetal force in the direction of the center of rotation.

> **learning objectives**
>
> - Develop an understanding of uniform circular motion as an indicator for net external force

Uniform circular motion describes the motion of an object along a circle or a circular arc at constant speed. It is the basic form of rotational motion in the same way that uniform linear motion is the basic form of translational motion. However, the two types of motion are different with respect to the force required to maintain the motion.

Let us consider Newton's first law of motion. It states that an object will maintain a constant velocity unless a net external force is applied. Therefore, uniform linear motion indicates the absence of a net external force. On the other hand, uniform circular motion requires that the velocity vector of an object constantly change direction. Since the velocity vector of the object is changing, an acceleration is occurring. Therefore, uniform *circular* motion indicates the *presence* of a net external force.

In uniform circular motion, the force is always perpendicular to the direction of the velocity. Since the direction of the velocity is continuously changing, the direction of the force must be as well.

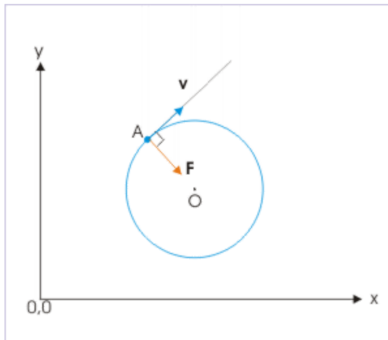
The direction of the velocity along the circular trajectory is tangential. The perpendicular direction to the circular trajectory is, therefore, the radial direction. Therefore, the force (and therefore the acceleration) in uniform direction motion is in the radial direction. For this reason, acceleration in uniform circular motion is recognized to "seek the center" — i.e., centripetal force.

The equation for the acceleration aa required to sustain uniform circular motion is:

$$a = \frac{v^2}{r} \tag{6.7.4}$$

where mm is the mass of the object, v is the velocity of the object, and r is the radius of the circle. Consequently, the net external force F_{net} required to sustain circular motion is:

$$F_{net} = \frac{m \cdot v^2}{r} \tag{6.7.5}$$



Uniform Circular Motion: In uniform circular motion, the centripetal force is perpendicular to the velocity. The centripetal force points toward the center of the circle, keeping the object on the circular track.

Key Points

- Friction is the force that resists relative motion between two surfaces sliding across each other. Friction converts kinetic energy into heat.
- Drag force is the force that resists motion of an object traveling through a fluid such as air or water. Drag force is proportional to the velocity of the object traveling.
- Deformation forces are forces caused by stretching or compressing a material. Some examples would be springs or elastics.
- Kinetic (or dynamic) friction occurs when two objects are moving relative to each other and rub together (like a sled on the ground).
- The force of friction can be represented by an equation $F_{friction} = \mu F_n$ where μ is the coefficient of friction and is a unitless number that represents the strength of the friction of the surface.
- Kinetic friction and static (stationary) friction use two different coefficients for the same material.
- Static friction is a force that acts to resist the start of motion. It is borne of macroscopic inconsistencies in the surfaces of materials in contact as well as intermolecular interactions between the materials, such as hydrogen bonding, Van der Waal's interactions and electrostatic interactions.
- Static friction uses a different, usually higher, coefficient than kinetic friction does.
- The force of static friction is $F_{fs} = \mu_s F_n$. Where μ_s is the coefficient of static friction which varies by material and FnFn is the normal force.
- Motion on an incline is resisted by friction.
- The frictional force on an incline is dependent on the angle of the incline. $F_f = \mu mg \cos(\theta)$ is the maximum friction force on an incline.
- If the friction force is greater than or equal to the forces in the direction of motion, then the net force is 0 and the object is in equilibrium.
- Objects moving through a fluid feel a force which resists their motion. This force is known as the drag force.
- The drag force is proportional to the square of the velocity of the object relative to the fluid.
- The equation for drag is $F_D = \frac{1}{2} C_\rho A v^2$. C is a constant called the drag coefficient. ρρ is the density of the fluid. A is the surface area in the direction of motion.

- The ratio of force to area $\frac{F}{A}$ is called stress and the ratio of change in length to length $\frac{\Delta L}{L}$ is called the strain.
- Stress and strain are related to each other by a constant called Young's Modulus or the elastic modulus which varies depending on the material. Using Young's Modulus the relation between stress and strain is given by: $\text{stress} = Y \cdot \text{strain}$.
- A material with a high elastic modulus is said to have high tensile strength. Such materials are very resistant to being stretched and require a large amount of force to deform a small amount.
- When there is no external net force on an object, the object is said to be in equilibrium.
- When an object is in equilibrium, it does not accelerate. If it had a velocity, the velocity remains constant; if it was at rest, it remains at rest.
- An equilibrium in motion is known as dynamic equilibrium; an equilibrium at rest is a static equilibrium.
- If two objects are connected, a force on one has an effect on the other.
- Connections can often be approximated as completely rigid. In completely rigid cases, the connection does not deform and the force is transferred instantaneously.
- Tension is the force of a rope or cable or other connector on the object it is connected to. It is one way force is transferred between objects.
- An object that is undergoing circular motion has a velocity vector that is constantly changing direction.
- The force that is needed to maintain circular motion points toward the center of the circular path. It is therefore known as the centripetal force.
- The velocity of an object in circular motion is always tangent to the circle, and the centripetal force is always perpendicular to the velocity.

Key Terms

- **kinetic energy**: The energy possessed by an object because of its motion, equal to one half the mass of the body times the square of its velocity.
- **static**: Fixed in place; having no motion.
- **kinetic**: Of or relating to motion
- **friction**: A force that resists the relative motion or tendency to such motion of two bodies in contact.
- **incline**: A slope.
- **equilibrium**: The state of a body at rest or in uniform motion, the resultant of all forces on which is zero.
- **fluid**: Any substance which can flow with relative ease, tends to assume the shape of its container, and obeys Bernoulli's principle; a liquid, gas or plasma.
- **strain**: The amount by which a material deforms under stress or force, given as a ratio of the deformation to the initial dimension of the material and typically symbolized by ϵϵ is termed the engineering strain. The true strain is defined as the natural logarithm of the ratio of the final dimension to the initial dimension.
- **stress**: The internal distribution of force per unit area (pressure) within a body reacting to applied forces which causes strain or deformation and is typically symbolized by σ.
- **dynamic**: Changing; active; in motion.
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **rigid**: Stiff, rather than flexible.
- **tangent**: a straight line touching a curve at a single point without crossing it at that point
- **perpendicular**: at or forming a right angle (to).

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6.8: Prelude to Newton's Laws of Motion

Figure 6.8.1: The Golden Gate Bridge, one of the greatest works of modern engineering, was the longest suspension bridge in the world in the year it opened, 1937. It is still among the 10 longest suspension bridges as of this writing. In designing and building a bridge, what physics must we consider? What forces act on the bridge? What forces keep the bridge from falling? How do the towers, cables, and ground interact to maintain stability?

When you drive across a bridge, you expect it to remain stable. You also expect to speed up or slow your car in response to traffic changes. In both cases, you deal with forces. The forces on the bridge are in equilibrium, so it stays in place. In contrast, the force produced by your car engine causes a change in motion. Isaac Newton discovered the laws of motion that describe these situations.

Forces affect every moment of your life. Your body is held to Earth by force and held together by the forces of charged particles. When you open a door, walk down a street, lift your fork, or touch a baby's face, you are applying forces. Zooming in deeper, your body's atoms are held together by electrical forces, and the core of the atom, called the nucleus, is held together by the strongest force we know—strong nuclear force.

6.9: Forces

Learning Objectives

- Distinguish between kinematics and dynamics
- Understand the definition of force
- Identify simple free-body diagrams
- Define the SI unit of force, the newton
- Describe force as a vector

The study of motion is called **kinematics**, but kinematics only describes the way objects move—their velocity and their acceleration. **Dynamics** is the study of how forces affect the motion of objects and systems. It considers the causes of motion of objects and systems of interest, where a system is anything being analyzed. The foundation of dynamics are the laws of motion stated by Isaac Newton (1642–1727). These laws provide an example of the breadth and simplicity of principles under which nature functions. They are also universal laws in that they apply to situations on Earth and in space.

Newton's laws of motion were just one part of the monumental work that has made him legendary (Figure $6.9.1$). The development of Newton's laws marks the transition from the Renaissance to the modern era. Not until the advent of modern physics was it discovered that Newton's laws produce a good description of motion only when the objects are moving at speeds much less than the speed of light and when those objects are larger than the size of most molecules (about 10^{-9} m in diameter). These constraints define the realm of Newtonian mechanics. At the beginning of the twentieth century, Albert Einstein (1879–1955) developed the theory of relativity and, along with many other scientists, quantum mechanics. Quantum mechanics does not have the constraints present in Newtonian physics. All of the situations we consider in this chapter, and all those preceding the introduction of relativity in Relativity, are in the realm of Newtonian physics.

Figure $6.9.1$: Isaac Newton (1642–1727) published his amazing work, Philosophiae Naturalis Principia Mathematica, in 1687. It proposed scientific laws that still apply today to describe the motion of objects (the laws of motion). Newton also discovered the law of gravity, invented calculus, and made great contributions to the theories of light and color.

Working Definition of Force

Dynamics is the study of the forces that cause objects and systems to move. To understand this, we need a working definition of force. An intuitive definition of **force**—that is, a push or a pull—is a good place to start. We know that a push or a pull has both magnitude and direction (therefore, it is a vector quantity), so we can define force as the push or pull on an object with a specific magnitude and direction. Force can be represented by vectors or expressed as a multiple of a standard force.

The push or pull on an object can vary considerably in either magnitude or direction. For example, a cannon exerts a strong force on a cannonball that is launched into the air. In contrast, Earth exerts only a tiny downward pull on a flea. Our everyday experiences also give us a good idea of how multiple forces add. If two people push in different directions on a third person, as illustrated in Figure $6.9.2$, we might expect the total force to be in the direction shown. Since force is a vector, it adds just like other vectors. Forces, like other vectors, are represented by arrows and can be added using the familiar head-to-tail method or trigonometric methods. These ideas were developed in Vectors.

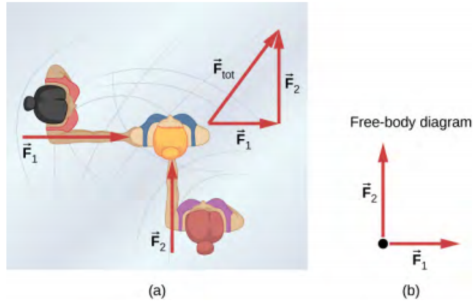


Figure 6.9.2: (a) An overhead view of two ice skaters pushing on a third skater. Forces are vectors and add like other vectors, so the total force on the third skater is in the direction shown. (b) A free-body diagram representing the forces acting on the third skater.

Figure 6.9.2(b) is our first example of a **free-body diagram**, which is a sketch showing all external forces acting on an object or system. The object or system is represented by a single isolated point (or free body), and only those forces acting on it that originate outside of the object or system—that is, **external forces**—are shown. (These forces are the only ones shown because only external forces acting on the free body affect its motion. We can ignore any internal forces within the body.) The forces are represented by vectors extending outward from the free body.

Free-body diagrams are useful in analyzing forces acting on an object or system, and are employed extensively in the study and application of Newton's laws of motion. You will see them throughout this text and in all your studies of physics. The following steps briefly explain how a free-body diagram is created; we examine this strategy in more detail in Drawing Free-Body Diagrams.

? Problem-Solving Strategy: Drawing Free-Body Diagrams

1. Draw the object under consideration. If you are treating the object as a particle, represent the object as a point. Place this point at the origin of an xy-coordinate system.
2. Include all forces that act on the object, representing these forces as vectors. However, do not include the net force on the object or the forces that the object exerts on its environment.
3. Resolve all force vectors into x- and y-components.
4. Draw a separate free-body diagram for each object in the problem.

We illustrate this strategy with two examples of free-body diagrams (Figure 6.9.3). The terms used in this figure are explained in more detail later in the chapter.

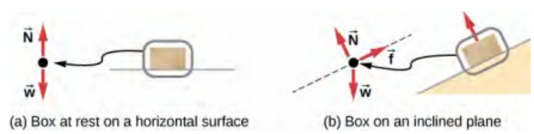


Figure 6.9.3: In these free-body diagrams, $\vec{N}$ is the normal force, $\vec{w}$ is the weight of the object, and $\vec{f}$ is the friction.

The steps given here are sufficient to guide you in this important problem-solving strategy. The final section of this chapter explains in more detail how to draw free-body diagrams when working with the ideas presented in this chapter.

Development of the Force Concept

A quantitative definition of force can be based on some standard force, just as distance is measured in units relative to a standard length. One possibility is to stretch a spring a certain fixed distance (Figure 6.9.4) and use the force it exerts to pull itself back to its relaxed shape—called a **restoring force**—as a standard. The magnitude of all other forces can be considered as multiples of this standard unit of force. Many other possibilities exist for standard forces. Some alternative definitions of force will be given later in this chapter.

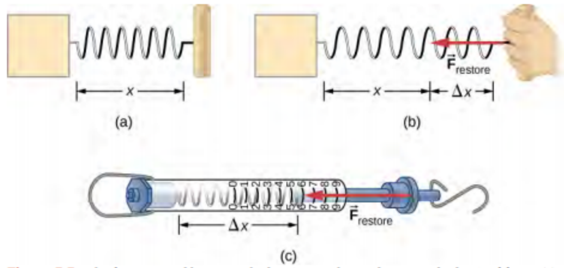


Figure 6.9.4: The force exerted by a stretched spring can be used as a standard unit of force. (a) This spring has a length x when undistorted. (b) When stretched a distance Δx , the spring exerts a restoring force $\vec{F}$ restore, which is reproducible. (c) A spring scale is one device that uses a spring to measure force. The force $\vec{F}$ restore is exerted on whatever is attached to the hook. Here, this force has a magnitude of six units of the force standard being employed.

Let's analyze force more deeply. Suppose a physics student sits at a table, working diligently on his homework (Figure 6.9.5). What external forces act on him? Can we determine the origin of these forces?

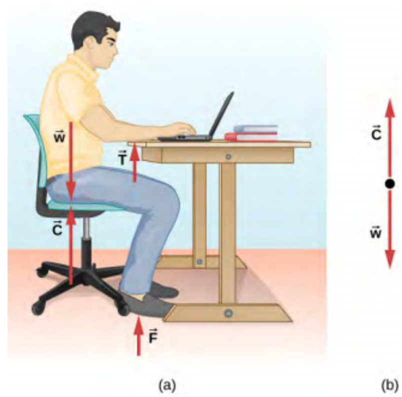


Figure 6.9.5 : (a) The forces acting on the student are due to the chair, the table, the floor, and Earth's gravitational attraction. (b) In solving a problem involving the student, we may want to consider the forces acting along the line running through his torso. A free-body diagram for this situation is shown.

In most situations, forces are grouped into two categories: **contact forces** and **field forces**. As you might guess, contact forces are due to direct physical contact between objects. For example, the student in Figure 6.9.5 experiences the contact forces $\vec{C}$, $\vec{F}$, and $\vec{T}$, which are exerted by the chair on his posterior, the floor on his feet, and the table on his forearms, respectively. Field forces, however, act without the necessity of physical contact between objects. They depend on the presence of a "field" in the region of space surrounding the body under consideration. Since the student is in Earth's gravitational field, he feels a gravitational force $\vec{w}$; in other words, he has weight.

You can think of a field as a property of space that is detectable by the forces it exerts. Scientists think there are only four fundamental force fields in nature. These are the gravitational, electromagnetic, strong nuclear, and weak fields (we consider these four forces in nature later in this text). As noted for $\vec{w}$ in Figure 6.9.5, the gravitational field is responsible for the weight of a body. The forces of the electromagnetic field include those of static electricity and magnetism; they are also responsible for the attraction among atoms in bulk matter. Both the strong nuclear and the weak force fields are effective only over distances roughly equal to a length of scale no larger than an atomic nucleus (10^{-15} m). Their range is so small that neither field has influence in the macroscopic world of Newtonian mechanics.

Contact forces are fundamentally electromagnetic. While the elbow of the student in Figure 6.9.5 is in contact with the tabletop, the atomic charges in his skin interact electromagnetically with the charges in the surface of the table. The net (total) result is the force $\vec{T}$. Similarly, when adhesive tape sticks to a piece of paper, the atoms of the tape are intermingled with those of the paper to cause a net electromagnetic force between the two objects. However, in the context of Newtonian mechanics, the electromagnetic origin of contact forces is not an important concern.

Vector Notation for Force

As previously discussed, force is a vector; it has both magnitude and direction. The SI unit of force is called the **newton** (abbreviated N), and 1 N is the force needed to accelerate an object with a mass of 1 kg at a rate of 1 m/s^2: 1 N = 1 kg • m/s^2. An easy way to remember the size of a newton is to imagine holding a small apple; it has a weight of about 1 N.

We can thus describe a two-dimensional force in the form $\vec{F}$ = a $\hat{i}$ + b $\hat{j}$ (the unit vectors $\hat{i}$ and $\hat{j}$ indicate the direction of these forces along the x-axis and the y-axis, respectively) and a three-dimensional force in the form $\vec{F}$ = a $\hat{i}$ + b $\hat{j}$ + c $\hat{k}$. In Figure 6.9.2, let's suppose that ice skater 1, on the left side of the figure, pushes horizontally with a force of 30.0 N to the right; we represent this as $\vec{F}_1$ = 30.0 $\hat{i}$ N. Similarly, if ice skater 2 pushes with a force of 40.0 N in the positive vertical direction shown, we would write $\vec{F}_2$ = 40.0 $\hat{j}$ N. The resultant of the two forces causes a mass to accelerate—in this case, the third ice skater. This resultant is called the **net external force** $\vec{F}_{net}$ and is found by taking the vector sum of all external forces acting on an object or system (thus, we can also represent net external force as $\sum \vec{F}$):

$$\vec{F}_{net} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \dots \tag{6.9.1}$$

This equation can be extended to any number of forces.

In this example, we have $\vec{F}_{net}$ = $\sum \vec{F}$ = $\vec{F}_1$ + $\vec{F}_2$ = 30.0 $\hat{i}$ + 40.0 $\hat{j}$. The hypotenuse of the triangle shown in Figure 6.9.2 is the resultant force, or net force. It is a vector. To find its magnitude (the size of the vector, without regard to direction), we use the rule given in Vectors, taking the square root of the sum of the squares of the components:

$$\vec{F}_{net} = \sqrt{(30.0\ N)^2 + (40.0\ N)^2} = 50.0\ N. \tag{6.9.2}$$

The direction is given by

$$\theta = \tan^{-1}\left(\frac{F_2}{F_1}\right) = \tan^{-1}\left(\frac{40.0}{30.0}\right) = 53.1^o, \tag{6.9.3}$$

measured from the positive x-axis, as shown in the free-body diagram in Figure 6.9.2(b).

Let's suppose the ice skaters now push the third ice skater with $\vec{F}_1$ = 3.0 $\hat{i}$ + 8.0 $\hat{j}$ N and $\vec{F}_2$ = 5.0 $\hat{i}$ + 4.0 $\hat{j}$ N. What is the resultant of these two forces? We must recognize that force is a vector; therefore, we must add using the rules for vector addition:

$$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2 = \left(3.0\hat{i} + 8.0\hat{j}\right) + \left(5.0\hat{i} + 4.0\hat{j}\right) = 8.0\hat{i} + 12\hat{j}\ N \tag{6.9.4}$$

Exercise 5.1

Find the magnitude and direction of the net force in the ice skater example just given.

Simulation

View this interactive simulation to learn how to add vectors. Drag vectors onto a graph, change their length and angle, and sum them together. The magnitude, angle, and components of each vector can be displayed in several formats.

6.10: Newton's First Law

Learning Objectives

- Describe Newton's first law of motion
- Recognize friction as an external force
- Define inertia
- Identify inertial reference frames
- Calculate equilibrium for a system

Experience suggests that an object at rest remains at rest if left alone and that an object in motion tends to slow down and stop unless some effort is made to keep it moving. However, Newton's first law gives a deeper explanation of this observation.

Newton's First Law of Motion

A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force.

Note the repeated use of the verb "remains." We can think of this law as preserving the status quo of motion. Also note the expression "constant velocity;" this means that the object maintains a path along a straight line, since neither the magnitude nor the direction of the velocity vector changes. We can use Figure 6.10.1 to consider the two parts of Newton's first law.

Figure 6.10.1: (a) A hockey puck is shown at rest; it remains at rest until an outside force such as a hockey stick changes its state of rest; (b) a hockey puck is shown in motion; it continues in motion in a straight line until an outside force causes it to change its state of motion. Although it is slick, an ice surface provides some friction that slows the puck.

Rather than contradicting our experience, Newton's first law says that there must be a cause for any change in velocity (a change in either magnitude or direction) to occur. This cause is a net external force, which we defined earlier in the chapter. An object sliding across a table or floor slows down due to the net force of friction acting on the object. If friction disappears, will the object still slow down?

The idea of cause and effect is crucial in accurately describing what happens in various situations. For example, consider what happens to an object sliding along a rough horizontal surface. The object quickly grinds to a halt. If we spray the surface with talcum powder to make the surface smoother, the object slides farther. If we make the surface even smoother by rubbing lubricating oil on it, the object slides farther yet. Extrapolating to a frictionless surface and ignoring air resistance, we can imagine the object sliding in a straight line indefinitely. Friction is thus the cause of slowing (consistent with Newton's first law). The object would not slow down if friction were eliminated.

Consider an air hockey table (Figure 6.10.2). When the air is turned off, the puck slides only a short distance before friction slows it to a stop. However, when the air is turned on, it creates a nearly frictionless surface, and the puck glides long distances without slowing down. Additionally, if we know enough about the friction, we can accurately predict how quickly the object slows down.

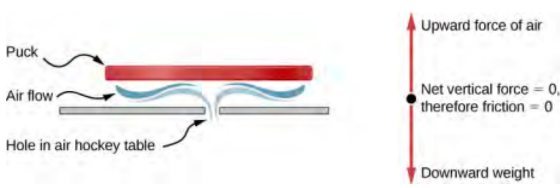


Figure 6.10.2: An air hockey table is useful in illustrating Newton's laws. When the air is off, friction quickly slows the puck; but when the air is on, it minimizes contact between the puck and the hockey table, and the puck glides far down the table.

Newton's first law is general and can be applied to anything from an object sliding on a table to a satellite in orbit to blood pumped from the heart. Experiments have verified that any change in velocity (speed or direction) must be caused by an external force. The idea of **generally applicable or universal laws** is important—it is a basic feature of all laws of physics. Identifying these laws is like recognizing patterns in nature from which further patterns can be discovered. The genius of Galileo, who first developed the idea for the first law of motion, and Newton, who clarified it, was to ask the fundamental question: "What is the cause?" Thinking in terms of cause and effect is fundamentally different from the typical ancient Greek approach, when questions such as "Why does a tiger have stripes?" would have been answered in Aristotelian fashion, such as "That is the nature of the beast." The ability to think in terms of cause and effect is the ability to make a connection between an observed behavior and the surrounding world.

Gravitation and Inertia

Regardless of the scale of an object, whether a molecule or a subatomic particle, two properties remain valid and thus of interest to physics: gravitation and inertia. Both are connected to mass. Roughly speaking, **mass** is a measure of the amount of matter in something. **Gravitation** is the attraction of one mass to another, such as the attraction between yourself and Earth that holds your feet to the floor. The magnitude of this attraction is your weight, and it is a force.

Mass is also related to **inertia**, the ability of an object to resist changes in its motion—in other words, to resist acceleration. Newton's first law is often called the **law of inertia**. As we know from experience, some objects have more inertia than others. It is more difficult to change the motion of a large boulder than that of a basketball, for example, because the boulder has more mass than the basketball. In other words, the inertia of an object is measured by its mass. The relationship between mass and weight is explored later in this chapter.

Inertial Reference Frames

Earlier, we stated Newton's first law as "A body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force." It can also be stated as "Every body remains in its state of uniform motion in a straight line unless it is compelled to change that state by forces acting on it." To Newton, "uniform motion in a straight line" meant constant velocity, which includes the case of zero velocity, or rest. Therefore, the first law says that the velocity of an object remains constant if the net force on it is zero.

Newton's first law is usually considered to be a statement about reference frames. It provides a method for identifying a special type of reference frame: the **inertial reference frame**. In principle, we can make the net force on a body zero. If its velocity relative to a given frame is constant, then that frame is said to be inertial. So by definition, an inertial reference frame is a reference frame in which Newton's first law is valid. Newton's first law applies to objects with constant velocity. From this fact, we can infer the following statement.

Inertial Reference Frame

A reference frame moving at constant velocity relative to an inertial frame is also inertial. A reference frame accelerating relative to an inertial frame is not inertial.

Are inertial frames common in nature? It turns out that well within experimental error, a reference frame at rest relative to the most distant, or "fixed," stars is inertial. All frames moving uniformly with respect to this fixed-star frame are also inertial. For example, a nonrotating reference frame attached to the Sun is, for all practical purposes, inertial, because its velocity relative to the fixed stars does not vary by more than one part in 10^{10}. Earth accelerates relative to the fixed stars because it rotates on its axis and revolves around the Sun; hence, a reference frame attached to its surface is not inertial. For most problems, however, such a frame serves as a sufficiently accurate approximation to an inertial frame, because the acceleration of a point on Earth's surface relative to the fixed stars is rather small ($< 3.4 \times 10^{-2}\ m/s^2$). Thus, unless indicated otherwise, we consider reference frames fixed on Earth to be inertial.

Finally, no particular inertial frame is more special than any other. As far as the laws of nature are concerned, all inertial frames are equivalent. In analyzing a problem, we choose one inertial frame over another simply on the basis of convenience.

Newton's First Law and Equilibrium

Newton's first law tells us about the equilibrium of a system, which is the state in which the forces on the system are balanced. Returning to Forces and the ice skaters in Figure 5.2.2, we know that the forces $\vec{F}_1$ and $\vec{F}_2$ combine to form a resultant force, or the net external force: $\vec{F}_R = \vec{F}_{net} = \vec{F}_1 + \vec{F}_2$. To create equilibrium, we require a balancing force that will produce a net force of zero. This force must be equal in magnitude but opposite in direction to $\vec{F}_R$, which means the vector must be $-\vec{F}_R$. Referring to the ice skaters, for which we found $\vec{F}_R$ to be 30.0 $\hat{i}$ + 40.0 $\hat{j}$ N, we can determine the balancing force by simply finding $-\vec{F}_R$ = −30.0 $\hat{i}$ − 40.0 $\hat{j}$ N. See the free-body diagram in Figure 5.2.2b.

We can give Newton's first law in vector form:

$$\vec{v} = constant\ when\ \vec{F}_{net} = \vec{0}\ N. \tag{6.10.1}$$

This equation says that a net force of zero implies that the velocity $\vec{v}$ of the object is constant. (The word "constant" can indicate zero velocity.)

Newton's first law is deceptively simple. If a car is at rest, the only forces acting on the car are weight and the contact force of the pavement pushing up on the car (Figure 6.10.3). It is easy to understand that a nonzero net force is required to change the state of motion of the car. However, if the car is in motion with constant velocity, a common misconception is that the engine force propelling the car forward is larger in magnitude than the friction force that opposes forward motion. In fact, the two forces have identical magnitude.

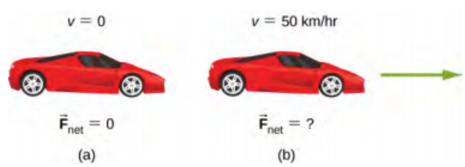


Figure 6.10.3: A car is shown (a) parked and (b) moving at constant velocity. How do Newton's laws apply to the parked car? What does the knowledge that the car is moving at constant velocity tell us about the net horizontal force on the car?

Example 5.1: When Does Newton's First Law Apply to Your Car?

Newton's laws can be applied to all physical processes involving force and motion, including something as mundane as driving a car.

a. Your car is parked outside your house. Does Newton's first law apply in this situation? Why or why not?
b. Your car moves at constant velocity down the street. Does Newton's first law apply in this situation? Why or why not?

Strategy

In (a), we are considering the first part of Newton's first law, dealing with a body at rest; in (b), we look at the second part of Newton's first law for a body in motion.

Solution

a. When your car is parked, all forces on the car must be balanced; the vector sum is 0 N. Thus, the net force is zero, and Newton's first law applies. The acceleration of the car is zero, and in this case, the velocity is also zero.
b. When your car is moving at constant velocity down the street, the net force must also be zero according to Newton's first law. The car's engine produces a forward force; friction, a force between the road and the tires of the car that opposes forward motion, has exactly the same magnitude as the engine force, producing the net force of zero. The body continues in its state of constant velocity until the net force becomes nonzero. Realize that **a net force of zero means that an object is either at rest or moving with constant velocity, that is, it is not accelerating**. What do you suppose happens when the car accelerates? We explore this idea in the next section.

Significance

As this example shows, there are two kinds of equilibrium. In (a), the car is at rest; we say it is in **static equilibrium**. In (b), the forces on the car are balanced, but the car is moving; we say that it is in **dynamic equilibrium**. (We examine this idea in more detail in Static Equilibrium and Elasticity.) Again, it is possible for two (or more) forces to act on an object yet for the object to move. In addition, a net force of zero cannot produce acceleration.

? Exercise 5.2

A skydiver opens his parachute, and shortly thereafter, he is moving at constant velocity. (a) What forces are acting on him? (b) Which force is bigger?

Simulation

Engage in this simulation to predict, qualitatively, how an external force will affect the speed and direction of an object's motion. Explain the effects with the help of a free-body diagram. Use free-body diagrams to draw position, velocity, acceleration, and force graphs, and vice versa. Explain how the graphs relate to one another. Given a scenario or a graph, sketch all four graphs.

6.11: Newton's Second Law

Learning Objectives

- Distinguish between external and internal forces
- Describe Newton's second law of motion
- Explain the dependence of acceleration on net force and mass

Newton’s second law is closely related to his first law. It mathematically gives the cause-and-effect relationship between force and changes in motion. Newton’s second law is quantitative and is used extensively to calculate what happens in situations involving a force. Before we can write down Newton’s second law as a simple equation that gives the exact relationship of force, mass, and acceleration, we need to sharpen some ideas we mentioned earlier.

Force and Acceleration

First, what do we mean by a change in motion? The answer is that a change in motion is equivalent to a change in velocity. A change in velocity means, by definition, that there is acceleration. Newton’s first law says that a net external force causes a change in motion; thus, we see that a **net external force causes nonzero acceleration**.

We defined external force in Forces as force acting on an object or system that originates outside of the object or system. Let’s consider this concept further. An intuitive notion of **external** is correct—it is outside the system of interest. For example, in Figure $6.11.1a$, the system of interest is the car plus the person within it. The two forces exerted by the two students are external forces. In contrast, an internal force acts between elements of the system. Thus, the force the person in the car exerts to hang on to the steering wheel is an internal force between elements of the system of interest. Only external forces affect the motion of a system, according to Newton’s first law. (The internal forces cancel each other out, as explained in the next section.) Therefore, we must define the boundaries of the system before we can determine which forces are external. Sometimes, the system is obvious, whereas at other times, identifying the boundaries of a system is more subtle. The concept of a system is fundamental to many areas of physics, as is the correct application of Newton’s laws. This concept is revisited many times in the study of physics.

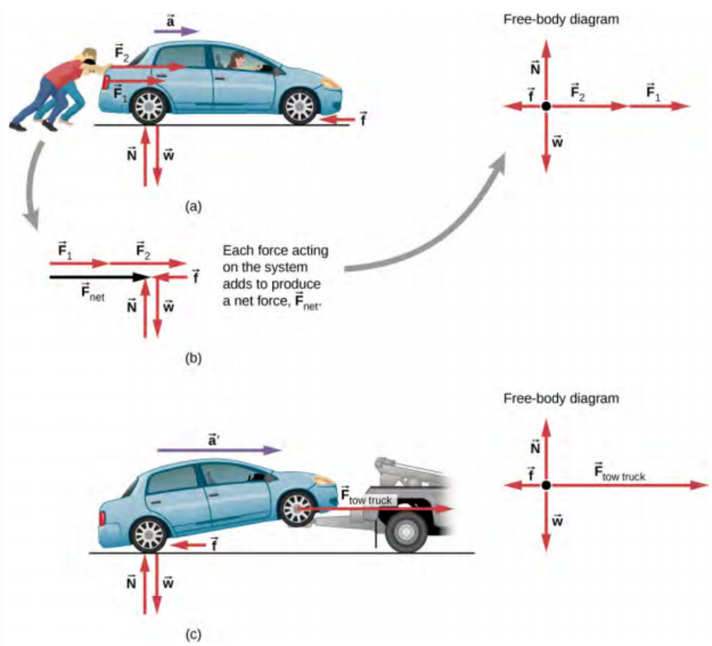


Figure 6.11.1: Different forces exerted on the same mass produce different accelerations. (a) Two students push a stalled car. All external forces acting on the car are shown. (b) The forces acting on the car are transferred to a coordinate plane (free-body diagram) for simpler analysis. (c) The tow truck can produce greater external force on the same mass, and thus greater acceleration.

From this example, you can see that different forces exerted on the same mass produce different accelerations. In Figure $6.11.1a$, the two students push a car with a driver in it. Arrows representing all external forces are shown. The system of interest is the car and its driver. The weight $\vec{w}$ of the system and the support of the ground $\vec{N}$ are also shown for completeness and are assumed to cancel (because there was no vertical motion and no imbalance of forces in the vertical direction to create a change in motion). The vector $\vec{f}$ represents the friction acting on the car, and it acts to the left, opposing the motion of the car. (We discuss friction in more detail in the next chapter.) In

Figure 6.11.1b all external forces acting on the system add together to produce the net force $\vec{F}_{net}$. The free-body diagram shows all of the forces acting on the system of interest. The dot represents the center of mass of the system. Each force vector extends from this dot. Because there are two forces acting to the right, the vectors are shown collinearly. Finally, in Figure 6.11.1c, a larger net external force produces a larger acceleration ($\vec{a}' > \vec{a}$) when the tow truck pulls the car.

It seems reasonable that acceleration would be directly proportional to and in the same direction as the net external force acting on a system. This assumption has been verified experimentally and is illustrated in Figure 6.11.1. To obtain an equation for Newton's second law, we first write the relationship of acceleration $\vec{a}$ and net external force $\vec{F}_{net}$ as the proportionality

$$\vec{a} \propto \vec{F}_{net} \tag{6.11.1}$$

where the symbol α means "proportional to." (Recall from Forces that the net external force is the vector sum of all external forces and is sometimes indicated as $\sum \vec{F}$.) This proportionality shows what we have said in words—acceleration is directly proportional to net external force. Once the system of interest is chosen, identify the external forces and ignore the internal ones. It is a tremendous simplification to disregard the numerous internal forces acting between objects within the system, such as muscular forces within the students' bodies, let alone the myriad forces between the atoms in the objects. Still, this simplification helps us solve some complex problems.

It also seems reasonable that acceleration should be inversely proportional to the mass of the system. In other words, the larger the mass (the inertia), the smaller the acceleration produced by a given force. As illustrated in Figure 6.11.2, the same net external force applied to a basketball produces a much smaller acceleration when it is applied to an SUV. The proportionality is written as

$$a \propto \frac{1}{m}, \tag{6.11.2}$$

where m is the mass of the system and a is the magnitude of the acceleration. Experiments have shown that acceleration is exactly inversely proportional to mass, just as it is directly proportional to net external force.

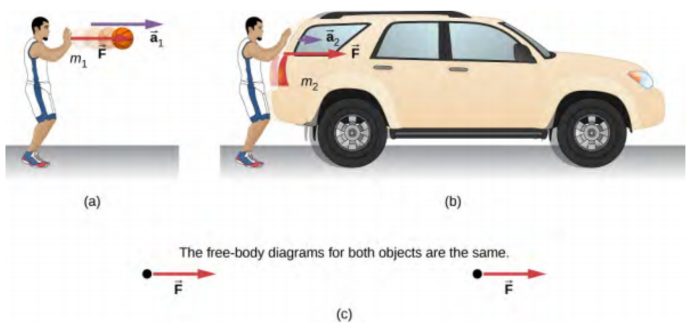


Figure 6.11.2: The same force exerted on systems of different masses produces different accelerations. (a) A basketball player pushes on a basketball to make a pass. (Ignore the effect of gravity on the ball.) (b) The same player exerts an identical force on a stalled SUV and produces far less acceleration. (c) The free-body diagrams are identical, permitting direct comparison of the two situations. A series of patterns for free-body diagrams will emerge as you do more problems and learn how to draw them in Drawing Free-Body Diagrams.

It has been found that the acceleration of an object depends only on the net external force and the mass of the object. Combining the two proportionalities just given yields **Newton's second law**.

Newton's Second Law of Motion

The acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system and is inversely proportion to its mass. In equation form, Newton's second law is

$$\vec{a} = \frac{\vec{F}_{net}}{m}, \tag{6.11.3}$$

where $\vec{a}$ is the acceleration, $\vec{F}_{net}$ is the net force, and m is the mass. This is often written in the more familiar form

$$\vec{F}_{net} = \sum \vec{F} = m\vec{a}, \tag{6.11.4}$$

but the first equation gives more insight into what Newton's second law means. When only the magnitude of force and acceleration are considered, this equation can be written in the simpler scalar form:

$$\vec{F}_{net} = ma. \tag{6.11.5}$$

The law is a cause-and-effect relationship among three quantities that is not simply based on their definitions. The validity of the second law is based on experimental verification. The free-body diagram, which you will learn to draw in Drawing Free-Body Diagrams, is the basis for writing Newton’s second law.

✓ Example 5.2: What Acceleration Can a Person Produce When Pushing a Lawn Mower?

Suppose that the net external force (push minus friction) exerted on a lawn mower is 51 N (about 11 lb.) parallel to the ground (Figure 6.11.3). The mass of the mower is 24 kg. What is its acceleration?

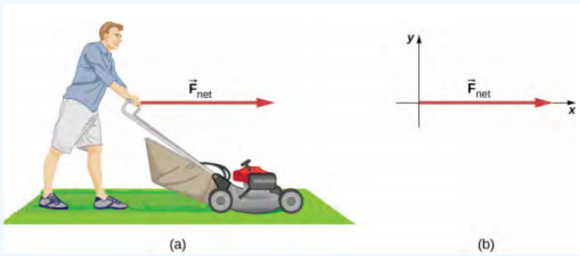


Figure 6.11.3: (a) The net force on a lawn mower is 51 N to the right. At what rate does the lawn mower accelerate to the right? (b) The free-body diagram for this problem is shown.

Strategy

This problem involves only motion in the horizontal direction; we are also given the net force, indicated by the single vector, but we can suppress the vector nature and concentrate on applying Newton’s second law. Since F_{net} and m are given, the acceleration can be calculated directly from Newton’s second law as F_{net} = ma.

Solution

The magnitude of the acceleration a is a = $\frac{F_{net}}{m}$. Entering known values gives

$$a = \frac{51\ N}{24\ kg}. \tag{6.11.6}$$

Substituting the unit of kilograms times meters per square second for newtons yields

$$a = \frac{51\ kg \cdot m/s^2}{24\ kg} = 2.1\ m/s^2. \tag{6.11.7}$$

Significance

The direction of the acceleration is the same direction as that of the net force, which is parallel to the ground. This is a result of the vector relationship expressed in Newton’s second law, that is, the vector representing net force is the scalar multiple of the acceleration vector. There is no information given in this example about the individual external forces acting on the system, but we can say something about their relative magnitudes. For example, the force exerted by the person pushing the mower must be greater than the friction opposing the motion (since we know the mower moved forward), and the vertical forces must cancel because no acceleration occurs in the vertical direction (the mower is moving only horizontally). The acceleration found is small enough to be reasonable for a person pushing a mower. Such an effort would not last too long, because the person’s top speed would soon be reached.

? Exercise 5.3

At the time of its launch, the HMS Titanic was the most massive mobile object ever built, with a mass of 6.0 x 10^7 kg. If a force of 6 MN (6 x 10^6 N) was applied to the ship, what acceleration would it experience?

In the preceding example, we dealt with net force only for simplicity. However, several forces act on the lawn mower. The weight $\vec{w}$ (discussed in detail in Mass and Weight) pulls down on the mower, toward the center of Earth; this produces a contact force on the ground. The ground must exert an upward force on the lawn mower, known as the normal force $\vec{N}$, which we define in Common Forces. These forces are balanced and therefore do not produce vertical acceleration. In the next example, we show both of these forces. As you continue to solve problems using Newton’s second law, be sure to show multiple forces.

Example 5.3: Which Force Is Bigger?

a. The car shown in Figure 6.11.4 is moving at a constant speed. Which force is bigger, $\vec{F}_{engine}$ or $\vec{F}_{friction}$? Explain.

b. The same car is now accelerating to the right. Which force is bigger, $\vec{F}_{engine}$ or $\vec{F}_{friction}$? Explain.

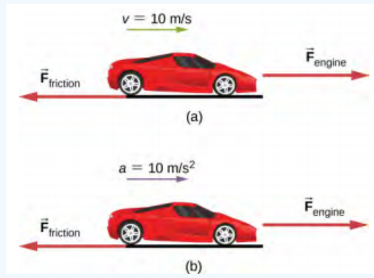


Figure 6.11.4: A car is shown (a) moving at constant speed and (b) accelerating. How do the forces acting on the car compare in each case? (a) What does the knowledge that the car is moving at constant velocity tell us about the net horizontal force on the car compared to the friction force? (b) What does the knowledge that the car is accelerating tell us about the horizontal force on the car compared to the friction force?

Strategy

We must consider Newton's first and second laws to analyze the situation. We need to decide which law applies; this, in turn, will tell us about the relationship between the forces.

Solution

a. The forces are equal. According to Newton's first law, if the net force is zero, the velocity is constant.

b. In this case, $\vec{F}_{engine}$ must be larger than $\vec{F}_{friction}$. According to Newton's second law, a net force is required to cause acceleration.

Significance

These questions may seem trivial, but they are commonly answered incorrectly. For a car or any other object to move, it must be accelerated from rest to the desired speed; this requires that the engine force be greater than the friction force. Once the car is moving at constant velocity, the net force must be zero; otherwise, the car will accelerate (gain speed). To solve problems involving Newton's laws, we must understand whether to apply Newton's first law (where $\sum \vec{F} = \vec{0}$) or Newton's second law (where $\sum \vec{F}$ is not zero). This will be apparent as you see more examples and attempt to solve problems on your own.

Example 5.4: What Rocket Thrust Accelerates This Sled?

Before manned space flights, rocket sleds were used to test aircraft, missile equipment, and physiological effects on human subjects at high speeds. They consisted of a platform that was mounted on one or two rails and propelled by several rockets.

Calculate the magnitude of force exerted by each rocket, called its thrust T, for the four-rocket propulsion system shown in Figure 6.11.5. The sled's initial acceleration is 49 m/s^2, the mass of the system is 2100 kg, and the force of friction opposing the motion is 650 N.

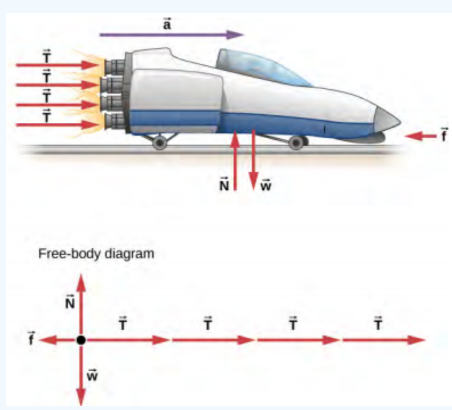


Figure 6.11.5: A sled experiences a rocket thrust that accelerates it to the right. Each rocket creates an identical thrust T. The system here is the sled, its rockets, and its rider, so none of the forces between these objects are considered. The arrow representing friction ($\vec{f}$) is drawn larger than scale.

Strategy

Although forces are acting both vertically and horizontally, we assume the vertical forces cancel because there is no vertical acceleration. This leaves us with only horizontal forces and a simpler one-dimensional problem. Directions are indicated with plus or minus signs, with right taken as the positive direction. See the free-body diagram in Figure $6.11.5$.

Solution

Since acceleration, mass, and the force of friction are given, we start with Newton's second law and look for ways to find the thrust of the engines. We have defined the direction of the force and acceleration as acting "to the right," so we need to consider only the magnitudes of these quantities in the calculations. Hence we begin with

$$F_{net} = ma \tag{6.11.8}$$

where F_{net} is the net force along the horizontal direction. We can see from the figure that the engine thrusts add, whereas friction opposes the thrust. In equation form, the net external force is

$$F_{net} = 4T - f. \tag{6.11.9}$$

Substituting this into Newton's second law gives us

$$F_{net} = ma = 4T - f. \tag{6.11.10}$$

Using a little algebra, we solve for the total thrust 4T:

$$4T = ma + f. \tag{6.11.11}$$

Substituting known values yields

$$4T = ma + f = (2100\ kg)(49\ m/s^2) + 650\ N. \tag{6.11.12}$$

Therefore, the total thrust is

$$4T = 1.0 \times 10^5\ N. \tag{6.11.13}$$

Significance

The numbers are quite large, so the result might surprise you. Experiments such as this were performed in the early 1960s to test the limits of human endurance, and the setup was designed to protect human subjects in jet fighter emergency ejections. Speeds of 1000 km/h were obtained, with accelerations of 45 g's. (Recall that g, acceleration due to gravity, is 9.80 m/s^2. When we say that acceleration is 45 g's, it is 45 x 9.8 m/s^2, which is approximately 440 m/s^2.) Although living subjects are not used anymore, land speeds of 10,000 km/h have been obtained with a rocket sled.

In this example, as in the preceding one, the system of interest is obvious. We see in later examples that choosing the system of interest is crucial—and the choice is not always obvious.

Newton's second law is more than a definition; it is a relationship among acceleration, force, and mass. It can help us make predictions. Each of those physical quantities can be defined independently, so the second law tells us something basic and universal about nature.

? Exercise 5.4

A 550-kg sports car collides with a 2200-kg truck, and during the collision, the net force on each vehicle is the force exerted by the other. If the magnitude of the truck's acceleration is 10 m/s^2, what is the magnitude of the sports car's acceleration?

Component Form of Newton's Second Law

We have developed Newton's second law and presented it as a vector equation in Equation 6.11.4. This vector equation can be written as three component equations:

$$\sum \vec{F}_x = m\vec{a}_x, \sum \vec{F}_y = m\vec{a}_y, \sum \vec{F}_z = m\vec{a}_z. \tag{6.11.14}$$

The second law is a description of how a body responds mechanically to its environment. The influence of the environment is the net force $\vec{F}_{net}$, the body's response is the acceleration $\vec{a}$, and the strength of the response is inversely proportional to the mass m. The larger the mass of an object, the smaller its response (its acceleration) to the influence of the environment (a given net force). Therefore, a body's mass is a measure of its inertia, as we explained in Newton's First Law.

Example 5.5: Force on a Soccer Ball

A 0.400-kg soccer ball is kicked across the field by a player; it undergoes acceleration given by $\vec{a} = 3.00\ \hat{i} + 7.00\ \hat{j}\ m/s^2$. Find (a) the resultant force acting on the ball and (b) the magnitude and direction of the resultant force.

Strategy

The vectors in $\hat{i}$ and $\hat{j}$ format, which indicate force direction along the x-axis and the y-axis, respectively, are involved, so we apply Newton's second law in vector form.

Solution

a. We apply Newton's second law:

$$\vec{F}_{net} = m\vec{a} = (0.400\ kg)\left(3.00\ \hat{i} + 7.00\ \hat{j}\ m/s^2\right) = 1.20\ \hat{i} + 2.80\ \hat{j}\ N. \tag{6.11.15}$$

b. . Magnitude and direction are found using the components of $\vec{F}_{net}$:

$$F_{net} = \sqrt{(1.20\ N)^2 + (2.80\ N)^2} = 3.05\ N\ and\ \theta = \tan^{-1}\left(\frac{2.80}{1.20}\right) = 66.8^o. \tag{6.11.16}$$

Significance

We must remember that Newton's second law is a vector equation. In (a), we are multiplying a vector by a scalar to determine the net force in vector form. While the vector form gives a compact representation of the force vector, it does not tell us how "big" it is, or where it goes, in intuitive terms. In (b), we are determining the actual size (magnitude) of this force and the direction in which it travels.

Example 5.6: Mass of a Car

Find the mass of a car if a net force of $-600.0\ \hat{j}$ N produces an acceleration of $-0.2\ \hat{j}\ m/s^2$.

Strategy

Vector division is not defined, so $m = \frac{\vec{F}_{net}}{\vec{a}}$ cannot be performed. However, mass m is a scalar, so we can use the scalar form of Newton's second law, $m = \frac{F_{net}}{a}$.

Solution

We use $m = \frac{F_{net}}{a}$ and substitute the magnitudes of the two vectors: $F_{net} = 600.0$ N and $a = 0.2\ m/s^2$. Therefore,

$$m = \frac{F_{net}}{a} = \frac{600.0\ N}{0.2\ m/s^2} = 3000\ kg.$$

Significance

Force and acceleration were given in the $\hat{i}$ and $\hat{j}$ format, but the answer, mass m, is a scalar and thus is not given in $\hat{i}$ and $\hat{j}$ form.

Example 5.7

Several Forces on a Particle A particle of mass m = 4.0 kg is acted upon by four forces of magnitudes. $F_1 = 10.0$ N, $F_2 = 40.0$ N, $F_3 = 5.0$ N, and $F_4 = 2.0$ N, with the directions as shown in the free-body diagram in Figure 6.11.6. What is the acceleration of the particle?

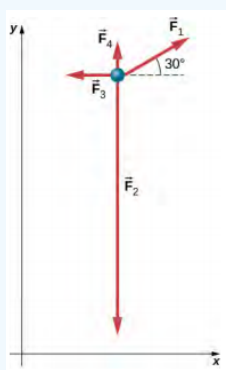


Figure 6.11.6: Four forces in the xy-plane are applied to a 4.0-kg particle.

Strategy

Because this is a two-dimensional problem, we must use a free-body diagram. First, $\vec{F}_1$ must be resolved into x- and y-components. We can then apply the second law in each direction.

Solution

We draw a free-body diagram as shown in Figure 6.11.6. Now we apply Newton's second law. We consider all vectors resolved into x- and y-components:

$$\sum F_x = ma_x \tag{6.11.17}$$

$$F_{1x} - F_{3x} = ma_x \tag{6.11.18}$$

$$F_1 \cos 30^o - F_{3x} = ma_x \tag{6.11.19}$$

$$(10.0\ N)(\cos 30^o) - 5.0\ N = (4.0\ kg)a_x \tag{6.11.20}$$

$$a_x = 0.92\ m/s^2. \tag{6.11.21}$$

$$\sum F_y = ma_y \tag{6.11.22}$$

$$F_{1y} + F_{4y} - F_{2y} = ma_y \tag{6.11.23}$$

$$F_1 \sin 30^o + F_{4y} - F_{2y} = ma_y \tag{6.11.24}$$

$$(10.0\ N)(\sin 30^o) + 2.0\ N - 40.0\ N = (4.0\ kg)a_y \qquad (6.11.$$

$$a_y = -8.3\ m/s^2. \tag{6.11.26}$$

Thus, the net acceleration is

$$\vec{a} = \left(0.92\ \hat{i} - 8.3\ \hat{j}\right) m/s^2, \tag{6.11.27}$$

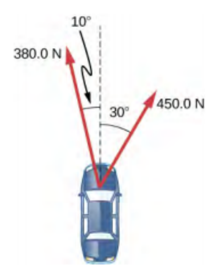
which is a vector of magnitude 8.4 m/s^2 directed at 276° to the positive x-axis.

Significance

Numerous examples in everyday life can be found that involve three or more forces acting on a single object, such as cables running from the Golden Gate Bridge or a football player being tackled by three defenders. We can see that the solution of this example is just an extension of what we have already done.

? Exercise 5.5

A car has forces acting on it, as shown below. The mass of the car is 1000.0 kg. The road is slick, so friction can be ignored. (a) What is the net force on the car? (b) What is the acceleration of the car?



Newton's Second Law and Momentum

Newton actually stated his second law in terms of momentum: "The instantaneous rate at which a body's momentum changes is equal to the net force acting on the body." ("Instantaneous rate" implies that the derivative is involved.) This can be given by the vector equation

$$\vec{F}_{net} = \frac{d\vec{p}}{dt}. \tag{6.11.28}$$

This means that Newton's second law addresses the central question of motion: What causes a change in motion of an object? Momentum was described by Newton as "quantity of motion," a way of combining both the velocity of an object and its mass. We devote Linear Momentum and Collisions to the study of momentum.

For now, it is sufficient to define momentum $\vec{p}$ as the product of the mass of the object m and its velocity $\vec{v}$:

$$\vec{p} = m\vec{v}. \tag{6.11.29}$$

Since velocity is a vector, so is momentum.

It is easy to visualize momentum. A train moving at 10 m/s has more momentum than one that moves at 2 m/s. In everyday life, we speak of one sports team as "having momentum" when they score points against the opposing team.

If we substitute Equation 6.11.29into Equation 6.11.28, we obtain

$$\vec{F}_{net} = \frac{d\vec{p}}{dt} = \frac{d(m\vec{v})}{dt}. \tag{6.11.30}$$

When m is constant, we have

$$\vec{F}_{net} = m\frac{d(\vec{v})}{dt} = m\vec{a}. \tag{6.11.31}$$

Thus, we see that the momentum form of Newton's second law reduces to the form given earlier in this section.

Simulation

Explore the forces at work when pulling a cart or pushing a refrigerator, crate, or person. Create an applied force and see how it makes objects move. Put an object on a ramp and see how it affects its motion.

6.12: Mass and Weight

Learning Objectives

- Explain the difference between mass and weight
- Explain why falling objects on Earth are never truly in free fall
- Describe the concept of weightlessness

Mass and weight are often used interchangeably in everyday conversation. For example, our medical records often show our weight in kilograms but never in the correct units of newtons. In physics, however, there is an important distinction. Weight is the pull of Earth on an object. It depends on the distance from the center of Earth. Unlike weight, mass does not vary with location. The mass of an object is the same on Earth, in orbit, or on the surface of the Moon.

Units of Force

The equation F_{net} = ma is used to define net force in terms of mass, length, and time. As explained earlier, the SI unit of force is the newton. Since F_{net} = ma,

$$1\ N = 1\ kg \cdot m/s^2.$$

Although almost the entire world uses the newton for the unit of force, in the United States, the most familiar unit of force is the pound (lb), where 1 N = 0.225 lb. Thus, a 225-lb person weighs 1000 N.

Weight and Gravitational Force

When an object is dropped, it accelerates toward the center of Earth. Newton's second law says that a net force on an object is responsible for its acceleration. If air resistance is negligible, the net force on a falling object is the gravitational force, commonly called its **weight** $\vec{w}$, or its force due to gravity acting on an object of mass m. Weight can be denoted as a vector because it has a direction; **down** is, by definition, the direction of gravity, and hence, weight is a downward force. The magnitude of weight is denoted as w. Galileo was instrumental in showing that, in the absence of air resistance, all objects fall with the same acceleration g. Using Galileo's result and Newton's second law, we can derive an equation for weight.

Consider an object with mass m falling toward Earth. It experiences only the downward force of gravity, which is the weight $\vec{w}$. Newton's second law says that the magnitude of the net external force on an object is $\vec{F}_{net} = m\vec{a}$. We know that the acceleration of an object due to gravity is $\vec{g}$, or $\vec{a} = \vec{g}$. Substituting these into Newton's second law gives us the following equations.

Defintion: Weight

The gravitational force on a mass is its weight. We can write this in vector form, where $\vec{w}$ is weight and m is mass, as

$$\vec{w} = m\vec{g}. \tag{6.12.1}$$

In scalar form, we can write

$$w = mg. \tag{6.12.2}$$

Since g = 9.80 m/s^2 on Earth, the weight of a 1.00-kg object on Earth is 9.80 N:

$$w = mg = (1.00\ kg)(9.80 m/s^2) = 9.80\ N. \tag{6.12.3}$$

When the net external force on an object is its weight, we say that it is in **free fall**, that is, the only force acting on the object is gravity. However, when objects on Earth fall downward, they are never truly in free fall because there is always some upward resistance force from the air acting on the object.

Acceleration due to gravity g varies slightly over the surface of Earth, so the weight of an object depends on its location and is not an intrinsic property of the object. Weight varies dramatically if we leave Earth's surface. On the Moon, for example, acceleration due to gravity is only 1.67 m/s^2. A 1.0-kg mass thus has a weight of 9.8 N on Earth and only about 1.7 N on the Moon.

The broadest definition of weight in this sense is that the weight of an object is the gravitational force on it from the nearest large body, such as Earth, the Moon, or the Sun. This is the most common and useful definition of weight in physics. It differs dramatically, however, from the definition of weight used by NASA and the popular media in relation to space travel and exploration. When they speak of “weightlessness” and “microgravity,” they are referring to the phenomenon we call “free fall” in physics. We use the preceding definition of weight, force $\vec{w}$ due to gravity acting on an object of mass m, and we make careful distinctions between free fall and actual weightlessness.

Be aware that weight and mass are different physical quantities, although they are closely related. Mass is an intrinsic property of an object: It is a quantity of matter. The quantity or amount of matter of an object is determined by the numbers of atoms and molecules of various types it contains. Because these numbers do not vary, in Newtonian physics, mass does not vary; therefore, its response to an applied force does not vary. In contrast, weight is the gravitational force acting on an object, so it does vary depending on gravity. For example, a person closer to the center of Earth, at a low elevation such as New Orleans, weighs slightly more than a person who is located in the higher elevation of Denver, even though they may have the same mass.

It is tempting to equate mass to weight, because most of our examples take place on Earth, where the weight of an object varies only a little with the location of the object. In addition, it is difficult to count and identify all of the atoms and molecules in an object, so mass is rarely determined in this manner. If we consider situations in which $\vec{g}$ is a constant on Earth, we see that weight $\vec{w}$ is directly proportional to mass m, since $\vec{w} = m\vec{g}$, that is, the more massive an object is, the more it weighs. Operationally, the masses of objects are determined by comparison with the standard kilogram, as we discussed in Units and Measurement. But by comparing an object on Earth with one on the Moon, we can easily see a variation in weight but not in mass. For instance, on Earth, a 5.0-kg object weighs 49 N; on the Moon, where g is 1.67 m/s^2, the object weighs 8.4 N. However, the mass of the object is still 5.0 kg on the Moon.

Example 6.12.1: Clearing a Field

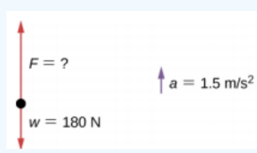
A farmer is lifting some moderately heavy rocks from a field to plant crops. He lifts a stone that weighs 40.0 lb. (about 180 N). What force does he apply if the stone accelerates at a rate of 1.5 m/s^2?

Strategy

We were given the weight of the stone, which we use in finding the net force on the stone. However, we also need to know its mass to apply Newton’s second law, so we must apply the equation for weight, w = mg, to determine the mass.

Solution

No forces act in the horizontal direction, so we can concentrate on vertical forces, as shown in the following free-body diagram. We label the acceleration to the side; technically, it is not part of the free-body diagram, but it helps to remind us that the object accelerates upward (so the net force is upward).



$$w = mg$$

$$m = \frac{w}{g} = \frac{180\ N}{9.8\ m/s^2} = 18\ kg$$

$$\sum F = ma$$

$$F - w = ma$$

$$F - 180\ N = (18\ kg)(1.5\ m/s^2)$$

$$F - 180\ N = 27\ N$$

$$F = 207\ N = 210\ N \text{ to two significant figures}$$

Significance

To apply Newton's second law as the primary equation in solving a problem, we sometimes have to rely on other equations, such as the one for weight or one of the kinematic equations, to complete the solution.

Exercise $6.12.1$

For $6.12.1$, find the acceleration when the farmer's applied force is 230.0 N

Simulation

Can you avoid the boulder field and land safely just before your fuel runs out, as Neil Armstrong did in 1969? This version of the classic video game accurately simulates the real motion of the lunar lander, with the correct mass, thrust, fuel consumption rate, and lunar gravity. The real lunar lander is hard to control.

6.13: Newton's Third Law

Learning Objectives

- State Newton's third law of motion
- Identify the action and reaction forces in different situations
- Apply Newton's third law to define systems and solve problems of motion

We have thus far considered force as a push or a pull; however, if you think about it, you realize that no push or pull ever occurs by itself. When you push on a wall, the wall pushes back on you. This brings us to Newton's third law.

Newton's Third Law of Motion

Whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts. Mathematically, if a body A exerts a force $\vec{F}$ on body B, then B simultaneously exerts a force $-\vec{F}$ on A, or in vector equation form,

$$\vec{F}_{AB} = -\vec{F}_{BA}. \tag{6.13.1}$$

Newton's third law represents a certain symmetry in nature: Forces always occur in pairs, and one body cannot exert a force on another without experiencing a force itself. We sometimes refer to this law loosely as "action-reaction," where the force exerted is the action and the force experienced as a consequence is the reaction. Newton's third law has practical uses in analyzing the origin of forces and understanding which forces are external to a system.

We can readily see Newton's third law at work by taking a look at how people move about. Consider a swimmer pushing off the side of a pool (Figure $6.13.1$). She pushes against the wall of the pool with her feet and accelerates in the direction opposite that of her push. The wall has exerted an equal and opposite force on the swimmer. You might think that two equal and opposite forces would cancel, but they do not **because they act on different systems**. In this case, there are two systems that we could investigate: the swimmer and the wall. If we select the swimmer to be the system of interest, as in the figure, then $F_{\text{wall on feet}}$ is an external force on this system and affects its motion. The swimmer moves in the direction of this force. In contrast, the force $F_{\text{feet on wall}}$ acts on the wall, not on our system of interest. Thus, $F_{\text{feet on wall}}$ does not directly affect the motion of the system and does not cancel $F_{\text{wall on feet}}$. The swimmer pushes in the direction opposite that in which she wishes to move. The reaction to her push is thus in the desired direction. In a free-body diagram, such as the one shown in Figure $6.13.1$, we never include both forces of an action-reaction pair; in this case, we only use $F_{\text{wall on feet}}$, not $F_{\text{feet on wall}}$.

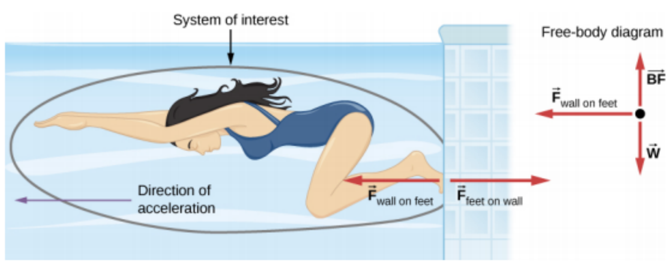


Figure $6.13.1$: When the swimmer exerts a force on the wall, she accelerates in the opposite direction; in other words, the net external force on her is in the direction opposite of $F_{\text{feet on wall}}$. This opposition occurs because, in accordance with Newton's third law, the wall exerts a force $F_{\text{wall on feet}}$ on the swimmer that is equal in magnitude but in the direction opposite to the one she exerts on it. The line around the swimmer indicates the system of interest. Thus, the free-body diagram shows only $F_{\text{wall on feet}}$, w (the gravitational force), and BF, which is the buoyant force of the water supporting the swimmer's weight. The vertical forces w and BF cancel because there is no vertical acceleration.

Other examples of Newton's third law are easy to find:

- As a professor paces in front of a whiteboard, he exerts a force backward on the floor. The floor exerts a reaction force forward on the professor that causes him to accelerate forward.
- A car accelerates forward because the ground pushes forward on the drive wheels, in reaction to the drive wheels pushing backward on the ground. You can see evidence of the wheels pushing backward when tires spin on a gravel road and throw the rocks backward.

- Rockets move forward by expelling gas backward at high velocity. This means the rocket exerts a large backward force on the gas in the rocket combustion chamber; therefore, the gas exerts a large reaction force forward on the rocket. This reaction force, which pushes a body forward in response to a backward force, is called **thrust**. It is a common misconception that rockets propel themselves by pushing on the ground or on the air behind them. They actually work better in a vacuum, where they can more readily expel the exhaust gases.
- Helicopters create lift by pushing air down, thereby experiencing an upward reaction force.
- Birds and airplanes also fly by exerting force on the air in a direction opposite that of whatever force they need. For example, the wings of a bird force air downward and backward to get lift and move forward.
- An octopus propels itself in the water by ejecting water through a funnel from its body, similar to a jet ski.
- When a person pulls down on a vertical rope, the rope pulls up on the person (Figure $6.13.2$).

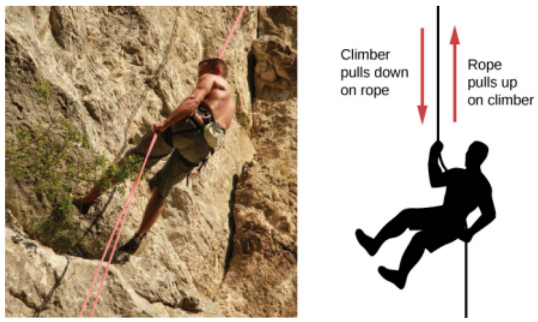


Figure $6.13.2$: When the mountain climber pulls down on the rope, the rope pulls up on the mountain climber.

There are two important features of Newton’s third law. First, the forces exerted (the action and reaction) are always equal in magnitude but opposite in direction. Second, these forces are acting on different bodies or systems: A’s force acts on B and B’s force acts on A. In other words, the two forces are distinct forces that do not act on the same body. Thus, they do not cancel each other.

For the situation shown in Figure 5.2.5, the third law indicates that because the chair is pushing upward on the boy with force $\vec{C}$, he is pushing downward on the chair with force $-\vec{C}$. Similarly, he is pushing downward with forces $-\vec{F}$ and $-\vec{T}$ on the floor and table, respectively. Finally, since Earth pulls downward on the boy with force $\vec{w}$, he pulls upward on Earth with force $-\vec{w}$. If that student were to angrily pound the table in frustration, he would quickly learn the painful lesson (avoidable by studying Newton’s laws) that the table hits back just as hard.

A person who is walking or running applies Newton’s third law instinctively. For example, the runner in Figure $6.13.3$ pushes backward on the ground so that it pushes him forward.

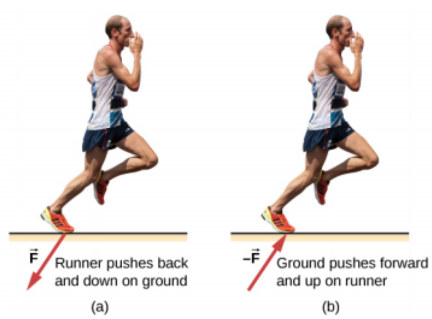


Figure $6.13.3$: The runner experiences Newton’s third law. (a) A force is exerted by the runner on the ground. (b) The reaction force of the ground on the runner pushes him forward.

Example 5.9: Forces on a Stationary Object

The package in Figure $6.13.4$ is sitting on a scale. The forces on the package are $\vec{S}$, which is due to the scale, and $-\vec{w}$, which is due to Earth’s gravitational field. The reaction forces that the package exerts are $-\vec{S}$ on the scale and $\vec{w}$ on Earth. Because the package is not accelerating, application of the second law yields

$$\vec{S}-\vec{w}=m\vec{a}=\vec{0}, \tag{6.13.2}$$

so

$$\vec{S} = \vec{w}. \tag{6.13.3}$$

Thus, the scale reading gives the magnitude of the package's weight. However, the scale does not measure the weight of the package; it measures the force $-\vec{S}$ on its surface. If the system is accelerating, $\vec{S}$ and $-\vec{w}$ would not be equal, as explained in Applications of Newton's Laws.

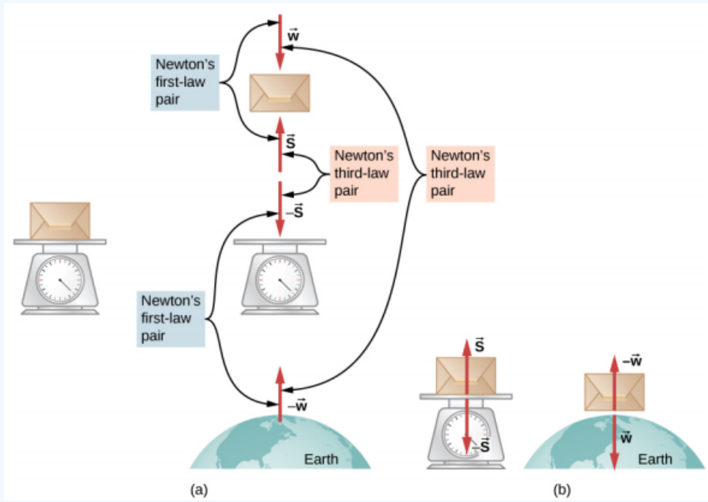


Figure 6.13.4: (a) The forces on a package sitting on a scale, along with their reaction forces. The force $\vec{w}$ is the weight of the package (the force due to Earth's gravity) and $\vec{S}$ is the force of the scale on the package. (b) Isolation of the package-scale system and the package-Earth system makes the action and reaction pairs clear.

✔ Example 5.10: Getting Up to Speed: Choosing the Correct System

A physics professor pushes a cart of demonstration equipment to a lecture hall (Figure 6.13.5). Her mass is 65.0 kg, the cart's mass is 12.0 kg, and the equipment's mass is 7.0 kg. Calculate the acceleration produced when the professor exerts a backward force of 150 N on the floor. All forces opposing the motion, such as friction on the cart's wheels and air resistance, total 24.0 N.

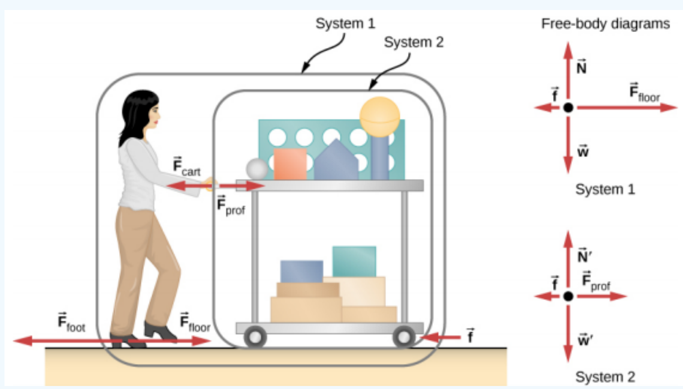


Figure 6.13.5: A professor pushes the cart with her demonstration equipment. The lengths of the arrows are proportional to the magnitudes of the forces (except for $\vec{f}$, because it is too small to drawn to scale). System 1 is appropriate for this example, because it asks for the acceleration of the entire group of objects. Only $\vec{F}_{floor}$ and $\vec{f}$ are external forces acting on System 1 along the line of motion. All other forces either cancel or act on the outside world. System 2 is chosen for the next example so that $\vec{F}_{prof}$ is an external force and enters into Newton's second law. The free-body diagrams, which serve as the basis for Newton's second law, vary with the system chosen.

Strategy

Since they accelerate as a unit, we define the system to be the professor, cart, and equipment. This is System 1 in Figure 6.13.5. The professor pushes backward with a force F_{foot} of 150 N. According to Newton's third law, the floor exerts a forward reaction force F_{floor} of 150 N on System 1. Because all motion is horizontal, we can assume there is no net force in the vertical direction. Therefore, the problem is one-dimensional along the horizontal direction. As noted, friction f opposes the motion and is thus in the opposite direction of F_{floor}. We do not include the forces F_{prof} or F_{cart} because these are internal forces, and we do not include F_{foot} because it acts on the floor, not on the system. There are no other significant forces acting on System 1. If the net external force can be found from all this information, we can use Newton's second law to find the acceleration as requested. See the free-body diagram in the figure.

Solution

Newton's second law is given by

$$a=\frac{F_{net}}{m}. \tag{6.13.4}$$

The net external force on System 1 is deduced from Figure $6.13.5$ and the preceding discussion to be

$$F_{net}=F_{floor}-f=150\ N-24.0\ N=126\ N. \tag{6.13.5}$$

The mass of System 1 is

$$m=(65.0+12.0+7.0)\ kg=84\ kg. \tag{6.13.6}$$

These values of F_{net} and m produce an acceleration of

$$a=\frac{F_{net}}{m}=\frac{126\ N}{84\ kg}=1.5\ m/s^2. \tag{6.13.7}$$

Significance

None of the forces between components of System 1, such as between the professor's hands and the cart, contribute to the net external force because they are internal to System 1. Another way to look at this is that forces between components of a system cancel because they are equal in magnitude and opposite in direction. For example, the force exerted by the professor on the cart results in an equal and opposite force back on the professor. In this case, both forces act on the same system and therefore cancel. Thus, internal forces (between components of a system) cancel. Choosing System 1 was crucial to solving this problem.

✔ Example 5.11: Force on the Cart: Choosing a New System

Calculate the force the professor exerts on the cart in Figure $6.13.5$, using data from the previous example if needed.

Strategy

If we define the system of interest as the cart plus the equipment (System 2 in Figure $6.13.5$), then the net external force on System 2 is the force the professor exerts on the cart minus friction. The force she exerts on the cart, F_{prof}, is an external force acting on System 2. F_{prof} was internal to System 1, but it is external to System 2 and thus enters Newton's second law for this system.

Solution

Newton's second law can be used to find F_{prof}. We start with

$$a=\frac{F_{net}}{m}. \tag{6.13.8}$$

The magnitude of the net external force on System 2 is

$$F_{net}=F_{prof}-f. \tag{6.13.9}$$

We solve for F_{prof}, the desired quantity:

$$F_{prof}=F_{net}+f. \tag{6.13.10}$$

The value of f is given, so we must calculate net F_{net}. That can be done because both the acceleration and the mass of System 2 are known. Using Newton's second law, we see that

$$F_{net}=ma, \tag{6.13.11}$$

where the mass of System 2 is 19.0 kg (m = 12.0 kg + 7.0 kg) and its acceleration was found to be a = 1.5 m/s^2 in the previous example. Thus,

$$F_{net} = ma = (19.0\ kg)(1.5\ m/s^2) = 29\ N. \tag{6.13.12}$$

Now we can find the desired force:

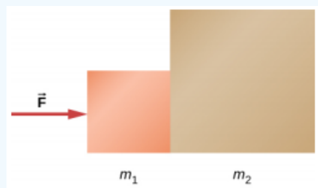
$$F_{prof} = F_{net} + f = 29\ N + 24.0\ N = 53\ N. \tag{6.13.13}$$

Significance

This force is significantly less than the 150-N force the professor exerted backward on the floor. Not all of that 150-N force is transmitted to the cart; some of it accelerates the professor. The choice of a system is an important analytical step both in solving problems and in thoroughly understanding the physics of the situation (which are not necessarily the same things).

Exercise 5.7

Two blocks are at rest and in contact on a frictionless surface as shown below, with m_1 = 2.0 kg, m_2 = 6.0 kg, and applied force 24 N. (a) Find the acceleration of the system of blocks. (b) Suppose that the blocks are later separated. What force will give the second block, with the mass of 6.0 kg, the same acceleration as the system of blocks?



Note

View this video to watch examples of action and reaction. View this video to watch examples of Newton’s laws and internal and external forces.

6.14: Common Forces

Learning Objectives

- Define normal and tension forces
- Distinguish between real and fictitious forces
- Apply Newton's laws of motion to solve problems involving a variety of forces

Forces are given many names, such as push, pull, thrust, and weight. Traditionally, forces have been grouped into several categories and given names relating to their source, how they are transmitted, or their effects. Several of these categories are discussed in this section, together with some interesting applications. Further examples of forces are discussed later in this text.

A Catalog of Forces: Normal, Tension, and Other Examples of Forces

A catalog of forces will be useful for reference as we solve various problems involving force and motion. These forces include normal force, tension, friction, and spring force.

Normal force

Weight (also called the force of gravity) is a pervasive force that acts at all times and must be counteracted to keep an object from falling. You must support the weight of a heavy object by pushing up on it when you hold it stationary, as illustrated in Figure 6.14.1(a). But how do inanimate objects like a table support the weight of a mass placed on them, such as shown in Figure 6.14.1(b)? When the bag of dog food is placed on the table, the table sags slightly under the load. This would be noticeable if the load were placed on a card table, but even a sturdy oak table deforms when a force is applied to it. Unless an object is deformed beyond its limit, it will exert a restoring force much like a deformed spring (or a trampoline or diving board). The greater the deformation, the greater the restoring force. Thus, when the load is placed on the table, the table sags until the restoring force becomes as large as the weight of the load. At this point, the net external force on the load is zero. That is the situation when the load is stationary on the table. The table sags quickly and the sag is slight, so we do not notice it. But it is similar to the sagging of a trampoline when you climb onto it.

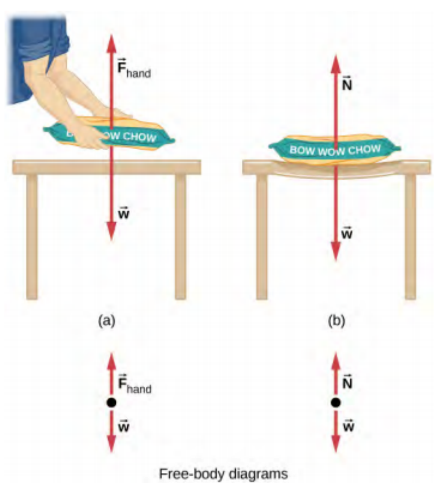


Figure 6.14.1: (a) The person holding the bag of dog food must supply an upward force $\vec{F}$ hand equal in magnitude and opposite in direction to the weight of the food $\vec{w}$ so that it doesn't drop to the ground. (b) The card table sags when the dog food is placed on it, much like a stiff trampoline. Elastic restoring forces in the table grow as it sags until they supply a force $\vec{N}$ equal in magnitude and opposite in direction to the weight of the load.

We must conclude that whatever supports a load, be it animate or not, must supply an upward force equal to the weight of the load, as we assumed in a few of the previous examples. If the force supporting the weight of an object, or a load, is perpendicular to the surface of contact between the load and its support, this force is defined as a **normal force** and here is given by the symbol $\vec{N}$. (This is not the newton unit for force, or N.) The word **normal** means perpendicular to a surface. This means that the normal force experienced by an object resting on a horizontal surface can be expressed in vector form as follows:

$$\vec{N} = -m\vec{g}.$$

In scalar form, this becomes

$$N = mg.$$

The normal force can be less than the object's weight if the object is on an incline.

✔ Example 5.12: Weight on an Incline

Consider the skier on the slope in Figure 6.14.2. Her mass including equipment is 60.0 kg. (a) What is her acceleration if friction is negligible? (b) What is her acceleration if friction is 45.0 N?

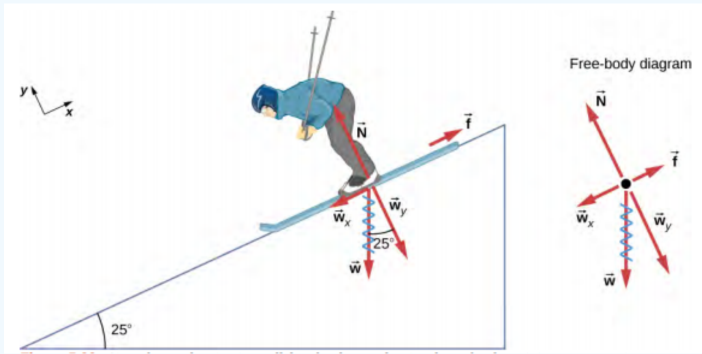


Figure 6.14.2: Since the acceleration is parallel to the slope and acting down the slope, it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular to it (axes shown to the left of the skier). $\vec{N}$ is perpendicular to the slope and $\vec{f}$ is parallel to the slope, but $\vec{w}$ has components along both axes, namely, w_y and w_x. Here, $\vec{w}$ has a squiggly line to show that it has been replaced by these components. The force $\vec{N}$ is equal in magnitude to w_y, so there is no acceleration perpendicular to the slope, but f is less than w_x, so there is a downslope acceleration (along the axis parallel to the slope).

Strategy

This is a two-dimensional problem, since not all forces on the skier (the system of interest) are parallel. The approach we have used in two-dimensional kinematics also works well here. Choose a convenient coordinate system and project the vectors onto its axes, creating two one-dimensional problems to solve. The most convenient coordinate system for motion on an incline is one that has one coordinate parallel to the slope and one perpendicular to the slope. (Motions along mutually perpendicular axes are independent.) We use x and y for the parallel and perpendicular directions, respectively. This choice of axes simplifies this type of problem, because there is no motion perpendicular to the slope and the acceleration is downslope. Regarding the forces, friction is drawn in opposition to motion (friction always opposes forward motion) and is always parallel to the slope, w_x is drawn parallel to the slope and downslope (it causes the motion of the skier down the slope), and w_y is drawn as the component of weight perpendicular to the slope. Then, we can consider the separate problems of forces parallel to the slope and forces perpendicular to the slope.

Solution

The magnitude of the component of weight parallel to the slope is

$$w_x = w \sin 25^o = mg \sin 25^o,$$

and the magnitude of the component of the weight perpendicular to the slope is

$$w_y = w \cos 25^o = mg \cos 25^o.$$

a. Neglect friction. Since the acceleration is parallel to the slope, we need only consider forces parallel to the slope. (Forces perpendicular to the slope add to zero, since there is no acceleration in that direction.) The forces parallel to the slope are the component of the skier's weight parallel to slope w_x and friction f. Using Newton's second law, with subscripts to denote quantities parallel to the slope,

$$a_x = \frac{F_{net\ x}}{m} \tag{6.14.1}$$

where $F_{net\ x} = w_x$ - mg sin 25°, assuming no friction for this part. Therefore,

$$a_x = \frac{F_{net\ x}}{m} = \frac{mg \sin 25^o}{m} = g \sin 25^o \tag{6.14.2}$$

$$(9.80\ m/s^2)(0.4226) = 4.14\ m/s^2 \tag{6.14.3}$$

is the acceleration.

b. Include friction. We have a given value for friction, and we know its direction is parallel to the slope and it opposes motion between surfaces in contact. So the net external force is

$$F_{net\ x} = w_x - f. \tag{6.14.4}$$

Substituting this into Newton's second law, $a_x = \frac{F_{net\ x}}{m}$, gives

$$a_x = \frac{F_{net\ x}}{m} = \frac{w_x - f}{m} = \frac{mg \sin 25^o - f}{m}. \tag{6.14.5}$$

We substitute known values to obtain

$$a_x = \frac{(60.0\ kg)(9.80\ m/s^2)(0.4226) - 45.0\ N}{60.0\ kg}. \tag{6.14.6}$$

This give us

$$a_x = 3.39\ m/s^2, \tag{6.14.7}$$

which is the acceleration parallel to the incline when there is 45.0 N of opposing friction.

Significance

Since friction always opposes motion between surfaces, the acceleration is smaller when there is friction than when there is none. It is a general result that if friction on an incline is negligible, then the acceleration down the incline is a = g sin θ, regardless of mass. As discussed previously, all objects fall with the same acceleration in the absence of air resistance. Similarly, all objects, regardless of mass, slide down a frictionless incline with the same acceleration (if the angle is the same).

When an object rests on an incline that makes an angle θ with the horizontal, the force of gravity acting on the object is divided into two components: a force acting perpendicular to the plane, wy , and a force acting parallel to the plane, wx (Figure 6.14.3). The normal force $\vec{N}$ is typically equal in magnitude and opposite in direction to the perpendicular component of the weight w_y. The force acting parallel to the plane, w_x, causes the object to accelerate down the incline.

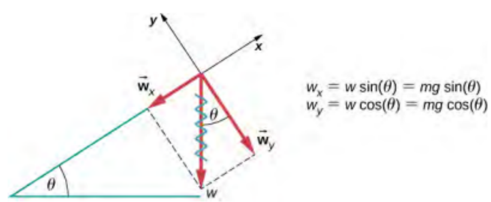


Figure 6.14.3: An object rests on an incline that makes an angle θ with the horizontal.

Be careful when resolving the weight of the object into components. If the incline is at an angle θ to the horizontal, then the magnitudes of the weight components are

$$w_x = w \sin \theta = mg \sin \theta$$

and

$$w_y = w \cos \theta = mg \cos \theta$$

We use the second equation to write the normal force experienced by an object resting on an inclined plane:

$$N = mg \cos \theta.$$

Instead of memorizing these equations, it is helpful to be able to determine them from reason. To do this, we draw the right angle formed by the three weight vectors. The angle θ of the incline is the same as the angle formed between w and w_y. Knowing this property, we can use trigonometry to determine the magnitude of the weight components:

$$\cos \theta = \frac{w_y}{w}, \quad w_y = w \cos \theta = mg \cos \theta$$

$$\sin \theta = \frac{w_x}{w}, \quad w_x = w \sin \theta = mg \sin \theta$$

? Exercise 5.8

A force of 1150 N acts parallel to a ramp to push a 250-kg gun safe into a moving van. The ramp is frictionless and inclined at 17°. (a) What is the acceleration of the safe up the ramp? (b) If we consider friction in this problem, with a friction force of 120 N, what is the acceleration of the safe?

Tension

A **tension** is a force along the length of a medium; in particular, it is a pulling force that acts along a stretched flexible connector, such as a rope or cable. The word "tension" comes from a Latin word meaning "to stretch." Not coincidentally, the flexible cords that carry muscle forces to other parts of the body are called tendons. Any flexible connector, such as a string, rope, chain, wire, or cable, can only exert a pull parallel to its length; thus, a force carried by a flexible connector is a tension with a direction parallel to the connector. Tension is a pull in a connector. Consider the phrase: "You can't push a rope." Instead, tension force pulls outward along the two ends of a rope. Consider a person holding a mass on a rope, as shown in Figure 6.14.4. If the 5.00-kg mass in the figure is stationary, then its acceleration is zero and the net force is zero. The only external forces acting on the mass are its weight and the tension supplied by the rope. Thus,

$$F_{net} = T - w = 0,$$

where T and w are the magnitudes of the tension and weight, respectively, and their signs indicate direction, with up being positive. As we proved using Newton's second law, the tension equals the weight of the supported mass:

$$T = w = mg.$$

Thus, for a 5.00-kg mass (neglecting the mass of the rope), we see that

$$T = mg = (5.00\ kg)(9.80\ m/s^2) = 49.0\ N.$$

If we cut the rope and insert a spring, the spring would extend a length corresponding to a force of 49.0 N, providing a direct observation and measure of the tension force in the rope.

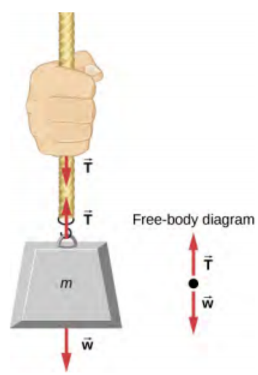


Figure 6.14.4: When a perfectly flexible connector (one requiring no force to bend it) such as this rope transmits a force $\vec{T}$, that force must be parallel to the length of the rope, as shown. By Newton's third law, the rope pulls with equal force but in opposite directions on the hand and the supported mass (neglecting the weight of the rope). The rope is the medium that carries the equal and opposite forces between the two objects. The tension anywhere in the rope between the hand and the mass is equal. Once you have determined the tension in one location, you have determined the tension at all locations along the rope.

Flexible connectors are often used to transmit forces around corners, such as in a hospital traction system, a tendon, or a bicycle brake cable. If there is no friction, the tension transmission is undiminished; only its direction changes, and it is always parallel to the flexible connector, as shown in Figure 6.14.5.

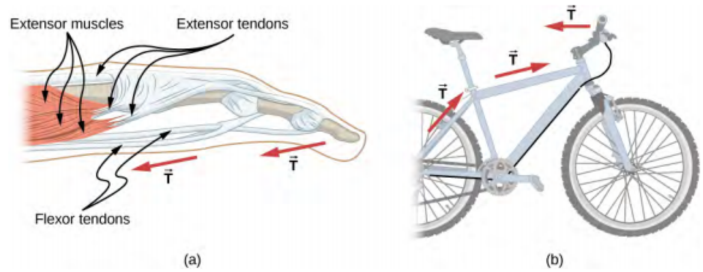


Figure 6.14.5: (a) Tendons in the finger carry force T from the muscles to other parts of the finger, usually changing the force's direction but not its magnitude (the tendons are relatively friction free). (b) The brake cable on a bicycle carries the tension T from the brake lever on the handlebars to the brake mechanism. Again, the direction but not the magnitude of T is changed.

✔ : What is the Tension in a Tightrope?

Calculate the tension in the wire supporting the 70.0-kg tightrope walker shown in Figure 6.14.6.

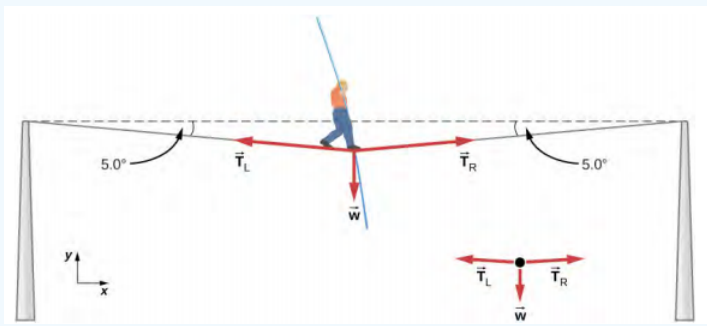


Figure 6.14.6: The weight of a tightrope walker causes a wire to sag by 5.0°. The system of interest is the point in the wire at which the tightrope walker is standing.

Strategy

As you can see in Figure 6.14.6, the wire is bent under the person's weight. Thus, the tension on either side of the person has an upward component that can support his weight. As usual, forces are vectors represented pictorially by arrows that have the same direction as the forces and lengths proportional to their magnitudes. The system is the tightrope walker, and the only external forces acting on him are his weight $\vec{w}$ and the two tensions $\vec{T}_L$ (left tension) and $\vec{T}_R$ (right tension). It is reasonable to neglect the weight of the wire. The net external force is zero, because the system is static. We can use trigonometry to find the tensions. One conclusion is possible at the outset—we can see from Figure 6.14.6(b) that the magnitudes of the tensions T_L and T_R must be equal. We know this because there is no horizontal acceleration in the rope and the only forces acting to the left and right are T_L and T_R. Thus, the magnitude of those horizontal components of the forces must be equal so that they cancel each other out.

Whenever we have two-dimensional vector problems in which no two vectors are parallel, the easiest method of solution is to pick a convenient coordinate system and project the vectors onto its axes. In this case, the best coordinate system has one horizontal axis (x) and one vertical axis (y).

Solution

First, we need to resolve the tension vectors into their horizontal and vertical components. It helps to look at a new free-body diagram showing all horizontal and vertical components of each force acting on the system (Figure 6.14.7).

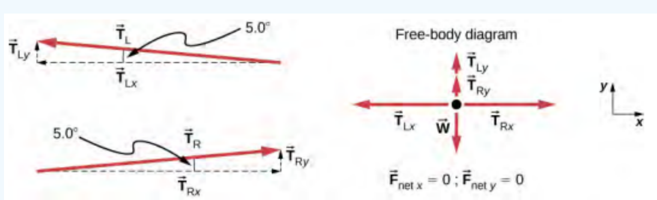


Figure 6.14.7: When the vectors are projected onto vertical and horizontal axes, their components along these axes must add to zero, since the tightrope walker is stationary. The small angle results in T being much greater than w.

Consider the horizontal components of the forces (denoted with a subscript x):

$$F_{netx} = T_{Rx} - T_{Lx}.$$

The net external horizontal force $F_{net\ x} = 0$, since the person is stationary. Thus,

$$F_{netx} = 0 = T_{Rx} - T_{Lx}.$$

$$T_{Lx} = T_{Rx}.$$

Now observe Figure 6.14.7. You can use trigonometry to determine the magnitude of T_L and T_R:

$$\cos 5.0^o = \frac{T_{Lx}}{T_L}, \quad T_{Lx} = T_L \cos 5.0^o$$

$$\cos 5.0^o = \frac{T_{Rx}}{T_R}, \quad T_{Rx} = T_R \cos 5.0^o.$$

Equating T_{Lx} and T_{Rx}:

$$T_L \cos 5.0^o = T_R \cos 5.0^o.$$

Thus,

$$T_L = T_R = T,$$

as predicted. Now, considering the vertical components (denoted by a subscript y), we can solve for T. Again, since the person is stationary, Newton's second law implies that $F_{net\ y} = 0$. Thus, as illustrated in the free-body diagram,

$$F_{nety} = T_{Ly} + T_{Ry} - w = 0.$$

We can use trigonometry to determine the relationships among T_{Ly}, T_{Ry}, and T. As we determined from the analysis in the horizontal direction, $T_L = T_R = T$:

$$\sin 5.0^o = \frac{T_{Ly}}{T_L}, \quad T_{Ly} = T_L \sin 5.0^o = T \sin 5.0^o$$

$$\sin 5.0^o = \frac{T_{Ry}}{T_R}, \quad T_{Ry} = T_R \sin 5.0^o = T \sin 5.0^o.$$

Now we can substitute the vales for T_{Ly} and T_{Ry}, into the net force equation in the vertical direction:

$$F_{nety} = T_{Ly} + T_{Ry} - w = 0$$

$$F_{nety} = 0 = T \sin 5.0^o + T \sin 5.0^o - w = 0$$

$$2T \sin 5.0^o - w = 0$$

$$2T \sin 5.0^o = w$$

and

$$T = \frac{w}{2 \sin 5.0^o} = \frac{mg}{2 \sin 5.0^o},$$

so

$$T = \frac{(70.0\ kg)(9.80\ m/s^2)}{2(0.0872)},$$

and the tension is

$$T = 3930\ N.$$

Significance

The vertical tension in the wire acts as a force that supports the weight of the tightrope walker. The tension is almost six times the 686-N weight of the tightrope walker. Since the wire is nearly horizontal, the vertical component of its tension is only a fraction of the tension in the wire. The large horizontal components are in opposite directions and cancel, so most of the tension in the wire is not used to support the weight of the tightrope walker.

If we wish to create a large tension, all we have to do is exert a force perpendicular to a taut flexible connector, as illustrated in Figure 6.14.6. As we saw in Example 5.13, the weight of the tightrope walker acts as a force perpendicular to the rope. We saw that the tension in the rope is related to the weight of the tightrope walker in the following way:

$$T = \frac{w}{2\sin\theta}.$$

We can extend this expression to describe the tension T created when a perpendicular force ($F_\perp$) is exerted at the middle of a flexible connector:

$$T = \frac{F_\perp}{2\sin\theta}.$$

The angle between the horizontal and the bent connector is represented by θ. In this case, T becomes large as θ approaches zero. Even the relatively small weight of any flexible connector will cause it to sag, since an infinite tension would result if it were horizontal (i.e., θ = 0 and sin θ = 0). For example, Figure 6.14.8 shows a situation where we wish to pull a car out of the mud when no tow truck is available. Each time the car moves forward, the chain is tightened to keep it as straight as possible. The tension in the chain is given by T = $\frac{F_\perp}{2\sin\theta}$, and since θ is small, T is large. This situation is analogous to the tightrope walker, except that the tensions shown here are those transmitted to the car and the tree rather than those acting at the point where $F_\perp$ is applied.

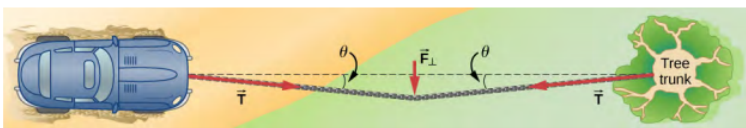


Figure 6.14.8: We can create a large tension in the chain—and potentially a big mess—by pushing on it perpendicular to its length, as shown.

? Exercise 5.9

One end of a 3.0-m rope is tied to a tree; the other end is tied to a car stuck in the mud. The motorist pulls sideways on the midpoint of the rope, displacing it a distance of 0.25 m. If he exerts a force of 200.0 N under these conditions, determine the force exerted on the car.

In Applications of Newton's Laws, we extend the discussion on tension in a cable to include cases in which the angles shown are not equal.

Friction

Friction is a resistive force opposing motion or its tendency. Imagine an object at rest on a horizontal surface. The net force acting on the object must be zero, leading to equality of the weight and the normal force, which act in opposite directions. If the surface is tilted, the normal force balances the component of the weight perpendicular to the surface. If the object does not slide downward, the component of the weight parallel to the inclined plane is balanced by friction. Friction is discussed in greater detail in the next chapter.

Spring force

A spring is a special medium with a specific atomic structure that has the ability to restore its shape, if deformed. To restore its shape, a spring exerts a restoring force that is proportional to and in the opposite direction in which it is stretched or compressed. This is the statement of a law known as Hooke's law, which has the mathematical form

$$\vec{F} = -k\vec{x}.$$

The constant of proportionality k is a measure of the spring's stiffness. The line of action of this force is parallel to the spring axis, and the sense of the force is in the opposite direction of the displacement vector (Figure 6.14.9). The displacement must be measured from the relaxed position; x = 0 when the spring is relaxed.

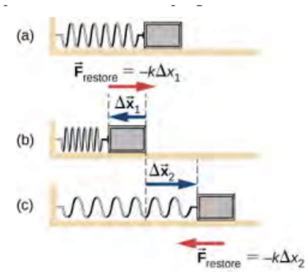


Figure 6.14.9: A spring exerts its force proportional to a displacement, whether it is compressed or stretched. (a) The spring is in a relaxed position and exerts no force on the block. (b) The spring is compressed by displacement $\Delta\vec{x}_1$ of the object and exerts restoring force $-k\Delta\vec{x}_1$. (c) The spring is stretched by displacement $\Delta\vec{x}_1$ of the object and exerts restoring force $-k\Delta\vec{x}_2$.

Real Forces and Inertial Frames

There is another distinction among forces: Some forces are real, whereas others are not. **Real forces** have some physical origin, such as a gravitational pull. In contrast, **fictitious forces** arise simply because an observer is in an accelerating or noninertial frame of reference, such as one that rotates (like a merry-go-round) or undergoes linear acceleration (like a car slowing down). For example, if a satellite is heading due north above Earth's Northern Hemisphere, then to an observer on Earth, it will appear to experience a force to the west that has no physical origin. Instead, Earth is rotating toward the east and moves east under the satellite. In Earth's frame, this looks like a westward force on the satellite, or it can be interpreted as a violation of Newton's first law (the law of inertia). We can identify a fictitious force by asking the question, "What is the reaction force?" If we cannot name the reaction force, then the force we are considering is fictitious. In the example of the satellite, the reaction force would have to be an eastward force on Earth. Recall that an inertial frame of reference is one in which all forces are real and, equivalently, one in which Newton's laws have the simple forms given in this chapter.

Earth's rotation is slow enough that Earth is nearly an inertial frame. You ordinarily must perform precise experiments to observe fictitious forces and the slight departures from Newton's laws, such as the effect just described. On a large scale, such as for the rotation of weather systems and ocean currents, the effects can be easily observed (Figure 6.14.10).

Figure 6.14.10: Hurricane Fran is shown heading toward the southeastern coast of the United States in September 1996. Notice the characteristic "eye" shape of the hurricane. This is a result of the Coriolis effect, which is the deflection of objects (in this case, air) when considered in a rotating frame of reference, like the spin of Earth.

The crucial factor in determining whether a frame of reference is inertial is whether it accelerates or rotates relative to a known inertial frame. Unless stated otherwise, all phenomena discussed in this text are in inertial frames.

The forces discussed in this section are real forces, but they are not the only real forces. Lift and thrust, for example, are more specialized real forces. In the long list of forces, are some more basic than others? Are some different manifestations of the same underlying force? The answer to both questions is yes, as you will see in the treatment of modern physics later in the text

Simulation

Explore forces and motion in this interactive simulation as you push household objects up and down a ramp. Lower and raise the ramp to see how the angle of inclination affects the parallel forces. Graphs show forces, energy, and work.

Simulation

Stretch and compress springs in this activity to explore the relationships among force, spring constant, and displacement. Investigate what happens when two springs are connected in series and in parallel.

6.15: Drawing Free-Body Diagrams

Learning Objectives

- Explain the rules for drawing a free-body diagram
- Construct free-body diagrams for different situations

The first step in describing and analyzing most phenomena in physics involves the careful drawing of a free-body diagram. Free-body diagrams have been used in examples throughout this chapter. Remember that a free-body diagram must only include the external forces acting on the body of interest. Once we have drawn an accurate free-body diagram, we can apply Newton's first law if the body is in equilibrium (balanced forces; that is, $F_{net} = 0$) or Newton's second law if the body is accelerating (unbalanced force; that is, $F_{net} \neq 0$).

In Forces, we gave a brief problem-solving strategy to help you understand free-body diagrams. Here, we add some details to the strategy that will help you in constructing these diagrams.

Problem-Solving Strategy: Constructing Free-Body Diagrams

Observe the following rules when constructing a free-body diagram:

1. Draw the object under consideration; it does not have to be artistic. At first, you may want to draw a circle around the object of interest to be sure you focus on labeling the forces acting on the object. If you are treating the object as a particle (no size or shape and no rotation), represent the object as a point. We often place this point at the origin of an xy-coordinate system.
2. Include all forces that act on the object, representing these forces as vectors. Consider the types of forces described in Common Forces—normal force, friction, tension, and spring force—as well as weight and applied force. Do not include the net force on the object. With the exception of gravity, all of the forces we have discussed require direct contact with the object. However, forces that the object exerts on its environment must not be included. We never include both forces of an action-reaction pair.
3. Convert the free-body diagram into a more detailed diagram showing the x- and y-components of a given force (this is often helpful when solving a problem using Newton's first or second law). In this case, place a squiggly line through the original vector to show that it is no longer in play—it has been replaced by its x- and y-components.
4. If there are two or more objects, or bodies, in the problem, draw a separate free-body diagram for each object.

Note: If there is acceleration, we do not directly include it in the free-body diagram; however, it may help to indicate acceleration outside the free-body diagram. You can label it in a different color to indicate that it is separate from the free-body diagram.

Let's apply the problem-solving strategy in drawing a free-body diagram for a sled. In Figure $6.15.1a$, a sled is pulled by force $\vec{P}$ at an angle of 30°. In part (b), we show a free-body diagram for this situation, as described by steps 1 and 2 of the problem-solving strategy. In part (c), we show all forces in terms of their x- and y-components, in keeping with step 3.

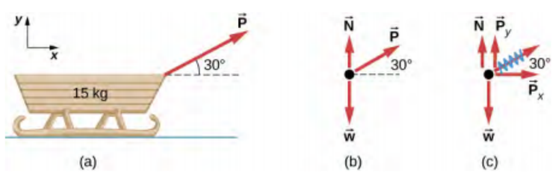


Figure 6.15.1: (a) A moving sled is shown as (b) a free-body diagram and (c) a free-body diagram with force components.

Example 6.15.1: Two Blocks on an Inclined Plane

Construct the free-body diagram for object A and object B in Figure $6.15.1$.

Strategy

We follow the four steps listed in the problem-solving strategy.

Solution

We start by creating a diagram for the first object of interest. In Figure $6.15.2a$, object A is isolated (circled) and represented by a dot.

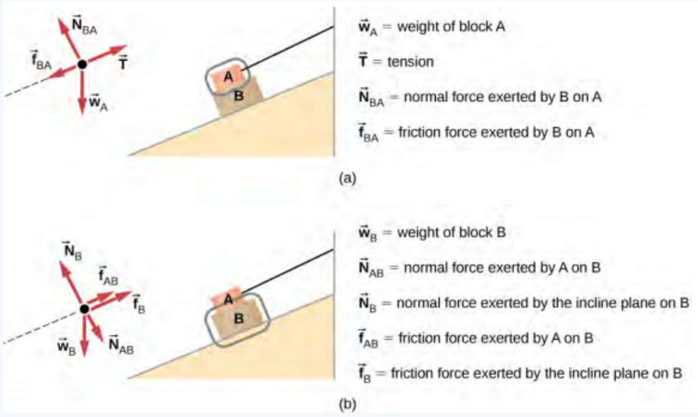


Figure $6.15.2$: (a) The free-body diagram for isolated object A. (b) The free-body diagram for isolated object B. Comparing the two drawings, we see that friction acts in the opposite direction in the two figures. Because object A experiences a force that tends to pull it to the right, friction must act to the left. Because object B experiences a component of its weight that pulls it to the left, down the incline, the friction force must oppose it and act up the ramp. Friction always acts opposite the intended direction of motion.

We now include any force that acts on the body. Here, no applied force is present. The weight of the object acts as a force pointing vertically downward, and the presence of the cord indicates a force of tension pointing away from the object. Object A has one interface and hence experiences a normal force, directed away from the interface. The source of this force is object B, and this normal force is labeled accordingly. Since object B has a tendency to slide down, object A has a tendency to slide up with respect to the interface, so the friction f_{BA} is directed downward parallel to the inclined plane.

As noted in step 4 of the problem-solving strategy, we then construct the free-body diagram in Figure 5.32(b) using the same approach. Object B experiences two normal forces and two friction forces due to the presence of two contact surfaces. The interface with the inclined plane exerts external forces of N_B and f_B, and the interface with object B exerts the normal force N_{AB} and friction f_{AB}; N_{AB} is directed away from object B, and f_{AB} is opposing the tendency of the relative motion of object B with respect to object A.

Significance

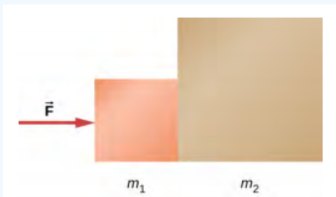
The object under consideration in each part of this problem was circled in gray. When you are first learning how to draw free-body diagrams, you will find it helpful to circle the object before deciding what forces are acting on that particular object. This focuses your attention, preventing you from considering forces that are not acting on the body

Example $6.15.2$: Two Blocks in Contact

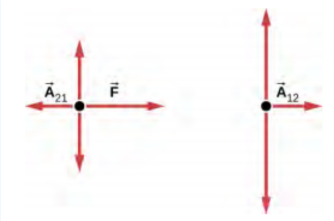
A force is applied to two blocks in contact, as shown.

Strategy

Draw a free-body diagram for each block. Be sure to consider Newton’s third law at the interface where the two blocks touch.



Solution



Significance

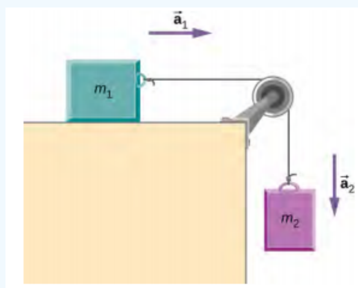
$\vec{A}_{21}$ is the action force of block 2 on block 1. $\vec{A}_{12}$ is the reaction force of block 1 on block 2. We use these free-body diagrams in Applications of Newton's Laws.

Example 6.15.3: Block on the Table (Coupled Blocks)

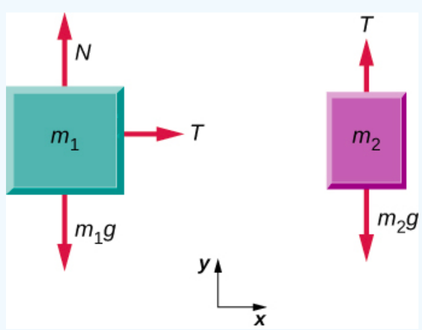
A block rests on the table, as shown. A light rope is attached to it and runs over a pulley. The other end of the rope is attached to a second block. The two blocks are said to be coupled. Block m_2 exerts a force due to its weight, which causes the system (two blocks and a string) to accelerate.

Strategy

We assume that the string has no mass so that we do not have to consider it as a separate object. Draw a free-body diagram for each block.



Solution

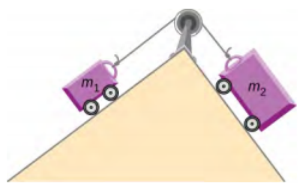


Significance

Each block accelerates (notice the labels shown for $\vec{a}_1$ and $\vec{a}_2$); however, assuming the string remains taut, they accelerate at the same rate. Thus, we have $|\vec{a}_1| = |\vec{a}_2|$. If we were to continue solving the problem, we could simply call the acceleration $\vec{a}$. Also, we use two free-body diagrams because we are usually finding tension T, which may require us to use a system of two equations in this type of problem. The tension is the same on both m_1 and m_2.

Exercise 6.15.1

a. Draw the free-body diagram for the situation shown.
b. Redraw it showing components; use x-axes parallel to the two ramps.



📌 Simulation

View this simulation to predict, qualitatively, how an external force will affect the speed and direction of an object’s motion. Explain the effects with the help of a free-body diagram. Use free-body diagrams to draw position, velocity, acceleration, and force graphs, and vice versa. Explain how the graphs relate to one another. Given a scenario or a graph, sketch all four graphs.

6.16: Newton's Laws of Motion (Exercises)

Conceptual Questions

5.1 Forces

1. What properties do forces have that allow us to classify them as vectors?

5.2 Newton's First Law

2. Taking a frame attached to Earth as inertial, which of the following objects cannot have inertial frames attached to them, and which are inertial reference frames?
   a. A car moving at constant velocity
   b. A car that is accelerating
   c. An elevator in free fall
   d. A space capsule orbiting Earth
   e. An elevator descending uniformly
3. A woman was transporting an open box of cupcakes to a school party. The car in front of her stopped suddenly; she applied her brakes immediately. She was wearing her seat belt and suffered no physical harm (just a great deal of embarrassment), but the cupcakes flew into the dashboard and became “smushcakes.” Explain what happened.

5.3 Newton's Second Law

4. Why can we neglect forces such as those holding a body together when we apply Newton’s second law?
5. A rock is thrown straight up. At the top of the trajectory, the velocity is momentarily zero. Does this imply that the force acting on the object is zero? Explain your answer.

5.4 Mass and Weight

6. What is the relationship between weight and mass? Which is an intrinsic, unchanging property of a body?
7. How much does a 70-kg astronaut weight in space, far from any celestial body? What is her mass at this location?
8. Which of the following statements is accurate?
   a. Mass and weight are the same thing expressed in different units.
   b. If an object has no weight, it must have no mass.
   c. If the weight of an object varies, so must the mass.
   d. Mass and inertia are different concepts.
   e. Weight is always proportional to mass.
9. When you stand on Earth, your feet push against it with a force equal to your weight. Why doesn’t Earth accelerate away from you?
10. How would you give the value of $\vec{g}$ in vector form?

5.5 Newton’s Third Law

11. Identify the action and reaction forces in the following situations:
    a. Earth attracts the Moon,
    b. a boy kicks a football,
    c. a rocket accelerates upward,
    d. a car accelerates forward,
    e. a high jumper leaps, and
    f. a bullet is shot from a gun.
12. Suppose that you are holding a cup of coffee in your hand. Identify all forces on the cup and the reaction to each force.
13. (a) Why does an ordinary rifle recoil (kick backward) when fired? (b) The barrel of a recoilless rifle is open at both ends. Describe how Newton’s third law applies when one is fired. (c) Can you safely stand close behind one when it is fired?

5.6 Common Forces

14. A table is placed on a rug. Then a book is placed on the table. What does the floor exert a normal force on?

15. A particle is moving to the right. (a) Can the force on it to be acting to the left? If yes, what would happen? (b) Can that force be acting downward? If yes, why?

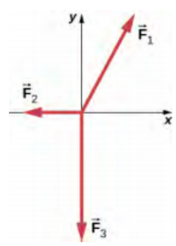
5.7 Drawing Free-Body Diagrams

16. In completing the solution for a problem involving forces, what do we do after constructing the free-body diagram? That is, what do we apply?
17. If a book is located on a table, how many forces should be shown in a free-body diagram of the book? Describe them.
18. If the book in the previous question is in free fall, how many forces should be shown in a free-body diagram of the book? Describe them.

Problems

5.1 Forces

19. Two ropes are attached to a tree, and forces of $\vec{F}_1 = 2.0\,\hat{i} + 4.0\,\hat{j}$ N and $\vec{F}_2 = 3.0\,\hat{i} + 6.0\,\hat{j}$ N are applied. The forces are coplanar (in the same plane). (a) What is the resultant (net force) of these two force vectors? (b) Find the magnitude and direction of this net force.
20. A telephone pole has three cables pulling as shown from above, with $\vec{F}_1 = (300.0\,\hat{i} + 500.0\,\hat{j})$, $\vec{F}_2 = -200.0\,\hat{i}$, and $\vec{F}_3 = -800.0\,\hat{j}$. (a) Find the net force on the telephone pole in component form. (b) Find the magnitude and direction of this net force.



21. Two teenagers are pulling on ropes attached to a tree. The angle between the ropes is 30.0°. David pulls with a force of 400.0 N and Stephanie pulls with a force of 300.0 N. (a) Find the component form of the net force. (b) Find the magnitude of the resultant (net) force on the tree and the angle it makes with David's rope.

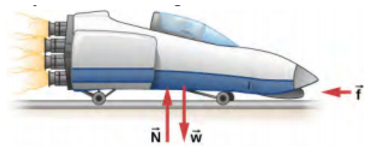
5.2 Newton's First Law

22. Two forces of $\vec{F}_1 = 75.0\ 2\,(\hat{i} - \hat{j})$ N and $\vec{F}_2 = \frac{150.0}{\sqrt{2}}(\hat{i} - \hat{j})$ N act on an object. Find the third force $\vec{F}_3$ that is needed to balance the first two forces.
23. While sliding a couch across a floor, Andrea and Jennifer exert forces $\vec{F}_A$ and $\vec{F}_J$ on the couch. Andrea's force is due north with a magnitude of 130.0 N and Jennifer's force is 32° east of north with a magnitude of 180.0 N. (a) Find the net force in component form. (b) Find the magnitude and direction of the net force. (c) If Andrea and Jennifer's housemates, David and Stephanie, disagree with the move and want to prevent its relocation, with what combined force $\vec{F}_{DS}$ should they push so that the couch does not move?

5.3 Newton's Second Law

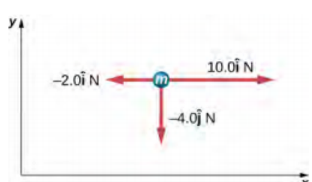
24. Andrea, a 63.0-kg sprinter, starts a race with an acceleration of 4.200 m/s^2. What is the net external force on her?
25. If the sprinter from the previous problem accelerates at that rate for 20.00 m and then maintains that velocity for the remainder of a 100.00-m dash, what will her time be for the race?
26. A cleaner pushes a 4.50-kg laundry cart in such a way that the net external force on it is 60.0 N. Calculate the magnitude of his cart's acceleration.
27. Astronauts in orbit are apparently weightless. This means that a clever method of measuring the mass of astronauts is needed to monitor their mass gains or losses, and adjust their diet. One way to do this is to exert a known force on an astronaut and measure the acceleration produced. Suppose a net external force of 50.0 N is exerted, and an astronaut's acceleration is measured to be 0.893 m/s^2. (a) Calculate her mass. (b) By exerting a force on the astronaut, the vehicle in which she orbits experiences an equal and opposite force. Use this knowledge to find an equation for the acceleration of the system (astronaut and spaceship) that would be measured by a nearby observer. (c) Discuss how this would affect the measurement of the astronaut's acceleration. Propose a method by which recoil of the vehicle is avoided.

28. In Figure 5.4.3, the net external force on the 24-kg mower is given as 51 N. If the force of friction opposing the motion is 24 N, what force F (in newtons) is the person exerting on the mower? Suppose the mower is moving at 1.5 m/s when the force F is removed. How far will the mower go before stopping?
29. The rocket sled shown below decelerates at a rate of 196 m/s^2. What force is necessary to produce this deceleration? Assume that the rockets are off. The mass of the system is 2.10 x 10^3 kg.

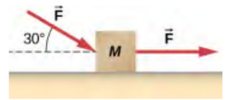


30. If the rocket sled shown in the previous problem starts with only one rocket burning, what is the magnitude of this acceleration? Assume that the mass of the system is 2.10 x 10^3 kg, the thrust T is 2.40 x 10^4 N, and the force of friction opposing the motion is 650.0 N. (b) Why is the acceleration not one-fourth of what it is with all rockets burning?
31. What is the deceleration of the rocket sled if it comes to rest in 1.10 s from a speed of 1000.0 km/h? (Such deceleration caused one test subject to black out and have temporary blindness.)
32. Suppose two children push horizontally, but in exactly opposite directions, on a third child in a wagon. The first child exerts a force of 75.0 N, the second exerts a force of 90.0 N, friction is 12.0 N, and the mass of the third child plus wagon is 23.0 kg. (a) What is the system of interest if the acceleration of the child in the wagon is to be calculated? (See the free-body diagram.) (b) Calculate the acceleration. (c) What would the acceleration be if friction were 15.0 N?

33. A powerful motorcycle can produce an acceleration of 3.50 m/s^2 while traveling at 90.0 km/h. At that speed, the forces resisting motion, including friction and air resistance, total 400.0 N. (Air resistance is analogous to air friction. It always opposes the motion of an object.) What is the magnitude of the force that motorcycle exerts backward on the ground to produce its acceleration if the mass of the motorcycle with rider is 245 kg?
34. A car with a mass of 1000.0 kg accelerates from 0 to 90.0 km/h in 10.0 s. (a) What is its acceleration? (b) What is the net force on the car?
35. The driver in the previous problem applies the brakes when the car is moving at 90.0 km/h, and the car comes to rest after traveling 40.0 m. What is the net force on the car during its deceleration?
36. An 80.0-kg passenger in an SUV traveling at 1.00 x 10^2 km/h is wearing a seat belt. The driver slams on the brakes and the SUV stops in 45.0 m. Find the force of the seat belt on the passenger.
37. A particle of mass 2.0 kg is acted on by a single force $\vec{F}_1 = 18\,\hat{i}$ N. (a) What is the particle's acceleration? (b) If the particle starts at rest, how far does it travel in the first 5.0 s?
38. Suppose that the particle of the previous problem also experiences forces $\vec{F}_2 = -15\,\hat{i}$ N and $\vec{F}_3 = 6.0\,\hat{j}$ N. What is its acceleration in this case?
39. Find the acceleration of the body of mass 5.0 kg shown below.



40. In the following figure, the horizontal surface on which this block slides is frictionless. If the two forces acting on it each have magnitude F = 30.0 N and M = 10.0 kg , what is the magnitude of the resulting acceleration of the block?

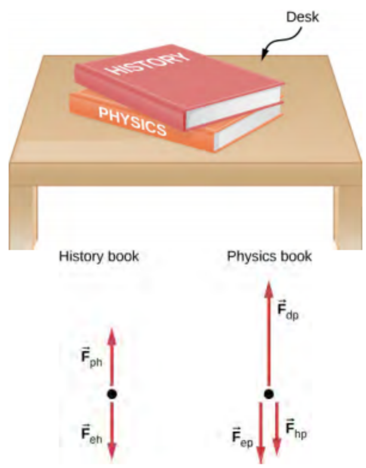


5.4 Mass and Weight

41. The weight of an astronaut plus his space suit on the Moon is only 250 N. (a) How much does the suited astronaut weigh on Earth? (b) What is the mass on the Moon? On Earth?
42. Suppose the mass of a fully loaded module in which astronauts take off from the Moon is 1.00 x 10^4 kg. The thrust of its engines is 3.00 x 10^4 N. (a) Calculate the module's magnitude of acceleration in a vertical takeoff from the Moon. (b) Could it lift off from Earth? If not, why not? If it could, calculate the magnitude of its acceleration.
43. A rocket sled accelerates at a rate of 49.0 m/s^2. Its passenger has a mass of 75.0 kg. (a) Calculate the horizontal component of the force the seat exerts against his body. Compare this with his weight using a ratio. (b) Calculate the direction and magnitude of the total force the seat exerts against his body.
44. Repeat the previous problem for a situation in which the rocket sled decelerates at a rate of 201 m/s^2. In this problem, the forces are exerted by the seat and the seat belt.
45. A body of mass 2.00 kg is pushed straight upward by a 25.0 N vertical force. What is its acceleration?
46. A car weighing 12,500 N starts from rest and accelerates to 83.0 km/h in 5.00 s. The friction force is 1350 N. Find the applied force produced by the engine.
47. A body with a mass of 10.0 kg is assumed to be in Earth's gravitational field with g = 9.80 m/s^2. What is the net force on the body if there are no other external forces acting on the object?
48. A fireman has mass m; he hears the fire alarm and slides down the pole with acceleration a (which is less than g in magnitude). (a) Write an equation giving the vertical force he must apply to the pole. (b) If his mass is 90.0 kg and he accelerates at 5.00 m/s^2, what is the magnitude of his applied force?
49. A baseball catcher is performing a stunt for a television commercial. He will catch a baseball (mass 145 g) dropped from a height of 60.0 m above his glove. His glove stops the ball in 0.0100 s. What is the force exerted by his glove on the ball?
50. When the Moon is directly overhead at sunset, the force by Earth on the Moon, F_{EM}, is essentially at 90° to the force by the Sun on the Moon, F_{SM}, as shown below. Given that F_{EM} = 1.98 x 10^{20} N and F_{SM} = 4.36 x 10^{20} N, all other forces on the Moon are negligible, and the mass of the Moon is 7.35 x 10^{22} kg, determine the magnitude of the Moon's acceleration.

5.5 Newton's Third Law

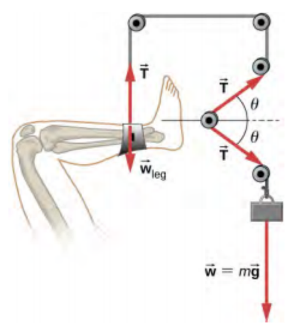
51. (a) What net external force is exerted on a 1100.0-kg artillery shell fired from a battleship if the shell is accelerated at 2.40 x 10^4 m/s^2? (b) What is the magnitude of the force exerted on the ship by the artillery shell, and why?
52. A brave but inadequate rugby player is being pushed backward by an opposing player who is exerting a force of 800.0 N on him. The mass of the losing player plus equipment is 90.0 kg, and he is accelerating backward at 1.20 m/s^2. (a) What is the force of friction between the losing player's feet and the grass? (b) What force does the winning player exert on the ground to move forward if his mass plus equipment is 110.0 kg?
53. A history book is lying on top of a physics book on a desk, as shown below; a free-body diagram is also shown. The history and physics books weigh 14 N and 18 N, respectively. Identify each force on each book with a double subscript notation (for instance, the contact force of the history book pressing against physics book can be described as $\vec{F}_{HP}$), and determine the value of each of these forces, explaining the process used.



54. A truck collides with a car, and during the collision, the net force on each vehicle is essentially the force exerted by the other. Suppose the mass of the car is 550 kg, the mass of the truck is 2200 kg, and the magnitude of the truck's acceleration is 10 m/s^2. Find the magnitude of the car's acceleration.

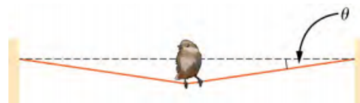
5.6 Common Forces

55. A leg is suspended in a traction system, as shown below. (a) Which part of the figure is used to calculate the force exerted on the foot? (b) What is the tension in the rope? Here $\vec{T}$ is the tension, $\vec{w}_{leg}$ is the weight of the leg, and $\vec{w}$ is the weight of the load that provides the tension.

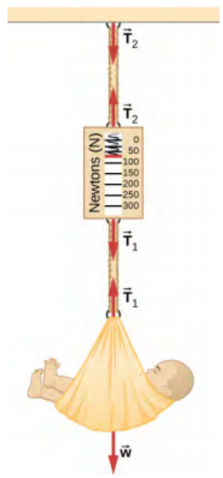


56. Suppose the shinbone in the preceding image was a femur in a traction setup for a broken bone, with pulleys and rope available. How might we be able to increase the force along the femur using the same weight?
57. A team of nine members on a tall building tug on a string attached to a large boulder on an icy surface. The boulder has a mass of 200 kg and is tugged with a force of 2350 N. (a) What is magnitude of the acceleration? (b) What force would be required to produce a constant velocity?
58. What force does a trampoline have to apply to Jennifer, a 45.0-kg gymnast, to accelerate her straight up at 7.50 m/s^2? The answer is independent of the velocity of the gymnast—she can be moving up or down or can be instantly stationary.
59. (a) Calculate the tension in a vertical strand of spider web if a spider of mass 2.00 x 10^{-5} kg hangs motionless on it. (b) Calculate the tension in a horizontal strand of spider web if the same spider sits motionless in the middle of it much like the tightrope walker in Figure 5.26. The strand sags at an angle of 12° below the horizontal. Compare this with the tension in the vertical strand (find their ratio).
60. Suppose Kevin, a 60.0-kg gymnast, climbs a rope. (a) What is the tension in the rope if he climbs at a constant speed? (b) What is the tension in the rope if he accelerates upward at a rate of 1.50 m/s^2?
61. Show that, as explained in the text, a force $F_\perp$ exerted on a flexible medium at its center and perpendicular to its length (such as on the tightrope wire in Figure 5.26) gives rise to a tension of magnitude $T = \frac{F_\perp}{2\sin\theta}$.
62. Consider Figure 5.28. The driver attempts to get the car out of the mud by exerting a perpendicular force of 610.0 N, and the distance she pushes in the middle of the rope is 1.00 m while she stands 6.00 m away from the car on the left and 6.00 m away from the tree on the right. What is the tension T in the rope, and how do you find the answer?
63. A bird has a mass of 26 g and perches in the middle of a stretched telephone line. (a) Show that the tension in the line can be calculated using the equation $T = \frac{mg}{2\sin\theta}$. Determine the tension when (b) θ = 5° and (c) θ = 0.5°. Assume that each

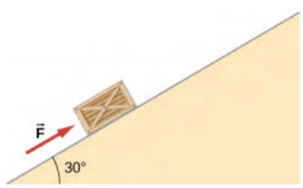
half of the line is straight.



64. One end of a 30-m rope is tied to a tree; the other end is tied to a car stuck in the mud. The motorist pulls sideways on the midpoint of the rope, displacing it a distance of 2 m. If he exerts a force of 80 N under these conditions, determine the force exerted on the car.
65. Consider the baby being weighed in the following figure. (a) What is the mass of the infant and basket if a scale reading of 55 N is observed? (b) What is tension T_1 in the cord attaching the baby to the scale? (c) What is tension T_2 in the cord attaching the scale to the ceiling, if the scale has a mass of 0.500 kg? (d) Sketch the situation, indicating the system of interest used to solve each part. The masses of the cords are negligible.



66. What force must be applied to a 100.0-kg crate on a frictionless plane inclined at 30° to cause an acceleration of 2.0 m/s^2 up the plane?

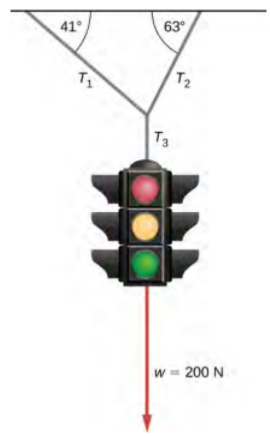


67. A 2.0-kg block is on a perfectly smooth ramp that makes an angle of 30° with the horizontal. (a) What is the block's acceleration down the ramp and the force of the ramp on the block? (b) What force applied upward along and parallel to the ramp would allow the block to move with constant velocity?

5.7 Drawing Free-Body Diagrams

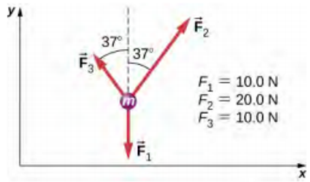
68. A ball of mass m hangs at rest, suspended by a string. (a) Sketch all forces. (b) Draw the free-body diagram for the ball.
69. A car moves along a horizontal road. Draw a free-body diagram; be sure to include the friction of the road that opposes the forward motion of the car.
70. A runner pushes against the track, as shown. (a) Provide a free-body diagram showing all the forces on the runner. (**Hint**: Place all forces at the center of his body, and include his weight.) (b) Give a revised diagram showing the xy-component form.

71. The traffic light hangs from the cables as shown. Draw a free-body diagram on a coordinate plane for this situation.

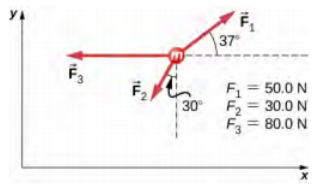


Additional Problems

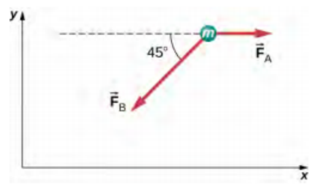
72. Two small forces, $\vec{F}_1 = -2.40\,\hat{i} - 6.10t\,\hat{j}$ N and $\vec{F}_2 = 8.50\,\hat{i} - 9.70\,\hat{j}$ N, are exerted on a rogue asteroid by a pair of space tractors. (a) Find the net force. (b) What are the magnitude and direction of the net force? (c) If the mass of the asteroid is 125 kg, what acceleration does it experience (in vector form)? (d) What are the magnitude and direction of the acceleration?
73. Two forces of 25 and 45 N act on an object. Their directions differ by 70°. The resulting acceleration has magnitude of 10.0 m/s^2. What is the mass of the body?
74. A force of 1600 N acts parallel to a ramp to push a 300-kg piano into a moving van. The ramp is inclined at 20°. (a) What is the acceleration of the piano up the ramp? (b) What is the velocity of the piano when it reaches the top if the ramp is 4.0 m long and the piano starts from rest?
75. Draw a free-body diagram of a diver who has entered the water, moved downward, and is acted on by an upward force due to the water which balances the weight (that is, the diver is suspended).
76. For a swimmer who has just jumped off a diving board, assume air resistance is negligible. The swimmer has a mass of 80.0 kg and jumps off a board 10.0 m above the water. Three seconds after entering the water, her downward motion is stopped. What average upward force did the water exert on her?
77. (a) Find an equation to determine the magnitude of the net force required to stop a car of mass m, given that the initial speed of the car is v_0 and the stopping distance is x. (b) Find the magnitude of the net force if the mass of the car is 1050 kg, the initial speed is 40.0 km/h, and the stopping distance is 25.0 m.
78. A sailboat has a mass of 1.50 x 10^3 kg and is acted on by a force of 2.00 x 10^3 N toward the east, while the wind acts behind the sails with a force of 3.00 x 10^3 N in a direction 45° north of east. Find the magnitude and direction of the resulting acceleration.
79. Find the acceleration of the body of mass 10.0 kg shown below.



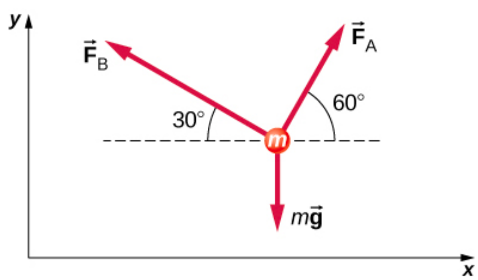
80. A body of mass 2.0 kg is moving along the x-axis with a speed of 3.0 m/s at the instant represented below. (a) What is the acceleration of the body? (b) What is the body's velocity 10.0 s later? (c) What is its displacement after 10.0 s?



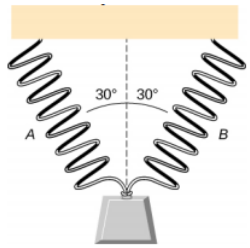
81. Force $\vec{F}_B$ has twice the magnitude of force $\vec{F}_A$. Find the direction in which the particle accelerates in this figure.



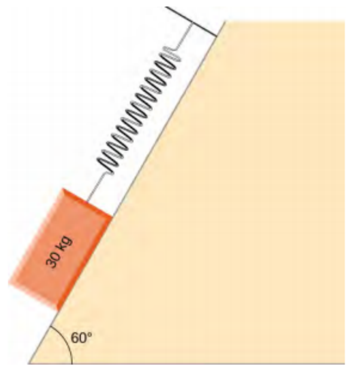
82. Shown below is a body of mass 1.0 kg under the influence of the forces $\vec{F}_A$, $\vec{F}_B$, and m$\vec{g}$. If the body accelerates to the left at 20 m/s^2, what are $\vec{F}_A$ and $\vec{F}_B$?



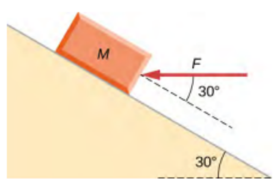
83. A force acts on a car of mass m so that the speed v of the car increases with position x as v = kx^2, where k is constant and all quantities are in SI units. Find the force acting on the car as a function of position.
84. A 7.0-N force parallel to an incline is applied to a 1.0-kg crate. The ramp is tilted at 20° and is frictionless. (a) What is the acceleration of the crate? (b) If all other conditions are the same but the ramp has a friction force of 1.9 N, what is the acceleration?
85. Two boxes, A and B, are at rest. Box A is on level ground, while box B rests on an inclined plane tilted at angle θ with the horizontal. (a) Write expressions for the normal force acting on each block. (b) Compare the two forces; that is, tell which one is larger or whether they are equal in magnitude. (c) If the angle of incline is 10°, which force is greater?
86. A mass of 250.0 g is suspended from a spring hanging vertically. The spring stretches 6.00 cm. How much will the spring stretch if the suspended mass is 530.0 g?
87. As shown below, two identical springs, each with the spring constant 20 N/m, support a 15.0-N weight. (a) What is the tension in spring A? (b) What is the amount of stretch of spring A from the rest position?



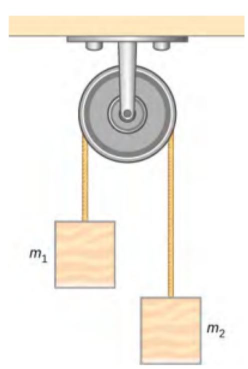
88. Shown below is a 30.0-kg block resting on a frictionless ramp inclined at 60° to the horizontal. The block is held by a spring that is stretched 5.0 cm. What is the force constant of the spring?



89. In building a house, carpenters use nails from a large box. The box is suspended from a spring twice during the day to measure the usage of nails. At the beginning of the day, the spring stretches 50 cm. At the end of the day, the spring stretches 30 cm. What fraction or percentage of the nails have been used?
90. A force is applied to a block to move it up a 30° incline. The incline is frictionless. If F = 65.0 N and M = 5.00 kg, what is the magnitude of the acceleration of the block?

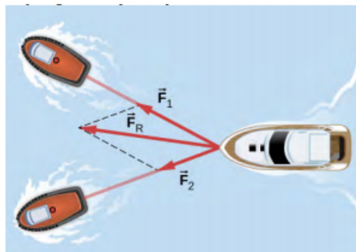


91. Two forces are applied to a 5.0-kg object, and it accelerates at a rate of 2.0 m/s^2 in the positive y-direction. If one of the forces acts in the positive x-direction with magnitude 12.0 N, find the magnitude of the other force.
92. The block on the right shown below has more mass than the block on the left ($m_2 > m_1$). Draw free-body diagrams for each block.



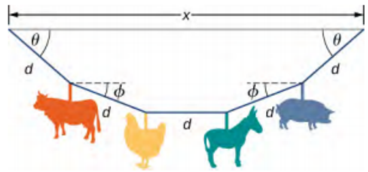
Challenge Problems

93. If two tugboats pull on a disabled vessel, as shown here in an overhead view, the disabled vessel will be pulled along the direction indicated by the result of the exerted forces. (a) Draw a free-body diagram for the vessel. Assume no friction or drag forces affect the vessel. (b) Did you include all forces in the overhead view in your free-body diagram? Why or why not?



94. A 10.0-kg object is initially moving east at 15.0 m/s. Then a force acts on it for 2.00 s, after which it moves northwest, also at 15.0 m/s. What are the magnitude and direction of the average force that acted on the object over the 2.00-s interval?
95. On June 25, 1983, shot-putter Udo Beyer of East Germany threw the 7.26-kg shot 22.22 m, which at that time was a world record. (a) If the shot was released at a height of 2.20 m with a projection angle of 45.0°, what was its initial velocity? (b) If while in Beyer's hand the shot was accelerated uniformly over a distance of 1.20 m, what was the net force on it?
96. A body of mass m moves in a horizontal direction such that at time t its position is given by $x(t) = at^4 + bt^3 + ct$, where a, b, and c are constants. (a) What is the acceleration of the body? (b) What is the time-dependent force acting on the body?
97. A body of mass m has initial velocity v_0 in the positive x-direction. It is acted on by a constant force F for time t until the velocity becomes zero; the force continues to act on the body until its velocity becomes $-v_0$ in the same amount of time. Write an expression for the total distance the body travels in terms of the variables indicated.
98. The velocities of a 3.0-kg object at t = 6.0 s and t = 8.0 s are $(3.0\,\hat{i} - 6.0\,\hat{j} + 4.0\,\hat{k})$ m/s and $(-2.0\,\hat{i} + 4.0\,\hat{k})$ m/s, respectively. If the object is moving at constant acceleration, what is the force acting on it?
99. A 120-kg astronaut is riding in a rocket sled that is sliding along an inclined plane. The sled has a horizontal component of acceleration of 5.0 m/s^2 and a downward component of 3.8 m/s^2. Calculate the magnitude of the force on the rider by the sled. (**Hint**: Remember that gravitational acceleration must be considered.)
100. Two forces are acting on a 5.0-kg object that moves with acceleration 2.0 m/s^2 in the positive y-direction. If one of the forces acts in the positive x-direction and has magnitude of 12 N, what is the magnitude of the other force?
101. Suppose that you are viewing a soccer game from a helicopter above the playing field. Two soccer players simultaneously kick a stationary soccer ball on the flat field; the soccer ball has mass 0.420 kg. The first player kicks with force 162 N at 9.0° north of west. At the same instant, the second player kicks with force 215 N at 15° east of south. Find the acceleration of the ball in $\hat{i}$ and $\hat{j}$ form.
102. A 10.0-kg mass hangs from a spring that has the spring constant 535 N/m. Find the position of the end of the spring away from its rest position. (Use g = 9.80 m/s^2.)
103. A 0.0502-kg pair of fuzzy dice is attached to the rearview mirror of a car by a short string. The car accelerates at constant rate, and the dice hang at an angle of 3.20° from the vertical because of the car's acceleration. What is the magnitude of the acceleration of the car?
104. At a circus, a donkey pulls on a sled carrying a small clown with a force given by $2.48\,\hat{i} + 4.33\,\hat{j}$ N . A horse pulls on the same sled, aiding the hapless donkey, with a force of $6.56\,\hat{i} + 5.33\,\hat{j}$ N. The mass of the sled is 575 kg. Using $\hat{i}$ and $\hat{j}$ form for the answer to each problem, find (a) the net force on the sled when the two animals act together, (b) the acceleration of the sled, and (c) the velocity after 6.50 s.
105. Hanging from the ceiling over a baby bed, well out of baby's reach, is a string with plastic shapes, as shown here. The string is taut (there is no slack), as shown by the straight segments. Each plastic shape has the same mass m, and they are equally spaced by a distance d, as shown. The angles labeled θ describe the angle formed by the end of the string and the ceiling at each end. The center length of sting is horizontal. The remaining two segments each form an angle with the horizontal, labeled ϕ. Let T_1 be the tension in the leftmost section of the string, T_2 be the tension in the section adjacent to it, and T_3 be the tension in the horizontal segment. (a) Find an equation for the tension in each section of the string in

terms of the variables m, g, and θ. (b) Find the angle ϕ in terms of the angle θ. (c) If θ = 5.10°, what is the value of ϕ? (d) Find the distance x between the endpoints in terms of d and θ.



106. A bullet shot from a rifle has mass of 10.0 g and travels to the right at 350 m/s. It strikes a target, a large bag of sand, penetrating it a distance of 34.0 cm. Find the magnitude and direction of the retarding force that slows and stops the bullet.
107. An object is acted on by three simultaneous forces: $\vec{F}_1$ =(−3.00 $\hat{i}$ + 2.00 $\hat{j}$) N, $\vec{F}_2$ = (6.00 $\hat{i}$ − 4.00 $\hat{i}$) N, and $\vec{F}_3$ = (2.00 $\hat{i}$ + 5.00 $\hat{i}$) N. The object experiences acceleration of 4.23 m/s^2. (a) Find the acceleration vector in terms of m. (b) Find the mass of the object. (c) If the object begins from rest, find its speed after 5.00 s. (d) Find the components of the velocity of the object after 5.00 s.
108. In a particle accelerator, a proton has mass 1.67 x 10^{-27} kg and an initial speed of 2.00 x 10^5 m/s. It moves in a straight line, and its speed increases to 9.00 x 10^5 m/s in a distance of 10.0 cm. Assume that the acceleration is constant. Find the magnitude of the force exerted on the proton.
109. A drone is being directed across a frictionless ice-covered lake. The mass of the drone is 1.50 kg, and its velocity is 3.00 $\hat{i}$ m/s. After 10.0 s, the velocity is 9.00 $\hat{i}$ + 4.00 $\hat{j}$ m/s. If a constant force in the horizontal direction is causing this change in motion, find (a) the components of the force and (b) the magnitude of the force.

Contributors and Attributions

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6.17: Newton's Laws of Motion (Summary)

Key Terms

dynamics	study of how forces affect the motion of objects and systems
external force	force acting on an object or system that originates outside of the object or system
force	push or pull on an object with a specific magnitude and direction; can be represented by vectors or expressed as a multiple of a standard force
free fall	situation in which the only force acting on an object is gravity
free-body diagram	sketch showing all external forces acting on an object or system; the system is represented by a single isolated point, and the forces are represented by vectors extending outward from that point
Hooke's law	in a spring, a restoring force proportional to and in the opposite direction of the imposed displacement
inertia	ability of an object to resist changes in its motion
inertial reference frame	reference frame moving at constant velocity relative to an inertial frame is also inertial; a reference frame accelerating relative to an inertial frame is not inertial
law of inertia	see Newton's first law of motion
net external force	vector sum of all external forces acting on an object or system; causes a mass to accelerate
newton	SI unit of force; 1 N is the force needed to accelerate an object with a mass of 1 kg at a rate of 1 m/s^2
Newton's first law of motion	body at rest remains at rest or, if in motion, remains in motion at constant velocity unless acted on by a net external force; also known as the law of inertia
Newton's second law of motion	acceleration of a system is directly proportional to and in the same direction as the net external force acting on the system and is inversely proportional to its mass
Newton's third law of motion	whenever one body exerts a force on a second body, the first body experiences a force that is equal in magnitude and opposite in direction to the force that it exerts
normal force	force supporting the weight of an object, or a load, that is perpendicular to the surface of contact between the load and its support; the surface applies this force to an object to support the weight of the object
tension	pulling force that acts along a stretched flexible connector, such as a rope or cable
thrust	reaction force that pushes a body forward in response to a backward force
weight	force $\vec{w}$ due to gravity acting on an object of mass m

Key Equations

Net external force	$\vec{F}_{net} = \sum \vec{F} = \vec{F}_1 + \vec{F}_2 + \ldots \quad (6.17.1)$
Newton's first law	$\vec{v} = constant\ when\ \vec{F}_{net} = \vec{0}\ N \quad (6.17.2)$
Newton’s second law, vector form	$\vec{F}_{net} = \sum \vec{F} = m\vec{a} \quad (6.17.3)$
Newton’s second law, scalar form	$\vec{F}_{net} = ma \quad (6.17.4)$
Newton’s second law, component form	$\sum \vec{F}_x = m\vec{a}_x, \sum \vec{F}_y = m\vec{a}_y, \sum \vec{F}_z = m\vec{a}_z \quad (6.17.5)$
Newton’s second law, momentum form	$\vec{F}_{net} = \frac{d\vec{p}}{dt} \quad (6.17.6)$
Definition of weight, vector form	$\vec{w} = m\vec{g} \quad (6.17.7)$
Definition of weight, scalar form	$w = mg \quad (6.17.8)$
Newton's third law	$\vec{F}_{AB} = -\vec{F}_{BA} \quad (6.17.9)$
Normal force on an object resting on a horizontal surface, vector form	$\vec{N} = -m\vec{g} \quad (6.17.10)$
Normal force on an object resting on a horizontal surface, scalar form	$N = mg \quad (6.17.11)$
Normal force on an object resting on an inclined plane, scalar form	$N = mg\cos\theta \quad (6.17.12)$
Tension in a cable supporting an object of mass m at rest, scalar form	$T = w = mg \quad (6.17.13)$

Summary

5.1 Forces

- Dynamics is the study of how forces affect the motion of objects, whereas kinematics simply describes the way objects move.
- Force is a push or pull that can be defined in terms of various standards, and it is a vector that has both magnitude and direction.
- External forces are any outside forces that act on a body. A free-body diagram is a drawing of all external forces acting on a body.
- The SI unit of force is the newton (N).

5.2 Newton's First Law

- According to Newton’s first law, there must be a cause for any change in velocity (a change in either magnitude or direction) to occur. This law is also known as the law of inertia.
- Friction is an external force that causes an object to slow down.
- Inertia is the tendency of an object to remain at rest or remain in motion. Inertia is related to an object’s mass.
- If an object’s velocity relative to a given frame is constant, then the frame is inertial. This means that for an inertial reference frame, Newton’s first law is valid.
- Equilibrium is achieved when the forces on a system are balanced.
- A net force of zero means that an object is either at rest or moving with constant velocity; that is, it is not accelerating.

5.3 Newton's Second Law

- An external force acts on a system from outside the system, as opposed to internal forces, which act between components within the system.
- Newton's second law of motion says that the net external force on an object with a certain mass is directly proportional to and in the same direction as the acceleration of the object.
- Newton's second law can also describe net force as the instantaneous rate of change of momentum. Thus, a net external force causes nonzero acceleration.

5.4 Mass and Weight

- Mass is the quantity of matter in a substance.
- The weight of an object is the net force on a falling object, or its gravitational force. The object experiences acceleration due to gravity.
- Some upward resistance force from the air acts on all falling objects on Earth, so they can never truly be in free fall.
- Careful distinctions must be made between free fall and weightlessness using the definition of weight as force due to gravity acting on an object of a certain mass.

5.5 Newton's Third Law

- Newton's third law of motion represents a basic symmetry in nature, with an experienced force equal in magnitude and opposite in direction to an exerted force.
- Two equal and opposite forces do not cancel because they act on different systems.
- Action-reaction pairs include a swimmer pushing off a wall, helicopters creating lift by pushing air down, and an octopus propelling itself forward by ejecting water from its body. Rockets, airplanes, and cars are pushed forward by a thrust reaction force.
- Choosing a system is an important analytical step in understanding the physics of a problem and solving it.

5.6 Common Forces

- When an object rests on a surface, the surface applies a force to the object that supports the weight of the object. This supporting force acts perpendicular to and away from the surface. It is called a normal force.
- When an object rests on a nonaccelerating horizontal surface, the magnitude of the normal force is equal to the weight of the object.
- When an object rests on an inclined plane that makes an angle θ with the horizontal surface, the weight of the object can be resolved into components that act perpendicular and parallel to the surface of the plane.
- The pulling force that acts along a stretched flexible connector, such as a rope or cable, is called tension. When a rope supports the weight of an object at rest, the tension in the rope is equal to the weight of the object. If the object is accelerating, tension is greater than weight, and if it is decelerating, tension is less than weight.
- The force of friction is a force experienced by a moving object (or an object that has a tendency to move) parallel to the interface opposing the motion (or its tendency).
- The force developed in a spring obeys Hooke's law, according to which its magnitude is proportional to the displacement and has a sense in the opposite direction of the displacement.
- Real forces have a physical origin, whereas fictitious forces occur because the observer is in an accelerating or noninertial frame of reference.

5.7 Drawing Free-Body Diagrams

- To draw a free-body diagram, we draw the object of interest, draw all forces acting on that object, and resolve all force vectors into x- and y-components. We must draw a separate free-body diagram for each object in the problem.
- A free-body diagram is a useful means of describing and analyzing all the forces that act on a body to determine equilibrium according to Newton's first law or acceleration according to Newton's second law.

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CHAPTER OVERVIEW

7: Applications of Newton

7.1: Prelude to Applications of Newton's Laws

Figure 7.1.1: Stock cars racing in the Grand National Divisional race at Iowa Speedway in May, 2015. Cars often reach speeds of 200 mph (320 km/h).

Car racing has grown in popularity in recent years. As each car moves in a curved path around the turn, its wheels also spin rapidly. The wheels complete many revolutions while the car makes only part of one (a circular arc). How can we describe the velocities, accelerations, and forces involved? What force keeps a racecar from spinning out, hitting the wall bordering the track? What provides this force? Why is the track banked? We answer all of these questions in this chapter as we expand our consideration of Newton’s laws of motion.

7.2: Solving Problems with Newton's Laws (Part 1)

Learning Objectives

- Apply problem-solving techniques to solve for quantities in more complex systems of forces
- Use concepts from kinematics to solve problems using Newton’s laws of motion
- Solve more complex equilibrium problems
- Solve more complex acceleration problems
- Apply calculus to more advanced dynamics problems

Success in problem solving is necessary to understand and apply physical principles. We developed a pattern of analyzing and setting up the solutions to problems involving Newton’s laws in Newton’s Laws of Motion; in this chapter, we continue to discuss these strategies and apply a step-by-step process.

Problem-Solving Strategies

We follow here the basics of problem solving presented earlier in this text, but we emphasize specific strategies that are useful in applying Newton’s laws of motion. Once you identify the physical principles involved in the problem and determine that they include Newton’s laws of motion, you can apply these steps to find a solution. These techniques also reinforce concepts that are useful in many other areas of physics. Many problem-solving strategies are stated outright in the worked examples, so the following techniques should reinforce skills you have already begun to develop.

Problem-Solving Strategy: Applying Newton’s Laws of Motion

1. Identify the physical principles involved by listing the givens and the quantities to be calculated.
2. Sketch the situation, using arrows to represent all forces.
3. Determine the system of interest. The result is a free-body diagram that is essential to solving the problem.
4. Apply Newton’s second law to solve the problem. If necessary, apply appropriate kinematic equations from the chapter on motion along a straight line.
5. Check the solution to see whether it is reasonable.

Let’s apply this problem-solving strategy to the challenge of lifting a grand piano into a second-story apartment. Once we have determined that Newton’s laws of motion are involved (if the problem involves forces), it is particularly important to draw a careful sketch of the situation. Such a sketch is shown in Figure $7.2.1a$. Then, as in Figure $7.2.1b$, we can represent all forces with arrows. Whenever sufficient information exists, it is best to label these arrows carefully and make the length and direction of each correspond to the represented force.

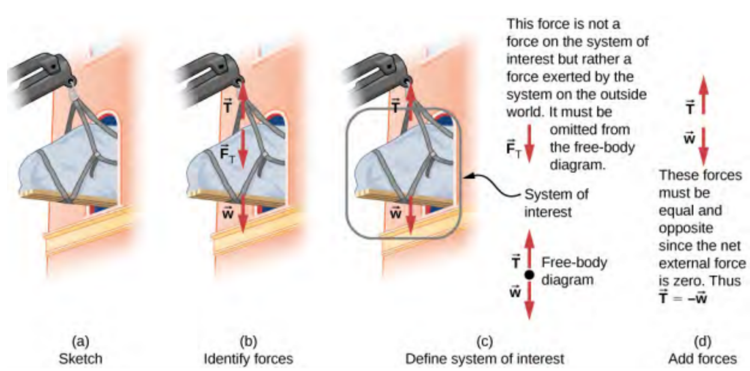


Figure $7.2.1$: (a) A grand piano is being lifted to a second-story apartment. (b) Arrows are used to represent all forces: $\vec{T}$ is the tension in the rope above the piano, $\vec{F}_T$ is the force that the piano exerts on the rope, and $\vec{w}$ is the weight of the piano. All other forces, such as the nudge of a breeze, are assumed to be negligible. (c) Suppose we are given the piano’s mass and asked to find the tension in the rope. We then define the system of interest as shown and draw a free-body diagram. Now

$\vec{F}_T$ is no longer shown, because it is not a force acting on the system of interest; rather, $\vec{F}_T$ acts on the outside world. (d) Showing only the arrows, the head-to-tail method of addition is used. It is apparent that if the piano is stationary, $\vec{T} = -\vec{w}$.

As with most problems, we next need to identify what needs to be determined and what is known or can be inferred from the problem as stated, that is, make a list of knowns and unknowns. It is particularly crucial to identify the system of interest, since Newton's second law involves only external forces. We can then determine which forces are external and which are internal, a necessary step to employ Newton's second law. (See Figure $7.2.1c$.) Newton's third law may be used to identify whether forces are exerted between components of a system (internal) or between the system and something outside (external). As illustrated in Newton's Laws of Motion, the system of interest depends on the question we need to answer. Only forces are shown in free-body diagrams, not acceleration or velocity. We have drawn several free-body diagrams in previous worked examples. Figure $7.2.1c$ shows a free-body diagram for the system of interest. Note that no internal forces are shown in a free-body diagram.

Once a free-body diagram is drawn, we apply Newton's second law. This is done in Figure $7.2.1d$ for a particular situation. In general, once external forces are clearly identified in free-body diagrams, it should be a straightforward task to put them into equation form and solve for the unknown, as done in all previous examples. If the problem is one-dimensional—that is, if all forces are parallel—then the forces can be handled algebraically. If the problem is two-dimensional, then it must be broken down into a pair of one-dimensional problems. We do this by projecting the force vectors onto a set of axes chosen for convenience. As seen in previous examples, the choice of axes can simplify the problem. For example, when an incline is involved, a set of axes with one axis parallel to the incline and one perpendicular to it is most convenient. It is almost always convenient to make one axis parallel to the direction of motion, if this is known. Generally, just write Newton's second law in components along the different directions. Then, you have the following equations:

$$\sum F_x = ma_x, \quad \sum F_y = ma_y. \tag{7.2.1}$$

(If, for example, the system is accelerating horizontally, then you can then set ay = 0.) We need this information to determine unknown forces acting on a system.

As always, we must check the solution. In some cases, it is easy to tell whether the solution is reasonable. For example, it is reasonable to find that friction causes an object to slide down an incline more slowly than when no friction exists. In practice, intuition develops gradually through problem solving; with experience, it becomes progressively easier to judge whether an answer is reasonable. Another way to check a solution is to check the units. If we are solving for force and end up with units of millimeters per second, then we have made a mistake.

There are many interesting applications of Newton's laws of motion, a few more of which are presented in this section. These serve also to illustrate some further subtleties of physics and to help build problem-solving skills. We look first at problems involving particle equilibrium, which make use of Newton's first law, and then consider particle acceleration, which involves Newton's second law.

Particle Equilibrium

Recall that a particle in equilibrium is one for which the external forces are balanced. Static equilibrium involves objects at rest, and dynamic equilibrium involves objects in motion without acceleration, but it is important to remember that these conditions are relative. For example, an object may be at rest when viewed from our frame of reference, but the same object would appear to be in motion when viewed by someone moving at a constant velocity. We now make use of the knowledge attained in Newton's Laws of Motion, regarding the different types of forces and the use of free-body diagrams, to solve additional problems in particle equilibrium.

✔ Example 6.1: Different Tensions at Different Angles

Consider the traffic light (mass of 15.0 kg) suspended from two wires as shown in Figure $7.2.2$. Find the tension in each wire, neglecting the masses of the wires.

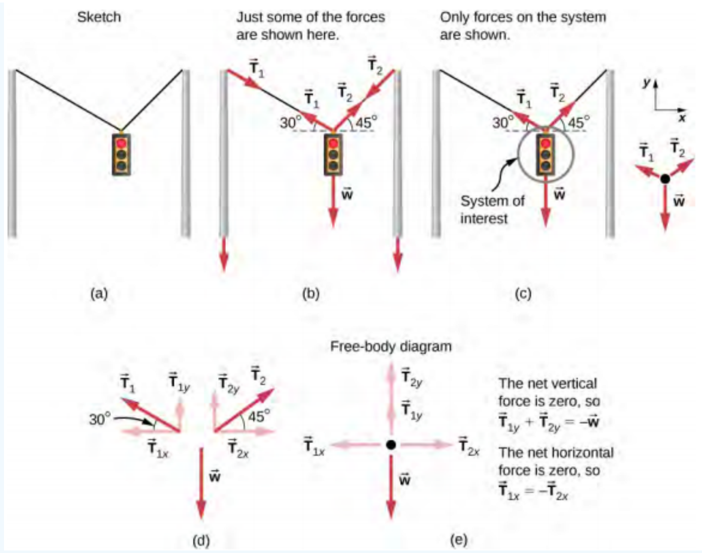


Figure $7.2.2$: A traffic light is suspended from two wires. (b) Some of the forces involved. (c) Only forces acting on the system are shown here. The free-body diagram for the traffic light is also shown. (d) The forces projected onto vertical (y) and horizontal (x) axes. The horizontal components of the tensions must cancel, and the sum of the vertical components of the tensions must equal the weight of the traffic light. (e) The free-body diagram shows the vertical and horizontal forces acting on the traffic light.

Strategy

The system of interest is the traffic light, and its free-body diagram is shown in Figure $7.2.2c$. The three forces involved are not parallel, and so they must be projected onto a coordinate system. The most convenient coordinate system has one axis vertical and one horizontal, and the vector projections on it are shown in Figure $7.2.2d$. There are two unknowns in this problem (T_1 and T_2), so two equations are needed to find them. These two equations come from applying Newton's second law along the vertical and horizontal axes, noting that the net external force is zero along each axis because acceleration is zero.

Solution

First consider the horizontal or x-axis:

$$F_{netx} = T_{2x} - T_{1x} = 0. \tag{7.2.2}$$

Thus, as you might expect,

$$T_{1x} = T_{2x}. \tag{7.2.3}$$

This give us the following relationship:

$$T_1 \cos 30^o = T_2 \cos 45^o. \tag{7.2.4}$$

Thus,

$$T_2 = 1.225 T_1. \tag{7.2.5}$$

Note that T_1 and T_2 are not equal in this case because the angles on either side are not equal. It is reasonable that T_2 ends up being greater than T_1 because it is exerted more vertically than T_1.

Now consider the force components along the vertical or y-axis:

$$F_{nety} = T_{1y} + T_{1x} - w = 0. \tag{7.2.6}$$

This implies

$$T_{1y} + T_{2y} = w. \tag{7.2.7}$$

Substituting the expressions for the vertical components gives

$$T_1 \sin 30^o + T_2 \sin 45^o = w. \tag{7.2.8}$$

There are two unknowns in this equation, but substituting the expression for T_2 in terms of T_1 reduces this to one equation with one unknown:

$$T_1(0.500) + (1.225T_1)(0.707) = w = mg, \tag{7.2.9}$$

which yields

$$1.366T_1 = (15.0\ kg)(9.80\ m/s^2). \tag{7.2.10}$$

Solving this last equation gives the magnitude of T_1 to be

$$T_1 = 108\ N. \tag{7.2.11}$$

Finally, we find the magnitude of T_2 by using the relationship between them, $T_2 = 1.225\ T_1$, found above. Thus we obtain

$$T_2 = 132\ N. \tag{7.2.12}$$

Significance

Both tensions would be larger if both wires were more horizontal, and they will be equal if and only if the angles on either side are the same (as they were in the earlier example of a tightrope walker in Newton's Laws of Motion.

✔ Example 6.2: Drag Force on a Barge

Two tugboats push on a barge at different angles (Figure 7.2.3). The first tugboat exerts a force of 2.7 x 10^5 N in the x-direction, and the second tugboat exerts a force of 3.6 x 10^5 N in the y-direction. The mass of the barge is 5.0 × 106 kg and its acceleration is observed to be 7.5 x 10^{-2} m/s^2 in the direction shown. What is the drag force of the water on the barge resisting the motion? (**Note**: Drag force is a frictional force exerted by fluids, such as air or water. The drag force opposes the motion of the object. Since the barge is flat bottomed, we can assume that the drag force is in the direction opposite of motion of the barge.)

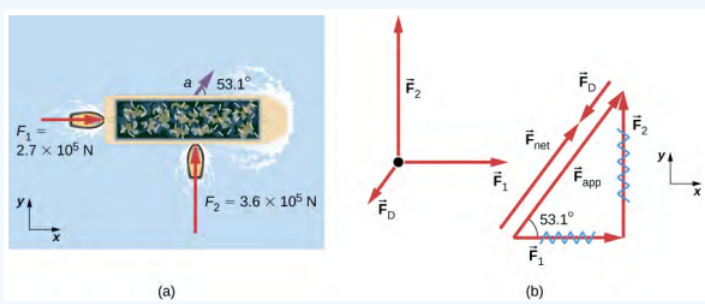


Figure 7.2.3: (a) A view from above of two tugboats pushing on a barge. (b) The free-body diagram for the ship contains only forces acting in the plane of the water. It omits the two vertical forces—the weight of the barge and the buoyant force of the water supporting it cancel and are not shown. Note that $\vec{F}_{app}$ is the total applied force of the tugboats.

Strategy

The directions and magnitudes of acceleration and the applied forces are given in Figure 7.2.3a. We define the total force of the tugboats on the barge as $\vec{F}_{app}$ so that

$$\vec{F}_{app} = \vec{F}_1 + \vec{F}_2. \tag{7.2.13}$$

The drag of the water $\vec{F}_D$ is in the direction opposite to the direction of motion of the boat; this force thus works against $\vec{F}_{app}$, as shown in the free-body diagram in Figure 7.2.3b. The system of interest here is the barge, since the forces on it are given as well as its acceleration. Because the applied forces are perpendicular, the x- and y-axes are in the same direction as $\vec{F}_1$ and $\vec{F}_2$. The problem quickly becomes a one-dimensional problem along the direction of $\vec{F}_{app}$, since friction is in the direction opposite to $\vec{F}_{app}$. Our strategy is to find the magnitude and direction of the net applied force $\vec{F}_{app}$ and then apply Newton's second law to solve for the drag force $\vec{F}_D$.

Solution

Since F_x and F_y are perpendicular, we can find the magnitude and direction of $\vec{F}_{app}$ directly. First, the resultant magnitude is given by the Pythagorean theorem:

$$\vec{F}_{app} = \sqrt{F_1^2 + F_2^2} = \sqrt{(2.7 \times 10^5\ N)^2 + (3.6 \times 10^5\ N)^2} = 4.5 \times 10^5\ N. \tag{7.2.14}$$

The angle is given by

$$\theta = \tan^{-1}\left(\frac{F_2}{F_1}\right) = \tan^{-1}\left(\frac{3.6 \times 10^5\ N}{2.7 \times 10^5\ N}\right) = 53.1^o. \tag{7.2.15}$$

From Newton’s first law, we know this is the same direction as the acceleration. We also know that $\vec{F}_D$ is in the opposite direction of $\vec{F}_{app}$, since it acts to slow down the acceleration. Therefore, the net external force is in the same direction as $\vec{F}_{app}$, but its magnitude is slightly less than $\vec{F}_{app}$. The problem is now one-dimensional. From the free-body diagram, we can see that

$$F_{net} = F_{app} - F_D. \tag{7.2.16}$$

However, Newton's second law states that

$$F_{net} = ma. \tag{7.2.17}$$

Thus,

$$F_{app} - F_D = ma. \tag{7.2.18}$$

This can be solved for the magnitude of the drag force of the water F_D in terms of known quantities:

$$F_D = F_{app} - ma. \tag{7.2.19}$$

Substituting known values gives

$$F_D = (4.5 \times 10^5\ N) - (5.0 \times 10^6\ kg)(7.5 \times 10^{-2}\ m/s^2) = 7.5 \times 10^4\ N. \tag{7.2.20}$$

The direction of $\vec{F}_D$ has already been determined to be in the direction opposite to $\vec{F}_{app}$, or at an angle of 53° south of west.

Significance

The numbers used in this example are reasonable for a moderately large barge. It is certainly difficult to obtain larger accelerations with tugboats, and small speeds are desirable to avoid running the barge into the docks. Drag is relatively small for a well-designed hull at low speeds, consistent with the answer to this example, where F_D is less than 1/600th of the weight of the ship.

In Newton’s Laws of Motion, we discussed the normal force, which is a contact force that acts normal to the surface so that an object does not have an acceleration perpendicular to the surface. The bathroom scale is an excellent example of a normal force acting on a body. It provides a quantitative reading of how much it must push upward to support the weight of an object. But can you predict what you would see on the dial of a bathroom scale if you stood on it during an elevator ride?

Will you see a value greater than your weight when the elevator starts up? What about when the elevator moves upward at a constant speed? Take a guess before reading the next example.

✓ Example 6.3: What does the Bathroom Scale Read in an Elevator?

Figure 7.2.4 shows a 75.0-kg man (weight of about 165 lb.) standing on a bathroom scale in an elevator. Calculate the scale reading: (a) if the elevator accelerates upward at a rate of 1.20 m/s^2, and (b) if the elevator moves upward at a constant speed of 1 m/s.

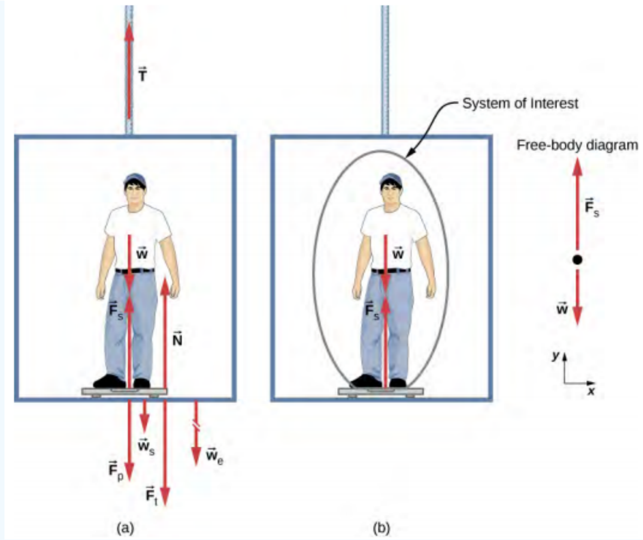


Figure $7.2.4$: (a) The various forces acting when a person stands on a bathroom scale in an elevator. The arrows are approximately correct for when the elevator is accelerating upward—broken arrows represent forces too large to be drawn to scale. $\vec{T}$ is the tension in the supporting cable, $\vec{w}$ is the weight of the person, $\vec{w}_s$ is the weight of the scale, $\vec{w}_e$ is the weight of the elevator, $\vec{F}_s$ is the force of the scale on the person, $\vec{F}_p$ is the force of the person on the scale, $\vec{F}_t$ is the force of the scale on the floor of the elevator, and $\vec{N}$ is the force of the floor upward on the scale. (b) The free-body diagram shows only the external forces acting on the designated system of interest—the person—and is the diagram we use for the solution of the problem.

Strategy

If the scale at rest is accurate, its reading equals $\vec{F}_p$, the magnitude of the force the person exerts downward on it. Figure $7.2.4a$ shows the numerous forces acting on the elevator, scale, and person. It makes this one-dimensional problem look much more formidable than if the person is chosen to be the system of interest and a free-body diagram is drawn, as in Figure $7.2.4b$. Analysis of the free-body diagram using Newton's laws can produce answers to both Figure $7.2.4a$ and (b) of this example, as well as some other questions that might arise. The only forces acting on the person are his weight $\vec{w}$ and the upward force of the scale $\vec{F}_s$. According to Newton's third law, $\vec{F}_p$ and $\vec{F}_s$ are equal in magnitude and opposite in direction, so that we need to find F_s in order to find what the scale reads. We can do this, as usual, by applying Newton's second law,

$$\vec{F}_{net} = m\vec{a}. \tag{7.2.21}$$

From the free-body diagram, we see that $\vec{F}_{net} = \vec{F}_s - \vec{w}$, so we have

$$F_s - w = ma. \tag{7.2.22}$$

Solving for F_s gives us an equation with only one unknown:

$$F_s = ma + w, \tag{7.2.23}$$

or, because w = mg, simply

$$F_s = ma + mg. \tag{7.2.24}$$

No assumptions were made about the acceleration, so this solution should be valid for a variety of accelerations in addition to those in this situation. (**Note**: We are considering the case when the elevator is accelerating upward. If the elevator is accelerating downward, Newton's second law becomes $F_s - w = -ma$.)

Solution

a. We have a = 1.20 m/s^2, so that

$$F_s = (75.0\ kg)(9.80\ m/s^2) + (75.0\ kg)(1.20\ m/s^2) \tag{7.2.25}$$

yielding

$$F_s = 825\ N. \tag{7.2.26}$$

b. Now, what happens when the elevator reaches a constant upward velocity? Will the scale still read more than his weight? For any constant velocity—up, down, or stationary—acceleration is zero because $a = \frac{\Delta v}{\Delta t}$ and $\Delta v = 0$. Thus,

$$F_s = ma + mg = 0 + mg \tag{7.2.27}$$

or

$$F_s = (75.0\ kg)(9.80\ m/s^2), \tag{7.2.28}$$

which gives

$$F_s = 735\ N. \tag{7.2.29}$$

Significance

The scale reading in Figure $7.2.4a$ is about 185 lb. What would the scale have read if he were stationary? Since his acceleration would be zero, the force of the scale would be equal to his weight:

$$F_{net} = ma = 0 = F_s - w \tag{7.2.30}$$

$$F_s = w = mg \tag{7.2.31}$$

$$F_s = (75.0\ kg)(9.80\ m/s^2) = 735\ N. \tag{7.2.32}$$

Thus, the scale reading in the elevator is greater than his 735-N (165-lb.) weight. This means that the scale is pushing up on the person with a force greater than his weight, as it must in order to accelerate him upward.

Clearly, the greater the acceleration of the elevator, the greater the scale reading, consistent with what you feel in rapidly accelerating versus slowly accelerating elevators. In Figure $7.2.4b$, the scale reading is 735 N, which equals the person's weight. This is the case whenever the elevator has a constant velocity—moving up, moving down, or stationary.

? Exercise 6.1

Now calculate the scale reading when the elevator accelerates downward at a rate of 1.20 m/s^2.

The solution to the previous example also applies to an elevator accelerating downward, as mentioned. When an elevator accelerates downward, a is negative, and the scale reading is **less** than the weight of the person. If a constant downward velocity is reached, the scale reading again becomes equal to the person's weight. If the elevator is in free fall and accelerating downward at g, then the scale reading is zero and the person appears to be weightless.

✓ Example 6.4: Two Attached Blocks

Figure $7.2.5$ shows a block of mass m_1 on a frictionless, horizontal surface. It is pulled by a light string that passes over a frictionless and massless pulley. The other end of the string is connected to a block of mass m_2. Find the acceleration of the blocks and the tension in the string in terms of m_1, m_2, and g.

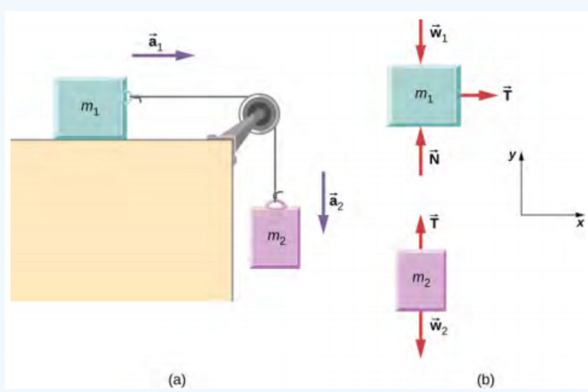


Figure $7.2.5$: (a) Block 1 is connected by a light string to block 2. (b) The free-body diagrams of the blocks.

Strategy

We draw a free-body diagram for each mass separately, as shown in Figure 7.2.5. Then we analyze each one to find the required unknowns. The forces on block 1 are the gravitational force, the contact force of the surface, and the tension in the string. Block 2 is subjected to the gravitational force and the string tension. Newton's second law applies to each, so we write two vector equations:

For block 1: $\vec{T}+\vec{w}_1+\vec{N}=m_1\vec{a}_1$

For block 2: $\vec{T}+\vec{w}_2=m_2\vec{a}_2$.

Notice that $\vec{T}$ is the same for both blocks. Since the string and the pulley have negligible mass, and since there is no friction in the pulley, the tension is the same throughout the string. We can now write component equations for each block. All forces are either horizontal or vertical, so we can use the same horizontal/vertical coordinate system for both objects.

Solution

The component equations follow from the vector equations above. We see that block 1 has the vertical forces balanced, so we ignore them and write an equation relating the x-components. There are no horizontal forces on block 2, so only the y-equation is written. We obtain these results:

Block 1	**Block 2**
$$\sum F_x=ma_x \tag{7.2.33}$$ $$T_x=m_1a_{1x} \tag{7.2.34}$$	$$\sum F_y=ma_y \tag{7.2.35}$$ $$T_y-m_2g=m_2a_{2y} \tag{7.2.36}$$

When block 1 moves to the right, block 2 travels an equal distance downward; thus, $a_{1x} = -a_{2y}$. Writing the common acceleration of the blocks as $a = a_{1x} = -a_{2y}$, we now have

$$T=m_1a \tag{7.2.37}$$

and

$$T-m_2g=-m_2a. \tag{7.2.38}$$

From these two equations, we can express a and T in terms of the masses m_1 and m_2, and g:

$$a=\frac{m_2}{m_1+m_2}g \tag{7.2.39}$$

and

$$T=\frac{m_1m_2}{m_1+m_2}g. \tag{7.2.40}$$

Significance

Notice that the tension in the string is less than the weight of the block hanging from the end of it. A common error in problems like this is to set $T = m_2g$. You can see from the free-body diagram of block 2 that cannot be correct if the block is accelerating.

Check Your Understanding 6.2

Calculate the acceleration of the system, and the tension in the string, when the masses are $m_1 = 5.00$ kg and $m_2 = 3.00$ kg.

Example 6.5: Atwood Machine

A classic problem in physics, similar to the one we just solved, is that of the Atwood machine, which consists of a rope running over a pulley, with two objects of different mass attached. It is particularly useful in understanding the connection between force and motion. In Figure 7.2.6, $m_1 = 2.00$ kg and $m_2 = 4.00$ kg. Consider the pulley to be frictionless. (a) If m_2 is released, what will its acceleration be? (b) What is the tension in the string?

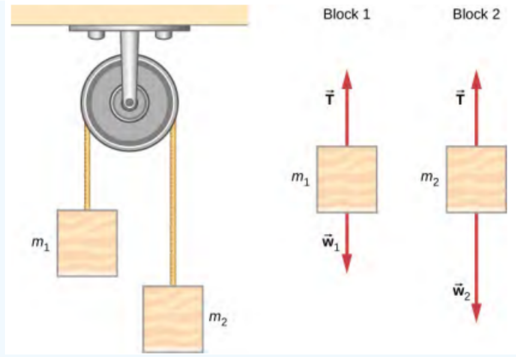


Figure 7.2.6: An Atwood machine and free-body diagrams for each of the two blocks.

Strategy

We draw a free-body diagram for each mass separately, as shown in the figure. Then we analyze each diagram to find the required unknowns. This may involve the solution of simultaneous equations. It is also important to note the similarity with the previous example. As block 2 accelerates with acceleration a_2 in the downward direction, block 1 accelerates upward with acceleration a_1. Thus, $a = a_1 = -a_2$.

Solution

a. We have

$$For\ m_1, \sum F_y = T - m_1 g = m_1 a. \quad For\ m_2, \sum F_y = T - m_2 g = -m_2 a. \tag{7.2.41}$$

(The negative sign in front of m_2 a indicates that m_2 accelerates downward; both blocks accelerate at the same rate, but in opposite directions.) Solve the two equations simultaneously (subtract them) and the result is

$$(m_2 - m_1)g = (m_1 + m_2)a. \tag{7.2.42}$$

Solving for a:

$$a = \frac{m_2 - m_1}{m_1 + m_2} g = \frac{4\ kg - 2\ kg}{4\ kg + 2\ kg}(9.8\ m/s^2) = 3.27\ m/s^2. \tag{7.2.43}$$

b. Observing the first block, we see that

$$T - m_1 g = m_1 a \tag{7.2.44}$$

$$T = m_1(g + a) = (2\ kg)(9.8\ m/s^2 + 3.27\ m/s^2) = 26.1\ N. \tag{7.2.45}$$

Significance

The result for the acceleration given in the solution can be interpreted as the ratio of the unbalanced force on the system, $(m_2 - m_1)g$, to the total mass of the system, $m_1 + m_2$. We can also use the Atwood machine to measure local gravitational field strength.

? Exercise 6.3

Determine a general formula in terms of m_1, m_2 and g for calculating the tension in the string for the Atwood machine shown above.

7.3: Solving Problems with Newton's Laws (Part 2)

Newton’s Laws of Motion and Kinematics

Physics is most interesting and most powerful when applied to general situations that involve more than a narrow set of physical principles. Newton’s laws of motion can also be integrated with other concepts that have been discussed previously in this text to solve problems of motion. For example, forces produce accelerations, a topic of kinematics, and hence the relevance of earlier chapters.

When approaching problems that involve various types of forces, acceleration, velocity, and/or position, listing the givens and the quantities to be calculated will allow you to identify the principles involved. Then, you can refer to the chapters that deal with a particular topic and solve the problem using strategies outlined in the text. The following worked example illustrates how the problem-solving strategy given earlier in this chapter, as well as strategies presented in other chapters, is applied to an integrated concept problem.

Example 6.6: What Force Must a Soccer Player Exert to Reach Top Speed?

A soccer player starts at rest and accelerates forward, reaching a velocity of 8.00 m/s in 2.50 s. (a) What is her average acceleration? (b) What average force does the ground exert forward on the runner so that she achieves this acceleration? The player’s mass is 70.0 kg, and air resistance is negligible.

Strategy

To find the answers to this problem, we use the problem-solving strategy given earlier in this chapter. The solutions to each part of the example illustrate how to apply specific problem-solving steps. In this case, we do not need to use all of the steps. We simply identify the physical principles, and thus the knowns and unknowns; apply Newton’s second law; and check to see whether the answer is reasonable.

Solution

a. We are given the initial and final velocities (zero and 8.00 m/s forward); thus, the change in velocity is Δv = 8.00 m/s . We are given the elapsed time, so Δt = 2.50 s. The unknown is acceleration, which can be found from its definition:

$$a=\frac{\Delta v}{\Delta t}. \tag{7.3.1}$$

Substituting the known values yields

$$a=\frac{8.00\ m/s}{2.50\ s}=3.20\ m/s^2. \tag{7.3.2}$$

b. Here we are asked to find the average force the ground exerts on the runner to produce this acceleration. (Remember that we are dealing with the force or forces acting on the object of interest.) This is the reaction force to that exerted by the player backward against the ground, by Newton’s third law. Neglecting air resistance, this would be equal in magnitude to the net external force on the player, since this force causes her acceleration. Since we now know the player’s acceleration and are given her mass, we can use Newton’s second law to find the force exerted. That is,

$$F_{net}=ma. \tag{7.3.3}$$

Substituting the known values of m and a gives

$$F_{net}=(70.0\ kg)(3.20\ m/s^2)=224\ N. \tag{7.3.4}$$

This is a reasonable result: The acceleration is attainable for an athlete in good condition. The force is about 50 pounds, a reasonable average force.

Significance

This example illustrates how to apply problem-solving strategies to situations that include topics from different chapters. The first step is to identify the physical principles, the knowns, and the unknowns involved in the problem. The second step is to solve for the unknown, in this case using Newton’s second law. Finally, we check our answer to ensure it is reasonable. These techniques for integrated concept problems will be useful in applications of physics outside of a physics course, such as in your profession, in other science disciplines, and in everyday life.

Exercise 6.4

The soccer player stops after completing the play described above, but now notices that the ball is in position to be stolen. If she now experiences a force of 126 N to attempt to steal the ball, which is 2.00 m away from her, how long will it take her to get to the ball?

Example 6.7: What Force Acts on a Model Helicopter?

A 1.50-kg model helicopter has a velocity of 5.00 $\hat{j}$ m/s at t = 0. It is accelerated at a constant rate for two seconds (2.00 s) after which it has a velocity of (6.00 $\hat{i}$ + 12.00 $\hat{j}$) m/s. What is the magnitude of the resultant force acting on the helicopter during this time interval?

Strategy

We can easily set up a coordinate system in which the x-axis ($\hat{i}$ direction) is horizontal, and the y-axis ($\hat{j}$ direction) is vertical. We know that Δt = 2.00s and Δv = (6.00 $\hat{i}$ + 12.00 $\hat{j}$ m/s) − (5.00 $\hat{j}$ m/s). From this, we can calculate the acceleration by the definition; we can then apply Newton's second law.

Solution

We have

$$a=\frac{\Delta v}{\Delta t}=\frac{(6.00\hat{i}+12.00\hat{j}\ m/s)-(5.00\hat{j}\ m/s)}{2.00\ s}=3.00\hat{i}+3.50\hat{j}\ m/s^2\$\$\$\$\sum\vec{F}=m\vec{a} \\ =(1.50\ kg)(3.00\hat{i}+3.50\hat{j}\ m/s^2)=4.50\hat{i}+5.25\hat{j}\ N. \tag{7.3.5}$$

The magnitude of the force is now easily found:

$$F=\sqrt{(4.50\ N)^2+(5.25\ N)^2}=6.91\ N. \tag{7.3.6}$$

Significance

The original problem was stated in terms of $\hat{i}$ − $\hat{j}$ vector components, so we used vector methods. Compare this example with the previous example.

Exercise 6.5

Find the direction of the resultant for the 1.50-kg model helicopter.

Example 6.8: Baggage Tractor

Figure $7.3.7$(a) shows a baggage tractor pulling luggage carts from an airplane. The tractor has mass 650.0 kg, while cart A has mass 250.0 kg and cart B has mass 150.0 kg. The driving force acting for a brief period of time accelerates the system from rest and acts for 3.00 s. (a) If this driving force is given by F = (820.0t) N, find the speed after 3.00 seconds. (b) What is the horizontal force acting on the connecting cable between the tractor and cart A at this instant?

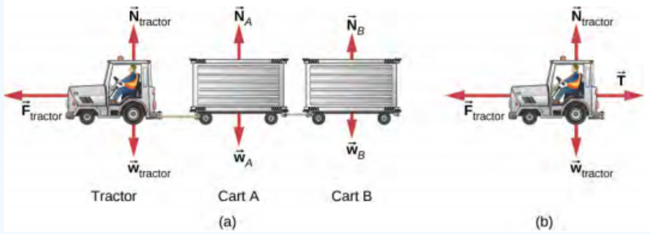


Figure 7.3.7: (a) A free-body diagram is shown, which indicates all the external forces on the system consisting of the tractor and baggage carts for carrying airline luggage. (b) A free-body diagram of the tractor only is shown isolated in order to calculate the tension in the cable to the carts.

Strategy

A free-body diagram shows the driving force of the tractor, which gives the system its acceleration. We only need to consider motion in the horizontal direction. The vertical forces balance each other and it is not necessary to consider them. For part b, we make use of a free-body diagram of the tractor alone to determine the force between it and cart A. This exposes the coupling force $\vec{T}$, which is our objective.

Solution

a.
$$\sum F_x = m_{system}\, a_x \; and \; \sum F_x = 820.0t, \tag{7.3.7}$$

so

$$820.0t = (650.0 + 250.0 + 150.0)a \tag{7.3.8}$$
$$a = 0.7809t. \tag{7.3.9}$$

Since acceleration is a function of time, we can determine the velocity of the tractor by using a = $\frac{dv}{dt}$ with the initial condition that v_0 = 0 at t = 0. We integrate from t = 0 to t = 3:

$$dv = adt$$
$$\int_0^3 dv = \int_0^{3.00} adt = \int_0^{3.00} 0.7809tdt$$
$$v = 0.3905t^2\Big]_0^{3.00} = 3.51\ m/s.$$

b. Refer to the free-body diagram in Figure 7.3.7(b)

$$\sum F_x = m_{tractor}\, a_x$$
$$820.0t - T = m_{tractor}(0.7805)t$$
$$(820.0)(3.00) - T = (650.0)(0.7805)(3.00)$$
$$T = 938\ N.$$

Significance

Since the force varies with time, we must use calculus to solve this problem. Notice how the total mass of the system was important in solving Figure 7.3.7(a), whereas only the mass of the truck (since it supplied the force) was of use in Figure 7.3.7(b).

Recall that v = $\frac{ds}{dt}$ and a = $\frac{dv}{dt}$. If acceleration is a function of time, we can use the calculus forms developed in Motion Along a Straight Line, as shown in this example. However, sometimes acceleration is a function of displacement. In this case, we can derive an important result from these calculus relations. Solving for dt in each, we have dt = $\frac{ds}{v}$ and dt = $\frac{dv}{a}$. Now, equating these expressions, we have $\frac{ds}{v}$ = $\frac{dv}{a}$. We can rearrange this to obtain a ds = v dv.

✔ Example 6.9: Motion of a Projectile Fired Vertically

A 10.0-kg mortar shell is fired vertically upward from the ground, with an initial velocity of 50.0 m/s (see Figure 7.3.8). Determine the maximum height it will travel if atmospheric resistance is measured as F_D = (0.0100 v^2) N, where v is the speed at any instant.

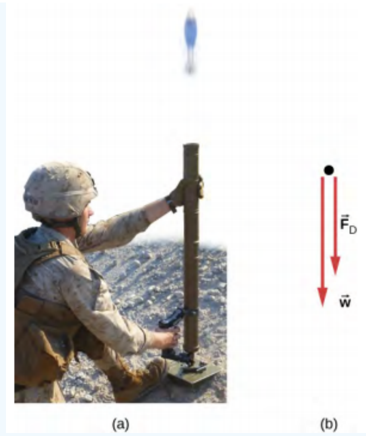


Figure 7.3.8: (a) The mortar fires a shell straight up; we consider the friction force provided by the air. (b) A free-body diagram is shown which indicates all the forces on the mortar shell.

Strategy

The known force on the mortar shell can be related to its acceleration using the equations of motion. Kinematics can then be used to relate the mortar shell's acceleration to its position.

Solution

Initially, $y_0 = 0$ and $v_0 = 50.0$ m/s. At the maximum height $y = h$, $v = 0$. The free-body diagram shows F_D to act downward, because it slows the upward motion of the mortar shell. Thus, we can write

$$\begin{aligned} \sum F_y &= ma_y \\ -F_D - w &= ma_y \\ -0.0100v^2 - 98.0 &= 10.0a \\ a &= -0.00100v^2 - 9.80. \end{aligned}$$

The acceleration depends on v and is therefore variable. Since a = f(v), we can relate a to v using the rearrangement described above,

$$ads = vdv. \tag{7.3.10}$$

We replace ds with dy because we are dealing with the vertical direction,

$$\begin{aligned} ady &= vdv \\ (-0.00100v^2 - 9.80)dy &= vdv. \end{aligned}$$

We now separate the variables (v's and dv's on one side; dy on the other):

$$\begin{aligned} \int_0^h dy &= \int_{50.0}^0 \frac{vdv}{(-0.00100v^2 - 9.80)} \\ &= -\int_{50.0}^0 \frac{vdv}{(-0.00100v^2 + 9.80)} \\ &= (-5 \times 10^3) \ln(0.00100v^2 + 9.80) \Big|_{50.0}^0. \end{aligned}$$

Thus, h = 114 m.

Significance

Notice the need to apply calculus since the force is not constant, which also means that acceleration is not constant. To make matters worse, the force depends on v (not t), and so we must use the trick explained prior to the example. The answer for the height indicates a lower elevation if there were air resistance. We will deal with the effects of air resistance and other drag forces in greater detail in Drag Force and Terminal Speed.

Exercise 6.6

If atmospheric resistance is neglected, find the maximum height for the mortar shell. Is calculus required for this solution?

Simulation

Explore the forces at work in this simulation when you try to push a filing cabinet. Create an applied force and see the resulting frictional force and total force acting on the cabinet. Charts show the forces, position, velocity, and acceleration vs. time. View a free-body diagram of all the forces (including gravitational and normal forces).

7.4: Friction (Part 1)

Learning Objectives

- Describe the general characteristics of friction
- List the various types of friction
- Calculate the magnitude of static and kinetic friction, and use these in problems involving Newton's laws of motion

When a body is in motion, it has resistance because the body interacts with its surroundings. This resistance is a force of friction. Friction opposes relative motion between systems in contact but also allows us to move, a concept that becomes obvious if you try to walk on ice. Friction is a common yet complex force, and its behavior still not completely understood. Still, it is possible to understand the circumstances in which it behaves.

Static and Kinetic Friction

The basic definition of friction is relatively simple to state.

Friction

Friction is a force that opposes relative motion between systems in contact.

There are several forms of friction. One of the simpler characteristics of sliding friction is that it is parallel to the contact surfaces between systems and is always in a direction that opposes motion or attempted motion of the systems relative to each other. If two systems are in contact and moving relative to one another, then the friction between them is called kinetic friction. For example, friction slows a hockey puck sliding on ice. When objects are stationary, static friction can act between them; the static friction is usually greater than the kinetic friction between two objects.

Static and Kinetic Friction

If two systems are in contact and stationary relative to one another, then the friction between them is called **static friction**. If two systems are in contact and moving relative to one another, then the friction between them is called **kinetic friction**.

Imagine, for example, trying to slide a heavy crate across a concrete floor—you might push very hard on the crate and not move it at all. This means that the static friction responds to what you do—it increases to be equal to and in the opposite direction of your push. If you finally push hard enough, the crate seems to slip suddenly and starts to move. Now static friction gives way to kinetic friction. Once in motion, it is easier to keep it in motion than it was to get it started, indicating that the kinetic frictional force is less than the static frictional force. If you add mass to the crate, say by placing a box on top of it, you need to push even harder to get it started and also to keep it moving. Furthermore, if you oiled the concrete you would find it easier to get the crate started and keep it going (as you might expect).

Figure $7.4.1$ is a crude pictorial representation of how friction occurs at the interface between two objects. Close-up inspection of these surfaces shows them to be rough. Thus, when you push to get an object moving (in this case, a crate), you must raise the object until it can skip along with just the tips of the surface hitting, breaking off the points, or both. A considerable force can be resisted by friction with no apparent motion. The harder the surfaces are pushed together (such as if another box is placed on the crate), the more force is needed to move them. Part of the friction is due to adhesive forces between the surface molecules of the two objects, which explains the dependence of friction on the nature of the substances. For example, rubber-soled shoes slip less than those with leather soles. Adhesion varies with substances in contact and is a complicated aspect of surface physics. Once an object is moving, there are fewer points of contact (fewer molecules adhering), so less force is required to keep the object moving. At small but nonzero speeds, friction is nearly independent of speed.

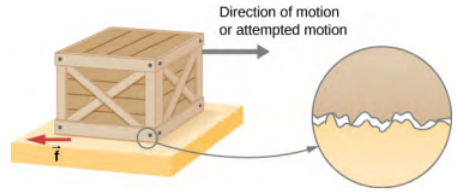


Figure 7.4.1: Frictional forces, such as $\vec{f}$, always oppose motion or attempted motion between objects in contact. Friction arises in part because of the roughness of the surfaces in contact, as seen in the expanded view. For the object to move, it must rise to where the peaks of the top surface can skip along the bottom surface. Thus, a force is required just to set the object in motion. Some of the peaks will be broken off, also requiring a force to maintain motion. Much of the friction is actually due to attractive forces between molecules making up the two objects, so that even perfectly smooth surfaces are not friction-free. (In fact, perfectly smooth, clean surfaces of similar materials would adhere, forming a bond called a "cold weld.")

The magnitude of the frictional force has two forms: one for static situations (static friction), the other for situations involving motion (kinetic friction). What follows is an approximate empirical (experimentally determined) model only. These equations for static and kinetic friction are not vector equations.

Magnitude of Static Friction

The magnitude of static friction f_s is

$$f_s \leq \mu_s N, \tag{7.4.1}$$

where μ_s is the coefficient of static friction and N is the magnitude of the normal force.

The symbol ≤ means **less than or equal to**, implying that static friction can have a maximum value of μ_sN. Static friction is a responsive force that increases to be equal and opposite to whatever force is exerted, up to its maximum limit. Once the applied force exceeds f_s (max), the object moves. Thus,

$$f_s(max) = \mu_s N. \tag{7.4.2}$$

Magnitude of Kinetic Friction

The magnitude of kinetic friction f_k is given by

$$f_k \leq \mu_k N, \tag{7.4.3}$$

where μ_k is the coefficient of kinetic friction.

A system in which $f_k = \mu_k$N is described as a system in which friction behaves simply. The transition from static friction to kinetic friction is illustrated in Figure 7.4.2

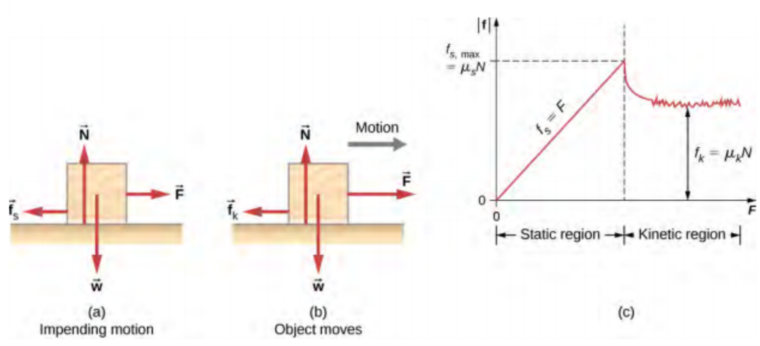


Figure 7.4.2: (a) The force of friction $\vec{f}$ between the block and the rough surface opposes the direction of the applied force $\vec{F}$. The magnitude of the static friction balances that of the applied force. This is shown in the left side of the graph in (c). (b) At some point, the magnitude of the applied force is greater than the force of kinetic friction, and the block moves to the right. This is shown in the right side of the graph. (c) The graph of the frictional force versus the applied force; note that fs (max) > f_k. This means that $\mu_s > \mu_k$

As you can see in Table 6.1, the coefficients of kinetic friction are less than their static counterparts. The approximate values of μ are stated to only one or two digits to indicate the approximate description of friction given by the preceding two equations.

Table 6.1 - Approximate Coefficients of Static and Kinetic Friction

System	Static Friction μ_s	Kinetic Friction μ_k
Rubber on dry concrete	1.0	0.7
Rubber on wet concrete	0.5-0.7	0.3-0.5
Wood on wood	0.5	0.3
Waxed wood on wet snow	0.14	0.1
Metal on wood	0.5	0.3
Steel on steel (dry)	0.6	0.3
Steel on steel (oiled)	0.05	0.03
Teflon on steel	0.04	0.04
Bone lubricated by synovial fluid	0.016	0.015
Shoes on wood	0.9	0.7
Shoes on ice	0.1	0.05
Ice on ice	0.1	0.03
Steel on ice	0.4	0.02

Equation 7.4.1 and Equation 7.4.3 include the dependence of friction on materials and the normal force. The direction of friction is always opposite that of motion, parallel to the surface between objects, and perpendicular to the normal force. For example, if the crate you try to push (with a force parallel to the floor) has a mass of 100 kg, then the normal force is equal to its weight,

$$w = mg = (100\ kg)(9.80\ m/s^2) = 980\ N, \tag{7.4.4}$$

perpendicular to the floor. If the coefficient of static friction is 0.45, you would have to exert a force parallel to the floor greater than

$$f_s(max) = \mu_s N = (0.45)(980\ N) = 440\ N \tag{7.4.5}$$

to move the crate. Once there is motion, friction is less and the coefficient of kinetic friction might be 0.30, so that a force of only

$$f_k = \mu_k N = (0.30)(980\ N) = 290\ N \tag{7.4.6}$$

keeps it moving at a constant speed. If the floor is lubricated, both coefficients are considerably less than they would be without lubrication. Coefficient of friction is a unitless quantity with a magnitude usually between 0 and 1.0. The actual value depends on the two surfaces that are in contact.

Many people have experienced the slipperiness of walking on ice. However, many parts of the body, especially the joints, have much smaller coefficients of friction—often three or four times less than ice. A joint is formed by the ends of two bones, which are connected by thick tissues. The knee joint is formed by the lower leg bone (the tibia) and the thighbone (the femur). The hip is a ball (at the end of the femur) and socket (part of the pelvis) joint. The ends of the bones in the joint are covered by cartilage, which provides a smooth, almost-glassy surface. The joints also produce a fluid (synovial fluid) that reduces friction and wear. A damaged or arthritic joint can be replaced by an artificial joint (Figure 7.4.3). These replacements can be made of metals (stainless steel or titanium) or plastic (polyethylene), also with very small coefficients of friction.

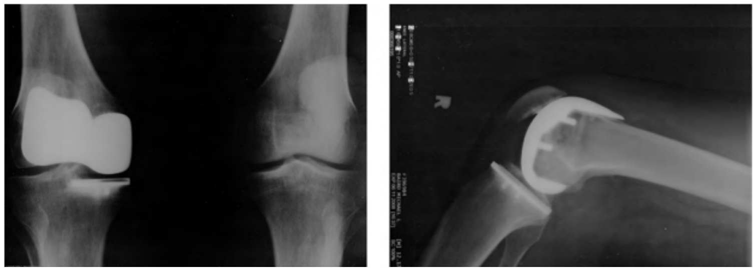


Figure 7.4.3: Artificial knee replacement is a procedure that has been performed for more than 20 years. These post-operative X-rays show a right knee joint replacement. (credit: Mike Baird)

Natural lubricants include saliva produced in our mouths to aid in the swallowing process, and the slippery mucus found between organs in the body, allowing them to move freely past each other during heartbeats, during breathing, and when a person moves. Hospitals and doctor's clinics commonly use artificial lubricants, such as gels, to reduce friction.

The equations given for static and kinetic friction are empirical laws that describe the behavior of the forces of friction. While these formulas are very useful for practical purposes, they do not have the status of mathematical statements that represent general principles (e.g., Newton's second law). In fact, there are cases for which these equations are not even good approximations. For instance, neither formula is accurate for lubricated surfaces or for two surfaces siding across each other at high speeds. Unless specified, we will not be concerned with these exceptions.

Example 6.10: Static and Kinetic Friction

A 20.0-kg crate is at rest on a floor as shown in Figure 7.4.4. The coefficient of static friction between the crate and floor is 0.700 and the coefficient of kinetic friction is 0.600. A horizontal force $\vec{P}$ is applied to the crate. Find the force of friction if (a) $\vec{P}$ = 20.0 N, (b) $\vec{P}$ = 30.0 N, (c) $\vec{P}$ = 120.0 N, and (d) $\vec{P}$ = 180.0 N.

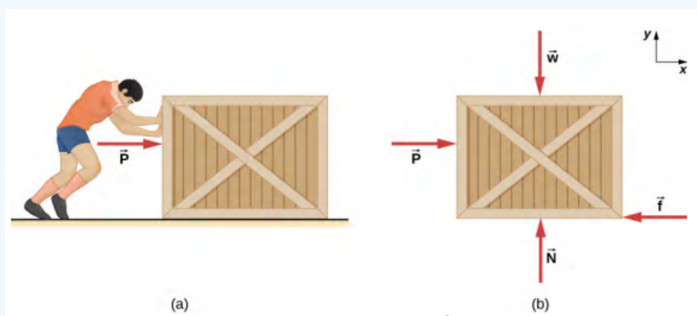


Figure 7.4.4: (a) A crate on a horizontal surface is pushed with a force $\vec{P}$. (b) The forces on the crate. Here, $\vec{f}$ may represent either the static or the kinetic frictional force.

Strategy

The free-body diagram of the crate is shown in Figure 7.4.4b. We apply Newton's second law in the horizontal and vertical directions, including the friction force in opposition to the direction of motion of the box.

Solution

Newton's second law gives

$$\sum F_x = ma_x \tag{7.4.7}$$

$$P - f = ma_x \tag{7.4.8}$$

$$\sum F_y = ma_y \tag{7.4.9}$$

$$N - w = 0. \tag{7.4.10}$$

Here we are using the symbol f to represent the frictional force since we have not yet determined whether the crate is subject to station friction or kinetic friction. We do this whenever we are unsure what type of friction is acting. Now the weight of the crate is

$$w = (20.0\ kg)(9.80\ m/s^2) = 196\ N, \tag{7.4.11}$$

which is also equal to N. The maximum force of static friction is therefore (0.700)(196 N) = 137 N. As long as $\vec{P}$ is less than 137 N, the force of static friction keeps the crate stationary and $f_s = \vec{P}$. Thus, (a) f_s = 20.0 N, (b) f_s = 30.0 N, and (c) f_s = 120.0 N. (d) If $\vec{P}$ = 180.0 N, the applied force is greater than the maximum force of static friction (137 N), so the crate can no longer remain at rest. Once the crate is in motion, kinetic friction acts. Then

$$f_k = \mu_k N = (0.600)(196\ N) = 118\ N, \tag{7.4.12}$$

and the acceleration is

$$a_x = \frac{\vec{P} - f_k}{m} = \frac{180.0\ N - 118\ N}{20.0\ kg} = 3.10\ m/s^2. \tag{7.4.13}$$

Significance

This example illustrates how we consider friction in a dynamics problem. Notice that static friction has a value that matches the applied force, until we reach the maximum value of static friction. Also, no motion can occur until the applied force equals the force of static friction, but the force of kinetic friction will then become smaller.

? Exercise 6.7

A block of mass 1.0 kg rests on a horizontal surface. The frictional coefficients for the block and surface are μ_s = 0.50 and μ_k = 0.40. (a) What is the minimum horizontal force required to move the block? (b) What is the block's acceleration when this force is applied?

7.5: Friction (Part 2)

Friction and the Inclined Plane

One situation where friction plays an obvious role is that of an object on a slope. It might be a crate being pushed up a ramp to a loading dock or a skateboarder coasting down a mountain, but the basic physics is the same. We usually generalize the sloping surface and call it an inclined plane but then pretend that the surface is flat. Let's look at an example of analyzing motion on an inclined plane with friction.

Example 7.5.1: Downhill Skier

A skier with a mass of 62 kg is sliding down a snowy slope at a constant velocity. Find the coefficient of kinetic friction for the skier if friction is known to be 45.0 N.

Strategy

The magnitude of kinetic friction is given as 45.0 N. Kinetic friction is related to the normal force N by $f_k = \mu_k N$; thus, we can find the coefficient of kinetic friction if we can find the normal force on the skier. The normal force is always perpendicular to the surface, and since there is no motion perpendicular to the surface, the normal force should equal the component of the skier's weight perpendicular to the slope. (See Figure 7.5.1, which repeats a figure from the chapter on Newton's laws of motion.)

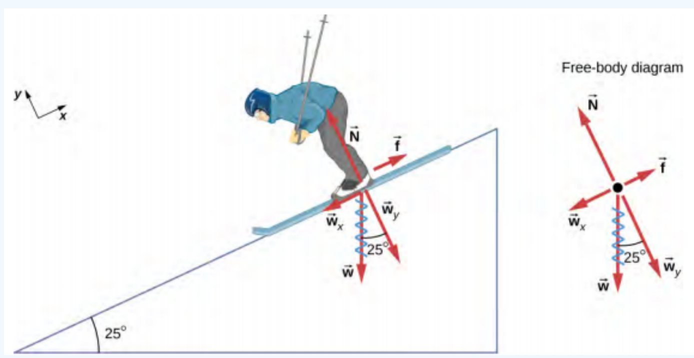


Figure 7.5.1: The motion of the skier and friction are parallel to the slope, so it is most convenient to project all forces onto a coordinate system where one axis is parallel to the slope and the other is perpendicular (axes shown to left of skier). The normal force $\vec{N}$ is perpendicular to the slope, and friction $\vec{f}$ is parallel to the slope, but the skier's weight $\vec{w}$ has components along both axes, namely $\vec{w}_y$ and $\vec{w}_x$. The normal force $\vec{N}$ is equal in magnitude to $\vec{w}_y$, so there is no motion perpendicular to the slope. However, $\vec{f}$ is less than $\vec{w}_x$ in magnitude, so there is acceleration down the slope (along the x-axis).

We have

$$N = w_y = w\cos 25^o = mg\cos 25^o. \tag{7.5.1}$$

Substituting this into our expression for kinetic friction, we obtain

$$f_k = \mu_k mg\cos 25^o, \tag{7.5.2}$$

which can now be solved for the coefficient of kinetic friction μ_k.

Solution

Solving for μ_k gives

$$\mu_k = \frac{f_k}{N} = \frac{f_k}{w\cos 25^o} = \frac{f_k}{mg\cos 25^o}. \tag{7.5.3}$$

Substituting known values on the right-hand side of the equation,

$$\mu_k = \frac{45.0\ N}{(62\ kg)(9.80\ m/s^2)(0.906)} = 0.082. \tag{7.5.4}$$

Significance

This result is a little smaller than the coefficient listed in Table 6.1 for waxed wood on snow, but it is still reasonable since values of the coefficients of friction can vary greatly. In situations like this, where an object of mass m slides down a slope that makes an angle θ with the horizontal, friction is given by $f_k = \mu_k$ mg cos θ. All objects slide down a slope with constant acceleration under these circumstances.

We have discussed that when an object rests on a horizontal surface, the normal force supporting it is equal in magnitude to its weight. Furthermore, simple friction is always proportional to the normal force. When an object is not on a horizontal surface, as with the inclined plane, we must find the force acting on the object that is directed perpendicular to the surface; it is a component of the weight.

We now derive a useful relationship for calculating coefficient of friction on an inclined plane. Notice that the result applies only for situations in which the object slides at constant speed down the ramp.

An object slides down an inclined plane at a constant velocity if the net force on the object is zero. We can use this fact to measure the coefficient of kinetic friction between two objects. As shown in Example $7.5.1$, the kinetic friction on a slope is $f_k = \mu_k$ mg cos θ. The component of the weight down the slope is equal to mg sin θ (see the free-body diagram in Figure $7.5.1$). These forces act in opposite directions, so when they have equal magnitude, the acceleration is zero. Writing these out,

$$\mu_k mg\cos\theta = mg\sin\theta. \tag{7.5.5}$$

Solving for μ_k, we find that

$$\mu_k = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta. \tag{7.5.6}$$

Put a coin on a book and tilt it until the coin slides at a constant velocity down the book. You might need to tap the book lightly to get the coin to move. Measure the angle of tilt relative to the horizontal and find μ_k. Note that the coin does not start to slide at all until an angle greater than θ is attained, since the coefficient of static friction is larger than the coefficient of kinetic friction. Think about how this may affect the value for μ_k and its uncertainty.

Atomic-Scale Explanations of Friction

The simpler aspects of friction dealt with so far are its macroscopic (large-scale) characteristics. Great strides have been made in the atomic-scale explanation of friction during the past several decades. Researchers are finding that the atomic nature of friction seems to have several fundamental characteristics. These characteristics not only explain some of the simpler aspects of friction—they also hold the potential for the development of nearly friction-free environments that could save hundreds of billions of dollars in energy which is currently being converted (unnecessarily) into heat.

Figure $7.5.2$ illustrates one macroscopic characteristic of friction that is explained by microscopic (small-scale) research. We have noted that friction is proportional to the normal force, but not to the amount of area in contact, a somewhat counterintuitive notion. When two rough surfaces are in contact, the actual contact area is a tiny fraction of the total area because only high spots touch. When a greater normal force is exerted, the actual contact area increases, and we find that the friction is proportional to this area.

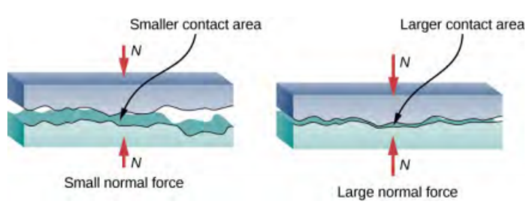


Figure $7.5.2$: Two rough surfaces in contact have a much smaller area of actual contact than their total area. When the normal force is larger as a result of a larger applied force, the area of actual contact increases, as does friction.

However, the atomic-scale view promises to explain far more than the simpler features of friction. The mechanism for how heat is generated is now being determined. In other words, why do surfaces get warmer when rubbed? Essentially, atoms are linked with one another to form lattices. When surfaces rub, the surface atoms adhere and cause atomic lattices to vibrate—essentially creating sound waves that penetrate the material. The sound waves diminish with distance, and their energy is converted into heat. Chemical reactions that are related to frictional wear can also occur between atoms and molecules on the surfaces. Figure $7.5.3$ shows how the tip of a probe drawn across another material is deformed by atomic-scale friction. The force needed to drag the tip can be measured and is found to be related to shear stress, which is discussed in Static Equilibrium and Elasticity. The variation in shear stress is remarkable (more than a factor of 1012) and difficult to predict theoretically, but shear stress is yielding a fundamental understanding of a large-scale phenomenon known since ancient times—friction.

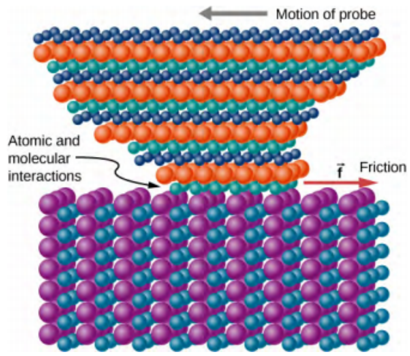


Figure 7.5.3: The tip of a probe is deformed sideways by frictional force as the probe is dragged across a surface. Measurements of how the force varies for different materials are yielding fundamental insights into the atomic nature of friction.

Simulation

Describe a model for friction on a molecular level. Describe matter in terms of molecular motion. The description should include diagrams to support the description; how the temperature affects the image; what are the differences and similarities between solid, liquid, and gas particle motion; and how the size and speed of gas molecules relate to everyday objects.

Example 7.5.2: Sliding Blocks

The two blocks of Figure 7.5.4 are attached to each other by a massless string that is wrapped around a frictionless pulley. When the bottom 4.00-kg block is pulled to the left by the constant force $\vec{P}$, the top 2.00-kg block slides across it to the right. Find the magnitude of the force necessary to move the blocks at constant speed. Assume that the coefficient of kinetic friction between all surfaces is 0.400.

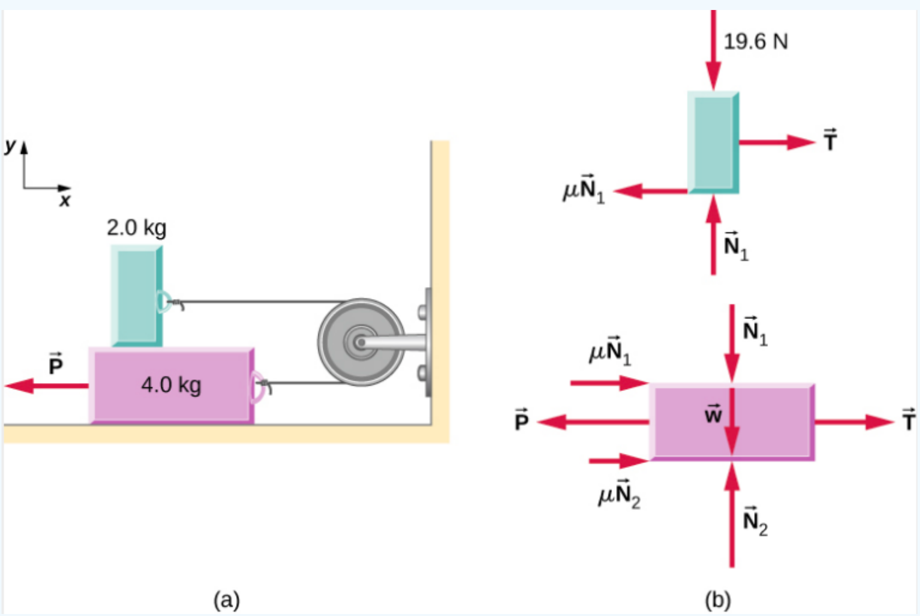


Figure 7.5.4: (a) Each block moves at constant velocity. (b) Free-body diagrams for the blocks.

Strategy

We analyze the motions of the two blocks separately. The top block is subjected to a contact force exerted by the bottom block. The components of this force are the normal force N_1 and the frictional force $-0.400\ N_1$. Other forces on the top block are the tension T in the string and the weight of the top block itself, 19.6 N. The bottom block is subjected to contact forces due to the top block and due to the floor. The first contact force has components $-N_1$ and $0.400\ N_1$, which are simply reaction forces to the contact forces that the bottom block exerts on the top block. The components of the contact force of the floor are N_2 and

0.400 N_2. Other forces on this block are −P, the tension T, and the weight –39.2 N. Solution Since the top block is moving horizontally to the right at constant velocity, its acceleration is zero in both the horizontal and the vertical directions. From Newton's second law,

$$\sum F_x = m_2 a_x \tag{7.5.7}$$

$$T - 0.400\ N_1 = 0 \tag{7.5.8}$$

$$\sum F_y = m_1 a_y \tag{7.5.9}$$

$$N_1 - 19.6\ N = 0. \tag{7.5.10}$$

Solving for the two unknowns, we obtain N_1 = 19.6 N and T = 0.40 N_1 = 7.84 N. The bottom block is also not accelerating, so the application of Newton's second law to this block gives

$$\sum F_x = m_2 a_x \tag{7.5.11}$$

$$T - P + 0.400\ N_1 + 0.400\ N_2 = 0 \tag{7.5.12}$$

$$\sum F_y = m_1 a_y \tag{7.5.13}$$

$$N_2 - 39.2\ N - N_1 = 0. \tag{7.5.14}$$

The values of N_1 and T were found with the first set of equations. When these values are substituted into the second set of equations, we can determine N_2 and P. They are

$$N_2 = 58.8\ N \text{ and } P = 39.2\ N. \tag{7.5.15}$$

Significance

Understanding what direction in which to draw the friction force is often troublesome. Notice that each friction force labeled in Figure 7.5.4 acts in the direction opposite the motion of its corresponding block.

✓ Example 7.5.3: A Crate on an Accelerating Truck

A 50.0-kg crate rests on the bed of a truck as shown in Figure 7.5.5. The coefficients of friction between the surfaces are μ_k = 0.300 and μ_s = 0.400. Find the frictional force on the crate when the truck is accelerating forward relative to the ground at (a) 2.00 m/s^2, and (b) 5.00 m/s^2.

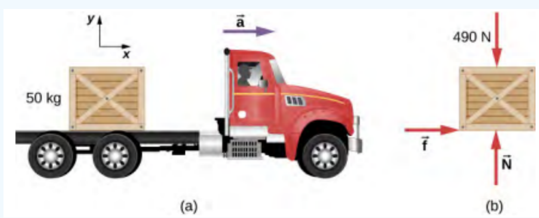


Figure 7.5.5: (a) A crate rests on the bed of the truck that is accelerating forward. (b) The free-body diagram of the crate.

Strategy

The forces on the crate are its weight and the normal and frictional forces due to contact with the truck bed. We start by assuming that the crate is not slipping. In this case, the static frictional force fs acts on the crate. Furthermore, the accelerations of the crate and the truck are equal.

Solution

a. Application of Newton's second law to the crate, using the reference frame attached to the ground, yields

$$\begin{aligned} \sum F_x &= ma_x \\ f_s &= (50.0\ kg)(2.00\ m/s^2) \\ &= 1.00 \times 10^2\ N \end{aligned}$$

$$\begin{aligned} \sum F_y &= ma_y \\ N - 4.90 \times 10^2\ N &= (50.0\ kg)(0) \\ N &= 4.90 \times 10^2\ N. \end{aligned}$$

We can now check the validity of our no-slip assumption. The maximum value of the force of static friction is

$$\mu_s N = (0.400)(4.90 \times 10^2\ N) = 196\ N, \tag{7.5.16}$$

whereas the **actual** force of static friction that acts when the truck accelerates forward at 2.00 m/s^2 is only 1.00 x 10^2 N. Thus, the assumption of no slipping is valid.

b. If the crate is to move with the truck when it accelerates at 5.0 m/s^2, the force of static friction must be

$$f_s = ma_x = (50.0\ kg)(5.00\ m/s^2) = 250\ N. \tag{7.5.17}$$

Since this exceeds the maximum of 196 N, the crate must slip. The frictional force is therefore kinetic and is

$$f_k = \mu_k N = (0.300)(4.90 \times 10^2\ N) = 147\ N. \tag{7.5.18}$$

The horizontal acceleration of the crate relative to the ground is now found from $$

$$\begin{aligned} \sum F_x &= ma_x \\ 147\ N &= (50.0\ kg)a_x, \\ so\ a_x &= 2.94\ m/s^2. \end{aligned}$$

\]

Significance

Relative to the ground, the truck is accelerating forward at 5.0 m/s^2 and the crate is accelerating forward at 2.94 m/s^2. Hence the crate is sliding backward relative to the bed of the truck with an acceleration 2.94 m/s^2 − 5.00 m/s^2 = −2.06 m/s^2.

Example 7.5.4: Snowboarding

Earlier, we analyzed the situation of a downhill skier moving at constant velocity to determine the coefficient of kinetic friction. Now let's do a similar analysis to determine acceleration. The snowboarder of Figure 7.5.6 glides down a slope that is inclined at θ = 13° to the horizontal. The coefficient of kinetic friction between the board and the snow is μ_k = 0.20. What is the acceleration of the snowboarder?

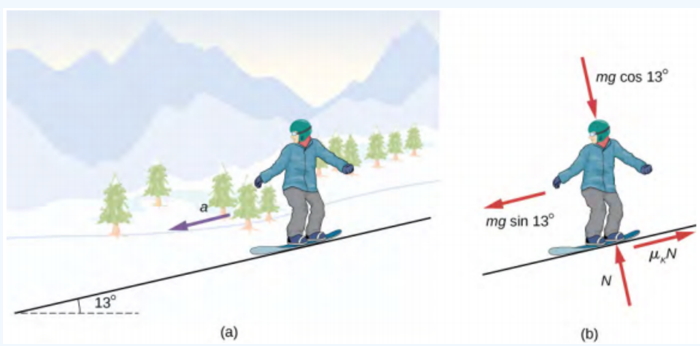


Figure 7.5.6: (a) A snowboarder glides down a slope inclined at 13° to the horizontal. (b) The free-body diagram of the snowboarder.

Strategy

The forces acting on the snowboarder are her weight and the contact force of the slope, which has a component normal to the incline and a component along the incline (force of kinetic friction). Because she moves along the slope, the most convenient reference frame for analyzing her motion is one with the x-axis along and the y-axis perpendicular to the incline. In this frame, both the normal and the frictional forces lie along coordinate axes, the components of the weight are mg sin θ along the slope and mg cos θ at right angles into the slope , and the only acceleration is along the x-axis ($a_y = 0$).

Solution

We can now apply Newton's second law to the snowboarder:

$\begin{aligned} \sum F_x &= ma_x \\ mg\sin\theta - \mu_k N &= ma_x \end{aligned}$	$\begin{aligned} \sum F_y &= ma_y \\ N - mg\cos\theta &= m(0). \end{aligned}$

From the second equation, N = mg cos θ. Upon substituting this into the first equation, we find

$$\begin{aligned} a_x &= g(\sin\theta - \mu_k \cos\theta) \\ &= g(\sin 13^o - 0.520 \cos 13^o) = 0.29\ m/s^2. \end{aligned}$$

Significance

Notice from this equation that if θ is small enough or μ_k is large enough, a_x is negative, that is, the snowboarder slows down.

? Exercise 7.5.4

The snowboarder is now moving down a hill with incline 10.0°. What is the skier's acceleration?

7.6: Centripetal Force

Learning Objectives

- Explain the equation for centripetal acceleration
- Apply Newton's second law to develop the equation for centripetal force
- Use circular motion concepts in solving problems involving Newton's laws of motion

In Motion in Two and Three Dimensions, we examined the basic concepts of circular motion. An object undergoing circular motion, like one of the race cars shown at the beginning of this chapter, must be accelerating because it is changing the direction of its velocity. We proved that this centrally directed acceleration, called centripetal acceleration, is given by the formula

$$a_c = \frac{v^2}{r} \tag{7.6.1}$$

where v is the velocity of the object, directed along a tangent line to the curve at any instant. If we know the angular velocity ω, then we can use

$$a_c = r\omega^2. \tag{7.6.2}$$

Angular velocity gives the rate at which the object is turning through the curve, in units of rad/s. This acceleration acts along the radius of the curved path and is thus also referred to as a radial acceleration.

An acceleration must be produced by a force. Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth's gravity on the Moon, friction between roller skates and a rink floor, a banked roadway's force on a car, and forces on the tube of a spinning centrifuge. Any net force causing uniform circular motion is called a **centripetal force**. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton's second law of motion, net force is mass times acceleration: F_{net} = ma. For uniform circular motion, the acceleration is the centripetal acceleration: a = a_c. Thus, the magnitude of centripetal force F_c is

$$F_c = ma_c. \tag{7.6.3}$$

By substituting the expressions for centripetal acceleration a_c ($a_c = \frac{v^2}{r}; a_c = r\omega^2$), we get two expressions for the centripetal force F_c in terms of mass, velocity, angular velocity, and radius of curvature:

$$F_c = m\frac{v^2}{r}; \quad F_c = mr\omega^2. \tag{7.6.4}$$

You may use whichever expression for centripetal force is more convenient. Centripetal force $\vec{F}_c$ is always perpendicular to the path and points to the center of curvature, because $\vec{a}_c$ is perpendicular to the velocity and points to the center of curvature. Note that if you solve the first expression for r, you get

$$r = \frac{mv^2}{F_c}. \tag{7.6.5}$$

This implies that for a given mass and velocity, a large centripetal force causes a small radius of curvature—that is, a tight curve, as in Figure 7.6.1.

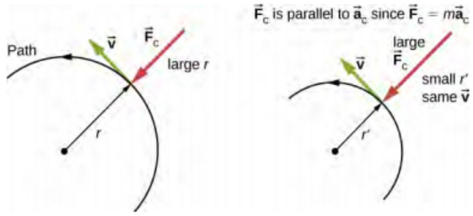


Figure 7.6.1: The frictional force supplies the centripetal force and is numerically equal to it. Centripetal force is perpendicular to velocity and causes uniform circular motion. The larger the F_c, the smaller the radius of curvature r and the sharper the curve. The second curve has the same v, but a larger F_c produces a smaller r′.

Example 7.6.1: What Coefficient of Friction Do Cars Need on a Flat Curve?

a. Calculate the centripetal force exerted on a 900.0-kg car that negotiates a 500.0-m radius curve at 25.00 m/s.
b. Assuming an unbanked curve, find the minimum static coefficient of friction between the tires and the road, static friction being the reason that keeps the car from slipping (Figure $7.6.2$).

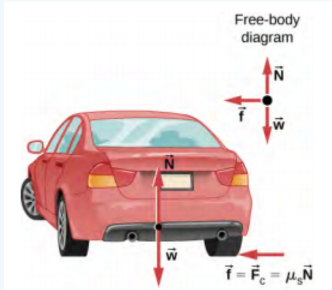


Figure $7.6.2$: This car on level ground is moving away and turning to the left. The centripetal force causing the car to turn in a circular path is due to friction between the tires and the road. A minimum coefficient of friction is needed, or the car will move in a larger-radius curve and leave the roadway.

Strategy

a. We know that $F_c = m\frac{v^2}{r}$. Thus

$$F_c = m\frac{v^2}{r} = \frac{(900.0\ kg)(25.00\ m/s)^2}{(500.0\ m)} = 1125\ N. \tag{7.6.6}$$

b. Figure $7.6.2$ shows the forces acting on the car on an unbanked (level ground) curve. Friction is to the left, keeping the car from slipping, and because it is the only horizontal force acting on the car, the friction is the centripetal force in this case. We know that the maximum static friction (at which the tires roll but do not slip) is μ_s N, where μ_s is the static coefficient of friction and N is the normal force. The normal force equals the car's weight on level ground, so N = mg. Thus the centripetal force in this situation is

$$F_c = f = \mu_s N = \mu_s mg. \tag{7.6.7}$$

Now we have a relationship between centripetal force and the coefficient of friction. Using the equation

$$F_c = m\frac{v^2}{r}. \tag{7.6.8}$$

we obtain

$$m\frac{v^2}{r} = \mu_s mg. \tag{7.6.9}$$

We solve this for μ_s, noting that mass cancels, and obtain

$$\mu_s = \frac{v^2}{rg}. \tag{7.6.10}$$

Substituting the knowns,

$$\mu_s = \frac{(25.00\ m/s)^2}{(500.0\ m)(9.80\ m/s^2)} = 0.13. \tag{7.6.11}$$

(Because coefficients of friction are approximate, the answer is given to only two digits.)

Significance

The coefficient of friction found in Figure $7.6.2b$ is much smaller than is typically found between tires and roads. The car still negotiates the curve if the coefficient is greater than 0.13, because static friction is a responsive force, able to assume a value less than but no more than μ_sN. A higher coefficient would also allow the car to negotiate the curve at a higher speed, but if the coefficient of friction is less, the safe speed would be less than 25 m/s. Note that mass cancels, implying that, in this example, it does not matter how heavily loaded the car is to negotiate the turn. Mass cancels because friction is assumed proportional to the normal force, which in turn is proportional to mass. If the surface of the road were banked, the normal force would be less, as discussed next.

Exercise 7.6.1

A car moving at 96.8 km/h travels around a circular curve of radius 182.9 m on a flat country road. What must be the minimum coefficient of static friction to keep the car from slipping?

Banked Curves

Let us now consider **banked curves**, where the slope of the road helps you negotiate the curve (Figure $7.6.3$). The greater the angle θ , the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve and consider an example related to it.

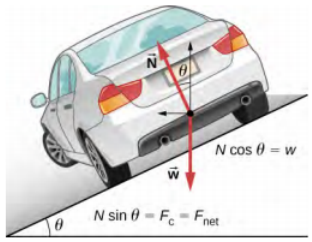


Figure $7.6.3$: The car on this banked curve is moving away and turning to the left.

For **ideal banking**, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.

Figure $7.6.3$ shows a free-body diagram for a car on a frictionless banked curve. If the angle θ is ideal for the speed and radius, then the net external force equals the necessary centripetal force. The only two external forces acting on the car are its weight $\vec{w}$ and the normal force of the road $\vec{N}$. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude $\frac{mv^2}{r}$. Because this is the crucial force and it is horizontal, we use a coordinate system with vertical and horizontal axes. Only the normal force has a horizontal component, so this must equal the centripetal force, that is,

$$N \sin\theta = \frac{mv^2}{r}. \tag{7.6.12}$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From Figure $7.6.3$, we see that the vertical component of the normal force is N cos θ, and the only other vertical force is the car's weight. These must be equal in magnitude; thus,

$$N \cos\theta = mg. \tag{7.6.13}$$

Now we can combine these two equations to eliminate N and get an expression for θ, as desired. Solving the second equation for N = $\frac{mg}{(\cos\theta)}$ and substituting this into the first yields

$$\begin{aligned} mg\frac{\sin\theta}{\cos\theta} &= \frac{mv^2}{r} \\ mg\tan\theta &= \frac{mv^2}{r} \\ \tan\theta &= \frac{v^2}{rg}. \end{aligned}$$

Taking the inverse tangent gives

$$\theta = \tan^{-1}\left(\frac{v^2}{rg}\right). \tag{7.6.14}$$

This expression can be understood by considering how θ depends on v and r. A large θ is obtained for a large v and a small r. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve were frictionless. Note that θ does not depend on the mass of the vehicle.

Example 7.6.2: What Is the Ideal Speed to Take a Steeply Banked Tight Curve?

Curves on some test tracks and race courses, such as Daytona International Speedway in Florida, are very steeply banked. This banking, with the aid of tire friction and very stable car configurations, allows the curves to be taken at very high speed. To illustrate, calculate the speed at which a 100.0-m radius curve banked at 31.0° should be driven if the road were frictionless.

Strategy

We first note that all terms in the expression for the ideal angle of a banked curve except for speed are known; thus, we need only rearrange it so that speed appears on the left-hand side and then substitute known quantities.

Solution

Starting with

$$\tan\theta = \frac{v^2}{rg}, \tag{7.6.15}$$

we get

$$v = \sqrt{rg\tan\theta}. \tag{7.6.16}$$

Noting that tan 31.0° = 0.609, we obtain

$$v = \sqrt{(100.0\ m)(9.80\ m/s^2)(0.609)} = 24.4\ m/s. \tag{7.6.17}$$

Significance

This is just about 165 km/h, consistent with a very steeply banked and rather sharp curve. Tire friction enables a vehicle to take the curve at significantly higher speeds.

Airplanes also make turns by banking. The lift force, due to the force of the air on the wing, acts at right angles to the wing. When the airplane banks, the pilot is obtaining greater lift than necessary for level flight. The vertical component of lift balances the airplane's weight, and the horizontal component accelerates the plane. The banking angle shown in Figure 7.6.4 is given by θ. We analyze the forces in the same way we treat the case of the car rounding a banked curve.

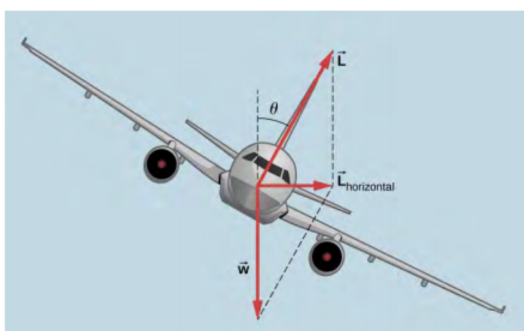


Figure 7.6.4: In a banked turn, the horizontal component of lift is unbalanced and accelerates the plane. The normal component of lift balances the plane's weight. The banking angle is given by θ. Compare the vector diagram with that shown in Figure 6.22.

Simulation

Join the ladybug in an exploration of rotational motion. Rotate the merry-go-round to change its angle or choose a constant angular velocity or angular acceleration. Explore how circular motion relates to the bug's xy-position, velocity, and acceleration using vectors or graphs.

Note

A circular motion requires a force, the so-called centripetal force, which is directed to the axis of rotation. This simplified model of a carousel demonstrates this force.

Inertial Forces and Noninertial (Accelerated) Frames: The Coriolis Force

What do taking off in a jet airplane, turning a corner in a car, riding a merry-go-round, and the circular motion of a tropical cyclone have in common? Each exhibits inertial forces—forces that merely seem to arise from motion, because the observer's frame of reference is accelerating or rotating. When taking off in a jet, most people would agree it feels as if you are being pushed back into the seat as the airplane accelerates down the runway. Yet a physicist would say that you tend to remain stationary while the seat pushes forward on you. An even more common experience occurs when you make a tight curve in your car—say, to the right (Figure $7.6.5$). You feel as if you are thrown (that is, forced) toward the left relative to the car. Again, a physicist would say that you are going in a straight line (recall Newton's first law) but the car moves to the right, not that you are experiencing a force from the left.

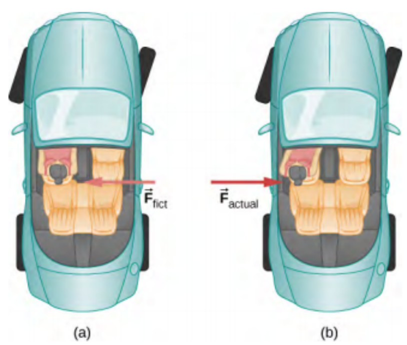


Figure $7.6.5$: (a) The car driver feels herself forced to the left relative to the car when she makes a right turn. This is an inertial force arising from the use of the car as a frame of reference. (b) In Earth's frame of reference, the driver moves in a straight line, obeying Newton's first law, and the car moves to the right. There is no force to the left on the driver relative to Earth. Instead, there is a force to the right on the car to make it turn.

We can reconcile these points of view by examining the frames of reference used. Let us concentrate on people in a car. Passengers instinctively use the car as a frame of reference, whereas a physicist might use Earth. The physicist might make this choice because Earth is nearly an inertial frame of reference, in which all forces have an identifiable physical origin. In such a frame of reference, Newton's laws of motion take the form given in Newton's Laws of Motion. The car is a **noninertial frame of reference** because it is accelerated to the side. The force to the left sensed by car passengers is an **inertial force** having no physical origin (it is due purely to the inertia of the passenger, not to some physical cause such as tension, friction, or gravitation). The car, as well as the driver, is actually accelerating to the right. This inertial force is said to be an inertial force because it does not have a physical origin, such as gravity.

A physicist will choose whatever reference frame is most convenient for the situation being analyzed. There is no problem to a physicist in including inertial forces and Newton's second law, as usual, if that is more convenient, for example, on a merry-go-round or on a rotating planet. Noninertial (accelerated) frames of reference are used when it is useful to do so. Different frames of reference must be considered in discussing the motion of an astronaut in a spacecraft traveling at speeds near the speed of light, as you will appreciate in the study of the special theory of relativity.

Let us now take a mental ride on a merry-go-round—specifically, a rapidly rotating playground merry-go-round (Figure $7.6.6$). You take the merry-go-round to be your frame of reference because you rotate together. When rotating in that noninertial frame of reference, you feel an inertial force that tends to throw you off; this is often referred to as a centrifugal force (not to be confused with centripetal force). Centrifugal force is a commonly used term, but it does not actually exist. You must hang on tightly to counteract your inertia (which people often refer to as centrifugal force). In Earth's frame of reference, there is no force trying to throw you off; we emphasize that centrifugal force is a fiction. You must hang on to make yourself go in a circle because otherwise you would go in a straight line, right off the merry-go-round, in keeping with Newton's first law. But the force you exert acts toward the center of the circle.

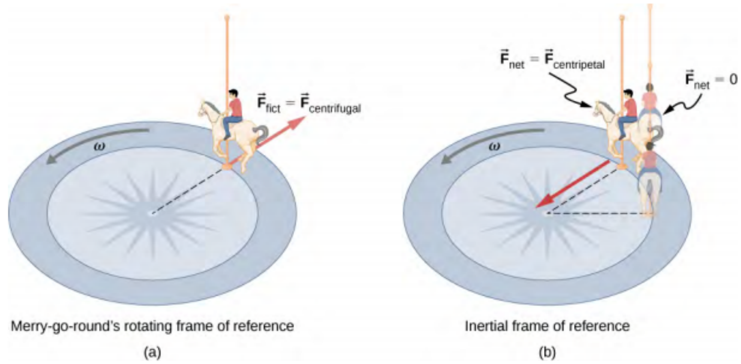


Figure 7.6.6: (a) A rider on a merry-go-round feels as if he is being thrown off. This inertial force is sometimes mistakenly called the centrifugal force in an effort to explain the rider's motion in the rotating frame of reference. (b) In an inertial frame of reference and according to Newton's laws, it is his inertia that carries him off (the unshaded rider has $F_{net} = 0$ and heads in a straight line). A force, $F_{centripetal}$, is needed to cause a circular path.

This inertial effect, carrying you away from the center of rotation if there is no centripetal force to cause circular motion, is put to good use in centrifuges (Figure 7.6.7). A centrifuge spins a sample very rapidly, as mentioned earlier in this chapter. Viewed from the rotating frame of reference, the inertial force throws particles outward, hastening their sedimentation. The greater the angular velocity, the greater the centrifugal force. But what really happens is that the inertia of the particles carries them along a line tangent to the circle while the test tube is forced in a circular path by a centripetal force.

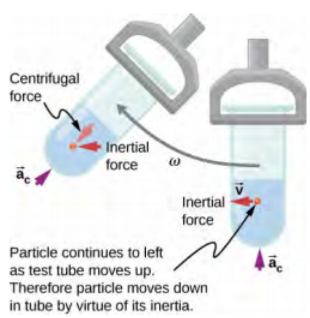


Figure 7.6.7: Centrifuges use inertia to perform their task. Particles in the fluid sediment settle out because their inertia carries them away from the center of rotation. The large angular velocity of the centrifuge quickens the sedimentation. Ultimately, the particles come into contact with the test tube walls, which then supply the centripetal force needed to make them move in a circle of constant radius.

Let us now consider what happens if something moves in a rotating frame of reference. For example, what if you slide a ball directly away from the center of the merry-go-round, as shown in Figure 7.6.8? The ball follows a straight path relative to Earth (assuming negligible friction) and a path curved to the right on the merry-go-round's surface. A person standing next to the merry-go-round sees the ball moving straight and the merry-go-round rotating underneath it. In the merry-go-round's frame of reference, we explain the apparent curve to the right by using an inertial force, called the **Coriolis force**, which causes the ball to curve to the right. The Coriolis force can be used by anyone in that frame of reference to explain why objects follow curved paths and allows us to apply Newton's laws in noninertial frames of reference.

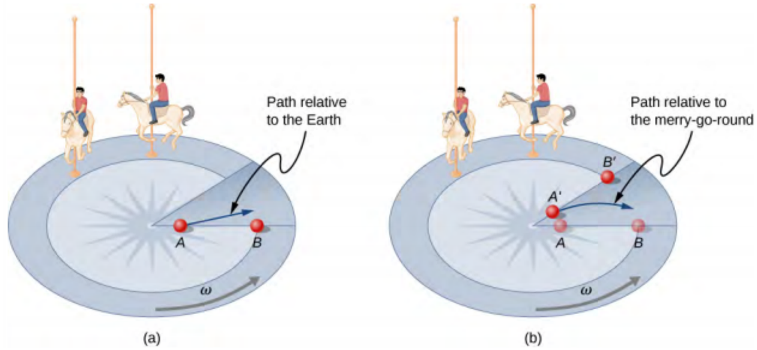


Figure 7.6.8: Looking down on the counterclockwise rotation of a merry-go-round, we see that a ball slid straight toward the edge follows a path curved to the right. The person slides the ball toward point B, starting at point A. Both points rotate to the shaded positions (A' and B') shown in the time that the ball follows the curved path in the rotating frame and a straight path in Earth's frame.

Up until now, we have considered Earth to be an inertial frame of reference with little or no worry about effects due to its rotation. Yet such effects **do** exist—in the rotation of weather systems, for example. Most consequences of Earth's rotation can be qualitatively understood by analogy with the merry-go-round. Viewed from above the North Pole, Earth rotates counterclockwise, as does the merry-go-round in Figure 7.6.8. As on the merry-go-round, any motion in Earth's Northern Hemisphere experiences a Coriolis force to the right. Just the opposite occurs in the Southern Hemisphere; there, the force is to the left. Because Earth's angular velocity is small, the Coriolis force is usually negligible, but for large-scale motions, such as wind patterns, it has substantial effects.

The Coriolis force causes hurricanes in the Northern Hemisphere to rotate in the counterclockwise direction, whereas tropical cyclones in the Southern Hemisphere rotate in the clockwise direction. (The terms hurricane, typhoon, and tropical storm are regionally specific names for cyclones, which are storm systems characterized by low pressure centers, strong winds, and heavy rains.) Figure 7.6.9 helps show how these rotations take place. Air flows toward any region of low pressure, and tropical cyclones contain particularly low pressures. Thus winds flow toward the center of a tropical cyclone or a low-pressure weather system at the surface. In the Northern Hemisphere, these inward winds are deflected to the right, as shown in the figure, producing a counterclockwise circulation at the surface for low-pressure zones of any type. Low pressure at the surface is associated with rising air, which also produces cooling and cloud formation, making low-pressure patterns quite visible from space. Conversely, wind circulation around high-pressure zones is clockwise in the Southern Hemisphere but is less visible because high pressure is associated with sinking air, producing clear skies.

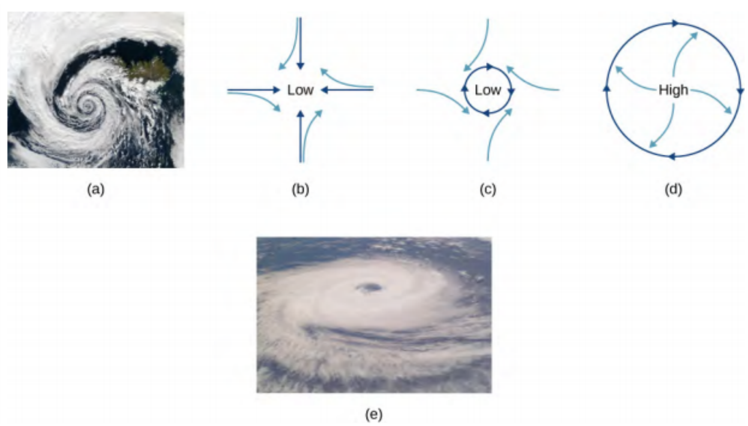


Figure 7.6.9: (a) The counterclockwise rotation of this Northern Hemisphere hurricane is a major consequence of the Coriolis force. (b) Without the Coriolis force, air would flow straight into a low-pressure zone, such as that found in tropical cyclones. (c) The Coriolis force deflects the winds to the right, producing a counterclockwise rotation. (d) Wind flowing away from a high-pressure zone is also deflected to the right, producing a clockwise rotation. (e) The opposite direction of rotation is produced by the Coriolis force in the Southern Hemisphere, leading to tropical cyclones. (credit a and credit e: modifications of work by NASA)

The rotation of tropical cyclones and the path of a ball on a merry-go-round can just as well be explained by inertia and the rotation of the system underneath. When noninertial frames are used, inertial forces, such as the Coriolis force, must be invented to explain the curved path. There is no identifiable physical source for these inertial forces. In an inertial frame, inertia explains the path, and no force is found to be without an identifiable source. Either view allows us to describe nature, but a view in an inertial frame is the simplest in the sense that all forces have origins and explanations.

7.7: Drag Force and Terminal Speed

Learning Objectives

- Express the drag force mathematically
- Describe applications of the drag force
- Define terminal velocity
- Determine an object’s terminal velocity given its mass

Another interesting force in everyday life is the force of drag on an object when it is moving in a fluid (either a gas or a liquid). You feel the drag force when you move your hand through water. You might also feel it if you move your hand during a strong wind. The faster you move your hand, the harder it is to move. You feel a smaller drag force when you tilt your hand so only the side goes through the air—you have decreased the area of your hand that faces the direction of motion.

Drag Forces

Like friction, the **drag force** always opposes the motion of an object. Unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid. This functionality is complicated and depends upon the shape of the object, its size, its velocity, and the fluid it is in. For most large objects such as cyclists, cars, and baseballs not moving too slowly, the magnitude of the drag force F_D is proportional to the square of the speed of the object. We can write this relationship mathematically as $F_D \propto v^2$. When taking into account other factors, this relationship becomes

$$F_D = \frac{1}{2} C \rho A v^2, \tag{7.7.1}$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid. (Recall that density is mass per unit volume.) This equation can also be written in a more generalized fashion as $F_D = bv^2$, where b is a constant equivalent to $0.5C\rho A$. We have set the exponent n for these equations as 2 because when an object is moving at high velocity through air, the magnitude of the drag force is proportional to the square of the speed. As we shall see in Fluid Mechanics, for small particles moving at low speeds in a fluid, the exponent n is equal to 1.

Definition: Drag Force

Drag force F_D is proportional to the square of the speed of the object. Mathematically,

$$F_D = \frac{1}{2} C \rho A v^2, \tag{7.7.2}$$

where C is the drag coefficient, A is the area of the object facing the fluid, and ρ is the density of the fluid.

Athletes as well as car designers seek to reduce the drag force to lower their race times (Figure $7.7.1A$). Aerodynamic shaping of an automobile can reduce the drag force and thus increase a car’s gas mileage. The value of the drag coefficient C is determined empirically, usually with the use of a wind tunnel (Figure $7.7.1B$).

Figure 7.7.1: (A) From racing cars to bobsled racers, aerodynamic shaping is crucial to achieving top speeds. Bobsleds are designed for speed and are shaped like a bullet with tapered fins. (credit: "U.S. Army"/Wikimedia Commons) (B): NASA researchers test a model plane in a wind tunnel. (credit: NASA/Ames).

The drag coefficient can depend upon velocity, but we assume that it is a constant here. Table 7.7.1 lists some typical drag coefficients for a variety of objects. Notice that the drag coefficient is a dimensionless quantity. At highway speeds, over 50% of the power of a car is used to overcome air drag. The most fuel-efficient cruising speed is about 70–80 km/h (about 45–50 mi/h). For this reason, during the 1970s oil crisis in the United States, maximum speeds on highways were set at about 90 km/h (55 mi/h).

Table 7.7.1: Typical Values of Drag Coefficient C

Object	C
Airfoil	0.05
Toyota Camry	0.28
Ford Focus	0.32
Honda Civic	0.36
Ferrari Testarossa	0.37
Dodge Ram Pickup	0.43
Sphere	0.45
Hummer H2 SUV	0.64
Skydiver (feet first)	0.70
Bicycle	0.90
Skydiver (horizontal)	1.0
Circular flat plate	1.12

Substantial research is under way in the sporting world to minimize drag. The dimples on golf balls are being redesigned, as are the clothes that athletes wear. Bicycle racers and some swimmers and runners wear full bodysuits. Australian Cathy Freeman wore a full body suit in the 2000 Sydney Olympics and won a gold medal in the 400-m race. Many swimmers in the 2008 Beijing Olympics wore (Speedo) body suits; it might have made a difference in breaking many world records (Figure 7.7.2). Most elite swimmers (and cyclists) shave their body hair. Such innovations can have the effect of slicing away milliseconds in a race, sometimes making the difference between a gold and a silver medal. One consequence is that careful and precise guidelines must be continuously developed to maintain the integrity of the sport.

Figure 7.7.2: Body suits, such as this LZR Racer Suit, have been credited with aiding in many world records after their release in 2008. Smoother “skin” and more compression forces on a swimmer’s body provide at least 10% less drag. (credit: NASA/Kathy Barnstorff)

Terminal Velocity

Some interesting situations connected to Newton’s second law occur when considering the effects of drag forces upon a moving object. For instance, consider a skydiver falling through air under the influence of gravity. The two forces acting on him are the force of gravity and the drag force (ignoring the small buoyant force). The downward force of gravity remains constant regardless of the velocity at which the person is moving. However, as the person’s velocity increases, the magnitude of the drag force increases until the magnitude of the drag force is equal to the gravitational force, thus producing a net force of zero. A zero net force means that there is no acceleration, as shown by Newton’s second law. At this point, the person’s velocity remains constant and we say that the person has reached his **terminal velocity** (v_T). Since F_D is proportional to the speed squared, a heavier skydiver must go faster for F_D to equal his weight. Let’s see how this works out more quantitatively.

At the terminal velocity,

$$F_{net} = mg - F_D = ma = 0. \tag{7.7.3}$$

Thus,

$$mg = F_D. \tag{7.7.4}$$

Using the equation for drag force, we have

$$mg = \frac{1}{2} C\rho A v_T^2. \tag{7.7.5}$$

Solving for the velocity, we obtain

$$v_T = \sqrt{\frac{2mg}{\rho CA}}. \tag{7.7.6}$$

Assume the density of air is ρ = 1.21 kg/m^3. A 75-kg skydiver descending head first has a cross-sectional area of approximately A = 0.18 m^2 and a drag coefficient of approximately C = 0.70. We find that

$$v_T = \sqrt{\frac{2(75\ kg)(9.80\ m/s^2)}{(1.21\ kg/m^3)(0.70)(0.18\ m^2)}} = 98\ m/s = 350\ km/h. \tag{7.7.7}$$

This means a skydiver with a mass of 75 kg achieves a terminal velocity of about 350 km/h while traveling in a pike (head first) position, minimizing the area and his drag. In a spread-eagle position, that terminal velocity may decrease to about 200 km/h as the area increases. This terminal velocity becomes much smaller after the parachute opens.

✓ Example 7.7.1: Terminal Velocity of a Skydiver

Find the terminal velocity of an 85-kg skydiver falling in a spread-eagle position.

Strategy

At terminal velocity, $F_{net} = 0$. Thus, the drag force on the skydiver must equal the force of gravity (the person’s weight). Using the equation of drag force, we find $mg = \frac{1}{2}\rho CAv^2$.

Solution

The terminal velocity v_T can be written as

$$v_T = \sqrt{\frac{2mg}{\rho CA}} = \sqrt{\frac{2(85\ kg)(9.80\ m/s^2)}{(1.21\ kg/m^3)(1.0)(0.70\ m^2)}} = 44\ m/s. \tag{7.7.8}$$

Significance

This result is consistent with the value for v_T mentioned earlier. The 75-kg skydiver going feet first had a terminal velocity of v_T = 98 m/s. He weighed less but had a smaller frontal area and so a smaller drag due to the air.

? Exercise 7.7.1

Find the terminal velocity of a 50-kg skydiver falling in spread-eagle fashion.

The size of the object that is falling through air presents another interesting application of air drag. If you fall from a 5-m-high branch of a tree, you will likely get hurt—possibly fracturing a bone. However, a small squirrel does this all the time, without getting hurt. You do not reach a terminal velocity in such a short distance, but the squirrel does.

The following interesting quote on animal size and terminal velocity is from a 1928 essay by a British biologist, J. B. S. Haldane, titled "On Being the Right Size."

> *"To the mouse and any smaller animal, [gravity] presents practically no dangers. You can drop a mouse down a thousand-yard mine shaft; and, on arriving at the bottom, it gets a slight shock and walks away, provided that the ground is fairly soft. A rat is killed, a man is broken, and a horse splashes. For the resistance presented to movement by the air is proportional to the surface of the moving object. Divide an animal's length, breadth, and height each by ten; its weight is reduced to a thousandth, but its surface only to a hundredth. So the resistance to falling in the case of the small animal is relatively ten times greater than the driving force."*

The above quadratic dependence of air drag upon velocity does not hold if the object is very small, is going very slow, or is in a denser medium than air. Then we find that the drag force is proportional just to the velocity. This relationship is given by Stokes' law.

Stokes' law

For a spherical object falling in a medium, the drag force is

$$F_s = 6\pi r\eta v, \tag{7.7.9}$$

where r is the radius of the object, η is the viscosity of the fluid, and v is the object's velocity.

Good examples of Stokes' law are provided by microorganisms, pollen, and dust particles. Because each of these objects is so small, we find that many of these objects travel unaided only at a constant (terminal) velocity. Terminal velocities for bacteria (size about \(1\, \mu m) can be about \(2\, \mu m/s. To move at a greater speed, many bacteria swim using flagella (organelles shaped like little tails) that are powered by little motors embedded in the cell.

Sediment in a lake can move at a greater terminal velocity (about 5 μm/s), so it can take days for it to reach the bottom of the lake after being deposited on the surface.

If we compare animals living on land with those in water, you can see how drag has influenced evolution. Fish, dolphins, and even massive whales are streamlined in shape to reduce drag forces. Birds are streamlined and migratory species that fly large distances often have particular features such as long necks. Flocks of birds fly in the shape of a spearhead as the flock forms a streamlined pattern (Figure 7.7.3). In humans, one important example of streamlining is the shape of sperm, which need to be efficient in their use of energy.

Figure $7.7.3$: Geese fly in a V formation during their long migratory travels. This shape reduces drag and energy consumption for individual birds, and also allows them a better way to communicate. (credit: “Julo”/Wikimedia Commons)

In lecture demonstrations, we do measurements of the drag force on different objects. The objects are placed in a uniform airstream created by a fan. Calculate the Reynolds number and the drag coefficient.

Video $7.7.1$: Fluid Mechanics - Drag force - Flow simulation

The Calculus of Velocity-Dependent Frictional Forces

When a body slides across a surface, the frictional force on it is approximately constant and given by $\mu_k N$. Unfortunately, the frictional force on a body moving through a liquid or a gas does not behave so simply. This drag force is generally a complicated function of the body’s velocity. However, for a body moving in a straight line at moderate speeds through a liquid such as water, the frictional force can often be approximated by

$$f_R = -bv, \tag{7.7.10}$$

where b is a constant whose value depends on the dimensions and shape of the body and the properties of the liquid, and v is the velocity of the body. Two situations for which the frictional force can be represented this equation are a motorboat moving through water and a small object falling slowly through a liquid.

Let’s consider the object falling through a liquid. The free-body diagram of this object with the positive direction downward is shown in Figure $7.7.4$. Newton’s second law in the vertical direction gives the differential equation

$$mg - bv = m\frac{dv}{dt}, \tag{7.7.11}$$

where we have written the acceleration as $\frac{dv}{dt}$. As v increases, the frictional force $-bv$ increases until it matches mg. At this point, there is no acceleration and the velocity remains constant at the terminal velocity v_T. From the previous equation,

$$mg - bv_T = 0, \tag{7.7.12}$$

so

$$v_T = \frac{mg}{b}. \tag{7.7.13}$$

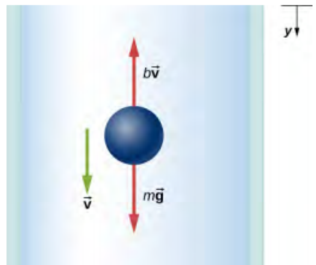


Figure 7.7.4: Free-body diagram of an object falling through a resistive medium.

We can find the object's velocity by integrating the differential equation for v. First, we rearrange terms in this equation to obtain

$$\frac{dv}{g - \left(\frac{b}{m}\right)v} = dt. \tag{7.7.14}$$

Assuming that $v = 0$ at \9t = 0\), integration of Equation 7.7.14yields

$$\int_0^v \frac{dv'}{g - \left(\frac{b}{m}\right)v'} = \int_0^t dt', \tag{7.7.15}$$

or

$$-\frac{m}{b}\ln\left(g - \frac{b}{m}v'\right)\Bigg|_0^v = t'\Big|_0^t, \tag{7.7.16}$$

where v' and t' are dummy variables of integration. With the limits given, we find

$$-\frac{m}{b}[\ln\left(g - \frac{b}{m}v\right) - \ln g] = t. \tag{7.7.17}$$

Since $\ln A - \ln B = \ln(\left(\frac{A}{B}\right)$, and $\ln(\left(\frac{A}{B}\right) = x$ implies $e^x = \dfrac{A}{B}$, we obtain

$$\frac{g - \left(\frac{bv}{m}\right)}{g} = e^{-\frac{bt}{m}}, \tag{7.7.18}$$

and

$$v = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right). \tag{7.7.19}$$

Notice that as t $\rightarrow \infty$, v $\rightarrow$ $\frac{mg}{b}$ = v_T, which is the terminal velocity.

The position at any time may be found by integrating the equation for v. With v = $\frac{dy}{dt}$,

$$dy = \frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)dt. \tag{7.7.20}$$

Assuming y = 0 when t = 0,

$$\int_0^y dy' = \frac{mg}{b}\int_0^t \left(1 - e^{-\frac{bt}{m}}\right)dt', \tag{7.7.21}$$

which integrates to

$$y = \frac{mg}{b}t + \frac{m^2 g}{b^2}\left(e^{-\frac{bt}{m}} - 1\right). \tag{7.7.22}$$

✔ Example 7.7.2: Effect of the Resistive Force on a Motorboat

A motorboat is moving across a lake at a speed v_0 when its motor suddenly freezes up and stops. The boat then slows down under the frictional force $f_R = -bv$.

a. What are the velocity and position of the boat as functions of time?
b. If the boat slows down from 4.0 to 1.0 m/s in 10 s, how far does it travel before stopping?

Solution

a. With the motor stopped, the only horizontal force on the boat is $f_R = -bv$, so from Newton's second law,

$$m\frac{dv}{dt} = -bv, \tag{7.7.23}$$

which we can write as

$$\frac{dv}{v} = -\frac{b}{m}dt. \tag{7.7.24}$$

Integrating this equation between the time zero when the velocity is v_0 and the time t when the velocity is v, we have

$$\int_0^v \frac{dv'}{v'} = -\frac{b}{m}\int_0^t dt'. \tag{7.7.25}$$

Thus,

$$\ln\frac{v}{v_0} = -\frac{b}{m}t, \tag{7.7.26}$$

which, since $\ln A = x$ implies $e^x = A$, we can write this as

$$v = v_0 e^{-\frac{bt}{m}}. \tag{7.7.27}$$

Now from the definition of velocity,

$$\frac{dx}{dt} = v_0 e^{-\frac{bt}{m}}, \tag{7.7.28}$$

so we have

$$dx = v_0 e^{-\frac{bt}{m}} dt. \tag{7.7.29}$$

With the initial position zero, we have

$$\int_0^x dx' = v_0 \int_0^t e^{-\frac{bt'}{m}} dt', \tag{7.7.30}$$

and

$$x = -\frac{mv_0}{b}e^{-\frac{bt}{m}}\Big|_0^t = \frac{mv_0}{b}\left(1 - e^{-\frac{bt}{m}}\right). \tag{7.7.31}$$

As time increases, $e^{-\frac{bt}{m}} \rightarrow 0$, and the position of the boat approaches a limiting value

$$x_{max} = \frac{mv_0}{b}. \tag{7.7.32}$$

Although this tells us that the boat takes an infinite amount of time to reach x_{max}, the boat effectively stops after a reasonable time. For example, at $t = \frac{10m}{b}$, we have

$$v = v_0 e^{-10} \simeq 4.5 \times 10^{-5} v_0, \tag{7.7.33}$$

whereas we also have

$$x = x_{max}\left(1 - e^{-10}\right) \simeq 0.99995 x_{max}. \tag{7.7.34}$$

Therefore, the boat's velocity and position have essentially reached their final values.

b. With v_0 = 4.0 m/s and v = 1.0 m/s, we have 1.0 m/s = (4.0 m/s) $e^{(-\frac{bt}{m})(10\ s)}$, so

$$\ln 0.25 = -\ln 4.0 = -\frac{b}{m}(10\ s), \tag{7.7.35}$$

and

$$\frac{b}{m} = \frac{1}{10}\ln 4.0\ s^{-1} = 0.14\ s^{-1}. \tag{7.7.36}$$

Now the boat's limiting position is

$$x_{max} = \frac{mv_0}{b} = \frac{4.0\ m/s}{0.14\ s^{-1}} = 29\ m. \tag{7.7.37}$$

Significance

In the both of the previous examples, we found "limiting" values. The terminal velocity is the same as the limiting velocity, which is the velocity of the falling object after a (relatively) long time has passed. Similarly, the limiting distance of the boat is the distance the boat will travel after a long amount of time has passed. Due to the properties of exponential decay, the time involved to reach either of these values is actually not too long (certainly not an infinite amount of time!) but they are quickly found by taking the limit to infinity.

? Exercise 7.7.2

Suppose the resistive force of the air on a skydiver can be approximated by $f = -bv^2$. If the terminal velocity of a 100-kg skydiver is 60 m/s, what is the value of b?

7.8: Applications of Newton's Laws (Exercises)

Conceptual Questions

6.1 Solving Problems with Newton's Laws

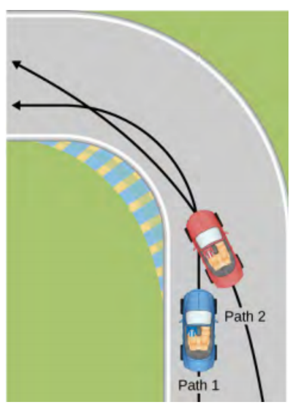
1. To simulate the apparent weightlessness of space orbit, astronauts are trained in the hold of a cargo aircraft that is accelerating downward at g. Why do they appear to be weightless, as measured by standing on a bathroom scale, in this accelerated frame of reference? Is there any difference between their apparent weightlessness in orbit and in the aircraft?

6.2 Friction

2. The glue on a piece of tape can exert forces. Can these forces be a type of simple friction? Explain, considering especially that tape can stick to vertical walls and even to ceilings.
3. When you learn to drive, you discover that you need to let up slightly on the brake pedal as you come to a stop or the car will stop with a jerk. Explain this in terms of the relationship between static and kinetic friction.
4. When you push a piece of chalk across a chalkboard, it sometimes screeches because it rapidly alternates between slipping and sticking to the board. Describe this process in more detail, in particular, explaining how it is related to the fact that kinetic friction is less than static friction. (The same slip-grab process occurs when tires screech on pavement.)
5. A physics major is cooking breakfast when she notices that the frictional force between her steel spatula and Teflon frying pan is only 0.200 N. Knowing the coefficient of kinetic friction between the two materials, she quickly calculates the normal force. What is it?

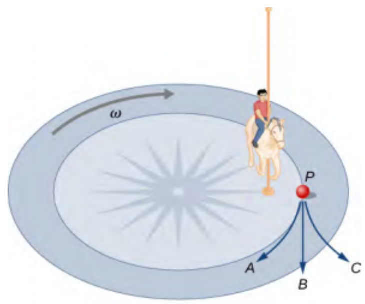
6.3 Centripetal Force

6. If you wish to reduce the stress (which is related to centripetal force) on high-speed tires, would you use large- or small-diameter tires? Explain.
7. Define centripetal force. Can any type of force (for example, tension, gravitational force, friction, and so on) be a centripetal force? Can any combination of forces be a centripetal force?
8. If centripetal force is directed toward the center, why do you feel that you are ‘thrown’ away from the center as a car goes around a curve? Explain.
9. Race car drivers routinely cut corners, as shown below (Path 2). Explain how this allows the curve to be taken at the greatest speed.

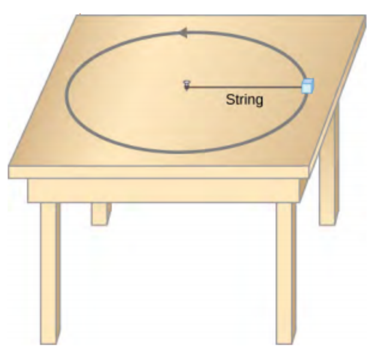


10. Many amusement parks have rides that make vertical loops like the one shown below. For safety, the cars are attached to the rails in such a way that they cannot fall off. If the car goes over the top at just the right speed, gravity alone will supply the centripetal force. What other force acts and what is its direction if:
    a. The car goes over the top at faster than this speed?
    b. The car goes over the top at slower than this speed?

11. What causes water to be removed from clothes in a spin-dryer?
12. As a skater forms a circle, what force is responsible for making his turn? Use a free-body diagram in your answer.
13. Suppose a child is riding on a merry-go-round at a distance about halfway between its center and edge. She has a lunch box resting on wax paper, so that there is very little friction between it and the merry-go-round. Which path shown below will the lunch box take when she lets go? The lunch box leaves a trail in the dust on the merry-go-round. Is that trail straight, curved to the left, or curved to the right? Explain your answer.



14. Do you feel yourself thrown to either side when you negotiate a curve that is ideally banked for your car's speed? What is the direction of the force exerted on you by the car seat?
15. Suppose a mass is moving in a circular path on a frictionless table as shown below. In Earth's frame of reference, there is no centrifugal force pulling the mass away from the center of rotation, yet there is a force stretching the string attaching the mass to the nail. Using concepts related to centripetal force and Newton's third law, explain what force stretches the string, identifying its physical origin.



16. When a toilet is flushed or a sink is drained, the water (and other material) begins to rotate about the drain on the way down. Assuming no initial rotation and a flow initially directly straight toward the drain, explain what causes the rotation and which direction it has in the Northern Hemisphere. (Note that this is a small effect and in most toilets the rotation is caused by directional water jets.) Would the direction of rotation reverse if water were forced up the drain?
17. A car rounds a curve and encounters a patch of ice with a very low coefficient of kinetic fiction. The car slides off the road. Describe the path of the car as it leaves the road.

18. In one amusement park ride, riders enter a large vertical barrel and stand against the wall on its horizontal floor. The barrel is spun up and the floor drops away. Riders feel as if they are pinned to the wall by a force something like the gravitational force. This is an inertial force sensed and used by the riders to explain events in the rotating frame of reference of the barrel. Explain in an inertial frame of reference (Earth is nearly one) what pins the riders to the wall, and identify all forces acting on them.
19. Two friends are having a conversation. Anna says a satellite in orbit is in free fall because the satellite keeps falling toward Earth. Tom says a satellite in orbit is not in free fall because the acceleration due to gravity is not 9.80 m/s^2. Who do you agree with and why?
20. A nonrotating frame of reference placed at the center of the Sun is very nearly an inertial one. Why is it not exactly an inertial frame?

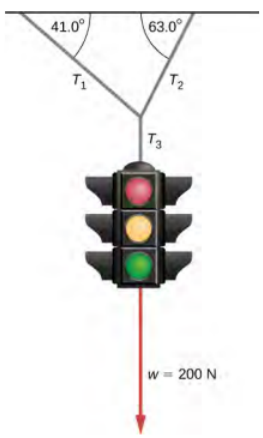
6.4 Drag Force and Terminal Speed

21. Athletes such as swimmers and bicyclists wear body suits in competition. Formulate a list of pros and cons of such suits.
22. Two expressions were used for the drag force experienced by a moving object in a liquid. One depended upon the speed, while the other was proportional to the square of the speed. In which types of motion would each of these expressions be more applicable than the other one?
23. As cars travel, oil and gasoline leaks onto the road surface. If a light rain falls, what does this do to the control of the car? Does a heavy rain make any difference?
24. Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?

Problems

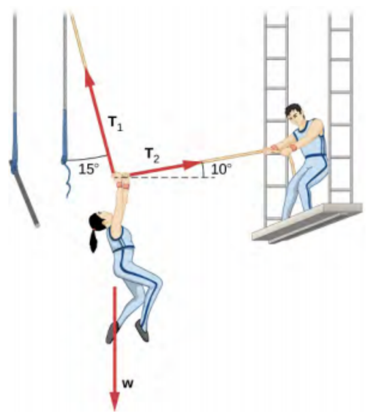
6.1 Solving Problems with Newton's Laws

25. A 30.0-kg girl in a swing is pushed to one side and held at rest by a horizontal force $\vec{F}$ so that the swing ropes are 30.0° with respect to the vertical. (a) Calculate the tension in each of the two ropes supporting the swing under these conditions. (b) Calculate the magnitude of $\vec{F}$.
26. Find the tension in each of the three cables supporting the traffic light if it weighs 2.00 x 10^2 N.



27. Three forces act on an object, considered to be a particle, which moves with constant velocity v = $(3\,\hat{i} - 2\,\hat{j})$ m/s. Two of the forces are $\vec{F}_1 = (3\,\hat{i} + 5\,\hat{j} - 6k\,\hat{k})$ N and $\vec{F}_2 = (4\,\hat{i} - 7\,\hat{j} + 2\,\hat{k})$ N. Find the third force.
28. A flea jumps by exerting a force of 1.20 x 10^{-5} N straight down on the ground. A breeze blowing on the flea parallel to the ground exerts a force of 0.500 x 10^{-6} N on the flea while the flea is still in contact with the ground. Find the direction and magnitude of the acceleration of the flea if its mass is 6.00 x 10^{-7} kg . Do not neglect the gravitational force.
29. Two muscles in the back of the leg pull upward on the Achilles tendon, as shown below. (These muscles are called the medial and lateral heads of the gastrocnemius muscle.) Find the magnitude and direction of the total force on the Achilles tendon. What type of movement could be caused by this force?

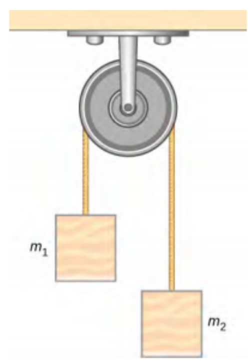
30. After a mishap, a 76.0-kg circus performer clings to a trapeze, which is being pulled to the side by another circus artist, as shown here. Calculate the tension in the two ropes if the person is momentarily motionless. Include a free-body diagram in your solution.



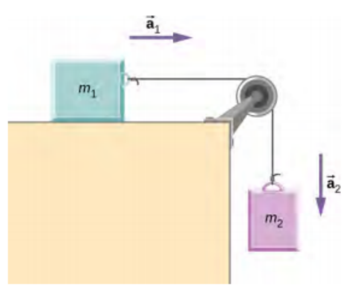
31. A 35.0-kg dolphin decelerates from 12.0 to 7.50 m/s in 2.30 s to join another dolphin in play. What average force was exerted to slow the first dolphin if it was moving horizontally? (The gravitational force is balanced by the buoyant force of the water.)
32. When starting a foot race, a 70.0-kg sprinter exerts an average force of 650 N backward on the ground for 0.800 s. (a) What is his final speed? (b) How far does he travel?
33. A large rocket has a mass of 2.00×10^6 kg at takeoff, and its engines produce a thrust of 3.50×10^7 N. (a) Find its initial acceleration if it takes off vertically. (b) How long does it take to reach a velocity of 120 km/h straight up, assuming constant mass and thrust?
34. A basketball player jumps straight up for a ball. To do this, he lowers his body 0.300 m and then accelerates through this distance by forcefully straightening his legs. This player leaves the floor with a vertical velocity sufficient to carry him 0.900 m above the floor. (a) Calculate his velocity when he leaves the floor. (b) Calculate his acceleration while he is straightening his legs. He goes from zero to the velocity found in (a) in a distance of 0.300 m. (c) Calculate the force he exerts on the floor to do this, given that his mass is 110.0 kg.
35. A 2.50-kg fireworks shell is fired straight up from a mortar and reaches a height of 110.0 m. (a) Neglecting air resistance (a poor assumption, but we will make it for this example), calculate the shell's velocity when it leaves the mortar. (b) The mortar itself is a tube 0.450 m long. Calculate the average acceleration of the shell in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the shell in the mortar? Express your answer in newtons and as a ratio to the weight of the shell.
36. A 0.500-kg potato is fired at an angle of 80.0° above the horizontal from a PVC pipe used as a "potato gun" and reaches a height of 110.0 m. (a) Neglecting air resistance, calculate the potato's velocity when it leaves the gun. (b) The gun itself is a tube 0.450 m long. Calculate the average acceleration of the potato in the tube as it goes from zero to the velocity found in (a). (c) What is the average force on the potato in the gun? Express your answer in newtons and as a ratio to the weight of the potato.
37. An elevator filled with passengers has a mass of 1.70×10^3 kg. (a) The elevator accelerates upward from rest at a rate of $1.20\ m/s^2$ for 1.50 s. Calculate the tension in the cable supporting the elevator. (b) The elevator continues upward at

constant velocity for 8.50 s. What is the tension in the cable during this time? (c) The elevator decelerates at a rate of 0.600 m/s^2 for 3.00 s. What is the tension in the cable during deceleration? (d) How high has the elevator moved above its original starting point, and what is its final velocity?

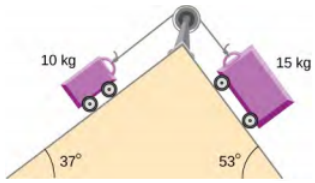
38. A 20.0-g ball hangs from the roof of a freight car by a string. When the freight car begins to move, the string makes an angle of 35.0° with the vertical. (a) What is the acceleration of the freight car? (b) What is the tension in the string?
39. A student's backpack, full of textbooks, is hung from a spring scale attached to the ceiling of an elevator. When the elevator is accelerating downward at 3.8 m/s^2, the scale reads 60 N. (a) What is the mass of the backpack? (b) What does the scale read if the elevator moves upward while slowing down at a rate 3.8 m/s^2? (c) What does the scale read if the elevator moves upward at constant velocity? (d) If the elevator had no brakes and the cable supporting it were to break loose so that the elevator could fall freely, what would the spring scale read?
40. A service elevator takes a load of garbage, mass 10.0 kg, from a floor of a skyscraper under construction, down to ground level, accelerating downward at a rate of 1.2 m/s^2. Find the magnitude of the force the garbage exerts on the floor of the service elevator?
41. A roller coaster car starts from rest at the top of a track 30.0 m long and inclined at 20.0° to the horizontal. Assume that friction can be ignored. (a) What is the acceleration of the car? (b) How much time elapses before it reaches the bottom of the track?
42. The device shown below is the Atwood's machine considered in Example 6.5. Assuming that the masses of the string and the frictionless pulley are negligible, (a) find an equation for the acceleration of the two blocks; (b) find an equation for the tension in the string; and (c) find both the acceleration and tension when block 1 has mass 2.00 kg and block 2 has mass 4.00 kg.



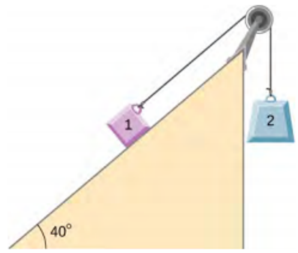
43. Two blocks are connected by a massless rope as shown below. The mass of the block on the table is 4.0 kg and the hanging mass is 1.0 kg. The table and the pulley are frictionless. (a) Find the acceleration of the system. (b) Find the tension in the rope. (c) Find the speed with which the hanging mass hits the floor if it starts from rest and is initially located 1.0 m from the floor.



44. Shown below are two carts connected by a cord that passes over a small frictionless pulley. Each cart rolls freely with negligible friction. Calculate the acceleration of the carts and the tension in the cord.

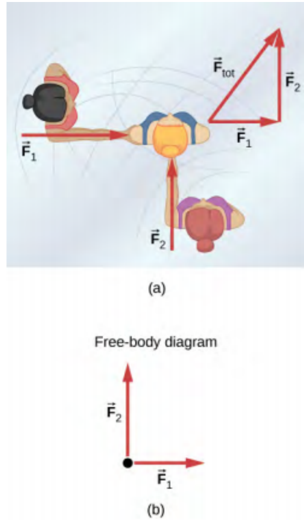


45. A 2.00 kg block (mass 1) and a 4.00 kg block (mass 2) are connected by a light string as shown; the inclination of the ramp is 40.0°. Friction is negligible. What is (a) the acceleration of each block and (b) the tension in the string?

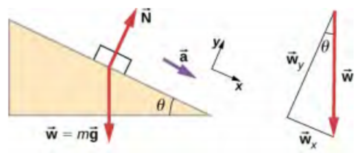


6.2 Friction

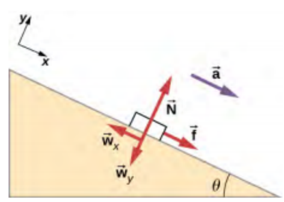
46. (a) When rebuilding his car's engine, a physics major must exert 3.00×10^2 N of force to insert a dry steel piston into a steel cylinder. What is the normal force between the piston and cylinder? (b) What force would he have to exert if the steel parts were oiled?
47. (a) What is the maximum frictional force in the knee joint of a person who supports 66.0 kg of her mass on that knee? (b) During strenuous exercise, it is possible to exert forces to the joints that are easily 10 times greater than the weight being supported. What is the maximum force of friction under such conditions? The frictional forces in joints are relatively small in all circumstances except when the joints deteriorate, such as from injury or arthritis. Increased frictional forces can cause further damage and pain.
48. Suppose you have a 120-kg wooden crate resting on a wood floor, with coefficient of static friction 0.500 between these wood surfaces. (a) What maximum force can you exert horizontally on the crate without moving it? (b) If you continue to exert this force once the crate starts to slip, what will its acceleration then be? The coefficient of sliding friction is known to be 0.300 for this situation.
49. (a) If half of the weight of a small 1.00×10^3-kg utility truck is supported by its two drive wheels, what is the maximum acceleration it can achieve on dry concrete? (b) Will a metal cabinet lying on the wooden bed of the truck slip if it accelerates at this rate? (c) Solve both problems assuming the truck has four-wheel drive.
50. A team of eight dogs pulls a sled with waxed wood runners on wet snow (mush!). The dogs have average masses of 19.0 kg, and the loaded sled with its rider has a mass of 210 kg. (a) Calculate the acceleration of the dogs starting from rest if each dog exerts an average force of 185 N backward on the snow. (b) Calculate the force in the coupling between the dogs and the sled.
51. Consider the 65.0-kg ice skater being pushed by two others shown below. (a) Find the direction and magnitude of F_{tot}, the total force exerted on her by the others, given that the magnitudes F_1 and F_2 are 26.4 N and 18.6 N, respectively. (b) What is her initial acceleration if she is initially stationary and wearing steel-bladed skates that point in the direction of F_{tot}? (c) What is her acceleration assuming she is already moving in the direction of F_{tot}? (Remember that friction always acts in the direction opposite that of motion or attempted motion between surfaces in contact.)



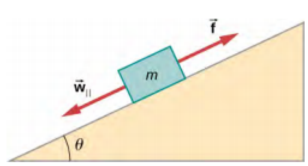
52. Show that the acceleration of any object down a frictionless incline that makes an angle θ with the horizontal is a = g sin θ. (Note that this acceleration is independent of mass.)



53. Show that the acceleration of any object down an incline where friction behaves simply (that is, where $f_k = \mu_k N$) is a = g(sin $\theta - \mu_k$ cos θ). Note that the acceleration is independent of mass and reduces to the expression found in the previous problem when friction becomes negligibly small (μ_k = 0).



54. Calculate the deceleration of a snow boarder going up a 5.00° slope, assuming the coefficient of friction for waxed wood on wet snow. The result of the preceding problem may be useful, but be careful to consider the fact that the snow boarder is going uphill.
55. A machine at a post office sends packages out a chute and down a ramp to be loaded into delivery vehicles. (a) Calculate the acceleration of a box heading down a 10.0° slope, assuming the coefficient of friction for a parcel on waxed wood is 0.100. (b) Find the angle of the slope down which this box could move at a constant velocity. You can neglect air resistance in both parts.
56. If an object is to rest on an incline without slipping, then friction must equal the component of the weight of the object parallel to the incline. This requires greater and greater friction for steeper slopes. Show that the maximum angle of an incline above the horizontal for which an object will not slide down is $\theta = \tan^{-1} \mu_s$. You may use the result of the previous problem. Assume that a = 0 and that static friction has reached its maximum value.



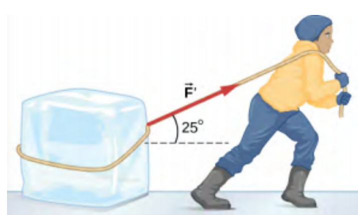
57. Calculate the maximum acceleration of a car that is heading down a 6.00° slope (one that makes an angle of 6.00° with the horizontal) under the following road conditions. You may assume that the weight of the car is evenly distributed on all

four tires and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the deceleration. (Ignore rolling.) Calculate for a car: (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that μ_s = 0.100, the same as for shoes on ice.

58. Calculate the maximum acceleration of a car that is heading up a 4.00° slope (one that makes an angle of 4.00° with the horizontal) under the following road conditions. Assume that only half the weight of the car is supported by the two drive wheels and that the coefficient of static friction is involved—that is, the tires are not allowed to slip during the acceleration. (Ignore rolling.) (a) On dry concrete. (b) On wet concrete. (c) On ice, assuming that μ_s = 0.100, the same as for shoes on ice.
59. Repeat the preceding problem for a car with four-wheel drive.
60. A freight train consists of two 8.00 x 10^5-kg engines and 45 cars with average masses of 5.50 x 10^5 kg. (a) What force must each engine exert backward on the track to accelerate the train at a rate of 5.00 x 10^{-2} m/s^2 if the force of friction is 7.50 x 10^5 N, assuming the engines exert identical forces? This is not a large frictional force for such a massive system. Rolling friction for trains is small, and consequently, trains are very energy-efficient transportation systems. (b) What is the force in the coupling between the 37th and 38th cars (this is the force each exerts on the other), assuming all cars have the same mass and that friction is evenly distributed among all of the cars and engines?
61. Consider the 52.0-kg mountain climber shown below. (a) Find the tension in the rope and the force that the mountain climber must exert with her feet on the vertical rock face to remain stationary. Assume that the force is exerted parallel to her legs. Also, assume negligible force exerted by her arms. (b) What is the minimum coefficient of friction between her shoes and the cliff?

62. A contestant in a winter sporting event pushes a 45.0-kg block of ice across a frozen lake as shown below. (a) Calculate the minimum force F he must exert to get the block moving. (b) What is its acceleration once it starts to move, if that force is maintained?

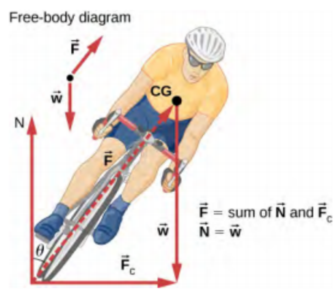
63. The contestant now pulls the block of ice with a rope over his shoulder at the same angle above the horizontal as shown below. Calculate the minimum force F he must exert to get the block moving. (b) What is its acceleration once it starts to move, if that force is maintained?



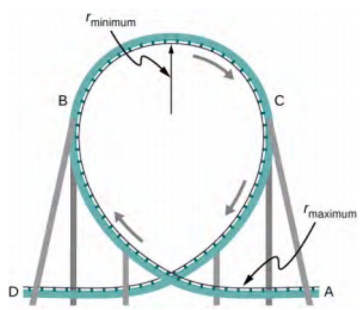
64. At a post office, a parcel that is a 20.0-kg box slides down a ramp inclined at 30.0° with the horizontal. The coefficient of kinetic friction between the box and plane is 0.0300. (a) Find the acceleration of the box. (b) Find the velocity of the box as it reaches the end of the plane, if the length of the plane is 2 m and the box starts at rest.

6.3 Centripetal Force

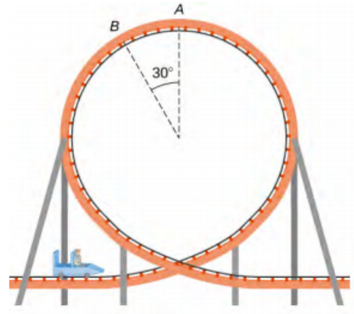
65. (a) A 22.0-kg child is riding a playground merry-go-round that is rotating at 40.0 rev/min. What centripetal force is exerted if he is 1.25 m from its center? (b) What centripetal force is exerted if the merry-go-round rotates at 3.00 rev/min and he is 8.00 m from its center? (c) Compare each force with his weight.
66. Calculate the centripetal force on the end of a 100-m (radius) wind turbine blade that is rotating at 0.5 rev/s. Assume the mass is 4 kg.
67. What is the ideal banking angle for a gentle turn of 1.20-km radius on a highway with a 10^5 km/h speed limit (about 65 mi/h), assuming everyone travels at the limit?
68. What is the ideal speed to take a 100.0-m-radius curve banked at a 20.0° angle?
69. (a) What is the radius of a bobsled turn banked at 75.0° and taken at 30.0 m/s, assuming it is ideally banked? (b) Calculate the centripetal acceleration. (c) Does this acceleration seem large to you?
70. Part of riding a bicycle involves leaning at the correct angle when making a turn, as seen below. To be stable, the force exerted by the ground must be on a line going through the center of gravity. The force on the bicycle wheel can be resolved into two perpendicular components—friction parallel to the road (this must supply the centripetal force) and the vertical normal force (which must equal the system's weight). (a) Show that θ (as defined as shown) is related to the speed v and radius of curvature r of the turn in the same way as for an ideally banked roadway—that is, $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$.
(b) Calculate θ for a 12.0-m/s turn of radius 30.0 m (as in a race).



71. If a car takes a banked curve at less than the ideal speed, friction is needed to keep it from sliding toward the inside of the curve (a problem on icy mountain roads). (a) Calculate the ideal speed to take a 100.0 m radius curve banked at 15.0°. (b) What is the minimum coefficient of friction needed for a frightened driver to take the same curve at 20.0 km/h?
72. Modern roller coasters have vertical loops like the one shown here. The radius of curvature is smaller at the top than on the sides so that the downward centripetal acceleration at the top will be greater than the acceleration due to gravity, keeping the passengers pressed firmly into their seats. What is the speed of the roller coaster at the top of the loop if the radius of curvature there is 15.0 m and the downward acceleration of the car is 1.50 g?



73. A child of mass 40.0 kg is in a roller coaster car that travels in a loop of radius 7.00 m. At point A the speed of the car is 10.0 m/s, and at point B, the speed is 10.5 m/s. Assume the child is not holding on and does not wear a seat belt. (a) What is the force of the car seat on the child at point A? (b) What is the force of the car seat on the child at point B? (c) What minimum speed is required to keep the child in his seat at point A?



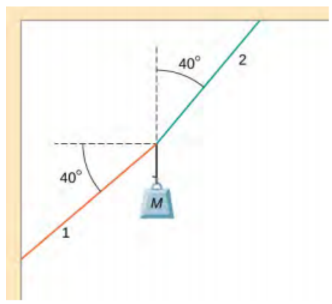
74. In the simple Bohr model of the ground state of the hydrogen atom, the electron travels in a circular orbit around a fixed proton. The radius of the orbit is 5.28 x 10^{-11} m, and the speed of the electron is 2.18 x 10^6 m/s. The mass of an electron is 9.11 x 10^{-31} kg. What is the force on the electron?
75. Railroad tracks follow a circular curve of radius 500.0 m and are banked at an angle of 5.0°. For trains of what speed are these tracks designed?
76. The CERN particle accelerator is circular with a circumference of 7.0 km. (a) What is the acceleration of the protons (m = 1.67 x 10^{-27} kg) that move around the accelerator at 5% of the speed of light? (The speed of light is v = 3.00 x 10^8 m/s.) (b) What is the force on the protons?
77. A car rounds an unbanked curve of radius 65 m. If the coefficient of static friction between the road and car is 0.70, what is the maximum speed at which the car traverse the curve without slipping?
78. A banked highway is designed for traffic moving at 90.0 km/h. The radius of the curve is 310 m. What is the angle of banking of the highway?

6.4 Drag Force and Terminal Speed

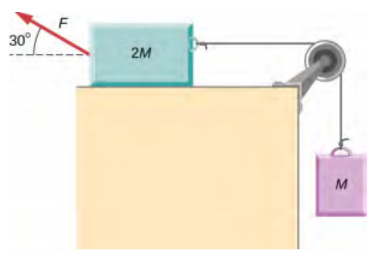
79. The terminal velocity of a person falling in air depends upon the weight and the area of the person facing the fluid. Find the terminal velocity (in meters per second and kilometers per hour) of an 80.0-kg skydiver falling in a pike (headfirst) position with a surface area of 0.140 m^2.
80. A 60.0-kg and a 90.0-kg skydiver jump from an airplane at an altitude of 6.00 x 10^3 m, both falling in the pike position. Make some assumption on their frontal areas and calculate their terminal velocities. How long will it take for each skydiver to reach the ground (assuming the time to reach terminal velocity is small)? Assume all values are accurate to three significant digits.
81. A 560-g squirrel with a surface area of 930 cm^2 falls from a 5.0-m tree to the ground. Estimate its terminal velocity. (Use a drag coefficient for a horizontal skydiver.) What will be the velocity of a 56-kg person hitting the ground, assuming no drag contribution in such a short distance?
82. To maintain a constant speed, the force provided by a car's engine must equal the drag force plus the force of friction of the road (the rolling resistance). (a) What are the drag forces at 70 km/h and 100 km/h for a Toyota Camry? (Drag area is 0.70 m^2) (b) What is the drag force at 70 km/h and 100 km/h for a Hummer H2? (Drag area is 2.44 m^2) Assume all values are accurate to three significant digits.
83. By what factor does the drag force on a car increase as it goes from 65 to 110 km/h?
84. Calculate the velocity a spherical rain drop would achieve falling from 5.00 km (a) in the absence of air drag (b) with air drag. Take the size across of the drop to be 4 mm, the density to be 1.00 x 10^3 kg/m^3, and the surface area to be πr^2.
85. Using Stokes' law, verify that the units for viscosity are kilograms per meter per second.
86. Find the terminal velocity of a spherical bacterium (diameter 2.00 μ_m) falling in water. You will first need to note that the drag force is equal to the weight at terminal velocity. Take the density of the bacterium to be 1.10 x 10^3 kg/m^3.
87. Stokes' law describes sedimentation of particles in liquids and can be used to measure viscosity. Particles in liquids achieve terminal velocity quickly. One can measure the time it takes for a particle to fall a certain distance and then use Stokes' law to calculate the viscosity of the liquid. Suppose a steel ball bearing (density 7.8 x 10^3 kg/m^3, diameter 3.0 mm) is dropped in a container of motor oil. It takes 12 s to fall a distance of 0.60 m. Calculate the viscosity of the oil.
88. Suppose that the resistive force of the air on a skydiver can be approximated by $f = -bv^2$. If the terminal velocity of a 50.0-kg skydiver is 60.0 m/s, what is the value of b?
89. A small diamond of mass 10.0 g drops from a swimmer's earring and falls through the water, reaching a terminal velocity of 2.0 m/s. (a) Assuming the frictional force on the diamond obeys $f = -bv$, what is b? (b) How far does the diamond fall before it reaches 90 percent of its terminal speed?

Additional Problems

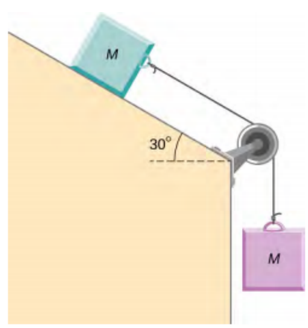
90. (a) What is the final velocity of a car originally traveling at 50.0 km/h that decelerates at a rate of 0.400 m/s^2 for 50.0 s? Assume a coefficient of friction of 1.0. (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?
91. A 75.0-kg woman stands on a bathroom scale in an elevator that accelerates from rest to 30.0 m/s in 2.00 s. (a) Calculate the scale reading in newtons and compare it with her weight. (The scale exerts an upward force on her equal to its reading.) (b) What is unreasonable about the result? (c) Which premise is unreasonable, or which premises are inconsistent?
92. (a) Calculate the minimum coefficient of friction needed for a car to negotiate an unbanked 50.0 m radius curve at 30.0 m/s. (b) What is unreasonable about the result? (c) Which premises are unreasonable or inconsistent?
93. As shown below, if M = 5.50 kg, what is the tension in string 1?



94. As shown below, if F = 60.0 N and M = 4.00 kg, what is the magnitude of the acceleration of the suspended object? All surfaces are frictionless.



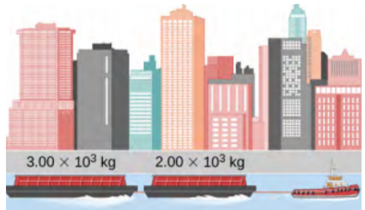
95. As shown below, if M = 6.0 kg, what is the tension in the connecting string? The pulley and all surfaces are frictionless.



96. A small space probe is released from a spaceship. The space probe has mass 20.0 kg and contains 90.0 kg of fuel. It starts from rest in deep space, from the origin of a coordinate system based on the spaceship, and burns fuel at the rate of 3.00 kg/s. The engine provides a constant thrust of 120.0 N. (a) Write an expression for the mass of the space probe as a function of time, between 0 and 30 seconds, assuming that the engine ignites fuel beginning at t = 0. (b) What is the velocity after 15.0 s? (c) What is the position of the space probe after 15.0 s, with initial position at the origin? (d) Write an expression for the position as a function of time, for t > 30.0 s.
97. A half-full recycling bin has mass 3.0 kg and is pushed up a 40.0° incline with constant speed under the action of a 26-N force acting up and parallel to the incline. The incline has friction. What magnitude force must act up and parallel to the incline for the bin to move down the incline at constant velocity?

98. A child has mass 6.0 kg and slides down a 35° incline with constant speed under the action of a 34-N force acting up and parallel to the incline. What is the coefficient of kinetic friction between the child and the surface of the incline?

99. The two barges shown here are coupled by a cable of negligible mass. The mass of the front barge is 2.00 x 10^3 kg and the mass of the rear barge is 3.00 x 10^3 kg. A tugboat pulls the front barge with a horizontal force of magnitude 20.0 x 10^3 N, and the frictional forces of the water on the front and rear barges are 8.00 x 10^3 N and 10.0 x 10^3 N, respectively. Find the horizontal acceleration of the barges and the tension in the connecting cable.



100. If the order of the barges of the preceding exercise is reversed so that the tugboat pulls the 3.00 x 10^3 -kg barge with a force of 20.0 x 10^3 N, what are the acceleration of the barges and the tension in the coupling cable?

101. An object with mass m moves along the x-axis. Its position at any time is given by x(t) = $pt^3 + qt^2$ where p and q are constants. Find the net force on this object for any time t.

102. A helicopter with mass 2.35 x 10^4 kg has a position given by $\vec{r}$(t) = (0.020 t^3) $\hat{i}$ + (2.2t) $\hat{j}$ − (0.060 t^2) |9\hat{k}\). Find the net force on the helicopter at t = 3.0 s.

103. Located at the origin, an electric car of mass m is at rest and in equilibrium. A time dependent force of $\vec{F}$(t) is applied at time t = 0, and its components are F_x(t) = p + nt and F_y(t) = qt where p, q, and n are constants. Find the position $\vec{r}$(t) and velocity $\vec{v}$(t) as functions of time t.

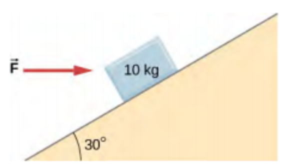
104. A particle of mass m is located at the origin. It is at rest and in equilibrium. A time-dependent force of $\vec{F}$(t) is applied at time t = 0, and its components are F_x(t) = pt and F_y(t) = n + qt where p, q, and n are constants. Find the position $\vec{r}$(t) and velocity $\vec{v}$(t) as functions of time t.

105. A 2.0-kg object has a velocity of 4.0 $\hat{i}$ m/s at t = 0. A constant resultant force of (2.0 $\hat{i}$ + 4.0 $\hat{j}$) N then acts on the object for 3.0 s. What is the magnitude of the object's velocity at the end of the 3.0-s interval?

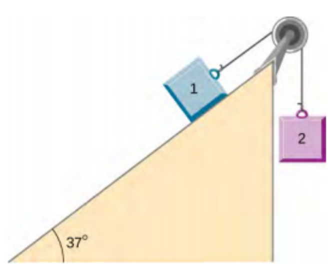
106. A 1.5-kg mass has an acceleration of (4.0 $\hat{i}$ − 3.0 $\hat{j}$) m/s^2. Only two forces act on the mass. If one of the forces is (2.0 $\hat{i}$ − 1.4 $\hat{j}$) N, what is the magnitude of the other force?

107. A box is dropped onto a conveyor belt moving at 3.4 m/s. If the coefficient of friction between the box and the belt is 0.27, how long will it take before the box moves without slipping?

108. Shown below is a 10.0-kg block being pushed by a horizontal force $\vec{F}$ of magnitude 200.0 N. The coefficient of kinetic friction between the two surfaces is 0.50. Find the acceleration of the block.



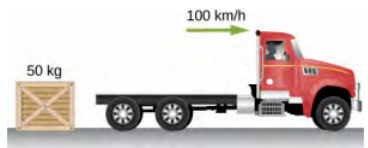
109. As shown below, the mass of block 1 is m_1 = 4.0 kg, while the mass of block 2 is m_2 = 8.0 kg. The coefficient of friction between m_1 and the inclined surface is μ_k = 0.40. What is the acceleration of the system?



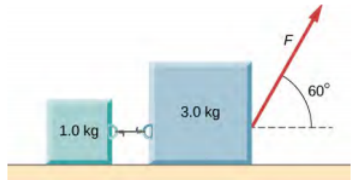
110. A student is attempting to move a 30-kg mini-fridge into her dorm room. During a moment of inattention, the mini-fridge slides down a 35 degree incline at constant speed when she applies a force of 25 N acting up and parallel to the incline.

What is the coefficient of kinetic friction between the fridge and the surface of the incline?

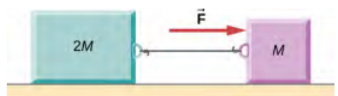
111. A crate of mass 100.0 kg rests on a rough surface inclined at an angle of 37.0° with the horizontal. A massless rope to which a force can be applied parallel to the surface is attached to the crate and leads to the top of the incline. In its present state, the crate is just ready to slip and start to move down the plane. The coefficient of friction is 80% of that for the static case. (a) What is the coefficient of static friction? (b) What is the maximum force that can be applied upward along the plane on the rope and not move the block? (c) With a slightly greater applied force, the block will slide up the plane. Once it begins to move, what is its acceleration and what reduced force is necessary to keep it moving upward at constant speed? (d) If the block is given a slight nudge to get it started down the plane, what will be its acceleration in that direction? (e) Once the block begins to slide downward, what upward force on the rope is required to keep the block from accelerating downward?
112. A car is moving at high speed along a highway when the driver makes an emergency braking. The wheels become locked (stop rolling), and the resulting skid marks are 32.0 meters long. If the coefficient of kinetic friction between tires and road is 0.550, and the acceleration was constant during braking, how fast was the car going when the wheels became locked?
113. A crate having mass 50.0 kg falls horizontally off the back of the flatbed truck, which is traveling at 100 km/h. Find the value of the coefficient of kinetic friction between the road and crate if the crate slides 50 m on the road in coming to rest. The initial speed of the crate is the same as the truck, 100 km/h.



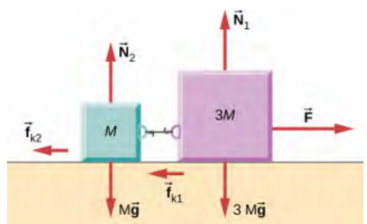
114. A 15-kg sled is pulled across a horizontal, snow-covered surface by a force applied to a rope at 30 degrees with the horizontal. The coefficient of kinetic friction between the sled and the snow is 0.20. (a) If the force is 33 N, what is the horizontal acceleration of the sled? (b) What must the force be in order to pull the sled at constant velocity?
115. A 30.0-g ball at the end of a string is swung in a vertical circle with a radius of 25.0 cm. The tangential velocity is 200.0 cm/s. Find the tension in the string: (a) at the top of the circle, (b) at the bottom of the circle, and (c) at a distance of 12.5 cm from the center of the circle (r = 12.5 cm).
116. A particle of mass 0.50 kg starts moves through a circular path in the xy-plane with a position given by $\vec{r}(t) = (4.0 \cos 3t)\hat{i} + (4.0 \sin 3t)\hat{j}$ where r is in meters and t is in seconds. (a) Find the velocity and acceleration vectors as functions of time. (b) Show that the acceleration vector always points toward the center of the circle (and thus represents centripetal acceleration). (c) Find the centripetal force vector as a function of time.
117. A stunt cyclist rides on the interior of a cylinder 12 m in radius. The coefficient of static friction between the tires and the wall is 0.68. Find the value of the minimum speed for the cyclist to perform the stunt.
118. When a body of mass 0.25 kg is attached to a vertical massless spring, it is extended 5.0 cm from its unstretched length of 4.0 cm. The body and spring are placed on a horizontal frictionless surface and rotated about the held end of the spring at 2.0 rev/s. How far is the spring stretched?
119. Railroad tracks follow a circular curve of radius 500.0 m and are banked at an angle of 5.00°. For trains of what speed are these tracks designed?
120. A plumb bob hangs from the roof of a railroad car. The car rounds a circular track of radius 300.0 m at a speed of 90.0 km/h. At what angle relative to the vertical does the plumb bob hang?
121. An airplane flies at 120.0 m/s and banks at a 30° angle. If its mass is 2.50×10^3 kg, (a) what is the magnitude of the lift force? (b) what is the radius of the turn?
122. The position of a particle is given by $\vec{r}(t) = A(\cos \omega t\, \hat{i} + \sin \omega t\, \hat{j})$, where ω is a constant. (a) Show that the particle moves in a circle of radius A. (b) Calculate $\frac{d\vec{r}}{dt}$ and then show that the speed of the particle is a constant $A\omega$. (c) Determine $\frac{d^2\vec{r}}{dt^2}$ and show that a is given by $a_c = r\omega^2$. (d) Calculate the centripetal force on the particle. [**Hint**: For (b) and (c), you will need to use $\left(\frac{d}{dt}\right)(\cos \omega t) = -\omega \sin \omega t$ and $\left(\frac{d}{dt}\right)(\sin \omega t) = \omega \cos \omega t$.
123. Two blocks connected by a string are pulled across a horizontal surface by a force applied to one of the blocks, as shown below. The coefficient of kinetic friction between the blocks and the surface is 0.25. If each block has an acceleration of $2.0\ m/s^2$ to the right, what is the magnitude F of the applied force?



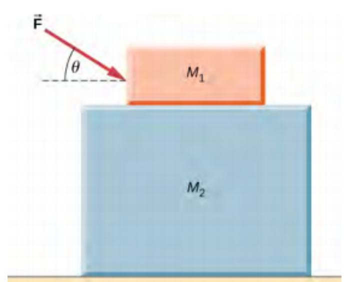
124. As shown below, the coefficient of kinetic friction between the surface and the larger block is 0.20, and the coefficient of kinetic friction between the surface and the smaller block is 0.30. If F = 10 N and M = 1.0 kg, what is the tension in the connecting string?



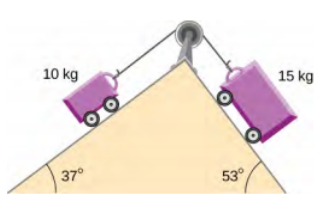
125. In the figure, the coefficient of kinetic friction between the surface and the blocks is μ_k. If M = 1.0 kg, find an expression for the magnitude of the acceleration of either block (in terms of F, μ_k, and g).



126. Two blocks are stacked as shown below, and rest on a frictionless surface. There is friction between the two blocks (coefficient of friction μ). An external force is applied to the top block at an angle θ with the horizontal. What is the maximum force F that can be applied for the two blocks to move together?



127. A box rests on the (horizontal) back of a truck. The coefficient of static friction between the box and the surface on which it rests is 0.24. What maximum distance can the truck travel (starting from rest and moving horizontally with constant acceleration) in 3.0 s without having the box slide?
128. A double-incline plane is shown below. The coefficient of friction on the left surface is 0.30, and on the right surface 0.16. Calculate the acceleration of the system.

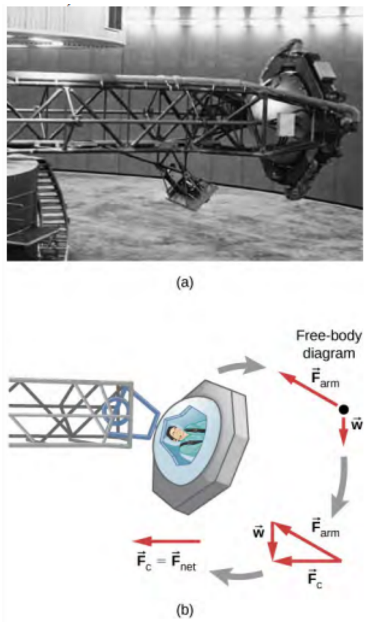


Challenge Problems

129. In a later chapter, you will find that the weight of a particle varies with altitude such that w = $\frac{mgr_0^2}{r^2}$ where r_0 is the radius of Earth and r is the distance from Earth's center. If the particle is fired vertically with velocity v_0 from Earth's surface,

determine its velocity as a function of position r. (Hint: use a dr = v dv, the rearrangement mentioned in the text.)

130. A large centrifuge, like the one shown below, is used to expose aspiring astronauts to accelerations similar to those experienced in rocket launches and atmospheric reentries. (a) At what angular velocity is the centripetal acceleration 10g if the rider is 15.0 m from the center of rotation? (b) The rider's cage hangs on a pivot at the end of the arm, allowing it to swing outward during rotation as shown in the bottom accompanying figure. At what angle θ below the horizontal will the cage hang when the centripetal acceleration is 10g? (Hint: The arm supplies centripetal force and supports the weight of the cage. Draw a free-body diagram of the forces to see what the angle θ should be.)



131. A car of mass 1000.0 kg is traveling along a level road at 100.0 km/h when its brakes are applied. Calculate the stopping distance if the coefficient of kinetic friction of the tires is 0.500. Neglect air resistance. (**Hint**: since the distance traveled is of interest rather than the time, x is the desired independent variable and not t. Use the Chain Rule to change the variable: $\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$.)
132. An airplane flying at 200.0 m/s makes a turn that takes 4.0 min. What bank angle is required? What is the percentage increase in the perceived weight of the passengers?
133. A skydiver is at an altitude of 1520 m. After 10.0 seconds of free fall, he opens his parachute and finds that the air resistance, F_D, is given by the formula $F_D = -bv$, where b is a constant and v is the velocity. If b = 0.750, and the mass of the skydiver is 82.0 kg, first set up differential equations for the velocity and the position, and then find: (a) the speed of the skydiver when the parachute opens, (b) the distance fallen before the parachute opens, (c) the terminal velocity after the parachute opens (find the limiting velocity), and (d) the time the skydiver is in the air after the parachute opens.
134. In a television commercial, a small, spherical bead of mass 4.00 g is released from rest at t = 0 in a bottle of liquid shampoo. The terminal speed is observed to be 2.00 cm/s. Find (a) the value of the constant b in the equation v = $\frac{mg}{b}\left(1 - e^{-\frac{bt}{m}}\right)$, and (b) the value of the resistive force when the bead reaches terminal speed.
135. A boater and motor boat are at rest on a lake. Together, they have mass 200.0 kg. If the thrust of the motor is a constant force of 40.0 N in the direction of motion, and if the resistive force of the water is numerically equivalent to 2 times the speed v of the boat, set up and solve the differential equation to find: (a) the velocity of the boat at time t; (b) the limiting velocity (the velocity after a long time has passed).

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7.9: Applications of Newton's Laws (Summary)

Key Terms

banked curve	curve in a road that is sloping in a manner that helps a vehicle negotiate the curve
centripetal force	any net force causing uniform circular motion
Coriolis force	inertial force causing the apparent deflection of moving objects when viewed in a rotating frame of reference
drag force	force that always opposes the motion of an object in a fluid; unlike simple friction, the drag force is proportional to some function of the velocity of the object in that fluid
friction	force that opposes relative motion or attempts at motion between systems in contact
ideal banking	sloping of a curve in a road, where the angle of the slope allows the vehicle to negotiate the curve at a certain speed without the aid of friction between the tires and the road; the net external force on the vehicle equals the horizontal centripetal force in the absence of friction
inertial force	force that has no physical origin
kinetic friction	force that opposes the motion of two systems that are in contact and moving relative to each other
noninertial frame of reference	accelerated frame of reference
static friction	force that opposes the motion of two systems that are in contact and are not moving relative to each other
terminal velocity	constant velocity achieved by a falling object, which occurs when the weight of the object is balanced by the upward drag force

Key Equations

Magnitude of static friction	$$f_s \leq \mu_s N \tag{7.9.1}$$
Magnitude of kinetic friction	$$f_k = \mu_k N \tag{7.9.2}$$
Centripetal force	$$\begin{aligned} F_c &= m\frac{v^2}{r} \\ &= mr\omega^2 \end{aligned}$$
Ideal angle of a banked curve	$$\tan\theta = \frac{v^2}{rg} \tag{7.9.3}$$
Drag force	$$F_D = \frac{1}{2}C\rho Av^2 \tag{7.9.4}$$
Stokes' law	$$F_s = 6\pi r\eta v \tag{7.9.5}$$

Summary

6.1 Solving Problems with Newton's Laws

- Newton's laws of motion can be applied in numerous situations to solve motion problems.
- Some problems contain multiple force vectors acting in different directions on an object. Be sure to draw diagrams, resolve all force vectors into horizontal and vertical components, and draw a free-body diagram. Always analyze the direction in which an object accelerates so that you can determine whether $F_{net} = ma$ or $F_{net} = 0$.
- The normal force on an object is not always equal in magnitude to the weight of the object. If an object is accelerating vertically, the normal force is less than or greater than the weight of the object. Also, if the object is on an inclined plane, the normal force is always less than the full weight of the object.
- Some problems contain several physical quantities, such as forces, acceleration, velocity, or position. You can apply concepts from kinematics and dynamics to solve these problems.

6.2 Friction

- Friction is a contact force that opposes the motion or attempted motion between two systems. Simple friction is proportional to the normal force N supporting the two systems.
- The magnitude of static friction force between two materials stationary relative to each other is determined using the coefficient of static friction, which depends on both materials.
- The kinetic friction force between two materials moving relative to each other is determined using the coefficient of kinetic friction, which also depends on both materials and is always less than the coefficient of static friction.

6.3 Centripetal Force

- Centripetal force $\vec{F}_c$ is a "center-seeking" force that always points toward the center of rotation. It is perpendicular to linear velocity and has the magnitude

$$F_c = ma_c. \tag{7.9.6}$$

- Rotating and accelerated frames of reference are noninertial. Inertial forces, such as the Coriolis force, are needed to explain motion in such frames.

6.4 Drag Force and Terminal Speed

- Drag forces acting on an object moving in a fluid oppose the motion. For larger objects (such as a baseball) moving at a velocity in air, the drag force is determined using the drag coefficient (typical values are given in Table 6.2), the area of the object facing the fluid, and the fluid density.
- For small objects (such as a bacterium) moving in a denser medium (such as water), the drag force is given by Stokes' law.

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CHAPTER OVERVIEW

7.10: Applications of Newton's Laws

Car racing has grown in popularity in recent years. As each car moves in a curved path around the turn, its wheels also spin rapidly. The wheels complete many revolutions while the car makes only part of one (a circular arc). How can we describe the velocities, accelerations, and forces involved? What force keeps a racecar from spinning out, hitting the wall bordering the track? What provides this force? Why is the track banked? We answer all of these questions in this chapter as we expand our consideration of Newton’s laws of motion.

Thumbnail Figure 6.1 - Stock cars racing in the Grand National Divisional race at Iowa Speedway in May, 2015. Cars often reach speeds of 200 mph (320 km/h).

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7.11: Introduction to UCM and Gravitation

Kinematics of UCM

Uniform circular motion is a motion in a circular path at constant speed.

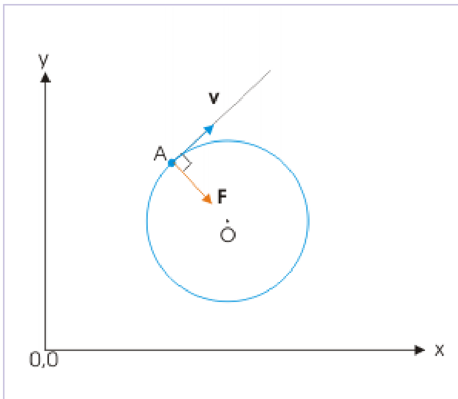
learning objectives

- Relate centripetal force and centripetal acceleration to uniform circular motion

Angular Quantities

Under uniform circular motion, angular and linear quantities have simple relations. When objects rotate about some axis, each point in the object follows a circular arc. The rotation angle is the amount of rotation and is analogous to linear distance. We define the rotation angle $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$$\Delta\theta = \frac{\Delta s}{r} \tag{7.11.1}$$



Angle θθ and Arc Length ss: The radius of a circle is rotated through an angle $\Delta\theta$. The arc length Δs is described on the circumference.

We define angular velocity ωω as the rate of change of an angle. In symbols, this is $\omega = \frac{\Delta\theta}{\Delta t}$, where an angular rotation $\Delta\theta$ takes place in a time Δt. From the relation of s and ($\Delta s = r\Delta\theta$), we see:

$$v = \frac{\Delta s}{\Delta t} = r\frac{\Delta\theta}{\Delta t} = r\omega \tag{7.11.2}$$

Under uniform circular motion, the angular velocity is constant. The acceleration can be written as:

$$ac = \frac{dv}{dt} = \omega\frac{dr}{dt} = \omega v = r\omega^2 = \frac{v^2}{r} \tag{7.11.3}$$

This acceleration, responsible for the uniform circular motion, is called centripetal acceleration.

Centripetal Force

Any force or combination of forces can cause a centripetal or radial acceleration. Just a few examples are the tension in the rope on a tether ball, the force of Earth’s gravity on the Moon, friction between roller skates and a rink floor, a banked roadway’s force on a car, and forces on the tube of a spinning centrifuge.

Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature, the same as the direction of centripetal acceleration. According to Newton’s second law of motion, net force is mass times acceleration. For uniform circular motion, the acceleration is the centripetal acceleration: $a = a_c$. Thus, the magnitude of centripetal force F_c is:

$$F_c = ma_c = m\frac{v^2}{r} = mr\omega^2 \tag{7.11.4}$$

Dynamics of UCM

Newton's universal law of gravitation states that every particle attracts every other particle with a force along a line joining them.

learning objectives

- Relate Kepler's laws to Newton's universal law of gravitation

Newton's Universal Law of Gravitation

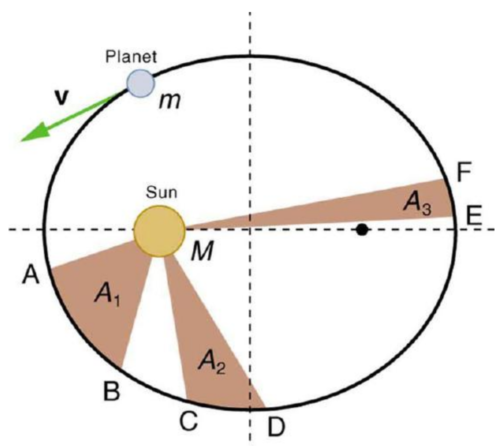
Newton's universal law of gravitation states that every particle in the universe attracts every other particle with a force along a line joining them. The force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them. For two bodies having masses mm and MM with a distance rr between their centers of mass, the equation for Newton's universal law of gravitation is:

$$\mathrm{F} = \mathrm{G}\frac{\mathrm{mM}}{\mathrm{r}^2} \tag{7.11.5}$$

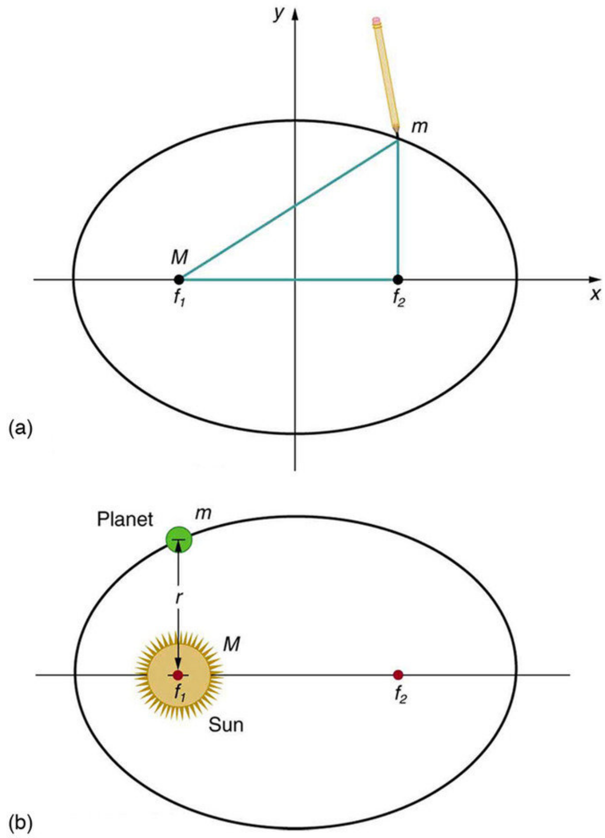
The gravitational force is responsible for artificial satellites orbiting the Earth. The Moon's orbit about Earth, the orbits of planets, asteroids, meteors, and comets about the Sun are other examples of gravitational orbits. Historically, Kepler discovered his 3 laws (called Kepler's law of planetary motion) long before the days of Newton. Kepler devised his laws after careful study (over some 20 years) of a large amount of meticulously recorded observations of planetary motion done by Tycho Brahe (1546–1601).

Kepler's Laws

- The orbit of each planet about the Sun is an ellipse with the Sun at one focus.
- Each planet moves so that an imaginary line drawn from the Sun to the planet sweeps out equal areas in equal times.
- The ratio of the squares of the periods of any two planets about the Sun is equal to the ratio of the cubes of their average distances from the Sun.



Kepler's Second Law: The shaded regions have equal areas. It takes equal times for mm to go from A to B, from C to D, and from E to F. The mass mm moves fastest when it is closest to M. Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.



Ellipses and Kepler's First Law: (a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci (f_1 and f_2) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, m follows an elliptical path with M at one focus. Kepler's first law states this fact for planets orbiting the Sun.

Derivation of Kepler's Third Law For Circular Orbits

Kepler's 3rd law is equivalent to:

$$\frac{T_1^2}{T_2^2} = \frac{r_3^1}{r_3^2} \tag{7.11.6}$$

T is the period (time for one orbit) and r is the average radius. We shall derive Kepler's third law, starting with Newton's laws of motion and his universal law of gravitation. We will assume a circular path (not an elliptical one) for simplicity.

Let us consider a circular orbit of a small mass mm around a large mass M, satisfying the two conditions stated at the beginning of this section. Gravity supplies the centripetal force to mass m. Therefore, for a uniform circular motion:

$$G\frac{mM}{r^2} = ma_c = m\frac{v^2}{r} \tag{7.11.7}$$

The mass m cancels, yielding:

$$G\frac{mM}{r^2} = v^2 \tag{7.11.8}$$

Now, to get at Kepler's third law, we must get the period T into the equation. By definition, period T is the time for one complete orbit. Now the average speed v is the circumference divided by the period:

$$v = \frac{2\pi r}{T} \tag{7.11.9}$$

Substituting this into the previous equation gives:

$$G\frac{M}{r}=\frac{4\pi^2 r^2}{T^2} \tag{7.11.10}$$

Solving for T^2 yields:

$$T^2=\frac{4\pi^2}{GM}r^3 \tag{7.11.11}$$

Since T^2 is proportional to r^3, their ratio is constant. This is Kepler's 3rd law.

Banked and Unbacked Highway Curves

In an "ideally banked curve," the angle θ is chosen such that one can negotiate the curve at a certain speed without the aid of friction.

learning objectives

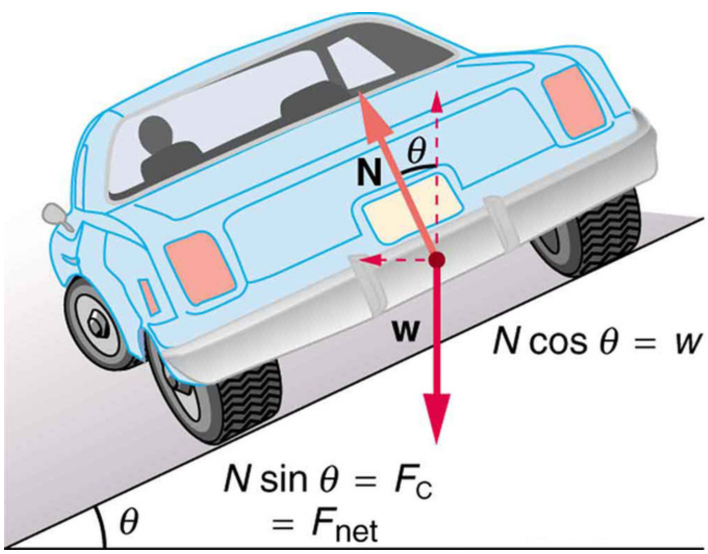
- Derive θ for an ideally banked curve for speed

Overview

As an example of a uniform circular motion and its application, let us now consider banked curves, where the slope of the road helps you negotiate the curve. The greater the angle θ, the faster you can take the curve. Race tracks for bikes as well as cars, for example, often have steeply banked curves. In an "ideally banked curve," the angle θ is such that you can negotiate the curve at a certain speed without the aid of friction between the tires and the road. We will derive an expression for θ for an ideally banked curve for speed $v$and consider an example related to it.

Uniform Circular Motion and Determining Ideal Banking Conditions

For ideal banking, the net external force equals the horizontal centripetal force in the absence of friction. The components of the normal force N in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively. In cases in which forces are not parallel, it is most convenient to consider components along perpendicular axes—in this case, the vertical and horizontal directions.



Car on a Banked Curve: The car on this banked curve is moving away and turning to the left.

Above is a free body diagram for a car on a frictionless banked curve. The only two external forces acting on the car are its weight ww and the normal force of the road N. (A frictionless surface can only exert a force perpendicular to the surface—that is, a normal force.) These two forces must add to give a net external force that is horizontal toward the center of curvature and has magnitude $\frac{mv^2}{r}$. Only the normal force has a horizontal component, and so this must equal the centripetal force—that is:

$$N \sin \theta = \frac{mv^2}{r} \tag{7.11.12}$$

Because the car does not leave the surface of the road, the net vertical force must be zero, meaning that the vertical components of the two external forces must be equal in magnitude and opposite in direction. From the figure, we see that the vertical component of the normal force is $N \cos \theta$, and the only other vertical force is the car's weight. These must be equal in magnitude, thus:

$$N \cos \theta = mg \tag{7.11.13}$$

Dividing the above equations yields:

$$\tan \theta = \frac{v^2}{rg} \tag{7.11.14}$$

Taking the inverse tangent gives:

$$\theta = \tan^{-1}(\frac{v^2}{rg}) \tag{7.11.15}$$

for an ideally banked curve with no friction.

This expression can be understood by considering how θ depends on v and r. A large θ will be obtained for a large $v$and a small r. That is, roads must be steeply banked for high speeds and sharp curves. Friction helps, because it allows you to take the curve at greater or lower speed than if the curve is frictionless. Note that θ does not depend on the mass of the vehicle.

Key Points

- Under uniform circular motion, angular and linear quantities have simple relations. The length of an arc is proportional to the rotation angle and the radius. Also, $v = r\omega$.
- The acceleration responsible for the uniform circular motion is called centripetal acceleration. It is given as $a_c = r\omega^2 = \frac{v^2}{r}$.
- Any net force causing uniform circular motion is called a centripetal force. The direction of a centripetal force is toward the center of curvature and its magnitude is $m\frac{v^2}{r} = mr\omega^2$.
- The gravitational force is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- Kepler discovered laws describing planetary motion long before the days of Newton, purely based on the observations of Tycho Brahe.
- Kepler's laws can be derived from the Newton's universal law of gravitation and his equation of motion.
- For ideal banking, the net external force equals the horizontal centripetal force in the absence of friction.
- For ideal banking, the components of the normal force NN in the horizontal and vertical directions must equal the centripetal force and the weight of the car, respectively.
- The ideal banking condition is given as $\theta = \tan^{-1}\left(\frac{v^2}{rg}\right)$.

Key Terms

- **centripetal**: Directed or moving towards a center.
- **asteroid**: A naturally occurring solid object, which is smaller than a planet and is not a comet, that orbits a star.
- **planet**: A large body which directly orbits any star (or star cluster) but which has not attained nuclear fusion.
- **normal force**: Any force acting normal, to a surface, or perpendicular to the tangent plane.

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7.12: Non-Uniform Circular Motion

Overview of Non-Uniform Circular Motion

Non-uniform circular motion denotes a change in the speed of a particle moving along a circular path.

learning objectives

- Explain when a particle undergoes non-uniform circular motion

What do we mean by non-uniform circular motion? The answer lies in the definition of uniform circular motion, which is a circular motion with constant speed. It follows then that non-uniform circular motion denotes a change in the speed of the particle moving along the circular path. Note especially the change in the velocity vector sizes, denoting change in the magnitude of velocity.

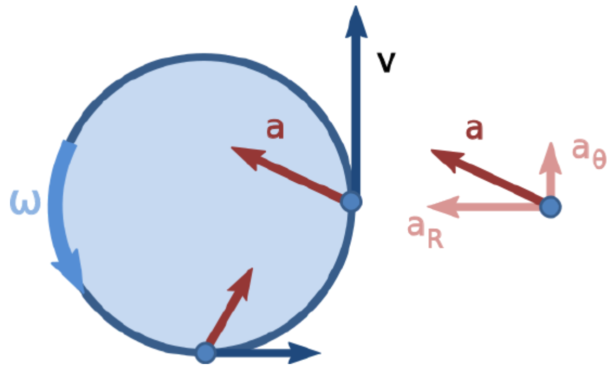

Diagram of non-uniform circular motion: In non-uniform circular motion, the magnitude of the angular velocity changes over time.

The change in direction is accounted by radial acceleration (centripetal acceleration), which is given by following relation: $a_r = \frac{v^2}{r}$. The change in speed has implications for radial (centripetal) acceleration. There are two possibilities:

1: The radius of circle is constant (like in the motion along a circular rail or motor track). A change in v will change the magnitude of radial acceleration. This means that the centripetal acceleration is not constant, as is the case with uniform circular motion. The greater the speed, the greater the radial acceleration. A particle moving at higher speed will need a greater radial force to change direction and vice-versa when the radius of the circular path is constant.

2: The radial (centripetal) force is constant (like a satellite rotating about the earth under the influence of a constant force of gravity). The circular motion adjusts its radius in response to changes in speed. This means that the radius of the circular path is variable, unlike the case of uniform circular motion. In any eventuality, the equation of centripetal acceleration in terms of "speed" and "radius" must be satisfied. The important thing to note here is that, although change in speed of the particle affects radial acceleration, the change in speed is not affected by radial or centripetal force. We need a tangential force to affect the change in the magnitude of a tangential velocity. The corresponding acceleration is called tangential acceleration.

In either case, the angular velocity in non-uniform circular motion is not constant as $\omega = \frac{v}{r}$ and v is varying.

Key Points

- In non- uniform circular motion, the size of the velocity vector (speed) changes, denoting change in the magnitude of velocity.
- The change in speed has implications for radial (centripetal) acceleration. There are two possibilities: 1) the radius of the circle is constant; or 2) the radial (centripetal) force is constant.
- In either case, the angular velocity in non-uniform circular motion is not constant, as $\omega = \frac{v}{r}$, and v varies.

Key Terms

- **radial**: Moving along a radius.
- **centripetal**: Directed or moving towards a center.

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7.13: Velocity, Acceleration, and Force

Rotational Angle and Angular Velocity

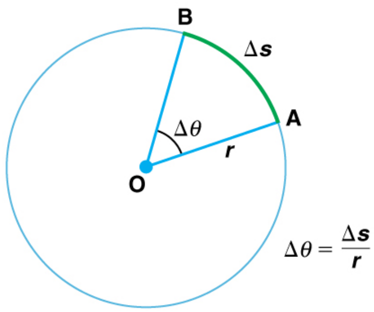
The rotational angle is a measure of how far an object rotates, and angular velocity measures how fast it rotates.

learning objectives

- Express the relationship between the rotational angle and the distance

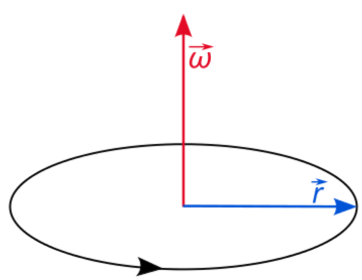
Rotational Angle and Angular Velocity

When an object rotates about an axis, as with a tire on a car or a record on a turntable, the motion can be described in two ways. A point on the edge of the rotating object will have some velocity and will be carried through an arc by riding the spinning object. The point will travel through a distance of $\Delta \mathrm{S}$, but it is often more convenient to talk about the extent the object has rotated. The amount the object rotates is called the rotational angle and may be measured in either degrees or radians. Since the rotational angle is related to the distance $\Delta \mathrm{S}$ and to the radius r by the equation $\Delta \theta = \frac{\Delta \mathrm{S}}{\mathrm{R}}$, it is usually more convenient to use radians.



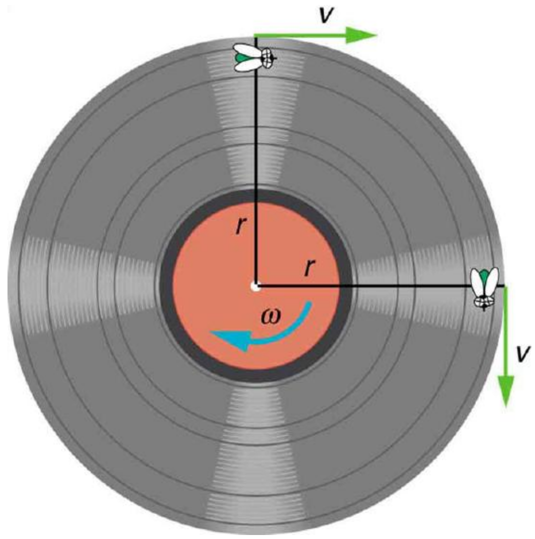
Angle θ and Arc Length s: The radius of a circle is rotated through an angle $\Delta\theta$. The arc length Δs is described on the circumference.

The speed at which the object rotates is given by the angular velocity, which is the rate of change of the rotational angle with respect to time. Although the angle itself is not a vector quantity, the angular velocity is a vector. The direction of the angular velocity vector is perpendicular to the plane of rotation, in a direction which is usually specified by the right-hand rule. Angular acceleration gives the rate of change of angular velocity. The angle, angular velocity, and angular acceleration are very useful in describing the rotational motion of an object.



The Direction of Angular Velocity: The angular velocity describes the speed of rotation and the orientation of the instantaneous axis about which the rotation occurs. The direction of the angular velocity will be along the axis of rotation. In this case (counter-clockwise rotation), the vector points upwards.

When the axis of rotation is perpendicular to the position vector, the angular velocity may be calculated by taking the linear velocity v of a point on the edge of the rotating object and dividing by the radius. This will give the angular velocity, usually denoted by ω, in terms of radians per second.



Angular Velocity: A fly on the edge of a rotating object records a constant velocity v. The object is rotating with an angular velocity equal to $\frac{v}{r}$.

Centripetial Acceleration

Centripetal acceleration is the constant change in velocity necessary for an object to maintain a circular path.

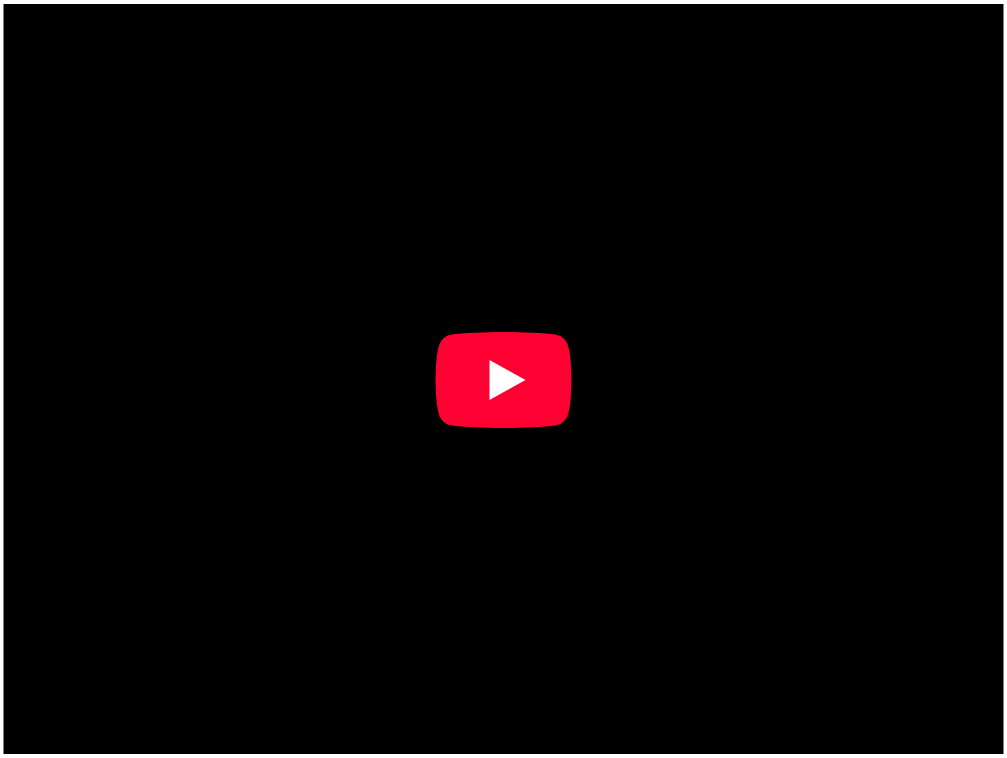
learning objectives

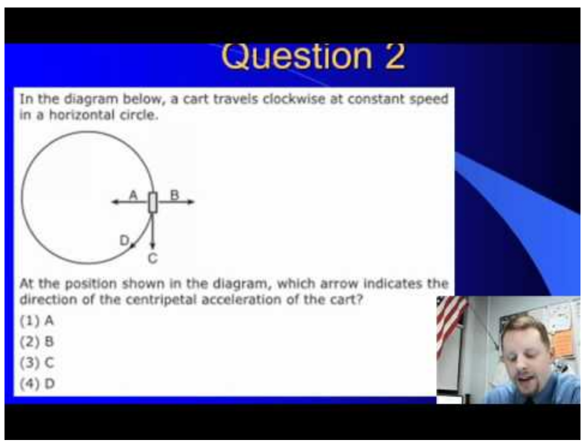
- Express the centripetal acceleration in terms of rotational velocity

Overview

As mentioned in previous sections on kinematics, any change in velocity is given by an acceleration. Often the changes in velocity are changes in magnitude. When an object speeds up or slows down this is a change in the objects velocity. Changes in the magnitude of the velocity match our intuitive and every day usage of the term accelerate. However, because velocity is a vector, it also has a direction. Therefore, any change in the direction of travel of an object must also be met with an acceleration.

Uniform circular motion involves an object traveling a circular path at constant speed. Since the speed is constant, one would not usually think that the object is accelerating. However, the direction is constantly changing as the object traverses the circle. Thus, it is said to be accelerating. One can feel this acceleration when one is on a roller coaster. Even if the speed is constant, a quick turn will provoke a feeling of force on the rider. This feeling is an acceleration.





Centripetal Acceleration: A brief overview of centripetal acceleration for high school physics students.

Calculating Centripetal Acceleration

To calculate the centripetal acceleration of an object undergoing uniform circular motion, it is necessary to have the speed at which the object is traveling and the radius of the circle about which the motion is taking place. The simple equation is:

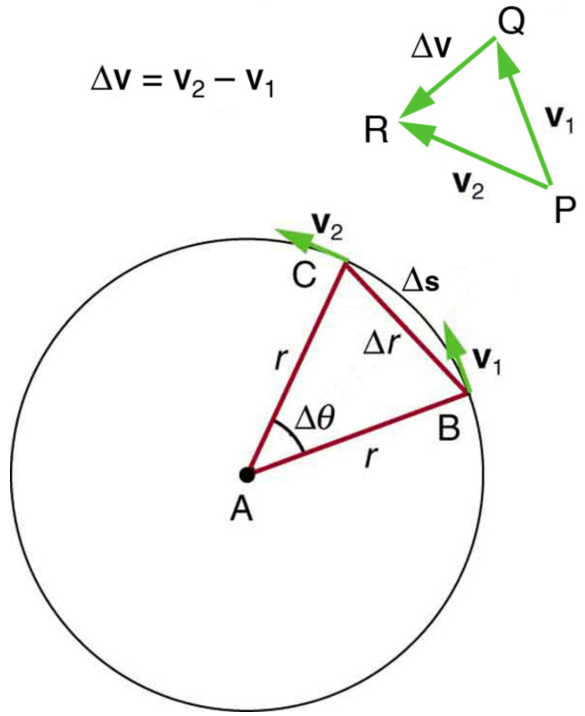
$$a_c = \frac{v^2}{r} \tag{7.13.1}$$

where v is the linear velocity of the object and r is the radius of the circle.

The centripetal acceleration may also be expressed in terms of rotational velocity as follows:

$$a_c = \omega^2 r \tag{7.13.2}$$

with omega being the rotational velocity given by $\frac{v}{r}$.



Centripetal Acceleration: As an object moves around a circle, the direction of the velocity vector constantly changes.

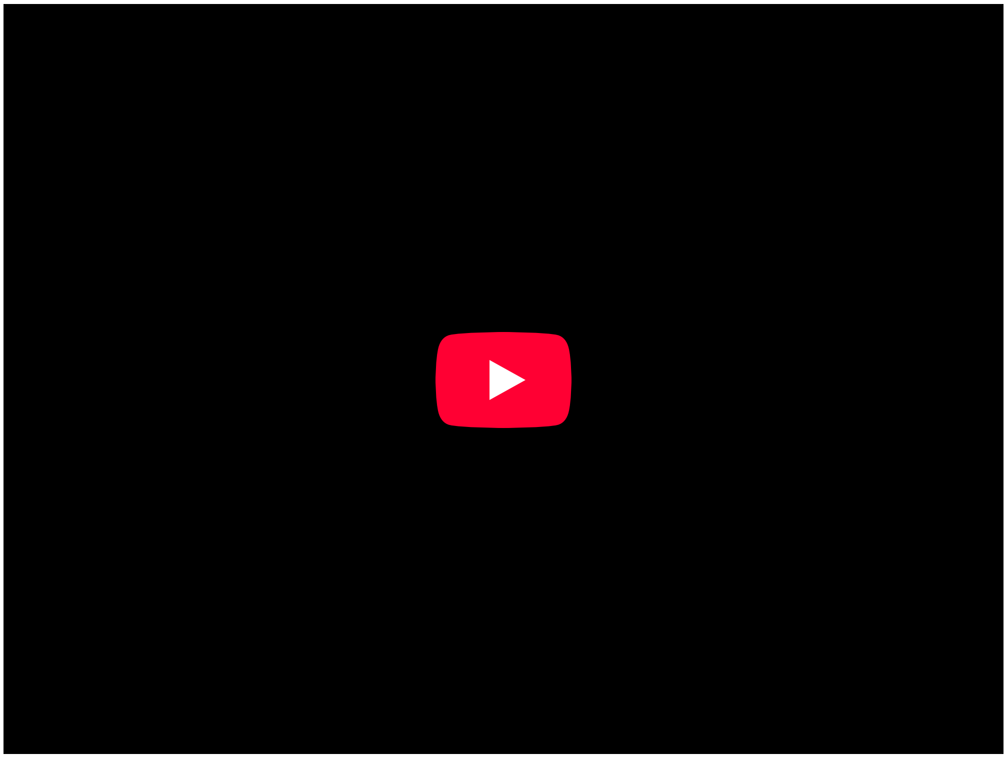
Centripetal Force

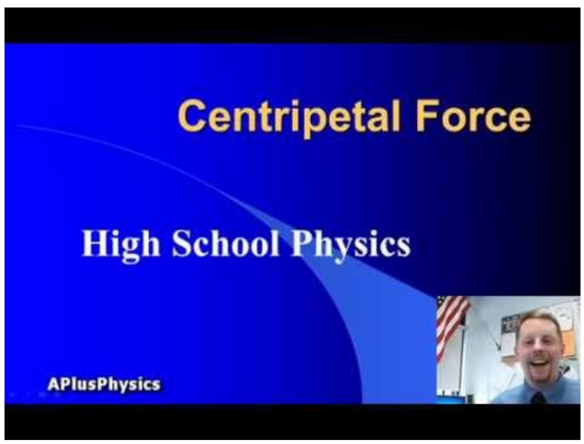
A force which causes motion in a curved path is called a centripetal force (uniform circular motion is an example of centripetal force).

learning objectives

- Express the equations for the centripetal force and acceleration

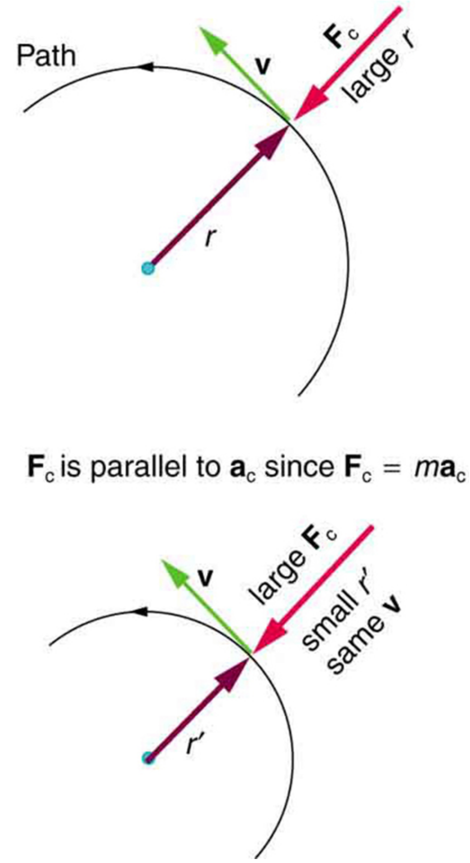
A force that causes motion in a curved path is called a centripetal force. Uniform circular motion is an example of centripetal force in action. It can be seen in the orbit of satellites around the earth, the tension in a rope in a game of tether ball, a roller coaster loop de loop, or in a bucket swung around the body.





Overview of centripetal force: A brief overview of centripetal force.

Previously, we learned that any change in a velocity is an acceleration. As the object moves through the circular path it is constantly changing direction, and therefore accelerating—causing constant force to be acting on the object. This centripetal force acts toward the center of curvature, toward the axis of rotation. Because the object is moving perpendicular to the force, the path followed by the object is a circular one. It is this force that keeps a ball from falling out of a bucket if you swing it in circular continuously.



Centripetal force: As an object travels around a circular path at a constant speed, it experiences a centripetal force accelerating it toward the center.

The equation for centripetal force is as follows:

$$\mathrm{Fc} = \frac{\mathrm{mv}^2}{\mathrm{r}} \tag{7.13.3}$$

where: $\mathrm{F_c}$ is centripetal force, m is mass, v is velocity, and r is the radius of the path of motion.

From Newton’s second law $\mathrm{F} = \mathrm{m} \cdot \mathrm{a}$, we can see that centripetal acceleration is:

$$\mathrm{a_c} = \frac{\mathrm{v}^2}{\mathrm{r}} \tag{7.13.4}$$

Centripetal force can also be expressed in terms of angular velocity. Angular velocity is the measure of how fast an object is traversing the circular path. As the object travels its path, it sweeps out an arc that can be measured in degrees or radians. The equation for centripetal force using angular velocity is:

$$\mathrm{F_c} = \mathrm{mr}\omega^2 \tag{7.13.5}$$

Key Points

- When an object rotates about an axis, the points on the edge of the object travel in arcs.
- The angle these arcs sweep out is called the rotational angle, and it is usually represented by the symbol *theta*.
- A measure of how quickly the object is rotating, with respect to time, is called the angular velocity. It is usually represented by a Greek *omega* symbol. Like its counterpart linear velocity, it is a vector.
- For an object to maintain circular motion it must constantly change direction.
- Since velocity is a vector, changes in direction constitute changes in velocity.
- A change in velocity is known as an acceleration. The change in velocity due to circular motion is known as centripetal acceleration.

- Centripetal acceleration can be calculated by taking the linear velocity squared divided by the radius of the circle the object is traveling along.
- When an object is in uniform circular motion, it is constantly changing direction, and therefore accelerating. This is angular acceleration.
- A force acting on the object in uniform circular motion (called centripetal force) is acting on the object from the center of the circle.

Key Terms

- **radians**: The angle subtended at the centre of a circle by an arc of the circle of the same length as the circle's radius.
- **acceleration**: The amount by which a speed or velocity increases (and so a scalar quantity or a vector quantity).
- **circular motion**: Motion in such a way that the path taken is that of a circle.
- **velocity**: A vector quantity that denotes the rate of change of position with respect to time, or a speed with a directional component.
- **centripetal**: Directed or moving towards a center.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.

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7.14: Types of Forces in Nature

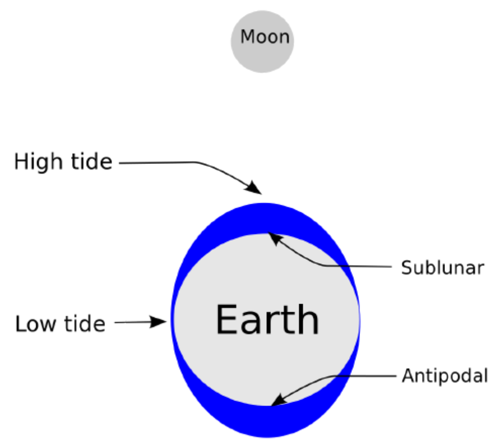
Tides

Tides are the rise and fall of sea levels due to the effects of the gravity exerted by the moon and the sun, and the rotation of the Earth.

learning objectives

- Explain factors that influence the times and amplitude of the tides at a locale

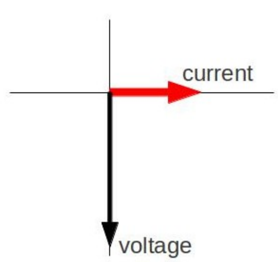
Tides are the rise and fall of sea levels due to the effects of gravitational forces exerted by the moon and the sun when combined with the rotation of the Earth. Tides occur to varying degrees and frequency, depending on location. Shorelines where two almost equally high tides and two low tides occur each day experience a semi- diurnal tide. The occurrence of only one high and one low tide each day is known as diurnal tide. A mixed tide referes to the daily occurrence of two uneven tides, or perhaps one high and one low tide. The times and amplitude of tides at various locales are influenced by the alignment of the sun and moon, the pattern of tides in the deep ocean, the shape of the coastline, and other forces.



Earth's tides.: Schematic of the lunar portion of earth's tides showing (exaggerated) high tides at the sublunar and antipodal points for the hypothetical case of an ocean of constant depth with no land. There would also be smaller, superimposed bulges on the sides facing toward and away from the sun.

Tidal Force

If we want to know the acceleration "felt" by an observer living on Earth due to the moon, a tricky part is that the Earth is not an inertial frame of reference because it is in "free fall" with respect to the moon. Given this, in order to figure out the force observed, we must subtract the acceleration of the (Earth) frame itself. The tidal force produced by the moon on a small particle located on Earth is the vector difference between the gravitational force exerted by the moon on the particle, and the gravitational force that would be exerted if it were located at the Earth's center of mass.



Moon's Gravity on the Earth: Top picture shows the gravity force due to the Moon at different locations F_r on Earth. Bottom picture shows the differential force $F_r - F_{center}$. This is the acceleration "felt" by an observer living on Earth.

As diagramed below, this is equivalent to subtracting the "red" vector from the "black" vectors on the surface of the Earth in the top picture, leading to the "differential" force represented by the bottom picture. Thus, the tidal force depends not on the strength of the lunar gravitational field, but on its gradient (which falls off approximately as the inverse cube of the distance to the originating gravitational body).

On average, the solar gravitational force on the Earth is 179 times stronger than the lunar, but because the sun is on average 389 times farther from the Earth its field gradient is weaker. The solar tidal force is 46% as large as the lunar. More precisely, the lunar tidal acceleration (along the moon-Earth axis, at the Earth's surface) is about $1.1 \cdot 10^{-7}$ g, while the solar tidal acceleration (along the sun-Earth axis, at the Earth's surface) is about $0.52 \cdot 10^{-7}$ g, where g is the gravitational acceleration at the Earth's surface. Venus has the largest effect of the other planets, at 0.000113 times the solar effect.

Tidal Energy

Energy of tides can be extracted by two means: inserting a water turbine into a tidal current, or building ponds that release/admit water through a turbine. In the first case, the energy amount is entirely determined by the timing and tidal current magnitude, but the best currents may be unavailable because the turbines would obstruct ships. In the second case, impoundment dams are expensive to construct, natural water cycles are completely disrupted, as is ship navigation. However, with multiple ponds, power can be generated at chosen times. Presently, there are few installed systems for tidal power generation (most famously, La Rance by Saint Malo, France), as many difficulties are involved. Aside from environmental issues, simply withstanding corrosion and biological fouling pose engineering challenges.

Tidal Energy Generator: Tidal energy generator that works like a wind turbine, but with the ocean currents providing the energy. The circle in the middle is the turbine. The contraption travels up and down the two legs just like a lift and sits on the sea floor when in use.

Unlike with wind power systems, tidal power proponents point out that generation levels can be reliably predicted (save for weather effects). While some generation is possible for most of the tidal cycle, in practice, turbines lose efficiency at lower operating rates. Since the power available from a flow is proportional to the cube of the flow speed, the times during which high power generation is possible are brief.

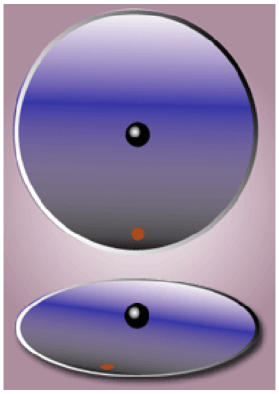
The Coriolois Force

The Coriolis effect is a deflection of moving objects when they are viewed in a rotating reference frame.

learning objectives

- Formulate relationship between the Coriolis force, mass of an object, and speed in the rotating frame

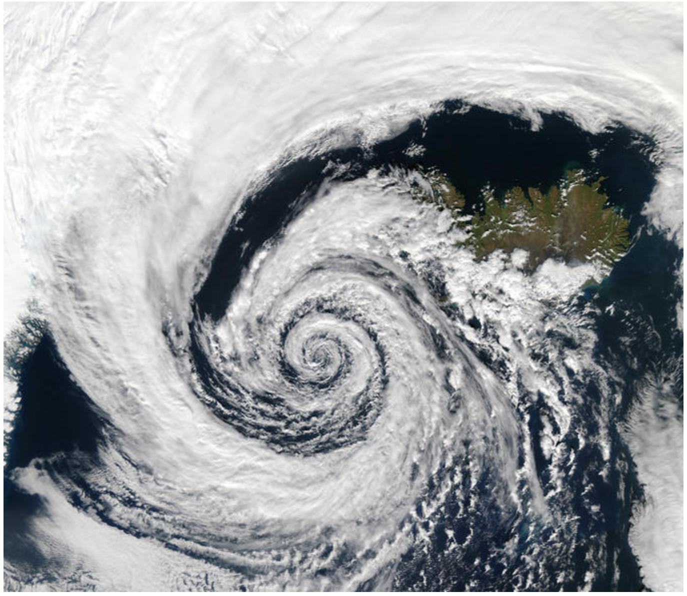
The Coriolis effect is a deflection of moving objects when they are viewed in a rotating reference frame. In a reference frame with clockwise rotation, the deflection is to the left of the motion of the object; in one with counter-clockwise rotation, the deflection is to the right. Although recognized previously by others, the mathematical expression for the Coriolis force appeared in an 1835 paper by French scientist Gaspard-Gustave Coriolis, in connection with the theory of water wheels. Early in the 20th century, the term "Coriolis force" began to be used in connection with meteorology.



Frames of Reference: In the inertial frame of reference (upper part of the picture), the black object moves in a straight line. However, the observer (red dot) who is standing in the rotating/non-inertial frame of reference (lower part of the picture) sees the object as following a curved path due to the Coriolis and centrifugal forces present in this frame.

Newton's laws of motion govern the motion of an object in a (non-accelerating) inertial frame of reference. When Newton's laws are transformed to a uniformly rotating frame of reference, the Coriolis and centrifugal forces appear. Both forces are proportional to the mass of the object. The Coriolis force is proportional to the rotation rate, and the centrifugal force is proportional to its square. The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame. It is proportional to the object's speed in the rotating frame. These additional forces are termed inertial forces, fictitious forces, or pseudo-forces. They allow the application of Newton's laws to a rotating system. They are correction factors that do not exist in a non-accelerating or inertial reference frame.

Perhaps the most commonly encountered rotating reference frame is the Earth. The Coriolis effect is caused by the rotation of the Earth and the inertia of the mass experiencing the effect. Because the Earth completes only one rotation per day, the Coriolis force is quite small. Its effects generally become noticeable only for motions occurring over large distances and long periods of time, such as large-scale movements of air in the atmosphere or water in the ocean. Such motions are constrained by the surface of the earth, so generally only the horizontal component of the Coriolis force is important. This force causes moving objects on the surface of the Earth to be deflected in a clockwise sense (with respect to the direction of travel) in the northern hemisphere and in a counter-clockwise sense in the southern hemisphere. Rather than flowing directly from areas of high pressure to low pressure, as they would in a non-rotating system, winds and currents tend to flow to the right of this direction north of the equator and to the left of this direction south of it. This effect is responsible for the rotation of large cyclones.



Coriolis Force: This low-pressure system over Iceland spins counter-clockwise due to balance between the Coriolis force and the pressure gradient force.

Other Geophysical Applications

Tidal and Coriolis forces may not be obvious over a small time-space scale, but they are important in meteorology, navigation, and fishing.

learning objectives

- Identify fields that have to take into account the tidal and Coriolis forces

We have studied tidal and Coriolis forces previously. To review, the tidal force is responsible for the tides — it is a "differential force," due to a secondary effect of the force of gravity. The Coriolis force is a fictitious force, representing a deflection of moving objects when they are viewed in a rotating reference frame of the Earth. Although their effects may not be obvious over a small time-space scale, these forces are important in such contexts as meteorology, navigation, fishing, and others.

The Tides

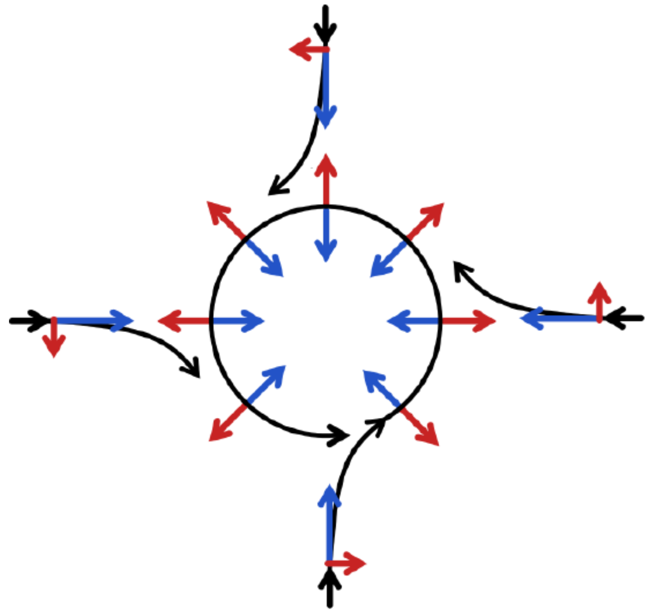
Tidal flows are important for marine navigation, and significant errors in position occur if they are not accounted for. Tidal heights are also important; for example, many rivers and harbors have a shallow "bar" at the entrance to prevent boats with significant draft from entering at low tide. Until the advent of automated navigation, competence in calculating tidal effects was important to naval officers. The certificate of examination for lieutenants in the Royal Navy once declared that the prospective officer was able to "shift his tides. "

Fig. 5.—Tidal indicator, Delaware River, Delaware.

Tidal Indicator: Tidal Indicator, Delaware River, Delaware c. 1897. In the moment pictured, the tide is 1.25 feet above mean low water and is still falling, as indicated by the pointing of the arrow. The indicator is powered by a system of pulleys, cables, and a float

The Coriolis Force

The Coriolis force is quite small, and its effects generally become noticeable only when we are dealing with motions occurring over large distances and long periods of time, such as large-scale movements of air in the atmosphere or water in the ocean. The Coriolis effects also became important in ballistics calculations — for example, calculating the trajectories of very long-range artillery shells. The most famous historical example is the Paris gun, used by the Germans during World War I to bombard Paris from a range of about 120 km.



Flow Representation: A schematic representation of flow around a low-pressure area in the Northern Hemisphere. The pressure-gradient force is represented by blue arrows and the Coriolis acceleration (always perpendicular to the velocity) by red arrows

Key Points

- The tidal force depends not on the strength of the lunar gravitational field itself, but on its gradient, which falls off approximately as the inverse cube of the distance to the originating gravitational body. This is because the tidal force felt by an observer on Earth is a differential force.
- The times and amplitude of the tides at a locale are influenced by several factors, such as the alignment of the sun and moon, the pattern of tides in the deep ocean, the shape of the coastline, and others forces.
- Energy of tides can be extracted by two means: inserting a water turbine into a tidal current, or building ponds that release/admit water through a turbine.
- When Newton's laws are transformed to a uniformly rotating frame of reference, the Coriolis and centrifugal forces appear.
- The Coriolis force acts in a direction perpendicular to the rotation axis and to the velocity of the body in the rotating frame; it is proportional to the object's mass and speed in the rotating frame.
- The Coriolis effect is caused by the rotation of the Earth and the inertia of the mass experiencing the effect.
- Tidal flows are important for marine navigation, and significant errors in position occur if they are not accounted for.
- The Coriolis force is quite small, and its effects generally become noticeable only when we are dealing with motions occurring over large distances and long periods of time, such as large-scale movements of air in the atmosphere or water in the ocean.
- The tidal force is responsible for the tides. It is a "differential force," due to a secondary effect of the force of gravity. The Coriolis force is a fictitious force, representing a deflection of moving objects when they are viewed in a rotating reference frame of the Earth.

Key Terms

- **inertial frame**: A frame of reference that describes time and space homogeneously, isotropically, and in a time-independent manner.
- **diurnal**: Having a daily cycle that is completed every 24 hours, usually referring to tasks, processes, tides, or sunrise to sunset.
- **gradient**: The rate at which a physical quantity increases or decreases relative to change in a given variable, especially distance.
- **fictitious force**: an apparent force that acts on all masses in a non-inertial frame of reference, such as a rotating reference frame
- **centrifugal force**: the apparent outward force that draws a rotating body away from the center of rotation
- **ballistics**: the science of mechanics that deals with the flight, behavior, and effects of projectiles, especially bullets, gravity bombs, rockets, or the like
- **meteorology**: the interdisciplinary scientific study of the atmosphere

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7.15: Newton's Law of Universal Gravitation

The Law of Universal Gravitation

Objects with mass feel an attractive force that is proportional to their masses and inversely proportional to the square of the distance.

learning objectives

- Express the Law of Universal Gravitation in mathematical form

While an apple might not have struck Sir Isaac Newton's head as myth suggests, the falling of one did inspire Newton to one of the great discoveries in mechanics: *The Law of Universal Gravitation*. Pondering why the apple never drops sideways or upwards or any other direction except perpendicular to the ground, Newton realized that the Earth itself must be responsible for the apple's downward motion.

Theorizing that this force must be proportional to the masses of the two objects involved, and using previous intuition about the inverse-square relationship of the force between the earth and the moon, Newton was able to formulate a general physical law by induction.

The Law of Universal Gravitation states that every point mass attracts every other point mass in the universe by a force pointing in a straight line between the centers-of-mass of both points, and this force is proportional to the masses of the objects and inversely proportional to their separation This attractive force always points inward, from one point to the other. The Law applies to all objects with masses, big or small. Two big objects can be considered as point-like masses, if the distance between them is very large compared to their sizes or if they are spherically symmetric. For these cases the mass of each object can be represented as a point mass located at its center-of-mass.

While Newton was able to articulate his Law of Universal Gravitation and verify it experimentally, he could only calculate the relative gravitational force in comparison to another force. It wasn't until Henry Cavendish's verification of the gravitational constant that the Law of Universal Gravitation received its final algebraic form:

$$\mathrm{F}=\mathrm{G}\frac{\mathrm{Mm}}{\mathrm{r}^2} \tag{7.15.1}$$

where F represents the force in Newtons, M and m represent the two masses in kilograms, and r represents the separation in meters. G represents the gravitational constant, which has a value of $6.674\cdot 10^{-11}\mathrm{N}(\mathrm{m}/\mathrm{kg})^2$. Because of the magnitude of G, gravitational force is very small unless large masses are involved.

image

Forces on two masses: All masses are attracted to each other. The force is proportional to the masses and inversely proportional to the square of the distance.

Gravitational Attraction of Spherical Bodies: A Uniform Sphere

The Shell Theorem states that a spherically symmetric object affects other objects as if all of its mass were concentrated at its center.

learning objectives

- Formulate the Shell Theorem for spherically symmetric objects

Universal Gravitation for Spherically Symmetric Bodies

The Law of Universal Gravitation states that the gravitational force between two points of mass is proportional to the magnitudes of their masses and the inverse-square of their separation, d:

$$\mathrm{F}=\frac{\mathrm{GmM}}{\mathrm{d}^2} \tag{7.15.2}$$

However, most objects are not point particles. Finding the gravitational force between three-dimensional objects requires treating them as points in space. For highly symmetric shapes such as spheres or spherical shells, finding this point is simple.

The Shell Theorem

Isaac Newton proved the Shell Theorem, which states that:

1. A spherically symmetric object affects other objects gravitationally as if all of its mass were concentrated at its center,
2. If the object is a spherically symmetric shell (i.e., a hollow ball) then the net gravitational force on a body *inside* of it is zero.

Since force is a vector quantity, the vector summation of all parts of the shell/sphere contribute to the net force, and this net force is the equivalent of one force measurement taken from the sphere's midpoint, or center of mass (COM). So when finding the force of gravity exerted on a ball of 10 kg, the distance measured from the ball is taken from the ball's center of mass to the earth's center of mass.

Given that a sphere can be thought of as a collection of infinitesimally thin, concentric, spherical shells (like the layers of an onion), then it can be shown that a corollary of the Shell Theorem is that the force exerted in an object inside of a solid sphere is only dependent on the mass of the sphere inside of the radius at which the object is. That is because shells at a greater radius than the one at which the object is, do *not* contribute a force to an object inside of them (Statement 2 of theorem).

When considering the gravitational force exerted on an object at a point *inside* or *outside* a uniform spherically symmetric object of radius RR, there are two simple and distinct situations that must be examined: the case of a hollow spherical shell, and that of a solid sphere with uniformly distributed mass.

Case 1: A hollow spherical shell

The gravitational force acting by a spherically symmetric shell upon a point mass *inside* it, is the vector sum of gravitational forces acted by each part of the shell, and this vector sum is equal to zero. That is, a mass mm *within* a spherically symmetric shell of mass M, will feel no net force (Statement 2 of Shell Theorem).

The net gravitational force that a spherical shell of mass M exerts on a body *outside* of it, is the vector sum of the gravitational forces acted by each part of the shell on the outside object, which add up to a net force acting as if mass M is concentrated on a point at the center of the sphere (Statement 1 of Shell Theorem).

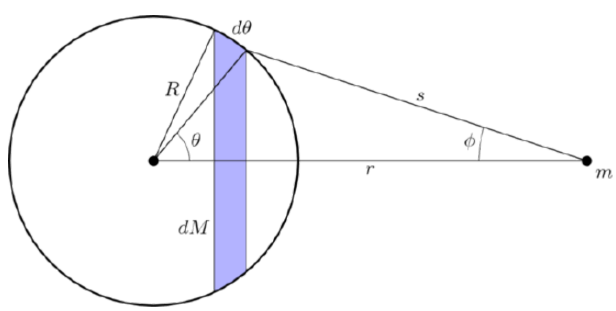


Diagram used in the proof of the Shell Theorem: This diagram outlines the geometry considered when proving The Shell Theorem. In particular, in this case a spherical shell of mass M (left side of figure) exerts a force on mass m (right side of the figure) outside of it. The surface area of a thin slice of the sphere is shown in color. (Note: The proof of the theorem is not presented here. Interested readers can explore further using the sources listed at the bottom of this article.)

Case 2: A solid, uniform sphere

The second situation we will examine is for a solid, uniform sphere of mass M and radius R, exerting a force on a body of mass m at a radius d *inside* of it (that is, $d < R$). We can use the results and corollaries of the Shell Theorem to analyze this case. The contribution of all shells of the sphere at a radius (or distance) greater than dd from the sphere's center-of-mass can be ignored (see above corollary of the Shell Theorem). Only the mass of the sphere within the desired radius $M < d$ (that is the mass of the sphere inside dd) is relevant, and can be considered as a point mass at the center of the sphere. So, the gravitational force acting upon point mass mm is:

$$F = \frac{GmM_{<d}}{d^2} \tag{7.15.3}$$

where it can be shown that $M_{<d}=\frac{4}{3}\pi d^3\rho$

(ρ is the mass density of the sphere and we are assuming that it does not depend on the radius. That is, the sphere's mass is uniformly distributed.)

Therefore, combining the above two equations we get:

$$F=\frac{4}{3}\pi G m\rho d \tag{7.15.4}$$

which shows that mass mm feels a force that is linearly proportional to its distance, dd, from the sphere's center of mass.

As in the case of hollow spherical shells, the net gravitational force that a solid sphere of uniformly distributed mass M exerts on a body *outside* of it, is the vector sum of the gravitational forces acted by each shell of the sphere on the outside object. The resulting net gravitational force acts as if mass M is concentrated on a point at the center of the sphere, which is the center of mass, or COM (Statement 1 of Shell Theorem). More generally, this result is true even if the mass M is *not* uniformly distributed, but its density varies radially (as is the case for planets).

Weight of the Earth

When the bodies have spatial extent, gravitational force is calculated by summing the contributions of point masses which constitute them.

learning objectives

- Describe how gravitational force is calculated for the bodies with spatial extent

Newton's law of universal gravitation states that every point mass in the universe attracts every other point mass with a force that is directly proportional to the product of their masses, and inversely proportional to the square of the distance between them.

In modern language, the law states the following: *Every point mass attracts every single other point mass by a force pointing along the line intersecting both points*. The force is proportional to the product of the two masses and inversely proportional to the square of the distance between them:

$$F=G\frac{m_1 m_2}{r^2} \tag{7.15.5}$$

where F is the force between the masses, G is the gravitational constant, m_1 is the first mass, m_2 is the second mass and r is the distance between the centers of the masses.

If the bodies in question have spatial extent (rather than being theoretical point masses), then the gravitational force between them is calculated by summing the contributions of the notional point masses which constitute the bodies. In the limit, as the component point masses become "infinitely small", this entails integrating the force (in vector form, see below) over the extents of the two bodies.

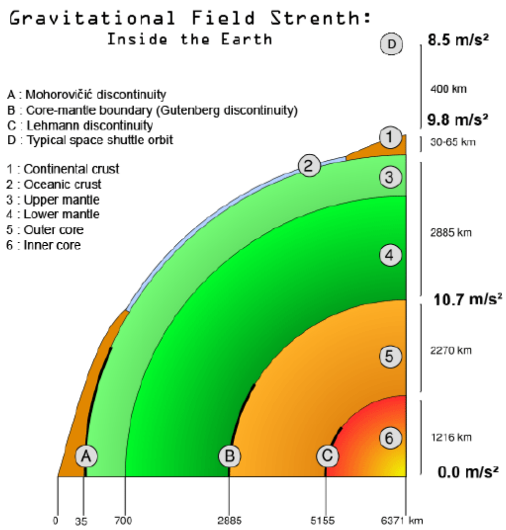
In this way it can be shown that an object with a spherically-symmetric distribution of mass exerts the same gravitational attraction on external bodies as if all the object's mass were concentrated at a point at its center.

For points inside a spherically-symmetric distribution of matter, Newton's Shell theorem can be used to find the gravitational force. The theorem tells us how different parts of the mass distribution affect the gravitational force measured at a point located a distance r_0 from the center of the mass distribution:

1. The portion of the mass that is located at radii $r<r_0$ causes the same force at r_0 as if all of the mass enclosed within a sphere of radius r_0 was concentrated at the center of the mass distribution (as noted above).
2. The portion of the mass that is located at radii $r>r_0$ exerts no net gravitational force at the distance r_0 from the center. That is, the individual gravitational forces exerted by the elements of the sphere out there, on the point at r_0 , cancel each other out.

As a consequence, for example, within a shell of uniform thickness and density there is no net gravitational acceleration anywhere within the hollow sphere. Furthermore, inside a uniform sphere the gravity increases linearly with the distance from the center; the increase due to the additional mass is 1.5 times the decrease due to the larger distance from the center. Thus, if a spherically symmetric body has a uniform core and a uniform mantle with a density that is less than $\frac{2}{3}$ of that of the core, then the gravity initially decreases outwardly beyond the boundary, and if the sphere is large enough, further outward the gravity increases again, and eventually it exceeds the gravity at the core/mantle boundary.

The gravity of the Earth may be highest at the core/mantle boundary, as shown in Figure 1:



Gravitational Field of Earth: Diagram of the gravitational field strength within the Earth.

Key Points

- Sir Isaac Newton's inspiration for the Law of Universal Gravitation was from the dropping of an apple from a tree.
- Newton's insight on the inverse-square property of gravitational force was from intuition about the motion of the earth and the moon.
- The mathematical formula for gravitational force is $\mathrm{F}=\mathrm{G}\frac{\mathrm{Mm}}{\mathrm{r}^2}$ where G is the gravitational constant.
- Since force is a vector quantity, the vector summation of all parts of the shell contribute to the net force, and this net force is the equivalent of one force measurement taken from the sphere's midpoint, or center of mass (COM).
- The gravitational force on an object within a hollow spherical shell is zero.
- The gravitational force on an object within a uniform spherical mass is linearly proportional to its distance from the sphere's center of mass (COM).
- Newton's law of universal gravitation states that every point mass in the universe attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- The second step in calculating earth's mass came with the development of Newton's law of universal gravitation.
- By equating Newton's second law with his law of universal gravitation, and inputting for the acceleration a the experimentally verified value of 9.8 $\frac{\mathrm{m}}{\mathrm{s}^2}$, the mass of earth is calculated to be $5.96\times10^{24}\mathrm{kg}$, making the earth's weight calculable given any gravitational field.
- The gravity of the Earth may be highest at the core/mantle boundary

Key Terms

- **induction**: Use inductive reasoning to generalize and interpret results from applying Newton's Law of Gravitation.
- **inverse**: Opposite in effect or nature or order.
- **center of mass**: The center of mass (COM) is the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero.
- **point mass**: A theoretical point with mass assigned to it.
- **weight**: The force on an object due to the gravitational attraction between it and the Earth (or whatever astronomical object it is primarily influenced by).
- **gravitational force**: A very long-range, but relatively weak fundamental force of attraction that acts between all particles that have mass; believed to be mediated by gravitons.

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7.16: Kepler's Laws

Kepler's First Law

Kepler's first law is: *The orbit of every planet is an ellipse with the Sun at one of the two foci*.

learning objectives

- Apply Kepler's first law to describe planetary motion

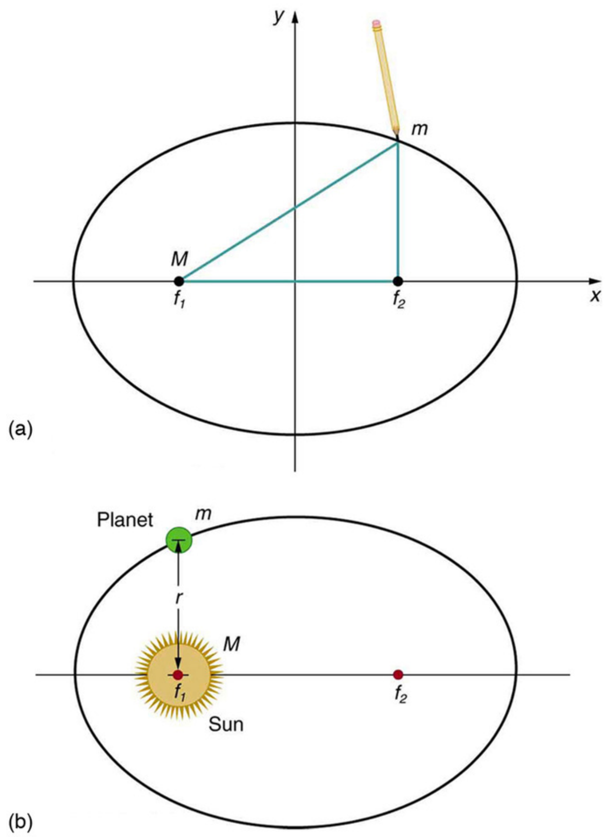
Kepler's First Law

Kepler's first law states that

Definition

The orbit of every planet is an ellipse with the Sun at one of the two foci.

An ellipse is a closed plane curve that resembles a stretched out circle. Note that the Sun is not at the center of the ellipse, but at one of its foci. The other focal point, f_2, has no physical significance for the orbit. The center of an ellipse is the midpoint of the line segment joining its focal points. A circle is a special case of an ellipse where both focal points coincide.



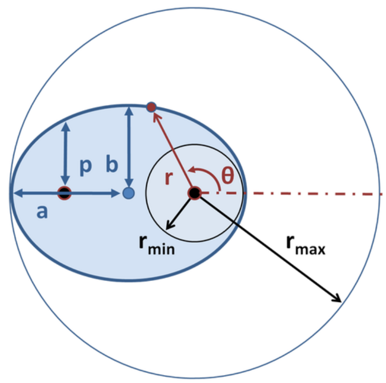
Ellipses and Kepler's First Law: (a) An ellipse is a closed curve such that the sum of the distances from a point on the curve to the two foci (f_1 and f_2) is a constant. You can draw an ellipse as shown by putting a pin at each focus, and then placing a string around a pencil and the pins and tracing a line on paper. A circle is a special case of an ellipse in which the two foci coincide (thus any point on the circle is the same distance from the center). (b) For any closed gravitational orbit, m follows an elliptical path with M at one focus. Kepler's first law states this fact for planets orbiting the Sun.

How stretched out an ellipse is from a perfect circle is known as its eccentricity: a parameter that can take any value greater than or equal to 0 (a circle) and less than 1 (as the eccentricity tends to 1, the ellipse tends to a parabola). The eccentricities of the planets known to Kepler varied from 0.007 (Venus) to 0.2 (Mercury). Minor bodies such as comets an asteroids (discovered after Kepler's time) can have very large eccentricities. The dwarf planet Pluto, discovered in 1929, has an eccentricity of 0.25.

Symbolically, an ellipse can be represented in polar coordinates as:

$$r=\frac{p}{1+\epsilon\cos\theta} \tag{7.16.1}$$

where (r,θ) are the polar coordinates (from the focus) for the ellipse, p is the semi-latus rectum, and ϵ is the eccentricity of the ellipse. For a planet orbiting the Sun, r is the distance from the Sun to the planet and θ is the angle between the planet's current position and its closest approach, with the Sun as the vertex.



Orbit As Ellipse: Heliocentric coordinate system (r,θ) for ellipse. Also shown are: semi-major axis a, semi-minor axis b and semi-latus rectum p; center of ellipse and its two foci marked by large dots. For $\theta=0^\circ$, $r=r_{\min}$ and for $\theta=180^\circ$, $r=r_{\max}$.

At $\theta=0^\circ$, perihelion, the distance is minimum

$$r_{\min}=\frac{p}{1+\epsilon}. \tag{7.16.2}$$

At $\theta=90^\circ$ and at $\theta=270^\circ$, the distance is p.

At $\theta=180^\circ$, aphelion, the distance is maximum

$$r_{\max}=\frac{p}{1-\epsilon}. \tag{7.16.3}$$

The semi-major axis a is the arithmetic mean between $r_{\min}$ and $r_{\max}$:

$$r_{\max}-a=a-r_{\min} \tag{7.16.4}$$

$$a=\frac{p}{1-\epsilon^2}. \tag{7.16.5}$$

The semi-minor axis b is the geometric mean between $r_{\min}$ and $r_{\max}$:

$$\frac{r_{\max}}{b}=\frac{b}{r_{\min}} \tag{7.16.6}$$

$$b=\frac{p}{\sqrt{1-\epsilon^2}} \tag{7.16.7}$$

The semi-latus rectum p is the harmonic mean between $r_{\min}$ and $r_{\max}$:

$$\frac{1}{r_{\min}}-\frac{1}{p}=\frac{1}{p}-\frac{1}{r_{\max}} \tag{7.16.8}$$

$$pa=r_{\max}\cdot r_{\min}=b^2. \tag{7.16.9}$$

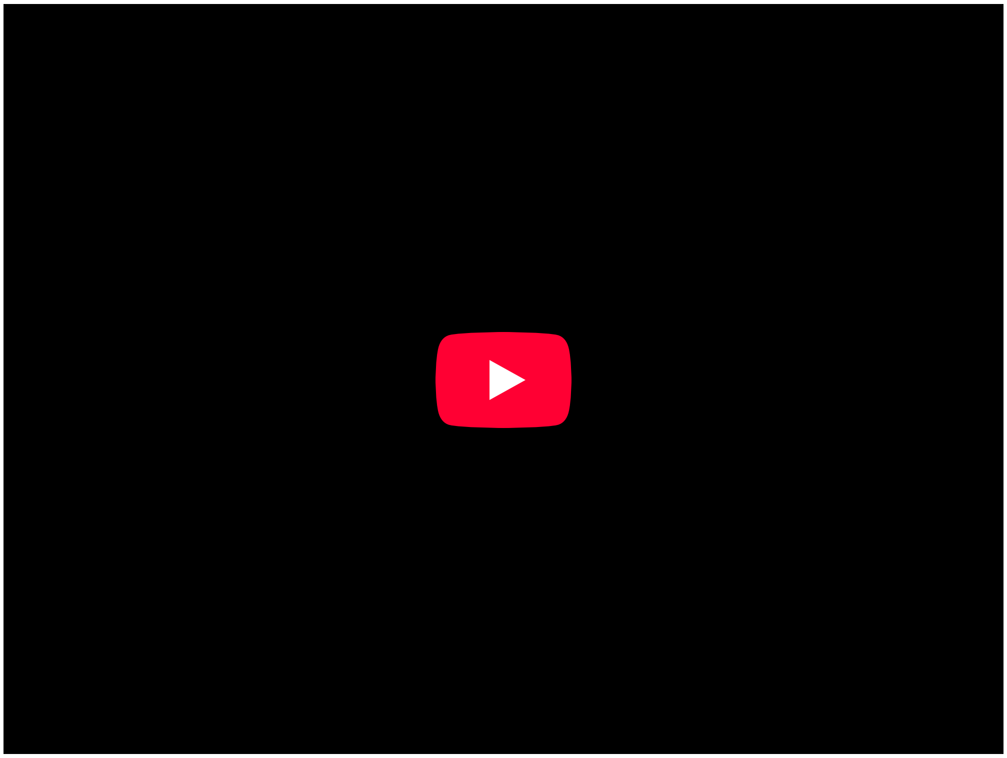
The eccentricity ϵ is the coefficient of variation between $r_{\min}$ and $r_{\max}$:

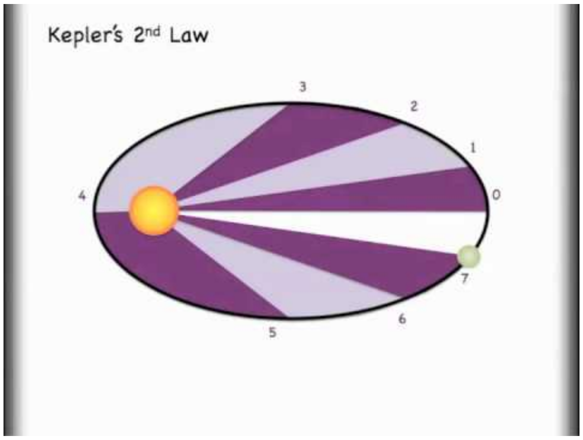
$$\epsilon=\frac{r_{\max}-r_{\min}}{r_{\max}+r_{\min}} \tag{7.16.10}$$

The area of the ellipse is

$$A = \pi ab \tag{7.16.11}$$

The special case of a circle is $\epsilon = 0$, resulting in $r = p = r_{min} = r_{max} = a = b$ and $A = \pi r^2$. The orbits of planets with very small eccentricities can be approximated as circles.





Understanding Kepler's 3 Laws and Orbits: In this video you will be introduced to Kepler's 3 laws and see how they are relevant to orbiting objects.

Kepler's Second Law

Kepler's second law states: A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.

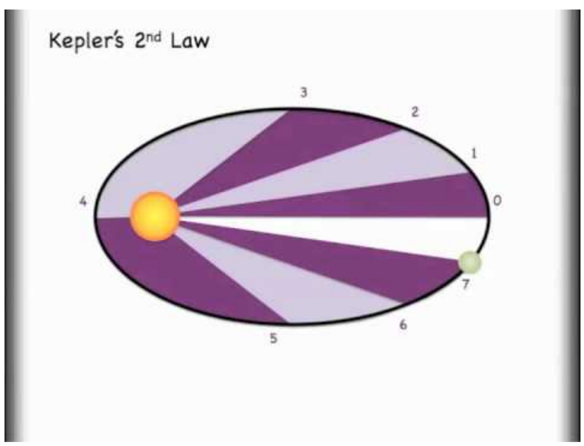
learning objectives

- Apply Kepler's second law to describe planetary motion

Kepler's second law states:

Definition

A line joining a planet and the Sun sweeps out equal areas during equal intervals of time.



Understanding Kepler's 3 Laws and Orbits: In this video you will be introduced to Kepler's 3 laws and see how they are relevant to orbiting objects.

In a small time the planet sweeps out a small triangle having base line and height. The area of this triangle is given by:

$$\mathrm{dA}=\frac{1}{2}\cdot\mathrm{r}\cdot\mathrm{rd}\theta \tag{7.16.12}$$

and so the constant areal velocity is:

$$\frac{\mathrm{dA}}{\mathrm{dt}}=\frac{1}{2}\mathrm{r}^{2}\frac{\mathrm{d}\theta}{\mathrm{dt}} \tag{7.16.13}$$

Now as the first law states that the planet follows an ellipse, the planet is at different distances from the Sun at different parts in its orbit. So the planet has to move faster when it is closer to the Sun so that it sweeps equal areas in equal times.

The total area enclosed by the elliptical orbit is:

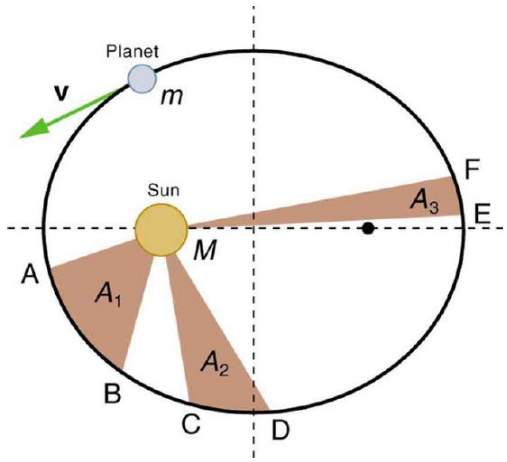
$$A = \pi ab \tag{7.16.14}$$

Therefore the period P satisfies:

$$\pi ab = P \cdot \frac{1}{2} r^2 \dot{\theta} \text{ or } r^2 \dot{\theta} = nab \tag{7.16.15}$$

Where $\dot{\theta} = \frac{d\theta}{dt}$ is the angular velocity, (using Newton notation for differentiation), and $n = \frac{2\pi}{P}$ is the mean motion of the planet around the Sun.

See below for an illustration of this effect. The planet traverses the distance between A and B, C and D, and E and F in equal times. When the planet is close to the Sun it has a larger velocity, making the base of the triangle larger, but the height of the triangle smaller, than when the planet is far from the Sun. One can see that the planet will travel fastest at perihelion and slowest at aphelion.



Kepler's Second Law: The shaded regions have equal areas. It takes equal times for m to go from A to B, from C to D, and from E to F. The mass m moves fastest when it is closest to M. Kepler's second law was originally devised for planets orbiting the Sun, but it has broader validity.

Kepler's Third Law

Kepler's third law states that *the square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit*.

learning objectives

- Apply Kepler's third law to describe planetary motion

Kepler's Third Law

Kepler's third law states:

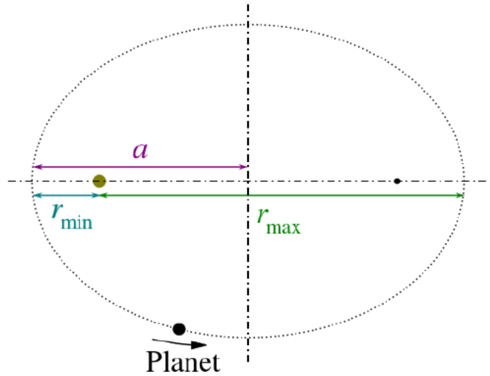
Definition

The square of the orbital period of a planet is directly proportional to the cube of the semi-major axis of its orbit.

The third law, published by Kepler in 1619, captures the relationship between the distance of planets from the Sun, and their orbital periods. Symbolically, the law can be expressed as

$$P^2 \propto a^3, \tag{7.16.16}$$

where P is the orbital period of the planet and a is the semi-major axis of the orbit (see).



Kepler's Third Law: Kepler's third law states that the square of the period of the orbit of a planet about the Sun is proportional to the cube of the semi-major axis of the orbit.

The constant of proportionality is

$$\frac{P^2_{\text{planet}}}{a^3_{\text{planet}}} = \frac{P^2_{\text{earth}}}{a^3_{\text{earth}}} = 1\,\frac{yr^2}{AU^3} \tag{7.16.17}$$

for a sidereal year (yr), and astronomical unit (AU).

Kepler enunciated this third law in a laborious attempt to determine what he viewed as the "music of the spheres" according to precise laws, and express it in terms of musical notation. Therefore, it used to be known as the harmonic law.

Derivation of Kepler's Third Law

We can derive Kepler's third law by starting with Newton's laws of motion and the universal law of gravitation. We can therefore demonstrate that the force of gravity is the cause of Kepler's laws.

Consider a circular orbit of a small mass m around a large mass M. Gravity supplies the centripetal force to mass m. Starting with Newton's second law applied to circular motion,

$$F_{\text{net}} = ma_c = m\frac{v^2}{r}. \tag{7.16.18}$$

The net external force on mass m is gravity, and so we substitute the force of gravity for F_{net}:

$$G\frac{mM}{r^2} = m\frac{v^2}{r} \tag{7.16.19}$$

The mass m cancels, as well as an r, yielding

$$G\frac{M}{r} = v^2 \tag{7.16.20}$$

The fact that m cancels out is another aspect of the oft-noted fact that at a given location all masses fall with the same acceleration. Here we see that at a given orbital radius r, all masses orbit at the same speed. This was implied by the result of the preceding worked example. Now, to get at Kepler's third law, we must get the period P into the equation. By definition, period P is the time for one complete orbit. Now the average speed v is the circumference divided by the period—that is,

$$v = \frac{2\pi r}{P}. \tag{7.16.21}$$

Substituting this into the previous equation gives

$$G\frac{M}{r} = \frac{4\pi^2 r^2}{P^2} \tag{7.16.22}$$

Solving for P^2 yields

$$P^2 = \frac{4\pi^2 r^3}{GM}. \tag{7.16.23}$$

Using subscripts 1 and 2 to denote two different satellites, and taking the ratio of the last equation for satellite 1 to satellite 2 yields

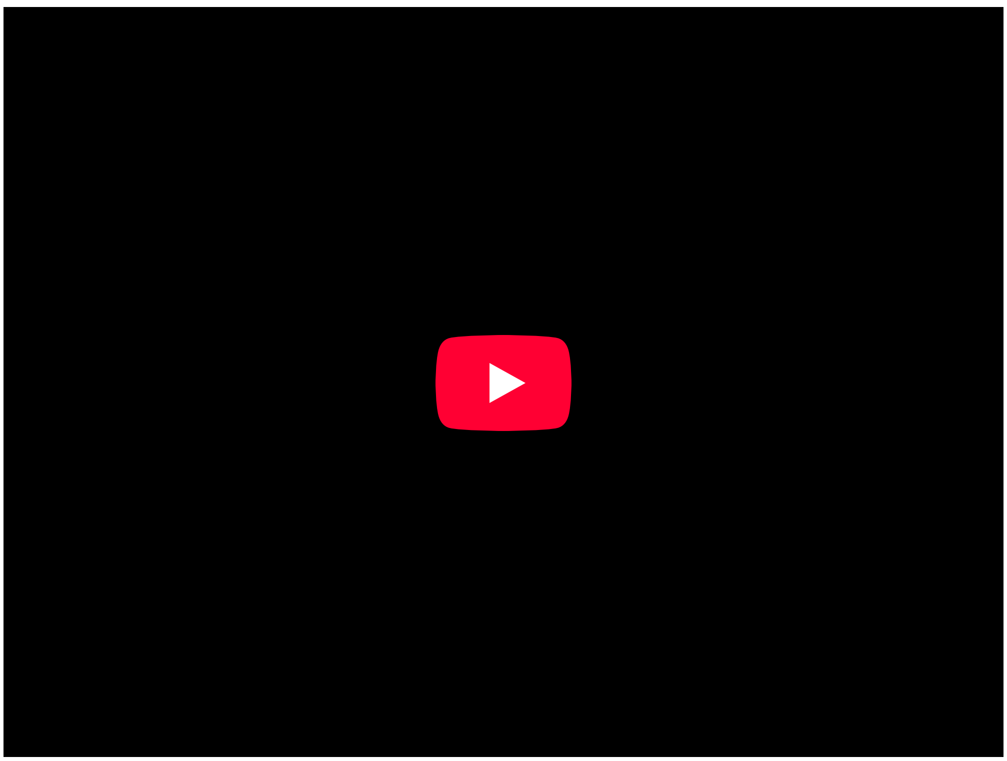
$$\frac{P_1^2}{P_2^2} = \frac{r_1^3}{r_2^3} \tag{7.16.24}$$

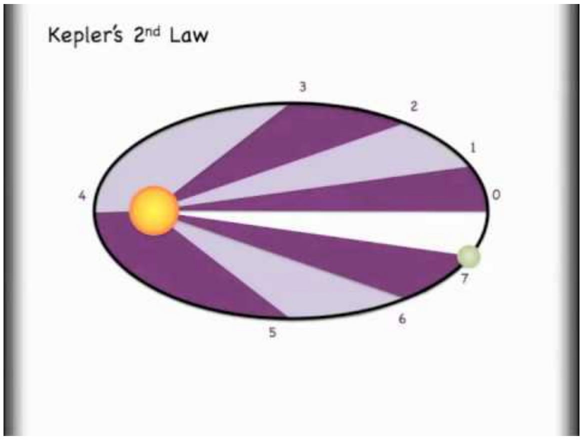
This is Kepler’s third law. Note that Kepler’s third law is valid only for comparing satellites of the same parent body, because only then does the mass of the parent body M cancel.

Now consider what one would get when solving $P^2 = \frac{4\pi^2 GM}{r^3}$ for the ratio $\frac{r^3}{P^2}$. We obtain a relationship that can be used to determine the mass M of a parent body from the orbits of its satellites:

$$M = \frac{4\pi^2 r^3}{GP^2} \tag{7.16.25}$$

If r and P are known for a satellite, then the mass M of the parent can be calculated. This principle has been used extensively to find the masses of heavenly bodies that have satellites. Furthermore, the ratio $\frac{r^3}{T^2}$ should be a constant for all satellites of the same parent body (because $\frac{r^3}{T^2} = \frac{GM}{4\pi^2}$).





Understanding Kepler's 3 Laws and Orbits: In this video you will be introduced to Kepler's 3 laws and see how they are relevant to orbiting objects.

Orbital Maneuvers

An orbital maneuver is the use of propulsion systems to change the orbit of a spacecraft (the rest of the flight is called "coasting").

learning objectives

- Explain purpose of an orbital maneuver

Orbital Maneuvers

In spaceflight, an orbital maneuver is the use of propulsion systems to change the orbit of a spacecraft. The rest of the flight, especially in a transfer orbit, is called coasting.

Rocket Equation

The Tsiolkovsky rocket equation or *ideal rocket equation* is an equation useful for considering vehicles that follow the basic principle of a rocket: a device that can apply acceleration to itself (a thrust) by expelling part of its mass with high speed and moving due to the conservation of momentum. Specifically, it is a mathematical equation relating the delta-v with the effective exhaust velocity and both the initial and final mass of a rocket (or other reaction engine).

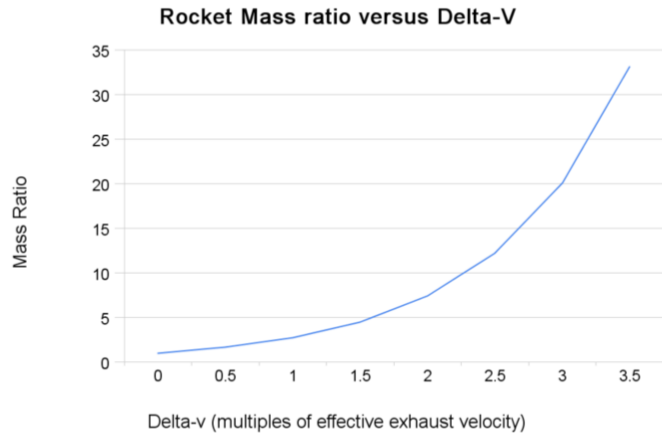
For any such maneuver (or journey involving a number of such maneuvers):

$$\Delta v = v_e \ln(\frac{m_0}{m_1}), \tag{7.16.26}$$

where:

- m_0 is the initial total mass, including propellant;
- m_1 is the final total mass;
- v_e is the effective exhaust velocity ($v_e = I_{sp} \cdot g_0$ where I_{sp} is the specific impulse expressed as a time period and g_0 is the gravitational constant); and
- Δv is delta-v the maximum change of speed of the vehicle (with no external forces acting).

See for an illustration plotting the relationship between final velocity and rocket mass ratios (according to the rocket equation).



Rocket Equation: Rocket mass ratios versus final velocity calculated from the rocket equation

Delta-v Budget:

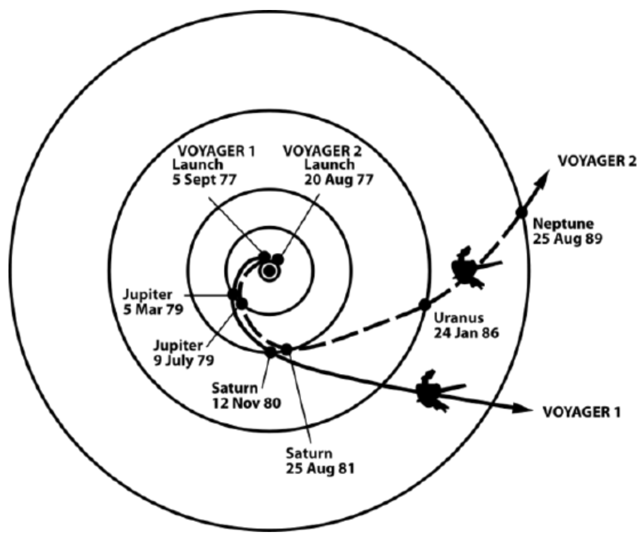
The total delta-v for each maneuver estimated for a mission is called a *delta-v budget*. With a good approximation of the delta-v budget, designers can estimate the fuel to payload requirements of the spacecraft using the rocket equation.

Oberth Effect and Gravitational Assist

In astronautics, the Oberth effect occurs when the use of a rocket engine travelling at high speed generates much more useful energy than one at low speed. This effect is the result of propellant having more usable energy (due to its kinetic energy on top of its chemical potential energy). The vehicle is able to employ this kinetic energy to generate more mechanical power.

Oberth effect is used in a powered flyby or Oberth maneuver in which the application of an impulse (typically from the use of a rocket engine) close to a gravitational body (where the gravity potential is low and the speed is high) allows for more change in kinetic energy and final speed (i.e. higher specific energy) than the same impulse applied further from the body for the same initial orbit.

In orbital mechanics, a gravitational slingshot (or gravity assist maneuver) is the use of the relative movement and gravity of a planet or other celestial body to alter the path and speed of a spacecraft, typically in an effort to save propellant, time, and expense. Gravity assistance can be used to accelerate, decelerate and/or re-direct the path of a spacecraft. This technique was used by the Voyager probes in their fly-bys of Jupiter and Saturn (see).

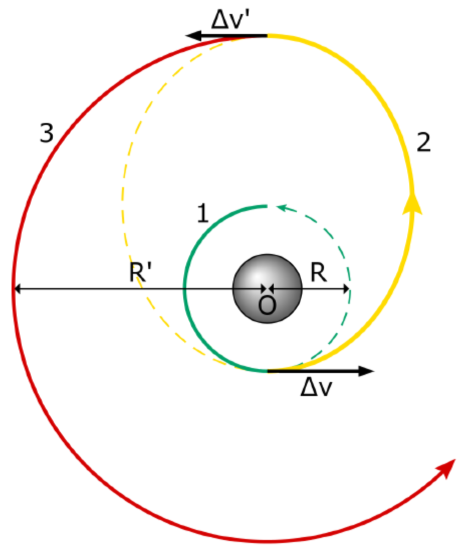


Voyager Path Using Gravity Assists: The trajectories that enabled NASA’s twin Voyager spacecraft to tour the four gas giant planets and achieve velocity to escape our solar system

Transfer Orbits

Orbit insertion is a general term used for a maneuver when it is more than a small correction. It may be used in a maneuver to change a transfer orbit or an ascent orbit into a stable one, but also to change a stable orbit into a descent (i.e., descent orbit insertion). Also, the term *orbit injection* is used, especially for changing a stable orbit into a transfer orbit—e.g., trans-lunar injection (TLI), trans-Mars injection (TMI) and trans-Earth injection (TEI).

The Hohmann transfer orbit is an elliptical orbit used to transfer between two circular orbits of different altitudes in the same plane. The orbital maneuver to perform the Hohmann transfer uses two engine impulses that move aspacecraft onto and off the transfer orbit, as diagramed in. Hohmann transfer orbits are the most efficient with fuel. Other non-Hohmann types of transfer orbits that are less efficient with fuel exist, but these may be more efficient with other resources (such as time).



Hohmann Transfer Orbit: A diagram of the Hohmann Transfer Orbit.

Orbital inclination change is an orbital maneuver aimed at changing an orbiting body's orbit inclination (this maneuver is also known as an orbital plane change as the plane of the orbit is tipped). The maneuver requires a change in the orbital velocity vector (delta-v) at the orbital nodes (i.e., the point at which the initial and desired orbits intersect: the line of orbital nodes is defined by the intersection of the two orbital planes).

Satellites

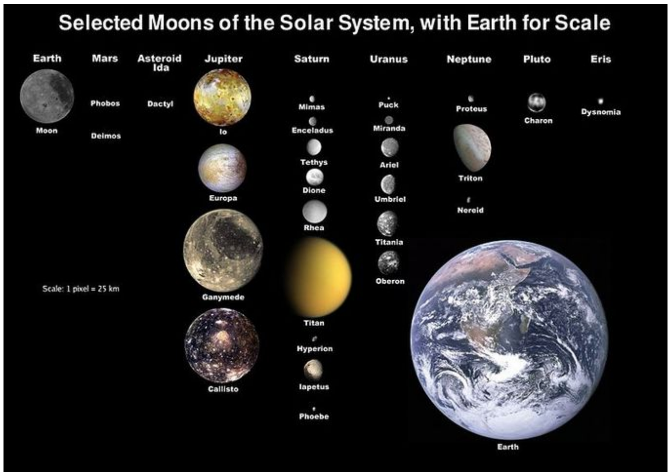
Natural satellites are celestial objects that orbit a larger body; artificial satellites are manmade objects put in the orbit of the Earth.

learning objectives

- Define the concept of a satellite, in the broadest possible terms

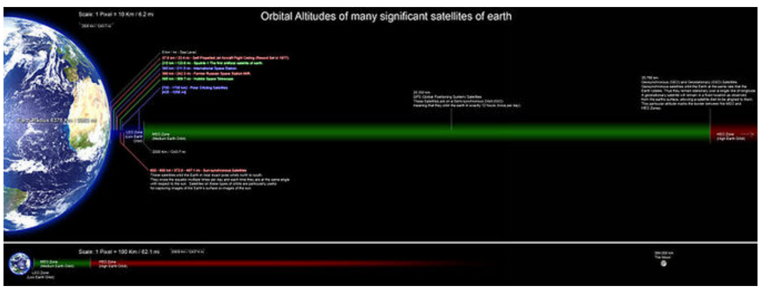
Satellites

The word "satellite" has a somewhat ambiguous definition. The broadest possible definition of a satellite is an object that orbits a larger one due to the force of gravity. Natural satellites, often called moons (see), are celestial bodies that orbit a larger body call a *primary* (often planet, though there are binary asteroids, too). It is technically correct to refer to a planet as a "satellite" of its parent star, though this is not common.



Moons of the Solar System: Nineteen natural satellites are large enough to be round, and one, Saturn's Titan, has a substantial atmosphere.

Artificial satellites (see) are man made objects put in orbit about the Earth or another planet in the Solar System. All satellites follow the laws of orbital mechanics, which can almost always be approximated with Newtonian physics.



Orbital Altitudes: Orbital Altitudes of several significant satellites of earth.

Natural satellites are often classified in terms of their size and composition, while artificial satellites are categorized in terms of their orbital parameters.

Natural Satellites

Formally classified natural satellites, or moons, include 176 planetary satellites orbiting six of the eight planets, and eight orbiting three of the five IAU-listed dwarf planets. As of January 2012, over 200 minor-planet moons have been discovered. There are 76 objects in the asteroid belt with satellites (five with two satellites each), four Jupiter trojans, 39 near-Earth objects, and 14 Mars-crossers. There are also 84 known natural satellites of trans-Neptunian objects. Planets around other stars are likely to have natural satellites as well, although none have yet been observed.

Of the inner planets, Mercury and Venus have no natural satellites; Earth has one large natural satellite, known as the Moon; and Mars has two tiny natural satellites, Phobos and Deimos. The large gas giants have extensive systems of natural satellites, including half a dozen comparable in size to Earth's Moon: the four Galilean moons, Saturn's Titan, and Neptune's Triton. Saturn has an additional six mid-sized natural satellites massive enough to have achieved hydrostatic equilibrium, and Uranus has five. It has been suggested that some satellites may potentially harbor life, though there is currently no direct evidence.

The Earth–Moon system is unique in that the ratio of the mass of the Moon to the mass of the Earth is much greater than that of any other natural satellite to planet ratio in the Solar System. Additionally the Moon's orbit with respect to the Sun is always concave.

The seven largest natural satellites in the Solar System (those bigger than 2,500 km across) are Jupiter's Galilean moons (Ganymede, Callisto, Io, and Europa), Saturn's moon Titan, Earth's moon, and Neptune's captured natural satellite Triton.

Artificial Satellites

The first satellite, Sputnik 1, was put into orbit around Earth and was therefore in geocentric orbit. By far this is the most common type of orbit with about 2,500 artificial satellites orbiting the Earth. Geocentric orbits may be further classified by their altitude, inclination and eccentricity.

The commonly used altitude classifications are Low Earth orbit (LEO), Medium Earth orbit (MEO) and High Earth orbit (HEO). Low Earth orbit is any orbit below 2000 km, and Medium Earth orbit is any orbit higher than that but still below the altitude for geosynchronous orbit at 35,786 km. High Earth orbit is any orbit higher than the altitude for geosynchronous orbit.

Altitude classifications

- Low Earth orbit (LEO): Geocentric orbits ranging in altitude from 0–2000 km (0–1240 miles)
- Medium Earth orbit (MEO): Geocentric orbits ranging in altitude from 2,000 km (1,200 mi) to just below geosynchronous orbit at 35,786 km (22,236 mi). Also known as an intermediate circular orbit.
- High Earth orbit (HEO): Geocentric orbits above the altitude of geosynchronous orbit 35,786 km (22,236 mi).

Inclination Classifications

- Inclined orbit: An orbit whose inclination in reference to the equatorial plane is not zero degrees.
- Polar orbit: An orbit that passes above or nearly above both poles of the planet on each revolution. Therefore it has an inclination of (or very close to) 90 degrees.
- Polar sun synchronous orbit: A nearly polar orbit that passes the equator at the same local time on every pass. Useful for image taking satellites because shadows will be nearly the same on every pass.

Eccentricity Classifications

- Circular orbit: An orbit that has an eccentricity of 0 and whose path traces a circle.
- Hohmann transfer orbit: An orbital maneuver that moves a spacecraft from one circular orbit to another using two engine impulses.
- Elliptic orbit: An orbit with an eccentricity greater than 0 and less than 1 whose orbit traces the path of an ellipse.
- Geosynchronous transfer orbit: An elliptic orbit where the perigee is at the altitude of a Low Earth orbit (LEO) and the apogee at the altitude of a geosynchronous orbit.
- Geostationary transfer orbit: An elliptic orbit where the perigee is at the altitude of a Low Earth orbit (LEO) and the apogee at the altitude of a geostationary orbit.

Key Points

- An ellipse is a closed plane curve that resembles a stretched out circle (The Sun is at one focus while the other focus has no physical significance. A circle is a special case of an ellipse where both focal points coincide.
- How stretched out an ellipse is from a perfect circle is known as its eccentricity: a parameter that can take any value greater than or equal to 0 (a circle) and less than 1 (as the eccentricity tends to 1, the ellipse tends to a parabola).
- Symbolically, an ellipse can be represented in polar coordinates as: $\mathrm{r}=\frac{\mathrm{p}}{1+\epsilon \cos \theta}$, where (r, θ) are the polar coordinates (from the focus) for the ellipse, p is the semi-latus rectum, and ϵ is the eccentricity of the ellipse.
- Perihelion is minimum distance from the Sun a planet achieves in its orbit and is given by$\mathrm{r}_{\max }=\frac{\mathrm{p}}{1-\epsilon}$. Aphelion is the largest distance from the Sun a planet reaches in his orbit and is given by $\mathrm{r}_{\max }=\dfrac{\mathrm{p}}{1-\epsilon}$.
- In a small time the planet sweeps out a small triangle having base line and height. The area of this triangle is given by $\mathrm{dA}=\frac{1}{2} \cdot \mathrm{r} \cdot \mathrm{r} \mathrm{d} \theta$ and so the constant areal velocity is: $\frac{\mathrm{dA}}{\mathrm{dt}}=\dfrac{1}{2} \mathrm{r}^{2} \dfrac{\mathrm{d} \theta}{\mathrm{dt}}$
- The period P satisfies: $\pi \mathrm{ab}=\mathrm{P} \cdot \frac{1}{2} \mathrm{r}^{2} \dot{\theta}$. One can see that the product of r^{2} and must be constant, so that when the planet is further from the Sun it travels at a slower rate and vise versa.
- A planet travels fastest at perihelion and slowest at aphelion.
- Kepler's third law can be represented symbolically as $\mathrm{P}^{2} \propto \mathrm{a}^{3}$, where P is the orbital period of the planet and a is the semi-major axis of the orbit (see.
- The constant of proportionality is $\dfrac{\mathrm{P}_{\text {planet }}^{2}}{\mathrm{a}_{\text {planet }}^{3}}=\frac{\mathrm{P}_{\text {earth }}^{2}}{\mathrm{a}_{\text {earth }}^{3}}=1 \frac{\mathrm{yr}^{2}}{\mathrm{AU}^{3}}$ for a sidereal year (yr), and astronomical unit (AU).
- Kepler's third law can be derived from Newton's laws of motion and the universal law of gravitation. Set the force of gravity equal to the centripetal force. After substituting an expression for the velocity of the planet, one can obtain: $\mathrm{G} \frac{\mathrm{M}}{\mathrm{r}}=\frac{4 \pi^{2}}{\mathrm{P}^{2}}$ which can also be written $\mathrm{P}^{2}=\frac{4 \pi^{2} \mathrm{a}^{3}}{\mathrm{GM}}$.
- Using the expression above we can obtain the mass of the parent body from the orbits of its satellites: $\mathrm{M}=\frac{4 \pi^{2} \mathrm{r}^{3}}{\mathrm{GP}^{2}}$

- The ideal rocket equation related the maximum change in velocity attainable by a rocket (delta-v or Δv) as a function of the exhaust velocity (v_e) and the ratio between the mass of the rocket with and without the propellant (m_0/m_1). The equation is given by $\Delta v = v_e \ln(\frac{m_0}{m_1})$.
- The Oberth effect: where the use of a rocket engine travelling at high speed generates more useful energy than one at low speed. Thus it is more efficient to apply thrust when the spacecraft is nearest to the planet (periastron).
- A gravity assist maneuver is the use of the relative movement and gravity of a planet (or other celestial body) to alter the velocity of a spacecraft—typically in order to save propellant, time, and expense. This technique was employed by the Voyager probes (see.
- The Hohmann transfer orbit is an elliptical orbit used to transfer between two circular orbits of different altitudes, in the same plane. The orbital maneuver to perform the Hohmann transfer uses two engine impulses which move aspacecraft onto and off the transfer orbit. See.
- The broadest possible definition of a satellite is an object that orbits a larger one due to the force of gravity.
- All satellites follow the laws of orbital mechanics, which can almost always be approximated with Newtonian physics.
- Natural satellites are often classified in terms of their size and composition, while artificial satellites are categorized in terms of their orbital parameters.
- Artificial Earth-orbiting satellites have orbits categorized by their altitudes, inclinations, and eccentricities.

Key Terms

- **eccentricity**: The coefficient of variation between r_{min} and r_{max}: $\epsilon = \frac{r_{max} - r_{min}}{r_{max} + r_{min}}$ The further appart the foci are, the stronger the eccentricity.
- **perihelion**: The point in the elliptical orbit of a planet or comet etc. where it is nearest to the Sun. The point farthest from the Sun is called aphelion.
- **semi-latus rectum**: The latus rectum is a chord perpendicular to the major axis and passing through the focus. The semi-latus rectum is half the latus rectrum. See distance p in.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.
- **mean motion**: An angle of 2π (radians) divided by the orbital period (of a celestial body in an elliptic orbit).
- **astronomical unit**: The mean distance from the Earth to the Sun (the semi-major axis of Earth's orbit), approximately 149,600,000 kilometres (symbol AU), used to measure distances in the solar system.
- **sidereal year**: The orbital period of the Earth; a measure of the time it takes for the Sun to return to the same position with respect to the stars of the celestial sphere. A sidereal year is about 20.4 minutes longer than the tropical year due to precession of the equinoxes.
- **Hohmann transfer orbit**: The Hohmann transfer orbit is an elliptical orbit used to transfer between two circular orbits of different altitudes, in the same plane. The orbital maneuver to perform the Hohmann transfer uses two engine impulses, one to move a spacecraft onto the transfer orbit and a second to move off it.
- **delta-v**: The maximum change in the scalar speed of a rocket if the rocket were operated in a vacuum away from external forces (i.e., if no other external forces act).
- **natural satellite**: A natural satellite, moon, or secondary planet is a celestial body that orbits a planet or smaller body, which is called its primary.
- **artificial satellite**: In the context of spaceflight, a satellite is an object which has been placed into orbit by human endeavour.

7.17: Gravitational Potential Energy

Defining Graviational Potential Energy

Gravitational energy is the potential energy associated with gravitational force, such as elevating objects against the Earth's gravity.

learning objectives

- Express gravitational potential energy for two masses

Gravitational energy is the potential energy associated with gravitational force, as work is required to elevate objects against Earth's gravity. The potential energy due to elevated positions is called gravitational potential energy, and is evidenced by water in an elevated reservoir or kept behind a dam. If an object falls from one point to another point inside a gravitational field, the force of gravity will do positive work on the object, and the gravitational potential energy will decrease by the same amount.

Consider a book placed on top of a table. As the book is raised from the floor to the table, some external force works against the gravitational force. If the book falls back to the floor, the "falling" energy the book receives is provided by the gravitational force. Thus, if the book falls off the table, this potential energy goes to accelerate the mass of the book and is converted into kinetic energy. When the book hits the floor, this kinetic energy is converted into heat and sound by the impact.

The factors that affect an object's gravitational potential energy are its height relative to some reference point, its mass, and the strength of the gravitational field it is in. Thus, a book lying on a table has less gravitational potential energy than the same book on top of a taller cupboard, and less gravitational potential energy than a heavier book lying on the same table. An object at a certain height above the Moon's surface has less gravitational potential energy than at the same height above the Earth's surface because the Moon's gravity is weaker. Note that "height" in the common sense of the term cannot be used for gravitational potential energy calculations when gravity is not assumed to be a constant. The following sections provide more detail.

Local Approximation

The strength of a gravitational field varies with location. However, when the change of distance is small in relation to the distances from the center of the source of the gravitational field, this variation in field strength is negligible and we can assume that the force of gravity on a particular object is constant. Near the surface of the Earth, for example, we assume that the acceleration due to gravity is a constant $g = 9.8 m/s^2$ ("standard gravity"). In this case, a simple expression for gravitational potential energy can be derived using the $W = Fd$ equation for work. The upward force required while moving at a constant velocity is equal to the weight, mg, of an object, so the work done in lifting it through a height h is the product mgh. Thus, when accounting only for mass, gravity, and altitude, the equation is:

$$U = mgh \tag{7.17.1}$$

where U is the potential energy of the object relative to its being on the Earth's surface, m is the mass of the object, g is the acceleration due to gravity, and h is the altitude of the object. If m is expressed in kilograms, g in m/s^2 and h in meters then U will be calculated in joules. In most situations, the change in potential energy is the relevant quantity:

$$\Delta U = mg\Delta h \tag{7.17.2}$$

General Formula

However, over large variations in distance, the approximation that gg is constant is no longer valid, and we have to use calculus and the general mathematical definition of work to determine gravitational potential energy. For the computation of the potential energy, we can integrate the gravitational force, whose magnitude is given by Newton's law of gravitation, with respect to the distance r between the two bodies. Using that definition, the gravitational potential energy of a system of masses m_1 and M_2 at a distance r using gravitational constant G is

$$U = -G\frac{m_1 M_2}{r} + K \tag{7.17.3}$$

where K is the constant of integration. Choosing the convention that $K = 0$ makes calculations simpler, albeit at the cost of making U negative. Note that in this case the potential energy becomes zero when r is infinite, and approaches negative infinity as r goes to zero.

Trebuchet: A trebuchet uses the gravitational potential energy of the counterweight to throw projectiles over long distances.

Key Points

- If an object falls from one point to another point inside a gravitational field, the force of gravity will do positive work on the object, and the gravitational potential energy will decrease by the same amount.
- Near the surface of the Earth, the work done in lifting an object through a height h is the product mgh, so $U = mgh$.
- The gravitational potential energy, U, of a system of masses m_1 and M_2 at a distance r using gravitational constant G is $U = -G\frac{m_1 M_2}{r} + K$.

Key Terms

- **Newton's law of gravitation**: This law states that every point mass in the universe attracts every other point mass with a force that is directly proportional to the product of their masses and inversely proportional to the square of the distance between them.
- **potential energy**: The energy an object has because of its position (in a gravitational or electric field) or its condition (as a stretched or compressed spring, as a chemical reactant, or by having rest mass)
- **gravity**: Resultant force on Earth's surface, of the attraction by the Earth's masses, and the centrifugal pseudo-force caused by the Earth's rotation.

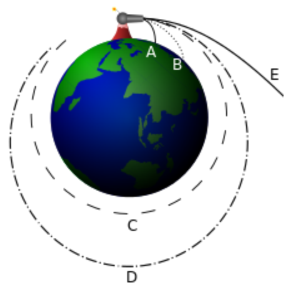
- **5.7: Gravitational Potential Energy** has no license indicated. Original source: https://ocw.mit.edu/courses/electrical-engineering-and-computer-science/6-013-electromagnetics-and-applications-spring-2009.

7.18: Energy Conservation

learning objectives

- Calculate the escape speed of an object given its kinetic energy and the gravitational potential energy

Escape speed is the required starting speed required by an object to go from a starting point in a gravitational potential field to an ending point that is infinitely far away. It is assumed that the velocity of the object at the ending point will be zero.



Isaac Newton's Analysis of Escape Speed: In this figure, Objects A and B don't have the required escape speed and so they fall back to Earth after launch. Objects C and D don't either, they achieve a circular and an elliptical orbit respectively. Object E is launched with sufficient escape velocity and escapes the Earth.

Imagine a situation in which a spaceship that does not have a propulsion system is launched straight away from a planet. (It is moot to discuss escape speed for objects with propulsion systems.) Let us assume that the only significant force that is acting on the spaceship is the force of gravity from the planet. The escape speed of the spaceship can calculated through a simple analysis of conservation of energy. The gravitational potential energy of the spaceship is:

$$\mathrm{U}=-\frac{\mathrm{GMm}}{\mathrm{r}} \tag{7.18.1}$$

Where G is the universal gravitational constant ($\mathrm{G}=6.67 \cdot 10^{-11} \mathrm{m}^3 \mathrm{kg}^{-1} \mathrm{s}^{-2}$), M is the mass of the planet, m is the mass of the spaceship, and r is the distance of the spaceship from the planet's center of gravity.

At the ending point of the spaceship, r goes to infinity. As r goes to infinity, the value of the gravitational potential energy expression goes to 0.

The kinetic energy of the spaceship can be found from:

$$\frac{1}{2} \mathrm{mv}^2 \tag{7.18.2}$$

Where mm is the mass of the spaceship and v is the velocity of the spaceship.

At the starting point of the spaceship, the velocity must have a magnitude equal to the escape speed ($\mathrm{s_e}$). The velocity of the spaceship is 0 at its ending point, and so consequently its kinetic energy is 0 in the end as well.

Summarizing the kinetic energy (K) and potential energy (U) of the spaceship at it's initial (i) and final (f) states:

$$(\mathrm{K}+\mathrm{U})_{\mathrm{i}}=\frac{1}{2} \mathrm{ms}_{\mathrm{e}}^2+\frac{-\mathrm{GMm}}{\mathrm{r}} \tag{7.18.3}$$

$$(\mathrm{K}+\mathrm{U})_{\mathrm{f}}=0+0 \tag{7.18.4}$$

Due to conservation of energy, the initial energy must equal the final energy and so we can solve for $\mathrm{s_e}$:

$$\mathrm{s}_{\mathrm{e}}=\sqrt{\frac{2 \mathrm{GM}}{\mathrm{r}}} \tag{7.18.5}$$

Interestingly, if the spaceship were to fall to the planet from a point infinitely far away it would obtain a final speed of $\mathrm{s_e}$ at the planet.

It should be noted that if an object is launched from a rotating body, such as the Earth, the speed at which the body rotates will affect the required velocity that an object must have relative to the surface of the body. If a rocket is launched tangentially from the Earth's equator in the same direction that the Earth is turning, it will require a lower velocity relative to the Earth than if it were launched in the opposite direction to meet escape speed requirements.

Additionally, it is a misconception that powered vehicles (such as rockets) require escape speed to leave orbit and travel through outer-space. If the vehicle has a propulsion system to provide it with energy once it has left the surface of the planet, it is not necessary to initially meet escape speed requirements.

Key Points

- It is assumed that the velocity of the object at the ending point will be zero.
- The requisite escape speed (se)(se) of an object to escape a spherically symmetric body is given by: $s_e = \sqrt{\frac{2GM}{r}}$, where G is the universal gravitational constant, M is the mass of the body, and r is the distance of the object from the body's center of gravity.
- Escape speed is the required speed that an object has to have to go from a starting point in a gravitational potential field to an ending point that is infinitely far away.
- The speed at which a body rotates will affect the required velocity that an object must have relative to the surface of the body.
- Objects that have propulsion systems do not need to reach escape velocity.

Key Terms

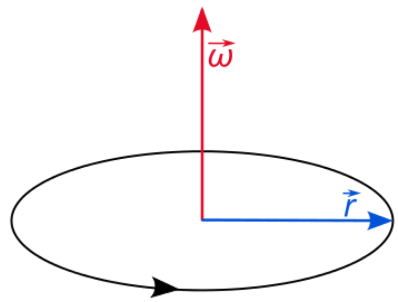
- **propulsion**: Force causing movement.
- **potential energy**: The energy an object has because of its position (in a gravitational or electric field) or its condition (as a stretched or compressed spring, as a chemical reactant, or by having rest mass)
- **kinetic energy**: The energy possessed by an object because of its motion, equal to one half the mass of the body times the square of its velocity.

7.19: Angular vs. Linear Quantities

learning objectives

- Describe properties characteristic to angular velocity and angular momentum

Linear motion is motion in a straight line. This type of motion has several familiar vector quantities associated with it, including linear velocity and momentum. These vector quantities each have a magnitude (a scalar, or number) and direction associated with them. Similarly, circular motion is motion in a circle. It has the same set of vector quantities associated with it, including angular velocity and angular momentum.



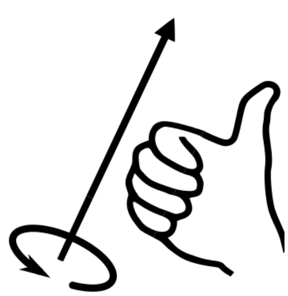
***Angular velocity diagram**: A vector diagram illustrating circular motion. The blue vector connects the origin (center) of the motion to the position of the particle. The red vector is the angular velocity vector, pointing perpendicular to the plane of motion and with magnitude equal to the instantaneous velocity. File:Angular velocity.svg - Wikipedia, the free encyclopedia. **Provided by**: Wikipedia. **Located at**: en.Wikipedia.org/w/index.php?title=File:Angular_velocity.svg&page=1. **License**: CC BY-SA: Attribution-ShareAlike*

Imagine a particle moving in a circle around a point at a constant speed. We will call that point the origin. At any instant in time, the particle is moving in a particular straight-line direction with that speed. In the next instant, the particle has the same speed, but the direction of its velocity has changed.

We recall from our study of linear velocity that a change in the direction of the velocity vector, is a change in velocity and a change in velocity is acceleration. However, we can define an angular momentum vector which is constant throughout this motion. The angular velocity has a direction perpendicular to the plane of circular motion, just like a bike axle points perpendicularly to the rotating wheel. This direction never changes as the object moves in its circle. The magnitude of the angular momentum is equal to the rate at which the angle of the particle advances:

$$\omega=\frac{\mathrm{d}\phi}{\mathrm{dt}} \tag{7.19.1}$$

Note that there are two vectors that are perpendicular to any plane. For example, imagine a vector pointing into your table and the opposite one pointing out of it. To remove this ambiguity, the convention in physics is to use the right hand rule: curl the fingers of your right hand in the direction of the circular motion, and your thumb will point toward the direction of the angular velocity and momentum vectors.



Right hand rule**: When determining the direction of an angular vector, use the right hand rule: curl the fingers of your right hand in the direction of the circular motion and your thumb points in the vector direction. File:Right-hand grip rule.svg - Wikipedia, the* *free encyclopedia.* ***Provided by*: Wikipedia.* ***Located at****:* ***en.Wikipedia.org/w/index.php?title=File:Right-hand_grip_rule.svg&page=1****.* ***License****: CC BY-SA: Attribution-ShareAlike*

The units of angular velocity are radians per second. Radian describes the plane angle subtended by a circular arc as the length of the arc divided by the radius of the arc. One radian is the angle subtended at the center of a circle by an arc that is equal in length to the radius of the circle. More generally, the magnitude in radians of such a subtended angle is equal to the ratio of the arc length to the radius of the circle; that is, $\theta = \frac{s}{r}$, where θ is the subtended angle in radians, s is arc length, and r is radius.

Thus, while the object moves in a circle at constant speed, it undergoes constant linear acceleration to keep it moving in a circle. However, it's angular velocity is constant since it continually sweeps out a constant arc length per unit time. Constant angular velocity in a circle is known as uniform circular motion.

Just as there is an angular version of velocity, so too is there an angular version of acceleration. When the object is going around a circle but its speed is changing, the object is undergoing angular acceleration. Just like with linear acceleration, angular acceleration is a change in the angular velocity vector. This change could be a change in the speed of the object or in the direction. Angular velocity can be clockwise or counterclockwise.

Key Points

- The direction of angular quantity vectors points perpendicular to the plane of the motion. You can determine this direction using the right hand rule.
- The direction of linear quantities such as velocity and momentum change as an object moves in a circle. We can instead define angular versions of these quantities which are constant throughout the circular motion.
- The units of angular quantities are per radian, a measurement of angle, rather than per linear distance (e.g. meter).

Key Terms

- **vector**: A directed quantity, one with both magnitude and direction; the between two points.
- **angular momentum**: A vector quantity describing an object in circular motion; its magnitude is equal to the momentum of the particle, and the direction is perpendicular to the plane of its circular motion.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.

CHAPTER OVERVIEW

8: Work and Energy

Topic hierarchy

8.1: Prelude to Work and Kinetic Energy

In this chapter, we discuss some basic physical concepts involved in every physical motion in the universe, going beyond the concepts of force and change in motion, which we discussed in Motion in Two and Three Dimensions and Newton’s Laws of Motion. These concepts are work, kinetic energy, and power. We explain how these quantities are related to one another, which will lead us to a fundamental relationship called the work-energy theorem. In the next chapter, we generalize this idea to the broader principle of conservation of energy.

Figure 8.1.1: A sprinter exerts her maximum power to do as much work on herself as possible in the short time that her foot is in contact with the ground. This adds to her kinetic energy, preventing her from slowing down during the race. Pushing back hard on the track generates a reaction force that propels the sprinter forward to win at the finish. (credit: modification of work by Marie-Lan Nguyen)

The application of Newton’s laws usually requires solving differential equations that relate the forces acting on an object to the accelerations they produce. Often, an analytic solution is intractable or impossible, requiring lengthy numerical solutions or simulations to get approximate results. In such situations, more general relations, like the work-energy theorem (or the conservation of energy), can still provide useful answers to many questions and require a more modest amount of mathematical calculation. In particular, you will see how the work-energy theorem is useful in relating the speeds of a particle at different points along its trajectory, to the forces acting on it, even when the trajectory is otherwise too complicated to deal with. Thus, some aspects of motion can be addressed with fewer equations and without vector decompositions.

8.2: Work

Learning Objectives

- Represent the work done by any force
- Evaluate the work done for various forces

In physics, **work** represents a type of energy. Work is done when a force acts on something that undergoes a displacement from one position to another. Forces can vary as a function of position, and displacements can be along various paths between two points. We first define the increment of work dW done by a force $\vec{F}$ acting through an infinitesimal displacement d$\vec{r}$ as the dot product of these two vectors:

$$dW = \vec{F} \cdot d\vec{r} = |\vec{F}||d\vec{r}| \cos \theta. \tag{8.2.1}$$

Then, we can add up the contributions for infinitesimal displacements, along a path between two positions, to get the total work.

Work Done by a Force

The work done by a force is the integral of the force with respect to displacement along the path of the displacement:

$$W_{AB} = \int_{path\ AB} \vec{F} \cdot d\vec{r}. \tag{8.2.2}$$

The vectors involved in the definition of the work done by a force acting on a particle are illustrated in Figure $8.2.1$.

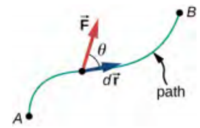


Figure $8.2.1$: Vectors used to define work. The force acting on a particle and its infinitesimal displacement are shown at one point along the path between A and B. The infinitesimal work is the dot product of these two vectors; the total work is the integral of the dot product along the path.

We choose to express the dot product in terms of the magnitudes of the vectors and the cosine of the angle between them, because the meaning of the dot product for work can be put into words more directly in terms of magnitudes and angles. We could equally well have expressed the dot product in terms of the various components introduced in Vectors. In two dimensions, these were the x- and y-components in Cartesian coordinates, or the r- and φ-components in polar coordinates; in three dimensions, it was just x-, y-, and z-components. Which choice is more convenient depends on the situation. In words, you can express Equation 8.2.1 for the work done by a force acting over a displacement as a product of one component acting parallel to the other component. From the properties of vectors, it doesn't matter if you take the component of the force parallel to the displacement or the component of the displacement parallel to the force—you get the same result either way.

Recall that the magnitude of a force times the cosine of the angle the force makes with a given direction is the component of the force in the given direction. The components of a vector can be positive, negative, or zero, depending on whether the angle between the vector and the component-direction is between 0° and 90° or 90° and 180°, or is equal to 90°. As a result, the work done by a force can be positive, negative, or zero, depending on whether the force is generally in the direction of the displacement, generally opposite to the displacement, or perpendicular to the displacement. The maximum work is done by a given force when it is along the direction of the displacement (cos θ = ± 1), and zero work is done when the force is perpendicular to the displacement (cos θ = 0).

The units of work are units of force multiplied by units of length, which in the SI system is newtons times meters, N • m. This combination is called a joule, for historical reasons that we will mention later, and is abbreviated as J. In the English system, still used in the United States, the unit of force is the pound (lb) and the unit of distance is the foot (ft), so the unit of work is the foot-pound (ft • lb).

Work Done by Constant Forces and Contact Forces

The simplest work to evaluate is that done by a force that is constant in magnitude and direction. In this case, we can factor out the force; the remaining integral is just the total displacement, which only depends on the end points A and B, but not on the path between them:

$$W_{AB} = \vec{F} \cdot \int_A^B d\vec{r} = \vec{F} \cdot (\vec{r}_B - \vec{r}_A) = |\vec{F}||\vec{r}_B - \vec{r}_A| \cos\theta \,(constant\ force).$$

We can also see this by writing out Equation 8.2.2 in Cartesian coordinates and using the fact that the components of the force are constant:

$$\begin{aligned} W_{AB} &= \int_{path\ AB} \vec{F} \cdot d\vec{r} = \int_{path\ AB} (F_x dx + F_y dy + F_z dz) = F_x \int_A^B dx + F_y \int_A^B dy + F_z \int_A^B dz \\ &= F_x(x_B - x_A) + F_y(y_B - y_A) + F_z(z_B - z_A) = \vec{F} \cdot (\vec{r}_B - \vec{r}_A). \end{aligned}$$

Figure $8.2.2a$ shows a person exerting a constant force $\vec{F}$ along the handle of a lawn mower, which makes an angle θ with the horizontal. The horizontal displacement of the lawn mower, over which the force acts, is $\vec{d}$. The work done on the lawn mower is

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta,$$

which the figure also illustrates as the horizontal component of the force times the magnitude of the displacement.

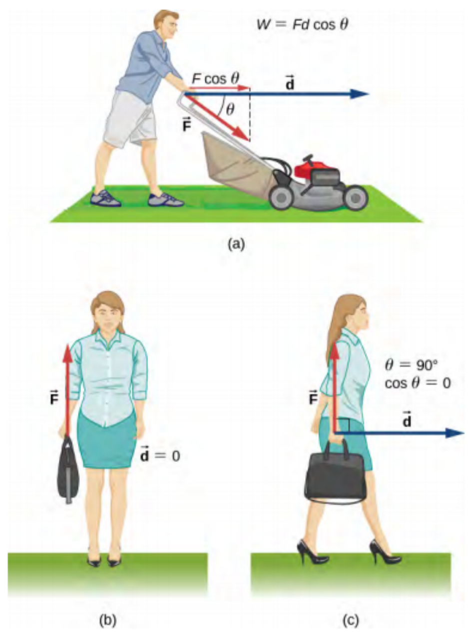


Figure $8.2.2$: Work done by a constant force. (a) A person pushes a lawn mower with a constant force. The component of the force parallel to the displacement is the work done, as shown in the equation in the figure. (b) A person holds a briefcase. No work is done because the displacement is zero. (c) The person in (b) walks horizontally while holding the briefcase. No work is done because cos θ is zero.

Figure $8.2.2b$ shows a person holding a briefcase. The person must exert an upward force, equal in magnitude to the weight of the briefcase, but this force does no work, because the displacement over which it acts is zero. So why do you eventually feel tired just holding the briefcase, if you're not doing any work on it? The answer is that muscle fibers in your arm are contracting and doing work inside your arm, even though the force your muscles exert externally on the briefcase doesn't do any work on it. (Part of the force you exert could also be tension in the bones and ligaments of your arm, but other muscles in your body would be doing work to maintain the position of your arm.)

In Figure $8.2.2c$, where the person in (b) is walking horizontally with constant speed, the work done by the person on the briefcase is still zero, but now because the angle between the force exerted and the displacement is 90° ($\vec{F}$ perpendicular to $\vec{d}$) and cos 90° = 0.

Example 8.2.1: Calculating the Work You Do to Push a Lawn Mower

How much work is done on the lawn mower by the person in Figure $8.2.2a$ if he exerts a constant force of 75.0 N at an angle 35° below the horizontal and pushes the mower 25.0 m on level ground?

Strategy

We can solve this problem by substituting the given values into the definition of work done on an object by a constant force, stated in the equation W = Fd cos θ. The force, angle, and displacement are given, so that only the work W is unknown.

Solution

The equation for the work is

$$W = Fd\cos\theta.$$

Substituting the known values gives

$$W = (75.0\ N)(25.0\ m)\cos(35.0^o) = 1.54\times 10^3\ J.$$

Significance

Even though one and a half kilojoules may seem like a lot of work, we will see in Potential Energy and Conservation of Energy that it's only about as much work as you could do by burning one sixth of a gram of fat.

When you mow the grass, other forces act on the lawn mower besides the force you exert—namely, the contact force of the ground and the gravitational force of Earth. Let's consider the work done by these forces in general. For an object moving on a surface, the displacement d $\vec{r}$ is tangent to the surface. The part of the contact force on the object that is perpendicular to the surface is the normal force $\vec{N}$. Since the cosine of the angle between the normal and the tangent to a surface is zero, we have

$$dW_N = \vec{N}\cdot d\vec{r} = \vec{0}.$$

The normal force never does work under these circumstances. (Note that if the displacement d $\vec{r}$ did have a relative component perpendicular to the surface, the object would either leave the surface or break through it, and there would no longer be any normal contact force. However, if the object is more than a particle, and has an internal structure, the normal contact force can do work on it, for example, by displacing it or deforming its shape. This will be mentioned in the next chapter.)

The part of the contact force on the object that is parallel to the surface is friction, $\vec{f}$. For this object sliding along the surface, kinetic friction $\vec{f}_k$ is opposite to d $\vec{r}$, relative to the surface, so the work done by kinetic friction is negative. If the magnitude of $\vec{f}_k$ is constant (as it would be if all the other forces on the object were constant), then the work done by friction is

$$W_{fr} = \int_A^B \vec{f}_k\cdot d\vec{r} = -f_k\int_A^B |dr| = -f_k|l_{AB}|. \tag{8.2.3}$$

where $|l_{AB}|$ is the path length on the surface. (Note that, especially if the work done by a force is negative, people may refer to the work done against this force, where $dW_{against} = -dW_{by}$. The work done against a force may also be viewed as the work required to overcome this force, as in "How much work is required to overcome…?") The force of static friction, however, can do positive or negative work. When you walk, the force of static friction exerted by the ground on your back foot accelerates you for part of each step. If you're slowing down, the force of the ground on your front foot decelerates you. If you're driving your car at the speed limit on a straight, level stretch of highway, the negative work done by kinetic friction of air resistance is balanced by the positive work done by the static friction of the road on the drive wheels. You can pull the rug out from under an object in such a way that it slides backward relative to the rug, but forward relative to the floor. In this case, kinetic friction exerted by the rug on the object could be in the same direction as the displacement of the object, relative to the floor, and do positive work. The bottom line is that you need to analyze each particular case to determine the work done by the forces, whether positive, negative or zero.

Example 8.2.2: Moving a Couch

You decide to move your couch to a new position on your horizontal living room floor. The normal force on the couch is 1 kN and the coefficient of friction is 0.6. (a) You first push the couch 3 m parallel to a wall and then 1 m perpendicular to the wall (A to B in Figure $8.2.3$). How much work is done by the frictional force? (b) You don't like the new position, so you move the couch straight back to its original position (B to A in Figure 8.2.3). What was the total work done against friction moving the couch away from its original position and back again?

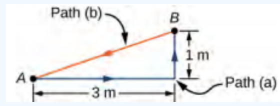


Figure 8.2.3: Top view of paths for moving a couch.

Strategy

The magnitude of the force of kinetic friction on the couch is constant, equal to the coefficient of friction times the normal force, $f_K = \mu_K N$. Therefore, the work done by it is $W_{fr} = -f_K d$, where d is the path length traversed. The segments of the paths are the sides of a right triangle, so the path lengths are easily calculated. In part (b), you can use the fact that the work done against a force is the negative of the work done by the force.

Solution

a. The work done by friction is

$$W = -(0.6)(1\ kN)(3\ m + 1\ m) = -2.4\ kJ.$$

b. The length of the path along the hypotenuse is $\sqrt{10}$ m, so the total work done against friction is

$$W = (0.6)(1\ kN)(3\ m + 1\ m + \sqrt{10}\ m) = 4.3\ kJ.$$

Significance

The total path over which the work of friction was evaluated began and ended at the same point (it was a closed path), so that the total displacement of the couch was zero. However, the total work was not zero. The reason is that forces like friction are classified as nonconservative forces, or dissipative forces, as we discuss in the next chapter.

? Exercise 7.1

Can kinetic friction ever be a constant force for all paths?

The other force on the lawn mower mentioned above was Earth's gravitational force, or the weight of the mower. Near the surface of Earth, the gravitational force on an object of mass m has a constant magnitude, mg, and constant direction, vertically down. Therefore, the work done by gravity on an object is the dot product of its weight and its displacement. In many cases, it is convenient to express the dot product for gravitational work in terms of the x-, y-, and z-components of the vectors. A typical coordinate system has the x-axis horizontal and the y-axis vertically up. Then the gravitational force is $-mg\,\hat{j}$, so the work done by gravity, over any path from A to B, is

$$W_{grav,\ AB} = -mg\hat{j} \cdot (\vec{r}_B - \vec{r}_A) = -mg(y_B - y_A). \tag{8.2.4}$$

The work done by a constant force of gravity on an object depends only on the object's weight and the difference in height through which the object is displaced. Gravity does negative work on an object that moves upward ($y_B > y_A$), or, in other words, you must do positive work against gravity to lift an object upward. Alternately, gravity does positive work on an object that moves downward ($y_B < y_A$), or you do negative work against gravity to "lift" an object downward, controlling its descent so it doesn't drop to the ground. ("Lift" is used as opposed to "drop".)

✓ Example 8.2.3: Shelving a Book

You lift an oversized library book, weighing 20 N, 1 m vertically down from a shelf, and carry it 3 m horizontally to a table (Figure 8.2.4). How much work does gravity do on the book? (b) When you're finished, you move the book in a straight line back to its original place on the shelf. What was the total work done against gravity, moving the book away from its original position on the shelf and back again?

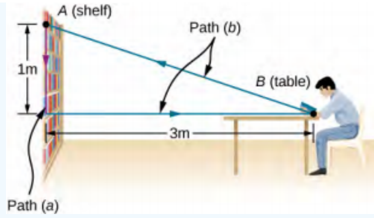


Figure 8.2.4: Side view of the paths for moving a book to and from a shelf.

Strategy

We have just seen that the work done by a constant force of gravity depends only on the weight of the object moved and the difference in height for the path taken, $W_{AB} = -mg(yB - y_A)$. We can evaluate the difference in height to answer (a) and (b).

Solution

a. Since the book starts on the shelf and is lifted down $y_B - y_A = -1$ m, we have

$$W = -(20\ N)(-1\ m) = 20J.$$

b. There is zero difference in height for any path that begins and ends at the same place on the shelf, so W = 0.

Significance

Gravity does positive work (20 J) when the book moves down from the shelf. The gravitational force between two objects is an attractive force, which does positive work when the objects get closer together. Gravity does zero work (0 J) when the book moves horizontally from the shelf to the table and negative work (−20 J) when the book moves from the table back to the shelf. The total work done by gravity is zero [20 J + 0 J + (−20 J) = 0].

Unlike friction or other dissipative forces, described in Example 8.2.2, the total work done against gravity, over any closed path, is zero. Positive work is done against gravity on the upward parts of a closed path, but an equal amount of negative work is done against gravity on the downward parts. In other words, work done **against** gravity, lifting an object **up**, is "given back" when the object comes back down. Forces like gravity (those that do zero work over any closed path) are classified as conservative forces and play an important role in physics.

? Exercise 7.2

Can Earth's gravity ever be a constant force for all paths?

Work Done by Forces that Vary

In general, forces may vary in magnitude and direction at points in space, and paths between two points may be curved. The infinitesimal work done by a variable force can be expressed in terms of the components of the force and the displacement along the path,

$$dW = F_x dx + F_y dy + F_z dz.$$

Here, the components of the force are functions of position along the path, and the displacements depend on the equations of the path. (Although we chose to illustrate dW in Cartesian coordinates, other coordinates are better suited to some situations.) Equation 8.2.2 defines the total work as a line integral, or the limit of a sum of infinitesimal amounts of work. The physical concept of work is straightforward: you calculate the work for tiny displacements and add them up. Sometimes the mathematics can seem complicated, but the following example demonstrates how cleanly they can operate.

✓ Example 8.2.4: Work Done by a Variable Force over a Curved Path

An object moves along a parabolic path $y = (0.5\ m^{-1})x^2$ from the origin A = (0, 0) to the point B = (2 m, 2 m) under the action of a force $\vec{F} = (5\ N/m)y\ \hat{i} + (10\ N/m)x\ \hat{j}$ (Figure 8.2.5). Calculate the work done.

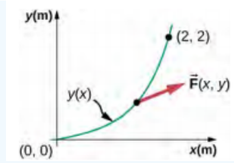


Figure 8.2.5: The parabolic path of a particle acted on by a given force.

Strategy

The components of the force are given functions of x and y. We can use the equation of the path to express y and dy in terms of x and dx; namely,

$$y = (0.5\ m^{-1})x^2\ and\ dy = 2(0.5\ m^{-1})xdx.$$

Then, the integral for the work is just a definite integral of a function of x.

Solution

The infinitesimal element of work is

$$\begin{aligned} dW &= F_x dx + F_y dy = (5\ N/m)ydx + (10\ N/m)xdy \\ &= (5\ N/m)(0.5\ m^{-1})x^2dx + (10\ N/m)2(0.5\ m^{-1})x^2dx = (12.5\ N/m^2)x^2dx. \end{aligned}$$

The integral of x^2 is $\frac{x^3}{3}$, so

$$W = \int_0^{2\ m} (12.5\ N/m^2)x^2dx = (12.5\ N/m^2)\frac{x^3}{3}\Big|_0^{2\ m} = (12.5\ N/m^2)\left(\frac{8}{3}\right) = 33.3\ J.$$

Significance

This integral was not hard to do. You can follow the same steps, as in this example, to calculate line integrals representing work for more complicated forces and paths. In this example, everything was given in terms of x- and y-components, which are easiest to use in evaluating the work in this case. In other situations, magnitudes and angles might be easier.

? Exercise 8.2.4

Find the work done by the same force in Example 8.2.4 over a cubic path, y = (0.25 m^{-2})x^3, between the same points A = (0, 0) and B = (2 m, 2 m).

You saw in Example 8.2.4 that to evaluate a line integral, you could reduce it to an integral over a single variable or parameter. Usually, there are several ways to do this, which may be more or less convenient, depending on the particular case. In Example 8.2.4, we reduced the line integral to an integral over x, but we could equally well have chosen to reduce everything to a function of y. We didn't do that because the functions in y involve the square root and fractional exponents, which may be less familiar, but for illustrative purposes, we do this now. Solving for x and dx, in terms of y, along the parabolic path, we get

$$x = \sqrt{\frac{y}{(0.5\ m^{-1})}} = \sqrt{(2\ m)y}\ and\ dx = \sqrt{(2\ m)} \times \frac{1}{2}\frac{dy}{\sqrt{y}} = \frac{dy}{\sqrt{(2\ m^{-1})y}}.$$

The components of the force, in terms of y, are

$$F_x = (5\ N/m)y\ and\ F_y = (10\ N/m)x = (10\ N/m)\sqrt{(2m)y},$$

so the infinitesimal work element becomes

$$\begin{aligned} dW &= F_x dx + F_y dy = \frac{(5\ N/m)ydy}{\sqrt{(2\ m^{-1})y}} + (10\ N/m)\sqrt{(2\ m)y}dy \\ &= (5\ N\cdot m^{-1/2})\left(\frac{1}{\sqrt{2}} + 2\sqrt{2}\right)\sqrt{y}dy = (17.7\ N\cdot m^{-1/2})y^{1/2}dy. \end{aligned}$$

The integral of $y^{1/2}$ is $\frac{2}{3}$ $y^{3/2}$, so the work done from A to B is

$$W = \int_0^{2\ m} (17.7\ N\cdot m^{-1/2}) y^{1/2} dy = (17.7\ N\cdot m^{-1/2}) \frac{2}{3} (2\ m)^{2/3} = 33.3\ J.$$

As expected, this is exactly the same result as before.

One very important and widely applicable variable force is the force exerted by a perfectly elastic spring, which satisfies Hooke's law $\vec{F} = -k\Delta\vec{x}$, where k is the spring constant, and $\Delta\vec{x} = \vec{x} - \vec{x}_{eq}$ is the displacement from the spring's unstretched (equilibrium) position (Newton's Laws of Motion). Note that the unstretched position is only the same as the equilibrium position if no other forces are acting (or, if they are, they cancel one another). Forces between molecules, or in any system undergoing small displacements from a stable equilibrium, behave approximately like a spring force.

To calculate the work done by a spring force, we can choose the x-axis along the length of the spring, in the direction of increasing length, as in Figure 8.2.6, with the origin at the equilibrium position $x_{eq} = 0$. (Then positive x corresponds to a stretch and negative x to a compression.) With this choice of coordinates, the spring force has only an x-component, $F_x = -kx$, and the work done when x changes from x_A to x_B is

$$W_{spring,\ AB} = \int_A^B F_x dx = -k \int_A^B x dx = -k \frac{x^2}{2}\Big|_A^B = -\frac{1}{2} k \left(x_B^2 - x_A^2\right). \tag{8.2.5}$$

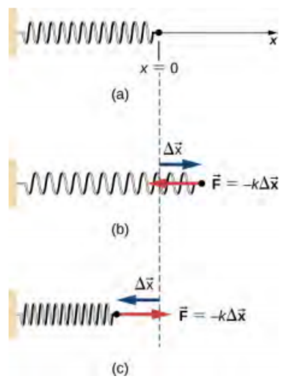


Figure 8.2.6: (a) The spring exerts no force at its equilibrium position. The spring exerts a force in the opposite direction to (b) an extension or stretch, and (c) a compression.

Notice that W_{AB} depends only on the starting and ending points, A and B, and is independent of the actual path between them, as long as it starts at A and ends at B. That is, the actual path could involve going back and forth before ending.

Another interesting thing to notice about Equation 8.2.5 is that, for this one-dimensional case, you can readily see the correspondence between the work done by a force and the area under the curve of the force versus its displacement. Recall that, in general, a one-dimensional integral is the limit of the sum of infinitesimals, f(x)dx , representing the area of strips, as shown in Figure 8.2.7. In Equation 8.2.5, since F = −kx is a straight line with slope −k, when plotted versus x, the "area" under the line is just an algebraic combination of triangular "areas," where "areas" above the x-axis are positive and those below are negative, as shown in Figure 8.2.8. The magnitude of one of these "areas" is just one-half the triangle's base, along the x-axis, times the triangle's height, along the force axis. (There are quotation marks around "area" because this base-height product has the units of work, rather than square meters.)

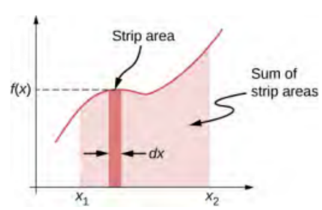


Figure 8.2.7: A curve of f(x) versus x showing the area of an infinitesimal strip, f(x)dx, and the sum of such areas, which is the integral of f(x) from x_1 to x_2.

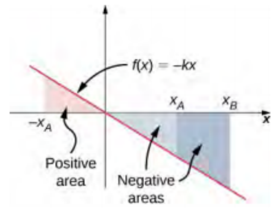


Figure 8.2.8: Curve of the spring force $f(x) = -kx$ versus x, showing areas under the line, between x_A and x_B, for both positive and negative values of x_A. When x_A is negative, the total area under the curve for the integral in Equation 8.2.5 is the sum of positive and negative triangular areas. When x_A is positive, the total area under the curve is the difference between two negative triangles.

Example 8.2.5: Work Done by a Spring Force

A perfectly elastic spring requires 0.54 J of work to stretch 6 cm from its equilibrium position, as in Figure 8.2.6b. (a) What is its spring constant k? (b) How much work is required to stretch it an additional 6 cm?

Strategy

Work "required" means work done against the spring force, which is the negative of the work in Equation 8.2.5, that is

$$W = \frac{1}{2}k(x_B^2 - x_A^2).$$

For part (a), $x_A = 0$ and $x_B = 6$ cm; for part (b), $x_B = 6$ cm and $x_B = 12$ cm. In part (a), the work is given and you can solve for the spring constant; in part (b), you can use the value of k, from part (a), to solve for the work.

Solution

a. \[W = 0.54\; J = \frac{1}{2} k [(6\; cm)^{2} - 0],\; so\; k = 3\; N/cm \ldotp \nonumber$$
b. \[W = \frac{1}{2} (3\; N/cm) [(12\; cm)^{2} - (6\; cm)^{2}],\; so\; k = 1.62\; J \ldotp \nonumber$$

Significance

Since the work done by a spring force is independent of the path, you only needed to calculate the difference in the quantity $\frac{1}{2}kx^2$ at the end points. Notice that the work required to stretch the spring from 0 to 12 cm is four times that required to stretch it from 0 to 6 cm, because that work depends on the square of the amount of stretch from equilibrium, $\frac{1}{2}kx^2$. In this circumstance, the work to stretch the spring from 0 to 12 cm is also equal to the work for a composite path from 0 to 6 cm followed by an additional stretch from 6 cm to 12 cm. Therefore, 4W(0 cm to 6 cm) = W(0 cm to 6 cm) + W(6 cm to 12 cm) , or W(6 cm to 12 cm) = 3W(0 cm to 6 cm) , as we found above.

Exercise 8.2.5

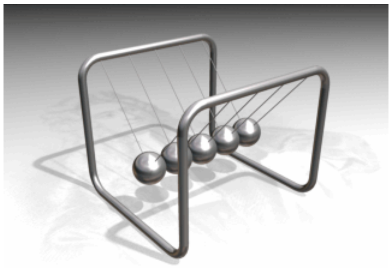
The spring in Example 8.2.5 is compressed 6 cm from its equilibrium length. (a) Does the spring force do positive or negative work and (b) what is the magnitude?

8.3: Kinetic Energy

Learning Objectives

- Calculate the kinetic energy of a particle given its mass and its velocity or momentum
- Evaluate the kinetic energy of a body, relative to different frames of reference

It's plausible to suppose that the greater the velocity of a body, the greater effect it could have on other bodies. This does not depend on the direction of the velocity, only its magnitude. At the end of the seventeenth century, a quantity was introduced into mechanics to explain collisions between two perfectly elastic bodies, in which one body makes a head-on collision with an identical body at rest. The first body stops, and the second body moves off with the initial velocity of the first body. (If you have ever played billiards or croquet, or seen a model of Newton's Cradle, you have observed this type of collision.) The idea behind this quantity was related to the forces acting on a body and was referred to as "the energy of motion." Later on, during the eighteenth century, the name **kinetic energy** was given to energy of motion.



Newton's cradle in motion. One ball is set in motion and soon collides with the rest, conveying the energy through the rest of the balls and eventually to the last ball, which in turn is set in motion. (CC SA-BY 3.0; Dominique Toussaint).

With this history in mind, we can now state the classical definition of kinetic energy. Note that when we say "classical," we mean non-relativistic, that is, at speeds much less that the speed of light. At speeds comparable to the speed of light, the special theory of relativity requires a different expression for the kinetic energy of a particle, as discussed in Relativity. Since objects (or systems) of interest vary in complexity, we first define the kinetic energy of a particle with mass m.

Kinetic Energy

The kinetic energy of a particle is one-half the product of the particle's mass m and the square of its speed v:

$$K = \frac{1}{2}mv^2. \tag{8.3.1}$$

We then extend this definition to any system of particles by adding up the kinetic energies of all the constituent particles:

$$K = \sum \frac{1}{2}mv^2. \tag{8.3.2}$$

Note that just as we can express Newton's second law in terms of either the rate of change of momentum or mass times the rate of change of velocity, so the kinetic energy of a particle can be expressed in terms of its mass and momentum ($\vec{p}$ = m $\vec{v}$), instead of its mass and velocity. Since v = $\frac{p}{m}$, we see that

$$K = \frac{1}{2}m\left(\frac{p}{m}\right)^2 = \frac{p^2}{2m} \tag{8.3.3}$$

also expresses the kinetic energy of a single particle. Sometimes, this expression is more convenient to use than Equation 8.3.1. The units of kinetic energy are mass times the square of speed, or kg • m^2/s^2. But the units of force are mass times acceleration, kg • m/s^2, so the units of kinetic energy are also the units of force times distance, which are the units of work, or joules. You will see in the next section that work and kinetic energy have the same units, because they are different forms of the same, more general, physical property.

Example 8.3.1: Kinetic Energy of an Object

a. What is the kinetic energy of an 80-kg athlete, running at 10 m/s?
b. The Chicxulub crater in Yucatan, one of the largest existing impact craters on Earth, is thought to have been created by an asteroid, traveling at 22 km/s and releasing 4.2 x 10^{23} J of kinetic energy upon impact. What was its mass?
c. In nuclear reactors, thermal neutrons, traveling at about 2.2 km/s, play an important role. What is the kinetic energy of such a particle?

Strategy

To answer these questions, you can use the definition of kinetic energy in Equation 8.3.1. You also have to look up the mass of a neutron.

Solution

Do not forget to convert km into m to do these calculations, although, to save space, we omitted showing these conversions.

a.
$$K = \frac{1}{2}(80\ kg)(10\ m/s)^2 = 4.0\ kJ.$$

b.
$$m = \frac{2K}{v^2} = \frac{2(4.2 \times 10^{23}\ J)}{22\ km/s)^2} = 1.7 \times 10^{15}\ kg.$$

c.
$$K = \frac{1}{2}(1.68 \times 110^{-27}\ kg)(2.2\ km/s)^2 = 4.1 \times 10^{-21}\ J.$$

Significance

In this example, we used the way mass and speed are related to kinetic energy, and we encountered a very wide range of values for the kinetic energies. Different units are commonly used for such very large and very small values. The energy of the impactor in part (b) can be compared to the explosive yield of TNT and nuclear explosions, 1 megaton = 4.18 x 10^{15} J. The Chicxulub asteroid's kinetic energy was about a hundred million megatons. At the other extreme, the energy of subatomic particle is expressed in electron-volts, 1 eV = 1.6 x 10^{-19} J. The thermal neutron in part (c) has a kinetic energy of about one fortieth of an electronvolt.

Exercise 8.3.1

a. A car and a truck are each moving with the same kinetic energy. Assume that the truck has more mass than the car. Which has the greater speed?
b. A car and a truck are each moving with the same speed. Which has the greater kinetic energy?

Because velocity is a relative quantity, you can see that the value of kinetic energy must depend on your frame of reference. You can generally choose a frame of reference that is suited to the purpose of your analysis and that simplifies your calculations. One such frame of reference is the one in which the observations of the system are made (likely an external frame). Another choice is a frame that is attached to, or moves with, the system (likely an internal frame). The equations for relative motion, discussed in Motion in Two and Three Dimensions, provide a link to calculating the kinetic energy of an object with respect to different frames of reference.

Example 8.3.2: Kinetic Energy Relative to Different Frames

A 75.0-kg person walks down the central aisle of a subway car at a speed of 1.50 m/s relative to the car, whereas the train is moving at 15.0 m/s relative to the tracks.

a. What is the person's kinetic energy relative to the car?
b. What is the person's kinetic energy relative to the tracks?
c. What is the person's kinetic energy relative to a frame moving with the person?

Strategy

Since speeds are given, we can use $\frac{1}{2}mv^2$ to calculate the person's kinetic energy. However, in part (a), the person's speed is relative to the subway car (as given); in part (b), it is relative to the tracks; and in part (c), it is zero. If we denote the car frame by C, the track frame by T, and the person by P, the relative velocities in part (b) are related by $\vec{v}_{PT} = \vec{v}_{PC} + \vec{v}_{CT}$. We can assume that the central aisle and the tracks lie along the same line, but the direction the person is walking relative to the car isn't specified, so we will give an answer for each possibility, $v_{PT} = v_{CT} \pm v_{PC}$, as shown in Figure 8.3.1.

Figure 8.3.1: The possible motions of a person walking in a train are (a) toward the front of the car and (b) toward the back of the car.

Solution

a.
$$K = \frac{1}{2}(75.0\ kg)(11.50\ m/s)^2 = 84.4\ J.$$

b.
$$v_{PT} = (15.0 \pm 1.50)7; m/s.$$

Therefore, the two possible values for kinetic energy relative to the car are

$$K = \frac{1}{2}(75.0\ kg)(13.5\ m/s)^2 = 6.83\ kJ$$

and

$$K = \frac{1}{2}(75.0\ kg)(16.5\ m/s)^2 = 10.2\ kJ.$$

c. In a frame where $v_P = 0$, K = 0 as well.

Significance

You can see that the kinetic energy of an object can have very different values, depending on the frame of reference. However, the kinetic energy of an object can never be negative, since it is the product of the mass and the square of the speed, both of which are always positive or zero.

Exercise 8.3.2

You are rowing a boat parallel to the banks of a river. Your kinetic energy relative to the banks is less than your kinetic energy relative to the water. Are you rowing with or against the current?

The kinetic energy of a particle is a single quantity, but the kinetic energy of a system of particles can sometimes be divided into various types, depending on the system and its motion. For example:

- If all the particles in a system have the same velocity, the system is undergoing translational motion and has **translational kinetic energy**.
- If an object is rotating, it could have **rotational kinetic energy**.
- If it is vibrating, it could have **vibrational kinetic energy**.

The kinetic energy of a system, relative to an internal frame of reference, may be called internal kinetic energy. The kinetic energy associated with random molecular motion may be called thermal energy. These names will be used in later chapters of the book, when appropriate. Regardless of the name, every kind of kinetic energy is the same physical quantity, representing energy associated with motion.

Example 8.3.3: Special Names for Kinetic Energy

a. A player lobs a mid-court pass with a 624-g basketball, which covers 15 m in 2 s. What is the basketball's horizontal translational kinetic energy while in flight?
b. An average molecule of air, in the basketball in part (a), has a mass of 29 u, and an average speed of 500 m/s, relative to the basketball. There are about 3 x 10^{23} molecules inside it, moving in random directions, when the ball is properly inflated. What is the average translational kinetic energy of the random motion of all the molecules inside, relative to the basketball?

c. How fast would the basketball have to travel relative to the court, as in part (a), so as to have a kinetic energy equal to the amount in part (b)?

Strategy

In part (a), first find the horizontal speed of the basketball and then use the definition of kinetic energy in terms of mass and speed, K = $\frac{1}{2}mv^2$. Then in part (b), convert unified units to kilograms and then use K = $\frac{1}{2}mv^2$ to get the average translational kinetic energy of one molecule, relative to the basketball. Then multiply by the number of molecules to get the total result. Finally, in part (c), we can substitute the amount of kinetic energy in part (b), and the mass of the basketball in part (a), into the definition K = $\frac{1}{2}mv^2$, and solve for v.

Solution

a. The horizontal speed is $\frac{(15\ m)}{(2\ s)}$, so the horizontal kinetic energy of the basketball is

$$\frac{1}{2}(0.624\ kg)(7.5\ m/s)^2 = 17.6\ J.$$

b. The average translational kinetic energy of a molecule is

$$\frac{1}{2}(29\ u)(1.66 \times 10^{-27}\ kg/u)(500\ m/s)^2 = 6.02 \times 10^{-21}\ J,$$

and the total kinetic energy of all the molecules is

$$(3 \times 10^{23})(6.02 \times 10^{-21}\ J) = 1.80\ kJ.$$

c.

$$v = \sqrt{\frac{2(1.8\ kJ)}{(0.624\ kg)}} = 76.0\ m/s.$$

Significance

In part (a), this kind of kinetic energy can be called the horizontal kinetic energy of an object (the basketball), relative to its surroundings (the court). If the basketball were spinning, all parts of it would have not just the average speed, but it would also have rotational kinetic energy. Part (b) reminds us that this kind of kinetic energy can be called internal or thermal kinetic energy. Notice that this energy is about a hundred times the energy in part (a). How to make use of thermal energy will be the subject of the chapters on thermodynamics. In part (c), since the energy in part (b) is about 100 times that in part (a), the speed should be about 10 times as big, which it is (76 compared to 7.5 m/s).

8.4: Work-Energy Theorem

Learning Objectives

- Apply the work-energy theorem to find information about the motion of a particle, given the forces acting on it
- Use the work-energy theorem to find information about the forces acting on a particle, given information about its motion

We have discussed how to find the work done on a particle by the forces that act on it, but how is that work manifested in the motion of the particle? According to Newton's second law of motion, the sum of all the forces acting on a particle, or the net force, determines the rate of change in the momentum of the particle, or its motion. Therefore, we should consider the work done by all the forces acting on a particle, or the **net work**, to see what effect it has on the particle's motion.

Let's start by looking at the net work done on a particle as it moves over an infinitesimal displacement, which is the dot product of the net force and the displacement:

$$dW_{net} = \vec{F}_{net} \cdot d\vec{r}.$$

Newton's second law tells us that

$$\vec{F}_{net} = m\left(\frac{d\vec{v}}{dt}\right)$$

so

$$dW_{net} = m\left(\frac{d\vec{v}}{dt}\right) \cdot d\vec{r}.$$

For the mathematical functions describing the motion of a physical particle, we can rearrange the differentials dt, etc., as algebraic quantities in this expression, that is,

$$\begin{aligned} dW_{net} &= m\left(\frac{d\vec{v}}{dt}\right) \cdot d\vec{r} \\ &= m\, d\vec{v} \cdot \left(\frac{d\vec{r}}{dt}\right) \\ &= m\vec{v} \cdot d\vec{v}, \end{aligned}$$

where we substituted the velocity for the time derivative of the displacement and used the commutative property of the dot product. Since derivatives and integrals of scalars are probably more familiar to you at this point, we express the dot product in terms of Cartesian coordinates before we integrate between any two points A and B on the particle's trajectory. This gives us the net work done on the particle:

$$W_{net,\ AB} = \int_A^B (mv_x dv_x + mv_y dv_y + mv_z dv_z \tag{8.4.1}$$

$$= \frac{1}{2} m \left| v_x^2 + v_y^2 + v_z^2 \right|_A^B = \left| \frac{1}{2} m v^2 \right|_A^B = K_B - K_A. \tag{8.4.2}$$

In the middle step, we used the fact that the square of the velocity is the sum of the squares of its Cartesian components, and in the last step, we used the definition of the particle's kinetic energy. This important result is called the work-energy theorem.

Work-Energy Theorem

The net work done on a particle equals the change in the particle's kinetic energy:

$$W_{net} = K_B - K_A. \tag{8.4.3}$$

Figure 8.4.1: Horse pulls are common events at state fairs. The work done by the horses pulling on the load results in a change in kinetic energy of the load, ultimately going faster. (credit: "Jassen"/ Flickr)

According to this theorem, when an object slows down, its final kinetic energy is less than its initial kinetic energy, the change in its kinetic energy is negative, and so is the net work done on it. If an object speeds up, the net work done on it is positive. When calculating the net work, you must include all the forces that act on an object. If you leave out any forces that act on an object, or if you include any forces that do not act on it, you will get a wrong result.

The importance of the work-energy theorem, and the further generalizations to which it leads, is that it makes some types of calculations much simpler to accomplish than they would be by trying to solve Newton's second law. For example, in the section on Newton's Laws of Motion, we found the speed of an object sliding down a frictionless plane by solving Newton's second law for the acceleration and using kinematic equations for constant acceleration, obtaining

$$v_f^2 = v_i^2 + 2g(s_f - s_i)\sin\theta, \tag{8.4.4}$$

where s is the displacement down the plane.

We can also get this result from the work-energy theorem (Equation 8.4.3). Since only two forces are acting on the object—gravity and the normal force—and the normal force does not do any work, the net work is just the work done by gravity. This only depends on the object's weight and the difference in height, so

$$W_{net} = W_{grav} = -mg(y_f - y_i), \tag{8.4.5}$$

where y is positive up. The work-energy theorem says that this equals the change in kinetic energy:

$$-mg(y_f - y_i) = \frac{1}{2}(v_f^2 - v_i^2). \tag{8.4.6}$$

Using a right triangle, we can see that

$$(y_f - y_i) = (s_f - s - i)\sin\theta,$$

so the result for the final speed is the same.

What is gained by using the work-energy theorem? The answer is that for a frictionless plane surface, not much. However, Newton's second law is easy to solve only for this particular case, whereas the work-energy theorem gives the final speed for any shaped frictionless surface. For an arbitrary curved surface, the normal force is not constant, and Newton's second law may be difficult or impossible to solve analytically. Constant or not, for motion along a surface, the normal force never does any work, because it's perpendicular to the displacement. A calculation using the work-energy theorem avoids this difficulty and applies to more general situations.

Problem-Solving Strategy: Work-Energy Theorem

1. Draw a free-body diagram for each force on the object.
2. Determine whether or not each force does work over the displacement in the diagram. Be sure to keep any positive or negative signs in the work done.
3. Add up the total amount of work done by each force.
4. Set this total work equal to the change in kinetic energy and solve for any unknown parameter.

5. Check your answers. If the object is traveling at a constant speed or zero acceleration, the total work done should be zero and match the change in kinetic energy. If the total work is positive, the object must have sped up or increased kinetic energy. If the total work is negative, the object must have slowed down or decreased kinetic energy

Example $8.4.1$: Loop-the-Loop

The frictionless track for a toy car includes a loop-the-loop of radius R. How high, measured from the bottom of the loop, must the car be placed to start from rest on the approaching section of track and go all the way around the loop?

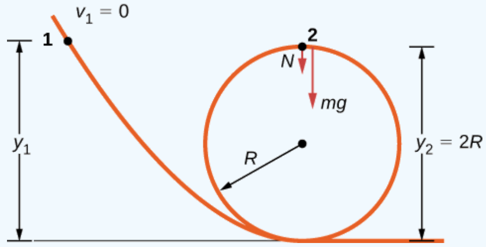


Figure $8.4.2$: A frictionless track for a toy car has a loop-the-loop in it. How high must the car start so that it can go around the loop without falling off?

Strategy

The free-body diagram at the final position of the object is drawn in Figure $8.4.2$. The gravitational work is the only work done over the displacement that is not zero. Since the weight points in the same direction as the net vertical displacement, the total work done by the gravitational force is positive. From the work-energy theorem, the starting height determines the speed of the car at the top of the loop,

$$mg(y_2 - y_1) = \frac{1}{2}mv_2^2,$$

where the notation is shown in the accompanying figure. At the top of the loop, the normal force and gravity are both down and the acceleration is centripetal, so

$$a_{top} = \frac{F}{m} = \frac{N+mg}{m} = \frac{v_2^2}{R}.$$

The condition for maintaining contact with the track is that there must be some normal force, however slight; that is, $N > 0$. Substituting for v_2^2 and N, we can find the condition for y_1.

Solution

Implement the steps in the strategy to arrive at the desired result:

$$N = -mg + \frac{mv_2^2}{R} = \frac{-mgR + 2mg(y_1 - 2R)}{R} > 0 \text{ or } y_1 > \frac{5R}{2}.$$

Significance

On the surface of the loop, the normal component of gravity and the normal contact force must provide the centripetal acceleration of the car going around the loop. The tangential component of gravity slows down or speeds up the car. A child would find out how high to start the car by trial and error, but now that you know the work-energy theorem, you can predict the minimum height (as well as other more useful results) from physical principles. By using the work-energy theorem, you did not have to solve a differential equation to determine the height.

Exercise $8.4.1$

Suppose the radius of the loop-the-loop in Example $8.4.1$ is 15 cm and the toy car starts from rest at a height of 45 cm above the bottom. What is its speed at the top of the loop?

In situations where the motion of an object is known, but the values of one or more of the forces acting on it are not known, you may be able to use the work-energy theorem to get some information about the forces. Work depends on the force and the distance over which it acts, so the information is provided via their product.

Example $8.4.2$: Determining a Stopping Force

A bullet has a mass of 40 grains (2.60 g) and a muzzle velocity of 1100 ft./s (335 m/s). It can penetrate eight 1-inch pine boards, each with thickness 0.75 inches. What is the average stopping force exerted by the wood, as shown in Figure $8.4.3$?

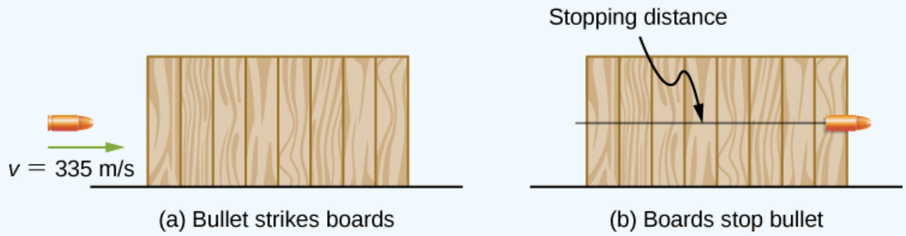


Figure $8.4.3$: The boards exert a force to stop the bullet. As a result, the boards do work and the bullet loses kinetic energy

Strategy

We can assume that under the general conditions stated, the bullet loses all its kinetic energy penetrating the boards, so the work-energy theorem says its initial kinetic energy is equal to the average stopping force times the distance penetrated. The change in the bullet's kinetic energy and the net work done stopping it are both negative, so when you write out the work-energy theorem, with the net work equal to the average force times the stopping distance, that's what you get. The total thickness of eight 1-inch pine boards that the bullet penetrates is 8 x $\frac{3}{4}$ in. = 6 in. = 15.2 cm.

Solution

Applying the work-energy theorem, we get

$$W_{net} = -F_{ave}\Delta s_{stop} = -K_{initial},$$

so

$$F_{ave} = \frac{\frac{1}{2}mv^2}{\Delta s_{stop}} = \frac{\frac{1}{2}(2.66\times 10^{-3}\ kg)(335\ m/s)^2}{0.152\ m} = 960\ N.$$

Significance

We could have used Newton's second law and kinematics in this example, but the work-energy theorem also supplies an answer to less simple situations. The penetration of a bullet, fired vertically upward into a block of wood, is discussed in one section of Asif Shakur's recent article ["Bullet-Block Science Video Puzzle." **The Physics Teacher** (January 2015) 53(1): 15-16]. If the bullet is fired dead center into the block, it loses all its kinetic energy and penetrates slightly farther than if fired off-center. The reason is that if the bullet hits off-center, it has a little kinetic energy after it stops penetrating, because the block rotates. The work-energy theorem implies that a smaller change in kinetic energy results in a smaller penetration. You will understand more of the physics in this interesting article after you finish reading Angular Momentum.

Learn more about work and energy in this PhET simulation (https://phet.colorado.edu/en/simulation/the-ramp) called "the ramp." Try changing the force pushing the box and the frictional force along the incline. The work and energy plots can be examined to note the total work done and change in kinetic energy of the box.

8.5: Power

Learning Objectives

- Relate the work done during a time interval to the power delivered
- Find the power expended by a force acting on a moving body

The concept of work involves force and displacement; the work-energy theorem relates the net work done on a body to the difference in its kinetic energy, calculated between two points on its trajectory. None of these quantities or relations involves time explicitly, yet we know that the time available to accomplish a particular amount of work is frequently just as important to us as the amount itself. In the chapter-opening figure, several sprinters may have achieved the same velocity at the finish, and therefore did the same amount of work, but the winner of the race did it in the least amount of time.

We express the relation between work done and the time interval involved in doing it, by introducing the concept of power. Since work can vary as a function of time, we first define **average power** as the work done during a time interval, divided by the interval,

$$P_{ave} = \frac{\Delta W}{\Delta t}. \tag{8.5.1}$$

Then, we can define the **instantaneous power** (frequently referred to as just plain **power**).

Definition: Power

Power is defined as the rate of doing work, or the limit of the average power for time intervals approaching zero,

$$P = \frac{dW}{dt}. \tag{8.5.2}$$

If the power is constant over a time interval, the average power for that interval equals the instantaneous power, and the work done by the agent supplying the power is

$$W = P\Delta t. \tag{8.5.3}$$

If the power during an interval varies with time (i.e., $P(t)$), then the work done is the time integral of the power,

$$W = \int P(t)dt. \tag{8.5.4}$$

The work-energy theorem relates how work can be transformed into kinetic energy. Since there are other forms of energy as well, as we discuss in the next chapter, we can also define power as the rate of transfer of energy. Work and energy are measured in units of joules, so power is measured in units of joules per second, which has been given the SI name watts, abbreviation W: 1 J/s = 1 W. Another common unit for expressing the power capability of everyday devices is horsepower: 1 hp = 746 W.

Example $8.5.1$: Pull-Up Power

An 80-kg army trainee does pull-ups on a horizontal bar (Figure $8.5.1$). It takes the trainee 0.8 seconds to raise the body from a lower position to where the chin is above the bar. How much power do the trainee's muscles supply moving his body from the lower position to where the chin is above the bar? (**Hint**: Make reasonable estimates for any quantities needed.)

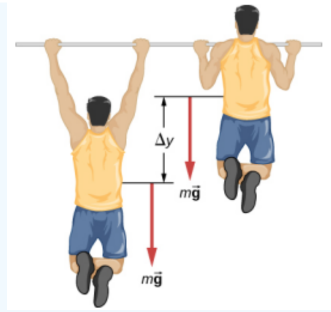


Figure 8.5.1: What is the power expended in doing ten pull-ups in ten seconds?

Strategy

The work done against gravity, going up or down a distance Δy, is mgΔy. Let's assume that Δy = 2 ft ≈ 60 cm. Also, assume that the arms comprise 10% of the body mass and are not included in the moving mass. With these assumptions, we can calculate the work done.

Solution

The result we get, applying our assumptions, is

$$P=\frac{\mathrm{mg}(\Delta y)}{t}=\frac{0.9(80\text{ kg})\left(9.8\text{ m/s}^2\right)(0.60\text{ m})}{0.8\text{ s}}=529\text{ W} \tag{8.5.5}$$

Significance

This is typical for power expenditure in strenuous exercise; in everyday units, it's somewhat more than one horsepower (1 hp = 746 W).

? Exercise 8.5.1

Estimate the power expended by a weightlifter raising a 150-kg barbell 2 m in 3 s.

Answer

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The power involved in moving a body can also be expressed in terms of the forces acting on it. If a force $\vec{F}$ acts on a body that is displaced d$\vec{r}$ in a time dt, the power expended by the force is

$$P=\frac{dW}{dt}=\frac{\vec{F}\cdot d\vec{r}}{dt}=\vec{F}\cdot\left(\frac{d\vec{r}}{dt}\right)=\vec{F}\cdot\vec{v}, \tag{8.5.6}$$

where $\vec{v}$ is the velocity of the body. The fact that the limits implied by the derivatives exist, for the motion of a real body, justifies the rearrangement of the infinitesimals.

✓ Example 8.5.2: Automotive Power Driving Uphill

How much power must an automobile engine expend to move a 1200-kg car up a 15% grade at 90 km/h (Figure 8.5.2)? Assume that 25% of this power is dissipated overcoming air resistance and friction.

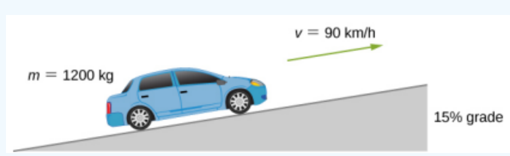


Figure 8.5.2: We want to calculate the power needed to move a car up a hill at constant speed.

Strategy

At constant velocity, there is no change in kinetic energy, so the net work done to move the car is zero. Therefore the power supplied by the engine to move the car equals the power expended against gravity and air resistance. By assumption, 75% of the power is supplied against gravity, which equals m$\vec{g} \cdot \vec{v}$ = mgv sin θ, where θ is the angle of the incline. A 15% grade means tan θ = 0.15. This reasoning allows us to solve for the power required.

Solution

Carrying out the suggested steps, we find

$$0.75P = mgv\sin(\tan^{-1} 0.15), \tag{8.5.7}$$

or

$$P = \frac{(1200 \times 9.8\ N)(\frac{90}{3.6}\frac{m}{s})\sin(8.53^o)}{0.75} = 58\ kW, \tag{8.5.8}$$

or about 78 hp. (You should supply the steps used to convert units.)

Significance

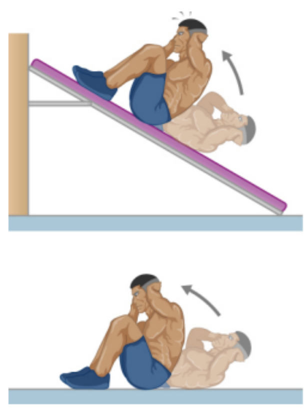
This is a reasonable amount of power for the engine of a small to mid-size car to supply (1 hp = 0.746 kW). Note that this is only the power expended to move the car. Much of the engine's power goes elsewhere, for example, into waste heat. That's why cars need radiators. Any remaining power could be used for acceleration, or to operate the car's accessories.

8.6: Work and Kinetic Energy (Exercises)

Conceptual Questions

7.1 Work

1. Give an example of something we think of as work in everyday circumstances that is not work in the scientific sense. Is energy transferred or changed in form in your example? If so, explain how this is accomplished without doing work.
2. Give an example of a situation in which there is a force and a displacement, but the force does no work. Explain why it does no work.
3. Describe a situation in which a force is exerted for a long time but does no work. Explain.
4. A body moves in a circle at constant speed. Does the centripetal force that accelerates the body do any work? Explain.
5. Suppose you throw a ball upward and catch it when it returns at the same height. How much work does the gravitational force do on the ball over its entire trip?
6. Why is it more difficult to do sit-ups while on a slant board than on a horizontal surface? (See below.)



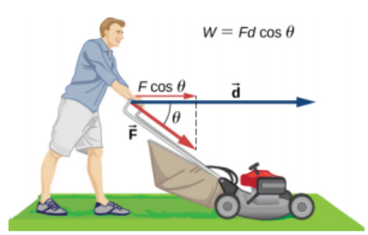
7. As a young man, Tarzan climbed up a vine to reach his tree house. As he got older, he decided to build and use a staircase instead. Since the work of the gravitational force mg is path independent, what did the King of the Apes gain in using stairs?

7.2 Kinetic Energy

8. A particle of m has a velocity of $v_x \hat{i} + v_y \hat{j} + v_z \hat{k}$. Is its kinetic energy given by $m(v_x^2 \hat{i} + v_y^2 \hat{j} + v_z^2 \hat{k})/2$? If not, what is the correct expression?
9. One particle has mass m and a second particle has mass 2m. The second particle is moving with speed v and the first with speed 2v. How do their kinetic energies compare?
10. A person drops a pebble of mass m_1 from a height h, and it hits the floor with kinetic energy K. The person drops another pebble of mass m_2 from a height of 2h, and it hits the floor with the same kinetic energy K. How do the masses of the pebbles compare?

7.3 Work-Energy Theorem

11. Under what conditions would it lose energy?



12. Work done on a system puts energy into it. Work done by a system removes energy from it. Give an example for each statement.

13. Two marbles of masses m and 2m are dropped from a height h. Compare their kinetic energies when they reach the ground.
14. Compare the work required to accelerate a car of mass 2000 kg from 30.0 to 40.0 km/h with that required for an acceleration from 50.0 to 60.0 km/h.
15. Suppose you are jogging at constant velocity. Are you doing any work on the environment and vice versa?
16. Two forces act to double the speed of a particle, initially moving with kinetic energy of 1 J. One of the forces does 4 J of work. How much work does the other force do?

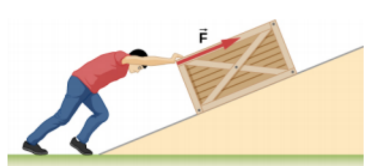
7.4 Power

17. Most electrical appliances are rated in watts. Does this rating depend on how long the appliance is on? (When off, it is a zero-watt device.) Explain in terms of the definition of power.
18. Explain, in terms of the definition of power, why energy consumption is sometimes listed in kilowatt-hours rather than joules. What is the relationship between these two energy units?
19. A spark of static electricity, such as that you might receive from a doorknob on a cold dry day, may carry a few hundred watts of power. Explain why you are not injured by such a spark.
20. Does the work done in lifting an object depend on how fast it is lifted? Does the power expended depend on how fast it is lifted?
21. Can the power expended by a force be negative?
22. How can a 50-W light bulb use more energy than a 1000-W oven?

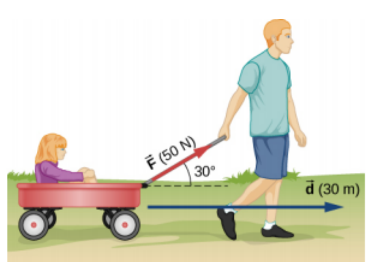
Problems

7.1 Work

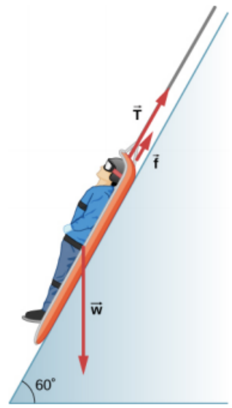
23. How much work does a supermarket checkout attendant do on a can of soup he pushes 0.600 m horizontally with a force of 5.00 N?
24. A 75.0-kg person climbs stairs, gaining 2.50 m in height. Find the work done to accomplish this task.
25. (a) Calculate the work done on a 1500-kg elevator car by its cable to lift it 40.0 m at constant speed, assuming friction averages 100 N. (b) What is the work done on the lift by the gravitational force in this process? (c) What is the total work done on the lift?
26. Suppose a car travels 108 km at a speed of 30.0 m/s, and uses 2.0 gal of gasoline. Only 30% of the gasoline goes into useful work by the force that keeps the car moving at constant speed despite friction. (The energy content of gasoline is about 140 MJ/gal.) (a) What is the magnitude of the force exerted to keep the car moving at constant speed? (b) If the required force is directly proportional to speed, how many gallons will be used to drive 108 km at a speed of 28.0 m/s?
27. Calculate the work done by an 85.0-kg man who pushes a crate 4.00 m up along a ramp that makes an angle of 20.0° with the horizontal (see below). He exerts a force of 500 N on the crate parallel to the ramp and moves at a constant speed. Be certain to include the work he does on the crate and on his body to get up the ramp.



28. How much work is done by the boy pulling his sister 30.0 m in a wagon as shown below? Assume no friction acts on the wagon.



29. A shopper pushes a grocery cart 20.0 m at constant speed on level ground, against a 35.0 N frictional force. He pushes in a direction 25.0° below the horizontal. (a) What is the work done on the cart by friction? (b) What is the work done on the cart by the gravitational force? (c) What is the work done on the cart by the shopper? (d) Find the force the shopper exerts, using energy considerations. (e) What is the total work done on the cart?
30. Suppose the ski patrol lowers a rescue sled and victim, having a total mass of 90.0 kg, down a 60.0° slope at constant speed, as shown below. The coefficient of friction between the sled and the snow is 0.100. (a) How much work is done by friction as the sled moves 30.0 m along the hill? (b) How much work is done by the rope on the sled in this distance? (c) What is the work done by the gravitational force on the sled? (d) What is the total work done?



31. A constant 20-N force pushes a small ball in the direction of the force over a distance of 5.0 m. What is the work done by the force?
32. A toy cart is pulled a distance of 6.0 m in a straight line across the floor. The force pulling the cart has a magnitude of 20 N and is directed at 37° above the horizontal. What is the work done by this force?
33. A 5.0-kg box rests on a horizontal surface. The coefficient of kinetic friction between the box and surface is $\mu_K = 0.50$. A horizontal force pulls the box at constant velocity for 10 cm. Find the work done by (a) the applied horizontal force, (b) the frictional force, and (c) the net force.
34. A sled plus passenger with total mass 50 kg is pulled 20 m across the snow (μk = 0.20) at constant velocity by a force directed 25° above the horizontal. Calculate (a) the work of the applied force, (b) the work of friction, and (c) the total work.
35. Suppose that the sled plus passenger of the preceding problem is pushed 20 m across the snow at constant velocity by a force directed 30° below the horizontal. Calculate (a) the work of the applied force, (b) the work of friction, and (c) the total work.
36. How much work does the force F(x) = (−2.0/x) N do on a particle as it moves from x = 2.0 m to x = 5.0 m?
37. How much work is done against the gravitational force on a 5.0-kg briefcase when it is carried from the ground floor to the roof of the Empire State Building, a vertical climb of 380 m?
38. It takes 500 J of work to compress a spring 10 cm. What is the force constant of the spring?
39. A bungee cord is essentially a very long rubber band that can stretch up to four times its unstretched length. However, its spring constant varies over its stretch [see Menz, P.G. “The Physics of Bungee Jumping.” **The Physics Teacher** (November 1993) 31: 483-487]. Take the length of the cord to be along the x-direction and define the stretch x as the length of the cord l minus its un-stretched length l0 ; that is, $x = l - l_0$ (see below). Suppose a particular bungee cord has a spring constant, for 0 ≤ x ≤ 4.88 m, of $k_1 = 204$ N/m and for 4.88 m ≤ x, of $k_2 = 111$ N/m. (Recall that the spring constant is the slope of the force F(x) versus its stretch x.) (a) What is the tension in the cord when the stretch is 16.7 m (the maximum desired for a given jump)? (b) How much work must be done against the elastic force of the bungee cord to stretch it 16.7 m?

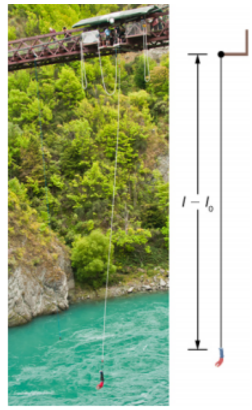


Figure 7.16 - (credit: Graeme Churchard)

40. A bungee cord exerts a nonlinear elastic force of magnitude $F(x) = k_1x + k_2x^3$, where x is the distance the cord is stretched, $k_1 = 204$ N/m and $k_2 = -0.233$ N/m^3. How much work must be done on the cord to stretch it 16.7 m?
41. Engineers desire to model the magnitude of the elastic force of a bungee cord using the equation \[F(x) = a \Bigg[\frac{x + 9\; m}{9\; m} − \left(\dfrac{9\; m}{x + 9\; m}\right)^{2} \Bigg],$$where x is the stretch of the cord along its length and a is a constant. If it takes 22.0 kJ of work to stretch the cord by 16.7 m, determine the value of the constant a.
42. A particle moving in the xy-plane is subject to a force

$$\vec{F}(x, y) = (50\ N \cdot m^2)\frac{(x\hat{i} + y\hat{j})}{(x^2 + y^2)^{3/2}}, \tag{8.6.1}$$

where x and y are in meters. Calculate the work done on the particle by this force, as it moves in a straight line from the point (3 m, 4 m) to the point (8 m, 6 m).
43. A particle moves along a curved path $y(x) = (10\ m)\{1 + \cos[(0.1\ m^{-1})x]\}$, from x = 0 to $x = 10\pi$ m, subject to a tangential force of variable magnitude $F(x) = (10\ N)\sin[(0.1\ m^{-1})x]$. How much work does the force do? (**Hint**: Consult a table of integrals or use a numerical integration program.)

7.2 Kinetic Energy

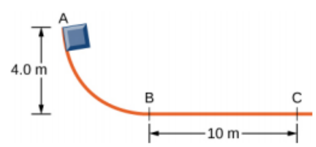
44. Compare the kinetic energy of a 20,000-kg truck moving at 110 km/h with that of an 80.0-kg astronaut in orbit moving at 27,500 km/h.
45. (a) How fast must a 3000-kg elephant move to have the same kinetic energy as a 65.0-kg sprinter running at 10.0 m/s? (b) Discuss how the larger energies needed for the movement of larger animals would relate to metabolic rates.
46. Estimate the kinetic energy of a 90,000-ton aircraft carrier moving at a speed of at 30 knots. You will need to look up the definition of a nautical mile to use in converting the unit for speed, where 1 knot equals 1 nautical mile per hour.
47. Calculate the kinetic energies of (a) a 2000.0-kg automobile moving at 100.0 km/h; (b) an 80.-kg runner sprinting at 10. m/s; and (c) a 9.1×10^{-31}-kg electron moving at 2.0×10^7 m/s.
48. A 5.0-kg body has three times the kinetic energy of an 8.0-kg body. Calculate the ratio of the speeds of these bodies.
49. An 8.0-g bullet has a speed of 800 m/s. (a) What is its kinetic energy? (b) What is its kinetic energy if the speed is halved?

7.3 Work-Energy Theorem

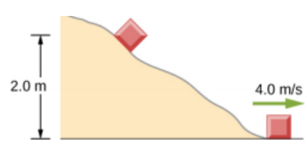
50. (a) Calculate the force needed to bring a 950-kg car to rest from a speed of 90.0 km/h in a distance of 120 m (a fairly typical distance for a non-panic stop). (b) Suppose instead the car hits a concrete abutment at full speed and is brought to a stop in 2.00 m. Calculate the force exerted on the car and compare it with the force found in part (a).
51. A car’s bumper is designed to withstand a 4.0-km/h (1.1-m/s) collision with an immovable object without damage to the body of the car. The bumper cushions the shock by absorbing the force over a distance. Calculate the magnitude of the average force on a bumper that collapses 0.200 m while bringing a 900-kg car to rest from an initial speed of 1.1 m/s.
52. Boxing gloves are padded to lessen the force of a blow. (a) Calculate the force exerted by a boxing glove on an opponent’s face, if the glove and face compress 7.50 cm during a blow in which the 7.00-kg arm and glove are brought to rest from an initial speed of 10.0 m/s. (b) Calculate the force exerted by an identical blow in the days when no gloves were used, and the knuckles and face would compress only 2.00 cm. Assume the change in mass by removing the glove

is negligible. (c) Discuss the magnitude of the force with glove on. Does it seem high enough to cause damage even though it is lower than the force with no glove?

53. Using energy considerations, calculate the average force a 60.0-kg sprinter exerts backward on the track to accelerate from 2.00 to 8.00 m/s in a distance of 25.0 m, if he encounters a headwind that exerts an average force of 30.0 N against him.
54. A 5.0-kg box has an acceleration of 2.0 m/s^2 when it is pulled by a horizontal force across a surface with μ_K = 0.50. Find the work done over a distance of 10 cm by (a) the horizontal force, (b) the frictional force, and (c) the net force. (d) What is the change in kinetic energy of the box?
55. A constant 10-N horizontal force is applied to a 20-kg cart at rest on a level floor. If friction is negligible, what is the speed of the cart when it has been pushed 8.0 m?
56. In the preceding problem, the 10-N force is applied at an angle of 45° below the horizontal. What is the speed of the cart when it has been pushed 8.0 m?
57. Compare the work required to stop a 100-kg crate sliding at 1.0 m/s and an 8.0-g bullet traveling at 500 m/s.
58. A wagon with its passenger sits at the top of a hill. The wagon is given a slight push and rolls 100 m down a 10° incline to the bottom of the hill. What is the wagon's speed when it reaches the end of the incline. Assume that the retarding force of friction is negligible.
59. An 8.0-g bullet with a speed of 800 m/s is shot into a wooden block and penetrates 20 cm before stopping. What is the average force of the wood on the bullet? Assume the block does not move.
60. A 2.0-kg block starts with a speed of 10 m/s at the bottom of a plane inclined at 37° to the horizontal. The coefficient of sliding friction between the block and plane is mu_k = 0.30. (a) Use the work-energy principle to determine how far the block slides along the plane before momentarily coming to rest. (b) After stopping, the block slides back down the plane. What is its speed when it reaches the bottom? (**Hint**: For the round trip, only the force of friction does work on the block.)
61. When a 3.0-kg block is pushed against a massless spring of force constant 4.5 x 10^3 N/m, the spring is compressed 8.0 cm. The block is released, and it slides 2.0 m (from the point at which it is released) across a horizontal surface before friction stops it. What is the coefficient of kinetic friction between the block and the surface?
62. A small block of mass 200 g starts at rest at A, slides to B where its speed is v_B = 8.0 m/s, then slides along the horizontal surface a distance 10 m before coming to rest at C. (See below.) (a) What is the work of friction along the curved surface? (b) What is the coefficient of kinetic friction along the horizontal surface?



63. A small object is placed at the top of an incline that is essentially frictionless. The object slides down the incline onto a rough horizontal surface, where it stops in 5.0 s after traveling 60 m. (a) What is the speed of the object at the bottom of the incline and its acceleration along the horizontal surface? (b) What is the height of the incline?
64. When released, a 100-g block slides down the path shown below, reaching the bottom with a speed of 4.0 m/s. How much work does the force of friction do?



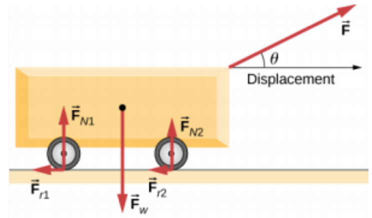
65. A 0.22LR-caliber bullet like that mentioned in Example 7.10 is fired into a door made of a single thickness of 1-inch pine boards. How fast would the bullet be traveling after it penetrated through the door?
66. A sled starts from rest at the top of a snow-covered incline that makes a 22° angle with the horizontal. After sliding 75 m down the slope, its speed is 14 m/s. Use the work-energy theorem to calculate the coefficient of kinetic friction between the runners of the sled and the snowy surface.

7.4 Power

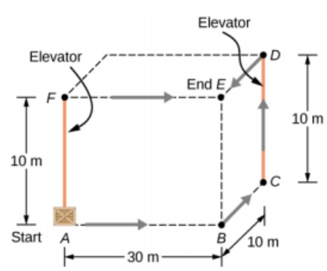
67. A person in good physical condition can put out 100 W of useful power for several hours at a stretch, perhaps by pedaling a mechanism that drives an electric generator. Neglecting any problems of generator efficiency and practical considerations such as resting time: (a) How many people would it take to run a 4.00-kW electric clothes dryer? (b) How many people would it take to replace a large electric power plant that generates 800 MW?
68. What is the cost of operating a 3.00-W electric clock for a year if the cost of electricity is \$0.0900 per kW • h?
69. A large household air conditioner may consume 15.0 kW of power. What is the cost of operating this air conditioner 3.00 h per day for 30.0 d if the cost of electricity is \$0.110 per kW • h?
70. (a) What is the average power consumption in watts of an appliance that uses 5.00 kW • h of energy per day? (b) How many joules of energy does this appliance consume in a year?
71. (a) What is the average useful power output of a person who does 6.00×10^6 J of useful work in 8.00 h? (b) Working at this rate, how long will it take this person to lift 2000 kg of bricks 1.50 m to a platform? (Work done to lift his body can be omitted because it is not considered useful output here.)
72. A 500-kg dragster accelerates from rest to a final speed of 110 m/s in 400 m (about a quarter of a mile) and encounters an average frictional force of 1200 N. What is its average power output in watts and horsepower if this takes 7.30 s?
73. (a) How long will it take an 850-kg car with a useful power output of 40.0 hp (1 hp equals 746 W) to reach a speed of 15.0 m/s, neglecting friction? (b) How long will this acceleration take if the car also climbs a 3.00-m high hill in the process?
74. (a) Find the useful power output of an elevator motor that lifts a 2500-kg load a height of 35.0 m in 12.0 s, if it also increases the speed from rest to 4.00 m/s. Note that the total mass of the counterbalanced system is 10,000 kg—so that only 2500 kg is raised in height, but the full 10,000 kg is accelerated. (b) What does it cost, if electricity is \$0.0900 per kW • h ?
75. (a) How long would it take a 1.50×10^5-kg airplane with engines that produce 100 MW of power to reach a speed of 250 m/s and an altitude of 12.0 km if air resistance were negligible? (b) If it actually takes 900 s, what is the power? (c) Given this power, what is the average force of air resistance if the airplane takes 1200 s? (**Hint**: You must find the distance the plane travels in 1200 s assuming constant acceleration.)
76. Calculate the power output needed for a 950-kg car to climb a 2.00° slope at a constant 30.0 m/s while encountering wind resistance and friction totaling 600 N.
77. A man of mass 80 kg runs up a flight of stairs 20 m high in 10 s. (a) how much power is used to lift the man? (b) If the man's body is 25% efficient, how much power does he expend?
78. The man of the preceding problem consumes approximately 1.05×10^7 J (2500 food calories) of energy per day in maintaining a constant weight. What is the average power he produces over a day? Compare this with his power production when he runs up the stairs.
79. An electron in a television tube is accelerated uniformly from rest to a speed of 8.4×10^7 m/s over a distance of 2.5 cm. What is the power delivered to the electron at the instant that its displacement is 1.0 cm?
80. Coal is lifted out of a mine a vertical distance of 50 m by an engine that supplies 500 W to a conveyer belt. How much coal per minute can be brought to the surface? Ignore the effects of friction.
81. A girl pulls her 15-kg wagon along a flat sidewalk by applying a 10-N force at 37° to the horizontal. Assume that friction is negligible and that the wagon starts from rest. (a) How much work does the girl do on the wagon in the first 2.0 s. (b) How much instantaneous power does she exert at t = 2.0 s ?
82. A typical automobile engine has an efficiency of 25%. Suppose that the engine of a 1000-kg automobile has a maximum power output of 140 hp. What is the maximum grade that the automobile can climb at 50 km/h if the frictional retarding force on it is 300 N?
83. When jogging at 13 km/h on a level surface, a 70-kg man uses energy at a rate of approximately 850 W. Using the facts that the "human engine" is approximately 25% efficient, determine the rate at which this man uses energy when jogging up a 5.0° slope at this same speed. Assume that the frictional retarding force is the same in both cases.

Additional Problems

84. A cart is pulled a distance D on a flat, horizontal surface by a constant force F that acts at an angle θ with the horizontal direction. The other forces on the object during this time are gravity (F_w), normal forces (F_{N1}) and (F_{N2}), and rolling frictions F_{r1} and F_{r2}, as shown below. What is the work done by each force?



85. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1$ = (3 N) $\hat{i}$ + (4 N) $\hat{j}$. As a result, the particle moves along the x-axis from x = 0 to x = 5 m in some time interval. What is the work done by $\vec{F}_1$?
86. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1$ = (3 N) $\hat{i}$ + (4 N) $\hat{j}$. As a result, the particle moves first along the x-axis from x = 0 to x = 5 m and then parallel to the y-axis from y = 0 to y = 6 m. What is the work done by $\vec{F}_1$?
87. Consider a particle on which several forces act, one of which is known to be constant in time: $\vec{F}_1$ = (3 N) $\hat{i}$ + (4 N) $\hat{j}$. As a result, the particle moves along a straight path from a Cartesian coordinate of (0 m, 0 m) to (5 m, 6 m). What is the work done by $\vec{F}_1$?
88. Consider a particle on which a force acts that depends on the position of the particle. This force is given by $\vec{F}_1$ = (2y) $\hat{i}$ + (3x) $\hat{j}$. Find the work done by this force when the particle moves from the origin to a point 5 meters to the right on the x-axis.
89. A boy pulls a 5-kg cart with a 20-N force at an angle of 30° above the horizontal for a length of time. Over this time frame, the cart moves a distance of 12 m on the horizontal floor. (a) Find the work done on the cart by the boy. (b) What will be the work done by the boy if he pulled with the same force horizontally instead of at an angle of 30° above the horizontal over the same distance?
90. A crate of mass 200 kg is to be brought from a site on the ground floor to a third floor apartment. The workers know that they can either use the elevator first, then slide it along the third floor to the apartment, or first slide the crate to another location marked C below, and then take the elevator to the third floor and slide it on the third floor a shorter distance. The trouble is that the third floor is very rough compared to the ground floor. Given that the coefficient of kinetic friction between the crate and the ground floor is 0.100 and between the crate and the third floor surface is 0.300, find the work needed by the workers for each path shown from A to E. Assume that the force the workers need to do is just enough to slide the crate at constant velocity (zero acceleration). Note: The work by the elevator against the force of gravity is not done by the workers.



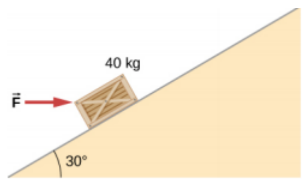
91. A hockey puck of mass 0.17 kg is shot across a rough floor with the roughness different at different places, which can be described by a position-dependent coefficient of kinetic friction. For a puck moving along the x-axis, the coefficient of kinetic friction is the following function of x, where x is in m: $\mu(x) = 0.1 + 0.05x$. Find the work done by the kinetic frictional force on the hockey puck when it has moved (a) from x = 0 to x = 2 m, and (b) from x = 2 m to x = 4 m.
92. A horizontal force of 20 N is required to keep a 5.0 kg box traveling at a constant speed up a frictionless incline for a vertical height change of 3.0 m. (a) What is the work done by gravity during this change in height? (b) What is the work done by the normal force? (c) What is the work done by the horizontal force?
93. A 7.0-kg box slides along a horizontal frictionless floor at 1.7 m/s and collides with a relatively massless spring that compresses 23 cm before the box comes to a stop. (a) How much kinetic energy does the box have before it collides with the spring? (b) Calculate the work done by the spring. (c) Determine the spring constant of the spring.
94. You are driving your car on a straight road with a coefficient of friction between the tires and the road of 0.55. A large piece of debris falls in front of your view and you immediate slam on the brakes, leaving a skid mark of 30.5 m (100-feet)

long before coming to a stop. A policeman sees your car stopped on the road, looks at the skid mark, and gives you a ticket for traveling over the 13.4 m/s (30 mph) speed limit. Should you fight the speeding ticket in court?

95. A crate is being pushed across a rough floor surface. If no force is applied on the crate, the crate will slow down and come to a stop. If the crate of mass 50 kg moving at speed 8 m/s comes to rest in 10 seconds, what is the rate at which the frictional force on the crate takes energy away from the crate?
96. Suppose a horizontal force of 20 N is required to maintain a speed of 8 m/s of a 50 kg crate. (a) What is the power of this force? (b) Note that the acceleration of the crate is zero despite the fact that 20 N force acts on the crate horizontally. What happens to the energy given to the crate as a result of the work done by this 20 N force?
97. Grains from a hopper falls at a rate of 10 kg/s vertically onto a conveyor belt that is moving horizontally at a constant speed of 2 m/s. (a) What force is needed to keep the conveyor belt moving at the constant velocity? (b) What is the minimum power of the motor driving the conveyor belt?
98. A cyclist in a race must climb a 5° hill at a speed of 8 m/s. If the mass of the bike and the biker together is 80 kg, what must be the power output of the biker to achieve the goal?

Challenge Problems

99. Shown below is a 40-kg crate that is pushed at constant velocity a distance 8.0 m along a 30° incline by the horizontal force $\vec{F}$. The coefficient of kinetic friction between the crate and the incline is μ_k = 0.40. Calculate the work done by (a) the applied force, (b) the frictional force, (c) the gravitational force, and (d) the net force.



100. The surface of the preceding problem is modified so that the coefficient of kinetic friction is decreased. The same horizontal force is applied to the crate, and after being pushed 8.0 m, its speed is 5.0 m/s. How much work is now done by the force of friction? Assume that the crate starts at rest.
101. The force F(x) varies with position, as shown below. Find the work done by this force on a particle as it moves from x = 1.0 m to x = 5.0 m.

102. Find the work done by the same force in Example 7.4, between the same points, A = (0, 0) and B = (2 m, 2 m) , over a circular arc of radius 2 m, centered at (0, 2 m). Evaluate the path integral using Cartesian coordinates. (**Hint**: You will probably need to consult a table of integrals.)
103. Answer the preceding problem using polar coordinates.
104. Find the work done by the same force in Example 7.4, between the same points, A = (0, 0) and B = (2 m, 2 m), over a circular arc of radius 2 m, centered at (2 m, 0). Evaluate the path integral using Cartesian coordinates. (**Hint**: You will probably need to consult a table of integrals.)
105. Answer the preceding problem using polar coordinates.
106. Constant power P is delivered to a car of mass m by its engine. Show that if air resistance can be ignored, the distance covered in a time t by the car, starting from rest, is given by s = $\left(\frac{8P}{9m}\right)^{1/2}$t $^{3/2}$.
107. Suppose that the air resistance a car encounters is independent of its speed. When the car travels at 15 m/ s, its engine delivers 20 hp to its wheels. (a) What is the power delivered to the wheels when the car travels at 30 m/ s? (b) How much energy does the car use in covering 10 km at 15 m/s? At 30 m/s? Assume that the engine is 25% efficient. (c) Answer the same questions if the force of air resistance is proportional to the speed of the automobile. (d) What do these results, plus your experience with gasoline consumption, tell you about air resistance?
108. Consider a linear spring, as in Figure 7.7(a), with mass M uniformly distributed along its length. The left end of the spring is fixed, but the right end, at the equilibrium position x = 0, is moving with speed v in the x-direction. What is the

total kinetic energy of the spring? (**Hint**: First express the kinetic energy of an infinitesimal element of the spring dm in terms of the total mass, equilibrium length, speed of the right-hand end, and position along the spring; then integrate.)

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8.7: Work and Kinetic Energy (Summary)

Key Terms

average power	work done in a time interval divided by the time interval
kinetic energy	energy of motion, one-half an object’s mass times the square of its speed
net work	work done by all the forces acting on an object
power	(or instantaneous power) rate of doing work
work	done when a force acts on something that undergoes a displacement from one position to another
work done by a force	integral, from the initial position to the final position, of the dot product of the force and the infinitesimal displacement along the path over which the force acts
work-energy theorem	net work done on a particle is equal to the change in its kinetic energy

Key Equations

Work done by a force over an infinitesimal displacement	$$dW = \vec{F} \cdot d\vec{r} = \lvert\vec{F}\rvert\lvert d\vec{r}\rvert \cos\theta \tag{8.7.1}$$
Work done by a force acting along a path from A to B	$$W_{AB} = \int_{path\ AB} \vec{F} \cdot d\vec{r} \tag{8.7.2}$$
Work done by a constant force of kinetic friction	$$W_{fr} = -f_k \lvert l_{AB}\rvert \tag{8.7.3}$$
Work done going from A to B by Earth’s gravity, near its surface	$$W_{grav,\ AB} = -mg(y_B - y_A) \tag{8.7.4}$$
Work done going from A to B by one-dimensional spring force	$$W_{spring,\ AB} = \left(\frac{1}{2}k\right)(x_B^2 - x_A^2) \tag{8.7.5}$$
Kinetic energy of a non-relativistic particle	$$K = \frac{1}{2}mv^2 = \frac{p^2}{2m} \tag{8.7.6}$$
Work-energy theorem	$$W_{net} = K_B - K_A \tag{8.7.7}$$
Power as rate of doing work	$$P = \frac{dW}{dt} \tag{8.7.8}$$
Power as the dot product of force and velocity	$$P = \vec{F} \cdot \vec{v} \tag{8.7.9}$$

Summary

7.1 Work

- The infinitesimal increment of work done by a force, acting over an infinitesimal displacement, is the dot product of the force and the displacement.
- The work done by a force, acting over a finite path, is the integral of the infinitesimal increments of work done along the path.
- The work done against a force is the negative of the work done by the force.

- The work done by a normal or frictional contact force must be determined in each particular case.
- The work done by the force of gravity, on an object near the surface of Earth, depends only on the weight of the object and the difference in height through which it moved.
- The work done by a spring force, acting from an initial position to a final position, depends only on the spring constant and the squares of those positions.

7.2 Kinetic Energy

- The kinetic energy of a particle is the product of one-half its mass and the square of its speed, for non-relativistic speeds.
- The kinetic energy of a system is the sum of the kinetic energies of all the particles in the system.
- Kinetic energy is relative to a frame of reference, is always positive, and is sometimes given special names for different types of motion.

7.3 Work-Energy Theorem

- Because the net force on a particle is equal to its mass times the derivative of its velocity, the integral for the net work done on the particle is equal to the change in the particle's kinetic energy. This is the work-energy theorem.
- You can use the work-energy theorem to find certain properties of a system, without having to solve the differential equation for Newton's second law.

7.4 Power

- Power is the rate of doing work; that is, the derivative of work with respect to time.
- Alternatively, the work done, during a time interval, is the integral of the power supplied over the time interval.
- The power delivered by a force, acting on a moving particle, is the dot product of the force and the particle's velocity

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CHAPTER OVERVIEW

8.8: Work and Kinetic Energy

The application of Newton's laws usually requires solving differential equations that relate the forces acting on an object to the accelerations they produce. Often, an analytic solution is intractable or impossible, requiring lengthy numerical solutions or simulations to get approximate results. In such situations, more general relations, like the work-energy theorem (or the conservation of energy), can still provide useful answers to many questions and require a more modest amount of mathematical calculation. In particular, you will see how the work-energy theorem is useful in relating the speeds of a particle, at different points along its trajectory, to the forces acting on it, even when the trajectory is otherwise too complicated to deal with. Thus, some aspects of motion can be addressed with fewer equations and without vector decompositions.

Thumbnail: One form of energy is mechanical work, the energy required to move an object of mass m a distance d when opposed by a force F, such as gravity.

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8.9: Introduction

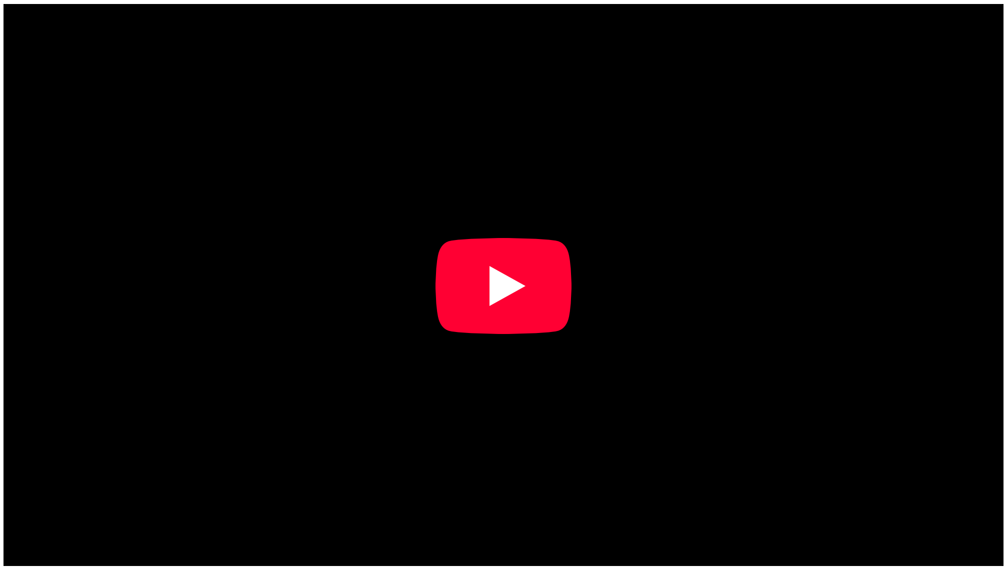
Introduction to Work and Energy

Work is the energy associated with the action of a force.

learning objectives

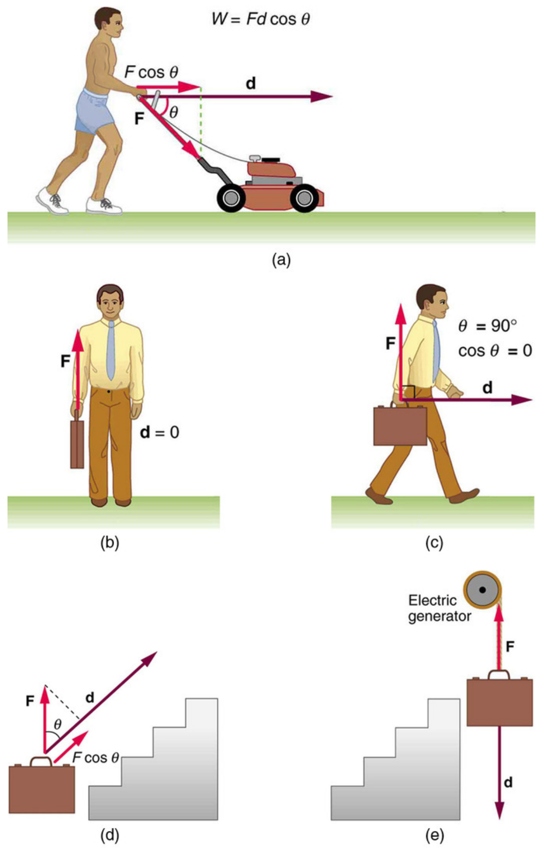
- Describe relationship between work, energy, and force

The work done on a system by a constant force is the product of the component of the force in the direction of motion times the distance through which the force acts. For one-way motion in one dimension, this is expressed in equation form as $\mathrm{W}=\mathrm{Fd}\cos\theta$, where W is work, F is the magnitude of the force on the system, d is the magnitude of the displacement of the system, and θ is the angle between the force vector F and the displacement vector d.



Work, Power, and Energy: Biology is useful.

Take this example of work in action from: (A) The work done by the force F on this lawn mower is $\mathrm{W}=\mathrm{Fd}\cos\theta$. Note that $\mathrm{W}=\mathrm{Fd}\cos\theta$ is the component of the force in the direction of motion. (B) A person holding a briefcase does no work on it, because there is no motion. No energy is transferred to or from the briefcase. (C) The person moving the briefcase horizontally at a constant speed does no work on it, and transfers no energy to it. (D) Work is done on the briefcase by carrying it up stairs at constant speed, because there is necessarily a component of force F in the direction of the motion. Energy is transferred to the briefcase and could, in turn, be used to do work. (E) When the briefcase is lowered, energy is transferred out of the briefcase and into an electric generator. Here the work done on the briefcase by the generator is negative, removing energy from the briefcase, because F and d are in opposite directions.



Examples of Work: This is how work in progress and energy co-exist and operate. Work is the energy associated with the action of a force.

In physics, a force is said to do work when it acts on a body so that there is a displacement of the point of application, however small, in the direction of the force. Thus a force does work when there is movement under the action of the force. The work done by a constant force of magnitude F on a point that moves a distance d in the direction of the force is the product:

$$\mathrm{W = Fd} \tag{8.9.1}$$

For example, if a force of 10 newton (F = 10 N) acts along point that travels two meters (d = 2 m), then it does the work W = (10 N)(2 m) = 20 N m = 20 J. This is approximately the work done lifting a 1 kg weight from the ground to over a person's head against the force of gravity. Notice that the work is doubled either by lifting twice the weight in the same distance or by lifting the same weight twice the distance.

Work is closely related to energy. The conservation of energy states that the change in total internal energy of a system equals the added heat minus the work performed by the system (see the first law of thermodynamics, and):

Baseball Pitcher: A baseball pitcher does work on a baseball by throwing the ball at some force, F, over some distance d, which for the average baseball field, is about 60 feet.

$$\delta E = \delta Q - \delta W \quad (8.9.2)$$

Also, from Newton's second law for rigid bodies, it can be shown that work on an object is equal to the change in kinetic energy of that object:

$$W = \Delta KE \quad (8.9.3)$$

The work of forces generated by a potential function is known as potential energy and the forces are said to be conservative. Therefore work on an object moving in a conservative force field is equal to minus the change of potential energy of the object:

$$W = -\Delta PE \quad (8.9.4)$$

This shows that work is the energy associated with the action of a force, and so has the physical dimensions and units of energy.

Key Points

- Work is the transfer of energy by a force acting on an object as it is displaced.
- The work done by a force is zero if the displacement is either zero or perpendicular to the force.
- The work done is positive if the force and displacement have the same direction, and the work done is negative if they have opposite direction.

Key Terms

- **energy**: A quantity that denotes the ability to do work and is measured in a unit dimensioned in mass × distance^2/time^2 (ML^2/T^2) or the equivalent.

8.10: Work Done by a Constant Force

Force in the Direction of Displacement

The work done by a constant force is proportional to the force applied times the displacement of the object.

learning objectives

- Contrast displacement and distance in constant force situations

Work Done by a Constant Force

When a force acts on an object over a distance, it is said to have done work on the object. Physically, the work done on an object is the change in kinetic energy that that object experiences. We will rigorously prove both of these claims.

The term work was introduced in 1826 by the French mathematician Gaspard-Gustave Coriolis as "weight lifted through a height," which is based on the use of early steam engines to lift buckets of water out of flooded ore mines. The SI unit of work is the newton-meter or joule (J).

Units

One way to validate if an expression is correct is to perform dimensional analysis. We have claimed that work is the change in kinetic energy of an object and that it is also equal to the force times the distance. The units of these two should agree. Kinetic energy – and all forms of energy – have units of joules (J). Likewise, force has units of newtons (N) and distance has units of meters (m). If the two statements are equivalent they should be equivalent to one another.

$$\mathrm{N \cdot m = kg\frac{m}{s^2} \cdot m = kg\frac{m^2}{s^2} = J} \tag{8.10.1}$$

Displacement versus Distance

Often times we will be asked to calculate the work done by a force on an object. As we have shown, this is proportional to the force and the distance which the object is displaced, not moved. We will investigate two examples of a box being moved to illustrate this.

Example Problems

Here are a few example problems:

(1.a) Consider a constant force of two newtons (F = 2 N) acting on a box of mass three kilograms (M = 3 kg). Calculate the work done on the box if the box is displaced 5 meters.

(1.b) Since the box is displaced 5 meters and the force is 2 N, we multiply the two quantities together. The object's mass will dictate how fast it is accelerating under the force, and thus the time it takes to move the object from point a to point b. Regardless of how long it takes, the object will have the same displacement and thus the same work done on it.

(2.a) Consider the same box (M = 3 kg) being pushed by a constant force of four newtons (F = 4 N). It begins at rest and is pushed for five meters (d = 5m). Assuming a frictionless surface, calculate the velocity of the box at 5 meters.

(2.b) We now understand that the work is proportional to the change in kinetic energy, from this we can calculate the final velocity. What do we know so far? We know that the block begins at rest, so the initial kinetic energy must be zero. From this we algebraically isolate and solve for the final velocity.

$$\mathrm{Fd = \Delta KE = KE_f - 0 = \frac{1}{2}mv_f^2} \tag{8.10.2}$$

$$\mathrm{v_f = \sqrt{2\frac{Fd}{m}} = \sqrt{2\frac{4N \cdot 5m}{2kg}} = \sqrt{10}m/s} \tag{8.10.3}$$

We see that the final velocity of the block is approximately 3.15 m/s.

Force at an Angle to Displacement

A force does not have to, and rarely does, act on an object parallel to the direction of motion.

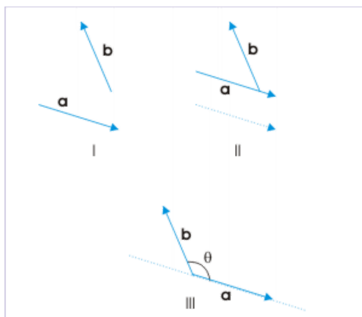
learning objectives

- Infer how to adjust one-dimensional motion for our three-dimensional world

The Fundamentals

Up until now, we have assumed that any force acting on an object has been parallel to the direction of motion. We have considered our motion to be one dimensional, only acting along the x or y axis. To best examine and understand how nature operators in our three-dimensional world, we will first discuss work in two dimensions in order to build our intuition.

A force does not have to, and rarely does, act on an object parallel to the direction of motion. In the past, we derived that $\mathbf{W}=\mathbf{Fd}$; such that the work done on an object is the force acting on the object multiplied by the displacement. But this is not the whole story. This expression contains an assumed cosine term, which we do not consider for forces parallel to the direction of motion. "Why would we do such a thing? " you may ask. We do this because the two are equivalent. If the angle of the force along the direction of motion is zero, such that the force is parallel to the direction of motion, then the cosine term equals one and does not change the expression. As we increase the force's angle with respect to the direction of motion, less and less work is done along the direction that we are considering; and more and more work is being done in another, perpendicular, direction of motion. This process continues until we are perpendicular to our original direction of motion, such that the angle is 90, and the cosine term would equal zero; resulting in zero work being done along our original direction. Instead, we are doing work in another direction!



Angle: Recall that both the force and direction of motion are vectors. When the angle is 90 degrees, the cosine term goes to zero. When along the same direction, they equal one.

Let's show this explicitly and then look at this phenomena in terms of a box moving along the x and y directions.

We have discussed that work is the integral of the force and the dot product respect to x. But in fact, dot product of force and a very small distance is equal to the two terms times cosine of the angle between the two. $\mathbf{F}\times\mathbf{dx}=\mathbf{Fd}\cos(\theta)$. Explicitly,

$$\int_{\mathrm{t}_2}^{\mathrm{t}_1}\mathbf{F}\cdot\mathbf{dx}=\int_{\mathrm{t}_2}^{\mathrm{t}_1}\mathbf{Fd}\cos\theta\mathbf{dx}=\mathbf{Fd}\cos\theta \tag{8.10.4}$$

A Box Being Pushed

Consider a coordinate system such that we have x as the abscissa and y as the ordinate. More so, consider a box being pushed along the x direction. What happens in the following three scenarios?

- The box is being pushed parallel to the x direction?
- The box is being pushed at an angle of 45 degrees to the x direction?
- The box is being pushed at an angle of 60 degrees to the x direction?
- The box is being pushed at an angle of 90 degrees to the x direction?

In the first scenario, we know that all of the force is acting on the box along the x-direction, which means that work will only be done along the x-direction. More so, a vertical perspective the box is not moving – it is unchanged in the y direction. Since the force is acting parallel to the direction of motion, the angle is equal to zero and our total work is simply the force times the displacement in the x-direction.

In the second scenario, the box is being pushed at an angle of 45 degrees to the x-direction; and thus also a 45 degree angle to the y-direction. When evaluated, the cosine of 45 degrees is equal to $\frac{1}{\sqrt{2}}$, or approximately 0.71. This means is that 71% of the force is contributing to the work along the x-direction. The other 29% is acting along the y-direction.

In the third scenario, we know that the force is acting at a 60 degree angle to the x-direction; and thus also a 30 degree angle to the y-direction. When evaluated, cosine of 60 degrees is equal to 1/2. This means that the force is equally acting in the x and y-direction! The work done is linear with respect to both x and y.

In the last scenario, the box is being pushed at an angle perpendicular to the x direction. In other words, we are pushing the box in the y-direction! Thus, the box's position will be unchanged and experience no displacement along the x-axis. The work done in the x direction will be zero.

Key Points

- Understanding work is quintessential to understanding systems in terms of their energy, which is necessary for higher level physics.
- Work is equivalent to the change in kinetic energy of a system.
- Distance is not the same as displacement. If a box is moved 3 meters forward and then 4 meters to the left, the total displacement is 5 meters, not 7 meters.
- Work done on an object along a given direction of motion is equal to the force times the displacement times the cosine of the angle.
- No work is done along a direction of motion if the force is perpendicular.
- When considering force parallel to the direction of motion, we omit the cosine term because it equals 1 which does not change the expression.

Key Terms

- **work**: A measure of energy expended in moving an object; most commonly, force times displacement. No work is done if the object does not move.
- **dot product**: A scalar product.

- **6.2: Work Done by a Constant Force** by Boundless has no license indicated.

8.11: Work Done by a Variable Force

Work Done by a Variable Force

Integration is used to calculate the work done by a variable force.

learning objectives

- Describe approaches used to calculate work done by a variable force

Using Integration to Calculate the Work Done by Variable Forces

A force is said to do work when it acts on a body so that there is a displacement of the point of application in the direction of the force. Thus, a force does work when it results in movement.

The work done by a constant force of magnitude F on a point that moves a displacement *[Math Processing Error]* in the direction of the force is simply the product

[Math Processing Error]

In the case of a variable force, integration is necessary to calculate the work done. For example, let's consider work done by a spring. According to the Hooke's law the restoring force (or spring force) of a perfectly elastic spring is proportional to its extension (or compression), but opposite to the direction of extension (or compression). So the spring force acting upon an object attached to a horizontal spring is given by:

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that is proportional to its displacement (extension or compression) in the x direction from the spring's equilibrium position, but its direction is opposite to the x direction. For a variable force, one must add all the infinitesimally small contributions to the work done during infinitesimally small time intervals dt (or equivalently, in infinitely small length intervals $dx=v_x dt$). In other words, an integral must be evaluated:

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This is the work done by a spring exerting a variable force on a mass moving from position x_o to x (from time 0 to time t). The work done is positive if the applied force is in the same direction as the direction of motion; so the work done by the object on spring from time 0 to time t, is:

[Math Processing Error]

in this relation *[Math Processing Error]* is the force acted upon spring by the object. *[Math Processing Error]* and *[Math Processing Error]* are in fact action- reaction pairs; and *[Math Processing Error]* is equal to the elastic potential energy stored in spring.

Using Integration to Calculate the Work Done by Constant Forces

The same integration approach can be also applied to the work done by a constant force. This suggests that *integrating* the product of force and distance is the general way of determining the work done by a force on a moving body.

Consider the situation of a gas sealed in a piston, the study of which is important in Thermodynamics. In this case, the Pressure (Pressure =Force/Area) is constant and can be taken out of the integral:

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Another example is the work done by gravity (a constant force) on a free-falling object (we assign the y-axis to vertical motion, in this case):

[Math Processing Error]

Notice that the result is *the same* as we would have obtained by simply evaluating the product of force and distance.

Units Used for Work

The SI unit of work is the joule (J), which is defined as the work done by a force of one newton moving an object through a distance of one meter.

Non-SI units of work include the erg, the foot-pound, the foot-pound, the kilowatt hour, the liter-atmosphere, and the horsepower-hour.

Key Points

- The work done by a constant force of magnitude F on a point that moves a displacement d in the direction of the force is the product: *[Math Processing Error]*.
- Integration approach can be used both to calculate work done by a variable force and work done by a constant force.
- The SI unit of work is the joule; non- SI units of work include the erg, the foot-pound, the foot-poundal, the kilowatt hour, the litre-atmosphere, and the horsepower-hour.

Key Terms

- **work**: A measure of energy expended in moving an object; most commonly, force times displacement. No work is done if the object does not move.
- **force**: A physical quantity that denotes ability to push, pull, twist or accelerate a body, which is measured in a unit dimensioned in mass × distance/time² (ML/T^2): SI: newton (N); CGS: dyne (dyn)

8.12: Work-Energy Theorem

Kinetic Energy and Work-Energy Theorem

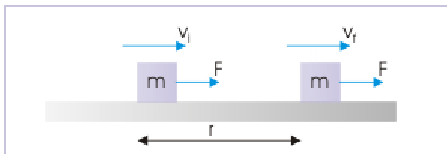
The work-energy theorem states that the work done by all forces acting on a particle equals the change in the particle's kinetic energy.

learning objectives

- Outline the derivation of the work-energy theorem

The Work-Energy Theorem

The principle of work and kinetic energy (also known as the work-energy theorem) states that the work done by the sum of all forces acting on a particle equals the change in the kinetic energy of the particle. This definition can be extended to rigid bodies by defining the work of the torque and rotational kinetic energy.



Kinetic Energy: A force does work on the block. The kinetic energy of the block increases as a result by the amount of work. This relationship is generalized in the work-energy theorem.

The work *W* done by the net force on a particle equals the change in the particle's kinetic energy K*E*:

$$\mathrm{W}=\Delta\mathrm{KE}=\frac{1}{2}\mathrm{mv_f^2}-\frac{1}{2}\mathrm{mv_i^2} \tag{8.12.1}$$

where v_i and v_f are the speeds of the particle before and after the application of force, and *m* is the particle's mass.

Derivation

For the sake of simplicity, we will consider the case in which the resultant force *F* is constant in both magnitude and direction and is parallel to the velocity of the particle. The particle is moving with constant acceleration *a* along a straight line. The relationship between the net force and the acceleration is given by the equation *F* = *ma* (Newton's second law), and the particle's displacement *d*, can be determined from the equation:

$$\mathrm{v_f^2}=\mathrm{v_i^2}+2\mathrm{ad} \tag{8.12.2}$$

obtaining,

$$\mathrm{d}=\frac{\mathrm{v_f^2}-\mathrm{v_i^2}}{2\mathrm{a}} \tag{8.12.3}$$

The work of the net force is calculated as the product of its magnitude (F=ma) and the particle's displacement. Substituting the above equations yields:

$$\mathrm{W}=\mathrm{Fd}=\mathrm{ma}\frac{\mathrm{v_f^2}-\mathrm{v_i^2}}{2\mathrm{a}}=\frac{1}{2}\mathrm{mv_f^2}-\frac{1}{2}\mathrm{mv_i^2}=\mathrm{KE_f}-\mathrm{KE_i}=\Delta\mathrm{KE} \tag{8.12.4}$$

Key Points

- The work *W* done by the net force on a particle equals the change in the particle's kinetic energy K*E*: $\mathrm{W}=\Delta\mathrm{KE}=\frac{1}{2}\mathrm{mv_f^2}-\frac{1}{2}\mathrm{mv_i^2}$.
- The work-energy theorem can be derived from Newton's second law.
- Work transfers energy from one place to another or one form to another. In more general systems than the particle system mentioned here, work can change the potential energy of a mechanical device, the heat energy in a thermal system, or the electrical energy in an electrical device.

Key Terms

- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)

8.13: Prelude to Potential Energy and Conservation of Energy

In George Rhoads' rolling ball sculpture, the principle of conservation of energy governs the changes in the ball's kinetic energy and relates them to changes and transfers for other types of energy associated with the ball's interactions. In this chapter, we introduce the important concept of potential energy. This will enable us to formulate the law of conservation of mechanical energy and to apply it to simple systems, making solving problems easier. In the final section on sources of energy, we will consider energy transfers and the general law of conservation of energy. Throughout this book, the law of conservation of energy will be applied in increasingly more detail, as you encounter more complex and varied systems, and other forms of energy.

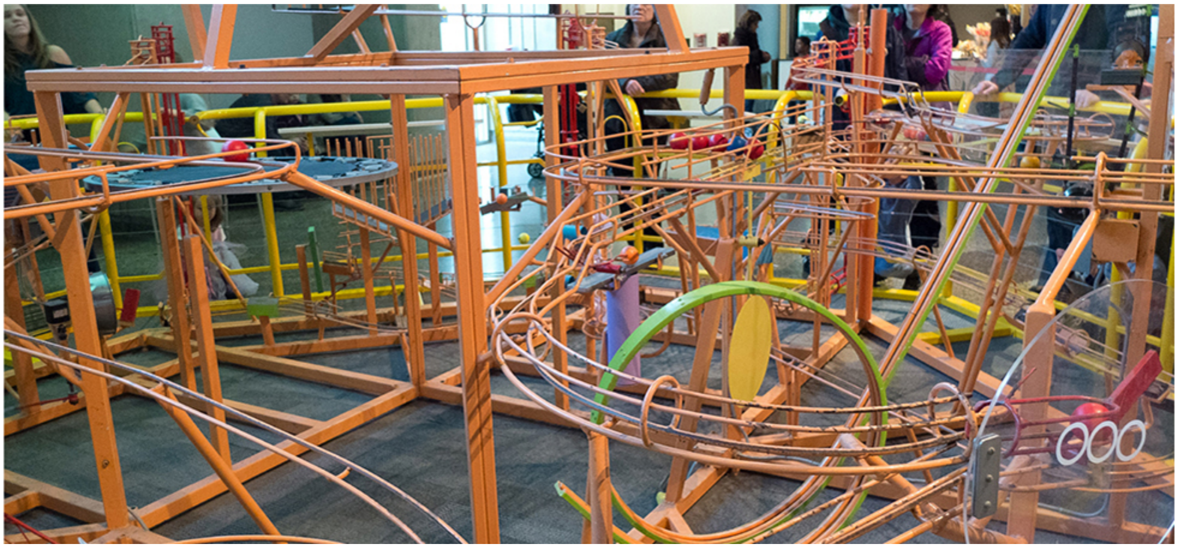


Figure 8.13.1: Shown here is part of a Ball Machine sculpture by George Rhoads. A ball in this contraption is lifted, rolls, falls, bounces, and collides with various objects, but throughout its travels, its kinetic energy changes in definite, predictable amounts, which depend on its position and the objects with which it interacts. (credit: modification of work by Roland Tanglao)

8.14: Potential Energy of a System

Learning Objectives

- Relate the difference of potential energy to work done on a particle for a system without friction or air drag
- Explain the meaning of the zero of the potential energy function for a system
- Calculate and apply the gravitational potential energy for an object near Earth's surface and the elastic potential energy of a mass-spring system

In Work, we saw that the work done on an object by the constant gravitational force, near the surface of Earth, over any displacement is a function only of the difference in the positions of the end-points of the displacement. This property allows us to define a different kind of energy for the system than its kinetic energy, which is called **potential energy**. We consider various properties and types of potential energy in the following subsections.

Potential Energy Basics

In Motion in Two and Three Dimensions, we analyzed the motion of a projectile, like kicking a football in Figure $8.14.1$. For this example, let's ignore friction and air resistance. As the football rises, the work done by the gravitational force on the football is negative, because the ball's displacement is positive vertically and the force due to gravity is negative vertically. We also noted that the ball slowed down until it reached its highest point in the motion, thereby decreasing the ball's kinetic energy. This loss in kinetic energy translates to a gain in gravitational potential energy of the football-Earth system.

As the football falls toward Earth, the work done on the football is now positive, because the displacement and the gravitational force both point vertically downward. The ball also speeds up, which indicates an increase in kinetic energy. Therefore, energy is converted from gravitational potential energy back into kinetic energy.

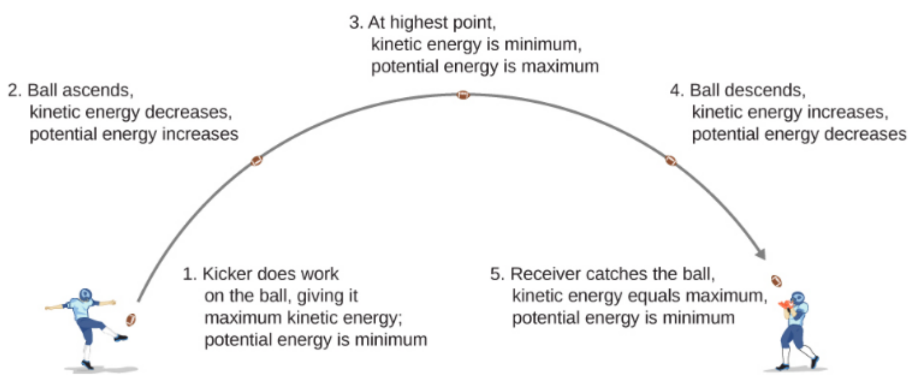


Figure 8.14.1: As a football starts its descent toward the wide receiver, gravitational potential energy is converted back into kinetic energy.

Based on this scenario, we can define the difference of potential energy from point A to point B as the negative of the work done:

$$\Delta U_{AB} = U_B - U_A = -W_{AB} \tag{8.14.1}$$

This formula explicitly states a **potential energy difference**, not just an absolute potential energy. Therefore, we need to define potential energy at a given position in such a way as to state standard values of potential energy on their own, rather than potential energy differences. We do this by rewriting the potential energy function in terms of an arbitrary constant,

$$\Delta U = U(\vec{\mathbf{r}}) - U\left(\vec{\mathbf{r}}_0\right) \tag{8.14.2}$$

The choice of the potential energy at a starting location of $\vec{r}_0$ is made out of convenience in the given problem. Most importantly, whatever choice is made should be stated and kept consistent throughout the given problem. There are some well-accepted choices of initial potential energy. For example, the lowest height in a problem is usually defined as zero potential energy, or if an object is in space, the farthest point away from the system is often defined as zero potential energy. Then, the potential energy, with respect to zero at $\vec{r}_0$, is just $U(\vec{r})$.

As long as there is no friction or air resistance, the change in kinetic energy of the football equals negative of the change in gravitational potential energy of the football. This can be generalized to any potential energy:

$$\Delta K_{AB} = -\Delta U_{AB} \tag{8.14.3}$$

Let's look at a specific example, choosing zero potential energy for gravitational potential energy at convenient points.

✔ Example 8.14.1: Basic Properties of Potential Energy

A particle moves along the *x*-axis under the action of a force given by F = -ax^2, where a = 3 N/m^2. (a) What is the difference in its potential energy as it moves from x_A = 1 m to x_B = 2 m? (b) What is the particle's potential energy at x = 1 m with respect to a given 0.5 J of potential energy at x=0?

Strategy

(a) The difference in potential energy is the negative of the work done, as defined by Equation 8.14.1. The work is defined in the previous chapter as the dot product of the force with the distance. Since the particle is moving forward in the *x*-direction, the dot product simplifies to a multiplication ($\hat{i} \cdot \hat{i}$ = 1). To find the total work done, we need to integrate the function between the given limits. After integration, we can state the work or the change in potential energy. (b) The potential energy function, with respect to zero at x=0, is the indefinite integral encountered in part (a), with the constant of integration determined from Equation 8.14.3. Then, we substitute the *x*-value into the function of potential energy to calculate the potential energy at x = 1.

Solution

a. The work done by the given force as the particle moves from coordinate x to x + dx in one dimension is

$$dW = \vec{\mathbf{F}} \cdot d\vec{\mathbf{r}} = F dx = -ax^2 dx$$

Substituting this expression into Equation 8.14.1, we obtain

$$\Delta U = -W = \int_{x_1}^{x_2} ax^2 dx = \frac{1}{3}\left(3\ \mathrm{N/m^2}\right) x^3 \Big|_{1\mathrm{m}}^{2\mathrm{m}} = 7\ \mathrm{J}$$

b. The indefinite integral for the potential energy function in part (a) is

$$U(x) = \frac{1}{3}ax^3 + \text{const.}\ ,$$

and we want the constant to be determined by

$$U(0) = 0.5\ J.$$

Thus, the potential energy with respect to zero at x = 0 is just

$$U(x) = \frac{1}{3}ax^3 + 0.5\ \mathrm{J}$$

Therefore, the potential energy at x = 1 m is

$$U(1\ \mathrm{m}) = \frac{1}{3}\left(3\ \mathrm{N/m^2}\right)(1\ \mathrm{m})^3 + 0.5\ \mathrm{J} = 1.5\ \mathrm{J}.$$

Significance

In this one-dimensional example, any function we can integrate, independent of path, is conservative. Notice how we applied the definition of potential energy difference to determine the potential energy function with respect to zero at a chosen point. Also notice that the potential energy, as determined in part (b), at x = 1 m is U(1 m) = 1 J and at x = 2 m is U(2 m) = 8 J; their difference is the result in part (a).

? Exercise 8.14.1

In Example 8.14.1, what are the potential energies of the particle at x = 1 m and x = 2 m with respect to zero at x = 1.5 m? Verify that the difference of potential energy is still 7 J.

Systems of Several Particles

In general, a system of interest could consist of several particles. The difference in the potential energy of the system is the negative of the work done by gravitational or elastic forces, which, as we will see in the next section, are conservative forces. The potential energy difference depends only on the initial and final positions of the particles, and on some parameters that characterize the interaction (like mass for gravity or the spring constant for a Hooke's law force).

It is important to remember that potential energy is a property of the interactions between objects in a chosen system, and not just a property of each object. This is especially true for electric forces, although in the examples of potential energy we consider below, parts of the system are either so big (like Earth, compared to an object on its surface) or so small (like a massless spring), that the changes those parts undergo are negligible when included in the system.

Types of Potential Energy

For each type of interaction present in a system, you can label a corresponding type of potential energy. The total potential energy of the system is the sum of the potential energies of all the types. (This follows from the additive property of the dot product in the expression for the work done.) Let's look at some specific examples of types of potential energy discussed in Work. First, we consider each of these forces when acting separately, and then when both act together.

Gravitational Potential Energy Near Earth's Surface

The system of interest consists of our planet, Earth, and one or more particles near its surface (or bodies small enough to be considered as particles, compared to Earth). The gravitational force on each particle (or body) is just its weight mg near the surface of Earth, acting vertically down. According to Newton's third law, each particle exerts a force on Earth of equal magnitude but in the opposite direction. Newton's second law tells us that the magnitude of the acceleration produced by each of these forces on Earth is mg divided by Earth's mass. Since the ratio of the mass of any ordinary object to the mass of Earth is vanishingly small, the motion of Earth can be completely neglected. Therefore, we consider this system to be a group of single-particle systems, subject to the uniform gravitational force of Earth.

In Work, the work done on a body by Earth's uniform gravitational force, near its surface, depended on the mass of the body, the acceleration due to gravity, and the difference in height the body traversed, as given by Equation 7.2.4. By definition, this work is the negative of the difference in the gravitational potential energy, so that difference is

$$\Delta U_{\text{grav}} = -W_{\text{grav},AB} = mg\left(y_B - y_A\right). \tag{8.14.4}$$

You can see from this that the gravitational potential energy function, near Earth's surface, is

$$U(y) = mgy + \text{const.} \tag{8.14.5}$$

You can choose the value of the constant, as described in the discussion of Equation 8.14.2; however, for solving most problems, the most convenient constant to choose is zero for when y=0, which is the lowest vertical position in the problem.

Example 8.14.2: Gravitational PotentIAL Energy of a hiker

The summit of Great Blue Hill in Milton, MA, is 147 m above its base and has an elevation above sea level of 195 m (Figure 8.14.2). (Its Native American name, *Massachusett*, was adopted by settlers for naming the Bay Colony and state near its location.) A 75-kg hiker ascends from the base to the summit. What is the gravitational potential energy of the hiker-Earth system with respect to zero gravitational potential energy at base height, when the hiker is (a) at the base of the hill, (b) at the summit, and (c) at sea level, afterward?

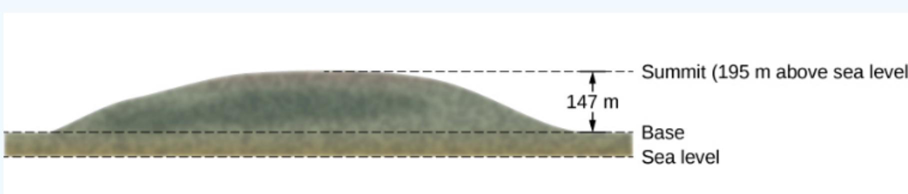


Figure 8.14.2: Sketch of the profile of Great Blue Hill, Milton, MA. The altitudes of the three levels are indicated.

Strategy

First, we need to pick an origin for the *y*-axis and then determine the value of the constant that makes the potential energy zero at the height of the base. Then, we can determine the potential energies from Equation 8.14.5, based on the relationship between the zero potential energy height and the height at which the hiker is located.

Solution

a. Let's choose the origin for the *y*-axis at base height, where we also want the zero of potential energy to be. This choice makes the constant equal to zero and

$$U(\text{ base }) = U(0) = 0$$

b. At the summit, y = 147 m, so

$$U(\text{ summit }) = U(147\text{ m}) = mgh = (75 \times 9.8\text{ N})(147\text{ m}) = 108\text{ kJ}.$$

c. At sea level, y = (147 - 195) m = -48 m, so

$$U\text{ (sea-level) } = (75 \times 9.8\text{N})(-48\text{m}) = -35.3\text{kJ}.$$

Significance

Besides illustrating the use of Equation 8.14.4 and Equation 8.14.5, the values of gravitational potential energy we found are reasonable. The gravitational potential energy is higher at the summit than at the base, and lower at sea level than at the base. Gravity does work on you on your way up, too! It does negative work and not quite as much (in magnitude), as your muscles do. But it certainly does work. Similarly, your muscles do work on your way down, as negative work. The numerical values of the potential energies depend on the choice of zero of potential energy, but the physically meaningful differences of potential energy do not. [Note that since Equation 8.14.2 is a difference, the numerical values do not depend on the origin of coordinates.]

Exercise 8.14.2

What are the values of the gravitational potential energy of the hiker at the base, summit, and sea level, with respect to a sea-level zero of potential energy?

Elastic Potential Energy

In Work, we saw that the work done by a perfectly elastic spring, in one dimension, depends only on the spring constant and the squares of the displacements from the unstretched position, as given in Equation 7.2.5. This work involves only the properties of a Hooke's law interaction and not the properties of real springs and whatever objects are attached to them. Therefore, we can define the difference of elastic potential energy for a spring force as the negative of the work done by the spring force in this equation, before we consider systems that embody this type of force. Thus,

$$\Delta U = -W_{AB} = \frac{1}{2}k\left(x_B^2 - x_A^2\right) \tag{8.14.6}$$

where the object travels from point *A* to point *B*. The potential energy function corresponding to this difference is

$$U(x) = \frac{1}{2}kx^2 + \text{const.} \tag{8.14.7}$$

If the spring force is the only force acting, it is simplest to take the zero of potential energy at x = 0, when the spring is at its unstretched length. Then, the constant is Equation 8.14.7 is zero. (Other choices may be more convenient if other forces are acting.)

Example 8.14.3: Spring Potential Energy

A system contains a perfectly elastic spring, with an unstretched length of 20 cm and a spring constant of 4 N/cm. (a) How much elastic potential energy does the spring contribute when its length is 23 cm? (b) How much more potential energy does it contribute if its length increases to 26 cm?

Strategy

When the spring is at its unstretched length, it contributes nothing to the potential energy of the system, so we can use Equation 8.14.7 with the constant equal to zero. The value of *x* is the length minus the unstretched length. When the spring is expanded, the spring's displacement or difference between its relaxed length and stretched length should be used for the *x*-value in calculating the potential energy of the spring.

Solution

a. The displacement of the spring is $x = 23\text{ cm} - 20\text{ cm} = 3\text{ cm}$, so the contributed potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}(4\text{ N/cm})(3\text{ cm})^2 = 0.18\text{ J}$.
b. When the spring's displacement is $x = 26\text{ cm} - 20\text{ cm} = 6\text{ cm}$, the potential energy is $U = \frac{1}{2}kx^2 = \frac{1}{2}(4\text{ N/cm})(6\text{ cm})^2 = 0.72\text{ J}$, which is a 0.54-J increase over the amount in part (a).

Significance

Calculating the elastic potential energy and potential energy differences from Equation 8.14.7 involves solving for the potential energies based on the given lengths of the spring. Since *U* depends on x^2, the potential energy for a compression (negative *x*) is the same as for an extension of equal magnitude.

? Exercise 8.14.3

When the length of the spring in Example 8.2.3 changes from an initial value of 22.0 cm to a final value, the elastic potential energy it contributes changes by −0.0800J. Find the final length.

Gravitational and Elastic Potential Energy

A simple system embodying both gravitational and elastic types of potential energy is a one-dimensional, vertical mass-spring system. This consists of a massive particle (or block), hung from one end of a perfectly elastic, massless spring, the other end of which is fixed, as illustrated in Figure 8.14.3.

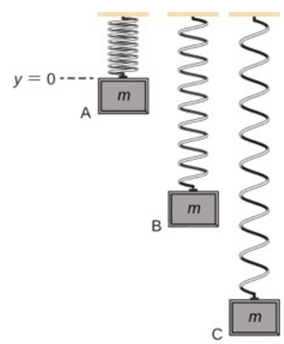


Figure 8.14.1: A vertical mass-spring system, with the positive *y*-axis pointing upward. The mass is initially at an unstretched spring length, point A. Then it is released, expanding past point B to point C, where it comes to a stop.

First, let's consider the potential energy of the system. We need to define the constant in the potential energy function of Equation 8.14.5. Often, the ground is a suitable choice for when the gravitational potential energy is zero; however, in this case, the highest point or when $y = 0$ is a convenient location for zero gravitational potential energy. Note that this choice is arbitrary, and the problem can be solved correctly even if another choice is picked.

We must also define the elastic potential energy of the system and the corresponding constant, as detailed in Equation 8.14.7. This is where the spring is unstretched, or at the $y = 0$ position.

If we consider that the total energy of the system is conserved, then the energy at point A equals point C. The block is placed just on the spring so its initial kinetic energy is zero. By the setup of the problem discussed previously, both the gravitational potential energy and elastic potential energy are equal to zero. Therefore, the initial energy of the system is zero. When the block arrives at point C, its kinetic energy is zero. However, it now has both gravitational potential energy and elastic potential energy. Therefore, we can solve for the distance y that the block travels before coming to a stop:

$$\begin{aligned} K_A + U_A &= K_C + U_C \\ 0 &= 0 + mgy_C + \frac{1}{2}k(y_C)^2 \\ y_C &= \frac{-2mg}{k} \end{aligned}$$

Figure 8.14.4: A bungee jumper transforms gravitational potential energy at the start of the jump into elastic potential energy at the bottom of the jump.

Example 8.14.4: Potential energy of a vertical mass-spring system

A block weighing 1.2 N is hung from a spring with a spring constant of 6.0 N/m, as shown in Figure 8.14.3. (a) What is the maximum expansion of the spring, as seen at point C? (b) What is the total potential energy at point B, halfway between A and C? (c) What is the speed of the block at point B?

Strategy

In part (a) we calculate the distance y_C as discussed in the previous text. Then in part (b), we use half of the y value to calculate the potential energy at point B using equations Equation 8.14.4 and Equation 8.14.6. This energy must be equal to the kinetic energy, Equation 7.3.1, at point B since the initial energy of the system is zero. By calculating the kinetic energy at point B, we can now calculate the speed of the block at point B.

Solution

a. Since the total energy of the system is zero at point A as discussed previously, the maximum expansion of the spring is calculated to be:

$$\begin{aligned} y_C &= \frac{-2mg}{k} \\ y_C &= \frac{-2(1.2\text{ N})}{(6.0\text{ N/m})} = -0.40\text{ m} \end{aligned} \tag{8.14.8}$$

b. The position of y_B is half of the position at y_C or -0.20 m. The total potential energy at point B would therefore be:

$$\begin{aligned} U_B &= mgy_B + \left(\frac{1}{2}ky_B\right)^2 \\ U_B &= (1.2\text{ N})(-0.20\text{ m}) + \frac{1}{2}(6\text{ N/m})(-0.20\text{ m})^2 \\ U_B &= -0.12\text{ J} \end{aligned}$$

c. The mass of the block is the weight divided by gravity.

$$m=\frac{F_w}{g}=\frac{1.2\text{ N}}{9.8\text{ m/s}^2}=0.12\text{ kg}$$

The kinetic energy at point B therefore is 0.12 J because the total energy is zero. Therefore, the speed of the block at point B is equal to

$$\begin{aligned} K &= \tfrac{1}{2}mv^2 \\ v &= \sqrt{\tfrac{2K}{m}}=\sqrt{\tfrac{2(0.12\text{ J})}{(0.12\text{ kg})}}=1.4\text{ m/s} \end{aligned} \tag{8.14.9}$$

Significance

Even though the potential energy due to gravity is relative to a chosen zero location, the solutions to this problem would be the same if the zero energy points were chosen at different locations.

? Exercise 8.14.4

Suppose the mass in Equation 8.14.6 is doubled while keeping the all other conditions the same. Would the maximum expansion of the spring increase, decrease, or remain the same? Would the speed at point B be larger, smaller, or the same compared to the original mass?

Simulation

View this simulation to learn about conservation of energy with a skater! Build tracks, ramps and jumps for the skater and view the kinetic energy, potential energy and friction as he moves. You can also take the skater to different planets or even space!

A sample chart of a variety of energies is shown in Table 8.14.1 to give you an idea about typical energy values associated with certain events. Some of these are calculated using kinetic energy, whereas others are calculated by using quantities found in a form of potential energy that may not have been discussed at this point.

Table 8.14.1: Energy of Various Objects and Phenomena

Object/phenomenon	Energy in joules
Big Bang	10^{68}
Annual world energy use	4.0×10^{20}
Large fusion bomb (9 megaton)	3.8×10^{16}
Hiroshima-size fission bomb (10 kiloton)	4.2×10^{13}
1 barrel crude oil	5.9×10^{9}
1 ton TNT	4.2×10^{9}
1 gallon of gasoline	1.2×10^{8}
Daily adult food intake (recommended)	1.2×10^{7}
1000-kg car at 90 km/h	3.1×10^{5}
Tennis ball at 100 km/h	22
Mosquito (10^{-2} g at 0.5 m/s)	1.3×10^{-6}
Single electron in a TV tube beam	4.0×10^{-15}
Energy to break one DNA strand	10^{-19}

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8.15: Conservative and Non-Conservative Forces

Learning Objectives

- Characterize a conservative force in several different ways
- Specify mathematical conditions that must be satisfied by a conservative force and its components
- Relate the conservative force between particles of a system to the potential energy of the system
- Calculate the components of a conservative force in various cases

In Potential Energy and Conservation of Energy, any transition between kinetic and potential energy conserved the total energy of the system. This was path independent, meaning that we can start and stop at any two points in the problem, and the total energy of the system—kinetic plus potential—at these points are equal to each other. This is characteristic of a **conservative force**. We dealt with conservative forces in the preceding section, such as the gravitational force and spring force. When comparing the motion of the football in Figure 8.2.1, the total energy of the system never changes, even though the gravitational potential energy of the football increases, as the ball rises relative to ground and falls back to the initial gravitational potential energy when the football player catches the ball. **Non-conservative forces** are dissipative forces such as friction or air resistance. These forces take energy away from the system as the system progresses, energy that you can't get back. These forces are path dependent; therefore it matters where the object starts and stops.

Definition: Conservative Force

The work done by a conservative force is independent of the path; in other words, the work done by a conservative force is the same for any path connecting two points:

$$W_{AB,\ path-1} = \int_{AB,\ path-1} \vec{F}_{cons} \cdot d\vec{r} = W_{AB,\ path-2} = \int_{AB,\ path-2} \vec{F}_{cons} \cdot d\vec{r}. \tag{8.15.1}$$

The work done by a non-conservative force depends on the path taken. Equivalently, a force is conservative if the work it does around any closed path is zero:

$$W_{closed\ path} = \oint \vec{E}_{cons} \cdot d\vec{r} = 0. \tag{8.15.2}$$

In Equation 8.15.2, we use the notation of a circle in the middle of the integral sign for a line integral over a closed path, a notation found in most physics and engineering texts.] Equations 8.15.1 and 8.15.2 are equivalent because any closed path is the sum of two paths: the first going from A to B, and the second going from B to A. The work done going along a path from B to A is the negative of the work done going along the same path from A to B, where A and B are any two points on the closed path:

$$\begin{aligned} 0 = \int \vec{F}_{cons} \cdot d\vec{r} &= \int_{AB,\ path-1} \vec{F}_{cons} \cdot d\vec{r} + \int_{BA,\ path-2} \vec{F}_{cons} \cdot d\vec{r} \\ &= \int_{AB,\ path-1} \vec{F}_{cons} \cdot d\vec{r} - \int_{AB,\ path-2} \vec{F}_{cons} \cdot d\vec{r} = 0. \end{aligned}$$

You might ask how we go about proving whether or not a force is conservative, since the definitions involve any and all paths from A to B, or any and all closed paths, but to do the integral for the work, you have to choose a particular path. One answer is that the work done is independent of path if the infinitesimal work $\vec{F} \cdot d\vec{r}$ is an **exact differential**, the way the infinitesimal net work was equal to the exact differential of the kinetic energy, $dW_{net} = m\vec{v} \cdot d\vec{v} = d\frac{1}{2}mv^2$, when we derived the work-energy theorem in Work-Energy Theorem. There are mathematical conditions that you can use to test whether the infinitesimal work done by a force is an exact differential, and the force is conservative. These conditions only involve differentiation and are thus relatively easy to apply. In two dimensions, the condition for $\vec{F} \cdot d\vec{r}$ = $F_x dx$ + $F_y dy$ to be an exact differential is

$$\frac{dF_x}{dy} = \frac{dF_y}{dx}. \tag{8.15.3}$$

You may recall that the work done by the force in Example 7.2.4 depended on the path. For that force,

$$F_x = (5\ N/m)y \text{ and } F_y = (10\ N/m)x. \tag{8.15.4}$$

Therefore,

$$\left(\frac{dF_x}{dy}\right) = 5\ N/m \neq \left(\frac{dF_y}{dx}\right) = 10\ N/m, \tag{8.15.5}$$

which indicates it is a non-conservative force. Can you see what you could change to make it a conservative force?

Figure 8.15.1: A grinding wheel applies a non-conservative force, because the work done depends on how many rotations the wheel makes, so it is path-dependent.

Example 8.15.1: Conservative or Not?

Which of the following two-dimensional forces are conservative and which are not? Assume a and b are constants with appropriate units:

a. $axy^3\hat{i} + ayx^3\hat{j}$,
b. $a\left[\left(\frac{y^2}{x}\right)\hat{i} + 2y\ln\left(\frac{x}{b}\right)\hat{j}\right]$,
c. $\frac{ax\hat{i}+ay\hat{j}}{x^2+y^2}$

Strategy

Apply the condition stated in Equation 8.15.3, namely, using the derivatives of the components of each force indicated. If the derivative of the y-component of the force with respect to x is equal to the derivative of the x-component of the force with respect to y, the force is a conservative force, which means the path taken for potential energy or work calculations always yields the same results.

Solution

a:

$$\frac{dF_x}{dy} = \frac{d(axy^3)}{dy} = 3axy^2$$

and

$$\frac{dF_y}{dx} = \frac{d(ayx^3)}{dx} = 3ayx^2,$$

so this force is non-conservative.

b:

$$\frac{dF_x}{dy} = \frac{d\left(\frac{ay^2}{x}\right)}{dy} = \frac{2ay}{x}$$

and

$$\frac{dF_y}{dx} = \frac{d(2ay\ln\left(\frac{x}{b}\right))}{dx} = \frac{2ay}{x},$$

so this force is conservative.

c:

$$\frac{dF_x}{dy} = \frac{d\left(\frac{ax}{(x^2+y^2)}\right)}{dy} = -\frac{ax(2y)}{(x^2+y^2)^2} = \frac{dF_y}{dx} = \frac{d\left(\frac{ay}{(x^2+y^2)}\right)}{dx}, \tag{8.15.6}$$

again conservative.

Significance

The conditions in Equation 8.15.3 are derivatives as functions of a single variable; in three dimensions, similar conditions exist that involve more derivatives.

Exercise 8.15.1

A two-dimensional, conservative force is zero on the x- and y-axes, and satisfies the condition $\left(\frac{dF_x}{dy}\right) = \left(\frac{dF_y}{dy}\right)$ = (4 N/m^3)xy. What is the magnitude of the force at the point $x = y = 1\,m$?

Before leaving this section, we note that non-conservative forces do not have potential energy associated with them because the energy is lost to the system and can't be turned into useful work later. So there is always a conservative force associated with every potential energy. We have seen that potential energy is defined in relation to the work done by conservative forces. That relation, Equation 8.2.1, involved an integral for the work; starting with the force and displacement, you integrated to get the work and the change in potential energy. However, integration is the inverse operation of differentiation; you could equally well have started with the potential energy and taken its derivative, with respect to displacement, to get the force. The infinitesimal increment of potential energy is the dot product of the force and the infinitesimal displacement,

$$dU = -\vec{F}\cdot d\vec{l} = -F_l dl. \tag{8.15.7}$$

Here, we chose to represent the displacement in an arbitrary direction by d$\vec{l}$, so as not to be restricted to any particular coordinate direction. We also expressed the dot product in terms of the magnitude of the infinitesimal displacement and the component of the force in its direction. Both these quantities are scalars, so you can divide by dl to get

$$F_l = -\frac{dU}{dl}. \tag{8.15.8}$$

This equation gives the relation between force and the potential energy associated with it. In words, the component of a conservative force, in a particular direction, equals the negative of the derivative of the corresponding potential energy, with respect to a displacement in that direction. For one-dimensional motion, say along the x-axis, Equation 8.15.8 give the entire vector force,

$$\bar{F} = F_x\hat{i} = -\frac{\partial U}{\partial x}\hat{i}. \tag{8.15.9}$$

In two dimensions,

$$\bar{F} = F_x\hat{i} + F_y\hat{j} \tag{8.15.10}$$

$$= -\left(\frac{\partial U}{\partial x}\right)\hat{i} - \left(\frac{\partial U}{\partial y}\right)\hat{j}. \tag{8.15.11}$$

From this equation, you can see why Equation 8.15.8 is the condition for the work to be an exact differential, in terms of the derivatives of the components of the force. In general, a partial derivative notation is used. If a function has many variables in it, the derivative is taken only of the variable the partial derivative specifies. The other variables are held constant. In three dimensions, you add another term for the z-component, and the result is that the force is the negative of the gradient of the potential energy. However, we won't be looking at three-dimensional examples just yet.

Example 8.15.2: Force due to a Quartic Potential Energy

The potential energy for a particle undergoing one-dimensional motion along the x-axis is

$$U(x) = \frac{1}{4}cx^4,$$

where c = 8 N/m^3. Its total energy at x = 0 is 2 J, and it is not subject to any non-conservative forces. Find (a) the positions where its kinetic energy is zero and (b) the forces at those positions.

Strategy

a. We can find the positions where K = 0, so the potential energy equals the total energy of the given system.
b. Using Equation 8.15.8, we can find the force evaluated at the positions found from the previous part, since the mechanical energy is conserved.

Solution

a. The total energy of the system of 2 J equals the quartic elastic energy as given in the problem

$$2\ J = \frac{1}{4}(8\ N/m^3)x_f^4. \tag{8.15.12}$$

Solving for x_f results in $x_f = \pm 1$ m.

b. From Equation 8.15.8,

$$F_x = -\frac{dU}{dx} = -cx^3. \tag{8.15.13}$$

Thus, evaluating the force at ±1 m , we get

$$\vec{F} = -(8\ N/m^3)(\pm 1\ m)^3\hat{i} = \pm 8\ N\hat{i}. \tag{8.15.14}$$

At both positions, the magnitude of the forces is 8 N and the directions are toward the origin, since this is the potential energy for a restoring force.

Significance

Finding the force from the potential energy is mathematically easier than finding the potential energy from the force, because differentiating a function is generally easier than integrating one.

Exercise 8.15.2

Find the forces on the particle in Example 8.15.2 when its kinetic energy is 1.0 J at $x = 0$.

8.16: Conservation of Energy

Learning Objectives

- Formulate the principle of conservation of mechanical energy, with or without the presence of non-conservative forces
- Use the conservation of mechanical energy to calculate various properties of simple systems

In this section, we elaborate and extend the result we derived in Potential Energy of a System, where we re-wrote the work-energy theorem in terms of the change in the kinetic and potential energies of a particle. This will lead us to a discussion of the important principle of the conservation of mechanical energy. As you continue to examine other topics in physics, in later chapters of this book, you will see how this conservation law is generalized to encompass other types of energy and energy transfers. The last section of this chapter provides a preview.

The terms ‘conserved quantity’ and ‘conservation law’ have specific, scientific meanings in physics, which are different from the everyday meanings associated with the use of these words. (The same comment is also true about the scientific and everyday uses of the word ‘work.’) In everyday usage, you could conserve water by not using it, or by using less of it, or by re-using it. Water is composed of molecules consisting of two atoms of hydrogen and one of oxygen. Bring these atoms together to form a molecule and you create water; dissociate the atoms in such a molecule and you destroy water. However, in scientific usage, a **conserved quantity** for a system stays constant, changes by a definite amount that is transferred to other systems, and/or is converted into other forms of that quantity. A conserved quantity, in the scientific sense, can be transformed, but not strictly created or destroyed. Thus, there is no physical law of conservation of water.

Systems with a Single Particle or Object

We first consider a system with a single particle or object. Returning to our development of Equation 8.2.2, recall that we first separated all the forces acting on a particle into conservative and non-conservative types, and wrote the work done by each type of force as a separate term in the work-energy theorem. We then replaced the work done by the conservative forces by the change in the potential energy of the particle, combining it with the change in the particle’s kinetic energy to get Equation 8.2.2. Now, we write this equation without the middle step and define the sum of the kinetic and potential energies, K + U = E; to be the **mechanical energy** of the particle.

Conservation of Energy

The mechanical energy E of a particle stays constant unless forces outside the system or non-conservative forces do work on it, in which case, the change in the mechanical energy is equal to the work done by the non-conservative forces:

$$W_{nc,\ AB} = \Delta(K+U)_{AB} = \Delta E_{AB}. \tag{8.16.1}$$

This statement expresses the concept of **energy conservation** for a classical particle as long as there is no non-conservative work. Recall that a classical particle is just a point mass, is nonrelativistic, and obeys Newton’s laws of motion. In Relativity, we will see that conservation of energy still applies to a non-classical particle, but for that to happen, we have to make a slight adjustment to the definition of energy.

It is sometimes convenient to separate the case where the work done by non-conservative forces is zero, either because no such forces are assumed present, or, like the normal force, they do zero work when the motion is parallel to the surface. Then

$$0 = W_{nc,\ AB} = \Delta(K+U)_{AB} = \Delta E_{AB}. \tag{8.16.2}$$

In this case, the conservation of mechanical energy can be expressed as follows: The mechanical energy of a particle does not change if all the non-conservative forces that may act on it do no work. Understanding the concept of energy conservation is the important thing, not the particular equation you use to express it.

Problem-Solving Strategy: Conservation of Energy

1. Identify the body or bodies to be studied (the system). Often, in applications of the principle of mechanical energy conservation, we study more than one body at the same time.
2. Identify all forces acting on the body or bodies.

3. Determine whether each force that does work is conservative. If a non-conservative force (e.g., friction) is doing work, then mechanical energy is not conserved. The system must then be analyzed with non-conservative work, Equation 8.16.2.
4. For every force that does work, choose a reference point and determine the potential energy function for the force. The reference points for the various potential energies do not have to be at the same location.
5. Apply the principle of mechanical energy conservation by setting the sum of the kinetic energies and potential energies equal at every point of interest.

Example 8.7: Simple pendulum

A particle of mass m is hung from the ceiling by a massless string of length 1.0 m, as shown in Figure 8.16.1. The particle is released from rest, when the angle between the string and the downward vertical direction is 30°. What is its speed when it reaches the lowest point of its arc?

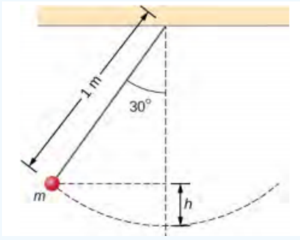


Figure 8.16.1: A particle hung from a string constitutes a simple pendulum. It is shown when released from rest, along with some distances used in analyzing the motion.

Strategy

Using our problem-solving strategy, the first step is to define that we are interested in the particle-Earth system. Second, only the gravitational force is acting on the particle, which is conservative (step 3). We neglect air resistance in the problem, and no work is done by the string tension, which is perpendicular to the arc of the motion. Therefore, the mechanical energy of the system is conserved, as represented by Equation 8.16.2, 0 = Δ(K + U). Because the particle starts from rest, the increase in the kinetic energy is just the kinetic energy at the lowest point. This increase in kinetic energy equals the decrease in the gravitational potential energy, which we can calculate from the geometry. In step 4, we choose a reference point for zero gravitational potential energy to be at the lowest vertical point the particle achieves, which is mid-swing. Lastly, in step 5, we set the sum of energies at the highest point (initial) of the swing to the lowest point (final) of the swing to ultimately solve for the final speed.

Solution

We are neglecting non-conservative forces, so we write the energy conservation formula relating the particle at the highest point (initial) and the lowest point in the swing (final) as

$$K_i + U_i = K_f + U_f. \tag{8.16.3}$$

Since the particle is released from rest, the initial kinetic energy is zero. At the lowest point, we define the gravitational potential energy to be zero. Therefore our conservation of energy formula reduces to

$$\begin{aligned} 0 + mgh &= \frac{1}{2}mv^2 + 0 \\ v &= \sqrt{2gh}. \end{aligned}$$

The vertical height of the particle is not given directly in the problem. This can be solved for by using trigonometry and two givens: the length of the pendulum and the angle through which the particle is vertically pulled up. Looking at the diagram, the vertical dashed line is the length of the pendulum string. The vertical height is labeled h. The other partial length of the vertical string can be calculated with trigonometry. That piece is solved for by

$$\cos\theta = \frac{x}{L} = L\cos\theta. \tag{8.16.4}$$

Therefore, by looking at the two parts of the string, we can solve for the height h,

$$\begin{aligned} x+h &= L \\ L\cos\theta + h &= L \\ h &= L - L\cos\theta \\ &= L(1-\cos\theta). \end{aligned}$$

We substitute this height into the previous expression solved for speed to calculate our result:

$$v = \sqrt{2gL(1-\cos\theta)} = \sqrt{2(9.8\ m/s^2)(1\ m)(1-\cos 30^o)} = 1.62\ m/s. \tag{8.16.5}$$

Significance

We found the speed directly from the conservation of mechanical energy, without having to solve the differential equation for the motion of a pendulum (see Oscillations). We can approach this problem in terms of bar graphs of total energy. Initially, the particle has all potential energy, being at the highest point, and no kinetic energy. When the particle crosses the lowest point at the bottom of the swing, the energy moves from the potential energy column to the kinetic energy column. Therefore, we can imagine a progression of this transfer as the particle moves between its highest point, lowest point of the swing, and back to the highest point (Figure 8.16.2). As the particle travels from the lowest point in the swing to the highest point on the far right hand side of the diagram, the energy bars go in reverse order from (c) to (b) to (a).

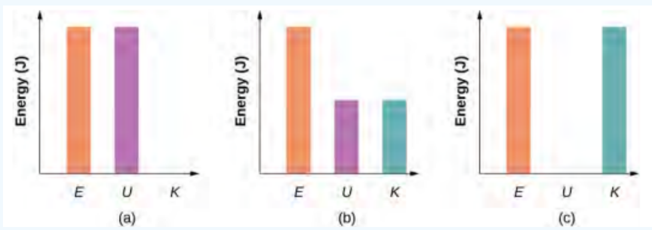


Figure 8.16.2: Bar graphs representing the total energy (E), potential energy (U), and kinetic energy (K) of the particle in different positions. (a) The total energy of the system equals the potential energy and the kinetic energy is zero, which is found at the highest point the particle reaches. (b) The particle is midway between the highest and lowest point, so the kinetic energy plus potential energy bar graphs equal the total energy. (c) The particle is at the lowest point of the swing, so the kinetic energy bar graph is the highest and equal to the total energy of the system.

? Exercise 8.7

How high above the bottom of its arc is the particle in the simple pendulum above, when its speed is 0.81 m/s?

✓ Example 8.8: Air resistance on a falling object

A helicopter is hovering at an altitude of 1 km when a panel from its underside breaks loose and plummets to the ground (Figure 8.16.3). The mass of the panel is 15 kg, and it hits the ground with a speed of 45 m/s. How much mechanical energy was dissipated by air resistance during the panel's descent?

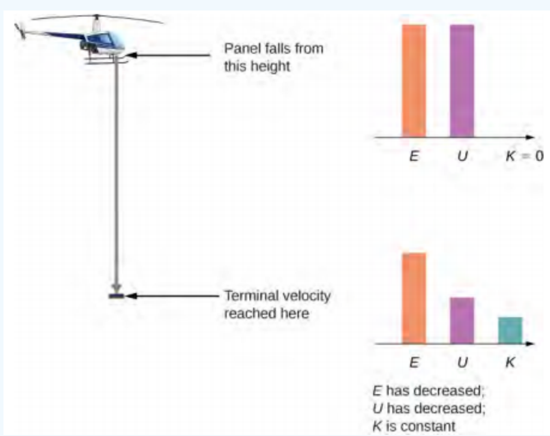


Figure 8.16.3: A helicopter loses a panel that falls until it reaches terminal velocity of 45 m/s. How much did air resistance contribute to the dissipation of energy in this problem?

Strategy

Step 1: Here only one body is being investigated.

Step 2: Gravitational force is acting on the panel, as well as air resistance, which is stated in the problem.

Step 3: Gravitational force is conservative; however, the non-conservative force of air resistance does negative work on the falling panel, so we can use the conservation of mechanical energy, in the form expressed by Equation 8.16.1, to find the energy dissipated. This energy is the magnitude of the work:

$$\Delta E_{diss} = |W_{nc,if}| = |\Delta(K+U)_{if}|. \tag{8.16.6}$$

Step 4: The initial kinetic energy, at yi = 1 km, is zero. We set the gravitational potential energy to zero at ground level out of convenience.

Step 5: The non-conservative work is set equal to the energies to solve for the work dissipated by air resistance.

Solution

The mechanical energy dissipated by air resistance is the algebraic sum of the gain in the kinetic energy and loss in potential energy. Therefore the calculation of this energy is

$$\begin{aligned}\Delta E_{diss} &= |K_f - K_i 9U_f - U_i| \\ &= \left|\frac{1}{2}(15\ kg)(45\ m/s)^2 - 0 + 0 - (15\ kg)(9.8\ m/s^2)(1000\ m)\right| \\ &= 130\ kJ.\end{aligned}$$

Significance

Most of the initial mechanical energy of the panel (U_i), 147 kJ, was lost to air resistance. Notice that we were able to calculate the energy dissipated without knowing what the force of air resistance was, only that it was dissipative.

? Exercise 8.8

You probably recall that, neglecting air resistance, if you throw a projectile straight up, the time it takes to reach its maximum height equals the time it takes to fall from the maximum height back to the starting height. Suppose you cannot neglect air resistance, as in Example 8.8. Is the time the projectile takes to go up (a) greater than, (b) less than, or (c) equal to the time it takes to come back down? Explain.

In these examples, we were able to use conservation of energy to calculate the speed of a particle just at particular points in its motion. But the method of analyzing particle motion, starting from energy conservation, is more powerful than that. More advanced treatments of the theory of mechanics allow you to calculate the full time dependence of a particle's motion, for a given potential energy. In fact, it is often the case that a better model for particle motion is provided by the form of its kinetic and potential energies, rather than an equation for force acting on it. (This is especially true for the quantum mechanical description of particles like electrons or atoms.)

We can illustrate some of the simplest features of this energy-based approach by considering a particle in one-dimensional motion, with potential energy U(x) and no non-conservative interactions present. Equation 8.16.1 and the definition of velocity require

$$K = \frac{1}{2}mv^2 = E - U(x) \tag{8.16.7}$$

$$v = \frac{dx}{dt} = \sqrt{\frac{2(E-U(x))}{m}}. \tag{8.16.8}$$

Separate the variables x and t and integrate, from an initial time t = 0 to an arbitrary time, to get

$$t = \int_0^t dt = \int_{x_0}^x \frac{dx}{\sqrt{\frac{2(E-U(x))}{m}}}. \tag{8.16.9}$$

If you can do the integral in Equation 8.16.9, then you can solve for x as a function of t.

✓ Example 8.9: Constant Acceleration

Use the potential energy $U(x) = -E\left(\frac{x}{x_0}\right)$, for $E > 0$, in Equation 8.16.9 to find the position x of a particle as a function of time t.

Strategy

Since we know how the potential energy changes as a function of x, we can substitute for U(x) in Equation 8.16.9, integrate, and then solve for x. This results in an expression of x as a function of time with constants of energy E, mass m, and the initial position x_0.

Solution

Following the first two suggested steps in the above strategy,

$$t=\int_{x_0}^{x}\frac{dx}{\sqrt{\left(\frac{2E}{mx_0}\right)(x_0-x)}}=\frac{1}{\sqrt{\left(\frac{2E}{mx_0}\right)}}\left|-2\sqrt{(x_0-x)}\right|_{x_0}^{x}=\frac{-2\sqrt{(x_0-x)}}{\sqrt{\left(\frac{2E}{mx_0}\right)}}. \tag{8.16.10}$$

Solving for the position, we obtain

$$x(t)=x_0-\frac{1}{2}\left(\frac{E}{mx_0}\right)t^2. \tag{8.16.11}$$

Significance

The position as a function of time, for this potential, represents one-dimensional motion with constant acceleration, $a = \left(\frac{E}{mx_0}\right)$, starting at rest from position x_0. This is not so surprising, since this is a potential energy for a constant force, $F = -\frac{dU}{dx} = \frac{E}{x_0}$, and $a = \frac{F}{m}$.

? Exercise 8.9

What potential energy U(x) can you substitute in Equation 8.16.2 that will result in motion with constant velocity of 2 m/s for a particle of mass 1 kg and mechanical energy 1 J?

We will look at another more physically appropriate example of the use of Equation 8.16.2 after we have explored some further implications that can be drawn from the functional form of a particle's potential energy.

Systems with Several Particles or Objects

Systems generally consist of more than one particle or object. However, the conservation of mechanical energy, in one of the forms in Equation 8.16.1 or Equation 8.16.2, is a fundamental law of physics and applies to any system. You just have to include the kinetic and potential energies of all the particles, and the work done by all the non-conservative forces acting on them. Until you learn more about the dynamics of systems composed of many particles, in Linear Momentum and Collisions, Fixed-Axis Rotation, and Angular Momentum, it is better to postpone discussing the application of energy conservation to then.

8.17: Potential Energy Diagrams and Stability

Learning Objectives

- Create and interpret graphs of potential energy
- Explain the connection between stability and potential energy

Often, you can get a good deal of useful information about the dynamical behavior of a mechanical system just by interpreting a graph of its potential energy as a function of position, called a **potential energy diagram**. This is most easily accomplished for a one-dimensional system, whose potential energy can be plotted in one two-dimensional graph—for example, U(x) versus x—on a piece of paper or a computer program. For systems whose motion is in more than one dimension, the motion needs to be studied in three-dimensional space. We will simplify our procedure for one-dimensional motion only.

First, let's look at an object, freely falling vertically, near the surface of Earth, in the absence of air resistance. The mechanical energy of the object is conserved, E = K + U, and the potential energy, with respect to zero at ground level, is U(y) = mgy, which is a straight line through the origin with slope mg . In the graph shown in Figure $8.17.1$, the x-axis is the height above the ground y and the y-axis is the object's energy.

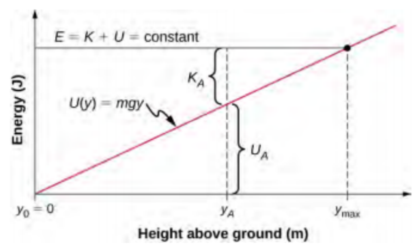


Figure 8.17.1: The potential energy graph for an object in vertical free fall, with various quantities indicated.

The line at energy E represents the constant mechanical energy of the object, whereas the kinetic and potential energies, K_A and U_A, are indicated at a particular height y_A. You can see how the total energy is divided between kinetic and potential energy as the object's height changes. Since kinetic energy can never be negative, there is a maximum potential energy and a maximum height, which an object with the given total energy cannot exceed:

$$K = E - U \geq 0, \tag{8.17.1}$$

$$U \leq E. \tag{8.17.2}$$

If we use the gravitational potential energy reference point of zero at y_0, we can rewrite the gravitational potential energy U as mgy. Solving for y results in

$$y \leq \frac{E}{mg} = y_{max}. \tag{8.17.3}$$

We note in this expression that the quantity of the total energy divided by the weight (mg) is located at the maximum height of the particle, or y_{max}. At the maximum height, the kinetic energy and the speed are zero, so if the object were initially traveling upward, its velocity would go through zero there, and y_{max}would be a turning point in the motion. At ground level, $y_0 = 0$, the potential energy is zero, and the kinetic energy and the speed are maximum:

$$U_0 = 0 = E - K_0, \tag{8.17.4}$$

$$E = K_0 = \frac{1}{2}mv_0^2, \tag{8.17.5}$$

$$v_0 = \pm\sqrt{\frac{2E}{m}}. \tag{8.17.6}$$

The maximum speed $\pm v_0$ gives the initial velocity necessary to reach y_{max}, the maximum height, and $-v_0$ represents the final velocity, after falling from y_{max}. You can read all this information, and more, from the potential energy diagram we have shown.

Consider a mass-spring system on a frictionless, stationary, horizontal surface, so that gravity and the normal contact force do no work and can be ignored (Figure $8.17.2$). This is like a one-dimensional system, whose mechanical energy E is a constant and whose potential energy, with respect to zero energy at zero displacement from the spring's unstretched length, x = 0, is $U(x) = \frac{1}{2}kx^2$.

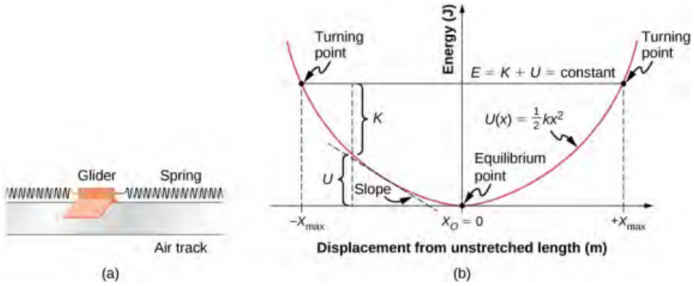


Figure 8.17.2: (a) A glider between springs on an air track is an example of a horizontal mass-spring system. (b) The potential energy diagram for this system, with various quantities indicated.

You can read off the same type of information from the potential energy diagram in this case, as in the case for the body in vertical free fall, but since the spring potential energy describes a variable force, you can learn more from this graph. As for the object in vertical free fall, you can deduce the physically allowable range of motion and the maximum values of distance and speed, from the limits on the kinetic energy, $0 \leq K \leq E$. Therefore, K = 0 and U = E at a **turning point**, of which there are two for the elastic spring potential energy,

$$x_{max} = \pm\sqrt{\frac{2E}{k}}. \tag{8.17.7}$$

The glider's motion is confined to the region between the turning points, $-x_{max} \leq x \leq x_{max}$. This is true for any (positive) value of E because the potential energy is unbounded with respect to x. For this reason, as well as the shape of the potential energy curve, U(x) is called an infinite potential well. At the bottom of the potential well, x = 0, U = 0 and the kinetic energy is a maximum, K = E, so $v_{max} = \pm\sqrt{\frac{2E}{m}}$.

However, from the slope of this potential energy curve, you can also deduce information about the force on the glider and its acceleration. We saw earlier that the negative of the slope of the potential energy is the spring force, which in this case is also the net force, and thus is proportional to the acceleration. When x = 0, the slope, the force, and the acceleration are all zero, so this is an **equilibrium point**. The negative of the slope, on either side of the equilibrium point, gives a force pointing back to the equilibrium point, F = ±kx, so the equilibrium is termed stable and the force is called a restoring force. This implies that U(x) has a relative minimum there. If the force on either side of an equilibrium point has a direction opposite from that direction of position change, the equilibrium is termed unstable, and this implies that U(x) has a relative maximum there.

✓ Example 8.10: Quartic and Quadratic Potential Energy Diagram

The potential energy for a particle undergoing one-dimensional motion along the x-axis is $U(x) = 2(x^4 - x^2)$, where U is in joules and x is in meters. The particle is not subject to any non-conservative forces and its mechanical energy is constant at E = −0.25 J. (a) Is the motion of the particle confined to any regions on the x-axis, and if so, what are they? (b) Are there any equilibrium points, and if so, where are they and are they stable or unstable?

Strategy

First, we need to graph the potential energy as a function of x. The function is zero at the origin, becomes negative as x increases in the positive or negative directions (x^2 is larger than x^4 for x < 1), and then becomes positive at sufficiently large |x|. Your graph should look like a double potential well, with the zeros determined by solving the equation U(x) = 0, and the extremes determined by examining the first and second derivatives of U(x), as shown in Figure 8.17.3.

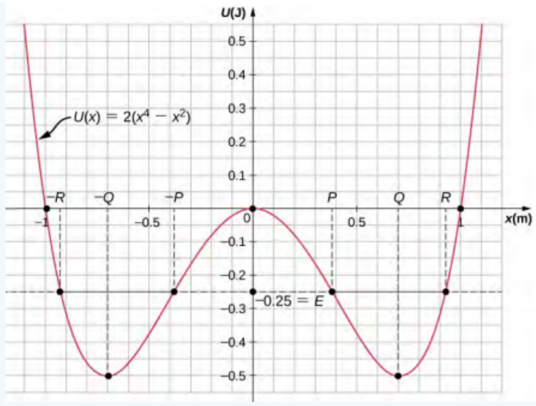


Figure 8.17.3: The potential energy graph for a one-dimensional, quartic and quadratic potential energy, with various quantities indicated.

You can find the values of (a) the allowed regions along the x-axis, for the given value of the mechanical energy, from the condition that the kinetic energy can't be negative, and (b) the equilibrium points and their stability from the properties of the force (stable for a relative minimum and unstable for a relative maximum of potential energy). You can just eyeball the graph to reach qualitative answers to the questions in this example. That, after all, is the value of potential energy diagrams.

You can see that there are two allowed regions for the motion (E > U) and three equilibrium points (slope $\frac{dU}{dx}$ = 0), of which the central one is unstable $\left(\frac{d^2U}{dx^2} < 0\right)$, and the other two are stable $\left(\frac{d^2U}{dx^2} > 0\right)$.

Solution

a. To find the allowed regions for x, we use the condition

$$K = E - U = -\frac{1}{4} - 2(x^4 - x^2) \geq 0. \tag{8.17.8}$$

If we complete the square in x 2 , this condition simplifies to $2\left(x^2 - \frac{1}{2}\right)^2 \leq \frac{1}{4}$, which we can solve to obtain

$$\frac{1}{2} - \sqrt{\frac{1}{8}} \leq x^2 \leq \frac{1}{2} + \sqrt{\frac{1}{8}}. \tag{8.17.9}$$

This represents two allowed regions, $x_p \leq x \leq x_R$ and $-x_R \leq x \leq -x_p$, where $x_p = 0.38$ and $x_R = 0.92$ (in meters).

b. To find the equilibrium points, we solve the equation

$$\frac{dU}{dx} = 8x^3 - 4x = 0 \tag{8.17.10}$$

and find x = 0 and $x = \pm x_Q$, where $x_Q = \frac{1}{\sqrt{2}} = 0.707$ (meters). The second derivative

$$\frac{d^2U}{dx^2} = 24x^2 - 4 \tag{8.17.11}$$

is negative at x = 0, so that position is a relative maximum and the equilibrium there is unstable. The second derivative is positive at $x = \pm x_Q$, so these positions are relative minima and represent stable equilibria.

Significance

The particle in this example can oscillate in the allowed region about either of the two stable equilibrium points we found, but it does not have enough energy to escape from whichever potential well it happens to initially be in. The conservation of mechanical energy and the relations between kinetic energy and speed, and potential energy and force, enable you to deduce much information about the qualitative behavior of the motion of a particle, as well as some quantitative information, from a graph of its potential energy.

? Exercise 8.10

Repeat Example 8.10 when the particle's mechanical energy is +0.25 J.

Before ending this section, let's practice applying the method based on the potential energy of a particle to find its position as a function of time, for the one-dimensional, mass-spring system considered earlier in this section.

✔ Example 8.11: Sinusoidal Oscillations

Find x(t) for a particle moving with a constant mechanical energy E > 0 and a potential energy $U(x) = \frac{1}{2}kx^2$, when the particle starts from rest at time t = 0.

Strategy

We follow the same steps as we did in Example 8.9. Substitute the potential energy U into Equation 8.4.9 and factor out the constants, like m or k. Integrate the function and solve the resulting expression for position, which is now a function of time.

Solution

Substitute the potential energy in Equation 8.4.9 and integrate using an integral solver found on a web search:

\[t = \int_{x_{0}}^{x} \frac{dx}{\sqrt{\left(\dfrac{k}{m}\right) \Big[\left(\dfrac{2E}{k}\right) - x^{2} \Big]}} = \sqrt{\frac{m}{k}} \Bigg[\sin^{-1} \left(\dfrac{x}{\sqrt{\frac{2E}{k}}}\right) - \sin^{-1} \left(\frac{x_{0}}{\sqrt{\frac{2E}{k}}}\right) \Bigg] \ldotp$$From the initial conditions at t = 0, the initial kinetic energy is zero and the initial potential energy is $\frac{1}{2}kx_0^2 = E$, from which you can see that $\frac{x_0}{\sqrt{\left(\frac{2E}{k}\right)}} = \pm 1$ and $\sin^{-1}(\pm) = \pm 90°$. Now you can solve for x:

$$x(t) = \sqrt{\left(\frac{2E}{k}\right)} \sin\left[\left(\sqrt{\frac{k}{m}}\right) t \pm 90^o\right] = \pm\sqrt{\left(\frac{2E}{k}\right)} \cos\left[\left(\sqrt{\frac{k}{m}}\right) t\right]. \tag{8.17.12}$$

Significance

A few paragraphs earlier, we referred to this mass-spring system as an example of a harmonic oscillator. Here, we anticipate that a harmonic oscillator executes sinusoidal oscillations with a maximum displacement of $\sqrt{\left(\frac{2E}{k}\right)}$ (called the amplitude) and a rate of oscillation of $\left(\frac{1}{2\pi}\right)\sqrt{\frac{k}{m}}$ (called the frequency). Further discussions about oscillations can be found in Oscillations.

? Exercise 8.11

Find x(t) for the mass-spring system in Example 8.11 if the particle starts from $x_0 = 0$ at t = 0. What is the particle's initial velocity?

8.18: Sources of Energy

Learning Objectives

- Describe energy transformations and conversions in general terms
- Explain what it means for an energy source to be renewable or nonrenewable

In this section, we have studied energy. We learned that energy can take different forms and can be transferred from one form to another. You will find that energy is discussed in many everyday, as well as scientific, contexts, because it is involved in all physical processes. It will also become apparent that many situations are best understood, or most easily conceptualized, by considering energy. So far, no experimental results have contradicted the conservation of energy. In fact, whenever measurements have appeared to conflict with energy conservation, new forms of energy have been discovered or recognized in accordance with this principle.

What are some other forms of energy? Many of these are covered in later chapters (also see Figure 8.18.1), but let's detail a few here:

- Atoms and molecules inside all objects are in random motion. The internal kinetic energy from these random motions is called **thermal energy**, because it is related to the temperature of the object. Note that thermal energy can also be transferred from one place to another, not transformed or converted, by the familiar processes of conduction, convection, and radiation. In this case, the energy is known as **heat energy**.
- **Electrical energy** is a common form that is converted to many other forms and does work in a wide range of practical situations.
- Fuels, such as gasoline and food, have **chemical energy**, which is potential energy arising from their molecular structure. Chemical energy can be converted into thermal energy by reactions like oxidation. Chemical reactions can also produce electrical energy, such as in batteries. Electrical energy can, in turn, produce thermal energy and light, such as in an electric heater or a light bulb.
- Light is just one kind of electromagnetic radiation, or **radiant energy**, which also includes radio, infrared, ultraviolet, X-rays, and gamma rays. All bodies with thermal energy can radiate energy in electromagnetic waves.
- **Nuclear energy** comes from reactions and processes that convert measurable amounts of mass into energy. Nuclear energy is transformed into radiant energy in the Sun, into thermal energy in the boilers of nuclear power plants, and then into electrical energy in the generators of power plants. These and all other forms of energy can be transformed into one another and, to a certain degree, can be converted into mechanical work.

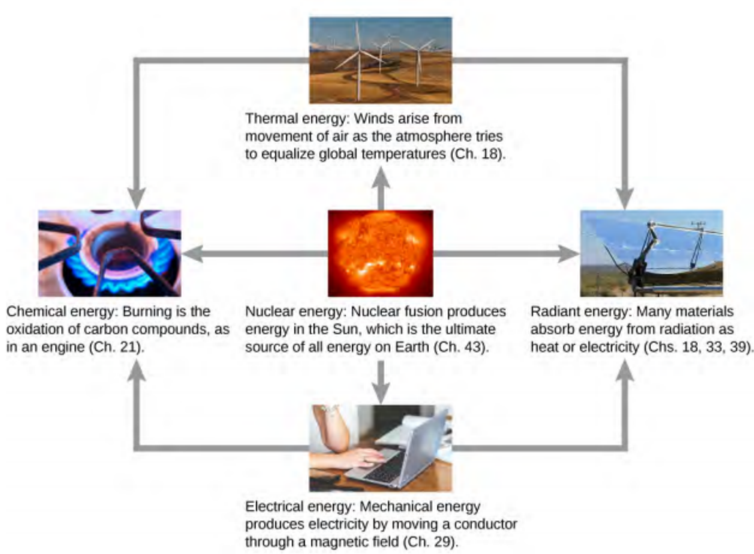


Figure 8.18.1: Energy that we use in society takes many forms, which be converted from one into another depending on the process involved. We will study many of these forms of energy in later chapters in this text. (credit "sun": EIT SOHO Consortium, ESA, NASA; credit "solar panels": "kjkolb"/Wikimedia Commons; credit "gas burner": Steven Depolo)

The transformation of energy from one form into another happens all the time. The chemical energy in food is converted into thermal energy through metabolism; light energy is converted into chemical energy through photosynthesis. Another example of energy conversion occurs in a solar cell. Sunlight impinging on a solar cell produces electricity, which can be used to run electric motors or heat water. In an example encompassing many steps, the chemical energy contained in coal is converted into thermal energy as it burns in a furnace, to transform water into steam, in a boiler. Some of the thermal energy in the steam is then converted into mechanical energy as it expands and spins a turbine, which is connected to a generator to produce electrical energy. In these examples, not all of the initial energy is converted into the forms mentioned, because some energy is always transferred to the environment.

Energy is an important element at all levels of society. We live in a very interdependent world, and access to adequate and reliable energy resources is crucial for economic growth and for maintaining the quality of our lives. The principal energy resources used in the world are shown in Figure 8.18.2. The figure distinguishes between two major types of energy sources: **renewable** and **non-renewable**, and further divides each type into a few more specific kinds. Renewable sources are energy sources that are replenished through naturally occurring, ongoing processes, on a time scale that is much shorter than the anticipated lifetime of the civilization using the source. Non-renewable sources are depleted once some of the energy they contain is extracted and converted into other kinds of energy. The natural processes by which non-renewable sources are formed typically take place over geological time scales.

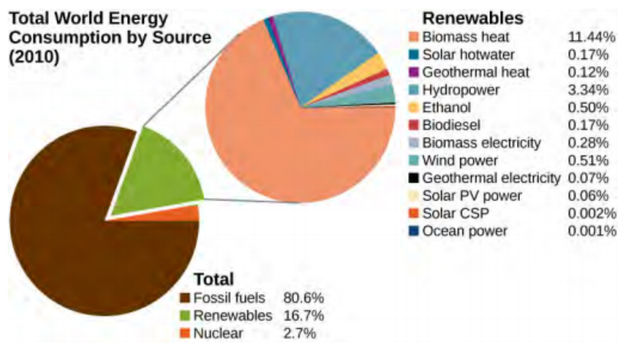


Figure 8.18.2: World energy consumption by source; the percentage of renewables is increasing, accounting for 19% in 2012.

Our most important non-renewable energy sources are fossil fuels, such as coal, petroleum, and natural gas. These account for about 81% of the world's energy consumption, as shown in the figure. Burning fossil fuels creates chemical reactions that transform potential energy, in the molecular structures of the reactants, into thermal energy and products. This thermal energy can be used to heat buildings or to operate steam-driven machinery. Internal combustion and jet engines convert some of the energy of rapidly expanding gases, released from burning gasoline, into mechanical work. Electrical power generation is mostly derived from transferring energy in expanding steam, via turbines, into mechanical work, which rotates coils of wire in magnetic fields to generate electricity. Nuclear energy is the other non-renewable source shown in Figure 8.18.2 and supplies about 3% of the world's consumption. Nuclear reactions release energy by transforming potential energy, in the structure of nuclei, into thermal energy, analogous to energy release in chemical reactions. The thermal energy obtained from nuclear reactions can be transferred and converted into other forms in the same ways that energy from fossil fuels are used.

An unfortunate byproduct of relying on energy produced from the combustion of fossil fuels is the release of carbon dioxide into the atmosphere and its contribution to global warming. Nuclear energy poses environmental problems as well, including the safety and disposal of nuclear waste. Besides these important consequences, reserves of non-renewable sources of energy are limited and, given the rapidly growing rate of world energy consumption, may not last for more than a few hundred years. Considerable effort is going on to develop and expand the use of renewable sources of energy, involving a significant percentage of the world's physicists and engineers.

Four of the renewable energy sources listed in Figure 8.18.2—those using material from plants as fuel (biomass heat, ethanol, biodiesel, and biomass electricity)—involve the same types of energy transformations and conversions as just discussed for fossil and nuclear fuels. The other major types of renewable energy sources are hydropower, wind power, geothermal power, and solar power.

Hydropower is produced by converting the gravitational potential energy of falling or flowing water into kinetic energy and then into work to run electric generators or machinery. Converting the mechanical energy in ocean surface waves and tides is in development. Wind power also converts kinetic energy into work, which can be used directly to generate electricity, operate mills, and propel sailboats.

The interior of Earth has a great deal of thermal energy, part of which is left over from its original formation (gravitational potential energy converted into thermal energy) and part of which is released from radioactive minerals (a form of natural nuclear energy). It will take a very long time for this geothermal energy to escape into space, so people generally regard it as a renewable source, when actually, it's just inexhaustible on human time scales.

The source of solar power is energy carried by the electromagnetic waves radiated by the Sun. Most of this energy is carried by visible light and infrared (heat) radiation. When suitable materials absorb electromagnetic waves, radiant energy is converted into thermal energy, which can be used to heat water, or when concentrated, to make steam and generate electricity (Figure $8.18.3$). However, in another important physical process, known as the photoelectric effect, energetic radiation impinging on certain materials is directly converted into electricity. Materials that do this are called photovoltaics (PV in Figure $8.18.2$). Some solar power systems use lenses or mirrors to concentrate the Sun's rays, before converting their energy through photovoltaics, and these are qualified as CSP in Figure $8.18.2$.

Figure $8.18.3$: Solar cell arrays found in a sunny area converting the solar energy into stored electrical energy. (credit: Sarah Swenty)

As we finish this chapter on energy and work, it is relevant to draw some distinctions between two sometimes misunderstood terms in the area of energy use. As we mentioned earlier, the "law of conservation of energy" is a very useful principle in analyzing physical processes. It cannot be proven from basic principles but is a very good bookkeeping device, and no exceptions have ever been found. It states that the total amount of energy in an isolated system always remains constant. Related to this principle, but remarkably different from it, is the important philosophy of energy conservation. This concept has to do with seeking to decrease the amount of energy used by an individual or group through reducing activities (e.g., turning down thermostats, diving fewer kilometers) and/or increasing conversion efficiencies in the performance of a particular task, such as developing and using more efficient room heaters, cars that have greater miles-per-gallon ratings, energy-efficient compact fluorescent lights, etc.

Since energy in an isolated system is not destroyed, created, or generated, you might wonder why we need to be concerned about our energy resources, since energy is a conserved quantity. The problem is that the final result of most energy transformations is waste heat, that is, work that has been "degraded" in the energy transformation. We will discuss this idea in more detail in the chapters on thermodynamics.

8.19: Potential Energy and Conservation of Energy (Exercises)

Conceptual Questions

8.1 Potential Energy of a System

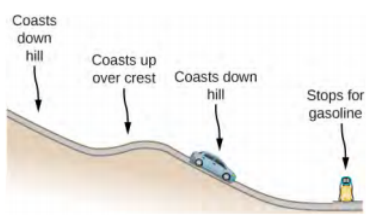
1. The kinetic energy of a system must always be positive or zero. Explain whether this is true for the potential energy of a system.
2. The force exerted by a diving board is conservative, provided the internal friction is negligible. Assuming friction is negligible, describe changes in the potential energy of a diving board as a swimmer drives from it, starting just before the swimmer steps on the board until just after his feet leave it.
3. Describe the gravitational potential energy transfers and transformations for a javelin, starting from the point at which an athlete picks up the javelin and ending when the javelin is stuck into the ground after being thrown.
4. A couple of soccer balls of equal mass are kicked off the ground at the same speed but at different angles. Soccer ball A is kicked off at an angle slightly above the horizontal, whereas ball B is kicked slightly below the vertical. How do each of the following compare for ball A and ball B? (a) The initial kinetic energy and (b) the change in gravitational potential energy from the ground to the highest point? If the energy in part (a) differs from part (b), explain why there is a difference between the two energies.
5. What is the dominant factor that affects the speed of an object that started from rest down a frictionless incline if the only work done on the object is from gravitational forces?
6. Two people observe a leaf falling from a tree. One person is standing on a ladder and the other is on the ground. If each person were to compare the energy of the leaf observed, would each person find the following to be the same or different for the leaf, from the point where it falls off the tree to when it hits the ground: (a) the kinetic energy of the leaf; (b) the change in gravitational potential energy; (c) the final gravitational potential energy?

8.2 Conservative and Non-Conservative Forces

7. What is the physical meaning of a non-conservative force?
8. A bottle rocket is shot straight up in the air with a speed 30 m/s. If the air resistance is ignored, the bottle would go up to a height of approximately 46 m. However, the rocket goes up to only 35 m before returning to the ground. What happened? Explain, giving only a qualitative response.
9. An external force acts on a particle during a trip from one point to another and back to that same point. This particle is only effected by conservative forces. Does this particle's kinetic energy and potential energy change as a result of this trip?

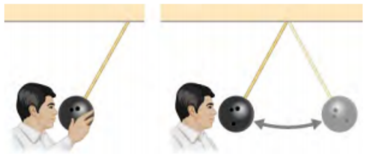
8.3 Conservation of Energy

10. When a body slides down an inclined plane, does the work of friction depend on the body's initial speed? Answer the same question for a body sliding down a curved surface.
11. Consider the following scenario. A car for which friction is not negligible accelerates from rest down a hill, running out of gasoline after a short distance (see below). The driver lets the car coast farther down the hill, then up and over a small crest. He then coasts down that hill into a gas station, where he brakes to a stop and fills the tank with gasoline. Identify the forms of energy the car has, and how they are changed and transferred in this series of events.



12. A dropped ball bounces to one-half its original height. Discuss the energy transformations that take place.
13. " E = K + U constant is a special case of the work-energy theorem." Discuss this statement.
14. In a common physics demonstration, a bowling ball is suspended from the ceiling by a rope. The professor pulls the ball away from its equilibrium position and holds it adjacent to his nose, as shown below. He releases the ball so that it swings

directly away from him. Does he get struck by the ball on its return swing? What is he trying to show in this demonstration?



15. A child jumps up and down on a bed, reaching a higher height after each bounce. Explain how the child can increase his maximum gravitational potential energy with each bounce.
16. Can a non-conservative force increase the mechanical energy of the system?
17. Neglecting air resistance, how much would I have to raise the vertical height if I wanted to double the impact speed of a falling object?
18. A box is dropped onto a spring at its equilibrium position. The spring compresses with the box attached and comes to rest. Since the spring is in the vertical position, does the change in the gravitational potential energy of the box while the spring is compressing need to be considered in this problem?

Problems

8.1 Potential Energy of a System

19. Using values from Table 8.2, how many DNA molecules could be broken by the energy carried by a single electron in the beam of an old-fashioned TV tube? (These electrons were not dangerous in themselves, but they did create dangerous X-rays. Later-model tube TVs had shielding that absorbed X-rays before they escaped and exposed viewers.)
20. If the energy in fusion bombs were used to supply the energy needs of the world, how many of the 9-megaton variety would be needed for a year's supply of energy (using data from Table 8.1)?
21. A camera weighing 10 N falls from a small drone hovering 20 m overhead and enters free fall. What is the gravitational potential energy change of the camera from the drone to the ground if you take a reference point of (a) the ground being zero gravitational potential energy? (b) The drone being zero gravitational potential energy? What is the gravitational potential energy of the camera (c) before it falls from the drone and (d) after the camera lands on the ground if the reference point of zero gravitational potential energy is taken to be a second person looking out of a building 30 m from the ground?
22. Someone drops a $50 - \mathrm{g}$ pebble off of a docked cruise ship, 70.0 m from the water line. A person on a dock 3.0 m from the water line holds out a net to catch the pebble. (a) How much work is done on the pebble by gravity during the drop? (b) What is the change in the gravitational potential energy during the drop? If the gravitational potential energy is zero at the water line, what is the gravitational potential energy (c) when the pebble is dropped? (d) When it reaches the net? What if the gravitational potential energy was 30.0 Joules at water level? (e) Find the answers to the same questions in (c) and (d).
23. A cat's crinkle ball toy of mass 15 g is thrown straight up with an initial speed of 3 m/s. Assume in this problem that air drag is negligible. (a) What is the kinetic energy of the ball as it leaves the hand? (b) How much work is done by the gravitational force during the ball's rise to its peak? (c) What is the change in the gravitational potential energy of the ball during the rise to its peak? (d) If the gravitational potential energy is taken to be zero at the point where it leaves your hand, what is the gravitational potential energy when it reaches the maximum height? (e) What if the gravitational potential energy is taken to be zero at the maximum height the ball reaches, what would the gravitational potential energy be when it leaves the hand? (f) What is the maximum height the ball reaches?

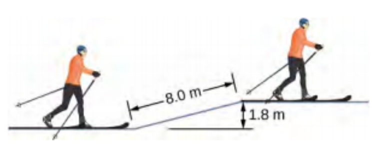
8.2 Conservative and Non-Conservative Forces

24. A force $F(x) = (3.0/x)$ N acts on a particle as it moves along the positive x-axis. (a) How much work does the force do on the particle as it moves from $x = 2.0$ m to $x = 5.0$ m? (b) Picking a convenient reference point of the potential energy to be zero at $x = \infty$, find the potential energy for this force.
25. A force $F(x) = (-5.0x^2 + 7.0x)$ N acts on a particle. (a) How much work does the force do on the particle as it moves from $x = 2.0$ m to $x = 5.0$ m? (b) Picking a convenient reference point of the potential energy to be zero at $x = \infty$, find the potential energy for this force.
26. Find the force corresponding to the potential energy $U(x) = -\frac{a}{x} + \frac{b}{x^2}$.

27. The potential energy function for either one of the two atoms in a diatomic molecule is often approximated by U(x) = $-\frac{a}{x^{12}} - \frac{b}{x^6}$ where x is the distance between the atoms. (a) At what distance of separation does the potential energy have a local minimum (not at x = ∞)? (b) What is the force on an atom at this separation? (c) How does the force vary with the separation distance?
28. A particle of mass 2.0 kg moves under the influence of the force F(x) = $\left(\frac{3}{\sqrt{x}}\right)$ N. If its speed at x = 2.0 m is v = 6.0 m/s, what is its speed at x = 7.0 m?
29. A particle of mass 2.0 kg moves under the influence of the force F(x) = $(-5x^2 + 7x)$ N. If its speed at x = −4.0 m is v = 20.0 m/s, what is its speed at x = 4.0 m?
30. A crate on rollers is being pushed without frictional loss of energy across the floor of a freight car (see the following figure). The car is moving to the right with a constant speed v0 . If the crate starts at rest relative to the freight car, then from the work-energy theorem, Fd = $\frac{mv^2}{2}$, where d, the distance the crate moves, and v, the speed of the crate, are both measured relative to the freight car. (a) To an observer at rest beside the tracks, what distance d′ is the crate pushed when it moves the distance d in the car? (b) What are the crate's initial and final speeds v_0' and v′ as measured by the observer beside the tracks? (c) Show that Fd′ = $\frac{m(v')^2}{2} - \frac{m(v_0')^2}{2}$ and, consequently, that work is equal to the change in kinetic energy in both reference systems.

8.3 Conservation of Energy

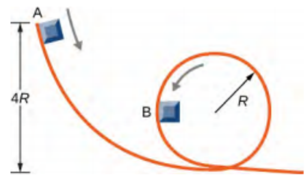
31. A boy throws a ball of mass 0.25 kg straight upward with an initial speed of 20 m/s When the ball returns to the boy, its speed is 17 m/s How much much work does air resistance do on the ball during its flight?
32. A mouse of mass 200 g falls 100 m down a vertical mine shaft and lands at the bottom with a speed of 8.0 m/s. During its fall, how much work is done on the mouse by air resistance?
33. Using energy considerations and assuming negligible air resistance, show that a rock thrown from a bridge 20.0 m above water with an initial speed of 15.0 m/s strikes the water with a speed of 24.8 m/s independent of the direction thrown. (**Hint**: show that $K_i + U_i = K_f + U_f$)
34. A 1.0-kg ball at the end of a 2.0-m string swings in a vertical plane. At its lowest point the ball is moving with a speed of 10 m/s. (a) What is its speed at the top of its path? (b) What is the tension in the string when the ball is at the bottom and at the top of its path?
35. Ignoring details associated with friction, extra forces exerted by arm and leg muscles, and other factors, we can consider a pole vault as the conversion of an athlete's running kinetic energy to gravitational potential energy. If an athlete is to lift his body 4.8 m during a vault, what speed must he have when he plants his pole?
36. Tarzan grabs a vine hanging vertically from a tall tree when he is running at 9.0 m/s. (a) How high can he swing upward? (b) Does the length of the vine affect this height?
37. Assume that the force of a bow on an arrow behaves like the spring force. In aiming the arrow, an archer pulls the bow back 50 cm and holds it in position with a force of 150 N. If the mass of the arrow is 50 g and the "spring" is massless, what is the speed of the arrow immediately after it leaves the bow?
38. A 100 − kg man is skiing across level ground at a speed of 8.0 m/s when he comes to the small slope 1.8 m higher than ground level shown in the following figure. (a) If the skier coasts up the hill, what is his speed when he reaches the top plateau? Assume friction between the snow and skis is negligible. (b) What is his speed when he reaches the upper level if an 80 − N frictional force acts on the skis?



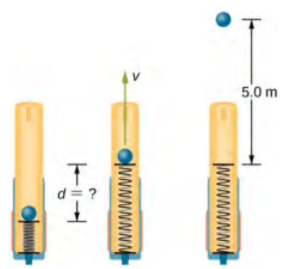
39. A sled of mass 70 kg starts from rest and slides down a 10° incline 80 m long. It then travels for 20 m horizontally before starting back up an 8° incline. It travels 80 m along this incline before coming to rest. What is the net work done on the

sled by friction?

40. A girl on a skateboard (total mass of 40 kg) is moving at a speed of 10 m/s at the bottom of a long ramp. The ramp is inclined at 20° with respect to the horizontal. If she travels 14.2 m upward along the ramp before stopping, what is the net frictional force on her?
41. A baseball of mass 0.25 kg is hit at home plate with a speed of 40 m/s. When it lands in a seat in the left-field bleachers a horizontal distance 120 m from home plate, it is moving at 30 m/s. If the ball lands 20 m above the spot where it was hit, how much work is done on it by air resistance?
42. A small block of mass m slides without friction around the loop-the-loop apparatus shown below. (a) If the block starts from rest at A, what is its speed at B? (b) What is the force of the track on the block at B?



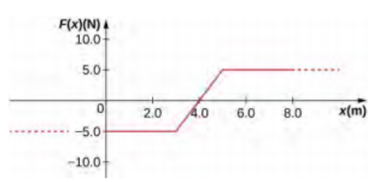
43. The massless spring of a spring gun has a force constant k = 12 N/cm. When the gun is aimed vertically, a 15-g projectile is shot to a height of 5.0 m above the end of the expanded spring. (See below.) How much was the spring compressed initially?



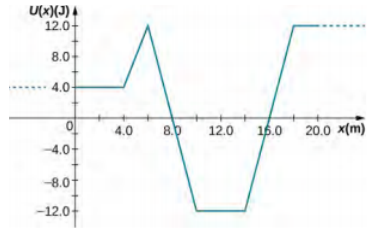
44. A small ball is tied to a string and set rotating with negligible friction in a vertical circle. If the ball moves over the top of the circle at its slowest possible speed (so that the tension in the string is negligible), what is the tension in the string at the bottom of the circle, assuming there is no additional energy added to the ball during rotation?

8.4 Potential Energy Diagrams and Stability

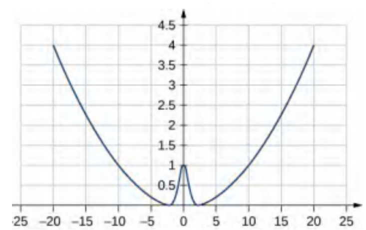
45. A mysterious constant force of 10 N acts horizontally on everything. The direction of the force is found to be always pointed toward a wall in a big hall. Find the potential energy of a particle due to this force when it is at a distance x from the wall, assuming the potential energy at the wall to be zero.
46. A single force F(x) = −4.0x (in newtons) acts on a 1.0-kg body. When x = 3.5 m, the speed of the body is 4.0 m/s. What is its speed at x = 2.0 m?
47. A particle of mass 4.0 kg is constrained to move along the x-axis under a single force $F(x) = -cx^3$, where $c = 8.0\ N/m^3$. The particle's speed at A, where $x_A = 1.0$ m, is 6.0 m/s. What is its speed at B, where $x_B = -2.0$ m?
48. The force on a particle of mass 2.0 kg varies with position according to $F(x) = -3.0x^2$ (x in meters, F(x) in newtons). The particle's velocity at x = 2.0 m is 5.0 m/s. Calculate the mechanical energy of the particle using (a) the origin as the reference point and (b) x = 4.0 m as the reference point. (c) Find the particle's velocity at x = 1.0 m. Do this part of the problem for each reference point.
49. A 4.0-kg particle moving along the x-axis is acted upon by the force whose functional form appears below. The velocity of the particle at x = 0 is v = 6.0 m/s. Find the particle's speed at x = (a) 2.0 m, (b) 4.0 m, (c) 10.0 m, (d) Does the particle turn around at some point and head back toward the origin? (e) Repeat part (d) if v = 2.0 m/s at x = 0.



50. A particle of mass 0.50 kg moves along the x-axis with a potential energy whose dependence on x is shown below. (a) What is the force on the particle at x = 2.0, 5.0, 8.0, and 12 m? (b) If the total mechanical energy E of the particle is −6.0 J, what are the minimum and maximum positions of the particle? (c) What are these positions if E = 2.0 J? (d) If E = 16 J, what are the speeds of the particle at the positions listed in part (a)?



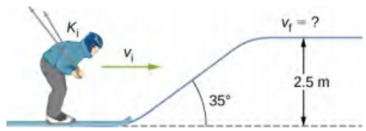
51. (a) Sketch a graph of the potential energy function U(x) = $\frac{kx^2}{2} + Ae^{-\alpha x^2}$, where k, A, and α are constants. (b) What is the force corresponding to this potential energy? (c) Suppose a particle of mass m moving with this potential energy has a velocity v_a when its position is x = a. Show that the particle does not pass through the origin unless $A \leq \frac{mv_a^2 + ka^2}{2\left(1-e^{-\alpha a^2}\right)}$.



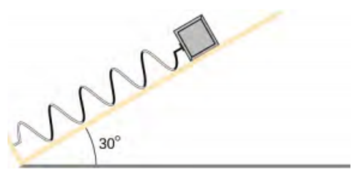
8.5 Sources of Energy

52. In the cartoon movie Pocahontas (https://openstaxcollege.org/l/21pocahontclip), Pocahontas runs to the edge of a cliff and jumps off, showcasing the fun side of her personality. (a) If she is running at 3.0 m/s before jumping off the cliff and she hits the water at the bottom of the cliff at 20.0 m/s, how high is the cliff? Assume negligible air drag in this cartoon. (b) If she jumped off the same cliff from a standstill, how fast would she be falling right before she hit the water?
53. In the reality television show “Amazing Race” (https://openstaxcollege.org/l/21amazraceclip), a contestant is firing 12-kg watermelons from a slingshot to hit targets down the field. The slingshot is pulled back 1.5 m and the watermelon is considered to be at ground level. The launch point is 0.3 m from the ground and the targets are 10 m horizontally away. Calculate the spring constant of the slingshot.
54. In the Back to the Future movies (https://openstaxcollege.org/l/21bactofutclip), a DeLorean car of mass 1230 kg travels at 88 miles per hour to venture back to the future. (a) What is the kinetic energy of the DeLorian? (b) What spring constant would be needed to stop this DeLorean in a distance of 0.1m?
55. In the Hunger Games movie (https://openstaxcollege.org/l/21HungGamesclip), Katniss Everdeen fires a 0.0200-kg arrow from ground level to pierce an apple up on a stage. The spring constant of the bow is 330 N/m and she pulls the arrow back a distance of 0.55 m. The apple on the stage is 5.00 m higher than the launching point of the arrow. At what speed does the arrow (a) leave the bow? (b) strike the apple?
56. In a “Top Fail” video (https://openstaxcollege.org/l/21topfailvideo), two women run at each other and collide by hitting exercise balls together. If each woman has a mass of 50 kg, which includes the exercise ball, and one woman runs to the right at 2.0 m/s and the other is running toward her at 1.0 m/s, (a) how much total kinetic energy is there in the system? (b) If energy is conserved after the collision and each exercise ball has a mass of 2.0 kg, how fast would the balls fly off toward the camera?
57. In a Coyote/Road Runner cartoon clip (https://openstaxcollege.org/l/21coyroadcarcl), a spring expands quickly and sends the coyote into a rock. If the spring extended 5 m and sent the coyote of mass 20 kg to a speed of 15 m/s, (a) what is the spring constant of this spring? (b) If the coyote were sent vertically into the air with the energy given to him by the spring, how high could he go if there were no non-conservative forces?
58. In an iconic movie scene, Forrest Gump (https://openstaxcollege.org/l/21ForrGumpvid) runs around the country. If he is running at a constant speed of 3 m/s, would it take him more or less energy to run uphill or downhill and why?

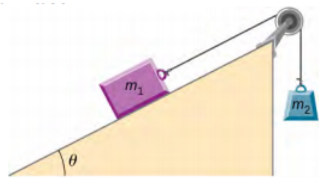
59. In the movie Monty Python and the Holy Grail (https://openstaxcollege.org/l/21monpytmovcl) a cow is catapulted from the top of a castle wall over to the people down below. The gravitational potential energy is set to zero at ground level. The cow is launched from a spring of spring constant 1.1×10^4 N/m that is expanded 0.5 m from equilibrium. If the castle is 9.1 m tall and the mass of the cow is 110 kg, (a) what is the gravitational potential energy of the cow at the top of the castle? (b) What is the elastic spring energy of the cow before the catapult is released? (c) What is the speed of the cow right before it lands on the ground?
60. A 60.0-kg skier with an initial speed of 12.0 m/s coasts up a 2.50-m high rise as shown. Find her final speed at the top, given that the coefficient of friction between her skis and the snow is 0.80.



61. (a) How high a hill can a car coast up (engines disengaged) if work done by friction is negligible and its initial speed is 110 km/h? (b) If, in actuality, a 750-kg car with an initial speed of 110 km/h is observed to coast up a hill to a height 22.0 m above its starting point, how much thermal energy was generated by friction? (c) What is the average force of friction if the hill has a slope of 2.5° above the horizontal?
62. A 5.00×10^5-kg subway train is brought to a stop from a speed of 0.500 m/s in 0.400 m by a large spring bumper at the end of its track. What is the spring constant k of the spring?
63. A pogo stick has a spring with a spring constant of 2.5×10^4 N/m, which can be compressed 12.0 cm. To what maximum height from the uncompressed spring can a child jump on the stick using only the energy in the spring, if the child and stick have a total mass of 40 kg?
64. A block of mass 500 g is attached to a spring of spring constant 80 N/m (see the following figure). The other end of the spring is attached to a support while the mass rests on a rough surface with a coefficient of friction of 0.20 that is inclined at angle of 30°. The block is pushed along the surface till the spring compresses by 10 cm and is then released from rest. (a) How much potential energy was stored in the block-spring-support system when the block was just released? (b) Determine the speed of the block when it crosses the point when the spring is neither compressed nor stretched. (c) Determine the position of the block where it just comes to rest on its way up the incline.

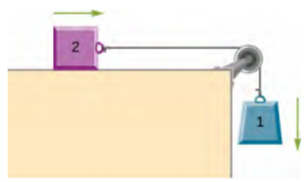


65. A block of mass 200 g is attached at the end of a massless spring at equilibrium length of spring constant 50 N/m. The other end of the spring is attached to the ceiling and the mass is released at a height considered to be where the gravitational potential energy is zero. (a) What is the net potential energy of the block at the instant the block is at the lowest point? (b) What is the net potential energy of the block at the midpoint of its descent? (c) What is the speed of the block at the midpoint of its descent?
66. A T-shirt cannon launches a shirt at 5.00 m/s from a platform height of 3.00 m from ground level. How fast will the shirt be traveling if it is caught by someone whose hands are (a) 1.00 m from ground level? (b) 4.00 m from ground level? Neglect air drag.
67. A child (32 kg) jumps up and down on a trampoline. The trampoline exerts a spring restoring force on the child with a constant of 5000 N/m. At the highest point of the bounce, the child is 1.0 m above the level surface of the trampoline. What is the compression distance of the trampoline? Neglect the bending of the legs or any transfer of energy of the child into the trampoline while jumping.
68. Shown below is a box of mass m1 that sits on a frictionless incline at an angle above the horizontal θ. This box is connected by a relatively massless string, over a frictionless pulley, and finally connected to a box at rest over the ledge, labeled m_2. If m_1 and m_2 are a height h above the ground and $m_2 >> m_1$: (a) What is the initial gravitational potential energy of the system? (b) What is the final kinetic energy of the system?

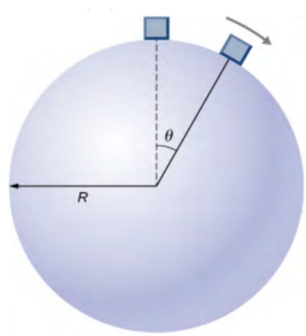


Additional Problems

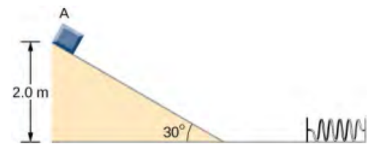
69. A massless spring with force constant k = 200 N/m hangs from the ceiling. A 2.0-kg block is attached to the free end of the spring and released. If the block falls 17 cm before starting back upwards, how much work is done by friction during its descent?
70. A particle of mass 2.0 kg moves under the influence of the force $F(x) = (-5x^2 + 7x)$ N. Suppose a frictional force also acts on the particle. If the particle's speed when it starts at x = −4.0 m is 0.0 m/s and when it arrives at x = 4.0 m is 9.0 m/s, how much work is done on it by the frictional force between x = −4.0 m and x = 4.0 m?
71. Block 2 shown below slides along a frictionless table as block 1 falls. Both blocks are attached by a frictionless pulley. Find the speed of the blocks after they have each moved 2.0 m. Assume that they start at rest and that the pulley has negligible mass. Use m_1 = 2.0 kg and m_2 = 4.0 kg.



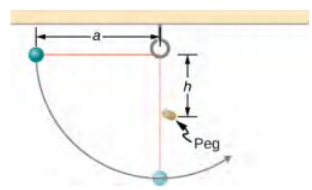
72. A body of mass m and negligible size starts from rest and slides down the surface of a frictionless solid sphere of radius R. (See below.) Prove that the body leaves the sphere when $\theta = \cos^{-1}(2/3)$.



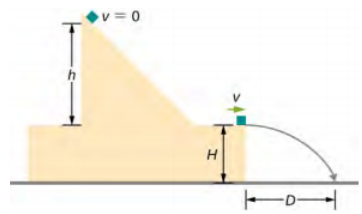
73. A mysterious force acts on all particles along a particular line and always points towards a particular point P on the line. The magnitude of the force on a particle increases as the cube of the distance from that point; that is $F \propto r^3$, if the distance from P to the position of the particle is r. Let b be the proportionality constant, and write the magnitude of the force as $F = br^3$. Find the potential energy of a particle subjected to this force when the particle is at a distance D from P, assuming the potential energy to be zero when the particle is at P.
74. An object of mass 10 kg is released at point A, slides to the bottom of the 30° incline, then collides with a horizontal massless spring, compressing it a maximum distance of 0.75 m. (See below.) The spring constant is 500 M/m, the height of the incline is 2.0 m, and the horizontal surface is frictionless. (a) What is the speed of the object at the bottom of the incline? (b) What is the work of friction on the object while it is on the incline? (c) The spring recoils and sends the object back toward the incline. What is the speed of the object when it reaches the base of the incline? (d) What vertical distance does it move back up the incline?



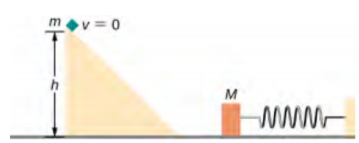
75. Shown below is a small ball of mass m attached to a string of length a. A small peg is located a distance h below the point where the string is supported. If the ball is released when the string is horizontal, show that h must be greater than 3a/5 if the ball is to swing completely around the peg.



76. A block leaves a frictionless inclined surface horizontally after dropping off by a height h. Find the horizontal distance D where it will land on the floor, in terms of h, H, and g



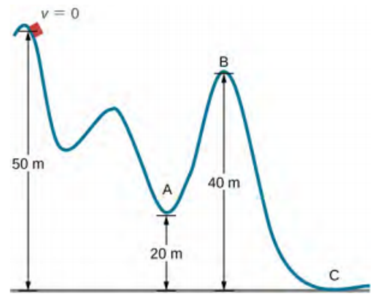
77. A block of mass m, after sliding down a frictionless incline, strikes another block of mass M that is attached to a spring of spring constant k (see below). The blocks stick together upon impact and travel together. (a) Find the compression of the spring in terms of m, M, h, g, and k when the combination comes to rest. (b) The loss of kinetic energy as a result of the bonding of the two masses upon impact is stored in the so-called binding energy of the two masses. Calculate the binding energy.



78. A block of mass 300 g is attached to a spring of spring constant 100 N/m. The other end of the spring is attached to a support while the block rests on a smooth horizontal table and can slide freely without any friction. The block is pushed horizontally till the spring compresses by 12 cm, and then the block is released from rest. (a) How much potential energy was stored in the block-spring support system when the block was just released? (b) Determine the speed of the block when it crosses the point when the spring is neither compressed nor stretched. (c) Determine the speed of the block when it has traveled a distance of 20 cm from where it was released.
79. Consider a block of mass 0.200 kg attached to a spring of spring constant 100 N/m. The block is placed on a frictionless table, and the other end of the spring is attached to the wall so that the spring is level with the table. The block is then pushed in so that the spring is compressed by 10.0 cm. Find the speed of the block as it crosses (a) the point when the spring is not stretched, (b) 5.00 cm to the left of point in (a), and (c) 5.00 cm to the right of point in (a).
80. A skier starts from rest and slides downhill. What will be the speed of the skier if he drops by 20 meters in vertical height? Ignore any air resistance (which will, in reality, be quite a lot), and any friction between the skis and the snow.
81. Repeat the preceding problem, but this time, suppose that the work done by air resistance cannot be ignored. Let the work done by the air resistance when the skier goes from A to B along the given hilly path be −2000 J. The work done by air resistance is negative since the air resistance acts in the opposite direction to the displacement. Supposing the mass of the skier is 50 kg, what is the speed of the skier at point B?
82. Two bodies are interacting by a conservative force. Show that the mechanical energy of an isolated system consisting of two bodies interacting with a conservative force is conserved. (**Hint**: Start by using Newton's third law and the definition

of work to find the work done on each body by the conservative force.)

83. In an amusement park, a car rolls in a track as shown below. Find the speed of the car at A, B, and C. Note that the work done by the rolling friction is zero since the displacement of the point at which the rolling friction acts on the tires is momentarily at rest and therefore has a zero displacement.



84. A 200-g steel ball is tied to a 2.00-m "massless" string and hung from the ceiling to make a pendulum, and then, the ball is brought to a position making a 30° angle with the vertical direction and released from rest. Ignoring the effects of the air resistance, find the speed of the ball when the string (a) is vertically down, (b) makes an angle of 20° with the vertical and (c) makes an angle of 10° with the vertical.
85. A hockey puck is shot across an ice-covered pond. Before the hockey puck was hit, the puck was at rest. After the hit, the puck has a speed of 40 m/s. The puck comes to rest after going a distance of 30 m. (a) Describe how the energy of the puck changes over time, giving the numerical values of any work or energy involved. (b) Find the magnitude of the net friction force.
86. A projectile of mass 2 kg is fired with a speed of 20 m/s at an angle of 30° with respect to the horizontal. (a) Calculate the initial total energy of the projectile given that the reference point of zero gravitational potential energy at the launch position. (b) Calculate the kinetic energy at the highest vertical position of the projectile. (c) Calculate the gravitational potential energy at the highest vertical position. (d) Calculate the maximum height that the projectile reaches. Compare this result by solving the same problem using your knowledge of projectile motion.
87. An artillery shell is fired at a target 200 m above the ground. When the shell is 100 m in the air, it has a speed of 100 m/s. What is its speed when it hits its target? Neglect air friction.
88. How much energy is lost to a dissipative drag force if a 60-kg person falls at a constant speed for 15 meters?
89. A box slides on a frictionless surface with a total energy of 50 J. It hits a spring and compresses the spring a distance of 25 cm from equilibrium. If the same box with the same initial energy slides on a rough surface, it only compresses the spring a distance of 15 cm, how much energy must have been lost by sliding on the rough surface?

Contributors and Attributions

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8.20: Potential Energy and Conservation of Energy (Summary)

Key Terms

conservative force	force that does work independent of path
conserved quantity	one that cannot be created or destroyed, but may be transformed between different forms of itself
energy conservation	total energy of an isolated system is constant
equilibrium point	position where the assumed conservative, net force on a particle, given by the slope of its potential energy curve, is zero
exact differential	is the total differential of a function and requires the use of partial derivatives if the function involves more than one dimension
mechanical energy	sum of the kinetic and potential energies
non-conservative force	force that does work that depends on path
non-renewable	energy source that is not renewable, but is depleted by human consumption
potential energy	function of position, energy possessed by an object relative to the system considered
potential energy diagram	graph of a particle's potential energy as a function of position
potential energy difference	negative of the work done acting between two points in space
renewable	energy source that is replenished by natural processes, over human time scales
turning point	position where the velocity of a particle, in one-dimensional motion, changes sign

Key Equations

Difference of potential energy	$\Delta U_{AB} = U_B - U_A = -W_{AB}$ (8.20.1)
Potential energy with respect to zero of potential energy at $\vec{r}$	$\vec{r}_0 \Delta U = U(\vec{r}) - U(\vec{r}_0)$ (8.20.2)
Gravitational potential energy near Earth's surface	$U(y) = mgy + const.$ (8.20.3)
Potential energy for an ideal spring	$U(x) = \frac{1}{2}kx^2 + const.$ (8.20.4)
Work done by conservative force over a closed path	$W_{closed\ path} = \oint \vec{E}_{cons} \cdot d\vec{r} = 0$ (8.20.5)
Condition for conservative force in two dimensions	$\left(\frac{dF_x}{dy}\right) = \left(\frac{dF_y}{dx}\right)$ (8.20.6)
Conservative force is the negative derivative of potential energy	$F_l = -\frac{dU}{dl}$ (8.20.7)
Conservation of energy with no non-conservative forces	$0 = W_{nc,\ AB} = \Delta(K+U)_{AB} = \Delta E_{AB}.$ (8.20.8)

Summary

8.1 Potential Energy of a System

- For a single-particle system, the difference of potential energy is the opposite of the work done by the forces acting on the particle as it moves from one position to another.
- Since only differences of potential energy are physically meaningful, the zero of the potential energy function can be chosen at a convenient location.
- The potential energies for Earth's constant gravity, near its surface, and for a Hooke's law force are linear and quadratic functions of position, respectively.

8.2 Conservative and Non-Conservative Forces

- A conservative force is one for which the work done is independent of path. Equivalently, a force is conservative if the work done over any closed path is zero.
- A non-conservative force is one for which the work done depends on the path.
- For a conservative force, the infinitesimal work is an exact differential. This implies conditions on the derivatives of the force's components.
- The component of a conservative force, in a particular direction, equals the negative of the derivative of the potential energy for that force, with respect to a displacement in that direction.

8.3 Conservation of Energy

- A conserved quantity is a physical property that stays constant regardless of the path taken.
- A form of the work-energy theorem says that the change in the mechanical energy of a particle equals the work done on it by non-conservative forces.
- If non-conservative forces do no work and there are no external forces, the mechanical energy of a particle stays constant. This is a statement of the conservation of mechanical energy and there is no change in the total mechanical energy.
- For one-dimensional particle motion, in which the mechanical energy is constant and the potential energy is known, the particle's position, as a function of time, can be found by evaluating an integral that is derived from the conservation of mechanical energy.

8.4 Potential Energy Diagrams and Stability

- Interpreting a one-dimensional potential energy diagram allows you to obtain qualitative, and some quantitative, information about the motion of a particle.
- At a turning point, the potential energy equals the mechanical energy and the kinetic energy is zero, indicating that the direction of the velocity reverses there.
- The negative of the slope of the potential energy curve, for a particle, equals the one-dimensional component of the conservative force on the particle. At an equilibrium point, the slope is zero and is a stable (unstable) equilibrium for a potential energy minimum (maximum).

8.5 Sources of Energy

- Energy can be transferred from one system to another and transformed or converted from one type into another. Some of the basic types of energy are kinetic, potential, thermal, and electromagnetic.
- Renewable energy sources are those that are replenished by ongoing natural processes, over human time scales. Examples are wind, water, geothermal, and solar power.
- Non-renewable energy sources are those that are depleted by consumption, over human time scales. Examples are fossil fuel and nuclear power.

Contributors and Attributions

8.21: Potential Energy and Conservation of Energy

Conservative and Nonconservative Forces

Conservative force—a force with the property that the work done in moving a particle between two points is independent of the path it takes.

learning objectives

- Describe properties of conservative and nonconservative forces

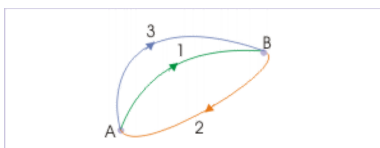
A conservative force is a force with the property that the work done in moving a particle between two points is independent of the path taken. Equivalently, if a particle travels in a closed loop, the net work done (the sum of the force acting along the path multiplied by the distance travelled) by a conservative force is zero.

A conservative force is dependent only on the position of the object. If a force is conservative, it is possible to assign a numerical value for the potential at any point. When an object moves from one location to another, the force changes the potential energy of the object by an amount that does not depend on the path taken. Gravity and spring forces are examples of conservative forces.

If a force is *not conservative*, then defining a scalar potential is not possible, because taking different paths would lead to conflicting potential differences between the start and end points. Nonconservative forces transfer energy *from* the object in motion (just like conservative force), but they do not transfer this energy back *to* the potential energy of the system to regain it during reverse motion. Instead, they transfer the energy from the system in an energy form which can not be used by the force to transfer it back to the object in motion. Friction is one such nonconservative force.

Path Independence of Conservative Force

Work done by the gravity in a closed path motion is zero. We can extend this observation to other conservative force systems as well. We imagine a closed path motion. We imagine this closed path motion be divided in two motions between points A and B as diagramed in Fig 1. Starting from point A to point B and then ending at point A via two work paths named 1 and 2 in the figure. The total work by the conservative force for the round trip is zero:



Motion Along Different Paths: Motion along different paths. For a conservative force, work done via different path is the same.

$$\mathrm{W} = \mathrm{W_{AB1}} + \mathrm{W_{BA2}} = 0. \tag{8.21.1}$$

Let us now change the path for motion from A to B by another path, shown as path 3. Again, the total work by the conservative force for the round trip via new route is zero: $\mathrm{W} = \mathrm{W_{AB3}} + \mathrm{W_{BA2}} = 0.$.

Comparing two equations, $\mathrm{W_{AB1}} = \mathrm{W_{AB3}}$. This is true for an arbitrary path. Therefore, work done for motion from A to B by conservative force along any paths are equal.

What is Potential Energy?

Potential energy is the energy difference between the energy of an object in a given position and its energy at a reference position.

learning objectives

- Relate the potential energy and the work

Potential energy is often associated with restoring forces such as a spring or the force of gravity. The action of stretching the spring or lifting the mass of an object is performed by an external force that works against the force field of the potential. This work is stored in the force field as potential energy. If the external force is removed the force field acts on the body to perform the work as it moves the body back to its initial position, reducing the stretch of the spring or causing the body to fall. The more formal definition is that potential energy is the energy difference between the energy of an object in a given position and its energy at a reference position.

Potential Energy in a Bow and Arrow: In the case of a bow and arrow, the energy is converted from the potential energy in the archer's arm to the potential energy in the bent limbs of the bow when the string is drawn back. When the string is released, the potential energy in the bow limbs is transferred back through the string to become kinetic energy in the arrow as it takes flight.

If the work for an applied force is independent of the path, then the work done by the force is evaluated at the start and end of the trajectory of the point of application. This means that there is a function $U(x)$, called a "potential," that can be evaluated at the two points $x(t=t_1)$ and $x(t_2)$ to obtain the work over any trajectory between these two points. It is tradition to define this function with a negative sign so that positive work is represented as a reduction in the potential:

$$W=\int_C F\cdot dx=\int_{x(t_1)}^{x(t_2)} F\cdot dx \tag{8.21.2}$$

$$=U(x(t_1))-U(x(t_2))=-\Delta U. \tag{8.21.3}$$

Examples of Potential Energy

There are various types of potential energy, each associated with a particular type of force. More specifically, every conservative force gives rise to potential energy. For example, the work of an elastic force is called elastic potential energy; work done by the gravitational force is called gravitational potential energy; and work done by the Coulomb force is called electric potential energy.

Gravity

Gravitational energy is the potential energy associated with gravitational force, as work is required to move objects against gravity.

learning objectives

- Generate an equation that can be used to express the gravitational potential energy near the earth

Gravitational energy is the potential energy associated with gravitational force (a conservative force), as work is required to elevate objects against Earth's gravity. The potential energy due to elevated positions is called gravitational potential energy, evidenced, for example, by water held in an elevated reservoir or behind a dam (as an example, shows Hoover Dam). If an object falls from one point to another point inside a gravitational field, the force of gravity will do positive work on the object, and the gravitational potential energy will decrease by the same amount.

Hoover Dam: Hoover dam uses the stored gravitational potential energy to generate electricity.

Potential Near Earth

Gravitational potential energy near the Earth can be expressed with respect to the height from the surface of the Earth. (The surface will be the zero point of the potential energy.) We can express the potential energy (gravitational potential energy) as:

$$PE = mgh, \tag{8.21.4}$$

where PE = potential energy measured in joules (J), m = mass of the object (measured in kg), and h = perpendicular height from the reference point (measured in m); g = gravitational acceleration ($9.8 m/s^2$). Near the surface of the Earth, g can be considered constant.

General Formula

However, over large variations in distance, the approximation that g is constant is no longer valid. Instead, we must use calculus and the general mathematical definition of work to determine gravitational potential energy. For the computation of the potential energy we can integrate the gravitational force, whose magnitude is given by Newton's law of gravitation (with respect to the distance r between the two bodies). Using that definition, the gravitational potential energy of a system of masses m and M at a distance r using gravitational constant G is:

$$U(r) = \int_r (G\frac{mM}{r'^2})dr' = -G\frac{mM}{r} + K, \tag{8.21.5}$$

where *K* is the constant of integration. Choosing the convention that *K*=0 makes calculations simpler, albeit at the cost of making *U* negative. For this choice, the potential at infinity is defined as 0.

Springs

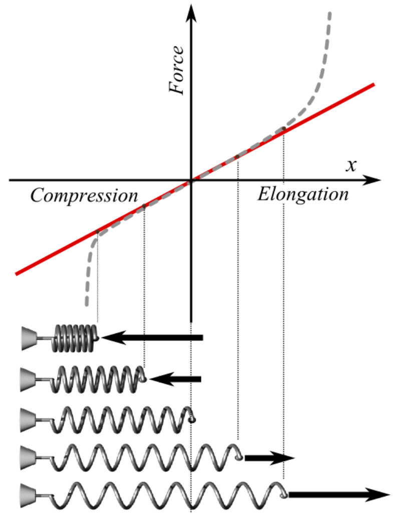
When a spring is stretched/compressed from its equilibrium position by x, its potential energy is give as $U = \frac{1}{2}kx^2$.

learning objectives

- Explain how potential energy is stored in springs

Spring force is conservative force, given by the Hooke's law: $F = -kx$, where k is spring constant, measured experimentally for a particular spring and x is the displacement. We would like to obtain an expression for the work done to the spring. From the conservation of mechanical energy (Check our Atom on "Conservation of Mechanical Energy), the work should be equal to the potential energy stored in spring. The displacement x is usually measured from the position of "neutral length" or "relaxed length"

– the length of spring corresponding to situation when spring is neither stretched nor compressed. We shall identify this position as the origin of coordinate reference (x=0).



Hooke's Law: Plot of applied force F vs. elongation X for a helical spring according to Hooke's law (solid line) and what the actual plot might look like (dashed line). Red is used extension, blue for compression. At bottom, schematic pictures of spring states corresponding to some points of the plot; the middle one is in the relaxed state (no force applied).

Let $\mathrm{x}=0$ and $\mathrm{x}=\mathrm{x_f}(>0)$ be the initial and final positions of the block attached to the string. As the block slowly moves, we do work W on the spring: $\mathrm{W}=\int_0^{\mathrm{x_f}}(\mathrm{kx})\mathrm{dx}=\frac{1}{2}\mathrm{kx_f^2}$. When we stretch the spring. We have to apply force in the same direction as the displacement. (Technically, work is given as the inner product of the two vectors: force and displacement. $\mathrm{W}=\mathrm{F}\cdot\Delta\mathrm{x}$). Therefore, the overall sign in the integral is +, not -.

If the block is gently released from the stretched position (\mathrm{x = x_f}\), the stored potential energy in the spring will start to be converted to the kinetic energy of the block, and vice versa. Neglecting frictional forces, Mechanical energy conservation demands that, at any point during its motion,

$$\text{Total Energy}=\frac{1}{2}\mathrm{mv}^2+\frac{1}{2}\mathrm{kx}^2 \tag{8.21.6}$$

$$=\frac{1}{2}\mathrm{kx_f^2}=\text{constant}. \tag{8.21.7}$$

From the energy conservation, we can estimate that, by the time the block reaches x=0 position, its speed will be $\mathrm{v}(\mathrm{x}=0)=\sqrt{\frac{\mathrm{k}}{\mathrm{m}}}\mathrm{x_f}$. The block will keep oscillating between x = $-x_f$ and x_f.

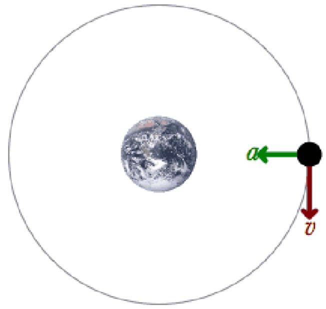
Conservation of Mechanical Energy

Conservation of mechanical energy states that the mechanical energy of an isolated system remains constant without friction.

learning objectives

- Formulate the principle of the conservation of the mechanical energy

Conservation of mechanical energy states that the mechanical energy of an isolated system remains constant in time, as long as the system is free of all frictional forces. In any real situation, frictional forces and other non-conservative forces are always present, but in many cases their effects on the system are so small that the principle of conservation of mechanical energy can be used as a fair approximation. An example of a such a system is shown in. Though energy cannot be created nor destroyed in an isolated system, it can be internally converted to any other form of energy.



A Mechanical System: An example of a mechanical system: A satellite is orbiting the Earth only influenced by the conservative gravitational force and the mechanical energy is therefore conserved. This acceleration is represented by a green acceleration vector and the velocity is represented by a red velocity vector.

Derivation

Let us consider what form the work -energy theorem takes when only conservative forces are involved (leading us to the conservation of energy principle). The work-energy theorem states that the net work done by all forces acting on a system equals its change in kinetic energy (KE). In equation form, this is:

$$\mathrm{W_{net}} = \frac{1}{2}\mathrm{mv}^2 - \frac{1}{2}\mathrm{mv}_0^2 = \Delta\mathrm{KE}. \tag{8.21.8}$$

If only conservative forces act, then $\mathrm{W_{net}} = \mathrm{W_c}$, where $\mathrm{W_c}$ is the total work done by all conservative forces. Thus, $\mathrm{W_c} = \Delta\mathrm{KE}$.

Now, if the conservative force, such as the gravitational force or a spring force, does work, the system loses potential energy (PE). That is, $\mathrm{W_c} = -\mathrm{PE}$. Therefore,

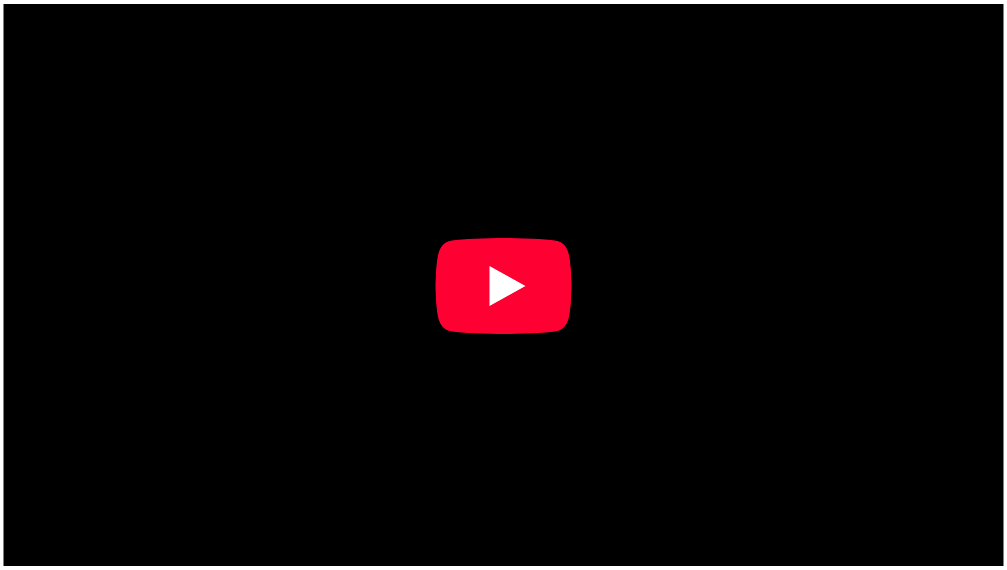
$$-\Delta\mathrm{PE} = \Delta\mathrm{KE} \tag{8.21.9}$$

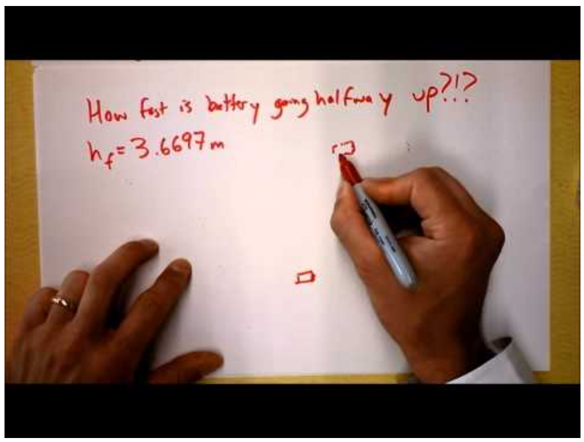
This equation means that the total kinetic and potential energy is constant for any process involving only conservative forces. That is,

$$\mathrm{KE} + \mathrm{PE} = \mathrm{const\ or\ KE_i} + \mathrm{PE_i} = \mathrm{KE_f} + \mathrm{PE_f}, \tag{8.21.10}$$

where i and f denote initial and final values. This equation is a form of the work-energy theorem for conservative forces; it is known as the conservation of mechanical energy principle.

Remember that the law applies to the extent that all the forces are conservative, so that friction is negligible. The total kinetic plus potential energy of a system is defined to be its mechanical energy $(\mathrm{KE} + \mathrm{PE})$. In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and various types of PE (with the total energy remaining constant).





Conservation of Mechanical Energy: Worked example.

Problem Solving With the Conservation of Energy

To solve a conservation of energy problem determine the system of interest, apply law of conservation of energy, and solve for the unknown.

learning objectives

- Identify steps necessary to solve a conservation of energy problem

Problem-solving Strategy

You should follow a series of steps whenever you are problem solving:

Step One

Determine the system of interest and identify what information is given and what quantity is to be calculated. For example, let's assume you have the problem with car on a roller coaster. You know that the cars of a roller coaster reach their maximum kinetic energy (KEKE) when at the bottom of their path. When they start rising, the kinetic energy begins to be converted to gravitational potential energy (PEgPEg). The sum of kinetic and potential energy in the system should remain constant, if losses to friction are ignored.

Determining Energy: The cars of a roller coaster reach their maximum kinetic energy when at the bottom of their path. When they start rising, the kinetic energy begins to be converted to gravitational potential energy. The sum of kinetic and potential energy in the system remains constant, ignoring losses to friction.

Step Two

Examine all the forces involved and determine whether you know or are given the potential energy from the work done by the forces. Then use step three or step four.

Step Three

If you know the potential energies (PE) for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of energy is:

$$\mathrm{KE_i + PE_i = KE_f + PE_f}. \tag{8.21.11}$$

Step Four

If you know the potential energy for only some of the forces, then the conservation of energy law in its most general form must be used:

$$\mathrm{KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f} \tag{8.21.12}$$

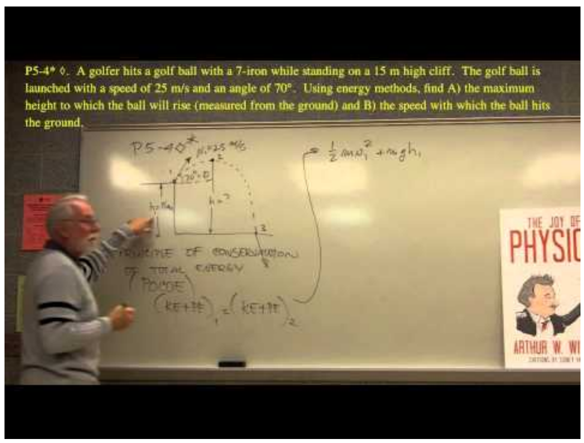
where $\mathrm{O_E}$ stand for all other energies, and $\mathrm{W_{nc}}$ stands for work done by non-conservative forces. In most problems, one or more of the terms is zero, simplifying its solution. Do *not* calculate $\mathrm{W_c}$, the work done by conservative forces; it is already incorporated in the PE terms.

Step Five

You have already identified the types of work and energy involved (in step two). Before solving for the unknown, eliminate terms wherever possible to simplify the algebra. For example, choose height $\mathrm{h=0}$ at either the initial or final point—this will allow to set PEg at zero. Then solve for the unknown in the customary manner.

Step Six

Check the answer to see if it is reasonable. Once you have solved a problem, reexamine the forms of work and energy to see if you have set up the conservation of energy equation correctly. For example, work done against friction should be negative, potential energy at the bottom of a hill should be less than that at the top, and so on.



Energy conservation: Part of a series of videos on physics problem-solving. The problems are taken from “The Joy of Physics. ” This one deals with energy conservation. The viewer is urged to pause the video at the problem statement and work the problem before watching the rest of the video.

Problem Solving with Dissipative Forces

In the presence of dissipative forces, total mechanical energy changes by exactly the amount of work done by nonconservative forces (W_c).

learning objectives

- Express the energy conservation relationship that can be applied to solve problems with dissipative forces

INTRODUCTION

We have seen a problem-solving strategy with the conservation of energy in the previous section. Here we will adopt the strategy for problems with dissipative forces. Since the work done by nonconservative (or dissipative) forces will irreversibly alter the energy of the system, the total mechanical energy (KE + PE) changes by exactly the amount of work done by nonconservative forces (W_c). Therefore, we obtain $\mathrm{KE_i + PE_i + W_{nc} = KE_f + PE_f}$, where KE and PE represent kinetic and potential energies respectively. Therefore, using the new energy conservation relationship, we can apply the same problem-solving strategy as with the case of conservative forces.

EXAMPLE

Consider the situation shown in, where a baseball player slides to a stop on level ground. Using energy considerations, calculate the distance the 65.0-kg baseball player slides, given that his initial speed is 6.00 m/s and the force of friction against him is a *constant 450 N.*

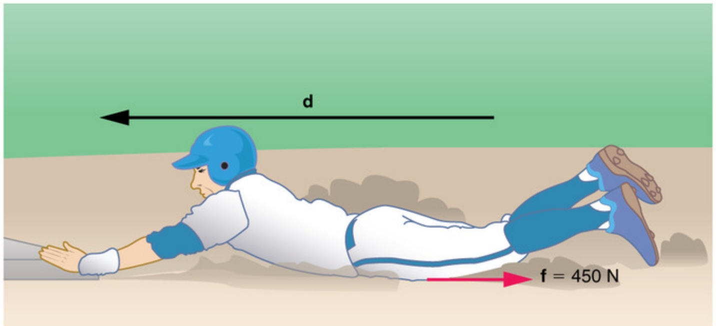


Fig 1: The baseball player slides to a stop in a distance d. In the process, friction removes the player's kinetic energy by doing an amount of work fd equal to the initial kinetic energy.

Strategy: Friction stops the player by converting his kinetic energy into other forms, including thermal energy. In terms of the work-energy theorem, the work done by friction (f), which is negative, is added to the initial kinetic energy to reduce it to zero. The work done by friction is negative, because f is in the opposite direction of the motion (that is, $\theta = 180^{\circ}$, and so $\cos\theta = -1$). Thus $W_{nc} = -fd$. The equation simplifies to $\frac{1}{2}mv_i^2 - fd = 0$.

Solution: Solving the previous equation for d and substituting known values yields, we get d = 2.60 m. The most important point of this example is that the amount of nonconservative work equals the change in mechanical energy.

Key Points

- If a particle travels in a closed loop, the net work done (the sum of the force acting along the path multiplied by the distance travelled) by a conservative force is zero.
- Conservative force is dependent only on the position of the object. If a force is conservative, it is possible to assign a numerical value for the potential at any point.
- Nonconservative force transfer the energy from the system in an energy form which can not be used by the force to transfer back to the object in motion.
- If the work for an applied force is independent of the path, then the work done by the force is evaluated at the start and end of the trajectory of the point of application. This means that there is a function U(x), called a " potential ".
- It is tradition to define the potential function with a negative sign so that positive work is represented as a reduction in the potential.
- Every conservative force gives rise to potential energy. Examples are elastic potential energy, gravitational potential energy, and electric potential energy.
- Gravitational potential energy near the earth can be expressed with respect to the height from the surface of the Earth as PE = mgh. g = gravitational acceleration (9.8m/s^2). Near the surface of the Earth, g can be considered constant.
- Over large variations in distance, the approximation that g is constant is no longer valid and a general formula should be used for the potential. It is given as: $U(r) = \int_r (G\frac{mM}{r'^2})dr' = -G\frac{mM}{r} + K$.
- Choosing the convention that the constant of integration K=0 assumes that the potential at infinity is defined to be 0.
- The displacement of spring x is usually measured from the position of "neutral length " or "relaxed length". Often, it is most convenient to identify this position as the origin of coordinate reference (x=0).
- If the block is gently released from the stretched position ($x = x_f$), energy conservation tells us that $\frac{1}{2}mv^2 + \frac{1}{2}kx^2 = \frac{1}{2}kx_f^2 = \text{constant}$.
- If the block is released from the stretched position ($x = x_f$), by the time the block reaches x=0 position, its speed will be $v(x=0) = \sqrt{\frac{k}{m}}x_f$. The block will keep oscillating between $x = -x_f$ and x_f.
- The conservation of mechanical energy can be written as "$KE + PE = \text{const}$ ".
- Though energy cannot be created nor destroyed in an isolated system, it can be internally converted to any other form of energy.
- In a system that experiences only conservative forces, there is a potential energy associated with each force, and the energy only changes form between KE and various types of PE, with the total energy remaining constant.
- If you know the potential energies for the forces that enter into the problem, then forces are all conservative, and you can apply conservation of mechanical energy simply in terms of potential and kinetic energy. The equation expressing conservation of

energy is: $\mathrm{KE_i + PE_i = KE_f + PE_f}$. .

- If you know the potential energy for only some of the forces, then the conservation of energy law in its most general form must be used: $\mathrm{KE_i + PE_i + W_{nc} + OE_i = KE_f + PE_f + OE_f}$, where OE stands for all other energies.
- Once you have solved a problem, always check the answer to see if it is reasonable.
- Using the new energy conservation relationship

$$\mathrm{KE_i + PE_i + W_{nc} = KE_f + PE_f} \tag{8.21.13}$$

, we can apply the same problem-solving strategy as with the case of conservative forces.
- The most important point is that the amount of nonconservative work equals the change in mechanical energy.
- The work done by nonconservative (or dissipative) forces will irreversibly dissipated in the system.

Key Terms

- **potential**: A curve describing the situation where the difference in the potential energies of an object in two different positions depends only on those positions.
- **Coulomb force**: the electrostatic force between two charges, as described by Coulomb's law
- **potential**: A curve describing the situation where the difference in the potential energies of an object in two different positions depends only on those positions.
- **conservative force**: A force with the property that the work done in moving a particle between two points is independent of the path taken.
- **Hooke's law**: the principle that the stress applied to a solid is directly proportional to the strain produced. This law describes the behavior of springs and solids stressed within their elastic limit.
- **conservation**: A particular measurable property of an isolated physical system does not change as the system evolves.
- **isolated system**: A system that does not interact with its surroundings, that is, its total energy and mass stay constant.
- **frictional force**: Frictional force is the force resisting the relative motion of solid surfaces, fluid layers, and material elements sliding against each other.
- **kinetic energy**: The energy possessed by an object because of its motion, equal to one half the mass of the body times the square of its velocity.
- **potential energy**: The energy an object has because of its position (in a gravitational or electric field) or its condition (as a stretched or compressed spring, as a chemical reactant, or by having rest mass)
- **conservative force**: A force with the property that the work done in moving a particle between two points is independent of the path taken.
- **dissipative force**: A force resulting in dissipation, a process in which energy (internal, bulk flow kinetic, or system potential) is transformed from some initial form to some irreversible final form.

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8.22: Power

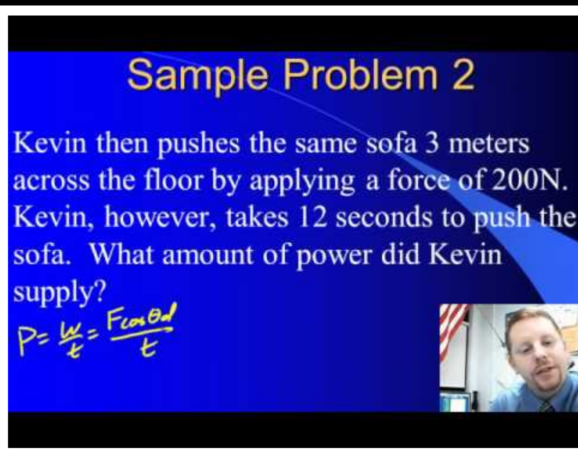
What is Power?

In physics, power is the rate of doing work—the amount of energy consumed per unit time.

learning objectives

- Relate power to the transfer, use, and transformation of different types of energy

In physics, power is the rate of doing work. It is the amount of energy consumed per unit time. The unit of power is the joule per second (J/s), known as the watt (in honor of James Watt, the eighteenth-century developer of the steam engine). For example, the rate at which a lightbulb transforms electrical energy into heat and light is measured in watts (W)—the more wattage, the more power, or equivalently the more electrical energy is used per unit time.



Power: A brief overview of power in an algebra-based physics course.

Energy transfer can be used to do work, so power is also the rate at which this work is performed. The same amount of work is done when carrying a load up a flight of stairs whether the person carrying it walks or runs, but more power is expended during the running because the work is done in a shorter amount of time. The output power of an electric motor is the product of the torque the motor generates and the angular velocity of its output shaft. The power expended to move a vehicle is the product of the traction force of the wheels and the velocity of the vehicle.

Examples of power are limited only by the imagination, because there are as many types as there are forms of work and energy. Sunlight reaching Earth's surface carries a maximum power of about 1.3 kilowatts per square meter (kW/m^2). A tiny fraction of this is retained by Earth over the long term. Our consumption rate of fossil fuels is far greater than the rate at which they are stored, so it is inevitable that they will be depleted. Power implies that energy is transferred, perhaps changing form. It is never possible to change one form completely into another without losing some of it as thermal energy. For example, a 60-W incandescent bulb converts only 5 W of electrical power to light, with 55 W dissipating into thermal energy. Furthermore, the typical electric power plant converts only 35 to 40 percent of its fuel into electricity. The remainder becomes a huge amount of thermal energy that must be dispersed as heat transfer, as rapidly as it is created. A coal-fired power plant may produce 1,000 megawatts; 1 megawatt (MW) is 106 W of electric power. But the power plant consumes chemical energy at a rate of about 2,500 MW, creating heat transfer to the surroundings at a rate of 1,500 MW.

Coal-fired Power Plant: Tremendous amounts of electric power are generated by coal-fired power plants such as this one in China, but an even larger amount of power goes into heat transfer to the surroundings. The large cooling towers here are needed to transfer heat as rapidly as it is produced. The transfer of heat is not unique to coal plants but is an unavoidable consequence of generating electric power from any fuel—nuclear, coal, oil, natural gas, or the like.

Humans: Work, Energy, and Power

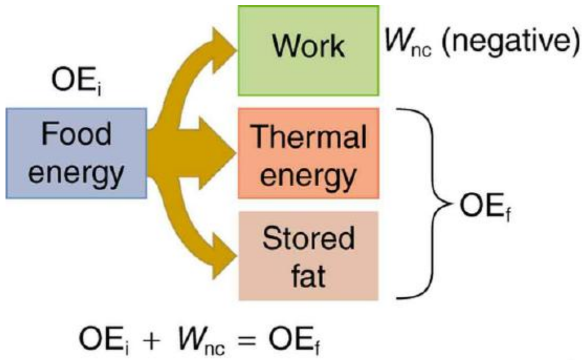
The human body converts energy stored in food into work, thermal energy, and/or chemical energy that is stored in fatty tissue.

learning objectives

- Identify what factors play a role in basal metabolic rate (BMR)

Humans: Work, Energy, and Power

Our own bodies, like all living organisms, are energy conversion machines. Conservation of energy implies that the chemical energy stored in food is converted into work, thermal energy, or stored as chemical energy in fatty tissue, as shown in. Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity. The fraction going into each form depends both on how much we eat and on our level of physical activity. If we eat more than is needed to do work and stay warm, the remainder goes into body fat.



Energy Conversion in Humans: Energy consumed by humans is converted to work, thermal energy, and stored fat. By far the largest fraction goes to thermal energy, although the fraction varies depending on the type of physical activity.

Functions that Require Energy

All bodily functions, from thinking to lifting weights, require energy. The many small muscle actions accompanying all quiet activity, from sleeping to head scratching, ultimately become thermal energy, as do less visible muscle actions by the heart, lungs, and digestive tract. Shivering, in fact, is an involuntary response to low body temperature that pits muscles against one another to produce thermal energy in the body (and do no work). The kidneys and liver consume a surprising amount of energy, but the biggest surprise of all it that a full 25% of all energy consumed by the body is used to maintain electrical potentials in all living cells. (Nerve cells use this electrical potential in nerve impulses.) This bioelectrical energy ultimately becomes mostly thermal energy, but some is utilized to power chemical processes such as in the kidneys and liver, and in fat production.

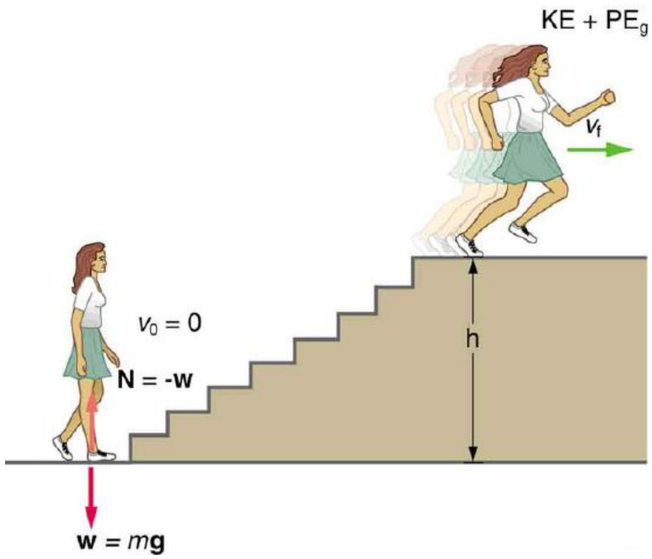
Basal Metabolic Rate

The rate at which the body uses food energy to sustain life and to do different activities is called the metabolic rate. The total energy conversion rate of a person at rest is called the basal metabolic rate (BMR) and is divided among various systems in the body. The largest fraction goes to the liver and spleen, with the brain coming next. Of course, during vigorous exercise, the energy consumption of the skeletal muscles and heart increase markedly. About 75% of the calories burned in a day go into these basic functions. The BMR is a function of age, gender, total body weight, and amount of muscle mass (which burns more calories than body fat). Athletes have a greater BMR due to this last factor.

Useful Work

Work done by a person is sometimes called useful work, which is work done on the outside world, such as lifting weights. Useful work requires a force exerted through a distance on the outside world, and so it excludes internal work, such as that done by the heart when pumping blood. Useful work does include that done in climbing stairs or accelerating to a full run, because these are accomplished by exerting forces on the outside world. Forces exerted by the body are nonconservative, so that they can change the mechanical energy (KE+PE) of the system worked upon, and this is often the goal.

For example, what is the power output for a 60.0-kg woman who runs up a 3.00 m high flight of stairs in 3.50 s, starting from rest but having a final speed of 2.00 m/s?.



Woman Running Up Stairs: When this woman runs upstairs starting from rest, she converts the chemical energy originally from food into kinetic energy and gravitational potential energy. Her power output depends on how fast she does this.

Her power output depends on how fast she does this. The work going into mechanical energy is W= KE + PE. At the bottom of the stairs, we take both KE and PE_g as initially zero; thus,

$$W = KE_f + PE_g = \frac{1}{2}mv_f^2 + mgh \tag{8.22.1}$$

where h is the vertical height of the stairs. Because all terms are given, we can calculate W and then divide it by time to get power. Substituting the expression for W into the definition of power given in the previous equation, $P = \frac{W}{t}$ yields

$$P = \frac{W}{t} = \frac{\frac{1}{2}mv_f^2 + mgh}{t} \tag{8.22.2}$$

Entering known values yields

$$P = \frac{0.5(60.0kg)(2.00m/s)^2 + (60.0kg)(9.80m/s^2)(3.00m)}{(3.50s)} = \frac{120J + 1764J}{3.50s} = 538W \tag{8.22.3}$$

The woman does 1764 J of work to move up the stairs compared with only 120 J to increase her kinetic energy; thus, most of her power output is required for climbing rather than accelerating.

Energy consumption is directly proportional to oxygen consumption because the digestive process is basically one of oxidizing food. We can measure the energy people use during various activities by measuring their oxygen use. Approximately 20 kJ of energy are produced for each liter of oxygen consumed, independent of the type of food.

Key Points

- Power implies that energy is transferred, perhaps changing form.
- Energy transfer can be used to do work, so power is also the rate at which this work is performed.
- The unit of power is the joule per second (J/s), known as the watt.
- The rate at which the body uses food energy to sustain life and to do different activities is called the metabolic rate, and the corresponding rate when at rest is called the basal metabolic rate (BMR).
- The energy included in the basal metabolic rate is divided among various systems in the body, with the largest fraction going to the liver and spleen, and the brain coming next.
- About 75% of food calories are used to sustain basic body functions included in the basal metabolic rate.
- Work done by a person is sometimes called useful work, which is work done on the outside world, such as lifting weights.
- The energy consumption of people during various activities can be determined by measuring their oxygen use, because the digestive process is basically one of oxidizing food.

Key Terms

- **power**: A measure of the rate of doing work or transferring energy.
- **watt**: In the International System of Units, the derived unit of power; the power of a system in which one joule of energy is transferred per second.
- **basal metabolic rate**: The amount of energy expended while at rest in a neutrally temperate environment, in the post-absorptive state.

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8.23: CASE STUDY: World Energy Use

World Energy Use

The most prominent sources of energy used in the world are non-renewable (i.e., unsustainable).

learning objectives

- Explain why renewable energy sources must be found and utilized

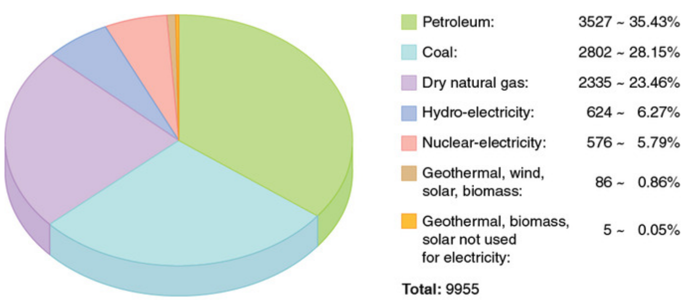
Energy Use

World energy consumption is the total amount of energy used by all humans on the planet (measured on a per-year basis). This measurement is the sum of all energy sources (and purposes) in use. Who measures this? Several organizations publish this data, including the International Energy Agency (IEA), the US Energy Information Administration (EIA), and the European Environment Agency. This data is useful because evaluating this information to discover trends might yield energy issues not currently being addressed, thereby encouraging the search for solutions. The IEA established a goal of limiting global warming to 2 degrees Celsius, but this goal is becoming more difficult to reach each year that the necessary action is not taken. In global energy use, fossil fuels make up a substantial portion. In 2011 they received over $500 billion in subsidies—six times more than that received by renewable energy sources.

Implementing new practices that will utilize different, renewable energy sources is important because having access to energy is important—it maintains our quality of life. Fossil fuels, however, are not sustainable at the rate they are currently used. About 40% of the world's energy comes from oil, but oil prices are dependent on uncertain factors (such as availability, politics, and world events). The United States alone uses 24% of the world's oil per year, yet it makes up only 4.5% of the world's population! In 2008, total worldwide energy consumption was 474 exajoules (474×10^{18} J=132,000 TWh)—equivalent to an average power usage of 15 terawatts (1.504×10^{13} W). Potential renewable energy sources include: solar energy at 1600 EJ (444,000 TWh), wind power at 600 EJ (167,000 TWh), geothermal energy at 500 EJ (139,000 TWh), biomass at 250 EJ (70,000 TWh), hydropower at 50 EJ (14,000 TWh) and ocean energy at 1 EJ (280 TWh).

Types of Energy

shows a pie chart of world energy usage by category—both renewable and nonrenewable sources. Renewable energy comes from sources with an unlimited supply. This includes energy from water, wind, the sun, and biomass. In the US, only 10% of energy comes from renewable sources (mostly hydroelectric energy). Nonrenewable sources makes up 85% of worldwide energy usage—from sources that eventually will be depleted, such as oil, natural gases and coal.



World Energy Use: This chart shows that the primary worldwide energy sources nonrenewable. If new practices are not put in place now, this model will not be sustainable.

Energy Needs

In the last 50 years, the global energy demand has tripled due to the number of developing countries and innovations in technology. It is projected to triple again over the next 30 years. In Europe, many in such developing areas recognize that the need for renewable energy sources, as the present course of energy usage cannot be sustained indefinitely. While renewable energy development makes up a only small percentage of the field, strides are being made in natural energy, particularly wind energy.

For example, by the year 2020 Germany plans to meet 10% of their total energy usage and 20% of its electricity usage with renewable resources. While some countries are making improvements in this field, coal usage is still a huge problem. In China, two thirds of the energy used each year is from commercial coal energy. India imports 50% of its oil, and 70% of its electricity is produced from coal, which is highly polluting.

Key Points

- The energy consumption increases with the increasing number of developing areas. In order for this development to continue, while maintaining quality of life, new and renewable energy sources must be found and utilized.
- Renewable energy comes from sources that will never deplete, no matter how much is used. An example of this is wind energy, which had been growing in popularity in countries like India and Germany.
- Nonrenewable energy makes up 85% the energy used on earth—the most popular form of energy being oil.

Key Terms

- **fossil fuel**: Any fuel derived from hydrocarbon deposits such as coal, petroleum, natural gas and, to some extent, peat; these fuels are irreplaceable, and their burning generates the greenhouse gas carbon dioxide.
- **renewable energy**: Energy that can be replenished at the same rate as it is used.

8.24: Further Topics

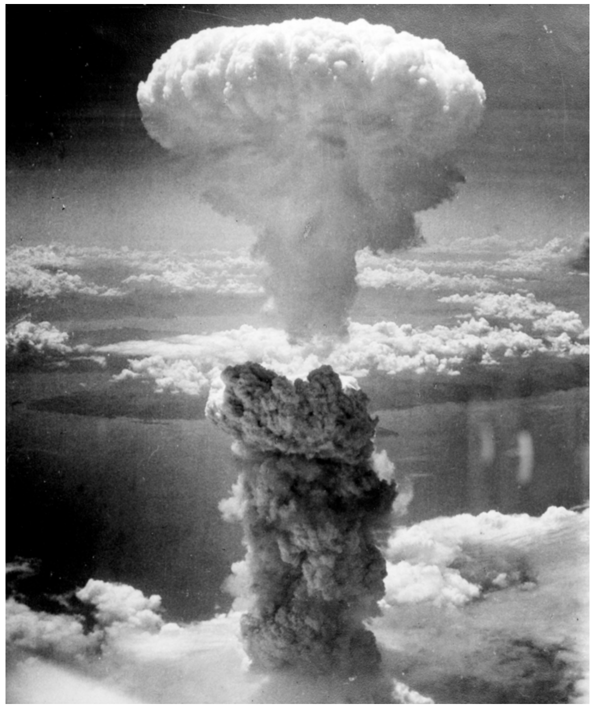
learning objectives

- Compare the different forms of energy interrelate to one another

Other Forms of Energy

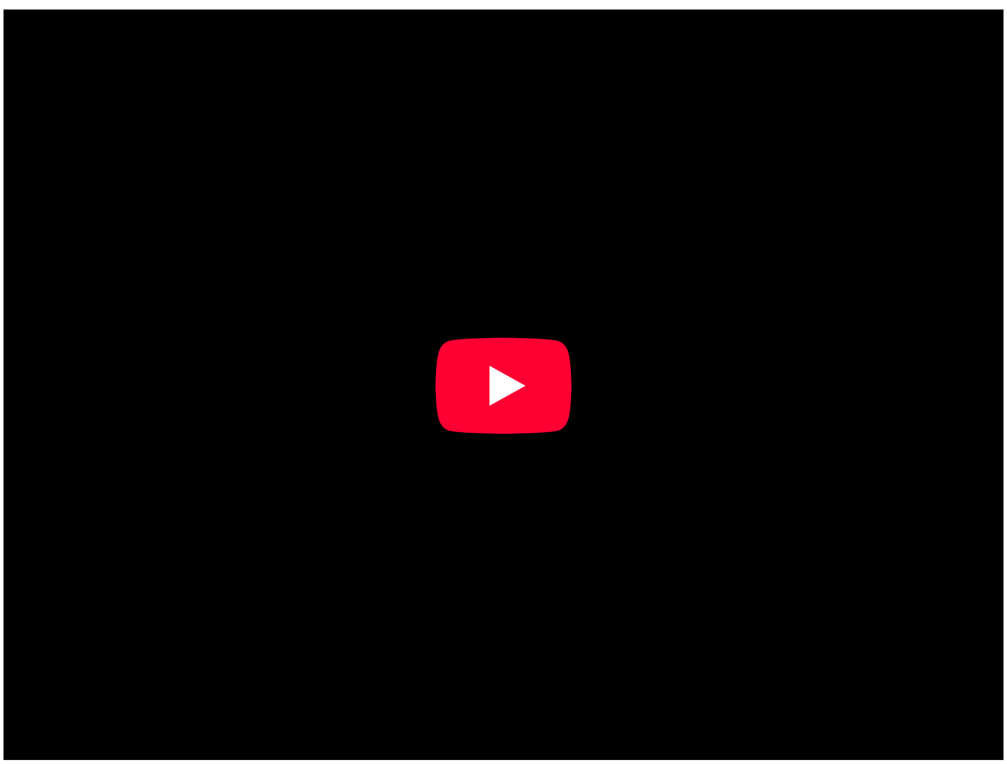
Thermal, chemical, electric, radiant, nuclear, magnetic, elastic, sound, mechanical, luminous, and mass are forms that energy can exist in. Energy can come in a variety of forms. These forms include:

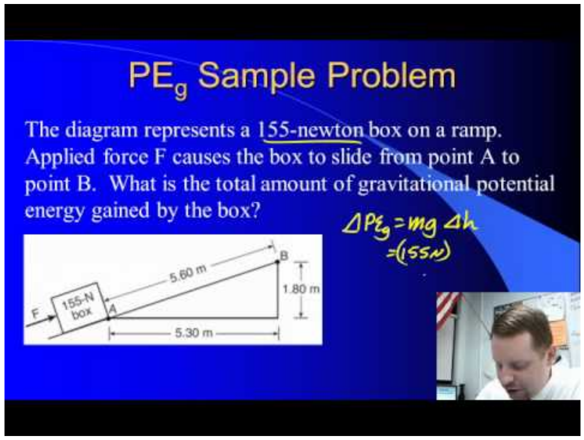
- Thermal Energy: This is energy associated with the microscopic random motion of particles in the media under consideration. An example of something that stores thermal energy is warm bath water.
- Chemical Energy: This is energy due to the way that atoms are arranged in molecules and various other collections of matter. An example of something that stores chemical energy is food. When your body digests and metabolizes food it utilizes its chemical energy.
- Electric Energy: This is energy that is from electrical potential energy, a result of Coulombic forces. Electrical potential energy is associated with the way that point charges in a system are arranged. An example of something that stores electric energy is a capacitor. A capacitor collects positive charge on one plate and negative charge on the other plate. Energy is thus stored in the resulting electrostatic field.
- Radiant Energy: This is any kind of electromagnetic radiation (see key term). An example of an electromagnetic wave is light.
- Nuclear Energy: This type of energy is liberated during the nuclear reactions of fusion and fission. Examples of things that utilize nuclear energy include nuclear power plants and nuclear weapons.
- Magnetic Energy: Technically magnetic energy is electric energy; the two are related by Maxwell's equations. An example of something that stores magnetic energy is a superconducting magnet used in an MRI.
- Elastic Energy: This is potential mechanical energy that is stored in the configuration of a material or physical system as work is performed to distort its volume or shape. An example of something that stores elastic energy is a stretched rubber band.
- Sound Energy: This is energy that is associated with the vibration or disturbance of matter. An example of something that creates sound energy is your voice box (larynx).
- Mechanical Energy: This is energy that is associated with the motion and position of an object. It is the sum of all of the kinetic and potential energy that the object has. An example of something that utilizes mechanical energy is a pendulum.
- Luminous Energy: This is energy that can be seen because it is visible light. An example of luminous energy is light from a flashlight.
- Mass: Can be converted to energy via: $\mathrm{E}=\mathrm{mc}^2$. For example, mass is converted into energy when a nuclear bomb explodes.



Atomic bomb explosion: The mushroom cloud of the atomic bombing of Nagasaki, Japan

In each of the aforementioned forms, energy exists as either kinetic energy, potential energy, or a combination of both. It is important to note that the above list is not necessarily complete as we may discover new forms of energy in the future such as “dark energy. ” Also, each of the forms that energy can take on (as listed above) are not necessarily mutually exclusive. For example, luminous energy is radiant energy.





Types of Energy: A brief overview of energy, kinetic energy, gravitational potential energy, and the work-energy theorem for algebra-based physics students.

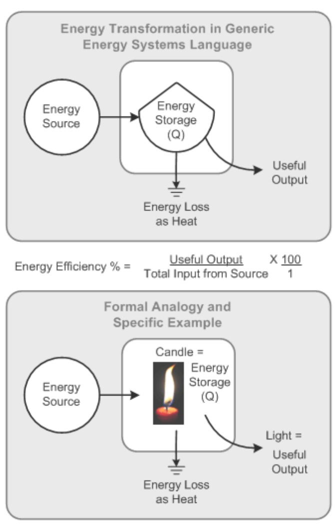
Energy Transformations

Energy transformation occurs when energy is changed from one form to another, and is a consequence of the first law of thermodynamics.

learning objectives

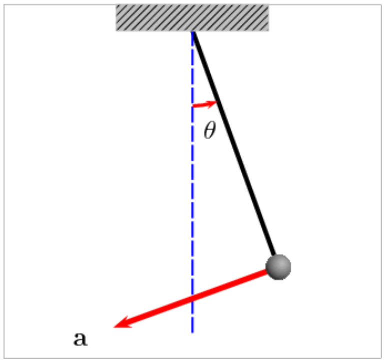
- Summarize the consequence of the first law of thermodynamics on the total energy of a system

Energy transformation occurs when energy is changed from one form to another. It is a consequence of the first law of thermodynamics that the total energy of a given system can only be changed when energy is added or subtracted from the system. Often it appears that energy has been lost from a system when it simply has been transformed. For example, an internal combustion engine converts the potential chemical energy in gasoline and oxygen into heat energy. This heat energy is then converted to kinetic energy, which is then used to propel the vehicle that is utilizing the engine. The technical term for a device that converts energy from one form to another is a *transducer*.



Energy Transformation: These figures illustrate the concepts of energy loss and useful energy output.

When analyzing energy transformations, it is important to consider the efficiency of conversion. The efficiency of conversions describes the ratio between the useful output and input of an energy conversion machine. Some energy transformations can occur with an efficiency of essentially 100%. For example, imagine a pendulum in a vacuum. As illustrated in, when the pendulum's mass reaches its maximum height, all if its energy exists in the form of potential energy. However, when the pendulum is at its lowest point, all of its energy exists in the form of kinetic energy.



Pendulum: This animation shows the velocity and acceleration vectors for a pendulum. One may note that at the maximum height of the pendulum's mass, the velocity is zero. This corresponds to zero kinetic energy and thus all of the energy of the pendulum is in the form of potential energy. When the pendulum's mass is at its lowest point, all of its energy is in the form of kinetic energy and we see its velocity vector has a maximum magnitude here.

Other energy transformations occur with a much lower efficiency of conversion. For example, the theoretical limit of the energy efficiency of a wind turbine (converting the kinetic energy of the wind to mechanical energy) is 59%. The process of photosynthesis is able to transform the light energy of the sun into chemical energy that can be used by a plant with an efficiency of conversion of a mere 6%.

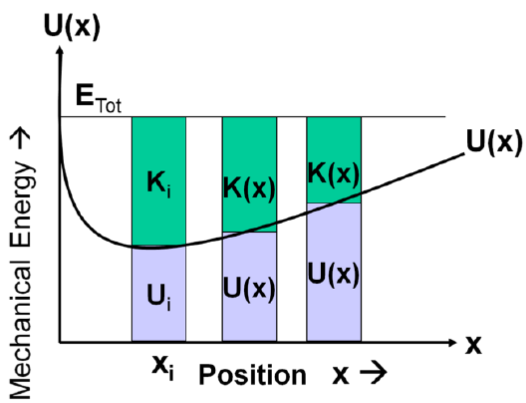
Potential Energy Curves and Equipotentials

A potential energy curve plots potential energy as a function of position; equipotential lines trace lines of equal potential energy.

learning objectives

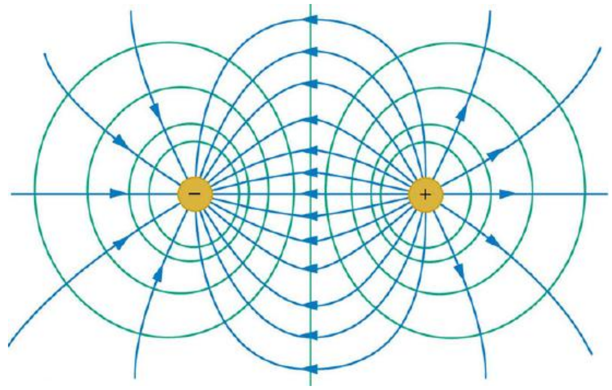
- Derive the potential of a point charge

A potential energy curve plots the potential energy of an object as a function of that object's position. For example, see. The system under consideration is a closed system, so the total energy of the system remains constant. This means that the kinetic and potential energy always have to sum to be the same value. We observe that the potential energy increases as the kinetic energy decreases and vice versa. The utility of a potential energy curve is that we can quickly determine the potential energy of the object in question at a given position.



Potential Energy Curve: This figure illustrates the potential energy of a particle as a function of position. The kinetic energy is also shown and is abbreviated K.

Equipotential lines trace lines of equal potential energy. In, if you were to draw a straight horizontal line through the center, that would be an equipotential line. In and, if you travel along an equipotential line, the electric potential will be constant.



Equipotential Lines for Two Equal and Opposite Point Charges: Electric field (blue) and equipotential lines (green) for two equal and opposite charges

Let us examine the physical explanation for the equipotentials lines in. The equation for the potential of a point charge is $\mathrm{V}=\frac{\mathrm{kQ}}{\mathrm{r}}$, where *V* is the potential, *k* is a constant with a value of $8.99 \cdot 10^9\ \mathrm{N\ m^2/C^2}$, *Q* is the magnitude of the point charge, and *r*is the distance from the charge. So, every point that is the same distance from the point charge will have the same electric potential energy. Therefore, if we draw a circle around the point charge, every point on the circle will have the same potential energy.

Work (*W*) is a measure of the change in potential energy $(\Delta \mathrm{PE}) : \mathrm{W} = -\backslash\mathrm{DeltaPE}$. Since the potential energy does not change along an equipotential line, you do not need to do any work to move along one. However, you *do* need to do work to move from one equipotential line to another. Recall that work is zero if force is perpendicular to motion; in the figures shown above, the forces resulting from the electric field are in the same direction as the electric field itself. So we note that each of the equipotential lines must be perpendicular to the electric field at every point.

Key Points

- Thermal, chemical, electric, radiant, nuclear, magnetic, elastic, sound, mechanical, luminous, and mass are forms that energy can exist in.
- Energy exists as either kinetic energy, potential energy, or a combination of both.
- We may discover new forms of energy (like "dark energy") in the future.
- The total energy of a given system can only be changed when energy is added or subtracted from the system.
- Often it appears that energy has been lost from a system when it simply has been transformed.
- The efficiency of conversions describes the ratio between the useful output of an energy conversion machine and the input.
- A potential energy curve plots the potential energy of an object as a function of its position.
- Equipotential lines trace lines of equal potential energy.
- You do not need to do any work to move along an equipotential line.

Key Terms

- **fusion**: A nuclear reaction in which nuclei combine to form more massive nuclei with the concomitant release of energy.
- **electromagnetic radiation**: radiation (quantized as photons) consisting of oscillating electric and magnetic fields oriented perpendicularly to each other, moving through space
- **fission**: The process of splitting the nucleus of an atom into smaller particles; nuclear fission.
- **pendulum**: A body suspended from a fixed support so that it swings freely back and forth under the influence of gravity; it is commonly used to regulate various devices such as clocks.
- **first law of thermodynamics**: A version of the law of conservation of energy, specialized for thermodynamical systems. It is usually formulated by stating that the change in the internal energy of a closed system is equal to the amount of heat supplied to the system, minus the amount of work done by the system on its surroundings.
- **potential energy**: The energy an object has because of its position (in a gravitational or electric field) or its condition (as a stretched or compressed spring, as a chemical reactant, or by having rest mass)
- **kinetic energy**: The energy possessed by an object because of its motion, equal to one half the mass of the body times the square of its velocity.

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CHAPTER OVERVIEW

9: Linear Momentum and Collisions

Thumbnail: A pool break-off shot. Image used with permission (CC-SA-BY; No-w-ay).

9.1: Prelude to Linear Momentum and Collisions

The concepts of work, energy, and the work-energy theorem are valuable for two primary reasons: First, they are powerful computational tools, making it much easier to analyze complex physical systems than is possible using Newton's laws directly (for example, systems with nonconstant forces); and second, the observation that the total energy of a closed system is conserved means that the system can only evolve in ways that are consistent with energy conservation. In other words, a system cannot evolve randomly; it can only change in ways that conserve energy.

Figure 9.1.1: The concepts of impulse, momentum, and center of mass are crucial for a major-league baseball player to successfully get a hit. If he misjudges these quantities, he might break his bat instead. (credit: modification of work by "Cathy T"/Flickr)

In this chapter, we develop and define another conserved quantity, called linear momentum, and another relationship (the impulse-momentum theorem), which will put an additional constraint on how a system evolves in time. Conservation of momentum is useful for understanding collisions, such as that shown in the above image. It is just as powerful, just as important, and just as useful as conservation of energy and the work-energy theorem.

9.2: Linear Momentum

Learning Objectives

- Explain what momentum is, physically
- Calculate the momentum of a moving object

Our study of kinetic energy showed that a complete understanding of an object's motion must include both its mass and its velocity

$$K = \left(\frac{1}{2}\right) mv^2. \tag{9.2.1}$$

However, as powerful as this concept is, it does not include any information about the direction of the moving object's velocity vector (e.g. the ball in Figure $9.2.1$). We'll now define a physical quantity that includes direction.

Figure $9.2.1$: The velocity and momentum vectors for the ball are in the same direction. The mass of the ball is about 0.5 kg, so the momentum vector is about half the length of the velocity vector because momentum is velocity time mass. (credit: modification of work by Ben Sutherland)

Like kinetic energy, this quantity includes both mass and velocity; like kinetic energy, it is a way of characterizing the "quantity of motion" of an object. It is given the name **momentum** (from the Latin word **movimentum**, meaning "movement"), and it is represented by the symbol p.

Definition: Momentum

The linear momentum p of an object is the product of its mass and its velocity:

$$\vec{p} = m\vec{v}. \tag{9.2.2}$$

As shown in Figure $9.2.1$, momentum is a vector quantity (since velocity is). This is one of the things that makes momentum useful and not a duplication of kinetic energy. It is perhaps most useful when determining whether an object's motion is difficult to change (Figure $9.2.1$) or easy to change (Figure $9.2.2$).

Figure 9.2.2: This supertanker transports a huge mass of oil; as a consequence, it takes a long time for a force to change its (comparatively small) velocity. (credit: modification of work by “the_tahoe_guy”/Flickr)

Unlike kinetic energy, momentum depends equally on an object’s mass and velocity. For example, as you will learn when you study thermodynamics, the average speed of an air molecule at room temperature (Figure 9.2.3) is approximately 500 m/s, with an average molecular mass of $6 \times 10^{-25}\ kg$; its momentum is thus

$$\begin{aligned} p_{molecule} &= (6 \times 10^{-25}\ kg)(500\ m/s) \\ &= 3 \times 10^{-22}\ kg \cdot m/s. \end{aligned}$$

For comparison, a typical automobile might have a speed of only 15 m/s, but a mass of 1400 kg, giving it a momentum of

$$\begin{aligned} p_{car} &= (1400\ kg)(15\ m/s) \\ &= 21,000\ kg \cdot m/s. \end{aligned}$$

These momenta are different by 27 orders of magnitude, or a factor of a billion billion billion!

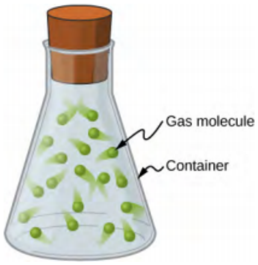


Figure 9.2.3: Gas molecules can have very large velocities, but these velocities change nearly instantaneously when they collide with the container walls or with each other. This is primarily because their masses are so tiny.

9.3: Impulse and Collisions (Part 1)

Learning Objectives

- Explain what an impulse is, physically
- Describe what an impulse does
- Relate impulses to collisions
- Apply the impulse-momentum theorem to solve problems

We have defined momentum to be the product of mass and velocity. Therefore, if an object's velocity should change (due to the application of a force on the object), then necessarily, its momentum changes as well. This indicates a connection between momentum and force. The purpose of this section is to explore and describe that connection.

Suppose you apply a force on a free object for some amount of time. Clearly, the larger the force, the larger the object's change of momentum will be. Alternatively, the more time you spend applying this force, again the larger the change of momentum will be, as depicted in Figure $9.3.1$. The amount by which the object's motion changes is therefore proportional to the magnitude of the force, and also to the time interval over which the force is applied.

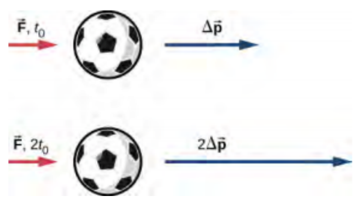


Figure $9.3.1$: The change in momentum of an object is proportional to the length of time during which the force is applied. If a force is exerted on the lower ball for twice as long as on the upper ball, then the change in the momentum of the lower ball is twice that of the upper ball.

Mathematically, if a quantity is proportional to two (or more) things, then it is proportional to the product of those things. The product of a force and a time interval (over which that force acts) is called impulse, and is given the symbol $\vec{J}$.

Definition: Impulse

Let $\vec{F}$(t) be the force applied to an object over some differential time interval dt (Figure $9.3.2$). The resulting impulse on the object is defined as

$$d\vec{J} \equiv \vec{F}(t)dt. \tag{9.3.1}$$

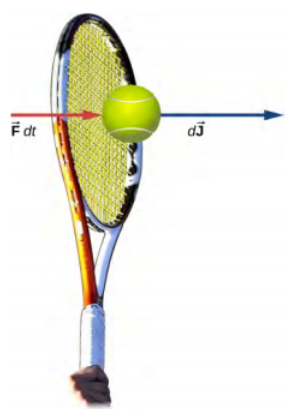


Figure $9.3.2$: A force applied by a tennis racquet to a tennis ball over a time interval generates an impulse acting on the ball.

The total impulse over the interval $t_f - t_i$ is

$$\vec{J} = \int_{t_i}^{t_f} d\vec{J} \tag{9.3.2}$$

or

$$\vec{J} \equiv \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{9.3.3}$$

Equations 9.3.1 and 9.3.3 together say that when a force is applied for an infinitesimal time interval dt, it causes an infinitesimal impulse $d\vec{J}$, and the total impulse given to the object is defined to be the sum (integral) of all these infinitesimal impulses.

To calculate the impulse using Equation 9.3.3, we need to know the force function F(t), which we often don't. However, a result from calculus is useful here: Recall that the average value of a function over some interval is calculated by

$$f(x)_{ave} = \frac{1}{\Delta x} \int_{x_i}^{x_f} f(x)dx \tag{9.3.4}$$

where $\Delta x = x_f - x_i$. Applying this to the time-dependent force function, we obtain

$$\vec{F}_{ave} = \frac{1}{\Delta t} \int_{t_i}^{t_f} \vec{F}(t)dt. \tag{9.3.5}$$

Therefore, from Equation 9.3.3,

$$\vec{J} = \vec{F}_{ave}\Delta t. \tag{9.3.6}$$

The idea here is that you can calculate the impulse on the object even if you don't know the details of the force as a function of time; you only need the average force. In fact, though, the process is usually reversed: You determine the impulse (by measurement or calculation) and then calculate the average force that caused that impulse.

To calculate the impulse, a useful result follows from writing the force in Equation 9.3.3 as $\vec{F}(t) = m\,\vec{a}(t)$:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt = m \int_{t_i}^{t_f} \vec{a}(t)dt = m\left[\vec{v}(t_f) - \vec{v}(t_i)\right]. \tag{9.3.7}$$

For a constant force $\vec{F}_{ave} = \vec{F} = m\vec{a}$, this simplifies to

$$\vec{J} = m\vec{a}\Delta t = m\vec{v}_f - m\vec{v}_i = m(\vec{v}_f - \vec{v}_i). \tag{9.3.8}$$

That is,

$$\vec{J} = m\Delta\vec{v}. \tag{9.3.9}$$

Note that the integral form, Equation 9.3.3, applies to constant forces as well; in that case, since the force is independent of time, it comes out of the integral, which can then be trivially evaluated.

Example 9.3.1: The Arizona Meteor Crater

Approximately 50,000 years ago, a large (radius of 25 m) iron-nickel meteorite collided with Earth at an estimated speed of 1.28×10^4 m/s in what is now the northern Arizona desert, in the United States. The impact produced a crater that is still visible today (Figure 9.3.3); it is approximately 1200 m (three-quarters of a mile) in diameter, 170 m deep, and has a rim that rises 45 m above the surrounding desert plain. Iron-nickel meteorites typically have a density of $\rho = 7970\ kg/m^3$. Use impulse considerations to estimate the average force and the maximum force that the meteor applied to Earth during the impact.

Figure 9.3.3: The Arizona Meteor Crater in Flagstaff, Arizona (often referred to as the Barringer Crater after the person who first suggested its origin and whose family owns the land). (credit: “Shane.torgerson”/Wikimedia Commons)

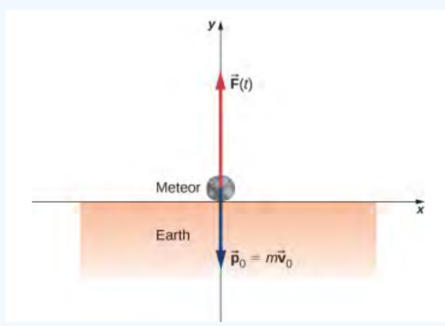
Strategy

It is conceptually easier to reverse the question and calculate the force that Earth applied on the meteor in order to stop it. Therefore, we’ll calculate the force on the meteor and then use Newton’s third law to argue that the force from the meteor on Earth was equal in magnitude and opposite in direction.

Using the given data about the meteor, and making reasonable guesses about the shape of the meteor and impact time, we first calculate the impulse using Equation 9.3.9. We then use the relationship between force and impulse Equation 9.3.6 to estimate the average force during impact. Next, we choose a reasonable force function for the impact event, calculate the average value of that function Equation 9.3.5, and set the resulting expression equal to the calculated average force. This enables us to solve for the maximum force.

Solution

Define upward to be the +y-direction. For simplicity, assume the meteor is traveling vertically downward prior to impact. In that case, its initial velocity is $\vec{v}_i = -v_i\,\hat{j}$, and the force Earth exerts on the meteor points upward, $\vec{F}(t) = +F(t)\,\hat{j}$. The situation at t = 0 is depicted below.



The average force during the impact is related to the impulse by

$$\vec{F}_{ave} = \frac{\vec{J}}{\Delta t}. \tag{9.3.10}$$

From Equation 9.3.9, $\vec{J} = m\Delta\vec{v}$, so we have

$$\vec{F}_{ave} = \frac{m\Delta\vec{v}}{\Delta t}. \tag{9.3.11}$$

The mass is equal to the product of the meteor’s density and its volume:

$$m = \rho V. \tag{9.3.12}$$

If we assume (guess) that the meteor was roughly spherical, we have

$$V = \frac{4}{3}\pi R^3. \tag{9.3.13}$$

Thus we obtain

$$\vec{F}_{ave}=\frac{\rho V \Delta \vec{v}}{\Delta t}=\frac{\rho\left(\frac{4}{3} \pi R^3\right)\left(\vec{v}_f-\vec{v}_i\right)}{\Delta t}. \tag{9.3.14}$$

The problem says the velocity at impact was −1.28 x 10^4 m/s $\hat{j}$ (the final velocity is zero); also, we guess that the primary impact lasted about t_{max} = 2 s. Substituting these values gives

$$\begin{aligned} \vec{F}_{ave} &= \frac{\left(7970\ kg/m^3\right)\left[\frac{4}{3}\pi (25\ m)^3\right]\left[0\ m/s-\left(-1.28\times 10^4\ m/s\ \hat{j}\right)\right]}{2\ s} \\ &= +\left(3.33\times 10^{12}\ N\right)\hat{j} \end{aligned}$$

This is the average force applied during the collision. Notice that this force vector points in the same direction as the change of velocity vector $\Delta\vec{v}$.

Next, we calculate the maximum force. The impulse is related to the force function by

$$\vec{J}=\int_{t_i}^{t_{max}} \vec{F}(t) dt. \tag{9.3.15}$$

We need to make a reasonable choice for the force as a function of time. We define t = 0 to be the moment the meteor first touches the ground. Then we assume the force is a maximum at impact, and rapidly drops to zero. A function that does this is

$$F(t)=F_{max} e^{\frac{-t^2}{2\tau^2}}. \tag{9.3.16}$$

The parameter τ represents how rapidly the force decreases to zero.) The average force is

$$F_{ave}=\frac{1}{\Delta t} \int_0^{t_{max}} F_{max} e^{\frac{-t^2}{2\tau^2}} dt \tag{9.3.17}$$

where $\Delta t = t_{max} - 0$ s. Since we already have a numeric value for F_{ave}, we can use the result of the integral to obtain F_{max}. Choosing $\tau = \frac{1}{e} t_{max}$ (this is a common choice, as you will see in later chapters), and guessing that t_{max} = 2 s, this integral evaluates to

$$F_{avg}=0.458\ F_{max}. \tag{9.3.18}$$

Thus, the maximum force has a magnitude of

$$\begin{aligned} 0.458\ F_{max} &= 3.33\times 10^{12}\ N \\ F_{max} &= 7.27\times 10^{12}\ N. \end{aligned}$$

The complete force function, including the direction, is

$$\vec{F}(t)=\left(7.27\times 10^{12}\ N\right) e^{\frac{-t^2}{8\ s^2}} \hat{y}. \tag{9.3.19}$$

This is the force Earth applied to the meteor; by Newton’s third law, the force the meteor applied to Earth is

$$\vec{F}(t)=-\left(7.27\times 10^{12}\ N\right) e^{\frac{-t^2}{8\ s^2}} \hat{y} \tag{9.3.20}$$

which is the answer to the original question.

Significance

The graph of this function contains important information. Let’s graph (the magnitude of) both this function and the average force together (Figure 9.3.4).

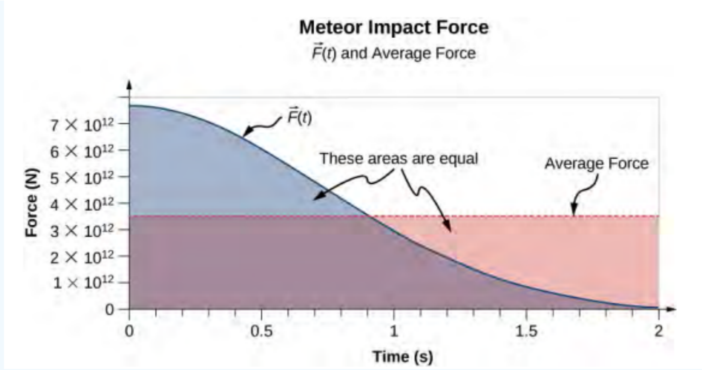


Figure 9.3.4: A graph of the average force (in red) and the force as a function of time (blue) of the meteor impact. The areas under the curves are equal to each other, and are numerically equal to the applied impulse.

Notice that the area under each plot has been filled in. For the plot of the (constant) force F_{ave}, the area is a rectangle, corresponding to F_{ave} Δt = J. As for the plot of F(t), recall from calculus that the area under the plot of a function is numerically equal to the integral of that function, over the specified interval; so here, that is $\int_0^{t_{max}} F(t)dt = J$. Thus, the areas are equal, and both represent the impulse that the meteor applied to Earth during the two-second impact. The average force on Earth sounds like a huge force, and it is. Nevertheless, Earth barely noticed it. The acceleration Earth obtained was just

$$\vec{a} = \frac{-\vec{F}_{ave}}{M_{Earth}} = \frac{-(3.33 \times 10^{12}\ N)\hat{j}}{5.97 \times 10^{24}\ kg} = -(5.6 \times 10^{-13} m/s^2)\hat{j} \tag{9.3.21}$$

which is completely immeasurable. That said, the impact created seismic waves that nowadays could be detected by modern monitoring equipment.

✓ Example 9.3.2: The Benefits of Impulse

A car traveling at 27 m/s collides with a building. The collision with the building causes the car to come to a stop in approximately 1 second. The driver, who weighs 860 N, is protected by a combination of a variable-tension seatbelt and an airbag (Figure 9.3.5). (In effect, the driver collides with the seatbelt and airbag and not with the building.) The airbag and seatbelt slow his velocity, such that he comes to a stop in approximately 2.5 s.

a. What average force does the driver experience during the collision?
b. Without the seatbelt and airbag, his collision time (with the steering wheel) would have been approximately 0.20 s. What force would he experience in this case?

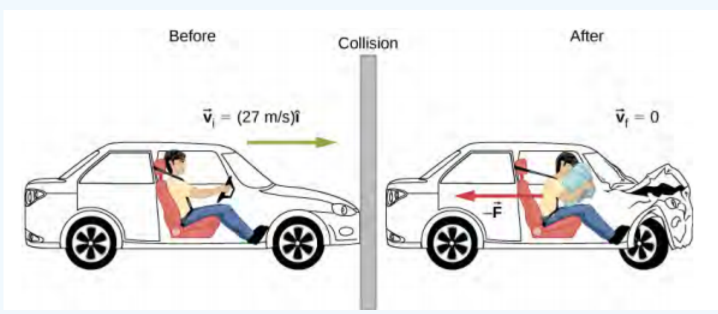


Figure 9.3.5: The motion of a car and its driver at the instant before and the instant after colliding with the wall. The restrained driver experiences a large backward force from the seatbelt and airbag, which causes his velocity to decrease to zero. (The forward force from the seatback is much smaller than the backward force, so we neglect it in the solution.)

Strategy

We are given the driver's weight, his initial and final velocities, and the time of collision; we are asked to calculate a force. Impulse seems the right way to tackle this; we can combine Equation 9.3.6 and Equation 9.3.9.

Solution

a. Define the +x-direction to be the direction the car is initially moving. We know

$$\vec{J}=\vec{F}\Delta t \tag{9.3.22}$$

and

$$\vec{J}=m\Delta\vec{v}. \tag{9.3.23}$$

Since J is equal to both those things, they must be equal to each other:

$$\vec{F}\Delta t=m\Delta\vec{v}. \tag{9.3.24}$$

We need to convert this weight to the equivalent mass, expressed in SI units:

$$\frac{860\ N}{9.8\ m/s^2}=87.8\ kg. \tag{9.3.25}$$

Remembering that $\Delta\vec{v}=\vec{v}_f-\vec{v}_i$, and noting that the final velocity is zero, we solve for the force:

$$\vec{F}=m\frac{0-v_i\ \hat{i}}{\Delta t}=(87.8\ kg)\left(\frac{-(27\ m/s)\hat{i}}{2.5\ s}\right)=-(948\ N)\hat{i}. \tag{9.3.26}$$

The negative sign implies that the force slows him down. For perspective, this is about 1.1 times his own weight.

b. Same calculation, just the different time interval:

$$\vec{F}=(87.8\ kg)\left(\frac{-(27\ m/s)\hat{i}}{0.20\ s}\right)=-(11,853\ N)\hat{i}. \tag{9.3.27}$$

which is about 14 times his own weight. Big difference!

Significance

You see that the value of an airbag is how greatly it reduces the force on the vehicle occupants. For this reason, they have been required on all passenger vehicles in the United States since 1991, and have been commonplace throughout Europe and Asia since the mid-1990s. The change of momentum in a crash is the same, with or without an airbag; the force, however, is vastly different.

9.4: Impulse and Collisions (Part 2)

Effect of Impulse

Since an impulse is a force acting for some amount of time, it causes an object's motion to change. Recall

$$\vec{J} = m\Delta\vec{v}. \tag{9.4.1}$$

Because m$\vec{v}$ is the momentum of a system, m$\Delta\vec{v}$ is the change of momentum $\Delta\vec{p}$. This gives us the following relation, called the **impulse-momentum theorem** (or relation).

Impulse-Momentum Theorem

An impulse applied to a system changes the system's momentum, and that change of momentum is exactly equal to the impulse that was applied:

$$\vec{J} = \Delta\vec{p}. \tag{9.4.2}$$

The impulse-momentum theorem is depicted graphically in Figure $9.4.1$.

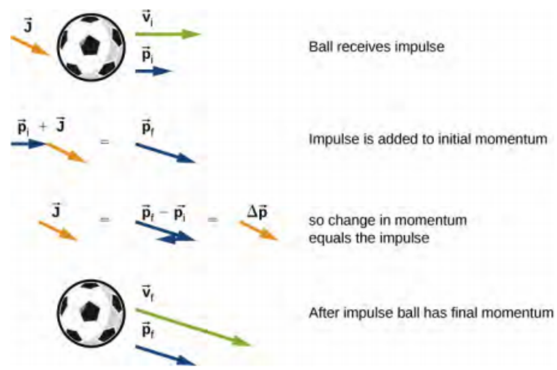


Figure $9.4.1$: Illustration of impulse-momentum theorem. (a) A ball with initial velocity $\vec{v}_0$ and momentum $\vec{p}_0$ receives an impulse $\vec{J}$. (b) This impulse is added vectorially to the initial momentum. (c) Thus, the impulse equals the change in momentum, $\vec{J} = \Delta\vec{p}$. (d) After the impulse, the ball moves off with its new momentum $\vec{p}_f$.

There are two crucial concepts in the impulse-momentum theorem:

1. Impulse is a vector quantity; an impulse of, say, −(10 N • s) $\hat{i}$ is very different from an impulse of +(10 N • s) $\hat{i}$; they cause completely opposite changes of momentum.
2. An impulse does not cause momentum; rather, it causes a change in the momentum of an object. Thus, you must subtract the final momentum from the initial momentum, and—since momentum is also a vector quantity—you must take careful account of the signs of the momentum vectors.

The most common questions asked in relation to impulse are to calculate the applied force, or the change of velocity that occurs as a result of applying an impulse. The general approach is the same.

Problem-Solving Strategy: Impulse-Momentum Theorem

1. Express the impulse as force times the relevant time interval.
2. Express the impulse as the change of momentum, usually mΔv.
3. Equate these and solve for the desired quantity.

Enterprise

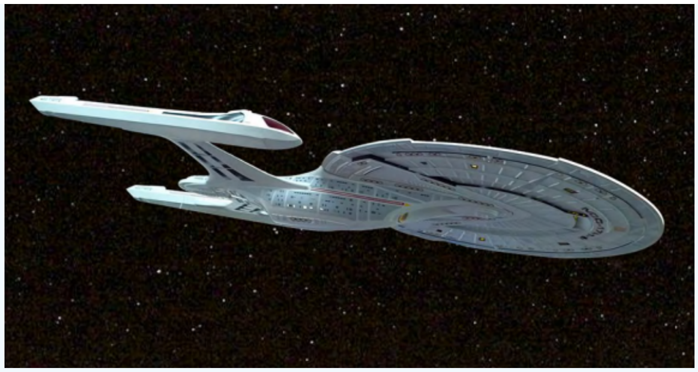


Figure 9.4.2: The fictional starship Enterprise from the Star Trek adventures operated on so-called "impulse engines" that combined matter with antimatter to produce energy.

"Mister Sulu, take us out; ahead one-quarter impulse." With this command, Captain Kirk of the starship **Enterprise** (Figure 9.4.2) has his ship start from rest to a final speed of $v_f = \frac{1}{4}(3.0 \times 10^8\ m/s)$. Assuming this maneuver is completed in 60 s, what average force did the impulse engines apply to the ship?

Strategy

We are asked for a force; we know the initial and final speeds (and hence the change in speed), and we know the time interval over which this all happened. In particular, we know the amount of time that the force acted. This suggests using the impulse-momentum relation. To use that, though, we need the mass of the **Enterprise**. An internet search gives a best estimate of the mass of the **Enterprise** (in the 2009 movie) as 2 x 10^9 kg.

Solution

Because this problem involves only one direction (i.e., the direction of the force applied by the engines), we only need the scalar form of the impulse-momentum theorem Equation 9.4.2, which is

$$\Delta p = J \tag{9.4.3}$$

with

$$\Delta p = m\Delta v \tag{9.4.4}$$

and

$$J = F\Delta t. \tag{9.4.5}$$

Equating these expressions gives

$$F\Delta t = m\Delta v. \tag{9.4.6}$$

Solving for the magnitude of the force and inserting the given values leads to

$$F = \frac{m\Delta v}{\Delta t} = \frac{(2\times 10^9\ kg)(7.35\times 10^7\ m/s)}{60\ s} = 2.5\times 10^{15}\ N. \tag{9.4.7}$$

Significance

This is an unimaginably huge force. It goes almost without saying that such a force would kill everyone on board instantly, as well as destroying every piece of equipment. Fortunately, the **Enterprise** has "inertial dampeners." It is left as an exercise for the reader's imagination to determine how these work.

? EXercise 9.4.1

The U.S. Air Force uses "10gs" (an acceleration equal to 10 x 9.8 m/s^2) as the maximum acceleration a human can withstand (but only for several seconds) and survive. How much time must the **Enterprise** spend accelerating if the humans on board are to experience an average of at most 10gs of acceleration? (Assume the inertial dampeners are offline.)

Example 9.4.2: The iPhone Drop

Apple released its iPhone 6 Plus in November 2014. According to many reports, it was originally supposed to have a screen made from sapphire, but that was changed at the last minute for a hardened glass screen. Reportedly, this was because the sapphire screen cracked when the phone was dropped. What force did the iPhone 6 Plus experience as a result of being dropped?

Strategy

The force the phone experiences is due to the impulse applied to it by the floor when the phone collides with the floor. Our strategy then is to use the impulse-momentum relationship. We calculate the impulse, estimate the impact time, and use this to calculate the force. We need to make a couple of reasonable estimates, as well as find technical data on the phone itself. First, let's suppose that the phone is most often dropped from about chest height on an average-height person. Second, assume that it is dropped from rest, that is, with an initial vertical velocity of zero. Finally, we assume that the phone bounces very little—the height of its bounce is assumed to be negligible.

Solution

Define upward to be the +y-direction. A typical height is approximately h = 1.5 m and, as stated, $\vec{v}_i$ = (0 m/s) $\hat{i}$. The average force on the phone is related to the impulse the floor applies on it during the collision:

$$\vec{F}_{ave} = \frac{\vec{J}}{\Delta t}. \tag{9.4.8}$$

The impulse $\vec{J}$ equals the change in momentum,

$$\vec{J} = \Delta \vec{p} \tag{9.4.9}$$

so

$$\vec{F}_{ave} = \frac{\Delta \vec{p}}{\Delta t}. \tag{9.4.10}$$

Next, the change of momentum is

$$\Delta \vec{p} = m\Delta \vec{v}. \tag{9.4.11}$$

We need to be careful with the velocities here; this is the change of velocity due to the collision with the floor. But the phone also has an initial drop velocity [$\vec{v}_i$ = (0 m/s) $\hat{j}$], so we label our velocities. Let:

- $\vec{v}_i$ = the initial velocity with which the phone was dropped (zero, in this example)
- $\vec{v}_1$ = the velocity the phone had the instant just before it hit the floor
- $\vec{v}_2$ = the final velocity of the phone as a result of hitting the floor

Figure 9.4.3 shows the velocities at each of these points in the phone's trajectory.

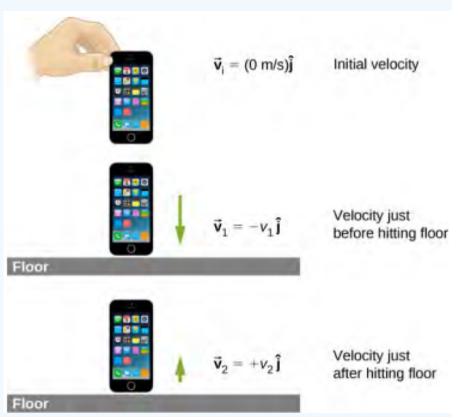


Figure 9.4.3: (a) The initial velocity of the phone is zero, just after the person drops it. (b) Just before the phone hits the floor, its velocity is $\vec{v}_1$, which is unknown at the moment, except for its direction, which is downward ($-\hat{j}$). (c) After bouncing off the floor, the phone has a velocity $\vec{v}_2$, which is also unknown, except for its direction, which is upward ($+\hat{j}$).

With these definitions, the change of momentum of the phone during the collision with the floor is

$$m\Delta\vec{v} = m(\vec{v}_2 - \vec{v}_1). \tag{9.4.12}$$

Since we assume the phone doesn't bounce at all when it hits the floor (or at least, the bounce height is negligible), then $\vec{v}_2$ is zero, so

$$m\Delta\vec{v} = m\left[0 - (-v_1\ \hat{j})\right] \tag{9.4.13}$$

$$m\Delta\vec{v} = +mv_1\ \hat{j}. \tag{9.4.14}$$

We can get the speed of the phone just before it hits the floor using either kinematics or conservation of energy. We'll use conservation of energy here; you should re-do this part of the problem using kinematics and prove that you get the same answer.

First, define the zero of potential energy to be located at the floor. Conservation of energy then gives us:

$$\begin{aligned} E_i &= E_1 \\ K_i + U_i &= K_1 + U_1 \\ \frac{1}{2}mv_i^2 + mgh_{drop} &= \frac{1}{2}mv_1^2 + mgh_{floor}. \end{aligned}$$

Defining $h_{floor} = 0$ and using $\vec{v}_i = (0\text{ m/s})\ \hat{j}$ gives

$$\begin{aligned} \frac{1}{2}mv_1^2 &= mgh_{drop} \\ v_1 &= \pm\sqrt{2gh_{drop}}. \end{aligned}$$

Because v_1 is a vector magnitude, it must be positive. Thus, $m\Delta v = mv_1 = m\sqrt{2gh_{drop}}$. Inserting this result into the expression for force gives

$$\begin{aligned} \vec{F} &= \frac{\Delta\vec{p}}{\Delta t} \\ &= \frac{m\Delta\vec{v}}{\Delta t} \\ &= \frac{+mv_1\ \hat{j}}{\Delta t} \\ &= \frac{m\sqrt{2gh}}{\Delta t}\ \hat{j}. \end{aligned}$$

Finally, we need to estimate the collision time. One common way to estimate a collision time is to calculate how long the object would take to travel its own length. The phone is moving at 5.4 m/s just before it hits the floor, and it is 0.14 m long, giving an estimated collision time of 0.026 s. Inserting the given numbers, we obtain

$$\vec{F} = \frac{(0.172\ kg)\sqrt{2(9.8\ m/s^2)(1.5\ m)}}{0.026\ s}\ \hat{j} = (36\ N)\hat{j}. \tag{9.4.15}$$

Significance

The iPhone itself weighs just (0.172 kg)(9.81 m/s^2) = 1.68 N; the force the floor applies to it is therefore over 20 times its weight.

Exercise 9.4.2

What if we had assumed the phone **did** bounce on impact? Would this have increased the force on the iPhone, decreased it, or made no difference?

Momentum and Force

In Example 9.4.1, we obtained an important relationship:

$$\vec{F}_{ave} = \frac{\Delta\vec{p}}{\Delta t}. \tag{9.4.16}$$

In words, the average force applied to an object is equal to the change of the momentum that the force causes, divided by the time interval over which this change of momentum occurs. This relationship is very useful in situations where the collision time Δt is small, but measurable; typical values would be 1/10th of a second, or even one thousandth of a second. Car crashes, punting a football, or collisions of subatomic particles would meet this criterion.

For a **continuously** changing momentum—due to a continuously changing force—this becomes a powerful conceptual tool. In the limit Δt → dt, Equation 9.3.1 becomes

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{9.4.17}$$

This says that the rate of change of the system's momentum (implying that momentum is a function of time) is exactly equal to the net applied force (also, in general, a function of time). This is, in fact, Newton's second law, written in terms of momentum rather than acceleration. This is the relationship Newton himself presented in his **Principia Mathematica** (although he called it "quantity of motion" rather than "momentum").

If the mass of the system remains constant, Equation 9.3.3 reduces to the more familiar form of Newton's second law. We can see this by substituting the definition of momentum:

$$\vec{F} = \frac{d(m\vec{v})}{dt} = m\frac{d\vec{v}}{dt} = m\vec{a}. \tag{9.4.18}$$

The assumption of constant mass allowed us to pull m out of the derivative. If the mass is not constant, we cannot use this form of the second law, but instead must start from Equation 9.3.3. Thus, one advantage to expressing force in terms of changing momentum is that it allows for the mass of the system to change, as well as the velocity; this is a concept we'll explore when we study the motion of rockets.

Newton's Second Law of Motion in Terms of Momentum

The net external force on a system is equal to the rate of change of the momentum of that system caused by the force:

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{9.4.19}$$

Although Equation 9.3.3 allows for changing mass, as we will see in Rocket Propulsion, the relationship between momentum and force remains useful when the mass of the system is constant, as in the following example.

Example 9.4.3: Calculating Force: Venus Williams' Tennis Serve

During the 2007 French Open, Venus Williams hit the fastest recorded serve in a premier women's match, reaching a speed of 58 m/s (209 km/h). What is the average force exerted on the 0.057-kg tennis ball by Venus Williams' racquet? Assume that the ball's speed just after impact is 58 m/s, as shown in Figure 9.4.4, that the initial horizontal component of the velocity before impact is negligible, and that the ball remained in contact with the racquet for 5.0 ms.

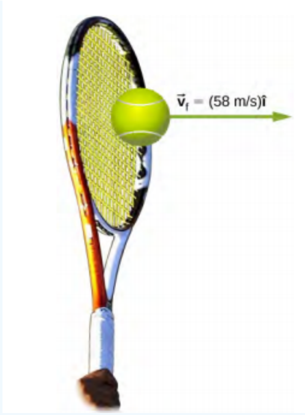


Figure 9.4.4: The final velocity of the tennis ball is $\vec{v}_f$ = (58 m/s) $\hat{i}$.

Strategy

This problem involves only one dimension because the ball starts from having no horizontal velocity component before impact. Newton's second law stated in terms of momentum is then written as

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{9.4.20}$$

As noted above, when mass is constant, the change in momentum is given by

$$\Delta p = m\Delta v = m(v_f - v_i) \tag{9.4.21}$$

where we have used scalars because this problem involves only one dimension. In this example, the velocity just after impact and the time interval are given; thus, once Δp is calculated, we can use F = $\frac{\Delta p}{\Delta t}$ to find the force.

Solution

To determine the change in momentum, insert the values for the initial and final velocities into the equation above:

$$\begin{aligned} \Delta p &= m(v_f - v_i) \\ &= (0.057\ kg)(58\ m/s - 0\ m/s) \\ &= 3.3\ kg \cdot m/s. \end{aligned}$$

Now the magnitude of the net external force can be determined by using

$$F = \frac{\Delta p}{\Delta t} = \frac{3.3\ kg \cdot m/s}{5.0 \times 10^{-3}\ s} = 6.6 \times 10^2\ N. \tag{9.4.22}$$

where we have retained only two significant figures in the final step.

Significance

This quantity was the average force exerted by Venus Williams' racquet on the tennis ball during its brief impact (note that the ball also experienced the 0.57-N force of gravity, but that force was not due to the racquet). This problem could also be solved by first finding the acceleration and then using F = ma, but one additional step would be required compared with the strategy used in this example.

9.5: Conservation of Linear Momentum (Part 1)

Learning Objectives

- Explain the meaning of “conservation of momentum”
- Correctly identify if a system is, or is not, closed
- Define a system whose momentum is conserved
- Mathematically express conservation of momentum for a given system
- Calculate an unknown quantity using conservation of momentum

Recall Newton’s third law: When two objects of masses m_1 and m_2 interact (meaning that they apply forces on each other), the force that object 2 applies to object 1 is equal in magnitude and opposite in direction to the force that object 1 applies on object 2. Let:

- $\vec{F}_{21}$ = the force on m_1 from m_2
- $\vec{F}_{12}$ = the force on m_2 from m_1

Then, in symbols, Newton’s third law says

$$\begin{aligned} \vec{F}_{21} &= -\vec{F}_{12} \\ m_1\vec{a}_1 &= -m_2\vec{a}_2. \end{aligned}$$

(Recall that these two forces do not cancel because they are applied to different objects. F_{21} causes m_1 to accelerate, and F_{12} causes m_2 to accelerate.)

Although the magnitudes of the forces on the objects are the same, the accelerations are not, simply because the masses (in general) are different. Therefore, the changes in velocity of each object are different:

$$\frac{d\vec{v}_1}{dt} \neq \frac{d\vec{v}_2}{dt}. \tag{9.5.1}$$

However, the products of the mass and the change of velocity are equal (in magnitude):

$$m_1\frac{d\vec{v}_1}{dt} = -m_2\frac{d\vec{v}_2}{dt}. \tag{9.5.2}$$

It’s a good idea, at this point, to make sure you’re clear on the physical meaning of the derivatives in Equation 9.3.3. Because of the interaction, each object ends up getting its velocity changed, by an amount dv. Furthermore, the interaction occurs over a time interval dt, which means that the change of velocities also occurs over dt. This time interval is the same for each object.

Let‘s assume, for the moment, that the masses of the objects do not change during the interaction. (We’ll relax this restriction later.) In that case, we can pull the masses inside the derivatives:

$$\frac{d}{dt}(m_1\vec{v}_1) = -\frac{d}{dt}(m_2\vec{v}_2) \tag{9.5.3}$$

and thus

$$\frac{d\vec{p}_1}{dt} = -\frac{d\vec{p}_2}{dt}. \tag{9.5.4}$$

This says that **the rate at which momentum changes is the same for both objects**. The masses are different, and the changes of velocity are different, but the rate of change of the product of m and $\vec{v}$ are the same.

Physically, this means that during the interaction of the two objects (m_1 and m_2), both objects have their momentum changed; but those changes are identical in magnitude, though opposite in sign. For example, the momentum of object 1 might increase, which means that the momentum of object 2 decreases by exactly the same amount.

In light of this, let’s re-write Equation 9.5.3 in a more suggestive form:

$$\frac{d\vec{p}_1}{dt} + \frac{d\vec{p}_2}{dt} = 0. \tag{9.5.5}$$

This says that during the interaction, although object 1's momentum changes, and object 2's momentum also changes, these two changes cancel each other out, so that the total change of momentum of the two objects together is zero.

Since the total combined momentum of the two objects together never changes, then we could write

$$\frac{d}{dt}(\vec{p}_1 + \vec{p}_2) = 0 \tag{9.5.6}$$

from which it follows that

$$\vec{p}_1 + \vec{p}_2 = constant. \tag{9.5.7}$$

As shown in Figure 9.5.1, the total momentum of the system before and after the collision remains the same.

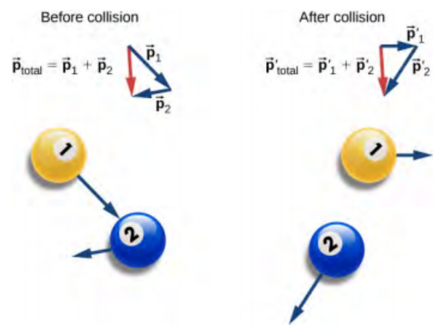


Figure 9.5.1: Before the collision, the two billiard balls travel with momenta $\vec{p}_1$ and $\vec{p}_2$. The total momentum of the system is the sum of these, as shown by the red vector labeled $\vec{p}_{total}$ on the left. After the collision, the two billiard balls travel with different momenta $\vec{p}\,'_1$ and $\vec{p}\,'_2$. The total momentum, however, has not changed, as shown by the red vector arrow $\vec{p}\,'_{total}$ on the right.

Generalizing this result to N objects, we obtain

$$\vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \cdots + \vec{p}_N = constant \tag{9.5.8}$$

$$\sum_{j=1}^{N} \vec{p}_j = constant. \tag{9.5.9}$$

Equation 9.5.9 is the definition of the total (or net) momentum of a system of N interacting objects, along with the statement that the total momentum of a system of objects is constant in time—or better, is conserved.

Conservation Laws

If the value of a physical quantity is constant in time, we say that the quantity is conserved.

Requirements for Momentum Conservation

There is a complication, however. A system must meet two requirements for its momentum to be conserved:

1. **The mass of the system must remain constant during the interaction.** As the objects interact (apply forces on each other), they may transfer mass from one to another; but any mass one object gains is balanced by the loss of that mass from another. The total mass of the system of objects, therefore, remains unchanged as time passes: \[\Big[\frac{dm}{dt} \Big]_{system} = 0 \ldotp$$
2. **The net external force on the system must be zero**. As the objects collide, or explode, and move around, they exert forces on each other. However, all of these forces are internal to the system, and thus each of these internal forces is balanced by another internal force that is equal in magnitude and opposite in sign. As a result, the change in momentum caused by each internal force is cancelled by another momentum change that is equal in magnitude and opposite in direction. Therefore, internal forces cannot change the total momentum of a system because the changes sum to zero. However, if there is some external force that acts on all of the objects (gravity, for example, or friction), then this force changes the momentum of the system as a whole; that is to say, the momentum of the system is changed by the external force. Thus, for the momentum of the system to be conserved, we must have

$$\vec{F}_{ext} = \vec{0}. \tag{9.5.10}$$

A system of objects that meets these two requirements is said to be a **closed system** (also called an isolated system). Thus, the more compact way to express this is shown below.

Law of Conservation of Momentum

The total momentum of a closed system is conserved:

$$\sum_{j=1}^{N} \vec{p}_j = constant. \tag{9.5.11}$$

This statement is called the **Law of Conservation of Momentum**. Along with the conservation of energy, it is one of the foundations upon which all of physics stands. All our experimental evidence supports this statement: from the motions of galactic clusters to the quarks that make up the proton and the neutron, and at every scale in between. In **a closed system, the total momentum never changes**.

Note that there absolutely **can** be external forces acting on the system; but for the system's momentum to remain constant, these external forces have to cancel, so that the **net** external force is zero. Billiard balls on a table all have a weight force acting on them, but the weights are balanced (canceled) by the normal forces, so there is no net force.

The Meaning of 'System'

A **system** (mechanical) is the collection of objects in whose motion (kinematics and dynamics) you are interested. If you are analyzing the bounce of a ball on the ground, you are probably only interested in the motion of the ball, and not of Earth; thus, the ball is your system. If you are analyzing a car crash, the two cars together compose your system (Figure $9.5.2$).

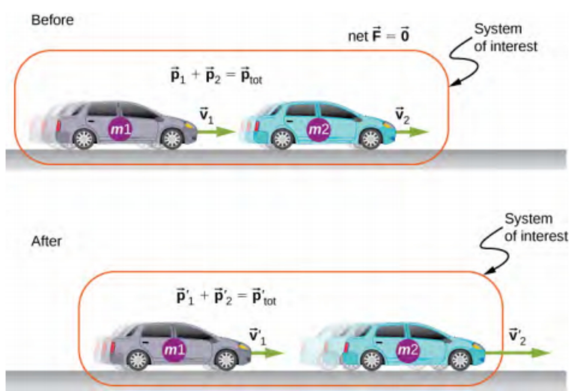


Figure $9.5.2$: The two cars together form the system that is to be analyzed. It is important to remember that the contents (the mass) of the system do not change before, during, or after the objects in the system interact.

9.6: Conservation of Linear Momentum (Part 2)

Problem-Solving Strategy: Conservation of Momentum

Using conservation of momentum requires four basic steps. The first step is crucial:

1. Identify a closed system (total mass is constant, no net external force acts on the system).
2. Write down an expression representing the total momentum of the system before the "event" (explosion or collision).
3. Write down an expression representing the total momentum of the system after the "event."
4. Set these two expressions equal to each other, and solve this equation for the desired quantity

Example 9.6.1: Colliding Carts

Two carts in a physics lab roll on a level track, with negligible friction. These carts have small magnets at their ends, so that when they collide, they stick together (Figure 9.6.1). The first cart has a mass of 675 grams and is rolling at 0.75 m/s to the right; the second has a mass of 500 grams and is rolling at 1.33 m/s, also to the right. After the collision, what is the velocity of the two joined carts?

Figure 9.6.1: Two lab carts collide and stick together after the collision.

Strategy

We have a collision. We're given masses and initial velocities; we're asked for the final velocity. This all suggests using conservation of momentum as a method of solution. However, we can only use it if we have a closed system. So we need to be sure that the system we choose has no net external force on it, and that its mass is not changed by the collision.

Defining the system to be the two carts meets the requirements for a closed system: The combined mass of the two carts certainly doesn't change, and while the carts definitely exert forces on each other, those forces are internal to the system, so they do not change the momentum of the system as a whole. In the vertical direction, the weights of the carts are canceled by the normal forces on the carts from the track.

Solution

Conservation of momentum is

$$\vec{p}_f = \vec{p}_i.$$

Define the direction of their initial velocity vectors to be the +x-direction. The initial momentum is then

$$\vec{p}_i = m_1 v_1 \ \hat{i} + m_2 v_2 \ \hat{i}.$$

The final momentum of the now-linked carts is

$$\vec{p}_f = (m_1 + m_2)\vec{v}_f.$$

Equating:

$$\begin{aligned}(m_1 + m_2)\vec{v}_f &= m_1 v_1 \ \hat{i} + m_2 v_2 \ \hat{i} \\ \vec{v}_f &= \left(\frac{m_1 v_1 + m_2 v_2}{m_1 + m_2}\right) \hat{i}.\end{aligned}$$

Substituting the given numbers:

$$\begin{aligned}\vec{v}_f &= \left[\frac{(0.675 \ kg)(0.75 \ m/s) + (0.5 \ kg)(1.33 \ m/s)}{1.175 \ kg}\right] \hat{i} \\ &= (0.997 \ m/s)\hat{i}.\end{aligned}$$

Significance

The principles that apply here to two laboratory carts apply identically to all objects of whatever type or size. Even for photons, the concepts of momentum and conservation of momentum are still crucially important even at that scale. (Since they are massless, the momentum of a photon is defined very differently from the momentum of ordinary objects. You will learn about this when you study quantum physics.)

? Exercise 9.6.1

Suppose the second, smaller cart had been initially moving to the left. What would the sign of the final velocity have been in this case?

✓ Example 9.6.2: A Bouncing Superball

A superball of mass 0.25 kg is dropped from rest from a height of h = 1.50 m above the floor. It bounces with no loss of energy and returns to its initial height (Figure 9.6.2).

a. What is the superball's change of momentum during its bounce on the floor?
b. What was Earth's change of momentum due to the ball colliding with the floor?
c. What was Earth's change of velocity as a result of this collision?

(This example shows that you have to be careful about defining your system.)

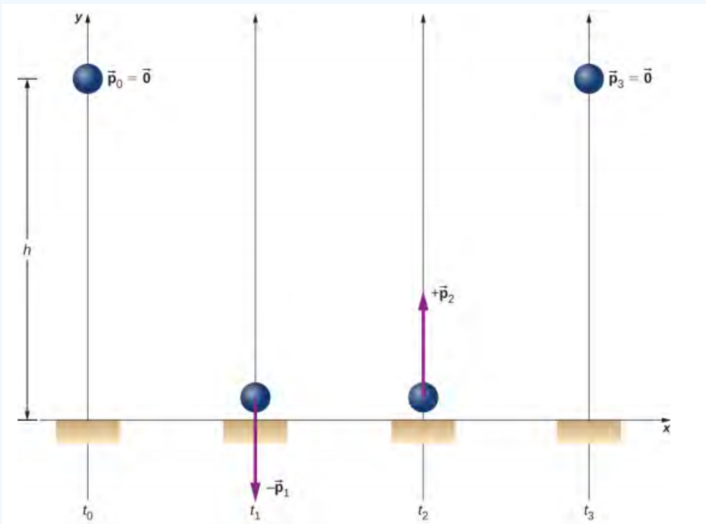


Figure 9.6.2: A superball is dropped to the floor (t_0), hits the floor (t_1), bounces (t_2), and returns to its initial height (t_3).

Strategy

Since we are asked only about the ball's change of momentum, we define our system to be the ball. But this is clearly not a closed system; gravity applies a downward force on the ball while it is falling, and the normal force from the floor applies a force during the bounce. Thus, we cannot use conservation of momentum as a strategy. Instead, we simply determine the ball's momentum just before it collides with the floor and just after, and calculate the difference. We have the ball's mass, so we need its velocities.

Solution

a. Since this is a one-dimensional problem, we use the scalar form of the equations. Let:
   - p_0 = the magnitude of the ball's momentum at time t_0, the moment it was released; since it was dropped from rest, this is zero.
   - p_1 = the magnitude of the ball's momentum at time t_1, the instant just before it hits the floor.
   - p_2 = the magnitude of the ball's momentum at time t_2, just after it loses contact with the floor after the bounce.

The ball's change of momentum is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= p_2\ \hat{j} - (-p_1\ \hat{j}) \\ &= (p_2 + p_1)\hat{j}.\end{aligned}$$

Its velocity just before it hits the floor can be determined from either conservation of energy or kinematics. We use kinematics here; you should re-solve it using conservation of energy and confirm you get the same result.

We want the velocity just before it hits the ground (at time t_1). We know its initial velocity $v_0 = 0$ (at time t_0), the height it falls, and its acceleration; we don't know the fall time. We could calculate that, but instead we use

$$\vec{v}_1 = -\hat{j}\sqrt{2gy} = -5.4\ m/s\ \hat{j}.$$

Thus the ball has a momentum of

$$\begin{aligned}\vec{p}_1 &= -(0.25\ kg)(-5.4\ m/s\ \hat{j}) \\ &= -(1.4\ kg\cdot m/s)\hat{j}.\end{aligned}$$

We don't have an easy way to calculate the momentum after the bounce. Instead, we reason from the symmetry of the situation.

Before the bounce, the ball starts with zero velocity and falls 1.50 m under the influence of gravity, achieving some amount of momentum just before it hits the ground. On the return trip (after the bounce), it starts with some amount of momentum, rises the same 1.50 m it fell, and ends with zero velocity. Thus, the motion after the bounce was the mirror image of the motion before the bounce. From this symmetry, it must be true that the ball's momentum after the bounce must be equal and opposite to its momentum before the bounce. (This is a subtle but crucial argument; make sure you understand it before you go on.) Therefore,

$$\vec{p}_2 = -\vec{p}_1 = +(1.4\ kg\cdot m/s)\hat{j}.$$

Thus, the ball's change of momentum during the bounce is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2 - \vec{p}_1 \\ &= (1.4\ kg\cdot m/s)\hat{j} - (-1.4\ kg\cdot m/s)\hat{j} \\ &= +(2.8\ kg\cdot m/s)\hat{j}.\end{aligned}$$

b. What was Earth's change of momentum due to the ball colliding with the floor? Your instinctive response may well have been either "zero; the Earth is just too massive for that tiny ball to have affected it" or possibly, "more than zero, but utterly negligible." But no—if we re-define our system to be the Superball + Earth, then this system is closed (neglecting the gravitational pulls of the Sun, the Moon, and the other planets in the solar system), and therefore the total change of momentum of this new system must be zero. Therefore, Earth's change of momentum is exactly the same magnitude:

$$\Delta\vec{p}_{Earth} = -2.8\ kg\cdot m/s\ \hat{j}. \tag{9.6.1}$$

c. What was Earth's change of velocity as a result of this collision? This is where your instinctive feeling is probably correct:

$$\begin{aligned}\Delta\vec{v}_{Earth} &= \frac{\Delta\vec{p}_{Earth}}{M_{Earth}} \\ &= -\frac{2.8\ kg\cdot m/s}{5.97\times 10^{24}\ kg}\ \hat{j} \\ &= -(4.7\times 10^{-25}\ m/s)\hat{j}.\end{aligned}$$

This change of Earth's velocity is utterly negligible

Significance

It is important to realize that the answer to part (c) is not a velocity; it is a change of velocity, which is a very different thing. Nevertheless, to give you a feel for just how small that change of velocity is, suppose you were moving with a velocity of 4.7 x 10^{-25} m/s. At this speed, it would take you about 7 million years to travel a distance equal to the diameter of a hydrogen atom.

? Exercise 9.6.2

Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)? Would the ball's change of momentum have been larger, smaller, or the same, if it had collided with the floor and stopped (without bouncing)?

✔ Example 9.6.3: Ice hockey 1

Two hockey pucks of identical mass are on a flat, horizontal ice hockey rink. The red puck is motionless; the blue puck is moving at 2.5 m/s to the left (Figure 9.6.3). It collides with the motionless red puck. The pucks have a mass of 15 g. After the collision, the red puck is moving at 2.5 m/s, to the left. What is the final velocity of the blue puck?

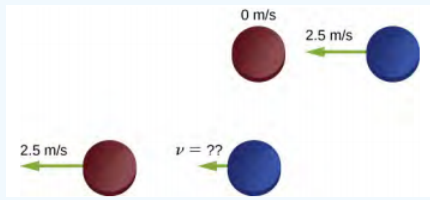


Figure 9.6.3: Two identical hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, we're told the masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy. Define the system to be the two pucks; there's no friction, so we have a closed system.

Before you look at the solution, what do you think the answer will be?

The blue puck final velocity will be:

a. zero
b. 2.5 m/s to the left
c. 2.5 m/s to the right
d. 1.25 m/s to the left
e. 1.25 m/s to the right
f. something else

Solution

Define the +x-direction to point to the right. Conservation of momentum then reads

$$\begin{aligned} \vec{p_f} &= \vec{p_i} \\ mv_{r_f}\ \hat{i} + mv_{b_f}\ \hat{i}\ &= mv_{r_i}\ \hat{i} - mv_{b_i}\ \hat{i}. \end{aligned}$$

Before the collision, the momentum of the system is entirely and only in the blue puck. Thus,

$$mv_{r_f}\ \hat{i} + mv_{b_f}\ \hat{i} = -mv_{b_i}\ \hat{i}$$

$$v_{r_f}\ \hat{i} + v_{b_f}\ \hat{i} = -v_{b_i}\ \hat{i}.$$

(Remember that the masses of the pucks are equal.) Substituting numbers:

$$\begin{aligned} -(2.5\ m/s)\hat{i} + \vec{v}_{b_f}\ &= -(2.5\ m/s)\hat{i} \\ \vec{v}_{b_f}\ &= 0. \end{aligned}$$

Significance

Evidently, the two pucks simply exchanged momentum. The blue puck transferred all of its momentum to the red puck. In fact, this is what happens in similar collision where $m_1 = m_2$.

Exercise 9.6.3

Even if there were some friction on the ice, it is still possible to use conservation of momentum to solve this problem, but you would need to impose an additional condition on the problem. What is that additional condition?

Philae

On November 12, 2014, the European Space Agency successfully landed a probe named **Philae** on Comet 67P/Churyumov/Gerasimenko (Figure 9.6.4). During the landing, however, the probe actually landed three times, because it bounced twice. Let's calculate how much the comet's speed changed as a result of the first bounce.

Figure 9.6.4: An artist's rendering of Philae landing on a comet. (credit: modification of work by "DLR German Aerospace Center"/Flickr)

Let's define upward to be the +y-direction, perpendicular to the surface of the comet, and y = 0 to be at the surface of the comet. Here's what we know:

- The mass of Comet 67P: $M_c = 1.0 \times 10^{13}$ kg
- The acceleration due to the comet's gravity: $\vec{a} = -(5.0 \times 10^{-3}\ \text{m/s}^2)\ \hat{j}$
- **Philae**'s mass: $M_p = 96$ kg
- Initial touchdown speed: $\vec{v}_1 = -(1.0\ \text{m/s})\ \hat{j}$
- Initial upward speed due to first bounce: $\vec{v}_2 = (0.38\ \text{m/s})\ \hat{j}$
- Landing impact time: $\Delta t = 1.3$ s

Strategy

We're asked for how much the comet's speed changed, but we don't know much about the comet, beyond its mass and the acceleration its gravity causes. However, we are told that the **Philae** lander collides with (lands on) the comet, and bounces off of it. A collision suggests momentum as a strategy for solving this problem.

If we define a system that consists of both **Philae** and Comet 67/P, then there is no net external force on this system, and thus the momentum of this system is conserved. (We'll neglect the gravitational force of the sun.) Thus, if we calculate the change of momentum of the lander, we automatically have the change of momentum of the comet. Also, the comet's change of velocity is directly related to its change of momentum as a result of the lander "colliding" with it.

Solution

Let $\vec{p}_1$ be **Philae**'s momentum at the moment just before touchdown, and $\vec{p}_2$ be its momentum just after the first bounce. Then its momentum just before landing was

$$\vec{p}_1 = M_p \vec{v}_1 = (96\ kg)(-1.0\ m/s\ \hat{j}) = -(96\ kg \cdot m/s)\hat{j}$$

and just after was

$$\vec{p}_2 = M_p \vec{v}_2 = (96\ kg)(+0.38\ m/s\ \hat{j}) = (36.5\ kg \cdot m/s)\hat{j}.$$

Therefore, the lander's change of momentum during the first bounce is

$$\begin{aligned}\Delta\vec{p} &= \vec{p}_2\vec{p}_1 \\ &= (36.5\ kg\cdot m/s)\hat{j} - (-96.0\ kg\cdot m/s\ \hat{j}) \\ &= (133\ kg\cdot m/s)\hat{j}\end{aligned}$$

Notice how important it is to include the negative sign of the initial momentum.

Now for the comet. Since momentum of the system must be conserved, the comet's momentum changed by exactly the negative of this:

$$\Delta\vec{p}_c = -\Delta\vec{p} = -(133\ kg\cdot m/s)\hat{j}.$$

Therefore, its change of velocity is

$$\Delta\vec{v}_c = \frac{\Delta\vec{p}_c}{M_c} = \frac{-(133\ kg\cdot m/s)\hat{j}}{1.0\times 10^{13}\ kg} = -(1.33\times 10^{-11}\ m/s)\hat{j}.$$

Significance

This is a very small change in velocity, about a thousandth of a billionth of a meter per second. Crucially, however, it is **not** zero.

? Exercise 9.6.4

The changes of momentum for **Philae** and for Comet 67/P were equal (in magnitude). Were the impulses experienced by **Philae** and the comet equal? How about the forces? How about the changes of kinetic energies?

9.7: Types of Collisions

Learning Objectives

- Identify the type of collision
- Correctly label a collision as elastic or inelastic
- Use kinetic energy along with momentum and impulse to analyze a collision

Although momentum is conserved in all interactions, not all interactions (collisions or explosions) are the same. The possibilities include:

- A single object can explode into multiple objects (explosions).
- Multiple objects can collide and stick together, forming a single object (inelastic).
- Multiple objects can collide and bounce off of each other, remaining as multiple objects (elastic). If they do bounce off each other, then they may recoil at the same speeds with which they approached each other before the collision, or they may move off more slowly.

It's useful, therefore, to categorize different types of interactions, according to how the interacting objects move before and after the interaction.

Explosions

The first possibility is that a single object may break apart into two or more pieces. An example of this is a firecracker, or a bow and arrow, or a rocket rising through the air toward space. These can be difficult to analyze if the number of fragments after the collision is more than about three or four; but nevertheless, the total momentum of the system before and after the explosion is identical.

Note that if the object is initially motionless, then the system (which is just the object) has no momentum and no kinetic energy. After the explosion, the net momentum of all the pieces of the object must sum to zero (since the momentum of this closed system cannot change). However, the system **will** have a great deal of kinetic energy after the explosion, although it had none before. Thus, we see that, although the momentum of the system is conserved in an explosion, the kinetic energy of the system most definitely is not; it increases. This interaction—one object becoming many, with an increase of kinetic energy of the system—is called an **explosion**.

Where does the energy come from? Does conservation of energy still hold? Yes; some form of potential energy is converted to kinetic energy. In the case of gunpowder burning and pushing out a bullet, chemical potential energy is converted to kinetic energy of the bullet, and of the recoiling gun. For a bow and arrow, it is elastic potential energy in the bowstring.

Inelastic

The second possibility is the reverse: that two or more objects collide with each other and stick together, thus (after the collision) forming one single composite object. The total mass of this composite object is the sum of the masses of the original objects, and the new single object moves with a velocity dictated by the conservation of momentum. However, it turns out again that, although the total momentum of the system of objects remains constant, the kinetic energy doesn't; but this time, the kinetic energy decreases. This type of collision is called **inelastic**.

Any collision where the objects stick together will result in the maximum loss of kinetic energy (i.e., K_f will be a minimum).

Such a collision is said to be **perfectly inelastic**. In the extreme case, multiple objects collide, stick together, and remain motionless after the collision. Since the objects are all motionless after the collision, the final kinetic energy is also zero; therefore, the loss of kinetic energy is a maximum.

- If $0 < K_f < K_i$, the collision is inelastic.
- If K_f is the lowest energy, or the energy lost by both objects is the most, the collision is perfectly inelastic (objects stick together).
- If $K_f = K_i$, the collision is elastic.

Elastic

The extreme case on the other end is if two or more objects approach each other, collide, and bounce off each other, moving away from each other at the same relative speed at which they approached each other. In this case, the total kinetic energy of the system is conserved. Such an interaction is called **elastic**.

In any interaction of a closed system of objects, the total momentum of the system is conserved ($\vec{p}_f = \vec{p}_i$) but the kinetic energy may not be:

- If $0 < K_f < K_i$, the collision is inelastic.
- If $K_f = 0$, the collision is perfectly inelastic.
- If $K_f = K_i$, the collision is elastic.
- If $K_f > K_i$, the interaction is an explosion.

The point of all this is that, in analyzing a collision or explosion, you can use both momentum and kinetic energy.

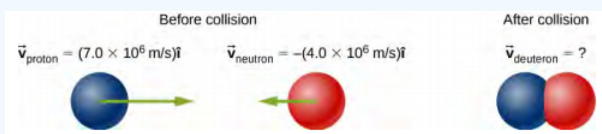
? Problem-Solving Strategy: Collisions

A closed system always conserves momentum; it might also conserve kinetic energy, but very often it doesn't. Energy-momentum problems confined to a plane (as ours are) usually have two unknowns. Generally, this approach works well:

1. Define a closed system.
2. Write down the expression for conservation of momentum.
3. If kinetic energy is conserved, write down the expression for conservation of kinetic energy; if not, write down the expression for the change of kinetic energy.
4. You now have two equations in two unknowns, which you solve by standard methods.

✓ Example 9.7.1: Formation of a deuteron

A proton (mass 1.67 x 10^{-27} kg) collides with a neutron (with essentially the same mass as the proton) to form a particle called a deuteron. What is the velocity of the deuteron if it is formed from a proton moving with velocity 7.0 x 10^6 m/s to the left and a neutron moving with velocity 4.0 x 10^6 m/s to the right?



Strategy

Define the system to be the two particles. This is a collision, so we should first identify what kind. Since we are told the two particles form a single particle after the collision, this means that the collision is perfectly inelastic. Thus, kinetic energy is not conserved, but momentum is. Thus, we use conservation of momentum to determine the final velocity of the system.

Solution

Treat the two particles as having identical masses M. Use the subscripts p, n, and d for proton, neutron, and deuteron, respectively. This is a one-dimensional problem, so we have

$$Mv_p - Mv_n = 2Mv_d. \tag{9.7.1}$$

The masses divide out:

$$\begin{aligned} v_p - v_n &= 2v_d \\ (7.0 \times 10^6\ m/s) - (4.0 \times 10^6\ m/s) &= 2v_d \\ v_d &= 1.5 \times 10^6\ m/s. \end{aligned}$$

The velocity is thus $\vec{v}_d = (1.5 \times 10^6\ m/s)\hat{i}$.

Significance

This is essentially how particle colliders like the Large Hadron Collider work: They accelerate particles up to very high speeds (large momenta), but in opposite directions. This maximizes the creation of so-called "daughter particles."

Example 9.7.2: Ice hockey 2

(This is a variation of an earlier example.)

Two ice hockey pucks of different masses are on a flat, horizontal hockey rink. The red puck has a mass of 15 grams, and is motionless; the blue puck has a mass of 12 grams, and is moving at 2.5 m/s to the left. It collides with the motionless red puck (Figure 9.7.1). If the collision is perfectly elastic, what are the final velocities of the two pucks?

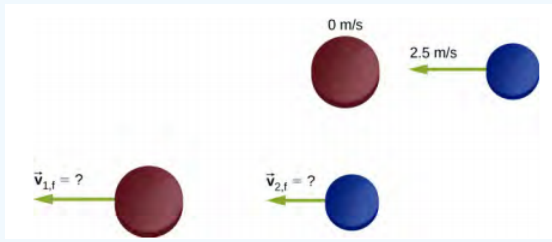


Figure 9.7.1: Two different hockey pucks colliding. The top diagram shows the pucks the instant before the collision, and the bottom diagram show the pucks the instant after the collision. The net external force is zero.

Strategy

We're told that we have two colliding objects, and we're told their masses and initial velocities, and one final velocity; we're asked for both final velocities. Conservation of momentum seems like a good strategy; define the system to be the two pucks. There is no friction, so we have a closed system. We have two unknowns (the two final velocities), but only one equation. The comment about the collision being perfectly elastic is the clue; it suggests that kinetic energy is also conserved in this collision. That gives us our second equation.

The initial momentum and initial kinetic energy of the system resides entirely and only in the second puck (the blue one); the collision transfers some of this momentum and energy to the first puck.

Solution

Conservation of momentum, in this case, reads

$$\begin{aligned} p_i &= p_f \\ m_2 v_{2,i} &= m_1 v_{1,f} + m_2 v_{2,f}. \end{aligned}$$

Conservation of kinetic energy reads

$$\begin{aligned} K_i &= K_f \\ \frac{1}{2} m_2 v_{2,i}^2 &= \frac{1}{2} m_1 v_{1,f}^2 + \frac{1}{2} m_2 v_{2,f}^2. \end{aligned}$$

There are our two equations in two unknowns. The algebra is tedious but not terribly difficult; you definitely should work it through. The solution is

$$v_{1,f} = \frac{(m_1 - m_2) v_{1,i} + 2 m_2 v_{2,i}}{m_1 + m_2} \tag{9.7.2}$$

$$v_{2,f} = \frac{(m_2 - m_1) v_{2,i} + 2 m_1 v_{1,i}}{m_1 + m_2} \tag{9.7.3}$$

Substituting the given numbers, we obtain

$$v_{1,f} = 2.22 \ m/s \tag{9.7.4}$$

$$v_{2,f} = -0.28 \ m/s. \tag{9.7.5}$$

Significance

Notice that after the collision, the blue puck is moving to the right; its direction of motion was reversed. The red puck is now moving to the left.

? Exercise 9.7.1

There is a second solution to the system of equations solved in this example (because the energy equation is quadratic): $v_{1,f}$ = −2.5 m/s, $v_{2,f}$ = 0. This solution is unacceptable on physical grounds; what's wrong with it?

✓ Example 9.7.3: Thor vs. iron man

The 2012 movie "The Avengers" has a scene where Iron Man and Thor fight. At the beginning of the fight, Thor throws his hammer at Iron Man, hitting him and throwing him slightly up into the air and against a small tree, which breaks. From the video, Iron Man is standing still when the hammer hits him. The distance between Thor and Iron Man is approximately 10 m, and the hammer takes about 1 s to reach Iron Man after Thor releases it. The tree is about 2 m behind Iron Man, which he hits in about 0.75 s. Also from the video, Iron Man's trajectory to the tree is very close to horizontal. Assuming Iron Man's total mass is 200 kg:

a. Estimate the mass of Thor's hammer
b. Estimate how much kinetic energy was lost in this collision

Strategy

After the collision, Thor's hammer is in contact with Iron Man for the entire time, so this is a perfectly inelastic collision. Thus, with the correct choice of a closed system, we expect momentum is conserved, but not kinetic energy. We use the given numbers to estimate the initial momentum, the initial kinetic energy, and the final kinetic energy. Because this is a one-dimensional problem, we can go directly to the scalar form of the equations.

Solution

a. First, we posit conservation of momentum. For that, we need a closed system. The choice here is the system (hammer + Iron Man), from the time of collision to the moment just before Iron Man and the hammer hit the tree. Let:
   - M_H = mass of the hammer
   - M_I = mass of Iron Man
   - v_H = velocity of the hammer before hitting Iron Man
   - v = combined velocity of Iron Man + hammer after the collision

Again, Iron Man's initial velocity was zero. Conservation of momentum here reads:

$$M_H v_H = (M_H + M_I)v. \tag{9.7.6}$$

We are asked to find the mass of the hammer, so we have

$$\begin{aligned} M_H v_H &= M_H v + M_1 v \\ M_H(v_H - v) &= M_I v \\ M_H &= \frac{M_I v}{v_H - v} \\ &= \frac{(200\ kg)\left(\frac{2\ m}{0.75\ s}\right)}{10\ m/s - \left(\frac{2\ m}{0.75\ s}\right)} \\ &= 73\ kg. \end{aligned}$$

Considering the uncertainties in our estimates, this should be expressed with just one significant figure; thus, $M_H = 7 \times 10^1$ kg.

b. The initial kinetic energy of the system, like the initial momentum, is all in the hammer:

$$\begin{aligned} K_i &= \frac{1}{2} M_H v_H^2 \\ &= \frac{1}{2}(70\ kg)(10\ m/s)^2 \\ &= 3500\ J. \end{aligned}$$

After the collision,

$$\begin{aligned} K_f &= \frac{1}{2}(M_H + M_I)v^2 \\ &= \frac{1}{2}(70\ kg + 200\ kg)(2.67\ m/s)^2 \\ &= 960\ J. \end{aligned}$$

Thus, there was a loss of 3500 J − 960 J = 2540 J.

Significance

From other scenes in the movie, Thor apparently can control the hammer's velocity with his mind. It is possible, therefore, that he mentally causes the hammer to maintain its initial velocity of 10 m/s while Iron Man is being driven backward toward the tree. If so, this would represent an external force on our system, so it would not be closed. Thor's mental control of his hammer is beyond the scope of this book, however.

Example 9.7.4: analyzing a car crash

At a stoplight, a large truck (3000 kg) collides with a motionless small car (1200 kg). The truck comes to an instantaneous stop; the car slides straight ahead, coming to a stop after sliding 10 meters. The measured coefficient of friction between the car's tires and the road was 0.62. How fast was the truck moving at the moment of impact?

Strategy

At first it may seem we don't have enough information to solve this problem. Although we know the initial speed of the car, we don't know the speed of the truck (indeed, that's what we're asked to find), so we don't know the initial momentum of the system. Similarly, we know the final speed of the truck, but not the speed of the car immediately after impact. The fact that the car eventually slid to a speed of zero doesn't help with the final momentum, since an external friction force caused that. Nor can we calculate an impulse, since we don't know the collision time, or the amount of time the car slid before stopping. A useful strategy is to impose a restriction on the analysis.

Suppose we define a system consisting of just the truck and the car. The momentum of this system isn't conserved, because of the friction between the car and the road. But if we could find the speed of the car the instant after impact—before friction had any measurable effect on the car—then we could consider the momentum of the system to be conserved, with that restriction.

Can we find the final speed of the car? Yes; we invoke the work-kinetic energy theorem.

Solution

First, define some variables. Let:

- M_c and M_T be the masses of the car and truck, respectively
- $v_{T,i}$ and $v_{T,f}$ be the velocities of the truck before and after the collision, respectively
- $v_{c,i}$ and $v_{c,f}$ be the velocities of the car before and after the collision, respectively
- K_i and K_f be the kinetic energies of the car immediately after the collision, and after the car has stopped sliding (so $K_f = 0$).
- d be the distance the car slides after the collision before eventually coming to a stop.

Since we actually want the initial speed of the truck, and since the truck is not part of the work-energy calculation, let's start with conservation of momentum. For the car + truck system, conservation of momentum reads

$$\begin{aligned} p_i &= p_f \\ M_c v_{c,i} + M_T v_{T,i} &= M_c v_{c,f} + M_T v_{T,f}. \end{aligned}$$

Since the car's initial velocity was zero, as was the truck's final velocity, this simplifies to

$$v_{T,i} = \frac{M_c}{M_T} v_{c,f}. \tag{9.7.7}$$

So now we need the car's speed immediately after impact. Recall that

$$W = \Delta K \tag{9.7.8}$$

where

$$\begin{aligned} \Delta K &= K_f - K_i \\ &= 0 - \frac{1}{2} M_c v_{c,f}^2. \end{aligned}$$

Also,

$$W = \vec{F} \cdot \vec{d} = Fd\cos\theta. \tag{9.7.9}$$

The work is done over the distance the car slides, which we've called d. Equating:

$$Fd\cos\theta = -\frac{1}{2} M_c v_{c,f}^2. \tag{9.7.10}$$

Friction is the force on the car that does the work to stop the sliding. With a level road, the friction force is

$$F = \mu_k M_c g. \tag{9.7.11}$$

Since the angle between the directions of the friction force vector and the displacement d is 180°, and cos(180°) = –1, we have

$$-(\mu_k M_c g)d = -\frac{1}{2} M_c v_{c,f}^2 \tag{9.7.12}$$

(Notice that the car's mass divides out; evidently the mass of the car doesn't matter.)

Solving for the car's speed immediately after the collision gives

$$v_{c,f} = \sqrt{2\mu_k g d}. \tag{9.7.13}$$

Substituting the given numbers:

$$\begin{aligned} v_{c,f} &= \sqrt{2(0.62)(9.81\ m/s^2)(10\ m)} \\ &= 11.0\ m/s. \end{aligned}$$

Now we can calculate the initial speed of the truck:

$$v_{T,i} = \left(\frac{1200\ kg}{3000\ kg}\right)(11.0\ m/s) = 4.4\ m/s. \tag{9.7.14}$$

Significance

This is an example of the type of analysis done by investigators of major car accidents. A great deal of legal and financial consequences depend on an accurate analysis and calculation of momentum and energy.

? Exercise 9.7.2

Suppose there had been no friction (the collision happened on ice); that would make μ_k zero, and thus $v_{c,f} = \sqrt{2\mu_k g d} = 0$, which is obviously wrong. What is the mistake in this conclusion?

Subatomic Collisions and Momentum

Conservation of momentum is crucial to our understanding of atomic and subatomic particles because much of what we know about these particles comes from collision experiments.

At the beginning of the twentieth century, there was considerable interest in, and debate about, the structure of the atom. It was known that atoms contain two types of electrically charged particles: negatively charged electrons and positively charged protons. (The existence of an electrically neutral particle was suspected, but would not be confirmed until 1932.) The question was, how were these particles arranged in the atom? Were they distributed uniformly throughout the volume of the atom (as J.J. Thomson proposed), or arranged at the corners of regular polygons (which was Gilbert Lewis' model), or rings of negative charge that surround the positively charged nucleus—rather like the planetary rings surrounding Saturn (as suggested by Hantaro Nagaoka), or something else?

The New Zealand physicist Ernest Rutherford (along with the German physicist Hans Geiger and the British physicist Ernest Marsden) performed the crucial experiment in 1909. They bombarded a thin sheet of gold foil with a beam of highenergy (that is, high-speed) alpha-particles (the nucleus of a helium atom). The alpha-particles collided with the gold atoms, and their subsequent velocities were detected and analyzed, using conservation of momentum and conservation of energy.

If the charges of the gold atoms were distributed uniformly (per Thomson), then the alpha-particles should collide with them and nearly all would be deflected through many angles, all small; the Nagaoka model would produce a similar result. If the atoms were arranged as regular polygons (Lewis), the alpha-particles would deflect at a relatively small number of angles.

What **actually** happened is that nearly **none** of the alpha-particles were deflected. Those that were, were deflected at large angles, some close to 180° —those alpha-particles reversed direction completely (Figure $9.7.2$). None of the existing atomic models could explain this. Eventually, Rutherford developed a model of the atom that was much closer to what we now have—again, using conservation of momentum and energy as his starting point.

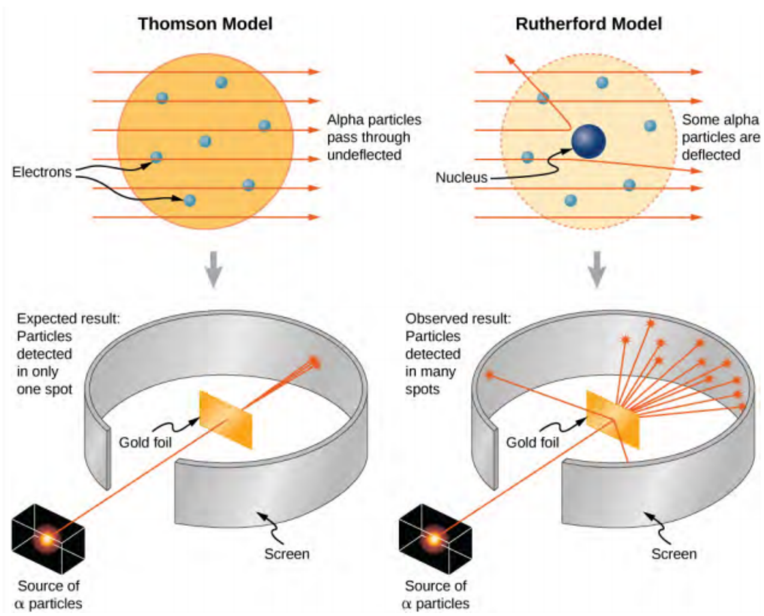


Figure $9.7.2$: The Thomson and Rutherford models of the atom. The Thomson model predicted that nearly all of the incident alpha-particles would be scattered and at small angles. Rutherford and Geiger found that nearly none of the alpha particles were scattered, but those few that were deflected did so through very large angles. The results of Rutherford's experiments were inconsistent with the Thomson model. Rutherford used conservation of momentum and energy to develop a new, and better model of the atom—the nuclear model.

9.8: Collisions in Multiple Dimensions

Learning Objectives

- Express momentum as a two-dimensional vector
- Write equations for momentum conservation in component form
- Calculate momentum in two dimensions, as a vector quantity

It is far more common for collisions to occur in two dimensions; that is, the angle between the initial velocity vectors is neither zero nor 180°. Let's see what complications arise from this.

The first idea we need is that momentum is a vector; like all vectors, it can be expressed as a sum of perpendicular components (usually, though not always, an x-component and a y-component, and a z-component if necessary). Thus, when we write down the statement of conservation of momentum for a problem, our momentum vectors can be, and usually will be, expressed in component form.

The second idea we need comes from the fact that momentum is related to force:

$$\vec{F} = \frac{d\vec{p}}{dt}. \tag{9.8.1}$$

Expressing both the force and the momentum in component form,

$$F_x = \frac{dp_x}{dt}, F_y = \frac{dp_y}{dt}, F_z = \frac{dp_z}{dt}. \tag{9.8.2}$$

Remember, these equations are simply Newton's second law, in vector form and in component form. We know that Newton's second law is true in each direction, independently of the others. It follows therefore (via Newton's third law) that conservation of momentum is also true in each direction independently.

These two ideas motivate the solution to two-dimensional problems: We write down the expression for conservation of momentum twice: once in the x-direction and once in the y-direction.

$$p_{f,x} = p_{1,i,x} + p_{2,i,x} \tag{9.8.3}$$

$$p_{f,y} = p_{1,i,y} + p_{2,i,y} \tag{9.8.4}$$

This procedure is shown graphically in Figure $9.8.1$.

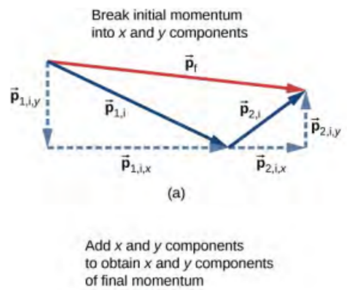


Figure $9.8.1$: (a) For two-dimensional momentum problems, break the initial momentum vectors into their x- and y-components. (b) Add the x- and y-components together separately. This gives you the x- and y-components of the final momentum, which are shown as red dashed vectors. (c) Adding these components together gives the final momentum.

We solve each of these two component equations independently to obtain the x- and y-components of the desired velocity vector:

$$v_{f,x} = \frac{m_1 v_{1,i,x} + m_2 v_{2,i,x}}{m} \tag{9.8.5}$$

$$v_{f,y} = \frac{m_1 v_{1,i,y} + m_2 v_{2,i,y}}{m} \tag{9.8.6}$$

(Here, m represents the total mass of the system.) Finally, combine these components using the Pythagorean theorem,

$$v_f = |\vec{v}_f| = \sqrt{v_{f,x}^2 + v_{f,y}^2}. \tag{9.8.7}$$

? Problem-Solving Strategy: Conservation of Momentum in Two Dimensions

The method for solving a two-dimensional (or even three-dimensional) conservation of momentum problem is generally the same as the method for solving a one-dimensional problem, except that you have to conserve momentum in both (or all three) dimensions simultaneously:

1. Identify a closed system.
2. Write down the equation that represents conservation of momentum in the x-direction, and solve it for the desired quantity. If you are calculating a vector quantity (velocity, usually), this will give you the x-component of the vector.
3. Write down the equation that represents conservation of momentum in the y-direction, and solve. This will give you the y-component of your vector quantity.
4. Assuming you are calculating a vector quantity, use the Pythagorean theorem to calculate its magnitude, using the results of steps 3 and 4.

✔ Example 9.14: Traffic Collision

A small car of mass 1200 kg traveling east at 60 km/hr collides at an intersection with a truck of mass 3000 kg that is traveling due north at 40 km/hr (Figure 9.8.2). The two vehicles are locked together. What is the velocity of the combined wreckage?

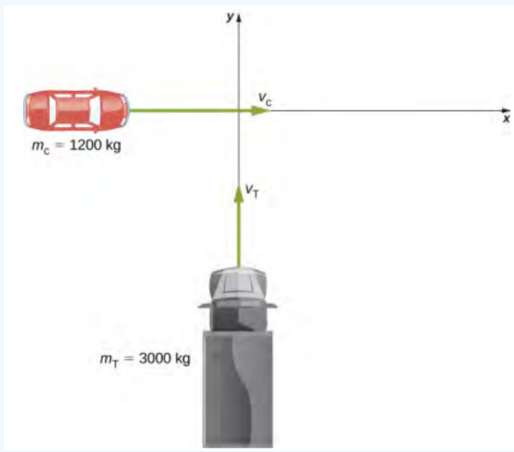


Figure 9.8.2: A large truck moving north is about to collide with a small car moving east. The final momentum vector has both x- and y-components.

Strategy

First off, we need a closed system. The natural system to choose is the (car + truck), but this system is not closed; friction from the road acts on both vehicles. We avoid this problem by restricting the question to finding the velocity at the instant just after the collision, so that friction has not yet had any effect on the system. With that restriction, momentum is conserved for this system.

Since there are two directions involved, we do conservation of momentum twice: once in the x-direction and once in the y-direction.

Solution

Before the collision the total momentum is

$$\vec{p} = m_c\vec{v}_c + m_T\vec{v}_T. \tag{9.8.8}$$

After the collision, the wreckage has momentum

$$\vec{p} = (m_c + m_T)\vec{v}_w. \tag{9.8.9}$$

Since the system is closed, momentum must be conserved, so we have

$$m_c\vec{v}_c + m_T\vec{v}_T = (m_c + m_T)\vec{v}_w. \tag{9.8.10}$$

We have to be careful; the two initial momenta are not parallel. We must add vectorially (Figure 9.8.3).

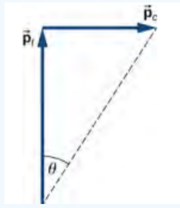


Figure 9.8.3: Graphical addition of momentum vectors. Notice that, although the car's velocity is larger than the truck's, its momentum is smaller.

If we define the +x-direction to point east and the +y-direction to point north, as in the figure, then (conveniently),

$$\vec{p}_c = p_c\ \hat{i} = m_c v_c\ \hat{i} \tag{9.8.11}$$

$$\vec{p}_T = p_T\ \hat{j} = m_T v_T\ \hat{j}. \tag{9.8.12}$$

Therefore, in the x-direction:

$$m_c v_c = (m_c + m_T) v_{w,x} \tag{9.8.13}$$

$$v_{w,x} = \left(\frac{m_c}{m_c + m_T}\right) v_c \tag{9.8.14}$$

and in the y-direction:

$$m_T v_T = (m_c + m_T) v_{w,y} \tag{9.8.15}$$

$$v_{w,y} = \left(\frac{m_T}{m_c + m_T}\right) v_T. \tag{9.8.16}$$

Applying the Pythagorean theorem gives

$$\begin{aligned} |\vec{v}_w| &= \sqrt{\left[\left(\frac{m_c}{m_c + m_T}\right) v_c\right]^2 + \left[\left(\frac{m_T}{m_c + m_T}\right) v_T\right]^2} \\ &= \sqrt{\left[\left(\frac{1200\ kg}{4200\ kg}\right)(16.67\ m/s)\right]^2 + \left[\left(\frac{3000\ kg}{4200\ kg}\right)(11.1\ m/s)\right]^2} \\ &= \sqrt{(4.76\ m/s)^2 + (7.93\ m/s)^2} \\ &= 9.25\ m/s \approx 33.3\ km/hr. \end{aligned}$$

As for its direction, using the angle shown in the figure,

$$\theta = \tan^{-1}\left(\frac{v_{w,x}}{v_{w,y}}\right) = \tan^{-1}\left(\frac{7.93\ m/s}{4.76\ m/s}\right) = 59^o. \tag{9.8.17}$$

This angle is east of north, or 31° counterclockwise from the +x-direction.

Significance

As a practical matter, accident investigators usually work in the "opposite direction"; they measure the distance of skid marks on the road (which gives the stopping distance) and use the work-energy theorem along with conservation of momentum to determine the speeds and directions of the cars prior to the collision. We saw that analysis in an earlier section.

? Exercise 9.9

Suppose the initial velocities were not at right angles to each other. How would this change both the physical result and the mathematical analysis of the collision?

Example 9.15: Exploding Scuba Tank

A common scuba tank is an aluminum cylinder that weighs 31.7 pounds empty (Figure $9.8.4$). When full of compressed air, the internal pressure is between 2500 and 3000 psi (pounds per square inch). Suppose such a tank, which had been sitting motionless, suddenly explodes into three pieces. The first piece, weighing 10 pounds, shoots off horizontally at 235 miles per hour; the second piece (7 pounds) shoots off at 172 miles per hour, also in the horizontal plane, but at a 19° angle to the first piece. What is the mass and initial velocity of the third piece? (Do all work, and express your final answer, in SI units.)

Figure $9.8.4$: A scuba tank explodes into three pieces.

Strategy

To use conservation of momentum, we need a closed system. If we define the system to be the scuba tank, this is not a closed system, since gravity is an external force. However, the problem asks for just the initial velocity of the third piece, so we can neglect the effect of gravity and consider the tank by itself as a closed system. Notice that, for this system, the initial momentum vector is zero.

We choose a coordinate system where all the motion happens in the xy-plane. We then write down the equations for conservation of momentum in each direction, thus obtaining the x- and y-components of the momentum of the third piece, from which we obtain its magnitude (via the Pythagorean theorem) and its direction. Finally, dividing this momentum by the mass of the third piece gives us the velocity.

Solution

First, let's get all the conversions to SI units out of the way:

$$31.7\ lb \times \frac{1\ kg}{2.2\ lb} \rightarrow 14.4\ kg \tag{9.8.18}$$

$$10\ lb \rightarrow 4.5\ kg \tag{9.8.19}$$

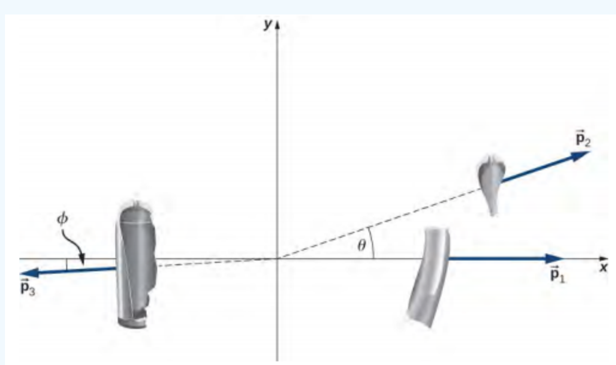
$$235\ \frac{miles}{hour} \times \frac{1\ hour}{3600\ s} \times \frac{1609\ m}{mile} = 105\ m/s \tag{9.8.20}$$

$$7\ lb \rightarrow 3.2\ kg \tag{9.8.21}$$

$$172 \frac{mile}{hour} = 77\ m/s \tag{9.8.22}$$

$$m_3 = 14.4\ kg - (4.5\ kg + 3.2\ kg) = 6.7\ kg. \tag{9.8.23}$$

Now apply conservation of momentum in each direction.



x-direction:

$$\begin{aligned} p_{f,x} &= p_{0,x} \\ p_{1,x} + p_{2,x} + p_{3,x} &= 0 \\ m_1 v_{1,x} + m_2 v_{2,x} + p_{3,x} &= 0 \\ p_{3,x} &= -m_1 v_{1,x} - m_2 v_{2,x} \end{aligned}$$

y-direction:

$$\begin{aligned} p_{f,y} &= p_{0,y} \\ p_{1,y} + p_{2,y} + p_{3,y} &= 0 \\ m_1 v_{1,y} + m_2 v_{2,y} + p_{3,y} &= 0 \\ p_{3,y} &= -m_1 v_{1,y} - m_2 v_{2,y} \end{aligned}$$

From our chosen coordinate system, we write the x-components as

$$\begin{aligned} p_{3,x} &= -m_1 v_1 - m_2 v_2 \cos\theta \\ &= -(4.5\ kg)(105\ m/s) - (3.2\ kg)(77\ m/s)\cos(19^o) \\ &= -705\ kg \cdot m/s. \end{aligned}$$

For the y-direction, we have

$$\begin{aligned} p_{3,y} &= 0 - m_2 v_2 \sin\theta \\ &= -(3.2\ kg)(77\ m/s)\sin(19^o) \\ &= -80.2\ kg \cdot m/s. \end{aligned}$$

This gives the magnitude of p_3:

$$\begin{aligned} p_3 &= \sqrt{p_{3,x}^2 + p_{3,y}^2} \\ &= \sqrt{(-705\ kg \cdot m/s)^2 + (-80.2\ kg \cdot m/s)} \\ &= 710\ kg \cdot m/s. \end{aligned}$$

The velocity of the third piece is therefore

$$v_3 = \frac{p_3}{m_3} = \frac{710\ kg \cdot m/s}{6.7\ kg} = 106\ m/s. \tag{9.8.24}$$

The direction of its velocity vector is the same as the direction of its momentum vector:

$$\phi = \tan^{-1}\left(\frac{p_{3,y}}{p_{3,x}}\right) = \tan^{-1}\left(\frac{80.2\ kg \cdot m/s}{705\ kg \cdot m/s}\right) = 6.49^o. \tag{9.8.25}$$

Because ϕ is below the −x -axis, the actual angle is 186.49° from the +x-direction.

Significance

The enormous velocities here are typical; an exploding tank of any compressed gas can easily punch through the wall of a house and cause significant injury, or death. Fortunately, such explosions are extremely rare, on a percentage basis.

? Exercise 9.10

Notice that the mass of the air in the tank was neglected in the analysis and solution. How would the solution method changed if the air was included? How large a difference do you think it would make in the final answer?

9.9: Center of Mass (Part 1)

Learning Objectives

- Explain the meaning and usefulness of the concept of center of mass
- Calculate the center of mass of a given system
- Apply the center of mass concept in two and three dimensions
- Calculate the velocity and acceleration of the center of mass

We have been avoiding an important issue up to now: When we say that an object moves (more correctly, accelerates) in a way that obeys Newton's second law, we have been ignoring the fact that all objects are actually made of many constituent particles. A car has an engine, steering wheel, seats, passengers; a football is leather and rubber surrounding air; a brick is made of atoms. There are many different types of particles, and they are generally not distributed uniformly in the object. How do we include these facts into our calculations?

Then too, an extended object might change shape as it moves, such as a water balloon or a cat falling (Figure $9.9.1$). This implies that the constituent particles are applying internal forces on each other, in addition to the external force that is acting on the object as a whole. We want to be able to handle this, as well.

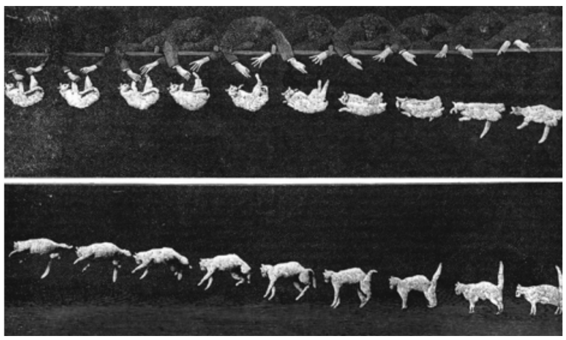


Figure $9.9.1$: As the cat falls, its body performs complicated motions so it can land on its feet, but one point in the system moves with the simple uniform acceleration of gravity.

The problem before us, then, is to determine what part of an extended object is obeying Newton's second law when an external force is applied and to determine how the motion of the object as a whole is affected by both the internal and external forces.

Be warned: To treat this new situation correctly, we must be rigorous and completely general. We won't make any assumptions about the nature of the object, or of its constituent particles, or either the internal or external forces. Thus, the arguments will be complex.

Internal and External Forces

Suppose we have an extended object of mass M, made of N interacting particles. Let's label their masses as m_j, where j = 1, 2, 3, …, N. Note that

$$M=\sum_{j=1}^{N} m_{j}. \tag{9.9.1}$$

If we apply some net **external force** $\vec{F}_{ext}$ on the object, every particle experiences some "share" or some fraction of that external force. Let:

$\vec{f}_{j}^{ext}$ = the fraction of the external force that the jth particle experiences

Notice that these fractions of the total force are not necessarily equal; indeed, they virtually never are. (They **can** be, but they usually aren't.) In general, therefore,

$$\vec{f}_{1}^{ext} \neq \vec{f}_{2}^{ext} \neq \cdots \neq \vec{f}_{N}^{ext}. \tag{9.9.2}$$

Next, we assume that each of the particles making up our object can interact (apply forces on) every other particle of the object. We won't try to guess what kind of forces they are; but since these forces are the result of particles of the object acting on other particles of the same object, we refer to them as **internal forces** $\vec{f}_j^{int}$; thus:

$\vec{f}_j^{int}$ = the net internal force that the jth particle experiences from all the other particles that make up the object.

Now, the **net** force, internal plus external, on the jth particle is the vector sum of these:

$$\vec{f}_j = \vec{f}_j^{int} + \vec{f}_j^{ext}. \tag{9.9.3}$$

where again, this is for all N particles; j = 1, 2, 3, … , N. As a result of this fractional force, the momentum of each particle gets changed:

$$\begin{aligned} \vec{f}_j &= \frac{d\vec{p}_j}{dt} \\ \vec{f}_j^{int} + \vec{f}_j^{ext} &= \frac{d\vec{p}_j}{dt}. \end{aligned}$$

The net force $\vec{F}$ on the object is the vector sum of these forces:

$$\begin{aligned} \vec{F}_{net} &= \sum_{j=1}^{N} (\vec{f}_j^{int} + \vec{f}_j^{ext}) \\ &= \sum_{j=1}^{N} \vec{f}_j^{int} + \sum_{j=1}^{N} \vec{f}_j^{ext}. \end{aligned}$$

This net force changes the momentum of the object as a whole, and the net change of momentum of the object must be the vector sum of all the individual changes of momentum of all of the particles:

$$\vec{F}_{net} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.9.4}$$

Combining Equation ??? and Equation 9.9.4 gives

$$\sum_{j=1}^{N} \vec{f}_j^{int} + \sum_{j=1}^{N} \vec{f}_j^{ext} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.9.5}$$

Let's now think about these summations. First consider the internal forces term; remember that each $\vec{f}_j^{int}$ is the force on the jth particle from the other particles in the object. But by Newton's third law, for every one of these forces, there must be another force that has the same magnitude, but the opposite sign (points in the opposite direction). These forces do not cancel; however, that's not what we're doing in the summation. Rather, we're simply **mathematically adding up** all the internal force vectors. That is, in general, the internal forces for any individual part of the object won't cancel, but when all the internal forces are added up, the internal forces must cancel in pairs. It follows, therefore, that the sum of all the internal forces must be zero:

$$\sum_{j=1}^{N} \vec{f}_j^{ext} = 0. \tag{9.9.6}$$

(This argument is subtle, but crucial; take plenty of time to completely understand it.)

For the external forces, this summation is simply the total external force that was applied to the whole object:

$$\sum_{j=1}^{N} \vec{f}_j^{ext} = \vec{F}_{ext}. \tag{9.9.7}$$

As a result,

$$\vec{F}_{ext} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.9.8}$$

This is an important result. Equation 9.9.8 tells us that the total change of momentum of the entire object (all N particles) is due only to the external forces; the internal forces do not change the momentum of the object as a whole. This is why you can't lift yourself in the air by standing in a basket and pulling up on the handles: For the system of you + basket, your upward pulling force is an internal force.

Force and Momentum

Remember that our actual goal is to determine the equation of motion for the entire object (the entire system of particles). To that end, let's define:

$\vec{p}_{CM}$ = the total momentum of the system of N particles (the reason for the subscript will become clear shortly)

Then we have

$$\vec{p}_{CM} == \equiv \sum_{j=1}^{N} \vec{p}_j. \tag{9.9.9}$$

and therefore Equation 9.9.8 can be written simply as

$$\vec{F} = \frac{d\vec{p}_{CM}}{dt}. \tag{9.9.10}$$

Since this change of momentum is caused by only the net external force, we have dropped the "ext" subscript. This is Newton's second law, but now for the entire extended object. If this feels a bit anticlimactic, remember what is hiding inside it: $\vec{p}_{CM}$ is the vector sum of the momentum of (in principle) hundreds of thousands of billions of billions of particles (6.02 x 10^{23}), all caused by one simple net external force—a force that you can calculate.

Center of Mass

Our next task is to determine what part of the extended object, if any, is obeying Equation 9.9.10.

It's tempting to take the next step; does the following equation mean anything?

$$\vec{F} = M\vec{a} \tag{9.9.11}$$

If it **does** mean something (acceleration of what, exactly?), then we could write

$$M\vec{a} = \frac{d\vec{p}_{CM}}{dt} \tag{9.9.12}$$

and thus

$$M\vec{a} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt} = \frac{d}{dt} \sum_{j=1}^{N} \vec{p}_j. \tag{9.9.13}$$

which follows because the derivative of a sum is equal to the sum of the derivatives.

Now, $\vec{p}_j$ is the momentum of the jth particle. Defining the positions of the constituent particles (relative to some coordinate system) as $\vec{r}_j$ = (x_j, y_j, z_j), we thus have

$$\vec{p}_j = m_j \vec{v}_j = m_j \frac{d\vec{r}_j}{dt}. \tag{9.9.14}$$

Substituting back, we obtain

$$M\vec{a} = \frac{d}{dt}\sum_{j=1}^{N} m_j \frac{d\vec{r}_j}{dt}$$
$$= \frac{d^2}{dt^2}\sum_{j=1}^{N} m_j\vec{r}_j.$$

Dividing both sides by M (the total mass of the extended object) gives us

$$\vec{a} = \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N} m_j\vec{r}_j\right). \tag{9.9.15}$$

Thus, the point in the object that traces out the trajectory dictated by the applied force in Equation 9.9.11 is inside the parentheses in Equation 9.9.15.

Looking at this calculation, notice that (inside the parentheses) we are calculating the product of each particle's mass with its position, adding all N of these up, and dividing this sum by the total mass of particles we summed. This is reminiscent of an average; inspired by this, we'll (loosely) interpret it to be the weighted average position of the mass of the extended object. It's actually called the **center of mass** of the object. Notice that the position of the center of mass has units of meters; that suggests a definition:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{N} m_j\vec{r}_j. \tag{9.9.16}$$

So, the point that obeys Equation 9.9.10 (and therefore Equation 9.9.11 as well) is the center of mass of the object, which is located at the position vector $\vec{r}_{CM}$.

It may surprise you to learn that there does not have to be any actual mass at the center of mass of an object. For example, a hollow steel sphere with a vacuum inside it is spherically symmetrical (meaning its mass is uniformly distributed about the center of the sphere); all of the sphere's mass is out on its surface, with no mass inside. But it can be shown that the center of mass of the sphere is at its geometric center, which seems reasonable. Thus, there is no mass at the position of the center of mass of the sphere. (Another example is a doughnut.) The procedure to find the center of mass is illustrated in Figure 9.9.2

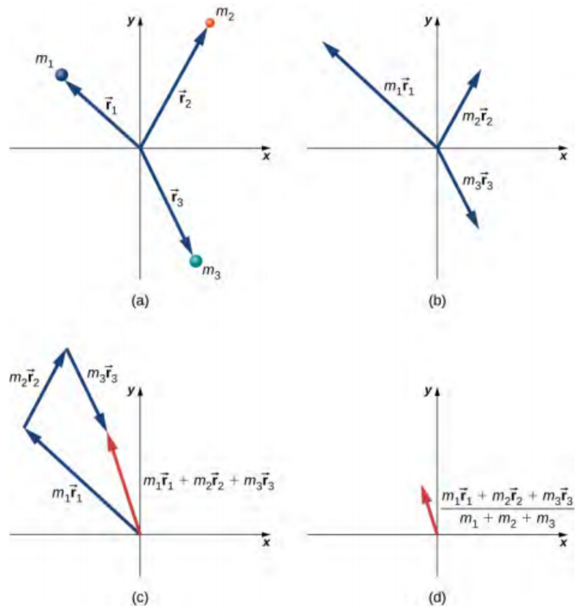


Figure 9.9.2: Finding the center of mass of a system of three different particles. (a) Position vectors are created for each object. (b) The position vectors are multiplied by the mass of the corresponding object. (c) The scaled vectors from part (b) are added together. (d) The final vector is divided by the total mass. This vector points to the center of mass of the system. Note that no mass is actually present at the center of mass of this system.

Since $\vec{r}_j = x_j\hat{i} + y_j\hat{j} + z_j\hat{k}$, it follows that:

$$r_{CM,x} = \frac{1}{m}\sum_{j=1}^{N} m_j x_j \tag{9.9.17}$$

$$r_{CM,y} = \frac{1}{m}\sum_{j=1}^{N} m_j y_j \tag{9.9.18}$$

$$r_{CM,z} = \frac{1}{m}\sum_{j=1}^{N} m_j z_j \tag{9.9.19}$$

and thus

$$\vec{r}_{CM} = r_{CM,x}\hat{i} + r_{CM,y}\hat{j} + r_{CM,z}\hat{k} \tag{9.9.20}$$

$$r_{CM} = |\vec{r}_{CM}| = (r^2_{CM,x} + r^2_{CM,y} + r^2_{CM,z})^{1/2}. \tag{9.9.21}$$

Therefore, you can calculate the components of the center of mass vector individually.

Finally, to complete the kinematics, the instantaneous velocity of the center of mass is calculated exactly as you might suspect:

$$\vec{v}_{CM} = \frac{d}{dt}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j\right) = \frac{1}{M}\sum_{j=1}^{N} m_j \vec{v}_j \tag{9.9.22}$$

and this, like the position, has x-, y-, and z-components.

To calculate the center of mass in actual situations, we recommend the following procedure:

Problem-Solving Strategy: Calculating the Center of Mass

The center of mass of an object is a position vector. Thus, to calculate it, do these steps:

1. Define your coordinate system. Typically, the origin is placed at the location of one of the particles. This is not required, however.
2. Determine the x, y, z-coordinates of each particle that makes up the object.
3. Determine the mass of each particle, and sum them to obtain the total mass of the object. Note that the mass of the object at the origin must be included in the total mass.
4. Calculate the x-, y-, and z-components of the center of mass vector, using Equation 9.9.17, Equation 9.9.18, and Equation 9.9.19.
5. If required, use the Pythagorean theorem to determine its magnitude.

Here are two examples that will give you a feel for what the center of mass is.

Example 9.16: Center of Mass of the Earth-Moon System

Using data from text appendix, determine how far the center of mass of the Earth-moon system is from the center of Earth. Compare this distance to the radius of Earth, and comment on the result. Ignore the other objects in the solar system.

Strategy

We get the masses and separation distance of the Earth and moon, impose a coordinate system, and use Equation 9.9.16 with just N = 2 objects. We use a subscript "e" to refer to Earth, and subscript "m" to refer to the moon.

Solution

Define the origin of the coordinate system as the center of Earth. Then, with just two objects, Equation 9.9.16 becomes

$$R = \frac{m_c r_c + m_m r_m}{m_c + m_m}. \tag{9.9.23}$$

From Appendix D,

$$m_c = 5.97 \times 10^{24}\ kg \tag{9.9.24}$$

$$m_m = 7.36 \times 10^{22}\ kg \tag{9.9.25}$$

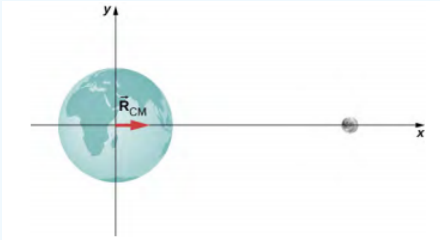
$$r_m = 3.82 \times 10^5\ m. \tag{9.9.26}$$

We defined the center of Earth as the origin, so $r_e = 0$ m. Inserting these into the equation for R gives

$$\begin{aligned} R &= \frac{(5.97 \times 10^{24}\ kg)(0\ m) + (7.36 \times 10^{22}\ kg)(3.82 \times 10^8\ m)}{(5.98 \times 10^{24}\ kg) + (7.36 \times 10^{22}\ kg)} \\ &= 4.64 \times 10^6\ m. \end{aligned}$$

Significance

The radius of Earth is 6.37 x 10^6 m, so the center of mass of the Earth-moon system is (6.37 − 4.64) x 10^6 m = 1.73 x 10^6 m = 1730 km (roughly 1080 miles) **below** the surface of Earth. The location of the center of mass is shown (not to scale).



? Exercise 9.11

Suppose we included the sun in the system. Approximately where would the center of mass of the Earth-moon-sun system be located? (Feel free to actually calculate it.)

✔ Example 9.17: Center of Mass of a Salt Crystal

Figure 9.9.3 shows a single crystal of sodium chloride—ordinary table salt. The sodium and chloride ions form a single unit, NaCl. When multiple NaCl units group together, they form a cubic lattice. The smallest possible cube (called the unit cell) consists of four sodium ions and four chloride ions, alternating. The length of one edge of this cube (i.e., the bond length) is 2.36 x 10^{-10} m. Find the location of the center of mass of the unit cell. Specify it either by its coordinates ($r_{CM,x}$, $r_{CM,y}$, $r_{CM,z}$), or by r_{CM} and two angles.

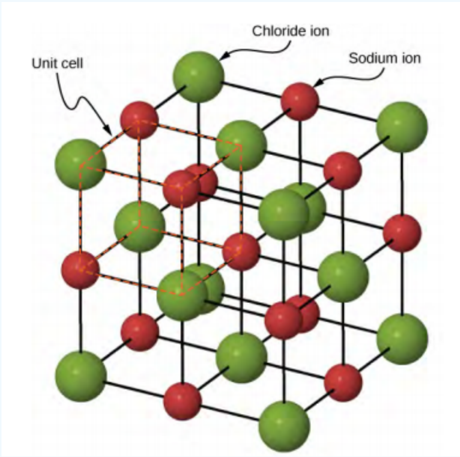


Figure 9.9.3: A drawing of a sodium chloride (NaCl) crystal.

Strategy

We can look up all the ion masses. If we impose a coordinate system on the unit cell, this will give us the positions of the ions. We can then apply Equation 9.9.17, Equation 9.9.18, and Equation 9.9.19 (along with the Pythagorean theorem).

Solution

Define the origin to be at the location of the chloride ion at the bottom left of the unit cell. Figure 9.9.4 shows the coordinate system.

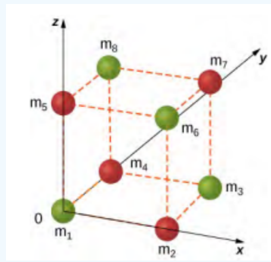


Figure 9.9.4: A single unit cell of a NaCl crystal.

There are eight ions in this crystal, so N = 8:

$$\vec{r}_{CM} = \frac{1}{M}\sum_{j=1}^{8} m_j \vec{r}_j. \tag{9.9.27}$$

The mass of each of the chloride ions is

$$35.453u \times \frac{1.660 \times 10^{-27}\ kg}{u} = 5.885 \times 10^{-26}\ kg \tag{9.9.28}$$

so we have

$$m_1 = m_3 = m_6 = m_8 = 5.885 \times 10^{-26}\ kg. \tag{9.9.29}$$

For the sodium ions,

$$m_2 = m_4 = m_5 = m_7 = 3.816 \times 10^{-26}\ kg. \tag{9.9.30}$$

The total mass of the unit cell is therefore

$$M = (4)(5.885 \times 10^{-26}\ kg) + (4)(3.816 \times 10^{-26}\ kg) = 3.880 \times 10^{-25}\ kg. \tag{9.9.31}$$

From the geometry, the locations are

$$\begin{aligned}
\vec{r}_1 &= 0 \\
\vec{r}_2 &= (2.36 \times 10^{-10}\ m)\hat{i} \\
\vec{r}_3 &= r_{3x}\hat{i} + r_{3y}\hat{j} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_4 &= (2.36 \times 10^{-10}\ m)\hat{j} \\
\vec{r}_5 &= (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_6 &= r_{6x}\hat{i} + r_{6z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_7 &= r_{7x}\hat{i} + r_{7y}\hat{j} + r_{7z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{i} + (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k} \\
\vec{r}_8 &= r_{8y}\hat{j} + r_{8z}\hat{k} = (2.36 \times 10^{-10}\ m)\hat{j} + (2.36 \times 10^{-10}\ m)\hat{k}.
\end{aligned}$$

Substituting:

$$\begin{aligned}|\vec{r}_{CM,x}| &= \sqrt{r^2_{CM,x}+r^2_{CM,y}+r^2_{CM,z}} \\ &= \frac{1}{M}\sum_{j=1}^{8} m_j(r_x)_j \\ &= \frac{1}{M}(m_1r_{1x}+m_2r_{2x}+m_3r_{3x}+m_4r_{4x}+m_5r_{5x}+m_6r_{6x}+m_7r_{7x}+m_8r_{8x}) \\ &= \frac{1}{3.8804\times10^{-25}\ kg}\Big[(5.885\times10^{-26}\ kg)(0\ m)+(3.816\times10^{-26}\ kg)(2.36\times10^{-10}\ m) \\ &+(5.885\times10^{-26}\ kg)(2.36\times10^{-10}\ m)+(3.816\times10^{-26}\ kg)(2.36\times10^{-10}\ m)+0+0 \\ &+(3.816\times10^{-26}\ kg)(2.36\times10^{-10}\ m)+0\Big] \\ &= 1.18\times10^{-10}\ m.\end{aligned}$$

Similar calculations give $r_{CM,y} = r_{CM,z} = 1.18 \times 10^{-10}$ m (you could argue that this must be true, by symmetry, but it's a good idea to check).

Significance

As it turns out, it was not really necessary to convert the mass from atomic mass units (u) to kilograms, since the units divide out when calculating r_{CM} anyway.

To express r_{CM} in terms of magnitude and direction, first apply the three-dimensional Pythagorean theorem to the vector components:

$$\begin{aligned}r_{CM} &= \sqrt{r^2_{CM,x}+r^2_{CM,y}+r^2_{CM,z}} \\ &= (1.18\times10^{-10}\ m)\sqrt{3} \\ &= 2.044\times10^{-10}\ m.\end{aligned}$$

Since this is a three-dimensional problem, it takes two angles to specify the direction of $\vec{r}_{CM}$. Let ϕ be the angle in the x,y-plane, measured from the +x-axis, counterclockwise as viewed from above; then:

$$\phi = \tan^{-1}\left(\frac{r_{CM,y}}{r_{CM,x}}\right) = 45^o. \tag{9.9.32}$$

Let θ be the angle in the y,z-plane, measured downward from the +z-axis; this is (not surprisingly):

$$\theta = \tan^{-1}\left(\frac{R_z}{R_y}\right) = 45^o. \tag{9.9.33}$$

Thus, the center of mass is at the geometric center of the unit cell. Again, you could argue this on the basis of symmetry

? Exercise 9.12

Suppose you have a macroscopic salt crystal (that is, a crystal that is large enough to be visible with your unaided eye). It is made up of a **huge** number of unit cells. Is the center of mass of this crystal necessarily at the geometric center of the crystal?

Two crucial concepts come out of these examples:

1. 1. As with all problems, you must define your coordinate system and origin. For center-of-mass calculations, it often makes sense to choose your origin to be located at one of the masses of your system. That choice automatically defines its distance in Equation 9.9.16to be zero. However, you must still include the mass of the object at your origin in your calculation of M, the total mass Equation 9.9.1. In the Earth-moon system example, this means including the mass of Earth. If you hadn't, you'd have ended up with the center of mass of the system being at the center of the moon, which is clearly wrong.
2. In the second example (the salt crystal), notice that there is no mass at all at the location of the center of mass. This is an example of what we stated above, that there does not have to be any actual mass at the center of mass of an object.

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9.10: Center of Mass (Part 2)

Center of Mass of Continuous Objects

If the object in question has its mass distributed uniformly in space, rather than as a collection of discrete particles, then $m_j \rightarrow dm$, and the summation becomes an integral:

$$\vec{r}_{CM} = \frac{1}{M} \int \vec{r} dm. \tag{9.10.1}$$

In this context, r is a characteristic dimension of the object (the radius of a sphere, the length of a long rod). To generate an integrand that can actually be calculated, you need to express the differential mass element dm as a function of the mass density of the continuous object, and the dimension r. An example will clarify this.

Example 9.10.1: CM of a Uniform Thin Hoop

Find the center of mass of a uniform thin hoop (or ring) of mass M and radius r.

Strategy

First, the hoop's symmetry suggests the center of mass should be at its geometric center. If we define our coordinate system such that the origin is located at the center of the hoop, the integral should evaluate to zero.

We replace dm with an expression involving the density of the hoop and the radius of the hoop. We then have an expression we can actually integrate. Since the hoop is described as "thin," we treat it as a one-dimensional object, neglecting the thickness of the hoop. Therefore, its density is expressed as the number of kilograms of material per meter. Such a density is called a **linear mass density**, and is given the symbol λ; this is the Greek letter "lambda," which is the equivalent of the English letter "l" (for "linear").

Since the hoop is described as uniform, this means that the linear mass density λ is constant. Thus, to get our expression for the differential mass element dm, we multiply λ by a differential length of the hoop, substitute, and integrate (with appropriate limits for the definite integral).

Solution

First, define our coordinate system and the relevant variables (Figure 9.10.1).

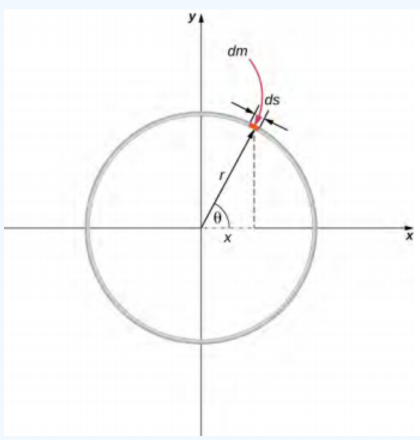


Figure 9.10.1: Finding the center of mass of a uniform hoop. We express the coordinates of a differential piece of the hoop, and then integrate around the hoop.

The center of mass is calculated with Equation 9.10.1:

$$\vec{r}_{CM} = \frac{1}{M} \int_a^b \vec{r} dm. \tag{9.10.2}$$

We have to determine the limits of integration a and b. Expressing $\vec{r}$ in component form gives us

$$\vec{r}_{CM}=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]dm. \tag{9.10.3}$$

In the diagram, we highlighted a piece of the hoop that is of differential length ds; it therefore has a differential mass dm = λds. Substituting:

$$\vec{r}_{CM}=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]\lambda ds. \tag{9.10.4}$$

However, the arc length ds subtends a differential angle d$theta$, so we have

$$ds=rd\theta \tag{9.10.5}$$

and thus

$$\vec{r}_{CM}=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]\lambda rd\theta. \tag{9.10.6}$$

One more step: Since λ is the linear mass density, it is computed by dividing the total mass by the length of the hoop:

$$\lambda=\frac{M}{2\pi r} \tag{9.10.7}$$

giving us

$$\begin{aligned}\vec{r}_{CM}&=\frac{1}{M}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]\left(\frac{M}{2\pi r}\right)rd\theta\\&=\frac{1}{2\pi}\int_a^b\left[(r\cos\theta)\hat{i}+(R\sin\theta)\hat{j}\right]d\theta.\end{aligned}$$

Notice that the variable of integration is now the angle θ. This tells us that the limits of integration (around the circular hoop) are θ = 0 to θ = 2π, so a = 0 and b = 2π. Also, for convenience, we separate the integral into the x- and y-components of $\vec{r}_{CM}$. The final integral expression is

$$\begin{aligned}\vec{r}_{CM}&=r_{CM,x}\hat{i}+r_{CM,y}\hat{j}\\&=\left[\frac{1}{2\pi}\int_0^{2\pi}(2\cos\theta d\theta\right]\hat{i}+\left[\frac{1}{2\pi}\int_0^{2\pi}(2\sin\theta d\theta\right]\hat{j}\\&=0\hat{i}+0\hat{j}=\vec{0}\end{aligned}$$

as expected.

Center of Mass and Conservation of Momentum

How does all this connect to conservation of momentum?

Suppose you have N objects with masses m_1, m_2, m_3, ...m_N and initial velocities $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$, ..., $\vec{v}_N$. The center of mass of the objects is

$$\vec{r}_{CM}=\frac{1}{M}\sum_{j=1}^{N}m_j\vec{r}_j. \tag{9.10.8}$$

Its velocity is

$$\vec{v}_{CM}=\frac{d\vec{r}_{CM}}{dt}=\frac{1}{M}\sum_{j=1}^{N}m_j\frac{d\vec{r}_j}{dt} \tag{9.10.9}$$

and thus the initial momentum of the center of mass is

$$\left[M \frac{d\vec{r}_{CM}}{dt}\right]_i = \sum_{j=1}^{N} m_j \frac{d\vec{r}_{j,i}}{dt}$$

$$M\vec{v}_{CM,i} = \sum_{j=1}^{N} m_j \vec{v}_{j,i}.$$

After these masses move and interact with each other, the momentum of the center of mass is

$$M\vec{v}_{CM,f} = \sum_{j=1}^{N} m_j \vec{v}_{j,f}. \tag{9.10.10}$$

But conservation of momentum tells us that the right-hand side of both equations must be equal, which says

$$M\vec{v}_{CM,f} = M\vec{v}_{CM,i}. \tag{9.10.11}$$

This result implies that conservation of momentum is expressed in terms of the center of mass of the system. Notice that as an object moves through space with no net external force acting on it, an individual particle of the object may accelerate in various directions, with various magnitudes, depending on the net internal force acting on that object at any time. (Remember, it is only the vector sum of all the internal forces that vanishes, not the internal force on a single particle.) Thus, such a particle's momentum will not be constant—but the momentum of the entire extended object will be, in accord with Equation 9.10.11.

Equation 9.10.11 implies another important result: Since M represents the mass of the entire system of particles, it is necessarily constant. (If it isn't, we don't have a closed system, so we can't expect the system's momentum to be conserved.) As a result, Equation 9.10.11 implies that, for a closed system,

$$\vec{v}_{CM,f} = \vec{v}_{CM,i}. \tag{9.10.12}$$

That is to say, **in the absence of an external force, the velocity of the center of mass never changes**.

You might be tempted to shrug and say, "Well yes, that's just Newton's first law," but remember that Newton's first law discusses the constant velocity of a particle, whereas Equation 9.10.12 applies to the center of mass of a (possibly vast) collection of interacting particles, and that there may not be any particle at the center of mass at all! So, this really is a remarkable result.

Example 9.10.2: Fireworks Display

When a fireworks rocket explodes, thousands of glowing fragments fly outward in all directions, and fall to Earth in an elegant and beautiful display (Figure 9.10.2). Describe what happens, in terms of conservation of momentum and center of mass.

Figure 9.10.2: These exploding fireworks are a vivid example of conservation of momentum and the motion of the center of mass.

The picture shows radial symmetry about the central points of the explosions; this suggests the idea of center of mass. We can also see the parabolic motion of the glowing particles; this brings to mind projectile motion ideas.

Solution

Initially, the fireworks rocket is launched and flies more or less straight upward; this is the cause of the more-or-less-straight, white trail going high into the sky below the explosion in the upper-right of the picture (the yellow explosion). This trail is not parabolic because the explosive shell, during its launch phase, is actually a rocket; the impulse applied to it by the ejection of the burning fuel applies a force on the shell during the rise-time interval. (This is a phenomenon we will study in the next section.) The shell has multiple forces on it; thus, it is not in free-fall prior to the explosion.

At the instant of the explosion, the thousands of glowing fragments fly outward in a radially symmetrical pattern. The symmetry of the explosion is the result of all the internal forces summing to zero $\left(\sum_j \vec{f}_j^{int} = 0\right)$; for every internal force, there is another that is equal in magnitude and opposite in direction.

However, as we learned above, these internal forces cannot change the momentum of the center of mass of the (now exploded) shell. Since the rocket force has now vanished, the center of mass of the shell is now a projectile (the only force on it is gravity), so its trajectory does become parabolic. The two red explosions on the left show the path of their centers of mass at a slightly longer time after explosion compared to the yellow explosion on the upper right.

In fact, if you look carefully at all three explosions, you can see that the glowing trails are not truly radially symmetric; rather, they are somewhat denser on one side than the other. Specifically, the yellow explosion and the lower middle explosion are slightly denser on their right sides, and the upper-left explosion is denser on its left side. This is because of the momentum of their centers of mass; the differing trail densities are due to the momentum each piece of the shell had at the moment of its explosion. The fragment for the explosion on the upper left of the picture had a momentum that pointed upward and to the left; the middle fragment's momentum pointed upward and slightly to the right; and the right-side explosion clearly upward and to the right (as evidenced by the white rocket exhaust trail visible below the yellow explosion).

Finally, each fragment is a projectile on its own, thus tracing out thousands of glowing parabolas.

Significance

In the discussion above, we said, "…the center of mass of the shell is now a projectile (the only force on it is gravity)…." This is not quite accurate, for there may not be any mass at all at the center of mass; in which case, there could not be a force acting on it. This is actually just verbal shorthand for describing the fact that the gravitational forces on all the particles act so that the center of mass changes position exactly as if all the mass of the shell were always located at the position of the center of mass.

? Exercise 9.10.2

How would the firework display change in deep space, far away from any source of gravity?

You may sometimes hear someone describe an explosion by saying something like, "the fragments of the exploded object always move in a way that makes sure that the center of mass continues to move on its original trajectory." This makes it sound as if the process is somewhat magical: how can it be that, in every explosion, it always works out that the fragments move in just the right way so that the center of mass' motion is unchanged? Phrased this way, it would be hard to believe no explosion ever does anything differently.

The explanation of this apparently astonishing coincidence is: We defined the center of mass precisely so this is exactly what we would get. Recall that first we defined the momentum of the system:

$$\vec{p}_{CM} = \sum_{j=1}^{N} \frac{d\vec{p}_j}{dt}. \tag{9.10.13}$$

We then concluded that the net external force on the system (if any) changed this momentum:

$$\vec{F} = \frac{d\vec{p}_{CM}}{dt} \tag{9.10.14}$$

and then—and here's the point—we defined an acceleration that would obey Newton's second law. That is, we demanded that we should be able to write

$$\vec{a} = \frac{\vec{F}}{M} \tag{9.10.15}$$

which requires that

$$\vec{a} = \frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N} m_j \vec{r}_j\right). \tag{9.10.16}$$

where the quantity inside the parentheses is the center of mass of our system. So, it’s not astonishing that the center of mass obeys Newton’s second law; we defined it so that it would.

9.11: Rocket Propulsion

Learning Objectives

- Describe the application of conservation of momentum when the mass changes with time, as well as the velocity
- Calculate the speed of a rocket in empty space, at some time, given initial conditions
- Calculate the speed of a rocket in Earth's gravity field, at some time, given initial conditions

Now we deal with the case where the mass of an object is changing. We analyze the motion of a rocket, which changes its velocity (and hence its momentum) by ejecting burned fuel gases, thus causing it to accelerate in the opposite direction of the velocity of the ejected fuel (Figure 9.11.1). Specifically: A fully fueled rocket ship in deep space has a total mass m_0 (this mass includes the initial mass of the fuel). At some moment in time, the rocket has a velocity $\vec{v}$ and mass m; this mass is a combination of the mass of the empty rocket and the mass of the remaining unburned fuel it contains. (We refer to m as the "instantaneous mass" and $\vec{v}$ as the "instantaneous velocity.") The rocket accelerates by burning the fuel it carries and ejecting the burned exhaust gases. If the burn rate of the fuel is constant, and the velocity at which the exhaust is ejected is also constant, what is the change of velocity of the rocket as a result of burning all of its fuel?

Figure 9.11.1: The space shuttle had a number of reusable parts. Solid fuel boosters on either side were recovered and refueled after each flight, and the entire orbiter returned to Earth for use in subsequent flights. The large liquid fuel tank was expended. The space shuttle was a complex assemblage of technologies, employing both solid and liquid fuel, and pioneering ceramic tiles as reentry heat shields. As a result, it permitted multiple launches as opposed to single-use rockets. (credit: modification of work by NASA)

Physical Analysis

Here's a description of what happens, so that you get a feel for the physics involved.

- As the rocket engines operate, they are continuously ejecting burned fuel gases, which have both mass and velocity, and therefore some momentum. By conservation of momentum, the rocket's momentum changes by this same amount (with the opposite sign). We will assume the burned fuel is being ejected at a constant rate, which means the rate of change of the rocket's momentum is also constant. By Equation 9.4.17, this represents a constant force on the rocket.
- However, as time goes on, the mass of the rocket (which includes the mass of the remaining fuel) continuously decreases. Thus, even though the force on the rocket is constant, the resulting acceleration is not; it is continuously increasing.
- So, the total change of the rocket's velocity will depend on the amount of mass of fuel that is burned, and that dependence is not linear.

The problem has the mass and velocity of the rocket changing; also, the total mass of ejected gases is changing. If we define our system to be the rocket + fuel, then this is a closed system (since the rocket is in deep space, there are no external forces acting on this system); as a result, momentum is conserved for this system. Thus, we can apply conservation of momentum to answer the question (Figure 9.11.2).

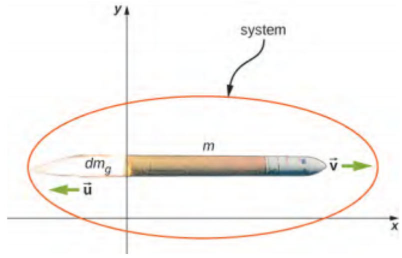


Figure 9.11.2: The rocket accelerates to the right due to the expulsion of some of its fuel mass to the left. Conservation of momentum enables us to determine the resulting change of velocity. The mass m is the instantaneous total mass of the rocket (i.e., mass of rocket body plus mass of fuel at that point in time). (credit: modification of work by NASA/Bill Ingalls)

At the same moment that the total instantaneous rocket mass is m (i.e., m is the mass of the rocket body plus the mass of the fuel at that point in time), we define the rocket's instantaneous velocity to be $\vec{v}$ = v $\hat{i}$ (in the +x-direction); this velocity is measured relative to an inertial reference system (the Earth, for example). Thus, the initial momentum of the system is $\vec{p}_i$ = mv $\hat{i}$.

The rocket's engines are burning fuel at a constant rate and ejecting the exhaust gases in the −x-direction. During an infinitesimal time interval dt, the engines eject a (positive) infinitesimal mass of gas dm_g at velocity $\vec{u}$ = −u $\hat{i}$; note that although the rocket velocity v $\hat{i}$ is measured with respect to Earth, the exhaust gas velocity is measured with respect to the (moving) rocket. Measured with respect to the Earth, therefore, the exhaust gas has velocity (v − u) $\hat{i}$.

As a consequence of the ejection of the fuel gas, the rocket's mass decreases by dm_g, and its velocity increases by dv $\hat{i}$. Therefore, including both the change for the rocket and the change for the exhaust gas, the final momentum of the system is

$$\begin{aligned}\vec{p}_f &= \vec{p}_{rocket} + \vec{p}_{gas}\\ &= (m - dm_g)(v + dv)\hat{i} + dm_g(v - u)\hat{i}.\end{aligned}$$

Since all vectors are in the x-direction, we drop the vector notation. Applying conservation of momentum, we obtain

$$\begin{aligned}p_i &= p_f\\ mv &= (m - dm_g)(v + dv) + dm_g(v - u)\\ mv &= mv + mdv - dm_g v - dm_g dv + dm_g v - dm_g u\\ mdv &= dm_g dv + dm_g u.\end{aligned}$$

Now, dm_g and dv are each very small; thus, their product $dm_g dv$ is very, very small, much smaller than the other two terms in this expression. We neglect this term, therefore, and obtain:

$$mdv = dm_g u. \tag{9.11.1}$$

Our next step is to remember that, since dm_grepresents an increase in the mass of ejected gases, it must also represent a decrease of mass of the rocket:

$$dm_g = -dm. \tag{9.11.2}$$

Replacing this, we have

$$mdv = -dmu \tag{9.11.3}$$

or

$$dv = -u\frac{dm}{m}. \tag{9.11.4}$$

Integrating from the initial mass m_0 to the final mass m of the rocket gives us the result we are after:

$$\begin{aligned}\int_{v_i}^{v} dv &= -u\int_{m_0}^{m}\frac{1}{m}dm\\ v - v_i &= u\ln\left(\frac{m_0}{m}\right)\end{aligned}$$

and thus our final answer is

$$\Delta v = u \ln\left(\frac{m_0}{m}\right). \tag{9.11.5}$$

This result is called the **rocket equation**. It was originally derived by the Soviet physicist Konstantin Tsiolkovsky in 1897. It gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass from m_0 down to m. As expected, the relationship between Δv and the change of mass of the rocket is nonlinear.

Problem-Solving Strategy: Rocket Propulsion

In rocket problems, the most common questions are finding the change of velocity due to burning some amount of fuel for some amount of time; or to determine the acceleration that results from burning fuel.

1. To determine the change of velocity, use the rocket equation Equation 9.11.5.
2. To determine the acceleration, determine the force by using the impulse-momentum theorem, using the rocket equation to determine the change of velocity

Example 9.11.1: Thrust on a Spacecraft

A spacecraft is moving in gravity-free space along a straight path when its pilot decides to accelerate forward. He turns on the thrusters, and burned fuel is ejected at a constant rate of $2.0 \times 10^2\ kg/s$, at a speed (relative to the rocket) of $2.5 \times 10^2\ m/s$. The initial mass of the spacecraft and its unburned fuel is $2.0 \times 10^4\ kg$, and the thrusters are on for 30 s.

a. What is the thrust (the force applied to the rocket by the ejected fuel) on the spacecraft?
b. What is the spacecraft's acceleration as a function of time?
c. What are the spacecraft's accelerations at t = 0, 15, 30, and 35 s?

Strategy

a. The force on the spacecraft is equal to the rate of change of the momentum of the fuel.
b. Knowing the force from part (a), we can use Newton's second law to calculate the consequent acceleration. The key here is that, although the force applied to the spacecraft is constant (the fuel is being ejected at a constant rate), the mass of the spacecraft isn't; thus, the acceleration caused by the force won't be constant. We expect to get a function $a(t)$, therefore.
c. We'll use the function we obtain in part (b), and just substitute the numbers given. Important: We expect that the acceleration will get larger as time goes on, since the mass being accelerated is continuously decreasing (fuel is being ejected from the rocket).

Solution

a. The momentum of the ejected fuel gas is

$$p = m_g v. \tag{9.11.6}$$

The ejection velocity v = 2.5 x 10^2 m/s is constant, and therefore the force is

$$F = \frac{dp}{dt} = v\frac{dm_g}{dt} = -v\frac{dm}{dt}. \tag{9.11.7}$$

Now, $\frac{dm_g}{dt}$ is the rate of change of the mass of the fuel; the problem states that this is 2.0 x 10^2 kg/s. Substituting, we get

$$\begin{aligned} F &= v\frac{dm_g}{dt} \\ &= (2.5 \times 10^2\ m/s)(2.0 \times 10^2\ kg/s) \\ &= 5 \times 10^4\ N. \end{aligned}$$

b. Above, we defined m to be the combined mass of the empty rocket plus however much unburned fuel it contained: $m = m_R + m_g$. From Newton's second law,

$$a = \frac{F}{m} = \frac{F}{m_R + m_g}. \tag{9.11.8}$$

The force is constant and the empty rocket mass m_R is constant, but the fuel mass m_g is decreasing at a uniform rate; specifically:

$$m_g = m_g(t) - m_{g_0} - \left(\frac{dm_g}{dt}\right)t. \tag{9.11.9}$$

This gives us

$$a(t) = \frac{F}{m_{g_1} - \left(\frac{dm_g}{dt}\right)t} = \frac{F}{M - \left(\frac{dm_g}{dt}\right)t}. \tag{9.11.10}$$

Notice that, as expected, the acceleration is a function of time. Substituting the given numbers:

$$a(t) = \frac{5\times 10^4\ N}{(2.0\times 10^4\ kg) - (2.0\times 10^2\ kg/s)t}. \tag{9.11.11}$$

c. At t = 0 s:

$$a(0\ s) = \frac{5\times 10^4\ N}{(2.0\times 10^4\ kg) - (2.0\times 10^2\ kg/s)(0\ s)} = 2.5\ m/s^2. \tag{9.11.12}$$

At t = 15 s, a(15 s) = 2.9 m/s^2.

At t = 30 s, a(30 s) = 3.6 m/s^2.

Acceleration is increasing, as we expected.

Significance

Notice that the acceleration is not constant; as a result, any dynamical quantities must be calculated either using integrals, or (more easily) conservation of total energy

? Excerise: 9.11.1

What is the physical difference (or relationship) between $\frac{dm}{dt}$ and $\frac{dm_g}{dt}$ in this example?

Rocket in a Gravitational Field

Let's now analyze the velocity change of the rocket during the launch phase, from the surface of Earth. To keep the math manageable, we'll restrict our attention to distances for which the acceleration caused by gravity can be treated as a constant g.

The analysis is similar, except that now there is an external force of $\vec{F} = -mg\ \hat{j}$ acting on our system. This force applies an impulse $d\vec{J} = \vec{F}dt = -mgdt\ \hat{j}$, which is equal to the change of momentum. This gives us

$$\begin{aligned} d\vec{p} &= d\vec{J} \\ \vec{p}_f - \vec{p}_i &= -mgdt\ \hat{j} \\ \left[(m - dm_g)(v + dv) + dm_g(v - u) - mv\right]\hat{j} &= -mgdt\ \hat{j} \end{aligned}$$

and so

$$mdv - dm_g u = -mgdt \tag{9.11.13}$$

where we have again neglected the term $dm_g dv$ and dropped the vector notation. Next we replace dm_g with $-dm$:

$$\begin{aligned} mdv + dmu &= -mgdt \\ mdv &= -dmu - mgdt. \end{aligned}$$

Dividing through by m gives

$$dv = -u\frac{dm}{m} - gdt \tag{9.11.14}$$

and integrating, we have

$$\Delta v = u \ln\left(\frac{m_0}{m}\right) - g\Delta t. \tag{9.11.15}$$

Unsurprisingly, the rocket's velocity is affected by the (constant) acceleration of gravity.

Remember that Δt is the burn time of the fuel. Now, in the absence of gravity, Equation 9.11.5 implies that it makes no difference how much time it takes to burn the entire mass of fuel; the change of velocity does not depend on Δt. However, in the presence of gravity, it matters a lot. The $-g\Delta t$ term in Equation 9.11.15 tells us that the **longer** the burn time is, the **smaller** the rocket's change of velocity will be. This is the reason that the launch of a rocket is so spectacular at the first moment of liftoff: It's essential to burn the fuel as quickly as possible, to get as large a Δv as possible.

9.12: Linear Momentum and Collisions (Exercises)

Conceptual Questions

9.1 Linear Momentum

1. An object that has a small mass and an object that has a large mass have the same momentum. Which object has the largest kinetic energy?
2. An object that has a small mass and an object that has a large mass have the same kinetic energy. Which mass has the largest momentum?

9.2 Impulse and Collisions

3. Is it possible for a small force to produce a larger impulse on a given object than a large force? Explain.
4. Why is a 10-m fall onto concrete far more dangerous than a 10-m fall onto water?
5. What external force is responsible for changing the momentum of a car moving along a horizontal road?
6. A piece of putty and a tennis ball with the same mass are thrown against a wall with the same velocity. Which object experience a greater force from the wall or are the forces equal? Explain.

9.3 Conservation of Linear Momentum

7. Under what circumstances is momentum conserved?
8. Can momentum be conserved for a system if there are external forces acting on the system? If so, under what conditions? If not, why not?
9. Explain in terms of momentum and Newton's laws how a car's air resistance is due in part to the fact that it pushes air in its direction of motion.
10. Can objects in a system have momentum while the momentum of the system is zero? Explain your answer.
11. A sprinter accelerates out of the starting blocks. Can you consider him as a closed system? Explain.
12. A rocket in deep space (zero gravity) accelerates by firing hot gas out of its thrusters. Does the rocket constitute a closed system? Explain.

9.4 Types of Collisions

13. Two objects of equal mass are moving with equal and opposite velocities when they collide. Can all the kinetic energy be lost in the collision?
14. Describe a system for which momentum is conserved but mechanical energy is not. Now the reverse: Describe a system for which kinetic energy is conserved but momentum is not.

9.5 Collisions in Multiple Dimensions

15. Momentum for a system can be conserved in one direction while not being conserved in another. What is the angle between the directions? Give an example.

9.6 Center of Mass

16. Suppose a fireworks shell explodes, breaking into three large pieces for which air resistance is negligible. How does the explosion affect the motion of the center of mass? How would it be affected if the pieces experienced significantly more air resistance than the intact shell?

9.7 Rocket Propulsion

17. It is possible for the velocity of a rocket to be greater than the exhaust velocity of the gases it ejects. When that is the case, the gas velocity and gas momentum are in the same direction as that of the rocket. How is the rocket still able to obtain thrust by ejecting the gases?

Problems

9.1 Linear Momentum

18. An elephant and a hunter are having a confrontation. A drawing of an elephant, on the left, and hunter, on the right. An xy-coordinate system has positive x to the right and positive y up. The elephant is labeled with m_E = 2000.0 kg, and vector v_E = 7.50 m/s $\hat{i}$. An arrow above the v_E vector points to the right. The hunter is labeled with m_{hunter} = 90.0 kg, and

vector v_{hunter} = 7.40 m/s $\hat{i}$. An arrow above the v_{hunter} vector points to the right. Between the hunter and elephant is a dart with a long arrow pointing to the left drawn near it and labeled vector v_{dart} = -600 m/s $\hat{i}$, and m_{dart} = 0.0400 kg.

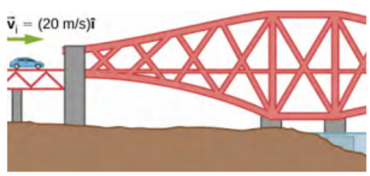
a. Calculate the momentum of the 2000.0-kg elephant charging the hunter at a speed of 7.50 m/s.
b. Calculate the ratio of the elephant's momentum to the momentum of a 0.0400-kg tranquilizer dart fired at a speed of 600 m/s.
c. What is the momentum of the 90.0-kg hunter running at 7.40 m/s after missing the elephant?

19. A skater of mass 40 kg is carrying a box of mass 5 kg. The skater has a speed of 5 m/s with respect to the floor and is gliding without any friction on a smooth surface. (a) Find the momentum of the box with respect to the floor. (b) Find the momentum of the box with respect to the floor after she puts the box down on the frictionless skating surface. (c) A car of mass 2000 kg is moving with a constant velocity of 10 m/s due east. What is the momentum of the car?
20. The mass of Earth is 5.97 x 10^{24} kg and its orbital radius is an average of 1.50 x 10^{11} m. Calculate the magnitude of its average linear momentum.
21. If a rainstorm drops 1 cm of rain over an area of 10 km^2 in the period of 1 hour, what is the momentum of the rain that falls in one second? Assume the terminal velocity of a raindrop is 10 m/s.
22. What is the average momentum of an avalanche that moves a 40-cm-thick layer of snow over an area of 100 m by 500 m over a distance of 1 km down a hill in 5.5 s? Assume a density of 350 kg/m^3 for the snow.
23. What is the average momentum of a 70.0-kg sprinter who runs the 100-m dash in 9.65 s?

9.2 Impulse and Collisions

25. A 75.0-kg person is riding in a car moving at 20.0 m/s when the car runs into a bridge abutment (see the following figure).

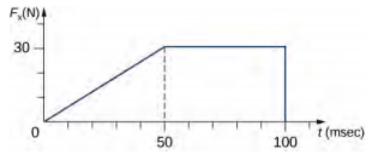
a. Calculate the average force on the person if he is stopped by a padded dashboard that compresses an average of 1.00 cm.
b. Calculate the average force on the person if he is stopped by an air bag that compresses an average of 15.0 cm.



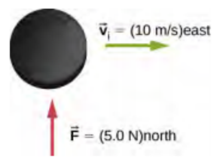
26. One hazard of space travel is debris left by previous missions. There are several thousand objects orbiting Earth that are large enough to be detected by radar, but there are far greater numbers of very small objects, such as flakes of paint. Calculate the force exerted by a 0.100-mg chip of paint that strikes a spacecraft window at a relative speed of 4.00 x 10^3 m/s, given the collision lasts 6.00 x 10^{-8} s.
27. A cruise ship with a mass of 1.00 x 10^7 kg strikes a pier at a speed of 0.750 m/s. It comes to rest after traveling 6.00 m, damaging the ship, the pier, and the tugboat captain's finances. Calculate the average force exerted on the pier using the concept of impulse. (**Hint**: First calculate the time it took to bring the ship to rest, assuming a constant force.)

28. Calculate the final speed of a 110-kg rugby player who is initially running at 8.00 m/s but collides head-on with a padded goalpost and experiences a backward force of 1.76 x 10^4 N for 5.50 x 10^{-2} s.
29. Water from a fire hose is directed horizontally against a wall at a rate of 50.0 kg/s and a speed of 42.0 m/s. Calculate the force exerted on the wall, assuming the water's horizontal momentum is reduced to zero.
30. A 0.450-kg hammer is moving horizontally at 7.00 m/ s when it strikes a nail and comes to rest after driving the nail 1.00 cm into a board. Assume constant acceleration of the hammer-nail pair. (a) Calculate the duration of the impact. (b) What was the average force exerted on the nail?
31. What is the momentum (as a function of time) of a 5.0-kg particle moving with a velocity $\vec{v}(t) = (2.0\ \hat{i} + 4.0t\ \hat{j})$ m/s? What is the net force acting on this particle?
32. The x-component of a force on a 46-g golf ball by a 7-iron versus time is plotted in the following figure:

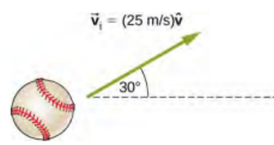
a. Find the x-component of the impulse during the intervals (i) [0, 50 ms], and (ii) [50 ms, 100 ms].
b. Find the change in the x-component of the momentum during the intervals (iii) [0, 50 ms], and (iv) [50 ms, 100 ms].



33. A hockey puck of mass 150 g is sliding due east on a frictionless table with a speed of 10 m/s. Suddenly, a constant force of magnitude 5 N and direction due north is applied to the puck for 1.5 s. Find the north and east components of the momentum at the end of the 1.5-s interval.

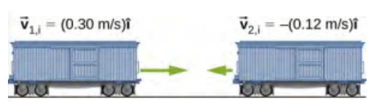


34. A ball of mass 250 g is thrown with an initial velocity of 25 m/s at an angle of 30° with the horizontal direction. Ignore air resistance. What is the momentum of the ball after 0.2 s? (Do this problem by finding the components of the momentum first, and then constructing the magnitude and direction of the momentum vector from the components.)

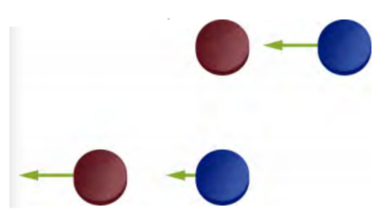


9.3 Conservation of Linear Momentum

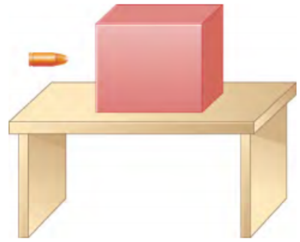
35. Train cars are coupled together by being bumped into one another. Suppose two loaded train cars are moving toward one another, the first having a mass of 1.50 x 10^5 kg and a velocity of (0.30 m/s) $\hat{i}$, and the second having a mass of 1.10 x 10^5 kg and a velocity of −(0.12 m/s) $\hat{i}$. What is their final velocity?



36. Two identical pucks collide elastically on an air hockey table. Puck 1 was originally at rest; puck 2 has an incoming speed of 6.00 m/s and scatters at an angle of 30° with respect to its incoming direction. What is the velocity (magnitude and direction) of puck 1 after the collision?



37. The figure below shows a bullet of mass 200 g traveling horizontally towards the east with speed 400 m/s, which strikes a block of mass 1.5 kg that is initially at rest on a frictionless table. After striking the block, the bullet is embedded in the block and the block and the bullet move together as one unit. (a) What is the magnitude and direction of the velocity of the block/bullet combination immediately after the impact? (b) What is the magnitude and direction of the impulse by the block on the bullet? (c) What is the magnitude and direction of the impulse from the bullet on the block? (d) If it took 3 ms for the bullet to change the speed from 400 m/s to the final speed after impact, what is the average force between the block and the bullet during this time?



38. A 20-kg child is coasting at 3.3 m/s over flat ground in a 4.0-kg wagon. The child drops a 1.0-kg ball out the back of the wagon. What is the final speed of the child and wagon?
39. A 5000-kg paving truck coasts over a road at 2.5 m/s and quickly dumps 1000 kg of gravel on the road. What is the speed of the truck after dumping the gravel?
40. Explain why a cannon recoils when it fires a shell.
41. Two figure skaters are coasting in the same direction, with the leading skater moving at 5.5 m/s and the trailing skating moving at 6.2 m/s. When the trailing skater catches up with the leading skater, he picks her up without applying any horizontal forces on his skates. If the trailing skater is 50% heavier than the 50-kg leading skater, what is their speed after he picks her up?
42. A 2000-kg railway freight car coasts at 4.4 m/s underneath a grain terminal, which dumps grain directly down into the freight car. If the speed of the loaded freight car must not go below 3.0 m/s, what is the maximum mass of grain that it can accept?

9.4 Types of Collisions

43. A 5.50-kg bowling ball moving at 9.00 m/s collides with a 0.850-kg bowling pin, which is scattered at an angle of 15.8° to the initial direction of the bowling ball and with a speed of 15.0 m/s. (a) Calculate the final velocity (magnitude and direction) of the bowling ball. (b) Is the collision elastic?
44. Ernest Rutherford (the first New Zealander to be awarded the Nobel Prize in Chemistry) demonstrated that nuclei were very small and dense by scattering helium-4 nuclei from gold-197 nuclei. The energy of the incoming helium nucleus was 8.00 x 10^{-13} J, and the masses of the helium and gold nuclei were 6.68 x 10^{-27} kg and 3.29 x 10^{-25} kg, respectively (note that their mass ratio is 4 to 197). (a) If a helium nucleus scatters to an angle of 120° during an elastic collision with a gold nucleus, calculate the helium nucleus's final speed and the final velocity (magnitude and direction) of the gold nucleus. (b) What is the final kinetic energy of the helium nucleus?

45. A 90.0-kg ice hockey player hits a 0.150-kg puck, giving the puck a velocity of 45.0 m/s. If both are initially at rest and if the ice is frictionless, how far does the player recoil in the time it takes the puck to reach the goal 15.0 m away?
46. A 100-g firecracker is launched vertically into the air and explodes into two pieces at the peak of its trajectory. If a 72-g piece is projected horizontally to the left at 20 m/s, what is the speed and direction of the other piece?
47. In an elastic collision, a 400-kg bumper car collides directly from behind with a second, identical bumper car that is traveling in the same direction. The initial speed of the leading bumper car is 5.60 m/s and that of the trailing car is 6.00 m/s. Assuming that the mass of the drivers is much, much less than that of the bumper cars, what are their final speeds?
48. Repeat the preceding problem if the mass of the leading bumper car is 30.0% greater than that of the trailing bumper car.
49. An alpha particle (^{4}He) undergoes an elastic collision with a stationary uranium nucleus (^{235}U). What percent of the kinetic energy of the alpha particle is transferred to the uranium nucleus? Assume the collision is one-dimensional.
50. You are standing on a very slippery icy surface and throw a 1-kg football horizontally at a speed of 6.7 m/ s. What is your velocity when you release the football? Assume your mass is 65 kg.
51. A 35-kg child sleds down a hill and then coasts along the flat section at the bottom, where a second 35-kg child jumps on the sled as it passes by her. If the speed of the sled is 3.5 m/s before the second child jumps on, what is its speed after she jumps on?
52. A boy sleds down a hill and onto a frictionless ice-covered lake at 10.0 m/s. In the middle of the lake is a 1000-kg boulder. When the sled crashes into the boulder, he is propelled over the boulder and continues sliding over the ice. If the

boy's mass is 40.0 kg and the sled's mass is 2.50 kg, what is the speed of the sled and the boulder after the collision?

9.5 Collisions in Multiple Dimensions

53. A 0.90-kg falcon is diving at 28.0 m/s at a downward angle of 35°. It catches a 0.325-kg pigeon from behind in midair. What is their combined velocity after impact if the pigeon's initial velocity was 7.00 m/s directed horizontally? Note that $\vec{v}_{1,i}$ is a unit vector pointing in the direction in which the falcon is initially flying.

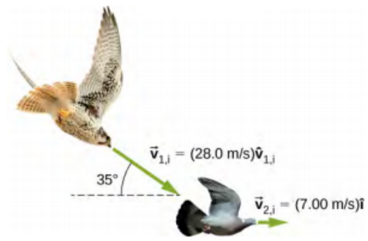
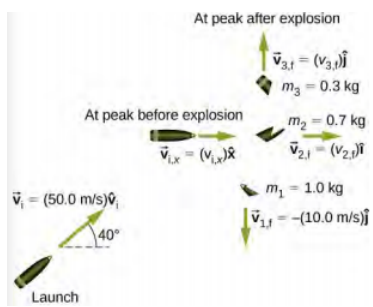


Figure 9.12.1 - (credit "hawk": modification of work by "USFWS Mountain-Prairie"/Flickr; credit "dove": modification of work by Jacob Spinks)

54. A billiard ball, labeled 1, moving horizontally strikes another billiard ball, labeled 2, at rest. Before impact, ball 1 was moving at a speed of 3.00 m/s, and after impact it is moving at 0.50 m/s at 50° from the original direction. If the two balls have equal masses of 300 g, what is the velocity of the ball 2 after the impact?
55. A projectile of mass 2.0 kg is fired in the air at an angle of 40.0 ° to the horizon at a speed of 50.0 m/s. At the highest point in its flight, the projectile breaks into three parts of mass 1.0 kg, 0.7 kg, and 0.3 kg. The 1.0-kg part falls straight down after breakup with an initial speed of 10.0 m/s, the 0.7-kg part moves in the original forward direction, and the 0.3-kg part goes straight up. (a) Find the speeds of the 0.3-kg and 0.7-kg pieces immediately after the break-up. (b) How high from the break-up point does the 0.3-kg piece go before coming to rest? (c) Where does the 0.7-kg piece land relative to where it was fired from?



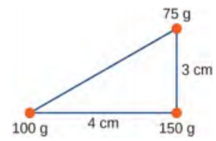
56. Two asteroids collide and stick together. The first asteroid has mass of 15 x 10^3 kg and is initially moving at 770 m/s. The second asteroid has mass of 20 x 10^3 kg and is moving at 1020 m/s. Their initial velocities made an angle of 20° with respect to each other. What is the final speed and direction with respect to the velocity of the first asteroid?
57. A 200-kg rocket in deep space moves with a velocity of (121 m/s) $\hat{i}$ + (38.0 m/s) $\hat{j}$. Suddenly, it explodes into three pieces, with the first (78 kg) moving at −(321 m/s) $\hat{i}$ + (228 m/s) $\hat{j}$ and the second (56 kg) moving at (16.0 m/s) $\hat{i}$ − (88.0 m/s) $\hat{j}$. Find the velocity of the third piece.
58. A proton traveling at 3.0 x 10^6 m/s scatters elastically from an initially stationary alpha particle and is deflected at an angle of 85° with respect to its initial velocity. Given that the alpha particle has four times the mass of the proton, what percent of its initial kinetic energy does the proton retain after the collision?
59. Three 70-kg deer are standing on a flat 200-kg rock that is on an ice-covered pond. A gunshot goes off and the deer scatter, with deer A running at (15 m/s) $\hat{i}$ + (5.0 m/s) $\hat{j}$, deer B running at (−12 m/s) $\hat{i}$ + (8.0 m/s) $\hat{j}$, and deer C running at (1.2 m/s) $\hat{i}$ − (18.0 m/s) $\hat{j}$. What is the velocity of the rock on which they were standing?
60. A family is skating. The father (75 kg) skates at 8.2 m/s and collides and sticks to the mother (50 kg), who was initially moving at 3.3 m/s and at 45° with respect to the father's velocity. The pair then collides with their daughter (30 kg), who was stationary, and the three slide off together. What is their final velocity?
61. An oxygen atom (mass 16 u) moving at 733 m/s at 15.0° with respect to the $\hat{i}$ direction collides and sticks to an oxygen molecule (mass 32 u) moving at 528 m/s at 128° with respect to the $\hat{i}$ direction. The two stick together to form ozone.

What is the final velocity of the ozone molecule?

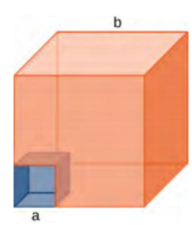
62. Two cars approach an extremely icy four-way perpendicular intersection. Car A travels northward at 30 m/s and car B is travelling eastward. They collide and stick together, traveling at 28° north of east. What was the initial velocity of car B?

9.6 Center of Mass

63. Three point masses are placed at the corners of a triangle as shown in the figure below. Find the center of mass of the three-mass system.

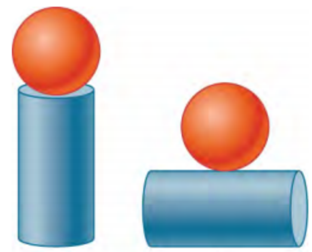


64. Two particles of masses m_1 and m_2 separated by a horizontal distance D are released from the same height h at the same time. Find the vertical position of the center of mass of these two particles at a time before the two particles strike the ground. Assume no air resistance.
65. Two particles of masses m_1 and m_2 separated by a horizontal distance D are let go from the same height h at different times. Particle 1 starts at t = 0 , and particle 2 is let go at t = T. Find the vertical position of the center of mass at a time before the first particle strikes the ground. Assume no air resistance.
66. Two particles of masses m_1 and m_2 move uniformly in different circles of radii R_1 and R_2 about origin in the x,y-plane. The x- and y-coordinates of the center of mass and that of particle 1 are given as follows (where length is in meters and t in seconds): $x_1(t) = 4\cos(2t)$, $y_1(t) = 4\sin(2t)$ and: $x_{CM}(t) = 3\cos(2t)$, $y_{CM}(t) = 3\sin(2t)$. (a) Find the radius of the circle in which particle 1 moves. (b) Find the x- and y-coordinates of particle 2 and the radius of the circle this particle moves.
67. Two particles of masses m_1 and m_2 move uniformly in different circles of radii R_1 and R_2 about the origin in the x, y-plane. The coordinates of the two particles in meters are given as follows (z = 0 for both). Here t is in seconds: $x_1(t) = 4\cos(2t)$, $y_1(t) = 4\sin(2t)$, $x_2(t) = 2\cos\left(3t - \frac{\pi}{2}\right)$, $y_2(t) = 2\sin\left(3t - \frac{\pi}{2}\right)$ (a) Find the radii of the circles of motion of both particles. (b) Find the x- and y-coordinates of the center of mass. (c) Decide if the center of mass moves in a circle by plotting its trajectory.
68. Find the center of mass of a one-meter long rod, made of 50 cm of iron (density 8 g/cm^3) and 50 cm of aluminum (density 2.7 g/cm^3).
69. Find the center of mass of a rod of length L whose mass density changes from one end to the other quadratically. That is, if the rod is laid out along the x-axis with one end at the origin and the other end at x = L, the density is given by $\rho(x) = \rho_0 + (\rho_1 - \rho_0)\left(\frac{x}{L}\right)^2$, where ρ_0 and ρ_1 are constant values.
70. Find the center of mass of a rectangular block of length a and width b that has a nonuniform density such that when the rectangle is placed in the x,y-plane with one corner at the origin and the block placed in the first quadrant with the two edges along the x- and y-axes, the density is given by $\rho(x, y) = \rho_0 x$, where ρ_0 is a constant.
71. Find the center of mass of a rectangular material of length a and width b made up of a material of nonuniform density. The density is such that when the rectangle is placed in the xy-plane, the density is given by $\rho(x, y) = \rho_0 xy$.
72. A cube of side a is cut out of another cube of side b as shown in the figure below. Find the location of the center of mass of the structure. (**Hint**: Think of the missing part as a negative mass overlapping a positive mass.)



73. Find the center of mass of cone of uniform density that has a radius R at the base, height h, and mass M. Let the origin be at the center of the base of the cone and have +z going through the cone vertex.
74. Find the center of mass of a thin wire of mass m and length L bent in a semicircular shape. Let the origin be at the center of the semicircle and have the wire arc from the +x axis, cross the +y axis, and terminate at the −x axis.
75. Find the center of mass of a uniform thin semicircular plate of radius R. Let the origin be at the center of the semicircle, the plate arc from the +x axis to the −x axis, and the z axis be perpendicular to the plate.

76. Find the center of mass of a sphere of mass M and radius R and a cylinder of mass m, radius r, and height h arranged as shown below. Express your answers in a coordinate system that has the origin at the center of the cylinder.



9.7 Rocket Propulsion

77. A 5.00-kg squid initially at rest ejects 0.250 kg of fluid with a velocity of 10.0 m/s. (a) What is the recoil velocity of the squid if the ejection is done in 0.100 s and there is a 5.00-N frictional force opposing the squid's movement? (b) How much energy is lost to work done against friction?
78. A rocket takes off from Earth and reaches a speed of 100 m/s in 10.0 s. If the exhaust speed is 1500 m/s and the mass of fuel burned is 100 kg, what was the initial mass of the rocket?
79. Repeat the preceding problem but for a rocket that takes off from a space station, where there is no gravity other than the negligible gravity due to the space station. 8
80. How much fuel would be needed for a 1000-kg rocket (this is its mass with no fuel) to take off from Earth and reach 1000 m/s in 30 s? The exhaust speed is 1000 m/s.
81. What exhaust speed is required to accelerate a rocket in deep space from 800 m/s to 1000 m/s in 5.0 s if the total rocket mass is 1200 kg and the rocket only has 50 kg of fuel left?
82. **Unreasonable Results** Squids have been reported to jump from the ocean and travel 30.0 m (measured horizontally) before re-entering the water. (a) Calculate the initial speed of the squid if it leaves the water at an angle of 20.0°, assuming negligible lift from the air and negligible air resistance. (b) The squid propels itself by squirting water. What fraction of its mass would it have to eject in order to achieve the speed found in the previous part? The water is ejected at 12.0 m/s; gravitational force and friction are neglected. (c) What is unreasonable about the results? (d) Which premise is unreasonable, or which premises are inconsistent?

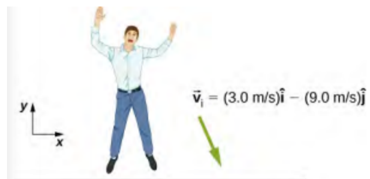
Additional Problems

83. Two 70-kg canoers paddle in a single, 50-kg canoe. Their paddling moves the canoe at 1.2 m/s with respect to the water, and the river they're in flows at 4 m/s with respect to the land. What is their momentum with respect to the land?
84. Which has a larger magnitude of momentum: a 3000-kg elephant moving at 40 km/h or a 60-kg cheetah moving at 112 km/h?
85. A driver applies the brakes and reduces the speed of her car by 20%, without changing the direction in which the car is moving. By how much does the car's momentum change?
86. You friend claims that momentum is mass multiplied by velocity, so things with more mass have more momentum. Do you agree? Explain.
87. Dropping a glass on a cement floor is more likely to break the glass than if it is dropped from the same height on a grass lawn. Explain in terms of the impulse.
88. Your 1500-kg sports car accelerates from 0 to 30 m/s in 10 s. What average force is exerted on it during this acceleration?
89. A ball of mass m is dropped. What is the formula for the impulse exerted on the ball from the instant it is dropped to an arbitrary time τ later? Ignore air resistance.
90. Repeat the preceding problem, but including a drag force due to air of $f_{drag} = -b\vec{v}$.
91. A 5.0-g egg falls from a 90-cm-high counter onto the floor and breaks. What impulse is exerted by the floor on the egg?
92. A car crashes into a large tree that does not move. The car goes from 30 m/s to 0 in 1.3 m. (a) What impulse is applied to the driver by the seatbelt, assuming he follows the same motion as the car? (b) What is the average force applied to the driver by the seatbelt?
93. Two hockey players approach each other head on, each traveling at the same speed v_i. They collide and get tangled together, falling down and moving off at a speed $\frac{v_i}{5}$. What is the ratio of their masses?

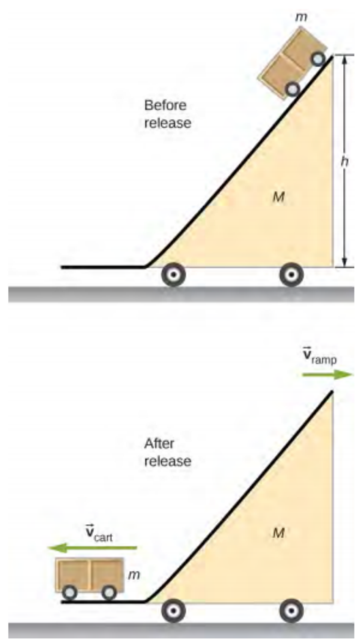
94. You are coasting on your 10-kg bicycle at 15 m/s and a 5.0-g bug splatters on your helmet. The bug was initially moving at 2.0 m/s in the same direction as you. If your mass is 60 kg, (a) what is the initial momentum of you plus your bicycle? (b) What is the initial momentum of the bug? (c) What is your change in velocity due to the collision with the bug? (d) What would the change in velocity have been if the bug were traveling in the opposite direction?
95. A load of gravel is dumped straight down into a 30 000-kg freight car coasting at 2.2 m/s on a straight section of a railroad. If the freight car's speed after receiving the gravel is 1.5 m/s, what mass of gravel did it receive?
96. Two carts on a straight track collide head on. The first cart was moving at 3.6 m/s in the positive x direction and the second was moving at 2.4 m/s in the opposite direction. After the collision, the second car continues moving in its initial direction of motion at 0.24 m/s. If the mass of the second car is 5.0 times that of the first, what is the final velocity of the first car?
97. A 100-kg astronaut finds himself separated from his spaceship by 10 m and moving away from the spaceship at 0.1 m/s. To get back to the spaceship, he throws a 10-kg tool bag away from the spaceship at 5.0 m/s. How long will he take to return to the spaceship?
98. Derive the equations giving the final speeds for two objects that collide elastically, with the mass of the objects being m_1 and m_2 and the initial speeds being $v_{1,i}$ and $v_{2,i} = 0$ (i.e., second object is initially stationary).
99. Repeat the preceding problem for the case when the initial speed of the second object is nonzero.
100. A child sleds down a hill and collides at 5.6 m/s into a stationary sled that is identical to his. The child is launched forward at the same speed, leaving behind the two sleds that lock together and slide forward more slowly. What is the speed of the two sleds after this collision?
101. For the preceding problem, find the final speed of each sled for the case of an elastic collision.
102. A 90-kg football player jumps vertically into the air to catch a 0.50-kg football that is thrown essentially horizontally at him at 17 m/s. What is his horizontal speed after catching the ball?
103. Three skydivers are plummeting earthward. They are initially holding onto each other, but then push apart. Two skydivers of mass 70 and 80 kg gain horizontal velocities of 1.2 m/s north and 1.4 m/s southeast, respectively. What is the horizontal velocity of the third skydiver, whose mass is 55 kg?
104. Two billiard balls are at rest and touching each other on a pool table. The cue ball travels at 3.8 m/s along the line of symmetry between these balls and strikes them simultaneously. If the collision is elastic, what is the velocity of the three balls after the collision?
105. A billiard ball traveling at (2.2 m/s) $\hat{i}$ − (0.4 m/s) $\hat{j}$ collides with a wall that is aligned in the $\hat{j}$ direction. Assuming the collision is elastic, what is the final velocity of the ball?
106. Two identical billiard balls collide. The first one is initially traveling at (2.2 m/s) $\hat{i}$ − (0.4 m/s) $\hat{j}$ and the second one at −(1.4 m/s) $\hat{i}$ + (2.4 m/s) $\hat{j}$. Suppose they collide when the center of ball 1 is at the origin and the center of ball 2 is at the point (2R, 0) where R is the radius of the balls. What is the final velocity of each ball?
107. Repeat the preceding problem if the balls collide when the center of ball 1 is at the origin and the center of ball 2 is at the point (0, 2R).
108. Repeat the preceding problem if the balls collide when the center of ball 1 is at the origin and the center of ball 2 is at the point $\left(\frac{\sqrt{3}R}{2}, \frac{R}{2}\right)$.
109. Where is the center of mass of a semicircular wire of radius R that is centered on the origin, begins and ends on the x axis, and lies in the x,y plane?
110. Where is the center of mass of a slice of pizza that was cut into eight equal slices? Assume the origin is at the apex of the slice and measure angles with respect to an edge of the slice. The radius of the pizza is R.
111. If the entire population of Earth were transferred to the Moon, how far would the center of mass of the Earth-Moon-population system move? Assume the population is 7 billion, the average human has a mass of 65 kg, and that the population is evenly distributed over both the Earth and the Moon. The mass of the Earth is 5.97×10^{24} kg and that of the Moon is 7.34×10^{22} kg. The radius of the Moon's orbit is about 3.84×10^{5} m.
112. You friend wonders how a rocket continues to climb into the sky once it is sufficiently high above the surface of Earth so that its expelled gasses no longer push on the surface. How do you respond?
113. To increase the acceleration of a rocket, should you throw rocks out of the front window of the rocket or out of the back window?

Challenge Problems

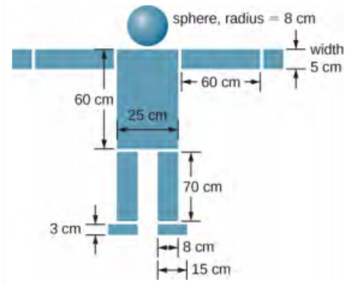
114. A 65-kg person jumps from the first floor window of a burning building and lands almost vertically on the ground with a horizontal velocity of 3 m/s and vertical velocity of −9 m/s. Upon impact with the ground he is brought to rest in a short time. The force experienced by his feet depends on whether he keeps his knees stiff or bends them. Find the force on his feet in each case. (a) First find the impulse on the person from the impact on the ground. Calculate both its magnitude and direction. (b) Find the average force on the feet if the person keeps his leg stiff and straight and his center of mass drops by only 1 cm vertically and 1 cm horizontally during the impact. (c) Find the average force on the feet if the person bends his legs throughout the impact so that his center of mass drops by 50 cm vertically and 5 cm horizontally during the impact. (d) Compare the results of part (b) and (c), and draw conclusions about which way is better. You will need to find the time the impact lasts by making reasonable assumptions about the deceleration. Although the force is not constant during the impact, working with constant average force for this problem is acceptable.



115. Two projectiles of mass m_1 and m_2 are fired at the same speed but in opposite directions from two launch sites separated by a distance D. They both reach the same spot in their highest point and strike there. As a result of the impact they stick together and move as a single body afterwards. Find the place they will land.
116. Two identical objects (such as billiard balls) have a one-dimensional collision in which one is initially motionless. After the collision, the moving object is stationary and the other moves with the same speed as the other originally had. Show that both momentum and kinetic energy are conserved.
117. A ramp of mass M is at rest on a horizontal surface. A small cart of mass m is placed at the top of the ramp and released. What are the velocities of the ramp and the cart relative to the ground at the instant the cart leaves the ramp?



118. Find the center of mass of the structure given in the figure below. Assume a uniform thickness of 20 cm, and a uniform density of 1 g/cm^3.



Contributors and Attributions

9.13: Linear Momentum and Collisions (Summary)

Key Terms

center of mass	weighted average position of the mass
closed system	system for which the mass is constant and the net external force on the system is zero
elastic	collision that conserves kinetic energy
explosion	single object breaks up into multiple objects; kinetic energy is not conserved in explosions
external force	force applied to an extended object that changes the momentum of the extended object as a whole
impulse	effect of applying a force on a system for a time interval; this time interval is usually small, but does not have to be
impulse-momentum theorem	change of momentum of a system is equal to the impulse applied to the system
inelastic	collision that does not conserve kinetic energy
internal force	force that the simple particles that make up an extended object exert on each other. Internal forces can be attractive or repulsive
Law of Conservation of Momentum	total momentum of a closed system cannot change
linear mass density	λ, expressed as the number of kilograms of material per meter
momentum	measure of the quantity of motion that an object has; it takes into account both how fast the object is moving, and its mass; specifically, it is the product of mass and velocity; it is a vector quantity
perfectly inelastic	collision after which all objects are motionless, the final kinetic energy is zero, and the loss of kinetic energy is a maximum
rocket equation	derived by the Soviet physicist Konstantin Tsiolkovsky in 1897, it gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass from m_i down to m
system	object or collection of objects whose motion is currently under investigation; however, your system is defined at the start of the problem, you must keep that definition for the entire problem

Key Equations

Definition of momentum	$$\vec{p} = m\vec{v} \tag{9.13.1}$$
Impulse	$$\vec{J} \equiv \int_{t_i}^{t_f} \vec{F}(t)dt \; or \; \vec{J} = \vec{F}_{ave}\Delta t \tag{9.13.2}$$
Impulse-momentum theorem	$$\vec{J} = \Delta\vec{p} \tag{9.13.3}$$
Average force from momentum	$$\vec{F} = \frac{\Delta\vec{p}}{\Delta t} \tag{9.13.4}$$

Instantaneous force from momentum (Newton’s second law)	$$\vec{F}(t)=\frac{d\vec{p}}{dt} \tag{9.13.5}$$
Conservation of momentum	$$\frac{d\vec{p}_1}{dt}+\frac{d\vec{p}_2}{dt}=0 \text{ or } \vec{p}_1+\vec{p}_2=constant \tag{9.13.6}$$
Generalized conservation of momentum	$$\sum_{j=1}^{N}\vec{p}_j=constant \tag{9.13.7}$$
Conservation of momentum in two dimensions	$$p_{f,x}=p_{1,i,x}+p_{2,i,x} \tag{9.13.8}$$ $$p_{f,y}=p_{1,i,y}+p_{2,i,y} \tag{9.13.9}$$
External forces	$$\vec{F}_{ext}=\sum_{j=1}^{N}\frac{d\vec{p}_j}{dt} \tag{9.13.10}$$
Newton’s second law for an extended object	$$\vec{F}=\frac{d\vec{p}_{CM}}{dt} \tag{9.13.11}$$
Acceleration of the center of mass	$$\vec{a}_{CM}=\frac{d^2}{dt^2}\left(\frac{1}{M}\sum_{j=1}^{N}m_j\vec{r}_j\right)=\frac{1}{M}\sum_{j=1}^{N}m_j\vec{a}_j \tag{9.13.12}$$
Position of the center of mass for a system of particles	$$\vec{r}_{CM}\equiv\sum_{j=1}^{N}m_j\vec{r}_j \tag{9.13.13}$$
Velocity of the center of mass	$$\vec{v}_{CM}=\frac{d}{dt}\left(\frac{1}{M}\sum_{j=1}^{N}m_j\vec{r}_j\right)=\frac{1}{M}\sum_{j=1}^{N}m_j\vec{v}_j \tag{9.13.14}$$
Position of the center of mass of a continuous object	$$\vec{r}_{CM}\equiv\frac{1}{M}\int\vec{r}dm \tag{9.13.15}$$
Rocket equation	$$\Delta v=u\ln\left(\frac{m_i}{m}\right) \tag{9.13.16}$$

Summary

9.1 Linear Momentum

- The motion of an object depends on its mass as well as its velocity. Momentum is a concept that describes this. It is a useful and powerful concept, both computationally and theoretically. The SI unit for momentum is kg • m/s.

9.2 Impulse and Collisions

- When a force is applied on an object for some amount of time, the object experiences an impulse.
- This impulse is equal to the object’s change of momentum.
- Newton’s second law in terms of momentum states that the net force applied to a system equals the rate of change of the momentum that the force causes.

9.3 Conservation of Linear Momentum

- The law of conservation of momentum says that the momentum of a closed system is constant in time (conserved).
- A closed (or isolated) system is defined to be one for which the mass remains constant, and the net external force is zero.

- The total momentum of a system is conserved only when the system is closed.

9.4 Types of Collisions

- An elastic collision is one that conserves kinetic energy.
- An inelastic collision does not conserve kinetic energy.
- Momentum is conserved regardless of whether or not kinetic energy is conserved.
- Analysis of kinetic energy changes and conservation of momentum together allow the final velocities to be calculated in terms of initial velocities and masses in one-dimensional, two-body collisions.

9.5 Collisions in Multiple Dimensions

- The approach to two-dimensional collisions is to choose a convenient coordinate system and break the motion into components along perpendicular axes.
- Momentum is conserved in both directions simultaneously and independently.
- The Pythagorean theorem gives the magnitude of the momentum vector using the x- and y-components, calculated using conservation of momentum in each direction.

9.6 Center of Mass

- An extended object (made up of many objects) has a defined position vector called the center of mass.
- The center of mass can be thought of, loosely, as the average location of the total mass of the object.
- The center of mass of an object traces out the trajectory dictated by Newton's second law, due to the net external force.
- The internal forces within an extended object cannot alter the momentum of the extended object as a whole.

9.7 Rocket Propulsion

- A rocket is an example of conservation of momentum where the mass of the system is not constant, since the rocket ejects fuel to provide thrust.
- The rocket equation gives us the change of velocity that the rocket obtains from burning a mass of fuel that decreases the total rocket mass.

9.14: Introduction

Linear Momentum

Linear momentum is the product of the mass and velocity of an object, it is conserved in elastic and inelastic collisions.

learning objectives

- Calculate the momentum of two colliding objects

In classical mechanics, linear momentum, or simply momentum (SI unit kg m/s, or equivalently N s), is the product of the mass and velocity of an object. Mathematically it is stated as:

$$\mathrm{p=mv} \tag{9.14.1}$$

(Note here that p and v are vectors.) Like velocity, linear momentum is a vector quantity, possessing a direction as well as a magnitude. Linear momentum is particularly important because it is a conserved quantity, meaning that in a closed system (without any external forces) its total linear momentum cannot change.

Because momentum has a direction, it can be used to predict the resulting direction of objects after they collide, as well as their speeds. Momentum is conserved in both inelastic and elastic collisions. (Kinetic energy is not conserved in inelastic collisions but is conserved in elastic collisions.) It important to note that if the collision takes place on a surface with friction, or if there is air resistance, we would need to account for the momentum of the bodies that would be transferred to the surface and/or air.

Let's take a look at a simple, one-dimensional example: The momentum of a system of two particles is the sum of their momenta. If two particles have masses m_1 and m_2, and velocities v_1 and v_2, the total momentum is:

$$\mathrm{p=p_1+p_2=m_1v_1+m_2v_2.} \tag{9.14.2}$$

Keep in mind that momentum and velocity are vectors. Therefore, in 1D, if two particles are moving in the same direction, v1 and v2 have the same sign. If the particles are moving in opposite directions they will have opposite signs.

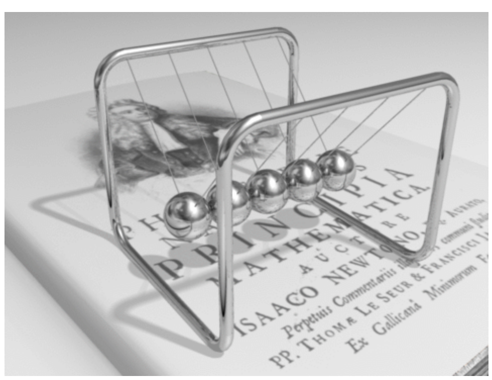
If two particles were moving on a plane we would choose our xy-plane to be on the plane of motion. We can then write the x and y component of the total momentum as:

$$\mathrm{p_x=p_{1x}+p_{2x}=m_1v_{1x}+m_2v_{2x}} \tag{9.14.3}$$

$$\mathrm{p_y=p_{1y}+p_{2y}=m_1v_{1y}+m_2v_{2y}.} \tag{9.14.4}$$

If the 2D momentum vector is decomposed into two components, the equations for each component are reduced to its 1D equivalents.

Momentum, like energy, is important because it is conserved. "Newton's cradle" shown in is an example of conservation of momentum. As we will discuss in the next concept (on Momentum, Force, and Newton's Second Law), in classical mechanics, conservation of linear momentum is implied by Newton's laws. Only a few physical quantities are conserved in nature. Studying these quantities yields fundamental insight into how nature works.



Newton's Cradle: Total momentum of the system (or Cradle) is conserved. (neglecting frictional loss in the system.)

Momentum, Force, and Newton's Second Law

In the most general form, Newton's 2nd law can be written as $F = \frac{dp}{dt}$.

learning objectives

- Relate Newton's Second Law to momentum and force

In a closed system (one that does not exchange any matter with the outside and is not acted on by outside forces), the total momentum is constant. This fact, known as the law of conservation of momentum, is implied by Newton's laws of motion. Suppose, for example, that two particles interact. Because of the third law, the forces between them are equal and opposite. If the particles are numbered 1 and 2, the second law states that

$$\frac{dp_1}{dt} = -\frac{dp_2}{dt} \tag{9.14.5}$$

or

$$\frac{d}{dt}(p_1 + p_2) = 0 \tag{9.14.6}$$

Therefore, total momentum (p_1+p_2) is constant. If the velocities of the particles are u_1 and u_2 before the interaction, and afterwards they are v_1 and v_2, then

$$m_1u_1 + m_2u_2 = m_1v_1 + m_2v_2 \tag{9.14.7}$$

This law holds regardless of the nature of the interparticle (or internal) force, no matter how complicated the force is between particles. Similarly, if there are several particles, the momentum exchanged between each pair of particles adds up to zero, so the total change in momentum is zero.

Newton's Second Law

Newton actually stated his second law of motion in terms of momentum: The net external force equals the change in momentum of a system divided by the time over which it changes. Using symbols, this law is

$$F_{net} = \frac{\Delta p}{\Delta t}, \tag{9.14.8}$$

where F_{net} is the net external force, Δp is the change in momentum, and Δt is the change in time.

This statement of Newton's second law of motion includes the more familiar $F_{net} = ma\backslash(asaspecialcase. Wecanderivethisformasfollows. First, notethatthechangeinmomentum\backslash(\Delta p$ is given by $\Delta p = \Delta(mv)$. If the mass of the system is constant, then $\Delta(mv) = m\Delta v$. So for constant mass, Newton's second law of motion becomes

$$F_{net} = \frac{\Delta p}{\Delta t} = \frac{m\Delta v}{\Delta t}. \tag{9.14.9}$$

Because $\frac{\Delta v}{\Delta t} = a$, we get the familiar equation $F_{net} = ma$ when the mass of the system is constant. Newton's second law of motion stated in terms of momentum is more generally applicable because it can be applied to systems where the mass is changing, such as rockets, as well as to systems of constant mass.

Momentum in a Closed System: In a game of pool, the system of entire balls can be considered a closed system. Therefore, the total momentum of the balls is conserved.

Impulse

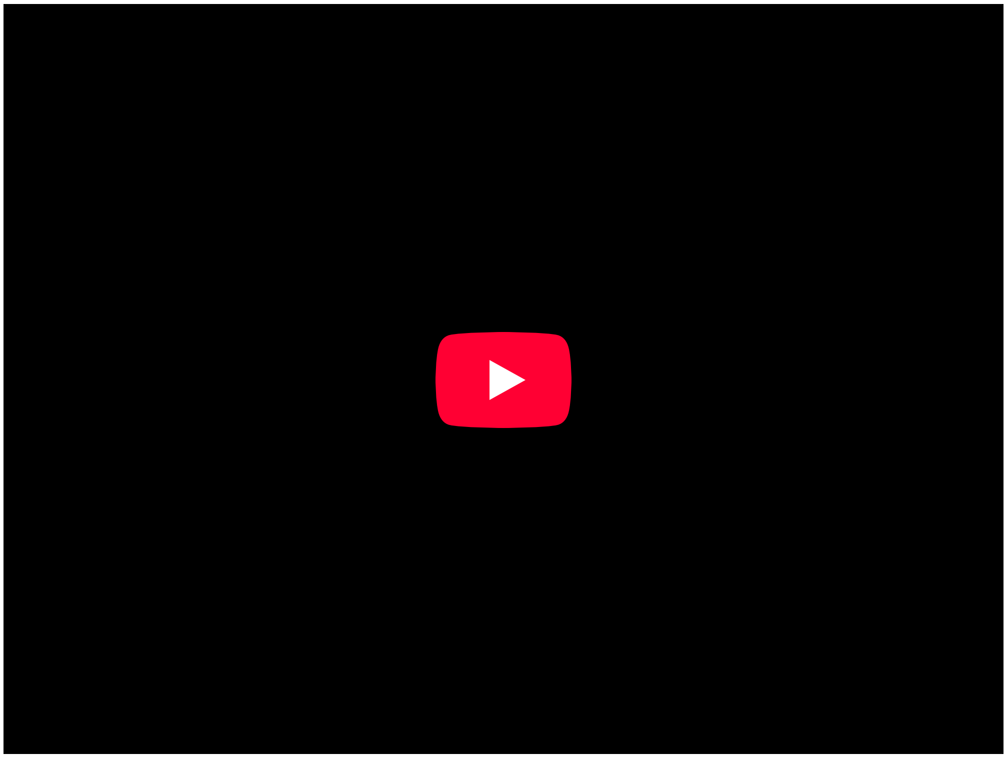
Impulse, or change in momentum, equals the average net external force multiplied by the time this force acts.

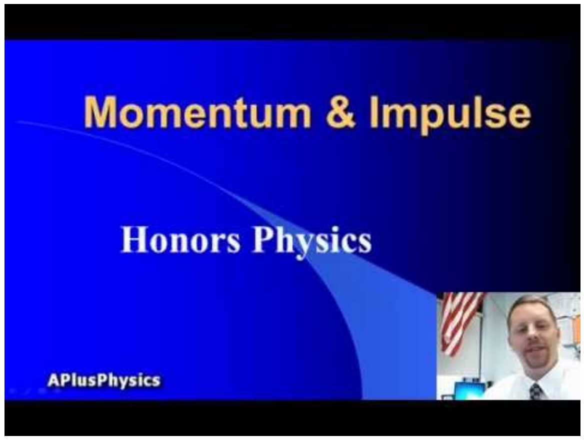
learning objectives

- Explain the relationship between change in momentum and the amount of time a force acts

Impulse

Forces produce either acceleration or deceleration on moving bodies, and the greater the force acting on an object, the greater its change in velocity and, hence, the greater its change in momentum. However, changing momentum is also related to how long a time the force acts. If a brief force is applied to a stalled automobile, a change in its momentum is produced. The same force applied over an extended period of time produces a greater change in the automobile's momentum. The quantity of impulse is *force* × *time interval*, or in shorthand notation:





Momentum & Impulse: A brief overview of momentum and impulse for high school physics students.

$$\mathrm{Impulse} = \mathrm{F}\Delta\mathrm{t}, \tag{9.14.10}$$

where F is the net force on the system, and Δt is the duration of the force.

From Newton's 2nd law:

$$\mathrm{F} = \frac{\Delta\mathrm{p}}{\Delta\mathrm{t}}\ (\Delta\mathrm{p} : \text{change in momentum}), \tag{9.14.11}$$

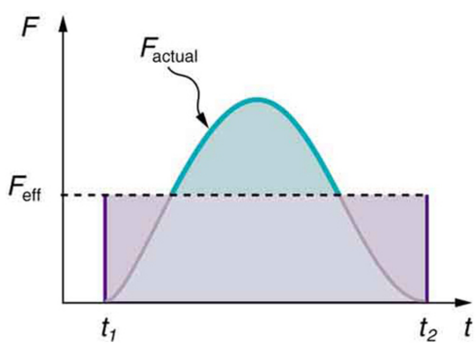
change in momentum equals the average net external force multiplied by the time this force acts.

$$\Delta\mathrm{p} = \mathrm{F}\Delta\mathrm{t}. \tag{9.14.12}$$

Therefore, impulse as defined in the previous paragraph is simply equivalent to p.

A force sustained over a long time produces more change in momentum than does the same force applied briefly. A small force applied for a long time can produce the same momentum change as a large force applied briefly because it is the product of the force and the time for which it is applied that is important. Impulse is always equal to change in momentum and is measured in Ns (Newton seconds), as both force and the time interval are important in changing momentum.

Our definition of impulse includes an assumption that the force is constant over the time interval $\Delta\mathrm{t}$. Forces are usually not constant. Forces vary considerably even during the brief time intervals considered. It is, however, possible to find an average effective force $\mathrm{F}_{\mathrm{eff}}$ that produces the same result as the corresponding time-varying force. shows a graph of what an actual force looks like as a function of time for a ball bouncing off the floor. The area under the curve has units of momentum and is equal to the impulse or change in momentum between times t_1 and t_2. That area is equal to the area inside the rectangle bounded by $\mathrm{F}_{\mathrm{eff}}, \mathrm{t}_1$, and t_2. Thus, the impulses and their effects are the same for both the actual and effective forces. Equivalently, we can simple find the area under the curve F(t) between t_1 and t_2 to compute the impulse in mathematical form:



Force vs. Time: A graph of force versus time with time along the x-axis and force along the y-axis for an actual force and an equivalent effective force. The areas under the two curves are equal.

$$\mathrm{Impulse} = \int_{\mathrm{t}_1}^{\mathrm{t}_2} \mathrm{F}(\mathrm{t})\mathrm{dt}. \tag{9.14.13}$$

Key Points

- Like velocity, linear momentum is a vector quantity, possessing a direction as well as a magnitude.
- Momentum, like energy, is important because it is a conserved quantity.
- The momentum of a system of particles is the sum of their momenta. If two particles have masses m_1 and m_2, and velocities v_1 and v_2, the total momentum is $p = p_1 + p_2 = m_1 v_1 + m_2 v_2.$.
- In a closed system, without any external forces, the total momentum is constant.
- The familiar equation $F = ma$ is a special case of the more general form of the 2nd law when the mass of the system is constant.
- Momentum conservation holds (in the absence of external force) regardless of the nature of the interparticle (or internal) force, no matter how complicated the force is between particles.
- A small force applied for a long time can produce the same momentum change as a large force applied briefly, because it is the product of the force and the time for which it is applied that is important.
- A force produces an acceleration, and the greater the force acting on an object, the greater its change in velocity and, hence, the greater its change in momentum. However, changing momentum is also related to how long a time the force acts.
- In case of a time-varying force, impulse can be calculated by integrating the force over the time duration. $\mathrm{Impulse} = \int_{t_1}^{t_2} F(t) dt.$.

Key Terms

- **inelastic**: (As referring to an inelastic collision, in contrast to an elastic collision.) A collision in which kinetic energy is not conserved.
- **elastic collision**: An encounter between two bodies in which the total kinetic energy of the two bodies after the encounter is equal to their total kinetic energy before the encounter. Elastic collisions occur only if there is no net conversion of kinetic energy into other forms.
- **conservation**: A particular measurable property of an isolated physical system does not change as the system evolves.
- **closed system**: A physical system that doesn't exchange any matter with its surroundings and isn't subject to any force whose source is external to the system.
- **momentum**: (of a body in motion) the product of its mass and velocity.
- **impulse**: The integral of force over time.

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9.15: Conservation of Momentum

Internal vs. External Forces

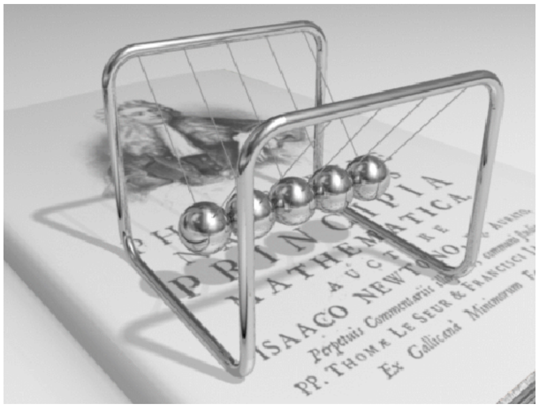
Net external forces (that are nonzero) change the total momentum of the system, while internal forces do not.

learning objectives

- Contrast the effects of external and internal forces on linear momentum and collisions

Linear Momentum and Collisions

Newton's 2nd law, applied to an isolated system composed of particles, $\mathrm{F_{tot}} = \frac{\mathrm{dp_{tot}}}{\mathrm{dt}} = 0$ indicates that the total momentum of the entire system $\mathrm{p_{tot}}$ should be constant in the absence of net external forces. Forces external to the system may change the total momentum when their sum is not 0, but internal forces, regardless of the nature of the forces, will not contribute to the change in the total momentum. To analyze a mechanical system, it is important to recognize which forces are internal and which are external. Once a mechanical system is clearly defined, it's not hard to understand what part should be considered external.



Newton's Cradle: Total momentum of the system (or Cradle) is conserved. (neglecting frictional loss in the system.)

- External forces: forces caused by external agent outside of the system.
- Internal forces: forces exchanged by the particles in the system.

To give you a better idea, let's consider a simple example. We have two hockey pucks sliding across a frictionless surface, and we neglect air resistance for simplicity. They collide with each other at t=0.

Let's first list all the forces present in the system. There are mainly three kinds of forces: Gravity, normal force (between ice & pucks), and frictional forces during the collision between the pucks

How should we define our system? In most cases, we would be interested in the motion of the pucks (and nothing else). Therefore, our system consists of two pucks (and nothing else). All the rest of the universe becomes external. With this in mind, we can see that gravity and normal forces are external, while the frictional forces between pucks are internal. Since all the external forces cancel out with each other, there are no net external forces. (Gravity and normal force on each puck have the same magnitude, but are in the opposite directions) Therefore, we conclude that the total momentum of the two pucks should be a conserved quantity.

- In the previous example, it is worthwhile to note that we didn't assume anything about the nature of the collision between the two pucks. Without knowing anything about the internal forces (frictional forces during contact), we learned that the total momentum of the system is a conserved quantity (p1 and p2 are momentum vectors of the pucks.) In fact, this relation holds true both in elastic or inelastic collisions. Whether the total kinetic energy of the pucks is conserved or not, total momentum is conserved.
- Also note that, in the previous example, if we include the rest of the Earth in our system, the gravity and normal forces themselves become internal.

Key Points

- External forces are forces caused by external agent outside of the system.
- Internal forces are forces exchanged by the objects in the system.
- To determine what part should be considered external and internal, mechanical system should be clearly defined.

Key Terms

- **inelastic**: (As referring to an inelastic collision, in contrast to an elastic collision.) A collision in which kinetic energy is not conserved.
- **elastic**: referring to elastic collision, in contrast to inelastic collision. A collision in which kinetic energy is conserved

9.16: Collisions

Conservation of Energy and Momentum

In an inelastic collision the total kinetic energy after the collision is not equal to the total kinetic energy before the collision.

learning objectives

- Assess the conservation of total momentum in an inelastic collision

At this point we will expand our discussion of inelastic collisions in one dimension to inelastic collisions in multiple dimensions. It is still true that the total kinetic energy after the collision is not equal to the total kinetic energy before the collision. While inelastic collisions may not conserve total kinetic energy, they do conserve total momentum.

We will consider an example problem in which one mass ($\mathrm{m_1}$) slides over a frictionless surface into another initially stationary mass ($\mathrm{m_2}$). Air resistance will be neglected. The following things are known:

$$\mathrm{m_1 = 0.250kg,} \tag{9.16.1}$$

$$\mathrm{m_2 = 0.400kg,} \tag{9.16.2}$$

$$\mathrm{v_1 = 2.00m/s,} \tag{9.16.3}$$

$$\mathrm{v_1\,' = 1.50m/s,} \tag{9.16.4}$$

$$\mathrm{v_2 = 0m/s,} \tag{9.16.5}$$

$$\theta_1\,' = 45.0\circ, \tag{9.16.6}$$

where $\mathrm{v_1}$ is the initial velocity of the first mass, $\mathrm{v_1\,'}$ is the final velocity of the first mass, $\mathrm{v_2}$ is the initial velocity of the second mass, and $\theta_1\,'$ is the angle between the velocity vector of the first mass and the x-axis.

The object is to calculate the magnitude and direction of the velocity of the second mass. After this, we will calculate whether this collision was inelastic or not.

Since there are no net forces at work (frictionless surface and negligible air resistance), there must be conservation of total momentum for the two masses. Momentum is equal to the product of mass and velocity. The initially stationary mass contributes no initial momentum. The components of velocities along the x-axis have the form $\mathrm{v \cdot \cos\theta}$, where θ is the angle between the velocity vector of the mass of interest and the x-axis.

Expressing these things mathematically:

$$\mathrm{m_1v_1 = m_1v_1\,'\cdot \cos(\theta_1) + m_2v`_2 \cdot \cos(\theta_2). \ (Eq. \ 2)} \tag{9.16.7}$$

The components of velocities along the y-axis have the form $\mathrm{v \cdot \sin\theta}$, where θ is the angle between the velocity vector of the mass of interest and the x-axis. By applying conservation of momentum in the y-direction we find:

$$\mathrm{0 = m_1v_1\,'\cdot \sin(\theta_1) + m_2v`_2 \cdot \sin(\theta_2). \ (Eq. \ 3)} \tag{9.16.8}$$

If we divide Eq. 3 by Eq. 2, we will find:

$$\tan\theta_2 = \frac{\mathrm{v_1\,'\cdot \sin\theta_1}}{\mathrm{v_1\,'\cos\theta\theta_1 - v_1}} \mathrm{(Eq. \ 4)} \tag{9.16.9}$$

Eq. 4 can then be solved to find $\theta_2$approx. 312º.

Now let' use Eq. 3 to solve for $\mathrm{v'_2}$. Re-arranging Eq. 3, we find:

$$\mathrm{v_2\,' = \frac{-m_1v_1\,'\cdot \sin\theta_1}{m_2 \cdot \sin\theta_2.}} \tag{9.16.10}$$

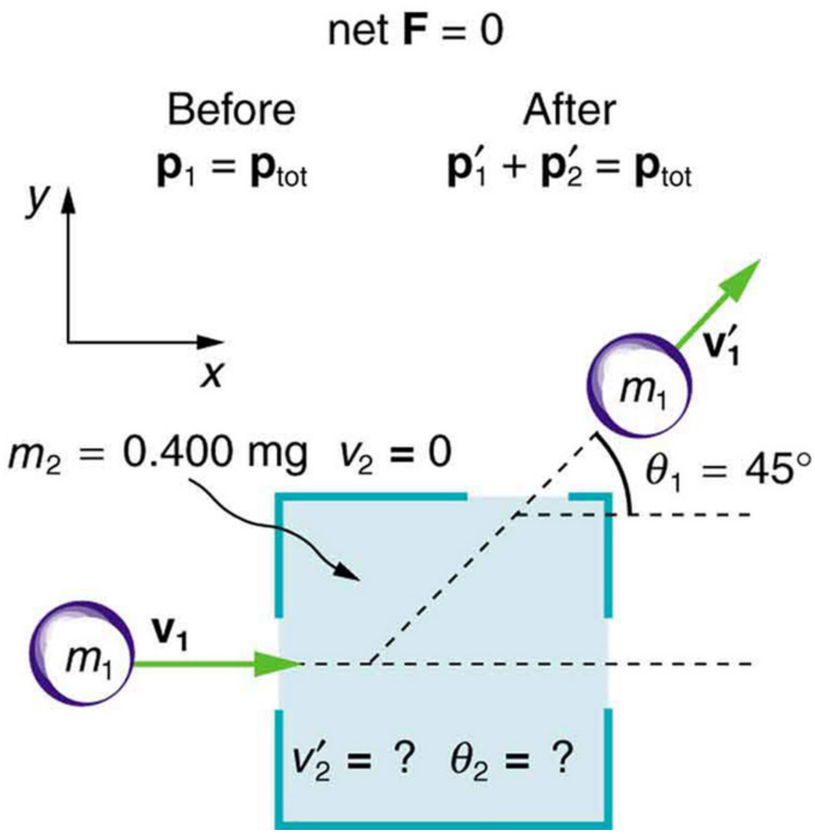
After plugging in our known values, we find that $\mathrm{v_2\,' = 0.886m/s}$.

We can now calculate the initial and final kinetic energy of the system to see if it the same.

$$\text{Initial Kinetic Energy} = \frac{1}{2}\mathrm{m}_1 \cdot \mathrm{v}_1^2 + \frac{1}{2}\mathrm{m}_2 \cdot \mathrm{v}_2^2 = 0.5\mathrm{J}. \tag{9.16.11}$$

$$\text{Final Kinetic Energy} = \frac{1}{2}\mathrm{m}_1 \cdot \mathrm{v}_1{}'^2 + \frac{1}{2}\mathrm{m}_2 \cdot \mathrm{v}_2{}'^2 \approx 0.43\mathrm{J}. \tag{9.16.12}$$

Since these values are not the same we know that it was an inelastic collision.



Collision Example: This illustrates the example problem in which one mass collides into another mass that is initially stationary.

Glancing Collisions

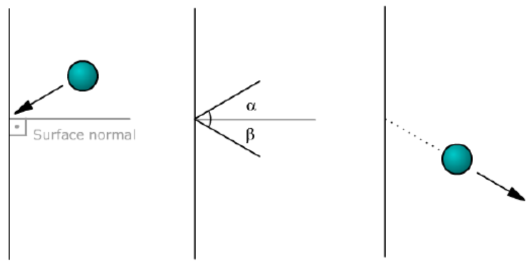
Glancing collision is a collision that takes place under a small angle, with the incident body being nearly parallel to the surface.

learning objectives

- Identify necessary conditions for a “glancing collision”

A collision is short duration interaction between two bodies or more than two bodies simultaneously causing change in motion of bodies involved due to internal forces acted between them during this. Collisions involve forces (there is a change in velocity). The magnitude of the velocity difference at impact is called the closing speed. All collisions conserve momentum. What distinguishes different types of collisions is whether they also conserve kinetic energy. Line of impact – It is the line which is common normal for surfaces are closest or in contact during impact. This is the line along which internal force of collision acts during impact and Newton’s coefficient of restitution is defined only along this line.

When dealing with an incident body that is nearly parallel to a surface, it is sometimes more useful to refer to the angle between the body and the surface, rather than that between the body and the surface normal (see), in other words 90° minus the angle of incidence. This small angle is called a glancing angle. Collision at glancing angle is called “glancing collision”.



Collision: Object is deflected after the collision withthe surface. The angles between the body and the surface normal areindicated as α and β. The angles between the body and the surface are $90-\alpha$ and $90-\beta$.

Collisions can either be elastic, meaning they conserve both momentum and kinetic energy, or inelastic, meaning they conserve momentum but not kinetic energy. An inelastic collision is sometimes also called a plastic collision.

A “perfectly-inelastic” collision (also called a “perfectly-plastic” collision) is a limiting case of inelastic collision in which the two bodies stick together after impact.

The degree to which a collision is elastic or inelastic is quantified by the coefficient of restitution, a value that generally ranges between zero and one. A perfectly elastic collision has a coefficient of restitution of one; a perfectly-inelastic collision has a coefficient of restitution of zero.

Elastic Collisions in One Dimension

An elastic collision is a collision between two or more bodies in which kinetic energy is conserved.

learning objectives

- Assess the relationship among the collision equations to derive elasticity

An elastic collision is a collision between two or more bodies in which the total kinetic energy of the bodies before the collision is equal to the total kinetic energy of the bodies after the collision. An elastic collision will not occur if kinetic energy is converted into other forms of energy. It important to understand how elastic collisions work, because atoms often undergo essentially elastic collisions when they collide. On the other hand, molecules do not undergo elastic collisions when they collide. In this atom we will review case of collision between two bodies.

The mathematics of an elastic collision is best demonstrated through an example. Consider a first particle with mass $\mathrm{m_1}$ and velocity $\mathrm{v_{1i}}$ and a second particle with mass $\mathrm{m_2}$ and velocity $\mathrm{v_{2i}}$. If these two particles collide, there must be conservation of momentum before and after the collision. If we know that this is an elastic collision, there must be conservation of kinetic energy by definition. Therefore, the velocities of particles 1 and 2 after the collision ($\mathrm{v_{1f}}$ and $\mathrm{v_{2f}}$ respectively) will be related to the initial velocities by:

$\mathrm{\frac{1}{2}m_1 \cdot v_{1i}^2 + \frac{1}{2}m_2 \cdot v_{2i}^2 = \frac{1}{2}m_1 \cdot v_{1f}^2 + \frac{1}{2}m_2 \cdot v_{2f}^2}$ (due to conservation of kinetic energy)

and

$\mathrm{m_1 \cdot v_{1i} + m_2 \cdot v_{2i} = m_1 \cdot v_{1f} + m_2 \cdot v_{2f}}$ (due to conservation of momentum).

Since we have two equations, we are able to solve for any two unknown variables. In our case, we will solve for the final velocities of the two particles.

By grouping like terms and canceling out the ½ terms, we can rewrite our conservation of kinetic energy equation as:

$$\mathrm{m_1 \cdot (v_{1i}^2 - v_{1f}^2) = m_2 \cdot (v_{2f}^2 - v_{2i}^2). \, (Eq.1)} \tag{9.16.13}$$

By grouping like terms from our conservation of momentum equation we can find:

$$\mathrm{m_1 \cdot (v_{1i} - v_{1f}) = m_2 \cdot (v_{2f} - v_{2i}). \, (Eq.\, 2)} \tag{9.16.14}$$

If we then divide Eq. 1 by Eq. 2 and perform some cancelations we will find:

$$\mathrm{v_{1i} + v_{1f} = v_{2f} + v_{2i}. \, (Eq.\, 3)} \tag{9.16.15}$$

We can solve for v_{1f} as:

$$v_{1f} = v_{2f} + v_{2i} - v_{1i}. \text{ (Eq. 4)} \tag{9.16.16}$$

At this point we see that v_{2f} is still an unknown variable. So we can fix this by plugging Eq. 4 into our initial conservation of momentum equation. Our conservation of momentum equation with Eq. 4 substituted in looks like:

$$m_1 \cdot v_{1i} + m_2 \cdot v_{2i} = m_1 \cdot (v_{2f} + v_{2i} - v_{1i}) + m_2 \cdot v_{2f}. \text{ (Eq.5)} \tag{9.16.17}$$

After doing a little bit of algebra on Eq. 5 we find:

$$v_{2f} = \frac{2 \cdot m_1}{(m_2 + m_1)} v_{1i} + \frac{(m_2 - m_1)}{(m_2 + m_1)} v_{2i}. \text{ (Eq.6)} \tag{9.16.18}$$

At this point we have successfully solved for the final velocity of the second particle. We still need to solve for the velocity of the first particle, so let us do that by plugging Eq. 6 into Eq. 4.

$$v_{1f} = [\frac{2 \cdot m_1}{(m_2 + m_1)} v_{1i} + \frac{(m_2 - m_1)}{(m_2 + m_1)} v_{2i}] + v_{2i} - v_{1i}. \text{ (Eq. 7)} \tag{9.16.19}$$

After performing some algebraic manipulation of Eq. 7, we finally find:

$$v_{1f} = \frac{(m_1 - m_2)}{(m_2 + m_1)} v_{1i} + \frac{2 \cdot m_2}{(m_2 + m_1)} v_{2i}. \text{ (Eq. 8)} \tag{9.16.20}$$

Elastic Collision of Two Unequal Masses: In this animation, two unequal masses collide and recoil.

Elastic Collisions in Multiple Dimensions

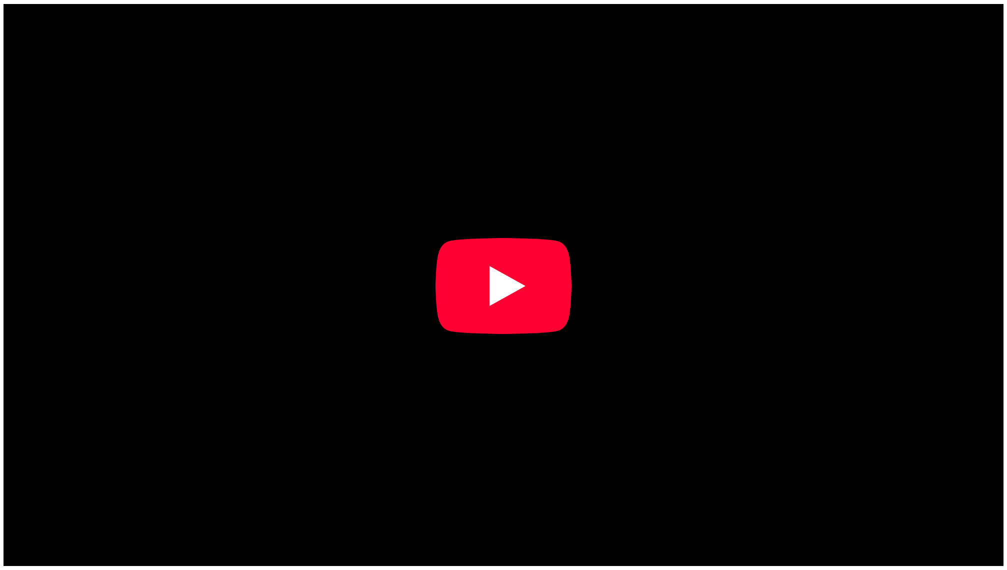
To solve a two dimensional elastic collision problem, decompose the velocity components of the masses along perpendicular axes.

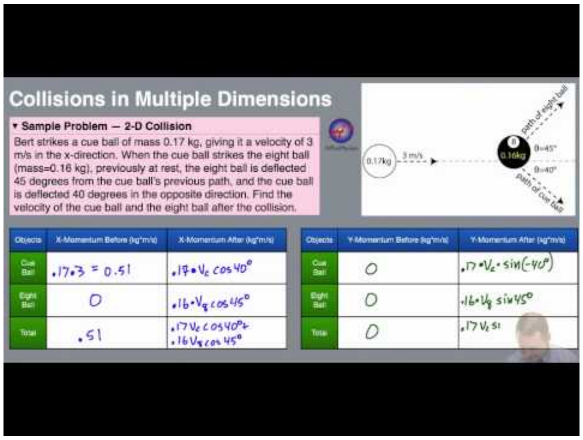
learning objectives

- Construct an equation for elastic collision

Overview

As stated previously, there is conservation of total kinetic energy before and after an elastic collision. If an elastic collision occurs in two dimensions, the colliding masses can travel side to side after the collision (not just along the same line as in a one dimensional collision). The general approach to solving a two dimensional elastic collision problem is to choose a coordinate system in which the velocity components of the masses can be decomposed along perpendicular axes.





Collisions in Multiple Dimensions: A brief introduction to problem solving of collisions in two dimensions using the law of conservation of momentum.

Example 9.16.1:

In this example, we consider only point masses. These are structure-less particles that cannot spin or rotate. We will consider a case in which no outside forces are acting on the system, meaning that momentum is conserved. We will consider a situation in which one particle is initially at rest. This situation is illustrated in.

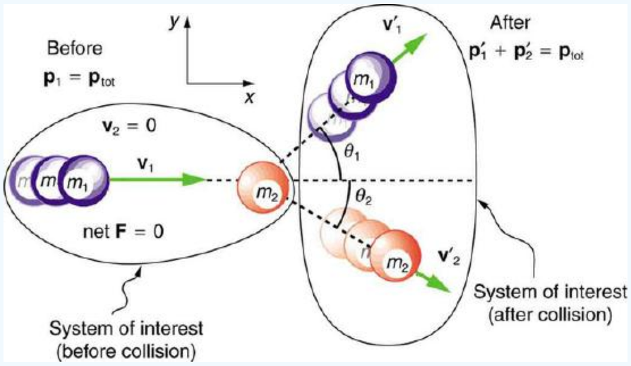


Illustration of Elastic Collision in Two Dimensions: In this illustration, we see the initial and final configurations of two masses that undergo an elastic collision in two dimensions.

By defining the x-axis to be along the direction of the incoming particle, we save ourselves time in breaking that velocity vector into its x- and y- components. Now let us consider conservation of momentum in the x direction:

$$p_{1x} + p_{2x} = p_{1x}{}' + p_{2x}{}' (\text{Eq. } 1) \tag{9.16.21}$$

In Eq. 1, the initial momentum of the incoming particle is represented by $p_{1x,}$ the initial momentum of the stationary particle is represented by $p_{2x,}$ the final momentum of the incoming particle is represented by $p'_{1x.}$ and the final momentum of the initially stationary particle is represented by $p'_{2x.}$

We can expand Eq. 1 by taking into account that momentum is equal to the product of mass and velocity. Also, we know that $p_{2x} = 0$ because the initial velocity of the stationary particle is 0.

The components of velocities along the x-axis have the form $v \cdot \cos\theta$, where θ is the angle between the velocity vector of the particle of interest and the x-axis.

Therefore:

$$m_1 v_1 = m_1 v_1{}' \cdot \cos(\theta_1) + m_2 v'_2 \cdot \cos(\theta_2) (\text{Eq. } 2) \tag{9.16.22}$$

The components of velocities along the y-axis have the form $v \cdot \sin\theta$, where θ is the angle between the velocity vector of the particle of interest (denoted in the following equations by subscript 1 or 2) and the x-axis. We can apply conservation of momentum in the y-direction in a similar way to yield:

$$0 = m_1 v'_1 \cdot \sin(\theta_1) + m_2 v'_2 \cdot \sin(\theta_2) (\text{Eq. } 3) \tag{9.16.23}$$

In finding Eq. 3, it was taken into consideration that the incoming particle had no component of velocity along the y-axis.

Solving for Two Unknowns

Now we have gotten to a point where we have two equations, this means that we can solve for any two unknowns that we want. We also know that because the collision is elastic that there must be conservation of kinetic energy before and after the collision. This means that we may also write Eq. 4, which gives us three equations to solve for three unknowns:

$$\frac{1}{2} m_1 \cdot v_1^2 + \frac{1}{2} m_2 \cdot v_2^2 = \frac{1}{2} m_1 \cdot v_1{}'^2 + \frac{1}{2} m_2 \cdot v_2{}'^2 \tag{9.16.24}$$

The general approach to finding the defining equations for an n-dimensional elastic collision problem is to apply conservation of momentum in each of the n- dimensions. You can generate an additional equation by utilizing conservation of kinetic energy.

Inelastic Collisions in One Dimension

Collisions may be classified as either inelastic or elastic collisions based on how energy is conserved in the collision.

learning objectives

- Distinguish examples of inelastic collision from elastic collisions

Overview

In an inelastic collision the total kinetic energy after the collision is not equal to the total kinetic energy before the collision. This is in contrast to an elastic collision in which conservation of total kinetic energy applies. While inelastic collisions may not conserve total kinetic energy, they do conserve total momentum.

Collisions

If two objects collide, there are many ways that kinetic energy can be transformed into other forms of energy. For example, in the collision of macroscopic bodies, some kinetic energy is turned into vibrational energy of the constituent atoms. This causes a heating effect and results in deformation of the bodies. Another example in which kinetic energy is transformed into another form of energy is when the molecules of a gas or liquid collide. When this happens, kinetic energy is often exchanged between the molecules' translational motion and their internal degrees of freedom.

A perfectly inelastic collision happens when the maximum amount of kinetic energy in a system is lost. In such a collision, the colliding particles stick together. The kinetic energy is used on the bonding energy of the two bodies.

Sliding Block Example

Let us consider an example of a two-body sliding block system. The first block slides into the second (initially stationary block). In this perfectly inelastic collision, the first block bonds completely to the second block as shown. We assume that the surface over which the blocks slide has no friction. We also assume that there is no air resistance. If the surface had friction or if there was air resistance, one would have to account for the bodies' momentum that would be transferred to the surface and/or air.

Inelastic Collision: In this animation, one mass collides into another initially stationary mass in a perfectly inelastic collision.

Writing about the equation for conservation of momentum, one finds:

$$\mathrm{m_a u_a + m_b u_b = (m_a + m_b) v} \tag{9.16.25}$$

where $m_a$is the mass of the incoming block, u_a is the velocity of the incoming block, $m_b$is the mass of the initially stationary block, $u_b$is the velocity of initially stationary block (0 m/s), and v is the final velocity the two body system. Solving for the final velocity,

$$\mathrm{v = \frac{m_a u_a + m_b u_b}{m_a + m_b}}. \tag{9.16.26}$$

Taking into account that the blocks have the same mass and that the one of the blocks is initially stationary, the expression for the final velocity of the system may be defined as:

$$\mathrm{v = \frac{u_a}{2}}. \tag{9.16.27}$$

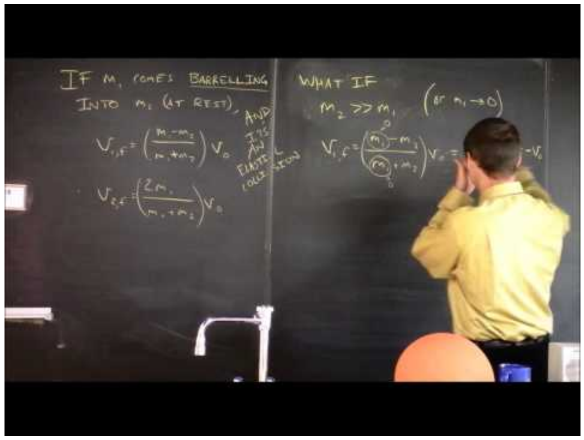
Inelastic Collisions in Multiple Dimensions

While inelastic collisions may not conserve total kinetic energy, they do conserve total momentum.

learning objectives

- Relate inelastic collision multiple dimension equations to the one dimension collisions you learned earlier

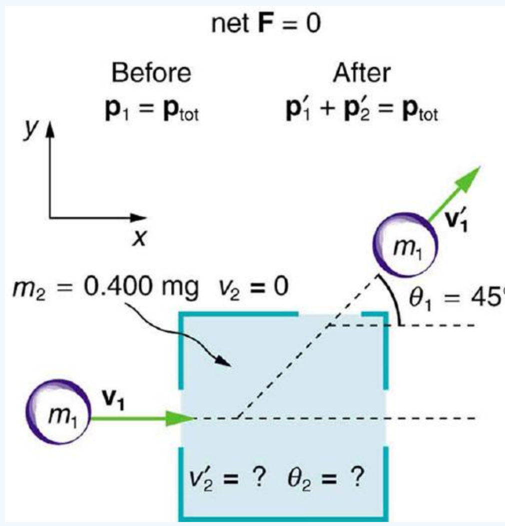
At this point we will expand our discussion of inelastic collisions in one dimension to inelastic collisions in multiple dimensions. It is still true that the total kinetic energy after the collision is not equal to the total kinetic energy before the collision. While inelastic collisions may not conserve total kinetic energy, they do conserve total momentum.



Example 9.16.2:

Examples of Collisions

We will consider an example problem, illustrated in, in which one mass (m1m1) slides over a frictionless surface into another initially stationary mass (m2m2). Air resistance will be neglected. The following quantities are known:



Collision Example: This illustrates the example problem in which one mass collides into another mass that is initially stationary.

$$m_1 = 0.250 kg, \quad (9.16.28)$$

$$m_2 = 0.400 kg, \quad (9.16.29)$$

$$v_1 = 2.00 m/s, \quad (9.16.30)$$

$$v_1 ' = 1.50 m/s, \quad (9.16.31)$$

$$v_2 = 0 m/s, \quad (9.16.32)$$

$$\theta_1 ' = 45.0\circ, \quad (9.16.33)$$

where v1v1 is the initial velocity of the first mass, v′1v1′ is the final velocity of the first mass, v2v2is the initial velocity of the second mass, and θ′1θ1′ is the angle between the velocity vector of the first mass and the x-axis.

The object is to calculate the magnitude and direction of the velocity of the second mass. After this, we will calculate whether this collision was inelastic or not.

Since there are no net forces at work (frictionless surface and negligible air resistance), there must be conservation of total momentum for the two masses. Momentum is equal to the product of mass and velocity. The (initially) stationary mass contributes no initial momentum. The components of velocities along the x-axis have the form v⋅cosθv⋅cosθ, where θ is the angle between the velocity vector of the mass of interest and the x-axis.

Expressing these things mathematically:

$$m_1 v_1 = m_1 v_1{'} \cdot \cos(\theta_1) + m_2 v`_2 \cdot \cos(\theta_2). \text{ (Eq. 2)} \tag{9.16.34}$$

The components of velocities along the y-axis have the form $v \cdot \sin\theta$, where θ is the angle between the velocity vector of the mass of interest and the x-axis. By applying conservation of momentum in the y-direction we find:

$$0 = m_1 v_1{'} \cdot \sin(\theta_1) + m_2 v`_2 \cdot \sin(\theta_2). \text{ (Eq. 3)} \tag{9.16.35}$$

If we divide Eq. 3 by Eq. 2, we will find:

$$\tan\theta_2 = \frac{v_1{'} \cdot \sin\theta_1}{v_1{'} \cos\theta\theta_1 - v_1} \text{(Eq. 4)} \tag{9.16.36}$$

Eq. 4 can then be solved to find $\theta_2$approx. 312º.

Now let' use Eq. 3 to solve for $v{'}_2$. Re-arranging Eq. 3, we find:

$$v_2{'} = \frac{-m_1 v_1{'} \cdot \sin\theta_1}{m_2 \cdot \sin\theta_2.} \tag{9.16.37}$$

After plugging in our known values, we find that $v_2{'} = 0.886 m/s$.

We can now calculate the initial and final kinetic energy of the system to see if it the same.

$$\text{Initial Kinetic Energy} = \frac{1}{2} m_1 \cdot v_1^2 + \frac{1}{2} m_2 \cdot v_2^2 = 0.5 J. \tag{9.16.38}$$

$$\text{Final Kinetic Energy} = \frac{1}{2} m_1 \cdot v_1{'^2} + \frac{1}{2} m_2 \cdot v_2{'^2} \approx 0.43 J. \tag{9.16.39}$$

As these values are not the same, we know this was an inelastic collision.

Key Points

- In an inelastic collision the total kinetic energy after the collision is not equal to the total kinetic energy before the collision.
- If there are no net forces at work (collision takes place on a frictionless surface and there is negligible air resistance), there must be conservation of total momentum for the two masses.
- The variable θ is the angle between the velocity vector of the mass of interest and the x-axis in traditional Cartesian coordinate systems.
- Collision is short duration interaction between two bodies or more than two bodies simultaneously causing change in motion of bodies involved due to internal forces acted between them during this.
- Collisions can either be elastic, meaning they conserve both momentum and kinetic energy, or inelastic, meaning they conserve momentum but not kinetic energy.
- When dealing with an incident body that is nearly parallel to a surface, it is sometimes more useful to refer to the angle between the body and the surface, rather than that between the body and the surface normal.
- An elastic collision will not occur if kinetic energy is converted into other forms of energy.
- While molecules do not undergo elastic collisions, atoms often undergo elastic collisions when they collide.
- If two particles are involved in an elastic collision, the velocity of the first particle after collision can be expressed as: $v_{1f} = \frac{(m_1 - m_2)}{(m_2 + m_1)} v_{1i} + \frac{2 \cdot m_2}{(m_2 + m_1)} v_{2i}$.
- If two particles are involved in an elastic collision, the velocity of the second particle after collision can be expressed as: $v_{2f} = \frac{2 \cdot m_1}{(m_2 + m_1)} v_{1i} + \frac{(m_2 - m_1)}{(m_2 + m_1)} v_{2i}$.
- If an elastic collision occurs in two dimensions, the colliding masses can travel side to side after the collision.
- By defining the x- axis to be along the direction of the incoming particle, we can simplify the defining equations.
- The general approach to finding the defining equations for an n-dimensional elastic collision problem is to apply conservation of momentum in each of the n- dimensions. You can generate an additional equation by utilizing conservation of kinetic energy.
- In an inelastic collision, the total kinetic energy after the collision is not equal to the total kinetic energy before the collision.
- While inelastic collisions may not conserve total kinetic energy, they do conserve total momentum.

- A perfectly inelastic collision happens when the maximum amount of kinetic energy in a system is lost.
- In an inelastic collision the total kinetic energy after the collision is not equal to the total kinetic energy before the collision.
- If there are no net forces at work (i.e., collision takes place on a frictionless surface and there is negligible air resistance), there must be conservation of total momentum for the two masses.
- The variable θ is the angle between the velocity vector of the mass of interest and the x-axis in traditional Cartesian coordinate systems.

Key Terms

- **kinetic energy**: The energy possessed by an object because of its motion, equal to one half the mass of the body times the square of its velocity.
- **momentum**: (of a body in motion) the product of its mass and velocity.
- **force**: A physical quantity that denotes ability to push, pull, twist or accelerate a body which is measured in a unit dimensioned in mass × distance/time^2 (ML/T^2): SI: newton (N); CGS: dyne (dyn)
- **elastic collision**: An encounter between two bodies in which the total kinetic energy of the two bodies after the encounter is equal to their total kinetic energy before the encounter. Elastic collisions occur only if there is no net conversion of kinetic energy into other forms.
- **dimension**: A measure of spatial extent in a particular direction, such as height, width or breadth, or depth.
- **degrees of freedom**: A degree of freedom is an independent physical parameter, often called a dimension, in the formal description of the state of a physical system. The set of all dimensions of a system is known as a phase space.
- **friction**: A force that resists the relative motion or tendency to such motion of two bodies in contact.

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9.17: Rocket Propulsion

Rocket Propulsion, Changing Mass, and Momentum

In rocket propulsion, matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains.

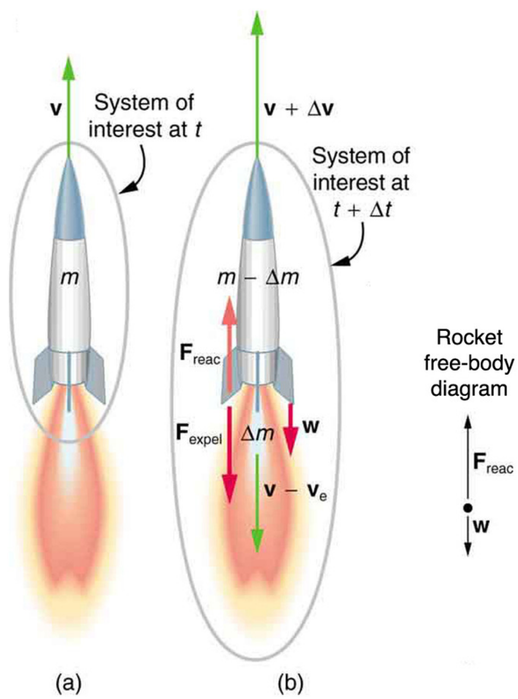
learning objectives

- Identify physical principles of rocket propulsion

Rocket Propulsion, Changing Mass, and Momentum

Rockets range in size from fireworks so small that ordinary people use them to immense Saturn Vs that once propelled massive payloads toward the Moon. The propulsion of all rockets, jet engines, deflating balloons, and even squids and octopuses is explained by the same physical principle: Newton's third law of motion. Matter is forcefully ejected from a system, producing an equal and opposite reaction on what remains. Another common example is the recoil of a gun. The gun exerts a force on a bullet to accelerate it and consequently experiences an equal and opposite force, causing the gun's recoil or kick.

shows a rocket accelerating straight up. In part (a), the rocket has a mass m and a velocity v relative to Earth, and hence a momentum mv. In part (b), a time Δt has elapsed in which the rocket has ejected a mass Δm of hot gas at a velocity v_e relative to the rocket. The remainder of the mass $(m - m)$ now has a greater velocity $(v + \Delta v)$. The momentum of the entire system (rocket plus expelled gas) has actually decreased because the force of gravity has acted for a time Δt, producing a negative impulse $\Delta p = -mg\Delta t$. (Remember that impulse is the net external force on a system multiplied by the time it acts, and it equals the change in momentum of the system.) So the center of mass of the system is in free fall but, by rapidly expelling mass, part of the system can accelerate upward. It is a commonly held misconception that the rocket exhaust pushes on the ground. If we consider thrust; that is, the force exerted on the rocket by the exhaust gases, then a rocket's thrust is greater in outer space than in the atmosphere or on the launch pad. In fact, gases are easier to expel into a vacuum.



Free-body diagram of rocket propulsion: (a) This rocket has a mass m and an upward velocity v. The net external force on the system is −mg, if air resistance is neglected. (b) A time Δt later the system has two main parts, the ejected gas and the remainder of the rocket. The reaction force on the rocket is what overcomes the gravitational force and accelerates it upward.

By calculating the change in momentum for the entire system over Δt, and equating this change to the impulse, the following expression can be shown to be a good approximation for the acceleration of the rocket.

$$a = \frac{v_e}{m}\frac{\Delta m}{\Delta t} - g \tag{9.17.1}$$

where a is the acceleration of the rocket, v_e is the escape velocity, m is the mass of the rocket, Δm is the mass of the ejected gas, and Δt is the time in which the gas is ejected.

Factors of Acceleration

A rocket's acceleration depends on three major factors, consistent with the equation for acceleration of a rocket. First, the greater the exhaust velocity of the gases relative to the rocket, v_e, the greater the acceleration is. The practical limit for v_e is about 2.5×10^3m/s for conventional (non-nuclear) hot-gas propulsion systems. The second factor is the rate at which mass is ejected from the rocket. This is the factor $\frac{\Delta m}{\Delta t}$ in the equation. The quantity $(\frac{\Delta m}{\Delta t})v_e$, with units of newtons, is called "thrust." The faster the rocket burns its fuel, the greater its thrust, and the greater its acceleration. The third factor is the mass m of the rocket. The smaller the mass is (all other factors being the same), the greater the acceleration. The rocket mass m decreases dramatically during flight because most of the rocket is fuel to begin with, so that acceleration increases continuously, reaching a maximum just before the fuel is exhausted.

To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth's gravity altogether, the mass of the rocket other than fuel must be as small as possible. It can be shown that, in the absence of air resistance and neglecting gravity, the final velocity of a one-stage rocket initially at rest is

$$v = v_e \ln\frac{m_0}{m_r} \tag{9.17.2}$$

where $\ln(m_0/m_r)$ is the natural logarithm of the ratio of the initial mass of the rocket (m_0) to what is left (m_r) after all of the fuel is exhausted. (Note that v is actually the change in velocity, so the equation can be used for any segment of the flight. If we start from rest, the change in velocity equals the final velocity.)

Key Points

- The propulsion of all rockets is explained by the same physical principle: Newton's third law of motion.
- A rocket's acceleration depends on three major factors: the exhaust velocity, the rate the exhaust is ejected, and the mass of the rocket.
- To achieve the high speeds needed to hop continents, obtain orbit, or escape Earth's gravity altogether, the mass of the rocket other than fuel must be as small as possible.

Key Terms

- **Newton's third law of motion**: states that all forces exist in pairs: if one object A exerts a force FA on a second object B, then B simultaneously exerts a force FB on A, and the two forces are equal and opposite: $FA = -FB$.

9.18: Center of Mass

learning objectives

- Identify the center of mass for an object with continuous mass distribution

In the previous modules on “Center of Mass and Translational Motion,” we learned why the concept of center of mass (COM) helps solving mechanics problems involving a rigid body. Here, we will study the rigorous definition of COM and how to determine the location of it. The position of COM is mass weighted average of the positions of particles.

Definition: center of mass

The *center of mass* is a statement of spatial arrangement of mass (i.e. distribution of mass within the system). The position of COM is given a mathematical formulation which involves distribution of mass in space:

$$\mathrm{r_{COM}} = \frac{\sum_i \mathrm{m_i r_i}}{\mathrm{M}}, \tag{9.18.1}$$

where $\mathrm{r_{COM}}$ and $\mathrm{r_i}$ are vectors representing the position of COM and i-th particle respectively, and M and mi are the total mass and mass of the i-th particle, respectively. This mean means that position of COM is mass weighted average of the positions of particles.

Object with Continuous Mass Distribution

If the mass distribution is continuous with the density ρ(r) within a volume V, the position of COM is given as

$$\mathrm{r_{COM}} = \frac{1}{\mathrm{M}} \int_{\mathrm{V}} \rho(\mathrm{r}) \mathrm{r dV}, \tag{9.18.2}$$

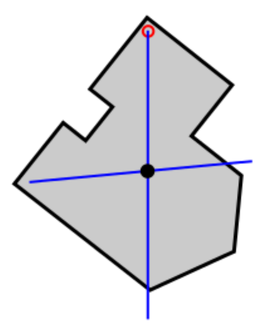
where M is the total mass in the volume. If a continuous mass distribution has uniform density, which means ρ is constant, then the center of mass is the same as the center of the volume.

Locating the Center of Mass

The experimental determination of the center of mass of a body uses gravity forces on the body and relies on the fact that in the parallel gravity field near the surface of Earth the center of mass is the same as the center of gravity.

The center of mass of a body with an axis of symmetry and constant density must lie on this axis. Thus, the center of mass of a circular cylinder of constant density has its center of mass on the axis of the cylinder. In the same way, the center of mass of a spherically symmetric body of constant density is at the center of the sphere. In general, for any symmetry of a body, its center of mass will be a fixed point of that symmetry.

In two dimensions: An experimental method for locating the center of mass is to suspend the object from two locations and to drop plumb lines from the suspension points. The intersection of the two lines is the center of mass.



Plumb Line Method for Center of Mass: Suspend the object from two locations and to drop plumb lines from the suspension points. The intersection of the two lines is the center of mass.

In three dimensions: By supporting an object at three points and measuring the forces that resist the weight of the object, COM of the three-dimensional coordinates of the center of mass can be determined.

Motion of the Center of Mass

We can describe the translational motion of a rigid body as if it is a point particle with the total mass located at the COM—center of mass.

learning objectives

- Derive the center of mass for the translational motion of a rigid body

We can describe the translational motion of a rigid body as if it is a point particle with the total mass located at the center of mass (COM). In this Atom. we will prove that the total mass (M) times the acceleration of the COM (a_{COM}), indeed, equals the sum of external forces. That is,

$$M \cdot a_{COM} = \sum F_{ext}. \tag{9.18.3}$$

You can see that the Newton's 2nd law applies as if we are describing the motion of a point particle (with mass M) under the influence of the external force.

Derivation

From the definition of the center of mass,

$$r_{COM} = \frac{\sum_i m_i r_i}{M}, \tag{9.18.4}$$

we get $M \cdot a_{COM} = \sum m_i a_i$ by taking time derivative twice on each side.

Note that $\sum m_i a_i = \sum F_i$.

In a system of particles, each particle may feel both external and internal forces. Here, external forces are forces from external sources, while internal forces are forces between particles in the system. Since the sum of all internal forces will be 0 due to the Newton's 3rd law,

$\sum F_i = \sum F_{i,ext}$. Therefore, we get $M \cdot a_{COM} = \sum F_{ext}$.

For example, when we confine our system to the Earth and the Moon, the gravitational force due to the Sun would be external, while the gravitational force on the Earth due to the Moon (and vice versa) would be internal. Since the gravitational forces between the Earth and the Moon are equal in magnitude and opposite in direction, they will cancel out each other in the sum (see).

COM of the Earth and Moon: Earth and Moon orbiting a COM inside the Earth. The red cross represents the COM of the two-body system. The COM will orbit around the Sun as if it is a point particle.

Corollary

When there is no external force, the COM momentum is conserved.

Proof: Since there is no external force, $M \cdot a_{COM} = 0$. Therefore,

$M \cdot v_{COM} = \text{constant}.$

Proof

Since there is no external force,

$$M \cdot a_{COM} = 0. \tag{9.18.5}$$

Therefore,

$M \cdot v_{COM} = constant$.

□

Center of Mass of the Human Body

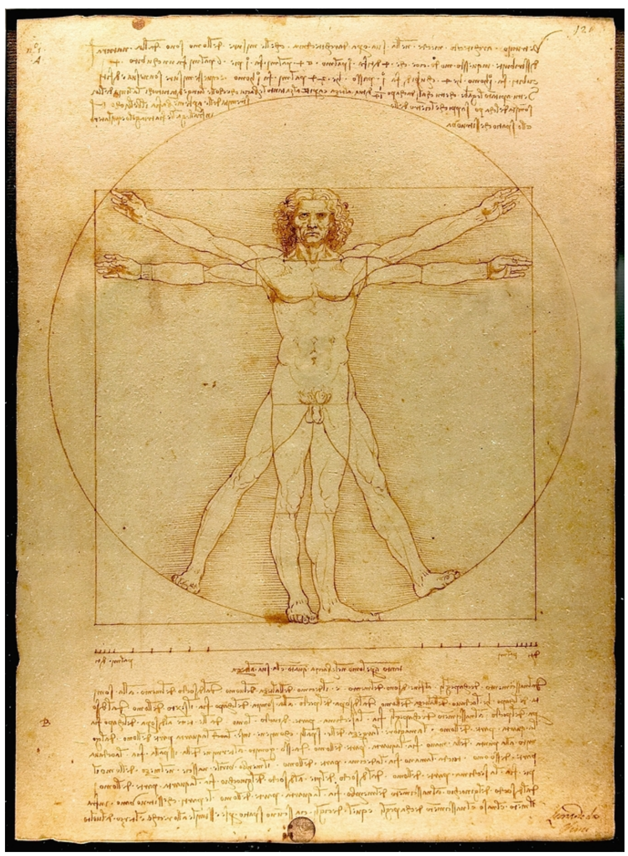
The center of mass (COM) is an important physical concept—it is the point about which objects rotate.

learning objectives

- Estimate the COM of a given object

The center of mass (COM) is an important physical concept. It is the point on an object at which the weighted relative position of the distributed mass sums to zero—the point about which objects rotate.

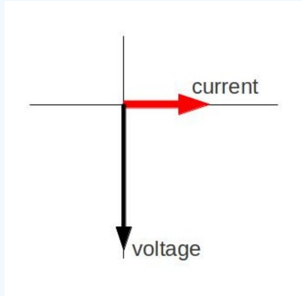
Human proportions have been important in art, measurement, and medicine (a well known drawing of the human body is seen in). Although the human body has complicated features, the location of the center of mass (COM) could be a good indicator of body proportions. The center of mass of the human body depends on the gender and the position of the limbs. In a standing posture, it is typically about 10 cm lower than the navel, near the top of the hip bones. In this Atom, we will learn how to measure the COM of a human body.



Leonardo da Vinci's "The Vitruvian Man": Vitruvian Man: A drawing created by Leonardo da Vinci. The drawing is based on the correlations of ideal human proportions with geometry described[4] by the ancient Roman architect Vitruvius in Book III of his treatise De Architectura.

Example 9.18.1:

First, let's take two scales and a wooden beam (H meter long), long enough to contain the entire body of the subject. Put the scales H meters apart, and place the beam across the scales, as illustrated in. Now, let the subject lie on the beam. Make sure that his/her heels are aligned with one end of the beam. Measure the readings (F_1, F_2) on the scale.



The COM of a Human Body: This figure demonstrates measuring the COM of a human body.

The system (person+beam) has three external forces: gravity on the subject (F_{CM}), and normal forces from the scales F_1 and F_2. The equation of motion for force (F=ma) will give us:

$$\mathrm{F1+F2=Mg,} \tag{9.18.6}$$

where M is mass of the subject. (We assume that the wooden beam has no mass.) This equation doesn't provide all the information to locate the COM. However, the equation of motion for torque $(\tau = \mathrm{I}\alpha)$ helps.

Since the net torque of the system is zero (hence no rotational acceleration),

$$\mathrm{hF_2-(H-h)F_1=0.(h: COM\ height)} \tag{9.18.7}$$

The COM is chosen as the origin for the torque. Therefore, gravity contributes nothing as a torque. Solving for h and using the equation of motion for force, we get

$$\mathrm{h=\frac{HF_1}{Mg}.} \tag{9.18.8}$$

Center of Mass and Translational Motion

The COM (center of mass) of a system of particles is a geometric point that assumes all the mass and external force(s) during motion.

learning objectives

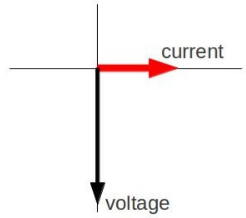
- Support the presence of COM in three dimensional bodies in motion

Introduction: COM, Linear Momentum, and Collisions

Our study of motion has been limited up to this point. We have referred to particle, object and body in the same way. We considered that actual three dimensional rigid bodies move such that all constituent particles had the same motion (i.e., same trajectory, velocity and acceleration). By doing this, we have essentially considered a rigid body as a point particle.

Center of Mass (COM)

An actual body, however, can move differently than this simplified paradigm. Consider a ball rolling down an incline plane or a stick thrown into air. Different parts of a body have different motions. While translating in the air, the stick rotates about a moving axis, as shown in. This means that such bodies may not behave like a point particle, as earlier suggested.



Forces on the COM: Left: The force appears to operate on the COM is “mgsinθ. Right: The force appears to operate on the COM is “mg”.

Describing motions of parts or particles that have different motions would be quite complicated to do in an integrated manner. However, such three dimensional bodies in motion have one surprising, simplifying characteristic—a geometric point that behaves like a particle. This point is known as center of mass, abbreviated COM (the mathematical definition of COM will be introduced in the next Atom on “Locating the Center of Mass”). It has the following two characterizing aspects:

- The center of mass appears to carry the whole mass of the body.
- At the center of mass, all external forces appear to apply.

Significantly, the center of a ball (the COM of a rolling ball) follows a straight linear path; whereas the COM of a stick follows a parabolic path (as shown in the figure above). Secondly, the forces appear to operate on the COMs in two cases (“mgsinθ and “mg”) as if they were indeed particle-like objects. This concept of COM, therefore, eliminate the complexities otherwise present in attempting to describe motions of rigid bodies.

Describing Motion in a Rigid Body

We can describe general motion of an object (with mass m) as follows:

- We describe the *translational motion* of a rigid body as if it is a point particle with mass m located at COM.
- *Rotation* of the particle, with respect to the COM, is described independently.

We “separate” the translational part of the motion from the rotational part. By introducing the concept of COM, the translational motion becomes that of a point particle with mass m. This simplifies significantly the mathematical complexity of the problem.

Key Points

- The center of mass (COM) is a statement of spatial arrangement of mass (i.e. distribution of mass within the system).
- The experimental determination of the center of mass of a body uses gravity forces on the body and relies on the fact that in the parallel gravity field near the surface of the earth the center of mass is the same as the center of gravity.
- For a 2D object, an experimental method for locating the center of mass is to suspend the object from two locations and to drop plumb lines from the suspension points. The intersection of the two lines is the center of mass.
- The total mass times the acceleration of the center of mass equals the sum of external forces.
- For the translational motion of a rigid body with mass M, Newton’s 2nd law applies as if we are describing the motion of a point particle (with mass M) under the influence of the external force.
- When there is no external force, the center of mass momentum is conserved.
- Although a human body has complicated features, the location of the center of mass (COM) could be a good indicator of the body proportions.
- We can measure the location of COM with two scales and a wooden beam. The linear and rotational equations of motion gives us the location.
- The center of mass of the human body depends on the gender and the position of the limbs. In a standing posture, it is typically about 10 cm lower than the navel, near the top of the hip bones.
- In a motion of a rigid body, different parts of the body have different motions.This means that these bodies may not behave like a point particle.
- There is a characteristic geometric point of the three dimensional body in motion. This point behaves as a particle, and is known as center of mass, abbreviated COM. COM appears to carry the whole mass of the body. All external forces appear to apply at COM.

- To describe the motion of a rigid body (with possibly a complicated geometry), we separate the translational part of the motion from the rotational part.

Key Terms

- **plumb line**: A cord with a weight attached, used to produce a vertical line.
- **rigid body**: An idealized solid whose size and shape are fixed and remain unaltered when forces are applied; used in Newtonian mechanics to model real objects.
- **center of mass**: The center of mass (COM) is the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero.
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **point particle**: An idealization of particles heavily used in physics. Its defining feature is that it lacks spatial extension, meaning that geometrically the particle is equivalent to a point.

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CHAPTER OVERVIEW

10: Static Equilibrium, Elasticity, and Torque

Topic hierarchy

10.1: Prelude to Static Equilibrium and Elasticity

In earlier chapters, you learned about forces and Newton's laws for translational motion. You then studied torques and the rotational motion of a body about a fixed axis of rotation. You also learned that static equilibrium means no motion at all and that dynamic equilibrium means motion without acceleration.

Figure 10.1.1: Two stilt walkers in standing position. All forces acting on each stilt walker balance out; neither changes its translational motion. In addition, all torques acting on each person balance out, and thus neither of them changes its rotational motion. The result is static equilibrium. (credit: modification of work by Stuart Redler)

In this chapter, we combine the conditions for static translational equilibrium and static rotational equilibrium to describe situations typical for any kind of construction. What type of cable will support a suspension bridge? What type of foundation will support an office building? Will this prosthetic arm function correctly? These are examples of questions that contemporary engineers must be able to answer.

The elastic properties of materials are especially important in engineering applications, including bioengineering. For example, materials that can stretch or compress and then return to their original form or position make good shock absorbers. In this chapter, you will learn about some applications that combine equilibrium with elasticity to construct real structures that last.

10.2: Conditions for Static Equilibrium

Learning Objectives

- Identify the physical conditions of static equilibrium.
- Draw a free-body diagram for a rigid body acted on by forces.
- Explain how the conditions for equilibrium allow us to solve statics problems.

We say that a rigid body is in **equilibrium** when both its linear and angular acceleration are zero relative to an inertial frame of reference. This means that a body in equilibrium can be moving, but if so, its linear and angular velocities must be constant. We say that a rigid body is in **static equilibrium** when it is at rest **in our selected frame of reference**. Notice that the distinction between the state of rest and a state of uniform motion is artificial—that is, an object may be at rest in our selected frame of reference, yet to an observer moving at constant velocity relative to our frame, the same object appears to be in uniform motion with constant velocity. Because the motion is **relative**, what is in static equilibrium to us is in dynamic equilibrium to the moving observer, and vice versa. Since the laws of physics are identical for all inertial reference frames, in an inertial frame of reference, there is no distinction between static equilibrium and equilibrium.

According to Newton's second law of motion, the linear acceleration of a rigid body is caused by a net force acting on it, or

$$\sum_k \vec{F}_k = m\vec{a}_{CM}. \tag{10.2.1}$$

Here, the sum is of all external forces acting on the body, where m is its mass and $\vec{a}_{CM}$ is the linear acceleration of its center of mass (a concept we discussed in Linear Momentum and Collisions on linear momentum and collisions). In equilibrium, the linear acceleration is zero. If we set the acceleration to zero in Equation 10.2.1, we obtain the following equation:

First Equilibrium Condition

The first equilibrium condition for the static equilibrium of a rigid body expresses **translational** equilibrium:

$$\sum_k \vec{F}_k = \vec{0}. \tag{10.2.2}$$

The first equilibrium condition, Equation 10.2.2, is the equilibrium condition for forces, which we encountered when studying applications of Newton's laws.

This vector equation is equivalent to the following three scalar equations for the components of the net force:

$$\sum_k F_{kx} = 0, \; \sum_k F_{ky} = 0, \; \sum_k F_{kz} = 0. \tag{10.2.3}$$

Analogously to Equation 10.2.1, we can state that the rotational acceleration $\vec{\alpha}$ of a rigid body about a fixed axis of rotation is caused by the net torque acting on the body, or

$$\sum_k \vec{\tau}_k = I\vec{\alpha}. \tag{10.2.4}$$

Here I is the rotational inertia of the body in rotation about this axis and the summation is over **all** torques $\vec{\tau}_k$ of external forces in Equation 10.2.2. In equilibrium, the rotational acceleration is zero. By setting to zero the right-hand side of Equation 10.2.4, we obtain the second equilibrium condition:

Second Equilibrium Condition

The second equilibrium condition for the static equilibrium of a rigid body expresses **rotational** equilibrium:

$$\sum_k \vec{\tau}_k = \vec{0}. \tag{10.2.5}$$

The second equilibrium condition, Equation 10.2.5, is the equilibrium condition for torques that we encountered when we studied rotational dynamics. It is worth noting that this equation for equilibrium is generally valid for rotational equilibrium about any axis of rotation (fixed or otherwise). Again, this vector equation is equivalent to three scalar equations for the vector components of the net torque:

$$\sum_k \tau_{kx} = 0, \; \sum_k \tau_{ky} = 0, \; \sum_k \tau_{kz} = 0. \tag{10.2.6}$$

The second equilibrium condition means that in equilibrium, there is no net external torque to cause rotation about any axis. The first and second equilibrium conditions are stated in a particular reference frame. The first condition involves only forces and is therefore independent of the origin of the reference frame. However, the second condition involves torque, which is defined as a cross product, $\vec{\tau}_k = \vec{r}_k \times \vec{F}_k$, where the position vector $\vec{r}_k$ with respect to the axis of rotation of the point where the force is applied enters the equation. Therefore, torque depends on the location of the axis in the reference frame. However, when rotational and translational equilibrium conditions hold simultaneously in one frame of reference, then they also hold in any other inertial frame of reference, so that the net torque about any axis of rotation is still zero. The explanation for this is fairly straightforward.

Suppose vector $\vec{R}$ is the position of the origin of a new inertial frame of reference S′ in the old inertial frame of reference S. From our study of relative motion, we know that in the new frame of reference S′, the position vector $\vec{r}\,'_k$ of the point where the force $\vec{F}_k$ is applied is related to $\vec{r}_k$ via the equation

$$\vec{r}'_k = \vec{r}_k - \vec{R}. \tag{10.2.7}$$

Now, we can sum all torques $\vec{\tau}'_k = \vec{r}'_k \times \vec{F}_k$ of all external forces in a new reference frame, S':

$$\sum_k \vec{\tau}'_k = \sum_k \vec{r}'_k \times \vec{F}_k = \sum_k (\vec{r}_k - \vec{R}) \times \vec{F}_k = \sum_k \vec{r}_k \times \vec{F}_k - \sum_k \vec{R} \times \vec{F}_k = \sum \vec{\tau}_k - \vec{R} \times \sum_k \vec{F}_k = \vec{0}. \tag{10.2.8}$$

In the final step in this chain of reasoning, we used the fact that in equilibrium in the old frame of reference, S, the first term vanishes because of Equation 10.2.5 and the second term vanishes because of Equation 10.2.2. Hence, we see that the net torque in any inertial frame of reference S′ is zero, provided that both conditions for equilibrium hold in an inertial frame of reference ???.

The practical implication of this is that when applying equilibrium conditions for a rigid body, we are free to choose any point as the origin of the reference frame. Our choice of reference frame is dictated by the physical specifics of the problem we are solving. In one frame of reference, the mathematical form of the equilibrium conditions may be quite complicated, whereas in another frame, the same conditions may have a simpler mathematical form that is easy to solve. The origin of a selected frame of reference is called the pivot point.

In the most general case, equilibrium conditions are expressed by the six scalar equations (Equations 10.2.3 and 10.2.6). For planar equilibrium problems with rotation about a fixed axis, which we consider in this chapter, we can reduce the number of equations to three. The standard procedure is to adopt a frame of reference where the z-axis is the axis of rotation. With this choice of axis, the net torque has only a z-component, all forces that have non-zero torques lie in the xy-plane, and therefore contributions to the net torque come from only the x- and y-components of external forces. Thus, for planar problems with the axis of rotation perpendicular to the xy-plane, we have the following three equilibrium conditions for forces and torques:

$$F_{1x} + F_{2x} + \dots + F_{Nx} = 0 \tag{10.2.9}$$

$$F_{1y} + F_{2y} + \dots + F_{Ny} = 0 \tag{10.2.10}$$

$$\tau_1 + \tau_2 + \dots + \tau_N = 0 \tag{10.2.11}$$

where the summation is over all N external forces acting on the body and over their torques. In Equation 10.2.11, we simplified the notation by dropping the subscript z, but we understand here that the summation is over all contributions along the z-axis, which is the axis of rotation. In Equation 10.2.11, the z-component of torque $\vec{\tau}_k$ from the force $\vec{F}_k$ is

$$\tau_k = r_k F_k \sin\theta \tag{10.2.12}$$

where r_k is the length of the lever arm of the force and F_k is the magnitude of the force (as you saw in Fixed-Axis Rotation). The angle θ is the angle between vectors $\vec{r}_k$ and $\vec{F}_k$, measuring **from vector $\vec{r}_k$ to vector $\vec{F}_k$** in the **counterclockwise** direction (Figure 10.2.1). When using Equation 10.2.12, we often compute the magnitude of torque and assign its sense as either positive (+) or negative (−), depending on the direction of rotation caused by this torque alone. In Equation 10.2.11, net torque is the sum of terms, with each term computed from Equation 10.2.12, and each term must have the correct **sense**. Similarly, in Equation 10.2.9, we assign the + sign to force components in the + x-direction and the − sign to components in the − x-direction. The same rule must be consistently followed in Equation 10.2.10, when computing force components along the y-axis.

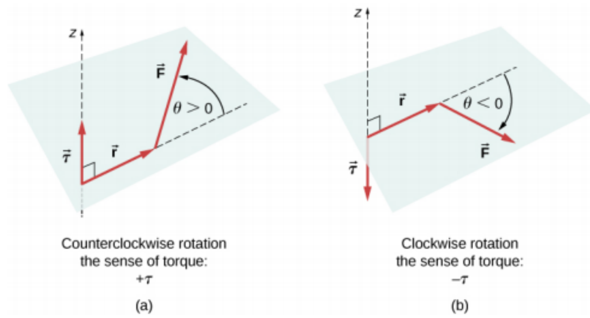


Figure 10.2.1: Torque of a force: (a) When the torque of a force causes counterclockwise rotation about the axis of rotation, we say that its sense is positive, which means the torque vector is parallel to the axis of rotation. (b) When torque of a force causes clockwise rotation about the axis, we say that its sense is negative, which means the torque vector is antiparallel to the axis of rotation.

> **Note**
>
> View this demonstration to see two forces act on a rigid square in two dimensions. At all times, the static equilibrium conditions given by Equation 10.2.9 through Equation 10.2.11 are satisfied. You can vary magnitudes of the forces and their lever arms and observe the effect these changes have on the square.

In many equilibrium situations, one of the forces acting on the body is its weight. In free-body diagrams, the weight vector is attached to the **center of gravity** of the body. For all practical purposes, the center of gravity is identical to the center of mass, as you learned in Linear Momentum and Collisions on linear momentum and collisions. Only in situations where a body has a large spatial extension so that the gravitational field is nonuniform throughout its volume, are the center of gravity and the center of mass located at different points. In practical situations, however, even objects as large as buildings or cruise ships are located in a uniform gravitational field on Earth’s surface, where the acceleration due to gravity has a constant magnitude of g = 9.8 m/s^2. In these situations, the center of gravity is identical to the center of mass. Therefore, throughout this chapter, we use the center of mass (CM) as the point where the weight vector is attached. Recall that the CM has a special physical meaning: When an external force is applied to a body at exactly its CM, the body as a whole undergoes translational motion and such a force does not cause rotation.

When the CM is located off the axis of rotation, a net **gravitational torque** occurs on an object. Gravitational torque is the torque caused by weight. This gravitational torque may rotate the object if there is no support present to balance it. The magnitude of the gravitational torque depends on how far away from the pivot the CM is located. For example, in the case of a tipping truck (Figure 10.2.2), the pivot is located on the line where the tires make contact with the road’s surface. If the CM is located high above the road’s surface, the gravitational torque may be large enough to turn the truck over. Passenger cars with a low-lying CM, close to the pavement, are more resistant to tipping over than are trucks.

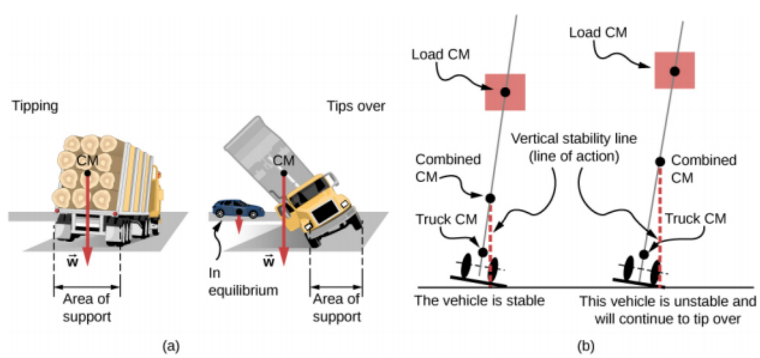


Figure 10.2.2: The distribution of mass affects the position of the center of mass (CM), where the weight vector $\vec{w}$ is attached. If the center of gravity is within the area of support, the truck returns to its initial position after tipping [see the left panel in (b)]. But if the center of gravity lies outside the area of support, the truck turns over [see the right panel in (b)]. Both vehicles in (b) are out of equilibrium. Notice that the car in (a) is in equilibrium: The low location of its center of gravity makes it hard to tip over.

Note

If you tilt a box so that one edge remains in contact with the table beneath it, then one edge of the base of support becomes a pivot. As long as the center of gravity of the box remains over the base of support, gravitational torque rotates the box back toward its original position of stable equilibrium. When the center of gravity moves outside of the base of support, gravitational torque rotates the box in the opposite direction, and the box rolls over. View this demonstration to experiment with stable and unstable positions of a box.

Example 12.1: Center of Gravity of a Car

A passenger car with a 2.5-m wheelbase has 52% of its weight on the front wheels on level ground, as illustrated in Figure 12.4. Where is the CM of this car located with respect to the rear axle?

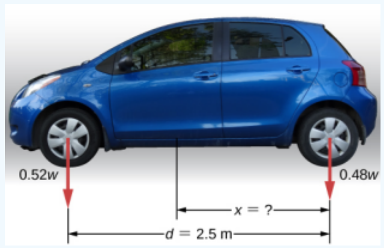


Figure 10.2.3: The weight distribution between the axles of a car. Where is the center of gravity located?

Strategy

We do not know the weight w of the car. All we know is that when the car rests on a level surface, 0.52w pushes down on the surface at contact points of the front wheels and 0.48w pushes down on the surface at contact points of the rear wheels. Also, the contact points are separated from each other by the distance d = 2.5 m. At these contact points, the car experiences normal reaction forces with magnitudes F_F = 0.52w and F_R = 0.48w on the front and rear axles, respectively. We also know that the car is an example of a rigid body in equilibrium whose entire weight w acts at its CM. The CM is located somewhere between the points where the normal reaction forces act, somewhere at a distance x from the point where F_R acts. Our task is to find x. Thus, we identify three forces acting on the body (the car), and we can draw a free-body diagram for the extended rigid body, as shown in Figure 10.2.4

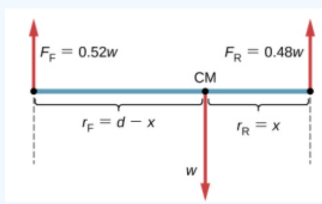


Figure 10.2.4: The free-body diagram for the car clearly indicates force vectors acting on the car and distances to the center of mass (CM). When CM is selected as the pivot point, these distances are lever arms of normal reaction forces. Notice that vector magnitudes and lever arms do not need to be drawn to scale, but all quantities of relevance must be clearly labeled.

We are almost ready to write down equilibrium conditions Equation 10.2.9 through Equation 10.2.11 for the car, but first we must decide on the reference frame. Suppose we choose the x-axis along the length of the car, the yaxis vertical, and the z-axis perpendicular to this xy-plane. With this choice we only need to write Equation 10.2.9 and Equation 10.2.11 because all the y-components are identically zero. Now we need to decide on the location of the pivot point. We can choose any point as the location of the axis of rotation (z-axis). Suppose we place the axis of rotation at CM, as indicated in the free-body diagram for the car. At this point, we are ready to write the equilibrium conditions for the car.

Solution

Each equilibrium condition contains only three terms because there are N = 3 forces acting on the car. The first equilibrium condition, Equation 10.2.9, reads

$$+F_F - w + F_R = 0. \tag{10.2.13}$$

This condition is trivially satisfied because when we substitute the data, Equation 10.2.13 becomes +0.52w − w + 0.48w = 0. The second equilibrium condition, Equation 10.2.11, reads

$$\tau_F + \tau_w + \tau_R = 0 \tag{10.2.14}$$

where τ_F is the torque of force F_F, τ_w is the gravitational torque of force w, and τ_R is the torque of force F_R. When the pivot is located at CM, the gravitational torque is identically zero because the lever arm of the weight with respect to an axis that passes through CM is zero. The lines of action of both normal reaction forces are perpendicular to their lever arms, so in Equation 10.2.12, we have $|\sin\theta| = 1$ for both forces. From the free-body diagram, we read that torque τ_F causes clockwise rotation about the pivot at CM, so its sense is negative; and torque τ_R causes counterclockwise rotation about the pivot at CM, so its sense is positive. With this information, we write the second equilibrium condition as

$$-r_F F_F + r_R F_R = 0. \tag{10.2.15}$$

With the help of the free-body diagram, we identify the force magnitudes F_R = 0.48w and F_F = 0.52w, and their corresponding lever arms r_R = x and r_F = d − x. We can now write the second equilibrium condition, Equation 10.2.15, explicitly in terms of the unknown distance x:

$$-0.52(d-x)w + 0.48xw = 0. \tag{10.2.16}$$

Here the weight w cancels and we can solve the equation for the unknown position x of the CM. The answer is x = 0.52d = 0.52(2.5 m) = 1.3 m. Solution Choosing the pivot at the position of the front axle does not change the result. The free-body diagram for this pivot location is presented in Figure 12.6. For this choice of pivot point, the second equilibrium condition is

$$-r_w w + r_R F_R = 0. \tag{10.2.17}$$

When we substitute the quantities indicated in the diagram, we obtain

$$-(d-x)w + 0.48dw = 0. \tag{10.2.18}$$

The answer obtained by solving Equation 10.2.15is, again, x = 0.52d = 1.3 m.

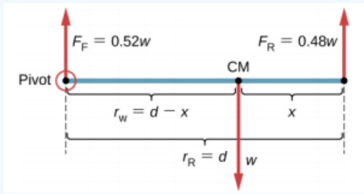


Figure 10.2.5: The equivalent free-body diagram for the car; the pivot is clearly indicated.

Significance

This example shows that when solving static equilibrium problems, we are free to choose the pivot location. For different choices of the pivot point we have different sets of equilibrium conditions to solve. However, all choices lead to the same solution to the problem.

? Exercise 12.1

Solve Example 12.1 by choosing the pivot at the location of the rear axle.

? Exercise 12.2

Explain which one of the following situations satisfies both equilibrium conditions: (a) a tennis ball that does not spin as it travels in the air; (b) a pelican that is gliding in the air at a constant velocity at one altitude; or (c) a crankshaft in the engine of a parked car.

A special case of static equilibrium occurs when all external forces on an object act at or along the axis of rotation or when the spatial extension of the object can be disregarded. In such a case, the object can be effectively treated like a point mass. In this special case, we need not worry about the second equilibrium condition, Equation 10.2.11, because all torques are identically zero and the first equilibrium condition (for forces) is the only condition to be satisfied. The free-body diagram and problem-solving strategy for this special case were outlined in Newton's Laws of Motion and Applications of Newton's Laws. You will see a typical equilibrium situation involving only the first equilibrium condition in the next example.

View this demonstration to see three weights that are connected by strings over pulleys and tied together in a knot. You can experiment with the weights to see how they affect the equilibrium position of the knot and, at the same time, see the vector-diagram representation of the first equilibrium condition at work.

Example 12.2: A Breaking Tension

A small pan of mass 42.0 g is supported by two strings, as shown in Figure 12.7. The maximum tension that the string can support is 2.80 N. Mass is added gradually to the pan until one of the strings snaps. Which string is it? How much mass must be added for this to occur?

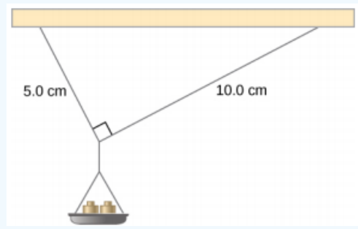


Figure 10.2.6: Mass is added gradually to the pan until one of the strings snaps.

Strategy

This mechanical system consisting of strings, masses, and the pan is in static equilibrium. Specifically, the knot that ties the strings to the pan is in static equilibrium. The knot can be treated as a point; therefore, we need only the first equilibrium condition. The three forces pulling at the knot are the tension $\vec{T}_1$ in the 5.0-cm string, the tension $\vec{T}_2$ in the 10.0-cm string, and the weight $\vec{w}$ of the pan holding the masses. We adopt a rectangular coordinate system with the y-axis pointing opposite to the direction of gravity and draw the free-body diagram for the knot (see Figure 12.8). To find the tension components, we must identify the direction angles α_1 and α_2 that the strings make with the horizontal direction that is the x-axis. As you can see in Figure 12.7, the strings make two sides of a right triangle. We can use the Pythagorean theorem to solve this triangle, shown in Figure 12.8, and find the sine and cosine of the angles α_1 and α_2. Then we can resolve the tensions into their rectangular components, substitute in the first condition for equilibrium (Equation 10.2.9 and Equation 10.2.10), and solve for the tensions in the strings. The string with a greater tension will break first.

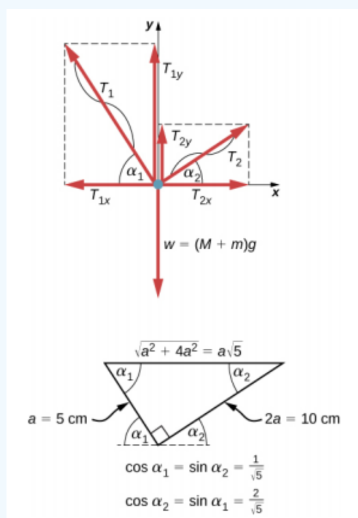


Figure 10.2.7: Free-body diagram for the knot in Example 12.2.

Solution

The weight w pulling on the knot is due to the mass M of the pan and mass m added to the pan, or w = (M + m)g. With the help of the free-body diagram in Figure 12.8, we can set up the equilibrium conditions for the knot:

in the x-direction,

$$-T_{1x} + T_{2x} = 0 \tag{10.2.19}$$

n the y-direction,

$$+T_{1y}+T_{2y}-w=0. \tag{10.2.20}$$

From the free-body diagram, the magnitudes of components in these equations are

$$\begin{aligned} T_{1x} &= T_1\cos\alpha_1 = \frac{T_1}{\sqrt{5}}, \quad T_{1y} = T_1\sin\alpha_1 = \frac{2T_1}{\sqrt{5}} \\ T_{2x} &= T_2\cos\alpha_2 = \frac{2T_2}{\sqrt{5}}, \quad T_{2y} = T_2\sin\alpha_2 = \frac{T_2}{\sqrt{5}}. \end{aligned}$$

We substitute these components into the equilibrium conditions and simplify. We then obtain two equilibrium equations for the tensions:

in x-direction,

$$T_1 = 2T_2 \tag{10.2.21}$$

in y-direction,

$$\frac{2T_1}{\sqrt{5}}+\frac{T_2}{\sqrt{5}}=(M+m)g. \tag{10.2.22}$$

The equilibrium equation for the x-direction tells us that the tension T_1 in the 5.0-cm string is twice the tension T_2 in the 10.0-cm string. Therefore, the shorter string will snap. When we use the first equation to eliminate T_2 from the second equation, we obtain the relation between the mass m on the pan and the tension T_1 in the shorter string:

$$\frac{2.5T_1}{\sqrt{5}}=(M+m)g. \tag{10.2.23}$$

The string breaks when the tension reaches the critical value of T_1 = 2.80 N. The preceding equation can be solved for the critical mass m that breaks the string:

$$m=\frac{2.5}{\sqrt{5}}\frac{T_1}{g}-M=\frac{2.5}{\sqrt{5}}\frac{2.80\ N}{9.8\ m/s^2}-0.042\ kg=0.277\ kg=277.0\ g. \tag{10.2.24}$$

Significance

Suppose that the mechanical system considered in this example is attached to a ceiling inside an elevator going up. As long as the elevator moves up at a constant speed, the result stays the same because the weight w does not change. If the elevator moves up with acceleration, the critical mass is smaller because the weight of M + m becomes larger by an apparent weight due to the acceleration of the elevator. Still, in all cases the shorter string breaks first.

10.3: Examples of Static Equilibrium

Learning Objectives

- Identify and analyze static equilibrium situations
- Set up a free-body diagram for an extended object in static equilibrium
- Set up and solve static equilibrium conditions for objects in equilibrium in various physical situations

All examples in this chapter are planar problems. Accordingly, we use equilibrium conditions in the component form of Equation 12.2.9 to Equation 12.2.11. We introduced a problem-solving strategy in Example 12.1 to illustrate the physical meaning of the equilibrium conditions. Now we generalize this strategy in a list of steps to follow when solving static equilibrium problems for extended rigid bodies. We proceed in five practical steps.

Problem-Solving Strategy: Static Equilibrium

1. Identify the object to be analyzed. For some systems in equilibrium, it may be necessary to consider more than one object. Identify all forces acting on the object. Identify the questions you need to answer. Identify the information given in the problem. In realistic problems, some key information may be implicit in the situation rather than provided explicitly.
2. Set up a free-body diagram for the object. (a) Choose the xy-reference frame for the problem. Draw a free-body diagram for the object, including only the forces that act on it. When suitable, represent the forces in terms of their components in the chosen reference frame. As you do this for each force, cross out the original force so that you do not erroneously include the same force twice in equations. Label all forces—you will need this for correct computations of net forces in the x- and y-directions. For an unknown force, the direction must be assigned arbitrarily; think of it as a 'working direction' or 'suspected direction.' The correct direction is determined by the sign that you obtain in the final solution. A plus sign (+) means that the working direction is the actual direction. A minus sign (−) means that the actual direction is opposite to the assumed working direction. (b) Choose the location of the rotation axis; in other words, choose the pivot point with respect to which you will compute torques of acting forces. On the free-body diagram, indicate the location of the pivot and the lever arms of acting forces—you will need this for correct computations of torques. In the selection of the pivot, keep in mind that the pivot can be placed anywhere you wish, but the guiding principle is that the best choice will simplify as much as possible the calculation of the net torque along the rotation axis.
3. Set up the equations of equilibrium for the object. (a) Use the free-body diagram to write a correct equilibrium condition Equation 12.2.9 for force components in the x-direction. (b) Use the free-body diagram to write a correct equilibrium condition Equation 12.2.13 for force components in the y-direction. (c) Use the free-body diagram to write a correct equilibrium condition Equation 12.2.11 for torques along the axis of rotation. Use Equation 12.2.12 to evaluate torque magnitudes and senses.
4. Simplify and solve the system of equations for equilibrium to obtain unknown quantities. At this point, your work involves algebra only. Keep in mind that the number of equations must be the same as the number of unknowns. If the number of unknowns is larger than the number of equations, the problem cannot be solved.
5. Evaluate the expressions for the unknown quantities that you obtained in your solution. Your final answers should have correct numerical values and correct physical units. If they do not, then use the previous steps to track back a mistake to its origin and correct it. Also, you may independently check for your numerical answers by shifting the pivot to a different location and solving the problem again, which is what we did in Example 12.1.

Note that setting up a free-body diagram for a rigid-body equilibrium problem is the most important component in the solution process. Without the correct setup and a correct diagram, you will not be able to write down correct conditions for equilibrium. Also note that a free-body diagram for an extended rigid body that may undergo rotational motion is different from a free-body diagram for a body that experiences only translational motion (as you saw in the chapters on Newton's laws of motion). In translational dynamics, a body is represented as its CM, where all forces on the body are attached and no torques appear. This does not hold true in rotational dynamics, where an extended rigid body cannot be represented by one point alone. The reason for this is that in analyzing rotation, we must identify torques acting on the body, and torque depends both on the acting force and on its lever arm. Here, the free-body diagram for an extended rigid body helps us identify external torques.

Example 12.3: The Torque Balance

Three masses are attached to a uniform meter stick, as shown in Figure 10.3.1. The mass of the meter stick is 150.0 g and the masses to the left of the fulcrum are $m_1 = 50.0$ g and $m_2 = 75.0$ g. Find the mass m3 that balances the system when it is attached at the right end of the stick, and the normal reaction force at the fulcrum when the system is balanced.

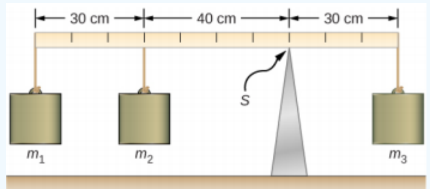


Figure 10.3.1: In a torque balance, a horizontal beam is supported at a fulcrum (indicated by S) and masses are attached to both sides of the fulcrum. The system is in static equilibrium when the beam does not rotate. It is balanced when the beam remains level.

Strategy

For the arrangement shown in the figure, we identify the following five forces acting on the meter stick:

1. $w_1 = m_1g$ is the weight of mass m_1;
2. $w_2 = m_2g$ is the weight of mass m_2;
3. w = mg is the weight of the entire meter stick;
4. $w_3 = m_3g$ is the weight of unknown mass m_3;
5. F_S is the normal reaction force at the support point S.

We choose a frame of reference where the direction of the y-axis is the direction of gravity, the direction of the xaxis is along the meter stick, and the axis of rotation (the z-axis) is perpendicular to the x-axis and passes through the support point S. In other words, we choose the pivot at the point where the meter stick touches the support. This is a natural choice for the pivot because this point does not move as the stick rotates. Now we are ready to set up the free-body diagram for the meter stick. We indicate the pivot and attach five vectors representing the five forces along the line representing the meter stick, locating the forces with respect to the pivot Figure 10.3.2. At this stage, we can identify the lever arms of the five forces given the information provided in the problem. For the three hanging masses, the problem is explicit about their locations along the stick, but the information about the location of the weight w is given implicitly. The key word here is "uniform." We know from our previous studies that the CM of a uniform stick is located at its midpoint, so this is where we attach the weight w, at the 50-cm mark.

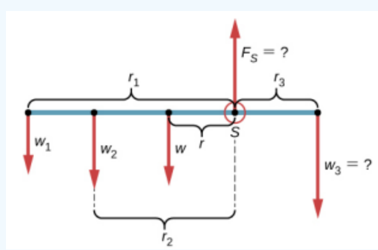


Figure 10.3.2: Free-body diagram for the meter stick. The pivot is chosen at the support point S.

Solution

With Figure 10.3.1 and Figure 10.3.2 for reference, we begin by finding the lever arms of the five forces acting on the stick:

$$\begin{aligned} r_1 &= 30.0\ cm + 40.0\ cm = 70.0\ cm \\ r_2 &= 40.0\ cm \\ r &= 50.0\ cm - 30.0\ cm = 20.0\ cm \\ r_S &= 0.0\ cm\ (because\ F_S\ is\ attached\ at\ the\ pivot) \\ r_3 &= 30.0\ cm. \end{aligned}$$

Now we can find the five torques with respect to the chosen pivot:

$$\begin{aligned}
\tau_1 &= +r_1 w_1 \sin 90^o = +r_1 m_1 g \quad (counterclockwise\ rotation,\ positive\ sense) \\
\tau_2 &= +r_2 w_2 \sin 90^o = +r_2 m_2 g \quad (counterclockwise\ rotation,\ positive\ sense) \\
\tau &= +rw \sin 90^o = +rmg \quad (gravitational\ torque) \\
\tau_S &= r_S F_S \sin \theta_S = 0 \quad (because\ r_S = 0\ cm) \\
\tau_3 &= -r_3 w_3 \sin 90^o = -r_3 m_3 g \quad (counterclockwise\ rotation,\ negative\ sense)
\end{aligned}$$

The second equilibrium condition (equation for the torques) for the meter stick is

$$\tau_1 + \tau_2 + \tau + \tau_S + \tau_3 = 0. \tag{10.3.1}$$

When substituting torque values into this equation, we can omit the torques giving zero contributions. In this way the second equilibrium condition is

$$+r_1 m_1 g + r_2 m_2 g + rmg - r_3 m_3 g = 0. \tag{10.3.2}$$

Selecting the +y-direction to be parallel to $\vec{F}_S$, the first equilibrium condition for the stick is

$$-w_1 - w_2 - w + F_S - w_3 = 0. \tag{10.3.3}$$

Substituting the forces, the first equilibrium condition becomes

$$-m_1 g - m_2 g - mg + F_S - m_3 g = 0. \tag{10.3.4}$$

We solve these equations simultaneously for the unknown values m_3 and F_S. In Equation 10.3.2, we cancel the g factor and rearrange the terms to obtain

$$r_3 m_3 = r_1 m_1 + r_2 m_2 + rm. \tag{10.3.5}$$

To obtain m_3 we divide both sides by r_3, so we have

$$\begin{aligned}
m_3 &= \frac{r_1}{r_3} m_1 + \frac{r_2}{r_3} m_2 + \frac{r}{r_3} m \\
&= \frac{70}{30}(50.0\ g) + \frac{40}{30}(75.0\ g) + \frac{20}{30}(150.0\ g) = 315.0 \left(\frac{2}{3}\right)\ g \simeq 317\ g.
\end{aligned}$$

To find the normal reaction force, we rearrange the terms in Equation 10.3.4, converting grams to kilograms:

$$\begin{aligned}
F_S &= (m_1 + m_2 + m + m_3) g \\
&= (50.0 + 75.0 + 150.0 + 316.7) \times (10^{-3}\ kg) \times (9.8\ m/s^2) = 5.8\ N.
\end{aligned}$$

Significance

Notice that Equation 10.3.2 is independent of the value of g. The torque balance may therefore be used to measure mass, since variations in g-values on Earth's surface do not affect these measurements. This is not the case for a spring balance because it measures the force.

? Exercise 12.3

Repeat Example 12.3 using the left end of the meter stick to calculate the torques; that is, by placing the pivot at the left end of the meter stick.

In the next example, we show how to use the first equilibrium condition (equation for forces) in the vector form given by Equation 12.2.9 and Equation 12.2.10. We present this solution to illustrate the importance of a suitable choice of reference frame. Although all inertial reference frames are equivalent and numerical solutions obtained in one frame are the same as in any other, an unsuitable choice of reference frame can make the solution quite lengthy and convoluted, whereas a wise choice of reference frame makes the solution straightforward. We show this in the equivalent solution to the same problem. This particular example illustrates an application of static equilibrium to biomechanics.

Example 12.4: Forces in the Forearm

A weightlifter is holding a 50.0-lb weight (equivalent to 222.4 N) with his forearm, as shown in Figure 10.3.3. His forearm is positioned at β = 60° with respect to his upper arm. The forearm is supported by a contraction of the biceps muscle, which causes a torque around the elbow. Assuming that the tension in the biceps acts along the vertical direction given by gravity, what tension must the muscle exert to hold the forearm at the position shown? What is the force on the elbow joint? Assume that the forearm's weight is negligible. Give your final answers in SI units.

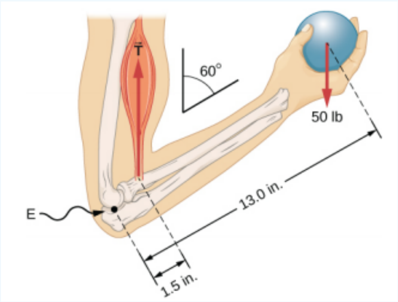


Figure 10.3.3: The forearm is rotated around the elbow (E) by a contraction of the biceps muscle, which causes tension $\vec{T}_M$.

Strategy

We identify three forces acting on the forearm: the unknown force $\vec{F}$ at the elbow; the unknown tension $\vec{T}_M$ in the muscle; and the weight $\vec{w}$ with magnitude w = 50 lb. We adopt the frame of reference with the x-axis along the forearm and the pivot at the elbow. The vertical direction is the direction of the weight, which is the same as the direction of the upper arm. The x-axis makes an angle β = 60° with the vertical. The y-axis is perpendicular to the x-axis. Now we set up the free-body diagram for the forearm. First, we draw the axes, the pivot, and the three vectors representing the three identified forces. Then we locate the angle β and represent each force by its x- and y-components, remembering to cross out the original force vector to avoid double counting. Finally, we label the forces and their lever arms. The free-body diagram for the forearm is shown in Figure 10.3.4. At this point, we are ready to set up equilibrium conditions for the forearm. Each force has x- and y-components; therefore, we have two equations for the first equilibrium condition, one equation for each component of the net force acting on the forearm.

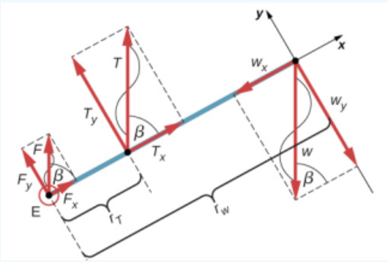


Figure 10.3.4: Free-body diagram for the forearm: The pivot is located at point E (elbow).

Notice that in our frame of reference, contributions to the second equilibrium condition (for torques) come only from the y-components of the forces because the x-components of the forces are all parallel to their lever arms, so that for any of them we have sin θ = 0 in Equation 12.2.12. For the y-components we have θ = ± 90° in Equation 12.2.12. Also notice that the torque of the force at the elbow is zero because this force is attached at the pivot. So the contribution to the net torque comes only from the torques of T_y and of w_y.

Solution

We see from the free-body diagram that the x-component of the net force satisfies the equation

$$+F_x + T_x - w_x = 0 \tag{10.3.6}$$

and the y-component of the net force satisfies

$$+F_y + T_y - w_y = 0. \tag{10.3.7}$$

Equation 10.3.6 and Equation 10.3.7 are two equations of the first equilibrium condition (for forces). Next, we read from the free-body diagram that the net torque along the axis of rotation is

$$+r_T T_y - r_w w_y = 0. \tag{10.3.8}$$

Equation 10.3.8 is the second equilibrium condition (for torques) for the forearm. The free-body diagram shows that the lever arms are r_T = 1.5 in. and r_w = 13.0 in. At this point, we do not need to convert inches into SI units, because as long as these units are consistent in Equation 12.23, they cancel out. Using the free-body diagram again, we find the magnitudes of the component forces:

$$\begin{aligned}
F_x &= F\cos\beta = F\cos 60^o = \frac{F}{2} \\
T_x &= T\cos\beta = T\cos 60^o = \frac{T}{2} \\
w_x &= w\cos\beta = w\cos 60^o = \frac{w}{2} \\
F_y &= F\sin\beta = F\sin 60^o = \frac{F\sqrt{3}}{2} \\
T_y &= T\sin\beta = T\sin 60^o = \frac{T\sqrt{3}}{2} \\
w_y &= w\sin\beta = w\sin 60^o = \frac{w\sqrt{3}}{2}.
\end{aligned}$$

We substitute these magnitudes into Equation 10.3.6, Equation 10.3.7, and Equation 10.3.8 to obtain, respectively,

$$\begin{aligned}
\frac{F}{2} + \frac{T}{2} - \frac{w}{2} &= 0 \\
\frac{F\sqrt{3}}{2} + \frac{T\sqrt{3}}{2} - \frac{w\sqrt{3}}{2} &= 0 \\
\frac{r_T T\sqrt{3}}{2} - \frac{r_w w\sqrt{3}}{2} &= 0.
\end{aligned}$$

When we simplify these equations, we see that we are left with only two independent equations for the two unknown force magnitudes, F and T, because Equation 10.3.6 for the x-component is equivalent to Equation 10.3.7 for the y-component. In this way, we obtain the first equilibrium condition for forces

$$F + T - w = 0 \tag{10.3.9}$$

and the second equilibrium condition for torques

$$r_T T - r_w w = 0. \tag{10.3.10}$$

The magnitude of tension in the muscle is obtained by solving Equation 10.3.10:

$$T = \frac{r_w}{r_T} w = frac13.01.5(50\ lb) = 433\frac{1}{3}\ lb \simeq 433.3\ lb. \tag{10.3.11}$$

The force at the elbow is obtained by solving Equation 10.3.9:

$$F = w - T = 50.0\ lb - 433.3\ lb = -383.3\ lb. \tag{10.3.12}$$

The negative sign in the equation tells us that the actual force at the elbow is antiparallel to the working direction adopted for drawing the free-body diagram. In the final answer, we convert the forces into SI units of force. The answer is

$$F = 383.3\ lb = 383.3(4.448\ N) = 1705\ N\ downward \tag{10.3.13}$$

$$T = 433.3\ lb = 433.3(4.448\ N) = 1927\ N\ upward. \tag{10.3.14}$$

Significance

Two important issues here are worth noting. The first concerns conversion into SI units, which can be done at the very end of the solution as long as we keep consistency in units. The second important issue concerns the hinge joints such as the elbow. In the initial analysis of a problem, hinge joints should always be assumed to exert a force in an **arbitrary direction**, and then you must solve for all components of a hinge force independently. In this example, the elbow force happens to be vertical because the problem assumes the tension by the biceps to be vertical as well. Such a simplification, however, is not a general rule.

Solution 2

Suppose we adopt a reference frame with the direction of the y-axis along the 50-lb weight and the pivot placed at the elbow. In this frame, all three forces have only y-components, so we have only one equation for the first equilibrium condition (for forces). We draw the free-body diagram for the forearm as shown in Figure 10.3.5, indicating the pivot, the acting forces and their lever arms with respect to the pivot, and the angles θ_T and θ_w that the forces $\vec{T}_M$ and $\vec{w}$ (respectively) make with their lever arms. In the definition of torque given by Equation 12.2.12, the angle θ_T is the direction angle of the vector $\vec{T}_M$, counted **counterclockwise** from the radial direction of the lever arm that always points away from the pivot. By the same convention, the angle θ_w is measured **counterclockwise** from the radial direction of the lever arm to the vector $\vec{w}$. Done this way, the non-zero torques are most easily computed by directly substituting into Equation 12.2.12 as follows:

$$\tau_T = r_T T \sin\theta_T = r_T T \sin\beta = r_T T \sin 60^o = +\frac{r_T T\sqrt{3}}{2} \tag{10.3.15}$$

$$\tau_w = r_w w \sin\theta_w = r_w w \sin(\beta + 180^o) = -r_w w \sin\beta = -\frac{r_w w\sqrt{3}}{2}. \tag{10.3.16}$$

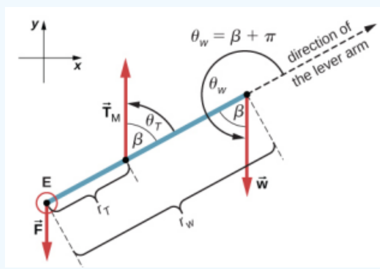


Figure 10.3.5: Free-body diagram for the forearm for the equivalent solution. The pivot is located at point E (elbow).

The second equilibrium condition, $\tau_T + \tau_w = 0$, can be now written as

$$\frac{r_T T\sqrt{3}}{2} - \frac{r_w w\sqrt{3}}{2} = 0. \tag{10.3.17}$$

From the free-body diagram, the first equilibrium condition (for forces) is

$$-F + T - w = 0. \tag{10.3.18}$$

Equation 10.3.17is identical to Equation 10.3.10and gives the result T = 433.3 lb. Equation 10.3.18gives

$$F = T - w = 433.3\ lb - 50.0\ lb = 383.3\ lb \tag{10.3.19}$$

We see that these answers are identical to our previous answers, but the second choice for the frame of reference leads to an equivalent solution that is simpler and quicker because it does not require that the forces be resolved into their rectangular components.

Exercise 12.4

Repeat Example 12.4 assuming that the forearm is an object of uniform density that weighs 8.896 N.

Example 12.5: A Ladder Resting Against a Wall

A uniform ladder is L = 5.0 m long and weighs 400.0 N. The ladder rests against a slippery vertical wall, as shown in Figure 10.3.6. The inclination angle between the ladder and the rough floor is β = 53°. Find the reaction forces from the floor and from the wall on the ladder and the coefficient of static friction μ_s at the interface of the ladder with the floor that prevents the ladder from slipping.

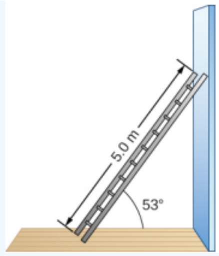


Figure 10.3.6 A 5.0-m-long ladder rests against a frictionless wall.

Strategy

We can identify four forces acting on the ladder. The first force is the normal reaction force N from the floor in the upward vertical direction. The second force is the static friction force f = μ_sN directed horizontally along the floor toward the wall—this force prevents the ladder from slipping. These two forces act on the ladder at its contact point with the floor. The third force is the weight w of the ladder, attached at its CM located midway between its ends. The fourth force is the normal reaction force F from the wall in the horizontal direction away from the wall, attached at the contact point with the wall. There are no other forces because the wall is slippery, which means there is no friction between the wall and the ladder. Based on this analysis, we adopt the frame of reference with the y-axis in the vertical direction (parallel to the wall) and the x-axis in the horizontal direction (parallel to the floor). In this frame, each force has either a horizontal component or a vertical component but not both, which simplifies the solution. We select the pivot at the contact point with the floor. In the free-body diagram for the ladder, we indicate the pivot, all four forces and their lever arms, and the angles between lever arms and the forces, as shown in Figure 10.3.7. With our choice of the pivot location, there is no torque either from the normal reaction force N or from the static friction f because they both act at the pivot.

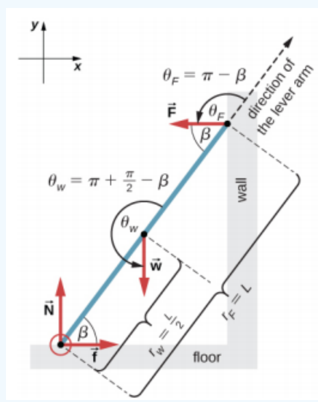


Figure 10.3.7: Free-body diagram for a ladder resting against a frictionless wall.

Solution

From the free-body diagram, the net force in the x-direction is

$$+f - F = 0 \tag{10.3.20}$$

the net force in the y-direction is

$$+N - w = 0 \tag{10.3.21}$$

and the net torque along the rotation axis at the pivot point is

$$\tau_w + \tau_F = 0. \tag{10.3.22}$$

where τ_w is the torque of the weight w and τ_F is the torque of the reaction F. From the free-body diagram, we identify that the lever arm of the reaction at the wall is r_F = L = 5.0 m and the lever arm of the weight is $r_w = \frac{L}{2}$ = 2.5 m. With the help of the free-body diagram, we identify the angles to be used in Equation 12.2.12 for torques: θ_F = 180° − β for the torque from the reaction force with the wall, and θ_w = 180° + (90° − β) for the torque due to the weight. Now we are ready to use Equation 12.2.12 to compute torques:

$$\tau_w = r_w w \sin\theta_w = r_w w \sin(180^o + 90^o - \beta) = -\frac{L}{2} w \sin(90^o - \beta) = -\frac{L}{2} w \cos\beta \tag{10.3.23}$$

$$tau_F = r_F F \sin\theta_F = r_F F \sin(180^o - \beta) = LF \sin\beta. \tag{10.3.24}$$

We substitute the torques into Equation 10.3.22and solve for F :

$$-\frac{L}{2} w \cos\beta + LF \sin\beta = 0 \tag{10.3.25}$$

$$F = \frac{w}{2}\cot\beta = \frac{400.0\ N}{2}\cot 53^o = 150.7\ N \tag{10.3.26}$$

We obtain the normal reaction force with the floor by solving Equation 10.3.21: N = w = 400.0 N. The magnitude of friction is obtained by solving Equation 10.3.20: f = F = 150.7 N. The coefficient of static friction is $\mu_s = \frac{f}{N} = \frac{150.7}{400.0} = 0.377$.

The net force on the ladder at the contact point with the floor is the vector sum of the normal reaction from the floor and the static friction forces:

$$\vec{F}_{floor} = \vec{f} + \vec{N} = (150.7\ N)(-\hat{i}) + (400.0\ N)(+\hat{j}) = (-150.7\ \hat{i} + 400.0\ \hat{j})N. \tag{10.3.27}$$

Its magnitude is

$$F_{floor} = \sqrt{f^2 + N^2} = \sqrt{150.7^2 + 400.0^2}\ N = 427.4\ N \tag{10.3.28}$$

and its direction is

$$\varphi = \tan^{-1}\left(\frac{N}{f}\right) = \tan^{-1}\left(\frac{400.0}{150.7}\right) = 69.3^o\ above\ the\ floor. \tag{10.3.29}$$

We should emphasize here two general observations of practical use. First, notice that when we choose a pivot point, there is no expectation that the system will actually pivot around the chosen point. The ladder in this example is not rotating at all but firmly stands on the floor; nonetheless, its contact point with the floor is a good choice for the pivot. Second, notice when we use Equation 12.2.12 for the computation of individual torques, we do not need to resolve the forces into their normal and parallel components with respect to the direction of the lever arm, and we do not need to consider a sense of the torque. As long as the angle in Equation 12.2.12 is correctly identified—with the help of a free-body diagram—as the angle measured counterclockwise from the direction of the lever arm to the direction of the force vector, Equation 12.2.12 gives both the magnitude and the sense of the torque. This is because torque is the vector product of the lever-arm vector crossed with the force vector, and Equation 12.2.12 expresses the rectangular component of this vector product along the axis of rotation.

Significance

This result is independent of the length of the ladder because L is canceled in the second equilibrium condition, Equation 10.3.25. No matter how long or short the ladder is, as long as its weight is 400 N and the angle with the floor is 53°, our results hold. But the ladder will slip if the net torque becomes negative in Equation 10.3.25. This happens for some angles when the coefficient of static friction is not great enough to prevent the ladder from slipping.

? Exercise 12.5

For the situation described in Example 12.5, determine the values of the coefficient μ_s of static friction for which the ladder starts slipping, given that β is the angle that the ladder makes with the floor.

✓ Example 12.6: Forces on Door Hinges

A swinging door that weighs w = 400.0 N is supported by hinges A and B so that the door can swing about a vertical axis passing through the hinges Figure 10.3.8. The door has a width of b = 1.00 m, and the door slab has a uniform mass density. The hinges are placed symmetrically at the door's edge in such a way that the door's weight is evenly distributed between them. The hinges are separated by distance a = 2.00 m. Find the forces on the hinges when the door rests half-open.

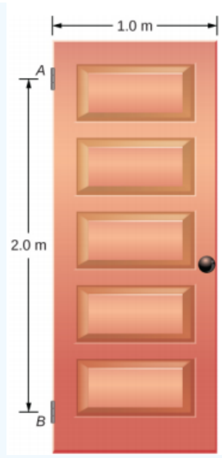


Figure 10.3.8: A 400-N swinging vertical door is supported by two hinges attached at points A and B.

Strategy

The forces that the door exerts on its hinges can be found by simply reversing the directions of the forces that the hinges exert on the door. Hence, our task is to find the forces from the hinges on the door. Three forces act on the door slab: an unknown force $\vec{A}$ from hinge A, an unknown force $\vec{B}$ from hinge B, and the known weight $\vec{w}$ attached at the center of mass of the door slab. The CM is located at the geometrical center of the door because the slab has a uniform mass density. We adopt a rectangular frame of reference with the y-axis along the direction of gravity and the x-axis in the plane of the slab, as shown in panel (a) of Figure $10.3.9$, and resolve all forces into their rectangular components. In this way, we have four unknown component forces: two components of force $\vec{A}$ (A_x and A_y), and two components of force $\vec{B}$ (B_x and B_y). In the free-body diagram, we represent the two forces at the hinges by their vector components, whose assumed orientations are arbitrary. Because there are four unknowns (A_x, B_x, A_y, and B_y), we must set up four independent equations. One equation is the equilibrium condition for forces in the x-direction. The second equation is the equilibrium condition for forces in the y-direction. The third equation is the equilibrium condition for torques in rotation about a hinge. Because the weight is evenly distributed between the hinges, we have the fourth equation, $A_y = B_y$. To set up the equilibrium conditions, we draw a free-body diagram and choose the pivot point at the upper hinge, as shown in panel (b) of Figure $10.3.9$. Finally, we solve the equations for the unknown force components and find the forces.

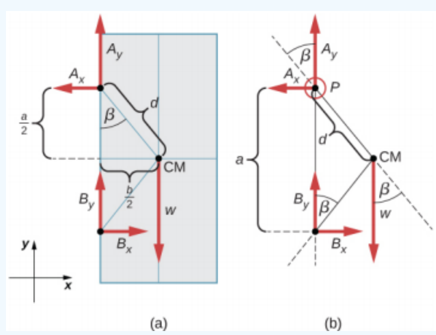


Figure 10.3.9: (a) Geometry and (b) free-body diagram for the door.

Solution

From the free-body diagram for the door we have the first equilibrium condition for forces:

in the x-direction,

$$-A_x + B_x = 0 \Rightarrow A_x + B_x \tag{10.3.30}$$

in y-direction, $$+ A_{y} + B_{y} - w = 0 \Rightarrow A_{y} = B_{y} = \frac{w}{2} = \frac{400.0\; N}{2} = 200.0\; N \ldotp\]

We select the pivot at point P (upper hinge, per the free-body diagram) and write the second equilibrium condition for torques in rotation about point P:

pivot at P: $$\tau_{w} + \tau_{Bx} + \tau_{By} = 0 \ldotp \label{12.32}\]

We use the free-body diagram to find all the terms in this equation:

$$\begin{aligned} \tau_{w} &= dw \sin(-\beta) = -dw \sin\beta = -dw \frac{\frac{b}{2}}{d} = -w \frac{b}{2} \\ \tau_{Bx} &= aB_{x} \sin 90^{o} = +aB_{x} \\ \tau_{By} &= aB_{y} \sin 180^{o} = 0. \end{aligned}$$

In evaluating sin β, we use the geometry of the triangle shown in part (a) of the figure. Now we substitute these torques into Equation ??? and compute B_x:

pivot at P: $$-w \frac{b}{2} + a B_{x} = 0 \Rightarrow B_{x} = w \frac{b}{2a} = (400.0\; N) \frac{1}{2\; \cdotp 2} = 100.0\; N \ldotp\]

Therefore the magnitudes of the horizontal component forces are $A_x = B_x = 100.0$ N. The forces on the door are

at the upper hinge:

$$\vec{F}_{A\ on\ door} = -100.0\ N\ \hat{i} + 200.0\ N\ \hat{j} \tag{10.3.31}$$

at the lower hinge: $$\vec{F}_{B\; on\; door} = +100.0\; N\; \hat{i} + 200.0\; N\; \hat{j} \ldotp\]

The forces on the hinges are found from Newton’s third law as

on the upper hinge:

$$\vec{F}_{door\ on\ A} = 100.0\ N\ \hat{i} - 200.0\ N\ \hat{j} \tag{10.3.32}$$

on the lower hinge: $$\vec{F}_{door\; on\; B} = - 100.0\; N\; \hat{i} - 200.0\; N\; \hat{j} \ldotp\]

Significance

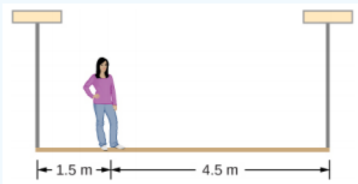
Note that if the problem were formulated without the assumption of the weight being equally distributed between the two hinges, we wouldn’t be able to solve it because the number of the unknowns would be greater than the number of equations expressing equilibrium conditions.

? Exercise 12.6

Solve the problem in Example 12.6 by taking the pivot position at the center of mass.

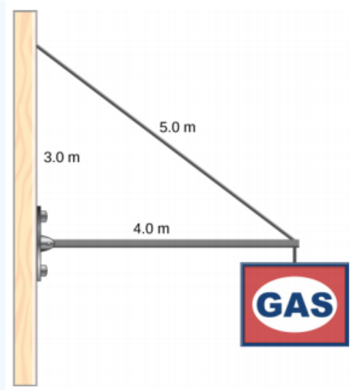
? Exercise 12.7

A 50-kg person stands 1.5 m away from one end of a uniform 6.0-m-long scaffold of mass 70.0 kg. Find the tensions in the two vertical ropes supporting the scaffold.



? Exercise 12.8

A 400.0-N sign hangs from the end of a uniform strut. The strut is 4.0 m long and weighs 600.0 N. The strut is supported by a hinge at the wall and by a cable whose other end is tied to the wall at a point 3.0 m above the left end of the strut. Find the tension in the supporting cable and the force of the hinge on the strut.


5.0 m
3.0 m
4.0 m
GAS

10.4: Stress, Strain, and Elastic Modulus (Part 1)

Learning Objectives

- Explain the concepts of stress and strain in describing elastic deformations of materials
- Describe the types of elastic deformation of objects and materials

A model of a rigid body is an idealized example of an object that does not deform under the actions of external forces. It is very useful when analyzing mechanical systems—and many physical objects are indeed rigid to a great extent. The extent to which an object can be **perceived** as rigid depends on the physical properties of the material from which it is made. For example, a ping-pong ball made of plastic is brittle, and a tennis ball made of rubber is elastic when acted upon by squashing forces. However, under other circumstances, both a ping-pong ball and a tennis ball may bounce well as rigid bodies. Similarly, someone who designs prosthetic limbs may be able to approximate the mechanics of human limbs by modeling them as rigid bodies; however, the actual combination of bones and tissues is an elastic medium.

For the remainder of this section, we move from consideration of forces that affect the motion of an object to those that affect an object's shape. A change in shape due to the application of a force is known as a deformation. Even very small forces are known to cause some deformation. Deformation is experienced by objects or physical media under the action of external forces—for example, this may be squashing, squeezing, ripping, twisting, shearing, or pulling the objects apart. In the language of physics, two terms describe the forces on objects undergoing deformation: **stress** and **strain**.

Stress is a quantity that describes the magnitude of forces that cause deformation. Stress is generally defined as **force per unit area**. When forces pull on an object and cause its elongation, like the stretching of an elastic band, we call such stress a **tensile stress**. When forces cause a compression of an object, we call it a **compressive stress**. When an object is being squeezed from all sides, like a submarine in the depths of an ocean, we call this kind of stress a **bulk stress** (or **volume stress**). In other situations, the acting forces may be neither tensile nor compressive, and still produce a noticeable deformation. For example, suppose you hold a book tightly between the palms of your hands, then with one hand you press-and-pull on the front cover away from you, while with the other hand you press-and-pull on the back cover toward you. In such a case, when deforming forces act tangentially to the object's surface, we call them 'shear' forces and the stress they cause is called **shear stress**.

The SI unit of stress is the pascal (Pa). When one newton of force presses on a unit surface area of one meter squared, the resulting stress is one pascal:

$$one\ pascal = 1.0\ Pa = \frac{1.0\ N}{1.0\ m^2}. \tag{10.4.1}$$

In the British system of units, the unit of stress is 'psi,' which stands for 'pound per square inch' (lb/in^2). Another unit that is often used for bulk stress is the atm (atmosphere). Conversion factors are

$$1\ psi = 6895\ Pa\ and\ 1\ Pa = 1.450 \times 10^{-4}\ psi \tag{10.4.2}$$

$$1\ atm = 1.013 \times 10^5\ Pa = 14.7\ psi. \tag{10.4.3}$$

An object or medium under stress becomes deformed. The quantity that describes this deformation is called **strain**. Strain is given as a fractional change in either length (under tensile stress) or volume (under bulk stress) or geometry (under shear stress). Therefore, strain is a dimensionless number. Strain under a tensile stress is called **tensile strain**, strain under bulk stress is called **bulk strain** (or **volume strain**), and that caused by shear stress is called **shear strain**.

The greater the stress, the greater the strain; however, the relation between strain and stress does not need to be linear. Only when stress is sufficiently low is the deformation it causes in direct proportion to the stress value. The proportionality constant in this relation is called the **elastic modulus**. In the linear limit of low stress values, the general relation between stress and strain is

$$stress = (elastic\ modulus) \times strain. \tag{10.4.4}$$

As we can see from dimensional analysis of this relation, the elastic modulus has the same physical unit as stress because strain is dimensionless.

We can also see from Equation 10.4.4 that when an object is characterized by a large value of elastic modulus, the effect of stress is small. On the other hand, a small elastic modulus means that stress produces large strain and noticeable deformation. For example, a stress on a rubber band produces larger strain (deformation) than the same stress on a steel band of the same dimensions because the elastic modulus for rubber is two orders of magnitude smaller than the elastic modulus for steel.

The elastic modulus for tensile stress is called **Young's modulus**; that for the bulk stress is called the **bulk modulus**; and that for shear stress is called the **shear modulus**. Note that the relation between stress and strain is an observed relation, measured in the laboratory. Elastic moduli for various materials are measured under various physical conditions, such as varying temperature, and collected in engineering data tables for reference (Table $10.4.1$). These tables are valuable references for industry and for anyone involved in engineering or construction. In the next section, we discuss strain-stress relations beyond the linear limit represented by Equation $10.4.4$, in the full range of stress values up to a fracture point. In the remainder of this section, we study the linear limit expressed by Equation $10.4.4$.

Table 10.4.1: Approximate Elastic Moduli for Selected Materials

Material	Young's modulus × 10^{10} Pa	Bulk modulus × 10^{10} Pa	Shear modulus × 10^{10} Pa
Aluminum	7.0	7.5	2.5
Bone (tension)	1.6	0.8	8.0
Bone (compression)	0.9		
Brass	9.0	6.0	3.5
Brick	1.5		
Concrete	2.0		
Copper	11.0	14.0	4.4
Crown glass	6.0	5.0	2.5
Granite	4.5	4.5	2.0
Hair (human)	1.0		
Hardwood	1.5		1.0
Iron	21.0	16.0	7.7
Lead	1.6	4.1	0.6
Marble	6.0	7.0	2.0
Nickel	21.0	17.0	7.8
Polystyrene	3.0		
Silk	6.0		
Spider thread	3.0		
Steel	20.0	16.0	7.5
Acetone		0.07	
Ethanol		0.09	
Glycerin		0.45	
Mercury		2.5	
Water		0.22	

Tensile or Compressive Stress, Strain, and Young's Modulus

Tension or compression occurs when two antiparallel forces of equal magnitude act on an object along only one of its dimensions, in such a way that the object does not move. One way to envision such a situation is illustrated in Figure $10.4.1$. A rod segment is either stretched or squeezed by a pair of forces acting along its length and perpendicular to its cross-section. The net effect of such forces is that the rod changes its length from the original length L_0 that it had before the forces appeared, to a new length L that it has under the action of the forces. This change in length $\Delta L = L - L_0$ may be either elongation (when L is larger than the original length L_o) or contraction (when L is smaller than the original length L_0). Tensile stress and strain occur when the forces are stretching an object, causing its elongation, and the length change ΔL is positive. Compressive stress and strain occur when the forces are contracting an object, causing its shortening, and the length change ΔL is negative.

In either of these situations, we define stress as the ratio of the deforming force $F_\perp$ to the cross-sectional area A of the object being deformed. The symbol $F_\perp$ that we reserve for the deforming force means that this force acts perpendicularly to the cross-section of the object. Forces that act parallel to the cross-section do not change the length of an object. The definition of the tensile stress is

$$tensile\ stress = \frac{F_\perp}{A}. \tag{10.4.5}$$

Tensile strain is the measure of the deformation of an object under tensile stress and is defined as the fractional change of the object's length when the object experiences tensile stress

$$tensile\ strain = \frac{\Delta L}{L_0}. \tag{10.4.6}$$

Compressive stress and strain are defined by the same formulas, Equations 10.4.5 and 10.4.6, respectively. The only difference from the tensile situation is that for compressive stress and strain, we take absolute values of the right-hand sides in Equation 10.4.5 and 10.4.6.

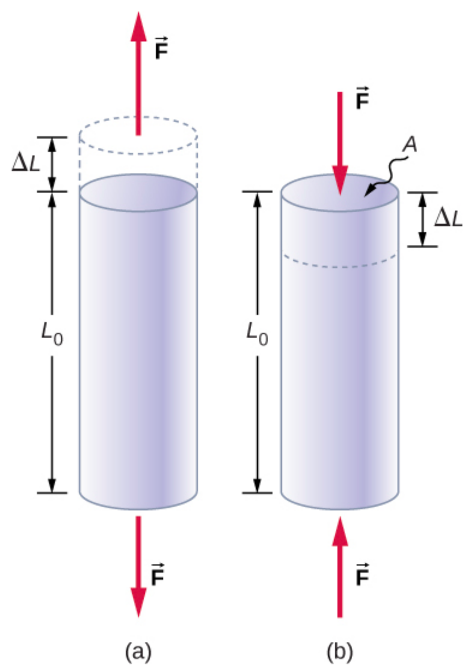


Figure 10.4.1: When an object is in either tension or compression, the net force on it is zero, but the object deforms by changing its original length L_0. (a) Tension: The rod is elongated by ΔL. (b) Compression: The rod is contracted by ΔL. In both cases, the deforming force acts along the length of the rod and perpendicular to its cross-section. In the linear range of low stress, the cross-sectional area of the rod does not change.

Young's modulus Y is the elastic modulus when deformation is caused by either tensile or compressive stress, and is defined by Equation 10.4.4. Dividing this equation by tensile strain, we obtain the expression for Young's modulus:

$$Y = \frac{tensile\ stress}{tensile\ strain} = \frac{\frac{F_\perp}{A}}{\frac{\Delta L}{L_0}} = \frac{F_\perp}{A} \times \frac{L_0}{\Delta L}. \tag{10.4.7}$$

✔ Example 10.4.1: Compressive Stress in a Pillar

A sculpture weighing 10,000 N rests on a horizontal surface at the top of a 6.0-m-tall vertical pillar Figure 10.4.1. The pillar's cross-sectional area is 0.20 m^2 and it is made of granite with a mass density of 2700 kg/m^3. Find the compressive stress at the cross-section located 3.0 m below the top of the pillar and the value of the compressive strain of the top 3.0-m segment of the pillar.

Figure 10.4.2: Nelson's Column in Trafalgar Square, London, England. (credit: modification of work by Cristian Bortes)

Strategy

First we find the weight of the 3.0-m-long top section of the pillar. The normal force that acts on the cross-section located 3.0 m down from the top is the sum of the pillar's weight and the sculpture's weight. Once we have the normal force, we use Equation 12.34 to find the stress. To find the compressive strain, we find the value of Young's modulus for granite in Table 10.4.1 and invert Equation 10.4.7.

Solution

The volume of the pillar segment with height h = 3.0 m and cross-sectional area A = 0.20 m^2 is

$$V = Ah = (0.20\ m^2)(3.0\ m) = 0.60\ m^3. \tag{10.4.8}$$

With the density of granite ρ = 2.7 x 10^3 kg/m^3, the mass of the pillar segment is

$$m = \rho V = (2.7 \times 10^3\ kg/m^3)(0.60\ m^3) = 1.60 \times 10^3\ kg. \tag{10.4.9}$$

The weight of the pillar segment is

$$w_p = mg = (1.60 \times 10^3\ kg)(9.80\ m/s^2) = 1.568 \times 10^4\ N. \tag{10.4.10}$$

The weight of the sculpture is w_s = 1.0 x 10^4 N, so the normal force on the cross-sectional surface located 3.0 m below the sculpture is

$$F_\perp = w_p + w_s = (1.568 + 1.0) \times 10^4\ N = 2.568 \times 10^4\ N. \tag{10.4.11}$$

Therefore, the stress is

$$stress = \frac{F_\perp}{A} = \frac{2.568 \times 10^4\ N}{0.20 m^2} = 1.284 \times 10^5\ Pa = 128.4\ kPa. \tag{10.4.12}$$

Young's modulus for granite is Y = 4.5 x 10^{10} Pa = 4.5 x 10^7 kPa. Therefore, the compressive strain at this position is

$$strain = \frac{stress}{Y} = \frac{128.4\ kPa}{4.5 \times 10^7\ kPa} = 2.85 \times 10^{-6}. \tag{10.4.13}$$

Significance

Notice that the normal force acting on the cross-sectional area of the pillar is not constant along its length, but varies from its smallest value at the top to its largest value at the bottom of the pillar. Thus, if the pillar has a uniform cross-sectional area along its length, the stress is largest at its base.

? Exercise 10.4.2

Find the compressive stress and strain at the base of Nelson's column.

✓ Example 10.4.2: Stretching a Rod

A 2.0-m-long steel rod has a cross-sectional area of 0.30 cm^2. The rod is a part of a vertical support that holds a heavy 550-kg platform that hangs attached to the rod's lower end. Ignoring the weight of the rod, what is the tensile stress in the rod and the elongation of the rod under the stress?

Strategy

First we compute the tensile stress in the rod under the weight of the platform in accordance with Equation 12.34. Then we invert Equation 12.36 to find the rod's elongation, using L_0 = 2.0 m. From Table 12.1, Young's modulus for steel is Y = 2.0 x 10^{11} Pa.

Solution

Substituting numerical values into the equations gives us

$$\begin{aligned} \frac{F_\perp}{A} &= \frac{(550\ kg)(9.8\ m/s^2)}{3.0 \times 10^{-5}\ m^2} = 1.8 \times 10^8\ Pa \\ \Delta L &= \frac{F_\perp}{A}\frac{L_0}{Y} = (1.8 \times 10^8\ Pa)\left(\frac{2.0\ m}{2.0 \times 10^{11}\ Pa}\right) = 1.8 \times 10^{-3}\ m = 1.8\ mm. \end{aligned}$$

Significance

Similarly as in the example with the column, the tensile stress in this example is not uniform along the length of the rod. Unlike in the previous example, however, if the weight of the rod is taken into consideration, the stress in the rod is largest at the top and smallest at the bottom of the rod where the equipment is attached.

? Exercise 10.4.2

A 2.0-m-long wire stretches 1.0 mm when subjected to a load. What is the tensile strain in the wire?

Objects can often experience both compressive stress and tensile stress simultaneously Figure 10.4.3. One example is a long shelf loaded with heavy books that sags between the end supports under the weight of the books. The top surface of the shelf is in compressive stress and the bottom surface of the shelf is in tensile stress. Similarly, long and heavy beams sag under their own weight. In modern building construction, such bending strains can be almost eliminated with the use of I-beams Figure 10.4.4.

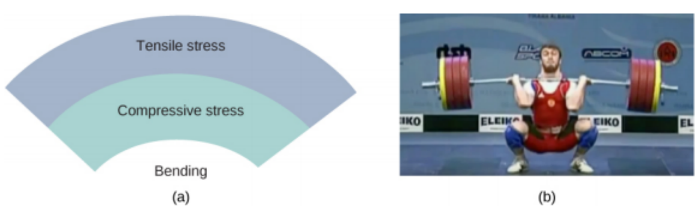


Figure 10.4.3: (a) An object bending downward experiences tensile stress (stretching) in the upper section and compressive stress (compressing) in the lower section. (b) Elite weightlifters often bend iron bars temporarily during lifting, as in the 2012 Olympics competition. (credit b: modification of work by Oleksandr Kocherzhenko)

Figure 10.4.4: Steel I-beams are used in construction to reduce bending strains. (credit: modification of work by “US Army Corps of Engineers Europe District”/Flickr)

Simulation

A heavy box rests on a table supported by three columns. View this demonstration to move the box to see how the compression (or tension) in the columns is affected when the box changes its position.

10.5: Stress, Strain, and Elastic Modulus (Part 2)

Bulk Stress, Strain, and Modulus

When you dive into water, you feel a force pressing on every part of your body from all directions. What you are experiencing then is bulk stress, or in other words, **pressure**. Bulk stress always tends to decrease the volume enclosed by the surface of a submerged object. The forces of this "squeezing" are always perpendicular to the submerged surface Figure 10.5.1. The effect of these forces is to decrease the volume of the submerged object by an amount ΔV compared with the volume V_0 of the object in the absence of bulk stress. This kind of deformation is called bulk strain and is described by a change in volume relative to the original volume:

$$bulk\ strain = \frac{\Delta V}{V_0} \tag{10.5.1}$$

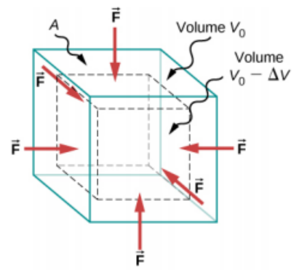


Figure 10.5.1: An object under increasing bulk stress always undergoes a decrease in its volume. Equal forces perpendicular to the surface act from all directions. The effect of these forces is to decrease the volume by the amount ΔV compared to the original volume, V_0.

The bulk strain results from the bulk stress, which is a force $F_\perp$ normal to a surface that presses on the unit surface area A of a submerged object. This kind of physical quantity, or pressure p, is defined as

$$pressure = p \equiv \frac{F_\perp}{A}. \tag{10.5.2}$$

We will study pressure in fluids in greater detail in Fluid Mechanics. An important characteristic of pressure is that it is a scalar quantity and does not have any particular direction; that is, pressure acts equally in all possible directions. When you submerge your hand in water, you sense the same amount of pressure acting on the top surface of your hand as on the bottom surface, or on the side surface, or on the surface of the skin between your fingers. What you are perceiving in this case is an increase in pressure Δp over what you are used to feeling when your hand is not submerged in water. What you feel when your hand is not submerged in the water is the **normal pressure** p_0 of one atmosphere, which serves as a reference point. The bulk stress is this increase in pressure, or Δp, over the normal level, p_0.

When the bulk stress increases, the bulk strain increases in response, in accordance with Equation 12.4.4. The proportionality constant in this relation is called the bulk modulus, B, or

$$B = \frac{bulk\ stress}{bulk\ strain} = \frac{\Delta p}{\frac{\Delta V}{V_0}} = -\Delta p \frac{V_0}{\Delta V}. \tag{10.5.3}$$

The minus sign that appears in Equation 10.5.3 is for consistency, to ensure that B is a positive quantity. Note that the minus sign (–) is necessary because an increase Δp in pressure (a positive quantity) always causes a decrease ΔV in volume, and decrease in volume is a negative quantity. The reciprocal of the bulk modulus is called **compressibility** k, or

$$k = \frac{1}{B} = -\frac{\frac{\Delta V}{V_0}}{\Delta p}. \tag{10.5.4}$$

The term 'compressibility' is used in relation to fluids (gases and liquids). Compressibility describes the change in the volume of a fluid per unit increase in pressure. Fluids characterized by a large compressibility are relatively easy to compress. For example, the compressibility of water is 4.64×10^{-5} /atm and the compressibility of acetone is 1.45×10^{-4} /atm. This means that under a 1.0-atm increase in pressure, the relative decrease in volume is approximately three times as large for acetone as it is for water.

Example 10.5.1: Hydraulic Press

In a hydraulic press Figure $10.5.2$, a 250-liter volume of oil is subjected to a 2300-psi pressure increase. If the compressibility of oil is 2.0 x 10^{-5} / atm, find the bulk strain and the absolute decrease in the volume of oil when the press is operating.

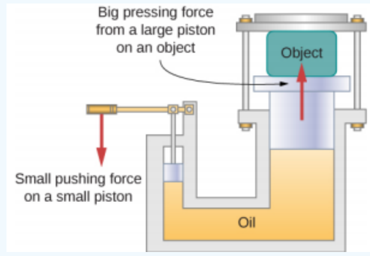


Figure 10.5.2: In a hydraulic press, when a small piston is displaced downward, the pressure in the oil is transmitted throughout the oil to the large piston, causing the large piston to move upward. A small force applied to a small piston causes a large pressing force, which the large piston exerts on an object that is either lifted or squeezed. The device acts as a mechanical lever.

Strategy

We must invert Equation 10.5.4 to find the bulk strain. First, we convert the pressure increase from psi to atm, Δp = 2300 psi = $\frac{2300}{14.7\ atm}$ ≈ 160 atm, and identify V_0 = 250 L.

Solution

Substituting values into the equation, we have

$$bulk\ strain = \frac{\Delta V}{V_0} = \frac{\Delta p}{B} = k\Delta p = (2.0 \times 10^{-5}\ /atm)(160\ atm) = 0.0032 \tag{10.5.5}$$

answer

$$\Delta V = 0.0032 V_0 = 0.0032(250\ L) = 0.78\ L. \tag{10.5.6}$$

Significance

Notice that since the compressibility of water is 2.32 times larger than that of oil, if the working substance in the hydraulic press of this problem were changed to water, the bulk strain as well as the volume change would be 2.32 times larger.

Exercise 10.5.1

If the normal force acting on each face of a cubical 1.0-m^3 piece of steel is changed by 1.0 x 10^7 N, find the resulting change in the volume of the piece of steel.

Shear Stress, Strain, and Modulus

The concepts of shear stress and strain concern only solid objects or materials. Buildings and tectonic plates are examples of objects that may be subjected to shear stresses. In general, these concepts do not apply to fluids.

Shear deformation occurs when two antiparallel forces of equal magnitude are applied tangentially to opposite surfaces of a solid object, causing no deformation in the transverse direction to the line of force, as in the typical example of shear stress illustrated in Figure $10.5.3$. Shear deformation is characterized by a gradual shift Δx of layers in the direction tangent to the acting forces. This gradation in Δx occurs in the transverse direction along some distance L_0. Shear strain is defined by the ratio of the largest displacement Δx to the transverse distance L_0

$$shear\ strain = \frac{\Delta x}{L_0}. \tag{10.5.7}$$

Shear strain is caused by shear stress. Shear stress is due to forces that act **parallel** to the surface. We use the symbol $F_{\parallel}$ for such forces. The magnitude $F_{\parallel}$ per surface area A where shearing force is applied is the measure of shear stress

$$shear\ stress = \frac{F_{\parallel}}{A}. \tag{10.5.8}$$

The shear modulus is the proportionality constant in Equation ??? and is defined by the ratio of stress to strain. Shear modulus is commonly denoted by S:

$$S=\frac{shear\ stress}{shear\ strain}=\frac{\frac{F_{\parallel}}{A}}{\frac{\Delta x}{L_0}}=\frac{F_{\parallel}}{A}\frac{L_0}{\Delta x}. \tag{10.5.9}$$

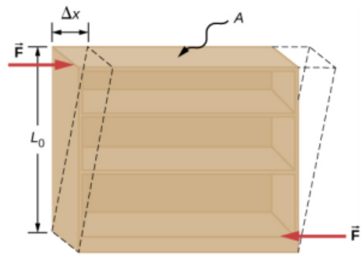


Figure 10.5.3: An object under shear stress: Two antiparallel forces of equal magnitude are applied tangentially to opposite parallel surfaces of the object. The dashed-line contour depicts the resulting deformation. There is no change in the direction transverse to the acting forces and the transverse length L_0 is unaffected. Shear deformation is characterized by a gradual shift Δx of layers in the direction tangent to the forces.

Example 10.5.2: An Old Bookshelf

A cleaning person tries to move a heavy, old bookcase on a carpeted floor by pushing tangentially on the surface of the very top shelf. However, the only noticeable effect of this effort is similar to that seen in Figure 10.5.2, and it disappears when the person stops pushing. The bookcase is 180.0 cm tall and 90.0 cm wide with four 30.0-cm-deep shelves, all partially loaded with books. The total weight of the bookcase and books is 600.0 N. If the person gives the top shelf a 50.0-N push that displaces the top shelf horizontally by 15.0 cm relative to the motionless bottom shelf, find the shear modulus of the bookcase.

Strategy

The only pieces of relevant information are the physical dimensions of the bookcase, the value of the tangential force, and the displacement this force causes. We identify $F_{\parallel}$ = 50.0 N, Δx = 15.0 cm, L_0 = 180.0 cm, and A = (30.0 cm)(90.0 cm) = 2700.0 cm^2, and we use Equation 10.5.9 to compute the shear modulus.

Solution

Substituting numbers into the equations, we obtain for the shear modulus

$$S=\frac{F_{\parallel}}{A}\frac{L_0}{\Delta x}=\frac{50.0\ N}{2700.0\ cm^2}\frac{180.0\ cm}{15.0\ cm}=\frac{2}{9}\frac{M}{cm^2}=\frac{2}{9}\times 10^4\ N/m^2=\frac{20}{9}\times 10^3\ Pa=2.222\ kPa.$$

We can also find shear stress and strain, respectively:

$$\frac{F_{\parallel}}{A}=\frac{50.0\ N}{2700.0\ cm^2}=\frac{5}{27}\ kPa=185.2\ Pa$$

$$\frac{\Delta x}{L_0}=\frac{15.0\ cm}{180.0\ cm}=\frac{1}{12}=0.083.$$

Significance

If the person in this example gave the shelf a healthy push, it might happen that the induced shear would collapse it to a pile of rubbish. Much the same shear mechanism is responsible for failures of earth-filled dams and levees; and, in general, for landslides.

Exercise 10.5.2

Explain why the concepts of Young's modulus and shear modulus do not apply to fluids.

- **12.5: Stress, Strain, and Elastic Modulus (Part 2)** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/university-physics-volume-1.

10.6: Elasticity and Plasticity

> **Learning Objectives**
>
> - Explain the limit where a deformation of material is elastic
> - Describe the range where materials show plastic behavior
> - Analyze elasticity and plasticity on a stress-strain diagram

We referred to the proportionality constant between stress and strain as the **elastic modulus**. But why do we call it that? What does it mean for an object to be elastic and how do we describe its behavior?

Elasticity is the tendency of solid objects and materials to return to their original shape after the external forces (load) causing a deformation are removed. An object is **elastic** when it comes back to its original size and shape when the load is no longer present. Physical reasons for elastic behavior vary among materials and depend on the microscopic structure of the material. For example, the elasticity of polymers and rubbers is caused by stretching polymer chains under an applied force. In contrast, the elasticity of metals is caused by resizing and reshaping the crystalline cells of the lattices (which are the material structures of metals) under the action of externally applied forces.

The two parameters that determine the elasticity of a material are its **elastic modulus** and its **elastic limit**. A high elastic modulus is typical for materials that are hard to deform; in other words, materials that require a high load to achieve a significant strain. An example is a steel band. A low elastic modulus is typical for materials that are easily deformed under a load; for example, a rubber band. If the stress under a load becomes too high, then when the load is removed, the material no longer comes back to its original shape and size, but relaxes to a different shape and size: The material becomes permanently deformed. The **elastic limit** is the stress value beyond which the material no longer behaves elastically but becomes permanently deformed.

Our perception of an elastic material depends on both its elastic limit and its elastic modulus. For example, all rubbers are characterized by a low elastic modulus and a high elastic limit; hence, it is easy to stretch them and the stretch is noticeably large. Among materials with identical elastic limits, the most elastic is the one with the lowest elastic modulus.

When the load increases from zero, the resulting stress is in direct proportion to strain in the way given by Equation 12.4.4, but only when stress does not exceed some limiting value. For stress values within this linear limit, we can describe elastic behavior in analogy with Hooke's law for a spring. According to Hooke's law, the stretch value of a spring under an applied force is directly proportional to the magnitude of the force. Conversely, the response force from the spring to an applied stretch is directly proportional to the stretch. In the same way, the deformation of a material under a load is directly proportional to the load, and, conversely, the resulting stress is directly proportional to strain. The linearity limit (or the **proportionality limit**) is the largest stress value beyond which stress is no longer proportional to strain. Beyond the linearity limit, the relation between stress and strain is no longer linear. When stress becomes larger than the linearity limit but still within the elasticity limit, behavior is still elastic, but the relation between stress and strain becomes nonlinear.

For stresses beyond the elastic limit, a material exhibits **plastic behavior**. This means the material deforms irreversibly and does not return to its original shape and size, even when the load is removed. When stress is gradually increased beyond the elastic limit, the material undergoes plastic deformation. Rubber-like materials show an increase in stress with the increasing strain, which means they become more difficult to stretch and, eventually, they reach a fracture point where they break. Ductile materials such as metals show a gradual decrease in stress with the increasing strain, which means they become easier to deform as stress-strain values approach the breaking point. Microscopic mechanisms responsible for plasticity of materials are different for different materials.

We can graph the relationship between stress and strain on a **stress-strain diagram**. Each material has its own characteristic strain-stress curve. A typical stress-strain diagram for a ductile metal under a load is shown in Figure $10.6.1$. In this figure, strain is a fractional elongation (not drawn to scale). When the load is gradually increased, the linear behavior (red line) that starts at the no-load point (the origin) ends at the linearity limit at point H. For further load increases beyond point H, the stress-strain relation is nonlinear but still elastic. In the figure, this nonlinear region is seen between points H and E. Ever larger loads take the stress to the elasticity limit E, where elastic behavior ends and plastic deformation begins. Beyond the elasticity limit, when the load is removed, for example at P, the material relaxes to a new shape and size along the green line. This is to say that the material becomes permanently deformed and does not come back to its initial shape and size when stress becomes zero.

The material undergoes plastic deformation for loads large enough to cause stress to go beyond the elasticity limit at E. The material continues to be plastically deformed until the stress reaches the fracture point (breaking point). Beyond the fracture point, we no longer have one sample of material, so the diagram ends at the fracture point. For the completeness of this qualitative description, it should be said that the linear, elastic, and plasticity limits denote a range of values rather than one sharp point.

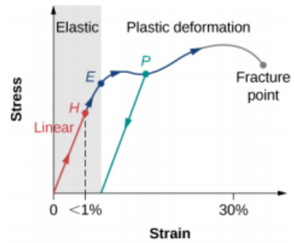


Figure 10.6.1: Typical stress-strain plot for a metal under a load: The graph ends at the fracture point. The arrows show the direction of changes under an ever-increasing load. Points H and E are the linearity and elasticity limits, respectively. Between points H and E, the behavior is nonlinear. The green line originating at P illustrates the metal’s response when the load is removed. The permanent deformation has a strain value at the point where the green line intercepts the horizontal axis.

The value of stress at the fracture point is called **breaking stress** (or **ultimate stress**). Materials with similar elastic properties, such as two metals, may have very different breaking stresses. For example, ultimate stress for aluminum is 2.2 x 10^8 Pa and for steel it may be as high as 20.0 x 10^8 Pa, depending on the kind of steel. We can make a quick estimate, based on Equation 12.4.5, that for rods with a 1-in^2 cross-sectional area, the breaking load for an aluminum rod is 3.2 x 10^4 lb, and the breaking load for a steel rod is about nine times larger.

10.7: Static Equilibrium and Elasticity (Exercises)

Conceptual Questions

12.1 Conditions for Static Equilibrium

1. What can you say about the velocity of a moving body that is in dynamic equilibrium?
2. Under what conditions can a rotating body be in equilibrium? Give an example.
3. What three factors affect the torque created by a force relative to a specific pivot point?
4. Mechanics sometimes put a length of pipe over the handle of a wrench when trying to remove a very tight bolt. How does this help? For the next four problems, evaluate the statement as either true or false and explain your answer.
5. If there is only one external force (or torque) acting on an object, it cannot be in equilibrium.
6. If an object is in equilibrium there must be an even number of forces acting on it.
7. If an odd number of forces act on an object, the object cannot be in equilibrium.
8. A body moving in a circle with a constant speed is in rotational equilibrium.
9. What purpose is served by a long and flexible pole carried by wire-walkers?

12.2 Examples of Static Equilibrium

10. Is it possible to rest a ladder against a rough wall when the floor is frictionless?
11. Show how a spring scale and a simple fulcrum can be used to weigh an object whose weight is larger than the maximum reading on the scale.
12. A painter climbs a ladder. Is the ladder more likely to slip when the painter is near the bottom or near the top?

12.3 Stress, Strain, and Elastic Modulus

Note: Unless stated otherwise, the weights of the wires, rods, and other elements are assumed to be negligible. Elastic moduli of selected materials are given in Table 12.1.

13. Why can a squirrel jump from a tree branch to the ground and run away undamaged, while a human could break a bone in such a fall?
14. When a glass bottle full of vinegar warms up, both the vinegar and the glass expand, but the vinegar expands significantly more with temperature than does the glass. The bottle will break if it is filled up to its very tight cap. Explain why and how a pocket of air above the vinegar prevents the bottle from breaking.
15. A thin wire strung between two nails in the wall is used to support a large picture. Is the wire likely to snap if it is strung tightly or if it is strung so that it sags considerably?
16. Review the relationship between stress and strain. Can you find any similarities between the two quantities?
17. What type of stress are you applying when you press on the ends of a wooden rod? When you pull on its ends?
18. Can compressive stress be applied to a rubber band?
19. Can Young's modulus have a negative value? What about the bulk modulus?
20. If a hypothetical material has a negative bulk modulus, what happens when you squeeze a piece of it?
21. Discuss how you might measure the bulk modulus of a liquid.

12.4 Elasticity and Plasticity

Note: Unless stated otherwise, the weights of the wires, rods, and other elements are assumed to be negligible. Elastic moduli of selected materials are given in Table 12.1.

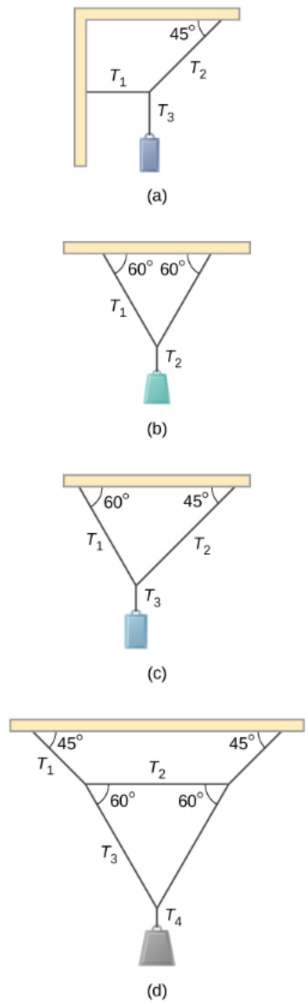
22. What is meant when a fishing line is designated as "a 10-lb test?"
23. Steel rods are commonly placed in concrete before it sets. What is the purpose of these rods?

Problems

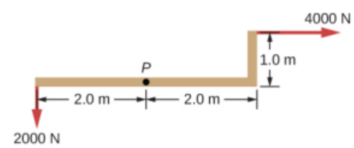
12.1 Conditions for Static Equilibrium

24. When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. How much torque are you exerting relative to the center of the bolt?
25. When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850 m from the hinges. What torque are you exerting relative to the hinges?

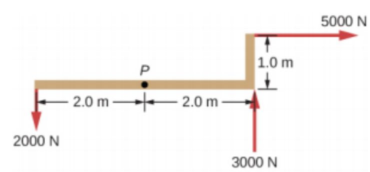
26. Find the magnitude of the tension in each supporting cable shown below. In each case, the weight of the suspended body is 100.0 N and the masses of the cables are negligible.



27. What force must be applied at point P to keep the structure shown in equilibrium? The weight of the structure is negligible.

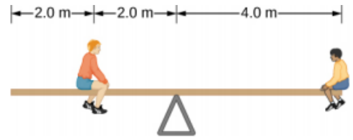


28. Is it possible to apply a force at P to keep in equilibrium the structure shown? The weight of the structure is negligible.



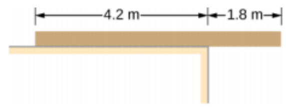
29. Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.
30. A small 1000-kg SUV has a wheel base of 3.0 m. If 60% if its weight rests on the front wheels, how far behind the front wheels is the wagon's center of mass?

31. The uniform seesaw is balanced at its center of mass, as seen below. The smaller boy on the right has a mass of 40.0 kg. What is the mass of his friend?

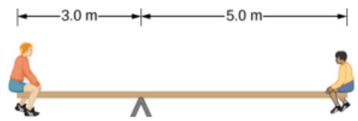


12.2 Examples of Static Equilibrium

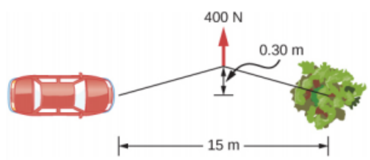
32. A uniform plank rests on a level surface as shown below. The plank has a mass of 30 kg and is 6.0 m long. How much mass can be placed at its right end before it tips? (**Hint**: When the board is about to tip over, it makes contact with the surface only along the edge that becomes a momentary axis of rotation.)



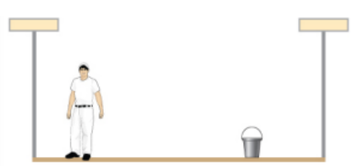
33. The uniform seesaw shown below is balanced on a fulcrum located 3.0 m from the left end. The smaller boy on the right has a mass of 40 kg and the bigger boy on the left has a mass 80 kg. What is the mass of the board?



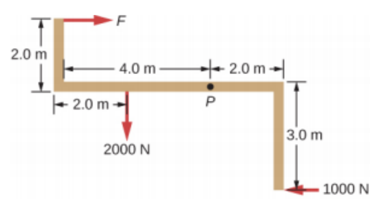
34. In order to get his car out of the mud, a man ties one end of a rope to the front bumper and the other end to a tree 15 m away, as shown below. He then pulls on the center of the rope with a force of 400 N, which causes its center to be displaced 0.30 m, as shown. What is the force of the rope on the car?



35. A uniform 40.0-kg scaffold of length 6.0 m is supported by two light cables, as shown below. An 80.0-kg painter stands 1.0 m from the left end of the scaffold, and his painting equipment is 1.5 m from the right end. If the tension in the left cable is twice that in the right cable, find the tensions in the cables and the mass of the equipment.

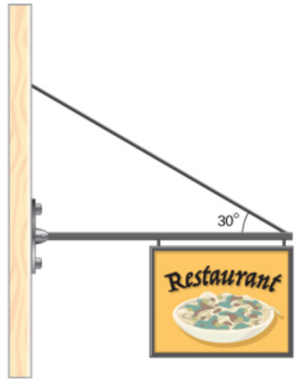


36. When the structure shown below is supported at point P, it is in equilibrium. Find the magnitude of force F and the force applied at P. The weight of the structure is negligible.

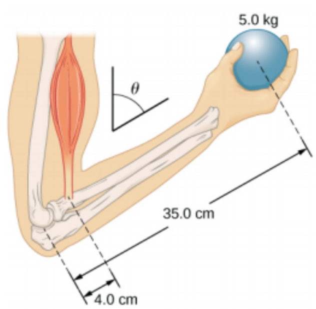


37. To get up on the roof, a person (mass 70.0 kg) places a 6.00-m aluminum ladder (mass 10.0 kg) against the house on a concrete pad with the base of the ladder 2.00 m from the house. The ladder rests against a plastic rain gutter, which we can assume to be frictionless. The center of mass of the ladder is 2.00 m from the bottom. The person is standing 3.00 m from the bottom. Find the normal reaction and friction forces on the ladder at its base.

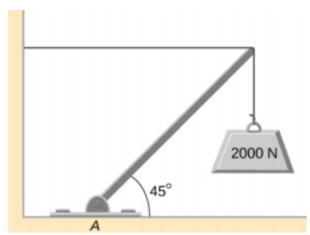
38. A uniform horizontal strut weighs 400.0 N. One end of the strut is attached to a hinged support at the wall, and the other end of the strut is attached to a sign that weighs 200.0 N. The strut is also supported by a cable attached between the end of the strut and the wall. Assuming that the entire weight of the sign is attached at the very end of the strut, find the tension in the cable and the force at the hinge of the strut.



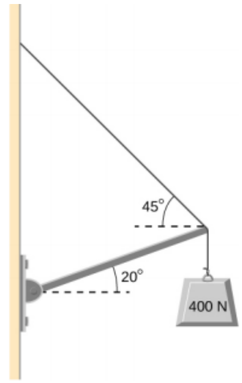
39. The forearm shown below is positioned at an angle θ with respect to the upper arm, and a 5.0-kg mass is held in the hand. The total mass of the forearm and hand is 3.0 kg, and their center of mass is 15.0 cm from the elbow. (a) What is the magnitude of the force that the biceps muscle exerts on the forearm for $\theta = 60°$? (b) What is the magnitude of the force on the elbow joint for the same angle? (c) How do these forces depend on the angle θ?



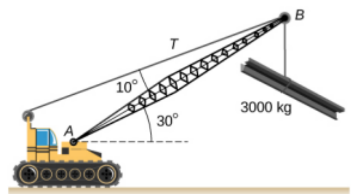
40. The uniform boom shown below weighs 3000 N. It is supported by the horizontal guy wire and by the hinged support at point A. What are the forces on the boom due to the wire and due to the support at A? Does the force at A act along the boom?



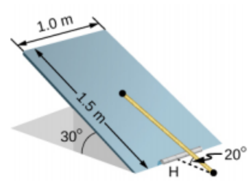
41. The uniform boom shown below weighs 700 N, and the object hanging from its right end weighs 400 N. The boom is supported by a light cable and by a hinge at the wall. Calculate the tension in the cable and the force on the hinge on the boom. Does the force on the hinge act along the boom?



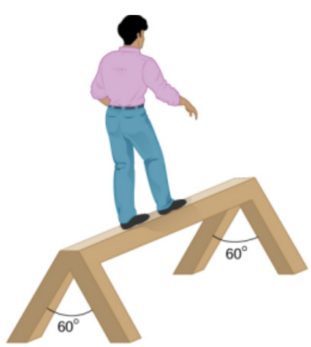
42. A 12.0-m boom, AB, of a crane lifting a 3000-kg load is shown below. The center of mass of the boom is at its geometric center, and the mass of the boom is 1000 kg. For the position shown, calculate tension T in the cable and the force at the axle A.



43. A uniform trapdoor shown below is 1.0 m by 1.5 m and weighs 300 N. It is supported by a single hinge (H), and by a light rope tied between the middle of the door and the floor. The door is held at the position shown, where its slab makes a 30° angle with the horizontal floor and the rope makes a 20° angle with the floor. Find the tension in the rope and the force at the hinge.



44. A 90-kg man walks on a sawhorse, as shown below. The sawhorse is 2.0 m long and 1.0 m high, and its mass is 25.0 kg. Calculate the normal reaction force on each leg at the contact point with the floor when the man is 0.5 m from the far end of the sawhorse. (**Hint**: At each end, find the total reaction force first. This reaction force is the vector sum of two reaction forces, each acting along one leg. The normal reaction force at the contact point with the floor is the normal (with respect to the floor) component of this force.)



12.3 Stress, Strain, and Elastic Modulus

45. The "lead" in pencils is a graphite composition with a Young's modulus of approximately $1.0 \times 10^9\ N/m^2$. Calculate the change in length of the lead in an automatic pencil if you tap it straight into the pencil with a force of 4.0 N. The lead is

0.50 mm in diameter and 60 mm long.

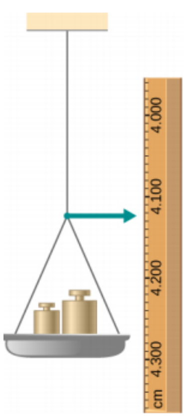
46. TV broadcast antennas are the tallest artificial structures on Earth. In 1987, a 72.0-kg physicist placed himself and 400 kg of equipment at the top of a 610-m-high antenna to perform gravity experiments. By how much was the antenna compressed, if we consider it to be equivalent to a steel cylinder 0.150 m in radius?
47. By how much does a 65.0-kg mountain climber stretch her 0.800-cm diameter nylon rope when she hangs 35.0 m below a rock outcropping? (For nylon, Y = 1.35×10^9 Pa.)
48. When water freezes, its volume increases by 9.05%. What force per unit area is water capable of exerting on a container when it freezes?
49. A farmer making grape juice fills a glass bottle to the brim and caps it tightly. The juice expands more than the glass when it warms up, in such a way that the volume increases by 0.2%. Calculate the force exerted by the juice per square centimeter if its bulk modulus is 1.8×10^9 N /m^2, assuming the bottle does not break.
50. A disk between vertebrae in the spine is subjected to a shearing force of 600.0 N. Find its shear deformation, using the shear modulus of 1.0×10^9 N/m^2. The disk is equivalent to a solid cylinder 0.700 cm high and 4.00 cm in diameter.
51. A vertebra is subjected to a shearing force of 500.0 N. Find the shear deformation, taking the vertebra to be a cylinder 3.00 cm high and 4.00 cm in diameter. How does your result compare with the result obtained in the preceding problem? Are spinal problems more common in disks than in vertebrae?
52. Calculate the force a piano tuner applies to stretch a steel piano wire by 8.00 mm, if the wire is originally 1.35 m long and its diameter is 0.850 mm.
53. A 20.0-m-tall hollow aluminum flagpole is equivalent in strength to a solid cylinder 4.00 cm in diameter. A strong wind bends the pole as much as a horizontal 900.0-N force on the top would do. How far to the side does the top of the pole flex?
54. A copper wire of diameter 1.0 cm stretches 1.0% when it is used to lift a load upward with an acceleration of 2.0 m/s^2. What is the weight of the load?
55. As an oil well is drilled, each new section of drill pipe supports its own weight and the weight of the pipe and the drill bit beneath it. Calculate the stretch in a new 6.00-m-long steel pipe that supports a 100-kg drill bit and a 3.00-km length of pipe with a linear mass density of 20.0 kg/m. Treat the pipe as a solid cylinder with a 5.00-cm diameter.
56. A large uniform cylindrical steel rod of density $\rho = 7.8$ g/cm^3 is 2.0 m long and has a diameter of 5.0 cm. The rod is fastened to a concrete floor with its long axis vertical. What is the normal stress in the rod at the cross-section located at (a) 1.0 m from its lower end? (b) 1.5 m from the lower end?
57. A 90-kg mountain climber hangs from a nylon rope and stretches it by 25.0 cm. If the rope was originally 30.0 m long and its diameter is 1.0 cm, what is Young’s modulus for the nylon?
58. A suspender rod of a suspension bridge is 25.0 m long. If the rod is made of steel, what must its diameter be so that it does not stretch more than 1.0 cm when a 2.5×10^4 -kg truck passes by it? Assume that the rod supports all of the weight of the truck.
59. A copper wire is 1.0 m long and its diameter is 1.0 mm. If the wire hangs vertically, how much weight must be added to its free end in order to stretch it 3.0 mm?
60. A 100-N weight is attached to a free end of a metallic wire that hangs from the ceiling. When a second 100-N weight is added to the wire, it stretches 3.0 mm. The diameter and the length of the wire are 1.0 mm and 2.0 m, respectively. What is Young’s modulus of the metal used to manufacture the wire?
61. The bulk modulus of a material is 1.0×10^{11} N/m^2. What fractional change in volume does a piece of this material undergo when it is subjected to a bulk stress increase of 10^7 N/m^2? Assume that the force is applied uniformly over the surface.
62. Normal forces of magnitude 1.0×10^6 N are applied uniformly to a spherical surface enclosing a volume of a liquid. This causes the radius of the surface to decrease from 50.000 cm to 49.995 cm. What is the bulk modulus of the liquid?
63. During a walk on a rope, a tightrope walker creates a tension of 3.94×10^3 N in a wire that is stretched between two supporting poles that are 15.0 m apart. The wire has a diameter of 0.50 cm when it is not stretched. When the walker is on the wire in the middle between the poles the wire makes an angle of 5.0° below the horizontal. How much does this tension stretch the steel wire when the walker is this position?
64. When using a pencil eraser, you exert a vertical force of 6.00 N at a distance of 2.00 cm from the hardwood-eraser joint. The pencil is 6.00 mm in diameter and is held at an angle of 20.0° to the horizontal. (a) By how much does the wood flex perpendicular to its length? (b) How much is it compressed lengthwise?

65. Normal forces are applied uniformly over the surface of a spherical volume of water whose radius is 20.0 cm. If the pressure on the surface is increased by 200 MPa, by how much does the radius of the sphere decrease?

12.4 Elasticity and Plasticity

66. A uniform rope of cross-sectional area 0.50 cm^2 breaks when the tensile stress in it reaches 6.00 x 10^6 N/m^2. (a) What is the maximum load that can be lifted slowly at a constant speed by the rope? (b) What is the maximum load that can be lifted by the rope with an acceleration of 4.00 m/s^2?
67. One end of a vertical metallic wire of length 2.0 m and diameter 1.0 mm is attached to a ceiling, and the other end is attached to a 5.0-N weight pan, as shown below. The position of the pointer before the pan is 4.000 cm. Different weights are then added to the pan area, and the position of the pointer is recorded in the table shown. Plot stress versus strain for this wire, then use the resulting curve to determine Young's modulus and the proportionality limit of the metal. What metal is this most likely to be?

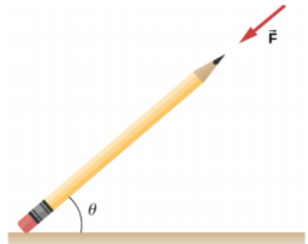
Added load (including pan) (N)	Scale reading (cm)
0	4.000
15	4.036
25	4.073
35	4.109
45	4.146
55	4.181
65	4.221
75	4.266
85	4.316



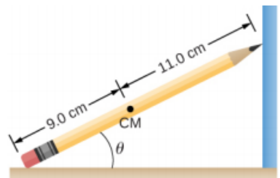
68. An aluminum (ρ = 2.7 g/cm^3) wire is suspended from the ceiling and hangs vertically. How long must the wire be before the stress at its upper end reaches the proportionality limit, which is 8.0 x 10^7 N/m^2?

Additional Problems

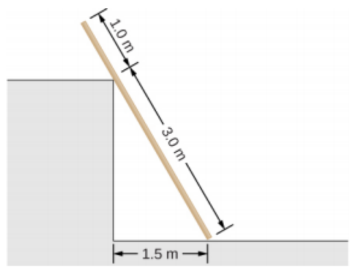
69. The coefficient of static friction between the rubber eraser of the pencil and the tabletop is μ_s = 0.80. If the force $\vec{F}$ is applied along the axis of the pencil, as shown below, what is the minimum angle at which the pencil can stand without slipping? Ignore the weight of the pencil.



70. A pencil rests against a corner, as shown below. The sharpened end of the pencil touches a smooth vertical surface and the eraser end touches a rough horizontal floor. The coefficient of static friction between the eraser and the floor is μ_s = 0.80. The center of mass of the pencil is located 9.0 cm from the tip of the eraser and 11.0 cm from the tip of the pencil lead. Find the minimum angle θ for which the pencil does not slip.



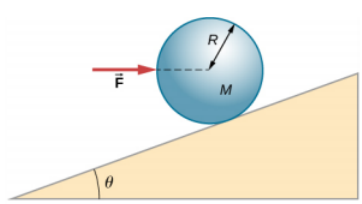
71. A uniform 4.0-m plank weighing 200.0 N rests against the corner of a wall, as shown below. There is no friction at the point where the plank meets the corner. (a) Find the forces that the corner and the floor exert on the plank. (b) What is the minimum coefficient of static friction between the floor and the plank to prevent the plank from slipping?



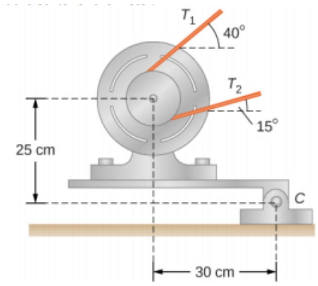
72. A 40-kg boy jumps from a height of 3.0 m, lands on one foot and comes to rest in 0.10 s after he hits the ground. Assume that he comes to rest with a constant deceleration. If the total cross-sectional area of the bones in his legs just above his ankles is 3.0 cm^2, what is the compression stress in these bones? Leg bones can be fractured when they are subjected to stress greater than 1.7 x 10^8 Pa. Is the boy in danger of breaking his leg?
73. Two thin rods, one made of steel and the other of aluminum, are joined end to end. Each rod is 2.0 m long and has cross-sectional area 9.1 mm^2. If a 10,000-N tensile force is applied at each end of the combination, find: (a) stress in each rod; (b) strain in each rod; and, (c) elongation of each rod.
74. Two rods, one made of copper and the other of steel, have the same dimensions. If the copper rod stretches by 0.15 mm under some stress, how much does the steel rod stretch under the same stress?

Challenge Problems

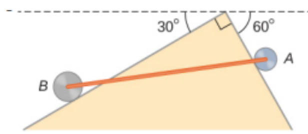
75. A horizontal force $\vec{F}$ is applied to a uniform sphere in direction exact toward the center of the sphere, as shown below. Find the magnitude of this force so that the sphere remains in static equilibrium. What is the frictional force of the incline on the sphere?



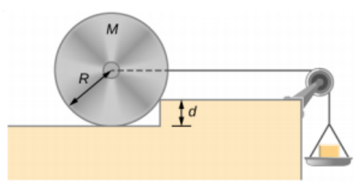
76. When a motor is set on a pivoted mount seen below, its weight can be used to maintain tension in the drive belt. When the motor is not running the tensions T_1 and T_2 are equal. The total mass of the platform and the motor is 100.0 kg, and the diameter of the drive belt pulley is 16.0 cm. when the motor is off, find: (a) the tension in the belt, and (b) the force at the hinged platform support at point C. Assume that the center of mass of the motor plus platform is at the center of the motor.



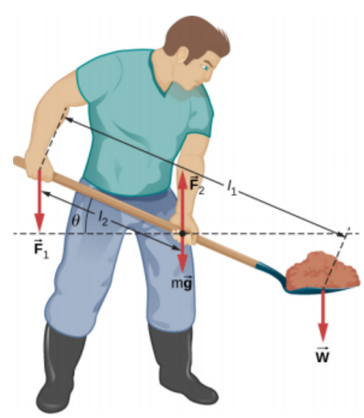
77. Two wheels A and B with weights w and 2w, respectively, are connected by a uniform rod with weight w/2, as shown below. The wheels are free to roll on the sloped surfaces. Determine the angle that the rod forms with the horizontal when the system is in equilibrium. **Hint**: There are five forces acting on the rod, which is two weights of the wheels, two normal reaction forces at points where the wheels make contacts with the wedge, and the weight of the rod.



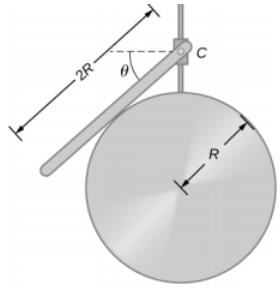
78. Weights are gradually added to a pan until a wheel of mass M and radius R is pulled over an obstacle of height d, as shown below. What is the minimum mass of the weights plus the pan needed to accomplish this?



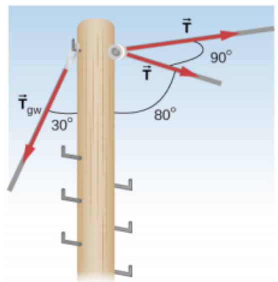
79. In order to lift a shovelful of dirt, a gardener pushes downward on the end of the shovel and pulls upward at distance l_2 from the end, as shown below. The weight of the shovel is $m\vec{g}$ and acts at the point of application of $\vec{F}_2$. Calculate the magnitudes of the forces $\vec{F}_1$ and $\vec{F}_2$ as functions of l_1, l_2, mg, and the weight W of the load. Why do your answers not depend on the angle $theta$ that the shovel makes with the horizontal?



80. A uniform rod of length 2R and mass M is attached to a small collar C and rests on a cylindrical surface of radius R, as shown below. If the collar can slide without friction along the vertical guide, find the angle θ for which the rod is in static equilibrium.



81. The pole shown below is at a 90.0° bend in a power line and is therefore subjected to more shear force than poles in straight parts of the line. The tension in each line is 4.00 x 10^4 N, at the angles shown. The pole is 15.0 m tall, has an 18.0 cm diameter, and can be considered to have half the strength of hardwood. (a) Calculate the compression of the pole. (b) Find how much it bends and in what direction. (c) Find the tension in a guy wire used to keep the pole straight if it is attached to the top of the pole at an angle of 30.0° with the vertical. The guy wire is in the opposite direction of the bend.



Contributors and Attributions

10.8: Static Equilibrium and Elasticity (Summary)

Key Terms

breaking stress (ultimate stress)	value of stress at the fracture point
bulk modulus	elastic modulus for the bulk stress
bulk strain (or volume strain)	strain under the bulk stress, given as fractional change in volume
bulk stress (or volume stress)	stress caused by compressive forces, in all directions
center of gravity	point where the weight vector is attached
compressibility	reciprocal of the bulk modulus
compressive strain	strain that occurs when forces are contracting an object, causing its shortening
compressive stress	stress caused by compressive forces, only in one direction
elastic	object that comes back to its original size and shape when the load is no longer present
elastic limit	stress value beyond which material no longer behaves elastically and becomes permanently deformed
elastic modulus	proportionality constant in linear relation between stress and strain, in SI pascals
equilibrium	body is in equilibrium when its linear and angular accelerations are both zero relative to an inertial frame of reference
first equilibrium condition	expresses translational equilibrium; all external forces acting on the body balance out and their vector sum is zero
gravitational torque	torque on the body caused by its weight; it occurs when the center of gravity of the body is not located on the axis of rotation
linearity limit (proportionality limit)	largest stress value beyond which stress is no longer proportional to strain
normal pressure	pressure of one atmosphere, serves as a reference level for pressure
pascal (Pa)	SI unit of stress, SI unit of pressure
plastic behavior	material deforms irreversibly, does not go back to its original shape and size when load is removed and stress vanishes
pressure	force pressing in normal direction on a surface per the surface area, the bulk stress in fluids
second equilibrium condition	expresses rotational equilibrium; all torques due to external forces acting on the body balance out and their vector sum is zero
shear modulus	elastic modulus for shear stress
shear strain	strain caused by shear stress
shear stress	stress caused by shearing forces
static equilibrium	body is in static equilibrium when it is at rest in our selected inertial frame of reference
strain	dimensionless quantity that gives the amount of deformation of an object or medium under stress
stress	quantity that contains information about the magnitude of force causing deformation, defined as force per unit area
stress-strain diagram	graph showing the relationship between stress and strain, characteristic of a material

tensile strain	strain under tensile stress, given as fractional change in length, which occurs when forces are stretching an object, causing its elongation
tensile stress	stress caused by tensile forces, only in one direction, which occurs when forces are stretching an object, causing its elongation
Young's modulus	elastic modulus for tensile or compressive stress

Key Equations

First Equilibrium Condition	$$\sum_k \vec{F}_k = \vec{0} \tag{10.8.1}$$
Second Equilibrium Condition	$$\sum_k \vec{\tau}_k = \vec{0} \tag{10.8.2}$$
Linear relation between stress and strain	$$stress = (elastic\ modulus) \times strain \tag{10.8.3}$$
Young's modulus	$$Y = \frac{tensile\ stress}{tensile\ strain} = \frac{F_\perp}{A}\frac{L_0}{\Delta L} \tag{10.8.4}$$
Bulk modulus	$$B = \frac{bulk\ stress}{bulk\ strain} = -\Delta p \frac{V_0}{\Delta V} \tag{10.8.5}$$
Shear modulus	$$S = \frac{shear\ stress}{shear\ strain} = \frac{F_\parallel}{A}\frac{L_0}{\Delta x} \tag{10.8.6}$$

Summary

12.1 Conditions for Static Equilibrium

- A body is in equilibrium when it remains either in uniform motion (both translational and rotational) or at rest. When a body in a selected inertial frame of reference neither rotates nor moves in translational motion, we say the body is in static equilibrium in this frame of reference.
- Conditions for equilibrium require that the sum of all external forces acting on the body is zero (first condition of equilibrium), and the sum of all external torques from external forces is zero (second condition of equilibrium). These two conditions must be simultaneously satisfied in equilibrium. If one of them is not satisfied, the body is not in equilibrium.
- The free-body diagram for a body is a useful tool that allows us to count correctly all contributions from all external forces and torques acting on the body. Free-body diagrams for the equilibrium of an extended rigid body must indicate a pivot point and lever arms of acting forces with respect to the pivot.

12.2 Examples of Static Equilibrium

- A variety of engineering problems can be solved by applying equilibrium conditions for rigid bodies.
- In applications, identify all forces that act on a rigid body and note their lever arms in rotation about a chosen rotation axis. Construct a free-body diagram for the body. Net external forces and torques can be clearly identified from a correctly constructed free-body diagram. In this way, you can set up the first equilibrium condition for forces and the second equilibrium condition for torques.
- In setting up equilibrium conditions, we are free to adopt any inertial frame of reference and any position of the pivot point. All choices lead to one answer. However, some choices can make the process of finding the solution unduly complicated. We reach the same answer no matter what choices we make. The only way to master this skill is to practice.

12.3 Stress, Strain, and Elastic Modulus

- External forces on an object (or medium) cause its deformation, which is a change in its size and shape. The strength of the forces that cause deformation is expressed by stress, which in SI units is measured in the unit of pressure (pascal). The extent of

deformation under stress is expressed by strain, which is dimensionless.
- For a small stress, the relation between stress and strain is linear. The elastic modulus is the proportionality constant in this linear relation.
- Tensile (or compressive) strain is the response of an object or medium to tensile (or compressive) stress. Here, the elastic modulus is called Young's modulus. Tensile (or compressive) stress causes elongation (or shortening) of the object or medium and is due to an external forces acting along only one direction perpendicular to the cross-section.
- Bulk strain is the response of an object or medium to bulk stress. Here, the elastic modulus is called the bulk modulus. Bulk stress causes a change in the volume of the object or medium and is caused by forces acting on the body from all directions, perpendicular to its surface. Compressibility of an object or medium is the reciprocal of its bulk modulus.
- Shear strain is the deformation of an object or medium under shear stress. The shear modulus is the elastic modulus in this case. Shear stress is caused by forces acting along the object's two parallel surfaces.

12.4 Elasticity and Plasticity

- An object or material is elastic if it comes back to its original shape and size when the stress vanishes. In elastic deformations with stress values lower than the proportionality limit, stress is proportional to strain. When stress goes beyond the proportionality limit, the deformation is still elastic but nonlinear up to the elasticity limit.
- An object or material has plastic behavior when stress is larger than the elastic limit. In the plastic region, the object or material does not come back to its original size or shape when stress vanishes but acquires a permanent deformation. Plastic behavior ends at the breaking point.

Contributors and Attributions

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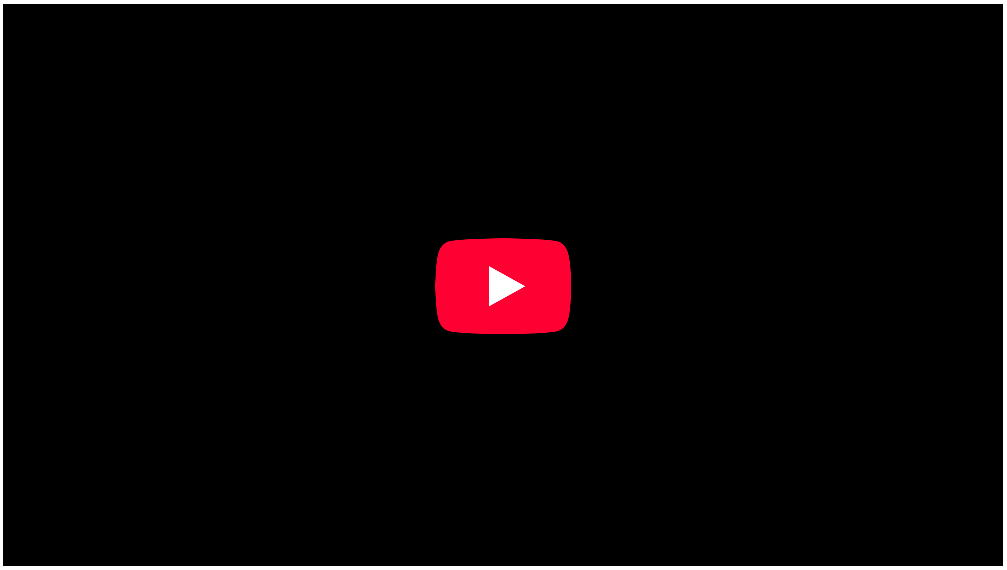
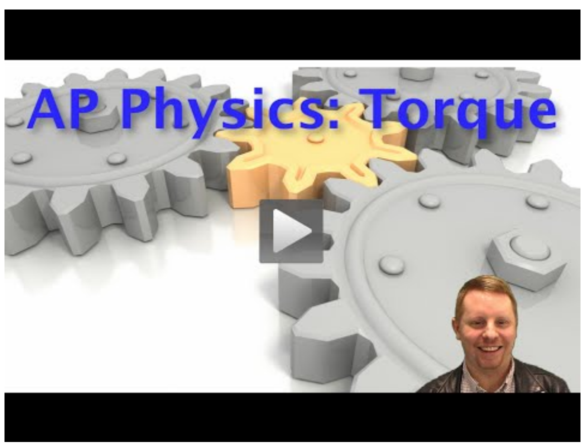
10.9: Introduction

learning objectives

- Describe the effect of the torque on an object

Torque about a point is a concept that denotes the tendency of force to turn or rotate an object in motion. This tendency is measured in general about a point, and is termed as *moment of force*. The torque in angular motion corresponds to force in translation. It is the "cause" whose effect is either angular acceleration or angular deceleration of a particle in general motion. Quantitatively, it is defined as a vector given by:





Torque: A brief introduction to torque for students studying rotational motion in algebra-based physics courses such as AP Physics 1 and Honors Physics.

$$mathrmT = r \times F \tag{10.9.1}$$

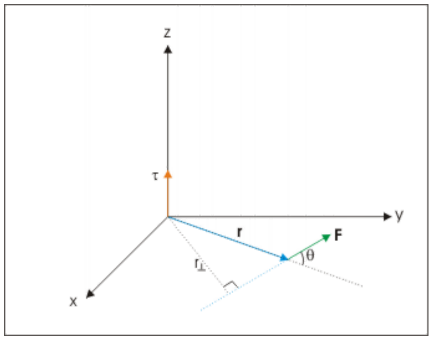
Rotation is a special case of angular motion. In the case of rotation, torque is defined with respect to an axis such that vector "r" is constrained as perpendicular to the axis of rotation. In other words, the plane of motion is perpendicular to the axis of rotation. Clearly, the torque in rotation corresponds to force in translation.

Torque is the cross product of force cross length of the moment arm; it is involved whenever there is a rotating object. Torque can also be expressed in terms of the angular acceleration of the object.

The determination of torque's direction is relatively easier than that of angular velocity. The reason for this is simple: the torque itself is equal to vector product of two vectors, unlike angular velocity which is one of the two operands of the vector product.

Clearly, if we know the directions of two operands here, the direction of torque can easily be interpreted.

Since torque depends on both the force and the distance from the axis of rotation, the SI units of torque are newton-meters.



Torque: Torque in terms of moment arm.

Key Points

- Torque is found by multiplying the applied force by the distance to the axis of rotation, called the moment arm.
- Torque is to rotation as force is to motion.
- The unit of torque is the newton-meter.

Key Terms

- **vector**: A directed quantity, one with both magnitude and direction; the between two points.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.
- **angular motion**: The motion of a body about a fixed point or fixed axis (as of a planet or pendulum). It is equal to the angle passed over at the point or axis by a line drawn to the body.

10.10: Conditions for Equilibrium

learning objectives

- Identify the first condition of equilibrium

First Condition of Equilibrium

For an object to be in equilibrium, it must be experiencing no acceleration. This means that both the net force and the net torque on the object must be zero. Here we will discuss the first condition, that of zero net force.

In the form of an equation, this first condition is:

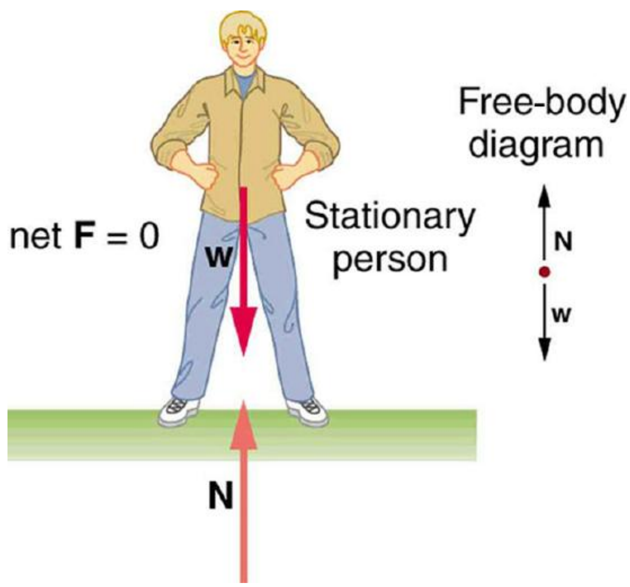
$$\mathrm{F_{net}}=0.$$

In order to achieve this conditon, the forces acting along *each* axis of motion must sum to zero. For example, the net external forces along the typical *x*– and *y*-axes are zero. This is written as

$$\text{net } \mathrm{F_x}=0 \text{ and net } \mathrm{F_y}=0.$$

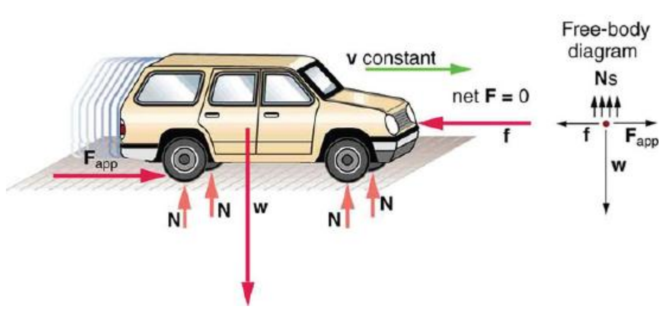
The condition $\mathrm{F_{net}}=0$ must be true for both static equilibrium, where the object's velocity is zero, and dynamic equilibrium, where the object is moving at a constant velocity.

Below, the motionless person is in static equilibrium. The forces acting on him add up to zero. Both forces are vertical in this case.



Person in Static Equilibrium: This motionless person is in static equilibrium.

Below, the car is in dynamic equilibrium because it is moving at constant velocity. There are horizontal and vertical forces, but the net external force in any direction is zero. The applied force between the tires and the road is balanced by air friction, and the weight of the car is supported by the normal forces, here shown to be equal for all four tires.



A Car in Dynamic Equilibrium: This car is in dynamic equilibrium because it is moving at constant velocity. The forces in all directions are balanced.

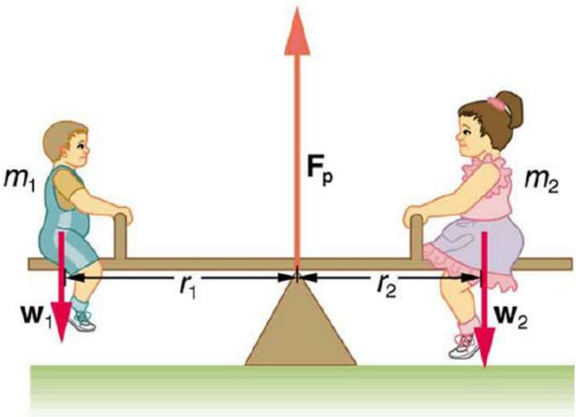
Second Condition

The second condition of static equilibrium says that the net torque acting on the object must be zero.

learning objectives

- Identify the second condition of static equilibrium

A child's seesaw, shown in, is an example of static equilibrium. An object in static equilibrium is one that has no acceleration in any direction. While there might be motion, such motion is constant.



Two children on a seesaw: The system is in static equilibrium, showing no acceleration in any direction.

If a given object is in static equilibrium, both the net force and the net torque on the object must be zero. Let's break this down:

Net Force Must Be Zero

The net force acting on the object must be zero. Therefore all forces balance in each direction. For example, a car moving along a highway at a constant speed is in equilibrium, as it is not accelerating in any forward or vertical direction. Mathematically, this is stated as $\mathrm{F_{net}} = \mathrm{ma} = 0$.

Net Torque Must Be Zero

The second condition necessary to achieve equilibrium involves avoiding accelerated rotation (maintaining a constant angular velocity). A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it.

To understand what factors affect rotation, let us think about what happens when you open an ordinary door by rotating it on its hinges.The magnitude, direction, and point of application of the force are incorporated into the definition of the physical quantity called torque—the rotational equivalent of a force. It is a measure of the effectiveness of a force in changing or accelerating a rotation (changing the angular velocity over a period of time).

In equation form, the magnitude of torque is defined to be $\tau = \mathrm{rF}\sin\theta$ where τ (the Greek letter tau) is the symbol for torque, r is the distance from the pivot point to the point where the force is applied, F is the magnitude of the force, and θ is the angle between the force and the vector directed from the point of application to the pivot point.

Two-Component Forces

In equilibrium, the net force and torque in any particular direction equal zero.

learning objectives

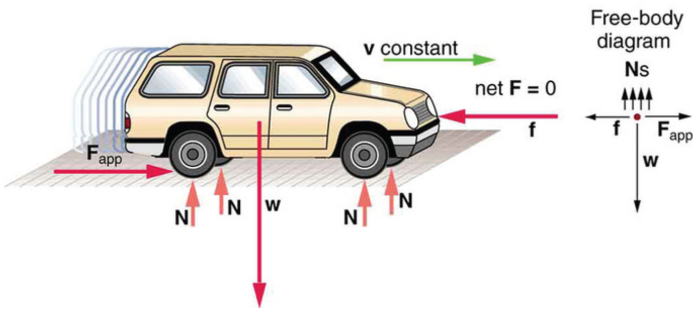
- Calculate the net force and the net torque for an object in equilibrium

An object with constant velocity has zero acceleration. A motionless object still has constant (zero) velocity, so motionless objects also have zero acceleration. Newton's second law states that:

$$\sum \mathrm{F} = \mathrm{ma} \tag{10.10.1}$$

so objects with constant velocity also have zero net external force. This means that all the forces acting on the object are balanced — that is to say, they are in equilibrium.

This rule also applies to motion in a specific direction. Consider an object moving along the *x*-axis. If no net force is applied to the object along the *x*-axis, it will continue to move along the *x*-axis at a constant velocity, with no acceleration.



Car Moving at Constant Velocity: A moving car for which the net x and y force components are zero

We can easily extend this rule to the *y*-axis. In any system, unless the applied forces cancel each other out (i.e., the resultant force is zero), there will be acceleration in the direction of the resultant force. In static systems, in which motion does not occur, the sum of the forces in all directions always equals zero. This concept can be represented mathematically with the following equations:

$$\sum \mathrm{F_x} = \mathrm{ma_x} = 0 \tag{10.10.2}$$

$$\sum \mathrm{F_y} = \mathrm{ma_y} = 0 \tag{10.10.3}$$

This rule also applies to rotational motion. If the resultant moment about a particular axis is zero, the object will have no rotational acceleration about the axis. If the object is not spinning, it will not start to spin. If the object is spinning, it will continue to spin at the same constant angular velocity. Again, we can extend this to moments about the *y*-axis as well. We can represent this rule mathematically with the following equations:

$$\sum \tau_{\mathrm{x}} = \mathrm{I}\alpha_{\mathrm{x}} = 0 \tag{10.10.4}$$

$$\sum \tau_{\mathrm{y}} = \mathrm{I}\alpha_{\mathrm{y}} = 0 \tag{10.10.5}$$

Key Points

- There are two conditions that must be met for an object to be in equilibrium.
- The first condition is that the net force on the object must be zero for the object to be in equilibrium.
- If net force is zero, then net force along any direction is zero.
- The second condition necessary to achieve equilibrium involves avoiding accelerated rotation.
- A rotating body or system can be in equilibrium if its rate of rotation is constant and remains unchanged by the forces acting on it.
- The magnitude of torque about a axis of rotation is defined to be $\tau = \mathrm{rF}\sin\theta$.
- In equilibrium, the net force in all directions is zero.
- If the net moment of inertia about an axis is zero, the object will have no rotational acceleration about the axis.
- In each direction, the net force takes the form: $\sum \mathrm{F} = \mathrm{ma} = 0$ and the net torque take the form: $\sum \tau = \mathrm{I}\alpha = 0$ where the sum represents the vector sum of all forces and torques acting.

Key Terms

- **force**: A physical quantity that denotes ability to push, pull, twist or accelerate a body which is measured in a unit dimensioned in mass × distance/time² (ML/T^2): SI: newton (N); CGS: dyne (dyn)
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **translation**: Motion of a body on a linear path, without deformation or rotation, i.e. such that every part of the body moves at the same speed and in the same direction; also (in physics), the linear motion of a body considered independently of its rotation.
- **equilibrium**: The state of a body at rest or in uniform motion, the resultant of all forces on which is zero.

10.11: Stability

learning objectives

- Explain the relationship between how center of mass is defined and static equilibrium

For an object to be in static equilibrium, we expect it to stay in the same state indefinitely. If it starts accelerating away from its current position, it would hardly be in equilibrium. To quantify equilibrium for a single object, there are two conditions:

1. The net external force on the object is zero: $\sum_i F_i = F_{net} = 0$
2. The net external torque, regardless of choice of origin, is also zero: $\sum_i r_i \times F_i = \sum_i \tau_i = \tau_{net} = 0$

Those two conditions hold regardless of whether the object we are talking about is a single point particle, a rigid body, or a collection of discrete particles. Being in equilibrium means that we expect no changes to the linear momentum or the angular momentum. Note that this does not mean that the system is not moving or rotating; instead it simply means that its movement will not change as time goes on.

In a special case when the external forces are governed by some potential (e.g. gravitational potential) we can gain insight into the nature of the equilibrium. From the definition of a potential we know that $F_{ext} = -\frac{dU(x)}{dx}|_{x_0}$. When the first derivative is zero, we can take the second derivative to find whether the equilibrium is stable or unstable. Explicitly, if the potential is concave-up at x_0, $\frac{d^2U(x)}{dx^2}|_{x_0} > 0$, then the system is stable; conversely, if the potential is concave-down, then the equilibrium is unstable. If the second derivative is zero or does not exist, then the equilibrium is neutral—neither stable nor unstable.

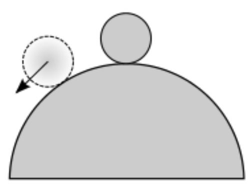
Mathematically, we can view this as a Taylor series expansion on the force slightly away from equilibrium,

$$F(x_0 + \delta x) = F(x_0) + \frac{dF(x)}{dx}|_{x_0} \delta x F(x0 + \delta x) = -\frac{dU(x)}{dx}|_{x_0} + (-\frac{d^2U(x)}{dx^2}|_{x_0})\delta x \quad ,$$

and when it is initially at equilibrium,

$$F(x_0) = 0 F(x_0) = 0 F(x_0 + \delta x) = -\frac{dU(x)}{dx}|_{x_0} + (-\frac{d^2U(x)}{dx^2}|_{x_0})\delta x U(y) = mgy \quad .$$

If the ball is at the top of the hill (where the potential is concave-down) it is possible for it to be perfectly balanced, and therefore at equilibrium. But if it gets pushed just slightly to the side, then it will roll down the hill with increasing speed, and the equilibrium is unstable.



Unstable Equilibrium: A ball on top of a hill can initially be balanced, but if it moves slightly left or right, it gets pushed further and further away from the initial equilibrium position. This is an example of unstable equilibrium.

Our notion of “balance” comes directly from the formulation of equilibrium. For something to be “balanced” means that the net external forces are zero. For example, a coin could balance standing up on a table. Initially the coin will feel no net external force or torque; it is in equilibrium. But if pushed slightly to the side, it will become “off-balance,” experiencing both a force and a torque causing it to fall to the table. It might have been initially “balanced” and at equilibrium, but it was an unstable equilibrium, prone to being disturbed. But why all this talk of external forces, with no mention of internal forces? The reason is that all the internal forces must sum to zero. This follows directly from Newton’s Third Law, $F_{12} = -F_{21}$. Every time we consider a force from particle 1 on particle 2 inside of a system, we know that it will later be cancelled out by the corresponding force from particle 2 on particle 1. We could include those forces in the sum, but it is unnecessary and internal forces are often more complicated than internal forces.

This differentiation between internal and external forces is a powerful one. It also implies that you can trace the motion of the system as a whole (ignoring motion inside the system) through the net external force acting on a center of mass. A center of mass acts as if it has the entire mass of the system, located at one point, and only feels external forces. Its position is defined as the weighted average of all the particles in the system: $\frac{R=\sum_i m_i r_i}{\sum_i m_i}$ or if we have a continuous density of mass, $\rho(r)$, then we can integrate: $R=\frac{\int V \rho(r) r dV}{\int V \rho(r) dV}$. The power of the center of mass is that it hides all the details of what is happening internally. We do not always want to lose the information of what is happening internally, but it is a useful tool to remember, when dealing with a number of complicated interactions.

Key Points

- Equilibrium is defined by no net forces or torques.
- Stability of an equilibrium can be determined by the second derivative of the potential.
- Defining a center of mass allows a simple way to study the behavior of a system or object as a whole.
- Stable equilibrium requires a restoring force. This restoring force can be derived by a Taylor expansion of the force, F(x).

Key Terms

- **stable equilibrium**: The response [of a system in static equilibrium] to a small perturbation is forces that tend to restore the equilibrium.
- **center of mass**: The center of mass (COM) is the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero.
- **static equilibrium**: the physical state in which all components of a system are at rest and the net force is equal to zero throughout the system

10.12: Solving Statics Problems

learning objectives

- Formulate and apply six steps to solve static problems

Statics is the study of forces in equilibrium. Recall that Newton's second law states:

$$\sum \mathrm{F} = \mathrm{ma} \tag{10.12.1}$$

Therefore, for all objects moving at constant velocity (including a velocity of 0 — stationary objects), the net external force is zero. There are forces acting, but they are balanced — that is to say, they are "in equilibrium. "

When solving equilibrium problems, it might help to use the following steps:

1. First, ensure that the problem you're solving is in fact a static problem—i.e., that no acceleration (including angular acceleration) is involved Remember: $\sum \mathrm{F} = \mathrm{ma} = 0$ for these situations. If rotational motion is involved, the condition $\sum \tau = \mathrm{I}\alpha = 0$ must also be satisfied, where is torque, is the moment of inertia, and is the angular acceleration.
2. Choose a pivot point. Often this is obvious because the problem involves a hinge or a fixed point. If the choice is not obvious, pick the pivot point as the location at which you have the most unknowns. This simplifies things because forces at the pivot point create no torque because of the cross product: $\tau = \mathrm{rF}$
3. Write an equation for the sum of torques, and then write equations for the sums of forces in the *x* and *y* directions. Set these sums equal to 0. Be careful with your signs.
4. Solve for your unknowns.
5. Insert numbers to find the final answer.
6. Check if the solution is reasonable by examining the magnitude, direction, and units of the answer. The importance of this last step cannot be overstated, although in unfamiliar applications, it can be more difficult to judge reasonableness. However, these judgments become progressively easier with experience.

Key Points

- First, ensure that the problem you're solving is in fact a static problem—i.e., that no acceleration (including angular acceleration) is involved.
- Choose a pivot point — use the location at which you have the most unknowns.
- Write equations for the sums of torques and forces in the *x* and *y* directions.
- Solve the equations for your unknowns algebraically, and insert numbers to find final answers.

Key Terms

- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **moment of inertia**: A measure of a body's resistance to a change in its angular rotation velocity

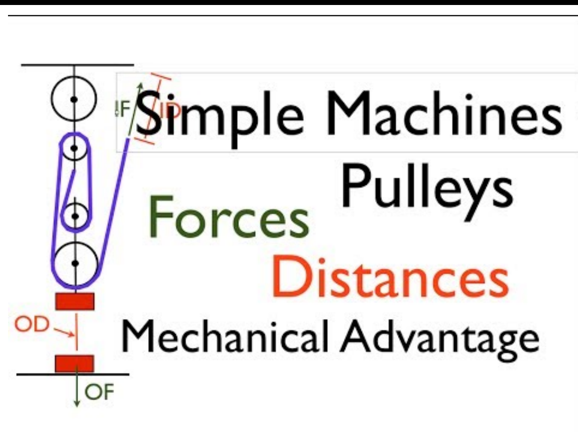
10.13: Applications of Statics

learning objectives

- Develop an understanding of how a machine applies force to work against a load force

Simple Machines

A simple machine is a device that changes the direction or magnitude of a force. They can be described as the simplest mechanisms that use mechanical advantage (or leverage) to multiply force. Usually, the term “simple machine” is referring to one of the six classical simple machines, defined by Renaissance scientists.



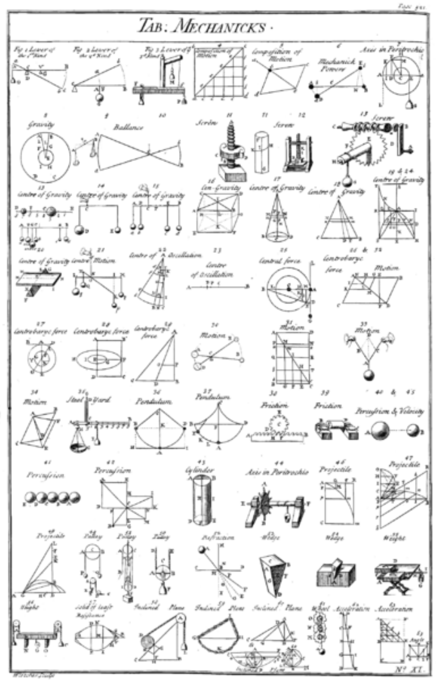
Simple Machines, Pulleys; Forces, Distances and MA: Describes the following terms as they relate to simple machine; input force, output force, input distance, output distance, mechanical advantage.

Simple machines are devices used to multiply or augment a force that we apply—often at the expense of a distance through which we apply the force. Some common examples include:

- Lever

- Wheel and Axle
- Pulley
- Inclined Plane
- Wedge
- Screw

When a device with a specific movement, called a mechanism, is joined with others to form a machine, these machines can be broken down into elementary movements. For example, a bicycle is a mechanism made up of wheels, levers, and pulleys.

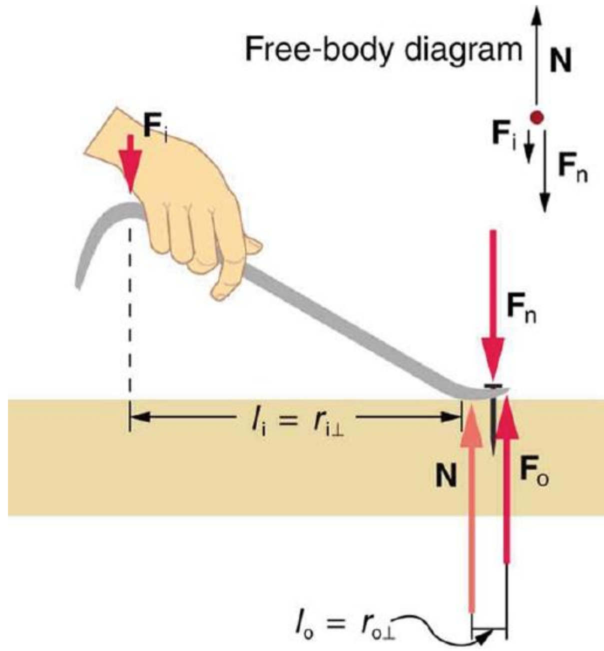


Simple Machines: Table of simple mechanisms, from Chambers' Cyclopedia, 1728. [1] Simple machines provide a "vocabulary" for understanding more complex machines.

Mechanics

A simple machine has an applied force that works against a load force. If there are no frictional losses, the work done on the load is equal to the work done by the applied force. This allows an increase in the output force at the cost of a proportional decrease in distance moved by the load. The ratio of the output force to the input force is the mechanical advantage of the machine. If the machine does not absorb energy, its mechanical advantage can be calculated from the machine's geometry. For instance, the mechanical advantage of a lever is equal to the ratio of its lever arms.

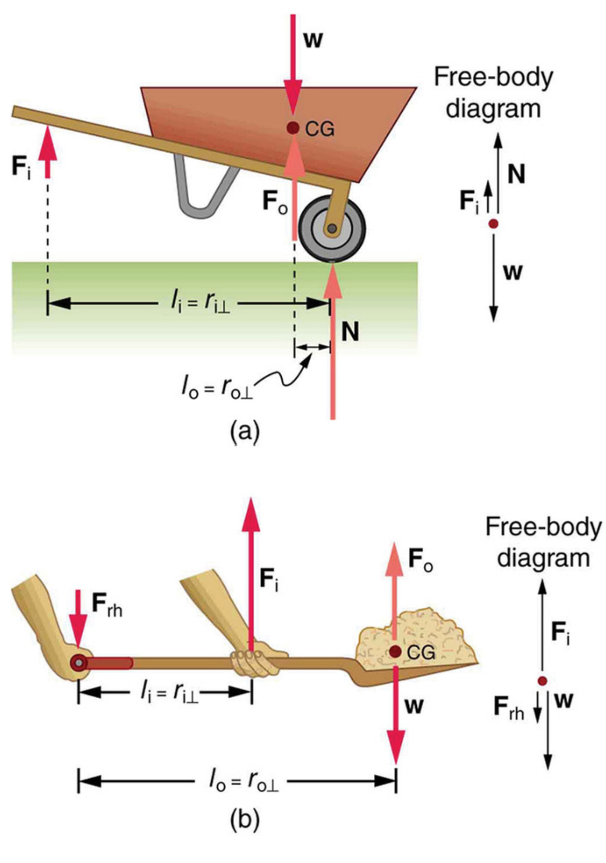
Simple machines which do not experience frictional losses are called ideal machines. For these ideal machines, the power in (rate of energy input) in equal to the power out (rate of energy output): Pin=PoutPin=Pout.



Lever: The amount of force produced by a machine can not be greater than the amount of force put into it.

Further Examples

Wheelbarrows and shovels are also examples of simple machines (these utilize levers). They use only three forces: the input force, output force, and force on the pivot. In the case of wheelbarrows, the output force is between the pivot (wheel's axle) and the input force. In the shovel, the input force is between the pivot and the load.



Examples of Simple Machines: Both of these machines use the concept of levers.

Arches and Domes

Arches and domes are structures that exhibit structural strength and can span large areas with no intermediate supports.

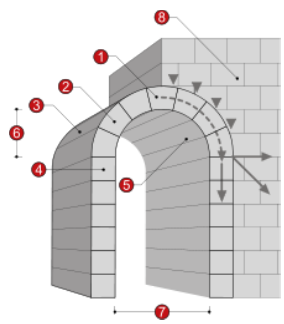
learning objectives

- Explain how an arch exhibits structural strength and how a dome can span a large area without intermediate supports

Arches and domes are structures that exhibit structural strength and can span large areas with no intermediate supports. In this atom, we will discuss the history and physics behind arches and domes.

Arches

An arch is a structure that spans a space, and supports structure and weight above it. Arches have been being built from as long ago as the second millennium, but were not used for a variety of structures until the Romans took advantage of their capabilities. Arches are a pure compression form. They span large areas by resolving forces into compressive stresses and eliminating tensile stresses (referred to as arch action). As the forces in an arch are carried toward the ground, the arch will push outward at the base (called thrust). As the height of the arch decreases, the outward thrust increases. To prevent the arch from collapsing, the thrust needs to be restrained, either with internal ties or external bracing. This external bracing is often called an abutment, as shown in.



Arches: A masonry arch1. Keystone 2. Voussoir 3. Extrados 4. Impost 5. Intrados 6. Rise 7. Clear span 8. Abutment

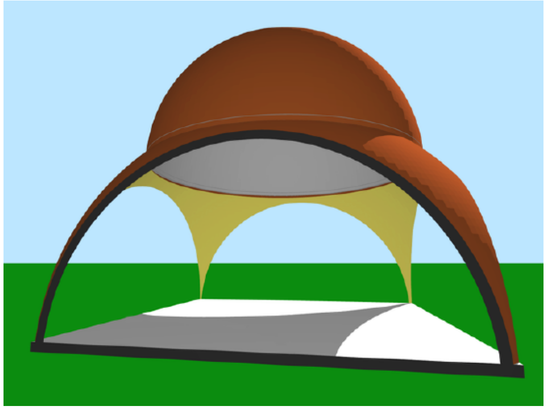
The most common true arch configurations are the fixed arch, the two-hinged arch and the three-hinged arch. The fixed arch is most often used in reinforced concrete bridge and tunnel construction, where the spans are short. Because it is subject to additional internal stress caused by thermal expansion and contraction, this type of arch is considered to be statically indeterminate. The two-hinged arch is most often used to bridge long spans. This type of arch has pinned connections at the base. Unlike the fixed arch, the pinned base is able to rotate, allowing the structure to move freely and compensate for the thermal expansion and contraction caused by changes in outdoor temperature. Because the structure is pinned between the two base connections, which can result in additional stresses, the two-hinged arch is also statically indeterminate, although not to the degree of the fixed arch.

Domes

A dome is an element of architecture that resembles the hollow upper half of a sphere. Dome structures made of various materials (from mud to stone, wood, brick, concrete, metal, glass and plastic) and have a long architectural lineage extending into prehistory.

A dome is basically an arch that has been rotated around its central vertical axis. Domes have the same properties and capabilities of arches, they can span large areas without intermediate supports and have a great deal of structural strength. When the base of a dome is not the same shape as its supporting walls, for example when a circular dome is on a square structure, techniques are employed to transition between the two. Pendentives are triangular sections of a sphere used to transition from the flat surfaces of supporting walls to the round base of a dome.

Domes can be divided into two kinds, simple and compound. Simple domes use pendentives that are part of the same sphere as the dome itself. Compound domes are part of the structure of a large sphere below that of the dome itself, forming a circular base, as shown in.



Compound Dome: A compound dome (red) with pendentives (yellow) from a sphere of greater radius than the dome.

Muscles and Joints

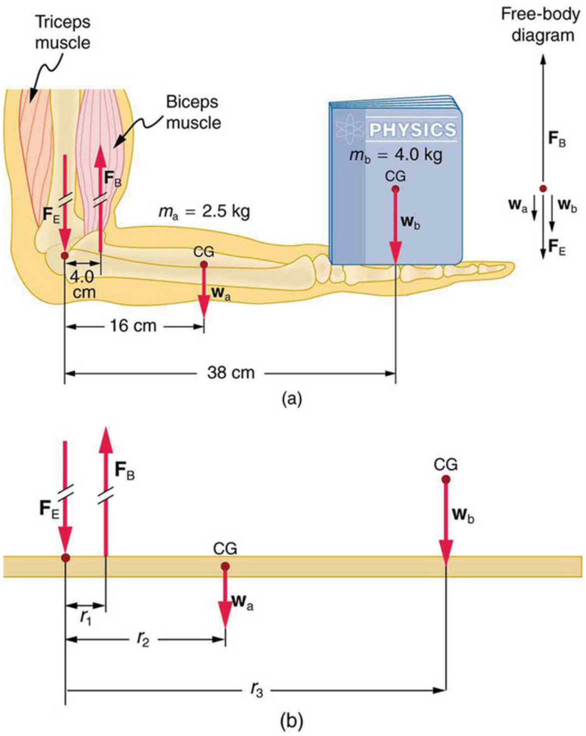
Most skeletal muscles and joints exert much larger forces within the body than the limbs will apply to the outside world.

learning objectives

- Explain the forces exerted by muscles

Muscles and Joints

Muscles and joints involve very interesting applications of statics. Muscles can only contract, so they occur in pairs. In the arm, the biceps muscle is a flexor: it closes the limb. The triceps muscle is an extensor that opens the limb. This configuration is typical of skeletal muscles, bones, and joints in humans and other vertebrates. Most skeletal muscles exert much larger forces within the body than the limbs will apply to the outside world. The reason is clear, since most muscles are attached to bones via tendons close to joints, causing these systems to have mechanical advantages much less than one. Viewing them as simple machines, the input force is much greater than the output force, as seen in.



The Forearm of a Person Holding a Book: (a.) The biceps exert a force FB to support the weight of the forearm and the book. The triceps are assumed to be relaxed. (b.) An approximately equivalent mechanical system with the pivot at the elbow joint

Very large forces are also created in the joints. Because muscles can contract but not expand beyond their resting length, joints and muscles often exert forces that act in opposite directions, and thus subtract. Forces in muscles and joints are largest when their load is far from the joint. For example, in racquet sports like tennis, the constant extension of the arm during game play creates large forces. The mass times the lever arm of a tennis racquet is an important factor, and many players use the heaviest racquet they can handle. It is no wonder that joint deterioration and damage to the tendons in the elbow, such as 'tennis elbow,' can result from repetitive motion, undue torques, and possible poor racquet selection in such sports.

Various tried techniques for holding and using a racquet, bat, or stick can not only increase sporting prowess but can minimize fatigue and long-term damage to the body. Training coaches and physical therapists use the knowledge of the relationships between forces and torques in the treatment of muscles and joints. In physical therapy, an exercise routine can apply a particular force and torque, which can revive muscles and joints in time. Some exercises should be performed under water, thus requiring the exertion of more force and further strengthening muscles.

Key Points

- The six classifications of simple machines were established by renaissance scientists; they are as follows: lever, wheel and axle, pulley, inclined plane, wedge and screw.
- Simple machines can be joined with other devices to create a more complicated machine. These building blocks are used to explain how machines work.
- The force output by a simple machine can exceed the force that was put into the machine.
- Arches span large areas by resolving forces into compressive stresses and eliminating tensile stresses.
- The most common true arch configurations are the fixed arch, the two-hinged arch, and the three-hinged arch.
- A dome is basically an arch that has been rotated around its central vertical axis.
- Domes are basically arches that have been rotated on their vertical axis, and have the same capabilities and properties of arches.
- Domes can be divided into two kinds, simple and compound.
- It is helpful to view muscles as a simple machines and draw them as free body diagrams.
- In muscles, the input force is often much greater than the output force.
- Very large forces are also created in the joints. Because muscles can contract but not expand beyond their resting length, joints and muscles often exert forces that act in opposite directions, and thus subtract.

Key Terms

- **machine**: A mechanical or electrical device that performs or assists in the performance of human tasks, whether physical or computational, laborious or for entertainment.
- **leverage**: A force amplified by means of a lever rotating around a pivot.
- **mechanical advantage**: In a simple machine, the ratio of the output force to the input force.
- **compressive stress**: Stress on materials that leads to a smaller volume.
- **tensile stress**: Stress state leading to expansion; that is, the length of a material tends to increase in the tensile direction while the volume remains constant.
- **pendentive**: The concave triangular sections of vaulting that provide the transition between a dome and the square base on which it is set and transfer the weight of the dome.
- **muscle**: A contractile form of tissue which animals use to effect movement.
- **joint**: Any part of the body where two bones join, in most cases allowing that part of the body to be bent or straightened.

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10.14: Elasticity, Stress, Strain, and Fracture

learning objectives

- Identify properties of elastic objects

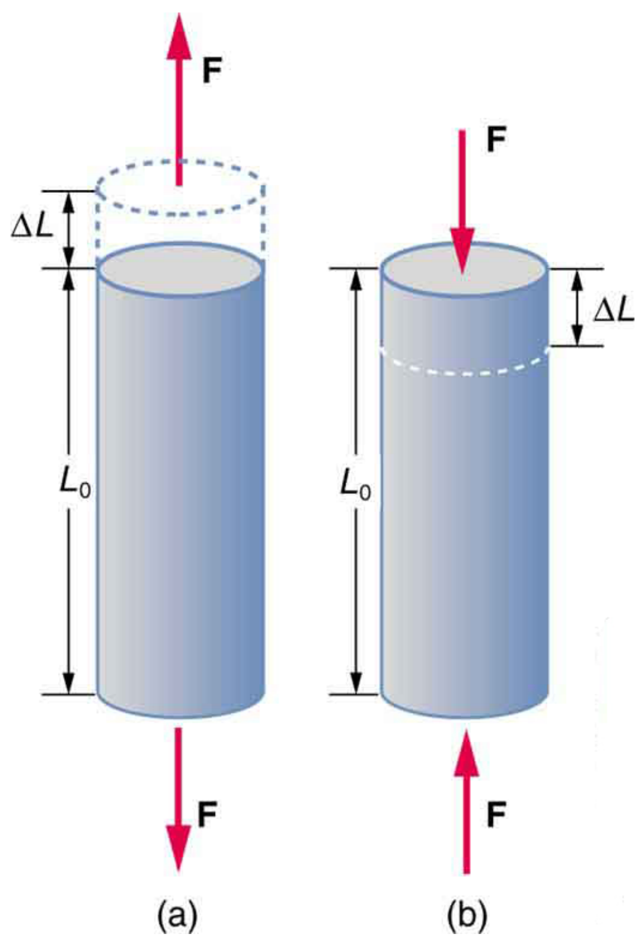
We now move from consideration of forces that affect the motion of an object (such as friction and drag) to those that affect an object's shape. If a bulldozer pushes a car into a wall, the car will not move once it hits the wall, but it will noticeably change shape. A change in shape due to the application of a force is a deformation. Even very small forces are known to cause some deformation. For small deformations, two important characteristics are observed. First, the object returns to its original shape when the force is removed—that is, the deformation is elastic for small deformations. Second, the size of the deformation is proportional to the force—that is, for small deformations, Hooke's law is obeyed. In equation form, Hooke's law is given by $\mathrm{F}=\mathrm{k}\Delta\mathrm{L}$, where $\Delta\mathrm{L}$ is the change in length.

Elasticity is a measure of how difficult it is to stretch an object. In other words it is a measure of how small kk is. Very elastic materials like rubber have small kk and thus will stretch a lot with only a small force.

Stress is a measure of the force put on the object over the area.

Strain is the change in length divided by the original length of the object.

Experiments have shown that the change in length (ΔL) depends on only a few variables. As already noted, ΔL is proportional to the force F and depends on the substance from which the object is made. Additionally, the change in length is proportional to the original length L0 and inversely proportional to the cross-sectional area of the wire or rod. For example, a long guitar string will stretch more than a short one, and a thick string will stretch less than a thin one.



Tension/Compression: Tension: The rod is stretched a length ΔL when a force is applied parallel to its length. (b) Compression: The same rod is compressed by forces with the same magnitude in the opposite direction. For very small deformations and uniform materials, ΔL is approximately the same for the same magnitude of tension or compression. For larger deformations, the cross-sectional area changes as the rod is compressed or stretched.

Fracture

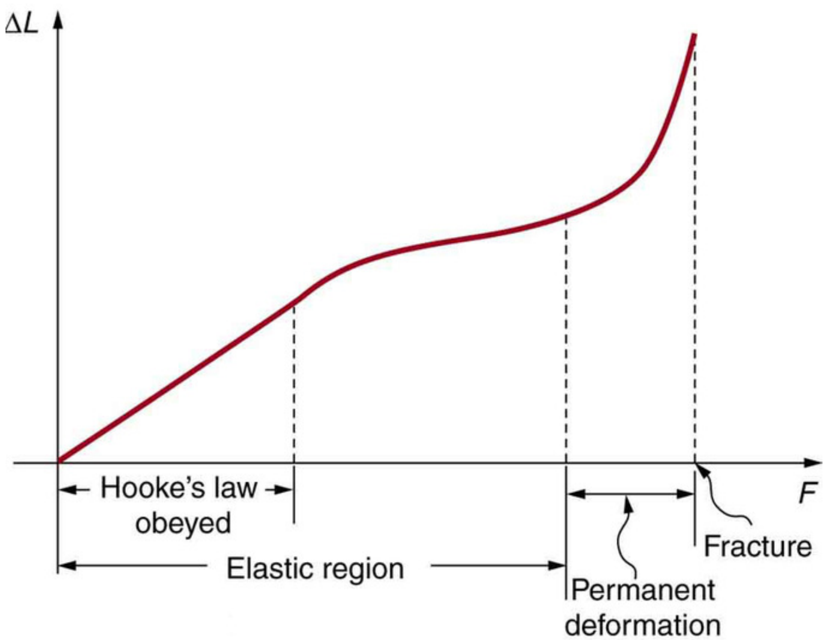
Fracture is caused by a strain placed on an object such that it deforms beyond its elastic limit and breaks.

learning objectives

- Relate fracture with the elastic limit of a material

Materials cannot stretch forever. When a strain is applied to a material it deforms elastically proportional to the force applied. However, after it has deformed a certain amount, the object can no longer take the strain and will break or fracture. The zone in which it bends under strain is called the elastic region. In that region the object will bend and then return to its original shape when the force is abated. Past that point, if more strain is added, the object may permanently deform and eventually fracture.

Fracture strength, also known as breaking strength, is the stress at which a specimen fails via fracture. This is usually determined for a given specimen by a tensile test, which charts the stress-strain curve. The final recorded point is the fracture strength.



Fracture: This is a graph of deformation ΔL versus applied force F. The straight segment is the linear region where Hooke's law is obeyed. The slope of the straight region is 1k. For larger forces, the graph is curved but the deformation is still elastic—L will return to zero if the force is removed. Still greater forces permanently deform the object until it finally fractures. The shape of the curve near fracture depends on several factors, including how the force F is applied. Note that in this graph the slope increases just before fracture, indicating that a small increase in F is producing a large increase in L near the fracture.

Bones, on the whole, do not fracture due to tension or compression. Rather they generally fracture due to sideways impact or bending, resulting in the bone shearing or snapping. The behavior of bones under tension and compression is important because it determines the load the bones can carry. Bones are classified as weight-bearing structures such as columns in buildings and trees. Weight-bearing structures have special features; columns in building have steel-reinforcing rods while trees and bones are fibrous. The bones in different parts of the body serve different structural functions and are prone to different stresses. Thus, the bone in the top of the femur is arranged in thin sheets separated by marrow while, in other places, the bones can be cylindrical and filled with marrow or just solid. Overweight people have a tendency toward bone damage due to sustained compressions in bone joints and tendons.

Key Points

- Elasticity is a measure of the deformation of an object when a force is applied. Objects that are very elastic like rubber have high elasticity and stretch easily.
- Stress is force over area.
- Strain is change in length over original length.
- Most objects behave elastically for small strains and return to their original shape after being bent.
- If the strain on an object is greater than the elastic limit of the object, it will permanently deform or eventually fracture.
- Fracture strength is a measure of the force needed to break an object.

Key Items

- **deformation**: A transformation; change of shape.
- **strain**: The amount by which a material deforms under stress or force, given as a ratio of the deformation to the initial dimension of the material and typically symbolized by ε is termed the engineering strain. The true strain is defined as the natural logarithm of the ratio of the final dimension to the initial dimension.
- **elastic**: Capable of stretching; particularly, capable of stretching so as to return to an original shape or size when force is released.

10.15: The Center of Gravity

learning objectives

- Describe how the center of mass of an oddly shaped object is found

Center of Gravity

When people think of objects, they think of them as singular particles of matter. In fact, every object is made up of millions of particles, all of which behave differently when moved. When people observe a stick being thrown in the air, it seems as though the entire object is moving at the same trajectory and velocity, but each particle is being subjected to a different motion in space and acceleration, depending on its place. The different parts of the body have different motions. shows the motion of a stick in the air: it seems to rotate around a single point. Three-dimensional bodies have a property called the center of mass, or center of gravity. This center of mass's main characteristic is that it appears to carry the whole mass of the body.

Center of Gravity: Although the center of mass is in the midpoint of the stick, all of the particles are moving as well.

The center of mass does not actually carry all the mass, despite appearances. Given a hollow sphere, the center is the center of mass, even though it does not actually have anything in it. As seen in, it looks as if the external forces of gravity appear to be working only on the center of mass, but each particle is being pushed or pulled by gravity. The center of mass is much easier to use when discussing bodies, because no one has to analyze each individual particle.

Mathematical Expression: The mathematical relation of center of gravity is read as: 'the position of the center of mass and weighted average of the position of the particles. '

Specifically: 'the total mass x the position of the center of mass= ∑ the mass of the individual particle x the position of the particle. ' The center of mass is a geometric point in three-dimensional volume. When using the definition above, it yields the following equation for center of mass:

$$\mathrm{r_{COM}} = \frac{\Sigma \mathrm{m_i r_i}}{\mathrm{M}} \tag{10.15.1}$$

where r is the reference axis x, y, or z; *m* is individual mass;$\mathrm{r_i}$ is the individual position; and M *M* is the total mass.

When taking the center of mass of an oddly shaped object, it is helpful to break it down into smaller sections whose mass and properties are easier to evaluate, and then add the products of the individual masses and positions and divide by the total mass.

Center of Mass: This child's toy uses the principles of 'center of mass' to stay balanced on a finger.

Key Points

- The center of mass 's main characteristic is that it appears to carry the whole mass of the body.
- The total mass x the position of the center of mass= ∑ mass of the individual particle x the position of the particle.
- The center of mass is a geometric point in three-dimensional volume. By using the definition above, the following equation for center of mass can be derived: $r_{COM} = \Sigma \frac{m_i r_i}{M}$.

Key Terms

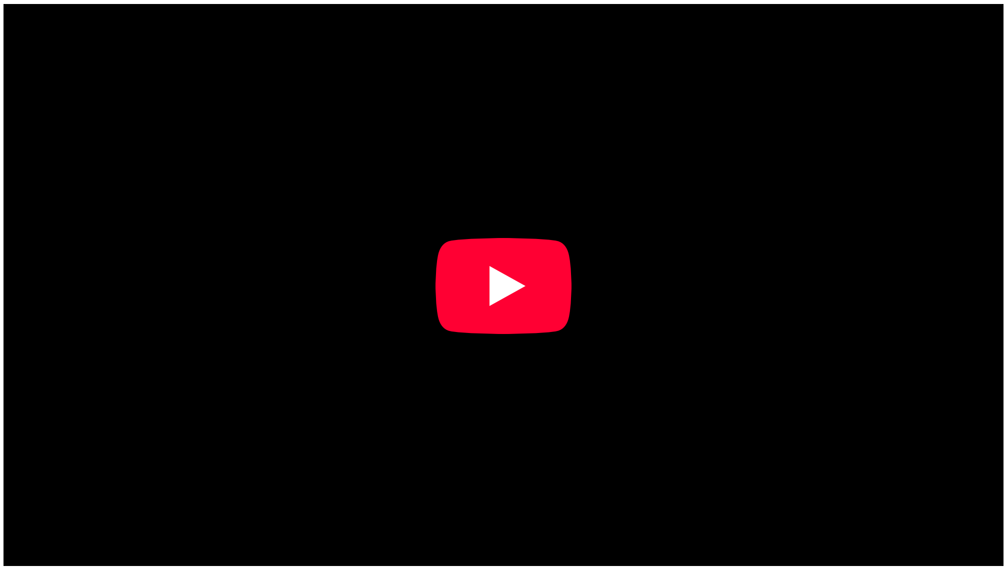
- **center of mass**: The center of mass (COM) is the unique point at the center of a distribution of mass in space that has the property that the weighted position vectors relative to this point sum to zero.

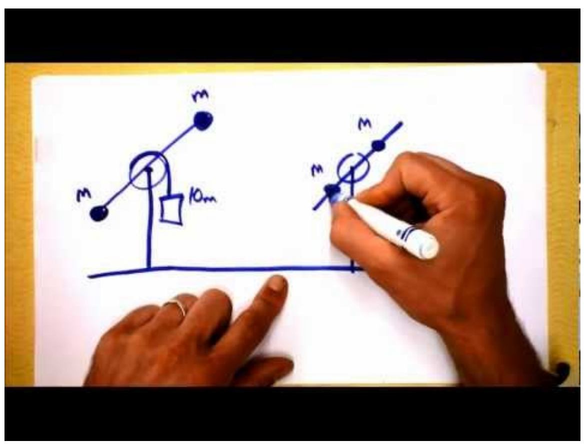
10.16: Torque and Angular Acceleration

learning objectives

- Express the relationship between the torque and the angular acceleration in a form of equation

Torque and angular acceleration are related by the following formula where is the objects moment of inertia and αα is the angular acceleration.





Torque, Angular Acceleration, and the Role of the Church in the French Revolution: Why do things change their angular velocity? Soon, you'll know.

Just like Newton's Second Law, which is force is equal to the mass times the acceleration, torque obeys a similar law. If you replace torque with force and rotational inertia with mass and angular acceleration with linear acceleration, you get Newton's Second Law back out. In fact, this equation is Newton's second law applied to a system of particles in rotation about a given axis. It makes no assumptions about constant rotational velocity.

The net torque about an axis of rotation is equal to the product of the rotational inertia about that axis and the angular acceleration, as shown in Figure 1.

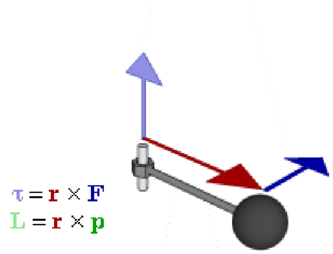


Figure 1: Relationship between force (F), torque (τ), momentum (p), and angular momentum (L) vectors in a rotating system

Similar to Newton's Second Law, angular motion also obeys Newton's First Law. If no outside forces act on an object, an object in motion remains in motion and an object at rest remains at rest. With rotating objects, we can say that unless an outside torque is applied, a rotating object will stay rotating and an object at rest will not begin rotating.

If a turntable were spinning counter clockwise (when viewed from the top), and you applied your fingers to opposite sides the turntable would begin to slow its spinning. From a translational viewpoint, at least, there would be no net force applied to the turntable. The force that points to one side would be cancelled by the force that points to the other. The forces of the two fingers would cancel. Therefore, the turntable would be in translational equilibrium. Despite that, the rotational velocity would be decreased meaning that the acceleration would no longer be zero. From this we might conclude that just because a rotating object is in translational equilibrium, it is not necessarily in rotational equilibrium.

Key Points

- When a torque is applied to an object it begins to rotate with an acceleration inversely proportional to its moment of inertia.
- This relation can be thought of as Newton's Second Law for rotation. The moment of inertia is the rotational mass and the torque is rotational force.
- Angular motion obeys Newton's First Law. If no outside forces act on an object, an object in motion remains in motion and an object at rest remains at rest.

Key Terms

- **angular acceleration**: The rate of change of angular velocity, often represented by α.
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **rotational inertia**: The tendency of a rotating object to remain rotating unless a torque is applied to it.

CHAPTER OVERVIEW

11: Rotational Kinematics, Angular Momentum, and Energy

11.1: Prelude to Fixed-Axis Rotation Introduction

In previous chapters, we described motion (kinematics) and how to change motion (dynamics), and we defined important concepts such as energy for objects that can be considered as point masses. Point masses, by definition, have no shape and so can only undergo translational motion. However, we know from everyday life that rotational motion is also very important and that many objects that move have both translation and rotation. The wind turbines in our chapter opening image are a prime example of how rotational motion impacts our daily lives, as the market for clean energy sources continues to grow.

Figure 11.1.1: Brazos wind farm in west Texas. As of 2012, wind farms in the US had a power output of 60 gigawatts, enough capacity to power 15 million homes for a year. (credit: modification of work by “ENERGY.GOV”/Flickr)

We begin to address rotational motion in this chapter, starting with fixed-axis rotation. Fixed-axis rotation describes the rotation around a fixed axis of a rigid body; that is, an object that does not deform as it moves. We will show how to apply all the ideas we’ve developed up to this point about translational motion to an object rotating around a fixed axis. In the next chapter, we extend these ideas to more complex rotational motion, including objects that both rotate and translate, and objects that do not have a fixed rotational axis.

11.2: Rotational Variables

Learning Objectives

- Describe the physical meaning of rotational variables as applied to fixed-axis rotation
- Explain how angular velocity is related to tangential speed
- Calculate the instantaneous angular velocity given the angular position function
- Find the angular velocity and angular acceleration in a rotating system
- Calculate the average angular acceleration when the angular velocity is changing
- Calculate the instantaneous angular acceleration given the angular velocity function

So far in this text, we have mainly studied translational motion, including the variables that describe it: displacement, velocity, and acceleration. Now we expand our description of motion to rotation—specifically, rotational motion about a fixed axis. We will find that rotational motion is described by a set of related variables similar to those we used in translational motion.

Angular Velocity

Uniform circular motion (discussed previously in Motion in Two and Three Dimensions) is motion in a circle at constant speed. Although this is the simplest case of rotational motion, it is very useful for many situations, and we use it here to introduce rotational variables.

In Figure $11.2.1$, we show a particle moving in a circle. The coordinate system is fixed and serves as a frame of reference to define the particle's position. Its position vector from the origin of the circle to the particle sweeps out the angle θ, which increases in the counterclockwise direction as the particle moves along its circular path. The angle θ is called the **angular position** of the particle. As the particle moves in its circular path, it also traces an arc length s.

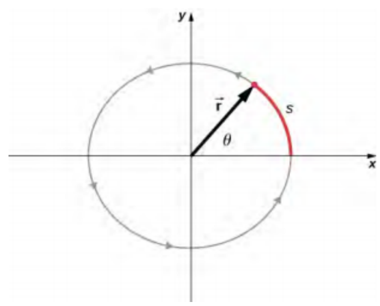


Figure 11.2.1: A particle follows a circular path. As it moves counterclockwise, it sweeps out a positive angle θ with respect to the x-axis and traces out an arc length s.

The angle is related to the radius of the circle and the arc length by

$$\theta = \frac{s}{r}. \tag{11.2.1}$$

The angle θ, the angular position of the particle along its path, has units of radians (rad). There are 2π radians in 360°. Note that the radian measure is a ratio of length measurements, and therefore is a dimensionless quantity. As the particle moves along its circular path, its angular position changes and it undergoes angular displacements $\Delta\theta$.

We can assign vectors to the quantities in Equation 11.2.1. The angle $\vec{\theta}$ is a vector out of the page in Figure $11.2.1$. The angular position vector $\vec{r}$ and the arc length $\vec{s}$ both lie in the plane of the page. These three vectors are related to each other by

$$\vec{s} = \vec{\theta} \times \vec{r}. \tag{11.2.2}$$

That is, the arc length is the cross product of the angle vector and the position vector, as shown in Figure $11.2.2$

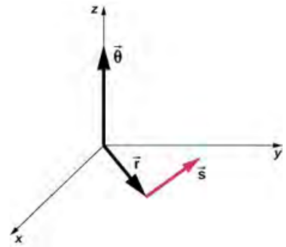


Figure 11.2.2: The angle vector points along the z-axis and the position vector and arc length vector both lie in the xy-plane. We see that $\vec{s} = \vec{\theta} \times \vec{r}$. All three vectors are perpendicular to each other.

The magnitude of the angular velocity, denoted by ω, is the time rate of change of the angle θ as the particle moves in its circular path. The instantaneous angular velocity is defined as the limit in which Δt → 0 in the average angular velocity $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$:

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}, \tag{11.2.3}$$

where θ is the angle of rotation (Figure 11.2.2). The units of angular velocity are radians per second (rad/s). Angular velocity can also be referred to as the rotation rate in radians per second. In many situations, we are given the rotation rate in revolutions/s or cycles/s. To find the angular velocity, we must multiply revolutions/s by 2π, since there are 2π radians in one complete revolution. Since the direction of a positive angle in a circle is counterclockwise, we take counterclockwise rotations as being positive and clockwise rotations as negative.

We can see how angular velocity is related to the tangential speed of the particle by differentiating Equation 11.2.1 with respect to time. We rewrite Equation 11.2.1 as

$$s = r\theta. \tag{11.2.4}$$

Taking the derivative with respect to time and noting that the radius r is a constant, we have

$$\frac{ds}{dt} = \frac{d}{dt}(r\theta) = \theta\frac{dr}{dt} + r\frac{d\theta}{dt} = r\frac{d\theta}{dt} \tag{11.2.5}$$

where $\theta\frac{dr}{dt}$ = 0. Here, $\frac{ds}{dt}$ is just the tangential speed v_t of the particle in Figure 11.2.1. Thus, by using Equation 11.2.3, we arrive at

$$v_t = r\omega. \tag{11.2.6}$$

That is, the tangential speed of the particle is its angular velocity times the radius of the circle. From Equation 11.2.6, we see that the tangential speed of the particle increases with its distance from the axis of rotation for a constant angular velocity. This effect is shown in Figure 11.2.3. Two particles are placed at different radii on a rotating disk with a constant angular velocity. As the disk rotates, the tangential speed increases linearly with the radius from the axis of rotation. In Figure 11.2.3, we see that $v_1 = r_1\omega_1$ and $v_2 = r_2\omega_2$. But the disk has a constant angular velocity, so $\omega_1 = \omega_2$. This means $\frac{v_1}{r_1} = \frac{v_2}{r_2}$ or $v_2 = \left(\frac{r_2}{r_1}\right)v_1$. Thus, since $r_2 > r_1$, $v_2 > v_1$.

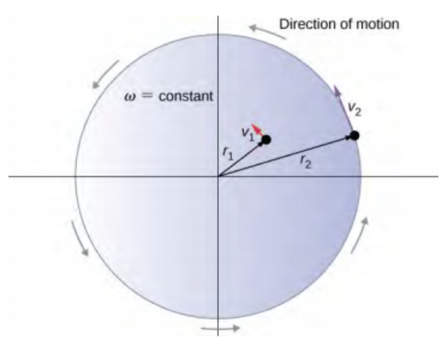


Figure 11.2.3: Two particles on a rotating disk have different tangential speeds, depending on their distance to the axis of rotation.

Up until now, we have discussed the magnitude of the angular velocity $\omega = \frac{d\theta}{dt}$, which is a scalar quantity—the change in angular position with respect to time. The vector $\vec{\omega}$ is the vector associated with the angular velocity and points along the axis of rotation. This is useful because when a rigid body is rotating, we want to know both the axis of rotation and the direction that the body is rotating about the axis, clockwise or counterclockwise. The angular velocity $\vec{\omega}$ gives us this information. The angular velocity $\vec{\omega}$ has a direction determined by what is called the right-hand rule. The right-hand rule is such that if the fingers of your right hand wrap counterclockwise from the x-axis (the direction in which θ increases) toward the y-axis, your thumb points in the direction of the positive z-axis (Figure 11.2.4). An angular velocity $\vec{\omega}$ that points along the positive z-axis therefore corresponds to a counterclockwise rotation, whereas an angular velocity $\vec{\omega}$ that points along the negative z-axis corresponds to a clockwise rotation.

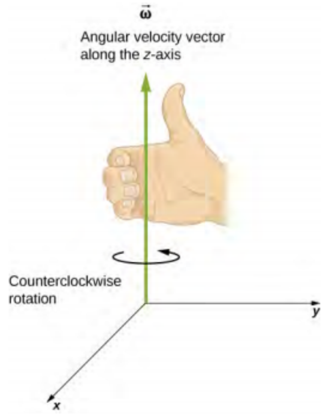


Figure 11.2.4: For counterclockwise rotation in the coordinate system shown, the angular velocity points in the positive z-direction by the right-hand-rule.

We can verify the right-hand-rule using the vector expression for the arc length $\vec{s} = \vec{\theta} \times \vec{r}$, Equation 11.2.2. If we differentiate this equation with respect to time, we find

$$\frac{d\vec{s}}{dt} = \frac{d}{dt}(\vec{\theta} \times \vec{r}) = \left(\frac{d\theta}{dt} \times \vec{r}\right) + \left(\vec{\theta} \times \frac{d\vec{r}}{dt}\right) = \frac{d\theta}{dt} \times \vec{r}. \tag{11.2.7}$$

Since $\vec{r}$ is constant, the term $\vec{\theta} \times \frac{d\vec{r}}{dt}$ = 0. Since $\vec{v} = \frac{d\vec{s}}{dt}$ is the tangential velocity and $\omega = \frac{d\vec{\theta}}{dt}$ is the angular velocity, we have

$$\vec{v} = \vec{\omega} \times \vec{r}. \tag{11.2.8}$$

That is, the tangential velocity is the cross product of the angular velocity and the position vector, as shown in Figure 11.2.5. From part (a) of this figure, we see that with the angular velocity in the positive z-direction, the rotation in the xy-plane is counterclockwise. In part (b), the angular velocity is in the negative z-direction, giving a clockwise rotation in the xy-plane.

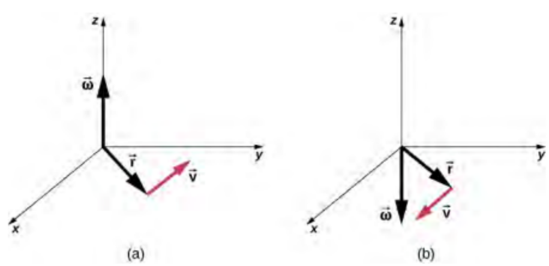


Figure 11.2.5: The vectors shown are the angular velocity, position, and tangential velocity. (a) The angular velocity points in the positive z-direction, giving a counterclockwise rotation in the xy-plane. (b) The angular velocity points in the negative z-direction, giving a clockwise rotation.

✓ Example 11.2.1: Rotation of a Flywheel

A flywheel rotates such that it sweeps out an angle at the rate of θ = ωt = (45.0 rad/s)t radians. The wheel rotates counterclockwise when viewed in the plane of the page. (a) What is the angular velocity of the flywheel? (b) What direction is the angular velocity? (c) How many radians does the flywheel rotate through in 30 s? (d) What is the tangential speed of a point on the flywheel 10 cm from the axis of rotation?

Strategy

The functional form of the angular position of the flywheel is given in the problem as θ(t) = ωt, so by taking the derivative with respect to time, we can find the angular velocity. We use the right-hand rule to find the angular velocity. To find the angular displacement of the flywheel during 30 s, we seek the angular displacement $\Delta\theta$, where the change in angular position is between 0 and 30 s. To find the tangential speed of a point at a distance from the axis of rotation, we multiply its distance times the angular velocity of the flywheel.

Solution

a. $\omega = \frac{d\theta}{dt}$ = 45 rad/s. We see that the angular velocity is a constant.
b. By the right-hand rule, we curl the fingers in the direction of rotation, which is counterclockwise in the plane of the page, and the thumb points in the direction of the angular velocity, which is out of the page.
c. $\Delta\theta = \theta(30\text{ s}) - \theta(0\text{ s}) = 45.0(30.0\text{ s}) - 45.0(0\text{ s}) = 1350.0$ rad.
d. $v_t = r\omega = (0.1\text{ m})(45.0\text{ rad/s}) = 4.5$ m/s.

Significance

In 30 s, the flywheel has rotated through quite a number of revolutions, about 215 if we divide the angular displacement by 2π. A massive flywheel can be used to store energy in this way, if the losses due to friction are minimal. Recent research has considered superconducting bearings on which the flywheel rests, with zero energy loss due to friction.

Angular Acceleration

We have just discussed angular velocity for uniform circular motion, but not all motion is uniform. Envision an ice skater spinning with his arms outstretched—when he pulls his arms inward, his angular velocity increases. Or think about a computer's hard disk slowing to a halt as the angular velocity decreases. We will explore these situations later, but we can already see a need to define an **angular acceleration** for describing situations where ω changes. The faster the change in ω, the greater the angular acceleration. We define the **instantaneous angular acceleration** α as the derivative of angular velocity with respect to time:

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}, \tag{11.2.9}$$

where we have taken the limit of the average angular acceleration, $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$ as $\Delta t \to 0$. The units of angular acceleration are (rad/s)/s, or rad/s^2.

In the same way as we defined the vector associated with angular velocity $\vec{\omega}$, we can define $\vec{\alpha}$, the vector associated with angular acceleration (Figure 11.2.6). If the angular velocity is along the positive z-axis, as in Figure 11.2.4, and $\frac{d\omega}{dt}$ is positive, then the angular acceleration $\vec{\alpha}$ is positive and points along the +z- axis. Similarly, if the angular velocity $\vec{\omega}$ is along the positive z-axis and $\frac{d\omega}{dt}$ is negative, then the angular acceleration is negative and points along the +z-axis.

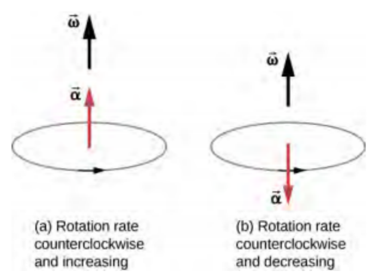


Figure 11.2.6: The rotation is counterclockwise in both (a) and (b) with the angular velocity in the same direction. (a) The angular acceleration is in the same direction as the angular velocity, which increases the rotation rate. (b) The angular acceleration is in the opposite direction to the angular velocity, which decreases the rotation rate.

We can express the tangential acceleration vector as a cross product of the angular acceleration and the position vector. This expression can be found by taking the time derivative of $\vec{v} = \vec{\omega} \times \vec{r}$ and is left as an exercise:

$$\vec{a} = \vec{\alpha} \times \vec{r}. \tag{11.2.10}$$

The vector relationships for the angular acceleration and tangential acceleration are shown in Figure 11.2.7.

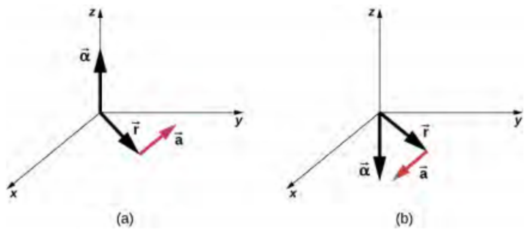


Figure 11.2.7: (a) The angular acceleration is the positive z-direction and produces a tangential acceleration in a counterclockwise sense. (b) The angular acceleration is in the negative z-direction and produces a tangential acceleration in the clockwise sense.

We can relate the tangential acceleration of a point on a rotating body at a distance from the axis of rotation in the same way that we related the tangential speed to the angular velocity. If we differentiate Equation 11.2.6 with respect to time, noting that the radius r is constant, we obtain

$$a_t = r\alpha. \tag{11.2.11}$$

Thus, the tangential acceleration a_t is the radius times the angular acceleration. Equations 11.2.6 and 11.2.11 are important for the discussion of rolling motion (see Angular Momentum).

Let's apply these ideas to the analysis of a few simple fixed-axis rotation scenarios. Before doing so, we present a problem-solving strategy that can be applied to rotational kinematics: the description of rotational motion.

? Problem-Solving Strategy: Rotational Kinematics

1. Examine the situation to determine that rotational kinematics (rotational motion) is involved.
2. Identify exactly what needs to be determined in the problem (identify the unknowns). A sketch of the situation is useful.
3. Make a complete list of what is given or can be inferred from the problem as stated (identify the knowns).
4. Solve the appropriate equation or equations for the quantity to be determined (the unknown). It can be useful to think in terms of a translational analog, because by now you are familiar with the equations of translational motion.
5. Substitute the known values along with their units into the appropriate equation and obtain numerical solutions complete with units. Be sure to use units of radians for angles.
6. Check your answer to see if it is reasonable: Does your answer make sense?

Now let's apply this problem-solving strategy to a few specific examples.

✓ Example 11.2.2: A Spinning Bicycle Wheel

A bicycle mechanic mounts a bicycle on the repair stand and starts the rear wheel spinning from rest to a final angular velocity of 250 rpm in 5.00 s. (a) Calculate the average angular acceleration in rad/s^2. (b) If she now hits the brakes, causing an angular acceleration of −87.3 rad/s^2, how long does it take the wheel to stop?

Strategy

The average angular acceleration can be found directly from its definition $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$ because the final angular velocity and time are given. We see that $\Delta\omega = \omega_{final} - \omega_{initial}$ = 250 rev/min and Δt is 5.00 s. For part (b), we know the angular acceleration and the initial angular velocity. We can find the stopping time by using the definition of average angular acceleration and solving for Δt, yielding

$$\Delta t = \frac{\Delta\omega}{\alpha}. \tag{11.2.12}$$

Solution

a. Entering known information into the definition of angular acceleration, we get

$$\bar{\alpha} = \frac{\Delta\omega}{\Delta t} = \frac{250\ rpm}{5.00\ s}. \tag{11.2.13}$$

Because $\Delta\omega$ is in revolutions per minute (rpm) and we want the standard units of rad/s^2 for angular acceleration, we need to convert from rpm to rad/s:

$$\Delta\omega = 250\frac{rev}{min}\cdot\frac{2\pi\ rad}{rev}\cdot\frac{1\ min}{60\ s} = 26.2\ rad/s. \tag{11.2.14}$$

Entering this quantity into the expression for α, we get

$$bar\alpha = \frac{\Delta\omega}{\Delta t} = \frac{26.2\ rpm}{5.00\ s} = 5.24\ rad/s^2. \tag{11.2.15}$$

b. Here the angular velocity decreases from 26.2 rad/s (250 rpm) to zero, so that $\Delta\omega$ is −26.2 rad/s, and α is given to be –87.3 rad/s^2. Thus

$$\Delta t = \frac{-26.2\ rad/s}{-87.3\ rad/s^2} = 0.300\ s. \tag{11.2.16}$$

Significance

Note that the angular acceleration as the mechanic spins the wheel is small and positive; it takes 5 s to produce an appreciable angular velocity. When she hits the brake, the angular acceleration is large and negative. The angular velocity quickly goes to zero.

? Exercise 11.2.1

The fan blades on a turbofan jet engine (shown below) accelerate from rest up to a rotation rate of 40.0 rev/s in 20 s. The increase in angular velocity of the fan is constant in time. (The GE90-110B1 turbofan engine mounted on a Boeing 777, as shown, is currently the largest turbofan engine in the world, capable of thrusts of 330–510 kN.) (a) What is the average angular acceleration? (b) What is the instantaneous angular acceleration at any time during the first 20 s?

✓ Example 11.2.3: Wind Turbine

A wind turbine (Figure 11.2.9) in a wind farm is being shut down for maintenance. It takes 30 s for the turbine to go from its operating angular velocity to a complete stop in which the angular velocity function is $\omega(t) = \left[\frac{(ts^{-1}-30.0)^2}{100.0}\right]$ rad/s. If the turbine is rotating counterclockwise looking into the page, (a) what are the directions of the angular velocity and acceleration vectors? (b) What is the average angular acceleration? (c) What is the instantaneous angular acceleration at t = 0.0, 15.0, 30.0 s?

Figure 11.2.9: A wind turbine that is rotating counterclockwise, as seen head on.

Strategy

a. We are given the rotational sense of the turbine, which is counterclockwise in the plane of the page. Using the right hand rule (Figure 10.5), we can establish the directions of the angular velocity and acceleration vectors.
b. We calculate the initial and final angular velocities to get the average angular acceleration. We establish the sign of the angular acceleration from the results in (a).
c. We are given the functional form of the angular velocity, so we can find the functional form of the angular acceleration function by taking its derivative with respect to time.

Solution

a. Since the turbine is rotating counterclockwise, angular velocity $\vec{\omega}$ points out of the page. But since the angular velocity is decreasing, the angular acceleration $\vec{\alpha}$ points into the page, in the opposite sense to the angular velocity.
b. The initial angular velocity of the turbine, setting t = 0, is ω = 9.0 rad/s. The final angular velocity is zero, so the average angular acceleration is

$$\bar{\alpha}\frac{\Delta\omega}{\Delta t} = \frac{\omega - \omega_0}{t - t_0} = \frac{0 - 9.0\ rad/s}{30.0 - 0\ s} = -0.3\ rad/s^2. \tag{11.2.17}$$

c. Taking the derivative of the angular velocity with respect to time gives $\alpha = \frac{d\omega}{dt} = \frac{(t-30.0)}{50.0}$ rad/s^2

$$\alpha(0.0; s) = -0.6\ rad/s^2, \alpha(15.0\ s) = -0.3\ rad/s^2, and\ \alpha(30.0\ s) = 0\ rad/s. \tag{11.2.18}$$

Significance

We found from the calculations in (a) and (b) that the angular acceleration α and the average angular acceleration $\bar{\alpha}$ are negative. The turbine has an angular acceleration in the opposite sense to its angular velocity.

We now have a basic vocabulary for discussing fixed-axis rotational kinematics and relationships between rotational variables. We discuss more definitions and connections in the next section.

11.3: Rotation with Constant Angular Acceleration

Learning Objectives

- Derive the kinematic equations for rotational motion with constant angular acceleration
- Select from the kinematic equations for rotational motion with constant angular acceleration the appropriate equations to solve for unknowns in the analysis of systems undergoing fixed-axis rotation
- Use solutions found with the kinematic equations to verify the graphical analysis of fixed-axis rotation with constant angular acceleration

In the preceding section, we defined the rotational variables of angular displacement, angular velocity, and angular acceleration. In this section, we work with these definitions to derive relationships among these variables and use these relationships to analyze rotational motion for a rigid body about a fixed axis under a constant angular acceleration. This analysis forms the basis for rotational kinematics. If the angular acceleration is constant, the equations of rotational kinematics simplify, similar to the equations of linear kinematics discussed in Motion along a Straight Line and Motion in Two and Three Dimensions. We can then use this simplified set of equations to describe many applications in physics and engineering where the angular acceleration of the system is constant. Rotational kinematics is also a prerequisite to the discussion of rotational dynamics later in this chapter.

Kinematics of Rotational Motion

Using our intuition, we can begin to see how the rotational quantities θ, ω, α, and t are related to one another. For example, we saw in the preceding section that if a flywheel has an angular acceleration in the same direction as its angular velocity vector, its angular velocity increases with time and its angular displacement also increases. On the contrary, if the angular acceleration is opposite to the angular velocity vector, its angular velocity decreases with time. We can describe these physical situations and many others with a consistent set of rotational kinematic equations under a constant angular acceleration. The method to investigate rotational motion in this way is called **kinematics of rotational motion**.

To begin, we note that if the system is rotating under a constant acceleration, then the average angular velocity follows a simple relation because the angular velocity is increasing linearly with time. The average angular velocity is just half the sum of the initial and final values:

$$\bar{\omega}=\frac{\omega_0+\omega_f}{2}. \tag{11.3.1}$$

From the definition of the average angular velocity, we can find an equation that relates the angular position, average angular velocity, and time:

$$\bar{\omega}=\frac{\Delta\theta}{\Delta t}. \tag{11.3.2}$$

Solving for θ, we have

$$\theta_f=\theta_0+\bar{\omega}t, \tag{11.3.3}$$

where we have set t_0 = 0. This equation can be very useful if we know the average angular velocity of the system. Then we could find the angular displacement over a given time period. Next, we find an equation relating ω, α, and t. To determine this equation, we start with the definition of angular acceleration:

$$\alpha=\frac{d\omega}{dt}. \tag{11.3.4}$$

We rearrange this to get αdt = dω and then we integrate both sides of this equation from initial values to final values, that is, from t_0 to t and ω_0 to ω_f. In uniform rotational motion, the angular acceleration is constant so it can be pulled out of the integral, yielding two definite integrals:

$$\alpha\int_{t_0}^{t}dt'=\int_{\omega_0}^{\omega_f}d\omega. \tag{11.3.5}$$

Setting t_0 = 0, we have

$$\alpha t=\omega_f-\omega_0. \tag{11.3.6}$$

We rearrange this to obtain

$$\omega_f=\omega_0+\alpha t, \tag{11.3.7}$$

where ω_0 is the initial angular velocity. Equation 11.3.7 is the rotational counterpart to the linear kinematics equation $v_f = v_0 + at$. With Equation 11.3.7, we can find the angular velocity of an object at any specified time t given the initial angular velocity and the angular acceleration.

Let's now do a similar treatment starting with the equation $\omega=\frac{d\theta}{dt}$. We rearrange it to obtain $\omega dt = d\theta$ and integrate both sides from initial to final values again, noting that the angular acceleration is constant and does not have a time dependence. However, this time, the angular velocity is not constant (in general), so we substitute in what we derived above:

$$\begin{aligned}\int_{t_0}^{t_f}(\omega_0+\alpha t')dt' &= \int_{\theta_0}^{\theta_f}d\theta; \\ \int_{t_0}^{t}\omega_0 dt+\int_{t_0}^{t}\alpha t dt &= \int_{\theta_0}^{\theta_f}d\theta=\left[\omega_0 t'+\alpha\left(\frac{(t')^2}{2}\right)^2\right]_{t_0}^{t}=\omega_0 t+\alpha\left(\frac{t^2}{2}\right)=\theta_f-\theta_0.\end{aligned}$$

where we have set $t_0 = 0$. Now we rearrange to obtain

$$\theta_f=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2. \tag{11.3.8}$$

Equation 11.3.8 is the rotational counterpart to the linear kinematics equation found in Motion Along a Straight Line for position as a function of time. This equation gives us the angular position of a rotating rigid body at any time t given the initial conditions (initial angular position and initial angular velocity) and the angular acceleration.

We can find an equation that is independent of time by solving for t in Equation 11.3.7 and substituting into Equation 11.3.8. Equation 11.3.8 becomes

$$\begin{aligned}\theta_f &= \theta_0+\omega_0\left(\frac{\omega_f-\omega_0}{\alpha}\right)+\frac{1}{2}\alpha\left(\frac{\omega_f-\omega_0}{\alpha}\right)^2 \\ &=\theta_0+\frac{\omega_0\omega_f}{\alpha}-\frac{\omega_0^2}{\alpha}+\frac{1}{2}\frac{\omega_f^2}{\alpha}-\frac{\omega_0\omega_f}{\alpha}+\frac{1}{2}\frac{\omega_0^2}{\alpha} \\ &=\theta_0+\frac{1}{2}\frac{\omega_f^2}{\alpha}-\frac{1}{2}\frac{\omega_0^2}{\alpha}, \\ \theta_f-\theta_0 &= \frac{\omega_f^2-\omega_0^2}{2\alpha}\end{aligned}$$

or

$$\omega_f^2=\omega_0^2+2\alpha(\Delta\theta). \tag{11.3.9}$$

Equation 11.3.3 through Equation 11.3.9 describe fixed-axis rotation for constant acceleration and are summarized in Table 10.1.

Table 10.1 - Kinematic Equations

Angular displacement from average angular velocity	$\theta_f=\theta_0+\bar{\omega}t$ (11.3.10)
Angular velocity from angular acceleration	$\omega_f=\omega_0+\alpha t$ (11.3.11)
Angular displacement from angular velocity and angular acceleration	$\theta_f=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2$ (11.3.12)
Angular velocity from angular displacement and angular acceleration	$\omega_f^2=\omega_0^2+2\alpha(\Delta\theta)$ (11.3.13)

Applying the Equations for Rotational Motion

Now we can apply the key kinematic relations for rotational motion to some simple examples to get a feel for how the equations can be applied to everyday situations.

Example 10.4: Calculating the Acceleration of a Fishing Reel

A deep-sea fisherman hooks a big fish that swims away from the boat, pulling the fishing line from his fishing reel. The whole system is initially at rest, and the fishing line unwinds from the reel at a radius of 4.50 cm from its axis of rotation. The reel is given an angular acceleration of 110 rad/s^2 for 2.00 s (Figure 11.3.1).

a. What is the final angular velocity of the reel after 2 s?
b. How many revolutions does the reel make?

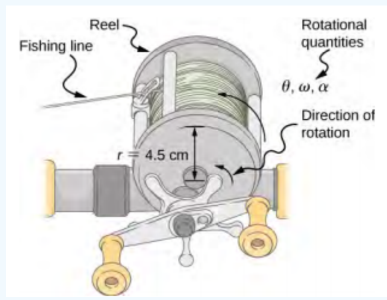


Figure 11.3.1: Fishing line coming off a rotating reel moves linearly

Strategy

Identify the knowns and compare with the kinematic equations for constant acceleration. Look for the appropriate equation that can be solved for the unknown, using the knowns given in the problem description.

Solution

a. We are given α and t and want to determine ω. The most straightforward equation to use is $\omega_f = \omega_0 + \alpha t$, since all terms are known besides the unknown variable we are looking for. We are given that ω_0 = 0 (it starts from rest), so

$$\omega_f = 0 + (110\ rad/s^2)(2.00\ s) = 220\ rad/s. \tag{11.3.14}$$

b. We are asked to find the number of revolutions. Because 1 rev = 2π rad, we can find the number of revolutions by finding θ in radians. We are given α and t, and we know ω_0 is zero, so we can obtain θ by using

$$\begin{aligned}\theta_f &= \theta_i + \omega_i t + \frac{1}{2}\alpha t^2 \\ &= 0 + 0 + (0.500)(110\ rad/s^2)(2.00\ s)^2 = 220\ rad.\end{aligned}$$

Converting radians to revolutions gives

$$Number\ of\ rev = (220\ rad)\left(\frac{1\ rev}{2\pi\ rad}\right) = 35.0\ rev. \tag{11.3.15}$$

Significance

This example illustrates that relationships among rotational quantities are highly analogous to those among linear quantities. The answers to the questions are realistic. After unwinding for two seconds, the reel is found to spin at 220 rad/s, which is 2100 rpm. (No wonder reels sometimes make high-pitched sounds.)

In the preceding example, we considered a fishing reel with a positive angular acceleration. Now let us consider what happens with a negative angular acceleration.

Example 10.5: Calculating the Duration When the Fishing Reel Slows Down and Stops

Now the fisherman applies a brake to the spinning reel, achieving an angular acceleration of −300 rad/s^2. How long does it take the reel to come to a stop?

Strategy

We are asked to find the time t for the reel to come to a stop. The initial and final conditions are different from those in the previous problem, which involved the same fishing reel. Now we see that the initial angular velocity is ω_0 = 220 rad/s and the final angular velocity ω is zero. The angular acceleration is given as α = −300 rad/s^2. Examining the available equations, we see all quantities but t are known in $\omega_f = \omega_0 + \alpha t$, making it easiest to use this equation.

Solution

The equation states

$$\omega_f = \omega_0 + \alpha t. \tag{11.3.16}$$

We solve the equation algebraically for t and then substitute the known values as usual, yielding

$$t = \frac{\omega_f - \omega_0}{\alpha} = \frac{0 - 220.0\ rad/s}{-300.0\ rad/s^2} = 0.733\ s. \tag{11.3.17}$$

Significance

Note that care must be taken with the signs that indicate the directions of various quantities. Also, note that the time to stop the reel is fairly small because the acceleration is rather large. Fishing lines sometimes snap because of the accelerations involved, and fishermen often let the fish swim for a while before applying brakes on the reel. A tired fish is slower, requiring a smaller acceleration.

Exercise 10.2

A centrifuge used in DNA extraction spins at a maximum rate of 7000 rpm, producing a “g-force” on the sample that is 6000 times the force of gravity. If the centrifuge takes 10 seconds to come to rest from the maximum spin rate: (a) What is the angular acceleration of the centrifuge? (b) What is the angular displacement of the centrifuge during this time?

Example 10.6: Angular Acceleration of a Propeller

Figure $11.3.2$ shows a graph of the angular velocity of a propeller on an aircraft as a function of time. Its angular velocity starts at 30 rad/s and drops linearly to 0 rad/s over the course of 5 seconds. (a) Find the angular acceleration of the object and verify the result using the kinematic equations. (b) Find the angle through which the propeller rotates during these 5 seconds and verify your result using the kinematic equations.

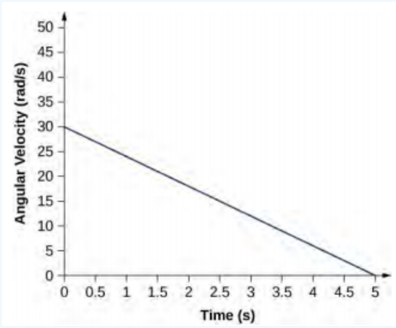


Figure 11.3.2: A graph of the angular velocity of a propeller versus time.

Strategy

a. Since the angular velocity varies linearly with time, we know that the angular acceleration is constant and does not depend on the time variable. The angular acceleration is the slope of the angular velocity vs. time graph, $\alpha = \frac{d\omega}{dt}$. To calculate the slope, we read directly from Figure $11.3.2$, and see that ω_0 = 30 rad/s at t = 0 s and ω_f = 0 rad/s at t = 5 s. Then, we can verify the result using $\omega = \omega_0 + \alpha t$.

b. We use the equation $\omega = \frac{d\theta}{dt}$; since the time derivative of the angle is the angular velocity, we can find the angular displacement by integrating the angular velocity, which from the figure means taking the area under the angular velocity graph. In other words:

$$\int_{\theta_0}^{\theta_f} d\theta = \theta_f - \theta_0 = \int_{t_0}^{t_f} \omega(t)dt. \tag{11.3.18}$$

Then we use the kinematic equations for constant acceleration to verify the result.

Solution

a. Calculating the slope, we get

$$\alpha = \frac{\omega - \omega_0}{t - t_0} = \frac{(0 - 30.0)\ rad/s}{(5.0 - 0)\ s} = -6.0\ rad/s^2. \tag{11.3.19}$$

We see that this is exactly Equation 11.3.7with a little rearranging of terms.

b. We can find the area under the curve by calculating the area of the right triangle, as shown in Figure 11.3.3.

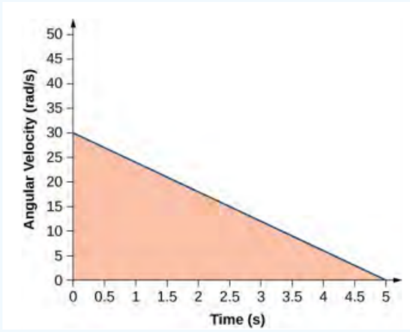


Figure 11.3.3: The area under the curve is the area of the right triangle.

\[\Delta \theta = area(triangle) = \frac{1}{2} (30\; rad/s)(5\; s) = 75\; rad \ldotp

$$Weverifythesolutionusing Equation 11.3.8: \tag{11.3.20}$$

\theta_{f} = \theta_{0} + \omega_{0} t + \frac{1}{2} \alpha t^{2} \ldotp

$$Setting\(\theta_0\) = 0, wehave \tag{11.3.21}$$

\theta_{0} = (30.0\; rad/s)(5.0\; s) + \frac{1}{2} (-6.0\; rad/s^{2})(5.0\; s)^{2} = 150.0 - 75.0 = 75.0\; rad \ldotp$$This verifies the solution found from finding the area under the curve.

Significance

We see from part (b) that there are alternative approaches to analyzing fixed-axis rotation with constant acceleration. We started with a graphical approach and verified the solution using the rotational kinematic equations. Since $\alpha = \frac{d\omega}{dt}$, we could do the same graphical analysis on an angular acceleration-vs.-time curve. The area under an α-vs.-t curve gives us the change in angular velocity. Since the angular acceleration is constant in this section, this is a straightforward exercise.

11.4: Relating Angular and Translational Quantities

Learning Objectives

- Given the linear kinematic equation, write the corresponding rotational kinematic equation
- Calculate the linear distances, velocities, and accelerations of points on a rotating system given the angular velocities and accelerations

In this section, we relate each of the rotational variables to the translational variables defined in Motion Along a Straight Line and Motion in Two and Three Dimensions. This will complete our ability to describe rigid-body rotations.

Angular vs. Linear Variables

In Rotational Variables, we introduced angular variables. If we compare the rotational definitions with the definitions of linear kinematic variables from Motion Along a Straight Line and Motion in Two and Three Dimensions, we find that there is a mapping of the linear variables to the rotational ones. Linear position, velocity, and acceleration have their rotational counterparts, as we can see when we write them side by side:

	Linear	Rotational
Position	$$x \tag{11.4.1}$$	$$\theta \tag{11.4.2}$$
Velocity	$$v=\frac{dx}{dt} \tag{11.4.3}$$	$$\omega=\frac{d\theta}{dt} \tag{11.4.4}$$
Acceleration	$$a=\frac{dv}{dt} \tag{11.4.5}$$	$$a=\frac{d\omega}{dt} \tag{11.4.6}$$

Let's compare the linear and rotational variables individually. The linear variable of position has physical units of meters, whereas the angular position variable has dimensionless units of radians, as can be seen from the definition of $\theta=\frac{s}{r}$, which is the ratio of two lengths. The linear velocity has units of m/s, and its counterpart, the angular velocity, has units of rad/s. In Rotational Variables, we saw in the case of circular motion that the linear tangential speed of a particle at a radius r from the axis of rotation is related to the angular velocity by the relation v_t = rω. This could also apply to points on a rigid body rotating about a fixed axis. Here, we consider only circular motion. In circular motion, both uniform and nonuniform, there exists a centripetal acceleration (Motion in Two and Three Dimensions). The centripetal acceleration vector points inward from the particle executing circular motion toward the axis of rotation. The derivation of the magnitude of the centripetal acceleration is given in Motion in Two and Three Dimensions. From that derivation, the magnitude of the centripetal acceleration was found to be

$$a_c=\frac{v_t^2}{r}, \tag{11.4.7}$$

where r is the radius of the circle.

Thus, in uniform circular motion when the angular velocity is constant and the angular acceleration is zero, we have a linear acceleration—that is, centripetal acceleration—since the tangential speed in Equation 11.4.7 is a constant. If nonuniform circular motion is present, the rotating system has an angular acceleration, and we have both a linear centripetal acceleration that is changing (because v_t is changing) as well as a linear tangential acceleration. These relationships are shown in Figure 11.4.1, where we show the centripetal and tangential accelerations for uniform and nonuniform circular motion.

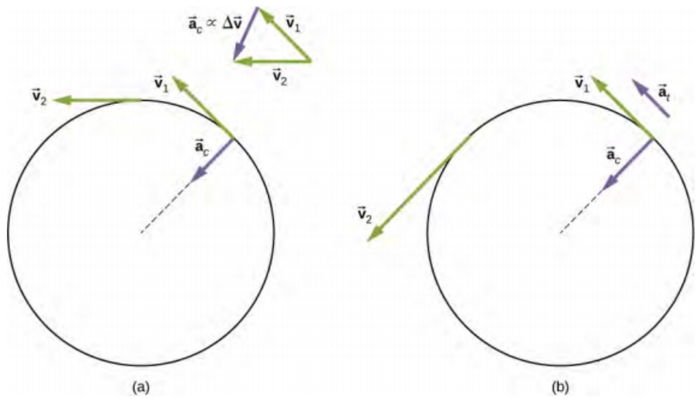


Figure 11.4.1: (a) Uniform circular motion: The centripetal acceleration a_c has its vector inward toward the axis of rotation. There is no tangential acceleration. (b) Nonuniform circular motion: An angular acceleration produces an inward centripetal acceleration that is changing in magnitude, plus a tangential acceleration a_t.

The centripetal acceleration is due to the change in the direction of tangential velocity, whereas the tangential acceleration is due to any change in the magnitude of the tangential velocity. The tangential and centripetal acceleration vectors $\vec{a}_t$ and $\vec{a}_c$ are always perpendicular to each other, as seen in Figure 11.4.1. To complete this description, we can assign a **total linear acceleration** vector to a point on a rotating rigid body or a particle executing circular motion at a radius r from a fixed axis. The total linear acceleration vector $\vec{a}$ is the vector sum of the centripetal and tangential accelerations,

$$\vec{a} = \vec{a}_c + \vec{a}_t. \tag{11.4.8}$$

The total linear acceleration vector in the case of nonuniform circular motion points at an angle between the centripetal and tangential acceleration vectors, as shown in Figure 11.4.2. Since $\vec{a}_c \perp \vec{a}_t$, the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2}. \tag{11.4.9}$$

Note that if the angular acceleration is zero, the total linear acceleration is equal to the centripetal acceleration.

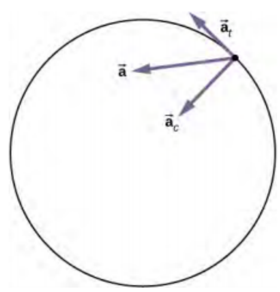


Figure 11.4.2: A particle is executing circular motion and has an angular acceleration. The total linear acceleration of the particle is the vector sum of the centripetal acceleration and tangential acceleration vectors. The total linear acceleration vector is at an angle in between the centripetal and tangential accelerations.

Relationships between Rotational and Translational Motion

We can look at two relationships between rotational and translational motion.

1. Generally speaking, the linear kinematic equations have their rotational counterparts. Table 10.2 lists the four linear kinematic equations and the corresponding rotational counterpart. The two sets of equations look similar to each other, but describe two different physical situations, that is, rotation and translation.

Table 10.2 - Rotational and Translational Kinematic Equations

Rotational	Translational
$$\theta_f = \theta_0 + \bar{\omega} t \tag{11.4.10}$$	$$x = x_0 + \bar{v} t \tag{11.4.11}$$

Rotational		Translational	
$\omega_f = \omega_0 + \alpha t$	(11.4.12)	$v_f = v_0 + at$	(11.4.13)
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(11.4.14)	$x_f = x_0 + v_0 t + \frac{1}{2}\omega t^2$	(11.4.15)
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	(11.4.16)	$v_f^2 = v_0^2 + 2a(\Delta x)$	(11.4.17)

2. The second correspondence has to do with relating linear and rotational variables in the special case of circular motion. This is shown in Table 10.3, where in the third column, we have listed the connecting equation that relates the linear variable to the rotational variable. The rotational variables of angular velocity and acceleration have subscripts that indicate their definition in circular motion.

Table 10.3 - Rotational and Translational Quantities: Circular Motion

Rotational		Translational		Relationship (r = radius	
θ	(11.4.18)	s	(11.4.19)	$\theta = \frac{s}{r}$	(11.4.20)
ω	(11.4.21)	v_t	(11.4.22)	$\omega = \frac{v_t}{r}$	(11.4.23)
α	(11.4.24)	a_t	(11.4.25)	$\alpha = \frac{a_t}{r}$	(11.4.26)
		a_c	(11.4.27)	$a_c = \frac{v_t^2}{r}$	(11.4.28)

Example 10.7: Linear Acceleration of a Centrifuge

A centrifuge has a radius of 20 cm and accelerates from a maximum rotation rate of 10,000 rpm to rest in 30 seconds under a constant angular acceleration. It is rotating counterclockwise. What is the magnitude of the total acceleration of a point at the tip of the centrifuge at t = 29.0s? What is the direction of the total acceleration vector?

Strategy

With the information given, we can calculate the angular acceleration, which then will allow us to find the tangential acceleration. We can find the centripetal acceleration at t = 0 by calculating the tangential speed at this time. With the magnitudes of the accelerations, we can calculate the total linear acceleration. From the description of the rotation in the problem, we can sketch the direction of the total acceleration vector.

Solution

The angular acceleration is

$$\alpha = \frac{\omega - \omega_0}{t} = \frac{0 - (1.0 \times 10^4)\left(\frac{2\pi\ rad}{60.0\ s}\right)}{30.0\ s} = -34.9\ rad/s^2. \tag{11.4.29}$$

Therefore, the tangential acceleration is

$$a_t = r\alpha = (0.2\ m)(-34.9\ rad/s^2) = -7.0\ m/s^2. \tag{11.4.30}$$

The angular velocity at t = 29.0 s is

$$\begin{aligned} \omega &= \omega_0 + \alpha t = (1.0 \times 10^4)\left(\frac{2\pi\ rad}{60.0\ s}\right) + (-39.49\ rad/s^2)(29.0\ s) \\ &= 1047.2\ rad/s - 1012.71\ rad/s = 35.1\ rad/s. \end{aligned}$$

Thus, the tangential speed at t = 29.0 s is

$$v_t = r\omega = (0.2\ m)(35.1\ rad/s) = 7.0\ m/s. \tag{11.4.31}$$

We can now calculate the centripetal acceleration at t = 29.0 s:

$$a_c = \frac{v^2}{r} = \frac{(7.0\ m/s)^2}{0.2\ m} = 245.0\ m/s^2. \tag{11.4.32}$$

Since the two acceleration vectors are perpendicular to each other, the magnitude of the total linear acceleration is

$$|\vec{a}| = \sqrt{a_c^2 + a_t^2} = \sqrt{(245.0)^2 + (-7.0)^2} = 245.1\ m/s^2. \tag{11.4.33}$$

Since the centrifuge has a negative angular acceleration, it is slowing down. The total acceleration vector is as shown in Figure $11.4.3$. The angle with respect to the centripetal acceleration vector is

$$\theta = \tan^{-1}\left(\frac{-7.0}{245.0}\right) = -1.6^o. \tag{11.4.34}$$

The negative sign means that the total acceleration vector is angled toward the clockwise direction.

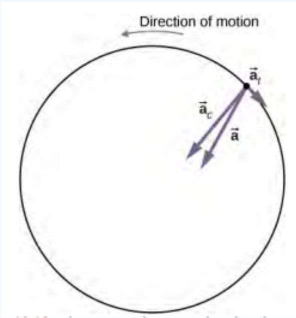


Figure 11.4.3: The centripetal, tangential, and total acceleration vectors. The centrifuge is slowing down, so the tangential acceleration is clockwise, opposite the direction of rotation (counterclockwise).

Significance

From Figure $11.4.3$, we see that the tangential acceleration vector is opposite the direction of rotation. The magnitude of the tangential acceleration is much smaller than the centripetal acceleration, so the total linear acceleration vector will make a very small angle with respect to the centripetal acceleration vector.

? Exercise 10.3

A boy jumps on a merry-go-round with a radius of 5 m that is at rest. It starts accelerating at a constant rate up to an angular velocity of 5 rad/s in 20 seconds. What is the distance travelled by the boy?

Simulation

Check out this PhET simulation to change the parameters of a rotating disk (the initial angle, angular velocity, and angular acceleration), and place bugs at different radial distances from the axis. The simulation then lets you explore how circular motion relates to the bugs' xy-position, velocity, and acceleration using vectors or graphs.

11.5: Moment of Inertia and Rotational Kinetic Energy

Learning Objectives

- Describe the differences between rotational and translational kinetic energy
- Define the physical concept of moment of inertia in terms of the mass distribution from the rotational axis
- Explain how the moment of inertia of rigid bodies affects their rotational kinetic energy
- Use conservation of mechanical energy to analyze systems undergoing both rotation and translation
- Calculate the angular velocity of a rotating system when there are energy losses due to nonconservative forces

So far in this chapter, we have been working with rotational kinematics: the description of motion for a rotating rigid body with a fixed axis of rotation. In this section, we define two new quantities that are helpful for analyzing properties of rotating objects: moment of inertia and rotational kinetic energy. With these properties defined, we will have two important tools we need for analyzing rotational dynamics.

Rotational Kinetic Energy

Any moving object has kinetic energy. We know how to calculate this for a body undergoing translational motion, but how about for a rigid body undergoing rotation? This might seem complicated because each point on the rigid body has a different velocity. However, we can make use of angular velocity—which is the same for the entire rigid body—to express the kinetic energy for a rotating object. Figure $11.5.1$ shows an example of a very energetic rotating body: an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are generated as the grindstone does its work. This system has considerable energy, some of it in the form of heat, light, sound, and vibration. However, most of this energy is in the form of **rotational kinetic energy**.

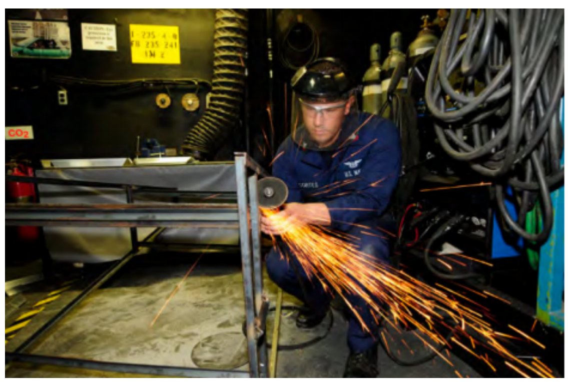


Figure $11.5.1$: The rotational kinetic energy of the grindstone is converted to heat, light, sound, and vibration. (credit: Zachary David Bell, US Navy)

Energy in rotational motion is not a new form of energy; rather, it is the energy associated with rotational motion, the same as kinetic energy in translational motion. However, because kinetic energy is given by $K = \frac{1}{2}mv^2$, and velocity is a quantity that is different for every point on a rotating body about an axis, it makes sense to find a way to write kinetic energy in terms of the variable ω, which is the same for all points on a rigid rotating body. For a single particle rotating around a fixed axis, this is straightforward to calculate. We can relate the angular velocity to the magnitude of the translational velocity using the relation $v_t = \omega r$, where r is the distance of the particle from the axis of rotation and v_t is its tangential speed. Substituting into the equation for kinetic energy, we find

$$K = \frac{1}{2}mv_t^2 = \frac{1}{2}m(\omega r)^2 = \frac{1}{2}\left(mr^2\right)\omega^2.$$

In the case of a rigid rotating body, we can divide up any body into a large number of smaller masses, each with a mass m_j and distance to the axis of rotation r_j, such that the total mass of the body is equal to the sum of the individual masses: $M = \sum_j m_j$. Each smaller mass has tangential speed v_j, where we have dropped the subscript t for the moment. The total kinetic energy of the rigid rotating body is

$$K = \sum_j \frac{1}{2}m_j v_j^2 = \sum_j \frac{1}{2}m_j(r_j\omega_j)^2$$

and since $\omega_j = \omega$ for all masses,

$$K = \frac{1}{2}\left(\sum_j m_j r_j^2\right)\omega^2. \tag{11.5.1}$$

The units of Equation 11.5.1 are joules (J). The equation in this form is complete, but awkward; we need to find a way to generalize it.

Moment of Inertia

If we compare Equation 11.5.1 to the way we wrote kinetic energy in Work and Kinetic Energy, ($\frac{1}{2}mv^2$), this suggests we have a new rotational variable to add to our list of our relations between rotational and translational variables. The quantity $\sum_j m_j r_j^2$ is the counterpart for mass in the equation for rotational kinetic energy. This is an important new term for rotational motion. This quantity is called the **moment of inertia** I, with units of kg·m^2:

$$I = \sum_j m_j r_j^2. \tag{11.5.2}$$

For now, we leave the expression in summation form, representing the moment of inertia of a system of point particles rotating about a fixed axis. We note that the moment of inertia of a single point particle about a fixed axis is simply mr^2, with r being the distance from the point particle to the axis of rotation. In the next section, we explore the integral form of this equation, which can be used to calculate the moment of inertia of some regular-shaped rigid bodies.

The moment of inertia is the quantitative measure of rotational inertia, just as in translational motion, and mass is the quantitative measure of linear inertia—that is, the more massive an object is, the more inertia it has, and the greater is its resistance to change in linear velocity. Similarly, the greater the moment of inertia of a rigid body or system of particles, the greater is its resistance to change in angular velocity about a fixed axis of rotation. It is interesting to see how the moment of inertia varies with *r*, the distance to the axis of rotation of the mass particles in Equation 11.5.2. Rigid bodies and systems of particles with more mass concentrated at a greater distance from the axis of rotation have greater moments of inertia than bodies and systems of the same mass, but concentrated near the axis of rotation. In this way, we can see that a hollow cylinder has more rotational inertia than a solid cylinder of the same mass when rotating about an axis through the center. Substituting Equation 11.5.2 into Equation 11.5.1, the expression for the kinetic energy of a rotating rigid body becomes

$$K = \frac{1}{2}I\omega^2. \tag{11.5.3}$$

We see from this equation that the kinetic energy of a rotating rigid body is directly proportional to the moment of inertia and the square of the angular velocity. This is exploited in flywheel energy-storage devices, which are designed to store large amounts of rotational kinetic energy. Many carmakers are now testing flywheel energy storage devices in their automobiles, such as the flywheel, or kinetic energy recovery system, shown in Figure 11.5.2.

Figure 11.5.2: A KERS (kinetic energy recovery system) flywheel used in cars. (credit: “cmonville”/Flickr)

The rotational and translational quantities for kinetic energy and inertia are summarized in Table 10.4. The relationship column is not included because a constant doesn't exist by which we could multiply the rotational quantity to get the translational quantity, as can be done for the variables in Table 10.3.

Table 10.4: Rotational and Translational Kinetic Energies and Inertia

Rotational	Translational
$I = \sum_j m_j r_j^2$	m
$K = \frac{1}{2} I \omega^2$	$K = \frac{1}{2} m v^2$

✔ Example 11.5.1: Moment of Inertia of a system of particles

Six small washers are spaced 10 cm apart on a rod of negligible mass and 0.5 m in length. The mass of each washer is 20 g. The rod rotates about an axis located at 25 cm, as shown in Figure 11.5.3. (a) What is the moment of inertia of the system? (b) If the two washers closest to the axis are removed, what is the moment of inertia of the remaining four washers? (c) If the system with six washers rotates at 5 rev/s, what is its rotational kinetic energy?

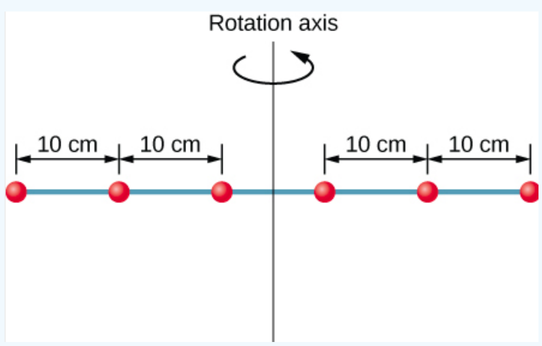


Figure 11.5.3: Six washers are spaced 10 cm apart on a rod of negligible mass and rotating about a vertical axis.

Strategy

a. We use the definition for moment of inertia for a system of particles and perform the summation to evaluate this quantity. The masses are all the same so we can pull that quantity in front of the summation symbol.
b. We do a similar calculation.
c. We insert the result from (a) into the expression for rotational kinetic energy.

Solution

a. $I=\sum m_j r_j^2=(0.02\text{ kg})\left(2\times(0.25\text{ m})^2+2\times(0.15\text{ m})^2+2\times(0.05\text{ m})^2\right)=0.0035\text{ kg}\cdot\text{m}^2$
b. $I=\sum_j m_j r_j^2=(0.02\text{ kg})\left(2\times(0.25\text{ m})^2+2\times(0.15\text{ m})^2\right)=0.0034\text{ kg}\cdot\text{m}^2$
c. $K=\frac{1}{2}I\omega^2=\frac{1}{2}\left(0.0035\text{ kg}\cdot\text{m}^2\right)(5.0\times 2\pi\text{ rad/s})^2=1.73\text{ J}$

Significance

We can see the individual contributions to the moment of inertia. The masses close to the axis of rotation have a very small contribution. When we removed them, it had a very small effect on the moment of inertia.

In the next section, we generalize the summation equation for point particles and develop a method to calculate moments of inertia for rigid bodies. For now, though, Figure 11.5.4gives values of rotational inertia for common object shapes around specified axes.

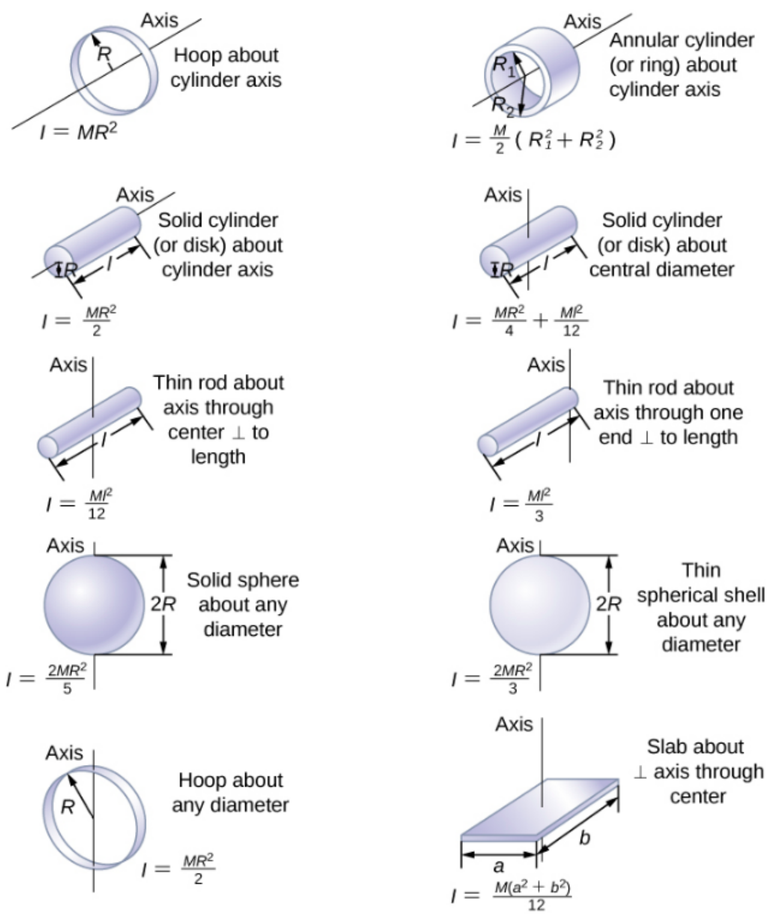


Figure 11.5.4: Values of rotational inertia for common shapes of objects.

Applying Rotational Kinetic Energy

Now let's apply the ideas of rotational kinetic energy and the moment of inertia table to get a feeling for the energy associated with a few rotating objects. The following examples will also help get you comfortable using these equations. First, let's look at a general problem-solving strategy for rotational energy.

PROBLEM-SOLVING STRATEGY: ROTATIONAL ENERGY

1. Determine that energy or work is involved in the rotation.
2. Determine the system of interest. A sketch usually helps.
3. Analyze the situation to determine the types of work and energy involved.
4. If there are no losses of energy due to friction and other nonconservative forces, mechanical energy is conserved, that is, $K_i+U_i=K_f+U_f$.

5. If nonconservative forces are present, mechanical energy is not conserved, and other forms of energy, such as heat and light, may enter or leave the system. Determine what they are and calculate them as necessary.
6. Eliminate terms wherever possible to simplify the algebra.
7. Evaluate the numerical solution to see if it makes sense in the physical situation presented in the wording of the problem.

Example 11.5.2: Calculating helicopter energies

A typical small rescue helicopter has four blades: Each is 4.00 m long and has a mass of 50.0 kg (Figure 11.5.5). The blades can be approximated as thin rods that rotate about one end of an axis perpendicular to their length. The helicopter has a total loaded mass of 1000 kg. (a) Calculate the rotational kinetic energy in the blades when they rotate at 300 rpm. (b) Calculate the translational kinetic energy of the helicopter when it flies at 20.0 m/s, and compare it with the rotational energy in the blades.

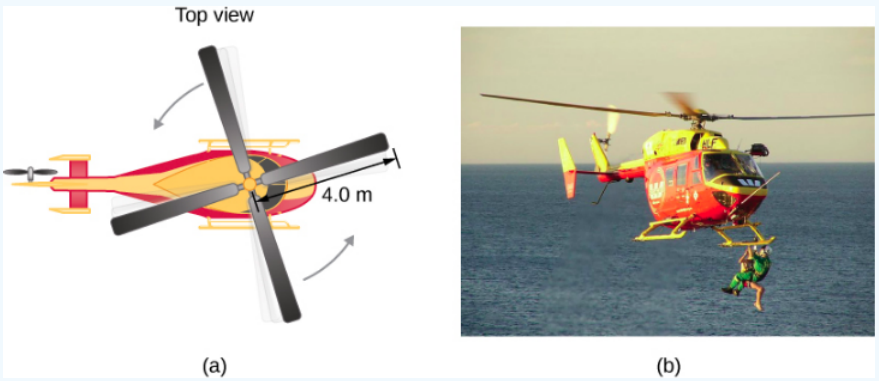


Figure 11.5.5: (a) Sketch of a four-blade helicopter. (b) A water rescue operation featuring a helicopter from the Auckland Westpac Rescue Helicopter Service. (credit b: modification of work by “111 Emergency”/Flickr)

Strategy

Rotational and translational kinetic energies can be calculated from their definitions. The wording of the problem gives all the necessary constants to evaluate the expressions for the rotational and translational kinetic energies.

Solution

a. The rotational kinetic energy is

$$K=\frac{1}{2}I\omega^2$$

We must convert the angular velocity to radians per second and calculate the moment of inertia before we can find K. The angular velocity ω is

$$\omega=\frac{300\text{ rev}}{1.00\text{ min}}\frac{2\pi\text{ rad}}{1\text{ rev}}\frac{1.00\text{ min}}{60.0\text{ s}}=31.4\frac{\text{rad}}{\text{s}}.$$

The moment of inertia of one blade is that of a thin rod rotated about its end, listed in Figure 11.5.4. The total I is four times this moment of inertia because there are four blades. Thus,

$$I=4\frac{Ml^2}{3}=4\times\frac{(50.0\text{ kg})(4.00\text{ m})^2}{3}=1067.0\text{ kg}\cdot\text{m}^2.$$

Entering ω and I into the expression for rotational kinetic energy gives

$$K=0.5\left(1067\text{ kg}\cdot\text{m}^2\right)(31.4\text{ rad/s})^2=5.26\times10^5\text{ J}.$$

b. Entering the given values into the equation for translational kinetic energy, we obtain

$$K=\frac{1}{2}mv^2=(0.5)(1000.0\text{ kg})(20.0\text{ m/s})^2=2.00\times10^5\text{ J}.$$

To compare kinetic energies, we take the ratio of translational kinetic energy to rotational kinetic energy. This ratio is

$$\frac{2.00 \times 10^5 \text{ J}}{5.26 \times 10^5 \text{ J}} = 0.380.$$

Significance

The ratio of translational energy to rotational kinetic energy is only 0.380. This ratio tells us that most of the kinetic energy of the helicopter is in its spinning blades.

✓ Example 11.5.3: Energy in a boomerang

A person hurls a boomerang into the air with a velocity of 30.0 m/s at an angle of 40.0° with respect to the horizontal (Figure 11.5.6). It has a mass of 1.0 kg and is rotating at 10.0 rev/s. The moment of inertia of the boomerang is given as $I = \frac{1}{12}mL^2$ where L = 0.7 m. (a) What is the total energy of the boomerang when it leaves the hand? (b) How high does the boomerang go from the elevation of the hand, neglecting air resistance?

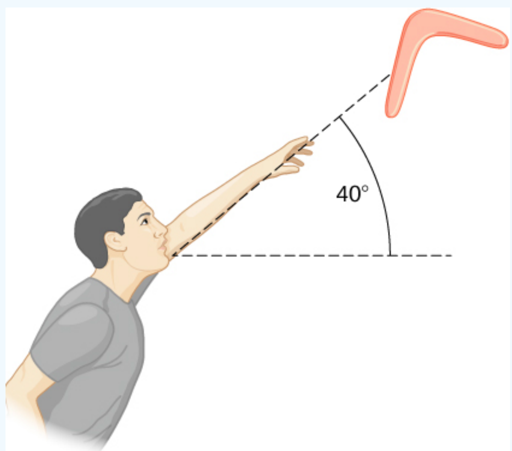


Figure 11.5.6: A boomerang is hurled into the air at an initial angle of 40°.

Strategy

We use the definitions of rotational and linear kinetic energy to find the total energy of the system. The problem states to neglect air resistance, so we don't have to worry about energy loss. In part (b), we use conservation of mechanical energy to find the maximum height of the boomerang.

Solution

a. Moment of inertia: $I = \frac{1}{12}mL^2 = \frac{1}{12}(1.0 \text{ kg})(0.7 \text{ m})^2 = 0.041 \text{ kg} \cdot \text{m}^2$.

Angular Velocity: $\omega = (10.0 \text{ rev/s})(2\pi) = 62.83 \text{ rad/s}$

The rotational kinetic energy is therefore

$$K_R = \frac{1}{2}\left(0.041 \text{ kg} \cdot \text{m}^2\right)(62.83 \text{ rad/s})^2 = 80.93 \text{ J}$$

The translational kinetic energy is

$$K_{\text{T}} = \frac{1}{2}mv^2 = \frac{1}{2}(1.0 \text{ kg})(30.0 \text{ m/s})^2 = 450.0 \text{ J}$$

Thus, the total energy in the boomerang is

$$K_{\text{Total}} = K_R + K_T = 80.93 + 450.0 = 530.93 \text{ J}.$$

b. We use conservation of mechanical energy. Since the boomerang is launched at an angle, we need to write the total energies of the system in terms of its linear kinetic energies using the velocity in the *x*- and *y*-directions. The total energy when the boomerang leaves the hand is

$$E_{\text{Before}} = \frac{1}{2}mv_x^2 + \frac{1}{2}mv_y^2 + \frac{1}{2}I\omega^2$$

The total energy at maximum height is

$$E_{\text{Final}} = \frac{1}{2}mv_x^2 + \frac{1}{2}I\omega^2 + mgh$$

By conservation of mechanical energy, $E_{Before} = E_{Final}$ so we have, after canceling like terms,

$$\frac{1}{2}mv_y^2 = mgh.$$

Since v_y = 30.0 m/s $(\sin 40^\circ)$ = 19.28 m/s, we find

$$h = \frac{(19.28\ \mathrm{m/s})^2}{2\,(9.8\ \mathrm{m/s^2})} = 18.97\ \mathrm{m}$$

Significance

In part (b), the solution demonstrates how energy conservation is an alternative method to solve a problem that normally would be solved using kinematics. In the absence of air resistance, the rotational kinetic energy was not a factor in the solution for the maximum height.

? Exercise 10.4

A nuclear submarine propeller has a moment of inertia of 800.0 kg • m^2. If the submerged propeller has a rotation rate of 4.0 rev/s when the engine is cut, what is the rotation rate of the propeller after 5.0 s when water resistance has taken 50,000 J out of the system?

11.6: Calculating Moments of Inertia

Learning Objectives

- Calculate the moment of inertia for uniformly shaped, rigid bodies
- Apply the parallel axis theorem to find the moment of inertia about any axis parallel to one already known
- Calculate the moment of inertia for compound objects

In the preceding subsection, we defined the moment of inertia but did not show how to calculate it. In this subsection, we show how to calculate the moment of inertia for several standard types of objects, as well as how to use known moments of inertia to find the moment of inertia for a shifted axis or for a compound object. This section is very useful for seeing how to apply a general equation to complex objects (a skill that is critical for more advanced physics and engineering courses).

Moment of Inertia

We defined the moment of inertia I of an object to be

$$I = \sum_i m_i r_i^2 \tag{11.6.1}$$

for all the point masses that make up the object. Because r is the distance to the axis of rotation from each piece of mass that makes up the object, the moment of inertia for any object depends on the chosen axis. To see this, let's take a simple example of two masses at the end of a massless (negligibly small mass) rod (Figure $11.6.1$) and calculate the moment of inertia about two different axes. In this case, the summation over the masses is simple because the two masses at the end of the barbell can be approximated as point masses, and the sum therefore has only two terms.

In the case with the axis in the center of the barbell, each of the two masses m is a distance R away from the axis, giving a moment of inertia of

$$I_1 = mR^2 + mR^2 = 2mR^2. \tag{11.6.2}$$

In the case with the axis at the end of the barbell—passing through one of the masses—the moment of inertia is

$$I_2 = m(0)^2 + m(2R)^2 = 4mR^2. \tag{11.6.3}$$

From this result, we can conclude that it is twice as hard to rotate the barbell about the end than about its center.

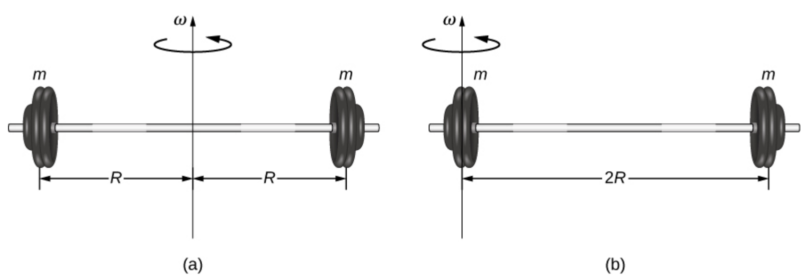


Figure 11.6.1: (a) A barbell with an axis of rotation through its center; (b) a barbell with an axis of rotation through one end.

In this example, we had two point masses and the sum was simple to calculate. However, to deal with objects that are not point-like, we need to think carefully about each of the terms in the equation. The equation asks us to sum over each 'piece of mass' a certain distance from the axis of rotation. But what exactly does each 'piece of mass' mean? Recall that in our derivation of this equation, each piece of mass had the same magnitude of velocity, which means the whole piece had to have a single distance r to the axis of rotation. However, this is not possible unless we take an infinitesimally small piece of mass dm, as shown in Figure $11.6.2$.

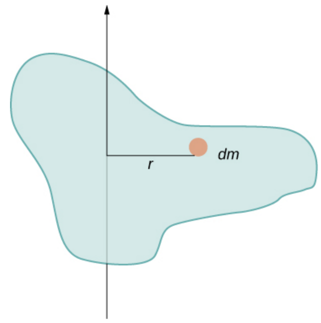


Figure 11.6.2: Using an infinitesimally small piece of mass to calculate the contribution to the total moment of inertia.

The need to use an infinitesimally small piece of mass dm suggests that we can write the moment of inertia by evaluating an integral over infinitesimal masses rather than doing a discrete sum over finite masses:

$$I = \sum_i m_i r_i^2 \tag{11.6.4}$$

becomes

$$I = \int r^2 dm. \tag{11.6.5}$$

This, in fact, is the form we need to generalize the equation for complex shapes. It is best to work out specific examples in detail to get a feel for how to calculate the moment of inertia for specific shapes. This is the focus of most of the rest of this section.

A uniform thin rod with an axis through the center

Consider a uniform (density and shape) thin rod of mass M and length L as shown in Figure $11.6.3$. We want a thin rod so that we can assume the cross-sectional area of the rod is small and the rod can be thought of as a string of masses along a one-dimensional straight line. In this example, the axis of rotation is perpendicular to the rod and passes through the midpoint for simplicity. Our task is to calculate the moment of inertia about this axis. We orient the axes so that the z-axis is the axis of rotation and the x-axis passes through the length of the rod, as shown in the figure. This is a convenient choice because we can then integrate along the x-axis.

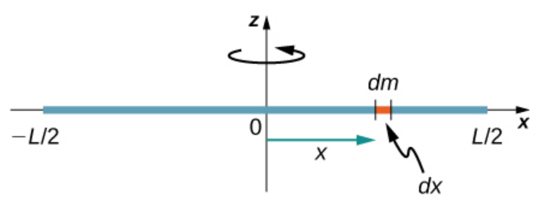


Figure 11.6.3: Calculation of the moment of inertia I for a uniform thin rod about an axis through the center of the rod.

We define dm to be a small element of mass making up the rod. The moment of inertia integral is an integral over the mass distribution. However, we know how to integrate over space, not over mass. We therefore need to find a way to relate mass to spatial variables. We do this using the **linear mass density** λ of the object, which is the mass per unit length. Since the mass density of this object is uniform, we can write

$$\lambda = \frac{m}{l} \; or \; m = \lambda l. \tag{11.6.6}$$

If we take the differential of each side of this equation, we find

$$dm = d(\lambda l) = \lambda (dl) \tag{11.6.7}$$

since λ is constant. We chose to orient the rod along the x-axis for convenience—this is where that choice becomes very helpful. Note that a piece of the rod dl lies completely along the x-axis and has a length dx; in fact, dl = dx in this situation. We can therefore write dm = λ(dx), giving us an integration variable that we know how to deal with. The distance of each piece of mass dm from the axis is given by the variable x, as shown in the figure. Putting this all together, we obtain

$$I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx. \tag{11.6.8}$$

The last step is to be careful about our limits of integration. The rod extends from x = $-\frac{L}{2}$ to x = $\frac{L}{2}$, since the axis is in the middle of the rod at x = 0. This gives us

$$\begin{aligned} I &= \int_{-\frac{L}{2}}^{\frac{L}{2}} x^2 \lambda dx = \lambda \frac{x^3}{3}\Bigg|_{-\frac{L}{2}}^{\frac{L}{2}} \\ &= \lambda \left(\frac{1}{3}\right)\left[\left(\frac{L}{2}\right)^3 - \left(-\frac{L}{2}\right)^3\right] = \lambda \left(\frac{1}{3}\right)\left(\frac{L^3}{8}\right)(2) = \left(\frac{M}{L}\right)\left(\frac{1}{3}\right)\left(\frac{L^3}{8}\right)(2) \\ &= \frac{1}{12} M L^2. \end{aligned}$$

Next, we calculate the moment of inertia for the same uniform thin rod but with a different axis choice so we can compare the results. We would expect the moment of inertia to be smaller about an axis through the center of mass than the endpoint axis, just as it was for the barbell example at the start of this section. This happens because more mass is distributed farther from the axis of rotation.

A Uniform Thin Rod with Axis at the End

Now consider the same uniform thin rod of mass M and length L, but this time we move the axis of rotation to the end of the rod. We wish to find the moment of inertia about this new axis (Figure $11.6.4$).

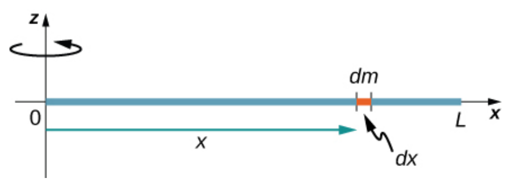


Figure 11.6.4: Calculation of the moment of inertia I for a uniform thin rod about an axis through the end of the rod.

The quantity dm is again defined to be a small element of mass making up the rod. Just as before, we obtain

$$I = \int r^2 dm = \int x^2 dm = \int x^2 \lambda dx. \tag{11.6.9}$$

However, this time we have different limits of integration. The rod extends from $x = 0$ to $x = L$, since the axis is at the end of the rod at $x = 0$. Therefore we find

$$I = \int_0^L x^2 \lambda \, dx \tag{11.6.10}$$

$$= \lambda \frac{x^3}{3}\Bigg|_0^L \tag{11.6.11}$$

$$= \lambda \left(\frac{1}{3}\right)\left[(L)^3 - (0)^3\right] \tag{11.6.12}$$

$$= \lambda \left(\frac{1}{3}\right) L^3 = \left(\frac{M}{L}\right)\left(\frac{1}{3}\right) L^3 \tag{11.6.13}$$

$$= \frac{1}{3} M L^2. \tag{11.6.14}$$

Note the rotational inertia of the rod about its endpoint is larger than the rotational inertia about its center (consistent with the barbell example) by a factor of four.

The Parallel-Axis Theorem

The similarity between the process of finding the moment of inertia of a rod about an axis through its middle and about an axis through its end is striking, and suggests that there might be a simpler method for determining the moment of inertia for a rod about any axis parallel to the axis through the center of mass. Such an axis is called a **parallel axis**. There is a theorem for this, called the **parallel-axis theorem**, which we state here but do not derive in this text.

Parallel-Axis Theorem

Let m be the mass of an object and let d be the distance from an axis through the object's center of mass to a new axis. Then we have

$$I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2. \tag{11.6.15}$$

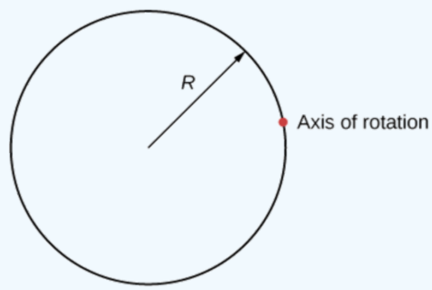
Let's apply this to the uniform thin rod with axis example solved above:

$$I_{parallel-axis} = I_{center\ of\ mass} + md^2 = \frac{1}{12}mL^2 + m\left(\frac{L}{2}\right)^2 = \left(\frac{1}{12} + \frac{1}{4}\right)mL^2 = \frac{1}{3}mL^2. \tag{11.6.16}$$

This result agrees with our more lengthy calculation (Equation 11.6.14). Equation 11.6.15 is a useful equation that we apply in some of the examples and problems.

Exrecise 11.6.1

What is the moment of inertia of a cylinder of radius R and mass m about an axis through a point on the surface, as shown below?



Answer

$$I_{parallel-axis} = I_{center\ of\ mass} + md^2 = mR^2 + mR^2 = 2mR^2$$

A Uniform Thin Disk about an Axis through the Center

Integrating to find the moment of inertia of a two-dimensional object is a little bit trickier, but one shape is commonly done at this level of study—a uniform thin disk about an axis through its center (Figure 11.6.5).

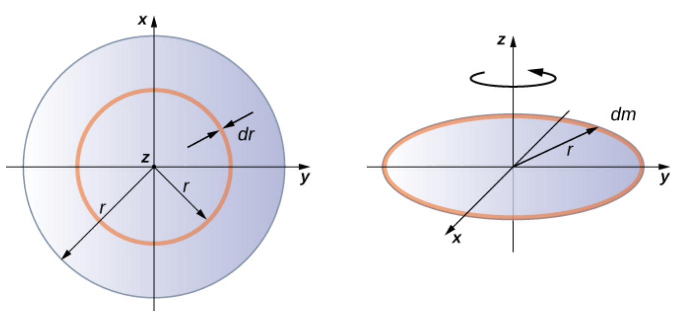


Figure 11.6.5: Calculating the moment of inertia for a thin disk about an axis through its center.

Since the disk is thin, we can take the mass as distributed entirely in the xy-plane. We again start with the relationship for the **surface mass density**, which is the mass per unit surface area. Since it is uniform, the surface mass density σ is constant:

$$\sigma = \frac{m}{A} \tag{11.6.17}$$

or

$$\sigma A = m \tag{11.6.18}$$

so

$$dm = \sigma(dA) \tag{11.6.19}$$

Now we use a simplification for the area. The area can be thought of as made up of a series of thin rings, where each ring is a mass increment dm of radius r equidistant from the axis, as shown in part (b) of the figure. The infinitesimal area of each ring dA is therefore given by the length of each ring ($2\pi r$) times the infinitesimmal width of each ring dr:

$$A = \pi r^2,\ dA = d(\pi r^2) = \pi dr^2 = 2\pi r dr. \tag{11.6.20}$$

The full area of the disk is then made up from adding all the thin rings with a radius range from 0 to R. This radius range then becomes our limits of integration for dr, that is, we integrate from $r = 0$ to $r = R$. Putting this all together, we have

$$\begin{aligned} I &= \int_0^R r^2 \sigma(2\pi r) dr = 2\pi\sigma \int_0^R r^3 dr = 2\pi\sigma \frac{r^4}{4}\Big|_0^R \\ &= 2\pi\sigma\left(\frac{R^4}{4} - 0\right) = 2\pi\left(\frac{m}{A}\right)\left(\frac{R^4}{4}\right) = 2\pi\left(\frac{m}{\pi R^2}\right)\left(\frac{R^4}{4}\right) = \frac{1}{2}mR^2. \end{aligned}$$

Note that this agrees with the value given in Figure 10.5.4.

Calculating the Moment of Inertia for Compound Objects

Now consider a compound object such as that in Figure $11.6.6$, which depicts a thin disk at the end of a thin rod. This cannot be easily integrated to find the moment of inertia because it is not a uniformly shaped object. However, if we go back to the initial definition of moment of inertia as a summation, we can reason that a compound object's moment of inertia can be found from the sum of each part of the object:

$$I_{total} = \sum_i I_i. \tag{11.6.21}$$

It is important to note that the moments of inertia of the objects in Equation $11.6.6$ are **about a common axis**. In the case of this object, that would be a rod of length L rotating about its end, and a thin disk of radius R rotating about an axis shifted off of the center by a distance $L + R$, where R is the radius of the disk. Let's define the mass of the rod to be m_r and the mass of the disk to be m_d.

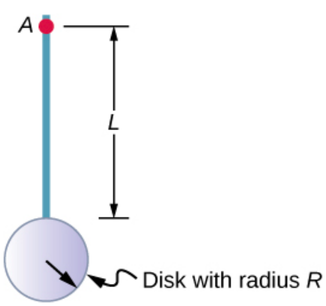


Figure $11.6.6$: Compound object consisting of a disk at the end of a rod. The axis of rotation is located at A.

The moment of inertia of the rod is simply $\frac{1}{3}m_r L^2$, but we have to use the parallel-axis theorem to find the moment of inertia of the disk about the axis shown. The moment of inertia of the disk about its center is $\frac{1}{2}m_d R^2$ and we apply the parallel-axis theorem (Equation 11.6.15) to find

$$I_{parallel-axis} = \frac{1}{2}m_d R^2 + m_d(L+R)^2. \tag{11.6.22}$$

Adding the moment of inertia of the rod plus the moment of inertia of the disk with a shifted axis of rotation, we find the moment of inertia for the compound object to be

$$I_{total} = \frac{1}{3}m_r L^2 + \frac{1}{2}m_d R^2 + m_d(L+R)^2. \tag{11.6.23}$$

Applying moment of inertia calculations to solve problems

Now let’s examine some practical applications of moment of inertia calculations.

Example 11.6.1: Person on a Merry-Go-Round

A 25-kg child stands at a distance $r = 1.0\,m$ from the axis of a rotating merry-go-round (Figure 11.6.7). The merry-go-round can be approximated as a uniform solid disk with a mass of 500 kg and a radius of 2.0 m. Find the moment of inertia of this system.

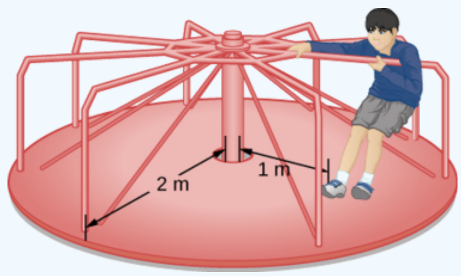


Figure 11.6.7: Calculating the moment of inertia for a child on a merry-go-round.

Strategy

This problem involves the calculation of a moment of inertia. We are given the mass and distance to the axis of rotation of the child as well as the mass and radius of the merry-go-round. Since the mass and size of the child are much smaller than the merry-go-round, we can approximate the child as a point mass. The notation we use is m_c = 25 kg, r_c = 1.0 m, m_m = 500 kg, r_m = 2.0 m. Our goal is to find $I_{total} = \sum_i I_i$ (Equation 11.6.21).

Solution

For the child, $I_c = m_c r^2$, and for the merry-go-round, $I_m = \frac{1}{2} m_m r^2$. Therefore

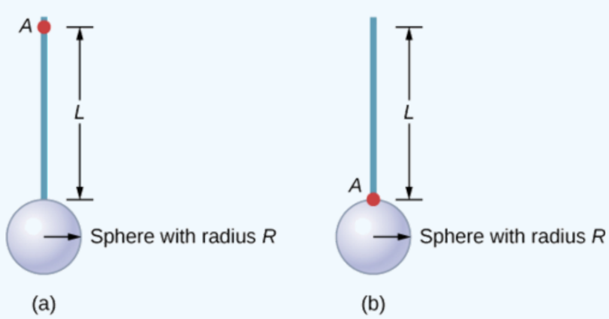
$$I_{total} = 25(1)^2 + \frac{1}{2}(500)(2)^2 = 25 + 1000 = 1025\ kg \cdot m^2.$$

Significance

The value should be close to the moment of inertia of the merry-go-round by itself because it has much more mass distributed away from the axis than the child does.

Example 11.6.2: Rod and Solid Sphere

Find the moment of inertia of the rod and solid sphere combination about the two axes as shown below. The rod has length 0.5 m and mass 2.0 kg. The radius of the sphere is 20.0 cm and has mass 1.0 kg.



Strategy

Since we have a compound object in both cases, we can use the parallel-axis theorem to find the moment of inertia about each axis. In (a), the center of mass of the sphere is located at a distance $L + R$ from the axis of rotation. In (b), the center of mass of the sphere is located a distance R from the axis of rotation. In both cases, the moment of inertia of the rod is about an axis at one end. Refer to Table 10.4 for the moments of inertia for the individual objects.

a.

$$\begin{aligned}
I_{total} &= \sum_i I_i = I_{Rod} + I_{Sphere};\\
I_{Sphere} &= I_{center\ of\ mass} + m_{Sphere}(L+R)^2 = \frac{2}{5}m_{Sphere}R^2 + m_{Sphere}(L+R)^2;\\
I_{total} &= I_{Rod} + I_{Sphere} = \frac{1}{3}m_{Rod}L^2 + \frac{2}{5}m_{Sphere}R^2 + m_{Sphere}(L+R)^2;\\
I_{total} &= \frac{1}{3}(20\ kg)(0.5\ m)^2 + \frac{2}{5}(1.0\ kg)(0.2\ m)^2 + (1.0\ kg)(0.5\ m + 0.2\ m)^2;\\
I_{total} &= (0.167 + 0.016 + 0.490)\ kg\cdot m^2 = 0.673\ kg\cdot m^2.
\end{aligned}$$

b.

$$\begin{aligned}
I_{Sphere} &= \frac{2}{5}m_{Sphere}R^2 + m_{Sphere}R^2;\\
I_{total} &= I_{Rod} + I_{Sphere} = \frac{1}{3}m_{Rod}L^2 + \frac{2}{5}(1.0\ kg)(0.2\ m)^2 + (1.0\ kg)(0.2\ m)^2;\\
I_{total} &= (0.167 + 0.016 + 0.04)\ kg\cdot m^2 = 0.223\ kg\cdot m^2.
\end{aligned}$$

Significance

Using the parallel-axis theorem eases the computation of the moment of inertia of compound objects. We see that the moment of inertia is greater in (a) than (b). This is because the axis of rotation is closer to the center of mass of the system in (b). The simple analogy is that of a rod. The moment of inertia about one end is $\frac{1}{3}\mathrm{mL}^2$, but the moment of inertia through the center of mass along its length is $\frac{1}{12}\mathrm{mL}^2$.

Example 11.6.3: Angular Velocity of a Pendulum

A pendulum in the shape of a rod (Figure 11.6.8) is released from rest at an angle of 30°. It has a length 30 cm and mass 300 g. What is its angular velocity at its lowest point?

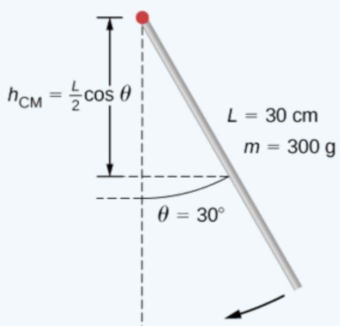


Figure 11.6.8: A pendulum in the form of a rod is released from rest at an angle of 30°.

Strategy

Use conservation of energy to solve the problem. At the point of release, the pendulum has gravitational potential energy, which is determined from the height of the center of mass above its lowest point in the swing. At the bottom of the swing, all of the gravitational potential energy is converted into rotational kinetic energy.

Solution

The change in potential energy is equal to the change in rotational kinetic energy, $\Delta U + \Delta K = 0$.

At the top of the swing:

$$U = mgh_{cm} = mgL^2(\cos\theta).$$

At the bottom of the swing,

$$U = mg\frac{L}{2}.$$

At the top of the swing, the rotational kinetic energy is K = 0. At the bottom of the swing, K = $\frac{1}{2}I\omega^2$. Therefore:

$$\Delta U+\Delta K=0\Rightarrow (mg\frac{L}{2}(1-\cos\theta)-0)+(0-\frac{1}{2}I\omega^2)=0$$

or

$$\frac{1}{2}I\omega^2=mg\frac{L}{2}(1-\cos\theta).$$

Solving for ω, we have

$$\omega=\sqrt{mg\frac{L}{I}(1-\cos\theta)}=\sqrt{mg\frac{L}{\frac{1}{3}mL^2}(1-\cos\theta)}=\sqrt{g\frac{3}{L}(1-\cos\theta)}.$$

Inserting numerical values, we have

$$\omega=\sqrt{(9.8\ m/s^2)\left(\frac{3}{0.3\ m}\right)(1-\cos 30)}=3.6\ rad/s.$$

Significance

Note that the angular velocity of the pendulum does not depend on its mass.

11.7: Torque

Learning Objectives

- Describe how the magnitude of a torque depends on the magnitude of the lever arm and the angle the force vector makes with the lever arm
- Determine the sign (positive or negative) of a torque using the right-hand rule
- Calculate individual torques about a common axis and sum them to find the net torque

An important quantity for describing the dynamics of a rotating rigid body is torque. We see the application of torque in many ways in our world. We all have an intuition about torque, as when we use a large wrench to unscrew a stubborn bolt. Torque is at work in unseen ways, as when we press on the accelerator in a car, causing the engine to put additional torque on the drive train. Or every time we move our bodies from a standing position, we apply a torque to our limbs. In this section, we define torque and make an argument for the equation for calculating torque for a rigid body with fixed-axis rotation.

Defining Torque

So far we have defined many variables that are rotational equivalents to their translational counterparts. Let's consider what the counterpart to force must be. Since forces change the translational motion of objects, the rotational counterpart must be related to changing the rotational motion of an object about an axis. We call this rotational counterpart **torque**.

In everyday life, we rotate objects about an axis all the time, so intuitively we already know much about torque. Consider, for example, how we rotate a door to open it. First, we know that a door opens slowly if we push too close to its hinges; it is more efficient to rotate a door open if we push far from the hinges. Second, we know that we should push perpendicular to the plane of the door; if we push parallel to the plane of the door, we are not able to rotate it. Third, the larger the force, the more effective it is in opening the door; the harder you push, the more rapidly the door opens. The first point implies that the farther the force is applied from the axis of rotation, the greater the angular acceleration; the second implies that the effectiveness depends on the angle at which the force is applied; the third implies that the magnitude of the force must also be part of the equation. Note that for rotation in a plane, torque has two possible directions. Torque is either clockwise or counterclockwise relative to the chosen pivot point. Figure $11.7.1$ shows counterclockwise rotations.

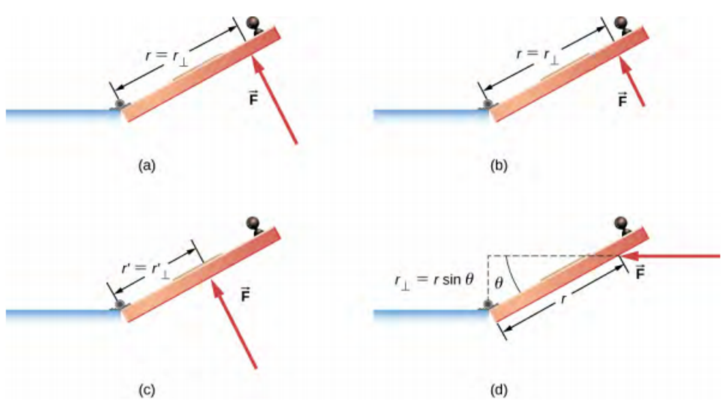


Figure 11.7.1: Torque is the turning or twisting effectiveness of a force, illustrated here for door rotation on its hinges (as viewed from overhead). Torque has both magnitude and direction. (a) A counterclockwise torque is produced by a force $\vec{F}$ acting at a distance r from the hinges (the pivot point). (b) A smaller counterclockwise torque is produced when a smaller force $\vec{F}\,'$ acts at the same distance r from the hinges. (c) The same force as in (a) produces a smaller counterclockwise torque when applied at a smaller distance from the hinges. (d) A smaller counterclockwise torque is produced by the same magnitude force as (a) acting at the same distance as (a) but at an angle θ that is less than 90°.

Now let's consider how to define torques in the general three-dimensional case.

Torque

When a force $\vec{F}$ is applied to a point P whose position is $\vec{r}$ relative to O (Figure $11.7.2$), the torque $\vec{\tau}$ around O is

$$\vec{\tau} = \vec{r} \times \vec{F}. \tag{11.7.1}$$

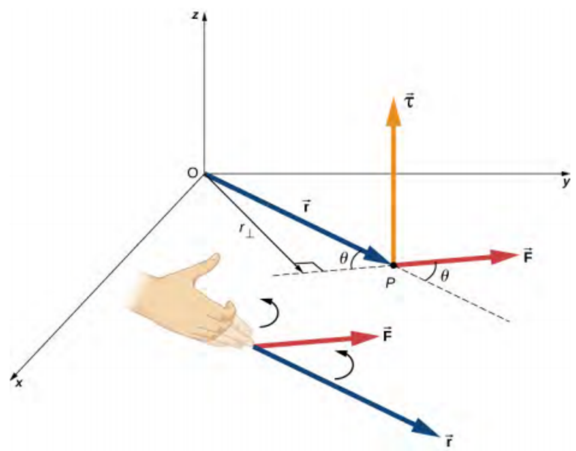


Figure 11.7.2: The torque is perpendicular to the plane defined by $\vec{r}$ and $\vec{F}$ and its direction is determined by the right-hand rule.

From the definition of the cross product, the torque $\vec{\tau}$ is perpendicular to the plane containing $\vec{r}$ and $\vec{F}$ and has magnitude

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = rF\sin\theta, \tag{11.7.2}$$

where θ is the angle between the vectors $\vec{r}$ and $\vec{F}$. The SI unit of torque is newtons times meters, usually written as N • m. The quantity $r_\perp$ = rsin θ is the perpendicular distance from O to the line determined by the vector $\vec{F}$ and is called the **lever arm**. Note that the greater the lever arm, the greater the magnitude of the torque. In terms of the lever arm, the magnitude of the torque is

$$|\vec{\tau}| = r_\perp F. \tag{11.7.3}$$

The cross product $\vec{r} \times \vec{F}$ also tells us the sign of the torque. In Figure 11.7.2, the cross product $\vec{r} \times \vec{F}$ is along the positive z-axis, which by convention is a positive torque. If $\vec{r} \times \vec{F}$ is along the negative z-axis, this produces a negative torque.

If we consider a disk that is free to rotate about an axis through the center, as shown in Figure 11.7.3, we can see how the angle between the radius $\vec{r}$ and the force $\vec{F}$ affects the magnitude of the torque. If the angle is zero, the torque is zero; if the angle is 90°, the torque is maximum. The torque in Figure 11.7.3 is positive because the direction of the torque by the right-hand rule is out of the page along the positive z-axis. The disk rotates counterclockwise due to the torque, in the same direction as a positive angular acceleration.

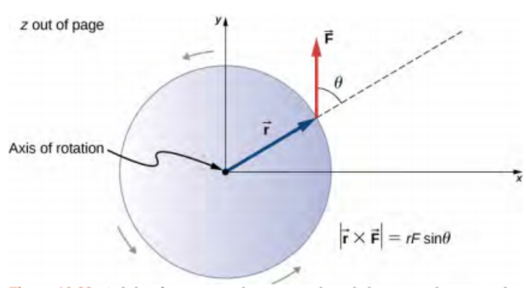


Figure 11.7.3: A disk is free to rotate about its axis through the center. The magnitude of the torque on the disk is rFsin θ.When θ = 0°, the torque is zero and the disk does not rotate. When θ = 90°, the torque is maximum and the disk rotates with maximum angular acceleration.

Any number of torques can be calculated about a given axis. The individual torques add to produce a net torque about the axis. When the appropriate sign (positive or negative) is assigned to the magnitudes of individual torques about a specified axis, the net torque about the axis is the sum of the individual torques:

$$\vec{\tau}_{net} = \sum_i |\vec{\tau}_i|. \tag{11.7.4}$$

Calculating Net Torque for Rigid Bodies on a Fixed Axis

In the following examples, we calculate the torque both abstractly and as applied to a rigid body. We first introduce a problem-solving strategy.

? Problem-Solving Strategy: Finding Net Torque

1. Choose a coordinate system with the pivot point or axis of rotation as the origin of the selected coordinate system.
2. Determine the angle between the lever arm $\vec{r}$ and the force vector.
3. Take the cross product of $\vec{r}$ and $\vec{F}$ to determine if the torque is positive or negative about the pivot point or axis.
4. Evaluate the magnitude of the torque using $r_{\perp}F$.
5. Assign the appropriate sign, positive or negative, to the magnitude.
6. Sum the torques to find the net torque.

✓ Example 10.14: Calculating Torque

Four forces are shown in Figure 11.7.4 at particular locations and orientations with respect to a given xy-coordinate system. Find the torque due to each force about the origin, then use your results to find the net torque about the origin.

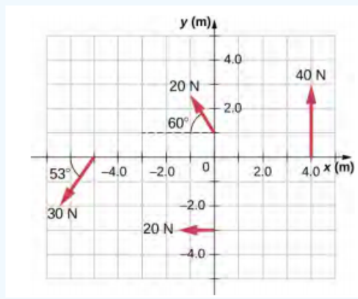


Figure 11.7.4: Four forces producing torques.

Strategy

This problem requires calculating torque. All known quantities—forces with directions and lever arms—are given in the figure. The goal is to find each individual torque and the net torque by summing the individual torques. Be careful to assign the correct sign to each torque by using the cross product of $\vec{r}$ and the force vector $\vec{F}$.

Solution

Use $|\vec{\tau}| = r_{\perp}F = rF\sin\theta$ to find the magnitude and $\vec{\tau} = \vec{r} \times \vec{F}$ to determine the sign of the torque.

The torque from force 40 N in the first quadrant is given by (4)(40)sin 90° = 160 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page, positive.

The torque from force 20 N in the third quadrant is given by −(3)(20)sin 90° = − 60 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is into the page, so it is negative.

The torque from force 30 N in the third quadrant is given by (5)(30)sin 53° = 120 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page, positive.

The torque from force 20 N in the second quadrant is given by (1)(20)sin 30° = 10 N • m.

The cross product of $\vec{r}$ and $\vec{F}$ is out of the page.

The net torque is therefore $\tau_{net} = \sum_i |\tau_i|$ = 160 − 60 + 120 + 10 = 230 N • m.

Significance

Note that each force that acts in the counterclockwise direction has a positive torque, whereas each force that acts in the clockwise direction has a negative torque. The torque is greater when the distance, force, or perpendicular components are greater.

✔ Example 10.15: Calculating Torque on a rigid body

Figure 11.7.5 shows several forces acting at different locations and angles on a flywheel. We have $|\vec{F}_1|$ = 20 N, $|\vec{F}_2|$ = 30 N, $|\vec{F}_3|$ = 30 N, and r = 0.5 m. Find the net torque on the flywheel about an axis through the center.

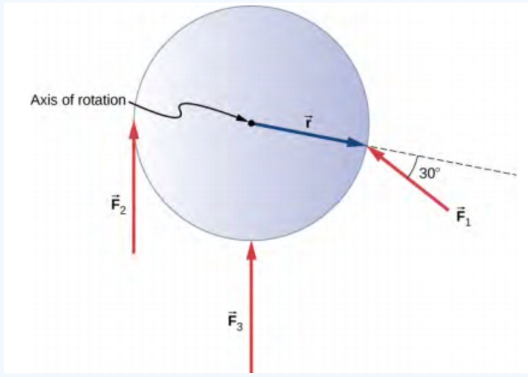


Figure 11.7.5: Three forces acting on a flywheel.

Strategy

We calculate each torque individually, using the cross product, and determine the sign of the torque. Then we sum the torques to find the net torque. Solution We start with $\vec{F}_1$. If we look at Figure 11.7.5, we see that $\vec{F}_1$ makes an angle of 90° + 60° with the radius vector $\vec{r}$. Taking the cross product, we see that it is out of the page and so is positive. We also see this from calculating its magnitude:

$$|\vec{\tau}_1| = rF_1 \sin 150^o = (0.5\ m)(20\ N)(0.5) = 5.0\ N \cdot m. \tag{11.7.5}$$

Next we look at $\vec{F}_2$. The angle between $\vec{F}_2$ and $\vec{r}$ is 90° and the cross product is into the page so the torque is negative. Its value is

$$|\vec{\tau}_2| = -rF_2 \sin 90^o = (-0.5\ m)(30\ N) = -15.0\ N \cdot m. \tag{11.7.6}$$

When we evaluate the torque due to $\vec{F}_3$, we see that the angle it makes with $\vec{r}$ is zero so $\vec{r} \times \vec{F}_3$ = 0. Therefore, $\vec{F}_3$ does not produce any torque on the flywheel.

We evaluate the sum of the torques:

$$\tau_{net} = \sum_i |\tau_i| = 5 - 15 = -10\ N \cdot m. \tag{11.7.7}$$

Significance

The axis of rotation is at the center of mass of the flywheel. Since the flywheel is on a fixed axis, it is not free to translate. If it were on a frictionless surface and not fixed in place, $\vec{F}_3$ would cause the flywheel to translate, as well as $\vec{F}_1$. Its motion would be a combination of translation and rotation.

? Exercise 10.6

A large ocean-going ship runs aground near the coastline, similar to the fate of the **Costa Concordia**, and lies at an angle as shown below. Salvage crews must apply a torque to right the ship in order to float the vessel for transport. A force of 5.0 x 10^5 N acting at point A must be applied to right the ship. What is the torque about the point of contact of the ship with the ground (Figure 11.7.6)?

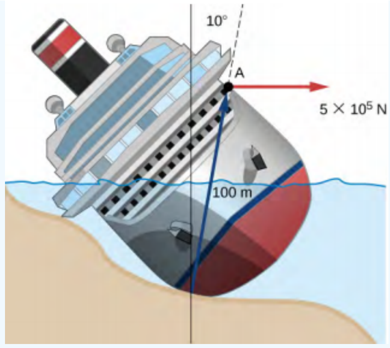


Figure 11.7.6: A ship runs aground and tilts, requiring torque to be applied to return the vessel to an upright position.

11.8: Newton’s Second Law for Rotation

Learning Objectives

- Calculate the torques on rotating systems about a fixed axis to find the angular acceleration
- Explain how changes in the moment of inertia of a rotating system affect angular acceleration with a fixed applied torque

In this subsection, we put together all the pieces learned so far in this chapter to analyze the dynamics of rotating rigid bodies. We have analyzed motion with kinematics and rotational kinetic energy but have not yet connected these ideas with force and/or torque. In this subsection, we introduce the rotational equivalent to Newton’s second law of motion and apply it to rigid bodies with fixed-axis rotation.

Newton’s Second Law for Rotation

We have thus far found many counterparts to the translational terms used throughout this text, most recently, torque, the rotational analog to force. This raises the question: Is there an analogous equation to Newton’s second law, $\sum \vec{F} = m\vec{a}$, which involves torque and rotational motion? To investigate this, we start with Newton’s second law for a single particle rotating around an axis and executing circular motion. Let’s exert a force $\vec{F}$ on a point mass m that is at a distance r from a pivot point (Figure 11.8.1). The particle is constrained to move in a circular path with fixed radius and the force is tangent to the circle. We apply Newton’s second law to determine the magnitude of the acceleration $a = \frac{F}{m}$ in the direction of $\vec{F}$. Recall that the magnitude of the tangential acceleration is proportional to the magnitude of the angular acceleration by $a = r\alpha$. Substituting this expression into Newton’s second law, we obtain

$$F = mr\alpha. \tag{11.8.1}$$

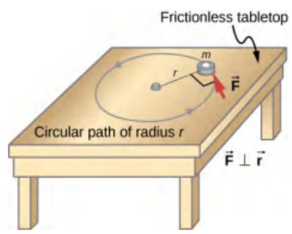


Figure 11.8.1: An object is supported by a horizontal frictionless table and is attached to a pivot point by a cord that supplies centripetal force. A force $\vec{F}$ is applied to the object perpendicular to the radius r, causing it to accelerate about the pivot point. The force is perpendicular to r.

Multiply both sides of this equation by r,

$$rF = mr^2\alpha. \tag{11.8.2}$$

Note that the left side of this equation is the torque about the axis of rotation, where r is the lever arm and F is the force, perpendicular to r. Recall that the moment of inertia for a point particle is $I = mr^2$. The torque applied perpendicularly to the point mass in Figure 11.8.1 is therefore

$$\tau = I\alpha. \tag{11.8.3}$$

The torque on the particle is equal to the moment of inertia about the rotation axis times the angular acceleration. We can generalize this equation to a rigid body rotating about a fixed axis.

Newton’s Second Law for Rotation

If more than one torque acts on a rigid body about a fixed axis, then the sum of the torques equals the moment of inertia times the angular acceleration:

$$\sum_i \tau_i = I\alpha. \tag{11.8.4}$$

The term Iα is a scalar quantity and can be positive or negative (counterclockwise or clockwise) depending upon the sign of the net torque. Remember the convention that counterclockwise angular acceleration is positive. Thus, if a rigid body is rotating clockwise and experiences a positive torque (counterclockwise), the angular acceleration is positive.

Equation 11.8.4 is **Newton's second law for rotation** and tells us how to relate torque, moment of inertia, and rotational kinematics. This is called the equation for **rotational dynamics**. With this equation, we can solve a whole class of problems involving force and rotation. It makes sense that the relationship for how much force it takes to rotate a body would include the moment of inertia, since that is the quantity that tells us how easy or hard it is to change the rotational motion of an object.

Deriving Newton's Second Law for Rotation in Vector Form

As before, when we found the angular acceleration, we may also find the torque vector. The second law $\sum \vec{F}$ = m$\vec{a}$ tells us the relationship between net force and how to change the translational motion of an object. We have a vector rotational equivalent of this equation, which can be found by using Equation 10.2.10 and Figure 10.2.7. Equation 10.2.10 relates the angular acceleration to the position and tangential acceleration vectors:

$$\vec{a} = \vec{\alpha} \times \vec{r}. \tag{11.8.5}$$

We form the cross product of this equation with $\vec{r}$ and use a cross product identity (note that $\vec{r} \cdot \vec{\alpha}$ = 0):

$$\vec{r} \times \vec{a} = \vec{r} \times (\vec{\alpha} \times \vec{r}) = \vec{\alpha}(\vec{r} \cdot \vec{r}) - \vec{r}(\vec{r} \cdot \vec{\alpha}) = \vec{\alpha}(\vec{r} \cdot \vec{r}) = \vec{\alpha} r^2. \tag{11.8.6}$$

We now form the cross product of Newton's second law with the position vector $\vec{r}$,

$$\sum(\vec{r} \times \vec{F}) = \vec{r} \times (m\vec{a}) = m\vec{r} \times \vec{a} = mr^2\vec{\alpha}. \tag{11.8.7}$$

Identifying the first term on the left as the sum of the torques, and mr^2 as the moment of inertia, we arrive at Newton's second law of rotation in vector form:

$$\sum \tau = I\alpha. \tag{11.8.8}$$

This equation is exactly Equation 11.8.4 but with the torque and angular acceleration as vectors. An important point is that the torque vector is in the same direction as the angular acceleration.

Applying the Rotational Dynamics Equation

Before we apply the rotational dynamics equation to some everyday situations, let's review a general problem-solving strategy for use with this category of problems.

Problem-Solving Strategy: Rotational Dynamics

1. Examine the situation to determine that torque and mass are involved in the rotation. Draw a careful sketch of the situation.
2. Determine the system of interest.
3. Draw a free-body diagram. That is, draw and label all external forces acting on the system of interest.
4. Identify the pivot point. If the object is in equilibrium, it must be in equilibrium for all possible pivot points—chose the one that simplifies your work the most.
5. Apply $\sum_i \tau_i = I\alpha$, the rotational equivalent of Newton's second law, to solve the problem. Care must be taken to use the correct moment of inertia and to consider the torque about the point of rotation.
6. As always, check the solution to see if it is reasonable.

Example 10.16: Calculating the Effect of Mass Distribution on a Merry-Go-Round

Consider the father pushing a playground merry-go-round in Figure 11.8.2. He exerts a force of 250 N at the edge of the 200.0-kg merry-go-round, which has a 1.50-m radius. Calculate the angular acceleration produced (a) when no one is on the merry-go-round and (b) when an 18.0-kg child sits 1.25 m away from the center. Consider the merry-go-round itself to be a uniform disk with negligible friction.

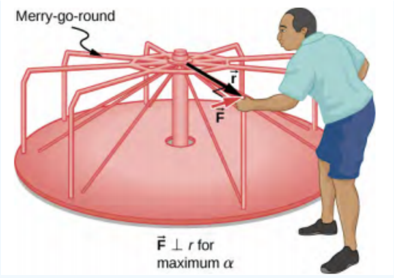


Figure 11.8.2: A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Strategy

The net torque is given directly by the expression $\sum_i \tau_i = I\alpha$, To solve for α, we must first calculate the net torque τ (which is the same in both cases) and moment of inertia I (which is greater in the second case).

Solution

a. The moment of inertia of a solid disk about this axis is given in Figure 10.5.4 to be

$$\frac{1}{2}MR^2. \tag{11.8.9}$$

We have M = 50.0 kg and R = 1.50 m, so

$$I = (0.500)(50.0\ kg)(1.50\ m)^2 = 56.25\ kg \cdot m^2. \tag{11.8.10}$$

To find the net torque, we note that the applied force is perpendicular to the radius and friction is negligible, so that

$$\tau = rF \sin\theta = (1.50\ m)(250.0\ N) - 375.0\ N \cdot m. \tag{11.8.11}$$

Now, after we substitute the known values, we find the angular acceleration to be

$$\alpha = \frac{\tau}{I} = \frac{375.0\ N \cdot m}{56.25\ kg \cdot m^2} = 6.67\ rad/s^2. \tag{11.8.12}$$

b. We expect the angular acceleration for the system to be less in this part because the moment of inertia is greater when the child is on the merry-go-round. To find the total moment of inertia I, we first find the child's moment of inertia I_c by approximating the child as a point mass at a distance of 1.25 m from the axis. Then

$$I_c = mR^2 = (18.0\ kg)(1.25\ m)^2 = 28.13\ kg \cdot m^2. \tag{11.8.13}$$

The total moment of inertia is the sum of the moments of inertia of the merry-go-round and the child (about the same axis):

$$I = (28.13\ kg \cdot m^2) + (56.25\ kg \cdot m^2) = 84.38\ kg \cdot m^2. \tag{11.8.14}$$

Substituting known values into the equation for α gives

$$\alpha = \frac{\tau}{I} = \frac{375.0\ N \cdot m}{84.38\ kg \cdot m^2} = 4.44\ rad/s. \tag{11.8.15}$$

Significance

The angular acceleration is less when the child is on the merry-go-round than when the merry-go-round is empty, as expected. The angular accelerations found are quite large, partly due to the fact that friction was considered to be negligible. If, for example, the father kept pushing perpendicularly for 2.00 s, he would give the merry-goround an angular velocity of 13.3 rad/s when it is empty but only 8.89 rad/s when the child is on it. In terms of revolutions per second, these angular velocities are 2.12 rev/s and 1.41 rev/s, respectively. The father would end up running at about 50 km/h in the first case.

? Exercise 10.7

The fan blades on a jet engine have a moment of inertia 30.0 kg • m^2. In 10 s, they rotate counterclockwise from rest up to a rotation rate of 20 rev/s. (a) What torque must be applied to the blades to achieve this angular acceleration? (b) What is the torque required to bring the fan blades rotating at 20 rev/s to a rest in 20 s?

11.9: Work and Power for Rotational Motion

Learning Objectives

- Use the work-energy theorem to analyze rotation to find the work done on a system when it is rotated about a fixed axis for a finite angular displacement
- Solve for the angular velocity of a rotating rigid body using the work-energy theorem
- Find the power delivered to a rotating rigid body given the applied torque and angular velocity
- Summarize the rotational variables and equations and relate them to their translational counterparts

Thus far in the section, we have extensively addressed kinematics and dynamics for rotating rigid bodies around a fixed axis. In this final subsection, we define work and power within the context of rotation about a fixed axis, which has applications to both physics and engineering. The discussion of work and power makes our treatment of rotational motion almost complete, with the exception of rolling motion and angular momentum, which are discussed in Angular Momentum. We begin this subsection with a treatment of the work-energy theorem for rotation.

Work for Rotational Motion

Now that we have determined how to calculate kinetic energy for rotating rigid bodies, we can proceed with a discussion of the work done on a rigid body rotating about a fixed axis. Figure 11.9.1 shows a rigid body that has rotated through an angle $d\theta$ from A to B while under the influence of a force $\vec{F}$. The external force $\vec{F}$ is applied to point P, whose position is $\vec{r}$, and the rigid body is constrained to rotate about a fixed axis that is perpendicular to the page and passes through O. The rotational axis is fixed, so the vector $\vec{r}$ moves in a circle of radius r, and the vector d $\vec{s}$ is perpendicular to $\vec{r}$.

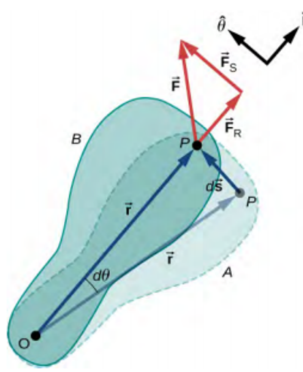


Figure 11.9.1: A rigid body rotates through an angle $d\theta$ from A to B by the action of an external force $\vec{F}$ applied to point P.

We have

$$\vec{s} = \vec{\theta} \times \vec{r}. \tag{11.9.1}$$

Thus,

$$d\vec{s} = d(\vec{\theta} \times \vec{r}) = d\vec{\theta} \times \vec{r} + d\vec{r} \times \vec{\theta} = d\vec{\theta} \times \vec{r}. \tag{11.9.2}$$

Note that $d\vec{r}$ is zero because $\vec{r}$ is fixed on the rigid body from the origin O to point P. Using the definition of work, we obtain

$$W = \int \sum \vec{F} \cdot d\vec{s} = \int \sum \vec{F} \cdot (d\vec{\theta} \times \vec{r}) = \int d\vec{\theta} \cdot (\vec{r} \times \sum \vec{F}) \tag{11.9.3}$$

where we used the identity $\vec{a} \cdot (\vec{b} \times \vec{c}) = \vec{b} \cdot (\vec{c} \times \vec{a})$. Noting that $(\vec{r} \times \sum \vec{F}) = \sum \vec{\tau}$, we arrive at the expression for the rotational **work** done on a rigid body:

$$W = \int \sum \vec{\tau} \cdot d\vec{\theta}. \tag{11.9.4}$$

The total work done on a rigid body is the sum of the torques integrated over the angle through which the body rotates. The incremental work is

$$dW = \left(\sum_i \tau_i\right) d\theta \tag{11.9.5}$$

where we have taken the dot product in Equation 11.9.4, leaving only torques along the axis of rotation. In a rigid body, all particles rotate through the same angle; thus the work of every external force is equal to the torque times the common incremental angle dθ. The quantity $\left(\sum_i \tau_i\right)$ is the net torque on the body due to external forces.

Similarly, we found the kinetic energy of a rigid body rotating around a fixed axis by summing the kinetic energy of each particle that makes up the rigid body. Since the work-energy theorem $W_i = \Delta K_i$ is valid for each particle, it is valid for the sum of the particles and the entire body.

Work-Energy Theorem for Rotation

The work-energy theorem for a rigid body rotating around a fixed axis is

$$W_{AB} = K_B - K_A \tag{11.9.6}$$

where

$$K = \frac{1}{2} I\omega^2 \tag{11.9.7}$$

and the rotational work done by a net force rotating a body from point A to point B is

$$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right) d\theta. \tag{11.9.8}$$

We give a strategy for using this equation when analyzing rotational motion.

Problem-Solving Strategy: Work-Energy Theorem for Rotational Motion

1. Identify the forces on the body and draw a free-body diagram. Calculate the torque for each force.
2. Calculate the work done during the body's rotation by every torque.
3. Apply the work-energy theorem by equating the net work done on the body to the change in rotational kinetic energy

Let's look at two examples and use the work-energy theorem to analyze rotational motion.

Example 10.17: Rotational Work and Energy

A 12.0 N • m torque is applied to a flywheel that rotates about a fixed axis and has a moment of inertia of 30.0 kg • m^2. If the flywheel is initially at rest, what is its angular velocity after it has turned through eight revolutions?

Strategy

We apply the work-energy theorem. We know from the problem description what the torque is and the angular displacement of the flywheel. Then we can solve for the final angular velocity.

Solution

The flywheel turns through eight revolutions, which is 16π radians. The work done by the torque, which is constant and therefore can come outside the integral in Equation 11.9.8, is

$$W_{AB} = \tau(\theta_B - \theta_A). \tag{11.9.9}$$

We apply the work-energy theorem:

$$W_{AB} = \tau(\theta_B - \theta_A) = \frac{1}{2} I\omega_B^2 - \frac{1}{2} I\omega_A^2. \tag{11.9.10}$$

With τ = 12.0 N • m, $\theta_B - \theta_A$ = 16.0π rad, I = 30.0 kg • m^2, and ω_A = 0, we have

$$(12.0\ N \cdot m)(16.0\pi\ rad) = \frac{1}{2}(30.0\ kg \cdot m^2)(\omega_B^2) - 0. \tag{11.9.11}$$

Therefore,

$$\omega_B = 6.3\ rad/s. \tag{11.9.12}$$

This is the angular velocity of the flywheel after eight revolutions.

Significance

The work-energy theorem provides an efficient way to analyze rotational motion, connecting torque with rotational kinetic energy.

Example 10.18: Rotational Work- A Pulley

A string wrapped around the pulley in Figure 11.9.2 is pulled with a constant downward force $\vec{F}$ of magnitude 50 N. The radius R and moment of inertia I of the pulley are 0.10 m and 2.5 x 10^{-3} kg • m^2, respectively. If the string does not slip, what is the angular velocity of the pulley after 1.0 m of string has unwound? Assume the pulley starts from rest.

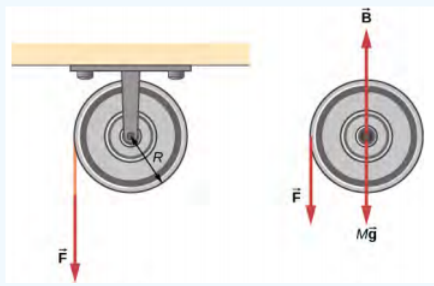


Figure 11.9.2: (a) A string is wrapped around a pulley of radius R. (b) The free-body diagram.

Strategy

Looking at the free-body diagram, we see that neither $\vec{B}$, the force on the bearings of the pulley, nor M$\vec{g}$, the weight of the pulley, exerts a torque around the rotational axis, and therefore does no work on the pulley. As the pulley rotates through an angle θ, $\vec{F}$ acts through a distance d such that d = Rθ.

Solution

Since the torque due to $\vec{F}$ has magnitude τ = RF, we have

$$W = \tau\theta = (FR)\theta = FD. \tag{11.9.13}$$

If the force on the string acts through a distance of 1.0 m, we have, from the work-energy theorem,

$$\begin{aligned} W_{AB} &= K_B - K_A \\ Fd &= \frac{1}{2}I\omega^2 - 0 \\ (50.0\ N)(1.0\ m) &= \frac{1}{2}(2.5\times 10^{-3}\ kg\cdot m^2)\omega^2. \end{aligned}$$

Solving for ω, we obtain

$$\omega = 200.0\ rad/s. \tag{11.9.14}$$

Power for Rotational Motion

Power always comes up in the discussion of applications in engineering and physics. Power for rotational motion is equally as important as power in linear motion and can be derived in a similar way as in linear motion when the force is a constant. The linear power when the force is a constant is P = $\vec{F}\cdot\vec{v}$. If the net torque is constant over the angular displacement, Equation 10.8.4 simplifies and the net torque can be taken out of the integral. In the following discussion, we assume the net torque is constant. We can apply the definition of power derived in Power to rotational motion. From Work and Kinetic Energy, the instantaneous power (or just power) is defined as the rate of doing work,

$$P = \frac{dW}{dt}. \tag{11.9.15}$$

If we have a constant net torque, Equation 10.8.4 becomes W = $\tau\theta$ and the power is

$$P = \frac{dW}{dt} = \frac{d}{dt}(\tau\theta) = \tau\frac{d\theta}{dt} \tag{11.9.16}$$

or

$$P = \tau\omega. \tag{11.9.17}$$

Example 10.19: Torque on a Boat Propeller

A boat engine operating at 9.0 x 10^4 W is running at 300 rev/min. What is the torque on the propeller shaft?

Strategy

We are given the rotation rate in rev/min and the power consumption, so we can easily calculate the torque.

Solution

$$300.0\ rev/min = 31.4\ rad/s; \tag{11.9.18}$$

$$\tau = \frac{P}{\omega} = \frac{9.0 \times 10^4\ N \cdot m/s}{31.4\ rad/s} = 2864.8\ N \cdot m. \tag{11.9.19}$$

Significance

It is important to note the radian is a dimensionless unit because its definition is the ratio of two lengths. It therefore does not appear in the solution.

Exercise 10.8

A constant torque of 500 kN • m is applied to a wind turbine to keep it rotating at 6 rad/s. What is the power required to keep the turbine rotating?

Rotational and Translational Relationships Summarized

The rotational quantities and their linear analog are summarized in three tables. Table 10.5 summarizes the rotational variables for circular motion about a fixed axis with their linear analogs and the connecting equation, except for the centripetal acceleration, which stands by itself. Table 10.6 summarizes the rotational and translational kinematic equations. Table 10.7 summarizes the rotational dynamics equations with their linear analogs.

Table 10.5 - Rotational and Translational Variables: Summary

Rotational	Translational	Relationship
θ (11.9.20)	x (11.9.21)	$\theta = \frac{s}{r}$ (11.9.22)
ω (11.9.23)	v_f (11.9.24)	$\omega = \frac{v_t}{r}$ (11.9.25)
α (11.9.26)	a_t (11.9.27)	$\alpha = \frac{a_t}{r}$ (11.9.28)
	a_c (11.9.29)	$a_c = \frac{v_t^2}{r}$ (11.9.30)

Table 10.6 - Rotational and Translational Kinematic Equations: Summary

Rotational		Translational	
$\theta_f = \theta_0 + \bar{\omega}t$	(11.9.31)	$x = x_0 + \bar{v}t$	(11.9.32)
$\omega_f = \omega_0 + \alpha t$	(11.9.33)	$v_f = v_0 + at$	(11.9.34)
$\theta_f = \theta_0 + \omega_0 t + \frac{1}{2}\alpha t^2$	(11.9.35)	$x_f = x_0 + v_0 t + \frac{1}{2}at^2$	(11.9.36)
$\omega_f^2 = \omega_0^2 + 2\alpha(\Delta\theta)$	(11.9.37)	$v_f^2 = v_0^2 + 2a(\Delta x)$	(11.9.38)

Table 10.7 - Rotational and Translational Equations: Dynamics

Rotational		Translational	
$I = \sum_i m_i r_i^2$	(11.9.39)	m	(11.9.40)
$K = \frac{1}{2}I\omega^2$	(11.9.41)	$K = \frac{1}{2}mv^2$	(11.9.42)
$\sum_i \tau_i = I\alpha$	(11.9.43)	$\sum_i \vec{F}_i = m\vec{a}$	(11.9.44)
$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right) d\theta$	(11.9.45)	$W = \int \vec{F} \cdot d\vec{s}$	(11.9.46)
$P = \tau\omega$	(11.9.47)	$P = \vec{F} \cdot \vec{v}$	(11.9.48)

11.10: Fixed-Axis Rotation Introduction (Exercises)

Conceptual Questions

10.1 Rotational Variables

1. A clock is mounted on the wall. As you look at it, what is the direction of the angular velocity vector of the second hand?
2. What is the value of the angular acceleration of the second hand of the clock on the wall?
3. A baseball bat is swung. Do all points on the bat have the same angular velocity? The same tangential speed?
4. The blades of a blender on a counter are rotating clockwise as you look into it from the top. If the blender is put to a greater speed what direction is the angular acceleration of the blades?

10.2 Rotation with Constant Angular Acceleration

5. If a rigid body has a constant angular acceleration, what is the functional form of the angular velocity in terms of the time variable?
6. If a rigid body has a constant angular acceleration, what is the functional form of the angular position?
7. If the angular acceleration of a rigid body is zero, what is the functional form of the angular velocity?
8. A massless tether with a masses tied to both ends rotates about a fixed axis through the center. Can the total acceleration of the tether/mass combination be zero if the angular velocity is constant?

10.3 Relating Angular and Translational Quantities

9. Explain why centripetal acceleration changes the direction of velocity in circular motion but not its magnitude.
10. In circular motion, a tangential acceleration can change the magnitude of the velocity but not its direction. Explain your answer.
11. Suppose a piece of food is on the edge of a rotating microwave oven plate. Does it experience nonzero tangential acceleration, centripetal acceleration, or both when: (a) the plate starts to spin faster? (b) The plate rotates at constant angular velocity? (c) The plate slows to a halt?

10.4 Moment of Inertia and Rotational Kinetic Energy

12. What if another planet the same size as Earth were put into orbit around the Sun along with Earth. Would the moment of inertia of the system increase, decrease, or stay the same?
13. A solid sphere is rotating about an axis through its center at a constant rotation rate. Another hollow sphere of the same mass and radius is rotating about its axis through the center at the same rotation rate. Which sphere has a greater rotational kinetic energy?

10.5 Calculating Moments of Inertia

14. If a child walks toward the center of a merry-go-round, does the moment of inertia increase or decrease?
15. A discus thrower rotates with a discus in his hand before letting it go. (a) How does his moment of inertia change after releasing the discus? (b) What would be a good approximation to use in calculating the moment of inertia of the discus thrower and discus?
16. Does increasing the number of blades on a propeller increase or decrease its moment of inertia, and why?
17. The moment of inertia of a long rod spun around an axis through one end perpendicular to its length is $\frac{mL^2}{3}$. Why is this moment of inertia greater than it would be if you spun a point mass m at the location of the center of mass of the rod (at $\frac{L}{2}$) (that would be $\frac{mL^2}{4}$)
18. Why is the moment of inertia of a hoop that has a mass M and a radius R greater than the moment of inertia of a disk that has the same mass and radius?

10.6 Torque

19. What three factors affect the torque created by a force relative to a specific pivot point?
20. Give an example in which a small force exerts a large torque. Give another example in which a large force exerts a small torque.
21. When reducing the mass of a racing bike, the greatest benefit is realized from reducing the mass of the tires and wheel rims. Why does this allow a racer to achieve greater accelerations than would an identical reduction in the mass of the bicycle’s frame?

22. Can a single force produce a zero torque?
23. Can a set of forces have a net torque that is zero and a net force that is not zero?
24. Can a set of forces have a net force that is zero and a net torque that is not zero?
25. In the expression $\vec{r} \times \vec{F}$ can $|\vec{r}|$ ever be less than the lever arm? Can it be equal to the lever arm?

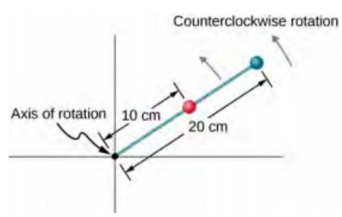
10.7 Newton's Second Law for Rotation

26. If you were to stop a spinning wheel with a constant force, where on the wheel would you apply the force to produce the maximum negative acceleration?
27. A rod is pivoted about one end. Two forces $\vec{F}$ and $-\vec{F}$ are applied to it. Under what circumstances will the rod not rotate?

Problems

10.1 Rotational Variables

28. Calculate the angular velocity of Earth.
29. A track star runs a 400-m race on a 400-m circular track in 45 s. What is his angular velocity assuming a constant speed?
30. A wheel rotates at a constant rate of 2.0 x 10^3 rev/min. (a) What is its angular velocity in radians per second? (b) Through what angle does it turn in 10 s? Express the solution in radians and degrees.
31. A particle moves 3.0 m along a circle of radius 1.5 m. (a) Through what angle does it rotate? (b) If the particle makes this trip in 1.0 s at a constant speed, what is its angular velocity? (c) What is its acceleration?
32. A compact disc rotates at 500 rev/min. If the diameter of the disc is 120 mm, (a) what is the tangential speed of a point at the edge of the disc? (b) At a point halfway to the center of the disc?
33. **Unreasonable results**. The propeller of an aircraft is spinning at 10 rev/s when the pilot shuts off the engine. The propeller reduces its angular velocity at a constant 2.0 rad/s^2 for a time period of 40 s. What is the rotation rate of the propeller in 40 s? Is this a reasonable situation?
34. A gyroscope slows from an initial rate of 32.0 rad/s at a rate of 0.700 rad/s^2. How long does it take to come to rest?
35. On takeoff, the propellers on a UAV (unmanned aerial vehicle) increase their angular velocity for 3.0 s from rest at a rate of ω = (25.0t) rad/s where t is measured in seconds. (a) What is the instantaneous angular velocity of the propellers at t = 2.0 s? (b) What is the angular acceleration?
36. The angular position of a rod varies as 20.0t^2 radians from time t = 0. The rod has two beads on it as shown in the following figure, one at 10 cm from the rotation axis and the other at 20 cm from the rotation axis. (a) What is the instantaneous angular velocity of the rod at t = 5 s? (b) What is the angular acceleration of the rod? (c) What are the tangential speeds of the beads at t = 5 s? (d) What are the tangential accelerations of the beads at t = 5 s? (e) What are the centripetal accelerations of the beads at t = 5 s?

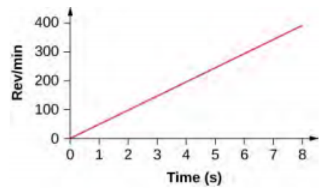


10.2 Rotation with Constant Angular Acceleration

37. A wheel has a constant angular acceleration of 5.0 rad/s^2. Starting from rest, it turns through 300 rad. (a) What is its final angular velocity? (b) How much time elapses while it turns through the 300 radians?
38. During a 6.0-s time interval, a flywheel with a constant angular acceleration turns through 500 radians that acquire an angular velocity of 100 rad/s. (a) What is the angular velocity at the beginning of the 6.0 s? (b) What is the angular acceleration of the flywheel?
39. The angular velocity of a rotating rigid body increases from 500 to 1500 rev/min in 120 s. (a) What is the angular acceleration of the body? (b) Through what angle does it turn in this 120 s?
40. A flywheel slows from 600 to 400 rev/min while rotating through 40 revolutions. (a) What is the angular acceleration of the flywheel? (b) How much time elapses during the 40 revolutions?
41. A wheel 1.0 m in radius rotates with an angular acceleration of 4.0 rad/s^2. (a) If the wheel's initial angular velocity is 2.0 rad/s, what is its angular velocity after 10 s? (b) Through what angle does it rotate in the 10-s interval? (c) What are the

tangential speed and acceleration of a point on the rim of the wheel at the end of the 10-s interval?

42. A vertical wheel with a diameter of 50 cm starts from rest and rotates with a constant angular acceleration of 5.0 rad/s^2 around a fixed axis through its center counterclockwise. (a) Where is the point that is initially at the bottom of the wheel at t = 10 s? (b) What is the point's linear acceleration at this instant?
43. A circular disk of radius 10 cm has a constant angular acceleration of 1.0 rad/s^2; at t = 0 its angular velocity is 2.0 rad/s. (a) Determine the disk's angular velocity at t = 5.0 s . (b) What is the angle it has rotated through during this time? (c) What is the tangential acceleration of a point on the disk at t = 5.0 s?
44. The angular velocity vs. time for a fan on a hovercraft is shown below. (a) What is the angle through which the fan blades rotate in the first 8 seconds? (b) Verify your result using the kinematic equations.



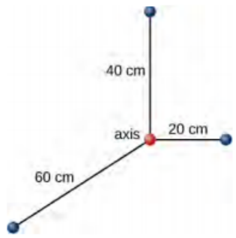
45. A rod of length 20 cm has two beads attached to its ends. The rod with beads starts rotating from rest. If the beads are to have a tangential speed of 20 m/s in 7 s, what is the angular acceleration of the rod to achieve this?

10.3 Relating Angular and Translational Quantities

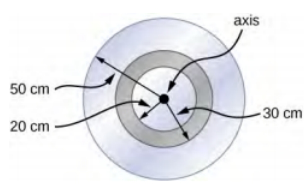
46. At its peak, a tornado is 60.0 m in diameter and carries 500 km/h winds. What is its angular velocity in revolutions per second?
47. A man stands on a merry-go-round that is rotating at 2.5 rad/s. If the coefficient of static friction between the man's shoes and the merry-go-round is $\mu S = 0.5$, how far from the axis of rotation can he stand without sliding?
48. An ultracentrifuge accelerates from rest to 100,000 rpm in 2.00 min. (a) What is the average angular acceleration in rad/s^2? (b) What is the tangential acceleration of a point 9.50 cm from the axis of rotation? (c) What is the centripetal acceleration in m/s^2 and multiples of g of this point at full rpm? (d) What is the total distance traveled by a point 9.5 cm from the axis of rotation of the ultracentrifuge?
49. A wind turbine is rotating counterclockwise at 0.5 rev/s and slows to a stop in 10 s. Its blades are 20 m in length. (a) What is the angular acceleration of the turbine? (b) What is the centripetal acceleration of the tip of the blades at t = 0 s? (c) What is the magnitude and direction of the total linear acceleration of the tip of the blades at t = 0 s?
50. What is (a) the angular speed and (b) the linear speed of a point on Earth's surface at latitude 30° N. Take the radius of the Earth to be 6309 km. (c) At what latitude would your linear speed be 10 m/s?
51. A child with mass 30 kg sits on the edge of a merrygo-round at a distance of 3.0 m from its axis of rotation. The merry-go-round accelerates from rest up to 0.4 rev/s in 10 s. If the coefficient of static friction between the child and the surface of the merry-go-round is 0.6, does the child fall off before 5 s?
52. A bicycle wheel with radius 0.3m rotates from rest to 3 rev/s in 5 s. What is the magnitude and direction of the total acceleration vector at the edge of the wheel at 1.0 s?
53. The angular velocity of a flywheel with radius 1.0 m varies according to $\omega(t) = 2.0t$. Plot $a_c(t)$ and $a_t(t)$ from t = 0 to 3.0 s for r = 1.0 m. Analyze these results to explain when $a_c >> a_t$ and when $a_c << a_t$ for a point on the flywheel at a radius of 1.0 m.

10.4 Moment of Inertia and Rotational Kinetic Energy

54. A system of point particles is shown in the following figure. Each particle has mass 0.3 kg and they all lie in the same plane. (a) What is the moment of inertia of the system about the given axis? (b) If the system rotates at 5 rev/s, what is its rotational kinetic energy?

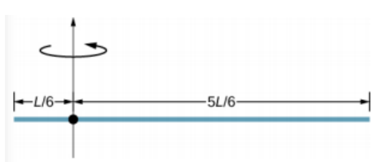


55. (a) Calculate the rotational kinetic energy of Earth on its axis. (b) What is the rotational kinetic energy of Earth in its orbit around the Sun?
56. Calculate the rotational kinetic energy of a 12-kg motorcycle wheel if its angular velocity is 120 rad/s and its inner radius is 0.280 m and outer radius 0.330 m.
57. A baseball pitcher throws the ball in a motion where there is rotation of the forearm about the elbow joint as well as other movements. If the linear velocity of the ball relative to the elbow joint is 20.0 m/s at a distance of 0.480 m from the joint and the moment of inertia of the forearm is 0.500 kg-m^2, what is the rotational kinetic energy of the forearm?
58. A diver goes into a somersault during a dive by tucking her limbs. If her rotational kinetic energy is 100 J and her moment of inertia in the tuck is 9.0 kg • m^2, what is her rotational rate during the somersault?
59. An aircraft is coming in for a landing at 300 meters height when the propeller falls off. The aircraft is flying at 40.0 m/s horizontally. The propeller has a rotation rate of 20 rev/s, a moment of inertia of 70.0 kg • m^2, and a mass of 200 kg. Neglect air resistance. (a) With what translational velocity does the propeller hit the ground? (b) What is the rotation rate of the propeller at impact?
60. If air resistance is present in the preceding problem and reduces the propeller's rotational kinetic energy at impact by 30%, what is the propeller's rotation rate at impact?
61. A neutron star of mass 2 x 10^{30} kg and radius 10 km rotates with a period of 0.02 seconds. What is its rotational kinetic energy?
62. An electric sander consisting of a rotating disk of mass 0.7 kg and radius 10 cm rotates at 15 rev/s. When applied to a rough wooden wall the rotation rate decreases by 20%. (a) What is the final rotational kinetic energy of the rotating disk? (b) How much has its rotational kinetic energy decreased?
63. A system consists of a disk of mass 2.0 kg and radius 50 cm upon which is mounted an annular cylinder of mass 1.0 kg with inner radius 20 cm and outer radius 30 cm (see below). The system rotates about an axis through the center of the disk and annular cylinder at 10 rev/s. (a) What is the moment of inertia of the system? (b) What is its rotational kinetic energy?



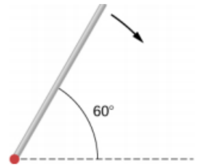
10.5 Calculating Moments of Inertia

64. While punting a football, a kicker rotates his leg about the hip joint. The moment of inertia of the leg is 3.75 kg • m^2 and its rotational kinetic energy is 175 J. (a) What is the angular velocity of the leg? (b) What is the velocity of tip of the punter's shoe if it is 1.05 m from the hip joint?
65. Using the parallel axis theorem, what is the moment of inertia of the rod of mass m about the axis shown below?

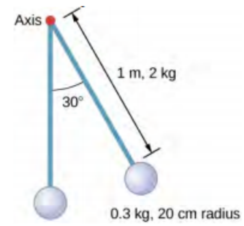


66. Find the moment of inertia of the rod in the previous problem by direct integration.
67. A uniform rod of mass 1.0 kg and length 2.0 m is free to rotate about one end (see the following figure). If the rod is released from rest at an angle of 60° with respect to the horizontal, what is the speed of the tip of the rod as it passes the

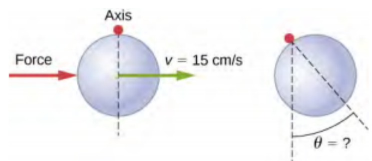
horizontal position?



68. A pendulum consists of a rod of mass 2 kg and length 1 m with a solid sphere at one end with mass 0.3 kg and radius 20 cm (see the following figure). If the pendulum is released from rest at an angle of 30°, what is the angular velocity at the lowest point?



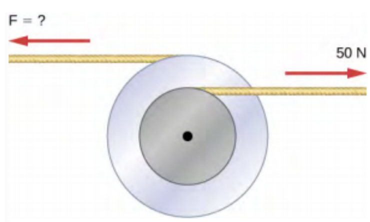
69. A solid sphere of radius 10 cm is allowed to rotate freely about an axis. The sphere is given a sharp blow so that its center of mass starts from the position shown in the following figure with speed 15 cm/s. What is the maximum angle that the diameter makes with the vertical?



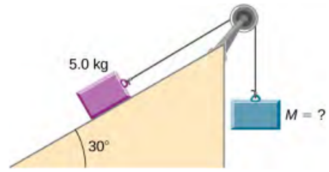
70. Calculate the moment of inertia by direct integration of a thin rod of mass M and length L about an axis through the rod at L/3, as shown below. Check your answer with the parallel-axis theorem.

10.6 Torque

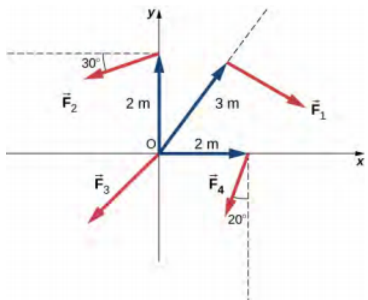
71. Two flywheels of negligible mass and differ3ent radii are bonded together and rotate about a common axis (see below). The smaller flywheel of radius 30 cm has a cord that has a pulling force of 50 N on it. What pulling force needs to be applied to the cord connecting the larger flywheel of radius 50 cm such that the combination does not rotate?



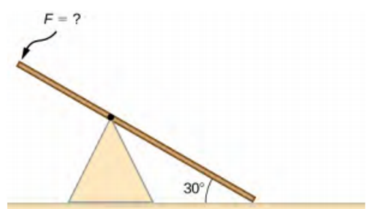
72. The cylinder head bolts on a car are to be tightened with a torque of 62.0 N·m. If a mechanic uses a wrench of length 20 cm, what perpendicular force must he exert on the end of the wrench to tighten a bolt correctly?
73. (a) When opening a door, you push on it perpendicularly with a force of 55.0 N at a distance of 0.850 m from the hinges. What torque are you exerting relative to the hinges? (b) Does it matter if you push at the same height as the hinges? There is only one pair of hinges.
74. When tightening a bolt, you push perpendicularly on a wrench with a force of 165 N at a distance of 0.140 m from the center of the bolt. How much torque are you exerting in newton-meters (relative to the center of the bolt)?
75. What hanging mass must be placed on the cord to keep the pulley from rotating (see the following figure)? The mass on the frictionless plane is 5.0 kg. The inner radius of the pulley is 20 cm and the outer radius is 30 cm.



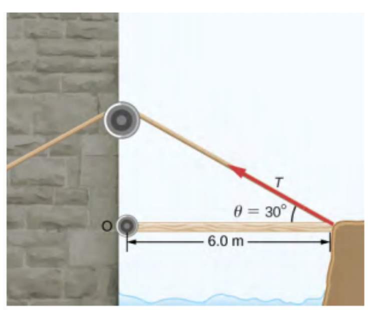
76. A simple pendulum consists of a massless tether 50 cm in length connected to a pivot and a small mass of 1.0 kg attached at the other end. What is the torque about the pivot when the pendulum makes an angle of 40° with respect to the vertical?
77. Calculate the torque about the z-axis that is out of the page at the origin in the following figure, given that $F_1 = 3$ N, $F_2 = 2$ N, $F_3 = 3$ N, $F_4 = 1.8$ N.



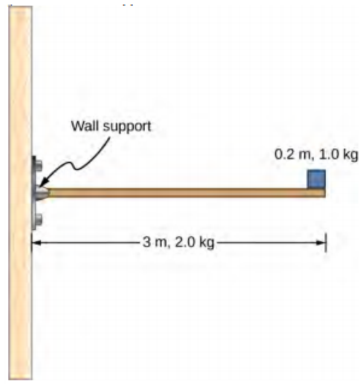
78. A seesaw has length 10.0 m and uniform mass 10.0 kg and is resting at an angle of 30° with respect to the ground (see the following figure). The pivot is located at 6.0 m. What magnitude of force needs to be applied perpendicular to the seesaw at the raised end so as to allow the seesaw to barely start to rotate?



79. A pendulum consists of a rod of mass 1 kg and length 1 m connected to a pivot with a solid sphere attached at the other end with mass 0.5 kg and radius 30 cm. What is the torque about the pivot when the pendulum makes an angle of 30° with respect to the vertical?
80. A torque of 5.00×10^3 N • m is required to raise a drawbridge (see the following figure). What is the tension necessary to produce this torque? Would it be easier to raise the drawbridge if the angle θ were larger or smaller?



81. A horizontal beam of length 3 m and mass 2.0 kg has a mass of 1.0 kg and width 0.2 m sitting at the end of the beam (see the following figure). What is the torque of the system about the support at the wall?

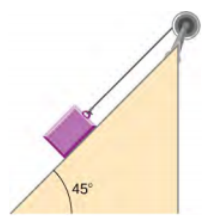


82. What force must be applied to end of a rod along the x-axis of length 2.0 m in order to produce a torque on the rod about the origin of $8.0\hat{k}$ N • m?
83. What is the torque about the origin of the force $(5.0\ \hat{i} - 2.0\ \hat{j} + 1.0\hat{k})$ N if it is applied at the point whose position is: $\vec{r} = (-2.0\ \hat{i} + 4.0\hat{j})$ m?

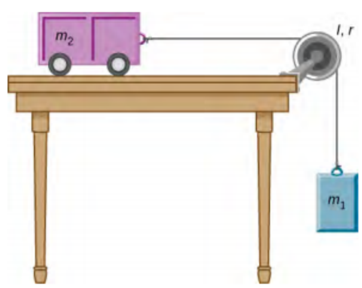
10.7 Newton’s Second Law for Rotation

84. You have a grindstone (a disk) that is 90.0 kg, has a 0.340-m radius, and is turning at 90.0 rpm, and you press a steel axe against it with a radial force of 20.0 N. (a) Assuming the kinetic coefficient of friction between steel and stone is 0.20, calculate the angular acceleration of the grindstone. (b) How many turns will the stone make before coming to rest?
85. Suppose you exert a force of 180 N tangential to a 0.280-m-radius, 75.0-kg grindstone (a solid disk). (a)What torque is exerted? (b) What is the angular acceleration assuming negligible opposing friction? (c) What is the angular acceleration if there is an opposing frictional force of 20.0 N exerted 1.50 cm from the axis?
86. A flywheel ($I = 50$ kg • m^2) starting from rest acquires an angular velocity of 200.0 rad/s while subject to a constant torque from a motor for 5 s. (a) What is the angular acceleration of the flywheel? (b) What is the magnitude of the torque?
87. A constant torque is applied to a rigid body whose moment of inertia is 4.0 kg • m^2 around the axis of rotation. If the wheel starts from rest and attains an angular velocity of 20.0 rad/s in 10.0 s, what is the applied torque?
88. A torque of 50.0 N • m is applied to a grinding wheel ($I = 20.0$ kg • m^2) for 20 s. (a) If it starts from rest, what is the angular velocity of the grinding wheel after the torque is removed? (b) Through what angle does the wheel move through while the torque is applied?
89. A flywheel ($I = 100.0$ kg • m^2) rotating at 500.0 rev/ min is brought to rest by friction in 2.0 min. What is the frictional torque on the flywheel?
90. A uniform cylindrical grinding wheel of mass 50.0 kg and diameter 1.0 m is turned on by an electric motor. The friction in the bearings is negligible. (a) What torque must be applied to the wheel to bring it from rest to 120 rev/min in 20 revolutions? (b) A tool whose coefficient of kinetic friction with the wheel is 0.60 is pressed perpendicularly against the wheel with a force of 40.0 N. What torque must be supplied by the motor to keep the wheel rotating at a constant angular velocity?
91. Suppose when Earth was created, it was not rotating. However, after the application of a uniform torque after 6 days, it was rotating at 1 rev/day. (a) What was the angular acceleration during the 6 days? (b) What torque was applied to Earth during this period? (c) What force tangent to Earth at its equator would produce this torque?
92. A pulley of moment of inertia 2.0 kg • m^2 is mounted on a wall as shown in the following figure. Light strings are wrapped around two circumferences of the pulley and weights are attached. What are (a) the angular acceleration of the pulley and (b) the linear acceleration of the weights? Assume the following data: $r_1 = 50$ cm, $r_2 = 20$ cm, $m_1 = 1.0$ kg, $m_2 = 2.0$ kg.

93. A block of mass 3 kg slides down an inclined plane at an angle of 45° with a massless tether attached to a pulley with mass 1 kg and radius 0.5 m at the top of the incline (see the following figure). The pulley can be approximated as a disk. The coefficient of kinetic friction on the plane is 0.4. What is the acceleration of the block?

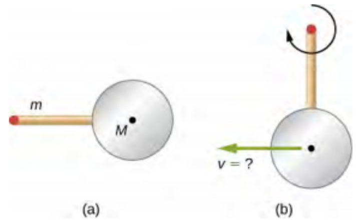


94. The cart shown below moves across the table top as the block falls. What is the acceleration of the cart? Neglect friction and assume the following data: $m_1 = 2.0$ kg, $m_2 = 4.0$ kg, $I = 0.4 \text{ kg} \cdot \text{m}^2$, $r = 20$ cm.



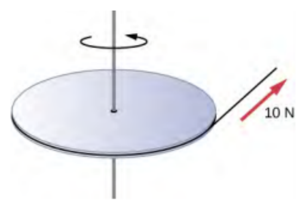
95. A uniform rod of mass and length is held vertically by two strings of negligible mass, as shown below. (a) Immediately after the string is cut, what is the linear acceleration of the free end of the stick? (b) Of the middle of the stick?

96. A thin stick of mass 0.2 kg and length L = 0.5 m is attached to the rim of a metal disk of mass M = 2.0 kg and radius R = 0.3 m. The stick is free to rotate around a horizontal axis through its other end (see the following figure). (a) If the combination is released with the stick horizontal, what is the speed of the center of the disk when the stick is vertical? (b) What is the acceleration of the center of the disk at the instant the stick is released? (c) At the instant the stick passes through the vertical?

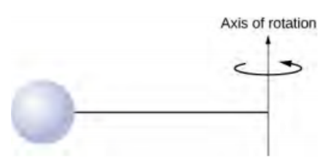


10.8 Work and Power for Rotational Motion

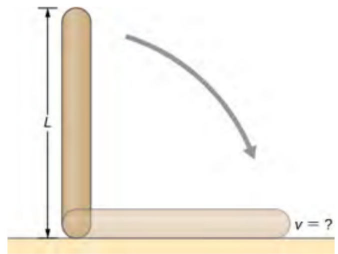
97. A wind turbine rotates at 20 rev/min. If its power output is 2.0 MW, what is the torque produced on the turbine from the wind?
98. A clay cylinder of radius 20 cm on a potter's wheel spins at a constant rate of 10 rev/s. The potter applies a force of 10 N to the clay with his hands where the coefficient of friction is 0.1 between his hands and the clay. What is the power that the potter has to deliver to the wheel to keep it rotating at this constant rate?
99. A uniform cylindrical grindstone has a mass of 10 kg and a radius of 12 cm. (a) What is the rotational kinetic energy of the grindstone when it is rotating at 1.5×10^3 rev/min? (b) After the grindstone's motor is turned off, a knife blade is pressed against the outer edge of the grindstone with a perpendicular force of 5.0 N. The coefficient of kinetic friction between the grindstone and the blade is 0.80. Use the work energy theorem to determine how many turns the grindstone makes before it stops.
100. A uniform disk of mass 500 kg and radius 0.25 m is mounted on frictionless bearings so it can rotate freely around a vertical axis through its center (see the following figure). A cord is wrapped around the rim of the disk and pulled with a force of 10 N. (a) How much work has the force done at the instant the disk has completed three revolutions, starting from rest? (b) Determine the torque due to the force, then calculate the work done by this torque at the instant the disk has completed three revolutions? (c) What is the angular velocity at that instant? (d) What is the power output of the force at that instant?



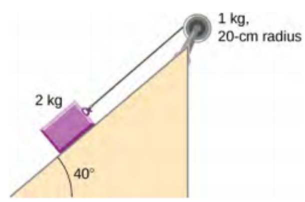
101. A propeller is accelerated from rest to an angular velocity of 1000 rev/min over a period of 6.0 seconds by a constant torque of 2.0×10^3 N • m. (a) What is the moment of inertia of the propeller? (b) What power is being provided to the propeller 3.0 s after it starts rotating?
102. A sphere of mass 1.0 kg and radius 0.5 m is attached to the end of a massless rod of length 3.0 m. The rod rotates about an axis that is at the opposite end of the sphere (see below). The system rotates horizontally about the axis at a constant 400 rev/min. After rotating at this angular speed in a vacuum, air resistance is introduced and provides a force 0.15 N on the sphere opposite to the direction of motion. What is the power provided by air resistance to the system 100.0 s after air resistance is introduced?



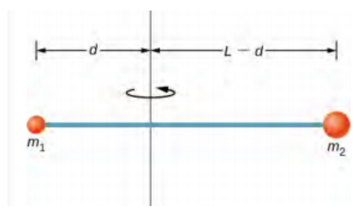
103. A uniform rod of length L and mass M is held vertically with one end resting on the floor as shown below. When the rod is released, it rotates around its lower end until it hits the floor. Assuming the lower end of the rod does not slip, what is the linear velocity of the upper end when it hits the floor?



104. An athlete in a gym applies a constant force of 50 N to the pedals of a bicycle at a rate of the pedals moving 60 rev/min. The length of the pedal arms is 30 cm. What is the power delivered to the bicycle by the athlete?
105. A 2-kg block on a frictionless inclined plane at 40° has a cord attached to a pulley of mass 1 kg and radius 20 cm (see the following figure). (a) What is the acceleration of the block down the plane? (b) What is the work done by the cord on the pulley?

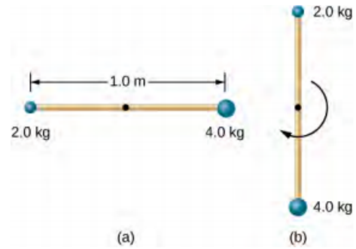


106. Small bodies of mass m_1 and m_2 are attached to opposite ends of a thin rigid rod of length L and mass M. The rod is mounted so that it is free to rotate in a horizontal plane around a vertical axis (see below). What distance d from m_1 should the rotational axis be so that a minimum amount of work is required to set the rod rotating at an angular velocity ω?

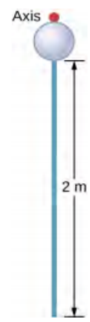


Additional Problems

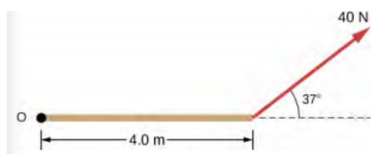
107. A cyclist is riding such that the wheels of the bicycle have a rotation rate of 3.0 rev/s. If the cyclist brakes such that the rotation rate of the wheels decrease at a rate of 0.3 rev/s^2, how long does it take for the cyclist to come to a complete stop?
108. Calculate the angular velocity of the orbital motion of Earth around the Sun.
109. A phonograph turntable rotating at $33\frac{1}{3}$ rev/min slows down and stops in 1.0 min. (a) What is the turntable's angular acceleration assuming it is constant? (b) How many revolutions does the turntable make while stopping?
110. With the aid of a string, a gyroscope is accelerated from rest to 32 rad/s in 0.40 s under a constant angular acceleration. (a) What is its angular acceleration in rad/s^2? (b) How many revolutions does it go through in the process?
111. Suppose a piece of dust has fallen on a CD. If the spin rate of the CD is 500 rpm, and the piece of dust is 4.3 cm from the center, what is the total distance traveled by the dust in 3 minutes? (Ignore accelerations due to getting the CD rotating.)
112. A system of point particles is rotating about a fixed axis at 4 rev/s. The particles are fixed with respect to each other. The masses and distances to the axis of the point particles are m_1 = 0.1 kg, r_1 = 0.2 m, m_2 = 0.05 kg, r_2 = 0.4 m, m_3 = 0.5 kg, r_3 = 0.01 m. (a) What is the moment of inertia of the system? (b) What is the rotational kinetic energy of the system?
113. Calculate the moment of inertia of a skater given the following information. (a) The 60.0-kg skater is approximated as a cylinder that has a 0.110-m radius. (b) The skater with arms extended is approximated by a cylinder that is 52.5 kg, has a 0.110-m radius, and has two 0.900-m-long arms which are 3.75 kg each and extend straight out from the cylinder like rods rotated about their ends.
114. A stick of length 1.0 m and mass 6.0 kg is free to rotate about a horizontal axis through the center. Small bodies of masses 4.0 and 2.0 kg are attached to its two ends (see the following figure). The stick is released from the horizontal position. What is the angular velocity of the stick when it swings through the vertical?



115. A pendulum consists of a rod of length 2 m and mass 3 kg with a solid sphere of mass 1 kg and radius 0.3 m attached at one end. The axis of rotation is as shown below. What is the angular velocity of the pendulum at its lowest point if it is released from rest at an angle of 30°?



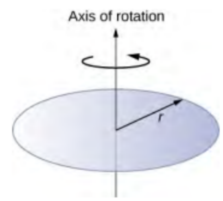
116. Calculate the torque of the 40-N force around the axis through O and perpendicular to the plane of the page as shown below.



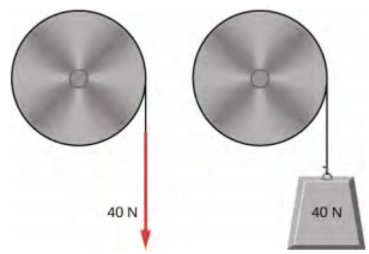
117. Two children push on opposite sides of a door during play. Both push horizontally and perpendicular to the door. One child pushes with a force of 17.5 N at a distance of 0.600 m from the hinges, and the second child pushes at a distance of 0.450 m. What force must the second child exert to keep the door from moving? Assume friction is negligible.
118. The force of 20 $\hat{j}$ N is applied at $\vec{r}$ = (4.0 $\hat{i}$ − 2.0 $\hat{j}$) m. What is the torque of this force about the origin? 119. An automobile engine can produce 200 N • m of torque. Calculate the angular acceleration produced if 95.0% of this torque is applied to the drive shaft, axle, and rear wheels of a car, given the following information. The car is suspended so that the wheels can turn freely. Each wheel acts like a 15.0-kg disk that has a 0.180-m radius. The walls of each tire act like a 2.00-kg annular ring that has inside radius of 0.180 m and outside radius of 0.320 m. The tread of each tire acts like a 10.0-kg hoop of radius 0.330 m. The 14.0-kg axle acts like a rod that has a 2.00-cm radius. The 30.0-kg drive shaft acts like a rod that has a 3.20-cm radius.
119. A grindstone with a mass of 50 kg and radius 0.8 m maintains a constant rotation rate of 4.0 rev/s by a motor while a knife is pressed against the edge with a force of 5.0 N. The coefficient of kinetic friction between the grindstone and the blade is 0.8. What is the power provided by the motor to keep the grindstone at the constant rotation rate?

Challenge Problems

121. The angular acceleration of a rotating rigid body is given by α = (2.0 − 3.0t) rad/s^2. If the body starts rotating from rest at t = 0, (a) what is the angular velocity? (b) Angular position? (c) What angle does it rotate through in 10 s? (d) Where does the vector perpendicular to the axis of rotation indicating 0° at t = 0 lie at t = 10 s?
122. Earth’s day has increased by 0.002 s in the last century. If this increase in Earth’s period is constant, how long will it take for Earth to come to rest?
123. A disk of mass m, radius R, and area A has a surface mass density $\sigma = \frac{mr}{AR}$ (see the following figure). What is the moment of inertia of the disk about an axis through the center?



124. Zorch, an archenemy of Rotation Man, decides to slow Earth's rotation to once per 28.0 h by exerting an opposing force at and parallel to the equator. Rotation Man is not immediately concerned, because he knows Zorch can only exert a force of 4.00 x 10^7 N (a little greater than a Saturn V rocket's thrust). How long must Zorch push with this force to accomplish his goal? (This period gives Rotation Man time to devote to other villains.)
125. A cord is wrapped around the rim of a solid cylinder of radius 0.25 m, and a constant force of 40 N is exerted on the cord shown, as shown in the following figure. The cylinder is mounted on frictionless bearings, and its moment of inertia is 6.0 kg • m^2. (a) Use the work energy theorem to calculate the angular velocity of the cylinder after 5.0 m of cord have been removed. (b) If the 40-N force is replaced by a 40-N weight, what is the angular velocity of the cylinder after 5.0 m of cord have unwound?



Contributors and Attributions

11.11: Fixed-Axis Rotation Introduction (Summary)

Key Terms

angular acceleration	time rate of change of angular velocity
angular position	angle a body has rotated through in a fixed coordinate system
angular velocity	time rate of change of angular position
instantaneous angular acceleration	derivative of angular velocity with respect to time
instantaneous angular velocity	derivative of angular position with respect to time
kinematics of rotational motion	describes the relationships among rotation angle, angular velocity, angular acceleration, and time
lever arm	perpendicular distance from the line that the force vector lies on to a given axis
linear mass density	the mass per unit length λ of a one dimensional object
moment of inertia	rotational mass of rigid bodies that relates to how easy or hard it will be to change the angular velocity of the rotating rigid body
Newton's second law for rotation	sum of the torques on a rotating system equals its moment of inertia times its angular acceleration
parallel axis	axis of rotation that is parallel to an axis about which the moment of inertia of an object is known
parallel-axis theorem	if the moment of inertia is known for a given axis, it can be found for any axis parallel to it
rotational dynamics	analysis of rotational motion using the net torque and moment of inertia to find the angular acceleration
rotational kinetic energy	kinetic energy due to the rotation of an object; this is part of its total kinetic energy
rotational work	work done on a rigid body due to the sum of the torques integrated over the angle through with the body rotates
surface mass density	mass per unit area σ of a two dimensional object
torque	cross product of a force and a lever arm to a given axis
total linear acceleration	vector sum of the centripetal acceleration vector and the tangential acceleration vector
work-energy theorem for rotation	the total rotational work done on a rigid body is equal to the change in rotational kinetic energy of the body

Key Equations

Angular position	$$\theta=\frac{s}{r} \tag{11.11.1}$$
Angular velocity	$$\omega=\lim_{\Delta t\to 0}\frac{\Delta\theta}{\Delta t}=\frac{d\theta}{dt} \tag{11.11.2}$$
Tangential speed	$$v_t=r\omega \tag{11.11.3}$$

Angular acceleration	$$\alpha=\lim_{\Delta t\to 0}\frac{\Delta\omega}{\Delta t}=\frac{d\omega}{dt}=\frac{d^2\theta}{dt^2} \tag{11.11.4}$$
Tangential acceleration	$$a_t=r\alpha \tag{11.11.5}$$
Average angular velocity	$$\bar{\omega}=\frac{\omega_0+\omega_f}{2} \tag{11.11.6}$$
Angular displacement	$$\theta_f=\theta_0+\bar{\omega}t \tag{11.11.7}$$
Angular velocity from constant angular acceleration	$$\omega_f=\omega_0+\alpha t \tag{11.11.8}$$
Angular velocity from displacement and constant angular acceleration	$$\theta_f=\theta_0+\omega_0 t+\frac{1}{2}\alpha t^2 \tag{11.11.9}$$
Change in angular velocity	$$\omega_f^2=\omega_0^2+2a(\Delta\theta) \tag{11.11.10}$$
Total acceleration	$$\vec{a}=\vec{a}_c+\vec{a}_t \tag{11.11.11}$$
Rotational kinetic energy	$$K=\frac{1}{2}\left(\sum_j m_j r_j^2\right)\omega^2 \tag{11.11.12}$$
Moment of inertia	$$I=\sum_j m_j r_j^2 \tag{11.11.13}$$
Rotational kinetic energy in terms of the moment of inertia of a rigid body	$$K=\frac{1}{2}I\omega^2 \tag{11.11.14}$$
Moment of inertia of a continuous object	$$I=\int r^2 dm \tag{11.11.15}$$
Parallel-axis theorem	$$I_{parallel-axis}=I_{initial}+md^2 \tag{11.11.16}$$
Moment of inertia of a compound object	$$I_{total}=\sum_i I_i \tag{11.11.17}$$
Torque vector	$$\vec{\tau}=\vec{r}\times\vec{F} \tag{11.11.18}$$
Magnitude of torque	$$\lvert\vec{\tau}\rvert=r_\perp F \tag{11.11.19}$$
Total torque	$$\vec{\tau}_{net}=\sum_i \lvert\vec{\tau}_i\rvert \tag{11.11.20}$$
Newton’s second law for rotation	$$\sum_i \tau_i=I\alpha \tag{11.11.21}$$

Incremental work done by a torque	$$dW = \left(\sum_i \tau_i\right) d\theta \tag{11.11.22}$$
Work-energy theorem	$$W_{AB} = K_B - K_A \tag{11.11.23}$$
Rotational work done by net force	$$W_{AB} = \int_{\theta_A}^{\theta_B} \left(\sum_i \tau_i\right) d\theta \tag{11.11.24}$$
Rotational power	$$P = \tau\omega \tag{11.11.25}$$

Summary

10.1 Rotational Variables

- The angular position θ of a rotating body is the angle the body has rotated through in a fixed coordinate system, which serves as a frame of reference.
- The angular velocity of a rotating body about a fixed axis is defined as ω(rad/s), the rotational rate of the body in radians per second. The instantaneous angular velocity of a rotating body $\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$ is the derivative with respect to time of the angular position θ, found by taking the limit $\Delta t \to 0$ in the average angular velocity $\bar{\omega} = \frac{\Delta\theta}{\Delta t}$. The angular velocity relates v_t to the tangential speed of a point on the rotating body through the relation $v_t = r\omega$, where r is the radius to the point and v_t is the tangential speed at the given point.
- The angular velocity $\vec{\omega}$ is found using the right-hand rule. If the fingers curl in the direction of rotation about a fixed axis, the thumb points in the direction of $\vec{\omega}$ (see Figure 10.5).
- If the system's angular velocity is not constant, then the system has an angular acceleration. The average angular acceleration over a given time interval is the change in angular velocity over this time interval, $\bar{\alpha} = \frac{\Delta\omega}{\Delta t}$. The instantaneous angular acceleration is the time derivative of angular velocity, $\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}$. The angular acceleration $\vec{\alpha}$ is found by locating the angular velocity. If a rotation rate of a rotating body is decreasing, the angular acceleration is in the opposite direction to $\vec{\omega}$. If the rotation rate is increasing, the angular acceleration is in the same direction as $\vec{\omega}$.
- The tangential acceleration of a point at a radius from the axis of rotation is the angular acceleration times the radius to the point.

10.2 Rotation with Constant Angular Acceleration

- The kinematics of rotational motion describes the relationships among rotation angle (angular position), angular velocity, angular acceleration, and time.
- For a constant angular acceleration, the angular velocity varies linearly. Therefore, the average angular velocity is 1/2 the initial plus final angular velocity over a given time period:

$$\bar{\omega} = \frac{\omega_0 + \omega_f}{2}. \tag{11.11.26}$$

- We used a graphical analysis to find solutions to fixed-axis rotation with constant angular acceleration. From the relation $\omega = \frac{d\theta}{dt}$, we found that the area under an angular velocity-vs.-time curve gives the angular displacement, $\theta_f - \theta_0 = \Delta\theta = \int_{t_0}^{t} \omega(t)dt$. The results of the graphical analysis were verified using the kinematic equations for constant angular acceleration. Similarly, since $\alpha = \frac{d\omega}{dt}$, the area under an angular acceleration-vs.-time graph gives the change in angular velocity: $\omega f - \omega 0 = \Delta\omega = \int_{t_0}^{t} \alpha(t)dt$.

10.3 Relating Angular and Translational Quantities

- The linear kinematic equations have their rotational counterparts such that there is a mapping $x \to \theta$, $v \to \omega$, $a \to \alpha$.
- A system undergoing uniform circular motion has a constant angular velocity, but points at a distance r from the rotation axis have a linear centripetal acceleration.

- A system undergoing nonuniform circular motion has an angular acceleration and therefore has both a linear centripetal and linear tangential acceleration at a point a distance r from the axis of rotation.
- The total linear acceleration is the vector sum of the centripetal acceleration vector and the tangential acceleration vector. Since the centripetal and tangential acceleration vectors are perpendicular to each other for circular motion, the magnitude of the total linear acceleration is $|\vec{a}| = \sqrt{a_c^2 + a_t^2}$.

10.4 Moment of Inertia and Rotational Kinetic Energy

- The rotational kinetic energy is the kinetic energy of rotation of a rotating rigid body or system of particles, and is given by $K = \frac{1}{2} I \omega^2$, where I is the moment of inertia, or "rotational mass" of the rigid body or system of particles.
- The moment of inertia for a system of point particles rotating about a fixed axis is $I = \sum_j m_j r_j^2$, where m_j is the mass of the point particle and r_j is the distance of the point particle to the rotation axis. Because of the r^2 term, the moment of inertia increases as the square of the distance to the fixed rotational axis. The moment of inertia is the rotational counterpart to the mass in linear motion.
- In systems that are both rotating and translating, conservation of mechanical energy can be used if there are no nonconservative forces at work. The total mechanical energy is then conserved and is the sum of the rotational and translational kinetic energies, and the gravitational potential energy.

10.5 Calculating Moments of Inertia

- Moments of inertia can be found by summing or integrating over every 'piece of mass' that makes up an object, multiplied by the square of the distance of each 'piece of mass' to the axis. In integral form the moment of inertia is $I = \int r^2 dm$.
- Moment of inertia is larger when an object's mass is farther from the axis of rotation.
- It is possible to find the moment of inertia of an object about a new axis of rotation once it is known for a parallel axis. This is called the parallel axis theorem given by $I_{\text{parallel-axis}} = I_{\text{center of mass}} + md^2$, where d is the distance from the initial axis to the parallel axis.
- Moment of inertia for a compound object is simply the sum of the moments of inertia for each individual object that makes up the compound object.

10.6 Torque

- The magnitude of a torque about a fixed axis is calculated by finding the lever arm to the point where the force is applied and using the relation $|\vec{\tau}| = r_\perp F$, where $r_\perp$ is the perpendicular distance from the axis to the line upon which the force vector lies.
- The sign of the torque is found using the right hand rule. If the page is the plane containing $\vec{r}$ and $\vec{F}$, then $\vec{r} \times \vec{F}$ is out of the page for positive torques and into the page for negative torques.
- The net torque can be found from summing the individual torques about a given axis.

10.7 Newton's Second Law for Rotation

- Newton's second law for rotation, $\sum_i \tau_i = I\alpha$, says that the sum of the torques on a rotating system about a fixed axis equals the product of the moment of inertia and the angular acceleration. This is the rotational analog to Newton's second law of linear motion.
- In the vector form of Newton's second law for rotation, the torque vector $\vec{\tau}$ is in the same direction as the angular acceleration $\vec{\alpha}$. If the angular acceleration of a rotating system is positive, the torque on the system is also positive, and if the angular acceleration is negative, the torque is negative.

10.8 Work and Power for Rotational Motion

- The incremental work dW in rotating a rigid body about a fixed axis is the sum of the torques about the axis times the incremental angle $d\theta$.
- The total work done to rotate a rigid body through an angle θ about a fixed axis is the sum of the torques integrated over the angular displacement. If the torque is a constant as a function of θ , then $W_{AB} = \tau(\theta_B - \theta_A)$.
- The work-energy theorem relates the rotational work done to the change in rotational kinetic energy: $W_{AB} = K_B - K_A$ where $K = \frac{1}{2} I \omega^2$.
- The power delivered to a system that is rotating about a fixed axis is the torque times the angular velocity, $P = \tau\omega$.

Contributors and Attributions

Samuel J. Ling (Truman State University), Jeff Sanny (Loyola Marymount University), and Bill Moebs with many contributing authors. This work is licensed by OpenStax University Physics under a Creative Commons Attribution License (by 4.0).

11.12: Conservation of Energy

learning objectives

- Conclude the interchangeability of force and radius with torque and angle of rotation in determining force

In this atom we will discuss work and energy associated with rotational motion. shows a worker using an electric grindstone propelled by a motor. Sparks are flying, and noise and vibration are created as layers of steel are pared from the pole. The stone continues to turn even after the motor is turned off, but it is eventually brought to a stop by friction. Clearly, the motor had to work to get the stone spinning. This work went into heat, light, sound, vibration, and considerable rotational kinetic energy.

Grindstone: The motor works in spinning the grindstone, giving it rotational kinetic energy. That energy is then converted to heat, light, sound, and vibration. (Credit: U.S. Navy photo by Mass Communication Specialist Seaman Zachary David Bell.)

Work must be done to rotate objects such as grindstones or merry-go-rounds. The simplest rotational situation is one in which the net force is exerted perpendicular to the radius of a disc and remains perpendicular as the disc starts to rotate. The force is parallel to the displacement, and so the net work (W) done is the product of the force (F) and the radius (r) of the disc (this is otherwise known as torque(τ)) times the angle (θ) of rotation:

$$\mathrm{W} = \mathrm{Fr}\theta = \tau\theta. \tag{11.12.1}$$

Work and energy in rotational motion are completely analogous to work and energy in translational motion and completely transferrable. Just as in translational motion (where kinetic energy equals $1/2mv^2$ where m is mass and v is velocity), energy is conserved in rotational motion. Kinetic energy (K.E.) in rotational motion is related to moment of rotational inertia (I) and angular velocity (ω):

$$\mathrm{KE} = \frac{1}{2}\mathrm{I}\omega^2. \tag{11.12.2}$$

The final rotational kinetic energy equals the work done by the torque:

$$\mathrm{W} = \tau\theta = \frac{1}{2}\mathrm{I}\omega^2 = \mathrm{KE}. \tag{11.12.3}$$

This confirms that the work done went into rotational kinetic energy. To return to the grindstone example, work was done to give the grindstone rotational energy, and work is done by friction so that it loses kinetic energy. However, the energy is never destroyed; it merely changes form from rotation of the grindstone to heat when friction is applied.

Key Points

- Rotating objects have rotational kinetic energy.
- Rotational kinetic energy can change form if work is done on the object.
- Energy is never destroyed, if rotational energy is gained or lost, something must have done work on it to change the form of the energy.

Key Terms

- **work**: A measure of energy expended in moving an object; most commonly, force times displacement. No work is done if the object does not move.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.
- **rotational inertia**: The tendency of a rotating object to remain rotating unless a torque is applied to it.

11.13: Quantities of Rotational Kinematics

learning objectives

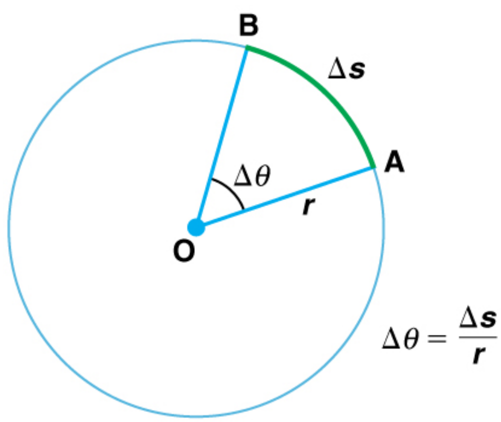
- Assess the relationship between radians the the revolution of a CD

When objects rotate about some axis—for example, when the CD (compact disc) rotates about its center—each point in the object follows a circular arc. Consider a line from the center of the CD to its edge. Each pit used to record sound along this line moves through the same angle in the same amount of time. The rotation angle is the amount of rotation, and is analogous to linear distance. We define the rotation angle $\Delta\theta$ to be the ratio of the arc length to the radius of curvature:

$\Delta\theta = \frac{\Delta s}{r}$ (illustrated in).

Rotation Angle: All points on a CD travel in circular arcs. The pits along a line from the center to the edge all move through the same angle Δ in a time Δt.

In mathematics, the angle of rotation (or angular position) is a measurement of the amount (i.e., the angle) that a figure is rotated about a fixed point (often the center of a circle, as shown in).



Angle θ and Arc Length s: The radius of a circle is rotated through an angle Δ. The arc length Δs is described on the circumference.

The arc length Δs is the distance traveled along a circular path. r is the radius of curvature of the circular path. We know that for one complete revolution, the arc length is the circumference of a circle of radius r. The circumference of a circle is 2πr. Thus, for one complete revolution the rotation angle is:

$$\Delta\theta = \frac{(2\pi r)}{r} = 2\pi. \tag{11.13.1}$$

This result is the basis for defining the units used to measure rotation angles to be radians (rad), defined so that:

$$2\pi \text{ rad} = 1 \text{ revolution.} \tag{11.13.2}$$

If $\Delta\theta = 2\pi$ rad, then the CD has made one complete revolution, and every point on the CD is back at its original position. Because there are 360º in a circle or one revolution, the relationship between radians and degrees is thus $2\pi \text{ rad} = 360^{\circ}$, so that:

$$1 \text{ rad} = \frac{360^{\circ}}{2\pi} = 57.3^{\circ}. \tag{11.13.3}$$

Angular Velocity, Omega

Angular velocity ω is the rate of change of an angle, mathematically defined as $\omega = \frac{\Delta\theta}{\Delta t}$.

learning objectives

- Examine how fast an object is rotating based on angular velocity

To examine how fast an object is rotating, we define angular velocity ω as the rate of change of an angle. In symbols, this is

$$\omega = \frac{\Delta\theta}{\Delta t}, \tag{11.13.4}$$

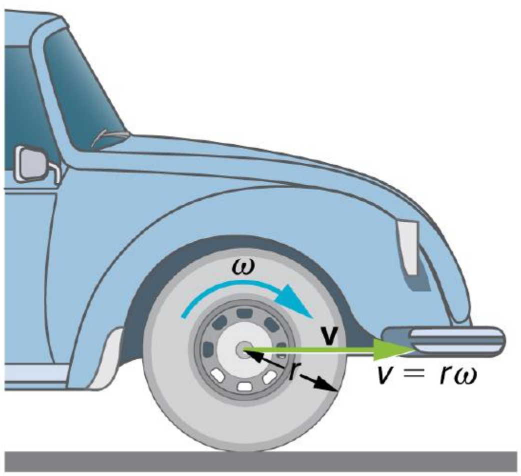
where an angular rotation Δ takes place in a time Δt. The greater the rotation angle in a given amount of time, the greater the angular velocity. The units for angular velocity are radians per second (rad/s).

Angular velocity ω is analogous to linear velocity v. To find the precise relationship between angular and linear velocity, we again consider a pit on the rotating CD. This pit moves an arc length Δs in a time Δt, and so it has a linear velocity $v = \frac{\Delta s}{\Delta t}$.

From $\Delta\theta = \frac{(\Delta s)}{r}$ we see that $\Delta s = r \cdot \Delta\theta$. Substituting this into the expression for v gives $v = \frac{(r \cdot \Delta\theta)}{(\Delta t)} = r(\frac{\Delta\theta}{\Delta t}) = r\omega$.

We can write this relationship in two different ways: $v = r\omega$ or $\omega = \frac{v}{r}$.

The first relationship states that the linear velocity v is proportional to the distance from the center of rotation, thus it is largest for a point on the rim (largest r), as you might expect. We can also call this linear speed v of a point on the rim the tangential speed. The second relationship can be illustrated by considering the tire of a moving car, as shown in the picture below. Note that the speed of the point at the center of the tire is the same as the speed v of the car. The faster the car moves, the faster the tire spins—large v means a large ω, because v=rω. Similarly, a larger-radius tire rotating at the same angular velocity (ω) will produce a greater linear speed (v) for the car.



Angular Velocity: A car moving at a velocity v to the right has a tire rotating with an angular velocity ω. The speed of the tread of the tire relative to the axle is v, the same as if the car were jacked up. Thus the car moves forward at linear velocity v=rω, where r is the tire radius. A larger angular velocity for the tire means a greater velocity for the car.

Angular Acceleration, Alpha

Angular acceleration is the rate of change of angular velocity, expressed mathematically as $\alpha = \frac{\Delta\omega}{\Delta t}$.

learning objectives

- Explain the relationship between angular acceleration and angular velocity

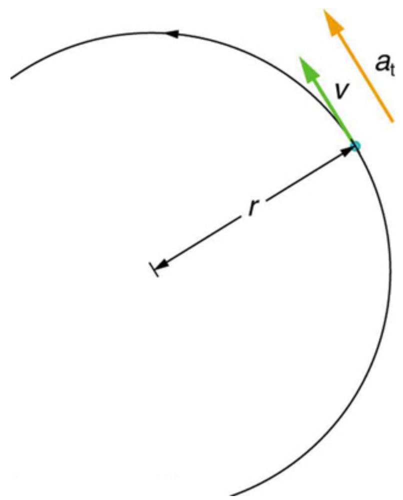
Angular acceleration is the rate of change of angular velocity. In SI units, it is measured in radians per second squared (rad/s^2), and is usually denoted by the Greek letter alpha (α).

Consider the following situations in which angular velocity is not constant: when a skater pulls in her arms, when a child starts up a merry-go-round from rest, or when a computer's hard disk slows to a halt when switched off. In all these cases, there is an angular acceleration in which ωω changes. The faster the change occurs, the greater the angular acceleration. Angular acceleration is defined as the rate of change of angular velocity. In equation form, angular acceleration is expressed as follows:

$$\alpha = \frac{\Delta\omega}{\Delta t} \tag{11.13.5}$$

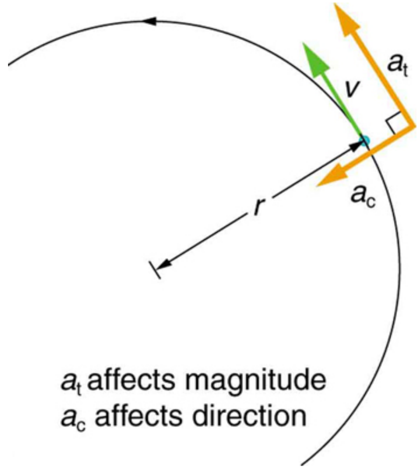
where $\Delta\omega$ is the change in angular velocity and Δt is the change in time. The units of angular acceleration are (rad/s)/s, or rad/s^2. If ωω increases, then αα is positive. If ωω decreases, then αα is negative.

It is useful to know how linear and angular acceleration are related. In circular motion, there is acceleration that is *tangent* to the circle at the point of interest (as seen in the diagram below). This acceleration is called *tangential acceleration*, a_t.



Tangential acceleration: In circular motion, acceleration can occur as the magnitude of the velocity changes: a is tangent to the motion. This acceleration is called tangential acceleration.

Tangential acceleration refers to changes in the magnitude of velocity but not its direction. In circular motion, centripetal acceleration, a_c, refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration (as seen in the diagram below.) Thus, a_t and a_c are perpendicular and independent of one another. Tangential acceleration a_t is directly related to the angular acceleration and is linked to an increase or decrease in the velocity (but not its direction).



Centripetal Acceleration: Centripetal acceleration occurs as the direction of velocity changes; it is perpendicular to the circular motion. Centripetal and tangential acceleration are thus perpendicular to each other.

Key Points

- The arc length Δs is the distance traveled along a circular path. r is the radius of curvature of the circular path.
- The rotation angle is the amount of rotation and is analogous to linear distance. We define the rotation angle $\Delta\theta$ to be the ratio of the arc length to the radius of curvature: $\Delta\theta = \frac{\Delta \mathrm{s}}{\mathrm{r}}$.
- For one complete revolution the rotation angle is 2π.
- The greater the rotation angle in a given amount of time, the greater the angular velocity.
- Angular velocity ω is analogous to linear velocity v.
- We can write the relationship between linear velocity and angular velocity in two different ways: $\mathrm{v} = \mathrm{r}\omega$ or $\omega = \frac{\mathrm{v}}{\mathrm{r}}$.
- The faster the change in angular velocity occurs, the greater the angular acceleration.
- In circular motion, linear acceleration is tangent to the circle at the point of interest, and is called tangential acceleration.
- In circular motion, centripetal acceleration refers to changes in the direction of the velocity but not its magnitude. An object undergoing circular motion experiences centripetal acceleration.

Key Terms

- **Angular position**: The angle in radians (degrees, revolutions) through which a point or line has been rotated in a specified sense about a specified axis.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.
- **angular acceleration**: The rate of change of angular velocity, often represented by α.
- **tangential acceleration**: The acceleration in a direction tangent to the circle at the point of interest in circular motion.

11.14: Angular Acceleration

learning objectives

- Relate angle of rotation, angular velocity, and angular acceleration to their equivalents in linear kinematics

Simply by using our intuition, we can begin to see the interrelatedness of rotational quantities like θ (angle of rotation), ω(angular velocity) and α (angular acceleration). For example, if a motorcycle wheel has a large angular acceleration for a fairly long time, it ends up spinning rapidly and rotating through many revolutions. The wheel's rotational motion is analogous to the fact that the motorcycle's large translational acceleration produces a large final velocity, and the distance traveled will also be large.

Kinematic Equations

Kinematics is the description of motion. We have already studied kinematic equations governing linear motion under constant acceleration:

$$v = v_0 + at \tag{11.14.1}$$

$$x = v_0 t + \frac{1}{2}at^2 \tag{11.14.2}$$

$$v^2 = v_0^2 + 2ax \tag{11.14.3}$$

Similarly, the kinematics of rotational motion describes the relationships among rotation angle, angular velocity, angular acceleration, and time. Let us start by finding an equation relating ω, α, and t. To determine this equation, we use the corresponding equation for linear motion:

$$v = v_0 + at. \tag{11.14.4}$$

As in linear kinematics where we assumed a is constant, here we assume that angular acceleration α is a constant, and can use the relation: $a = r\alpha$ Where r – radius of curve.Similarly, we have the following relationships between linear and angular values:

$$v = r\omega \tag{11.14.5}$$

$$x = r\theta \tag{11.14.6}$$

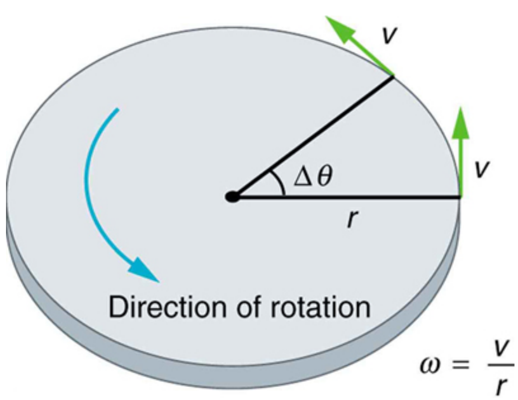
By using the relationships $a = r\alpha, v = r\omega,$ and $x = r\theta$, we derive all the other kinematic equations for rotational motion under constant acceleration:

$$\omega = \omega_0 + \alpha t \tag{11.14.7}$$

$$\theta = \omega_0 t + \frac{1}{2}\alpha t^2 \tag{11.14.8}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta \tag{11.14.9}$$

The equations given above can be used to solve any rotational or translational kinematics problem in which a and α are constant. shows the relationship between some of the quantities discussed in this atom.



Linear and Angular: This figure shows uniform circular motion and some of its defined quantities.

Key Points

- The kinematic equations for rotational and/or linear motion given here can be used to solve any rotational or translational kinematics problem in which a and α are constant.
- By using the relationships between velocity and angular velocity, distance and angle of rotation, and acceleration and angular acceleration, rotational kinematic equations can be derived from their linear motion counterparts.
- To derive rotational equations from the linear counterparts, we used the relationships $a = r\alpha, v = r\omega,$ and $x = r\theta$.

Key Terms

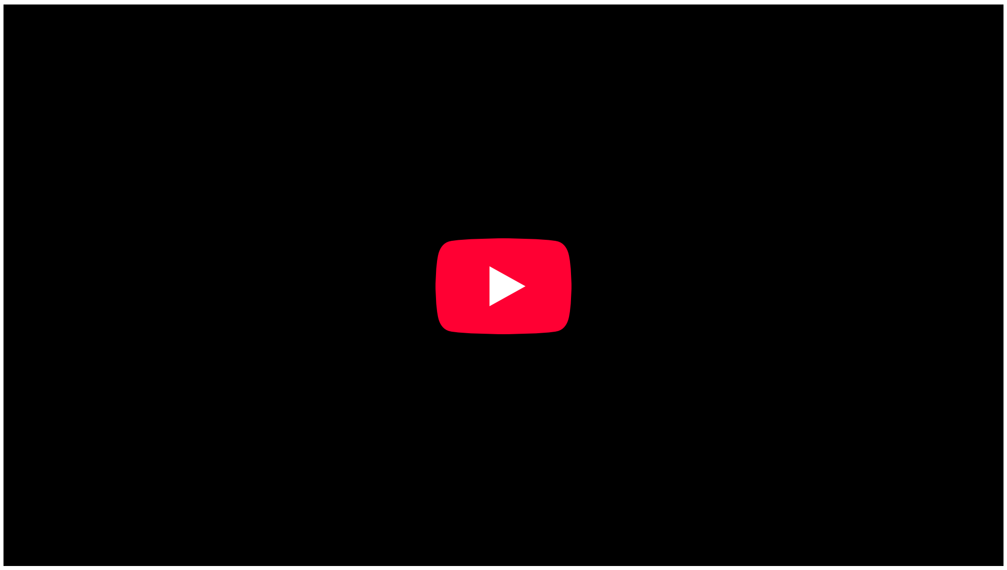
- **kinematics**: The branch of mechanics concerned with objects in motion, but not with the forces involved.
- **angular**: Relating to an angle or angles; having an angle or angles; forming an angle or corner; sharp-cornered; pointed; as in, an angular figure.

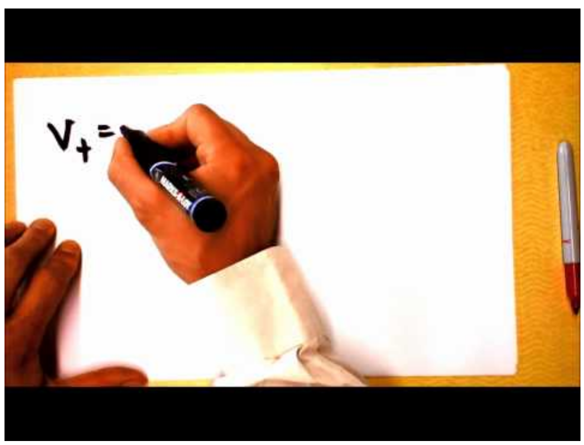
11.15: Rotational Kinematics

learning objectives

- Distinguish the two different motions in which rolling without slipping is broken down

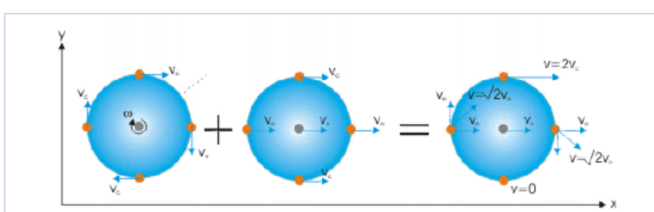
Rolling without slipping generally occurs when an object rolls without skidding. To relate this to a real world phenomenon, we can consider the example of a wheel rolling on a flat, horizontal surface.





Connecting Linear and Rotational Motion! Rolling without Slipping!: How fast does the axle of a bike wheel move? How fast does the BOTTOM of a wheel move?

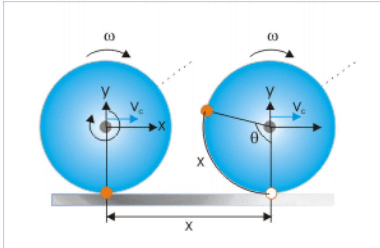
Rolling without slipping can be better understood by breaking it down into two different motions: 1) Motion of the center of mass, with linear velocity *v* (translational motion); and 2) rotational motion around its center, with angular velocity *w*.



Rolling Motion: Rolling motion is a combination of rotational and translational motion.

When an object is rolling on a plane without slipping, the point of contact of the object with the plane does not move. If we imagine a wheel moving forward by rolling on a plane at speed v, it must also be rotating about its axis at an angular speed ωω since it is rolling.

The object's angular velocity ωω is directly proportional to its velocity, because as we know, the faster we are driving in a car, the faster the wheels have to turn. So, to determine the exact relationship between linear velocity and angular velocity, we can imagine the scenario in which the wheel travels a distance of x while rotating through an angle θ.



Rolling Without Slipping: A body rolling a distance of x on a plane without slipping.

In mathematical terms, the length of the arc is equal to the angle of the segment multiplied by the object's radius, R. Therefore, we can say that the length of the arc of the wheel that has rotated an angle θ, is equal to Rθ. Furthermore, since the wheel is in constant contact with the ground, the length of the arc correlating to the angle is also equal to x. Therefore,

$$\mathrm{x}=\mathrm{R}\theta \tag{11.15.1}$$

Since xx and θθ depend on time, we can take the derivative of both sides to obtain:

$$\frac{\mathrm{dx}}{\mathrm{dt}}=\mathrm{R}\frac{\mathrm{d}\theta}{\mathrm{dt}} \tag{11.15.2}$$

where $\frac{\mathrm{dx}}{\mathrm{dt}}$ is equal to the linear velocity v, and $\frac{\mathrm{d}\theta}{\mathrm{dt}}$ is equal to the angular velocity ωω. We can then simplify this equation to:

$$\mathrm{v}=\omega\mathrm{R} \tag{11.15.3}$$

Key Points

- Rolling without slipping can be better understood by breaking it down into translational motion and rotational motion.
- When an object is rolling on a plane without slipping, the point of contact of the object with the plane does not move.
- A rolling object's velocity v is directly related to its angular velocity ω, and is mathematically expressed as $\mathrm{v}=\omega\mathrm{R}$, where R is the object's radius and v is its linear velocity.

Key Terms

- **angular velocity**: A vector quantity describing the motion of an object in circular motion; its magnitude is equal to the angular speed (ωω) of the particle, and the direction is perpendicular to the plane of its circular motion.
- **linear velocity**: A vector quantity that denotes the rate of change of position with respect to time of the object's center of mass.

- Sunil Kumar Singh, Rolling Motion. November 8, 2012. **Provided by**: OpenStax CNX. **Located at**: **http://cnx.org/content/m14311/latest/**. **License**: ***CC BY: Attribution***
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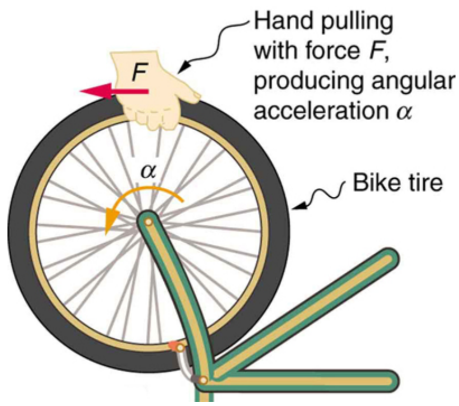
11.16: Dynamics

learning objectives

- Explain the relationship between the force, mass, radius, and angular acceleration

If you have ever spun a bike wheel or pushed a merry-go-round, you have experienced the force needed to change angular velocity. Our intuition is reliable in predicting many of the factors that are involved. For example, we know that a door opens slowly if we push too close to its hinges. Furthermore, we know that the more massive the door, the more slowly it opens. The first example implies that the farther the force is applied from the pivot, the greater the angular acceleration; another implication is that angular acceleration is inversely proportional to mass. These relationships should seem very similar to the familiar relationships among force, mass, and acceleration embodied in Newton's second law of motion. There are, in fact, precise rotational analogs to both force and mass.

Rotational inertia, as illustrated in, is the resistance of objects to changes in their rotation. In other words, a rotating object will stay rotating and a non-rotating object will stay non-rotating unless acted on by a torque. This should remind you of Newton's First Law.



Rotational Inertia: Force is required to spin the bike wheel. The greater the force, the greater the angular acceleration produced. The more massive the wheel, the smaller the angular acceleration. If you push on a spoke closer to the axle, the angular acceleration will be smaller.

To develop the precise relationship among force, mass, radius, and angular acceleration, consider what happens if we exert a force F on a point mass m that is at a distance r from a pivot point. Because the force is perpendicular to r, an acceleration $\mathrm{a} = \frac{\mathrm{F}}{\mathrm{m}}$ is obtained in the direction of F. We can rearrange this equation such that F=ma and then look for ways to relate this expression to expressions for rotational quantities. We note that a=rα, and we substitute this expression into F=ma, yielding:

$$\mathrm{F} = \mathrm{mr}\alpha. \tag{11.16.1}$$

Recall that torque is the turning effectiveness of a force. In this case, because F is perpendicular to r, torque is simply $\tau = \mathrm{Fr}$. So, if we multiply both sides of the equation above by r, we get torque on the left-hand side. That is,

$$\mathrm{rF} = \mathrm{mr}^2\alpha \tag{11.16.2}$$

, or

$$\tau = \mathrm{mr}^2\alpha. \tag{11.16.3}$$

This equation is the rotational analog of Newton's second law ($\mathrm{F} = \mathrm{ma}$), where torque is analogous to force, angular acceleration is analogous to translational acceleration, and mr^2 is analogous to mass (or inertia). The quantity mr^2 is called the rotational inertia or moment of inertia of a point mass m a distance r from the center of rotation.

Different shapes of objects have different rotational inertia which depend on the distribution of their mass.

Key Points

- The farther the force is applied from the pivot, the greater the angular acceleration.
- Angular acceleration is inversely proportional to mass.
- The equation $\tau = \mathrm{mr}^2\alpha.$ is the rotational analog of Newton's second law ($\mathrm{F} = \mathrm{ma}$), where torque is analogous to force, angular acceleration is analogous to translational acceleration, and mr^2 is analogous to mass (or inertia).

Key Terms

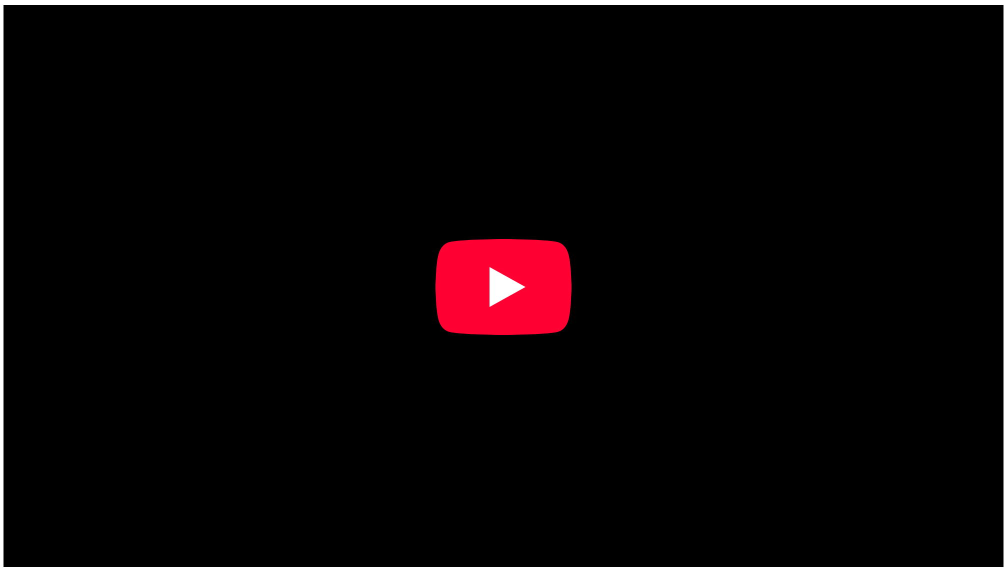
- **rotational inertia**: The tendency of a rotating object to remain rotating unless a torque is applied to it.
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)

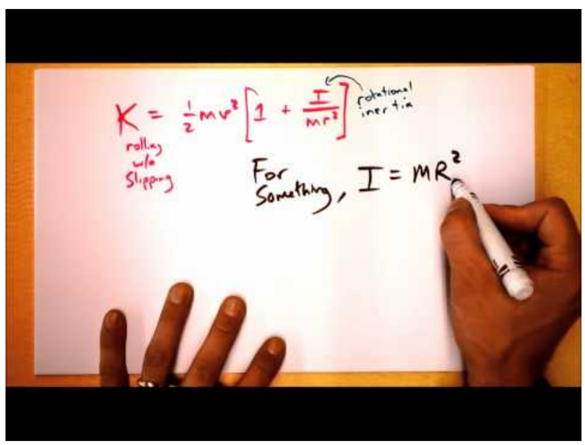
11.17: Rotational Kinetic Energy

learning objectives

- Express the rotational kinetic energy as a function of the angular velocity and the moment of inertia, and relate it to the total kinetic energy

Rotational kinetic energy is the kinetic energy due to the rotation of an object and is part of its total kinetic energy. Looking at rotational energy separately around an object's axis of rotation yields the following dependence on the object's moment of inertia:





Kinetic Energy of Rotation: Things that roll without slipping have some fraction of their energy as translational kinetic and the remainder as rotational kinetic. The ratio depends on the moment of inertia of the object that's rolling.

$$\mathrm{E_{rotational}} = \frac{1}{2}\mathrm{I}\omega^2, \tag{11.17.1}$$

where ω is the angular velocity and I is the moment of inertia around the axis of rotation.

The mechanical work applied during rotation is the torque (τ) times the rotation angle (θ) : $\mathrm{W} = \tau\theta$.

The instantaneous power of an angularly accelerating body is the torque times the angular velocity: $\mathrm{P} = \tau\omega$.

Note the close relationship between the result for rotational energy and the energy held by linear (or translational) motion:

$$\mathrm{E_{translational}} = \frac{1}{2}\mathrm{mv}^2. \tag{11.17.2}$$

In the rotating system, the moment of inertia takes the role of the mass and the angular velocity takes the role of the linear velocity.

As an example, let us calculate the rotational kinetic energy of the Earth (animated in Figure 1). As the Earth has a period of about 23.93 hours, it has an angular velocity of 7.29×10^{-5} rad/s. The Earth has a moment of inertia, $I = 8.04 \times 10^{37}$ kg·m2. Therefore, it has a rotational kinetic energy of 2.138×10^{29} J.

The Rotating Earth: The earth's rotation is a prominent example of rotational kinetic energy.

This can be partially tapped using tidal power. Additional friction of the two global tidal waves creates energy in a physical manner, infinitesimally slowing down Earth's angular velocity. Due to conservation of angular momentum this process transfers angular momentum to the Moon's orbital motion, increasing its distance from Earth and its orbital period.

Moment of Inertia

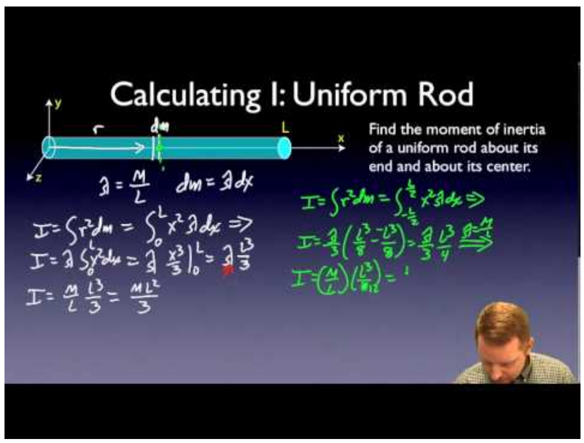
The moment of inertia is a property of a mass that measures its resistance to rotational acceleration about one or more axes.

learning objectives

- Identify a property of a mass described by the moment of inertia

The Moment of Inertia

The moment of inertia is a property of the distribution of mass in space that measures mass's resistance to rotational acceleration about one or more axes. Newton's first law, which describes the inertia of a body in linear motion, can be extended to the inertia of a body rotating about an axis using the moment of inertia. That is, an object that is rotating at constant angular velocity will remain rotating unless it is acted upon by an external torque. In this way, the moment of inertia plays the same role in rotational dynamics as mass does in linear dynamics: it describes the relationship between angular momentum and angular velocity as well as torque and angular acceleration.



Moment of Inertia: A brief introduction to moment of inertia (rotational inertia) for calculus-based physics students.

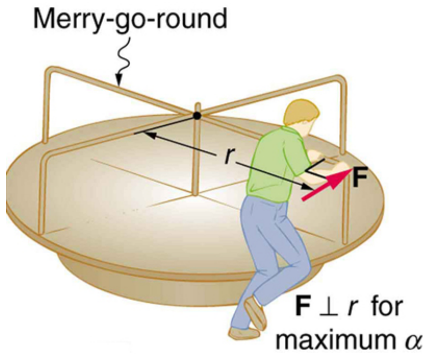
The moment of inertia *I* of an object can be defined as the sum of mr^2 for all the point masses of which it is composed, where m is the mass and r is the distance of the mass from the center of mass. It can be expressed mathematically as: $\mathrm{I}=\sum \mathrm{mr}^2$. Here, I is analogous to m in translational motion.

As an example, consider a hoop of radius r. Assuming that the hoop material is uniform, the hoop's moment of inertia can be found by summing up all the mass of the hoop and multiplying by the distance of that mass from the center of mass. Since the hoop is a circle and the mass is uniform around the circle, the moment of inertia is mr^2. All of the mass m is at a distance r from the center.

Moment of inertia also depends on the axis about which you rotate an object. Objects will usually rotate about their center of mass, but can be made to rotate about any axis. The moment of inertia in the case of rotation about a different axis other than the center of mass is given by the parallel axis theorem. The theorem states that the moment of inertia for an object rotated about a different axis parallel to the axis passing through the center of mass is $\mathrm{I}_{\mathrm{cm}}+\mathrm{mr}^2$ where r is now the distance between the two axes and IcmIcmis the moment of inertia when rotated about the center of mass which you learned how to calculate in the previous paragraph.

A general relationship among the torque, moment of inertia, and angular acceleration is: net $\tau=\mathrm{I}\alpha$, or $\alpha=\frac{(\text{net } \tau)}{\mathrm{I}}$. Net τ is the total torque from all forces relative to a chosen axis. Such torques are either positive or negative and add like ordinary numbers. The relationship in $\tau=\mathrm{I}\alpha$ is the rotational analog to Newton's second law and is very applicable. This equation is actually valid for any torque, applied to any object, and relative to any axis.

As can be expected, the larger the torque, the larger the angular acceleration. For example, the harder a child pushes on a merry-go-round, the slower it accelerates for the same torque. The basic relationship between the moment of inertia and the angular acceleration is that the larger the moment of inertia, the smaller the angular acceleration. The moment of inertia depends not only on the mass of an object, but also on its distribution of mass relative to the axis around which it rotates. For example, it wil be much easier to accelerate a merry-go-round full of children if they stand close to its axis than if they all stand at the outer edge.



Moment of Inertia on a Merry-Go-Round: A father pushes a playground merry-go-round at its edge and perpendicular to its radius to achieve maximum torque.

Key Points

- Rotational kinetic energy can be expressed as: $\mathrm{E_{rotational}} = \frac{1}{2}\mathrm{I}\omega^2$ where ω is the angular velocity and I is the moment of inertia around the axis of rotation.
- The mechanical work applied during rotation is the torque times the rotation angle: (θ) : $\mathrm{W} = \tau\theta$.
- The instantaneous power of an angularly accelerating body is the torque times the angular velocity: $\mathrm{P} = \tau\omega$.
- There is a close relationship between the result for rotational energy and the energy held by linear (or translational) motion.
- Newton's first law, which describes the inertia of a body in linear motion, can be extended to the inertia of a body rotating about an axis using the moment of inertia.
- An object that is rotating at constant angular velocity will remain rotating unless it is acted upon by an external torque.
- The larger the torque, the larger the angular acceleration.

Key Items

- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **inertia**: The property of a body that resists any change to its uniform motion; equivalent to its mass.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.

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11.18: Prelude to Angular Momentum

Angular momentum is the rotational counterpart of linear momentum. Any massive object that rotates about an axis carries angular momentum, including rotating flywheels, planets, stars, hurricanes, tornadoes, whirlpools, and so on. The helicopter shown below can be used to illustrate the concept of angular momentum. The lift blades spin about a vertical axis through the main body and carry angular momentum. The body of the helicopter tends to rotate in the opposite sense in order to conserve angular momentum. The small rotors at the tail of the aircraft provide a counter thrust against the body to prevent this from happening, and the helicopter stabilizes itself. The concept of conservation of angular momentum is discussed later in this chapter. In the main part of this chapter, we explore the intricacies of angular momentum of rigid bodies such as a top, and also of point particles and systems of particles. But to be complete, we start with a discussion of rolling motion, which builds upon the concepts of the previous chapter.

Figure 11.18.1: A helicopter has its main lift blades rotating to keep the aircraft airborne. Due to conservation of angular momentum, the body of the helicopter would want to rotate in the opposite sense to the blades, if it were not for the small rotor on the tail of the aircraft, which provides thrust to stabilize it.

11.19: Rolling Motion

Learning Objectives

- Describe the physics of rolling motion without slipping
- Explain how linear variables are related to angular variables for the case of rolling motion without slipping
- Find the linear and angular accelerations in rolling motion with and without slipping
- Calculate the static friction force associated with rolling motion without slipping
- Use energy conservation to analyze rolling motion

Rolling motion is that common combination of rotational and translational motion that we see everywhere, every day. Think about the different situations of wheels moving on a car along a highway, or wheels on a plane landing on a runway, or wheels on a robotic explorer on another planet. Understanding the forces and torques involved in **rolling motion** is a crucial factor in many different types of situations.

For analyzing rolling motion in this chapter, refer to Figure 10.5.4 in Fixed-Axis Rotation to find moments of inertia of some common geometrical objects. You may also find it useful in other calculations involving rotation.

Rolling Motion without Slipping

People have observed rolling motion without slipping ever since the invention of the wheel. For example, we can look at the interaction of a car's tires and the surface of the road. If the driver depresses the accelerator to the floor, such that the tires spin without the car moving forward, there must be kinetic friction between the wheels and the surface of the road. If the driver depresses the accelerator slowly, causing the car to move forward, then the tires roll without slipping. It is surprising to most people that, in fact, the bottom of the wheel is at rest with respect to the ground, indicating there must be static friction between the tires and the road surface. In Figure $11.19.1$, the bicycle is in motion with the rider staying upright. The tires have contact with the road surface, and, even though they are rolling, the bottoms of the tires deform slightly, do not slip, and are at rest with respect to the road surface for a measurable amount of time. There must be static friction between the tire and the road surface for this to be so.

Figure 11.19.1: (a) The bicycle moves forward, and its tires do not slip. The bottom of the slightly deformed tire is at rest with respect to the road surface for a measurable amount of time. (b) This image shows that the top of a rolling wheel appears blurred by its motion, but the bottom of the wheel is instantaneously at rest. (credit a: modification of work by Nelson Lourenço; credit b: modification of work by Colin Rose)

To analyze rolling without slipping, we first derive the linear variables of velocity and acceleration of the center of mass of the wheel in terms of the angular variables that describe the wheel's motion. The situation is shown in Figure $11.19.2$

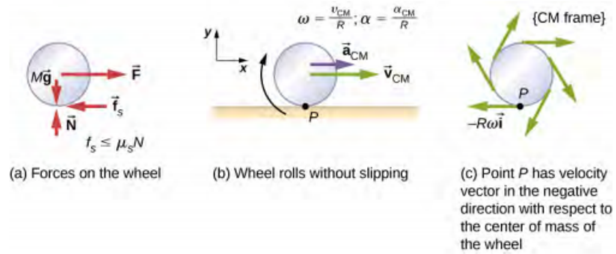


Figure 11.19.2: (a) A wheel is pulled across a horizontal surface by a force $\vec{F}$. The force of static friction $\vec{f}_s$, $|\vec{f}_s| \leq \mu_s$N is large enough to keep it from slipping. (b) The linear velocity and acceleration vectors of the center of mass and the relevant expressions for ω and α. Point P is at rest relative to the surface. (c) Relative to the center of mass (CM) frame, point P has linear velocity −R $\omega\hat{i}$.

From Figure 11.19.2(a), we see the force vectors involved in preventing the wheel from slipping. In (b), point P that touches the surface is at rest relative to the surface. Relative to the center of mass, point P has velocity −R$\omega\hat{i}$, where R is the radius of the wheel and ω is the wheel's angular velocity about its axis. Since the wheel is rolling, the velocity of P with respect to the surface is its velocity with respect to the center of mass plus the velocity of the center of mass with respect to the surface:

$$\vec{v}_P = -R\omega\hat{i} + v_{CM}\hat{i}. \tag{11.19.1}$$

Since the velocity of P relative to the surface is zero, $v_P = 0$, this says that

$$v_{CM} = R\omega. \tag{11.19.2}$$

Thus, the velocity of the wheel's center of mass is its radius times the angular velocity about its axis. We show the correspondence of the linear variable on the left side of the equation with the angular variable on the right side of the equation. This is done below for the linear acceleration.

If we differentiate Equation 11.19.2on the left side of the equation, we obtain an expression for the linear acceleration of the center of mass. On the right side of the equation, R is a constant and since $\alpha = \frac{d\omega}{dt}$, we have

$$a_{CM} = R\alpha. \tag{11.19.3}$$

Furthermore, we can find the distance the wheel travels in terms of angular variables by referring to Figure 11.19.3. As the wheel rolls from point A to point B, its outer surface maps onto the ground by exactly the distance traveled, which is d_{CM}.

We see from Figure 11.19.3that the length of the outer surface that maps onto the ground is the arc length Rθ. Equating the two distances, we obtain

$$d_{CM} = R\theta. \tag{11.19.4}$$

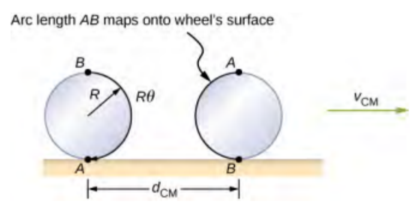


Figure 11.19.3: As the wheel rolls on the surface, the arc length Rθ from A to B maps onto the surface, corresponding to the distance d_{CM} that the center of mass has moved.

✔ Example 11.19.1: Rolling Down an Inclined Plane

A solid cylinder rolls down an inclined plane without slipping, starting from rest. It has mass m and radius r. (a) What is its acceleration? (b) What condition must the coefficient of static friction μ_S satisfy so the cylinder does not slip?

Strategy

Draw a sketch and free-body diagram, and choose a coordinate system. We put x in the direction down the plane and y upward perpendicular to the plane. Identify the forces involved. These are the normal force, the force of gravity, and the force due to friction. Write down Newton's laws in the x- and y-directions, and Newton's law for rotation, and then solve for the acceleration and force due to friction.

Solution

a. The free-body diagram and sketch are shown in Figure 11.19.4, including the normal force, components of the weight, and the static friction force. There is barely enough friction to keep the cylinder rolling without slipping. Since there is no slipping, the magnitude of the friction force is less than or equal to μ_SN. Writing down Newton's laws in the x- and y-directions, we have

$$\sum F_x = ma_x; \sum F_y = ma_y. \tag{11.19.5}$$

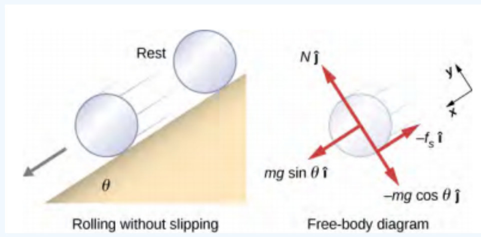


Figure 11.19.4: A solid cylinder rolls down an inclined plane without slipping from rest. The coordinate system has x in the direction down the inclined plane and y perpendicular to the plane. The free-body diagram is shown with the normal force, the static friction force, and the components of the weight $m\vec{g}$. Friction makes the cylinder roll down the plane rather than slip.

Substituting in from the free-body diagram

$$\begin{aligned} mg\sin\theta - f_s &= m(a_{CM})x, \\ N - mg\cos\theta &= 0 \end{aligned}$$

we can then solve for the linear acceleration of the center of mass from these equations:

$$a_{CM} = g\sin\theta - \frac{f_s}{m}. \tag{11.19.6}$$

However, it is useful to express the linear acceleration in terms of the moment of inertia. For this, we write down Newton's second law for rotation,

$$\sum \tau_{CM} = I_{CM}\alpha. \tag{11.19.7}$$

The torques are calculated about the axis through the center of mass of the cylinder. The only nonzero torque is provided by the friction force. We have

$$f_s r = I_{CM}\alpha. \tag{11.19.8}$$

Finally, the linear acceleration is related to the angular acceleration by

$$(a_{CM})_x = r\alpha. \tag{11.19.9}$$

These equations can be used to solve for a_{CM}, α, and f_S in terms of the moment of inertia, where we have dropped the x-subscript. We write a_{CM} in terms of the vertical component of gravity and the friction force, and make the following substitutions.

$$f_S = \frac{I_{CM}\alpha}{r} = \frac{I_{CM}a_{CM}}{r^2} \tag{11.19.10}$$

From this we obtain

$$\begin{aligned} a_{CM} &= g\sin\theta - \frac{I_{CM}a_{CM}}{mr^2}, \\ &= \frac{mg\sin\theta}{m + \left(\frac{I_{CM}}{r^2}\right)}. \end{aligned}$$

Note that this result is independent of the coefficient of static friction, μ_s.

Since we have a solid cylinder, from Figure 10.5.4, we have $I_{CM} = \frac{mr^2}{2}$ and

$$a_{CM} = \frac{mg\sin\theta}{m + \left(\frac{mr^2}{2r^2}\right)} = \frac{2}{3}g\sin\theta. \tag{11.19.11}$$

Therefore, we have

$$\alpha = \frac{a_{CM}}{r} = \frac{2}{3r}g\sin\theta. \tag{11.19.12}$$

b. Because slipping does not occur, $f_S \leq \mu_s N$. Solving for the friction force,

$$f_s = I_{CM}\frac{\alpha}{r} = I_{CM}\frac{(a_{CM})}{r^2} = \left(\frac{I_{CM}}{r^2}\right)\left(\frac{mg\sin\theta}{m + \left(\frac{I_{CM}}{r^2}\right)}\right) = \frac{mgI_{CM}\sin\theta}{mr^2 + I_{CM}}. \tag{11.19.13}$$

Substituting this expression into the condition for no slipping, and noting that N = mg cos θ, we have

$$\frac{mgI_{CM}\sin\theta}{mr^2 + I_{CM}} \leq \mu_s mg\cos\theta \tag{11.19.14}$$

or

$$\mu_s \geq \frac{\tan\theta}{1 + \left(\frac{mr^2}{I_{CM}}\right)}. \tag{11.19.15}$$

For the solid cylinder, this becomes

$$\mu_s \geq \frac{\tan\theta}{1 + \left(\frac{2mr^2}{mr^2}\right)} = \frac{1}{3}\tan\theta. \tag{11.19.16}$$

Significance

a. The linear acceleration is linearly proportional to sin θ. Thus, the greater the angle of the incline, the greater the linear acceleration, as would be expected. The angular acceleration, however, is linearly proportional to sin θ and inversely proportional to the radius of the cylinder. Thus, the larger the radius, the smaller the angular acceleration.
b. For no slipping to occur, the coefficient of static friction must be greater than or equal to $\frac{1}{3}\tan\theta$. Thus, the greater the angle of incline, the greater the coefficient of static friction must be to prevent the cylinder from slipping.

? Exercise 11.19.2

A hollow cylinder is on an incline at an angle of 60°. The coefficient of static friction on the surface is μ_s = 0.6. (a) Does the cylinder roll without slipping? (b) Will a solid cylinder roll without slipping?

It is worthwhile to repeat the equation derived in this example for the acceleration of an object rolling without slipping:

$$a_{CM} = \frac{mg\sin\theta}{m + \left(\frac{I_{CM}}{r^2}\right)}. \tag{11.19.17}$$

This is a very useful equation for solving problems involving rolling without slipping. Note that the acceleration is less than that for an object sliding down a frictionless plane with no rotation. The acceleration will also be different for two rotating cylinders with different rotational inertias.

Rolling Motion with Slipping

In the case of rolling motion with slipping, we must use the coefficient of kinetic friction, which gives rise to the kinetic friction force since static friction is not present. The situation is shown in Figure 11.19.5 In the case of slipping, $v_{CM} - R\omega \neq 0$, because point P on the wheel is not at rest on the surface, and $v_P \neq 0$. Thus, $\omega \neq \frac{v_{CM}}{R}$, $\alpha \neq \frac{a_{CM}}{R}$.

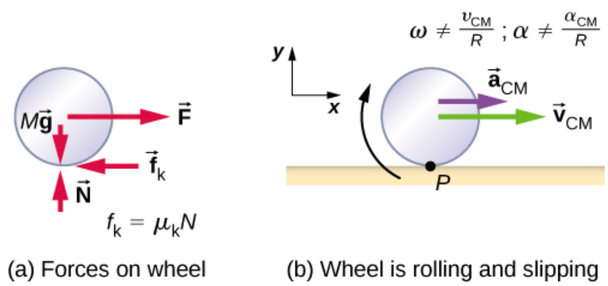


Figure 11.19.5: (a) Kinetic friction arises between the wheel and the surface because the wheel is slipping. (b) The simple relationships between the linear and angular variables are no longer valid.

Example 11.19.2: Rolling Down an Inclined Plane with Slipping

A solid cylinder rolls down an inclined plane from rest and undergoes slipping (Figure 11.19.6). It has mass m and radius r. (a) What is its linear acceleration? (b) What is its angular acceleration about an axis through the center of mass?

Strategy

Draw a sketch and free-body diagram showing the forces involved. The free-body diagram is similar to the no-slipping case except for the friction force, which is kinetic instead of static. Use Newton's second law to solve for the acceleration in the x-direction. Use Newton's second law of rotation to solve for the angular acceleration.

Solution

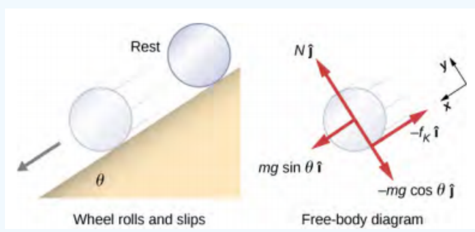


Figure 11.19.6: A solid cylinder rolls down an inclined plane from rest and undergoes slipping. The coordinate system has x in the direction down the inclined plane and y upward perpendicular to the plane. The free-body diagram shows the normal force, kinetic friction force, and the components of the weight $m\vec{g}$.

The sum of the forces in the y-direction is zero, so the friction force is now $f_k = \mu_k N = \mu_k mg\cos\theta$. Newton's second law in the x-direction becomes

$$\sum F_x = ma_x,$$

$$mg\sin\theta - \mu_k mg\cos\theta = m(a_{CM})_x,$$

or

$$(a_{CM})_x = g(\sin\theta - \mu_k\cos\theta).$$

The friction force provides the only torque about the axis through the center of mass, so Newton's second law of rotation becomes

$$\sum \tau_{CM} = I_{CM}\alpha,$$

$$f_k r = I_{CM}\alpha = \frac{1}{2}mr^2\alpha.$$

Solving for α, we have

$$\alpha = \frac{2f_k}{mr} = \frac{2\mu_k g\cos\theta}{r}.$$

Significance

We write the linear and angular accelerations in terms of the coefficient of kinetic friction. The linear acceleration is the same as that found for an object sliding down an inclined plane with kinetic friction. The angular acceleration about the axis of rotation is linearly proportional to the normal force, which depends on the cosine of the angle of inclination. As $\theta \rightarrow 90°$, this force goes to zero, and, thus, the angular acceleration goes to zero.

Conservation of Mechanical Energy in Rolling Motion

In the preceding chapter, we introduced rotational kinetic energy. Any rolling object carries rotational kinetic energy, as well as translational kinetic energy and potential energy if the system requires. Including the gravitational potential energy, the total mechanical energy of an object rolling is

$$E_T = \frac{1}{2}mv^2_{CM} + \frac{1}{2}I_{CM}\omega^2 + mgh. \tag{11.19.18}$$

In the absence of any nonconservative forces that would take energy out of the system in the form of heat, the total energy of a rolling object without slipping is conserved and is constant throughout the motion. Examples where energy is not conserved are a rolling object that is slipping, production of heat as a result of kinetic friction, and a rolling object encountering air resistance.

You may ask why a rolling object that is not slipping conserves energy, since the static friction force is nonconservative. The answer can be found by referring back to Figure $11.19.2$ Point P in contact with the surface is at rest with respect to the surface. Therefore, its infinitesimal displacement $d\vec{r}$ with respect to the surface is zero, and the incremental work done by the static friction force is zero. We can apply energy conservation to our study of rolling motion to bring out some interesting results.

Example 11.19.3: Curiosity Rover

The **Curiosity** rover, shown in Figure $11.19.7$, was deployed on Mars on August 6, 2012. The wheels of the rover have a radius of 25 cm. Suppose astronauts arrive on Mars in the year 2050 and find the now-inoperative **Curiosity** on the side of a basin. While they are dismantling the rover, an astronaut accidentally loses a grip on one of the wheels, which rolls without slipping down into the bottom of the basin 25 meters below. If the wheel has a mass of 5 kg, what is its velocity at the bottom of the basin?

Figure 11.19.7: The NASA Mars Science Laboratory rover Curiosity during testing on June 3, 2011. The location is inside the Spacecraft Assembly Facility at NASA's Jet Propulsion Laboratory in Pasadena, California. (credit: NASA/JPL-Caltech)

Strategy

We use mechanical energy conservation to analyze the problem. At the top of the hill, the wheel is at rest and has only potential energy. At the bottom of the basin, the wheel has rotational and translational kinetic energy, which must be equal to the initial potential energy by energy conservation. Since the wheel is rolling without slipping, we use the relation $v_{CM} = r\omega$ to relate the translational variables to the rotational variables in the energy conservation equation. We then solve for the velocity. From Figure $11.19.7$, we see that a hollow cylinder is a good approximation for the wheel, so we can use this moment of inertia to simplify the calculation.

Solution

Energy at the top of the basin equals energy at the bottom:

$$mgh = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}I_{CM}\omega^2.$$

The known quantities are $I_{CM} = mr^2$, r = 0.25 m, and h = 25.0 m.

We rewrite the energy conservation equation eliminating ω by using $\omega = v_{CM}r$. We have

$$mgh = \frac{1}{2}mv_{CM}^2 + \frac{1}{2}mr^2\frac{v_{CM}^2}{r^2}$$

or

$$gh = \frac{1}{2}v_{CM}^2 + \frac{1}{2}v_{CM}^2 \Rightarrow v_{CM} = \sqrt{gh}.$$

On Mars, the acceleration of gravity is 3.71 m/s^2, which gives the magnitude of the velocity at the bottom of the basin as

$$v_{CM} = \sqrt{(3.71\ m/s^2)(25.0\ m)} = 9.63\ m/s.$$

Significance

This is a fairly accurate result considering that Mars has very little atmosphere, and the loss of energy due to air resistance would be minimal. The result also assumes that the terrain is smooth, such that the wheel wouldn't encounter rocks and bumps along the way.

Also, in this example, the kinetic energy, or energy of motion, is equally shared between linear and rotational motion. If we look at the moments of inertia in Figure 10.5.4, we see that the hollow cylinder has the largest moment of inertia for a given radius and mass. If the wheels of the rover were solid and approximated by solid cylinders, for example, there would be more kinetic energy in linear motion than in rotational motion. This would give the wheel a larger linear velocity than the hollow cylinder approximation. Thus, the solid cylinder would reach the bottom of the basin faster than the hollow cylinder.

11.20: Angular Momentum

Learning Objectives

- Describe the vector nature of angular momentum
- Find the total angular momentum and torque about a designated origin of a system of particles
- Calculate the angular momentum of a rigid body rotating about a fixed axis
- Calculate the torque on a rigid body rotating about a fixed axis
- Use conservation of angular momentum in the analysis of objects that change their rotation rate

Why does Earth keep on spinning? What started it spinning to begin with? Why doesn't Earth's gravitational attraction not bring the Moon crashing in toward Earth? And how does an ice skater manage to spin faster and faster simply by pulling her arms in? Why does she not have to exert a torque to spin faster?

The answer to these questions is that just as the total linear motion (momentum) in the universe is conserved, so is the total rotational motion conserved. We call the total rotational motion angular momentum, the rotational counterpart to linear momentum. In this chapter, we first define and then explore angular momentum from a variety of viewpoints. First, however, we investigate the angular momentum of a single particle. This allows us to develop angular momentum for a system of particles and for a rigid body.

Angular Momentum of a Single Particle

Figure $11.20.1$shows a particle at a position $\vec{r}$ with linear momentum $\vec{p}$ = m$\vec{v}$ with respect to the origin. Even if the particle is not rotating about the origin, we can still define an angular momentum in terms of the position vector and the linear momentum.

Angular Momentum of a Particle

The angular momentum $\vec{l}$ of a particle is defined as the cross-product of $\vec{r}$ and $\vec{p}$, and is perpendicular to the plane containing $\vec{r}$ and $\vec{p}$:

$$\vec{l} = \vec{r} \times \vec{p}. \tag{11.20.1}$$

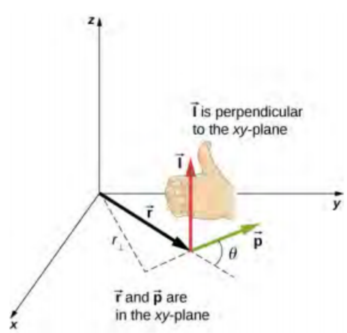


Figure 11.20.1: In three-dimensional space, the position vector $\vec{r}$ locates a particle in the xy-plane with linear momentum $\vec{p}$. The angular momentum with respect to the origin is $\vec{l} = \vec{r} \times \vec{p}$, which is in the z-direction. The direction of $\vec{l}$ is given by the right-hand rule, as shown.

The intent of choosing the direction of the angular momentum to be perpendicular to the plane containing $\vec{r}$ and $\vec{p}$ is similar to choosing the direction of torque to be perpendicular to the plane of $\vec{r}$ and $\vec{F}$, as discussed in Fixed-Axis Rotation. The magnitude of the angular momentum is found from the definition of the cross-product,

$$l = rp\sin\theta, \tag{11.20.2}$$

where θ is the angle between $\vec{r}$ and $\vec{p}$. The units of angular momentum are kg • m^2/s. As with the definition of torque, we can define a lever arm $r_\perp$ that is the perpendicular distance from the momentum vector $\vec{p}$ to the origin, $r_\perp = r\sin\theta$. With this definition, the magnitude of the angular momentum becomes

$$l = r_\perp p = r_\perp mv. \tag{11.20.3}$$

We see that if the direction of $\vec{p}$ is such that it passes through the origin, then $\theta = 0$, and the angular momentum is zero because the lever arm is zero. In this respect, the magnitude of the angular momentum depends on the choice of origin. If we take the time derivative of the angular momentum, we arrive at an expression for the torque on the particle:

$$\begin{aligned}\frac{d\vec{l}}{dt} &= \frac{d\vec{r}}{dt}\times\vec{p}+\vec{r}\times\frac{d\vec{p}}{dt}\\ &=\vec{v}\times m\vec{v}+\vec{r}\times\frac{d\vec{p}}{dt}\\ &=\vec{r}\times\frac{d\vec{p}}{dt}.\end{aligned}$$

Here we have used the definition of $\vec{p}$ and the fact that a vector crossed into itself is zero. From Newton's second law $\frac{d\vec{p}}{dt}=\sum\vec{F}$, the net force acting on the particle, and the definition of the net torque, we can write

$$\frac{d\vec{l}}{dt}=\sum\vec{\tau}. \tag{11.20.4}$$

Note the similarity with the linear result of Newton's second law, $\frac{d\vec{p}}{dt}=\sum\vec{F}$. The following problem-solving strategy can serve as a guideline for calculating the angular momentum of a particle.

Problem-Solving Strategy: Angular Momentum of a Particle

1. Choose a coordinate system about which the angular momentum is to be calculated.
2. Write down the radius vector to the point particle in unit vector notation.
3. Write the linear momentum vector of the particle in unit vector notation.
4. Take the cross product $\vec{l}=\vec{r}\times\vec{p}$ and use the right-hand rule to establish the direction of the angular momentum vector.
5. See if there is a time dependence in the expression of the angular momentum vector. If there is, then a torque exists about the origin, and use $\frac{d\vec{l}}{dt}=\sum\vec{\tau}$ to calculate the torque. If there is no time dependence in the expression for the angular momentum, then the net torque is zero.

Example 11.20.2: Angular Momentum and Torque on a Meteor

A meteor enters Earth's atmosphere (Figure 11.20.2) and is observed by someone on the ground before it burns up in the atmosphere. The vector $\vec{r}$ = 25 km $\hat{i}$ + 25 km $\hat{j}$ gives the position of the meteor with respect to the observer. At the instant the observer sees the meteor, it has linear momentum $\vec{p}$ = (15.0 kg)(−2.0 km/s $\hat{j}$), and it is accelerating at a constant 2.0 m/s^2 $(-\hat{j})$ along its path, which for our purposes can be taken as a straight line.

a. What is the angular momentum of the meteor about the origin, which is at the location of the observer?
b. What is the torque on the meteor about the origin?

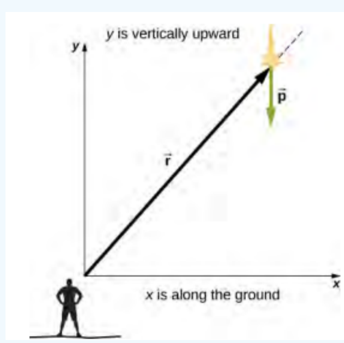


Figure 11.20.2: An observer on the ground sees a meteor at position $\vec{r}$ with linear momentum $\vec{p}$.

Strategy

We resolve the acceleration into x- and y-components and use the kinematic equations to express the velocity as a function of acceleration and time. We insert these expressions into the linear momentum and then calculate the angular momentum using the cross-product. Since the position and momentum vectors are in the xy-plane, we expect the angular momentum vector to be along the z-axis. To find the torque, we take the time derivative of the angular momentum.

Solution

The meteor is entering Earth's atmosphere at an angle of 90.0° below the horizontal, so the components of the acceleration in the x- and y-directions are

$$a_x = 0,\ a_y = -2.0\ m/s^2.$$

We write the velocities using the kinematic equations.

$$v_x = 0,\ v_y = (-2.0\times 10^3\ m/s) - (2.0\ m/s^2)t.$$

a. The angular momentum is

$$\begin{aligned}\vec{l}\ &= \vec{r}\times\vec{p} = (25.0\ km\ \hat{i} + 25.0\ km\ \hat{j})\times(15.0\ kg)(0\hat{i}+v_y\hat{j})\\ &= 15.0\ kg[25.0\ km(v_y)\hat{k}]\\ &= 15.0\ kg\{(2.50\times 10^4\ m)[(-2.0\times 10^3\ m/s)-(2.0\ m/s^2)t]\hat{k}\}.\end{aligned}$$

At t = 0, the angular momentum of the meteor about the origin is

$$\vec{l}_0 = 15.0\ kg[(2.50\times 10^4\ m)(-2.0\times 10^3\ m/s)\hat{k}] = 7.50\times 10^8\ kg\cdot m^2/s(-\hat{k}).$$

This is the instant that the observer sees the meteor.

b. To find the torque, we take the time derivative of the angular momentum. Taking the time derivative of $\vec{l}$ as a function of time, which is the second equation immediately above, we have

$$\frac{d\vec{l}}{dt} = (-15.0\ kg)(2.50\times 10^4\ m)(2.0\ m/s^2)\hat{k}.$$

Then, since $\frac{d\vec{l}}{dt} = \sum\vec{\tau}$, we have

$$\sum\vec{\tau} = -7.5\times 10^5\ N\cdot m\ \hat{k}.$$

The units of torque are given as newton-meters, not to be confused with joules. As a check, we note that the lever arm is the x-component of the vector $\vec{r}$ in Figure $11.20.2$since it is perpendicular to the force acting on the meteor, which is along its path. By Newton's second law, this force is

$$\vec{F} = ma(-\hat{j}) = (15.0\ kg)(2.0\ m/s^2)(-\hat{j}) = 30.0\ kg\cdot m/s^2(-\hat{j}). \tag{11.20.5}$$

The lever arm is

$$\vec{r}_\perp = 2.5\times 10^4\ m\ \hat{i}. \tag{11.20.6}$$

Thus, the torque is

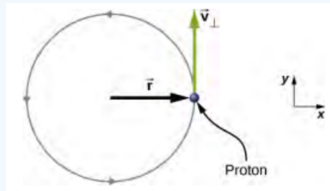
$$\begin{aligned}\sum\vec{\tau} = \vec{r}_\perp\times\vec{F}\ &= (2.5\times 10^4\ m\ \hat{i})\times(-30.0\ kg\cdot m/s^2\ \hat{j}),\\ &= 7.5\times 10^5\ N\cdot m(-\hat{k}).\end{aligned}$$

Significance

Since the meteor is accelerating downward toward Earth, its radius and velocity vector are changing. Therefore, since $\vec{l} = \vec{r}\times\vec{p}$, the angular momentum is changing as a function of time. The torque on the meteor about the origin, however, is constant, because the lever arm $\vec{r}_\perp$ and the force on the meteor are constants. This example is important in that it illustrates that the angular momentum depends on the choice of origin about which it is calculated. The methods used in this example are also important in developing angular momentum for a system of particles and for a rigid body.

Exercise 11.20.1

A proton spiraling around a magnetic field executes circular motion in the plane of the paper, as shown below. The circular path has a radius of 0.4 m and the proton has velocity 4.0 x 10^6 m/s. What is the angular momentum of the proton about the origin?



Angular Momentum of a System of Particles

The angular momentum of a system of particles is important in many scientific disciplines, one being astronomy. Consider a spiral galaxy, a rotating island of stars like our own Milky Way. The individual stars can be treated as point particles, each of which has its own angular momentum. The vector sum of the individual angular momenta give the total angular momentum of the galaxy. In this section, we develop the tools with which we can calculate the total angular momentum of a system of particles.

In the preceding section, we introduced the angular momentum of a single particle about a designated origin. The expression for this angular momentum is $\vec{l} = \vec{r} \times \vec{p}$, where the vector $\vec{r}$ is from the origin to the particle, and $\vec{p}$ is the particle's linear momentum. If we have a system of N particles, each with position vector from the origin given by $\vec{r}_i$ and each having momentum $\vec{p}_i$, then the total angular momentum of the system of particles about the origin is the vector sum of the individual angular momenta about the origin. That is,

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_N. \tag{11.20.7}$$

Similarly, if particle i is subject to a net torque $\vec{\tau_i}$ about the origin, then we can find the net torque about the origin due to the system of particles by differentiating Equation 11.7:

$$\frac{d\vec{L}}{dt} = \sum_i \frac{d\vec{l}_i}{dt} = \sum_i \tau_i. \tag{11.20.8}$$

The sum of the individual torques produces a net external torque on the system, which we designate $\sum \vec{\tau}$. Thus,

$$\frac{d\vec{L}}{dt} = \sum_i \tau_i. \tag{11.20.9}$$

Equation 11.20.9states that **the rate of change of the total angular momentum of a system is equal to the net external torque acting on the system when both quantities are measured with respect to a given origin**. Equation 11.20.9can be applied to any system that has net angular momentum, including rigid bodies, as discussed in the next section.

Example 11.20.2: Angular Momentum of Three Particles

Referring to Figure $11.20.1a$

a. Determine the total angular momentum due to the three particles about the origin.
b. What is the rate of change of the angular momentum?

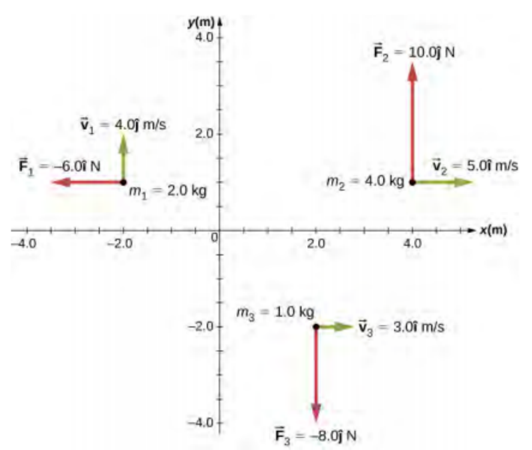


Figure 11.20.3: Three particles in the xy-plane with different position and momentum vectors.

Strategy

Write down the position and momentum vectors for the three particles. Calculate the individual angular momenta and add them as vectors to find the total angular momentum. Then do the same for the torques.

Solution

a. Particle 1:

$$\vec{r}_1 = -2.0\ m\ \hat{i} + 1.0\ m\ \hat{j},\ \vec{p}_1 = (2.0\ kg)(4.0\ m/s\ \hat{j}) = 8.0\ kg \cdot m/s\ \hat{j}, \tag{11.20.10}$$

$$\vec{l}_1 = \vec{r}_1 \times \vec{p}_1 = -16.0\ kg \cdot m^2/s\ \hat{k}. \tag{11.20.11}$$

Particle 2:

$$\vec{r}_2 = 4.0\ m\ \hat{i} + 1.0\ m\ \hat{j},\ \vec{p}_2 = (4.0\ kg)(5.0\ m/s\ \hat{i}) = 20.0\ kg \cdot m/s\ \hat{i}, \tag{11.20.12}$$

$$\vec{l}_2 = \vec{r}_2 \times \vec{p}_2 = -20.0\ kg \cdot m^2/s\ \hat{k}. \tag{11.20.13}$$

Particle 3:

$$\vec{r}_3 = 2.0\ m\ \hat{i} - 2.0\ m\ \hat{j},\ \vec{p}_3 = (1.0\ kg)(3.0\ m/s\ \hat{i}) = 3.0\ kg \cdot m/s\ \hat{j}, \tag{11.20.14}$$

$$\vec{l}_3 = \vec{r}_3 \times \vec{p}_3 = 6.0\ kg \cdot m^2/s\ \hat{k}. \tag{11.20.15}$$

We add the individual angular momenta to find the total about the origin:

$$\vec{l}_T = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 = -30\ kg \cdot m^2/s\ \hat{k}. \tag{11.20.16}$$

b. The individual forces and lever arms are

$$\begin{aligned} \vec{r}_{1\perp} &= 1.0\ m\ \hat{j},\ \vec{F}_1 = -6.0\ N\ \hat{i},\ \vec{\tau}_1 = 6.0\ N \cdot m\ \hat{k} \\ \vec{r}_{2\perp} &= 4.0\ m\ \hat{i},\ \vec{F}_2 = 10.0\ N\ \hat{i},\ \vec{\tau}_2 = 40.0\ N \cdot m\ \hat{k} \\ \vec{r}_{3\perp} &= 2.0\ m\ \hat{j},\ \vec{F}_3 = -8.0\ N\ \hat{j},\ \vec{\tau}_3 = -16.0\ N \cdot m\ \hat{k}. \end{aligned}$$

Therefore:

$$\sum_i \vec{\tau}_i = \vec{\tau}_1 + \vec{\tau}_2 + \vec{\tau}_3 = 30\ N \cdot m\ \hat{k}. \tag{11.20.17}$$

Significance

This example illustrates the superposition principle for angular momentum and torque of a system of particles. Care must be taken when evaluating the radius vectors $\vec{r}_i$ of the particles to calculate the angular momenta, and the lever arms, $\vec{r}_{i\perp}$ to calculate the torques, as they are completely different quantities.

Angular Momentum of a Rigid Body

We have investigated the angular momentum of a single particle, which we generalized to a system of particles. Now we can use the principles discussed in the previous section to develop the concept of the angular momentum of a rigid body. Celestial objects such as planets have angular momentum due to their spin and orbits around stars. In engineering, anything that rotates about an axis carries angular momentum, such as flywheels, propellers, and rotating parts in engines. Knowledge of the angular momenta of these objects is crucial to the design of the system in which they are a part.

To develop the angular momentum of a rigid body, we model a rigid body as being made up of small mass segments, Δm_i. In Figure $11.20.4$, a rigid body is constrained to rotate about the z-axis with angular velocity ω. All mass segments that make up the rigid body undergo circular motion about the z-axis with the same angular velocity. Part (a) of the figure shows mass segment Δm_i with position vector $\vec{r}_i$ from the origin and radius R_i to the z-axis. The magnitude of its tangential velocity is $v_i = R_i\omega$. Because the vectors $\vec{v}_i$ and $\vec{r}_i$ are perpendicular to each other, the magnitude of the angular momentum of this mass segment is

$$l_i = r_i(\Delta m v_i)\sin 90^o. \tag{11.20.18}$$

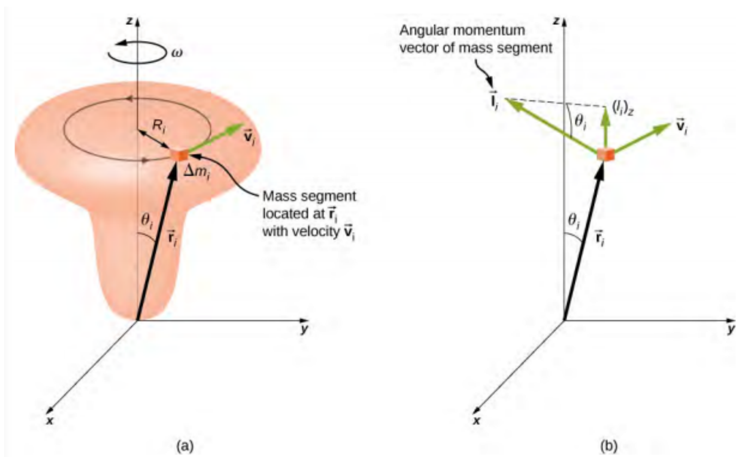


Figure 11.20.4: (a) A rigid body is constrained to rotate around the z-axis. The rigid body is symmetrical about the z-axis. A mass segment Δm_i is located at position $\vec{r}_i$, which makes angle θ_i with respect to the z-axis. The circular motion of an infinitesimal mass segment is shown. (b) $\vec{l}_i$ is the angular momentum of the mass segment and has a component along the z-axis $(\vec{l}_i)_z$.

Using the right-hand rule, the angular momentum vector points in the direction shown in Figure $11.20.4b$. The sum of the angular momenta of all the mass segments contains components both along and perpendicular to the axis of rotation. Every mass segment has a perpendicular component of the angular momentum that will be cancelled by the perpendicular component of an identical mass segment on the opposite side of the rigid body. Thus, the component along the axis of rotation is the only component that gives a nonzero value when summed over all the mass segments. From part (b), the component of $\vec{l}_i$ along the axis of rotation is

$$\begin{aligned}(l_i)_z &= l_i \sin\theta_i = (r_i\Delta m_i v_i)\sin\theta_i, \\ &= (r_i\sin\theta_i)(\Delta m_i v_i) = R_i\Delta m_i v_i.\end{aligned}$$

The net angular momentum of the rigid body along the axis of rotation is

$$L = \sum_i (\vec{l}_i)_z = \sum_i R_i\Delta m_i v_i = \sum_i R_i\Delta m_i(R_i\omega) = \omega\sum_i \Delta m_i(R_i)^2. \tag{11.20.19}$$

The summation $\sum_i \Delta(R_i)^2$ is simply the moment of inertia I of the rigid body about the axis of rotation. For a thin hoop rotating about an axis perpendicular to the plane of the hoop, all of the R_i's are equal to R so the summation reduces to $R^2\sum_i \Delta m_i = mR^2$, which is the moment of inertia for a thin hoop found in Figure 10.20. Thus, the magnitude of the angular momentum along the axis of rotation of a rigid body rotating with angular velocity ω about the axis is

$$L = I\omega. \tag{11.20.20}$$

This equation is analogous to the magnitude of the linear momentum $p = mv$. The direction of the angular momentum vector is directed along the axis of rotation given by the right-hand rule.

Example 11.20.3: Angular Momentum of a Robot Arm

A robot arm on a Mars rover like **Curiosity** shown in Figure 11.20.5 is 1.0 m long and has forceps at the free end to pick up rocks. The mass of the arm is 2.0 kg and the mass of the forceps is 1.0 kg (Figure 11.20.5). The robot arm and forceps move from rest to ω = 0.1π rad/s in 0.1 s. It rotates down and picks up a Mars rock that has mass 1.5 kg. The axis of rotation is the point where the robot arm connects to the rover.

a. What is the angular momentum of the robot arm by itself about the axis of rotation after 0.1 s when the arm has stopped accelerating?
b. What is the angular momentum of the robot arm when it has the Mars rock in its forceps and is rotating upwards?
c. When the arm does not have a rock in the forceps, what is the torque about the point where the arm connects to the rover when it is accelerating from rest to its final angular velocity?

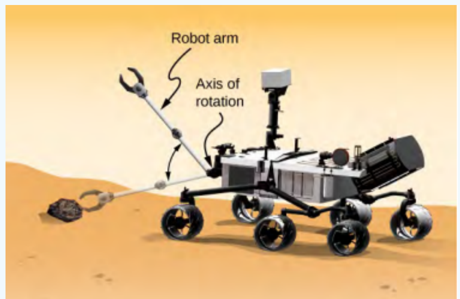


Figure 11.20.5: A robot arm on a Mars rover swings down and picks up a Mars rock. (credit: modification of work by NASA/JPL-Caltech)

Strategy

We use Equation 11.20.20 to find angular momentum in the various configurations. When the arm is rotating downward, the right-hand rule gives the angular momentum vector directed out of the page, which we will call the positive z-direction. When the arm is rotating upward, the right-hand rule gives the direction of the angular momentum vector into the page or in the negative z-direction. The moment of inertia is the sum of the individual moments of inertia. The arm can be approximated with a solid rod, and the forceps and Mars rock can be approximated as point masses located at a distance of 1 m from the origin. For part (c), we use Newton's second law of motion for rotation to find the torque on the robot arm.

Solution

a. Writing down the individual moments of inertia, we have

- Robot arm:

$$I_R = \frac{1}{3}m_R r^2 = \frac{1}{3}(2.00\ kg)(1.00\ m)^2 = \frac{2}{3}\ kg\cdot m^2. \tag{11.20.21}$$

- Forceps:

$$I_F = m_F r^2 = (1.0\ kg)(1.0\ m)^2 = 1.0\ kg\cdot m^2. \tag{11.20.22}$$

- Mars rock:

$$I_{MR} = m_{MR} r^2 = (1.5\ kg)(1.0\ m)^2 = 1.5\ kg\cdot m^2. \tag{11.20.23}$$

Therefore, without the Mars rock, the total moment of inertia is

$$I_{Total} = I_R + I_F = 1.67\ kg\cdot m^2 \tag{11.20.24}$$

and the magnitude of the angular momentum is

$$L = I\omega = (1.67\ kg\cdot m^2)(0.1\pi\ rad/s) = 0.17\pi\ kg\cdot m^2/s. \tag{11.20.25}$$

The angular momentum vector is directed out of the page in the $\hat{k}$ direction since the robot arm is rotating counterclockwise.

b. We must include the Mars rock in the calculation of the moment of inertia, so we have

$$I_{Total} = I_R + I_F + I_{MR} = 3.17\ kg \cdot m^2 \tag{11.20.26}$$

and

$$L = I\omega = (3.17\ kg \cdot m^2)(0.1\pi\ rad/s) = 0.32\pi\ kg \cdot m^2/s. \tag{11.20.27}$$

Now the angular momentum vector is directed into the page in the $-\hat{k}$ direction, by the right-hand rule, since the robot arm is now rotating clockwise.

c. We find the torque when the arm does not have the rock by taking the derivative of the angular momentum using Equation 11.20.9 $\frac{d\vec{L}}{dt} = \sum \vec{\tau}$. But since L = I$\omega$, and understanding that the direction of the angular momentum and torque vectors are along the axis of rotation, we can suppress the vector notation and find

$$\frac{dL}{dt} = \frac{d(I\omega)}{dt} = I\frac{d\omega}{dt} = I\alpha = \sum \tau, \tag{11.20.28}$$

which is Newton's second law for rotation. Since $\alpha = \frac{0.1\pi\ rad/s}{0.1\ s} = \pi\ rad/s^2$, we can calculate the net torque:

$$\sum \tau = I\alpha = (1.67\ kg \cdot m^2)(\pi\ rad/s^2) = 1.67\pi\ N \cdot m. \tag{11.20.29}$$

Significance

The angular momentum in (a) is less than that of (b) due to the fact that the moment of inertia in (b) is greater than (a), while the angular velocity is the same.

? Exercise 11.20.3

Which has greater angular momentum: a solid sphere of mass m rotating at a constant angular frequency ω_0 about the z-axis, or a solid cylinder of same mass and rotation rate about the z-axis?

Simulation

Visit the University of Colorado's Interactive Simulation of Angular Momentum to learn more about angular momentum.

11.21: Conservation of Angular Momentum

Learning Objectives

- Apply conservation of angular momentum to determine the angular velocity of a rotating system in which the moment of inertia is changing
- Explain how the rotational kinetic energy changes when a system undergoes changes in both moment of inertia and angular velocity

So far, we have looked at the angular momentum of systems consisting of point particles and rigid bodies. We have also analyzed the torques involved, using the expression that relates the external net torque to the change in angular momentum. Examples of systems that obey this equation include a freely spinning bicycle tire that slows over time due to torque arising from friction, or the slowing of Earth's rotation over millions of years due to frictional forces exerted on tidal deformations.

However, suppose there is no net external torque on the system, $\sum \vec{\tau} = 0$. In this case, we can introduce the **law of conservation of angular momentum**.

Law of Conservation of Angular Momentum

The angular momentum of a system of particles around a point in a fixed inertial reference frame is conserved if there is no net external torque around that point:

$$\frac{d\vec{L}}{dt} = 0 \tag{11.21.1}$$

or

$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_N = constant. \tag{11.21.2}$$

Note that the total angular momentum $\vec{L}$ is conserved. Any of the individual angular momenta can change as long as their sum remains constant. This law is analogous to linear momentum being conserved when the external force on a system is zero.

As an example of conservation of angular momentum, Figure $11.21.1$shows an ice skater executing a spin. The net torque on her is very close to zero because there is relatively little friction between her skates and the ice. Also, the friction is exerted very close to the pivot point. Both $|\vec{F}|$ and $|\vec{r}|$ are small, so $|\vec{\tau}|$ is negligible. Consequently, she can spin for quite some time. She can also increase her rate of spin by pulling her arms and legs in. Why does pulling her arms and legs in increase her rate of spin? The answer is that her angular momentum is constant, so that

$$L' = L \tag{11.21.3}$$

or

$$I'\omega' = I\omega, \tag{11.21.4}$$

where the primed quantities refer to conditions after she has pulled in her arms and reduced her moment of inertia. Because I′ is smaller, the angular velocity ω' must increase to keep the angular momentum constant.

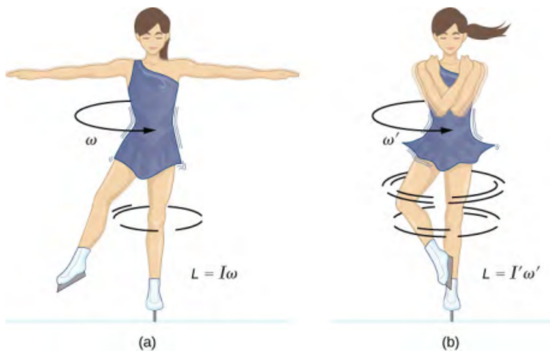


Figure 11.21.1: (a) An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. (b) Her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

It is interesting to see how the rotational kinetic energy of the skater changes when she pulls her arms in. Her initial rotational energy is

$$K_{Rot} = \frac{1}{2} I\omega^2, \tag{11.21.5}$$

whereas her final rotational energy is

$$K'_{Rot} = \frac{1}{2} I(\omega')^2. \tag{11.21.6}$$

Since I′ω' = Iω, we can substitute for ω' and find

$$K'_{Rot} = \frac{1}{2} I'(\omega')^2 = \frac{1}{2} I' \left(\frac{I}{I'}\omega\right)^2 = \frac{1}{2} I\omega^2 \left(\frac{I}{I'}\right) = K_{Rot} \left(\frac{I}{I'}\right). \tag{11.21.7}$$

Because her moment of inertia has decreased, $I' < I$, her final rotational kinetic energy has increased. The source of this additional rotational kinetic energy is the work required to pull her arms inward. Note that the skater's arms do not move in a perfect circle—they spiral inward. This work causes an increase in the rotational kinetic energy, while her angular momentum remains constant. Since she is in a frictionless environment, no energy escapes the system. Thus, if she were to extend her arms to their original positions, she would rotate at her original angular velocity and her kinetic energy would return to its original value.

The solar system is another example of how conservation of angular momentum works in our universe. Our solar system was born from a huge cloud of gas and dust that initially had rotational energy. Gravitational forces caused the cloud to contract, and the rotation rate increased as a result of conservation of angular momentum (Figure 11.21.2).

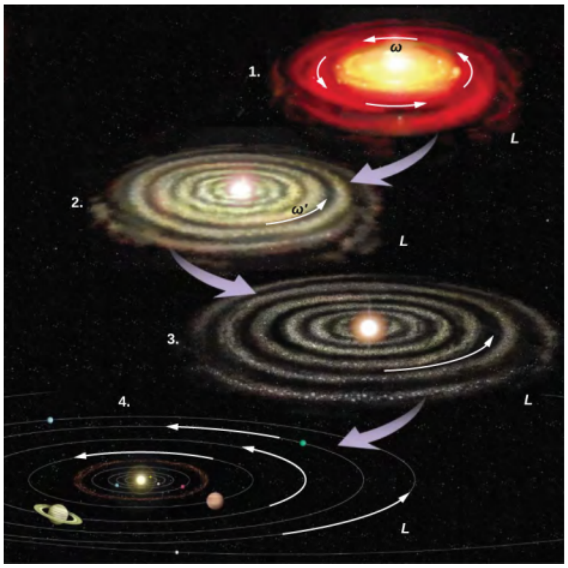


Figure 11.21.2: The solar system coalesced from a cloud of gas and dust that was originally rotating. The orbital motions and spins of the planets are in the same direction as the original spin and conserve the angular momentum of the parent cloud. (credit: modification of work by NASA)

We continue our discussion with an example that has applications to engineering.

Example 11.21.1: Coupled Flywheels

A flywheel rotates without friction at an angular velocity ω_0 = 600 rev/min on a frictionless, vertical shaft of negligible rotational inertia. A second flywheel, which is at rest and has a moment of inertia three times that of the rotating flywheel, is dropped onto it (Figure 11.21.3). Because friction exists between the surfaces, the flywheels very quickly reach the same rotational velocity, after which they spin together.

a. Use the law of conservation of angular momentum to determine the angular velocity ω of the combination.
b. What fraction of the initial kinetic energy is lost in the coupling of the flywheels?

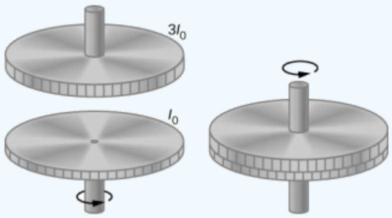


Figure 11.21.3: Two flywheels are coupled and rotate together.

Strategy

Part (a) is straightforward to solve for the angular velocity of the coupled system. We use the result of (a) to compare the initial and final kinetic energies of the system in part (b).

Solution

a. No external torques act on the system. The force due to friction produces an internal torque, which does not affect the angular momentum of the system. Therefore conservation of angular momentum gives

$$I_0\omega_0 = (I_0 + 3I_0)\omega,$$

$$\omega = \frac{1}{4}\omega_0 = 150\ rev/min = 15.7\ rad/s.$$

b. Before contact, only one flywheel is rotating. The rotational kinetic energy of this flywheel is the initial rotational kinetic energy of the system, $\frac{1}{2}I_0\omega_0^2$. The final kinetic energy is

$$\frac{1}{2}(4I_0)\omega^2 = \frac{1}{2}(4I_0)\left(\frac{\omega_0}{4}\right)^2 = \frac{1}{8}I_0\omega_0^2.$$

Therefore, the ratio of the final kinetic energy to the initial kinetic energy is

$$\frac{\frac{1}{8}I_0\omega_0^2}{\frac{1}{2}I_0\omega_0^2} = \frac{1}{4}.$$

Thus, 3/4 of the initial kinetic energy is lost to the coupling of the two flywheels.

Significance

Since the rotational inertia of the system increased, the angular velocity decreased, as expected from the law of conservation of angular momentum. In this example, we see that the final kinetic energy of the system has decreased, as energy is lost to the coupling of the flywheels. Compare this to the example of the skater in Figure $11.21.1$ doing work to bring her arms inward and adding rotational kinetic energy.

? Exercise $11.21.1$

A merry-go-round at a playground is rotating at 4.0 rev/min. Three children jump on and increase the moment of inertia of the merry-go-round/children rotating system by 25%. What is the new rotation rate?

✓ Example $11.21.2A$: Dismount from a High Bar

An 80.0-kg gymnast dismounts from a high bar. He starts the dismount at full extension, then tucks to complete a number of revolutions before landing. His moment of inertia when fully extended can be approximated as a rod of length 1.8 m and when in the tuck a rod of half that length. If his rotation rate at full extension is 1.0 rev/s and he enters the tuck when his center of mass is at 3.0 m height moving horizontally to the floor, how many revolutions can he execute if he comes out of the tuck at 1.8 m height? See Figure $11.21.4$

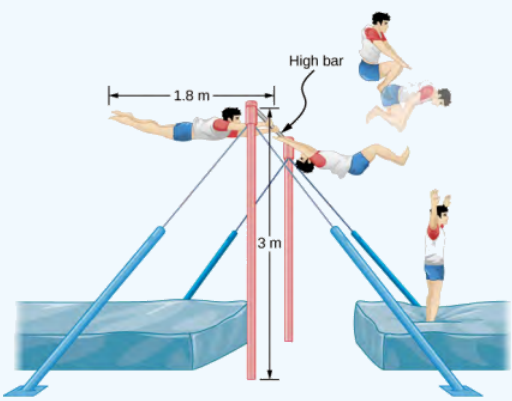


Figure 11.21.4: A gymnast dismounts from a high bar and executes a number of revolutions in the tucked position before landing upright.

Strategy

Using conservation of angular momentum, we can find his rotation rate when in the tuck. Using the equations of kinematics, we can find the time interval from a height of 3.0 m to 1.8 m. Since he is moving horizontally with respect to the ground, the equations of free fall simplify. This will allow the number of revolutions that can be executed to be calculated. Since we are using a ratio, we can keep the units as rev/s and don't need to convert to radians/s.

Solution

The moment of inertia at full extension is

$$I_0 = \frac{1}{12}mL^2 = \frac{1}{12}(80.0\ kg)(1.8\ m)^2 = 21.6\ kg \cdot m^2.$$

The moment of inertia in the tuck is

$$I_f = \frac{1}{12} m L_f^2 = \frac{1}{12}(80.0\ kg)(0.9\ m)^2 = 5.4\ kg \cdot m^2.$$

Conservation of angular momentum:

$$I_f \omega_f = I_0 \omega_0 \Rightarrow \omega_f = \frac{I_0 \omega_0}{I_f} = \frac{(21.6\ kg \cdot m^2)(1.0\ rev/s)}{5.4\ kg \cdot m^2} = 4.0\ rev/s.$$

Time interval in the tuck:

$$t = \sqrt{\frac{2h}{g}} = \sqrt{\frac{2(3.0 - 1.8)m}{9.8\ m/s}} = 0.5\ s.$$

In 0.5 s, he will be able to execute two revolutions at 4.0 rev/s.

Significance

Note that the number of revolutions he can complete will depend on how long he is in the air. In the problem, he is exiting the high bar horizontally to the ground. He could also exit at an angle with respect to the ground, giving him more or less time in the air depending on the angle, positive or negative, with respect to the ground. Gymnasts must take this into account when they are executing their dismounts.

Example 11.21.2B: Conservation of Angular Momentum of a Collision

A bullet of mass m = 2.0 g is moving horizontally with a speed of 500.0 m/s. The bullet strikes and becomes embedded in the edge of a solid disk of mass M = 3.2 kg and radius R = 0.5 m. The cylinder is free to rotate around its axis and is initially at rest (Figure 11.21.5). What is the angular velocity of the disk immediately after the bullet is embedded?

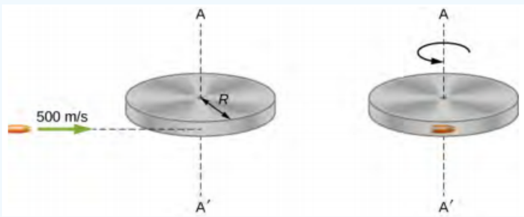


Figure 11.21.5: A bullet is fired horizontally and becomes embedded in the edge of a disk that is free to rotate about its vertical axis.

Strategy

For the system of the bullet and the cylinder, no external torque acts along the vertical axis through the center of the disk. Thus, the angular momentum along this axis is conserved. The initial angular momentum of the bullet is mvR, which is taken about the rotational axis of the disk the moment before the collision. The initial angular momentum of the cylinder is zero. Thus, the net angular momentum of the system is mvR. Since angular momentum is conserved, the initial angular momentum of the system is equal to the angular momentum of the bullet embedded in the disk immediately after impact.

Solution

The initial angular momentum of the system is

$$L_i = mvR.$$

The moment of inertia of the system with the bullet embedded in the disk is

$$I = mR^2 + \frac{1}{2}MR^2 = \left(m + \frac{M}{2}\right)R^2.$$

The final angular momentum of the system is

$$L_f = I\omega_f.$$

Thus, by conservation of angular momentum, $L_i = L_f$ and

$$mvR = \left(m + \frac{M}{2}\right) R^2 \omega_f.$$

Solving for ω_f,

$$\omega_f = \frac{mvR}{\left(m + \frac{M}{2}\right) R^2} = \frac{(2.0 \times 10^{-3}\ kg)(500.0\ m/s)}{(2.0 \times 10^{-3}\ kg + 1.6\ kg)(0.50\ m)} = 1.2\ rad/s.$$

Significance

The system is composed of both a point particle and a rigid body. Care must be taken when formulating the angular momentum before and after the collision. Just before impact the angular momentum of the bullet is taken about the rotational axis of the disk.

11.22: Precession of a Gyroscope

Learning Objectives

- Describe the physical processes underlying the phenomenon of precession
- Calculate the precessional angular velocity of a gyroscope

Figure $11.22.1$shows a gyroscope, defined as a spinning disk in which the axis of rotation is free to assume any orientation. When spinning, the orientation of the spin axis is unaffected by the orientation of the body that encloses it. The body or vehicle enclosing the gyroscope can be moved from place to place and the orientation of the spin axis will remain the same. This makes gyroscopes very useful in navigation, especially where magnetic compasses can't be used, such as in manned and unmanned spacecraft, intercontinental ballistic missiles, unmanned aerial vehicles, and satellites like the Hubble Space Telescope.

Figure 11.22.1: A gyroscope consists of a spinning disk about an axis that is free to assume any orientation.

We illustrate the **precession** of a gyroscope with an example of a top in the next two figures. If the top is placed on a flat surface near the surface of Earth at an angle to the vertical and is not spinning, it will fall over, due to the force of gravity producing a torque acting on its center of mass. This is shown in Figure $11.22.2a$. However, if the top is spinning on its axis, rather than topple over due to this torque, it precesses about the vertical, shown in $11.22.2b$. This is due to the torque on the center of mass, which provides the change in angular momentum.

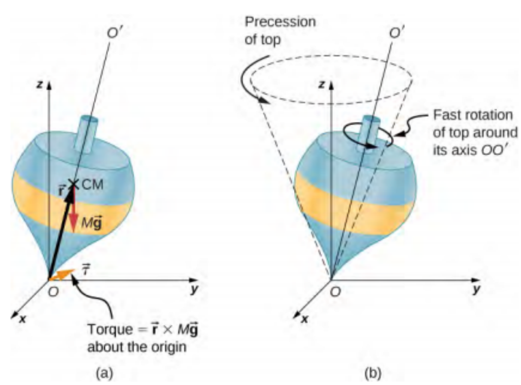


Figure 11.22.2: (a) If the top is not spinning, there is a torque $\vec{r} \times M\vec{g}$ about the origin, and the top falls over. (b) If the top is spinning about its axis OO′, it doesn't fall over but precesses about the z-axis.

Figure $11.22.3$shows the forces acting on a spinning top. The torque produced is perpendicular to the angular momentum vector. This changes the direction of the angular momentum vector $\vec{L}$ according to $d\vec{L} = \vec{\tau}dt$, but not its magnitude. The top precesses around a vertical axis, since the torque is always horizontal and perpendicular to $\vec{L}$. If the top is **not** spinning, it acquires angular momentum in the direction of the torque, and it rotates around a horizontal axis, falling over just as we would expect.

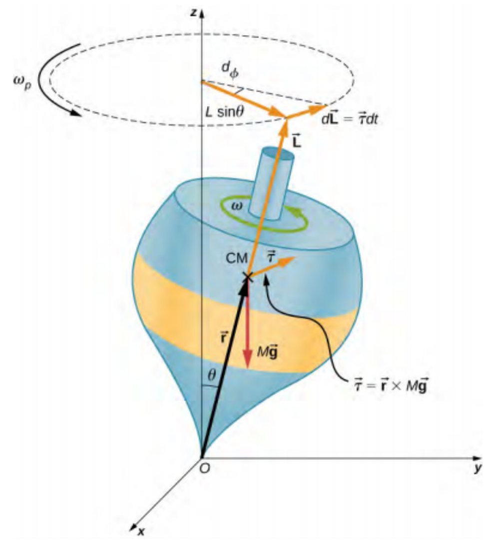


Figure 11.22.3: The force of gravity acting on the center of mass produces a torque $\vec{\tau}$ in the direction perpendicular to $\vec{L}$. The magnitude of $\vec{L}$ doesn't change but its direction does, and the top precesses about the z-axis.

We can experience this phenomenon first hand by holding a spinning bicycle wheel and trying to rotate it about an axis perpendicular to the spin axis. As shown in Figure $11.22.4$, the person applies forces perpendicular to the spin axis in an attempt to rotate the wheel, but instead, the wheel axis starts to change direction to her left due to the applied torque.

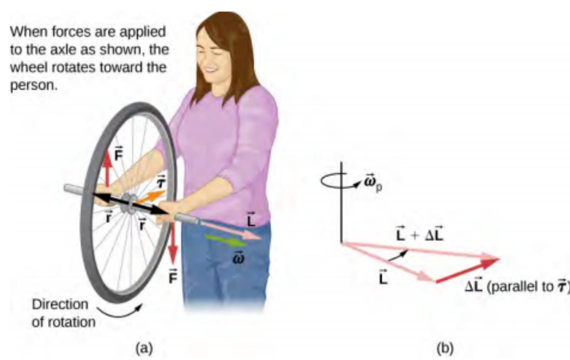


Figure 11.22.4: (a) A person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum $\Delta\vec{L}$ in exactly the same direction. (b) A vector diagram depicting how $\Delta\vec{L}$ and $\vec{L}$ add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.

We all know how easy it is for a bicycle to tip over when sitting on it at rest. But when riding the bicycle at a good pace, it is harder to tip it over because we must change the angular momentum vector of the spinning wheels.

Note

View this video on gyroscope precession for a complete demonstration of precession of the bicycle wheel.

Also, when a spinning disk is put in a box such as a Blu-Ray player, try to move it. It is easy to translate the box in a given direction but difficult to rotate it about an axis perpendicular to the axis of the spinning disk, since we are putting a torque on the box that will cause the angular momentum vector of the spinning disk to precess.

We can calculate the precession rate of the top in Figure $11.22.3$. From Figure $11.22.3$, we see that the magnitude of the torque is

$$\tau = rMg\sin\theta. \tag{11.22.1}$$

Thus,

$$dL = rMg\sin\theta dt. \tag{11.22.2}$$

The angle the top precesses through in time dt is

$$d\phi = \frac{dL}{L\sin\theta} = \frac{rMg\sin\theta}{L\sin\theta}dt = \frac{rMg}{L}dt. \tag{11.22.3}$$

The precession angular velocity is $\omega_P = \frac{d\phi}{dt}$ and from this equation we see that

$$\omega_P = \frac{rMg}{L}. \tag{11.22.4}$$

or, since L = Iω,

$$\omega_P = \frac{rMg}{I\omega}. \tag{11.22.5}$$

In this derivation, we assumed that $\omega_P << \omega$, that is, that the precession angular velocity is much less than the angular velocity of the gyroscope disk. The precession angular velocity adds a small component to the angular momentum along the z-axis. This is seen in a slight bob up and down as the gyroscope precesses, referred to as nutation.

Earth itself acts like a gigantic gyroscope. Its angular momentum is along its axis and currently points at Polaris, the North Star. But Earth is slowly precessing (once in about 26,000 years) due to the torque of the Sun and the Moon on its nonspherical shape.

Example 11.22.1: Period of Precession

A gyroscope spins with its tip on the ground and is spinning with negligible frictional resistance. The disk of the gyroscope has mass 0.3 kg and is spinning at 20 rev/s. Its center of mass is 5.0 cm from the pivot and the radius of the disk is 5.0 cm. What is the precessional period of the gyroscope?

Strategy

We use Equation 11.22.5 to find the precessional angular velocity of the gyroscope. This allows us to find the period of precession.

Solution

The moment of inertia of the disk is

$$I = \frac{1}{2}mr^2 = \frac{1}{2}(0.30\ kg)(0.05\ m)^2 = 3.75 \times 10^{-4}\ kg \cdot m^2.$$

The angular velocity of the disk is

$$20.0\ rev/s = (20.0)(2\pi)\ rad/s = 125.66\ rad/s.$$

We can now substitute in Equation 11.22.5. The precessional angular velocity is

$$\omega_P = \frac{rMg}{I\omega} = \frac{(0.05\ m)(0.3\ kg)(9.8\ m/s^2)}{(3.75 \times 10^{-4}\ kg \cdot m^2)(125.66\ rad/s)} = 3.12\ rad/s.$$

The precessional period of the gyroscope is

$$T_P = \frac{2\pi}{3.12\ rad/s} = 2.0\ s.$$

Significance

The precessional angular frequency of the gyroscope, 3.12 rad/s, or about 0.5 rev/s, is much less than the angular velocity 20 rev/s of the gyroscope disk. Therefore, we don't expect a large component of the angular momentum to arise due to precession, and Equation 11.12 is a good approximation of the precessional angular velocity.

Exercises 11.22.1

A top has a precession frequency of 5.0 rad/s on Earth. What is its precession frequency on the Moon?

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11.23: Angular Momentum (Exercises)

Conceptual Questions

11.1 Rolling Motion

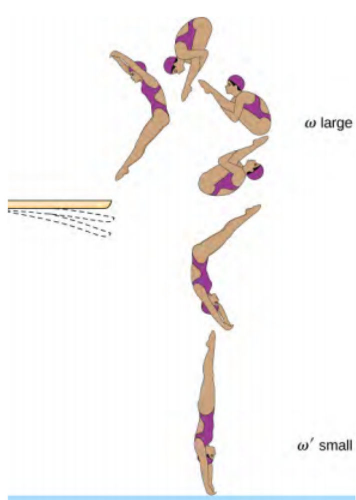
1. Can a round object released from rest at the top of a frictionless incline undergo rolling motion?
2. A cylindrical can of radius R is rolling across a horizontal surface without slipping. (a) After one complete revolution of the can, what is the distance that its center of mass has moved? (b) Would this distance be greater or smaller if slipping occurred?
3. A wheel is released from the top on an incline. Is the wheel most likely to slip if the incline is steep or gently sloped?
4. Which rolls down an inclined plane faster, a hollow cylinder or a solid sphere? Both have the same mass and radius.
5. A hollow sphere and a hollow cylinder of the same radius and mass roll up an incline without slipping and have the same initial center of mass velocity. Which object reaches a greater height before stopping?

11.2 Angular Momentum

6. Can you assign an angular momentum to a particle without first defining a reference point?
7. For a particle traveling in a straight line, are there any points about which the angular momentum is zero? Assume the line intersects the origin.
8. Under what conditions does a rigid body have angular momentum but not linear momentum?
9. If a particle is moving with respect to a chosen origin it has linear momentum. What conditions must exist for this particle's angular momentum to be zero about the chosen origin?
10. If you know the velocity of a particle, can you say anything about the particle's angular momentum?

11.3 Conservation of Angular Momentum

11. What is the purpose of the small propeller at the back of a helicopter that rotates in the plane perpendicular to the large propeller?
12. Suppose a child walks from the outer edge of a rotating merry-go-round to the inside. Does the angular velocity of the merry-go-round increase, decrease, or remain the same? Explain your answer. Assume the merry-go-round is spinning without friction.
13. As the rope of a tethered ball winds around a pole, what happens to the angular velocity of the ball?
14. Suppose the polar ice sheets broke free and floated toward Earth's equator without melting. What would happen to Earth's angular velocity?
15. Explain why stars spin faster when they collapse.
16. Competitive divers pull their limbs in and curl up their bodies when they do flips. Just before entering the water, they fully extend their limbs to enter straight down (see below). Explain the effect of both actions on their angular velocities. Also explain the effect on their angular momentum.



11.4 Precession of a Gyroscope

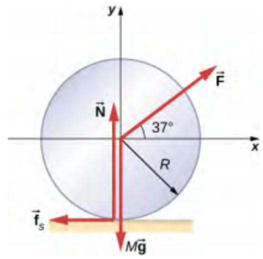
17. Gyroscopes used in guidance systems to indicate directions in space must have an angular momentum that does not change in direction. When placed in the vehicle, they are put in a compartment that is separated from the main fuselage, such that changes in the orientation of the fuselage does not affect the orientation of the gyroscope. If the space vehicle is subjected to large forces and accelerations how can the direction of the gyroscopes angular momentum be constant at all times?
18. Earth precesses about its vertical axis with a period of 26,000 years. Discuss whether Equation 11.12 can be used to calculate the precessional angular velocity of Earth.

Problems

11.1 Rolling Motion

19. What is the angular velocity of a 75.0-cm-diameter tire on an automobile traveling at 90.0 km/h?
20. A boy rides his bicycle 2.00 km. The wheels have radius 30.0 cm. What is the total angle the tires rotate through during his trip?
21. If the boy on the bicycle in the preceding problem accelerates from rest to a speed of 10.0 m/s in 10.0 s, what is the angular acceleration of the tires?
22. Formula One race cars have 66-cm-diameter tires. If a Formula One averages a speed of 300 km/h during a race, what is the angular displacement in revolutions of the wheels if the race car maintains this speed for 1.5 hours?
23. A marble rolls down an incline at 30° from rest. (a) What is its acceleration? (b) How far does it go in 3.0 s?
24. Repeat the preceding problem replacing the marble with a solid cylinder. Explain the new result.
25. A rigid body with a cylindrical cross-section is released from the top of a 30° incline. It rolls 10.0 m to the bottom in 2.60 s. Find the moment of inertia of the body in terms of its mass m and radius r.
26. A yo-yo can be thought of a solid cylinder of mass m and radius r that has a light string wrapped around its circumference (see below). One end of the string is held fixed in space. If the cylinder falls as the string unwinds without slipping, what is the acceleration of the cylinder?

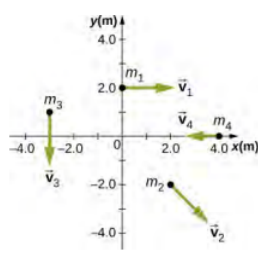
27. A solid cylinder of radius 10.0 cm rolls down an incline with slipping. The angle of the incline is 30°. The coefficient of kinetic friction on the surface is 0.400. What is the angular acceleration of the solid cylinder? What is the linear acceleration?
28. A bowling ball rolls up a ramp 0.5 m high without slipping to storage. It has an initial velocity of its center of mass of 3.0 m/s. (a) What is its velocity at the top of the ramp? (b) If the ramp is 1 m high does it make it to the top?
29. A 40.0-kg solid cylinder is rolling across a horizontal surface at a speed of 6.0 m/s. How much work is required to stop it?
30. A 40.0-kg solid sphere is rolling across a horizontal surface with a speed of 6.0 m/s. How much work is required to stop it? Compare results with the preceding problem.
31. A solid cylinder rolls up an incline at an angle of 20°. If it starts at the bottom with a speed of 10 m/s, how far up the incline does it travel?
32. A solid cylindrical wheel of mass M and radius R is pulled by a force $\vec{F}$ applied to the center of the wheel at 37° to the horizontal (see the following figure). If the wheel is to roll without slipping, what is the maximum value of $|\vec{F}|$? The coefficients of static and kinetic friction are $\mu_S = 0.40$ and $\mu_k = 0.30$.



33. A hollow cylinder that is rolling without slipping is given a velocity of 5.0 m/s and rolls up an incline to a vertical height of 1.0 m. If a hollow sphere of the same mass and radius is given the same initial velocity, how high vertically does it roll up the incline?

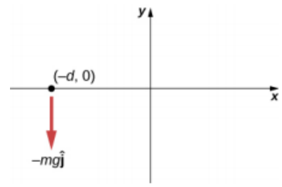
11.2 Angular Momentum

34. A 0.2-kg particle is travelling along the line y = 2.0 m with a velocity 5.0 m/s. What is the angular momentum of the particle about the origin?
35. A bird flies overhead from where you stand at an altitude of 300.0 m and at a speed horizontal to the ground of 20.0 m/s. The bird has a mass of 2.0 kg. The radius vector to the bird makes an angle θ with respect to the ground. The radius vector to the bird and its momentum vector lie in the xy-plane. What is the bird's angular momentum about the point where you are standing?
36. A Formula One race car with mass 750.0 kg is speeding through a course in Monaco and enters a circular turn at 220.0 km/h in the counterclockwise direction about the origin of the circle. At another part of the course, the car enters a second circular turn at 180 km/h also in the counterclockwise direction. If the radius of curvature of the first turn is 130.0 m and that of the second is 100.0 m, compare the angular momenta of the race car in each turn taken about the origin of the circular turn.
37. A particle of mass 5.0 kg has position vector $\vec{r} = (2.0\ \hat{i} - 3.0\ \hat{j})$m at a particular instant of time when its velocity is $\vec{v} = (3.0\ \hat{i})$m/s with respect to the origin. (a) What is the angular momentum of the particle? (b) If a force $\vec{F} = 5.0\ \hat{j}$ N acts on the particle at this instant, what is the torque about the origin?
38. Use the right-hand rule to determine the directions of the angular momenta about the origin of the particles as shown below. The z-axis is out of the page.



39. Suppose the particles in the preceding problem have masses $m_1 = 0.10$ kg, $m_2 = 0.20$ kg, $m_3 = 0.30$ kg, $m_4 = 0.40$ kg. The velocities of the particles are $v_1 = 2.0\ \hat{i}$ m/s, $v_2 = (3.0\ \hat{i} - 3.0\ \hat{j})$m/s, $v_3 = -1.5\ \hat{j}$ m/s, $v_4 = -4.0\ \hat{i}$ m/s. (a) Calculate the angular momentum of each particle about the origin. (b) What is the total angular momentum of the four-particle system about the origin?
40. Two particles of equal mass travel with the same speed in opposite directions along parallel lines separated by a distance d. Show that the angular momentum of this two-particle system is the same no matter what point is used as the reference for calculating the angular momentum.
41. An airplane of mass 4.0×10^4 kg flies horizontally at an altitude of 10 km with a constant speed of 250 m/s relative to Earth. (a) What is the magnitude of the airplane's angular momentum relative to a ground observer directly below the plane? (b) Does the angular momentum change as the airplane flies along its path?
42. At a particular instant, a 1.0-kg particle's position is $\vec{r} = (2.0\ \hat{i} - 4.0\ \hat{j} + 6.0\ \hat{k})$m, its velocity is $\vec{v} = (-1.0\ \hat{i} + 4.0\ \hat{j} + 1.0\ \hat{k})$m/s, and the force on it is $\vec{F} = (10.0\ \hat{i} + 15.0\ \hat{j})$N. (a) What is the angular momentum of the particle about the origin? (b) What is the torque on the particle about the origin? (c) What is the time rate of change of the particle's angular momentum at this instant?

43. A particle of mass m is dropped at the point (−d, 0) and falls vertically in Earth's gravitational field $-g\,\hat{j}$. (a) What is the expression for the angular momentum of the particle around the z-axis, which points directly out of the page as shown below? (b) Calculate the torque on the particle around the z-axis. (c) Is the torque equal to the time rate of change of the angular momentum?

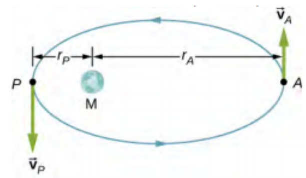


44. (a) Calculate the angular momentum of Earth in its orbit around the Sun. (b) Compare this angular momentum with the angular momentum of Earth about its axis.
45. A boulder of mass 20 kg and radius 20 cm rolls down a hill 15 m high from rest. What is its angular momentum when it is half way down the hill? (b) At the bottom?
46. A satellite is spinning at 6.0 rev/s. The satellite consists of a main body in the shape of a sphere of radius 2.0 m and mass 10,000 kg, and two antennas projecting out from the center of mass of the main body that can be approximated with rods of length 3.0 m each and mass 10 kg. The antenna's lie in the plane of rotation. What is the angular momentum of the satellite?
47. A propeller consists of two blades each 3.0 m in length and mass 120 kg each. The propeller can be approximated by a single rod rotating about its center of mass. The propeller starts from rest and rotates up to 1200 rpm in 30 seconds at a constant rate. (a) What is the angular momentum of the propeller at t = 10 s; t = 20 s? (b) What is the torque on the propeller?
48. A pulsar is a rapidly rotating neutron star. The Crab nebula pulsar in the constellation Taurus has a period of 33.5 x 10^{-3} s, radius 10.0 km, and mass 2.8 x 10^{30} kg. The pulsar's rotational period will increase over time due to the release of electromagnetic radiation, which doesn't change its radius but reduces its rotational energy. (a) What is the angular momentum of the pulsar? (b) Suppose the angular velocity decreases at a rate of 10^{-14} rad/s^2. What is the torque on the pulsar?
49. The blades of a wind turbine are 30 m in length and rotate at a maximum rotation rate of 20 rev/min. (a) If the blades are 6000 kg each and the rotor assembly has three blades, calculate the angular momentum of the turbine at this rotation rate. (b) What is the torque require to rotate the blades up to the maximum rotation rate in 5 minutes?
50. A roller coaster has mass 3000.0 kg and needs to make it safely through a vertical circular loop of radius 50.0 m. What is the minimum angular momentum of the coaster at the bottom of the loop to make it safely through? Neglect friction on the track. Take the coaster to be a point particle.
51. A mountain biker takes a jump in a race and goes airborne. The mountain bike is traveling at 10.0 m/s before it goes airborne. If the mass of the front wheel on the bike is 750 g and has radius 35 cm, what is the angular momentum of the spinning wheel in the air the moment the bike leaves the ground?

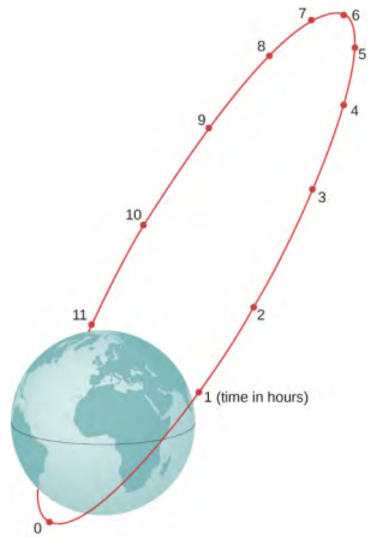
11.3 Conservation of Angular Momentum

52. A disk of mass 2.0 kg and radius 60 cm with a small mass of 0.05 kg attached at the edge is rotating at 2.0 rev/s. The small mass suddenly separates from the disk. What is the disk's final rotation rate?
53. The Sun's mass is 2.0 x 10^{30} kg, its radius is 7.0 x 10^5 km, and it has a rotational period of approximately 28 days. If the Sun should collapse into a white dwarf of radius 3.5 x 10^3 km, what would its period be if no mass were ejected and a sphere of uniform density can model the Sun both before and after?
54. A cylinder with rotational inertia I_1 = 2.0 kg • m^2 rotates clockwise about a vertical axis through its center with angular speed ω_1 = 5.0 rad/s. A second cylinder with rotational inertia I_2 = 1.0 kg • m^2 rotates counterclockwise about the same axis with angular speed ω_2 = 8.0 rad/s. If the cylinders couple so they have the same rotational axis what is the angular speed of the combination? What percentage of the original kinetic energy is lost to friction?
55. A diver off the high board imparts an initial rotation with his body fully extended before going into a tuck and executing three back somersaults before hitting the water. If his moment of inertia before the tuck is 16.9 kg • m^2 and after the tuck during the somersaults is 4.2 kg • m^2, what rotation rate must he impart to his body directly off the board and before the tuck if he takes 1.4 s to execute the somersaults before hitting the water?

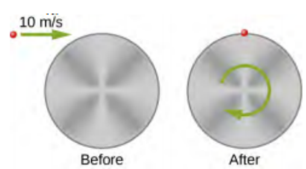
56. An Earth satellite has its apogee at 2500 km above the surface of Earth and perigee at 500 km above the surface of Earth. At apogee its speed is 730 m/s. What is its speed at perigee? Earth's radius is 6370 km (see below).



57. A Molniya orbit is a highly eccentric orbit of a communication satellite so as to provide continuous communications coverage for Scandinavian countries and adjacent Russia. The orbit is positioned so that these countries have the satellite in view for extended periods in time (see below). If a satellite in such an orbit has an apogee at 40,000.0 km as measured from the center of Earth and a velocity of 3.0 km/s, what would be its velocity at perigee measured at 200.0 km altitude?

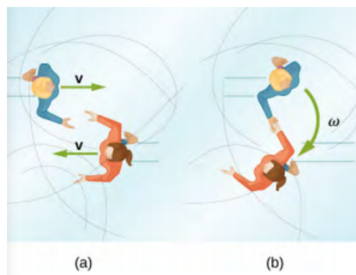


58. Shown below is a small particle of mass 20 g that is moving at a speed of 10.0 m/s when it collides and sticks to the edge of a uniform solid cylinder. The cylinder is free to rotate about its axis through its center and is perpendicular to the page. The cylinder has a mass of 0.5 kg and a radius of 10 cm, and is initially at rest. (a) What is the angular velocity of the system after the collision? (b) How much kinetic energy is lost in the collision?



59. A bug of mass 0.020 kg is at rest on the edge of a solid cylindrical disk (M = 0.10 kg, R = 0.10 m) rotating in a horizontal plane around the vertical axis through its center. The disk is rotating at 10.0 rad/s. The bug crawls to the center of the disk. (a) What is the new angular velocity of the disk? (b) What is the change in the kinetic energy of the system? (c) If the bug crawls back to the outer edge of the disk, what is the angular velocity of the disk then? (d) What is the new kinetic energy of the system? (e) What is the cause of the increase and decrease of kinetic energy?
60. A uniform rod of mass 200 g and length 100 cm is free to rotate in a horizontal plane around a fixed vertical axis through its center, perpendicular to its length. Two small beads, each of mass 20 g, are mounted in grooves along the rod. Initially, the two beads are held by catches on opposite sides of the rod's center, 10 cm from the axis of rotation. With the beads in this position, the rod is rotating with an angular velocity of 10.0 rad/s. When the catches are released, the beads slide outward along the rod. (a) What is the rod's angular velocity when the beads reach the ends of the rod? (b) What is the rod's angular velocity if the beads fly off the rod?
61. A merry-go-round has a radius of 2.0 m and a moment of inertia 300 kg • m^2. A boy of mass 50 kg runs tangent to the rim at a speed of 4.0 m/s and jumps on. If the merry-go-round is initially at rest, what is the angular velocity after the boy jumps on?

62. A playground merry-go-round has a mass of 120 kg and a radius of 1.80 m and it is rotating with an angular velocity of 0.500 rev/s. What is its angular velocity after a 22.0-kg child gets onto it by grabbing its outer edge? The child is initially at rest.
63. Three children are riding on the edge of a merry-go-round that is 100 kg, has a 1.60-m radius, and is spinning at 20.0 rpm. The children have masses of 22.0, 28.0, and 33.0 kg. If the child who has a mass of 28.0 kg moves to the center of the merry-go-round, what is the new angular velocity in rpm?
64. (a) Calculate the angular momentum of an ice skater spinning at 6.00 rev/s given his moment of inertia is 0.400 kg • m^2. (b) He reduces his rate of spin (his angular velocity) by extending his arms and increasing his moment of inertia. Find the value of his moment of inertia if his angular velocity decreases to 1.25 rev/s. (c) Suppose instead he keeps his arms in and allows friction of the ice to slow him to 3.00 rev/s. What average torque was exerted if this takes 15.0 s?
65. Twin skaters approach one another as shown below and lock hands. (a) Calculate their final angular velocity, given each had an initial speed of 2.50 m/s relative to the ice. Each has a mass of 70.0 kg, and each has a center of mass located 0.800 m from their locked hands. You may approximate their moments of inertia to be that of point masses at this radius. (b) Compare the initial kinetic energy and final kinetic energy.

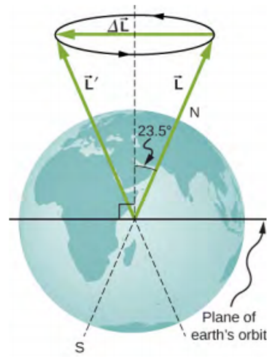


66. A baseball catcher extends his arm straight up to catch a fast ball with a speed of 40 m/s. The baseball is 0.145 kg and the catcher's arm length is 0.5 m and mass 4.0 kg. (a) What is the angular velocity of the arm immediately after catching the ball as measured from the arm socket? (b) What is the torque applied if the catcher stops the rotation of his arm 0.3 s after catching the ball?
67. In 2015, in Warsaw, Poland, Olivia Oliver of Nova Scotia broke the world record for being the fastest spinner on ice skates. She achieved a record 342 rev/min, beating the existing Guinness World Record by 34 rotations. If an ice skater extends her arms at that rotation rate, what would be her new rotation rate? Assume she can be approximated by a 45-kg rod that is 1.7 m tall with a radius of 15 cm in the record spin. With her arms stretched take the approximation of a rod of length 130 cm with 10% of her body mass aligned perpendicular to the spin axis. Neglect frictional forces.
68. A satellite in a geosynchronous circular orbit is 42,164.0 km from the center of Earth. A small asteroid collides with the satellite sending it into an elliptical orbit of apogee 45,000.0 km. What is the speed of the satellite at apogee? Assume its angular momentum is conserved.
69. A gymnast does cartwheels along the floor and then launches herself into the air and executes several flips in a tuck while she is airborne. If her moment of inertia when executing the cartwheels is 13.5 kg • m^2 and her spin rate is 0.5 rev/s, how many revolutions does she do in the air if her moment of inertia in the tuck is 3.4 kg • m^2 and she has 2.0 s to do the flips in the air?
70. The centrifuge at NASA Ames Research Center has a radius of 8.8 m and can produce forces on its payload of 20 gs or 20 times the force of gravity on Earth. (a) What is the angular momentum of a 20-kg payload that experiences 10 gs in the centrifuge? (b) If the driver motor was turned off in (a) and the payload lost 10 kg, what would be its new spin rate, taking into account there are no frictional forces present?
71. A ride at a carnival has four spokes to which pods are attached that can hold two people. The spokes are each 15 m long and are attached to a central axis. Each spoke has mass 200.0 kg, and the pods each have mass 100.0 kg. If the ride spins at 0.2 rev/s with each pod containing two 50.0-kg children, what is the new spin rate if all the children jump off the ride?
72. An ice skater is preparing for a jump with turns and has his arms extended. His moment of inertia is 1.8 kg • m^2 while his arms are extended, and he is spinning at 0.5 rev/s. If he launches himself into the air at 9.0 m/s at an angle of 45° with respect to the ice, how many revolutions can he execute while airborne if his moment of inertia in the air is 0.5 kg • m^2?
73. A space station consists of a giant rotating hollow cylinder of mass 10^6 kg including people on the station and a radius of 100.00 m. It is rotating in space at 3.30 rev/min in order to produce artificial gravity. If 100 people of an average mass of 65.00 kg spacewalk to an awaiting spaceship, what is the new rotation rate when all the people are off the station?

74. Neptune has a mass of 1.0 x 10^{26} kg and is 4.5 x 10^{9} km from the Sun with an orbital period of 165 years. Planetesimals in the outer primordial solar system 4.5 billion years ago coalesced into Neptune over hundreds of millions of years. If the primordial disk that evolved into our present day solar system had a radius of 10^{11} km and if the matter that made up these planetesimals that later became Neptune was spread out evenly on the edges of it, what was the orbital period of the outer edges of the primordial disk?

11.4 Precession of a Gyroscope

75. A gyroscope has a 0.5-kg disk that spins at 40 rev/s. The center of mass of the disk is 10 cm from a pivot which is also the radius of the disk. What is the precession angular velocity?
76. The precession angular velocity of a gyroscope is 1.0 rad/s. If the mass of the rotating disk is 0.4 kg and its radius is 30 cm, as well as the distance from the center of mass to the pivot, what is the rotation rate in rev/s of the disk?
77. The axis of Earth makes a 23.5° angle with a direction perpendicular to the plane of Earth's orbit. As shown below, this axis precesses, making one complete rotation in 25,780 y. (a) Calculate the change in angular momentum in half this time. (b) What is the average torque producing this change in angular momentum? (c) If this torque were created by a pair of forces acting at the most effective point on the equator, what would the magnitude of each force be?



Additional Problems

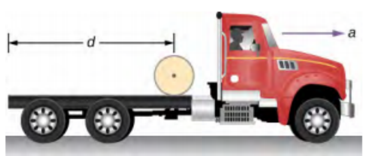
78. A marble is rolling across the floor at a speed of 7.0 m/ s when it starts up a plane inclined at 30° to the horizontal. (a) How far along the plane does the marble travel before coming to a rest? (b) How much time elapses while the marble moves up the plane?
79. Repeat the preceding problem replacing the marble with a hollow sphere. Explain the new results.
80. The mass of a hoop of radius 1.0 m is 6.0 kg. It rolls across a horizontal surface with a speed of 10.0 m/s. (a) How much work is required to stop the hoop? (b) If the hoop starts up a surface at 30° to the horizontal with a speed of 10.0 m/s, how far along the incline will it travel before stopping and rolling back down?
81. Repeat the preceding problem for a hollow sphere of the same radius and mass and initial speed. Explain the differences in the results.
82. A particle has mass 0.5 kg and is traveling along the line x = 5.0 m at 2.0 m/s in the positive y-direction. What is the particle's angular momentum about the origin?
83. A 4.0-kg particle moves in a circle of radius 2.0 m. The angular momentum of the particle varies in time according to l = $5.0t^2$. (a) What is the torque on the particle about the center of the circle at t = 3.4 s? (b) What is the angular velocity of the particle at t = 3.4 s?
84. A proton is accelerated in a cyclotron to 5.0 x 10^{6} m/s in 0.01 s. The proton follows a circular path. If the radius of the cyclotron is 0.5 km, (a) What is the angular momentum of the proton about the center at its maximum speed? (b) What is the torque on the proton about the center as it accelerates to maximum speed?
85. (a) What is the angular momentum of the Moon in its orbit around Earth? (b) How does this angular momentum compare with the angular momentum of the Moon on its axis? Remember that the Moon keeps one side toward Earth at all times.
86. A DVD is rotating at 500 rpm. What is the angular momentum of the DVD if has a radius of 6.0 cm and mass 20.0 g?
87. A potter's disk spins from rest up to 10 rev/s in 15 s. The disk has a mass 3.0 kg and radius 30.0 cm. What is the angular momentum of the disk at t = 5 s, t = 10 s?
88. Suppose you start an antique car by exerting a force of 300 N on its crank for 0.250 s. What is the angular momentum given to the engine if the handle of the crank is 0.300 m from the pivot and the force is exerted to create maximum torque

the entire time?

89. A solid cylinder of mass 2.0 kg and radius 20 cm is rotating counterclockwise around a vertical axis through its center at 600 rev/min. A second solid cylinder of the same mass is rotating clockwise around the same vertical axis at 900 rev/min. If the cylinders couple so that they rotate about the same vertical axis, what is the angular velocity of the combination?
90. A boy stands at the center of a platform that is rotating without friction at 1.0 rev/s. The boy holds weights as far from his body as possible. At this position the total moment of inertia of the boy, platform, and weights is 5.0 kg • m^2. The boy draws the weights in close to his body, thereby decreasing the total moment of inertia to 1.5 kg • m^2. (a) What is the final angular velocity of the platform? (b) By how much does the rotational kinetic energy increase?
91. Eight children, each of mass 40 kg, climb on a small merry-go-round. They position themselves evenly on the outer edge and join hands. The merry-go-round has a radius of 4.0 m and a moment of inertia 1000.0 kg • m^2. After the merry-go-round is given an angular velocity of 6.0 rev/min, the children walk inward and stop when they are 0.75 m from the axis of rotation. What is the new angular velocity of the merry-go-round? Assume there is negligible frictional torque on the structure.
92. A thin meter stick of mass 150 g rotates around an axis perpendicular to the stick's long axis at an angular velocity of 240 rev/min. What is the angular momentum of the stick if the rotation axis (a) passes through the center of the stick? (b) Passes through one end of the stick?
93. A satellite in the shape of a sphere of mass 20,000 kg and radius 5.0 m is spinning about an axis through its center of mass. It has a rotation rate of 8.0 rev/s. Two antennas deploy in the plane of rotation extending from the center of mass of the satellite. Each antenna can be approximated as a rod has mass 200.0 kg and length 7.0 m. What is the new rotation rate of the satellite?
94. A top has moment of inertia 3.2 x 10^{-4} kg • m^2 and radius 4.0 cm from the center of mass to the pivot point. If it spins at 20.0 rev/s and is precessing, how many revolutions does it precess in 10.0 s?

Challenge Problems

95. The truck shown below is initially at rest with solid cylindrical roll of paper sitting on its bed. If the truck moves forward with a uniform acceleration a, what distance s does it move before the paper rolls off its back end? (**Hint**: If the roll accelerates forward with a′, then is accelerates backward relative to the truck with an acceleration a − a′. Also, Rα = a − a′.)



96. A bowling ball of radius 8.5 cm is tossed onto a bowling lane with speed 9.0 m/s. The direction of the toss is to the left, as viewed by the observer, so the bowling ball starts to rotate counterclockwise when in contact with the floor. The coefficient of kinetic friction on the lane is 0.3. (a) What is the time required for the ball to come to the point where it is not slipping? What is the distance d to the point where the ball is rolling without slipping?
97. A small ball of mass 0.50 kg is attached by a massless string to a vertical rod that is spinning as shown below. When the rod has an angular velocity of 6.0 rad/s, the string makes an angle of 30° with respect to the vertical. (a) If the angular velocity is increased to 10.0 rad/s, what is the new angle of the string? (b) Calculate the initial and final angular momenta of the ball. (c) Can the rod spin fast enough so that the ball is horizontal?

98. A bug flying horizontally at 1.0 m/s collides and sticks to the end of a uniform stick hanging vertically. After the impact, the stick swings out to a maximum angle of 5.0° from the vertical before rotating back. If the mass of the stick is 10 times that of the bug, calculate the length of the stick.

Contributors and Attributions

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11.24: Angular Momentum (Summary)

Key Terms

angular momentum	rotational analog of linear momentum, found by taking the product of moment of inertia a
law of conservation of angular momentum	angular momentum is conserved, that is, the initial angular momentum is equal to the final angular momentum whe
precession	circular motion of the pole of the axis of a spinning object around another axis due
rolling motion	combination of rotational and translational motion with or without slippi

Key Equations

Velocity of center of mass of rolling object	$$v_{CM} = R\omega$$	(11.24.1)
Acceleration of center of mass of rolling object	$$a_{CM} = R\alpha$$	(11.24.2)
Displacement of center of mass of rolling object	$$d_{CM} = R\theta$$	(11.24.3)
Acceleration of an object rolling without slipping	$$a_{CM} = \frac{mg\sin\theta}{m + \left(\frac{I_{CM}}{r^2}\right)}$$	(11.24.4)
Angular momentum	$$\vec{l} = \vec{r} \times \vec{p}$$	(11.24.5)
Derivative of angular momentum equals torque	$$\frac{d\vec{l}}{dt} = \sum \vec{\tau}$$	(11.24.6)
Angular momentum of a system of particles	$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_N$$	(11.24.7)
For a system of particles, derivative of angular momentum equals torque	$$\frac{d\vec{L}}{dt} = \sum \vec{\tau}$$	(11.24.8)
Angular momentum of a rotating rigid body	$$L = I\omega$$	(11.24.9)
Conservation of angular momentum	$$\frac{d\vec{L}}{dt} = 0$$	(11.24.10)
Conservation of angular momentum	$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \cdots + \vec{l}_N = constant$$	(11.24.11)
Precessional angular velocity	$$\omega_P = \frac{rMg}{I\omega}$$	(11.24.12)

Summary

11.1 Rolling Motion

- In rolling motion without slipping, a static friction force is present between the rolling object and the surface. The relations $v_{CM} = R\omega$, $a_{CM} = R\alpha$, and $d_{CM} = R\theta$ all apply, such that the linear velocity, acceleration, and distance of the center of mass are the angular variables multiplied by the radius of the object.
- In rolling motion with slipping, a kinetic friction force arises between the rolling object and the surface. In this case, $v_{CM} \neq R\omega$, $a_{CM} \neq R\alpha$, and $d_{CM} \neq R\theta$.
- Energy conservation can be used to analyze rolling motion. Energy is conserved in rolling motion without slipping. Energy is not conserved in rolling motion with slipping due to the heat generated by kinetic friction.

11.2 Angular Momentum

- The angular momentum $\vec{l} = \vec{r} \times \vec{p}$ of a single particle about a designated origin is the vector product of the position vector in the given coordinate system and the particle's linear momentum.
- The angular momentum $\vec{l} = \sum_i \vec{l}_i$ of a system of particles about a designated origin is the vector sum of the individual momenta of the particles that make up the system.
- The net torque on a system about a given origin is the time derivative of the angular momentum about that origin: $\frac{d\vec{L}}{dt} = \sum \vec{\tau}$
- A rigid rotating body has angular momentum L = Iω directed along the axis of rotation. The time derivative of the angular momentum $\frac{dL}{dt} = \sum \tau$ gives the net torque on a rigid body and is directed along the axis of rotation.

11.3 Conservation of Angular Momentum

- In the absence of external torques, a system's total angular momentum is conserved. This is the rotational counterpart to linear momentum being conserved when the external force on a system is zero.
- For a rigid body that changes its angular momentum in the absence of a net external torque, conservation of angular momentum gives $I_f \omega_f = I_i \omega_i$. This equation says that the angular velocity is inversely proportional to the moment of inertia. Thus, if the moment of inertia decreases, the angular velocity must increase to conserve angular momentum.
- Systems containing both point particles and rigid bodies can be analyzed using conservation of angular momentum. The angular momentum of all bodies in the system must be taken about a common axis.

11.4 Precession of a Gyroscope

- When a gyroscope is set on a pivot near the surface of Earth, it precesses around a vertical axis, since the torque is always horizontal and perpendicular to $\vec{L}$. If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque, and it rotates about a horizontal axis, falling over just as we would expect.
- The precessional angular velocity is given by $\omega_P = \frac{rMg}{I\omega}$, where r is the distance from the pivot to the center of mass of the gyroscope, I is the moment of inertia of the gyroscope's spinning disk, M is its mass, and ω is the angular frequency of the gyroscope disk.

Contributors and Attributions

11.25: Conservation of Angular Momentum

learning objectives

- Evaluate the implications of net torque on conservation of energy

Let us consider some examples of momentum: the Earth continues to spin at the same rate it has for billions of years; a high-diver who is "rotating" when jumping off the board does not need to make any physical effort to continue rotating, and indeed would be unable to stop rotating before hitting the water. These examples have the hallmarks of a *conservation law*. Following are further observations to consider:

1. *A closed system is involved*. Nothing is making an effort to twist the Earth or the high-diver. They are isolated from rotation changing influences (hence the term "closed system").
2. *Something remains unchanged*. There appears to be a numerical quantity for measuring rotational motion such that the total amount of that quantity remains constant in a closed system.
3. *Something can be transferred back and forth without changing the total amount*. A diver rotates faster with arms and legs pulled toward the chest from a fully stretched posture.

Angular Momentum

The conserved quantity we are investigating is called angular momentum. The symbol for angular momentum is the letter L. Just as linear momentum is conserved when there is no net external forces, angular momentum is constant or conserved when the net torque is zero. We can see this by considering Newton's 2nd law for rotational motion:

$\vec{\tau} = \frac{d\vec{L}}{dt}$, where τ is the torque. For the situation in which the net torque is zero, $\frac{d\vec{L}}{dt} = 0$.

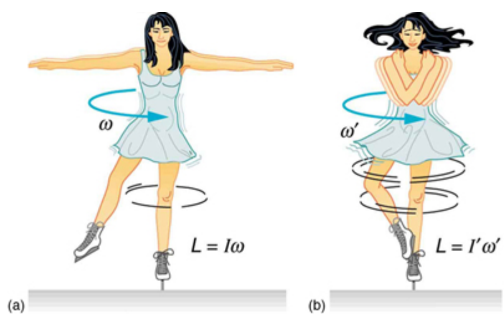
If the change in angular momentum ΔL is zero, then the angular momentum is constant; therefore,

$\vec{L} = \text{constant}$ (when net τ=0).

This is an expression for the law of conservation of angular momentum.

Example and Implications

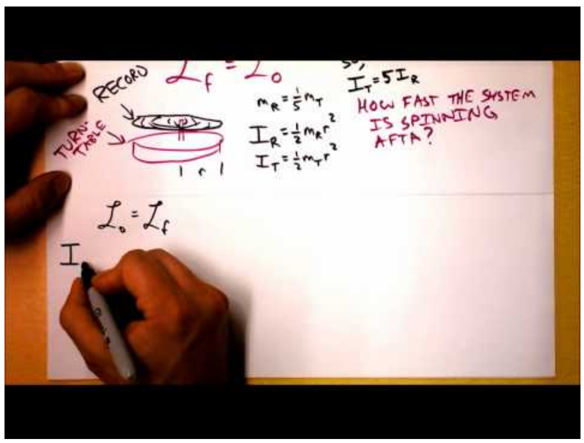
An example of conservation of angular momentum is seen in an ice skater executing a spin, as shown in. The net torque on her is very close to zero, because 1) there is relatively little friction between her skates and the ice, and 2) the friction is exerted very close to the pivot point.



Conservation of Angular Momentum: An ice skater is spinning on the tip of her skate with her arms extended. Her angular momentum is conserved because the net torque on her is negligibly small. In the next image, her rate of spin increases greatly when she pulls in her arms, decreasing her moment of inertia. The work she does to pull in her arms results in an increase in rotational kinetic energy.

(Both F and r are small, and so $\vec{\tau} = \vec{r} \times \vec{F}$ is negligibly small.) Consequently, she can spin for quite some time. She can also increase her rate of spin by pulling in her arms and legs. When she does this, the rotational inertia decreases and the rotation rate increases in order to keep the angular momentum $L = I\omega$ constant. (I: rotational inertia, ω: angular velocity)

Conservation of angular momentum is one of the key conservation laws in physics, along with the conservation laws for energy and (linear) momentum. These laws are applicable even in microscopic domains where quantum mechanics governs; they exist due to inherent symmetries present in nature.



Conservation of Angular Momentum Theory: What it do?

Rotational Collisions

In a closed system, angular momentum is conserved in a similar fashion as linear momentum.

learning objectives

- Evaluate the difference in equation variables in rotational versus angular momentum

During a collision of objects in a closed system, momentum is always conserved. This fact is readily seen in linear motion. When an object of mass m and velocity v collides with another object of mass m_2 and velocity v_2, the net momentum after the collision, $mv_{1f} + mv_{2f}$, is the same as the momentum before the collision, $mv_{1i} + mv_{2i}$.

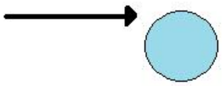
What if an rotational component of motion is introduced? Is momentum still conserved ?

Bowling ball and pi: When a bowling ball collides with a pin, linear and angular momentum is conserved

Yes. For objects with a rotational component, there exists angular momentum. Angular momentum is defined, mathematically, as $L = I\omega$, or $L = r \times p$. This equation is an analog to the definition of linear momentum as $p = mv$. Units for linear momentum are kg⋅m/s while units for angular momentum are kg⋅m^2/s. As we would expect, an object that has a large moment of inertia *I*, such as Earth, has a very large angular momentum. An object that has a large angular velocity ω, such as a centrifuge, also has a rather large angular momentum.

So rotating objects that collide in a closed system conserve not only linear momentum p in all directions, but also angular momentum L in all directions.

For example, take the case of an archer who decides to shoot an arrow of mass m_1 at a stationary cylinder of mass m_2 and radius r, lying on its side. If the archer releases the arrow with a velocity v_{1i} and the arrow hits the cylinder at its radial edge, what's the final momentum ?



Arrow hitting cyclinde: The arrow hits the edge of the cylinder causing it to roll.

Initially, the cylinder is stationary, so it has no momentum linearly or radially. Once the arrow is released, it has a linear momentum $p = m_{1i}$ and an angular component relative to the cylinders rotating axis, $L = rp = rm_1 v_{1i}$. After the collision, the arrow sticks to the rolling cylinder and the system has a net angular momentum equal to the original angular momentum of the arrow before the collision.

Key Points

- When an object is spinning in a closed system and no external torques are applied to it, it will have no change in angular momentum.
- The conservation of angular momentum explains the angular acceleration of an ice skater as she brings her arms and legs close to the vertical axis of rotation.
- If the net torque is zero, then angular momentum is constant or conserved.
- Angular momentum is defined, mathematically, as $L = I\omega$, or $L = r \times p$. Which is the moment of inertia times the angular velocity, or the radius of the object crossed with the linear momentum.
- In a closed system, angular momentum is conserved in all directions after a collision.
- Since momentum is conserved, part of the momentum in a collision may become angular momentum as an object starts to spin after a collision.

Key Items

- **quantum mechanics**: The branch of physics that studies matter and energy at the level of atoms and other elementary particles; it substitutes probabilistic mechanisms for classical Newtonian ones.

- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **angular momentum**: A vector quantity describing an object in circular motion; its magnitude is equal to the momentum of the particle, and the direction is perpendicular to the plane of its circular motion.
- **momentum**: (of a body in motion) the product of its mass and velocity.
- **rotation**: The act of turning around a centre or an axis.

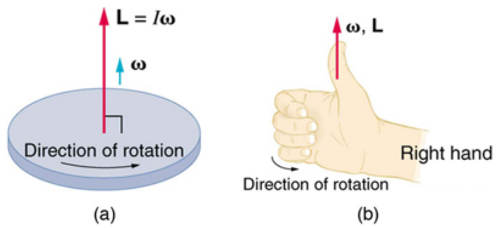
11.26: Vector Nature of Rotational Kinematics

learning objectives

- Identify the direction of a vector using the Right Hand Rule

Angular momentum and angular velocity have both magnitude and direction and, therefore, are vector quantities. The direction of these quantities is inherently difficult to track—a point on a rotating wheel is constantly rotating and changing direction. The axis of rotation of a rotating wheel is the only place that has a fixed direction. The direction of angular momentum and velocity can be determined along this axis.

Imagine the axis of rotation as a pole through the center of a wheel. The pole protrudes on both sides of the wheel and, depending on which side you're looking at, the wheel is turning either clockwise or counterclockwise. This dependency on perspective makes determining the angle of rotation slightly more difficult. As with all physical quantities, there is a standard for measurement that makes these types of quantities consistent. For angular quantities, the direction of the vector is determined using the Right Hand Rule, illustrated in.



The Right Hand Rule: Figure (a) shows a disk is rotating counterclockwise when viewed from above. Figure (b) shows the right-hand rule. The direction of angular velocity ω size and angular momentum L are defined to be the direction in which the thumb of your right hand points when you curl your fingers in the direction of the disk's rotation as shown.

The right hand rule can be used to find the direction of both the angular momentum and the angular velocity. From a spinning disc, for example, let's again imagine a pole through the center of the disc, at the axis of rotation. Using the right hand rule, your right hand would be grasping the pole so that your four fingers (index, middle, ring, and pinky) are following the direction of rotation. That is, an imaginary arrow from your wrist to your fingertips points in the same direction as the disc is rotating. In addition, your thumb is pointing straight out in the axis, perpendicular to your other fingers (or parallel to the 'pole' at the axis of rotation). Using this right hand rule, the direction of angular velocity ω and angular momentum L are defined as the direction in which the thumb of your right hand points when you curl your fingers in the direction of the disc's rotation.

Gyroscopes

A gyroscope is a spinning wheel or disk in which the axle is free to assume any orientation.

learning objectives

- Compare the concept of a rotating wheel with a gyroscope

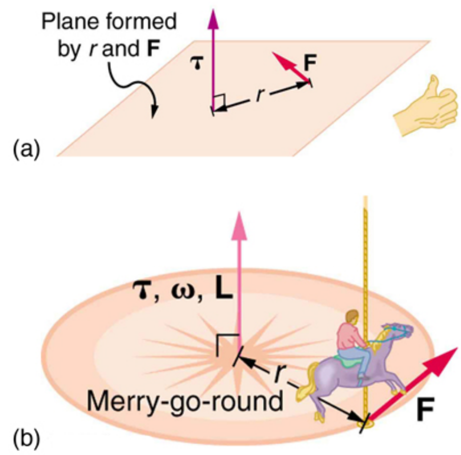
A gyroscope is a device for measuring or maintaining orientation based on the principles of angular momentum. Mechanically, a gyroscope is a spinning wheel or disk in which the axle is free to assume any orientation. Although this orientation does not remain fixed, it changes in response to an external torque much less and in a different direction than it would without the large angular momentum associated with the disk's high rate of spin and moment of inertia. The device's orientation remains nearly fixed, regardless of the mounting platform's motion, because mounting the device in a gimbal minimizes external torque.

How It Works: Examples

Torque: Torque changes angular momentum as expressed by the equation,

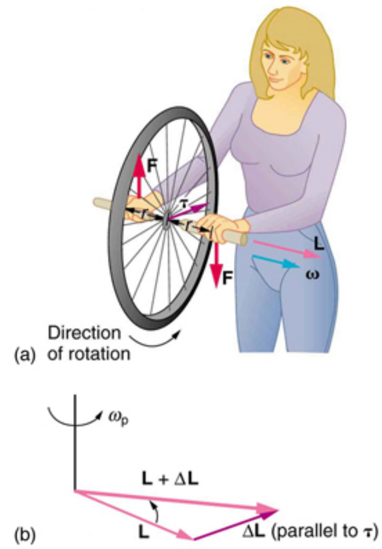
$$\tau = \frac{\Delta \mathrm{L}}{\Delta \mathrm{t}}. \tag{11.26.1}$$

This equation means that the direction of ΔL is the same as the direction of the torque that creates it, as illustrated in. This direction can be determined using the right hand rule, which says that the fingers on your hand curl towards the direction of rotation or force exerted, and your thumb points towards the direction of angular momentum, torque, and angular velocity.



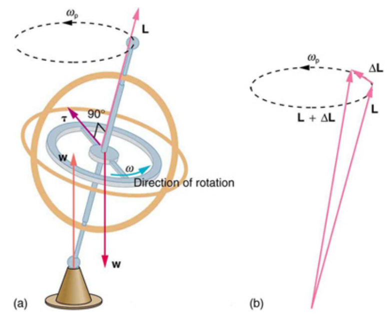
Direction of Torque and Angular Momentum: In figure (a), the torque is perpendicular to the plane formed by r and F and is the direction your right thumb would point to if you curled your fingers in the direction of F. Figure (b) shows that the direction of the torque is the same as that of the angular momentum it produces.

Rotating wheel: Consider a bicycle wheel with handles attached to it, as in. With the wheel rotating as shown, its angular momentum is to the woman's left. Suppose the person holding the wheel tries to rotate it as in the figure. Her natural expectation is that the wheel will rotate in the direction she pushes it, however, what happens is quite different. The forces exerted create a torque that is horizontal toward the person, and this torque creates a change in angular momentum L in the same direction, perpendicular to the original angular momentum L, thus changing the direction of L but not the magnitude of L. ΔL and L add, giving a new angular momentum with direction that is inclined more toward the person than before. The axis of the wheel has thus moved perpendicular to the forces exerted on it, instead of in the expected direction.



Gyroscopic Effect: In figure (a), a person holding the spinning bike wheel lifts it with her right hand and pushes down with her left hand in an attempt to rotate the wheel. This action creates a torque directly toward her. This torque causes a change in angular momentum ΔL in exactly the same direction. Figure (b) shows a vector diagram depicting how ΔL and L add, producing a new angular momentum pointing more toward the person. The wheel moves toward the person, perpendicular to the forces she exerts on it.

Gyroscope: This same logic explains the behavior of gyroscopes (see). There are two forces acting on a spinning gyroscope. The torque produced is perpendicular to the angular momentum, thus the direction of the angular momentum is changed, but not its magnitude. The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to L. If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque (L=ΔL), and it rotates around a horizontal axis, falling over just as we would expect.



Gyroscopes: As seen in figure (a), the forces on a spinning gyroscope are its weight and the supporting force from the stand. These forces create a horizontal torque on the gyroscope, which create a change in angular momentum ΔL that is also horizontal. In figure (b), ΔL and L add to produce a new angular momentum with the same magnitude, but different direction, so that the gyroscope precesses in the direction shown instead of falling over.

Applications

Gyroscopes serve as rotational sensors. For this reason, applications of gyroscopes include inertial navigation systems where magnetic compasses would not work (as in the Hubble telescope) or would not be precise enough (as in ICBMs). Another application is the stabilization of flying vehicles, such as radio-controlled helicopters or unmanned aerial vehicles.

Key Points

- Angular velocity and angular momentum are vector quantities and have both magnitude and direction.
- The direction of angular velocity and angular momentum are perpendicular to the plane of rotation.
- Using the right hand rule, the direction of both angular velocity and angular momentum is defined as the direction in which the thumb of your right hand points when you curl your fingers in the direction of rotation.
- Torque is perpendicular to the plane formed by r and F and is the direction your right thumb would point if you curled the fingers of your right hand in the direction of F.
- The direction of the torque is thus the same as that of the angular momentum it produces.
- The gyroscope precesses around a vertical axis, since the torque is always horizontal and perpendicular to L. If the gyroscope is not spinning, it acquires angular momentum in the direction of the torque, and it rotates about a horizontal axis, falling over just as we would expect.

Key Items

- **angular momentum**: A vector quantity describing an object in circular motion; its magnitude is equal to the momentum of the particle, and the direction is perpendicular to the plane of its circular motion.
- **right hand rule**: Direction of angular velocity ω and angular momentum L in which the thumb of your right hand points when you curl your fingers in the direction of rotation.
- **angular velocity**: A vector quantity describing an object in circular motion; its magnitude is equal to the speed of the particle and the direction is perpendicular to the plane of its circular motion.
- **gimbal**: A device for suspending something, such as a ship's compass, so that it will remain level when its support is tipped.
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)

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11.27: Problem Solving

learning objectives

- Develop and apply a strong problem-solving strategy for rotational kinematics

Problem-Solving Strategy For Rotational Kinematics

When solving problems on rotational kinematics:

- Examine the situation to determine that rotational kinematics (rotational motion) is involved. Rotation must be involved, but without the need to consider forces or masses that affect the motion.
- Identify exactly what needs to be determined in the problem (identify the unknowns). A sketch of the situation is useful.
- Make a list of what is given or can be inferred from the problem as stated (identify the knowns).
- Solve the appropriate equation or equations for the quantity to be determined (the unknown). It can be useful to think in terms of a translational analog because by now you are familiar with such motion.
- Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units. Be sure to use units of radians for angles.
- Check your answer to see if it is reasonable: Does your answer make sense?

Example 11.27.1:

Suppose a large freight train accelerates from rest, giving its 0.350 m radius wheels an angular acceleration of 0.250 rad/s^2. After the wheels have made 200 revolutions (assume no slippage): (a) How far has the train moved down the track? (b) What are the final angular velocity of the wheels and the linear velocity of the train?

In part (a), we are asked to find x, and in (b) we are asked to find ω and v. We are given the number of revolutions θ, the radius of the wheels r, and the angular acceleration α.

The distance x is very easily found from the relationship between distance and rotation angle: $\theta = \frac{x}{r}$.

Solving this equation for x yields $x = r\theta$.

Before using this equation, we must convert the number of revolutions into radians, because we are dealing with a relationship between linear and rotational quantities:

$$\theta = (200\text{rev})\left(\frac{2\pi \text{ rad}}{1 \text{ rev}}\right) = 1257 \text{ rad}. \tag{11.27.1}$$

Substitute the known values into $x = r\theta$ to find the distance the train moved down the track:

$$x = r\theta = (0.350 \text{ m})(1257 \text{ rad}) = 440 \text{ m}. \tag{11.27.2}$$

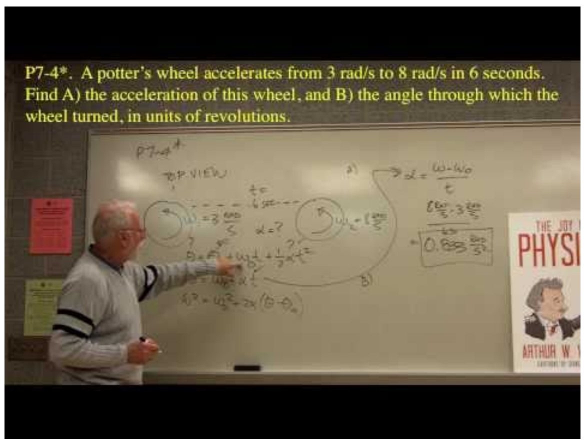
We cannot use any equation that incorporates t to find ω, because the equation would have at least two unknown values. The equation $\omega^2 = \omega_0^2 + 2\alpha\theta$ will work, because we know the values for all variables except ω. Taking the square root of this equation and entering the known values gives

$$\omega = \sqrt{0 + 2(0.250 \text{ rad/s}^2)(1257 \text{ rad})} \tag{11.27.3}$$

$$= 25.1 \text{ rad/s} \tag{11.27.4}$$

One may find the linear velocity of the train, v, through its relationship to ω:

$$v = r\omega = (0.350 \text{ m})(25.1 \text{ rad/s}) = 8.77 \text{ m/s} \tag{11.27.5}$$



Rotational motion: Part of a series of videos on physics problem-solving. The problems are taken from "The Joy of Physics. " This one deals with angular motion. The viewer is urged to pause the video at the problem statement and work the problem before watching the rest of the video.

Rotational	Translational	
$\theta = \overline{\omega} t$	$x = \overline{v} t$	
$\omega = \omega_0 + \alpha t$	$v = v_0 + at$	(constant α, a)
$\theta = \omega_0 t + \frac{1}{2}\alpha t^2$	$x = v_0 t + \frac{1}{2} a t^2$	(constant α, a)
$\omega^2 = {\omega_0}^2 + 2\alpha\theta$	$v^2 = {v_0}^2 + 2ax$	(constant α, a)

Table 1: **Rotational Kinematic Equations**

Equation list: Rotational and translational kinematic equations.

Key Points

- Examine the situation to determine that rotational kinematics (rotational motion) is involved, and identify exactly what needs to be determined.
- Make a list of what is given or can be inferred from the problem as stated and solve the appropriate equations.

- Substitute the known values along with their units into the appropriate equation, and obtain numerical solutions complete with units.

Key Terms

- **kinematics**: The branch of mechanics concerned with objects in motion, but not with the forces involved.

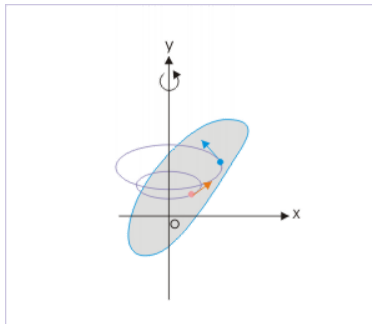
11.28: Linear and Rotational Quantities

learning objectives

- Derive uniform circular motion from linear equations

Defining Circular Motion

The description of circular motion is described better in terms of angular quantity than its linear counter part. The reasons are easy to understand. For example, consider the case of uniform circular motion. Here, the velocity of particle is changing – though the motion is "uniform". The two concepts do not go together. The general connotation of the term "uniform" indicates "constant", but the velocity is actually changing all the time.



A Rotating Body: Each particle constituting the body executes a uniform circular motion about the fixed axis. For the description of the motion, angular quantities are the better choice.

When we describe the uniform circular motion in terms of angular velocity, there is no contradiction. The velocity (i.e. angular velocity) is indeed constant. This is the first advantage of describing uniform circular motion in terms of angular velocity.

Second advantage is that angular velocity conveys the physical sense of the rotation of the particle as against linear velocity, which indicates translational motion. Alternatively, angular description emphasizes the distinction between two types of motion (translational and rotational).

Relationship Between Linear and Angular Speed

For simplicity, let's consider a uniform circular motion. For the length of the arc subtending angle " at the origin and "r" is the radius of the circle containing the position of the particle, we have $\mathrm{s} = \mathrm{r}\theta$.

Differentiating with respect to time, we have

$$\frac{\mathrm{ds}}{\mathrm{dt}} = \frac{\mathrm{dr}}{\mathrm{dt}}\theta + \mathrm{r}\frac{\mathrm{d}\theta}{\mathrm{dt}}. \tag{11.28.1}$$

Because $\frac{\mathrm{dr}}{\mathrm{dt}} = 0$ for a uniform circular motion, we get $\mathrm{v} = \omega\mathrm{r}$. Similarly, we also get $\mathrm{a} = \alpha\mathrm{r}$ where a stands for linear acceleration, while α refers to angular acceleration (In a more general case, the relationship between angular and linear quantities are given as $\mathrm{v} = \omega \times \mathrm{r}, \mathrm{a} = \alpha \times \mathrm{r} + \omega \times \mathrm{v}$

Rotational Kinematic Equations

With the relationship of the linear and angular speed/acceleration, we can derive the following four rotational kinematic equations for constant aa and αα:

$$\omega = \omega_0 + \alpha\mathrm{t} : \mathrm{v} = \mathrm{v}_0 + \mathrm{at} \tag{11.28.2}$$

$$\theta = \omega_0\mathrm{t} + (\frac{1}{2})\alpha\mathrm{t}^2 : \mathrm{x} = \mathrm{v}_0\mathrm{t} + (\frac{1}{2})\mathrm{at}^2 \tag{11.28.3}$$

$$\omega^2 = \omega_0^2 + 2\alpha\theta : \mathrm{v}^2 = \mathrm{v}_0^2 + 2\mathrm{ax} \tag{11.28.4}$$

Mass, Momentum, Energy, and Newton's Second Law

As we use mass, linear momentum, translational kinetic energy, and Newton's 2nd law to describe linear motion, we can describe a general rotational motion using corresponding scalar/vector/tensor quantities:

- Mass/ Rotational inertia:
- Linenar/angular momentum:
- Force/ Torque:
- Kinetic energy:

For example, just as we use the equation of motion F=maF=ma to describe a linear motion, we can use its counterpart $\tau = \frac{dL}{dt} = r \times F$ to describe an angular motion. The descriptions are equivalent, and the choice can be made purely for the convenience of use.

Key Points

- As we use mass, linear momentum, translational kinetic energy, and Newton's 2nd law to describe linear motion, we can describe a general rotational motion using corresponding scalar/vector/tensor quantities.
- Angular and linear velocity have the following relationship: $v = \omega \times r$.
- As we use the equation of motion $F = ma$ to describe a linear motion, we can use its counterpart $\tau = \frac{dL}{dt} = r \times F$, to describe angular motion. The descriptions are equivalent, and the choice can be made purely for the convenience of use.

Key Items

- **uniform circular motion**: Movement around a circular path with constant speed.
- **torque**: A rotational or twisting effect of a force; (SI unit newton-meter or Nm; imperial unit foot-pound or ft-lb)
- **rotational inertia**: The tendency of a rotating object to remain rotating unless a torque is applied to it.

CHAPTER OVERVIEW

12: Temperature and Kinetic Theory

Topic hierarchy

CHAPTER OVERVIEW

12.1: Temperature and Heat

In this chapter, we explore heat and temperature. It is not always easy to distinguish these terms. Heat is the flow of energy from one object to another. This flow of energy is caused by a difference in temperature. The transfer of heat can change temperature, as can work, another kind of energy transfer that is central to thermodynamics. We return to these basic ideas several times throughout the next four chapters, and you will see that they affect everything from the behavior of atoms and molecules to cooking to our weather on Earth to the life cycles of stars.

Thumbnail: Natural convection plays an important role in heat transfer inside this pot of water. Once conducted to the inside, heat transfer to other parts of the pot is mostly by convection. The hotter water expands, decreases in density, and rises to transfer heat to other regions of the water, while colder water sinks to the bottom. This process keeps repeating.

Heat Transfer, Specific Heat, and Calorimetry

Learning Objectives

By the end of this section, you will be able to:

- Explain phenomena involving heat as a form of energy transfer
- Solve problems involving heat transfer

We have seen in previous chapters that energy is one of the fundamental concepts of physics. **Heat** is a type of energy transfer that is caused by a temperature difference, and it can change the temperature of an object. As we learned earlier in this chapter, **heat** transfer is the movement of energy from one place or material to another as a result of a difference in temperature. Heat transfer is fundamental to such everyday activities as home heating and cooking, as well as many industrial processes. It also forms a basis for the topics in the remainder of this chapter.

We also introduce the concept of internal energy, which can be increased or decreased by heat transfer. We discuss another way to change the internal energy of a system, namely doing work on it. Thus, we are beginning the study of the relationship of heat and work, which is the basis of engines and refrigerators and the central topic (and origin of the name) of thermodynamics.

Internal Energy and Heat

A thermal system has **internal energy** (also called **thermal energy)**, which is the sum of the mechanical energies of its molecules. A system's internal energy is proportional to its temperature. As we saw earlier in this chapter, if two objects at different temperatures are brought into contact with each other, energy is transferred from the hotter to the colder object until the bodies reach thermal equilibrium (that is, they are at the same temperature). No work is done by either object because no force acts through a distance (as we discussed in Work and Kinetic Energy). These observations reveal that heat is energy transferred spontaneously due to a temperature difference. Figure 1 shows an example of heat transfer.

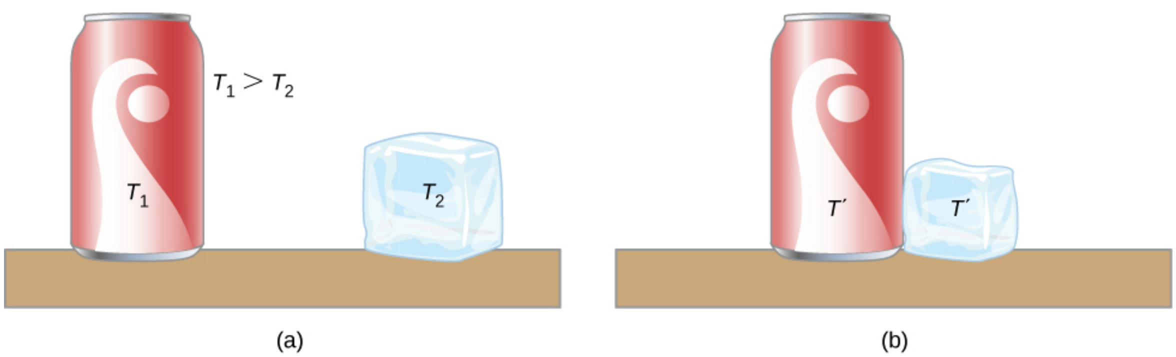


Figure 1: (a) Here, the soft drink has a higher temperature than the ice, so they are not in thermal equilibrium. (b) When the soft drink and ice are allowed to interact, heat is transferred from the drink to the ice due to the difference in temperatures until they reach the same temperature, T', achieving equilibrium. In fact, since the soft drink and ice are both in contact with the surrounding air and the bench, the ultimate equilibrium temperature will be the same as that of the surroundings.

The meaning of "heat" in physics is different from its ordinary meaning. For example, in conversation, we may say "the heat was unbearable," but in physics, we would say that the temperature was high. Heat is a form of energy flow, whereas temperature is not. Incidentally, humans are sensitive to **heat flow** rather than to temperature.

Since heat is a form of energy, its SI unit is the joule (J). Another common unit of energy often used for heat is the calorie (cal), defined as the energy needed to change the temperature of 1.00 g of water by 1.00^oC—specifically, between 14.5^oC and 15.5^oC since there is a slight temperature dependence. Also commonly used is the kilocalorie (kcal), which is the energy needed to change the temperature of 1.00 kg of water by 1.00^oC. Since mass is most often specified in kilograms, the kilocalorie is convenient. Confusingly, food calories (sometimes called "big calories," abbreviated Cal) are actually kilocalories, a fact not easily determined from package labeling.

Mechanical Equivalent of Heat

It is also possible to change the temperature of a substance by doing work, which transfers energy into or out of a system. This realization helped establish that heat is a form of energy. James Prescott Joule (1818–1889) performed many experiments to establish the **mechanical equivalent of heat—the work needed to produce the same effects as heat transfer**. In the units used for these two quantities, the value for this equivalence is

$$1.000\,kcal = 4186\,J.$$

We consider this equation to represent the conversion between two units of energy. (Other numbers that you may see refer to calories defined for temperature ranges other than 14.5^oC to 15.5^oC.)

Figure 2 shows one of Joule's most famous experimental setups for demonstrating that work and heat can produce the same effects and measuring the mechanical equivalent of heat. It helped establish the principle of conservation of energy. Gravitational potential energy (**U**) was converted into kinetic energy (**K**), and then randomized by viscosity and turbulence into increased average kinetic energy of atoms and molecules in the system, producing a temperature increase. Joule's contributions to thermodynamics were so significant that the SI unit of energy was named after him.

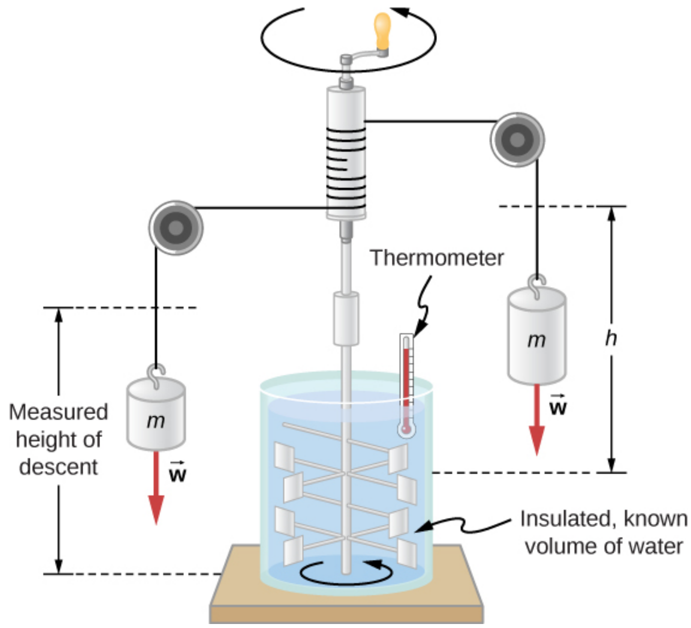


Figure 2: Joule's experiment established the equivalence of heat and work. As the masses descended, they caused the paddles to do work, $W = mgh$, on the water. The result was a temperature increase, ΔT, measured by the thermometer. Joule found that ΔT was proportional to **W** and thus determined the mechanical equivalent of heat.

Increasing internal energy by heat transfer gives the same result as increasing it by doing work. Therefore, although a system has a well-defined internal energy, we cannot say that it has a certain "heat content" or "work content." A well-defined quantity that depends only on the current state of the system, rather than on the history of that system, is known as a **state variable**. Temperature and internal energy are state variables. To sum up this paragraph, **heat and work are not state variables**.

Incidentally, increasing the internal energy of a system does not necessarily increase its temperature. As we'll see in the next section, the temperature does not change when a substance changes from one phase to another. An example is the melting of ice, which can be accomplished by adding heat or by doing frictional work, as when an ice cube is rubbed against a rough surface.

Temperature Change and Heat Capacity

We have noted that heat transfer often causes temperature change. Experiments show that with no phase change and no work done on or by the system, the transferred heat is typically directly proportional to the change in temperature and to the mass of the system, to a good approximation. (Below we show how to handle situations where the approximation is not valid.) The constant of proportionality depends on the substance and its phase, which may be gas, liquid, or solid. We omit discussion of the fourth phase, plasma, because although it is the most common phase in the universe, it is rare and short-lived on Earth.

We can understand the experimental facts by noting that the transferred heat is the change in the internal energy, which is the total energy of the molecules. Under typical conditions, the total kinetic energy of the molecules K_{total} is a constant fraction of the internal energy (for reasons and with exceptions that we'll see in the next chapter). The average kinetic energy of a molecule K_{ave} is proportional to the absolute temperature. Therefore, the change in internal energy of a system is typically proportional to the change in temperature and to the number of molecules, **N**. Mathematically, $\Delta U \propto \Delta K_{total} = NK_{ave} \propto N\Delta T$. The dependence on the substance results in large part from the different masses of atoms and molecules. We are considering its heat capacity in terms of its mass, but as we will see in the next chapter, in some cases, heat capacities **per molecule** are similar for different substances. The dependence on substance and phase also results from differences in the potential energy associated with interactions between atoms and molecules.

Heat Transfer and Temperature Change

A practical approximation for the relationship between heat transfer and temperature change is:

$$Q = mc\Delta T,$$

where Q is the symbol for heat transfer ("quantity of heat"), **m** is the mass of the substance, and ΔT is the change in temperature. The symbol **c** stands for the **specific heat** (also called "**specific heat capacity**") and depends on the material and phase. The specific heat is numerically equal to the amount of heat necessary to change the temperature of $1.00\,kg$ of mass by 1.00^oC. The SI unit for specific heat is $J/(kg \times K)$ or $J/(kg \times ^o C)$. (Recall that the temperature change ΔT is the same in units of kelvin and degrees Celsius.)

Values of specific heat must generally be measured, because there is no simple way to calculate them precisely. Table 1 lists representative values of specific heat for various substances. We see from this table that the specific heat of water is five times that of glass and 10 times that of iron, which means that it takes five times as much heat to raise the temperature of water a given amount as for glass, and 10 times as much as for iron. In fact, water has one of the largest specific heats of any material, which is important for sustaining life on Earth.

The specific heats of gases depend on what is maintained constant during the heating—typically either the volume or the pressure. In the table, the first specific heat value for each gas is measured at constant volume, and the second (in parentheses) is measured at constant pressure. We will return to this topic in the chapter on the kinetic theory of gases.

Table 1: Specific Heats of Various Substances

Substance	Specific Heat (c) $J/kg \cdot ^o C$	Specific Heat (c) $kcal/kg \cdot ^o C$[2]
	Solids	
Aluminum	900	0.215
Asbestos	800	0.19
Concrete, granite (average)	840	0.20
Copper	387	0.0924
Glass	840	0.20
Gold	129	0.0308
Human body (average at 37^oC)	3500	0.83
Ice (average -50^oC to 0^oC)	2090	0.50
Iron, steel	452	0.108
Lead	128	0.0305
Silver	235	0.0562
Wood	1700	0.40
	Liquids	

Substance	Specific Heat (c) $J/kg \cdot^o C$	Specific Heat (c) $kcal/kg \cdot^o C$[2]
Benzene	1740	0.415
Ethanol	2450	0.586
Glycerin	2410	0.576
Mercury	139	0.0333
Water (15.0^oC)	4186	1.000
	Gases[3]	
Air (dry)	721 (1015)	0.172 (0.242)
Ammonia	1670 (2190)	0.399 (0.523)
Carbon dioxide	638 (833)	0.152 (0.199)
Nitrogen	739 (1040)	0.177 (0.248)
Oxygen **1**	651 (913)	0.156 (0.218)
Steam (100^oC)	1520 (2020)	0.363 (0.482)

The values for solids and liquids are at constant volume and 25^oC, except as noted. [2]These values are identical in units of $cal/g \cdot\ ^oC$. [3] Specific heats at constant volume and at 20.0^oC except as noted, and at 1.00 atm pressure. Values in parentheses are specific heats at a constant pressure of 1.00 atm.

In general, specific heat also depends on temperature. Thus, a precise definition of **c** for a substance must be given in terms of an infinitesimal change in temperature. To do this, we note that $c = \frac{1}{m}\frac{\Delta Q}{\Delta T}$ and replace Δ with d:

$$c = \frac{1}{m}\frac{dQ}{dT}.$$

Except for gases, the temperature and volume dependence of the specific heat of most substances is weak at normal temperatures. Therefore, we will generally take specific heats to be constant at the values given in the table.

✓ Example 1: Calculating the Required Heat

A 0.500-kg aluminum pan on a stove and 0.250 L of water in it are heated from 20.0^oC to 80.0^oC. (a) How much heat is required? What percentage of the heat is used to raise the temperature of (b) the pan and (c) the water?

Strategy

We can assume that the pan and the water are always at the same temperature. When you put the pan on the stove, the temperature of the water and that of the pan are increased by the same amount. We use the equation for the heat transfer for the given temperature change and mass of water and aluminum. The specific heat values for water and aluminum are given in Table 1.

Solution

1. Calculate the temperature difference:

$$\Delta t = T_f - T_i = 60.0^oC.$$

2. Calculate the mass of water. Because the density of water is 1000 kg/m^3, 1 L of water has a mass of 1 kg, and the mass of 0.250 L of water is $m_w = 0.250\,kg$.
3. Calculate the heat transferred to the water. Use the specific heat of water in Table 1:

$$Q_w = m_w c_w \Delta T = (0.250\,kg)(4186\,J/kg^oC)(60.0^oC) = 62.8\,kJ.$$

4. Calculate the heat transferred to the aluminum. Use the specific heat for aluminum in Table 1:

$$Q_{A1} = m_{A1} c_{A1} \Delta T = (0.500\,kg)(900\,J/kg^oC)(60.0^oC) = 27.0\,kJ.$$

5. Find the total transferred heat:

$$Q_{Total} = Q_W + Q_{A1} = 89.8\, kJ.$$

Significance

In this example, the heat transferred to the container is a significant fraction of the total transferred heat. Although the mass of the pan is twice that of the water, the specific heat of water is over four times that of aluminum. Therefore, it takes a bit more than twice as much heat to achieve the given temperature change for the water as for the aluminum pan.

Example 2 illustrates a temperature rise caused by doing work. (The result is the same as if the same amount of energy had been added with a blowtorch instead of mechanically.)

Calculating the Temperature Increase from the Work Done on a Substance.

Truck brakes used to control speed on a downhill run do work, converting gravitational potential energy into increased internal energy (higher temperature) of the brake material (Figure 3). This conversion prevents the gravitational potential energy from being converted into kinetic energy of the truck. Since the mass of the truck is much greater than that of the brake material absorbing the energy, the temperature increase may occur too fast for sufficient heat to transfer from the brakes to the environment; in other words, the brakes may overheat.

Figure 3: The smoking brakes on a braking truck are visible evidence of the mechanical equivalent of heat.

Calculate the temperature increase of 10 kg of brake material with an average specific heat of 800 $J/kg\cdot^{C}$ if the material retains 10% of the energy from a 10,000-kg truck descending 75.0 m (in vertical displacement) at a constant speed.

Strategy

We calculate the gravitational potential energy (**Mgh**) that the entire truck loses in its descent, equate it to the increase in the brakes' internal energy, and then find the temperature increase produced in the brake material alone.

Solution

First we calculate the change in gravitational potential energy as the truck goes downhill:

$$Mgh = (10,000\, kg)(9.80\, m/s^2)(75.0\, m) = 7.35 \times 10^6\, J.$$

Because the kinetic energy of the truck does not change, conservation of energy tells us the lost potential energy is dissipated, and we assume that 10% of it is transferred to internal energy of the brakes, so take $Q = Mgh/10$. Then we calculate the temperature change from the heat transferred, using

$$\Delta T = \frac{7.35 \times 10^5\, J}{(10\, kg)(800\, J/kg^oC)} = 92^oC.$$

Significance

If the truck had been traveling for some time, then just before the descent, the brake temperature would probably be higher than the ambient temperature. The temperature increase in the descent would likely raise the temperature of the brake material very high, so this technique is not practical. Instead, the truck would use the technique of engine braking. A different idea underlies the recent technology of hybrid and electric cars, where mechanical energy (kinetic and gravitational potential energy) is converted by the brakes into electrical energy in the battery, a process called regenerative braking.

In a common kind of problem, objects at different temperatures are placed in contact with each other but isolated from everything else, and they are allowed to come into equilibrium. A container that prevents heat transfer in or out is called a **calorimeter**, and the use of a calorimeter to make measurements (typically of heat or specific heat capacity) is called **calorimetry**.

We will use the term "calorimetry problem" to refer to any problem in which the objects concerned are thermally isolated from their surroundings. An important idea in solving calorimetry problems is that during a heat transfer between objects isolated from their surroundings, the heat gained by the colder object must equal the heat lost by the hotter object, due to conservation of energy:

$$Q_{cold} + Q_{hot} = 0.$$

We express this idea by writing that the sum of the heats equals zero because the heat gained is usually considered positive; the heat lost, negative.

✓ Calculating the Final Temperature in Calorimetry

Suppose you pour 0.250 kg of 20.0^oC water (about a cup) into a 0.500-kg aluminum pan off the stove with a temperature of 150^oC. Assume no heat transfer takes place to anything else: The pan is placed on an insulated pad, and heat transfer to the air is neglected in the short time needed to reach equilibrium. Thus, this is a calorimetry problem, even though no isolating container is specified. Also assume that a negligible amount of water boils off. What is the temperature when the water and pan reach thermal equilibrium?

Strategy

Originally, the pan and water are not in thermal equilibrium: The pan is at a higher temperature than the water. Heat transfer restores thermal equilibrium once the water and pan are in contact; it stops once thermal equilibrium between the pan and the water is achieved. The heat lost by the pan is equal to the heat gained by the water—that is the basic principle of calorimetry.

Solution

1. Use the equation for heat transfer $Q = mc\Delta T$ to express the heat lost by the aluminum pan in terms of the mass of the pan, the specific heat of aluminum, the initial temperature of the pan, and the final temperature:
$$Q_{hot} = m_{A1}c_{A1}(T_f - 150^oC).$$
2. Express the heat gained by the water in terms of the mass of the water, the specific heat of water, the initial temperature of the water, and the final temperature:
$$Q_{cold} = m_w c_w (T_f - 20.0^oC).$$
3. Note that $Q_{hot} < 0$ and $Q_{cold} > 0$ and that as stated above, they must sum to zero:
$$Q_{cold} + Q_{hot} = 0$$
$$Q_{cold} = -Q_{hot}$$
$$m_w c_w (T_f - 20.0^C) = -m_{A1}c_{A1}(T_f - 150^oC).$$
4. This a linear equation for the unknown final temperature, T_f. Solving for T_f,
$$T_f = \frac{m_{A1}c_{A1}(150^oC) + m_w c_w (20.0^oC)}{m_{A1}c_{A1} + m_w c_w},$$
and insert the numerical values:
$$T_f = \frac{(0.500\,kg)(900\,J/kg^oC)(150^oC) + (0.250\,kg)(4186\,J/kg^oC)(20.0^oC)}{(0.500\,kg)(900\,J/kg^oC) + (0.250\,kg)(4186\,J/kg^oC)} = 59.1\,^oC.$$

Significance Why is the final temperature so much closer to 20.0^oC than to 150^oC? The reason is that water has a greater specific heat than most common substances and thus undergoes a smaller temperature change for a given heat transfer. A large body of water, such as a lake, requires a large amount of heat to increase its temperature appreciably. This explains why the temperature of a lake stays relatively constant during the day even when the temperature change of the air is large. However, the water temperature does change over longer times (e.g., summer to winter).

? Exercise 3

If 25 kJ is necessary to raise the temperature of a rock from 25^oC to 30^oC, how much heat is necessary to heat the rock from 45^oC to 50^oC?

Answer

To a good approximation, the heat transfer depends only on the temperature difference. Since the temperature differences are the same in both cases, the same 25 kJ is necessary in the second case. (As we will see in the next section, the answer would have been different if the object had been made of some substance that changes phase anywhere between 30^oC and 50^oC.)

✓ Temperature-Dependent Heat Capacity

At low temperatures, the specific heats of solids are typically proportional to T^3. The first understanding of this behavior was due to the Dutch physicist Peter Debye, who in 1912, treated atomic oscillations with the quantum theory that Max Planck had recently used for radiation. For instance, a good approximation for the specific heat of salt, NaCl, is $c = 3.33 \times 10^4 \frac{J}{kg \cdot k} \left(\frac{T}{321\ K}\right)^3$. The constant 321 K is called the **Debye temperature** of NaCl, Θ_D and the formula works well when $T < 0.04\Theta_D$. Using this formula, how much heat is required to raise the temperature of 24.0 g of NaCl from 5 K to 15 K?

Solution

Because the heat capacity depends on the temperature, we need to use the equation

$$c = \frac{1}{m}\frac{dQ}{dT}.$$

We solve this equation for **Q** by integrating both sides: $Q = m\int_{T_1}^{T_2} cdT$.

Then we substitute the given values in and evaluate the integral:

$$Q = (0.024\ kg)\int_{T1}^{T2} 3.33 \times 10^4 \frac{J}{kg \cdot K}\left(\frac{T}{321\ K}\right)^3 dT = \left(6.04 \times 10^{-6} \frac{J}{K^4}\right) T^4 |_{5\ K}^{15\ K} = 0.302\ J.$$

Significance If we had used the equation $Q = mc\Delta T$ and the room-temperature specific heat of salt, 880 $J/kg \cdot K$, we would have gotten a very different value.

Mechanisms of Heat Transfer

Learning Objectives

By the end of this section, you will be able to:

- Explain some phenomena that involve conductive, convective, and radiative heat transfer
- Solve problems on the relationships between heat transfer, time, and rate of heat transfer
- Solve problems using the formulas for conduction and radiation

Just as interesting as the effects of heat transfer on a system are the methods by which it occurs. Whenever there is a temperature difference, heat transfer occurs. It may occur rapidly, as through a cooking pan, or slowly, as through the walls of a picnic ice chest. So many processes involve heat transfer that it is hard to imagine a situation where no heat transfer occurs. Yet every heat transfer takes place by only three methods:

1. **Conduction** is heat transfer through stationary matter by physical contact. (The matter is stationary on a macroscopic scale—we know that thermal motion of the atoms and molecules occurs at any temperature above absolute zero.) Heat transferred from the burner of a stove through the bottom of a pan to food in the pan is transferred by **conduction**.
2. **Convection** is the heat transfer by the macroscopic movement of a fluid. This type of transfer takes place in a forced-air furnace and in weather systems, for example.
3. Heat transfer by **radiation** occurs when microwaves, infrared radiation, visible light, or another form of electromagnetic radiation is emitted or absorbed. An obvious example is the warming of Earth by the Sun. A less obvious example is thermal radiation from the human body.

In the illustration at the beginning of this chapter, the fire warms the snowshoers' faces largely by radiation. Convection carries some heat to them, but most of the air flow from the fire is upward (creating the familiar shape of flames), carrying heat to the food being cooked and into the sky. The snowshoers wear clothes designed with low conductivity to prevent heat flow out of their bodies.

In this section, we examine these methods in some detail. Each method has unique and interesting characteristics, but all three have two things in common: They transfer heat solely because of a temperature difference, and the greater the temperature difference, the faster the heat transfer (Figure 1).

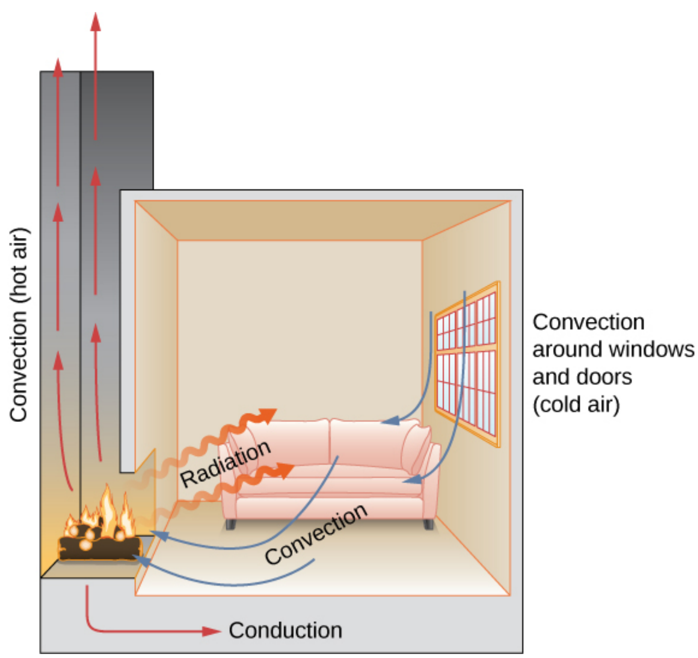


Figure 1: In a fireplace, heat transfer occurs by all three methods: conduction, convection, and radiation. Radiation is responsible for most of the heat transferred into the room. Heat transfer also occurs through conduction into the room, but much slower. Heat transfer by convection also occurs through cold air entering the room around windows and hot air leaving the room by rising up the chimney.

? Exercise 1

Name an example from daily life (different from the text) for each mechanism of heat transfer.

Solution

Conduction: Heat transfers into your hands as you hold a hot cup of coffee. Convection: Heat transfers as the barista "steams" cold milk to make hot cocoa. Radiation: Heat transfers from the Sun to a jar of water with tea leaves in it to make "Sun tea." A great many other answers are possible.

Conduction

As you walk barefoot across the living room carpet in a cold house and then step onto the kitchen tile floor, your feet feel colder on the tile. This result is intriguing, since the carpet and tile floor are both at the same temperature. The different sensation is explained by the different rates of heat transfer: The heat loss is faster for skin in contact with the tiles than with the carpet, so the sensation of cold is more intense.

Some materials conduct thermal energy faster than others. Figure 2 shows a material that conducts heat slowly—it is a good thermal insulator, or poor heat conductor—used to reduce heat flow into and out of a house.

Figure 2: Insulation is used to limit the conduction of heat from the inside to the outside (in winter) and from the outside to the inside (in summer). (credit: Giles Douglas)

A molecular picture of heat conduction will help justify the equation that describes it. Figure 3 shows molecules in two bodies at different temperatures, T_h and T_c for "hot" and "cold." The average kinetic energy of a molecule in the hot body is higher than in the colder body. If two molecules collide, energy transfers from the high-energy to the low-energy molecule. In a metal, the picture would also include free valence electrons colliding with each other and with atoms, likewise transferring energy. The cumulative effect of all collisions is a net flux of heat from the hotter body to the colder body. Thus, the rate of heat transfer increases with increasing temperature difference $\Delta T = T_h - T_c$. If the temperatures are the same, the net heat transfer rate is zero. Because the number of collisions increases with increasing area, heat conduction is proportional to the cross-sectional area—a second factor in the equation.

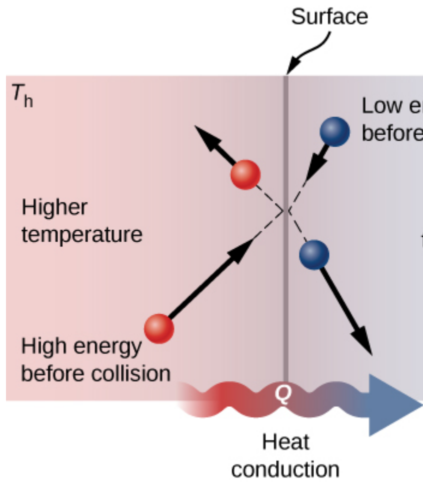


Figure 3: Molecules in two bodies at different temperatures have different average kinetic energies. Collisions occurring at the contact surface tend to transfer energy from high-temperature regions to low-temperature regions. In this illustration, a molecule in the lower-temperature region (right side) has low energy before collision, but its energy increases after colliding with a high-energy molecule at the contact surface. In contrast, a molecule in the higher-temperature region (left side) has high energy before collision, but its energy decreases after colliding with a low-energy molecule at the contact surface.

A third quantity that affects the conduction rate is the thickness of the material through which heat transfers. Figure 4 shows a slab of material with a higher temperature on the left than on the right. Heat transfers from the left to the right by a series of molecular collisions. The greater the distance between hot and cold, the more time the material takes to transfer the same amount of heat.

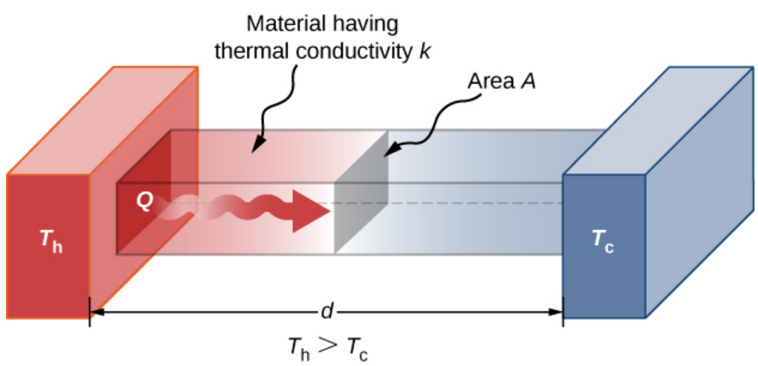


Figure 4: Heat conduction occurs through any material, represented here by a rectangular bar, whether window glass or walrus blubber.

All four of these quantities appear in a simple equation deduced from and confirmed by experiments. The rate of conductive heat transfer through a slab of material, such as the one in Figure 4, is given by

$$P = \frac{dQ}{dt} = \frac{kA(T_h - T_c)}{d}$$

where **P** is the power or rate of heat transfer in watts or in kilocalories per second, **A** and **d** are its surface area and thickness, as shown in Figure 4, $T_h - T_c$ is the temperature difference across the slab, and **k** is the thermal conductivity of the material. Table 1 gives representative values of thermal conductivity.

More generally, we can write

$$P = -kA\frac{dT}{dx},$$

where **x** is the coordinate in the direction of heat flow. Since in Figure 4, the power and area are constant, **dT/dx** is constant, and the temperature decreases linearly from T_h to T_c.

Table 1: Thermal Conductivities of Common Substances Values are given for temperatures near 0^oC.

Substance	Thermal Conductivity $k(W/m^oC)$
Diamond	2000

Substance	Thermal Conductivity $k (W/m^oC)$
Silver	420
Copper	390
Gold	318
Aluminum	220
Steel iron	80
Steel (stainless)	14
Ice	2.2
Glass (average)	0.84
Concrete brick	0.84
Water	0.6
Fatty tissue (without blood)	0.2
Asbestos	0.16
Plasterboard	0.16
Wood	0.08–0.16
Snow (dry)	0.10
Cork	0.042
Glass wool	0.042
Wool	0.04
Down feathers	0.025
Air	0.023
Polystyrene foam	0.010

Example 1: Calculating Heat Transfer through Conduction

A polystyrene foam icebox has a total area of $0.950\, m^2$ and walls with an average thickness of 2.50 cm. The box contains ice, water, and canned beverages at 0^oC. The inside of the box is kept cold by melting ice. How much ice melts in one day if the icebox is kept in the trunk of a car at 35.0^oC?

Strategy

This question involves both heat for a phase change (melting of ice) and the transfer of heat by conduction. To find the amount of ice melted, we must find the net heat transferred. This value can be obtained by calculating the rate of heat transfer by conduction and multiplying by time.

Solution

First we identify the knowns.

$k = 0.010\, W/m \cdot {}^oC$ for polystyrene foam; $A = 0.950\, m^2$; $d = 2.50\, cm = 0.0250\, m$; $T_c = 0^oC$; $T_h = 35.0^oC$; $t = 1\, day = 24\, hour = 86,400\, s$.

Then we identify the unknowns. We need to solve for the mass of the ice, **m**. We also need to solve for the net heat transferred to melt the ice, **Q**. The rate of heat transfer by conduction is given by

$$P = \frac{dQ}{dT} = \frac{kA(T_h - T_c)}{d}.$$

The heat used to melt the ice is $Q = mL_f$. We insert the known values:

$$P = \frac{(0.010\,W/m\cdot^oC)(0.950\,m^2)(35.0^oC - 0^oC)}{0.0250\,m} = 13.3\,W.$$

Multiplying the rate of heat transfer by the time we obtain

$$Q = Pt = (13.3\,W)(86,400\,s) = 1.15 \times 10^6\,J.$$

We set this equal to the heat transferred to melt the ice, $Q = mL_f$ and solve for the mass **m**:

$$m = \frac{Q}{L_f} = \frac{1.15 \times 10^6\,J}{334 \times 10^3\,J/kg} = 3.44\,kg.$$

Significance

The result of 3.44 kg, or about 7.6 lb, seems about right, based on experience. You might expect to use about a 4 kg (7–10 lb) bag of ice per day. A little extra ice is required if you add any warm food or beverages.

Table 1 shows that polystyrene foam is a very poor conductor and thus a good insulator. Other good insulators include fiberglass, wool, and goosedown feathers. Like polystyrene foam, these all contain many small pockets of air, taking advantage of air's poor thermal conductivity.

In developing insulation, the smaller the conductivity **k** and the larger the thickness **d**, the better. Thus, the ratio **d/k**, called the **R factor**, is large for a good insulator. The rate of conductive heat transfer is inversely proportional to **R**. **R** factors are most commonly quoted for household insulation, refrigerators, and the like. Unfortunately, in the United States, **R** is still in non-metric units of $ft^2\cdot^oF\cdot h/Btu$, although the unit usually goes unstated [1 British thermal unit (Btu) is the amount of energy needed to change the temperature of 1.0 lb of water by 1.0^oF which is 1055.1 J]. A couple of representative values are an **R** factor of 11 for 3.5-inch-thick fiberglass batts (pieces) of insulation and an **R** factor of 19 for 6.5-inch-thick fiberglass batts (Figure 5). In the US, walls are usually insulated with 3.5-inch batts, whereas ceilings are usually insulated with 6.5-inch batts. In cold climates, thicker batts may be used.

Figure 5: The fiberglass batt is used for insulation of walls and ceilings to prevent heat transfer between the inside of the building and the outside environment. (credit: Tracey Nicholls)

Note that in Table 1, most of the best thermal conductors—silver, copper, gold, and aluminum—are also the best electrical conductors, because they contain many free electrons that can transport thermal energy. (Diamond, an electrical insulator, conducts heat by atomic vibrations.) Cooking utensils are typically made from good conductors, but the handles of those used on the stove are made from good insulators (bad conductors).

✔ Two Conductors End to End

A steel rod and an aluminum rod, each of diameter 1.00 cm and length 25.0 cm, are welded end to end. One end of the steel rod is placed in a large tank of boiling water at 100^oC, while the far end of the aluminum rod is placed in a large tank of water at 20^oC. The rods are insulated so that no heat escapes from their surfaces. What is the temperature at the joint, and what is the rate of heat conduction through this composite rod?

Strategy

The heat that enters the steel rod from the boiling water has no place to go but through the steel rod, then through the aluminum rod, to the cold water. Therefore, we can equate the rate of conduction through the steel to the rate of conduction through the aluminum.

We repeat the calculation with a second method, in which we use the thermal resistance **R** of the rod, since it simply adds when two rods are joined end to end. (We will use a similar method in the chapter on direct-current circuits.)

Solution

1. Identify the knowns and convert them to SI units. The length of each rod is $L_{A1} = L_{steel} = 0.25\,m$, the cross-sectional area of each rod is $A_{A1} = A_{steel} = 7.85 \times 10^{-5} m^2$, the thermal conductivity of aluminum is $k_{A1} = 220\,W/m \cdot ^o C$, the thermal conductivity of steel is $k_{steel} = 80\,W/m \cdot ^o C$ the temperature at the hot end is $T = 100^oC$ and the temperature at the cold end is $T = 20^oC$.
2. Calculate the heat-conduction rate through the steel rod and the heat-conduction rate through the aluminum rod in terms of the unknown temperature **T** at the joint:

$$\begin{aligned} P_{steel} &= \frac{k_{steel}A_{steel}\Delta T_{steel}}{L_{steel}} \\ &= \frac{(80\,W/m \cdot ^oC)(7.85\times 10^{-5}m^2)(100\,^oC - T)}{0.25} \\ &= (0.0251\,W/^oC)(100^oC - T); \\ P_{A1} &= \frac{k_{A1}A_{A1}\Delta T_{A1}}{L_{A1}} \\ &= \frac{(220\,W/m \cdot ^oC)(7.85\times 10^{-5}m^2)(T - 20^oC)}{0.25\,m} \\ &= (0.0691\,W/^oC)(T - 20^oC). \end{aligned}$$

3. Set the two rates equal and solve for the unknown temperature:

$$(0.0691\,W/^oC)(T - 20^oC) = (0.0251\,W/^oC)(100^oC - T)$$

$$T = 41.3^oC.$$

4. Calculate either rate:

$$P_{steel} = (0.0251\,W/^oC)(100^oC - 41.3^oC) = 1.47\,W.$$

5. If desired, check your answer by calculating the other rate.

Solution

1. Recall that $R = L/k$. Now $P = A\Delta T/R$, or $\Delta T = PR/A$.
2. We know that $\Delta T_{steel} + \Delta T_{A1} = 100^oC - 20^oC = 80^oC$. We also know that $P_{steel} = P_{A1}$, and we denote that rate of heat flow by **P**. Combine the equations:

$$\frac{PR_{steel}}{A} + \frac{PR_{A1}}{A} = 80^oC.$$

Thus, we can simply add **R** factors. Now, $P = \frac{80^oC}{A(R_{steel}+R_{A1})}$.

3. Find the R_s from the known quantities:

$$R_{steel} = 3.13 \times 10^{-3} m^2 \cdot {}^oC/W$$

and

$$R_{A1} = 1.14 \times 10^{-3} m^2 \cdot {}^oC/W.$$

4. Substitute these values in to find $P = 1.47\,W$ as before.
5. Determine ΔT for the aluminum rod (or for the steel rod) and use it to find **T** at the joint.

$$\Delta T_{A1} = \frac{PR_{A1}}{A} = \frac{(1.47\,W)(1.14 \times 10^{-3} m^2 \cdot {}^oC/W)}{7.85 \times 10^{-5} m^2} = 21.3^oC,$$

so $T = 20^oC + 21.3^oC = 41.3^oC$, as in Solution 1.

6. If desired, check by determining ΔT for the other rod.

Significance

In practice, adding **R** values is common, as in calculating the **R** value of an insulated wall. In the analogous situation in electronics, the resistance corresponds to **AR** in this problem and is additive even when the areas are unequal, as is common in electronics. Our equation for heat conduction can be used only when the areas are equal; otherwise, we would have a problem in three-dimensional heat flow, which is beyond our scope.

Exercise 2

How does the rate of heat transfer by conduction change when all spatial dimensions are doubled?

Answer

Because area is the product of two spatial dimensions, it increases by a factor of four when each dimension is doubled $(A_{final} = (2d)^2 = 4d^2 = 4A_{initial})$. The distance, however, simply doubles. Because the temperature difference and the coefficient of thermal conductivity are independent of the spatial dimensions, the rate of heat transfer by conduction increases by a factor of four divided by two, or two:

$$P_{final} = \frac{kA_{final}(T_h - T_c)}{d_{final}} = \frac{k(4A_{final}(T_h - T_c))}{2d_{initial}} = 2\frac{kA_{final}(T_h - T_c)}{d_{initial}} = 2P_{initial}.$$

Conduction is caused by the random motion of atoms and molecules. As such, it is an ineffective mechanism for heat transport over macroscopic distances and short times. For example, the temperature on Earth would be unbearably cold during the night and extremely hot during the day if heat transport in the atmosphere were only through conduction. Also, car engines would overheat unless there was a more efficient way to remove excess heat from the pistons. The next module discusses the important heat-transfer mechanism in such situations.

Convection

In **convection**, thermal energy is carried by the large-scale flow of matter. It can be divided into two types. In **forced convection**, the flow is driven by fans, pumps, and the like. A simple example is a fan that blows air past you in hot surroundings and cools you by replacing the air heated by your body with cooler air. A more complicated example is the cooling system of a typical car, in which a pump moves coolant through the radiator and engine to cool the engine and a fan blows air to cool the radiator.

In **free** or **natural convection**, the flow is driven by buoyant forces: hot fluid rises and cold fluid sinks because density decreases as temperature increases. The house in Figure 6 is kept warm by natural convection, as is the pot of water on the stove in Figure 7. Ocean currents and large-scale atmospheric circulation, which result from the buoyancy of warm air and water, transfer hot air from the tropics toward the poles and cold air from the poles toward the tropics. (Earth's rotation interacts with those flows, causing the observed eastward flow of air in the temperate zones.)

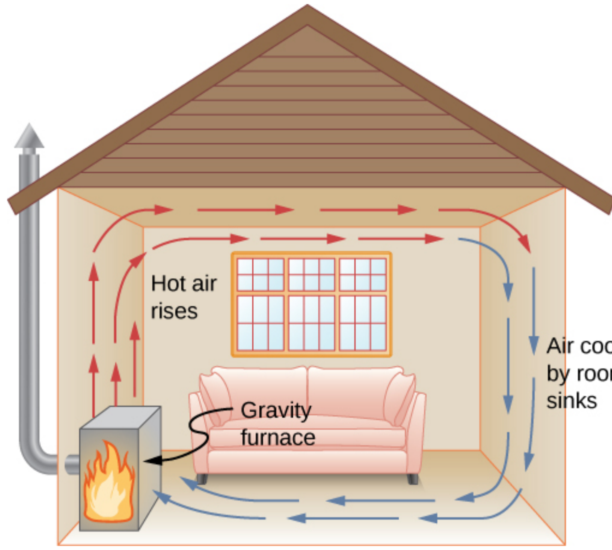


Figure 6: Air heated by a so-called gravity furnace expands and rises, forming a convective loop that transfers energy to other parts of the room. As the air is cooled at the ceiling and outside walls, it contracts, eventually becoming denser than room air and sinking to the floor. A properly designed heating system using natural convection, like this one, can heat a home quite efficiently.

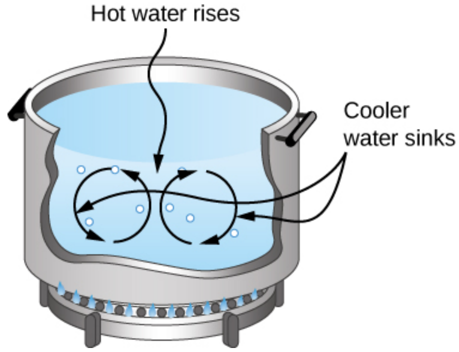


Figure 7: Natural convection plays an important role in heat transfer inside this pot of water. Once conducted to the inside, heat transfer to other parts of the pot is mostly by convection. The hotter water expands, decreases in density, and rises to transfer heat to other regions of the water, while colder water sinks to the bottom. This process keeps repeating.

Note

Natural convection like that of Figures 6 and 7, but acting on rock in Earth's mantle, drives plate tectonics that are the motions that have shaped Earth's surface.

Convection is usually more complicated than conduction. Beyond noting that the convection rate is often approximately proportional to the temperature difference, we will not do any quantitative work comparable to the formula for conduction. However, we can describe convection qualitatively and relate convection rates to heat and time. However, air is a poor conductor. Therefore, convection dominates heat transfer by air, and the amount of available space for airflow determines whether air transfers heat rapidly or slowly. There is little heat transfer in a space filled with air with a small amount of other material that prevents flow. The space between the inside and outside walls of a typical American house, for example, is about 9 cm (3.5 in.)—large enough for convection to work effectively. The addition of wall insulation prevents airflow, so heat loss (or gain) is decreased. On the other hand, the gap between the two panes of a double-paned window is about 1 cm, which largely prevents convection and takes advantage of air's low conductivity reduce heat loss. Fur, cloth, and fiberglass also take advantage of the low conductivity of air by trapping it in spaces too small to support convection (Figure 8).

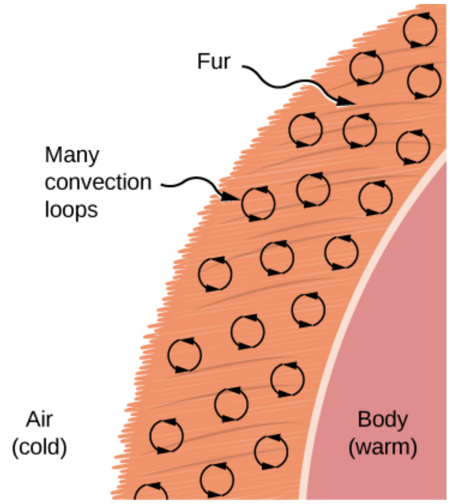


Figure 8: Fur is filled with air, breaking it up into many small pockets. Convection is very slow here, because the loops are so small. The low conductivity of air makes fur a very good lightweight insulator.

Some interesting phenomena happen when convection is accompanied by a phase change. The combination allows us to cool off by sweating even if the temperature of the surrounding air exceeds body temperature. Heat from the skin is required for sweat to evaporate from the skin, but without air flow, the air becomes saturated and evaporation stops. Air flow caused by convection replaces the saturated air by dry air and evaporation continues.

✓ Example 3: Calculating the Flow of Mass during Convection

The average person produces heat at the rate of about 120 W when at rest. At what rate must water evaporate from the body to get rid of all this energy? (For simplicity, we assume this evaporation occurs when a person is sitting in the shade and surrounding temperatures are the same as skin temperature, eliminating heat transfer by other methods.)

Strategy

Energy is needed for this phase change ($Q = mL_v$). Thus, the energy loss per unit time is

$$\frac{Q}{t} = \frac{mL_v}{t} = 120\,W = 120\,J/s.$$

We divide both sides of the equation by L_v to find that the mass evaporated per unit time is

$$\frac{m}{t} = \frac{120\,J/s}{L_v}.$$

Solution

Insert the value of the latent heat from [link], $L_v = 2430\,kJ/kg = 2430\,J/g$. This yields

$$\frac{m}{t} = \frac{120\,J/s}{2430\,J/g} = 0.0494\,g/s = 2.96\,g/min.$$

Significance

Evaporating about 3 g/min seems reasonable. This would be about 180 g (about 7 oz.) per hour. If the air is very dry, the sweat may evaporate without even being noticed. A significant amount of evaporation also takes place in the lungs and breathing passages.

Another important example of the combination of phase change and convection occurs when water evaporates from the oceans. Heat is removed from the ocean when water evaporates. If the water vapor condenses in liquid droplets as clouds form, possibly far from the ocean, heat is released in the atmosphere. Thus, there is an overall transfer of heat from the ocean to the atmosphere. This process is the driving power behind thunderheads, those great cumulus clouds that rise as much as 20.0 km into the stratosphere (Figure 9). Water vapor carried in by convection condenses, releasing tremendous amounts of energy. This energy causes the air to expand and rise to colder altitudes. More condensation occurs in these regions, which in turn drives the cloud even higher. This mechanism is an example of positive feedback, since the process reinforces and accelerates itself. It sometimes produces violent storms, with lightning and hail. The same mechanism drives hurricanes.

Note

This time-lapse video shows convection currents in a thunderstorm, including “rolling” motion similar to that of boiling water.

Figure 9: Cumulus clouds are caused by water vapor that rises because of convection. The rise of clouds is driven by a positive feedback mechanism. (credit: “Amada44”/Wikimedia Commons)

Exercise 3

Explain why using a fan in the summer feels refreshing.

Answer

Using a fan increases the flow of air: Warm air near your body is replaced by cooler air from elsewhere. Convection increases the rate of heat transfer so that moving air “feels” cooler than still air.

Radiation

You can feel the heat transfer from the Sun. The space between Earth and the Sun is largely empty, so the Sun warms us without any possibility of heat transfer by convection or conduction. Similarly, you can sometimes tell that the oven is hot without touching its door or looking inside—it may just warm you as you walk by. In these examples, heat is transferred by radiation (Figure 10). That is, the hot body emits electromagnetic waves that are absorbed by the skin. No medium is required for electromagnetic waves to propagate. Different names are used for electromagnetic waves of different wavelengths: radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and gamma rays.

Figure 10: Most of the heat transfer from this fire to the observers occurs through infrared radiation. The visible light, although dramatic, transfers relatively little thermal energy. Convection transfers energy away from the observers as hot air rises, while conduction is negligibly slow here. Skin is very sensitive to infrared radiation, so you can sense the presence of a fire without looking at it directly. (credit: Daniel O’Neil)

The energy of electromagnetic radiation varies over a wide range, depending on the wavelength: A shorter wavelength (or higher frequency) corresponds to a higher energy. Because more heat is radiated at higher temperatures, higher temperatures produce more intensity at every wavelength but especially at shorter wavelengths. In visible light, wavelength determines color—red has the longest wavelength and violet the shortest—so a temperature change is accompanied by a color change. For example, an electric heating element on a stove glows from red to orange, while the higher-temperature steel in a blast furnace glows from yellow to white. Infrared radiation is the predominant form radiated by objects cooler than the electric element and the steel. The radiated energy as a function of wavelength depends on its intensity, which is represented in Figure 11 by the height of the distribution. (Electromagnetic Waves explains more about the electromagnetic spectrum, and Photons and Matter Waves discusses why the decrease in wavelength corresponds to an increase in energy.)

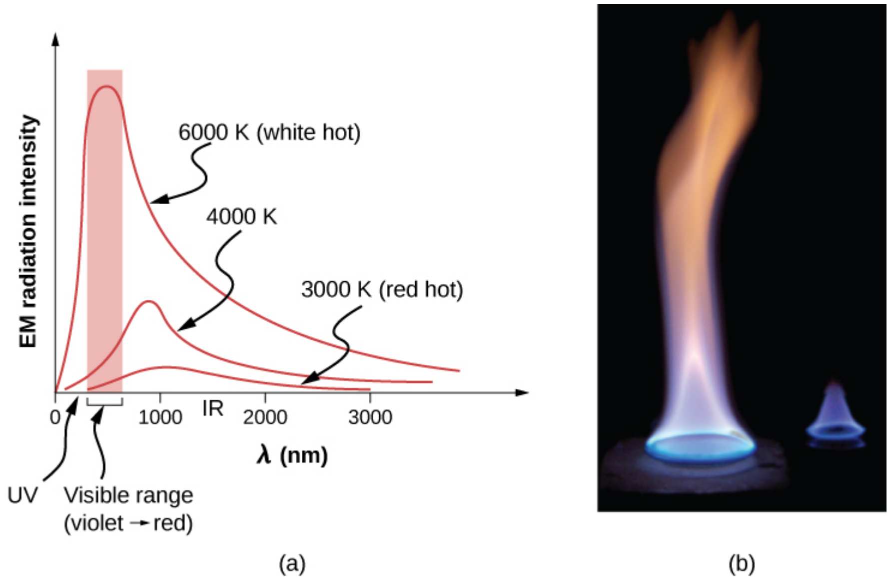


Figure 11: (a) A graph of the spectrum of electromagnetic waves emitted from an ideal radiator at three different temperatures. The intensity or rate of radiation emission increases dramatically with temperature, and the spectrum shifts down in wavelength toward the visible and ultraviolet parts of the spectrum. The shaded portion denotes the visible part of the spectrum. It is apparent that the shift toward the ultraviolet with temperature makes the visible appearance shift from red to white to blue as temperature increases. (b) Note the variations in color corresponding to variations in flame temperature.

The rate of heat transfer by radiation also depends on the object's color. Black is the most effective, and white is the least effective. On a clear summer day, black asphalt in a parking lot is hotter than adjacent gray sidewalk, because black absorbs better than gray (Figure 12). The reverse is also true—black radiates better than gray. Thus, on a clear summer night, the asphalt is colder than the gray sidewalk, because black radiates the energy more rapidly than gray. A perfectly black object would be an **ideal radiator** and an **ideal absorber**, as it would capture all the radiation that falls on it. In contrast, a perfectly white object or a perfect mirror would reflect all radiation, and a perfectly transparent object would transmit it all (Figure 13). Such objects would not emit any radiation. Mathematically, the color is represented by the **emissivity e**. A "blackbody" radiator would have an $e=1$, whereas a perfect reflector or transmitter would have $e=0$. For real examples, tungsten light bulb filaments have an **e** of about 0.5, and carbon black (a material used in printer toner) has an emissivity of about 0.95.

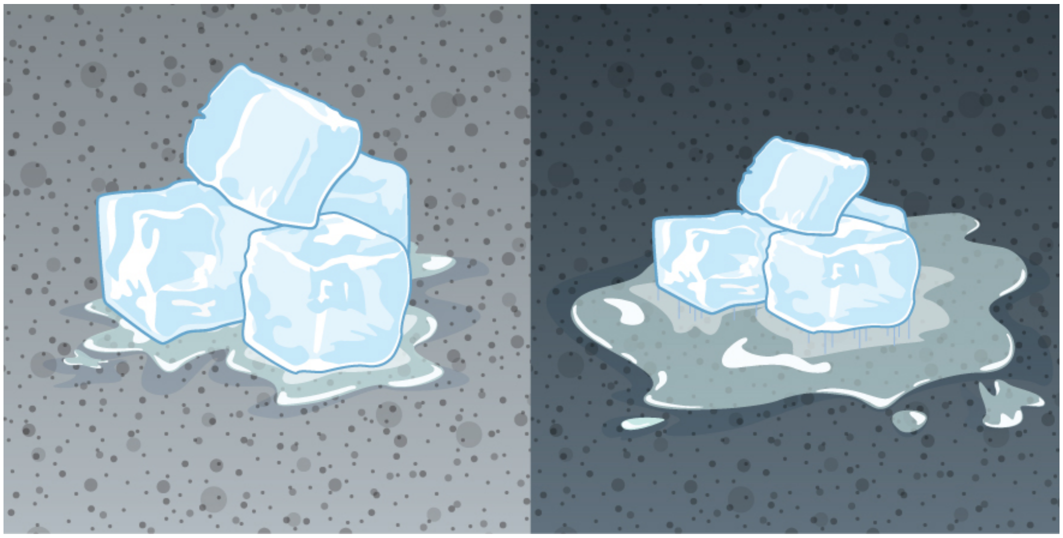


Figure 12: The darker pavement is hotter than the lighter pavement (much more of the ice on the right has melted), although both have been in the sunlight for the same time. The thermal conductivities of the pavements are the same.

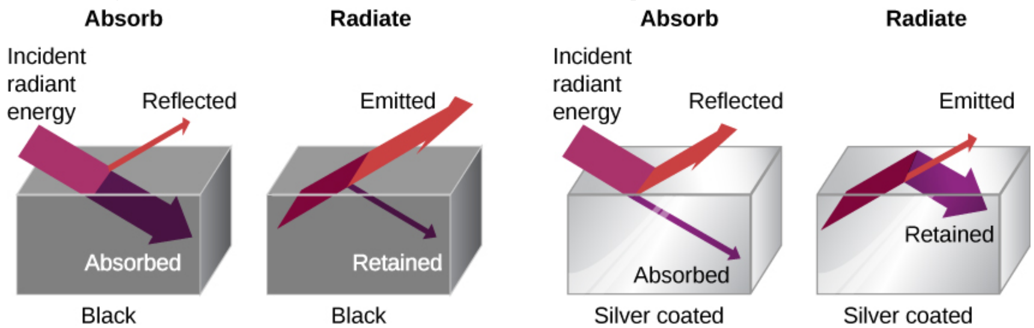


Figure 13: A black object is a good absorber and a good radiator, whereas a white, clear, or silver object is a poor absorber and a poor radiator.

To see that, consider a silver object and a black object that can exchange heat by radiation and are in thermal equilibrium. We know from experience that they will stay in equilibrium (the result of a principle that will be discussed at length in Second Law of Thermodynamics). For the black object's temperature to stay constant, it must emit as much radiation as it absorbs, so it must be as good at radiating as absorbing. Similar considerations show that the silver object must radiate as little as it absorbs. Thus, one property, emissivity, controls both radiation and absorption.

Finally, the radiated heat is proportional to the object's surface area, since every part of the surface radiates. If you knock apart the coals of a fire, the radiation increases noticeably due to an increase in radiating surface area.

The rate of heat transfer by emitted radiation is described by the **Stefan-Boltzmann law of radiation**:

$$P = \sigma A e T^4,$$

where $\sigma = 5.67 \times 10^{-8}\ J/s \cdot m^2 \cdot K^4$ is the Stefan-Boltzmann constant, a combination of fundamental constants of nature; **A** is the surface area of the object; and **T** is its temperature in kelvins.

The proportionality to the **fourth power** of the absolute temperature is a remarkably strong temperature dependence. It allows the detection of even small temperature variations. Images called **thermographs** can be used medically to detect regions of abnormally high temperature in the body, perhaps indicative of disease. Similar techniques can be used to detect heat leaks in homes (Figure 14), optimize performance of blast furnaces, improve comfort levels in work environments, and even remotely map Earth's temperature profile.

Figure 14: A thermograph of part of a building shows temperature variations, indicating where heat transfer to the outside is most severe. Windows are a major region of heat transfer to the outside of homes. (credit: US Army)

The Stefan-Boltzmann equation needs only slight refinement to deal with a simple case of an object's absorption of radiation from its surroundings. Assuming that an object with a temperature T_1 is surrounded by an environment with uniform temperature T_2, the **net rate of heat transfer by radiation** is

$$P_{net} = \sigma e A (T_2^4 - T_1^4),$$

where **e** is the emissivity of the object alone. In other words, it does not matter whether the surroundings are white, gray, or black: The balance of radiation into and out of the object depends on how well it emits and absorbs radiation. When $T_2 > T_1$, the quantity P_{net} is positive, that is, the net heat transfer is from hot to cold.

Before doing an example, we have a complication to discuss: different emissivities at different wavelengths. If the fraction of incident radiation an object reflects is the same at all visible wavelengths, the object is gray; if the fraction depends on the wavelength, the object has some other color. For instance, a red or reddish object reflects red light more strongly than other visible wavelengths. Because it absorbs less red, it radiates less red when hot. Differential reflection and absorption of wavelengths outside the visible range have no effect on what we see, but they may have physically important effects. Skin is a very good absorber and emitter of infrared radiation, having an emissivity of 0.97 in the infrared spectrum. Thus, in spite of the obvious variations in skin color, we are all nearly black in the infrared. This high infrared emissivity is why we can so easily feel radiation on our skin. It is also the basis for the effectiveness of night-vision scopes used by law enforcement and the military to detect human beings.

Example 4: Calculating the Net Heat Transfer of a Person

What is the rate of heat transfer by radiation of an unclothed person standing in a dark room whose ambient temperature is 22.0^oC? The person has a normal skin temperature of 33.0^oC and a surface area of $1.50\,m^2$. The emissivity of skin is 0.97 in the infrared, the part of the spectrum where the radiation takes place.

Strategy

We can solve this by using the equation for the rate of radiative heat transfer.

Solution

Insert the temperature values $T_2 = 295\,K$ and $T_1 = 306\,K$, so that

$$\frac{Q}{t} = \sigma e A (T_2^4 - T_1^4)$$

$$= (5.67 \times 10^{-8}\,J/s \cdot m^2 \cdot K^4)(0.97)(1.50\,m^2)[(295\,K)^4 - (306\,K)^4]$$

$$= -99\,J/s = -99\,W.$$

Significance

> This value is a significant rate of heat transfer to the environment (note the minus sign), considering that a person at rest may produce energy at the rate of 125 W and that conduction and convection are also transferring energy to the environment. Indeed, we would probably expect this person to feel cold. Clothing significantly reduces heat transfer to the environment by all mechanisms, because clothing slows down both conduction and convection, and has a lower emissivity (especially if it is light-colored) than skin.

The average temperature of Earth is the subject of much current discussion. Earth is in radiative contact with both the Sun and dark space, so we cannot use the equation for an environment at a uniform temperature. Earth receives almost all its energy from radiation of the Sun and reflects some of it back into outer space. Conversely, dark space is very cold, about 3 K, so that Earth radiates energy into the dark sky. The rate of heat transfer from soil and grasses can be so rapid that frost may occur on clear summer evenings, even in warm latitudes.

The average temperature of Earth is determined by its energy balance. To a first approximation, it is the temperature at which Earth radiates heat to space as fast as it receives energy from the Sun.

An important parameter in calculating the temperature of Earth is its emissivity (**e**). On average, it is about 0.65, but calculation of this value is complicated by the great day-to-day variation in the highly reflective cloud coverage. Because clouds have lower emissivity than either oceans or land masses, they reflect some of the radiation back to the surface, greatly reducing heat transfer into dark space, just as they greatly reduce heat transfer into the atmosphere during the day. There is negative feedback (in which a change produces an effect that opposes that change) between clouds and heat transfer; higher temperatures evaporate more water to form more clouds, which reflect more radiation back into space, reducing the temperature.

The often-mentioned **greenhouse effect** is directly related to the variation of Earth's emissivity with wavelength (Figure 15). The greenhouse effect is a natural phenomenon responsible for providing temperatures suitable for life on Earth and for making Venus unsuitable for human life. Most of the infrared radiation emitted from Earth is absorbed by carbon dioxide (CO_2) and water (H_2O) in the atmosphere and then re-radiated into outer space or back to Earth. Re-radiation back to Earth maintains its surface temperature about 40^oC higher than it would be if there were no atmosphere. (The glass walls and roof of a greenhouse increase the temperature inside by blocking convective heat losses, not radiative losses.)

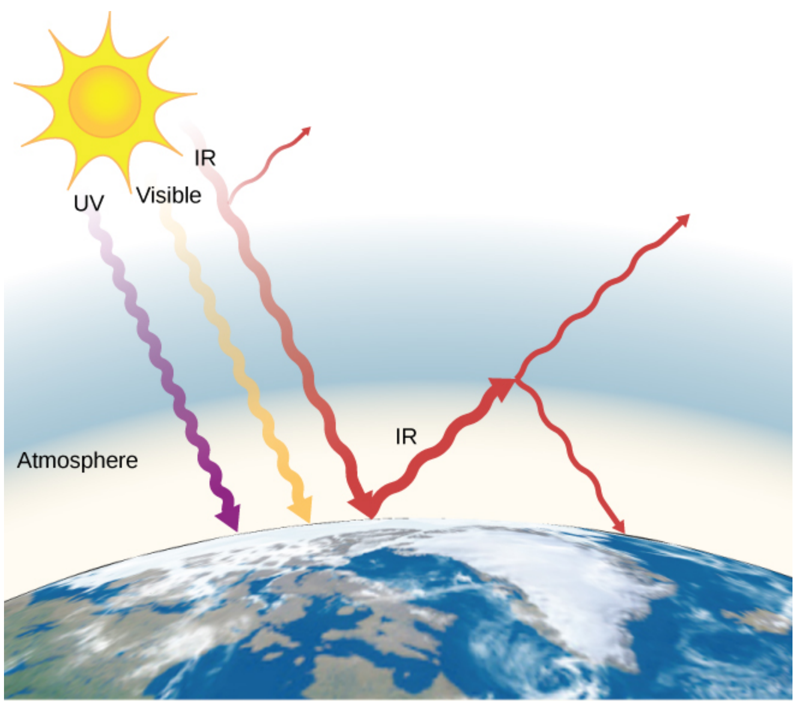


Figure 15: The greenhouse effect is the name given to the increase of Earth's temperature due to absorption of radiation in the atmosphere. The atmosphere is transparent to incoming visible radiation and most of the Sun's infrared. The Earth absorbs that energy and re-emits it. Since Earth's temperature is much lower than the Sun's, it re-emits the energy at much longer wavelengths, in the infrared. The atmosphere absorbs much of that infrared radiation and radiates about half of the energy back down, keeping Earth warmer than it would otherwise be. The amount of trapping depends on concentrations of trace gases such as carbon dioxide, and an increase in the concentration of these gases increases Earth's surface temperature.

The greenhouse effect is central to the discussion of global warming due to emission of carbon dioxide and methane (and other greenhouse gases) into Earth's atmosphere from industry, transportation, and farming. Changes in global climate could lead to more intense storms, precipitation changes (affecting agriculture), reduction in rain forest biodiversity, and rising sea levels.

You can explore a simulation of the greenhouse effect that takes the point of view that the atmosphere scatters (redirects) infrared radiation rather than absorbing it and reradiating it. You may want to run the simulation first with no greenhouse gases in the atmosphere and then look at how adding greenhouse gases affects the infrared radiation from the Earth and the Earth's temperature.

Problem-Solving Strategy: Effects of Heat Transfer

1. Examine the situation to determine what type of heat transfer is involved.
2. Identify the type(s) of heat transfer—conduction, convection, or radiation.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
4. Make a list of what is given or what can be inferred from the problem as stated (identify the knowns).
5. Solve the appropriate equation for the quantity to be determined (the unknown).
6. For conduction, use the equation $P = \frac{kA\Delta T}{d}$. Table 1 lists thermal conductivities. For convection, determine the amount of matter moved and the equation $Q = mc\Delta T$, along with $Q = mL_f$ or $Q = mL_v$ if a substance changes phase. For radiation, the equation $P_{net} = \sigma eA(T_2^4 - T_1^4)$ gives the net heat transfer rate.
7. Substitute the knowns along with their units into the appropriate equation and obtain numerical solutions complete with units.
8. Check the answer to see if it is reasonable. Does it make sense?

Exercise 4

How much greater is the rate of heat radiation when a body is at the temperature 40^oC than when it is at the temperature 20^oC?

[Hide solution]

The radiated heat is proportional to the fourth power of the **absolute temperature**. Because $T_1 = 293\ K$ and $T_2 = 313\ K$, the rate of heat transfer increases by about 30% of the original rate.

Phase Changes

Learning Objectives

By the end of this section, you will be able to:

- Describe phase transitions and equilibrium between phases
- Solve problems involving latent heat
- Solve calorimetry problems involving phase changes

Phase transitions play an important theoretical and practical role in the study of heat flow. In **melting** (or "**fusion**"), a solid turns into a liquid; the opposite process is **freezing**. In evaporation, a liquid turns into a gas; the opposite process is condensation.

A substance melts or freezes at a temperature called its melting point, and boils (evaporates rapidly) or condenses at its boiling point. These temperatures depend on pressure. High pressure favors the denser form, so typically, high pressure raises the melting point and boiling point, and low pressure lowers them. For example, the boiling point of water is 100^oC at 1.00 atm. At higher pressure, the boiling point is higher, and at lower pressure, it is lower. The main exception is the melting and freezing of water, discussed in the next section.

Phase Diagrams

The phase of a given substance depends on the pressure and temperature. Thus, plots of pressure versus temperature showing the phase in each region provide considerable insight into thermal properties of substances. Such a **pT** graph is called a **phase diagram**.

Figure 1 shows the phase diagram for water. Using the graph, if you know the pressure and temperature, you can determine the phase of water. The solid curves—boundaries between phases—indicate phase transitions, that is, temperatures and pressures at which the phases coexist. For example, the boiling point of water is 100^oC at 1.00 atm. As the pressure increases, the boiling temperature rises gradually to 374^oC at a pressure of 218 atm. A pressure cooker (or even a covered pot) cooks food faster than an open pot, because the water can exist as a liquid at temperatures greater than 100^oC without all boiling away. (As we'll see in the next section, liquid water conducts heat better than steam or hot air.) The boiling point curve ends at a certain point called the **critical point**—that is, a critical temperature, above which the liquid and gas phases cannot be distinguished; the substance is called a **supercritical fluid**. At sufficiently high pressure above the critical point, the gas has the density of a liquid but does not condense. Carbon dioxide, for example, is supercritical at all temperatures above 31^oC. **Critical pressure** is the pressure of the critical point.

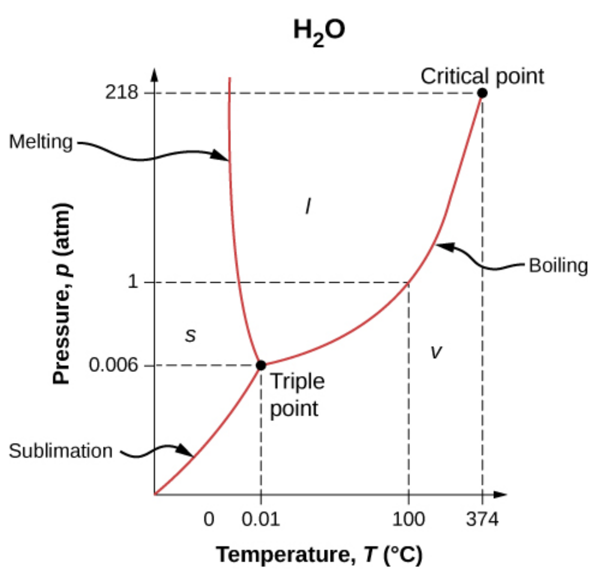


Figure 1: The phase diagram (pT graph) for water shows solid (s), liquid (l), and vapor (v) phases. At temperatures and pressure above those of the critical point, there is no distinction between liquid and vapor. Note that the axes are nonlinear and the graph is not to scale. This graph is simplified—it omits several exotic phases of ice at higher pressures. The phase diagram of water is unusual because the melting-point curve has a negative slope, showing that you can melt ice by increasing the pressure.

Similarly, the curve between the solid and liquid regions in Figure 1 gives the melting temperature at various pressures. For example, the melting point is 0^oC at 1.00 atm, as expected. Water has the unusual property that ice is less dense than liquid water at the melting point, so at a fixed temperature, you can change the phase from solid (ice) to liquid (water) by increasing the pressure. That is, the melting temperature of ice falls with increased pressure, as the phase diagram shows. For example, when a car is driven over snow, the increased pressure from the tires melts the snowflakes; afterwards, the water refreezes and forms an ice layer.

As you learned in the earlier section on thermometers and temperature scales, the **triple point** is the combination of temperature and pressure at which ice, liquid water, and water vapor can coexist stably—that is, all three phases exist in equilibrium. For water, the triple point occurs at $273.16\,K\,(0.01^oC)$ and 611.2 Pa; that is a more accurate calibration temperature than the melting point of water at 1.00 atm, or $273.15\,K\,(0.0^oC)$.

> **Note**
>
> View this video to see a substance at its triple point.

At pressures below that of the triple point, there is no liquid phase; the substance can exist as either gas or solid. For water, there is no liquid phase at pressures below 0.00600 atm. The phase change from solid to gas is called **sublimation**. You may have noticed that snow can disappear into thin air without a trace of liquid water, or that ice cubes can disappear in a freezer. Both are examples of sublimation. The reverse also happens: Frost can form on very cold windows without going through the liquid stage. Figure 2 shows the result, as well as showing a familiar example of sublimation. Carbon dioxide has no liquid phase at atmospheric pressure. Solid CO_2 is known as **dry ice** because instead of melting, it sublimes. Its sublimation temperature at atmospheric pressure is -78^oC. Certain air fresheners use the sublimation of a solid to spread a perfume around a room. Some solids, such as osmium tetroxide, are so toxic that they must be kept in sealed containers to prevent human exposure to their sublimation-produced vapors.

(a) (b)

Figure 2: Direct transitions between solid and vapor are common, sometimes useful, and even beautiful. (a) Dry ice sublimes directly to carbon dioxide gas. The visible “smoke” consists of water droplets that condensed in the air cooled by the dry ice. (b) Frost forms patterns on a very cold window, an example of a solid formed directly from a vapor. (credit a: modification of work by Windell Oskay; credit b: modification of work by Liz West)

Equilibrium

At the melting temperature, the solid and liquid phases are in equilibrium. If heat is added, some of the solid will melt, and if heat is removed, some of the liquid will freeze. The situation is somewhat more complex for liquid-gas equilibrium. Generally, liquid and gas are in equilibrium at any temperature. We call the gas phase a **vapor** when it exists at a temperature below the boiling temperature, as it does for water at 20.0^oC. Liquid in a closed container at a fixed temperature evaporates until the pressure of the gas reaches a certain value, called the **vapor pressure**, which depends on the gas and the temperature. At this equilibrium, if heat is added, some of the liquid will evaporate, and if heat is removed, some of the gas will condense; molecules either join the liquid or form suspended droplets. If there is not enough liquid for the gas to reach the vapor pressure in the container, all the liquid eventually evaporates.

If the vapor pressure of the liquid is greater than the **total** ambient pressure, including that of any air (or other gas), the liquid evaporates rapidly; in other words, it boils. Thus, the boiling point of a liquid at a given pressure is the temperature at which its vapor pressure equals the ambient pressure. Liquid and gas phases are in equilibrium at the boiling temperature (Figure 3). If a substance is in a closed container at the boiling point, then the liquid is boiling and the gas is condensing at the same rate without net change in their amounts.

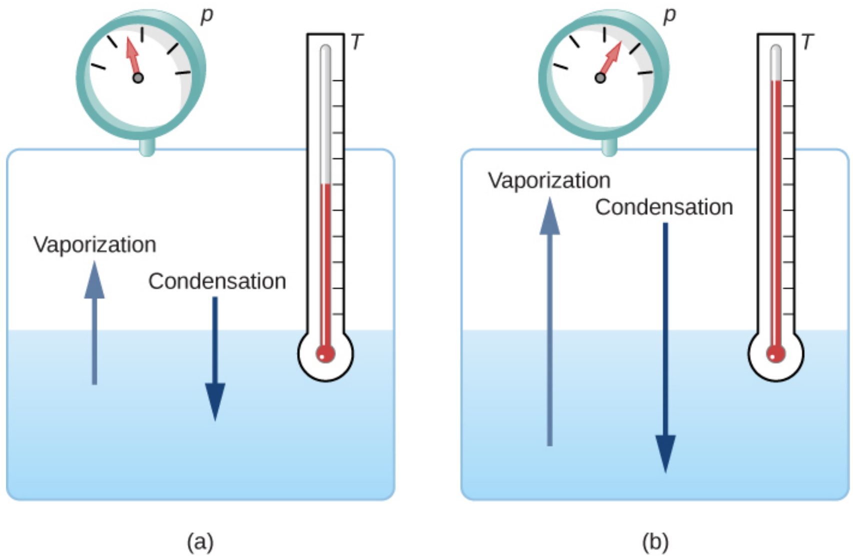


Figure 3: Equilibrium between liquid and gas at two different boiling points inside a closed container. (a) The rates of boiling and condensation are equal at this combination of temperature and pressure, so the liquid and gas phases are in equilibrium. (b) At a higher temperature, the boiling rate is faster, that is, the rate at which molecules leave the liquid and enter the gas is faster. This increases the number of molecules in the gas, which increases the gas pressure, which in turn increases the rate at which gas molecules condense and enter the liquid. The pressure stops increasing when it reaches the point where the boiling rate and the condensation rate are equal. The gas and liquid are in equilibrium again at this higher temperature and pressure.

For water, $100^o C$ is the boiling point at 1.00 atm, so water and steam should exist in equilibrium under these conditions. Why does an open pot of water at $100^o C$ boil completely away? The gas surrounding an open pot is not pure water: it is mixed with air. If pure water and steam are in a closed container at $100^o C$ and 1.00 atm, they will coexist—but with air over the pot, there are fewer water molecules to condense, and water boils away. Another way to see this is that at the boiling point, the vapor pressure equals the ambient pressure. However, part of the ambient pressure is due to air, so the pressure of the steam is less than the vapor pressure at that temperature, and evaporation continues. Incidentally, the equilibrium vapor pressure of solids is not zero, a fact that accounts for sublimation.

? Exercise 1

Explain why a cup of water (or soda) with ice cubes stays at $0^o C$ even on a hot summer day.

Answer

The ice and liquid water are in thermal equilibrium, so that the temperature stays at the freezing temperature as long as ice remains in the liquid. (Once all of the ice melts, the water temperature will start to rise.)

Phase Change and Latent Heat

So far, we have discussed heat transfers that cause temperature change. However, in a phase transition, heat transfer does not cause any temperature change.

For an example of phase changes, consider the addition of heat to a sample of ice at $-20^o C$ (Figure 4) and atmospheric pressure. The temperature of the ice rises linearly, absorbing heat at a constant rate of 2090 $J/kg \cdot^o C$ until it reaches $0^o C$. Once at this temperature, the ice begins to melt and continues until it has all melted, absorbing 333 kJ/kg of heat. The temperature remains constant at $0^o C$ during this phase change. Once all the ice has melted, the temperature of the liquid water rises, absorbing heat at a new constant rate of 4186 $J/kg \cdot^o C$. At $100^o C$ the water begins to boil. The temperature again remains constant during this phase change while the water absorbs 2256 kJ/kg of heat and turns into steam. When all the liquid has become steam, the temperature rises again, absorbing heat at a rate of $2020\ J/kg \cdot^o C$. If we started with steam and cooled it to make it condense into liquid water and freeze into ice, the process would exactly reverse, with the temperature again constant during each phase transition.

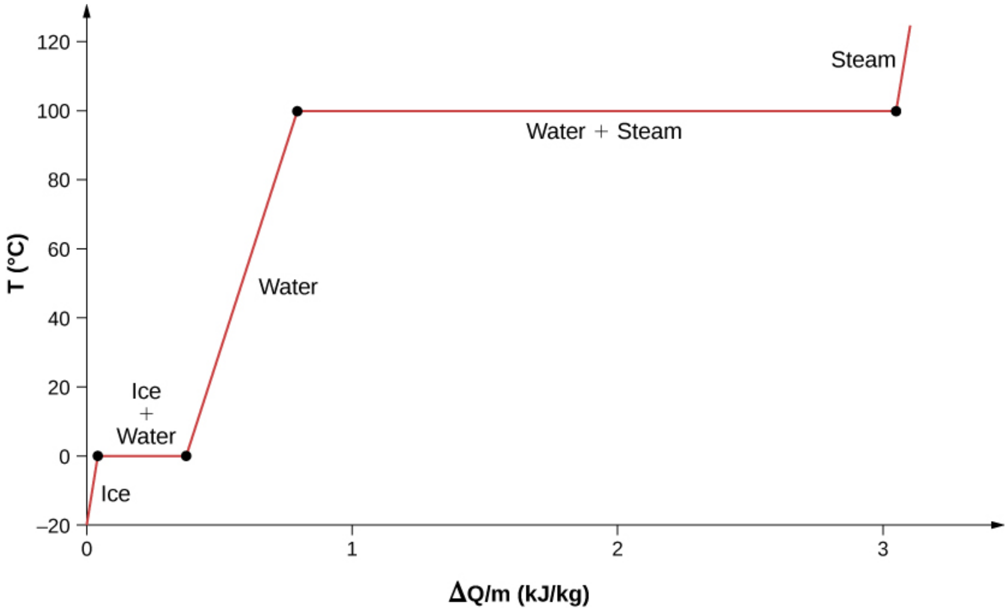


Figure 4: Temperature versus heat. The system is constructed so that no vapor evaporates while ice warms to become liquid water, and so that, when vaporization occurs, the vapor remains in the system. The long stretches of constant temperatures at $0^o C$ and $100^o C$ reflect the large amounts of heat needed to cause melting and vaporization, respectively.

Where does the heat added during melting or boiling go, considering that the temperature does not change until the transition is complete? Energy is required to melt a solid, because the attractive forces between the molecules in the solid must be broken apart, so that in the liquid, the molecules can move around at comparable kinetic energies; thus, there is no rise in temperature. Energy is needed to vaporize a liquid for similar reasons. Conversely, work is done by attractive forces when molecules are brought together during freezing and condensation. That energy must be transferred out of the system, usually in the form of heat, to allow the molecules to stay together (Figure 4). Thus, condensation occurs in association with cold objects—the glass in Figure 5, for example.

Figure 5: Condensation forms on this glass of iced tea because the temperature of the nearby air is reduced. The air cannot hold as much water as it did at room temperature, so water condenses. Energy is released when the water condenses, speeding the melting of the ice in the glass. (credit: Jenny Downing)

The energy released when a liquid freezes is used by orange growers when the temperature approaches $0^o C$. Growers spray water on the trees so that the water freezes and heat is released to the growing oranges. This prevents the temperature inside the orange from dropping below freezing, which would damage the fruit (Figure 6).

Figure 6: The ice on these trees released large amounts of energy when it froze, helping to prevent the temperature of the trees from dropping below 0^oC. Water is intentionally sprayed on orchards to help prevent hard frosts. (credit: Hermann Hammer)

The energy involved in a phase change depends on the number of bonds or force pairs and their strength. The number of bonds is proportional to the number of molecules and thus to the mass of the sample. The energy per unit mass required to change a substance from the solid phase to the liquid phase, or released when the substance changes from liquid to solid, is known as the **heat of fusion**. The energy per unit mass required to change a substance from the liquid phase to the vapor phase is known as the **heat of vaporization**. The strength of the forces depends on the type of molecules. The heat **Q** absorbed or released in a phase change in a sample of mass **m** is given by

$$Q = mL_f (melting/freezing) \tag{1}$$

$$Q = mL_v (vaporization/condensation) \tag{2}$$

where the latent heat of fusion L_f and latent heat of vaporization L_v are material constants that are determined experimentally. (Latent heats are also called **latent heat coefficients** and heats of transformation.) These constants are "latent," or hidden, because in phase changes, energy enters or leaves a system without causing a temperature change in the system, so in effect, the energy is hidden.

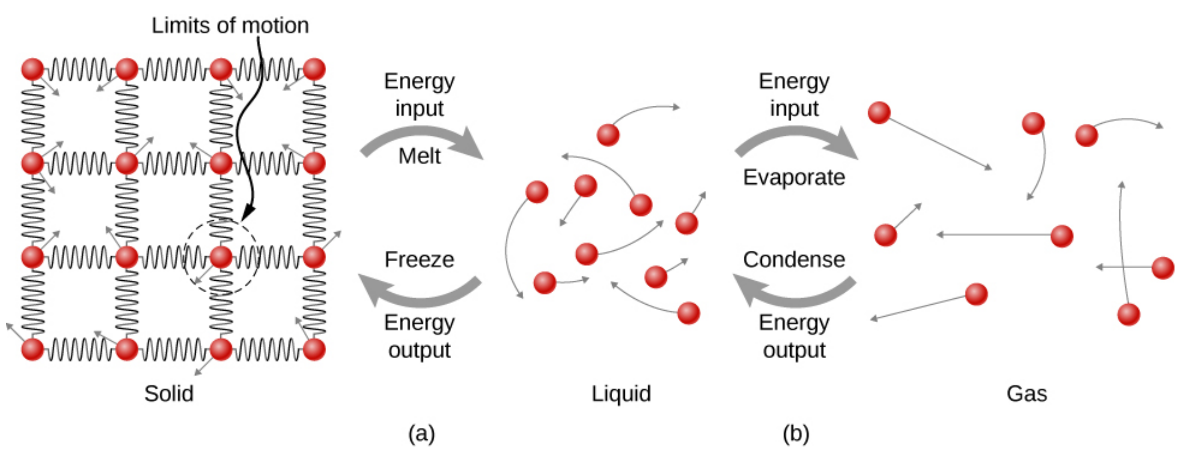


Figure 7: (a) Energy is required to partially overcome the attractive forces (modeled as springs) between molecules in a solid to form a liquid. That same energy must be removed from the liquid for freezing to take place. (b) Molecules become separated by large distances when going from liquid to vapor, requiring significant energy to completely overcome molecular attraction. The same energy must be removed from the vapor for condensation to take place.

Table 1 lists representative values of L_f and L_v in kJ/kg, together with melting and boiling points. Note that in general, $L_v > L_f$. The table shows that the amounts of energy involved in phase changes can easily be comparable to or greater than those involved in temperature changes, as Figure 7 and the accompanying discussion also showed.

Figure 1: Heats of Fusion and Vaporization

		L_f			L_v	
Substance	**Melting Point (^oC)**	**kJ/kg**	**kcal/kg**	**Boiling Point (^oC)**	**kJ/kg**	**kcal/kg**

		L_f			L_v	
Helium[2]	$-272.2\,(0.95\,K)$	5.23	1.25	$-268.9\,(4.2\,K)$	20.9	4.99
Hydrogen	$-259.3\,(13.9\,K)$	58.6	14.0	$-252.9\,(20.2\,K)$	452	108
Nitrogen	$-210.0\,(63.2\,K)$	25.5	6.09	$-195.8\,(77.4\,K)$	201	48.0
Oxygen	$-218.8\,(54.4\,K)$	13.8	3.30	$-183.0\,(90.2\,K)$	213	50.9
Ethanol	–114	104	24.9	78.3	854	204
Ammonia	–75	332	79.3	–33.4	1370	327
Mercury	–38.9	11.8	2.82	357	272	65.0
Water	0.00	334	79.8	100.0	2256[3]	539[4]
Sulfur	119	38.1	9.10	444.6	326	77.9
Lead	327	24.5	5.85	1750	871	208
Antimony	631	165	39.4	1440	561	134
Aluminum	660	380	90	2450	11400	2720
Silver	961	88.3	21.1	2193	2336	558
Gold	1063	64.5	15.4	2660	1578	377
Copper	1083	134	32.0	2595	5069	1211
Uranium	1133	84	20	3900	1900	454
Tungsten	3410	184	44	5900	4810	1150

Values quoted at the normal melting and boiling temperatures at standard atmospheric pressure ($1\,atm$). [2]Helium has no solid phase at atmospheric pressure. The melting point given is at a pressure of 2.5 MPa. [3]At 37.0^oC (body temperature), the heat of vaporization L_v for water is 2430 kJ/kg or 580 kcal/kg. [4]At 37.0^oC (body temperature), the heat of vaporization, L_v for water is 2430 kJ/kg or 580 kcal/kg.

Phase changes can have a strong stabilizing effect on temperatures that are not near the melting and boiling points, since evaporation and condensation occur even at temperatures below the boiling point. For example, air temperatures in humid climates rarely go above approximately 38.0^oC because most heat transfer goes into evaporating water into the air. Similarly, temperatures in humid weather rarely fall below the dew point—the temperature where condensation occurs given the concentration of water vapor in the air—because so much heat is released when water vapor condenses.

More energy is required to evaporate water below the boiling point than at the boiling point, because the kinetic energy of water molecules at temperatures below 100^oC is less than that at 100^oC, so less energy is available from random thermal motions. For example, at body temperature, evaporation of sweat from the skin requires a heat input of 2428 kJ/kg, which is about 10% higher than the latent heat of vaporization at 100^oC. This heat comes from the skin, and this evaporative cooling effect of sweating helps reduce the body temperature in hot weather. However, high humidity inhibits evaporation, so that body temperature might rise, while unevaporated sweat might be left on your brow.

✓ Exercise 1: Calculating Final Temperature from Phase Change

Three ice cubes are used to chill a soda at 20^oC with mass $m_{soda} = 0.25\,kg$. The ice is at 0^oC and each ice cube has a mass of 6.0 g. Assume that the soda is kept in a foam container so that heat loss can be ignored and that the soda has the same specific heat as water. Find the final temperature when all ice has melted.

Strategy

The ice cubes are at the melting temperature of 0^oC. Heat is transferred from the soda to the ice for melting. Melting yields water at 0^oC, so more heat is transferred from the soda to this water until the water plus soda system reaches thermal equilibrium.

The heat transferred to the ice is

$$Q_{ice} = m_{ice}L_f + m_{ice}c_w(T_f - 0^oC).$$

The heat given off by the soda is

$$Q_{soda} = m_{soda}c_w(T_f - 20^oC).$$

Since no heat is lost, $Q_{ice} = -Q_{soda}$, as in Example 1.5.3, so that

$$m_{ice}L_f + m_{ice}c_w(T_f - 0^oC) = -m_{soda}c_w(T_f - 20^oC).$$

Solve for the unknown quantity

$$T_f = \frac{m_{soda}c_w(20^oC) - m_{ice}L_f}{(m_{soda} + m_{ice})c_w}$$

Solution

First we identify the known quantities. The mass of ice is $m_{ice} = 3 \times 6.0\,g = 0.018\,kg$ and the mass of soda is $m_{soda} = 0.25\,kg$. Then we calculate the final temperature:

$$T_f = \frac{20,930\,J - 6012\,J}{1122\,J/^oC} = 13^oC.$$

Significance This example illustrates the large energies involved during a phase change. The mass of ice is about 7% of the mass of the soda but leads to a noticeable change in the temperature of the soda. Although we assumed that the ice was at the freezing temperature, this is unrealistic for ice straight out of a freezer: The typical temperature is -6^oC. However, this correction makes no significant change from the result we found. Can you explain why?

Like solid-liquid and liquid-vapor transitions, direct solid-vapor transitions or sublimations involve heat. The energy transferred is given by the equation $Q = mL_s$, where L_s is the **heat of sublimation**, analogous to L_f and L_v. The heat of sublimation at a given temperature is equal to the heat of fusion plus the heat of vaporization at that temperature.

We can now calculate any number of effects related to temperature and phase change. In each case, it is necessary to identify which temperature and phase changes are taking place. Keep in mind that heat transfer and work can cause both temperature and phase changes.

Problem-Solving Strategy: The Effects of Heat Transfer

1. Examine the situation to determine that there is a change in the temperature or phase. Is there heat transfer into or out of the system? When it is not obvious whether a phase change occurs or not, you may wish to first solve the problem as if there were no phase changes, and examine the temperature change obtained. If it is sufficient to take you past a boiling or melting point, you should then go back and do the problem in steps—temperature change, phase change, subsequent temperature change, and so on.
2. Identify and list all objects that change temperature or phase.
3. Identify exactly what needs to be determined in the problem (identify the unknowns). A written list is useful.
4. Make a list of what is given or what can be inferred from the problem as stated (identify the knowns). If there is a temperature change, the transferred heat depends on the specific heat of the substance (Heat Transfer, Specific Heat, and Calorimetry), and if there is a phase change, the transferred heat depends on the latent heat of the substance (Table 1).
5. Solve the appropriate equation for the quantity to be determined (the unknown).
6. Substitute the knowns along with their units into the appropriate equation and obtain numerical solutions complete with units. You may need to do this in steps if there is more than one state to the process, such as a temperature change followed by a phase change. However, in a calorimetry problem, each step corresponds to a term in the single equation $Q_{hot} + Q_{cold} = 0$.
7. Check the answer to see if it is reasonable. Does it make sense? As an example, be certain that any temperature change does not also cause a phase change that you have not taken into account.

Exercise 2

Why does snow often remain even when daytime temperatures are higher than the freezing temperature?

Snow is formed from ice crystals and thus is the solid phase of water. Because enormous heat is necessary for phase changes, it takes a certain amount of time for this heat to be transferred from the air, even if the air is above 0^oC

Prelude to Temperature and Heat

Heat and temperature are important concepts for each of us, every day. How we dress in the morning depends on whether the day is hot or cold, and most of what we do requires energy that ultimately comes from the Sun. The study of heat and temperature is part of an area of physics known as thermodynamics. The laws of thermodynamics govern the flow of energy throughout the universe. They are studied in all areas of science and engineering, from chemistry to biology to environmental science.

Figure 1: These snowshoers on Mount Hood in Oregon are enjoying the heat flow and light caused by high temperature. All three mechanisms of heat transfer are relevant to this picture. The heat flowing out of the fire also turns the solid snow to liquid water and vapor. (credit: “Mt. Hood Territory”/Flickr)

In this chapter, we explore heat and temperature. It is not always easy to distinguish these terms. Heat is the flow of energy from one object to another. This flow of energy is caused by a difference in temperature. The transfer of heat can change temperature, as can work, another kind of energy transfer that is central to thermodynamics. We return to these basic ideas several times throughout the next four chapters, and you will see that they affect everything from the behavior of atoms and molecules to cooking to our weather on Earth to the life cycles of stars.

Temperature and Heat (Answer)

Check Your Understanding

1.1. The actual amount (mass) of gasoline left in the tank when the gauge hits "empty" is less in the summer than in the winter. The gasoline has the same volume as it does in the winter when the "add fuel" light goes on, but because the gasoline has expanded, there is less mass.

1.2. Not necessarily, as the thermal stress is also proportional to Young's modulus.

1.3. To a good approximation, the heat transfer depends only on the temperature difference. Since the temperature differences are the same in both cases, the same 25 kJ is necessary in the second case. (As we will see in the next section, the answer would have been different if the object had been made of some substance that changes phase anywhere between **30°C** and **50°C**.)

1.4. The ice and liquid water are in thermal equilibrium, so that the temperature stays at the freezing temperature as long as ice remains in the liquid. (Once all of the ice melts, the water temperature will start to rise.)

1.5. Snow is formed from ice crystals and thus is the solid phase of water. Because enormous heat is necessary for phase changes, it takes a certain amount of time for this heat to be transferred from the air, even if the air is above **0°C**.

1.6. Conduction: Heat transfers into your hands as you hold a hot cup of coffee. Convection: Heat transfers as the barista "steams" cold milk to make hot cocoa. Radiation: Heat transfers from the Sun to a jar of water with tea leaves in it to make "Sun tea." A great many other answers are possible.

1.7. Because area is the product of two spatial dimensions, it increases by a factor of four when each dimension is doubled ($A_{final} = (2d)^2 = 4d^2 = 4A_{initial}$). The distance, however, simply doubles. Because the temperature difference and the coefficient of thermal conductivity are independent of the spatial dimensions, the rate of heat transfer by conduction increases by a factor of four divided by two, or two: $P_{final} = \frac{kA_{final}(T_h - T_c)}{d_{final}} = \frac{k(4A_{final}(T_h - T_c))}{2d_{initial}} = 2\frac{kA_{final}(T_h - T_c)}{d_{initial}} = 2P_{initial}$.

1.8. Using a fan increases the flow of air: Warm air near your body is replaced by cooler air from elsewhere. Convection increases the rate of heat transfer so that moving air "feels" cooler than still air.

1.9. The radiated heat is proportional to the fourth power of the absolute temperature. Because $T_1 = 293K$ and $T_2 = 313K$, the rate of heat transfer increases by about 30% of the original rate.

Conceptual Questions

1. They are at the same temperature, and if they are placed in contact, no net heat flows between them.

3. The reading will change.

5. The cold water cools part of the inner surface, making it contract, while the rest remains expanded. The strain is too great for the strength of the material. Pyrex contracts less, so it experiences less strain.

7. In principle, the lid expands more than the jar because metals have higher coefficients of expansion than glass. That should make unscrewing the lid easier. (In practice, getting the lid and jar wet may make gripping them more difficult.)

9. After being heated, the length is $(1 + 300\alpha)(1m)$. After being cooled, the length is $(1 - 300\alpha)(1 + 300\alpha)(1m)$. That answer is not 1 m, but it should be. The explanation is that even if α is exactly constant, the relation $\Delta L = \alpha L \Delta T$ is strictly true only in the limit of small ΔT. Since α values are small, the discrepancy is unimportant in practice.

11. Temperature differences cause heat transfer.

13. No, it is stored as thermal energy. A thermodynamic system does not have a well-defined quantity of heat.

15. It raises the boiling point, so the water, which the food gains heat from, is at a higher temperature.

17. Yes, by raising the pressure above 56 atm.

19. work

21. **0°C** (at or near atmospheric pressure)

23. Condensation releases heat, so it speeds up the melting.

25. Because of water's high specific heat, it changes temperature less than land. Also, evaporation reduces temperature rises. The air tends to stay close to equilibrium with the water, so its temperature does not change much where there's a lot of water around, as in San Francisco but not Sacramento.

27. The liquid is oxygen, whose boiling point is above that of nitrogen but whose melting point is below the boiling point of liquid nitrogen. The crystals that sublime are carbon dioxide, which has no liquid phase at atmospheric pressure. The crystals that melt are water, whose melting point is above carbon dioxide's sublimation point. The water came from the instructor's breath.

29. Increasing circulation to the surface will warm the person, as the temperature of the water is warmer than human body temperature. Sweating will cause no evaporative cooling under water or in the humid air immediately above the tub.

31. It spread the heat over the area above the heating elements, evening the temperature there, but does not spread the heat much beyond the heating elements.

33. Heat is conducted from the fire through the fire box to the circulating air and then convected by the air into the room (forced convection).

35. The tent is heated by the Sun and transfers heat to you by all three processes, especially radiation.

37. If shielded, it measures the air temperature. If not, it measures the combined effect of air temperature and net radiative heat gain from the Sun.

39. Turn the thermostat down. To have the house at the normal temperature, the heating system must replace all the heat that was lost. For all three mechanisms of heat transfer, the greater the temperature difference between inside and outside, the more heat is lost and must be replaced. So the house should be at the lowest temperature that does not allow freezing damage.

41. Air is a good insulator, so there is little conduction, and the heated air rises, so there is little convection downward.

Problems

43. That must be Celsius. Your Fahrenheit temperature is **102°F**. Yes, it is time to get treatment.

45. a. $\Delta T_C = 22.2^\circ C$;

b. We know that $\Delta T_F = T_{F2} - T_{F1}$. We also know that $T_{F2} = \frac{9}{5}T_{C2} + 32$ and $T_{F1} = \frac{9}{5}T_{C1} + 32$. So, substituting, we have $\Delta T_F = (\frac{9}{5}T_{C2} + 32) - (\frac{9}{5}T_{C1} + 32)$. Partially solving and rearranging the equation, we have $\Delta T_F = \frac{9}{5}(T_{C2} - T_{C1})$. Therefore, $\Delta T_F = \frac{9}{5}\Delta T_C$ ΔTF=95ΔTC.

47. a. **−40°**; b. 575 K

49. Using Table 1.2 to find the coefficient of thermal expansion of marble:

$L = L_0 + \Delta L = L_0(1 + \alpha\Delta T) = 170m[1 + (2.5 \times 10^{-6}/^\circ C)(-45.0^\circ C)] = 169.98m$

(Answer rounded to five significant figures to show the slight difference in height.)

51. We use β instead of α since this is a volume expansion with constant surface area. Therefore:

$\Delta L = \alpha L \Delta T = (6.0 \times 10^{-5}/^\circ C)(0.0300m)(3.00^\circ C) = 5.4 \times 10^{-6}m.$

53. On the warmer day, our tape measure will expand linearly. Therefore, each measured dimension will be smaller than the actual dimension of the land. Calling these measured dimensions l'l' and w'w', we will find a new area, A. Let's calculate these measured dimensions:

$$l' = l_0 - \Delta l = (20m) - (20^\circ C)(20m)(\frac{1.2 \times 10^{-5}}{^\circ C}) = 19.9952m$$

$$A' = l \times w' = (29.9928m)(19.9952m) = 599.71m^2$$

Cost change = $(A - A')(\frac{\$60,000}{m^2}) = ((600 - 599.71)m^2)(\frac{\$60,000}{m^2}) = \$17,000$

Because the area gets smaller, the price of the land **decreases** by about \$17,000.

55. a. Use Table 1.2 to find the coefficients of thermal expansion of steel and aluminum. Then $\Delta L_{Al} - \Delta L_{steel} = (\alpha_{Al} - \alpha_{steel})L_0\Delta T = (\frac{2.5 \times 10^{-5}}{^\circ C} - \frac{1.2 \times 10^{-5}}{^\circ C})(1.00m)(22^\circ C) = 2.9 \times 10^{-4}m$

b. By the same method with $L_0 = 30.0m$, we have $\Delta L = 8.6 \times 10^{-3}m$.

57. $\Delta V = 0.475L$

59. If we start with the freezing of water, then it would expand to $(1m^3)(\frac{1000kg/m^3}{917kg/m^3}) = 1.09m^3 = 1.98 \times 10^8 N/m^2$ of ice.

61. $m = 5.20 \times 10^8 J$

63. $Q = mc\Delta T \Rightarrow \Delta T = \frac{Q}{mc}$; a. **21.0°C**; b. **25.0°C**; c. **29.3°C**; d. **50.0°C**

65. $Q = mc\Delta T \Rightarrow c = \frac{Q}{m\Delta T} = \frac{1.04kcal}{(0.250kg)(45.0^\circ C)} = 0.0924kcal/kg \cdot {}^\circ C$. It is copper.

67. a. $Q = m_w c_w \Delta T + m_{A1} c_{A1} \Delta T = (m_w c_w + m_{A1} c_{A1})\Delta T$; \($Q = [(0.500kg)(1.00kcal/kg \cdot {}^\circ C) + (0.100kg)(0.215kcal/kg \cdot {}^\circ C)](54.9^\circ C) = 28.63kcal$; $\frac{Q}{m_p} = \frac{28.63kcal}{5.00g} = 5.73kcal/g$;

b. $\frac{Q}{m_p} = \frac{200kcal}{33g} = 6kcal/g$, which is consistent with our results to part (a), to one significant figure.

69. 0.139°C

71. It should be lower. The beaker will not make much difference: **16.3°C**

73. a. $1.00 \times 10^5 J$;

b. $3.68 \times 10^5 J$;

c. The ice is much more effective in absorbing heat because it first must be melted, which requires a lot of energy, and then it gains the same amount of heat as the bag that started with water. The first $2.67 \times 10^5 J$ of heat is used to melt the ice, then it absorbs the $1.00 \times 10^5 J$ of heat as water.

75. 58.1 g

77. Let **M** be the mass of pool water and m be the mass of pool water that evaporates.

$$Mc\Delta T = mLV_{(37^\circ C)} \Rightarrow \frac{m}{M} = \frac{c\Delta T}{L_{V(37^\circ C)}} = \frac{(1.00kcal/kg\cdot{}^\circ C)(1.50^\circ C)}{580kcal/kg} = 2.59 \times 10^{-3} \ ;$$

(Note that L_V for water at **37°C** is used here as a better approximation than L_V for **100°C** water.)

79. a. $1.47 \times 10^{15} kg$;

b. $4.90 \times 10^{20} J$;

c. $48.5y$

81. a. 9.35 L;

b. Crude oil is less dense than water, so it floats on top of the water, thereby exposing it to the oxygen in the air, which it uses to burn. Also, if the water is under the oil, it is less able to absorb the heat generated by the oil.

83. a. 319 kcal; b. **2.00°C**

85. First bring the ice up to **0°C** and melt it with heat Q_1: 4.74 kcal. This lowers the temperature of water by $\Delta T_2 : 23.15^\circ C$. Now, the heat lost by the hot water equals that gained by the cold water (T_f is the final temperature): **20.6°C**

87. Let the subscripts r, e, v, and w represent rock, equilibrium, vapor, and water, respectively.

$$m_r c_r (T_1 - T_e) = m_V L_V + m_W c_W (T_e - T_2) \ ;$$

$$m_r = \frac{m_V L_V + m_W c_W (T_e - T_2)}{c_r (T_1 - T_e)} = \frac{(0.0250kg)(2256 \times 10^3 J/kg) + (3.975kg)(4186 \times 10^3 J/kg\cdot{}^\circ C)(100^\circ C - 15^\circ C)}{(840J/kg\cdot{}^\circ C)(500^\circ C - 100^\circ C)} = 4.38kg$$

89. a. $1.01 \times 10^3 W$;

b. One 1-kilowatt room heater is needed.

91. 84.0 W

93. 2.59 kg

95. a. 39.7 W; b. 820 kcal

97. $\frac{Q}{t} = \frac{kA(T_2 - T_1)}{d}, so that \frac{(Q/t)_{wall}}{(Q/t)_{window}} = \frac{k_{wall} A_{wall} d_{window}}{k_{window} A_{window} d_{wall}} = \frac{(2 \times 0.042 J/s\cdot m\cdot{}^\circ C)(10.0m^2)(0.750 \times 10^{-2} m)}{(0.84 J/s\cdot m\cdot{}^\circ C)(2.00m^2)(13.0 \times 10^{-2} m)}$

This gives 0.0288 wall: window, or 35:1 window: wall

99. $\frac{Q}{t} = \frac{kA(T_2 - T_1)}{d} = \frac{kA\Delta T}{d} \Rightarrow \Delta T = \frac{d(Q/t)}{kA} = \frac{(6.00 \times 10^{-3} m)(2256W)}{(0.84 J/s\cdot m\cdot{}^\circ C)(1.54 \times 10^{-2} m^2)} = 1046^\circ C = 1.05 \times 10^3 K$

101. We found in the preceding problem that $P = 126\Delta T W\cdot{}^\circ C$ as baseline energy use. So the total heat loss during this period is $Q = (126 J/s\cdot{}^\circ C)(15.0^\circ C)(120 days)(86.4 \times 10^3 s/day) = 1960 \times 10^6 J$. At the cost of \$1/MJ, the cost is \$1960. From an earlier problem, the savings is 12% or \$235/y. We need $150m^2$ of insulation in the attic. At $\$4/m^2$, this is a \$500 cost. So the payback period is $\$600/(\$235/y) = 2.6 years$(excluding labor costs).

Additional Problems

103. 7.39%

105. $\frac{F}{A} = (210 \times 10^9 Pa)(12 \times 10^{-6}/{}^\circ C)(40^\circ C - (-15^\circ C)) = 1.4 \times 10^8 N/m^2$

107. a. 1.06 cm;

b. 1.11 cm

109. $1.7kJ/(kg\cdot{}^{\circ}C)$

111. a. $1.57\times10^4 kcal$;

b. $18.3kW\cdot h$;

c. 1.29×10^4 kcal

113. 6.3°C. All of the ice melted.

115. 63.9°C, all the ice melted

117. a. 83 W;

b. $1.97\times10^3 W$; The single-pane window has a rate of heat conduction equal to 1969/83, or 24 times that of a double-pane window.

119. The rate of heat transfer by conduction is 20.0 W. On a daily basis, this is 1,728 kJ/day. Daily food intake is **2400kcal/d×4186J/kcal=10,050kJ/day**. So only 17.2% of energy intake goes as heat transfer by conduction to the environment at this ΔT.

121. 620 K

Challenge Problems

123. Denoting the period by **P**, we know $P=2\pi\sqrt{L/g}$. When the temperature increases by dT, the length increases by αLdT. Then the new length is a. $P=2\pi\sqrt{L+\alpha LdT}\,g=2\pi\sqrt{\frac{L}{g}(1+\alpha dT)}=2\pi\sqrt{\frac{L}{g}}(1+\frac{1}{2}\alpha dT)=P(1+\frac{1}{2}\alpha dT)$ by the binomial expansion. b. The clock runs slower, as its new period is 1.00019 s. It loses 16.4 s per day.

125. The amount of heat to melt the ice and raise it to **100°C** is not enough to condense the steam, but it is more than enough to lower the steam's temperature by **50°C**, so the final state will consist of steam and liquid water in equilibrium, and the final temperature is **100°C**; 9.5 g of steam condenses, so the final state contains 49.5 g of steam and 40.5 g of liquid water.

127. a. $dL/dT=kT/\rho L$;

b. $L=\sqrt{2kTt/\rho L_f}$;

c. yes

129. a. $\sigma(\pi R^2)T_s^4$;

b. $e\sigma\pi R^2T_s^4$;

c. $2e\sigma\pi R^2T_e^4$;

d. $T_s^4=2T_e^4T_s^4=2T_e^4$;

e. $e\sigma T_s^4+\frac{1}{4}(1-A)S=\sigma T_s^4$;

f. 288 K

Temperature and Heat (Exercises)

Conceptual Questions

1.1: Temperature and Thermal Equilibrium

1. What does it mean to say that two systems are in thermal equilibrium?

2. Give an example in which A has some kind of non-thermal equilibrium relationship with B, and B has the same relationship with C, but A does not have that relationship with C.

1.2: Thermometers and Temperature Scales

3. If a thermometer is allowed to come to equilibrium with the air, and a glass of water is not in equilibrium with the air, what will happen to the thermometer reading when it is placed in the water?

4. Give an example of a physical property that varies with temperature and describe how it is used to measure temperature.

1.3: Thermal Expansion

5. Pouring cold water into hot glass or ceramic cookware can easily break it. What causes the breaking? Explain why Pyrex®, a glass with a small coefficient of linear expansion, is less susceptible.

6. One method of getting a tight fit, say of a metal peg in a hole in a metal block, is to manufacture the peg slightly larger than the hole. The peg is then inserted when at a different temperature than the block. Should the block be hotter or colder than the peg during insertion? Explain your answer.

7. Does it really help to run hot water over a tight metal lid on a glass jar before trying to open it? Explain your answer.

8. When a cold alcohol thermometer is placed in a hot liquid, the column of alcohol goes down slightly before going up. Explain why.

9. Calculate the length of a 1-meter rod of a material with thermal expansion coefficient αα when the temperature is raised from 300 K to 600 K. Taking your answer as the new initial length, find the length after the rod is cooled back down to 300 K. Is your answer 1 meter? Should it be? How can you account for the result you got?

10. Noting the large stresses that can be caused by thermal expansion, an amateur weapon inventor decides to use it to make a new kind of gun. He plans to jam a bullet against an aluminum rod inside a closed invar tube. When he heats the tube, the rod will expand more than the tube and a very strong force will build up. Then, by a method yet to be determined, he will open the tube in a split second and let the force of the rod launch the bullet at very high speed. What is he overlooking?

1.4: Heat Transfer, Specific Heat, and Calorimetry

11. How is heat transfer related to temperature?

12. Describe a situation in which heat transfer occurs.

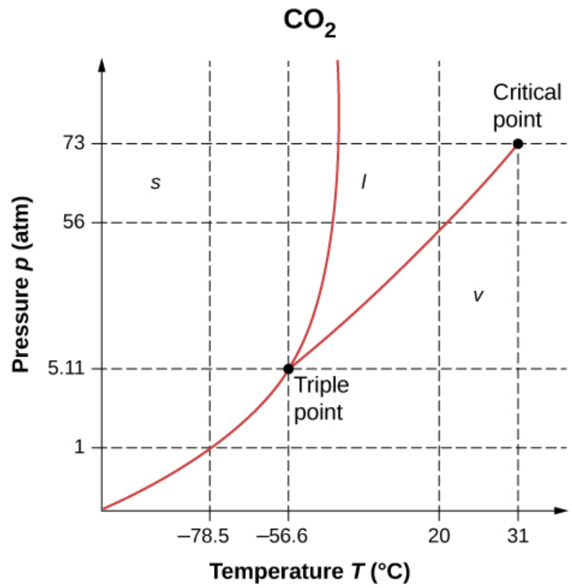
13. When heat transfers into a system, is the energy stored as heat? Explain briefly.

14. The brakes in a car increase in temperature by $\mathbf{\Delta T}$ when bringing the car to rest from a speed v. How much greater would $\mathbf{\Delta T}$ be if the car initially had twice the speed? You may assume the car stops fast enough that no heat transfers out of the brakes.

1.5: Phase Changes

15. A pressure cooker contains water and steam in equilibrium at a pressure greater than atmospheric pressure. How does this greater pressure increase cooking speed?

16. As shown below, which is the phase diagram for carbon dioxide, what is the vapor pressure of solid carbon dioxide (dry ice) at −78.5°C?−78.5°C? (Note that the axes in the figure are nonlinear and the graph is not to scale.)



17. Can carbon dioxide be liquefied at room temperature (**20°C**)? If so, how? If not, why not? (See the phase diagram in the preceding problem.)

18. What is the distinction between gas and vapor?

19. Heat transfer can cause temperature and phase changes. What else can cause these changes?

20. How does the latent heat of fusion of water help slow the decrease of air temperatures, perhaps preventing temperatures from falling significantly below **0°C,** in the vicinity of large bodies of water?

21. What is the temperature of ice right after it is formed by freezing water?

22. If you place **0°C** ice into **0°C** water in an insulated container, what will the net result be? Will there be less ice and more liquid water, or more ice and less liquid water, or will the amounts stay the same?

23. What effect does condensation on a glass of ice water have on the rate at which the ice melts? Will the condensation speed up the melting process or slow it down?

24. In Miami, Florida, which has a very humid climate and numerous bodies of water nearby, it is unusual for temperatures to rise above about **38°C** (**100°F**). In the desert climate of Phoenix, Arizona, however, temperatures rise above that almost every day in July and August. Explain how the evaporation of water helps limit high temperatures in humid climates.

25. In winter, it is often warmer in San Francisco than in Sacramento, 150 km inland. In summer, it is nearly always hotter in Sacramento. Explain how the bodies of water surrounding San Francisco moderate its extreme temperatures.

26. Freeze-dried foods have been dehydrated in a vacuum. During the process, the food freezes and must be heated to facilitate dehydration. Explain both how the vacuum speeds up dehydration and why the food freezes as a result.

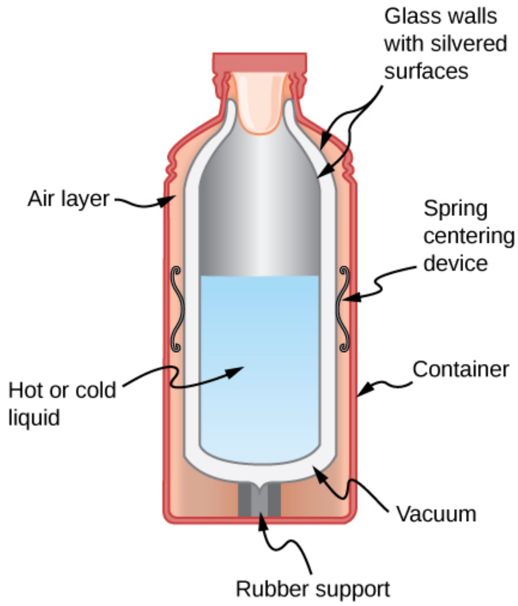
27. In a physics classroom demonstration, an instructor inflates a balloon by mouth and then cools it in liquid nitrogen. When cold, the shrunken balloon has a small amount of light blue liquid in it, as well as some snow-like crystals. As it warms up, the liquid boils, and part of the crystals sublime, with some crystals lingering for a while and then producing a liquid. Identify the blue liquid and the two solids in the cold balloon. Justify your identifications using data from Table 1.4.

1.6: Mechanisms of Heat Transfer

28. What are the main methods of heat transfer from the hot core of Earth to its surface? From Earth's surface to outer space?

29. When our bodies get too warm, they respond by sweating and increasing blood circulation to the surface to transfer thermal energy away from the core. What effect will those processes have on a person in a **40.0-°C** hot tub?

30. Shown below is a cut-away drawing of a thermos bottle (also known as a Dewar flask), which is a device designed specifically to slow down all forms of heat transfer. Explain the functions of the various parts, such as the vacuum, the silvering of the walls, the thin-walled long glass neck, the rubber support, the air layer, and the



31. Some electric stoves have a flat ceramic surface with heating elements hidden beneath. A pot placed over a heating element will be heated, while the surface only a few centimeters away is safe to touch. Why is ceramic, with a conductivity less than that of a metal but greater than that of a good insulator, an ideal choice for the stove top?

32. Loose-fitting white clothing covering most of the body, shown below, is ideal for desert dwellers, both in the hot Sun and during cold evenings. Explain how such clothing is advantageous during both day and night.

33. One way to make a fireplace more energy-efficient is to have room air circulate around the outside of the fire box and back into the room. Detail the methods of heat transfer involved.

34. On cold, clear nights horses will sleep under the cover of large trees. How does this help them keep warm?

35. When watching a circus during the day in a large, dark-colored tent, you sense significant heat transfer from the tent. Explain why this occurs.

36. Satellites designed to observe the radiation from cold (3 K) dark space have sensors that are shaded from the Sun, Earth, and the Moon and are cooled to very low temperatures. Why must the sensors be at low temperature?

37. Why are thermometers that are used in weather stations shielded from the sunshine? What does a thermometer measure if it is shielded from the sunshine? What does it measure if it is not?

38. Putting a lid on a boiling pot greatly reduces the heat transfer necessary to keep it boiling. Explain why.

39. Your house will be empty for a while in cold weather, and you want to save energy and money. Should you turn the thermostat down to the lowest level that will protect the house from damage such as freezing pipes, or leave it at the normal temperature? (If you don't like coming back to a cold house, imagine that a timer controls the heating system so the house will be warm when you get back.) Explain your answer.

40. You pour coffee into an unlidded cup, intending to drink it 5 minutes later. You can add cream when you pour the cup or right before you drink it. (The cream is at the same temperature either way. Assume that the cream and coffee come into thermal equilibrium with each other very quickly.) Which way will give you hotter coffee? What feature of this question is different from the previous one?

41. Broiling is a method of cooking by radiation, which produces somewhat different results from cooking by conduction or convection. A gas flame or electric heating element produces a very high temperature close to the food and above it. Why is radiation the dominant heat-transfer method in this situation?

42. On a cold winter morning, why does the metal of a bike feel colder than the wood of a porch?

Problems

1.2: Thermometers and Temperature Scales

43. While traveling outside the United States, you feel sick. A companion gets you a thermometer, which says your temperature is 39. What scale is that on? What is your Fahrenheit temperature? Should you seek medical help?

44. What are the following temperatures on the Kelvin scale?

(a) **68.0°F**, an indoor temperature sometimes recommended for energy conservation in winter

(b) **134°F**, one of the highest atmospheric temperatures ever recorded on Earth (Death Valley, California, 1913)

(c) **9890°F**, the temperature of the surface of the Sun

45. (a) Suppose a cold front blows into your locale and drops the temperature by 40.0 Fahrenheit degrees. How many degrees Celsius does the temperature decrease when it decreases by **40.0°F**? (b) Show that any change in temperature in Fahrenheit degrees is nine-fifths the change in Celsius degrees

46. An Associated Press article on climate change said, "Some of the ice shelf's disappearance was probably during times when the planet was 36 degrees Fahrenheit (2 degrees Celsius) to 37 degrees Fahrenheit (3 degrees Celsius) warmer than it is today." What mistake did the reporter make?

47. (a) At what temperature do the Fahrenheit and Celsius scales have the same numerical value? (b) At what temperature do the Fahrenheit and Kelvin scales have the same numerical value?

48. A person taking a reading of the temperature in a freezer in Celsius makes two mistakes: first omitting the negative sign and then thinking the temperature is Fahrenheit. That is, the person reads **–x°C** as **x°F**. Oddly enough, the result is the correct Fahrenheit temperature. What is the original Celsius reading? Round your answer to three significant figures.

1.4: Heat Transfer, Specific Heat, and Calorimetry

49. The height of the Washington Monument is measured to be 170.00 m on a day when the temperature is **35.0°C**. What will its height be on a day when the temperature falls to **−10.0°C**? Although the monument is made of limestone, assume that its coefficient of thermal expansion is the same as that of marble. Give your answer to five significant figures.

50. How much taller does the Eiffel Tower become at the end of a day when the temperature has increased by **15°C**? Its original height is 321 m and you can assume it is made of steel.

51. What is the change in length of a 3.00-cm-long column of mercury if its temperature changes from **37.0°C** to **40.0°C**, assuming the mercury is constrained to a cylinder but unconstrained in length? Your answer will show why thermometers contain bulbs at the bottom instead of simple columns of liquid.

52. How large an expansion gap should be left between steel railroad rails if they may reach a maximum temperature **35.0°C** greater than when they were laid? Their original length is 10.0 m.

53. You are looking to buy a small piece of land in Hong Kong. The price is "only" $60,000 per square meter. The land title says the dimensions are **20m×30m**. By how much would the total price change if you measured the parcel with a steel tape measure on a day when the temperature was **20°C** above the temperature that the tape measure was designed for? The dimensions of the land do not change.

54. Global warming will produce rising sea levels partly due to melting ice caps and partly due to the expansion of water as average ocean temperatures rise. To get some idea of the size of this effect, calculate the change in length of a column of water 1.00 km high for a temperature increase of **1.00°C**. Assume the column is not free to expand sideways. As a model of the ocean, that is a reasonable approximation, as only parts of the ocean very close to the surface can expand sideways onto land, and only to a limited degree. As another approximation, neglect the fact that ocean warming is not uniform with depth.

55. (a) Suppose a meter stick made of steel and one made of aluminum are the same length at **0°C**. What is their difference in length at **22.0°C**?

(b) Repeat the calculation for two 30.0-m-long surveyor's tapes.

56. (a) If a 500-mL glass beaker is filled to the brim with ethyl alcohol at a temperature of **5.00°C**, how much will overflow when the alcohol's temperature reaches the room temperature of **22.0°C**?

(b) How much less water would overflow under the same conditions?

57. Most cars have a coolant reservoir to catch radiator fluid that may overflow when the engine is hot. A radiator is made of copper and is filled to its 16.0-L capacity when at **10.0°C.** What volume of radiator fluid will overflow when the radiator and fluid reach a temperature of **95.0°C**, given that the fluid's volume coefficient of expansion is **β=400×10−6/°C**? (Your answer will be a conservative estimate, as most car radiators have operating temperatures greater than **95.0°C**).

58. A physicist makes a cup of instant coffee and notices that, as the coffee cools, its level drops 3.00 mm in the glass cup. Show that this decrease cannot be due to thermal contraction by calculating the decrease in level if the $\boldsymbol{350 cm^3}$ of coffee is in a 7.00-cm-diameter cup and decreases in temperature from **95.0°C** to **45.0°C**. (Most of the drop in level is actually due to escaping bubbles of air.)

59. The density of water at **0°C** is very nearly $1000 kg/m^3$ (it is actually $999.84 kg/m^3$), whereas the density of ice at **0°C**0°C is $917 kg/m^3$. Calculate the pressure necessary to keep ice from expanding when it freezes, neglecting the effect such a large pressure would have on the freezing temperature. (This problem gives you only an indication of how large the forces associated with freezing water might be.)

60. Show that **β=3α**, by calculating the infinitesimal change in volume dV of a cube with sides of length L when the temperature changes by dT.

1.4: Thermal Expansion

61. On a hot day, the temperature of an 80,000-L swimming pool increases by **1.50°C**. What is the net heat transfer during this heating? Ignore any complications, such as loss of water by evaporation.

62. To sterilize a 50.0-g glass baby bottle, we must raise its temperature from **22.0°C** to **95.0°C**. How much heat transfer is required?

63. The same heat transfer into identical masses of different substances produces different temperature changes. Calculate the final temperature when 1.00 kcal of heat transfers into 1.00 kg of the following, originally at **20.0°C**:

(a) water;

(b) concrete;

(c) steel; and

(d) mercury.

64. Rubbing your hands together warms them by converting work into thermal energy. If a woman rubs her hands back and forth for a total of 20 rubs, at a distance of 7.50 cm per rub, and with an average frictional force of 40.0 N, what is the temperature increase? The mass of tissues warmed is only 0.100 kg, mostly in the palms and fingers.

65. A **0.250-kg** block of a pure material is heated from **20.0°C** to **65.0°C** by the addition of 4.35 kJ of energy. Calculate its specific heat and identify the substance of which it is most likely composed.

66. Suppose identical amounts of heat transfer into different masses of copper and water, causing identical changes in temperature. What is the ratio of the mass of copper to water?

67. (a) The number of kilocalories in food is determined by calorimetry techniques in which the food is burned and the amount of heat transfer is measured. How many kilocalories per gram are there in a 5.00-g peanut if the energy from burning it is transferred to 0.500 kg of water held in a 0.100-kg aluminum cup, causing a **54.9-°C**temperature increase? Assume the process takes place in an ideal calorimeter, in other words a perfectly insulated container.

(b) Compare your answer to the following labeling information found on a package of dry roasted peanuts: a serving of 33 g contains 200 calories. Comment on whether the values are consistent.

68. Following vigorous exercise, the body temperature of an 80.0 kg person is **40.0°C.** At what rate in watts must the person transfer thermal energy to reduce the body temperature to **37.0°C** in 30.0 min, assuming the body continues to produce energy at the rate of 150 W? **(1watt=1joule/secondor1W=1J/s)**

69. In a study of healthy young men^1, doing 20 push-ups in 1 minute burned an amount of energy per kg that for a 70.0-kg man corresponds to 8.06 calories (kcal). How much would a 70.0-kg man's temperature rise if he did not lose any heat during that time?

70. A 1.28-kg sample of water at **10.0°C** is in a calorimeter. You drop a piece of steel with a mass of 0.385 kg at **215°C** into it. After the sizzling subsides, what is the final equilibrium temperature? (Make the reasonable assumptions that any steam produced condenses into liquid water during the process of equilibration and that the evaporation and condensation don't affect the outcome, as we'll see in the next section.)

71. Repeat the preceding problem, assuming the water is in a glass beaker with a mass of 0.200 kg, which in turn is in a calorimeter. The beaker is initially at the same temperature as the water. Before doing the problem, should the answer be higher or lower than the preceding answer? Comparing the mass and specific heat of the beaker to those of the water, do you think the beaker will make much difference?

1.5: Phase Changes

72. How much heat transfer (in kilocalories) is required to thaw a 0.450-kg package of frozen vegetables originally at **0°C** if their heat of fusion is the same as that of water?

73. A bag containing **0°C** ice is much more effective in absorbing energy than one containing the same amount of **0°C** water.

(a) How much heat transfer is necessary to raise the temperature of 0.800 kg of water from **0°C** to **30.0°C**?

(b) How much heat transfer is required to first melt 0.800 kg of **0°C** ice and then raise its temperature?

(c) Explain how your answer supports the contention that the ice is more effective.

74. (a) How much heat transfer is required to raise the temperature of a 0.750-kg aluminum pot containing 2.50 kg of water from **30.0°C** to the boiling point and then boil away 0.750 kg of water?

(b) How long does this take if the rate of heat transfer is 500 W?

75. Condensation on a glass of ice water causes the ice to melt faster than it would otherwise. If 8.00 g of vapor condense on a glass containing both water and 200 g of ice, how many grams of the ice will melt as a result? Assume no other heat transfer occurs. Use (L_v\) for water at **37°C** as a better approximation than L_v for water at **100°C**.)

76. On a trip, you notice that a 3.50-kg bag of ice lasts an average of one day in your cooler. What is the average power in watts entering the ice if it starts at **0°C** and completely melts to **0°C** water in exactly one day?

77. On a certain dry sunny day, a swimming pool's temperature would rise by **1.50°C** if not for evaporation. What fraction of the water must evaporate to carry away precisely enough energy to keep the temperature constant?

78. (a) How much heat transfer is necessary to raise the temperature of a 0.200-kg piece of ice from **−20.0°C** to**130.0°C**, including the energy needed for phase changes?

(b) How much time is required for each stage, assuming a constant 20.0 kJ/s rate of heat transfer? (c) Make a graph of temperature versus time for this process.

79. In 1986, an enormous iceberg broke away from the Ross Ice Shelf in Antarctica. It was an approximately rectangular prism 160 km long, 40.0 km wide, and 250 m thick.

(a) What is the mass of this iceberg, given that the density of ice is $917 kg/m^3$?

(b) How much heat transfer (in joules) is needed to melt it?

(c) How many years would it take sunlight alone to melt ice this thick, if the ice absorbs an average of $100 W/m^2$, 12.00 h per day?

80. How many grams of coffee must evaporate from 350 g of coffee in a 100-g glass cup to cool the coffee and the cup from **95.0°C** to **45.0°C**? Assume the coffee has the same thermal properties as water and that the average heat of vaporization is 2340 kJ/kg (560 kcal/g). Neglect heat losses through processes other than evaporation, as well as the change in mass of the coffee as it cools. Do the latter two assumptions cause your answer to be higher or lower than the true answer?

81. (a) It is difficult to extinguish a fire on a crude oil tanker, because each liter of crude oil releases $2.80 \times 10^7 J$ of energy when burned. To illustrate this difficulty, calculate the number of liters of water that must be expended to absorb the energy released by burning 1.00 L of crude oil, if the water's temperature rises from **20.0°C** to **100°C**, it boils, and the resulting steam's temperature rises to **300°C** at constant pressure.

(b) Discuss additional complications caused by the fact that crude oil is less dense than water.

82. The energy released from condensation in thunderstorms can be very large. Calculate the energy released into the atmosphere for a small storm of radius 1 km, assuming that 1.0 cm of rain is precipitated uniformly over this area.

83. To help prevent frost damage, 4.00 kg of water at **0°C**is sprayed onto a fruit tree.

(a) How much heat transfer occurs as the water freezes?

(b) How much would the temperature of the 200-kg tree decrease if this amount of heat transferred from the tree? Take the specific heat to be **3.35kJ/kg·°C**, and assume that no phase change occurs in the tree.

84. A 0.250-kg aluminum bowl holding **0.800kg** of soup at **25.0°C** is placed in a freezer. What is the final temperature if 388 kJ of energy is transferred from the bowl and soup, assuming the soup's thermal properties are the same as that of water?

85. A 0.0500-kg ice cube at **−30.0°C** is placed in 0.400 kg of **35.0-°C** water in a very well-insulated container. What is the final temperature?

86. If you pour 0.0100 kg of **20.0°C** water onto a 1.20-kg block of ice (which is initially at **−15.0°C**), what is the final temperature? You may assume that the water cools so rapidly that effects of the surroundings are negligible.

87. Indigenous people sometimes cook in watertight baskets by placing hot rocks into water to bring it to a boil. What mass of **500-°C** granite must be placed in 4.00 kg of **15.0-°C** water to bring its temperature to **100°C**, if 0.0250 kg of water escapes as vapor from the initial sizzle? You may neglect the effects of the surroundings.

88. What would the final temperature of the pan and water be in Example 1.7 if 0.260 kg of water were placed in the pan and 0.0100 kg of the water evaporated immediately, leaving the remainder to come to a common temperature with the pan?

1.6: Mechanisms of Heat Transfer

89. (a) Calculate the rate of heat conduction through house walls that are 13.0 cm thick and have an average thermal conductivity twice that of glass wool. Assume there are no windows or doors. The walls' surface area is $120 m^2$ and their inside surface is at **18.0°C**, while their outside surface is at **5.00°C**.

(b) How many 1-kW room heaters would be needed to balance the heat transfer due to conduction?

90. The rate of heat conduction out of a window on a winter day is rapid enough to chill the air next to it. To see just how rapidly the windows transfer heat by conduction, calculate the rate of conduction in watts through a $3.00 - m^2$window that is 0.634 cm thick (1/4 in.) if the temperatures of the inner and outer surfaces are **5.00°C** and **−10.0°C−**, respectively. (This rapid rate will not be maintained—the inner surface will cool, even to the point of frost formation.)

91. Calculate the rate of heat conduction out of the human body, assuming that the core internal temperature is **37.0°C**, the skin temperature is **34.0°C**, the thickness of the fatty tissues between the core and the skin averages 1.00 cm, and the surface area is $1.40 m^2$.

92. Suppose you stand with one foot on ceramic flooring and one foot on a wool carpet, making contact over an area of $80.0cm^2$ with each foot. Both the ceramic and the carpet are 2.00 cm thick and are **10.0°C** on their bottom sides. At what rate must heat transfer occur from each foot to keep the top of the ceramic and carpet at **33.0°C**?

93. A man consumes 3000 kcal of food in one day, converting most of it to thermal energy to maintain body temperature. If he loses half this energy by evaporating water (through breathing and sweating), how many kilograms of water evaporate?

94. A firewalker runs across a bed of hot coals without sustaining burns. Calculate the heat transferred by conduction into the sole of one foot of a firewalker given that the bottom of the foot is a 3.00-mm-thick callus with a conductivity at the low end of the range for wood and its density is $300kg/m^3$. The area of contact is $25.0cm^2$, the temperature of the coals is **700°C**, and the time in contact is 1.00 s. Ignore the evaporative cooling of sweat.

95. (a) What is the rate of heat conduction through the 3.00-cm-thick fur of a large animal having a $1.40-m^2$ surface area? Assume that the animal's skin temperature is **32.0°C**, that the air temperature is **−5.00°C**, and that fur has the same thermal conductivity as air.

(b) What food intake will the animal need in one day to replace this heat transfer?

96. A walrus transfers energy by conduction through its blubber at the rate of 150 W when immersed in **−1.00°C** water. The walrus's internal core temperature is **37.0°C**, and it has a surface area of $2.00m^2$. What is the average thickness of its blubber, which has the conductivity of fatty tissues without blood?

97. Compare the rate of heat conduction through a 13.0-cm-thick wall that has an area of $10.0m^2$ and a thermal conductivity twice that of glass wool with the rate of heat conduction through a 0.750-cm-thick window that has an area of $2.00m^2$, assuming the same temperature difference across each.

98. Suppose a person is covered head to foot by wool clothing with average thickness of 2.00 cm and is transferring energy by conduction through the clothing at the rate of 50.0 W. What is the temperature difference across the clothing, given the surface area is $1.40m^2$?

99. Some stove tops are smooth ceramic for easy cleaning. If the ceramic is 0.600 cm thick and heat conduction occurs through the same area and at the same rate as computed in Example 1.11, what is the temperature difference across it? Ceramic has the same thermal conductivity as glass and brick.

100. One easy way to reduce heating (and cooling) costs is to add extra insulation in the attic of a house. Suppose a single-story cubical house already had 15 cm of fiberglass insulation in the attic and in all the exterior surfaces. If you added an extra 8.0 cm of fiberglass to the attic, by what percentage would the heating cost of the house drop? Take the house to have dimensions 10 m by 15 m by 3.0 m. Ignore air infiltration and heat loss through windows and doors, and assume that the interior is uniformly at one temperature and the exterior is uniformly at another.

101. Many decisions are made on the basis of the payback period: the time it will take through savings to equal the capital cost of an investment. Acceptable payback times depend upon the business or philosophy one has. (For some industries, a payback period is as small as 2 years.) Suppose you wish to install the extra insulation in the preceding problem. If energy cost **$1.00** per million joules and the insulation was $4.00 per square meter, then calculate the simple payback time. Take the average **ΔT** for the 120-day heating season to be **15.0°C.**

Additional Problems

102. In 1701, the Danish astronomer Ole Rømer proposed a temperature scale with two fixed points, freezing water at 7.5 degrees, and boiling water at 60.0 degrees. What is the boiling point of oxygen, 90.2 K, on the Rømer scale?

103. What is the percent error of thinking the melting point of tungsten is **3695°C** instead of the correct value of 3695 K?

104. An engineer wants to design a structure in which the difference in length between a steel beam and an aluminum beam remains at 0.500 m regardless of temperature, for ordinary temperatures. What must the lengths of the beams be?

105. How much stress is created in a steel beam if its temperature changes from **–15°C** to **40°C** but it cannot expand? For steel, the Young's modulus $Y = 210 \times 10^9 N/m^2$ from Stress, Strain, and Elastic Modulus. (Ignore the change in area resulting from the expansion.)

106. A brass rod ($Y = 90 \times 10^9 N/m^2$), with a diameter of 0.800 cm and a length of 1.20 m when the temperature is **25°C**, is fixed at both ends. At what temperature is the force in it at 36,000 N?

107. A mercury thermometer still in use for meteorology has a bulb with a volume of $0.780 cm^3$ and a tube for the mercury to expand into of inside diameter 0.130 mm. (a) Neglecting the thermal expansion of the glass, what is the spacing between marks **1°C** apart? (b) If the thermometer is made of ordinary glass (not a good idea), what is the spacing?

108. Even when shut down after a period of normal use, a large commercial nuclear reactor transfers thermal energy at the rate of 150 MW by the radioactive decay of fission products. This heat transfer causes a rapid increase in temperature if the cooling system fails (**1watt=1joule/second** or **1W=1J/s** and **1MW=1megawatt**).

(a) Calculate the rate of temperature increase in degrees Celsius per second (**°C/s**) if the mass of the reactor core is $1.60 \times 10^5 kg$and it has an average specific heat of **0.3349kJ/kg·°C**.

(b) How long would it take to obtain a temperature increase of **2000°C**, which could cause some metals holding the radioactive materials to melt? (The initial rate of temperature increase would be greater than that calculated here because the heat transfer is concentrated in a smaller mass. Later, however, the temperature increase would slow down because the 500,000-kg steel containment vessel would also begin to heat up.)

109. You leave a pastry in the refrigerator on a plate and ask your roommate to take it out before you get home so you can eat it at room temperature, the way you like it. Instead, your roommate plays video games for hours. When you return, you notice that the pastry is still cold, but the game console has become hot. Annoyed, and knowing that the pastry will not be good if it is microwaved, you warm up the pastry by unplugging the console and putting it in a clean trash bag (which acts as a perfect calorimeter) with the pastry on the plate. After a while, you find that the equilibrium temperature is a nice, warm **38.3°C**. You know that the game console has a mass of 2.1 kg. Approximate it as having a uniform initial temperature of **45°C**. The pastry has a mass of 0.16 kg and a specific heat of **3.0kJ/(kg·ºC),** and is at a uniform initial temperature of **4.0°C**. The plate is at the same temperature and has a mass of 0.24 kg and a specific heat of **0.90J/(kg·ºC)**. What is the specific heat of the console?

110. Two solid spheres, A and B, made of the same material, are at temperatures of **0°C** and **100°C**, respectively. The spheres are placed in thermal contact in an ideal calorimeter, and they reach an equilibrium temperature of **20°C**. Which is the bigger sphere? What is the ratio of their diameters?

111. In some countries, liquid nitrogen is used on dairy trucks instead of mechanical refrigerators. A 3.00-hour delivery trip requires 200 L of liquid nitrogen, which has a density of $808 kg/m^3$..

(a) Calculate the heat transfer necessary to evaporate this amount of liquid nitrogen and raise its temperature to **3.00°C**. (Use c_P and assume it is constant over the temperature range.) This value is the amount of cooling the liquid nitrogen supplies.

(b) What is this heat transfer rate in kilowatt-hours?

(c) Compare the amount of cooling obtained from melting an identical mass of **0-°C** ice with that from evaporating the liquid nitrogen.

112. Some gun fanciers make their own bullets, which involves melting lead and casting it into lead slugs. How much heat transfer is needed to raise the temperature and melt 0.500 kg of lead, starting from **25.0°C**?

113. A 0.800-kg iron cylinder at a temperature of $1.00 \times 10^3 {}^\circ C$ is dropped into an insulated chest of 1.00 kg of ice at its melting point. What is the final temperature, and how much ice has melted?

114. Repeat the preceding problem with 2.00 kg of ice instead of 1.00 kg.

115. Repeat the preceding problem with 0.500 kg of ice, assuming that the ice is initially in a copper container of mass 1.50 kg in equilibrium with the ice.

116. A 30.0-g ice cube at its melting point is dropped into an aluminum calorimeter of mass 100.0 g in equilibrium at **24.0°C** with 300.0 g of an unknown liquid. The final temperature is **4.0°C**. What is the heat capacity of the liquid?

117. (a) Calculate the rate of heat conduction through a double-paned window that has a $1.50 - m^2$ area and is made of two panes of 0.800-cm-thick glass separated by a 1.00-cm air gap. The inside surface temperature is 15.0°C,15.0°C, while that on the outside is **−10.0°C**.(**Hint:** There are identical temperature drops across the two glass panes. First find these and then the temperature drop across the air gap. This problem ignores the increased heat transfer in the air gap due to convection.)

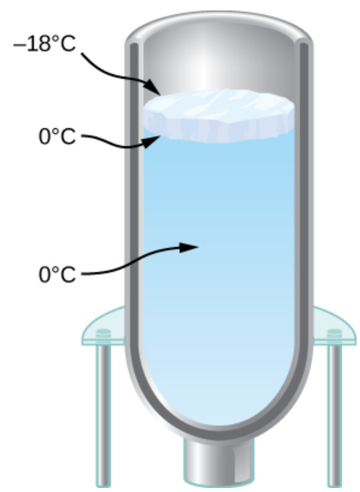
(b) Calculate the rate of heat conduction through a 1.60-cm-thick window of the same area and with the same temperatures. Compare your answer with that for part (a).

118. (a) An exterior wall of a house is 3 m tall and 10 m wide. It consists of a layer of drywall with an R factor of 0.56, a layer 3.5 inches thick filled with fiberglass batts, and a layer of insulated siding with an R factor of 2.6. The wall is built so well that there are no leaks of air through it. When the inside of the wall is at **22°C** and the outside is at **−2°C**, what is the rate of heat flow through the wall?

(b) More realistically, the 3.5-inch space also contains 2-by-4 studs—wooden boards 1.5 inches by 3.5 inches oriented so that 3.5-inch dimension extends from the drywall to the siding. They are “on 16-inch centers,” that is, the centers of the studs are 16 inches apart. What is the heat current in this situation? Don’t worry about one stud more or less.

119. For the human body, what is the rate of heat transfer by conduction through the body’s tissue with the following conditions: the tissue thickness is 3.00 cm, the difference in temperature is **2.00°C**, and the skin area is1.50m21.50m2. How does this compare with the average heat transfer rate to the body resulting from an energy intake of about 2400 kcal per day? (No exercise is included.)

120. You have a Dewar flask (a laboratory vacuum flask) that has an open top and straight sides, as shown below. You fill it with water and put it into the freezer. It is effectively a perfect insulator, blocking all heat transfer, except on the top. After a time, ice forms on the surface of the water. The liquid water and the bottom surface of the ice, in contact with the liquid water, are at **0°C**. The top surface of the ice is at the same temperature as the air in the freezer, **−18°C**. Set the rate of heat flow through the ice equal to the rate of loss of heat of fusion as the water freezes. When the ice layer is 0.700 cm thick, find the rate in m/s at which the ice is thickening.



121. An infrared heater for a sauna has a surface area of $0.050m^2$ and an emissivity of 0.84. What temperature must it run at if the required power is 360 W? Neglect the temperature of the environment.

122. (a) Determine the power of radiation from the Sun by noting that the intensity of the radiation at the distance of Earth is \(1370W/m^2\. Hint: That intensity will be found everywhere on a spherical surface with radius equal to that of Earth’s orbit.

(b) Assuming that the Sun’s temperature is 5780 K and that its emissivity is 1, find its radius.

Challenge Problems

123. A pendulum is made of a rod of length L and negligible mass, but capable of thermal expansion, and a weight of negligible size.

(a) Show that when the temperature increases by dT, the period of the pendulum increases by a fraction **αLdT/2**.

(b) A clock controlled by a brass pendulum keeps time correctly at **10°C.** If the room temperature is **30°C**, does the clock run faster or slower? What is its error in seconds per day?

124. At temperatures of a few hundred kelvins the specific heat capacity of copper approximately follows the empirical formula $c=\alpha+\beta T+\delta T^{-2}$, where **α=349J/kg·K**, $\beta=0.107 J/kg\cdot K^2$, and $\delta=4.58\times 10^5 J\cdot kg\cdot K$. How much heat is needed to raise the temperature of a 2.00-kg piece of copper from **20°C** to **250°C**?

125. In a calorimeter of negligible heat capacity, 200 g of steam at **150°C** and 100 g of ice at **−40°C** are mixed. The pressure is maintained at 1 atm. What is the final temperature, and how much steam, ice, and water are present?

126. An astronaut performing an extra-vehicular activity (space walk) shaded from the Sun is wearing a spacesuit that can be approximated as perfectly white **(e=0)** except for a **5cm×8cm** patch in the form of the astronaut's national flag. The patch has emissivity 0.300. The spacesuit under the patch is 0.500 cm thick, with a thermal conductivity **k=0.0600W/m°C**, and its inner surface is at a temperature of **20.0°C**. What is the temperature of the patch, and what is the rate of heat loss through it? Assume the patch is so thin that its outer surface is at the same temperature as the outer surface of the spacesuit under it. Also assume the temperature of outer space is 0 K. You will get an equation that is very hard to solve in closed form, so you can solve it numerically with a graphing calculator, with software, or even by trial and error with a calculator.

127. The goal in this problem is to find the growth of an ice layer as a function of time. Call the thickness of the ice layer L.

(a) Derive an equation for dL/dt in terms of L , the temperature T above the ice, and the properties of ice (which you can leave in symbolic form instead of substituting the numbers).

(b) Solve this differential equation assuming that at **t=0**, you have **L=0**. If you have studied differential equations, you will know a technique for solving equations of this type: manipulate the equation to get dL/dt multiplied by a (very simple) function of L on one side, and integrate both sides with respect to time. Alternatively, you may be able to use your knowledge of the derivatives of various functions to guess the solution, which has a simple dependence on t.

(c) Will the water eventually freeze to the bottom of the flask?

128. As the very first rudiment of climatology, estimate the temperature of Earth. Assume it is a perfect sphere and its temperature is uniform. Ignore the greenhouse effect. Thermal radiation from the Sun has an intensity (the "solar constant" S) of about $1370 W/m^2$ at the radius of Earth's orbit.

(a) Assuming the Sun's rays are parallel, what area must S be multiplied by to get the total radiation intercepted by Earth? It will be easiest to answer in terms of Earth's radius, R.

(b) Assume that Earth reflects about 30% of the solar energy it intercepts. In other words, Earth has an albedo with a value of **A=0.3**. In terms of S, A, and R, what is the rate at which Earth absorbs energy from the Sun?

(c) Find the temperature at which Earth radiates energy at the same rate. Assume that at the infrared wavelengths where it radiates, the emissivity **e** is 1. Does your result show that the greenhouse effect is important?

(d) How does your answer depend on the the area of Earth?

129. Let's stop ignoring the greenhouse effect and incorporate it into the previous problem in a very rough way. Assume the atmosphere is a single layer, a spherical shell around Earth, with an emissivity **e=0.77** (chosen simply to give the right answer) at infrared wavelengths emitted by Earth and by the atmosphere. However, the atmosphere is transparent to the Sun's radiation (that is, assume the radiation is at visible wavelengths with no infrared), so the Sun's radiation reaches the surface. The greenhouse effect comes from the difference between the atmosphere's transmission of visible light and its rather strong absorption of infrared. Note that the atmosphere's radius is not significantly different from Earth's, but since the atmosphere is a layer above Earth, it emits radiation both upward and downward, so it has twice Earth's area. There are three radiative energy transfers in this problem: solar radiation absorbed by Earth's surface; infrared radiation from the surface, which is absorbed by the atmosphere according to its emissivity; and infrared radiation from the atmosphere, half of which is absorbed by Earth and half of which goes out into space. Apply the method of the previous problem to get an equation for Earth's surface and one for the atmosphere, and solve them for the two unknown temperatures, surface and atmosphere.

a. In terms of Earth's radius, the constant σσ, and the unknown temperature T_s of the surface, what is the power of the infrared radiation from the surface?

b. What is the power of Earth's radiation absorbed by the atmosphere?

c. In terms of the unknown temperature T_e of the atmosphere, what is the power radiated from the atmosphere?

d. Write an equation that says the power of the radiation the atmosphere absorbs from Earth equals the power of the radiation it emits.

e. Half of the power radiated by the atmosphere hits Earth. Write an equation that says that the power Earth absorbs from the atmosphere and the Sun equals the power that it emits.

f. Solve your two equations for the unknown temperature of Earth.

For steps that make this model less crude, see for example the lectures by Paul O’Gorman.

Temperature and Heat (Summary)

Key Terms

absolute temperature scale	scale, such as Kelvin, with a zero point that is absolute zero
absolute zero	temperature at which the average kinetic energy of molecules is zero
calorie (cal)	energy needed to change the temperature of 1.00 g of water by $1.00°C$
calorimeter	container that prevents heat transfer in or out
calorimetry	study of heat transfer inside a container impervious to heat
Celsius scale	temperature scale in which the freezing point of water is $0°C$ and the boiling point of water is $100°C$
coefficient of linear expansion	(α) material property that gives the change in length, per unit length, per $1-{}^{\circ}C$ change in temperature; a constant used in the calculation of linear expansion; the coefficient of linear expansion depends to some degree on the temperature of the material
coefficient of volume expansion	(β) similar to αα but gives the change in volume, per unit volume, per $1-{}^{\circ}C$ change in temperature
conduction	heat transfer through stationary matter by physical contact
convection	heat transfer by the macroscopic movement of fluid
critical point	for a given substance, the combination of temperature and pressure above which the liquid and gas phases are indistinguishable
critical pressure	pressure at the critical point
critical temperature	temperature at the critical point
degree Celsius	$(°C)$ unit on the Celsius temperature scale
degree Fahrenheit	$(°F)$ unit on the Fahrenheit temperature scale
emissivity	measure of how well an object radiates
Fahrenheit scale	temperature scale in which the freezing point of water is $32°F$ and the boiling point of water is $212°F$
greenhouse effect	warming of the earth that is due to gases such as carbon dioxide and methane that absorb infrared radiation from Earth’s surface and reradiate it in all directions, thus sending some of it back toward Earth
heat	energy transferred solely due to a temperature difference
heat of fusion	energy per unit mass required to change a substance from the solid phase to the liquid phase, or released when the substance changes from liquid to solid
heat of sublimation	energy per unit mass required to change a substance from the solid phase to the vapor phase
heat of vaporization	energy per unit mass required to change a substance from the liquid phase to the vapor phase

heat transfer	movement of energy from one place or material to another as a result of a difference in temperature
Kelvin scale (K)	temperature scale in which 0 K is the lowest possible temperature, representing absolute zero
kilocalorie (kcal)	energy needed to change the temperature of 1.00 kg of water between **14.5°C** and **15.5°C**
latent heat coefficient	general term for the heats of fusion, vaporization, and sublimation
mechanical equivalent of heat	work needed to produce the same effects as heat transfer
net rate of heat transfer by radiation	$P_{net} = \sigma e A(T_2^4 - T_1^4)$
phase diagram	graph of pressure vs. temperature of a particular substance, showing at which pressures and temperatures the phases of the substance occur
radiation	energy transferred by electromagnetic waves directly as a result of a temperature difference
rate of conductive heat transfer	rate of heat transfer from one material to another
specific heat	amount of heat necessary to change the temperature of 1.00 kg of a substance by **1.00°C**; also called "specific heat capacity"
Stefan-Boltzmann law of radiation	$P = \sigma A e T^4$, where $\sigma = 5.67 \times 10^{-8} J/s \cdot m^2 \cdot K^4$ is the Stefan-Boltzmann constant, **A** is the surface area of the object, **T** is the absolute temperature, and **e** is the emissivity
sublimation	phase change from solid to gas
temperature	quantity measured by a thermometer, which reflects the mechanical energy of molecules in a system
thermal conductivity	property of a material describing its ability to conduct heat
thermal equilibrium	condition in which heat no longer flows between two objects that are in contact; the two objects have the same temperature
thermal expansion	change in size or volume of an object with change in temperature
thermal stress	stress caused by thermal expansion or contraction
triple point	pressure and temperature at which a substance exists in equilibrium as a solid, liquid, and gas
vapor	gas at a temperature below the boiling temperature
vapor pressure	pressure at which a gas coexists with its solid or liquid phase
zeroth law of thermodynamics	law that states that if two objects are in thermal equilibrium, and a third object is in thermal equilibrium with one of those objects, it is also in thermal equilibrium with the other object

Key Equations

Linear thermal expansion	$\Delta L = \alpha L \Delta T$
Thermal expansion in two dimensions	$\Delta A = 2\alpha A \Delta T$
Thermal expansion in three dimensions	$\Delta V = \beta V \Delta T$
Heat transfer	$Q = mc\Delta T$

Transfer of heat in a calorimeter	$Q_{cold} + Q_{hot} = 0$
Heat due to phase change (melting and freezing)	$Q = mL_f$
Heat due to phase change (evaporation and condensation)	$Q = mLv$
Rate of conductive heat transfer	$P = \frac{kA(T_h - T_c)}{d}$
Net rate of heat transfer by radiation	$P_{net} = \sigma eA(T_2^4 - T_1^4)$

Summary

1.2 Temperature and Thermal Equilibrium

- Temperature is operationally defined as the quantity measured by a thermometer. It is proportional to the average kinetic energy of atoms and molecules in a system.
- Thermal equilibrium occurs when two bodies are in contact with each other and can freely exchange energy. Systems are in thermal equilibrium when they have the same temperature.
- The zeroth law of thermodynamics states that when two systems, **A** and **B**, are in thermal equilibrium with each other, and B is in thermal equilibrium with a third system **C**, then **A** is also in thermal equilibrium with **C**.

1.3 Thermometers and Temperature Scales

- Three types of thermometers are alcohol, liquid crystal, and infrared radiation (pyrometer).
- The three main temperature scales are Celsius, Fahrenheit, and Kelvin. Temperatures can be converted from one scale to another using temperature conversion equations.
- The three phases of water (ice, liquid water, and water vapor) can coexist at a single pressure and temperature known as the triple point.

1.4 Thermal Expansion

- Thermal expansion is the increase of the size (length, area, or volume) of a body due to a change in temperature, usually a rise. Thermal contraction is the decrease in size due to a change in temperature, usually a fall in temperature.
- Thermal stress is created when thermal expansion or contraction is constrained.

1.5 Heat Transfer, Specific Heat, and Calorimetry

- Heat and work are the two distinct methods of energy transfer.
- Heat transfer to an object when its temperature changes is often approximated well by $Q = mc\Delta T$,,where m is the object’s mass and cis the specific heat of the substance.

1.6 Phase Changes

- Most substances have three distinct phases (under ordinary conditions on Earth), and they depend on temperature and pressure.
- Two phases coexist (i.e., they are in thermal equilibrium) at a set of pressures and temperatures.
- Phase changes occur at fixed temperatures for a given substance at a given pressure, and these temperatures are called boiling, freezing (or melting), and sublimation points.

1.7 Mechanisms of Heat Transfer

- Heat is transferred by three different methods: conduction, convection, and radiation.
- Heat conduction is the transfer of heat between two objects in direct contact with each other.
- The rate of heat transfer P (energy per unit time) is proportional to the temperature difference $T_h - T_c$ and the contact area A and inversely proportional to the distance d between the objects.
- Convection is heat transfer by the macroscopic movement of mass. Convection can be natural or forced, and generally transfers thermal energy faster than conduction. Convection that occurs along with a phase change can transfer energy from cold regions to warm ones.
- Radiation is heat transfer through the emission or absorption of electromagnetic waves.
- The rate of radiative heat transfer is proportional to the emissivity e. For a perfect blackbody, $e = 1$, whereas a perfectly white, clear, or reflective body has $e = 0$, with real objects having values of e between 1 and 0.
- The rate of heat transfer depends on the surface area and the fourth power of the absolute temperature:

$$P = \sigma e A T^4,$$

where $\sigma = 5.67 \times 10^{-8} J/s \cdot m^2 \cdot K^4$ is the Stefan-Boltzmann constant and **e** is the emissivity of the body. The net rate of heat transfer from an object by radiation is

$$\frac{Q_{net}}{t} = \sigma e A (T_2^4 - T_1^4),$$

where T_1 is the temperature of the object surrounded by an environment with uniform temperature T_2 and **e** is the emissivity of the object.

Temperature and Thermal Equilibrium

Learning Objectives

By the end of this section, you will be able to:

- Define temperature and describe it qualitatively
- Explain thermal equilibrium
- Explain the zeroth law of thermodynamics

Heat is familiar to all of us. We can feel heat entering our bodies from the summer Sun or from hot coffee or tea after a winter stroll. We can also feel heat leaving our bodies as we feel the chill of night or the cooling effect of sweat after exercise.

What is heat? How do we define it and how is it related to temperature? What are the effects of heat and how does it flow from place to place? We will find that, in spite of the richness of the phenomena, a small set of underlying physical principles unites these subjects and ties them to other fields. We start by examining temperature and how to define and measure it.

Temperature

The concept of temperature has evolved from the common concepts of hot and cold. The scientific definition of temperature explains more than our senses of hot and cold. As you may have already learned, many physical quantities are defined solely in terms of how they are observed or measured, that is, they are defined **operationally**. **Temperature** is operationally defined as the quantity of what we measure with a thermometer. As we will see in detail in a later chapter on the kinetic theory of gases, temperature is proportional to the average kinetic energy of translation, a fact that provides a more physical definition. Differences in temperature maintain the transfer of heat, or **heat transfe**r, throughout the universe. **Heat transfer** is the movement of energy from one place or material to another as a result of a difference in temperature. (You will learn more about heat transfer later in this chapter.)

Thermal Equilibrium

An important concept related to temperature is **thermal equilibrium**. Two objects are in thermal equilibrium if they are in close contact that allows either to gain energy from the other, but nevertheless, no net energy is transferred between them. Even when not in contact, they are in thermal equilibrium if, when they are placed in contact, no net energy is transferred between them. If two objects remain in contact for a long time, they typically come to equilibrium. In other words, two objects in thermal equilibrium do not exchange energy.

Experimentally, if object **A** is in equilibrium with object **B**, and object **B** is in equilibrium with object **C**, then (as you may have already guessed) object **A** is in equilibrium with object **C**. That statement of transitivity is called the **zeroth law of thermodynamics.** (The number "zeroth" was suggested by British physicist Ralph Fowler in the 1930s. The first, second, and third laws of thermodynamics were already named and numbered then. The zeroth law had seldom been stated, but it needs to be discussed before the others, so Fowler gave it a smaller number.) Consider the case where **A** is a thermometer. The zeroth law tells us that if **A** reads a certain temperature when in equilibrium with **B**, and it is then placed in contact with **C**, it will not exchange energy with **C**; therefore, its temperature reading will remain the same (Figure 1). In other words, **if two objects are in thermal equilibrium, they have the same temperature**.

Figure 1: If thermometer A is in thermal equilibrium with object B, and B is in thermal equilibrium with C, then A is in thermal equilibrium with C. Therefore, the reading on A stays the same when A is moved over to make contact with C.

A thermometer measures its own temperature. It is through the concepts of thermal equilibrium and the zeroth law of thermodynamics that we can say that a thermometer measures the temperature of **something else,** and to make sense of the statement that two objects are at the same temperature.

In the rest of this chapter, we will often refer to “systems” instead of “objects.” As in the chapter on linear momentum and collisions, a system consists of one or more objects—but in thermodynamics, we require a system to be macroscopic, that is, to consist of a huge number (such as 10^{23}) of molecules. Then we can say that a system is in thermal equilibrium with itself if all parts of it are at the same temperature. (We will return to the definition of a thermodynamic system in the chapter on the first law of thermodynamics.)

Thermal Expansion

Learning Objectives

By the end of this section, you will be able to:

- Answer qualitative questions about the effects of thermal expansion
- Solve problems involving thermal expansion, including those involving thermal stress

The expansion of alcohol in a thermometer is one of many commonly encountered examples of **thermal expansion**, which is the change in size or volume of a given system as its temperature changes. The most visible example is the expansion of hot air. When air is heated, it expands and becomes less dense than the surrounding air, which then exerts an (upward) force on the hot air and makes steam and smoke rise, hot air balloons float, and so forth. The same behavior happens in all liquids and gases, driving natural heat transfer upward in homes, oceans, and weather systems, as we will discuss in an upcoming section. Solids also undergo thermal expansion. Railroad tracks and bridges, for example, have expansion joints to allow them to freely expand and contract with temperature changes, as shown in Figure 1.

(a) (b)

Figure 1: (a) Thermal expansion joints like these in the (b) Auckland Harbour Bridge in New Zealand allow bridges to change length without buckling. (credit: "ŠJů"/Wikimedia Commons).

What is the underlying cause of thermal expansion? As previously mentioned, an increase in temperature means an increase in the kinetic energy of individual atoms. In a solid, unlike in a gas, the molecules are held in place by forces from neighboring molecules; as we saw in Oscillations, the forces can be modeled as in harmonic springs described by the Lennard-Jones potential. Energy in Simple Harmonic Motion shows that such potentials are asymmetrical in that the potential energy increases more steeply when the molecules get closer to each other than when they get farther away. Thus, at a given kinetic energy, the distance moved is greater when neighbors move away from each other than when they move toward each other. The result is that increased kinetic energy (increased temperature) increases the average distance between molecules—the substance expands.

For most substances under ordinary conditions, it is an excellent approximation that there is no preferred direction (that is, the solid is "isotropic"), and an increase in temperature increases the solid's size by a certain fraction in each dimension. Therefore, if the solid is free to expand or contract, its proportions stay the same; only its overall size changes.

Definition: Thermal Expansion in One Dimension

According to experiments, the dependence of thermal expansion on temperature, substance, and original length is summarized in the equation

$$\frac{dL}{dT} = \alpha L \tag{1}$$

where L is the original length $\frac{dL}{dT}$ is the change in length with respect to temperature, and α is the **coefficient of linear expansion**, a material property that varies slightly with temperature. As α is nearly constant and also very small, for practical purposes, we use the linear approximation:

$$\Delta L \approx \alpha L \Delta T. \tag{2}$$

Table 1 lists representative values of the coefficient of linear expansion. As noted earlier, ΔT is the same whether it is expressed in units of degrees Celsius or kelvins; thus, α may have units of $1/^oC$ or 1/K with the same value in either case. Approximating α as a constant is quite accurate for small changes in temperature and sufficient for most practical purposes, even for large changes in temperature. We examine this approximation more closely in the next example.

Table 1: Thermal Expansion Coefficients

Material	Coefficient of Linear Expansion $\alpha(1/^oC)$	Coefficient of Volume Expansion $\beta(1/^oC)$
	Solids	
Aluminum	25×10^{-6}	75×10^{-6}
Brass	19×10^{-6}	56×10^{-6}
Copper	17×10^{-6}	51×10^{-6}
Gold	14×10^{-6}	42×10^{-6}
Iron or steel	12×10^{-6}	35×10^{-6}
Invar (nickel-iron alloy)	0.9×10^{-6}	2.7×10^{-6}
Lead	29×10^{-6}	87×10^{-6}
Silver	18×10^{-6}	54×10^{-6}
Glass (ordinary)	9×10^{-6}	27×10^{-6}
Glass (Pyrex®)	3×10^{-6}	9×10^{-6}
Quartz	0.4×10^{-6}	1×10^{-6}
Concrete, brick	-12×10^{-6}	-36×10^{-6}
Marble (average)	2.5×10^{-6}	7.5×10^{-6}
	Liquids	
Ether		1650×10^{-6}
Ethyl alcohol		1100×10^{-6}
Gasoline		950×10^{-6}
Glycerin		500×10^{-6}
Mercury		180×10^{-6}
Water		210×10^{-6}
	Gases	
Air and most other gases at atmospheric pressure		3400×10^{-6}

Bimetallic Strip as thermometers

Thermal expansion is exploited in the bimetallic strip (Figure 2). This device can be used as a thermometer if the curving strip is attached to a pointer on a scale. It can also be used to automatically close or open a switch at a certain temperature, as in older or analog thermostats.

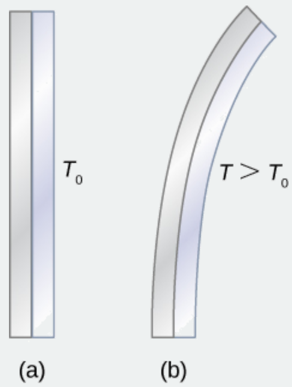


Figure 2: The curvature of a bimetallic strip depends on temperature. (a) The strip is straight at the starting temperature, where its two components have the same length. (b) At a higher temperature, this strip bends to the right, because the metal on the left has expanded more than the metal on the right. At a lower temperature, the strip would bend to the left.

Example 1: Calculating Linear Thermal Expansion

The main span of San Francisco's Golden Gate Bridge is 1275 m long at its coldest. The bridge is exposed to temperatures ranging from -15^oC to 40^oC. What is its change in length between these temperatures? Assume that the bridge is made entirely of steel.

Strategy

Use the equation for linear thermal expansion $\Delta L = \alpha L \Delta T$ to calculate the change in length, ΔL. Use the coefficient of linear expansion α for steel from Table 1, and note that the change in temperature ΔT is 55^oC.

Solution

Substitute all of the known values into the equation to solve for ΔL:

$$\begin{aligned} \Delta L &= \alpha L \Delta T \\ &= \left(\frac{12 \times 10^{-6}}{^oC}\right)(1275\, m)(55^oC) \\ &= 0.84\, m. \end{aligned}$$

Significance

Although not large compared with the length of the bridge, this change in length is observable. It is generally spread over many expansion joints so that the expansion at each joint is small.

Thermal Expansion in Two and Three Dimensions

Unconstrained objects expand in all dimensions, as illustrated in Figure 3. That is, their areas and volumes, as well as their lengths, increase with temperature. Because the proportions stay the same, holes and container volumes also get larger with temperature. If you cut a hole in a metal plate, the remaining material will expand exactly as it would if the piece you removed were still in place. The piece would get bigger, so the hole must get bigger too.

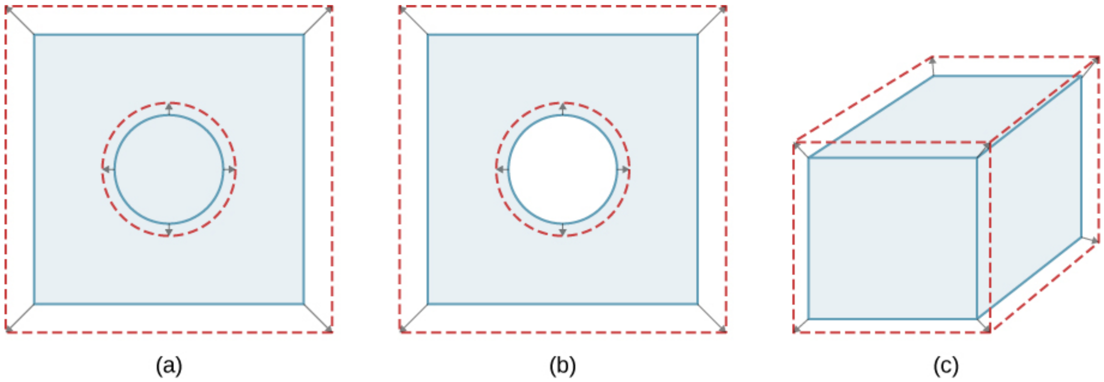


Figure 3: In general, objects expand in all directions as temperature increases. In these drawings, the original boundaries of the objects are shown with solid lines, and the expanded boundaries with dashed lines. (a) Area increases because both length and width increase. The area of a circular plug also increases. (b) If the plug is removed, the hole it leaves becomes larger with increasing temperature, just as if the expanding plug were still in place. (c) Volume also increases, because all three dimensions increase.

Definition: Thermal Expansion in Two Dimensions

For small temperature changes, the change in area ΔA is given by

$$\Delta A = 2\alpha A \Delta T \tag{3}$$

where ΔA is the range area A, ΔT is the change in temperature, and α is the coefficient of linear expansion, which varies slightly with temperature.

Definition: Thermal Expansion in Three Dimensions

The relationship between volume and temperature $\frac{dV}{dT}$ is given by $\frac{dV}{dT} = \beta V \Delta T$, where β is the **coefficient of volume expansion**. As you can show in Exercise, $\beta = 3\alpha$. This equation is usually written as

$$\Delta V = \beta V \Delta T. \tag{4}$$

Note that the values of β in Table 1 are equal to 3α except for rounding.

Volume expansion is defined for liquids, but linear and area expansion are not, as a liquid's changes in linear dimensions and area depend on the shape of its container. Thus, Table 1 shows liquids' values of β but not α.

In general, objects expand with increasing temperature. Water is the most important exception to this rule. Water does expand with increasing temperature (its density **decreases**) at temperatures greater than $4^oC (40^oF)$. However, it is densest at $+4^oC$ and expands with **decreasing** temperature between $+4^oC$ and 0^oC $(40^oF \, to \, 32^oF)$, as shown in Figure 4. A striking effect of this phenomenon is the **freezing of water** in a pond. When water near the surface cools down to 4^oC, it is denser than the remaining water and thus sinks to the bottom. This "turnover" leaves a layer of warmer water near the surface, which is then cooled. However, if the temperature in the surface layer drops below 4^oC, that water is less dense than the water below, and thus stays near the top. As a result, the pond surface can freeze over. The layer of ice insulates the liquid water below it from low air temperatures. Fish and other aquatic life can survive in 4^oC water beneath ice, due to this unusual characteristic of water.

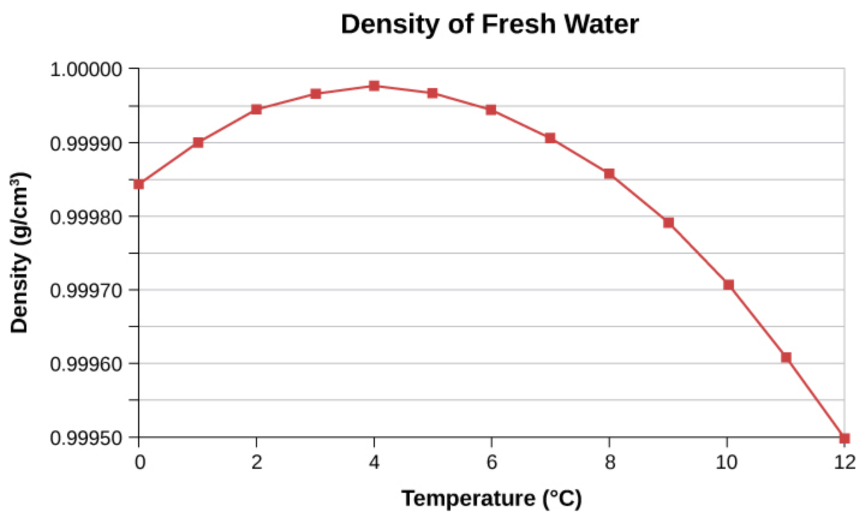


Figure 4: This curve shows the density of water as a function of temperature. Note that the thermal expansion at low temperatures is very small. The maximum density at 4^oC is only 0.0075% greater than the density at 2^oC, and 0.012% greater than that at 0^oC. The decrease of density below 4^oC occurs because the liquid water approachs the solid crystal form of ice, which contains more empty space than the liquid.

Example 2: Calculating Thermal Expansion

Suppose your 60.0-L (15.9-gal) steel gasoline tank is full of gas that is cool because it has just been pumped from an underground reservoir. Now, both the tank and the gasoline have a temperature of 15.0^oC. How much gasoline has spilled by the time they warm to 35.0^oC?

Strategy

The tank and gasoline increase in volume, but the gasoline increases more, so the amount spilled is the difference in their volume changes. We can use the equation for volume expansion to calculate the change in volume of the gasoline and of the tank. (The gasoline tank can be treated as solid steel.)

Solution

1. Use the equation for volume expansion to calculate the increase in volume of the steel tank:

$$\Delta V_s = \beta_s V_s \Delta T.$$

2. The increase in volume of the gasoline is given by this equation:

$$\Delta V_{gas} = \beta_{gas} V_{gas} \Delta T.$$

3. Find the difference in volume to determine the amount spilled as

$$V_{spill} = \Delta V_{gas} - \Delta V_s.$$

Alternatively, we can combine these three equations into a single equation. (Note that the original volumes are equal.)

$$\begin{aligned} V_{spill} &= (\beta_{gas} - \beta_s) V \Delta T \\ &= [(950 - 35) \times 10^{-6} / ^oC](60.0\,L)(20.0\,^oC) \\ &= 1.10\,L. \end{aligned}$$

Significance

This amount is significant, particularly for a 60.0-L tank. The effect is so striking because the gasoline and steel expand quickly. The rate of change in thermal properties is discussed later in this chapter.

If you try to cap the tank tightly to prevent overflow, you will find that it leaks anyway, either around the cap or by bursting the tank. Tightly constricting the expanding gas is equivalent to compressing it, and both liquids and solids resist compression with extremely large forces. To avoid rupturing rigid containers, these containers have air gaps, which allow them to expand and contract without stressing them.

? Exercises 1

Does a given reading on a gasoline gauge indicate more gasoline in cold weather or in hot weather, or does the temperature not matter?

Answer

The actual amount (mass) of gasoline left in the tank when the gauge hits "empty" is less in the summer than in the winter. The gasoline has the same volume as it does in the winter when the "add fuel" light goes on, but because the gasoline has expanded, there is less mass.

Thermal Stress

If you change the temperature of an object while preventing it from expanding or contracting, the object is subjected to stress that is compressive if the object would expand in the absence of constraint and tensile if it would contract. This stress resulting from temperature changes is known as **thermal stress**. It can be quite large and can cause damage.

To avoid this stress, engineers may design components so they can expand and contract freely. For instance, in highways, gaps are deliberately left between blocks to prevent thermal stress from developing. When no gaps can be left, engineers must consider thermal stress in their designs. Thus, the reinforcing rods in concrete are made of steel because steel's coefficient of linear expansion is nearly equal to that of concrete.

To calculate the thermal stress in a rod whose ends are both fixed rigidly, we can think of the stress as developing in two steps. First, let the ends be free to expand (or contract) and find the expansion (or contraction). Second, find the stress necessary to compress (or extend) the rod to its original length by the methods you studied in Static Equilibrium and Elasticity on static equilibrium and elasticity. In other words, the ΔL of the thermal expansion equals the ΔL of the elastic distortion (except that the signs are opposite).

✔ Example 3: Calculating Thermal Stress

Concrete blocks are laid out next to each other on a highway without any space between them, so they cannot expand. The construction crew did the work on a winter day when the temperature was 5^oC. Find the stress in the blocks on a hot summer day when the temperature is 38^oC. The compressive Young's modulus of concrete is $Y = 20 \times 10^9\ N/m^2$.

Strategy

According to the chapter on static equilibrium and elasticity, the stress **F**/**A** is given by

$$\frac{F}{A} = Y\frac{\Delta L}{L_0},$$

where Y is the Young's modulus of the material—concrete, in this case. In thermal expansion, $\Delta L = \alpha L_0 \delta T$. We combine these two equations by noting that the two ΔL's are equal, as stated above. Because we are not given L_0 or **A**, we can obtain a numerical answer only if they both cancel out.

Solution

We substitute the thermal-expansion equation into the elasticity equation to get

$$\begin{aligned}\frac{F}{A} &= Y\frac{\alpha L_0 \Delta T}{L_0} \\ &= Y\alpha\Delta T,\end{aligned}$$

and as we hoped, L_0 has canceled and A appears only in F/A, the notation for the quantity we are calculating.

Now we need only insert the numbers:

$$\begin{aligned}\frac{F}{A} &= (20 \times 10^6\ N/m^2)(12 \times 10^{-6}/^oC)(38^oC - 5^oC) \\ &= 7.9 \times 10^6\ N/m^2.\end{aligned}$$

Significance The ultimate compressive strength of concrete is $20 \times 10^6 \ N/m^2$, so the blocks are unlikely to break. However, the ultimate shear strength of concrete is only $2 \times 10^6 \ N/m^2$, so some might chip off.

Exercise 2

Two objects **A** and **B** have the same dimensions and are constrained identically. **A** is made of a material with a higher thermal expansion coefficient than **B**. If the objects are heated identically, will **A** feel a greater stress than **B**?

Answer

Not necessarily, as the thermal stress is also proportional to Young's modulus.

Thermometers and Temperature Scales

> **Learning Objectives**
>
> By the end of this section, you will be able to:
>
> - Describe several different types of thermometers
> - Convert temperatures between the Celsius, Fahrenheit, and Kelvin scales

Any physical property that depends consistently and reproducibly on temperature can be used as the basis of a thermometer. For example, volume increases with temperature for most substances. This property is the basis for the common alcohol thermometer and the original mercury thermometers. Other properties used to measure temperature include electrical resistance, color, and the emission of infrared radiation (Figure 1).

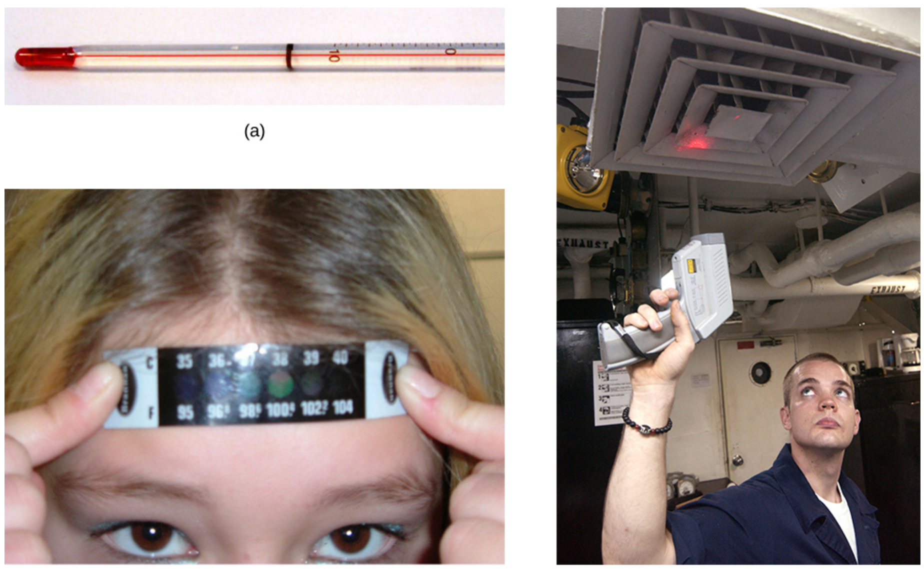


Figure 1: Because many physical properties depend on temperature, the variety of thermometers is remarkable. (a) In this common type of thermometer, the alcohol, containing a red dye, expands more rapidly than the glass encasing it. When the thermometer's temperature increases, the liquid from the bulb is forced into the narrow tube, producing a large change in the length of the column for a small change in temperature. (b) Each of the six squares on this plastic (liquid crystal) thermometer contains a film of a different heat-sensitive liquid crystal material. Below 95^oF, all six squares are black. When the plastic thermometer is exposed to a temperature of 95^oF, the first liquid crystal square changes color. When the temperature reaches above 96.8^oF, the second liquid crystal square also changes color, and so forth. (c) A firefighter uses a pyrometer to check the temperature of an aircraft carrier's ventilation system. The pyrometer measures infrared radiation (whose emission varies with temperature) from the vent and quickly produces a temperature readout. Infrared thermometers are also frequently used to measure body temperature by gently placing them in the ear canal. Such thermometers are more accurate than the alcohol thermometers placed under the tongue or in the armpit. (credit b: modification of work by Tess Watson; credit c: modification of work by Lamel J. Hinton)

Thermometers measure temperature according to well-defined scales of measurement. The three most common temperature scales are Fahrenheit, Celsius, and Kelvin. Temperature scales are created by identifying two reproducible temperatures. The freezing and boiling temperatures of water at standard atmospheric pressure are commonly used.

On the **Celsius scale**, the freezing point of water is 0^oC and the boiling point is 100^oC. The unit of temperature on this scale is **the degree Celsius (^oC).** The **Fahrenheit scale** (still the most frequently used for common purposes in the United States) has the freezing point of water at 32^oF and the boiling point a. 212^oF Its unit is the **degree Fahrenheit (^oF).** You can see that 100 Celsius degrees span the same range as 180 Fahrenheit degrees. Thus, a temperature difference of one degree on the Celsius scale is 1.8 times as large as a difference of one degree on the Fahrenheit scale, or

$$\Delta T_F = \frac{9}{5}\Delta T_C.$$

The definition of temperature in terms of molecular motion suggests that there should be a lowest possible temperature, where the average kinetic energy of molecules is zero (or the minimum allowed by quantum mechanics). Experiments confirm the existence of such a temperature, called **absolute zero**. An **absolute temperature scale** is one whose zero point is absolute zero. Such scales are convenient in science because several physical quantities, such as the volume of an ideal gas, are directly related to absolute temperature.

The **Kelvin scale** is the absolute temperature scale that is commonly used in science. The SI temperature unit is the **kelvin**, which is abbreviated K (not accompanied by a degree sign). Thus 0 K is absolute zero. The freezing and boiling points of water are 273.15 K and 373.15 K, respectively. Therefore, temperature differences are the same in units of kelvins and degrees Celsius, or $\Delta T_C = \Delta T_K$.

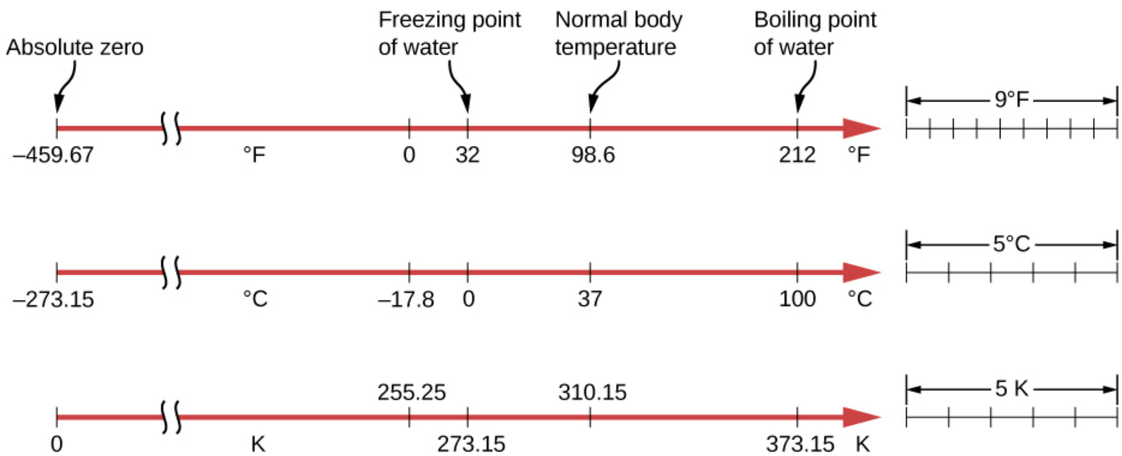


Figure 2: Relationships between the Fahrenheit, Celsius, and Kelvin temperature scales are shown. The relative sizes of the scales are also shown.

The relationships between the three common temperature scales are shown in Figure 2. Temperatures on these scales can be converted using the equations in Table 1.

Table 1

To convert from…	Use this equation…
Celsius to Fahrenheit	$T_F = \frac{9}{5} T_C + 32$
Fahrenheit to Celsius	$T_C = \frac{5}{9}(T_F - 32)$
Celsius to Kelvin	$T_K = T_C + 273.15$
Kelvin to Celsius	$T_C = T_K - 273.15$
Fahrenheit to Kelvin	$T_K = \frac{5}{9}(T_F - 32) + 273.15$
Kelvin to Fahrenheit	$T_F = \frac{9}{5}(T_K - 273.15) + 32$

To convert between Fahrenheit and Kelvin, convert to Celsius as an intermediate step.

Converting between Temperature Scales - Room Temperature

"Room temperature" is generally defined in physics to be 25^oC. (a) What is room temperature in ^oF? (b) What is it in K?

Strategy To answer these questions, all we need to do is choose the correct conversion equations and substitute the known values.

Solution

To convert from ^oC to ^oF, use the equation

$$T_F = \frac{9}{5} T_C + 32.$$

Substitute the known value into the equation and solve:

$$T_F = \frac{9}{5}(25^o C) + 32 = 77^o F.$$

Similarly, we find that $T_K = T_C + 273.15 = 298\ K$.

The Kelvin scale is part of the SI system of units, so its actual definition is more complicated than the one given above. First, it is not defined in terms of the freezing and boiling points of water, but in terms of the triple point. The triple point is the unique combination of temperature and pressure at which ice, liquid water, and water vapor can coexist stably. As will be discussed in the section on phase changes, the coexistence is achieved by lowering the pressure and consequently the boiling point to reach the freezing point. The triple-point temperature is defined as 273.16 K. This definition has the advantage that although the freezing temperature and boiling temperature of water depend on pressure, there is only one triple-point temperature.

Second, even with two points on the scale defined, different thermometers give somewhat different results for other temperatures. Therefore, a standard thermometer is required. Metrologists (experts in the science of measurement) have chosen the **constant-volume gas thermometer** for this purpose. A vessel of constant volume filled with gas is subjected to temperature changes, and the measured temperature is proportional to the change in pressure. Using "TP" to represent the triple point,

$$T = \frac{p}{p_{TP}} T_{TP}.$$

The results depend somewhat on the choice of gas, but the less dense the gas in the bulb, the better the results for different gases agree. If the results are extrapolated to zero density, the results agree quite well, with zero pressure corresponding to a temperature of absolute zero.

Constant-volume gas thermometers are big and come to equilibrium slowly, so they are used mostly as standards to calibrate other thermometers.

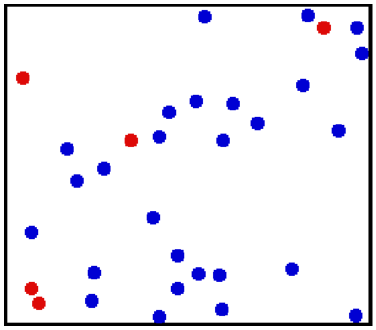
12.2: Introduction

learning objectives

- Describe gas using the kinetic theory of gases

Introduction to Temperature and Kinetic Theory

The kinetic theory of gases describes a gas as a large number of small particles (atoms or molecules), all of which are in constant, random motion. The rapidly moving particles constantly collide with each other, and with the walls of the container. Kinetic theory explains macroscopic properties of gases (such as pressure, temperature, and volume) by considering their molecular composition and motion. Essentially, the theory posits that pressure is due not to static repulsion between molecules (as was Isaac Newton's conjecture) but rather due to collisions between molecules moving at different velocities through Brownian motion. Also, the temperature of an ideal monatomic gas is a measure of the average kinetic energy of its atoms, as illustrated in.



Translational Motion of Helium: Real gases do not always behave according to the ideal model under certain conditions, such as high pressure. Here, the size of helium atoms relative to their spacing is shown to scale under 1950 atmospheres of pressure.

The kinetic theory of gases uses the model of the ideal gas to relate temperature to the average translational kinetic energy of the molecules in a container of gas in thermodynamic equilibrium. Classical mechanics defines the translational kinetic energy of a gas molecule as follows:

$$\mathrm{E_k} = \frac{1}{2}\mathrm{mv}^2, \tag{12.2.1}$$

where m is the particle mass and v its speed (the magnitude of its velocity). The distribution of the speeds (which determine the translational kinetic energies) of the particles in a classical ideal gas is called the Maxwell-Boltzmann distribution. In kinetic theory, the temperature of a classical ideal gas is related to its average kinetic energy per degree of freedom $\mathrm{E_k}$ via the equation:

$$\bar{\mathrm{E}}_\mathrm{k} = \frac{1}{2}\mathrm{kT}, \tag{12.2.2}$$

(k: Boltzmann's constant). We will derive this relationship in the following atoms. We will also derive the ideal gas law:

$$\mathrm{pV} = \mathrm{nRT}, \tag{12.2.3}$$

(R: ideal gas constant, n: number of moles of gas) from a microscopic theory.

Atomic Theory of Matter

Atomic theory is a scientific theory of the nature of matter which states that matter is composed of discrete units called atoms.

learning objectives

- Formulate five postulates of John Dalton's atomic theory

Atomic theory is a scientific theory of the nature of matter which states that matter is composed of discrete units called atoms, as opposed to the obsolete notion that matter could be divided into any arbitrarily small quantity. Although physicists discovered that the so-called "indivisible atom" was actually a conglomerate of various subatomic particles, the concept of atoms is still important because they are building blocks of matter and form the basis of chemistry.

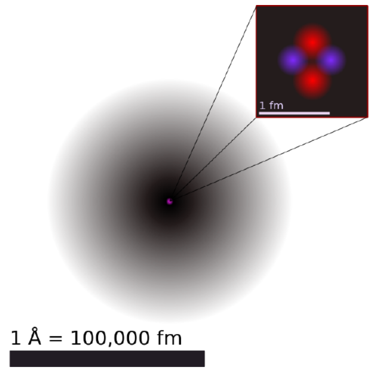


Illustration of the Helium Atom: This is an illustration of the helium atom, depicting the nucleus (pink) and the electron cloud distribution (black). The nucleus (upper right) in helium-4 is in reality spherically symmetric and closely resembles the electron cloud, although for more complicated nuclei this is not always the case. The black bar is one angstrom (10-10 m, or 100 pm).

Dalton's Atomic Hypothesis

Philosophical proposals regarding atoms have been suggested since the years of the ancient Greeks, but John Dalton was the first to propose a scientific theory of atoms. He based his study on two laws about chemical reactions that emerged (without referring to the notion of an atomic theory) in the late 18^{th}century. The first was the law of conservation of mass, formulated by Antoine Lavoisier in 1789, which states that the total mass in a chemical reaction remains constant (that is, the reactants have the same mass as the products). The second was the law of definite proportions, first proven by the French chemist Joseph Louis Proust.

Dalton proposed that each chemical element is composed of atoms of a single, unique type, and though they cannot be altered or destroyed by chemical means, they can combine to form more complex structures (chemical compounds). This marked the first truly scientific theory of the atom, since Dalton reached his conclusions by experimentation and examination of the results in an empirical fashion. For this reason, Dalton is considered the originator of modern atomic theory.

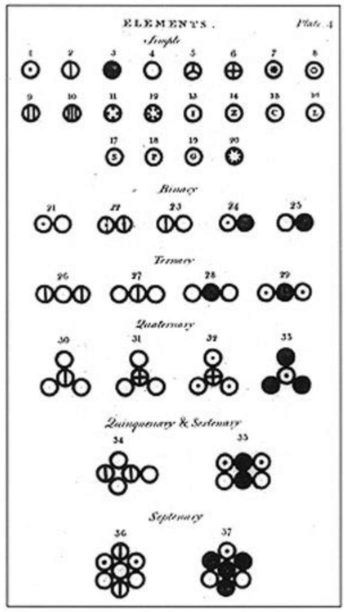
5 Main Points

Dalton's atomic theory had 5 main points:

1. Elements are made of extremely small particles called atoms.
2. Atoms of a given element are identical in size, mass, and other properties; atoms of different elements differ in size, mass, and other properties.
3. Atoms cannot be subdivided, created, or destroyed.
4. Atoms of different elements combine in simple whole-number ratios to form chemical compounds.
5. In chemical reactions, atoms are combined, separated, or rearranged.

Of these five, only three are still considered valid today. 1, 4, and 5 are valid, while 2 and 3 have turned out not to be the case. Atoms can be broken down into smaller pieces, and atoms of a given element can vary in mass and other properties (see isotopes and ions).

Knowing that a gas is composed of small atomic and molecular particles, it is natural to try to explain properties of the gas from a microscopic point of view. This effort led to the development of the kinetic theory of gases, where macroscopic properties of gases, such as pressure, temperature, and volume, are explained by considering their molecular composition and motion.



John Dalton's A New System of Chemical Philosophy: Various atoms and molecules as depicted in John Dalton's A New System of Chemical Philosophy (1808).

Key Points

- The kinetic theory posits that pressure is due to collisions between molecules moving at different velocities through Brownian motion.
- The temperature of an ideal monatomic gas is a measure of the average kinetic energy of its atoms. In kinetic theory, it is related to its average kinetic energy per degree of freedom E_k via the equation: $\bar{E}_k = \frac{1}{2}kT$.
- The kinetic theory of gases uses the model of the ideal gas to relate temperature to the average translational kinetic energy of the molecules in a container of gas in thermodynamic equilibrium.
- John Dalton was the first to propose a scientific theory of atoms. He based his study on two laws: the law of conservation of mass and the law of definite proportions.
- Dalton proposed that each chemical element is composed of atoms of a single, unique type, and though they cannot be altered or destroyed by chemical means, they can combine to form more complex structures.
- Kinetic theory of gases explain macroscopic properties of gases, such as pressure, temperature, and volume, by considering their molecular composition and motion.
- While Dalton's idea of matter being composed of various atoms was correct, he was wrong about some of their properties. Atoms can be broken down into smaller parts. Atoms of the same element can have slightly different masses and behave differently. See isotopes and ions for examples.

Key Terms

- **ideal gas**: A hypothetical gas whose molecules exhibit no interaction and undergo elastic collision with each other and with the walls of the container.
- **degree of freedom**: Any of the coordinates, a minimum number of which are needed to specify the motion of a mechanical system.
- **Brownian motion**: Random motion of particles suspended in a fluid, arising from those particles being struck by individual molecules of the fluid.
- **atom**: The smallest possible amount of matter which still retains its identity as a chemical element, now known to consist of a nucleus surrounded by electrons.
- **kinetic theory of gases**: The kinetic theory of gases describes a gas as a large number of small particles (atoms or molecules), all of which are in constant, random motion.
- **chemical reaction**: A process, involving the breaking or making of interatomic bonds, in which one or more substances are changed into others.

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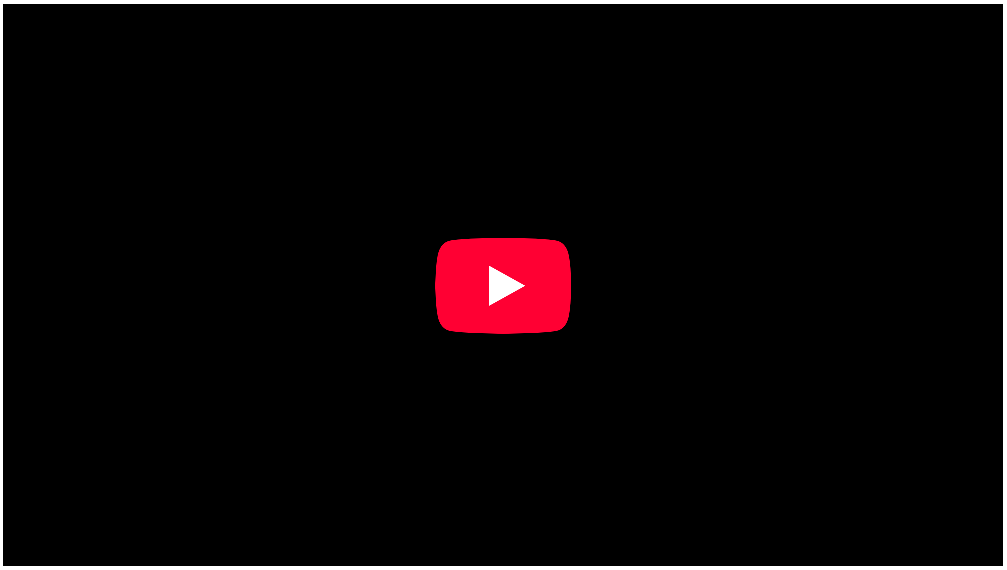
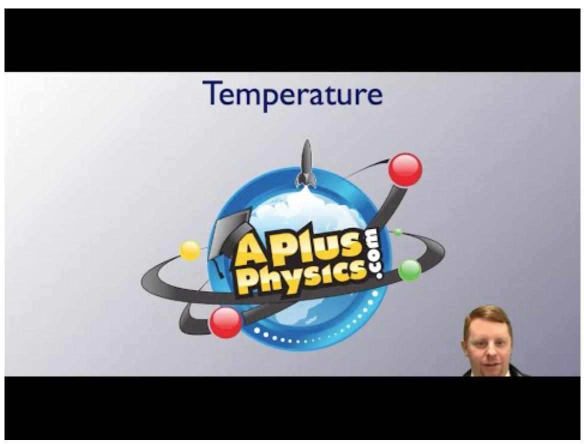
12.3: Temperature and Temperature Scales

learning objectives

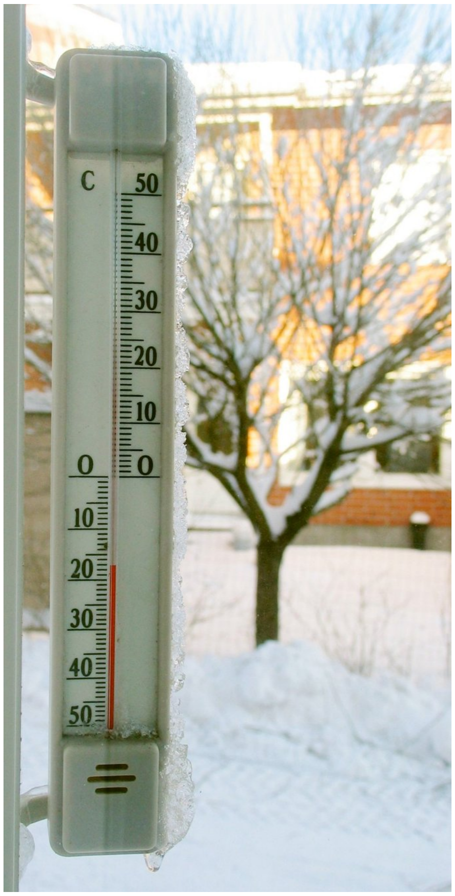
- Explain how the Celsius scale is defined

Celsius, also known as centigrade, is a scale to measure temperature. The unit of measurement is the degree Celsius (°C). It is one of the most commonly used temperature units in the world. The unit system is named after the Swedish astronomer Anders Celsius (1701-1744), who developed a similar temperature scale.



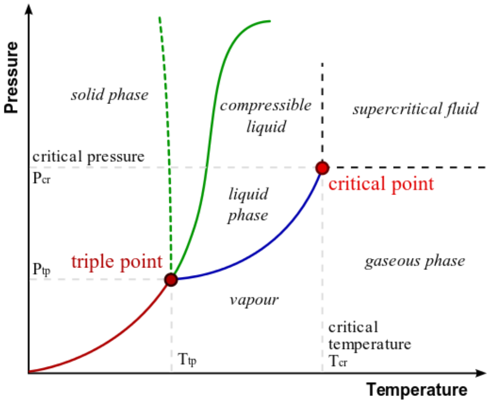


Temperature Scales: A brief introduction to temperature and temperature scales for students studying thermal physics or thermodynamics.



Thermometer: A thermometer calibrated in degrees Celsius

From 1743 until 1954, 0°C was defined as the freezing point of water, and 100°C was defined as the boiling point of water, both at a pressure of one standard atmosphere, with mercury as the working material. Although these defining correlations are commonly taught in schools today, by international agreement the unit "degree Celsius" and the Celsius scale are currently defined by two different temperatures: absolute zero and the triple point of Vienna Standard Mean Ocean Water (VSMOW; specially purified water). This definition also precisely relates the Celsius scale to the Kelvin scale, which defines the SI base unit of thermodynamic temperature and which uses the symbol K. Absolute zero, the lowest temperature possible (the temperature at which matter reaches minimum entropy), is defined as being precisely 0K and -273.15°C. The temperature of the triple point of water is defined as precisely 273.16K and 0.01°C. Based on this, the relationship between degree Celsius and Kelvin is as follows:



Phase Diagram of Water: In this typical phase diagram of water, the green lines mark the freezing point, and the blue line marks the boiling point, showing how they vary with pressure. The dotted line illustrates the anomalous behavior of water. Note that water changes states based on the pressure and temperature.

$$T_{\text{Celsius}} = T_{\text{Kelvin}} - 273.15. \tag{12.3.1}$$

Besides expressing specific temperatures along its scale (e.g., “Gallium melts at 29.7646°C” and “The temperature outside is 23 degrees Celsius”), the degree Celsius is also suitable for expressing temperature intervals — differences between temperatures, or their uncertainties (e.g. “The output of the heat exchanger is hotter by 40 degrees Celsius” and “Our standard uncertainty is ±3°C”). Because of this dual usage, one must not rely upon the unit name or its symbol to denote that a quantity is a temperature interval; it must be clear through context or explicit statement that the quantity is an interval.

Farenheit Scale

In the Fahrenheit scale, the freezing of water is defined at 32 degrees, while the boiling point of water is defined to be 212 degrees.

learning objectives

- Explain how the Fahrenheit scale is defined and convert between it and Celsius

The Fahrenheit scale measures temperature. It is based on a scale proposed in 1724 by physicist Daniel Gabriel Fahrenheit (1686-1736). The unit of this scale is the degree Fahrenheit (°F). On this scale, water’s freezing point is defined to be 32 degrees, while water’s boiling point is defined to be 212 degrees.

Historically, the zero point of the Fahrenheit scale was determined by evaluating a thermometer placed in brine. Fahrenheit himself used a mixture of ice, water, and ammonium chloride (a salt) at a 1:1:1 ratio. This is a frigorific mixture, which stabilizes its temperature automatically; the stable temperature of this mixture was defined as 0 °F (-17.78 °C). The second determining point, 32 degrees, was a mixture of just ice and water at a 1:1 ratio. The third determining point, 96 degrees, was approximately the temperature of the human body, then called “blood-heat. ”

The Fahrenheit system puts the boiling and freezing points of water exactly 180 degrees apart. Therefore, a degree on the Fahrenheit scale is 1/180 of the interval between the freezing point and the boiling point. On the Celsius scale, the freezing and boiling points of water are 100 degrees apart. A temperature interval of 1 °F is equal to an interval of 5/9 degrees Celsius (°C). To convert °F to °C, you can use the following formula:

$$T_{\text{Celsius}} = \frac{5}{9}(T_{\text{Fahrenheit}} - 32) \tag{12.3.2}$$

The Fahrenheit and Celsius scales intersect at -40° (-40 °F and -40 °C represent the same temperature). Absolute zero (-273.15 °C, or 0K) is defined as -459.67 °F.

The Fahrenheit scale was replaced by the Celsius scale in most countries in the mid- to late-20th century, though Canada retains it as a supplementary scale that can be used alongside the Celsius scale. The Fahrenheit scale remains the official scale of the United States, the Cayman Islands, Palau, the Bahamas, and Belize.

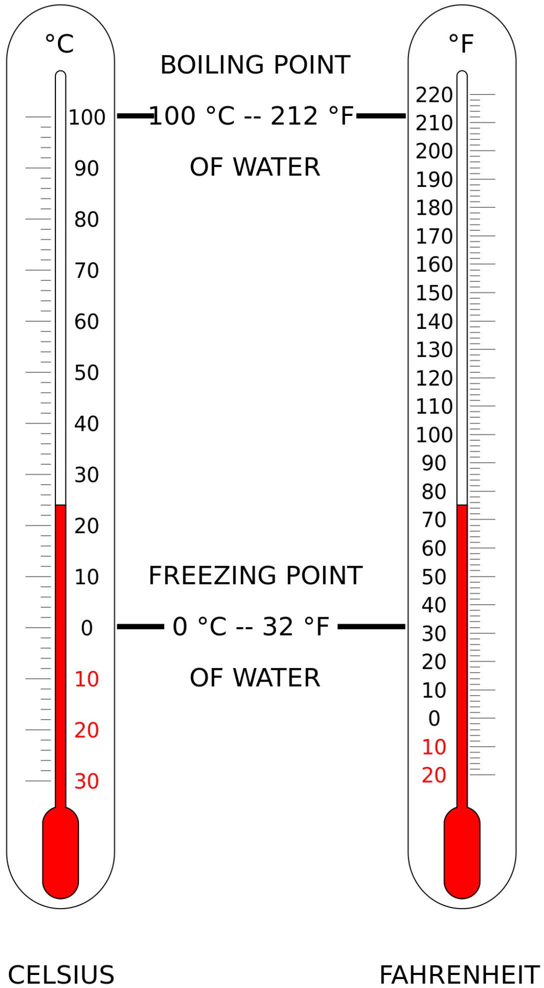
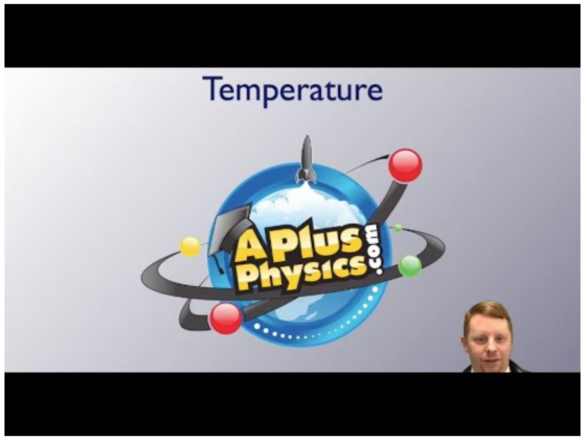


Fig 2: Comparison of Celsius vs Fahrenheit scales.



Temperature Scales: A brief introduction to temperature and temperature scales for students studying thermal physics or thermodynamics.

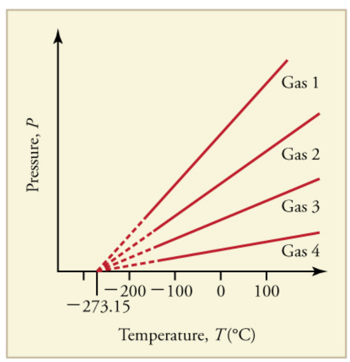
Absolute Zero

Absolute zero is the coldest possible temperature; formally, it is the temperature at which entropy reaches its minimum value.

learning objectives

- Explain why absolute zero is a natural choice as the null point for a temperature unit system

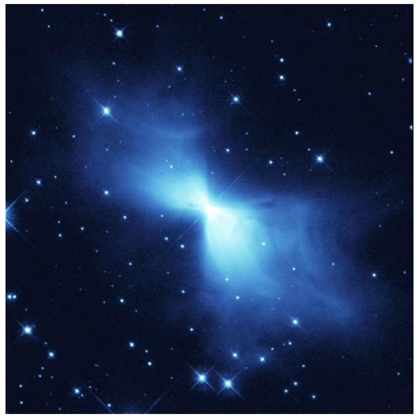
Absolute zerois the coldest possible temperature. Formally, it is the temperature at which entropy reaches its minimum value. More simply put, absolute zero refers to a state in which all the energy of a system is extracted (by definition, the lowest energy state the system can have). Absolute zero is universal in the sense that all matteris in ground state at this temperature. Therefore, it is a natural choice as the null point for a temperature unit system.



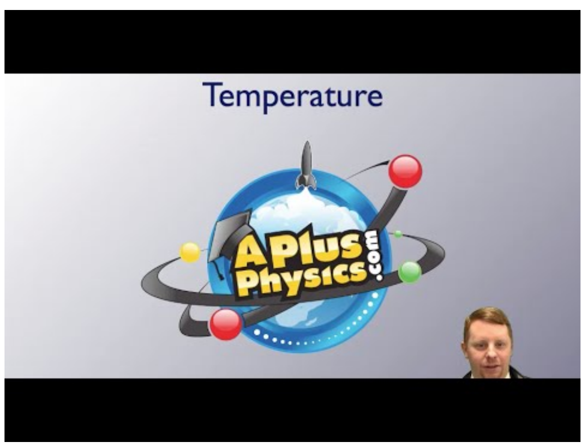
Graph of Pressure Versus Temperature: Graph of pressure versus temperature for various gases kept at a constant volume. Note that all of the graphs extrapolate to zero pressure at the same temperature

To be precise, a system at absolute zero still possesses quantum mechanical zero-point energy, the energy of its ground state. The uncertainty principle states that the position of a particle cannot be determined with absolute precision; therefore a particle is in motion even if it is at absolute zero, and a ground state still carries a minimal amount of kinetic energy. However, in the interpretation of classical thermodynamics, kinetic energy can be zero, and the thermal energy of matter vanishes.

The zero point of a thermodynamic temperature scale, such as the Kelvin scale, is set at absolute zero. By international agreement, absolute zero is defined as 0K on the Kelvin scale and as -273.15° on the Celsius scale (equivalent to -459.67° on the Fahrenheit scale). Scientists have brought systems to temperatures very close to absolute zero, at which point matter exhibits quantum effects such as superconductivity and superfluidity. The lowest temperature that has been achieved in the laboratory is in the 100 pK range, where pK (pico-Kelvin) is equivalent to 10^{-12} K. The lowest natural temperature ever recorded is approximately 1K, observed in the rapid expansion of gases leaving the Boomerang Nebula, shown below.



Boomerang Nebula: The rapid expansion of gases resulting in the Boomerang Nebula causes the lowest observed temperature outside a laboratory.



Temperature Scales: A brief introduction to temperature and temperature scales for students studying thermal physics or thermodynamics.

Kelvin Scale

The kelvin is a unit of measurement for temperature; the null point of the Kelvin scale is absolute zero, the lowest possible temperature.

learning objectives

- Explain how the Kelvin scale is defined

The kelvin is a unit of measurement for temperature. It is one of the seven base units in the International System of Units (SI) and is assigned the unit symbol K. The Kelvin scale is an absolute, thermodynamic temperature scale using absolute zero as its null point. In the classical description of thermodynamics, absolute zero is the temperature at which all thermal motion ceases.

The choice of absolute zero as null point for the Kelvin scale is logical. Different types of matter boil or freeze at different temperatures, but at 0K (absolute zero), *all* thermal motions of *any* matter are maximally suppressed. The Kelvin scale is used extensively in scientific work because a number of physical quantities, such as the volume of an ideal gas, are directly related to absolute temperature.

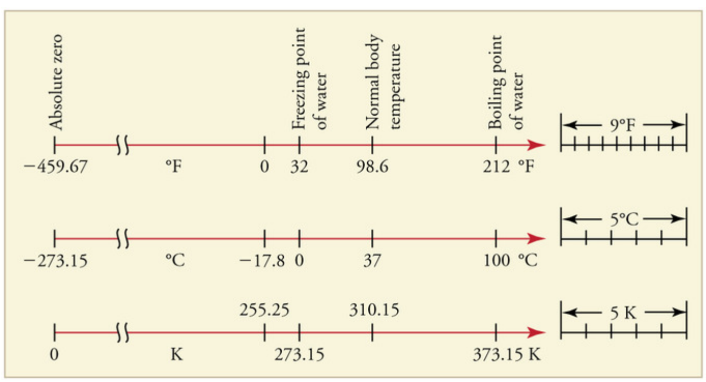
The Kelvin scale is named after Glasgow University engineer and physicist William Thomson, 1st Baron Kelvin (1824-1907), who wrote of the need for an “absolute thermometric scale. ” Unlike the degree Fahrenheit and the degree Celsius, the kelvin is not referred to or typeset as a degree. The kelvin is the primary unit of measurement in the physical sciences, but it is often used in conjunction with the degree Celsius, which has the same magnitude. The kelvin is defined as the fraction 1/273.16 of the thermodynamic temperature of the triple point of water (exactly 0.01°C, or 32.018°F). To convert kelvin to degrees Celsius, we use the following formula:

$$\mathrm{T_{Celsius}} = \mathrm{T_{Kelvin}} - 273.15 \tag{12.3.3}$$

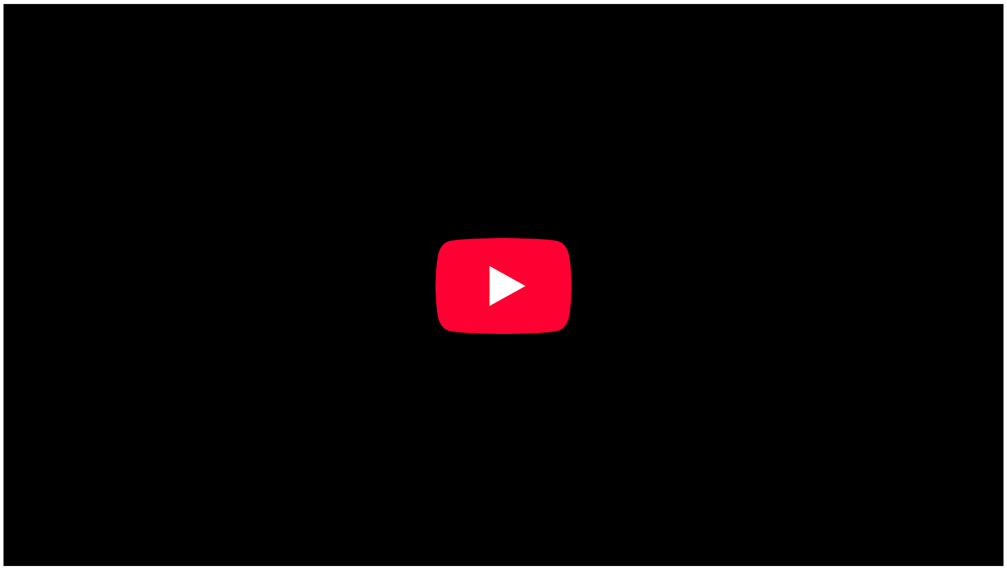
Subtracting 273.16K from the temperature of the triple point of water, 0.01°C, makes absolute zero (0K) equivalent to -273.15°C and -460°F.

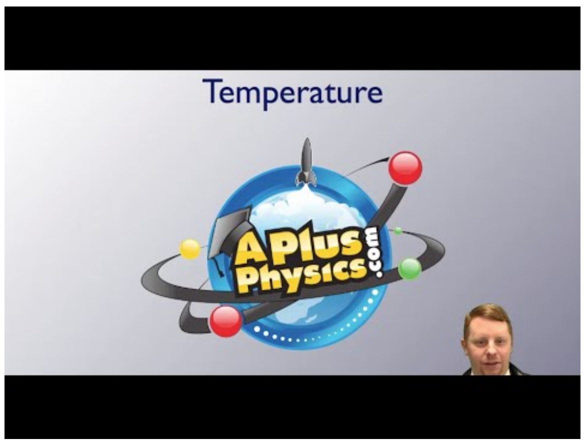
where n_1 = # of scores in group 1
n_2 = # of scores in group 2
R_1 = sum of ranks for group 1
R_2 = sum of ranks for group 2

Calculating U



Relationships Between the Temperature Scales: Relationships between the Fahrenheit, Celsius, and Kelvin temperature scales, rounded to the nearest degree. The relative sizes of the scales are also shown





Temperature Scales: A brief introduction to temperature and temperature scales for students studying thermal physics or thermodynamics.

Key Points

- The degree Celsius (°C) can refer to a specific temperature on the Celsius scale as well as a unit to indicate a temperature interval, a difference between two temperatures or an uncertainty.
- The Celsius scale is currently defined by two different temperatures: absolute zero and the triple point of Vienna Standard Mean Ocean Water (VSMOW; specially purified water).
- Based on this, the relationship between Celsius and Kelvin is as follows: $\mathrm{T_{Celsius}=T_{Kelvin}-273.15}$.
- The Fahrenheit system puts the boiling and freezing points of water exactly 180 degrees apart. Therefore, a degree on the Fahrenheit scale is 1⁄180 of the interval between the freezing point and the boiling point.
- To convert °F to °C, you can use the following formula: $\mathrm{T_{Celsius}=\frac{5}{9}(T_{Fahrenheit}-32)}$. The Fahrenheit and Celsius scales intersect at −40°.
- The Fahrenheit scale was replaced by the Celsius scale in most countries during the mid to late 20th century. Fahrenheit remains the official scale of the United States, Cayman Islands, Palau, Bahamas and Belize.
- Absolute zero is universal in the sense that all matter is in ground state at this temperature. Therefore, it is a natural choice as the null point for a temperature unit system.
- K system at absolute zero still possesses quantum mechanical zero-point energy, the energy of its ground state. However, in the interpretation of classical thermodynamics, kinetic energy can be zero, and the thermal energy of matter vanishes.
- The lowest temperature that has been achieved in the laboratory is in the 100 pK range, where pK (pico- Kelvin) is equivalent to 10-12 K. The lowest natural temperature ever recorded is approximately 1K, observed in the rapid expansion of gases leaving

the Boomerang Nebula.
- 0K (absolute zero) is universal because all thermal motions of all matter are maximally suppressed at this temperature. Absolute zero is therefore the natural choice as the null point of the Kelvin scale.
- The Kelvin scale is used extensively in scientific work because a number of physical quantities, such as the volume of an ideal gas, are directly related to absolute temperature.
- To convert kelvin to degree Celsius, we use the following formula: $T_{Celsius} = T_{Kelvin} - 273.15$.

Key Terms

- **kelvin**: in the International System of Units, the base unit of thermodynamic temperature; 1/273.16 of the thermodynamic temperature of the triple point of water; symbolized as K
- **absolute zero**: The coldest possible temperature: zero on the Kelvin scale and approximately -273.15°C and -459.67°F. The total absence of heat; the temperature at which motion of all molecules would cease.
- **standard atmosphere**: an international reference pressure defined as 101.325 kPa and formerly used as a unit of pressure
- **brine**: a solution of salt (usually sodium chloride) in water
- **frigorific mixture**: A mixture of two or more chemicals that reaches an equilibrium temperature independent of the temperature of any of its constituent chemicals. The temperature is also relatively independent of the quantities of mixtures as long as a significant amount of each original chemical is present in its pure form
- **entropy**: A measure of how evenly energy (or some analogous property) is distributed in a system.
- **thermodynamics**: a branch of natural science concerned with heat and its relation to energy and work
- **absolute zero**: The coldest possible temperature: zero on the Kelvin scale and approximately -273.15°C and -459.67°F. The total absence of heat; the temperature at which motion of all molecules would cease.
- **Triple point**: The unique temperature and pressure at which the solid, liquid and gas phases of a substance are all in equilibrium.
- **ideal gas**: A hypothetical gas whose molecules exhibit no interaction and undergo elastic collision with each other and with the walls of the container.

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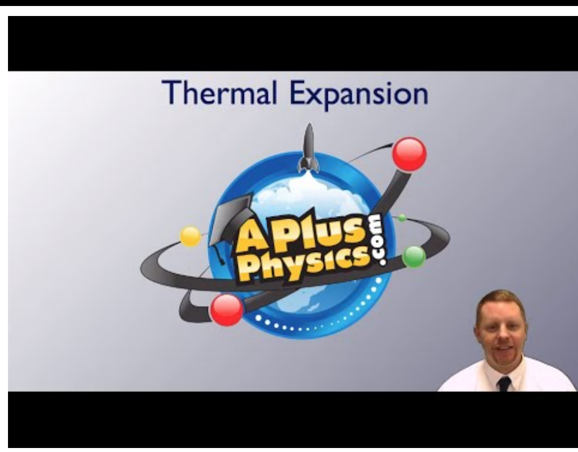
12.4: Thermal Expansion

learning objectives

- Describe volume changes that take place in response to a temperature change

Thermal expansion is the tendency of matter to change in volume in response to a change in temperature. (An example of this is the buckling of railroad track, as seen in.) Atoms and molecules in a solid, for instance, constantly oscillate around its equilibrium point. This kind of excitation is called thermal motion. When a substance is heated, its constituent particles begin moving more, thus maintaining a greater average separation with their neighboring particles. The degree of expansion divided by the change in temperature is called the material's coefficient of thermal expansion; it generally varies with temperature.

Fig 1: Thermal expansion of long continuous sections of rail tracks is the driving force for rail buckling. This phenomenon resulted in 190 train derailments during 1998–2002 in the US alone.



Thermal Expansion: A brief introduction to thermal expansion for students.

Expansion, Not Contraction

Why does matter usually expand when heated? The answer can be found in the shape of the typical particle-particle potential in matter. Particles in solids and liquids constantly feel the presence of other neighboring particles. This interaction can be represented mathematically as a potential curve. Fig 2 illustrates how this inter-particle potential usually takes an asymmetric form rather than a symmetric form, as a function of particle-particle distance. Note that the potential curve is steeper for shorter distance. In the diagram, (b) shows that as the substance is heated, the equilibrium (or average) particle-particle distance increases. Materials which contract or maintain their shape with increasing temperature are rare. This effect is limited in size, and only occurs within limited temperature ranges.

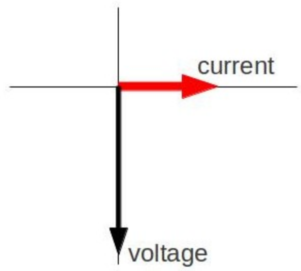


Fig 2: Typical inter-particle potential in condensed matter (such as solid or liquid).

Linear Expansion

To a first approximation, the change in length measurements of an object (*linear*dimension as opposed to, for example, volumetric dimension) due to thermal expansion is related to temperature change by a *linear expansion coefficient*. It is the fractional change in length per degree of temperature change. Assuming negligible effect of pressure, we may write:

$$\alpha_{\mathrm{L}} = \frac{1}{\mathrm{L}} \frac{\mathrm{dL}}{\mathrm{dT}}, \tag{12.4.1}$$

where L is a particular length measurement and $\frac{\mathrm{dL}}{\mathrm{dT}}$ is the rate of change of that linear dimension per unit change in temperature. From the definition of the expansion coefficient, the change in the linear dimension $\Delta \mathrm{L}$ over a temperature range $\Delta \mathrm{T}$ can be estimated to be:

$$\frac{\Delta \mathrm{L}}{\mathrm{L}} = \alpha_{\mathrm{L}} \Delta \mathrm{T}. \tag{12.4.2}$$

This equation works well as long as the linear-expansion coefficient does not change much over the change in temperature. If it does, the equation must be integrated.

Area Expansion

Objects expand in all dimensions. That is, their areas and volumes, as well as their lengths, increase with temperature.

learning objectives

- Express the area thermal expansion coefficient in the form of an equation

We learned about the linear expansion (in one dimension) in the previous Atom. Objects expand in all dimensions, and we can extend the thermal expansion for 1D to two (or three) dimensions. That is, their areas and volumes, as well as their lengths, increase with temperature.

Quiz

Before we look into details, here is an interesting question. Imagine that we have a rectangular sheet of metal with a circular hole in the middle. If the metal is heated, we can guess that the the piece, in general, will get larger due to thermal expansion. Now, what is going to happen with the circular hole in the middle? Is the hole going to be larger or smaller? Answer: Imagine that we have a similar metal sheet but without a hole. Draw an imaginary circular line representing the circular hole in our quiz. How does this imaginary circle change as the metal is heated? Yes. It will get bigger. Therefore, you can guess that the hole in our quiz will get larger.

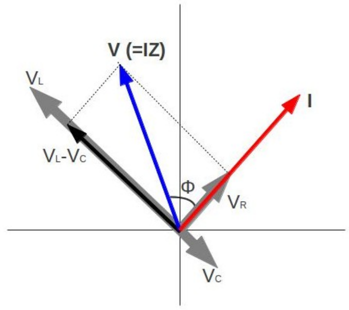


Fig 1: In general, objects expand in all directions as temperature increases. In these drawings, the original boundaries of the objects are shown with solid lines, and the expanded boundaries with dashed lines. (a) Area increases because both length and width increase. The area of a circular plug also increases. (b) If the plug is removed, the hole it leaves becomes larger with increasing temperature, just as if the expanding plug were still in place.

Area thermal expansion coefficient

The area thermal expansion coefficient relates the change in a material's area dimensions to a change in temperature. It is the fractional change in area per degree of temperature change. Ignoring pressure, we may write: $\alpha_A = \frac{1}{A}\frac{dA}{dT}$, where is some area of interest on the object, and $\frac{dA}{dT}$ is the rate of change of that area per unit change in temperature. The change in the linear dimension can be estimated as: $\frac{\Delta A}{A} = \alpha_A \Delta T$. This equation works well as long as the linear expansion coefficient does not change much over the change in temperature ΔT. If it does, the equation must be integrated.

Relationship to linear thermal expansion coefficient

For isotropic materials, and for small expansions, the linear thermal expansion coefficient is one half of the area coefficient. To derive the relationship, let's take a square of steel that has sides of length *L*. The original area will be $A = L^2$,and the new area, after a temperature increase, will be

$$A + \Delta A = (L + \Delta L)^2 \tag{12.4.3}$$

$$= L^2 + 2L\Delta L + (\Delta L)^2 \tag{12.4.4}$$

$$\approx L^2 + 2L\Delta L \tag{12.4.5}$$

$$= A + 2A\frac{\Delta L}{L} \tag{12.4.6}$$

The approximation holds for a sufficiently small ΔL campared to L. Since $\frac{\Delta A}{A} = 2\frac{\Delta L}{L}$ from the equation above (and from the definitions of the thermal coefficients), we get $\alpha_A = 2_{\alpha L}$.

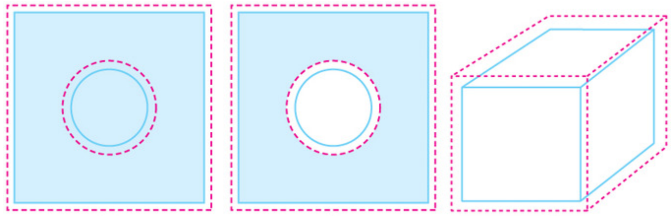
Volume Expansion

Substances expand or contract when their temperature changes, with expansion or contraction occurring in all directions.

learning objectives

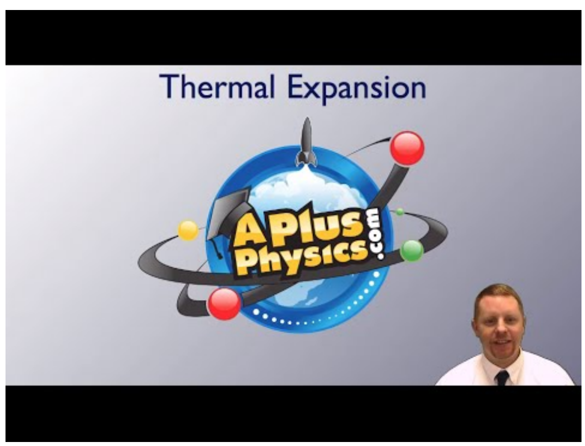
- Compare the effects of the pressure on the expansion of gaseous and solid materials

The volumetric thermal expansion coefficient is the most basic thermal expansion coefficient. illustrates that, in general, substances expand or contract when their temperature changes, with expansion or contraction occurring in all directions. Such substances that expand in all directions are called isotropic. For isotropic materials, the area and linear coefficients may be calculated from the volumetric coefficient (discussed below).



(a) (b) (c)

Volumetric Expansion: In general, objects expand in all directions as temperature increases. In these drawings, the original boundaries of the objects are shown with solid lines, and the expanded boundaries with dashed lines. (a) Area increases because both length and width increase. The area of a circular plug also increases. (b) If the plug is removed, the hole it leaves becomes larger with increasing temperature, just as if the expanding plug were still in place. (c) Volume also increases, because all three dimensions increase.



Thermal Expansion – Volume Expansion: A brief introduction to thermal expansion for students.

Mathematical definitions of these coefficients are defined below for solids, liquids, and gasses:

$$\alpha_V = \frac{1}{V}\left(\frac{\partial V}{\partial T}\right)p. \tag{12.4.7}$$

The subscript *p* indicates that the pressure is held constant during the expansion. In the case of a gas, the fact that the pressure is held constant is important, as the volume of a gas will vary appreciably with pressure as well as with temperature.

For a solid, we can ignore the effects of pressure on the material, thus the volumetric thermal expansion coefficient can be written:

$$\alpha_V = \frac{1}{V}\frac{dV}{dT}, \tag{12.4.8}$$

where V is the volume of the material, and is dV/dT the rate of change of that volume with temperature. This means that the volume of a material changes by some fixed fractional amount. For example, a steel block with a volume of 1 cubic meter might expand to 1.002 cubic meters when the temperature is raised by 50 °C. This is an expansion of 0.2%. The volumetric expansion coefficient would be 0.2% for 50 °C, or 0.004% per degree C.

Relationship to Linear Thermal Expansion Coefficient

For isotropic material, and for small expansions, the linear thermal expansion coefficient is one third the volumetric coefficient. To derive the relationship, let's take a cube of steel that has sides of length L. The original volume will be $V = L^3$,and the new volume, after a temperature increase, will be:

$$V + \Delta V = (L + \Delta L)^3 \tag{12.4.9}$$

$$= L^3 + 3L^2\Delta L + 3L(\Delta L)^2 + (\Delta L)^3 \tag{12.4.10}$$

$$\approx L^3 + 3L^2\Delta L \tag{12.4.11}$$

$$= V + 3V\frac{\Delta L}{L}. \tag{12.4.12}$$

The approximation holds for a sufficiently small ΔL compared to L. Since:

$$\frac{\Delta V}{V} = 3\frac{\Delta L}{L} \tag{12.4.13}$$

(and from the definitions of the thermal coefficients), we arrive at:

$$\alpha_V = 3\alpha_L \tag{12.4.14}$$

Special Properties of Water

Objects will expand with increasing temperature, but water is the most important exception to the general rule.

learning objectives

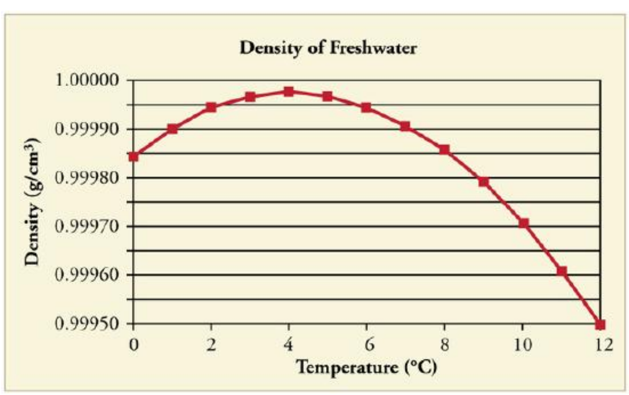
- Describe thermal expansion properties of water

Special Properties of Water

In general, objects will expand with increasing temperature. However, a number of materials contract on heating within certain temperature ranges; this is usually called negative thermal expansion, rather than "thermal contraction. " Water is the most important exception to the general rule. Water has this unique characteristic because of the particular nature of the hydrogen bond in H_2O.

Density of Water as Temperature Changes

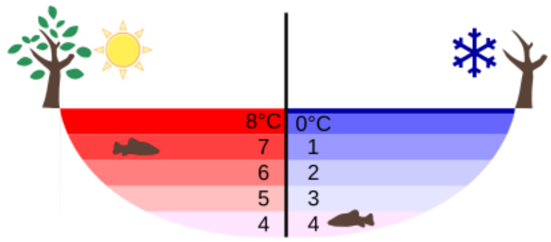
At temperatures greater than 4ºC (40ºF) water expands with increasing temperature (its density decreases). However, it expands with decreasing temperature when it is between +4ºC and 0ºC (40ºF to 32ºF). Water is densest at +4ºC.



Water Density vs. Temperature: The density of water as a function of temperature. Note that the thermal expansion is actually very small. The maximum density at +4ºC is only 0.0075% greater than the density at 2ºC, and 0.012% greater than that at 0ºC.

Perhaps the most striking effect of this phenomenon is the freezing of water in a pond. When water near the surface cools down to 4ºC it is denser than the remaining water and thus will sink to the bottom. This “turnover” results in a layer of warmer water near the surface, which is then cooled. Eventually the pond has a uniform temperature of 4ºC. If the temperature in the surface layer drops below 4ºC, the water is less dense than the water below, and thus stays near the top.

As a result, the pond surface can completely freeze over, while the bottom may remain at 4ºC. The ice on top of liquid water provides an insulating layer from winter’s harsh exterior air temperatures. Fish and other aquatic life can survive in 4ºC water beneath ice, due to this unusual characteristic of water. It also produces circulation of water in the pond that is necessary for a healthy ecosystem of the body of water.



Temperature in a Lake: Temperature distribution in a lake on warm and cold days in winter

Ice Versus Water

The solid form of most substances is denser than the liquid phase; thus, a block of most solids will sink in the liquid. However, a block of ice floats in liquid water because ice is less dense. Upon freezing, the density of water decreases by about 9%.

Key Points

- Inter-particle potential usually takes an asymmetric form, rather than a symmetric form as a function of particle-particle distance. This is why matters expands and contracts as temperature changes.
- The change in length measurements of an object due to thermal expansion is related to temperature change by a “linear expansion coefficient”, which is given as $\alpha_{\mathrm{L}}=\frac{1}{\mathrm{L}} \frac{\mathrm{dL}}{\mathrm{dT}}$.
- The linear expansion coefficient is as an approximation over a narrow temperature interval only.
- The area thermal expansion coefficient relates the change in a material’s area dimensions to a change in temperature. It is defined as $\alpha_{\mathrm{A}}=\frac{1}{\mathrm{A}} \frac{\mathrm{dA}}{\mathrm{dT}}$.
- The relationship between the area and linear thermal expansion coefficient is given as the following: $\alpha_{\mathrm{A}}=2 \alpha_{\mathrm{L}}$.
- Just like the linear expansion coefficient, the area thermal expansion coefficient works as an approximation over a narrow temperature interval only.
- Substances that expand at the same rate in every direction are called isotropic.
- In the case of a gas, expansion depends on how the pressure changed in the process because the volume of a gas will vary appreciably with pressure as well as temperature.
- For a solid, we can ignore the effects of pressure on the material, and the volumetric thermal expansion coefficient can be written as $\alpha_{\mathrm{V}}=\frac{1}{\mathrm{V}} \frac{\mathrm{dV}}{\mathrm{dT}}$. For isotropic materials, $\alpha_{\mathrm{V}}=3 \alpha_{\mathrm{L}}$.
- Water expands with increasing temperature (its density decreases) when it is at temperatures greater than 4ºC (40ºF). However, it expands with decreasing temperature when it is between +4ºC and 0ºC (40ºF to 32ºF). Water is densest at +4ºC.
- Due to the peculiar thermal expansion property of water, a pond surface can completely freeze over, while the bottom may remain at 4ºC. Fish and other aquatic life can survive in 4ºC water beneath ice, due to this unusual characteristic of water.
- The solid form of most substances is denser than the liquid phase; thus, a block of most solids will sink in the liquid. However, a block of ice floats in liquid water because ice is less dense.

Key Terms

- **potential**: A curve describing the situation where the difference in the potential energies of an object in two different positions depends only on those positions.
- **linear thermal expansion coefficient**: The fractional change in length per degree of temperature change.
- **isotropic**: Having properties that are identical in all directions; exhibiting isotropy.
- **hydrogen bond**: A weak bond in which a hydrogen atom in one molecule is attracted to an electronegative atom (usually nitrogen or oxygen) in the same or different molecule.

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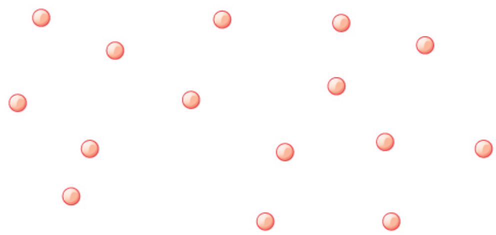
12.5: Ideal Gas Law

learning objectives

- Describe how ideal gas law was derived.

The ideal gas law is the equation of state of a hypothetical ideal gas (an illustration is offered in). In an ideal gas, there is no molecule-molecule interaction, and only elastic collisions are allowed. It is a good approximation to the behavior of many gases under many conditions, although it has several limitations. It was first stated by Émile Clapeyron in 1834 as a combination of Boyle's law and Charles' law.



Atoms and Modules in a Gas: Atoms and molecules in a gas are typically widely separated, as shown. Because the forces between them are quite weak at these distances, they are often described by the ideal gas law.

Empirical Derivation

Boyle's law states that pressure P and volume V of a given mass of confined gas are inversely proportional:

$$P \propto \frac{1}{V}, \tag{12.5.1}$$

while Charles' law states that volume of a gas is proportional to the absolute temperature T of the gas at constant pressure

$$V \propto T. \tag{12.5.2}$$

By combining the two laws, we get

$$\frac{PV}{T} = C, \tag{12.5.3}$$

where C is a constant which is directly proportional to the amount of gas, n (representing the number of moles).

The proportionality factor is the universal gas constant, R, i.e. $C = nR$.

Hence the ideal gas law

$$PV = nRT \tag{12.5.4}$$

Equivalently, it can be written as $PV = NkT$,

where k is Boltzmann's constant and N is the number of molecules.

(Since N = nN_A, you can see that $R = N_{Ak}$, where N_A is Avogadro's number.)

Note that the empirical derivation does not consider microscopic details. However, the equation can be derived from first principles in the classical thermodynamics (which goes beyond the scope of this Atom).

Microscopic version

We have seen in the Atom on "Origin of Pressure" that

$$P = \frac{Nm\bar{v^2}}{3V}, \tag{12.5.5}$$

where P is the pressure, N is the number of molecules, m is the mass of the molecule, v is the speed of molecules, and V is the volume of the gas. Therefore, we derive a microscopic version of the ideal gas law

$$PV = \frac{1}{3} Nm\bar{v}^2 \tag{12.5.6}$$

Isotherms

An isothermal process is a change of a system in which the temperature remains constant: $\Delta T = 0$.

learning objectives

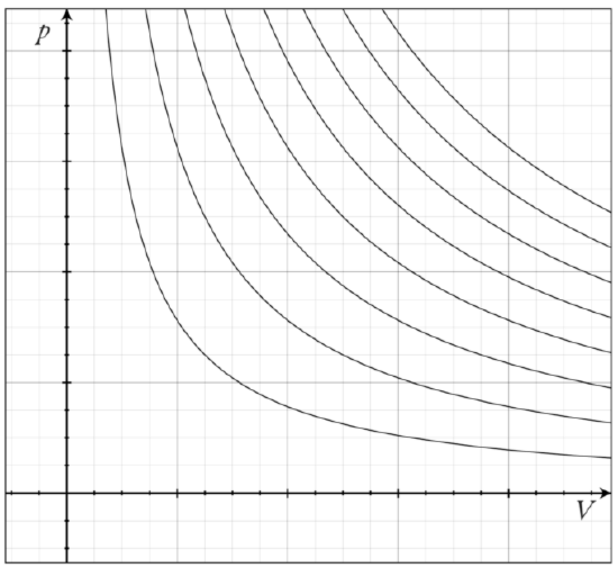
- Identify conditions at which isothermal processes can occur.

An isothermal process is a change of a system in which the temperature remains constant: $\Delta T = 0$. Typically this occurs when a system is in contact with an outside thermal reservoir (heat bath), and the change occurs slowly enough to allow the system to adjust continually to the temperature of the reservoir through heat exchange. In contrast, an adiabatic process occurs when a system exchanges no heat with its surroundings ($Q = 0$). In other words, in an isothermal process, the value $\Delta T = 0$ but $Q \neq 0$, while in an adiabatic process, $\Delta T \neq 0$ but $Q = 0$.

For an ideal gas, the product PV (P: pressure, V: volume) is a constant if the gas is kept at isothermal conditions (Boyle's law). According to the ideal gas law, the value of the constant is NkT, where N is the number of molecules of gas and k is Boltzmann's constant.

This means that $p = \frac{NkT}{V} = \frac{\text{Constant}}{V}$ holds.

The family of curves generated by this equation is shown in the graph presented in. Each curve is called an isotherm. Such graphs are termed indicator diagrams—first used by James Watt and others to monitor the efficiency of engines. The temperature corresponding to each curve in the figure increases from the lower left to the upper right.



Isotherms of an Ideal Gas: Several isotherms of an ideal gas on a PV diagram.

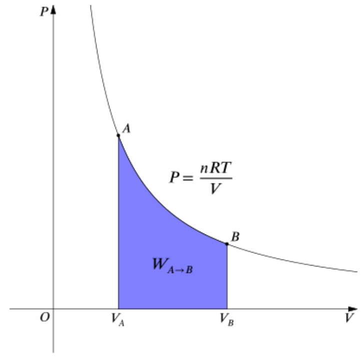
Calculation of Work

In thermodynamics, the work involved when a gas changes from state A to state B is simply:

$$W_{A \to B} = \int_{V_A}^{V_B} PdV. \tag{12.5.7}$$

(This equation is derived in our Atom on "Constant Pressure" under kinetic theory. Note that $P = \frac{F}{A}$. This definition is consistent with our definition of work being force times distance.)

For an isothermal, reversible process, this integral equals the area under the relevant pressure-volume isotherm, and is indicated in blue in for an ideal gas. Again, $P = \frac{nRT}{V}$ applies and with T being constant (as this is an isothermal process), we have:



Work Done by Gas During Expansion: The blue area represents "work" done by the gas during expansion for this isothermal change.

$$W_{A \to B} = \int_{V_A}^{V_B} pdV = \int_{V_A}^{V_B} \frac{NkT}{V} dV \tag{12.5.8}$$

$$= NkT \ln \frac{V_B}{V_A}. \tag{12.5.9}$$

By convention, work is defined as the work the system does on its environment. If, for example, the system expands by a piston moving in the direction of force applied by the internal pressure of a gas, then the work is counted as positive. As this work is done by using internal energy of the system, the result is that the internal energy decreases. Conversely, if the environment does work on the system so that its internal energy increases, the work is counted as negative (for details on internal energy, check our Atom on "Internal Energy of an Ideal Gas").

Constant Pressure

Isobaric processis a thermodynamic process in which the pressure stays constant (at constant pressure, work done by a gas is $P\Delta V$).

learning objectives

- Describe behavior of monatomic gas during isobaric processes.

Under a certain constraint (e.g., pressure), gases can expand or contract; depending on the type of constraint, the final state of the gas may change. For example, an ideal gas that expands while its temperature is kept constant (called isothermal process) will exist in a different state than a gas that expands while pressure stays constant (called isobaric process). This Atom addresses isobaric process and correlated terms. We will discuss isothermal process in a subsequent Atom.

Isobaric Process

An isobaric process is a thermodynamic process in which pressure stays constant: $\Delta P = 0$. For an ideal gas, this means the volume of a gas is proportional to its temperature (historically, this is called Charles' law). Let's consider a case in which a gas does work on a piston at constant pressure P, referring to Fig 1 as illustration. Since the pressure is constant, the force exerted is constant and the work done is given as W=Fd, where F (=PA) is the force on the piston applied by the pressure and d is the displacement of the piston. Therefore, the work done by the gas (W) is:

$$W = PAd \tag{12.5.10}$$

Because the change in volume of a cylinder is its cross-sectional area A times the displacement d, we see that Ad=ΔV, the change in volume. Thus,

$$W = P\Delta V \tag{12.5.11}$$

(as seen in Fig 2—isobaric process). Note: if ΔV is positive, then W is positive, meaning that work is done by the gas on the outside world. Using the ideal gas law $PV = NkT(P = \text{const})$,

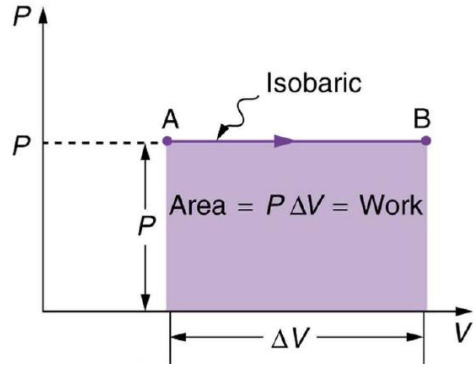


Fig 2: A graph of pressure versus volume for a constant-pressure, or isobaric process. The area under the curve equals the work done by the gas, since W=PΔV.

$$W = Nk\Delta T \tag{12.5.12}$$

(Eq. 1) for an ideal gas undergoing an isobaric process.

Monatomic Gas

According to the first law of thermodynamics,

$$Q = \Delta U + W \tag{12.5.13}$$

(Eq. 2), where W is work done by the system, U is internal energy, and Q is heat. The law says that the heat transferred to the system does work but also changes the internal energy of the system. Since,

$U = \frac{3}{2}NkT$ for a monatomic gas, we get $\Delta U = \frac{3}{2}Nk\Delta T$

(Eq. 3; for the details on internal energy, see our Atom on "Internal Energy of an Ideal Gas"). By using the Equations 1 and 3, Eq. 2 can be written as:

$Q = \frac{5}{2}Nk\Delta T$ for monatomic gas in an isobaric process.

Specific Heat

Specific heat at constant pressure is defined by the following equation:

$Q = ncP\Delta T$

Here n is the amount of particles in a gas represented in moles. By noting that $N = N_A n$ and $R = kN_A$ (N_A: Avogadro's number, R: universal gas constant), we derive:

$c_P = \frac{5}{2}kN_A = \frac{5}{2}R$ for a monatomic gas.

Problem Solving

With the ideal gas law we can figure pressure, volume or temperature, and the number of moles of gases under ideal thermodynamic conditions.

learning objectives

- Identify steps used to solve the ideal gas equation.

The Ideal Gas Law is the equation of state of a hypothetical ideal gas. It is a good approximation to the behavior of many gases under many conditions, although it has several limitations. It is most accurate for monatomic gases at high temperatures and low pressures.

The ideal gas law has the form:

$$PV = nRT, \tag{12.5.14}$$

where R is the universal gas constant, and with it we can find values of the pressure P, volume V, temperature T, or number of moles n under a certain ideal thermodynamic condition. Typically, you are given enough parameters to calculate the unknown. Variations of the ideal gas equation may help solving the problem easily. Here are some general tips.

The ideal gas law can also come in the form:

$$\mathrm{PV} = \mathrm{NkT}, \tag{12.5.15}$$

where N is the number of particles in the gas and k is the Boltzmann constant.

To solve the ideal gas equation:

1. Write down all the information that you know about the gas.
2. If necessary, convert the known values to SI units.
3. Choose a relevant gas law equation that will allow you to calculate the unknown variable.
4. Substitute the known values into the equation. Calculate the unknown variable.

Remember that the general gas equation only applies if the molar quantity of the gas is fixed. For example, if a gas is mixed with another gas, you may have to apply the equation separately for individual gases.

Example

Let's imagine that at the beginning of a journey a truck tire has a volume of 30,000 cm^3 and an internal pressure of 170 kPa. The temperature of the tire is 16°C. By the end of the trip, the volume of the tire has increased to 32,000 cm^3 and the temperature of the air inside the tire is 40°C. What is the tire pressure at the end of the journey?

Tire Pressure: Tire pressure may change significantly during the operation of the vehicle. This is mostly due to the temperature change of the air in tires.

Solution:

Step 1. Write down all the information that you know about the gas: P_1= 170 kPa and P_2 is unknown. V_1 = 30,000 cm^3 and V_2 = 32,000 cm^3. T_1 = 16°C and T_2 = 40°C.

Step 2. Convert the known values to SI units if necessary: Here, temperature must be converted into Kelvin. Therefore, T_1 = 16 + 273 = 289 K, T_2 = 40 + 273 = 313 K

Step 3. Choose a relevant gas law equation that will allow you to calculate the unknown variable: We can use the general gas equation to solve this problem: $\frac{\mathrm{P_1 V_1}}{\mathrm{T_1}} = \frac{\mathrm{P_2 V_2}}{\mathrm{T_2}}$.

Therefore, $\mathrm{P_2} = \frac{\mathrm{P_1 \times V_1 \times T_2}}{\mathrm{T_1 \times V_2}}$.

Step 4. Substitute the known values into the equation. Calculate the unknown variable:

$$P_2 = \frac{170 \times 30,000 \times 313}{289 \times 32,000} = 173 \text{ kPa} \tag{12.5.16}$$

The pressure of the tire at the end of the journey is 173 kPa.

Note that in Step 2 we did not bother to convert the volume values to m^3. In Step 4, pressure appears both in the numerator and denominator. In this case the conversion was not necessary.

Avogador’s Number

The number of molecules in a mole is called Avogadro’s number (N_A)—defined as 6.02x 10^{23} mol^{-1}.

learning objectives

- Explain relationship between Avogadro’s number and mole.

When measuring the amount of substance, it is sometimes easier to work with a unit other than the number of molecules. A mole (abbreviated mol) is a base unit in the International System of Units (SI). It is defined as any substance containing as many atoms or molecules as there are in exactly 12 grams (0.012 kg) of carbon-12. The actual number of atoms or molecules in one mole is called *Avogadro’s constant (N_A)*, in recognition of Italian scientist Amedeo Avogadro.

Amadeo Avogadro: Amedeo Avogadro (1776–1856). He established the relationship between the masses of the same volume of different gases (at the same temperature and pressure) corresponds to the relationship between their respective molecular weights.

Avogadro’s number (N) refers to the number of molecules in one gram-molecule of oxygen. This indicates an amount of substance as opposed to an independent dimension of measurement. In 1811 Amedeo Avogadro first proposed that the volume of a gas (at a given pressure and temperature) is proportional to the number of atoms or molecules, regardless of the nature of the gas (i.e., this number is universal and independent of the type of gas). In 1926, Jean Perrin won the Nobel Prize in Physics, largely for his work in determining the Avogadro constant (by several different methods). The value of Avogadro’s constant, N_A , has been found to equal $6.02 \times 10^{23} mol^{-1}$.

Role in Science

Avogadro’s constant is a scaling factor between macroscopic and microscopic (atomic scale) observations of nature. As such, it provides the relation between other physical constants and properties. For example, it establishes a relationship between the gas constant R and the Boltzmann constant k,

$$R = kN_A = 8.314472(15)J\ mol^{-1}K^{-1}; \tag{12.5.17}$$

and the Faraday constant F and the elementary charge e,

$$F = N_A e = 96485.3383(83)C\ mol^{-1}. \tag{12.5.18}$$

Measuring N_A

The determination of N_A is crucial to the calculation of an atom's mass, since the latter is obtained by dividing the mass of a mole of the gas by Avogadro's constant. In his study on Brownian motion in 1905, Albert Einstein proposed that this constant could be determined based on the quantities observable in Brownian motion. Subsequently, Einstein's idea was verified, leading to the first determination of N_A in 1908 through the experimental work of Jean Baptiste Perrin.

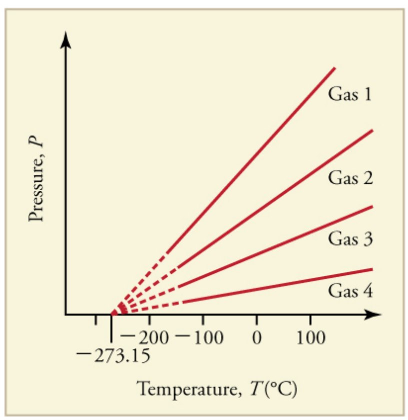
Absolute Temperature

Absolute temperature is the most commoly used thermodyanmic temperature unit and is the standard unit of temperature.

learning objectives

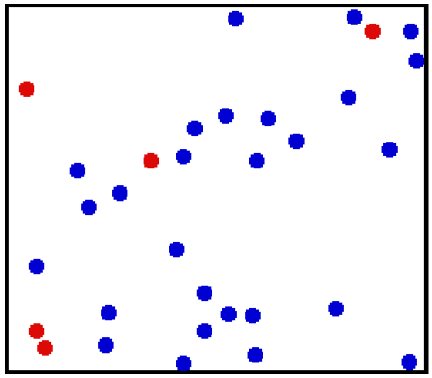
- Describe relationship between absolute temperature and kinetic energy.

Thermodynamic temperature is the absolute measure of temperature. It is one of the principal parameters of thermodynamics and kinetic theory of gases. Thermodynamic temperature is an "absolute" scale because it is the measure of the fundamental property underlying temperature: its null or zero point ("absolute zero") is the temperature at which the particle constituents of matter have minimal motion and cannot become any colder. That is, they have minimal motion, retaining only quantum mechanical motion, as diagramed in.



Graph of Pressure Versus Temperature: Graph of pressure versus temperature for various gases kept at a constant volume. Note that all of the graphs extrapolate to zero pressure at the same temperature

At its simplest, "temperature" arises from the kinetic energy of the random motions of matter's particle constituents such as molecules or atoms, as seen in. Therefore, it is reasonable to choose absolute zero, where all classical motion ceases, as the reference point (T=0) of our temperature system. By using the absolute temperature scale (Kelvin system), which is the most commonly used thermodynamic temperature, we have shown that the average translational kinetic energy (KE) of a particle in a gas has a simple relationship to the temperature:



Translational Motion of Helium: Real gases do not always behave according to the ideal model under certain conditions, such as high pressure. Here, the size of helium atoms relative to their spacing is shown to scale under 1950 atmospheres of pressure.

$$\bar{KE} = \frac{3}{2}kT. \tag{12.5.19}$$

Note that this equation would not look this elegant if the Fahrenheit scale were used instead.

The Kelvin scale

The kelvin (or "absolute temperature") is the standard thermodyanmic temperature unit. It is one of the seven base units in the International System of Units (SI) and is assigned the unit symbol K. By international agreement, the unit kelvin and its scale are defined by two points: absolute zero and the triple point of Vienna Standard Mean Ocean Water (water with a specified blend of hydrogen and oxygen isotopes). Absolute zero, the lowest possible temperature, is defined precisely as 0 K and −273.15 °C. The triple point of water is defined precisely as 273.16 K and 0.01 °C.

Key Points

- Ideal gas law was derived empirically by combining Boyle's law and Charles' law.
- Although the empirical derivation of the equation does not consider microscopic details, the ideal gas law can be derived from first principles in the classical thermodynamics.
- Pressure and volume of a gas can be related to the average velocity of molecues: $PV = \frac{1}{3}Nm\bar{v^2}$.
- Isothermal processes typically occur when a system is in contact with an outside thermal reservoir (heat bath), and the change occurs slowly enough to allow the system to adjust continually to the temperature of the reservoir through heat exchange.
- For an ideal gas, from the ideal gas law $PV = NkT, PV$ remains constant through an isothermal process. A curve in a P-V diagram generated by the equation $PV = const$ is called an isotherm.
- For an isothermal, reversible process, the work done by the gas is equal to the area under the relevant pressure -volume isotherm. It is given as $W_A \rightarrow B = NkT \ln \frac{V_B}{V_A}$.
- Gases can expand or contract under a certain constraint. Depending on the constraint, the final state of the gas may change.
- The heat transferred to the system does work but also changes the internal energy of the system. In an isobaric process for a monatomic gas, heat and the temperature change satisfy the following equation: $Q = \frac{5}{2}Nk\Delta T$.
- For a monatomic ideal gas, specific heat at constant pressure is $\frac{5}{2}R$.
- Write down all the information that you know about the gas and convert the known values to SI units if necessary.
- Choose a relevant gas law equation that will allow you to calculate the unknown variable, and substitute the known values into the equation. Then calculate the unknown variable.
- The general gas equation only applies if the molar quantity of the gas is fixed.
- Avogadro hypothesized that equal volumes of gas, at the same pressure and temperature, contain equal numbers of molecules, regardless of the type of gas.
- Avogadro's constant is a scaling factor between macroscopic and microscopic (atomic scale) observations of nature. It provides the relation between other physical constants and properties.
- Albert Einstein proposed that Avogadro's number could be determined based on the quantities observable in Brownian motion. NA was measured for the first time by Jean Baptiste Perrin in 1908.
- Temperature arises from the kinetic energy of the random motions of matter 's particle constituents such as molecules or atoms. Therefore, it is reasonable to choose absolute zero, where all classical motion ceases, as the reference point.

- By international agreement, the unit kelvin and its scale are defined by two points: absolute zero and the triple point of the standardized water.
- At absolute zero, the particle constituents of matter have minimal motion and cannot become any colder. They retain minimal, quantum mechanical motion.

Key Terms

- **mole**: In the International System of Units, the base unit of amount of substance; the amount of substance of a system which contains as many elementary entities as there are atoms in 12 g of carbon-12. Symbol: mol.
- **ideal gas**: A hypothetical gas whose molecules exhibit no interaction and undergo elastic collision with each other and with the walls of the container.
- **Avogadro's number**: the number of constituent particles (usually atoms or molecules) in one mole of a given substance. It has dimensions of reciprocal mol and its value is equal to 6.02214129·1023 mol-1
- **adiabatic**: Occurring without gain or loss of heat.
- **internal energy**: The sum of all energy present in the system, including kinetic and potential energy; equivalently, the energy needed to create a system, excluding the energy necessary to displace its surroundings.
- **the first law of thermodynamics**: A version of the law of energy conservation: the change in the internal energy of a closed system is equal to the amount of heat supplied to the system, minus the amount of work done by the system on its surroundings.
- **specific heat**: The ratio of the amount of heat needed to raise the temperature of a unit mass of substance by a unit degree to the amount of heat needed to raise that of the same mass of water by the same amount.
- **SI units**: International System of Units (abbreviated SI from French: Le Système international d'unités). It is the modern form of the metric system.
- **gas constant**: A universal constant, R, that appears in the ideal gas law, (PV = nRT), derived from two fundamental constants, the Boltzman constant and Avogadro's number, (R = NAk).
- **Faraday constant**: The magnitude of electric charge per mole of electrons.
- **Brownian motion**: Random motion of particles suspended in a fluid, arising from those particles being struck by individual molecules of the fluid.
- **absolute zero**: The coldest possible temperature: zero on the Kelvin scale and approximately -273.15°C and -459.67°F. The total absence of heat; the temperature at which motion of all molecules would cease.
- **International System of Units**: (SI): The standard set of basic units of measurement used in scientific literature worldwide.
- **Vienna Standard Mean Ocean Water**: A standard defining a standardized isotopic composition of water.

CHAPTER OVERVIEW

12.6: The Kinetic Theory of Gases

Gases are literally all around us—the air that we breathe is a mixture of gases. Other gases include those that make breads and cakes soft, those that make drinks fizzy, and those that burn to heat many homes. Engines and refrigerators depend on the behaviors of gases, as we will see in later chapters. As we discussed in the preceding chapter, the study of heat and temperature is part of an area of physics known as thermodynamics, in which we require a system to be *macroscopic*, that is, to consist of a huge number (such as 10^{23}) of molecules. We begin by considering some macroscopic properties of gases: volume, pressure, and temperature. The simple model of a hypothetical "ideal gas" describes these properties of a gas very accurately under many conditions. We move from the ideal gas model to a more widely applicable approximation, called the Van der Waals model. To understand gases even better, we must also look at them on the *microscopic* scale of molecules. In gases, the molecules interact weakly, so the microscopic behavior of gases is relatively simple, and they serve as a good introduction to systems of many molecules. The molecular model of gases is called the kinetic theory of gases and is one of the classic examples of a molecular model that explains everyday behavior.

Thumbnail: In an ordinary gas, so many molecules move so fast that they collide billions of times every second

Distribution of Molecular Speeds

Learning Objectives

By the end of this section, you will be able to:

- Describe the distribution of molecular speeds in an ideal gas
- Find the average and most probable molecular speeds in an ideal gas

Particles in an ideal gas all travel at relatively high speeds, but they do not travel at the same speed. The rms speed is one kind of average, but many particles move faster and many move slower. The actual distribution of speeds has several interesting implications for other areas of physics, as we will see in later chapters.

The Maxwell-Boltzmann Distribution

The motion of molecules in a gas is random in magnitude and direction for individual molecules, but a gas of many molecules has a predictable distribution of molecular speeds. This predictable distribution of molecular speeds is known as the **Maxwell-Boltzmann distribution**, after its originators, who calculated it based on kinetic theory, and it has since been confirmed experimentally (Figure 1).

To understand this figure, we must define a distribution function of molecular speeds, since with a finite number of molecules, the probability that a molecule will have exactly a given speed is 0.

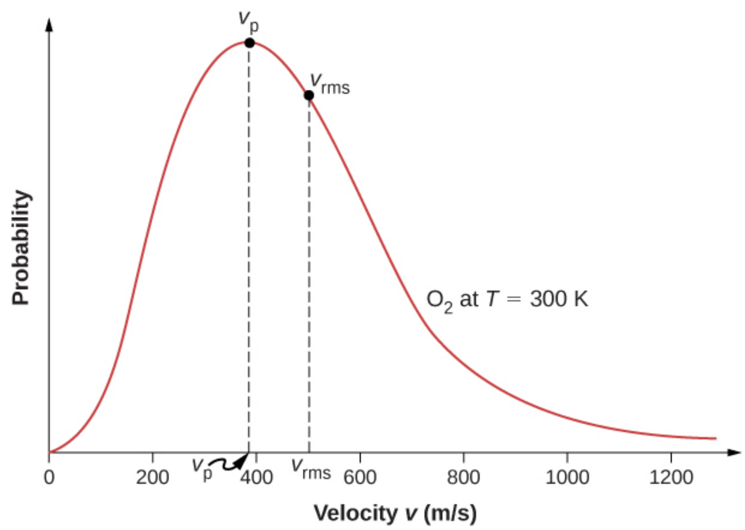


Figure 1: The Maxwell-Boltzmann distribution of molecular speeds in an ideal gas. The most likely speed v_p is less than the rms speed v_{rms}. Although very high speeds are possible, only a tiny fraction of the molecules have speeds that are an order of magnitude greater than v_{rms}.

We define the distribution function $f(v)$ by saying that the expected number $N(v_1, v_2)$ of particles with speeds between v_1 and v_2 is given by

$$N(v_1, v_2) = N\int_{v_1}^{v_2} f(v)dv.$$

[Since **N** is dimensionless, the unit of **f(v)** is seconds per meter.] We can write this equation conveniently in differential form:

$$dN = Nf(v)dv.$$

In this form, we can understand the equation as saying that the number of molecules with speeds between **v** and $v+dv$ is the total number of molecules in the sample times **f(v)** times **dv**. That is, the probability that a molecule's speed is between **v** and $v+dv$ is f(v)dv.

We can now quote Maxwell's result, although the proof is beyond our scope.

Maxwell-Boltzmann Distribution of Speeds

The distribution function for speeds of particles in an ideal gas at temperature T is

$$f(v)=\frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{3/2}v^2e^{-mv^2/2k_BT}.$$

The factors before the v^2 are a normalization constant; they make sure that $N(0,\infty)=N$ by making sure that $\int_0^{|infty} f(v)dv=1$. Let's focus on the dependence on **v**. The factor of v^2 means that $f(0)=0$ and for small **v**, the curve looks like a parabola. The factor of $e^{-m_0v^2/2k_BT}$ means that $\lim_{v\to\infty} f(v)=0$ and the graph has an exponential tail, which indicates that a few molecules may move at several times the rms speed. The interaction of these factors gives the function the single-peaked shape shown in the figure.

Example 1: Calculating the Ratio of Numbers of Molecules Near Given Speeds

In a sample of nitrogen N_2 with a molar mass of 28.0 g/mol) at a temperature of 273^oC find the ratio of the number of molecules with a speed very close to 300 m/s to the number with a speed very close to 100 m/s.

Strategy

Since we're looking at a small range, we can approximate the number of molecules near 100 m/s as $dN_{100}=f(100\,m/s)dv$. Then the ratio we want is

$$\frac{dN_{300}}{dN_{100}}=\frac{f(300\,m/s)dv}{f(100\,m/s)dv}=\frac{f(300\,m/s)}{f(100\,m/s)}.$$

All we have to do is take the ratio of the two **f** values.

Solution

1. Identify the knowns and convert to SI units if necessary.

$$T=300\,K,\;k_B=1.38\times 10^{-23}J/K$$

$$M=0.0280\,kg/mol\,so\,m=4.65\times 10^{-26}\,kg$$

2. Substitute the values and solve.

$$\begin{aligned}\frac{f(300\,m/s)}{f(100\,m/s)}&=\frac{\frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{3/2}(300\,m/s)^2exp[-m(300\,m/s)^2/2k_BT]}{\frac{4}{\sqrt{\pi}}\left(\frac{m}{2k_BT}\right)^{3/2}(100\,m/s)^2exp[-m(100\,m/s)^2/2k_BT]}\\&=\frac{(300\,m/s)^2exp[-(4.65\times 10^{-26}\,kg)(300\,m/s)^2/2(1.38\times 10^{-23}J/K)(300\,K)]}{(100\,m/s)^2exp[-(4.65\times 10^{-26}\,kg)(100\,m/s)^2/2(1.38\times 10^{-23}J/K)(300\,K)]}\\&=3^2exp\left[-\frac{(4.65\times 10^{-26}kg)[(300\,m/s)^2-(100\,m/s)^2]}{2(1.38\times 10^{-23}\,J/K)(300\,K)}\right]=5.74\end{aligned}$$

Figure 2 shows that the curve is shifted to higher speeds at higher temperatures, with a broader range of speeds.

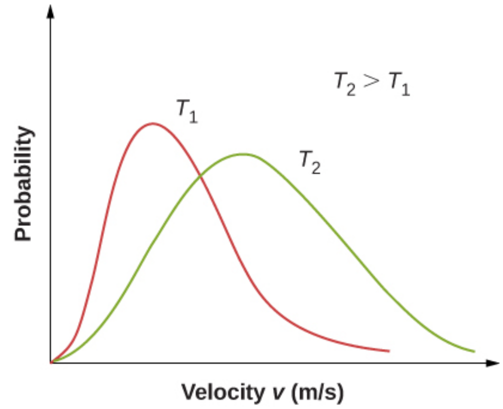


Figure 2: The Maxwell-Boltzmann distribution is shifted to higher speeds and broadened at higher temperatures.

> With only a relatively small number of molecules, the distribution of speeds fluctuates around the Maxwell-Boltzmann distribution. However, you can view this simulation to see the essential features that more massive molecules move slower and have a narrower distribution. Use the set-up “2 Gases, Random Speeds”. Note the display at the bottom comparing histograms of the speed distributions with the theoretical curves.

We can use a probability distribution to calculate average values by multiplying the distribution function by the quantity to be averaged and integrating the product over all possible speeds. (This is analogous to calculating averages of discrete distributions, where you multiply each value by the number of times it occurs, add the results, and divide by the number of values. The integral is analogous to the first two steps, and the normalization is analogous to dividing by the number of values.) Thus the average velocity is

$$\bar{v} = \int_0^\infty v f(v) dv = \sqrt{\frac{8}{\pi}\frac{k_B T}{m}} = \sqrt{\frac{8}{\pi}\frac{RT}{M}}.$$

Similarly,

$$v_{rms} = \sqrt{\bar{v}^2} = \sqrt{\int_0^\infty v^2 f(v) dv} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3RT}{M}}$$

as in Pressure, Temperature, and RMS Speed. The **most probable speed**, also called the **peak speed** v_p is the speed at the peak of the velocity distribution. (In statistics it would be called the mode.) It is less than the rms speed v_{rms}. The most probable speed can be calculated by the more familiar method of setting the derivative of the distribution function, with respect to **v**, equal to 0. The result is

$$v_p = \sqrt{\frac{2k_B T}{m}} = \sqrt{\frac{2RT}{M}},$$

which is less than v_{rms}. In fact, the rms speed is greater than both the most probable speed and the average speed.

The peak speed provides a sometimes more convenient way to write the Maxwell-Boltzmann distribution function:

$$f(v)=\frac{4v^2}{\sqrt{\pi}v_p^3}e^{-v^2/v_p^2}$$

In the factor $e^{-mv^2/2k_BT}$, it is easy to recognize the translational kinetic energy. Thus, that expression is equal to e^{-K/k_BT}. The distribution **f**(**v**) can be transformed into a kinetic energy distribution by requiring that $f(K)dK=f(v)dv$. Boltzmann showed that the resulting formula is much more generally applicable if we replace the kinetic energy of translation with the total mechanical energy **E**. Boltzmann's result is

$$f(E)=\frac{2}{\sqrt{\pi}}(k_BT)^{-3/2}\sqrt{E}e^{-E/k_BT}=\frac{2}{\sqrt{\pi}(k_BT)^{3/2}}\frac{\sqrt{E}}{e^{E/k_BT}}.$$

The first part of this equation, with the negative exponential, is the usual way to write it. We give the second part only to remark that e^{E/k_BT} in the denominator is ubiquitous in quantum as well as classical statistical mechanics.

Problem-Solving Strategy: Speed Distribution

- **Step 1.** Examine the situation to determine that it relates to the distribution of molecular speeds.
- **Step 2.** Make a list of what quantities are given or can be inferred from the problem as stated (identify the known quantities).
- **Step 3.** Identify exactly what needs to be determined in the problem (identify the unknown quantities). A written list is useful.
- **Step 4.** Convert known values into proper SI units (K for temperature, Pa for pressure, m^3 for volume, molecules for **N**, and moles for **n**). In many cases, though, using **R** and the molar mass will be more convenient than using k_B and the molecular mass.
- **Step 5.** Determine whether you need the distribution function for velocity or the one for energy, and whether you are using a formula for one of the characteristic speeds (average, most probably, or rms), finding a ratio of values of the distribution function, or approximating an integral.
- **Step 6.** Solve the appropriate equation for the ideal gas law for the quantity to be determined (the unknown quantity). Note that if you are taking a ratio of values of the distribution function, the normalization factors divide out. Or if approximating an integral, use the method asked for in the problem.
- **Step 7.** Substitute the known quantities, along with their units, into the appropriate equation and obtain numerical solutions complete with units.

We can now gain a qualitative understanding of a puzzle about the composition of Earth's atmosphere. Hydrogen is by far the most common element in the universe, and helium is by far the second-most common. Moreover, helium is constantly produced on Earth by radioactive decay. Why are those elements so rare in our atmosphere? The answer is that gas molecules that reach speeds above Earth's escape velocity, about 11 km/s, can escape from the atmosphere into space. Because of the lower mass of hydrogen and helium molecules, they move at higher speeds than other gas molecules, such as nitrogen and oxygen. Only a few exceed escape velocity, but far fewer heavier molecules do. Thus, over the billions of years that Earth has existed, far more hydrogen and helium molecules have escaped from the atmosphere than other molecules, and hardly any of either is now present.

We can also now take another look at evaporative cooling, which we discussed in the chapter on temperature and heat. Liquids, like gases, have a distribution of molecular energies. The highest-energy molecules are those that can escape from the intermolecular attractions of the liquid. Thus, when some liquid evaporates, the molecules left behind have a lower average energy, and the liquid has a lower temperature.

Heat Capacity and Equipartition of Energy

Learning Objectives

By the end of this section, you will be able to:

- Solve problems involving heat transfer to and from ideal monatomic gases whose volumes are held constant
- Solve similar problems for non-monatomic ideal gases based on the number of degrees of freedom of a molecule
- Estimate the heat capacities of metals using a model based on degrees of freedom

In the chapter on temperature and heat, we defined the specific heat capacity with the equation $Q = mc\Delta T$, or $c = (1/m)Q/\Delta T$. However, the properties of an ideal gas depend directly on the number of moles in a sample, so here we define specific heat capacity in terms of the number of moles, not the mass. Furthermore, when talking about solids and liquids, we ignored any changes in volume and pressure with changes in temperature—a good approximation for solids and liquids, but for gases, we have to make some condition on volume or pressure changes. Here, we focus on the heat capacity with the volume held constant. We can calculate it for an ideal gas.

Heat Capacity of an Ideal Monatomic Gas at Constant Volume

We define the **molar heat capacity at constant volume** C_V as

$$\underbrace{C_V = \frac{1}{n}\frac{Q}{\Delta T}}_{\text{with constant V}}$$

This is often expressed in the form

$$Q = nC_V\Delta T$$

If the volume does not change, there is no overall displacement, so no work is done, and the only change in internal energy is due to the heat flow $\Delta E_{int} = Q$. (This statement is discussed further in the next chapter.) We use the equation $E_{int} = 3nRT/2$ to write $\Delta E_{int} = 3nR\Delta T/2$ and substitute ΔE for **Q** to find $Q = 3nR\Delta T/2$, which gives the following simple result for an ideal monatomic gas:

$$C_V = \frac{3}{2}R.$$

It is independent of temperature, which justifies our use of finite differences instead of a derivative. This formula agrees well with experimental results.

In the next chapter we discuss the molar specific heat at constant pressure C_p, which is always greater than C_V.

Example 1: Calculating Temperature

A sample of 0.125 kg of xenon is contained in a rigid metal cylinder, big enough that the xenon can be modeled as an ideal gas, at a temperature of 20.0^oC. The cylinder is moved outside on a hot summer day. As the xenon comes into equilibrium by reaching the temperature of its surroundings, 180 J of heat are conducted to it through the cylinder walls. What is the equilibrium temperature? Ignore the expansion of the metal cylinder.

Solution

1. Identify the knowns: We know the initial temperature T_1 is 20.0^oC, the heat **Q** is 180 J, and the mass **m** of the xenon is 0.125 kg.
2. Identify the unknown. We need the final temperature, so we'll need ΔT.
3. Determine which equations are needed. Because xenon gas is monatomic, we can use $Q = 3nR\Delta T/2$. Then we need the number of moles $n = m/M$.
4. Substitute the known values into the equations and solve for the unknowns.

The molar mass of xenon is 131.3 g, so we obtain

$$n = \frac{125\,g}{131.3\,g/mol} = 0.952\,mol,$$

$$\Delta T = \frac{2Q}{3nR} = \frac{2(180\,J)}{3(0.952\,mol)(8.31\,J/mol \cdot {}^oC)} = 15.2^oC.$$

Therefore, the final temperature is 35.2^oC. The problem could equally well be solved in kelvin; as a kelvin is the same size as a degree Celsius of temperature change, you would get $\Delta T = 15.2\,K$.

Significance

The heating of an ideal or almost ideal gas at constant volume is important in car engines and many other practical systems.

Exercise 1

Suppose 2 moles of helium gas at 200 K are mixed with 2 moles of krypton gas at 400 K in a calorimeter. What is the final temperature?

Answer

As the number of moles is equal and we know the molar heat capacities of the two gases are equal, the temperature is halfway between the initial temperatures, 300 K.

We would like to generalize our results to ideal gases with more than one atom per molecule. In such systems, the molecules can have other forms of energy beside translational kinetic energy, such as rotational kinetic energy and vibrational kinetic and potential energies. We will see that a simple rule lets us determine the average energies present in these forms and solve problems in much the same way as we have for monatomic gases.

Degrees of Freedom

In the previous section, we found that $\frac{1}{2}mv^2 = \frac{3}{2}k_BT$ and $v^2 = 3v_x^2$, from which it follows that $\frac{1}{2}mv_x^2 = \frac{1}{2}k_BT$. The same equation holds for $\frac{3}{2}k_BT$ as the sum of contributions of $\frac{1}{2}k_BT$ from each of the three dimensions of translational motion. Shifting to the gas as a whole, we see that the 3 in the formula $C_V = \frac{3}{2}R$ also reflects those three dimensions. We define a **degree of freedom** as an independent possible motion of a molecule, such as each of the three dimensions of translation. Then, letting **d** represent the number of degrees of freedom, the molar heat capacity at constant volume of a monatomic ideal gas is $C_V = \frac{d}{2}R$, where $d = 3$.

The branch of physics called **statistical mechanics** tells us, and experiment confirms, that C_V of any ideal gas is given by this equation, regardless of the number of degrees of freedom. This fact follows from a more general result, the equipartition theorem, which holds in classical (non-quantum) thermodynamics for systems in thermal equilibrium under technical conditions that are beyond our scope. Here, we mention only that in a system, the energy is shared among the degrees of freedom by collisions.

Equipartition Theorem

The energy of a thermodynamic system in equilibrium is partitioned equally among its degrees of freedom. Accordingly, the molar heat capacity of an ideal gas is proportional to its number of degrees of freedom, **d**:

$$C_V = \frac{d}{2}R.$$

This result is due to the Scottish physicist James Clerk **Maxwell** (1831−1871), whose name will appear several more times in this book.

For example, consider a diatomic ideal gas (a good model for nitrogen, N_2, and oxygen, O_2). Such a gas has more degrees of freedom than a monatomic gas. In addition to the three degrees of freedom for translation, it has two degrees of freedom for rotation perpendicular to its axis. Furthermore, the molecule can vibrate along its axis. This motion is often modeled by imagining a spring connecting the two atoms, and we know from simple harmonic motion that such motion has both kinetic and potential energy. Each of these forms of energy corresponds to a degree of freedom, giving two more.

We might expect that for a diatomic gas, we should use 7 as the number of degrees of freedom; classically, if the molecules of a gas had only translational kinetic energy, collisions between molecules would soon make them rotate and vibrate. However, as explained in the previous module, quantum mechanics controls which degrees of freedom are active. The result is shown in Figure 1. Both rotational and vibrational energies are limited to discrete values. For temperatures below about 60 K, the energies of hydrogen molecules are too low for a collision to bring the rotational state or vibrational state of a molecule from the lowest energy to the second lowest, so the only form of energy is translational kinetic energy, and $d = 3$ or $C_V = 3R/2$ as in a monatomic gas. Above that temperature, the two rotational degrees of freedom begin to contribute, that is, some molecules are excited to the rotational state with the second-lowest energy. (This temperature is much lower than that where rotations of monatomic gases contribute, because diatomic molecules have much higher rotational inertias and hence much lower rotational energies.) From about room temperature (a bit less than 300 K) to about 600 K, the rotational degrees of freedom are fully active, but the vibrational ones are not, and $d = 5$. Then, finally, above about 3000 K, the vibrational degrees of freedom are fully active, and $d = 7$ as the classical theory predicted.

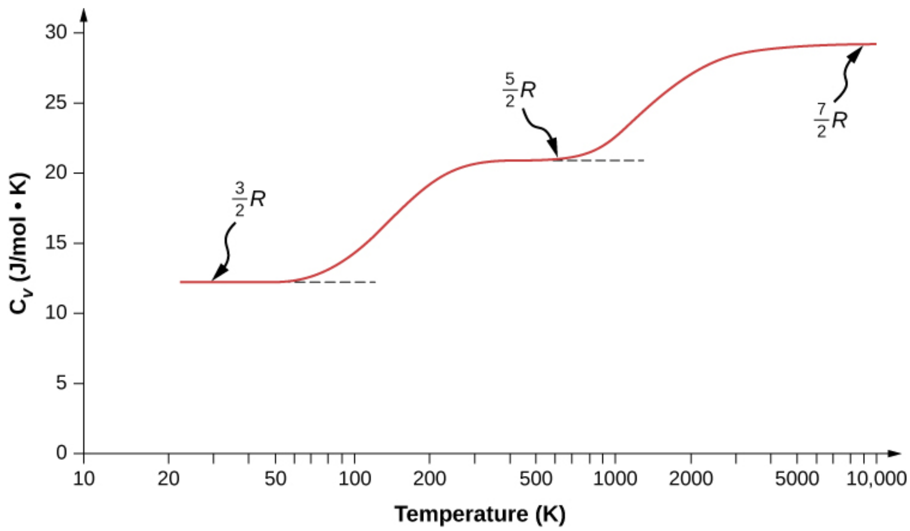


Figure 1: The molar heat capacity of hydrogen as a function of temperature (on a logarithmic scale). The three "steps" or "plateaus" show different numbers of degrees of freedom that the typical energies of molecules must achieve to activate. Translational kinetic energy corresponds to three degrees of freedom, rotational to another two, and vibrational to yet another two.

Polyatomic molecules typically have one additional rotational degree of freedom at room temperature, since they have comparable moments of inertia around any axis. Thus, at room temperature, they have $d = 6$ and at high temperature, $d = 8$. We usually assume that gases have the theoretical room-temperature values of **d**.

As shown in Table 1, the results agree well with experiments for many monatomic and diatomic gases, but the agreement for triatomic gases is only fair. The differences arise from interactions that we have ignored between and within molecules.

Table 1: C_V/R for Various Monatomic, Diatomic, and Triatomic Gases

Gas	C_V/R at 25^oC and 1 atm
Ar	1.50
He	1.50
Ne	1.50
CO	2.50
H_2	2.47
N_2	2.50
O_2	2.53
F_2	2.8
CO_2	3.48

Gas	C_V/R at 25^oC and 1 atm
H_2S	3.13
N_2O	3.66

What about internal energy for diatomic and polyatomic gases? For such gases, C_V is a function of temperature (Figure 1), so we do not have the kind of simple result we have for monatomic ideal gases.

Molar Heat Capacity of Solid Elements

The idea of equipartition leads to an estimate of the molar heat capacity of solid elements at ordinary temperatures. We can model the atoms of a solid as attached to neighboring atoms by springs (Figure 2).

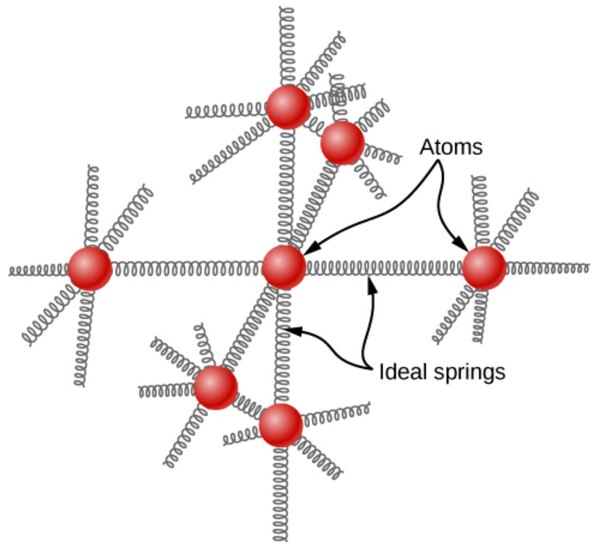


Figure 2: In a simple model of a solid element, each atom is attached to others by six springs, two for each possible motion: x, y, and z. Each of the three motions corresponds to two degrees of freedom, one for kinetic energy and one for potential energy. Thus $d = 6$.

Analogously to the discussion of vibration in the previous module, each atom has six degrees of freedom: one kinetic and one potential for each of the **x**-, **y**-, and **z**-directions. Accordingly, the molar specific heat of a metal should be 3**R**. This result, known as the **Law of Dulong and Petit**, works fairly well experimentally at room temperature. (For every element, it fails at low temperatures for quantum-mechanical reasons. Since quantum effects are particularly important for low-mass particles, the Law of Dulong and Petit already fails at room temperature for some light elements, such as beryllium and carbon. It also fails for some heavier elements for various reasons beyond what we can cover.)

Problem-Solving Strategy: Heat Capacity and Equipartition

The strategy for solving these problems is the same as the one in Phase Changes for the effects of heat transfer. The only new feature is that you should determine whether the case just presented—ideal gases at constant volume—applies to the problem. (For solid elements, looking up the specific heat capacity is generally better than estimating it from the Law of Dulong and Petit.) In the case of an ideal gas, determine the number **d** of degrees of freedom from the number of atoms in the gas molecule and use it to calculate C_V (or use C_V to solve for **d**).

Example 2: Calculating Temperature: Calorimetry with an Ideal Gas

A 300-g piece of solid gallium (a metal used in semiconductor devices) at its melting point of only 30.0^oC is in contact with 12.0 moles of air (assumed diatomic) at 95.0^oC in an insulated container. When the air reaches equilibrium with the gallium, 202 g of the gallium have melted. Based on those data, what is the heat of fusion of gallium? Assume the volume of the air does not change and there are no other heat transfers.

Strategy

We'll use the equation $Q_{hot} + Q_{cold} = 0$. As some of the gallium doesn't melt, we know the final temperature is still the melting point. Then the only Q_{hot} is the heat lost as the air cools, $Q_{hot} = n_{air} C_V \Delta T$, where $C_V = 5R/2$. The only Q_{cold} is the latent heat of fusion of the gallium, $Q_{cold} = m_{Ga} L_f$. It is positive because heat flows into the gallium.

Solution

1. Set up the equation:

$$n_{air} C_V \Delta T + m_{Ga} L_f = 0.$$

2. Substitute the known values and solve:

$$(12.0\, mol) \left(\frac{5}{2}\right) \left(8.31 \frac{J}{mol \cdot {}^oC}\right) (30.0^oC - 95.0^oC) + (0.202\, kg) L_f = 0.$$

We solve to find that the heat of fusion of gallium is 80.2 kJ/kg.

Molecular Model of an Ideal Gas

Learning Objectives

By the end of this section, you will be able to:

- Apply the ideal gas law to situations involving the pressure, volume, temperature, and the number of molecules of a gas
- Use the unit of moles in relation to numbers of molecules, and molecular and macroscopic masses
- Explain the ideal gas law in terms of moles rather than numbers of molecules
- Apply the van der Waals gas law to situations where the ideal gas law is inadequate

In this section, we explore the thermal behavior of gases. Our word "gas" comes from the Flemish word meaning "chaos," first used for vapors by the seventeenth-century chemist J. B. van Helmont. The term was more appropriate than he knew, because gases consist of molecules moving and colliding with each other at random. This randomness makes the connection between the microscopic and macroscopic domains simpler for gases than for liquids or solids.

How do gases differ from solids and liquids? Under ordinary conditions, such as those of the air around us, the difference is that the molecules of gases are much farther apart than those of solids and liquids. Because the typical distances between molecules are large compared to the size of a molecule, as illustrated in Figure 1, the forces between them are considered negligible, except when they come into contact with each other during collisions. Also, at temperatures well above the boiling temperature, the motion of molecules is fast, and the gases expand rapidly to occupy all of the accessible volume. In contrast, in liquids and solids, molecules are closer together, and the behavior of molecules in liquids and solids is highly constrained by the molecules' interactions with one another. The macroscopic properties of such substances depend strongly on the forces between the molecules, and since many molecules are interacting, the resulting "many-body problems" can be extremely complicated (see section on Condensed Matter Physics).

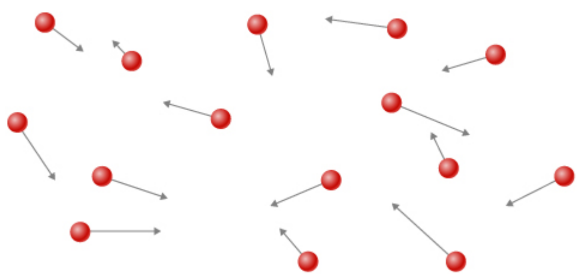


Figure 1: Atoms and molecules in a gas are typically widely separated. Because the forces between them are quite weak at these distances, the properties of a gas depend more on the number of atoms per unit volume and on temperature than on the type of atom.

The Gas Laws

In the previous chapter, we saw one consequence of the large intermolecular spacing in gases: Gases are easily compressed. Table 1.4.1 shows that gases have larger coefficients of volume expansion than either solids or liquids. These large coefficients mean that gases expand and contract very rapidly with temperature changes. We also saw (in the section on thermal expansion) that most gases expand at the same rate or have the same coefficient of volume expansion, β. This raises a question: Why do all gases act in nearly the same way, when all the various liquids and solids have widely varying expansion rates?

To study how the pressure, temperature, and volume of a gas relate to one another, consider what happens when you pump air into a deflated car tire. The tire's volume first increases in direct proportion to the amount of air injected, without much increase in the tire pressure. Once the tire has expanded to nearly its full size, the tire's walls limit its volume expansion. If we continue to pump air into the tire, the pressure increases. When the car is driven and the tires flex, their temperature increases, and therefore the pressure increases even further (Figure 2).

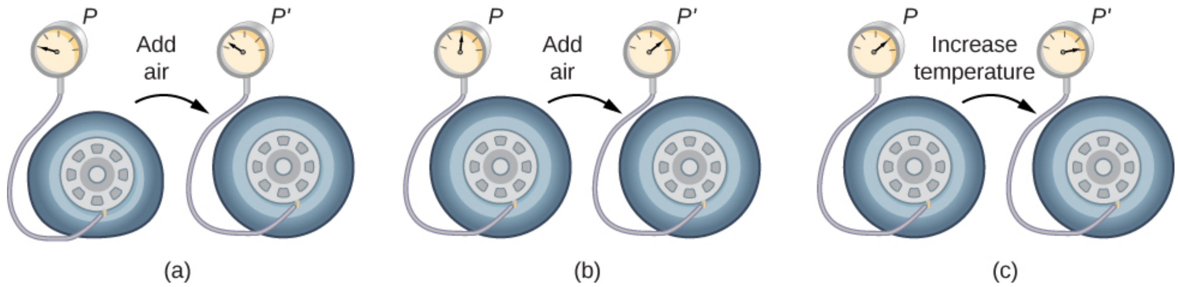


Figure 2: When air is pumped into a deflated tire, its volume first increases without much increase in pressure. (b) When the tire is filled to a certain point, the tire walls resist further expansion, and the pressure increases with more air. (c) Once the tire is inflated, its pressure increases with temperature.

Figure 3 shows data from the experiments of Robert **Boyle** (1627–1691), illustrating what is now called **Boyle's law**: At constant temperature and number of molecules, the absolute pressure of a gas and its volume are inversely proportional. (Recall from the section on Fluid Mechanics that the absolute pressure is the true pressure and the gauge pressure is the absolute pressure minus the ambient pressure, typically atmospheric pressure.) The graph in Figure 3 displays this relationship as an inverse proportionality of volume to pressure.

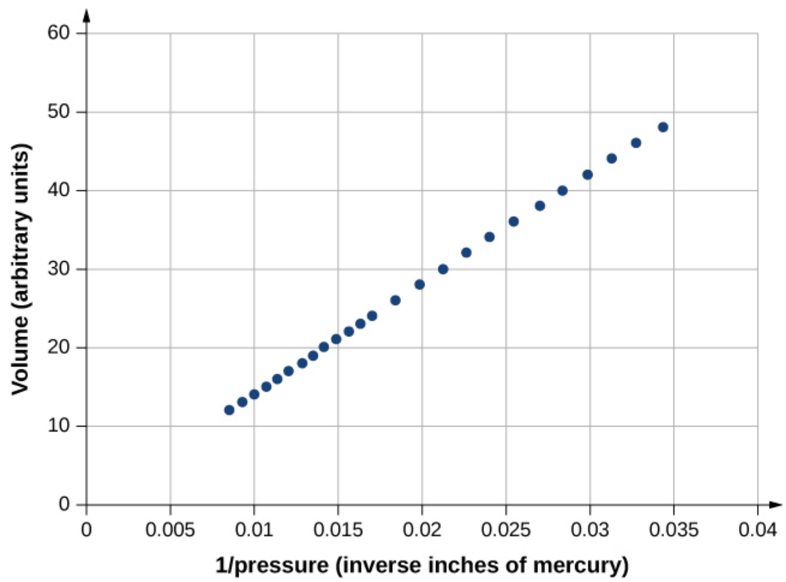


Figure 3: Robert Boyle and his assistant found that volume and pressure are inversely proportional. Here their data are plotted as V versus 1/p; the linearity of the graph shows the inverse proportionality. The number shown as the volume is actually the height in inches of air in a cylindrical glass tube. The actual volume was that height multiplied by the cross-sectional area of the tube, which Boyle did not publish. The data are from Boyle's book A Defence of the Doctrine Touching the Spring and Weight of the Air…, p. 60.

Figure 4 shows experimental data illustrating what is called **Charles's law**, after Jacques **Charles** (1746–1823). Charles's law states that at constant pressure and number of molecules, the volume of a gas is proportional to its absolute temperature.

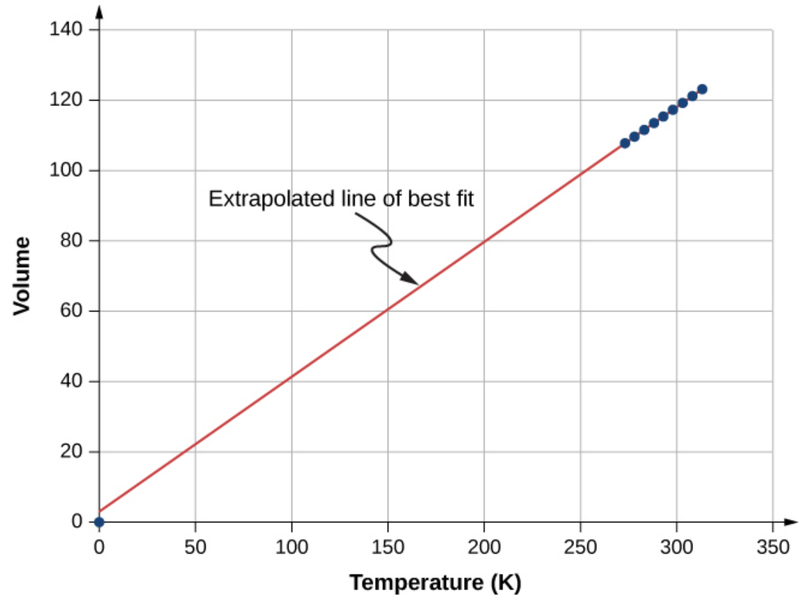


Figure 4: Experimental data showing that at constant pressure, volume is approximately proportional to temperature. The best-fit line passes approximately through the origin.[2]

Similar is Amonton's or **Gay-Lussac's law**, which states that at constant volume and number of molecules, the pressure is proportional to the temperature. That law is the basis of the constant-volume gas thermometer, discussed in the previous chapter. (The histories of these laws and the appropriate credit for them are more complicated than can be discussed here.)

It is known experimentally that for gases at low density (such that their molecules occupy a negligible fraction of the total volume) and at temperatures well above the boiling point, these proportionalities hold to a good approximation. Not surprisingly, with the other quantities held constant, either pressure or volume is proportional to the number of molecules. More surprisingly, when the proportionalities are combined into a single equation, the constant of proportionality is independent of the composition of the gas. The resulting equation for all gases applies in the limit of low density and high temperature; it's the same for oxygen as for helium or uranium hexafluoride. A gas at that limit is called an **ideal gas**; it obeys the **ideal gas law**, which is also called the equation of state of an ideal gas.

Ideal Gas Law

The ideal gas law states that

$$pV = Nk_BT,$$

where **p** is the absolute pressure of a gas, **V** is the volume it occupies, **N** is the number of molecules in the gas, and **T** is its absolute temperature.

The constant k_B is called the **Boltzmann** constant in honor of the Austrian physicist Ludwig **Boltzmann** (1844–1906) and has the value

$$k_B = 1.38 \times 10^{-23}\, J/K.$$

The ideal gas law describes the behavior of any real gas when its density is low enough or its temperature high enough that it is far from liquefaction. This encompasses many practical situations. In the next section, we'll see why it's independent of the type of gas.

In many situations, the ideal gas law is applied to a sample of gas with a constant number of molecules; for instance, the gas may be in a sealed container. If **N** is constant, then solving for **N** shows that **pV /T** is constant. We can write that fact in a convenient form:

$$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2},$$

where the subscripts 1 and 2 refer to any two states of the gas at different times. Again, the temperature must be expressed in kelvin and the pressure must be absolute pressure, which is the sum of gauge pressure and atmospheric pressure.

✓ Example 1: Calculating Pressure Changes Due to Temperature Changes

Suppose your bicycle tire is fully inflated, with an absolute pressure of $7.00 \times 10^5\,Pa$ (a gauge pressure of just under $90.0\,lb/in^2$.) at a temperature of 18.0^oC. What is the pressure after its temperature has risen to 35.0^oC on a hot day? Assume there are no appreciable leaks or changes in volume.

Strategy

The pressure in the tire is changing only because of changes in temperature. We know the initial pressure $p_0 = 7.00 \times 10^5\,Pa$, the initial temperature $T_0 = 18.0^oC$, and the final temperature $T_f = 35.0^oC$. We must find the final pressure p_f. Since the number of molecules is constant, we can use the equation

$$\frac{p_fV_f}{T_f} = \frac{p_0V_0}{T_0}.$$

Since the volume is constant, V_f and V_0 are the same and they divide out. Therefore,

$$\frac{p_f}{T_f} = \frac{p_0}{T_0}.$$

We can then rearrange this to solve for p_f:

$$p_f = p_0\frac{T_f}{T_0},$$

where the temperature must be in kelvin.

Solution

1. Convert temperatures from degrees Celsius to kelvin

$$T_0 = (18.0 + 273)K = 291\,K,$$

$$T_f = (35.0 + 273)K = 308\,K.$$

2. Substitute the known values into the equation,

$$p_f = p_0\frac{T_f}{T_0} = 7.00 \times 10^5\,Pa\left(\frac{308\,K}{291\,K}\right) = 7.41 \times 10^5\,Pa.$$

Significance

The final temperature is about 6% greater than the original temperature, so the final pressure is about 6% greater as well. Note that **absolute pressure** (see Fluid Mechanics) and **absolute temperature** (see Temperature and Heat) must be used in the ideal gas law.

✓ Example 2: Calculating the Number of Molecules in a Cubic Meter of Gas

How many molecules are in a typical object, such as gas in a tire or water in a glass? This calculation can give us an idea of how large **N** typically is. Let's calculate the number of molecules in the air that a typical healthy young adult inhales in one breath, with a volume of 500 mL, at **standard temperature and pressure (STP)**, which is defined as 0^oC and atmospheric pressure. (Our young adult is apparently outside in winter.)

Strategy

Because pressure, volume, and temperature are all specified, we can use the ideal gas law, \(pV = k_BT\}, to find N.

Solution

1. Identify the knowns.

$$T = 0^oC = 273\,K,\ p = 1.01 \times 10^5\,Pa,\ V = 500\,mL = 5 \times 10^{-4}\,m^3,\ k_B = 1.38 \times 10^{-23}\,J/K$$

2. Substitute the known values into the equation and solve for **N**.

$$N = \frac{pV}{k_BT} = \frac{(1.01\times10^5\,Pa)(5\times10^{-4}\,m^3)}{(1.38\times10^{-23}\,J/K)(273\,K)} = 1.34\times10^{22}\,molecules$$

Significance **N** is huge, even in small volumes. For example, $1\,cm^3$ of a gas at STP contains 2.68×10^{19} molecules. Once again, note that our result for **N** is the same for all types of gases, including mixtures.

As we observed in the chapter on fluid mechanics, pascals are N/m^2, so $Pa\cdot m^3 = N\cdot m = J$. Thus, our result for **N** is dimensionless, a pure number that could be obtained by counting (in principle) rather than measuring. As it is the number of molecules, we put “molecules” after the number, keeping in mind that it is an aid to communication rather than a unit.

Moles and Avogadro’s Number

It is often convenient to measure the amount of substance with a unit on a more human scale than molecules. The SI unit for this purpose was developed by the Italian scientist Amedeo Avogadro (1776–1856). (He worked from the hypothesis that equal volumes of gas at equal pressure and temperature contain equal numbers of molecules, independent of the type of gas. As mentioned above, this hypothesis has been confirmed when the ideal gas approximation applies.) A mole (abbreviated mol) is defined as the amount of any substance that contains as many molecules as there are atoms in exactly 12 grams (0.012 kg) of carbon-12. (Technically, we should say “formula units,” not “molecules,” but this distinction is irrelevant for our purposes.) The number of molecules in one mole is called Avogadro’s number (N_A) and the value of Avogadro’s number is now known to be

$$N_A = 6.02\times10^{-23}\,mol^{-1}$$

. We can now write $N = N_An$, where **n** represents the number of moles of a substance.

Avogadro’s number relates the mass of an amount of substance in grams to the number of protons and neutrons in an atom or molecule (12 for a carbon-12 atom), which roughly determine its mass. It’s natural to define a unit of mass such that the mass of an atom is approximately equal to its number of neutrons and protons. The unit of that kind accepted for use with the SI is the **unified atomic mass unit (u)**, also called the **dalton**. Specifically, a carbon-12 atom has a mass of exactly 12 u, so that its molar mass **M** in grams per mole is numerically equal to the mass of one carbon-12 atom in u. That equality holds for any substance. In other words, N_A is not only the conversion from numbers of molecules to moles, but it is also the conversion from u to grams: $6.02\times10^{23}\,u = 1\,g$. See Figure 5.

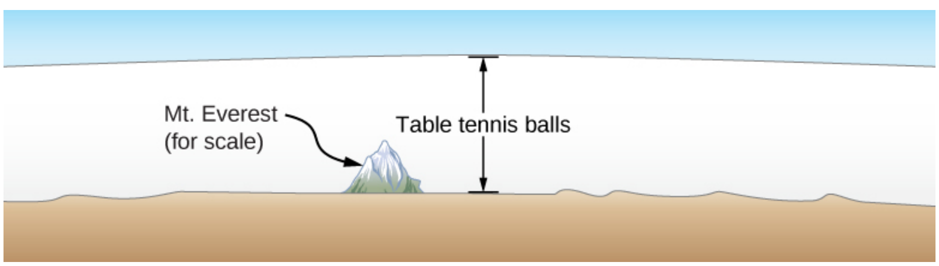


Figure 5: How big is a mole? On a macroscopic level, Avogadro’s number of table tennis balls would cover Earth to a depth of about 40 km.

Exercise $1A$

The recommended daily amount of vitamin B_3 or niacin, $C_6NH_5O_2$, for women who are not pregnant or nursing, is 14 mg. Find the number of molecules of niacin in that amount.

Answer

We first need to calculate the molar mass (the mass of one mole) of niacin. To do this, we must multiply the number of atoms of each element in the molecule by the element’s molar mass.

(6 mol of carbon)(12.0 g/mol) + (5 mol hydrogen)(1.0 g/mol) + (1 mol of nitrogen)(14 g/mol) + (2 mol oxygen)(16.0 g/mol) = 123 g/mol

Then we need to calculate the number of moles in 14 mg.

$$\left(\frac{14\,mg}{123\,g/mol}\right)\left(\frac{1\,g}{1000\,mg}\right) = 1.14\times10^{-4}\,mol.$$

Then, we use Avogadro's number to calculate the number of molecules:

$N = nN_A = (1.14\times10^{-4}\,mol)(6.02\times10^{23}\,molecules/mol) = 6.85\times10^{19} molecules$

Exercise $1B$

The density of air in a classroom ($p = 1.00\,atm$ and $T = 20^oC$) is $1.28\,kg/m^3$. At what pressure is the density $0.600\,kg/m^3$ if the temperature is kept constant?

Answer

The density of a gas is equal to a constant, the average molecular mass, times the number density **N**/**V**. From the ideal gas law, $pV = Nk_BT$, we see that $N/V = p/k_BT$. Therefore, at constant temperature, if the density and, consequently, the number density are reduced by half, the pressure must also be reduced by half, and $p_f = 0.500\,atm$.

The Ideal Gas Law Restated using Moles

A very common expression of the ideal gas law uses the number of moles in a sample, **n**, rather than the number of molecules, **N**. We start from the ideal gas law,

$$pV = Nk_BT,$$

and multiply and divide the right-hand side of the equation by Avogadro's number N_A.This gives us

$$pV = \frac{N}{N_A}N_Ak_BT.$$

Note that $n = N/N_A$ is the number of moles. We define the universal gas constant as $R = N_Ak_B$, and obtain the ideal gas law in terms of moles.

Ideal Gas Law (in terms of moles)

In terms of number of moles **n**, the ideal gas law is written as

$$pV = nRT.$$

In SI units,

$$R = N_Ak_B = (6.02\times10^{23}\,mol^{-1})(1.38\times10^{-23}J/K) = 8.31\,J/mol\cdot K$$

In other units,

$$R = 1.99\frac{cal}{mol\cdot K} = 0.0821\frac{L\cdot atm}{mol\cdot K}.$$

You can use whichever value of **R** is most convenient for a particular problem.

Example 3: Density of Air at STP and in a Hot Air Balloon

Calculate the density of dry air (a) under standard conditions and (b) in a hot air balloon at a temperature of 120^oC. Dry air is approximately $78\% N_2$, $21\% O_2$,and $1\%\,Ar$.

Strategy and Solution

1. We are asked to find the density, or mass per cubic meter. We can begin by finding the molar mass. If we have a hundred molecules, of which 78 are nitrogen, 21 are oxygen, and 1 is argon, the average molecular mass is $frac78\,m_{N_2} + 21\,m_{O_2} + m_{Ar}100$, or the mass of each constituent multiplied by its percentage. The same applies to the molar mass, which therefore is

$$M = 0.78\, M_{N_2} + 0,21\, M_{O_2} + 0.01\, M_{Ar} = 29.0\, g/mol.$$

Now we can find the number of moles per cubic meter. We use the ideal gas law in terms of moles, $pV = nRT$, with $p = 1.00\, atm$, $T = 273\, K$, $V = 1\, m^3$, and $R = 8.31\, J/mol \cdot K$. The most convenient choice for **R** in this case is $R = 8.31\, J/mol \cdot K$ because the known quantities are in SI units:

$$n = \frac{pV}{RT} = \frac{(1.01 \times 10^5\, Pa)(1\, m^3)}{(8.31\, J/mol \cdot K)(273\, K)} = 44.5\, mol.$$

Then, the mass m_s of that air is

$$m_s = nM = (44.5\, mol)(29.0\, g/mol) = 1290\, g = 1.29\, kg.$$

Finally the density of air at STP is

$$\rho = \frac{m_s}{V} = \frac{1.29\, kg}{1\, m^3} = 1.29\, kg/m^3.$$

2. The air pressure inside the balloon is still 1 atm because the bottom of the balloon is open to the atmosphere. The calculation is the same except that we use a temperature of 120^oC, which is 393 K. We can repeat the calculation in (a), or simply observe that the density is proportional to the number of moles, which is inversely proportional to the temperature. Then using the subscripts 1 for air at STP and 2 for the hot air, we have

$$\rho_2 = \frac{T_1}{T_2}\rho_1 = \frac{273\, K}{393\, K}(1.29\, kg/m^3) = 0.896\, kg/m^3.$$

Significance

Using the methods of Archimedes' Principle and Buoyancy, we can find that the net force on $2200\, m^3$ of air at 120^oC is $F_b - F_g = \rho_{atmosphere} V_g - \rho_{hot\ air} V_g = 8.49 \times 10^3\, N$, or enough to lift about 867 kg. The mass density and molar density of air at STP, found above, are often useful numbers. From the molar density, we can easily determine another useful number, the volume of a mole of any ideal gas at STP, which is 22.4 L.

? Exercise 3

Liquids and solids have densities on the order of 1000 times greater than gases. Explain how this implies that the distances between molecules in gases are on the order of 10 times greater than the size of their molecules.

Answer

Density is mass per unit volume, and volume is proportional to the size of a body (such as the radius of a sphere) cubed. So if the distance between molecules increases by a factor of 10, then the volume occupied increases by a factor of 1000, and the density decreases by a factor of 1000. Since we assume molecules are in contact in liquids and solids, the distance between their centers is on the order of their typical size, so the distance in gases is on the order of 10 times as great.

The ideal gas law is closely related to energy: The units on both sides of the equation are joules. The right-hand side of the ideal gas law equation is Nk_BT. This term is roughly the total translational kinetic energy (which, when discussing gases, refers to the energy of translation of a molecule, not that of vibration of its atoms or rotation) of **N** molecules at an absolute temperature **T**, as we will see formally in the next section. The left-hand side of the ideal gas law equation is **pV**. As mentioned in the example on the number of molecules in an ideal gas, pressure multiplied by volume has units of energy. The energy of a gas can be changed when the gas does work as it increases in volume, something we explored in the preceding chapter, and the amount of work is related to the pressure. This is the process that occurs in gasoline or steam engines and turbines, as we'll see in the next chapter.

Problem-Solving Strategy: The Ideal Gas Law

- **Step 1.** Examine the situation to determine that an ideal gas is involved. Most gases are nearly ideal unless they are close to the boiling point or at pressures far above atmospheric pressure.
- **Step 2.** Make a list of what quantities are given or can be inferred from the problem as stated (identify the known quantities).

- **Step 3.** Identify exactly what needs to be determined in the problem (identify the unknown quantities). A written list is useful.
- **Step 4.** Determine whether the number of molecules or the number of moles is known or asked for to decide whether to use the ideal gas law as $pV = Nk_BT$, where **N** is the number of molecules, or $pV = nRT$, where **n** is the number of moles.
- **Step 5.** Convert known values into proper SI units (K for temperature, Pa for pressure, m^3 for volume, molecules for **N**, and moles for **n**). If the units of the knowns are consistent with one of the non-SI values of **R**, you can leave them in those units. Be sure to use absolute temperature and absolute pressure.
- **Step 6.** Solve the ideal gas law for the quantity to be determined (the unknown quantity). You may need to take a ratio of final states to initial states to eliminate the unknown quantities that are kept fixed.
- **Step 7.** Substitute the known quantities, along with their units, into the appropriate equation and obtain numerical solutions complete with units.
- **Step 8.** Check the answer to see if it is reasonable: Does it make sense?

The Van der Waals Equation of State

We have repeatedly noted that the ideal gas law is an approximation. How can it be improved upon? The **van der Waals equation of state** (named after the Dutch physicist Johannes **van der Waals**, 1837−1923) improves it by taking into account two factors. First, the attractive forces between molecules, which are stronger at higher density and reduce the pressure, are taken into account by adding to the pressure a term equal to the square of the molar density multiplied by a positive coefficient **a**. Second, the volume of the molecules is represented by a positive constant **b**, which can be thought of as the volume of a mole of molecules. This is subtracted from the total volume to give the remaining volume that the molecules can move in. The constants **a** and **b** are determined experimentally for each gas. The resulting equation is

$$\left[p + a\left(\frac{n}{V}\right)^2\right](V - nb) = nRT.$$

In the limit of low density (small **n**), the **a** and **b** terms are negligible, and we have the ideal gas law, as we should for low density. On the other hand, if $V - nb$ is small, meaning that the molecules are very close together, the pressure must be higher to give the same **nRT**, as we would expect in the situation of a highly compressed gas. However, the increase in pressure is less than that argument would suggest, because at high density the $(n/V)^2$ term is significant. Since it's positive, it causes a lower pressure to give the same **nRT**.

The van der Waals equation of state works well for most gases under a wide variety of conditions. As we'll see in the next module, it even predicts the gas-liquid transition.

pV Diagrams

We can examine aspects of the behavior of a substance by plotting a **pV diagram**, which is a graph of pressure versus volume. When the substance behaves like an ideal gas, the ideal gas law $pV = nRT$ describes the relationship between its pressure and volume. On a **pV** diagram, it's common to plot an **isotherm**, which is a curve showing **p** as a function of **V** with the number of molecules and the temperature fixed. Then, for an ideal gas, $pV = constant.$ For example, the volume of the gas decreases as the pressure increases. The resulting graph is a hyperbola.

However, if we assume the van der Waals equation of state, the isotherms become more interesting, as shown in Figure 6. At high temperatures, the curves are approximately hyperbolas, representing approximately ideal behavior at various fixed temperatures. At lower temperatures, the curves look less and less like hyperbolas—that is, the gas is not behaving ideally. There is a **critical temperature** T_c at which the curve has a point with zero slope. Below that temperature, the curves do not decrease monotonically; instead, they each have a "hump," meaning that for a certain range of volume, increasing the volume increases the pressure.

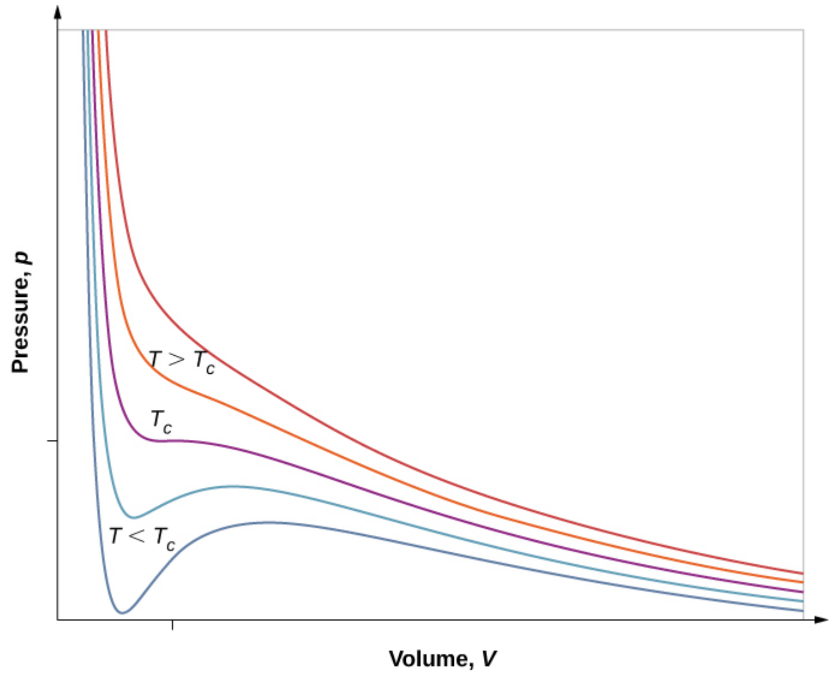


Figure 6: pV diagram for a Van der Waals gas at various temperatures. The red curves are calculated at temperatures above the critical temperature and the blue curves at temperatures below it. The blue curves have an oscillation in which volume (**V**) increases with increasing temperature (**T**), an impossible situation, so they must be corrected as in Figure 7. (credit: "Eman"/Wikimedia Commons)

Such behavior would be completely unphysical. Instead, the curves are understood as describing a liquid-gas **phase transition**. The oscillating part of the curve is replaced by a horizontal line, showing that as the volume increases at constant temperature, the pressure stays constant. That behavior corresponds to boiling and condensation; when a substance is at its boiling temperature for a particular pressure, it can increase in volume as some of the liquid turns to gas, or decrease as some of the gas turns to liquid, without any change in temperature or pressure.

Figure 7 shows similar isotherms that are more realistic than those based on the van der Waals equation. The steep parts of the curves to the left of the transition region show the liquid phase, which is almost incompressible—a slight decrease in volume requires a large increase in pressure. The flat parts show the liquid-gas transition; the blue regions that they define represent combinations of pressure and volume where liquid and gas can coexist.

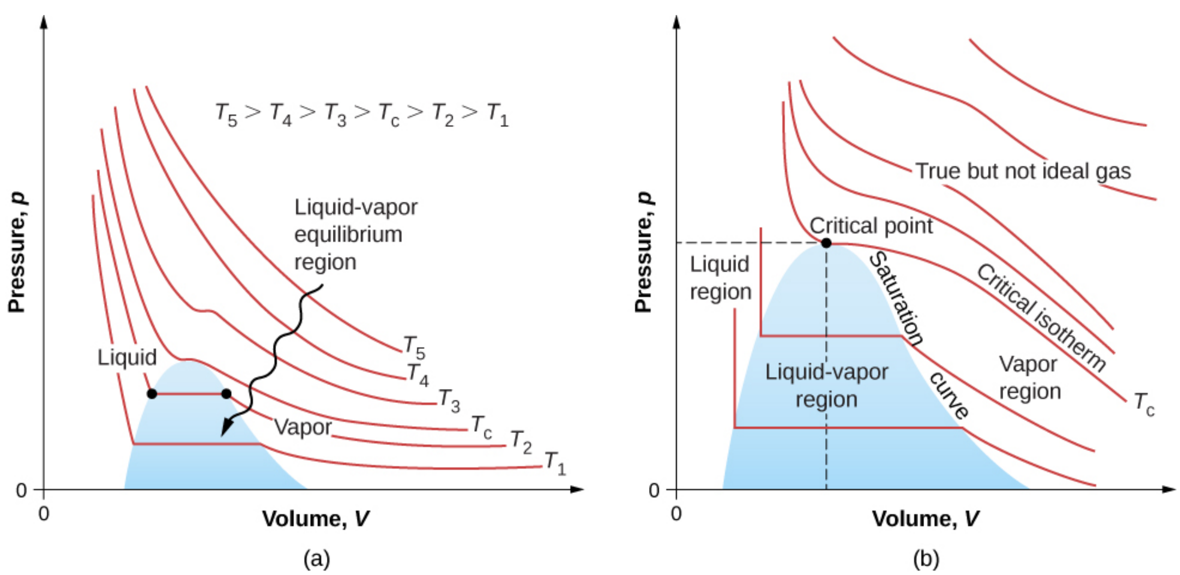


Figure 7: pV diagrams. (a) Each curve (isotherm) represents the relationship between **p** and **V** at a fixed temperature; the upper curves are at higher temperatures. The lower curves are not hyperbolas because the gas is no longer an ideal gas. (b) An expanded portion of the **pV** diagram for low temperatures, where the phase can change from a gas to a liquid. The term "vapor" refers to the gas phase when it exists at a temperature below the boiling temperature.

The isotherms above T_c do not go through the liquid-gas transition. Therefore, liquid cannot exist above that temperature, which is the critical temperature (described in the chapter on temperature and heat). At sufficiently low pressure above that temperature, the gas has the density of a liquid but will not condense; the gas is said to be **supercritical**. At higher pressure, it is solid. Carbon dioxide, for example, has no liquid phase at a temperature above 31.0^oC. The critical pressure is the maximum pressure at which the liquid can exist. The point on the **pV** diagram at the critical pressure and temperature is the critical point (which you learned about in the chapter on temperature and heat). Table lists representative critical temperatures and pressures.

Table 1: Critical Temperatures and Pressures for Various Substances

Substance	**Critical temperature**	T_c	**Critical pressure**	
	K	^oC	Pa	atm
Water	647.4	374.3	22.12×10^6	219.0
Sulfur dioxide	430.7	157.6	7.88×10^6	78.0
Ammonia	405.5	132.4	11.28×10^6	111.7
Carbon dioxide	304.2	31.1	7.39×10^6	73.2
Oxygen	154.8	–118.4	5.08×10^6	50.3
Nitrogen	126.2	–146.9	3.39×10^6	33.6
Hydrogen	33.3	–239.9	1.30×10^6	12.9
Helium	5.3	–267.9	0.229×10^6	2.27

Prelude to The Kinetic Theory of Gases

As we discussed in the preceding chapter, the study of heat and temperature is part of an area of physics known as thermodynamics, in which we require a system to be macroscopic, that is, to consist of a huge number (such as 10^{23}) of molecules. We begin by considering some macroscopic properties of gases: volume, pressure, and temperature. The simple model of a hypothetical "ideal gas" describes these properties of a gas very accurately under many conditions. We move from the ideal gas model to a more widely applicable approximation, called the Van der Waals model.

Figure 1: A volcanic eruption releases tons of gas and dust into the atmosphere. Most of the gas is water vapor, but several other gases are common, including greenhouse gases such as carbon dioxide and acidic pollutants such as sulfur dioxide. However, the emission of volcanic gas is not all bad: Many geologists believe that in the earliest stages of Earth's formation, volcanic emissions formed the early atmosphere. (credit: modification of work by "Boaworm"/Wikimedia Commons)

Gases are literally all around us—the air that we breathe is a mixture of gases. Other gases include those that make breads and cakes soft, those that make drinks fizzy, and those that burn to heat many homes. Engines and refrigerators depend on the behaviors of gases, as we will see in later chapters.

To understand gases even better, we must also look at them on the microscopic scale of molecules. In gases, the molecules interact weakly, so the microscopic behavior of gases is relatively simple, and they serve as a good introduction to systems of many molecules. The molecular model of gases is called the kinetic theory of gases and is one of the classic examples of a molecular model that explains everyday behavior.

Pressure, Temperature, and RMS Speed

Learning Objectives

By the end of this section, you will be able to:

- Explain the relations between microscopic and macroscopic quantities in a gas
- Solve problems involving mixtures of gases
- Solve problems involving the distance and time between a gas molecule's collisions

We have examined pressure and temperature based on their macroscopic definitions. Pressure is the force divided by the area on which the force is exerted, and temperature is measured with a thermometer. We can gain a better understanding of pressure and temperature from the **kinetic theory of gases**, the theory that relates the macroscopic properties of gases to the motion of the molecules they consist of. First, we make two assumptions about molecules in an ideal gas.

1. There is a very large number **N** of molecules, all identical and each having mass **m**.
2. The molecules obey Newton's laws and are in continuous motion, which is random and isotropic, that is, the same in all directions.

To derive the ideal gas law and the connection between microscopic quantities such as the energy of a typical molecule and macroscopic quantities such as temperature, we analyze a sample of an ideal gas in a rigid container, about which we make two further assumptions:

3. The molecules are much smaller than the average distance between them, so their total volume is much less than that of their container (which has volume **V**). In other words, we take the Van der Waals constant **b**, the volume of a mole of gas molecules, to be negligible compared to the volume of a mole of gas in the container.
4. The molecules make perfectly elastic collisions with the walls of the container and with each other. Other forces on them, including gravity and the attractions represented by the Van der Waals constant **a**, are negligible (as is necessary for the assumption of isotropy).

The collisions between molecules do not appear in the derivation of the ideal gas law. They do not disturb the derivation either, since collisions between molecules moving with random velocities give new random velocities. Furthermore, if the velocities of gas molecules in a container are initially not random and isotropic, molecular collisions are what make them random and isotropic.

We make still further assumptions that simplify the calculations but do not affect the result. First, we let the container be a rectangular box. Second, we begin by considering **monatomic** gases, those whose molecules consist of single atoms, such as helium. Then, we can assume that the atoms have no energy except their translational kinetic energy; for instance, they have neither rotational nor vibrational energy. (Later, we discuss the validity of this assumption for real monatomic gases and dispense with it to consider diatomic and polyatomic gases.)

Figure 1 shows a collision of a gas molecule with the wall of a container, so that it exerts a force on the wall (by Newton's third law). These collisions are the source of pressure in a gas. As the number of molecules increases, the number of collisions, and thus the pressure, increases. Similarly, if the average velocity of the molecules is higher, the gas pressure is higher.

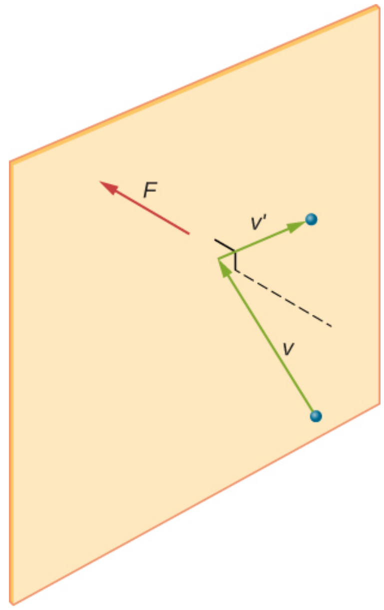


Figure 1: When a molecule collides with a rigid wall, the component of its momentum perpendicular to the wall is reversed. A force is thus exerted on the wall, creating pressure.

In a sample of gas in a container, the randomness of the molecular motion causes the number of collisions of molecules with any part of the wall in a given time to fluctuate. However, because a huge number of molecules collide with the wall in a short time, the number of collisions on the scales of time and space we measure fluctuates by only a tiny, usually unobservable fraction from the average. We can compare this situation to that of a casino, where the outcomes of the bets are random and the casino's takings fluctuate by the minute and the hour. However, over long times such as a year, the casino's takings are very close to the averages expected from the odds. A tank of gas has enormously more molecules than a casino has bettors in a year, and the molecules make enormously more collisions in a second than a casino has bets.

A calculation of the average force exerted by molecules on the walls of the box leads us to the ideal gas law and to the connection between temperature and molecular kinetic energy. (In fact, we will take two averages: one over time to get the average force exerted by one molecule with a given velocity, and then another average over molecules with different velocities.) This approach was developed by Daniel **Bernoulli** (1700–1782), who is best known in physics for his work on fluid flow (hydrodynamics). Remarkably, Bernoulli did this work before Dalton established the view of matter as consisting of atoms.

Figure 2 shows a container full of gas and an expanded view of an elastic collision of a gas molecule with a wall of the container, broken down into components. We have assumed that a molecule is small compared with the separation of molecules in the gas, and that its interaction with other molecules can be ignored. Under these conditions, the ideal gas law is experimentally valid. Because we have also assumed the wall is rigid and the particles are points, the collision is elastic (by conservation of energy—there's nowhere for a particle's kinetic energy to go). Therefore, the molecule's kinetic energy remains constant, and hence, its speed and the magnitude of its momentum remain constant as well. This assumption is not always valid, but the results in the rest of this module are also obtained in models that let the molecules exchange energy and momentum with the wall.

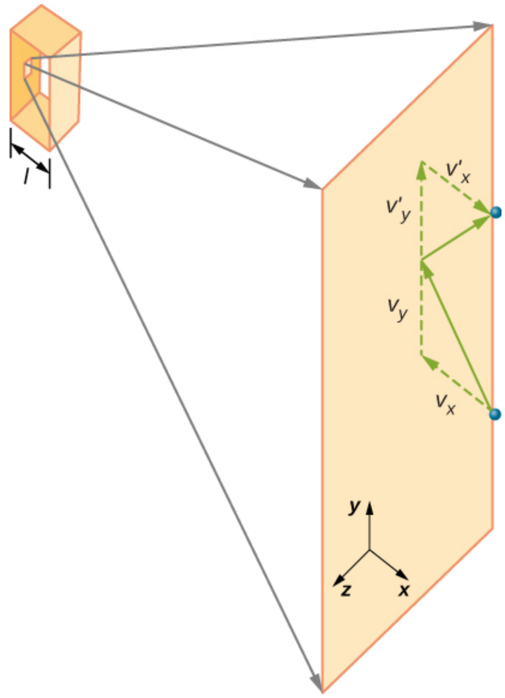


Figure 2: Gas in a box exerts an outward pressure on its walls. A molecule colliding with a rigid wall has its velocity and momentum in the x-direction reversed. This direction is perpendicular to the wall. The components of its velocity momentum in the y- and z-directions are not changed, which means there is no force parallel to the wall.

If the molecule's velocity changes in the **x**-direction, its momentum changes from $-m_x$ to $+mv_x$. Thus, its change in momentum is $\Delta mv = +mv_x - (-mv_x) = 2mv_x$. According to the impulse-momentum theorem given in the chapter on linear momentum and collisions, the force exerted on the **i**th molecule, where **i** labels the molecules from 1 to **N**, is given by

$$F_i = \frac{\Delta P_i}{\Delta t} = \frac{2mv_{ix}}{\Delta t}.$$

(In this equation alone, **p** represents momentum, not pressure.) There is no force between the wall and the molecule except while the molecule is touching the wall. During the short time of the collision, the force between the molecule and wall is relatively large, but that is not the force we are looking for. We are looking for the average force, so we take Δt to be the average time between collisions of the given molecule with this wall, which is the time in which we expect to find one collision. Let **l** represent the length of the box in the **x**-direction. Then Δt is the time the molecule would take to go across the box and back, a distance 2**l**, at a speed of v_x. Thus $\delta t = 2l/v_x$, and the expression for the force becomes

$$F_i = \frac{2mv_{ix}}{2l/v_{ix}} = \frac{mv_{ix}^2}{l}.$$

This force is due to **one** molecule. To find the total force on the wall, **F**, we need to add the contributions of all **N** molecules:

$$F = \sum_{i=1}^{N} F_i = \sum_{i=1}^{N} \frac{mv_{ix}^2}{l} = \frac{m}{l} \sum_{i=1}^{N} v_{ix}^2.$$

We now use the definition of the average, which we denote with a bar, to find the force:

$$F = N\frac{m}{l}\left(\frac{1}{N}\sum_{i=1}^{N} v_{ix}^2\right) = N\frac{mv_{ix}^2}{l}.$$

We want the force in terms of the speed **v**, rather than the **x**-component of the velocity. Note that the total velocity squared is the sum of the squares of its components, so that

$$\bar{v}^2 = \bar{v}_x^2 + \bar{v}_y^2 + \bar{v}_z^2.$$

With the assumption of isotropy, the three averages on the right side are equal, so

$$\bar{v}^2 = 3\bar{v}_{ix}^2.$$

Substituting this into the expression for **F** gives

$$F = N\frac{m\bar{v}^2}{3l}.$$

The pressure is **F**/**A**, so we obtain

$$p = \frac{F}{A} = N\frac{m\bar{v}^2}{3Al} = \frac{Nm\bar{v}^2}{3V},$$

where we used $V = Al$ for the volume. This gives the important result

$$pV = \frac{1}{3}Nm\bar{v}^2.$$

Combining this equation with $pV = Nk_BT$ gives

$$\frac{1}{3}Nm\bar{v}^2 = Nk_BT.$$

We can get the average kinetic energy of a molecule, $\frac{1}{2}m\bar{v}^2$, from the left-hand side of the equation by dividing out **N** and multiplying by 3/2.

Average Kinetic Energy per Molecule

The average kinetic energy of a molecule is directly proportional to its absolute temperature:

$$\overline{K} = \frac{1}{2}m\bar{v}^2 = \frac{3}{2}k_BT.$$

The equation $\overline{K} = \frac{3}{2}k_BT$ is the average kinetic energy per molecule. Note in particular that nothing in this equation depends on the molecular mass (or any other property) of the gas, the pressure, or anything but the temperature. If samples of helium and xenon gas, with very different molecular masses, are at the same temperature, the molecules have the same average kinetic energy.

The **internal energy** of a thermodynamic system is the sum of the mechanical energies of all of the molecules in it. We can now give an equation for the internal energy of a monatomic ideal gas. In such a gas, the molecules' only energy is their translational kinetic energy. Therefore, denoting the internal energy by E_{int} we simply have $E_{int} = N\overline{K}$, or

$$E_{int} = \frac{3}{2}Nk_BT.$$

Often we would like to use this equation in terms of moles:

$$E_{int} = \frac{3}{2}nRT.$$

We can solve $\overline{K} = \frac{1}{2}m\bar{v}^2 = \frac{3}{2}k_BT$ for a typical speed of a molecule in an ideal gas in terms of temperature to determine what is known as the **root-mean-square** (**rms**) **speed** of a molecule.

RMS Speed of a Molecule

The root-mean-square (rms) speed of a molecule, or the square root of the average of the square of the speed $\bar{v}^2$, is

$$v_{rms} = \sqrt{\bar{v}^2} = \sqrt{\frac{3k_BT}{m}}.$$

The rms speed is not the average or the most likely speed of molecules, as we will see in Distribution of Molecular Speeds, but it provides an easily calculated estimate of the molecules' speed that is related to their kinetic energy. Again we can write this equation in terms of the gas constant **R** and the molar mass **M** in kg/mol:

$$v_{rms} = \sqrt{\frac{3\,RT}{M}}.$$

We digress for a moment to answer a question that may have occurred to you: When we apply the model to atoms instead of theoretical point particles, does rotational kinetic energy change our results? To answer this question, we have to appeal to quantum mechanics. In quantum mechanics, rotational kinetic energy cannot take on just any value; it's limited to a discrete set of values, and the smallest value is inversely proportional to the rotational inertia. The rotational inertia of an atom is tiny because almost all of its mass is in the nucleus, which typically has a radius less than $10^{-14}m$. Thus the minimum rotational energy of an atom is much more than $\frac{1}{2}k_BT$ for any attainable temperature, and the energy available is not enough to make an atom rotate. We will return to this point when discussing diatomic and polyatomic gases in the next section.

Example 1: Calculating Kinetic Energy and Speed of a Gas Molecule

a. What is the average kinetic energy of a gas molecule at 20.0^oC (room temperature)?
b. Find the rms speed of a nitrogen molecule (N_2) at this temperature.

Strategy

(a) The known in the equation for the average kinetic energy is the temperature:

$$\overline{K} = \frac{1}{2}m\bar{v}^2 = \frac{3}{2}k_BT.$$

Before substituting values into this equation, we must convert the given temperature into kelvin: $T = (20.0 + 273)\,K = 293\,K$. We can find the rms speed of a nitrogen molecule by using the equation

$$v_{rms} = \sqrt{\bar{v}^2} = \sqrt{\frac{3k_BT}{m}},$$

but we must first find the mass of a nitrogen molecule. Obtaining the molar mass of nitrogen N_2 from the periodic table, we find

$$m = \frac{M}{N_A} = \frac{2(14.0067 \times 10^{-3}\,kg/mol)}{6.02 \times 10^{23} mol^{-1}} = 4.65 \times 10^{-26}\,kg.$$

Solution

a. The temperature alone is sufficient for us to find the average translational kinetic energy. Substituting the temperature into the translational kinetic energy equation gives

$$\overline{K} = \frac{3}{2}k_BT = \frac{3}{2}(1.38 \times 10^{-23}\,J/K)(293\,K) = 6.07 \times 10^{-21}\,J.$$

b. Substituting this mass and the value for k_B into the equation for v_{rms} yields

$$v_{rms} = \sqrt{\frac{3k_BT}{m}} = \sqrt{\frac{3(1.38 \times 10^{-23}\,J/K)(293\,K)}{4.65 \times 10^{-26}\,kg}} = 511\,m/s.$$

Significance

Note that the average kinetic energy of the molecule is independent of the type of molecule. The average translational kinetic energy depends only on absolute temperature. The kinetic energy is very small compared to macroscopic energies, so that we do not feel when an air molecule is hitting our skin. On the other hand, it is much greater than the typical difference in gravitational potential energy when a molecule moves from, say, the top to the bottom of a room, so our neglect of gravitation is justified in typical real-world situations. The rms speed of the nitrogen molecule is surprisingly large. These large molecular velocities do not yield macroscopic movement of air, since the molecules move in all directions with equal likelihood. The **mean free path** (the distance a molecule moves on average between collisions, discussed a bit later in this section) of molecules in air is very small, so the molecules move rapidly but do not get very far in a second. The high value for rms speed is reflected in the speed of sound, which is about 340 m/s at room temperature. The higher the rms speed of air molecules, the faster sound vibrations can be transferred through the air. The speed of sound increases with temperature and is greater in gases with small molecular masses, such as helium (see Figure 3).

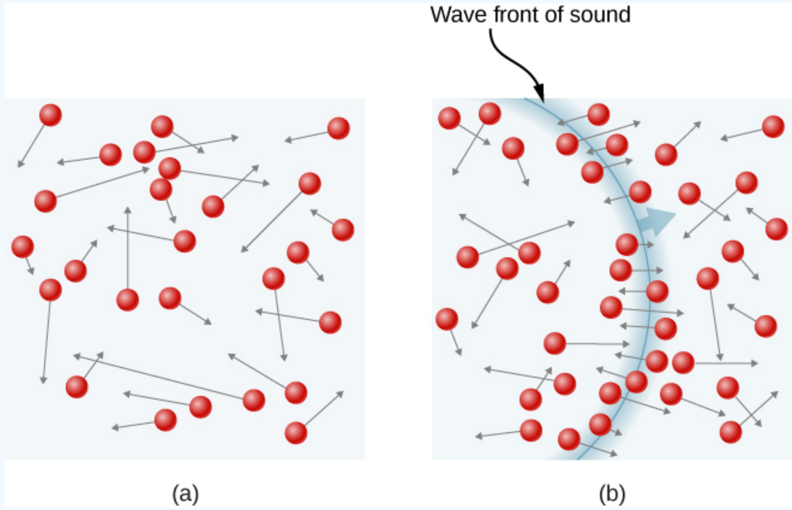


Figure 3: (a) In an ordinary gas, so many molecules move so fast that they collide billions of times every second. (b) Individual molecules do not move very far in a small amount of time, but disturbances like sound waves are transmitted at speeds related to the molecular speeds.

Example 2: Calculating Temperature: Escape Velocity of Helium Atoms

To escape Earth's gravity, an object near the top of the atmosphere (at an altitude of 100 km) must travel away from Earth at 11.1 km/s. This speed is called the **escape velocity**. At what temperature would helium atoms have an rms speed equal to the escape velocity?

Strategy

Identify the knowns and unknowns and determine which equations to use to solve the problem.

Solution

1. Identify the knowns: **v** is the escape velocity, 11.1 km/s.
2. Identify the unknowns: We need to solve for temperature, **T**. We also need to solve for the mass **m** of the helium atom.
3. Determine which equations are needed.
   - To get the mass **m** of the helium atom, we can use information from the periodic table:
   $$m=\frac{M}{N_A},$$
   - To solve for temperature **T**, we can rearrange
   $$\frac{1}{2}m\bar{v}^2=\frac{3}{2}k_BT$$
   to yield
   $$T=\frac{m\bar{v}^2}{3k_B}.$$
4. Substitute the known values into the equations and solve for the unknowns,
$$m=\frac{M}{N_A}=\frac{4.0026\times 10^{-3}kg/mol}{6.02\times 10^{23}mol}=6.65\times 10^{-27}kg$$
and

$$T = \frac{(6.65 \times 10^{-27} kg)(11.1 \times 10^3 m/s)^2}{3(1.38 \times 10^{-23} J/K)} = 1.98 \times 10^4 \, K.$$

Significance

This temperature is much higher than atmospheric temperature, which is approximately 250 K (-25^oC or -10^oF) at high elevation. Very few helium atoms are left in the atmosphere, but many were present when the atmosphere was formed, and more are always being created by radioactive decay (see the chapter on nuclear physics). The reason for the loss of helium atoms is that a small number of helium atoms have speeds higher than Earth's escape velocity even at normal temperatures. The speed of a helium atom changes from one collision to the next, so that at any instant, there is a small but nonzero chance that the atom's speed is greater than the escape velocity. The chance is high enough that over the lifetime of Earth, almost all the helium atoms that have been in the atmosphere have reached escape velocity at high altitudes and escaped from Earth's gravitational pull. Heavier molecules, such as oxygen, nitrogen, and water, have smaller rms speeds, and so it is much less likely that any of them will have speeds greater than the escape velocity. In fact, the likelihood is so small that billions of years are required to lose significant amounts of heavier molecules from the atmosphere. Figure 4 shows the effect of a lack of an atmosphere on the Moon. Because the gravitational pull of the Moon is much weaker, it has lost almost its entire atmosphere. The atmospheres of Earth and other bodies are compared in this chapter's exercises.

Figure 4:This photograph of Apollo 17 Commander Eugene Cernan driving the lunar rover on the Moon in 1972 looks as though it was taken at night with a large spotlight. In fact, the light is coming from the Sun. Because the acceleration due to gravity on the Moon is so low (about 1/6 that of Earth), the Moon's escape velocity is much smaller. As a result, gas molecules escape very easily from the Moon, leaving it with virtually no atmosphere. Even during the daytime, the sky is black because there is no gas to scatter sunlight. (credit: Harrison H. Schmitt/NASA)

? Exercise 2

If you consider a very small object, such as a grain of pollen, in a gas, then the number of molecules striking its surface would also be relatively small. Would you expect the grain of pollen to experience any fluctuations in pressure due to statistical fluctuations in the number of gas molecules striking it in a given amount of time?

Answer

Yes. Such fluctuations actually occur for a body of any size in a gas, but since the numbers of molecules are immense for macroscopic bodies, the fluctuations are a tiny percentage of the number of collisions, and the averages spoken of in this section vary imperceptibly. Roughly speaking, the fluctuations are inversely proportional to the square root of the number of collisions, so for small bodies, they can become significant. This was actually observed in the nineteenth century for pollen grains in water and is known as Brownian motion.

Vapor Pressure, Partial Pressure, and Dalton's Law

The pressure a gas would create if it occupied the total volume available is called the gas's **partial pressure**. If two or more gases are mixed, they will come to thermal equilibrium as a result of collisions between molecules; the process is analogous to heat conduction as described in the chapter on temperature and heat. As we have seen from kinetic theory, when the gases have the same temperature, their molecules have the same average kinetic energy. Thus, each gas obeys the ideal gas law separately and exerts the same pressure on the walls of a container that it would if it were alone. Therefore, in a mixture of gases, **the total pressure is the sum of partial pressures of the component gases**, assuming ideal gas behavior and no chemical reactions between the components. This law is known as **Dalton's law of partial pressures**, after the English scientist John **Dalton** (1766–1844) who proposed it. Dalton's law is consistent with the fact that pressures add according to Pascal's principle.

In a mixture of ideal gases in thermal equilibrium, the number of molecules of each gas is proportional to its partial pressure. This result follows from applying the ideal gas law to each in the form $p/n = RT/V$. Because the right-hand side is the same for any gas at a given temperature in a container of a given volume, the left-hand side is the same as well.

- Partial pressure is the pressure a gas would create if it existed alone.
- Dalton's law states that the total pressure is the sum of the partial pressures of all of the gases present.
- For any two gases (labeled 1 and 2) in equilibrium in a container, $\frac{p_1}{n_1} = \frac{p_2}{n_2}$.

An important application of partial pressure is that, in chemistry, it functions as the concentration of a gas in determining the rate of a reaction. Here, we mention only that the partial pressure of oxygen in a person's lungs is crucial to life and health. Breathing air that has a partial pressure of oxygen below 0.16 atm can impair coordination and judgment, particularly in people not acclimated to a high elevation. Lower partial pressures of O_2 have more serious effects; partial pressures below 0.06 atm can be quickly fatal, and permanent damage is likely even if the person is rescued. However, the sensation of needing to breathe, as when holding one's breath, is caused much more by high concentrations of carbon dioxide in the blood than by low concentrations of oxygen. Thus, if a small room or closet is filled with air having a low concentration of oxygen, perhaps because a leaking cylinder of some compressed gas is stored there, a person will not feel any "choking" sensation and may go into convulsions or lose consciousness without noticing anything wrong. Safety engineers give considerable attention to this danger.

Another important application of partial pressure is **vapor pressure**, which is the partial pressure of a vapor at which it is in equilibrium with the liquid (or solid, in the case of sublimation) phase of the same substance. At any temperature, the partial pressure of the water in the air cannot exceed the vapor pressure of the water at that temperature, because whenever the partial pressure reaches the vapor pressure, water condenses out of the air. Dew is an example of this condensation. The temperature at which condensation occurs for a sample of air is called the **dew point**. It is easily measured by slowly cooling a metal ball; the dew point is the temperature at which condensation first appears on the ball.

The vapor pressures of water at some temperatures of interest for meteorology are given in Table 1.

Table 1: Vapor Pressure of Water at Various Temperatures

T(^oC)	Vapor Pressure (Pa)
0	610.5
3	757.9
5	872.3
8	1073
10	1228
13	1497
15	1705
18	2063
20	2338
23	2809
25	3167

T(^oC)	Vapor Pressure (Pa)
30	4243
35	5623
40	7376

The **relative humidity** (R.H.) at a temperature **T** is defined by

$$R.H. = \frac{Partial\,pressure\,of\,water\,vapor\,at\,T}{Vapor\,pressure\,of\,water\,at\,T} \times 100\%.$$

A relative humidity of 100% means that the partial pressure of water is equal to the vapor pressure; in other words, the air is saturated with water.

Example 3: Calculating Relative Humidity

What is the relative humidity when the air temperature is 25^oC and the dew point is 15^oC?

Strategy

We simply look up the vapor pressure at the given temperature and that at the dew point and find the ratio.

Solution

$$R.H. = \frac{Partial\,pressure\,of\,water\,vapor\,at\,15^oC}{Vapor\,pressure\,of\,water\,at\,25^oC} \times 100\% = \frac{1705\ Pa}{3167\ Pa} \times 100\% = 53.8\%.$$

Significance

R.H. is important to our comfort. The value of 53.8% is within the range of 40% to 60% recommended for comfort indoors.

As noted in the chapter on temperature and heat, the temperature seldom falls below the dew point, because when it reaches the dew point or frost point, water condenses and releases a relatively large amount of latent heat of vaporization.

Mean Free Path and Mean Free Time

We now consider collisions explicitly. The usual first step (which is all we'll take) is to calculate the mean free path, λ, the average distance a molecule travels between collisions with other molecules, and the **mean free time** τ, the average time between the collisions of a molecule. If we assume all the molecules are spheres with a radius r then a molecule will collide with another if their centers are within a distance 2**r** of each other. For a given particle, we say that the area of a circle with that radius, $4\pi r^2$, is the "cross-section" for collisions. As the particle moves, it traces a cylinder with that cross-sectional area. The mean free path is the length λ such that the expected number of other molecules in a cylinder of length λ and cross-section $4\pi r^2$ is 1. If we temporarily ignore the motion of the molecules other than the one we're looking at, the expected number is the number density of molecules, **N/V**, times the volume, and the volume is $4\pi r^2\lambda$, so we have $(N/V)4\pi r^2\lambda = 1$, or

$$\lambda = \frac{V}{4\pi r^2 N}.$$

Taking the motion of all the molecules into account makes the calculation much harder, but the only change is a factor of $\sqrt{2}$. The result is

$$\lambda = \frac{V}{4\sqrt{2}\pi r^2 N}.$$

In an ideal gas, we can substitute $V/N = k_BT/p$ to obtain

$$\lambda = \frac{k_BT}{4\sqrt{2}\pi r^2 p}.$$

The **mean free time** τ is simply the mean free path divided by a typical speed, and the usual choice is the rms speed. Then

$$tau = \frac{k_B T}{4\sqrt{2}\pi r^2 p v_{rms}}.$$

✓ Calculating Mean Free Time

Find the mean free time for argon atoms ($M = 39.9\, g/mol$) at a temperature of $0^o C$ and a pressure of 1.00 atm. Take the radius of an argon atom to be $1.70 \times 10^{-10}\ m$.

Solution

1. Identify the knowns and convert into SI units. We know the molar mass is 0.0399 kg/mol, the temperature is 273 K, the pressure is $1.01 \times 10^5\ Pa$, and the radius is $1.70 \times 10^{-10}\ m$.
2. Find the rms speed: $v_{rms} = \sqrt{\frac{3RT}{M}} = 413\, m/s$.
3. Substitute into the equation for the mean free time:

$$\tau = \frac{k_B T}{4\sqrt{2}\pi r^2 p v_{rms}} = \frac{(1.38 \times 10^{-23}\ J/K)(273\ K)}{4\sqrt{2}\pi(1.70 \times 10^{-10} m)^2(1.01 \times 10^5\ Pa)(413\, m/s)} = 1.76 \times 10^{-10}\ s.$$

Significance

We can hardly compare this result with our intuition about gas molecules, but it gives us a picture of molecules colliding with extremely high frequency.

? Exercise 4

Which has a longer mean free path, liquid water or water vapor in the air?

Answer

In a liquid, the molecules are very close together, constantly colliding with one another. For a gas to be nearly ideal, as air is under ordinary conditions, the molecules must be very far apart. Therefore the mean free path is much longer in the air.

The Kinetic Theory of Gases (Answer)

Check Your Understanding

2.1. We first need to calculate the molar mass (the mass of one mole) of niacin. To do this, we must multiply the number of atoms of each element in the molecule by the element's molar mass.

$$(6 mol of carbon)(12.0 g/mol) + (5 mol hydrogen)(1.0 g/mol) + (1 mol nitrogen)(14 g/mol) \\ + (2 mol oxygen)(16.0 g/mol) = 123 g/mol$$

Then we need to calculate the number of moles in 14 mg.

$$(\frac{14 mg}{123 g/mol})(\frac{1 g}{1000 mg}) = 1.14 \times 10^{-4} mol.$$

Then, we use Avogadro's number to calculate the number of molecules:

$$N = nN_A = (1.14 \times 10^{-4} mol)(6.02 \times 10^{23} molecules/mol) = 6.85 \times 10^{19} molecules.$$

2.2. The density of a gas is equal to a constant, the average molecular mass, times the number density N/V. From the ideal gas law, $pV = Nk_BT$, we see that $N/V = p/k_BT$. Therefore, at constant temperature, if the density and, consequently, the number density are reduced by half, the pressure must also be reduced by half, and $p_f = 0.500 atm$.

2.3. Density is mass per unit volume, and volume is proportional to the size of a body (such as the radius of a sphere) cubed. So if the distance between molecules increases by a factor of 10, then the volume occupied increases by a factor of 1000, and the density decreases by a factor of 1000. Since we assume molecules are in contact in liquids and solids, the distance between their centers is on the order of their typical size, so the distance in gases is on the order of 10 times as great.

2.4. Yes. Such fluctuations actually occur for a body of any size in a gas, but since the numbers of molecules are immense for macroscopic bodies, the fluctuations are a tiny percentage of the number of collisions, and the averages spoken of in this section vary imperceptibly. Roughly speaking, the fluctuations are inversely proportional to the square root of the number of collisions, so for small bodies, they can become significant. This was actually observed in the nineteenth century for pollen grains in water and is known as Brownian motion.

2.5. In a liquid, the molecules are very close together, constantly colliding with one another. For a gas to be nearly ideal, as air is under ordinary conditions, the molecules must be very far apart. Therefore the mean free path is much longer in the air.

2.6. As the number of moles is equal and we know the molar heat capacities of the two gases are equal, the temperature is halfway between the initial temperatures, 300 K.

Conceptual Questions

1. 2 moles, as that will contain twice as many molecules as the 1 mole of oxygen

3. pressure

5. The flame contains hot gas (heated by combustion). The pressure is still atmospheric pressure, in mechanical equilibrium with the air around it (or roughly so). The density of the hot gas is proportional to its number density **N/V** (neglecting the difference in composition between the gas in the flame and the surrounding air). At higher temperature than the surrounding air, the ideal gas law says that $N/V = p/k_BT$ is less than that of the surrounding air. Therefore the hot air has lower density than the surrounding air and is lifted by the buoyant force.

7. The mean free path is inversely proportional to the square of the radius, so it decreases by a factor of 4. The mean free time is proportional to the mean free path and inversely proportional to the rms speed, which in turn is inversely proportional to the square root of the mass. That gives a factor of $\sqrt{8}$ in the numerator, so the mean free time decreases by a factor of $\sqrt{2}$.

9. Since they're more massive, their gravity is stronger, so the escape velocity from them is higher. Since they're farther from the Sun, they're colder, so the speeds of atmospheric molecules including hydrogen and helium are lower. The combination of those facts means that relatively few hydrogen and helium molecules have escaped from the outer planets.

11. One where nitrogen is stored, as excess CO_2 will cause a feeling of suffocating, but excess nitrogen and insufficient oxygen will not.

13. Less, because at lower temperatures their heat capacity was only **3RT/2.**

15. a. false; b. true; c. true; d. true

17. 1200 K

Problems

19. a. 0.137 atm;

b. $p_g = (1atm)\frac{T_2V_1}{T_1V_2} - 1atm$. Because of the expansion of the glass, $V_2 = 0.99973$. Multiplying by that factor does not make any significant difference.

21. a. $1.79 \times 10^{-3} mol$;

b. 0.227 mol;

c. 1.08×10^{21} molecules for the nitrogen, 1.37×10^{23} molecules for the carbon dioxide

23. $7.84 \times 10^{-2} mol$

25. 1.87×10^{3}

27. 2.47×10^{7} molecules

29. $6.95 \times 10^{5} Pa$;; 6.86 atm

31. a. $9.14 \times 10^{6} Pa$;

b. $8.22 \times 10^{6} Pa$;

c. 2.15 K;

d. no

33. 40.7 km

35. a. 0.61 N;

b. 0.20 Pa

37. a. 5.88 m/s;

b. 5.89 m/s

39. 177 m/s

41. 4.54×10^{3}

43. a. 0.0352 mol;

b. $5.65 \times 10^{-21} J$;

c. 139 J

45. 21.1 kPa

47. 458 K

49. $3.22 \times 10^{3} K$

51. a. 1.004;

b. 764 K;

c. This temperature is equivalent to **915ºF**, which is high but not impossible to achieve. Thus, this process is feasible. At this temperature, however, there may be other considerations that make the process difficult. (In general, uranium enrichment by gaseous diffusion is indeed difficult and requires many passes.)

53. 65 mol

55. a. 0.76 atm;

b. 0.29 atm;

c. The pressure there is barely above the quickly fatal level.

57. $4.92 \times 10^5 K$; Yes, that's an impractically high temperature.

59. polyatomic

61. 3.08×10^3J\)

63. $29.2^\circ C$

65. $-1.6^\circ C$

67. 0.00157

69. About 0.072. Answers may vary slightly. A more accurate answer is 0.074.

71. a. 419 m/s;

b. 472 m/s;

c. 513 m/s

73. 541 K

75. 2400 K for all three parts

Additional Problems

77. a. $1.20 kg/m^3$;

b. $65.9 kg/m^3$

79. 7.9 m

81. a. supercritical fluid;

b. $3.00 \times 10^7 Pa$

83. 40.18%

85. a. $2.21 \times 10^{27} molecules/m^3$;

b. $3.67 \times 10^3 mol/m^3$

87. 8.2 mm

89. a. **1080J/kg°C**;

b. **12%**

91. **\(\displaystyle\displaystyle** 2\sqrt{e}/3\) about 1.10

93. a. 411 m/s;

b. According to Table 2.3, the C_V of H_2 is significantly different from the theoretical value, so the ideal gas model does not describe it very well at room temperature and pressure, and the Maxwell-Boltzmann speed distribution for ideal gases may not hold very well, even less well at a lower temperature.

Challenge Problems

95. 29.5 N/m

97. Substituting $v = \sqrt{\frac{2k_BT}{m}}u$ and $dv = \sqrt{\frac{2k_BT}{m}}du$ gives

$$\int_0^\infty \frac{4}{\sqrt{\pi}}(\frac{m}{2k_BT})^{3/2}v^2e^{-mv^2/2k_BT}dv = \int_0^\infty \frac{4}{\sqrt{\pi}}(\frac{m}{2k_BT})^{3/2}(\frac{2k_BT}{m})u^2e^{-u^2}\sqrt{\frac{2k_BT}{m}}du = \int_0^\infty \frac{4}{\sqrt{\pi}}u^2e^{-u^2}du$$

$$= \frac{4}{\sqrt{\pi}}\frac{\sqrt{\pi}}{4} = 1$$

99. Making the scaling transformation as in the previous problems, we find that $\bar{v^2} = \int_0^\infty \frac{4}{\sqrt{\pi}} (\frac{m}{2k_BT})^{3/2} v^2 v^2 e^{-mv^2/2k_BT} dv = \int_0^\infty \frac{4}{\sqrt{\pi}} \frac{2k_BT}{m} u^4 e^{-u^2} du$. As in the previous problem, we integrate by parts: $\int_0^\infty u^4 e^{-u^2} du = [-\frac{1}{2} u^3 e^{-u^2}]_0^\infty + \frac{3}{2} \int_0^\infty u^2 e^{-u^2} du$. Again, the first term is 0, and we were given in an earlier problem that the integral in the second term equals $\frac{\sqrt{\pi}}{4}$. We now have $\bar{v^2} = \frac{4}{\sqrt{\pi}} \frac{2k_BT}{m} \frac{3}{2} \frac{\sqrt{\pi}}{4} = \frac{3k_BT}{m}$. Taking the square root of both sides gives the desired result: $v_{rms} = \sqrt{\frac{3k_BT}{m}}$.

The Kinetic Theory of Gases (Summary)

Key Terms

Avogadro's number	N_A, the number of molecules in one mole of a substance; $N_A = 6.02 \times 10^{23}$ particles/mole
Boltzmann constant	k_B, a physical constant that relates energy to temperature and appears in the ideal gas law; $k_B = 1.38 \times 10^{-23} J/K$
critical temperature	T_c at which the isotherm has a point with zero slope
Dalton's law of partial pressures	physical law that states that the total pressure of a gas is the sum of partial pressures of the component gases
degree o+f freedom	independent kind of motion possessing energy, such as the kinetic energy of motion in one of the three orthogonal spatial directions
equipartition theorem	theorem that the energy of a classical thermodynamic system is shared equally among its degrees of freedom
ideal gas	gas at the limit of low density and high temperature
ideal gas law	physical law that relates the pressure and volume of a gas, far from liquefaction, to the number of gas molecules or number of moles of gas and the temperature of the gas
internal energy	sum of the mechanical energies of all of the molecules in it
kinetic theory of gases	theory that derives the macroscopic properties of gases from the motion of the molecules they consist of
Maxwell-Boltzmann distribution	function that can be integrated to give the probability of finding ideal gas molecules with speeds in the range between the limits of integration
mean free path	average distance between collisions of a particle
mean free time	average time between collisions of a particle
mole	quantity of a substance whose mass (in grams) is equal to its molecular mass
most probable speed	speed near which the speeds of most molecules are found, the peak of the speed distribution function
partial pressure	pressure a gas would create if it occupied the total volume of space available
peak speed	same as "most probable speed"
pV diagram	graph of pressure vs. volume
root-mean-square (rms) speed	square root of the average of the square (of a quantity)
supercritical	condition of a fluid being at such a high temperature and pressure that the liquid phase cannot exist
universal gas constant	**R**, the constant that appears in the ideal gas law expressed in terms of moles, given by $R = N_A k_B$
van der Waals equation of state	equation, typically approximate, which relates the pressure and volume of a gas to the number of gas molecules or number of moles of gas and the temperature of the gas

vapor pressure	partial pressure of a vapor at which it is in equilibrium with the liquid (or solid, in the case of sublimation) phase of the same substance

Key Equations

Ideal gas law in terms of molecules	$pV = Nk_BT$
Ideal gas law ratios if the amount of gas is constant	$\frac{p_1V_1}{T_1} = \frac{p_2V_2}{T_2}$
Ideal gas law in terms of moles	$pV = nRT$
Van der Waals equation	$[p + a(\frac{n}{V})^2](V - nb) = nRT$
Pressure, volume, and molecular speed	$pV = \frac{1}{3}Nm\bar{v^2}$
Root-mean-square speed	$v_{rms} = \sqrt{\frac{3RT}{M}} = \sqrt{\frac{3k_BT}{m}}$
Mean free path	$\lambda = \frac{V}{4\sqrt{2}\pi r^2 N} = \frac{k_BT}{4\sqrt{2}\pi r^2}$
Mean free time	$\tau = \frac{k_BT}{4\sqrt{2}\pi r^2 p v_{rms}}$

The following two equations apply only to a monatomic ideal gas:

Average kinetic energy of a molecule	$\bar{K} = \frac{3}{2}k_BT$
Internal energy	$E_{int} = \frac{3}{2}Nk_BT.$
Heat in terms of molar heat capacity at constant volume	$Q = nCV\Delta T$
Molar heat capacity at constant volume for an ideal gas with d degrees of freedom	$C_V = \frac{d}{2}R$
Maxwell–Boltzmann speed distribution	$f(v) = \frac{4}{\sqrt{\pi}}(\frac{m}{2k_BT})^{3/2}v^2e^{-mv^2/2k_BT}$
Average velocity of a molecule	$\bar{v} = \sqrt{\frac{8}{\pi}\frac{k_BT}{m}} = \sqrt{\frac{8}{\pi}\frac{RT}{M}}$
Peak velocity of a molecule	$v_p = \sqrt{\frac{2k_BT}{m}} = \sqrt{\frac{2RT}{M}}$

Summary

2.2 Molecular Model of an Ideal Gas

- The ideal gas law relates the pressure and volume of a gas to the number of gas molecules and the temperature of the gas.
- A mole of any substance has a number of molecules equal to the number of atoms in a 12-g sample of carbon-12. The number of molecules in a mole is called Avogadro's number N_A,

$$N_A = 6.02 \times 10^{23} mol^{-1}.$$

- A mole of any substance has a mass in grams numerically equal to its molecular mass in unified mass units, which can be determined from the periodic table of elements. The ideal gas law can also be written and solved in terms of the number of moles of gas:

$$pV = nRT,$$

  where **n** is the number of moles and **R** is the universal gas constant,

$$R = 8.31 J/mol \cdot K.$$

- The ideal gas law is generally valid at temperatures well above the boiling temperature.
- The van der Waals equation of state for gases is valid closer to the boiling point than the ideal gas law.

- Above the critical temperature and pressure for a given substance, the liquid phase does not exist, and the sample is “supercritical.”

2.3 Pressure, Temperature, and RMS Speed

- Kinetic theory is the atomic description of gases as well as liquids and solids. It models the properties of matter in terms of continuous random motion of molecules.
- The ideal gas law can be expressed in terms of the mass of the gas’s molecules and $\bar{v^2}$, the average of the molecular speed squared, instead of the temperature.
- The temperature of gases is proportional to the average translational kinetic energy of molecules. Hence, the typical speed of gas molecules v_{rms} is proportional to the square root of the temperature and inversely proportional to the square root of the molecular mass.
- In a mixture of gases, each gas exerts a pressure equal to the total pressure times the fraction of the mixture that the gas makes up.
- The mean free path (the average distance between collisions) and the mean free time of gas molecules are proportional to the temperature and inversely proportional to the molar density and the molecules’ cross-sectional area.

2.4 Heat Capacity and Equipartition of Energy

- Every degree of freedom of an ideal gas contributes $\frac{1}{2}k_BT$ per atom or molecule to its changes in internal energy.
- Every degree of freedom contributes $\frac{1}{2}R$ to its molar heat capacity at constant volume C_V.
- Degrees of freedom do not contribute if the temperature is too low to excite the minimum energy of the degree of freedom as given by quantum mechanics. Therefore, at ordinary temperatures, **d=3** for monatomic gases, **d=5** for diatomic gases, and **d≈6** for polyatomic gases.

2.5 Distribution of Molecular Speeds

- The motion of individual molecules in a gas is random in magnitude and direction. However, a gas of many molecules has a predictable distribution of molecular speeds, known as the Maxwell-Boltzmann distribution.
- The average and most probable velocities of molecules having the Maxwell-Boltzmann speed distribution, as well as the rms velocity, can be calculated from the temperature and molecular mass.

The Kinetic Theory of Gases Introduction (Exercises)

Conceptual Questions

2.1 Molecular Model of an Ideal Gas

1. Two H_2 molecules can react with one (O_2\) molecule to produce two H_2O molecules. How many moles of hydrogen molecules are needed to react with one mole of oxygen molecules?

2. Under what circumstances would you expect a gas to behave significantly differently than predicted by the ideal gas law?

3. A constant-volume gas thermometer contains a fixed amount of gas. What property of the gas is measured to indicate its temperature?

4. Inflate a balloon at room temperature. Leave the inflated balloon in the refrigerator overnight. What happens to the balloon, and why?

5. In the last chapter, free convection was explained as the result of buoyant forces on hot fluids. Explain the upward motion of air in flames based on the ideal gas law.

2.2 Pressure, Temperature, and RMS Speed

6. How is momentum related to the pressure exerted by a gas? Explain on the molecular level, considering the behavior of molecules.

7. If one kind of molecule has double the radius of another and eight times the mass, how do their mean free paths under the same conditions compare? How do their mean free times compare?

8. What is the average velocity of the air molecules in the room where you are right now?

9. Why do the atmospheres of Jupiter, Saturn, Uranus, and Neptune, which are much more massive and farther from the Sun than Earth is, contain large amounts of hydrogen and helium?

10. Statistical mechanics says that in a gas maintained at a constant temperature through thermal contact with a bigger system (a “reservoir”) at that temperature, the fluctuations in internal energy are typically a fraction $1/\sqrt{N}$ of the internal energy. As a fraction of the total internal energy of a mole of gas, how big are the fluctuations in the internal energy? Are we justified in ignoring them?

11. Which is more dangerous, a closet where tanks of nitrogen are stored, or one where tanks of carbon dioxide are stored?

2.3 Heat Capacity and Equipartition of Energy

12. Experimentally it appears that many polyatomic molecules’ vibrational degrees of freedom can contribute to some extent to their energy at room temperature. Would you expect that fact to increase or decrease their heat capacity from the value R? Explain.

13. One might think that the internal energy of diatomic gases is given by $E_{int} = 5RT/2.$. Do diatomic gases near room temperature have more or less internal energy than that? Hint: Their internal energy includes the total energy added in raising the temperature from the boiling point (very low) to room temperature.

14. You mix 5 moles of H_2 at 300 K with 5 moles of He at 360 K in a perfectly insulated calorimeter. Is the final temperature higher or lower than 330 K?

2.4 Distribution of Molecular Speeds

15. One cylinder contains helium gas and another contains krypton gas at the same temperature. Mark each of these statements true, false, or impossible to determine from the given information.

(a) The rms speeds of atoms in the two gases are the same.

(b) The average kinetic energies of atoms in the two gases are the same.

(c) The internal energies of 1 mole of gas in each cylinder are the same.

(d) The pressures in the two cylinders are the same.

16. Repeat the previous question if one gas is still helium but the other is changed to fluorine, F_2.

17. An ideal gas is at a temperature of 300 K. To double the average speed of its molecules, what does the temperature need to be changed to?

Problems

2.1 Molecular Model of an Ideal Gas

18. The gauge pressure in your car tires is $2.50 \times 10^5 N/m^2$ a temperature of **35.0°C** when you drive it onto a ship in Los Angeles to be sent to Alaska. What is their gauge pressure on a night in Alaska when their temperature has dropped to **−40.0°C**? Assume the tires have not gained or lost any air.

19. Suppose a gas-filled incandescent light bulb is manufactured so that the gas inside the bulb is at atmospheric pressure when the bulb has a temperature of **20.0°C**.

(a) Find the gauge pressure inside such a bulb when it is hot, assuming its average temperature is **60.0°C** (an approximation) and neglecting any change in volume due to thermal expansion or gas leaks.

(b) The actual final pressure for the light bulb will be less than calculated in part (a) because the glass bulb will expand. Is this effect significant?

20. People buying food in sealed bags at high elevations often notice that the bags are puffed up because the air inside has expanded. A bag of pretzels was packed at a pressure of 1.00 atm and a temperature of **22.0°C**.When opened at a summer picnic in Santa Fe, New Mexico, at a temperature of **32.0°C**, the volume of the air in the bag is 1.38 times its original volume. What is the pressure of the air?

21. How many moles are there in

(a) 0.0500 g of N_2 gas (**M=28.0g/mol**)?

(b) 10.0 g of CO_2 gas (**M=44.0g/mol**)?

(c) How many molecules are present in each case?

22. A cubic container of volume 2.00 L holds 0.500 mol of nitrogen gas at a temperature of **25.0°C**. What is the net force due to the nitrogen on one wall of the container? Compare that force to the sample's weight.

23. Calculate the number of moles in the 2.00-L volume of air in the lungs of the average person. Note that the air is at **37.0°C** (body temperature) and that the total volume in the lungs is several times the amount inhaled in a typical breath as given in Example 2.2.

24. An airplane passenger has $100 cm^3$ of air in his stomach just before the plane takes off from a sea-level airport. What volume will the air have at cruising altitude if cabin pressure drops to $7.50 \times 10^4 N/m^2$?

25. A company advertises that it delivers helium at a gauge pressure of $1.72 \times 10^7 Pa$ in a cylinder of volume 43.8 L. How many balloons can be inflated to a volume of 4.00 L with that amount of helium? Assume the pressure inside the balloons is 1.01×105Pa1.01×105Pa and the temperature in the cylinder and the balloons is 25.0°C25.0°C.

26. According to http://hyperphysics.phy-astr.gsu.edu.../venusenv.html, the atmosphere of Venus is approximately 96.5 and 3.5 by volume. On the surface, where the temperature is about 750 K and the pressure is about 90 atm, what is the density of the atmosphere?

27. An expensive vacuum system can achieve a pressure as low as $1.00 \times 10^{-7} N/m^2$ at **20.0°C**. How many molecules are there in a cubic centimeter at this pressure and temperature?

28. The number density N/V of gas molecules at a certain location in the space above our planet is about $1.00 \times 10^{11} m^{-3}$, and the pressure is $2.75 \times 10^{-10N}/m^2$ in this space. What is the temperature there?

29. A bicycle tire contains 2.00 L of gas at an absolute pressure of $7.00 \times 10^5 N/m^2$ and a temperature of **18.0°C**. What will its pressure be if you let out an amount of air that has a volume of $100 cm^3$ at atmospheric pressure? Assume tire temperature and volume remain constant.

30. In a common demonstration, a bottle is heated and stoppered with a hard-boiled egg that's a little bigger than the bottle's neck. When the bottle is cooled, the pressure difference between inside and outside forces the egg into the bottle. Suppose the bottle has a volume of 0.500 L and the temperature inside it is raised to **80.0°C** while the pressure remains constant at 1.00 atm because the bottle is open.

(a) How many moles of air are inside?

(b) Now the egg is put in place, sealing the bottle. What is the gauge pressure inside after the air cools back to the ambient temperature of **25°C** but before the egg is forced into the bottle?

31. A high-pressure gas cylinder contains 50.0 L of toxic gas at a pressure of $1.40 \times 10^7 N/m^2$ and a temperature of **25.0°C**. The cylinder is cooled to dry ice temperature (**−78.5°C**) to reduce the leak rate and pressure so that it can be safely repaired.

(a) What is the final pressure in the tank, assuming a negligible amount of gas leaks while being cooled and that there is no phase change?

(b) What is the final pressure if one-tenth of the gas escapes? (c) To what temperature must the tank be cooled to reduce the pressure to 1.00 atm (assuming the gas does not change phase and that there is no leakage during cooling)? (d) Does cooling the tank as in part

(c) appear to be a practical solution?

32. Find the number of moles in 2.00 L of gas at **35.0°C**and under $7.41 \times 10^7 N/m^2$ of pressure.

33. Calculate the depth to which Avogadro's number of table tennis balls would cover Earth. Each ball has a diameter of 3.75 cm. Assume the space between balls adds an extra **25.0%** to their volume and assume they are not crushed by their own weight.

34. a) What is the gauge pressure in a **25.0°C** car tire containing 3.60 mol of gas in a 30.0-L volume?

(b) What will its gauge pressure be if you add 1.00 L of gas originally at atmospheric pressure and **25.0°C** ? Assume the temperature remains at **25.0°C** and the volume remains constant.

2.2 Pressure, Temperature, and RMS Speed

In the problems in this section, assume all gases are ideal.

35. A person hits a tennis ball with a mass of 0.058 kg against a wall. The average component of the ball's velocity perpendicular to the wall is 11 m/s, and the ball hits the wall every 2.1 s on average, rebounding with the opposite perpendicular velocity component.

(a) What is the average force exerted on the wall?

(b) If the part of the wall the person hits has an area of $3.0m^2$, what is the average pressure on that area?

36. A person is in a closed room (a racquetball court) with $V = 453m^3$ hitting a ball (**m=42.0g**)(m=42.0g) around at random without any pauses. The average kinetic energy of the ball is 2.30 J.

(a) What is the average value of v_x^2? Does it matter which direction you take to be x?

(b) Applying the methods of this chapter, find the average pressure on the walls?

(c) Aside from the presence of only one "molecule" in this problem, what is the main assumption in Pressure, Temperature, and RMS Speed that does not apply here?

37. Five bicyclists are riding at the following speeds: 5.4 m/s, 5.7 m/s, 5.8 m/s, 6.0 m/s, and 6.5 m/s. (a) What is their average speed? (b) What is their rms speed?

38. Some incandescent light bulbs are filled with argon gas. What is v_{rms} for argon atoms near the filament, assuming their temperature is 2500 K?

39. Typical molecular speeds (v_{rms}) are large, even at low temperatures. What is v_{rms} for helium atoms at 5.00 K, less than one degree above helium's liquefaction temperature?

40. What is the average kinetic energy in joules of hydrogen atoms on the **5500°C** surface of the Sun?

(b) What is the average kinetic energy of helium atoms in a region of the solar corona where the temperature is $6.00 \times 10^5 K$?

41. What is the ratio of the average translational kinetic energy of a nitrogen molecule at a temperature of 300 K to the gravitational potential energy of a nitrogen-molecule−Earth system at the ceiling of a 3-m-tall room with respect to the same system with the molecule at the floor?

42. What is the total translational kinetic energy of the air molecules in a room of volume $23m^3$ if the pressure is $9.5 \times 10^4 Pa$ (the room is at fairly high elevation) and the temperature is **21°C**? Is any item of data unnecessary for the solution?

43. The product of the pressure and volume of a sample of hydrogen gas at **0.00°C** is 80.0 J.

(a) How many moles of hydrogen are present?

(b) What is the average translational kinetic energy of the hydrogen molecules?

(c) What is the value of the product of pressure and volume at **200°C?**

44. What is the gauge pressure inside a tank of $4.86 \times 10^4 mol$ of compressed nitrogen with a volume of $6.56m^3$ if the rms speed is 514 m/s?

45. If the rms speed of oxygen molecules inside a refrigerator of volume $22.0 ft.^3$ is 465 m/s, what is the partial pressure of the oxygen? There are 5.71 moles of oxygen in the refrigerator, and the molar mass of oxygen is 32.0 g/mol.

46. The escape velocity of any object from Earth is 11.1 km/s. At what temperature would oxygen molecules (molar mass is equal to 32.0 g/mol) have root-mean-square velocity v_{rms} equal to Earth's escape velocity of 11.1 km/s?

47. The escape velocity from the Moon is much smaller than that from the Earth, only 2.38 km/s. At what temperature would hydrogen molecules (molar mass is equal to 2.016 g/mol) have a root-mean-square velocity vrmsvrms equal to the Moon's escape velocity?

48. Nuclear fusion, the energy source of the Sun, hydrogen bombs, and fusion reactors, occurs much more readily when the average kinetic energy of the atoms is high—that is, at high temperatures. Suppose you want the atoms in your fusion experiment to have average kinetic energies of $6.40 \times 10^{-14} J$. What temperature is needed?

49. Suppose that the typical speed (v_{rms}) of carbon dioxide molecules (molar mass is 44.0 g/mol) in a flame is found to be 1350 m/s. What temperature does this indicate?

50. (a) Hydrogen molecules (molar mass is equal to 2.016 g/mol) have vrmsvrms equal to 193 m/s. What is the temperature? (b) Much of the gas near the Sun is atomic hydrogen (H rather than H_2). Its temperature would have to be $1.5 \times 10^7 K$ for the rms speed vrmsvrms to equal the escape velocity from the Sun. What is that velocity?

51. There are two important isotopes of uranium, ^{235}U and ^{238}U; these isotopes are nearly identical chemically but have different atomic masses. Only ^{235}U is very useful in nuclear reactors. Separating the isotopes is called uranium enrichment (and is often in the news as of this writing, because of concerns that some countries are enriching uranium with the goal of making nuclear weapons.) One of the techniques for enrichment, gas diffusion, is based on the different molecular speeds of uranium hexafluoride gas, UF_6.

(a) The molar masses of ^{235}U and $^{238}UF_6$ are 349.0 g/mol and 352.0 g/mol, respectively. What is the ratio of their typical speeds vrmsvrms?

(b) At what temperature would their typical speeds differ by 1.00 m/s?

(c) Do your answers in this problem imply that this technique may be difficult?

52. The partial pressure of carbon dioxide in the lungs is about 470 Pa when the total pressure in the lungs is 1.0 atm. What percentage of the air molecules in the lungs is carbon dioxide? Compare your result to the percentage of carbon dioxide in the atmosphere, about 0.033%.

53. Dry air consists of approximately **78%nitrogen,21%oxygen,and 1%argon** by mole, with trace amounts of other gases. A tank of compressed dry air has a volume of 1.76 cubic feet at a gauge pressure of 2200 pounds per square inch and a temperature of 293 K. How much oxygen does it contain in moles?

54. (a) Using data from the previous problem, find the mass of nitrogen, oxygen, and argon in 1 mol of dry air. The molar mass of N_2 is 28.0 g/mol, that of O_2 is 32.0 g/mol, and that of argon is 39.9 g/mol.

(b) Dry air is mixed with pentane (C_5H_{12}, molar mass 72.2 g/mol), an important constituent of gasoline, in an air-fuel ratio of 15:1 by mass (roughly typical for car engines). Find the partial pressure of pentane in this mixture at an overall pressure of 1.00 atm.

55. (a) Given that air is **21%** oxygen, find the minimum atmospheric pressure that gives a relatively safe partial pressure of oxygen of 0.16 atm.

(b) What is the minimum pressure that gives a partial pressure of oxygen above the quickly fatal level of 0.06 atm?

(c) The air pressure at the summit of Mount Everest (8848 m) is 0.334 atm. Why have a few people climbed it without oxygen, while some who have tried, even though they had trained at high elevation, had to turn back?

56. (a) If the partial pressure of water vapor is 8.05 torr, what is the dew point? **(760torr=1atm=101,325Pa)**

(b) On a warm day when the air temperature is **35°C** and the dew point is **25°C**, what are the partial pressure of the water in the air and the relative humidity?

2.3 Heat Capacity and Equipartition of Energy

57. To give a helium atom nonzero angular momentum requires about 21.2 eV of energy (that is, 21.2 eV is the difference between the energies of the lowest-energy or ground state and the lowest-energy state with angular momentum). The electron-volt or eV is defined as $1.60 \times 10^{-19} J$. Find the temperature T where this amount of energy equals $k_BT/2$. Does this explain why we can ignore the rotational energy of helium for most purposes? (The results for other monatomic gases, and for diatomic gases rotating around the axis connecting the two atoms, have comparable orders of magnitude.)

58. (a) How much heat must be added to raise the temperature of 1.5 mol of air from **25.0°C** to **33.0°C** at constant volume? Assume air is completely diatomic.

(b) Repeat the problem for the same number of moles of xenon, Xe.

59. A sealed, rigid container of 0.560 mol of an unknown ideal gas at a temperature of **30.0°C** is cooled to **−40.0°C**. In the process, 980 J of heat are removed from the gas. Is the gas monatomic, diatomic, or polyatomic?

60. A sample of neon gas (Ne, molar mass **M=20.2g/mol**) at a temperature of **13.0°C** is put into a steel container of mass 47.2 g that's at a temperature of **−40.0°C**. The final temperature is **−28.0°C**. (No heat is exchanged with the surroundings, and you can neglect any change in the volume of the container.) What is the mass of the sample of neon?

61. A steel container of mass 135 g contains 24.0 g of ammonia, NH_3, which has a molar mass of 17.0 g/mol. The container and gas are in equilibrium at **12.0°C**. How much heat has to be removed to reach a temperature of **−20.0°C**? Ignore the change in volume of the steel.

62. A sealed room has a volume of $24m^3$. It's filled with air, which may be assumed to be diatomic, at a temperature of **24°C** and a pressure of 9.83×104Pa.9.83×104Pa. A 1.00-kg block of ice at its melting point is placed in the room. Assume the walls of the room transfer no heat. What is the equilibrium temperature?

63. Heliox, a mixture of helium and oxygen, is sometimes given to hospital patients who have trouble breathing, because the low mass of helium makes it easier to breathe than air. Suppose helium at **25°C** is mixed with oxygen at **35°C** to make a mixture that is **70%** helium by mole. What is the final temperature? Ignore any heat flow to or from the surroundings, and assume the final volume is the sum of the initial volumes.

64. Professional divers sometimes use heliox, consisting of **79%** helium and **21%** oxygen by mole. Suppose a perfectly rigid scuba tank with a volume of 11 L contains heliox at an absolute pressure of $2.1 \times 10^7 Pa$ at a temperature of **31°C**

(a) How many moles of helium and how many moles of oxygen are in the tank?

(b) The diver goes down to a point where the sea temperature is **27°C** while using a negligible amount of the mixture. As the gas in the tank reaches this new temperature, how much heat is removed from it?

65. In car racing, one advantage of mixing liquid nitrous oxide (N_2O) with air is that the boiling of the "nitrous" absorbs latent heat of vaporization and thus cools the air and ultimately the fuel-air mixture, allowing more fuel-air mixture to go into each cylinder. As a very rough look at this process, suppose 1.0 mol of nitrous oxide gas at its boiling point, **−88°C**, is mixed with 4.0 mol of air (assumed diatomic) at 30°C30°C. What is the final temperature of the mixture? Use the measured heat capacity of N_2O at **25°C**, which is **30.4J/mol°C**. (The primary advantage of nitrous oxide is that it consists of 1/3 oxygen, which is more than air contains, so it supplies more oxygen to burn the fuel. Another advantage is that its decomposition into nitrogen and oxygen releases energy in the cylinder.)

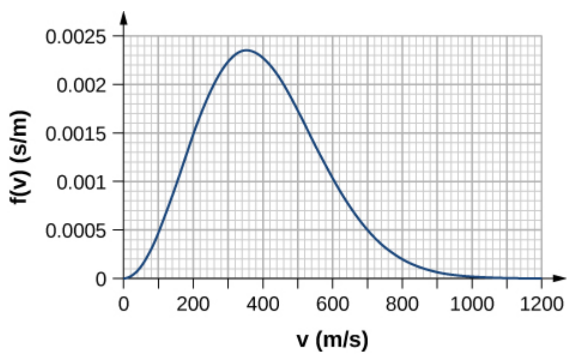
2.4 Distribution of Molecular Speeds

66. In a sample of hydrogen sulfide (**M=34.1g/mol**) at a temperature of 3.00×10^2K, estimate the ratio of the number of molecules that have speeds very close to v_{rms} to the number that have speeds very close to $2v_{rms}$.

67. Using the approximation $\int_{v_1}^{v_1+\Delta v} f(v)dv \approx f(v1)\Delta v$ for small **Δv**, estimate the fraction of nitrogen molecules at a temperature of 3.00×10^2K that have speeds between 290 m/s and 291 m/s.

68. Using the method of the preceding problem, estimate the fraction of nitric oxide (NO) molecules at a temperature of 250 K that have energies between $3.45\times10^{-21}J$ and $3.50\times10^{-21}J$.

69. By counting squares in the following figure, estimate the fraction of argon atoms at **T=300K** that have speeds between 600 m/s and 800 m/s. The curve is correctly normalized. The value of a square is its length as measured on the x-axis times its height as measured on the y-axis, with the units given on those axes.



70. Using a numerical integration method such as Simpson's rule, find the fraction of molecules in a sample of oxygen gas at a temperature of 250 K that have speeds between 100 m/s and 150 m/s. The molar mass of oxygen (O_2) is 32.0 g/mol. A precision to two significant digits is enough.

71. Find (a) the most probable speed,

(b) the average speed, and

(c) the rms speed for nitrogen molecules at 295 K.

72. Repeat the preceding problem for nitrogen molecules at 2950 K.

73. At what temperature is the average speed of carbon dioxide molecules (**M=44.0g/mol**) 510 m/s?

74. The most probable speed for molecules of a gas at 296 K is 263 m/s. What is the molar mass of the gas? (You might like to figure out what the gas is likely to be.)

75. a) At what temperature do oxygen molecules have the same average speed as helium atoms (**M=4.00g/mol**) have at 300 K?

b) What is the answer to the same question about most probable speeds?

c) What is the answer to the same question about rms speeds?

Additional Problems

76. In the deep space between galaxies, the density of molecules (which are mostly single atoms) can be as low as $10^6 atoms/m^3$, and the temperature is a frigid 2.7 K. What is the pressure?

(b) What volume (in m^3) is occupied by 1 mol of gas?

(c) If this volume is a cube, what is the length of its sides in kilometers?

77. (a) Find the density in SI units of air at a pressure of 1.00 atm and a temperature of **20°C**20°C, assuming that air is 78 and 1

(b) Find the density of the atmosphere on Venus, assuming that it's 96 and 4, with a temperature of 737 K and a pressure of 92.0 atm.

78. The air inside a hot-air balloon has a temperature of 370 K and a pressure of 101.3 kPa, the same as that of the air outside. Using the composition of air as 78 and 1, find the density of the air inside the balloon.

79. When an air bubble rises from the bottom to the top of a freshwater lake, its volume increases by **80%**. If the temperatures at the bottom and the top of the lake are 4.0 and 10 **°C**, respectively, how deep is the lake?

80. (a) Use the ideal gas equation to estimate the temperature at which 1.00 kg of steam (molar mass **M=18.0g/mol**) at a pressure of $1.50 \times 10^6 Pa$ occupies a volume of $0.220 m^3$.

(b) The van der Waals constants for water are $a = 0.5537 Pa \cdot m^6/mol^2$ and $b = 3.049 \times 10^{-5} m^3/mol$. Use the Van der Waals equation of state to estimate the temperature under the same conditions.

(c) The actual temperature is 779 K. Which estimate is better?

81. One process for decaffeinating coffee uses carbon dioxide (**M=44.0g/mol**) at a molar density of about $14,600 mol/m^3$ and a temperature of about **60°C**.

(a) Is CO_2 a solid, liquid, gas, or supercritical fluid under those conditions?

(b) The van der Waals constants for carbon dioxide are $a = 0.3658 Pa \cdot m^6/mol^2$ and $b = 4.286 \times 10^{-5} m^3/mol$. Using the van der Waals equation, estimate the pressure of CO_2 at that temperature and density.

82. On a winter day when the air temperature is **0°C**, the relative humidity is **50%**. Outside air comes inside and is heated to a room temperature of **20°C**. What is the relative humidity of the air inside the room. (Does this problem show why inside air is so dry in winter?)

83. On a warm day when the air temperature is **30°C**, a metal can is slowly cooled by adding bits of ice to liquid water in it. Condensation first appears when the can reaches **15°C**. What is the relative humidity of the air?

84. (a) People often think of humid air as "heavy." Compare the densities of air with **0%** relative humidity and **100%**100% relative humidity when both are at 1 atm and **30°C**. Assume that the dry air is an ideal gas composed of molecules with a molar mass of 29.0 g/mol and the moist air is the same gas mixed with water vapor.

(b) As discussed in the chapter on the applications of Newton's laws, the air resistance felt by projectiles such as baseballs and golf balls is approximately $F_D = C\rho A v^2/2$, where **ρ** is the mass density of the air, A is the cross-sectional area of the projectile, and C is the projectile's drag coefficient. For a fixed air pressure, describe qualitatively how the range of a projectile changes with the relative humidity.

(c) When a thunderstorm is coming, usually the humidity is high and the air pressure is low. Do those conditions give an advantage or disadvantage to home-run hitters?

85. The mean free path for helium at a certain temperature and pressure is $2.10 \times 10^{-7} m$. The radius of a helium atom can be taken as $1.10 \times 10^{-11} m$. What is the measure of the density of helium under those conditions

(a) in molecules per cubic meter and

(b) in moles per cubic meter?

86. The mean free path for methane at a temperature of 269 K and a pressure of $1.11 \times 10^5 Pa$ is $4.81 \times 10^{-8} m$. Find the effective radius r of the methane molecule.

87. In the chapter on fluid mechanics, Bernoulli's equation for the flow of incompressible fluids was explained in terms of changes affecting a small volume dV of fluid. Such volumes are a fundamental idea in the study of the flow of compressible fluids such as gases as well. For the equations of hydrodynamics to apply, the mean free path must be much less than the linear size of such a volume, $a \approx dV^{1/3}$. For air in the stratosphere at a temperature of 220 K and a pressure of 5.8 kPa, how big should a be for it to be 100 times the mean free path? Take the effective radius of air molecules to be $1.88 \times 10^{-11} m$, which is roughly correct for N_2.

88. Find the total number of collisions between molecules in 1.00 s in 1.00 L of nitrogen gas at standard temperature and pressure (**0°C**, 1.00 atm). Use $1.88 \times 10^{-10} m$ as the effective radius of a nitrogen molecule. (The number of collisions per second is the reciprocal of the collision time.) Keep in mind that each collision involves two molecules, so if one molecule collides once in a certain period of time, the collision of the molecule it hit cannot be counted.

89. (a) Estimate the specific heat capacity of sodium from the Law of Dulong and Petit. The molar mass of sodium is 23.0 g/mol.

(b) What is the percent error of your estimate from the known value, **1230J/kg·°C**?

90. A sealed, perfectly insulated container contains 0.630 mol of air at **20.0°C** and an iron stirring bar of mass 40.0 g. The stirring bar is magnetically driven to a kinetic energy of 50.0 J and allowed to slow down by air resistance. What is the equilibrium temperature?

91. Find the ratio $f(v_p)/f(v_{rms})$ for hydrogen gas (**M=2.02g/mol**) at a temperature of 77.0 K.

92. Unreasonable results. (a) Find the temperature of 0.360 kg of water, modeled as an ideal gas, at a pressure of $1.01\times10^5 Pa$ if it has a volume of $0.615m^3$.

(b) What is unreasonable about this answer? How could you get a better answer?

93. Unreasonable results. (a) Find the average speed of hydrogen sulfide, H_2S, molecules at a temperature of 250 K. Its molar mass is 31.4 g/mol

(b) The result isn't very unreasonable, but why is it less reliable than those for, say, neon or nitrogen?

Challenge Problems

94. An airtight dispenser for drinking water is **25cm×10cm** in horizontal dimensions and 20 cm tall. It has a tap of negligible volume that opens at the level of the bottom of the dispenser. Initially, it contains water to a level 3.0 cm from the top and air at the ambient pressure, 1.00 atm, from there to the top. When the tap is opened, water will flow out until the gauge pressure at the bottom of the dispenser, and thus at the opening of the tap, is 0. What volume of water flows out? Assume the temperature is constant, the dispenser is perfectly rigid, and the water has a constant density of $1000kg/m^3$.

95. Eight bumper cars, each with a mass of 322 kg, are running in a room 21.0 m long and 13.0 m wide. They have no drivers, so they just bounce around on their own. The rms speed of the cars is 2.50 m/s. Repeating the arguments of Pressure, Temperature, and RMS Speed, find the average force per unit length (analogous to pressure) that the cars exert on the walls.

96. Verify that $v_p=\sqrt{\dfrac{2k_BT}{m}}$.

97. Verify the normalization equation $\displaystyle\int_0^\infty f(v)dv=1$. In doing the integral, first make the substitution $u=\sqrt{\dfrac{m}{2k_BT}}v=\dfrac{v}{v_p}$ .This "scaling" transformation gives you all features of the answer except for the integral, which is a dimensionless numerical factor. You'll need the formula $\displaystyle\int_0^\infty x^2e^{-x^2}dx=\frac{\sqrt{\pi}}{4}$ find the numerical factor and verify the normalization.

98. Verify that $\bar{v}=\sqrt{\dfrac{8}{\pi}\dfrac{k_BT}{m}}$.. Make the same scaling transformation as in the preceding problem.

99. Verify that \(\displaystyle v_{rms}=\sqrt{\bar{v^2}=\sqrt{\frac{3k_BT}{m}}\).

12.7: Kinetic Theory

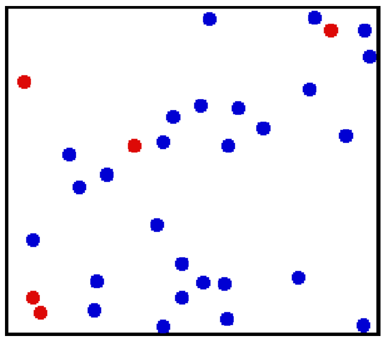
learning objectives

- Express the relationship between the pressure and the average kinetic energy of gas molecules in the form of equation

In Newtonian mechanics, if pressure is the force divided by the area on which the force is exerted, then what is the origin of pressure in a gas? What forces create the pressure? We can gain a better understanding of pressure (and temperature as well) from the kinetic theory of gases, which assumes that atoms and molecules are in continuous random motion.

Microscopic Origin of Pressure

Pressure is explained by kinetic theory as arising from the force exerted by molecules or atoms impacting on the walls of a container, as illustrated in the figure below. Consider a gas of N molecules, each of mass m, enclosed in a cubical container of volume $V=L^3$. When a gas molecule collides with the wall of the container perpendicular to the x coordinate axis and bounces off in the opposite direction with the same speed (an elastic collision), then the momentum lost by the particle and gained by the wall (Δp) is:



Translational Motion of Helium: Real gases do not always behave according to the ideal model under certain conditions, such as high pressure. Here, the size of helium atoms relative to their spacing is shown to scale under 1950 atmospheres of pressure.

$$\Delta p = p_{i,x} - p_{f,x} = p_{i,x} - (-p_{i,x}) \tag{12.7.1}$$

$$= 2p_{i,x} = 2mv_x \tag{12.7.2}$$

where v_x is the x-component of the initial velocity of the particle.

The particle impacts one specific side wall once every $\Delta t = \frac{2L}{v_x}$,

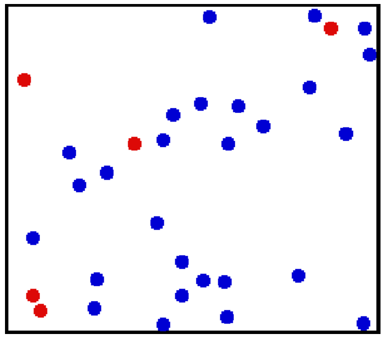
(where L is the distance between opposite walls). The force due to this particle is:

$$F = \frac{\Delta p}{\Delta t} = \frac{mv_x^2}{L}. \tag{12.7.3}$$

The total force on the wall, therefore, is:

$$F = \frac{Nm\bar{v}_x^2}{L} \tag{12.7.4}$$

where the bar denotes an average over the *N* particles. Since the assumption is that the particles move in random directions, if we divide the velocity vectors of all particles in three mutually perpendicular directions, the average value of the squared velocity along each direction must be same. (This does not mean that each particle always travel in 45 degrees to the coordinate axes.)



Pressure: Pressure arises from the force exerted by molecules or atoms impacting on the walls of a container.

This gives $\bar{v_x^2} = \frac{\bar{v^2}}{3}$. We can rewrite the force as $F = \frac{Nm\bar{v^2}}{3L}$.

This force is exerted on an area L^2. Therefore the pressure of the gas is:

$$P = \frac{F}{L^2} = \frac{Nm\bar{v^2}}{3V} = \frac{nm\bar{v^2}}{3}, \tag{12.7.5}$$

where $V=L^3$ is the volume of the box. The fraction n=N/V is the number density of the gas. This is a first non-trivial result of the kinetic theory because it relates pressure (a macroscopic property) to the average (translational) kinetic energy per molecule which is a microscopic property.

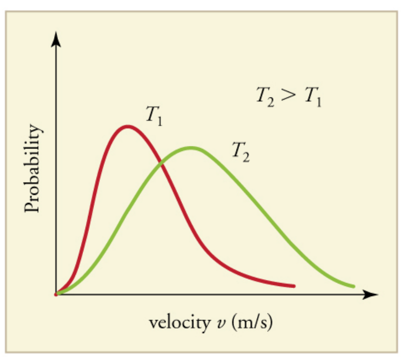
Speed Distribution of Molecules

A gas of many molecules has a predictable distribution of molecular speeds, known as the Maxwell-Boltzmann distribution.

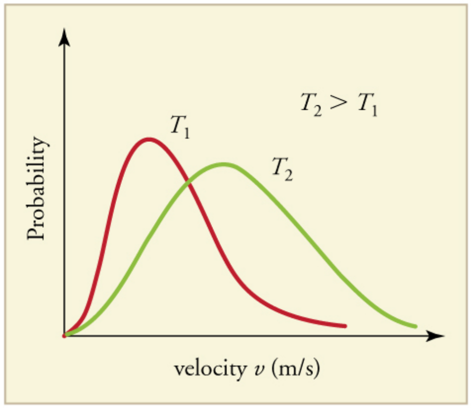
learning objectives

- Describe the shape and temperature dependence of the Maxwell-Boltzmann distribution curve

The motion of molecules in a gas is random in magnitude and direction for individual molecules, but a gas of many molecules has a predictable distribution of molecular speeds, known as the Maxwell-Boltzmann distribution (illustrated in). The distribution has a long tail because some molecules may go several times the rms speed. The most probable speed v_p (at the peak of the curve) is less than the rms speed v_{rms}. As shown in, the curve is shifted to higher speeds at higher temperatures, with a broader range of speeds.



Maxwell-Boltzmann Distribution at Higher Temperatures: The Maxwell-Boltzmann distribution is shifted to higher speeds and is broadened at higher temperatures.



Maxwell-Boltzmann Distribution: The Maxwell-Boltzmann distribution of molecular speeds in an ideal gas. The most likely speed v_p is less than the rms speed v_rms. Although very high speeds are possible, only a tiny fraction of the molecules have speeds that are an order of magnitude greater than v_rms.

Maxwell-Boltzmann Distribution

Maxwell-Boltzmann distribution is a probability distribution. It applies to ideal gases close to thermodynamic equilibrium, and is given as the following equation:

$$f_v(v_x, v_y, v_z) = (\frac{m}{2\pi kT})^{3/2} \exp[-\frac{m(v_x^2 + v_y^2 + v_z^2)}{2kT}], \tag{12.7.6}$$

where fv is the velocity probability density function. (Derivation of the formula goes beyond the scope of introductory physics.) The formula calculates the probability of finding a particle with velocity in the infinitesimal element [dv_x, dv_y, dv_z] about velocity v = [v_x, v_y, v_z] is:

$$f_v(v_x, v_y, v_z) dv_x dv_y dv_z. \tag{12.7.7}$$

It can also be shown that the Maxwell–Boltzmann velocity distribution for the vector velocity [v_x, v_y, v_z] is the product of the distributions for each of the three directions:

$$f_v(v_x, v_y, v_z) = f_v(v_x) f_v(v_y) f_v(v_z) \tag{12.7.8}$$

where the distribution for a single direction is,

$$f_v(v_i) = \sqrt{\frac{m}{2\pi kT}} \exp[\frac{-mv_i^2}{2kT}]. \tag{12.7.9}$$

This makes sense because particles are moving randomly, meaning that each component of the velocity should be independent.

Distribution for the Speed

Usually, we are more interested in the speeds of molecules rather than their component velocities. The Maxwell–Boltzmann distribution for the speed follows immediately from the distribution of the velocity vector, above. Note that the speed is:

$$v = \sqrt{v_x^2 + v_y^2 + v_z^2} \tag{12.7.10}$$

and the increment of volume is:

$$dv_x\ dv_y\ dv_z = v^2 \sin\phi\ dv\ d\theta\ d\phi, \tag{12.7.11}$$

where θ and ϕ are the "course" (azimuth of the velocity vector) and "path angle" (elevation angle of the velocity vector). Integration of the normal probability density function of the velocity, above, over the course (from 0 to 2π) and path angle (from 0 to π), with substitution of the speed for the sum of the squares of the vector components, yields the following probability density function (known simply as the Maxwell distribution):

$f(v) = \sqrt{(\frac{m}{2\pi kT})^3} 4\pi v^2\ \exp(\frac{-mv^2}{2kT})$ for speed v.

Temperature

Temperature is directly proportional to the average translational kinetic energy of molecules in an ideal gas.

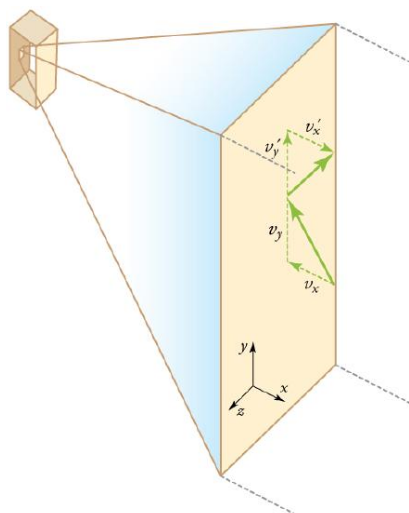
learning objectives

- Describe relationship between temperature and energy of molecules in an ideal gas

Intuitively, hotter air suggests faster movement of air molecules. In this atom, we will derive an equation relating the temperature of a gas (a macroscopic quantity) to the average kinetic energy of individual molecules (a microscopic quantity). This is a basic and extremely important relationship in the kinetic theory of gases.

Microscopic View

We assume that a molecule is small compared with the separation of molecules in the gas (confined in a three dimensional container), and that its interaction with other molecules can be ignored. Also, we assume elastic collisions when molecules hit the wall of the container, as illustrated in.



Elastic Collisions When Molecules Hit the Wall of the Container: Gas in a box exerts an outward pressure on its walls. A molecule colliding with a rigid wall has the direction of its velocity and momentum in the x-direction reversed. This direction is perpendicular to the wall. The components of its velocity momentum in the y- and z-directions are not changed, which means there is no force parallel to the wall.

We have seen in the Atom on “Origin of Pressure ” that, for an ideal gas under our assumptions:

$$\mathrm{P}=\frac{\mathrm{Nm}\bar{\mathrm{v}}^2}{3\mathrm{V}}, \tag{12.7.12}$$

where P is the pressure, N is the number of molecules, m is the mass of the molecule, v is the speed of molecules, and V is the volume of the gas. From the equation, we get:

$$\mathrm{PV}=\frac{1}{3}\mathrm{Nm}\bar{\mathrm{v}}^2\ (\mathrm{Eq.}\ 1). \tag{12.7.13}$$

What can we learn from this atomic and molecular version of the ideal gas law ? We can derive a relationship between temperature and the average translational kinetic energy of molecules in a gas. Recall the macroscopic expression of the ideal gas law:

$$\mathrm{PV}=\mathrm{NkT}\ (\mathrm{Eq.}\ 2), \tag{12.7.14}$$

where N is the number of molecules, T is the temperature of the gas, and k is the Boltzmann constant.

Equating the right hand sides of the macroscopic and microscopic versions of the ideal gas law (Eq. 1 & 2) gives:

$$\frac{1}{3}\mathrm{m}\bar{\mathrm{v}}^2=\mathrm{kT}. \tag{12.7.15}$$

Thermal Energy

Note that the average kinetic energy (KE) of a molecule in the gas is:

$$\frac{1}{2}\mathrm{m}\bar{\mathrm{v}}^2. \tag{12.7.16}$$

Therefore, we derive the relation between average KE and temperature as follows:

$$\bar{\mathrm{KE}}=\frac{1}{2}\mathrm{m}\bar{\mathrm{v}}^2=\frac{3}{2}\mathrm{kT},\ (\mathrm{Eq.}\ 3). \tag{12.7.17}$$

The average translational kinetic energy of a molecule is called thermal energy.

RMS Speed

Eq. 3 is a molecular interpretation of temperature. It has been found to be valid for gases and reasonably accurate in liquids and solids. It is another definition of temperature based on an expression of the molecular energy. It is sometimes useful to know the average speed of molecules in a gas in terms of temperature:

$$\bar{\mathrm{v}}^2=\mathrm{v}_{\mathrm{rms}}=\sqrt{\frac{3\mathrm{kT}}{\mathrm{m}}}, \tag{12.7.18}$$

where $\mathrm{v}_{\mathrm{rms}}$ stands for root-mean-square (rms) speed.

Internal Energy of an Ideal Gas

Internal energy is the total energy contained by a thermodynamic system, and has two major components: kinetic energy and potential energy.

learning objectives

- Determine the number of degrees of freedom and calculate the internal energy for an ideal gas molecule

In thermodynamics, internal energy is the total energy contained by a thermodynamic system. Internal energy has two major components: kinetic energy and potential energy. The kinetic energy is due to the motion of the system's particles (e.g., translations, rotations, vibrations). In ideal gases, there is no inter-particle interaction. Therefore, we will disregard potential energy and only focus on the kinetic energy contribution to the internal energy.

Monatomic Gases

A monatomic gas is one in which atoms are not bound to each other. Noble gases (He, Ne, etc.) are typical examples. A helium balloon is shown in the following figure. In this case, the kinetic energy consists only of the translational energy of the individual atoms. Monoatomic particles do not vibrate, and their rotational energy can be neglected because atomic moment of inertia is so small. Also, they are not electronically excited to higher energies except at very high temperatures. Therefore, practical internal energy changes in an ideal gas may be described solely by changes in its translational kinetic energy.

Helium Blimp: Helium, like other noble gases, is a monatomic gas, which often can be described by the ideal gas law. It is the gas of choice to fill airships such as the Goodyear blimp.

The average kinetic energy (KE) of a particle in an ideal gas is given as:

$$\bar{\mathrm{KE}}=\frac{1}{2}\mathrm{m}\bar{\mathrm{v}}^2=\frac{3}{2}\mathrm{kT}, \tag{12.7.19}$$

where k is the Boltzmann's constant. (See the Atom on "Temperature" in kinetic theory.) With N atoms in the gas, its total internal energy U is given as:

$$U=\frac{3}{2}NkT \tag{12.7.20}$$

where N is the number of atoms in the gas. Note that there are three degrees of freedom in monatomic gases: translation in x, y and z directions.

Since atomic motion is random (and therefore isotropic), each degrees of freedom contribute $\frac{1}{2}kT$ per atom to the internal energy.

Diatomic gases

A diatomic molecule (H_2, O_2, N_2, etc.) has 5 degrees of freedom (3 for translation in x, y and z directions, and 2 for rotation). Therefore, the internal energy for diatomic gases is $U=\frac{5}{2}NkT$.

Key Points

- We can gain a better understanding of pressure (and temperature as well) from the kinetic theory of gases, which assumes that atoms and molecules are in continuous random motion.
- Pressure, a macroscopic property, can be related to the average (translational) kinetic energy per molecule which is a microscopic property by $P=\frac{nm\bar{v^2}}{3}$.
- Since the assumption is that the particles move in random directions, the average value of velocity squared along each direction must be same. This gives: $\bar{v_x^2}=\bar{v_y^2}=\bar{v_z^2}=\bar{v^2}/3$.
- The Maxwell-Boltzmann distribution has a long tail, and the most probable speed v_p is less than the rms speed v_{rms}. The distribution curve is shifted to higher speeds at higher temperatures, with a broader range of speeds.
- Maxwell-Boltzmann distribution is given as follows: $f_v(v_x, v_y, v_z)=(\frac{m}{2\pi kT})^{3/2}\exp[-\frac{m(v_x^2+v_y^2+v_z^2)}{2kT}]$. It is a product of three independent 1D Maxwell-Boltzmann distributions.
- Molecular speed distribution is given as $f(v)=\sqrt{(\frac{m}{2\pi kT})^3}4\pi v^2\exp(\frac{-mv^2}{2kT})$. This is simply called Maxwell distribution.
- The average translational kinetic energy of a molecule is equivalent to $\frac{3}{2}kT$ and is called thermal energy.
- In kinematic theory of gases, macroscopic quantities (such as press and temperature) are explained by considering microscopic (random) motion of molecules.
- The rms speed of molecules in a gas is given as $\sqrt{\frac{3kT}{m}}$.
- In ideal gases, there is no inter-particle interaction. Therefore, only the kinetic energy contribute to the internal energy.
- Each degrees of freedom contribute $\frac{1}{2}kT$ per atom to the internal energy.
- For monatomic ideal gases with N atoms, its total internal energy U is given as $U=\frac{3}{2}NkT$. For diatomic gases, $U=\frac{5}{2}NkT$.

Key Terms

- **kinetic theory of gases**: The kinetic theory of gases describes a gas as a large number of small particles (atoms or molecules), all of which are in constant, random motion.
- **Newtonian mechanics**: Early classical mechanics as propounded by Isaac Newton, especially that based on his laws of motion and theory of gravity.
- **rms**: Root mean square: a statistical measure of the magnitude of a varying quantity.
- **ideal gas**: A hypothetical gas whose molecules exhibit no interaction and undergo elastic collision with each other and with the walls of the container.
- **Boltzmann's constant**: The physical constant relating energy at the particle level with temperature observed at the bulk level. It is the gas constant R divided by Avogadro's number, NA.
- **moment of inertia**: A measure of a body's resistance to a change in its angular rotation velocity
- **noble gas**: Any of the elements of group 18 of the periodic table, being monatomic and (with very limited exceptions) inert.

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12.8: Phase Changes

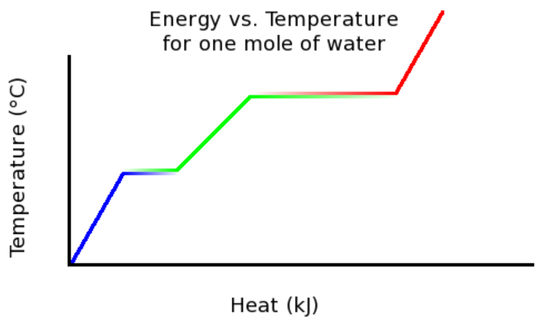
learning objectives

- Describe behavior of the medium during a phase transition

A phase of a thermodynamic system and the states of matter have uniform physical properties. During a phase transition of a given medium certain properties of the medium change, often discontinuously, as a result of some external condition, such as temperature or pressure. For example, a liquid may become gas upon heating to the boiling point, resulting in an abrupt change in volume. The measurement of the external conditions at which the transformation occurs is termed the phase transition. The term is most commonly used to describe transitions between solid, liquid and gaseous states of matter and, in rare cases, plasma.

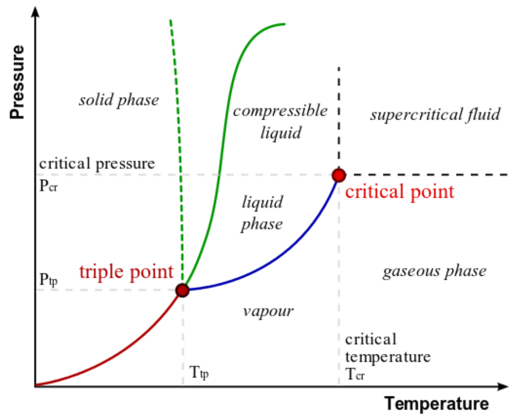
As an example, if you boil water, it never goes above 100 degrees Celsius. Only after it has completely evaporated will it get any hotter. This is because once water reaches the boiling point, extra energy is used to change the state of matter and increase the potential energy instead of the kinetic energy. The opposite happens when water freezes. To boil or melt one mole of a substance, a certain amount of energy is required. These amounts of energy are the molar heat of vaporization and molar heat of fusion. If that amount of energy is added to a mole of that substance at boiling or freezing point, all of it will melt or boil, but the temperature won't change.

Temperature increases linearly with heat, until the melting point. But the heat added does not change the temperature; that heat energy is instead used to break intermolecular bonds and convert ice into water. At this point, there is a mixture of both ice and water. Once all ice has been melted, the temperature again rises linearly with heat added. At the boiling point, temperature no longer rises with heat added because the energy is once again being used to break intermolecular bonds. Once all water has been boiled to steam, the temperature will continue to rise linearly as heat is added.



Temperature vs. Heat: This graph shows the temperature of ice as heat is added.

The plots of pressure versus temperatures provide considerable insight into thermal properties of substances. There are well-defined regions on these graphs that correspond to various phases of matter, so PT graphs are called phase diagrams. Using the graph, if you know the pressure and temperature you can determine the phase of water. The solid lines—boundaries between phases—indicate temperatures and pressures at which the phases coexist (that is, they exist together in ratios, depending on pressure and temperature). For example, the boiling point of water is 100º C at 1.00 atm. As the pressure increases, the boiling temperature rises steadily to 374º C at a pressure of 218 atm. A pressure cooker (or even a covered pot) will cook food faster because the water can exist as a liquid at temperatures greater than 100º C without all boiling away. The curve ends at a point called the critical point, because at higher temperatures the liquid phase does not exist at any pressure. The critical temperature for oxygen is -118ºC, so oxygen cannot be liquefied above this temperature.



Phase Diagram of Water: In this typical phase diagram of water, the green lines mark the freezing point, and the blue line marks the boiling point, showing how they vary with pressure. The dotted line illustrates the anomalous behavior of water. Note that water changes states based on the pressure and temperature.

Humidity, Evaporation, and Boiling

The amount of water vapor in air is a result of evaporation or boiling, until an equilibrium is reached.

learning objectives

- Explain why water boils at 100 °C

Overview

The term relative humidity refers to how much water vapor is in the air compared with the maximum possible. At its maximum, denoted as saturation, the relative humidity is 100%, and evaporation is inhibited. The amount of water vapor the air can hold depends on its temperature. For example, relative humidity rises in the evening, as air temperature declines, sometimes reaching the dew point. At the dew point temperature, relative humidity is 100%, and fog may result from the condensation of water droplets if they are small enough to stay in suspension. Conversely, if one wished to dry something, it is more effective to blow hot air over it rather than cold air, because, among other things, hot air can hold more water vapor.

Evaporation

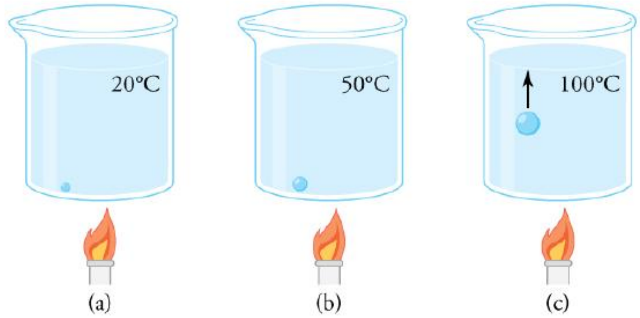
The capacity of air to hold water vapor is based on vapor pressure of water. The liquid and solid phases are continuously giving off vapor because some of the molecules have high enough speeds to enter the gas phase, a process called evaporation; see (a). For the molecules to evaporate, they must be located near the surface, be moving in the proper direction, and have sufficient kinetic energy to overcome liquid-phase intermolecular forces. When only a small proportion of the molecules meet these criteria, the rate of evaporation is low. Since the kinetic energy of a molecule is proportional to its temperature, evaporation proceeds more quickly at higher temperatures.

If a lid is placed over the container, as in (b), evaporation continues, increasing the pressure, until sufficient vapor has built up for condensation to balance evaporation. Then equilibrium has been achieved, and the vapor pressure is equal to the partial pressure of water in the container. Vapor pressure increases with temperature because molecular speeds are higher as temperature increases.

As the faster-moving molecules escape, the remaining molecules have lower average kinetic energy, and the temperature of the liquid decreases. This phenomenon is also called evaporative cooling. This is why evaporating sweat cools the human body. Evaporation also tends to proceed more quickly with higher flow rates between the gaseous and liquid phase and in liquids with higher vapor pressure. For example, laundry on a clothes line will dry (by evaporation) more rapidly on a windy day than on a still day.

Application for Boiling Water

Why does water boil at 100ºC? The vapor pressure of water at 100ºC is 1.01×10^5 Pa, or 1.00 atm. Thus, it can evaporate without limit at this temperature and pressure. But why does it form bubbles when it boils? This is because water ordinarily contains significant amounts of dissolved air and other impurities, which are observed as small bubbles of air in a glass of water. If a bubble starts out at the bottom of the container at 20ºC, it contains water vapor (about 2.30%). The pressure inside the bubble is fixed at 1.00 atm (we ignore the slight pressure exerted by the water around it). As the temperature rises, the amount of air in the bubble stays the same, but the water vapor increases; the bubble expands to keep the pressure at 1.00 atm. At 100ºC, water vapor enters the bubble continuously since the partial pressure of water is equal to 1.00 atm in equilibrium. It cannot reach this pressure, however, since the bubble also contains air and total pressure is 1.00 atm. The bubble grows in size and thereby increases the buoyant force. The bubble breaks away and rises rapidly to the surface, resulting in boiling. (See.)



Close-up of the Boiling Process: (a) An air bubble in water starts out saturated with water vapor at 20ºC. (b) As the temperature rises, water vapor enters the bubble because its vapor pressure increases. The bubble expands to keep its pressure at 1.00 atm. (c) At 100ºC, water vapor enters the bubble continuously because water's vapor pressure exceeds its partial pressure in the bubble, which must be less than 1.00 atm. The bubble grows and rises to the surface.

Key Points

- The term is most commonly used to describe transitions between solid, liquid and gaseous states of matter and, in rare cases, plasma.
- Once water reaches the boiling point, extra energy is used to change the state of matter and increase the potential energy instead of the kinetic energy.
- Plots of pressure versus temperatures, an example of a phase diagram, provide considerable insight into thermal properties of substances.
- Relative humidity is the fraction of water vapor in a gas compared to the saturation value.
- Since the kinetic energy of a molecule is proportional to its temperature, evaporation proceeds more quickly at higher temperatures.
- Vapor pressure increases with temperature because molecular speeds are higher as temperature increases.
- Water boils at 100 °C because the vapor pressure exceeds atmospheric pressure at this temperature.

Key Terms

- **intermolecular**: from one molecule to another; between molecules
- **plasma**: a state of matter consisting of partially ionized gas
- **thermodynamic**: Relating to the conversion of heat into other forms of energy.
- **equilibrium**: The state of a body at rest or in uniform motion, the resultant of all forces on which is zero.
- **vapor pressure**: The pressure that a vapor exerts, or the partial pressure if it is mixed with other gases.
- **humidity**: The amount of water vapor in the air.

12.9: The Zeroth Law of Thermodynamics

learning objectives

- Identify major implications of the Zeroth Law of Thermodynamics

The Zeroth Law of Thermodynamics

There are a few ways to state the Zeroth Law of Thermodynamics, but the simplest is as follows: systems that are in thermal equilibrium exist at the same temperature.

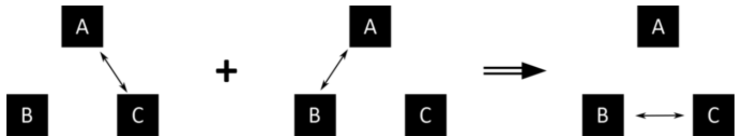
Systems are in thermal equilibrium if they do not transfer heat, even though they are in a position to do so, based on other factors. For example, food that’s been in the refrigerator overnight is in thermal equilibrium with the air in the refrigerator: heat no longer flows from one source (the food) to the other source (the air) or back.

What the Zeroth Law of Thermodynamics means is that temperature is something worth measuring, because it indicates whether heat will move between objects. This will be true regardless of how the objects interact. Even if two objects don’t touch, heat may still flow between them, such as by radiation (as from a heat lamp). However, according to the Zeroth Law of Thermodynamics, if the systems are in thermal equilibrium, no heat flow will take place.

There are more formal ways to state the Zeroth Law of Thermodynamics, which is commonly stated in the following manner:

Let A, B, and C be three systems. If A and C are in thermal equilibrium, and A and B are in thermal equilibrium, then B and C are in thermal equilibrium.

This statement is represented symbolically in. Temperature is not mentioned explicitly, but it’s implied that temperature exists. Temperature is the quantity that is always the same for all systems in thermal equilibrium with one another.



Zeroth Law of Thermodynamics: The double arrow represents thermal equilibrium between systems. If systems A and C are in equilibrium, and systems A and B are in equilibrium, then systems B and C are in equilibrium. The systems A, B, and C are at the same temperature.

Key Points

- Assuming A, B, and C are three systems, if A and C are in thermal equilibrium, and A and B are in thermal equilibrium, then B and C are in thermal equilibrium.
- Two systems are in thermal equilibrium if they could transfer heat between each other, but do not.
- The Zeroth Law of Thermodynamics implies that temperature is a quantity worth measuring.

Key Terms

- **thermal equilibrium**: Two systems are in thermal equilibrium if they could transfer heat between each other, but don’t.
- **zeroth law of thermodynamics**: Let A, B and C be three systems. If A and C are in thermal equilibrium, and A and B are in thermal equilibrium, then B and C are in thermal equilibrium.

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12.10: Thermal Stresses

learning objectives

- Formulate relationship between thermal stress and thermal expansion

Thermal Expansion

Thermal expansion is the change in size or volume of a given mass with temperature. The expansion of alcohol in a thermometer is one of many commonly encountered examples of this. Hot air rises because its volume increases, which causes the hot air's density to be smaller than the density of surrounding air, causing a buoyant (upward) force on the hot air. The same happens in all liquids and gases, driving natural heat transfer upward in homes, oceans, and weather systems. Solids also undergo thermal expansion. Railroad tracks and bridges, for example, have expansion joints to allow them to freely expand and contract with temperature changes.

Thermal Expansion Joints: Thermal expansion joints like these in the Auckland Harbour Bridge in New Zealand allow bridges to change length without buckling. (credit: Ingolfson, Wikimedia Commons)

What are the basic properties of thermal expansion? First, thermal expansion is clearly related to temperature change. The greater the temperature change, the more a bimetallic strip will bend. Second, it depends on the material. In a thermometer, for example, the expansion of alcohol is much greater than the expansion of the glass containing it.

What is the underlying cause of thermal expansion? An increase in temperature implies an increase in the kinetic energy of the individual atoms. In a solid, unlike in a gas, the atoms or molecules are closely packed together, but their kinetic energy (in the form of small, rapid vibrations) pushes neighboring atoms or molecules apart from each other. This neighbor-to-neighbor pushing results in a slightly greater distance, on average, between neighbors, and adds up to a larger size for the whole body. For most substances under ordinary conditions, there is no preferred direction, and an increase in temperature will increase the solid's size by a certain fraction in each dimension.

To be more quantitative, the change in length ΔL is proportional to length L. The dependence of thermal expansion on temperature, substance, and length is summarized in the equation

$$\Delta L = \alpha L \Delta T \tag{12.10.1}$$

where ΔL is the change in length L, ΔT is the change in temperature, and α is the coefficient of linear expansion, which varies slightly with temperature.

Thermal Stress

Thermal stress is created by thermal expansion or contraction. Thermal stress can be destructive, such as when expanding gasoline ruptures a tank. It can also be useful, for example, when two parts are joined together by heating one in manufacturing, then slipping it over the other and allowing the combination to cool. Thermal stress can explain many phenomena, such as the weathering of rocks and pavement by the expansion of ice when it freezes.

Forces and pressures created by thermal stress can be quite large. Railroad tracks and roadways can buckle on hot days if they lack sufficient expansion joints. Power lines sag more in the summer than in the winter, and will snap in cold weather if there is insufficient slack. Cracks open and close in plaster walls as a house warms and cools. Glass cooking pans will crack if cooled rapidly or unevenly, because of differential contraction and the stresses it creates. (Pyrex® is less susceptible because of its small coefficient of thermal expansion.) Nuclear reactor pressure vessels are threatened by overly rapid cooling, and although none have failed, several have been cooled faster than considered desirable. Biological cells are ruptured when foods are frozen, detracting from their taste. Repeated thawing and freezing accentuates the damage. Even the oceans can be affected. A significant portion of the rise in sea level that is resulting from global warming is due to the thermal expansion of sea water.

Metal is regularly used in the human body for hip and knee implants. Most implants need to be replaced over time because, among other things, metal does not bond with bone. Researchers are trying to find better metal coatings that would allow metal-to-bone bonding. One challenge is to find a coating that has an expansion coefficient similar to that of metal. If the expansion coefficients are too different, the thermal stresses during the manufacturing process lead to cracks at the coating-metal interface.

Another example of thermal stress is found in the mouth. Dental fillings can expand differently from tooth enamel. It can give pain when eating ice cream or having a hot drink. Cracks might occur in the filling. Metal fillings (gold, silver, etc.) are being replaced by composite fillings (porcelain), which have smaller coefficients of expansion, and are closer to those of teeth.

Key Points

- Thermal expansion is the change in size or volume of a given mass with changing temperature.
- An increase in temperature implies an increase in the kinetic energy of the individual atoms, which will increase a solid's size by a certain fraction in each dimension.
- Thermal stress is created when thermal expansion is constrained.

Key Terms

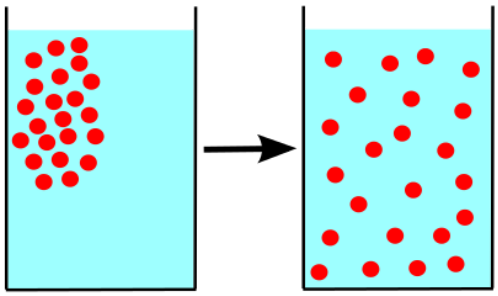
- **stress**: The internal distribution of force per unit area (pressure) within a body reacting to applied forces which causes strain or deformation and is typically symbolized by σ.
- **differential**: A qualitative or quantitative difference between similar or comparable things.

12.11: Diffusion

learning objectives

- Discuss the process and results of diffusion, identifying factors that affect its rate

Diffusion is the movement of particles move from an area of high concentration to an area of low concentration until equilibrium is reached. A distinguishing feature of diffusion is that it results in mixing or mass transport without requiring bulk motion. Thus, diffusion should not be confused with convection or advection, which are other transport mechanisms that use bulk motion to move particles from one place to another.



Diffusion: Particles moving from areas of high concentration to areas of low concentration.

Molecular diffusion, often called simply diffusion, is the thermal motion of all (liquid or gas) particles at temperatures above absolute zero. The rate of this movement is a function of temperature, viscosity of the fluid and the size (mass) of the particles. Diffusion explains the net flux of molecules from a region of higher concentration to one of lower concentration. However, diffusion can still occur in the absence of a concentration gradient.

The result of diffusion is a gradual mixing of material. In a phase with uniform temperature, absent external net forces acting on the particles, the diffusion process will eventually result in complete mixing.

Key Points

- Molecular diffusion, often called simply diffusion, is the thermal motion of all (liquid or gas) particles at temperatures above absolute zero.
- The result of diffusion is a gradual mixing of material. In a phase with uniform temperature, absent external net forces acting on the particles, the diffusion process will eventually result in complete mixing.
- Diffusion can also occur in the absence of a concentration gradient — equilibrium particles are still moving around their container.

Key Terms

- **equilibrium**: The state of a body at rest or in uniform motion, the resultant of all forces on which is zero.
- **diffusion**: Diffusion is the movement of particles from regions of high concentration toward regions of lower concentration.
- **concentration**: The proportion of a substance in a mixture.

CHAPTER OVERVIEW

13: Heat and Heat Transfer

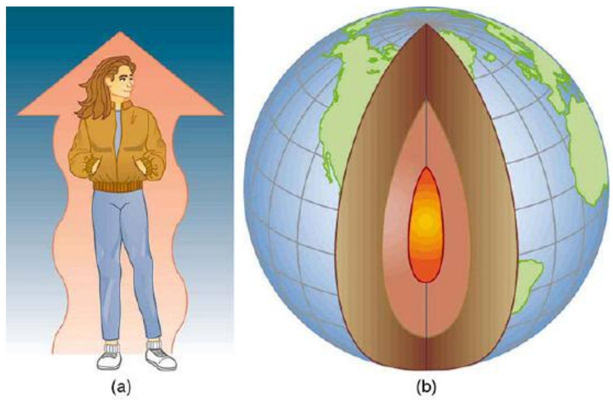
13.1: Introduction

learning objectives

- Distinguish three modes of heat transfer

Introduction to Heat and Heat Transfer

Energy can exist in many forms and heat is one of the most intriguing. Heat is often hidden, as it only exists when in transit, and is transferred by a number of distinctly different methods. Heat transfer touches every aspect of our lives and helps us understand how the universe functions. It explains the chill we feel on a clear breezy night, or why Earth's core has yet to cool. This module defines and explores heat transfer, its effects, and the methods by which heat is transferred. These topics are fundamental, as well as practical, and will often be referred to in the modules ahead.



Examples of Heat Transfer: (a) The chilling effect of a clear breezy night is produced by the wind and by radiative heat transfer to cold outer space. (b) There was once great controversy about the Earth's age, but it is now generally accepted to be about 4.5 billion years old. Much of the debate is centered on the Earth's molten interior. According to our understanding of heat transfer, if the Earth is really that old, its center should have cooled off long ago. The discovery of radioactivity in rocks revealed the source of energy that keeps the Earth's interior molten, despite heat transfer to the surface, and from there to cold outer space.

Definitions

Scottish physicist James Clerk Maxwell, in his 1871 classic *Theory of Heat*, was one of many who began to build on the already established idea that heat was something to do with matter in motion. This was the same idea put forwards by Sir Benjamin Thompson in 1798, who said he was only following on from the work of many others. One of Maxwell's recommended books was *Heat as a Mode of Motion* by John Tyndall. Maxwell outlined four stipulations for the definition of heat:

- It is something which may be transferred from one body to another.
- It is a measurable quantity, and thus treated mathematically.
- It cannot be treated as a substance, because it may be transformed into something that is not a substance, such as mechanical work.
- Heat is one of the forms of energy.

In the following sections, we will define heat more rigorously, paying particular attention to how it can be measured and quantified.

Estimation of Quantity of Heat

The quantity of heat transferred by some process can either be directly measured, or determined indirectly through calculations based on other quantities. Direct measurement is by calorimetry and is the primary empirical basis of the idea of quantity of heat transferred in a process. The transferred heat is measured by changes in a body of known properties, for example, temperature rise, change in volume or length, or phase change, such as melting of ice. Indirect estimations of quantity of heat transferred rely on the law of conservation of energy, and, in particular cases, on the first law of thermodynamics (explored in the following sections). Indirect estimation is the primary approach of many theoretical studies of quantity of heat transferred.

Heat Transfer Methods

After defining and quantifying heat transfer and its effects on physical systems, we will discuss the methods by which heat is transferred. So many processes involve heat transfer, so that it is hard to imagine a situation where no heat transfer occurs. Yet every process involving heat transfer takes place by only three methods:

1. Conduction is heat transfer through stationary matter by physical contact. (The matter is stationary on a macroscopic scale—we know there is thermal motion of the atoms and molecules at any temperature above absolute zero.) Heat transferred between the electric burner of a stove and the bottom of a pan is transferred by conduction.
2. Convection is the heat transfer by the macroscopic movement of a fluid. This type of transfer takes place in a forced-air furnace and in weather systems, for example.
3. Heat transfer by radiation occurs when microwaves, infrared radiation, visible light, or another form of electromagnetic radiation is emitted or absorbed. An obvious example is the warming of the Earth by the Sun. A less obvious example is thermal radiation from the human body.

Heat as Energy Transfer

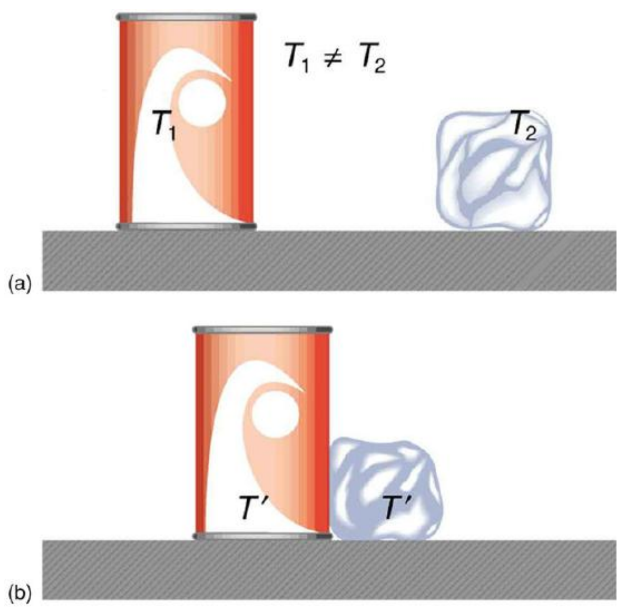
Heat is the spontaneous transfer of energy due to a temperature difference.

learning objectives

- Identify SI and common units of heat

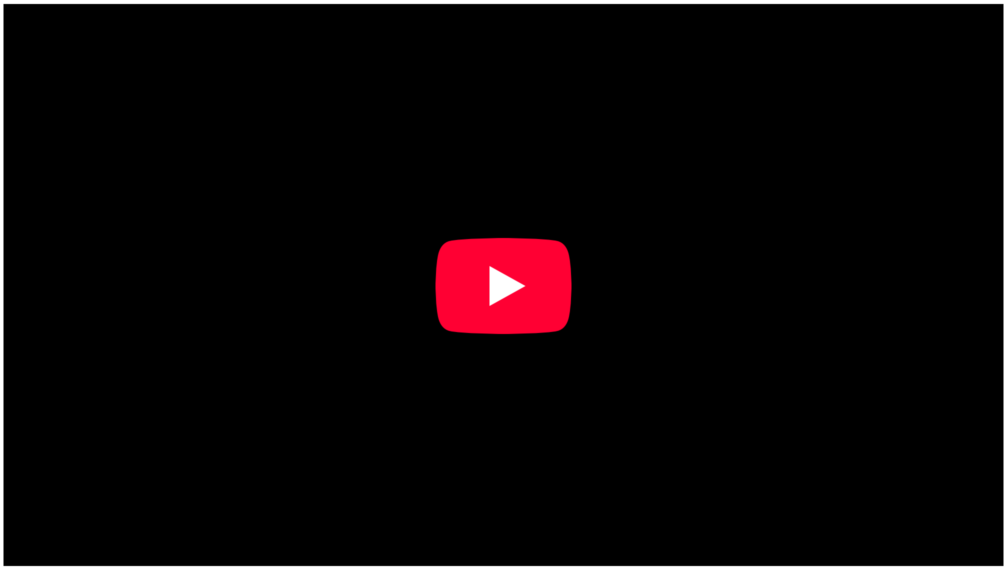
Heat as Energy Transfer

Consider two objects at different temperatures that are brought together. Energy is transferred from the hotter object to the cooler one, until both objects reach thermal equilibrium (i.e., both become the same temperature). How is this energy transferred? No work is done by either object, because no force acts through a distance. The transfer of energy is caused by the temperature difference, and ceases once the temperatures are equal. This observation leads to the following definition of heat: Heat is the spontaneous transfer of energy due to a temperature difference.



Heat Transfer and Equilibrium: (a) The soft drink and the ice have different temperatures, T1 and T2, and are not in thermal equilibrium. (b) When the soft drink and ice are allowed to interact, energy is transferred until they reach the same temperature T, achieving equilibrium. Heat transfer occurs due to the difference in temperatures. In fact, since the soft drink and ice are both in contact with the surrounding air and bench, the equilibrium temperature will be the same for both.

Where Is the Most Heat Lost?: Use movable thermometers to discover where a house has poor insulation.



Heat Transfer: A brief introduction to heat transfer for students.

Heat is often confused with temperature. For example, we may say the heat was unbearable, when we actually mean that the temperature was high. Heat is a form of energy, whereas temperature is not. The misconception arises because we are sensitive to the flow of heat, rather than the temperature.

Units

Owing to the fact that heat is a form of energy, it has the SI unit of *joule* (J). The *calorie* (cal) is a common unit of energy, defined as the energy needed to change the temperature of 1.00 g of water by 1.00ºC —specifically, between 14.5ºC and 15.5ºC, since there is a slight temperature dependence. Another common unit of heat is the kilocalorie (kcal), which is the energy needed to change the temperature of 1.00 kg of water by 1.00ºC. Since mass is often specified in kilograms, kilocalorie is commonly used. Food calories (given the notation Cal, and sometimes called “big calorie”) are actually kilocalories (1kilocalorie=1000 calories), a fact not easily determined from package labeling in the United States, but more common in Europe and elsewhere. In some engineering fields, the British Thermal Unit (BTU), equal to about 1.055 kilo-joules, is widely used.

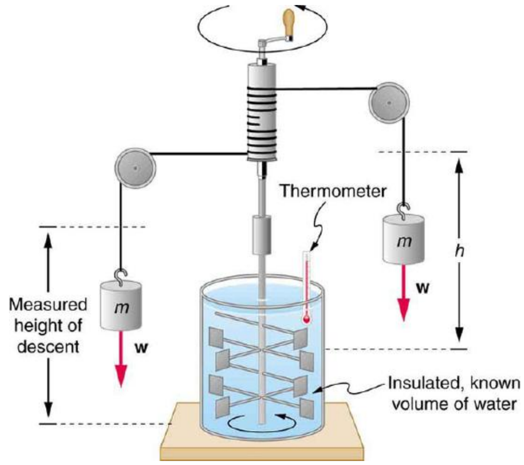


Figure 1 Equivalence of Heat and Work: Schematic depiction of Joule's experiment that established the equivalence of heat and work

The total amount of energy transferred as heat is conventionally written as Q for algebraic purposes. Heat released by a system into its surroundings is by convention a negative quantity ($Q < 0$); when a system absorbs heat from its surroundings, it is positive ($Q > 0$).

Mechanical Equivalent of Heat

It is also possible to change the temperature of a substance by doing work. Work can transfer energy into or out of a system. This realization helped establish the fact that heat is a form of energy. James Prescott Joule (1818–1889) performed many experiments to establish the mechanical equivalent of heat—*the work needed to produce the same effects as heat transfer*. In terms of the units used for these two terms, the best modern value for this equivalence is 1.000 kcal = 4186 J. We consider this equation as the conversion between two different units of energy.

Figure 1 shows one of Joule's most famous experimental setups for demonstrating the mechanical equivalent of heat. It demonstrated that work and heat can produce the same effects, and helped establish the principle of conservation of energy. Gravitational potential energy (PE) (work done by the gravitational force) is converted into kinetic energy (KE), and then randomized by viscosity and turbulence into increased average kinetic energy of atoms and molecules in the system, producing a temperature increase. His contributions to the field of thermodynamics were so significant that the SI unit of energy was named after him.

Heat added or removed from a system changes its internal energy (a concept we will discuss in the following section) and thus its temperature. Such a temperature increase is observed while cooking. However, adding heat does not necessarily increase the temperature. An example is melting of ice; that is, when a substance changes from one phase to another. Work done on the system or by the system can also change the internal energy of the system. Joule demonstrated that the temperature of a system can be increased by stirring. If an ice cube is rubbed against a rough surface, work is done by the frictional force. A system has a well-defined internal energy, but we cannot say that it has a certain "heat content" or "work content." We use the phrase "heat transfer" to emphasize its nature.

Internal Energy

The internal energy of a system is the sum of all kinetic and potential energy in a system.

learning objectives

- Express the internal energy in terms of kinetic and potential energy

Internal Energy

James Joule showed that both heat and work can produce the same change in the internal energy of a substance, establishing the principle of the mechanical equivalence of heat. Heat is emphatically a quantity that solely describes energy being transferred. It makes no sense to speak of the total 'heat' an object or system contains. However, a system does contain a quantifiable amount of energy called the internal energy of a system. The internal energy of a system is the quantity that changes with the addition or subtraction of work or heat. It is closely related to temperature.

Definition

The internal energy is the energy required to create a system, excluding the energy necessary to displace its surroundings. Internal energy has two components: kinetic energy and potential energy. The kinetic energy consists of all the energy involving the motions of the particles constituting the system, including translation, vibration, and rotation. The potential energy is associated with the static constituents of matter, static electric energy of atoms within molecules or crystals, and the energy from chemical bonds. The equation describing the total internal energy of a system is then:

$$U = U_{kinetic} + U_{potential}. \tag{13.1.1}$$

We can also think of the internal energy as the sum of all the energy states of each component in the system:

$$U = \sum_i E_i. \tag{13.1.2}$$

At any finite temperature, kinetic and potential energies are constantly converted into each other, but the total energy remains constant in an isolated system. The kinetic energy portion of internal energy gives rise to the temperature of the system. We can use statistical mechanics to relate the (somewhat) random motions of particles in a system to the mean kinetic energy of the ensemble of particles, and thus the empirically measurable quantity expressed as temperature.

We can see that internal energy is an extensive property: it depends on the size of the system or on the amount of substance it contains.

In most cases, we are not concerned with the *total* amount of internal energy in the system, as it is rarely convenient or necessary to consider all energies belonging to the system. Rather, we are far more interested in the *change* in internal energy, given some transfer of work or heat. This can be expressed as:

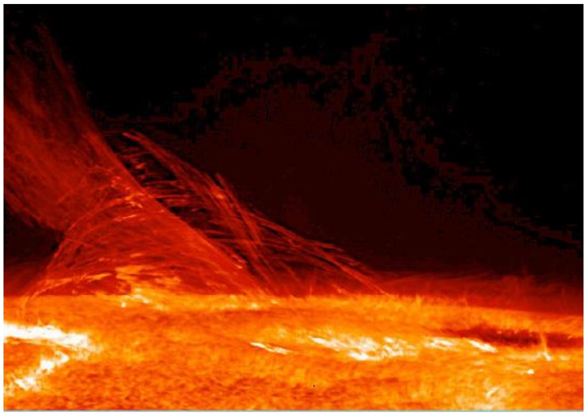
$$\Delta U = Q + W_{mech} + W_{other}. \tag{13.1.3}$$

Q is heat added to a system and W_{mech} is the mechanical work performed by the surroundings due to pressure or volume changes in the system. All other perturbations and energies added by other processes, such as an electric current introduced into an electronic circuit, is summarized as the term W_{extra}.

We can calculate a small change in internal energy of the system by considering the infinitesimal amount of heat δQ added to the system minus the infinitesimal amount of work δW done by the system:

$$dU = \delta Q - \delta W. \tag{13.1.4}$$

This expression is the first law of thermodynamics.



The Sun and Internal Energy: Nuclear fusion in the sun converts nuclear potential energy into available internal energy and keeps the temperature of the Sun very high.

Key Points

- Heat is a crucial concept that touches every aspect of our lives. James Clerk Maxwell set down important principles that couple into the definition of heat.
- The quantity of heat transfer can be directly measured or estimated indirectly through the science of calorimetry.
- There are three modes of heat transfer: conduction, convection, and radiation.
- If two objects at different temperature are brought together, energy will transfer from the hotter object to the cooler one until both are at the same temperature. This transfer of energy is known as heat.
- Heat should not be confused with temperature. Temperature describes the internal state of an object, while heat refers to the energy transferred to or from the object.
- Since heat is a form of energy, its SI unit is the joule. Other common units of heat energy include the calorie and kilocalorie, equal to 4.186 and 4,186 joules, respectively.
- Because heat and work both involve the transfer of energy, they can each produce the same effects. The concept of the mechanical equivalent of heat was instrumental in establishing the principle of conservation of energy.
- While a system does not contain ' heat,' it does contain a total amount of energy called internal energy.
- The internal energy is the energy necessary to create a system, minus the energy necessary to displace its surroundings.
- Most of the time, we are interested in the change in internal energy rather than the total internal energy.
- The first law of thermodynamics, $\mathrm{dU} = \Delta\mathrm{Q} - \Delta\mathrm{W}$, describes small changes in internal energy.

Key Terms

- **heat transfer**: The transmission of thermal energy via conduction, convection, or radiation.
- **calorimetry**: The science of measuring the heat absorbed or evolved during the course of a chemical reaction or change of state.
- **kilocalorie**: A non-SI unit of energy equal to 1,000 calories or 4,186 joules; equal to the "calorie" or "Calorie" used in nutritional labeling. Symbol: kcal.
- **thermal equilibrium**: Two systems are in thermal equilibrium if they could transfer heat between each other, but don't.
- **mechanical equivalent of heat**: The work needed to produce the same effects as heat transfer.
- **internal energy**: The sum of all energy present in the system, including kinetic and potential energy; equivalently, the energy needed to create a system, excluding the energy necessary to displace its surroundings.
- **isolated system**: A system that does not interact with its surroundings, that is, its total energy and mass stay constant.

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13.2: Specific Heat

learning objectives

- Explain the enthalpy in a system with constant volume and pressure

Heat capacity (usually denoted by a capital C, often with subscripts), or thermal capacity, is the measurable physical quantity that characterizes the amount of heat required to change a substance's temperature by a given amount. In SI units, heat capacity is expressed in units of joules per kelvin (J/K).

An object's heat capacity (symbol *C*) is defined as the ratio of the amount of heat energy transferred to an object to the resulting increase in temperature of the object.

$$C = \frac{Q}{\Delta T}. \tag{13.2.1}$$

Heat capacity is an extensive property, so it scales with the size of the system. A sample containing twice the amount of substance as another sample requires the transfer of twice as much heat (Q) to achieve the same change in temperature (ΔT). For example, if it takes 1,000 J to heat a block of iron, it would take 2,000 J to heat a second block of iron with twice the mass as the first.

The Measurement of Heat Capacity

The heat capacity of most systems is not a constant. Rather, it depends on the state variables of the thermodynamic system under study. In particular, it is dependent on temperature itself, as well as on the pressure and the volume of the system, and the ways in which pressures and volumes have been allowed to change while the system has passed from one temperature to another. The reason for this is that pressure-volume work done to the system raises its temperature by a mechanism other than heating, while pressure-volume work done by the system absorbs heat without raising the system's temperature. (The temperature dependence is why the definition a calorie is formally the energy needed to heat 1 g of water from 14.5 to 15.5 °C instead of generally by 1 °C.)

Different measurements of heat capacity can therefore be performed, most commonly at constant pressure and constant volume. The values thus measured are usually subscripted (by p and V, respectively) to indicate the definition. Gases and liquids are typically also measured at constant volume. Measurements under constant pressure produce larger values than those at constant volume because the constant pressure values also include heat energy that is used to do work to expand the substance against the constant pressure as its temperature increases. This difference is particularly notable in gases where values under constant pressure are typically 30% to 66.7% greater than those at constant volume.

Thermodynamic Relations and Definition of Heat Capacity

The internal energy of a closed system changes either by adding heat to the system or by the system performing work. Recalling the first law of thermodynamics,

$$dU = \delta Q - \delta W. \tag{13.2.2}$$

For work as a result of an increase of the system volume we may write,

$$dU = \delta Q - PdV. \tag{13.2.3}$$

If the heat is added at constant volume, then the second term of this relation vanishes and one readily obtains

$$(\frac{\partial U}{\partial T})_V = (\frac{\partial Q}{\partial T})_V = C_V. \tag{13.2.4}$$

This defines the *heat capacity at constant volume*, C_V. Another useful quantity is the *heat capacity at constant pressure*, C_P. With the *enthalpy* of the system given by

$$H = U + PV, \tag{13.2.5}$$

our equation for d*U* changes to

$$dH = \delta Q + VdP, \tag{13.2.6}$$

and therefore, at constant pressure, we have

$$(\frac{\partial H}{\partial T})_P = (\frac{\partial Q}{\partial T})_P = C_P. \tag{13.2.7}$$

Specific Heat

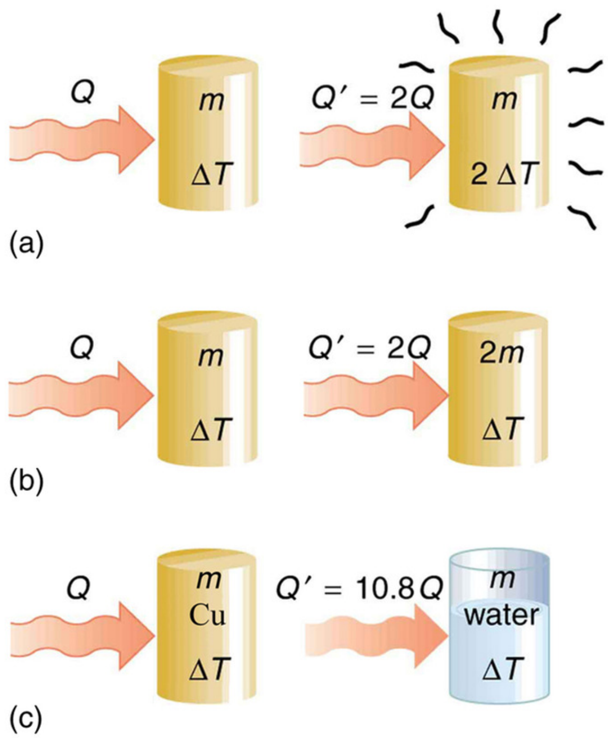
The specific heat is an intensive property that describes how much heat must be added to a particular substance to raise its temperature.

learning objectives

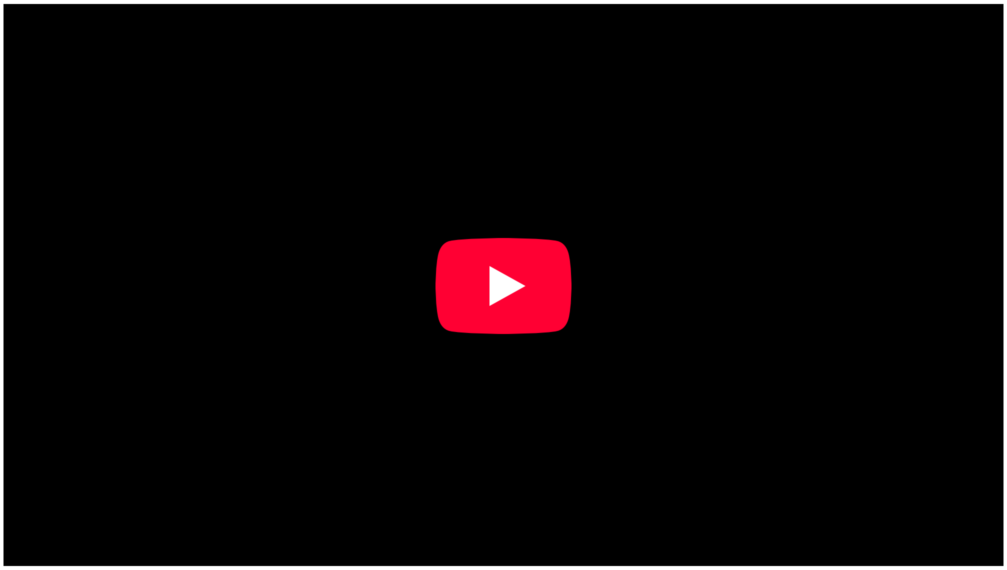
- Summarize the quantitative relationship between heat transfer and temperature change

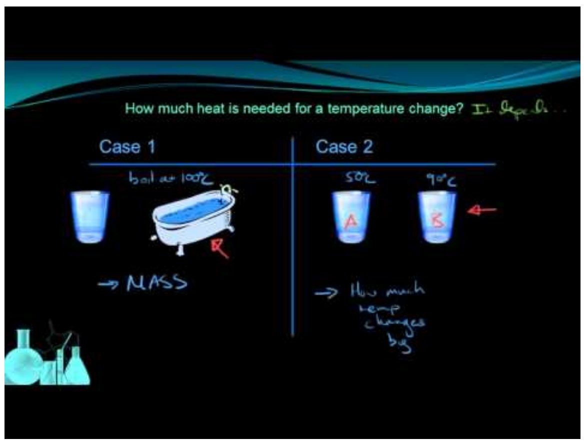
Specific Heat

The heat capacity is an extensive property that describes how much heat energy it takes to raise the temperature of a given system. However, it would be pretty inconvenient to measure the heat capacity of every unit of matter. What we want is an intensive property that depends only on the type and phase of a substance and can be applied to systems of arbitrary size. This quantity is known as the specific heat capacity (or simply, the specific heat), which is the heat capacity per unit mass of a material. Experiments show that the transferred heat depends on three factors: (1) The change in temperature, (2) the mass of the system, and (3) the substance and phase of the substance. The last two factors are encapsulated in the value of the specific heat.



Heat Transfer and Specific Heat Capacity: The heat Q transferred to cause a temperature change depends on the magnitude of the temperature change, the mass of the system, and the substance and phase involved. (a) The amount of heat transferred is directly proportional to the temperature change. To double the temperature change of a mass m, you need to add twice the heat. (b) The amount of heat transferred is also directly proportional to the mass. To cause an equivalent temperature change in a doubled mass, you need to add twice the heat. (c) The amount of heat transferred depends on the substance and its phase. If it takes an amount Q of heat to cause a temperature change ΔT in a given mass of copper, it will take 10.8 times that amount of heat to cause the equivalent temperature change in the same mass of water assuming no phase change in either substance.





Specific Heat Capacity: This lesson relates heat to a change in temperature. We discuss how the amount of heat needed for a temperature change is dependent on mass and the substance involved, and that relationship is represented by the specific heat capactiy of the substance, C.

The dependence on temperature change and mass are easily understood. Because the (average) kinetic energy of an atom or molecule is proportional to the absolute temperature, the internal energy of a system is proportional to the absolute temperature and the number of atoms or molecules. Since the transferred heat is equal to the change in the internal energy, the heat is proportional to the mass of the substance and the temperature change. The transferred heat also depends on the substance so that, for example, the heat necessary to raise the temperature is less for alcohol than for water. For the same substance, the transferred heat also depends on the phase (gas, liquid, or solid).

The quantitative relationship between heat transfer and temperature change contains all three factors:

$$\mathrm{Q=mc\Delta T,} \tag{13.2.8}$$

where Q is the symbol for heat transfer, m is the mass of the substance, and ΔT is the change in temperature. The symbol c stands for specific heat and depends on the material and phase.

The specific heat is the amount of heat necessary to change the temperature of 1.00 kg of mass by 1.00ºC. The specific heat c is a property of the substance; its SI unit is J/(kg⋅K) or J/(kg⋅C). Recall that the temperature change (ΔT) is the same in units of kelvin and degrees Celsius. Note that the total heat capacity C is simply the product of the specific heat capacity c and the mass of the substance m, i.e.,

$$\mathrm{C=mc \text{ or } c=\frac{C}{m}=\frac{C}{\rho V},} \tag{13.2.9}$$

where ϱ is the density of the substance and V is its volume.

Values of specific heat must generally be looked up in tables, because there is no simple way to calculate them. Instead, they are measured empirically. In general, the specific heat also depends on the temperature. The table below lists representative values of specific heat for various substances. Except for gases, the temperature and volume dependence of the specific heat of most substances is weak. The specific heat of water is five times that of glass and ten times that of iron, which means that it takes five times as much heat to raise the temperature of water the same amount as for glass and ten times as much heat to raise the temperature of water as for iron. In fact, water has one of the largest specific heats of any material, which is important for sustaining life on Earth.

Substances	**Specific heat (*c*)**	
Solids	J/kg·°C	kcal/kg·°C[2]
Aluminum	900	0.215
Asbestos	800	0.19
Concrete, granite (average)	840	0.20
Copper	387	0.0924
Glass	840	0.20
Gold	129	0.0308
Human body (average at 37 °C)	3500	0.83
Ice (average, -50°C to 0°C)	2090	0.50
Iron, steel	452	0.108
Lead	128	0.0305
Silver	235	0.0562
Wood	1700	0.4
Liquids		
Benzene	1740	0.415
Ethanol	2450	0.586
Glycerin	2410	0.576
Mercury	139	0.0333
Water (15.0 °C)	4186	1.000
Gases [3]		
Air (dry)	721 (1015)	0.172 (0.242)
Ammonia	1670 (2190)	0.399 (0.523)
Carbon dioxide	638 (833)	0.152 (0.199)
Nitrogen	739 (1040)	0.177 (0.248)
Oxygen	651 (913)	0.156 (0.218)
Steam (100°C)	1520 (2020)	0.363 (0.482)

Specific Heats: Listed are the specific heats of various substances. These values are identical in units of cal/(g·C).3. cv at constant volume and at 20.0ºC, except as noted, and at 1.00 atm average pressure. Values in parentheses are cp at a constant pressure of 1.00 atm.

Calorimetry

Calorimetry is the measurement of the heat of chemical reactions or physical changes.

learning objectives

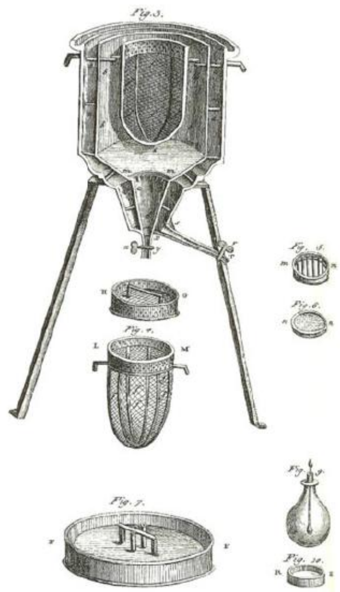
- Analyze the relationship between the gas constant for an ideal gas yield and volume

Calorimetry

Overview

Calorimetry is the science of measuring the heat of chemical reactions or physical changes. Calorimetry is performed with a calorimeter. A simple calorimeter just consists of a thermometer attached to a metal container full of water suspended above a combustion chamber. The word calorimetry is derived from the Latin word *calor*, meaning heat. Scottish physician and scientist Joseph Black, who was the first to recognize the distinction between heat and temperature, is said to be the founder of calorimetry.

Calorimetry requires that the material being heated have known thermal properties, i.e. specific heat capacities. The classical rule, recognized by Clausius and by Kelvin, is that the pressure exerted by the calorimetric material is fully and rapidly determined solely by its temperature and volume; this rule is for changes that do not involve phase change, such as melting of ice. There are many materials that do not comply with this rule, and for them, more complex equations are required than those below.



Ice Calorimeter: The world's first ice-calorimeter, used in the winter of 1782-83, by Antoine Lavoisier and Pierre-Simon Laplace, to determine the heat evolved in variouschemical changes; calculations which were based on Joseph Black's prior discovery of latent heat. These experiments mark the foundation of thermochemistry.

Basic Calorimetry at Constant Value

Constant-volume calorimetry is calorimetry performed at a constant volume. This involves the use of a constant-volume calorimeter (one type is called a Bomb calorimeter). For constant-volume calorimetry:

$$\delta \mathrm{Q} = \mathrm{CV}\Delta \mathrm{T} = \mathrm{mcV}\Delta \mathrm{T} \tag{13.2.10}$$

where δQ is the increment of heat gained by the sample, C_V is the heat capacity at constant volume, c_v is the specific heat at constant volume, and ΔT is the change in temperature.

Measuring Enthalpy Change

To find the enthalpy change per mass (or per mole) of a substance A in a reaction between two substances A and B, the substances are added to a calorimeter and the initial and final temperatures (before the reaction started and after it has finished) are noted. Multiplying the temperature change by the mass and specific heat capacities of the substances gives a value for the energy given off or absorbed during the reaction:

$$\delta \mathrm{Q} = \Delta \mathrm{T}(\mathrm{m_A c_A} + \mathrm{m_B c_B}) \tag{13.2.11}$$

Dividing the energy change by how many grams (or moles) of A were present gives its enthalpy change of reaction. This method is used primarily in academic teaching as it describes the theory of calorimetry. It does not account for the heat loss through the container or the heat capacity of the thermometer and container itself. In addition, the object placed inside the calorimeter shows that the objects transferred their heat to the calorimeter and into the liquid, and the heat absorbed by the calorimeter and the liquid is equal to the heat given off by the metals.

Constant-Pressure Calorimetry

A constant-pressure calorimeter measures the change in enthalpy of a reaction occurring in solution during which the atmospheric pressure remains constant. An example is a coffee-cup calorimeter, which is constructed from two nested Styrofoam cups and a lid with two holes, allowing insertion of a thermometer and a stirring rod. The inner cup holds a known amount of a solute, usually water, that absorbs the heat from the reaction. When the reaction occurs, the outer cup provides insulation. Then

$$C_P = \frac{W\Delta H}{M\Delta T} \tag{13.2.12}$$

where C_p is the specific heat at constant pressure, ΔH is the enthalpy of the solution, ΔT is the change in temperature, W is the mass of the solute, and M is the molecular mass of the solute. The measurement of heat using a simple calorimeter, like the coffee cup calorimeter, is an example of constant-pressure calorimetry, since the pressure (atmospheric pressure) remains constant during the process. Constant-pressure calorimetry is used in determining the changes in enthalpy occurring in solution. Under these conditions the change in enthalpy equals the heat (Q=ΔH).

Specific Heat for an Ideal Gas at Constant Pressure and Volume

An ideal gas has different specific heat capacities under constant volume or constant pressure conditions.

learning objectives

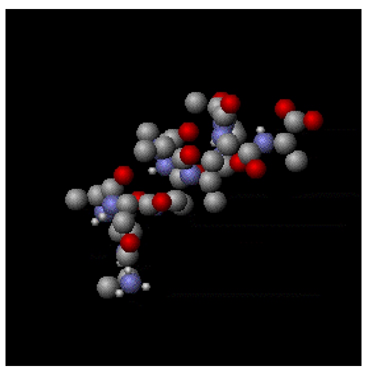
- Explain how to derive the adiabatic index

Specific Heat for an Ideal Gas at Constant Pressure and Volume

The heat capacity at constant volume of nR = 1 J·K^{-1} of any gas, including an ideal gas is:

$$(\frac{\partial U}{\partial T})_V = c_v \tag{13.2.13}$$

This represents the dimensionless heat capacity at constant volume; it is generally a function of temperature due to intermolecular forces. For moderate temperatures, the constant for a monoatomic gas is c_v=3/2 while for a diatomic gas it is c_v=5/2 (see). Macroscopic measurements on heat capacity provide information on the microscopic structure of the molecules.



Molecular internal vibrations: When a gas is heated, translational kientic energy of molecules in the gas will increase. In addition, molecules in the gas may pick up many characteristic internal vibrations. Potential energy stored in these internal degrees of freedom contributes to specific heat of the gas.

The heat capacity at constant pressure of 1 J·K^{-1} ideal gas is:

$$(\frac{\partial H}{\partial T})_V = c_p = c_v + R \tag{13.2.14}$$

where H=U+pV is the enthalpy of the gas.

Measuring the heat capacity at constant volume can be prohibitively difficult for liquids and solids. That is, small temperature changes typically require large pressures to maintain a liquid or solid at constant volume (this implies the containing vessel must be nearly rigid or at least very strong). It is easier to measure the heat capacity at constant pressure (allowing the material to expand or contract freely) and solve for the heat capacity at constant volume using mathematical relationships derived from the basic thermodynamic laws.

Utilizing the Fundamental Thermodynamic Relation we can show:

$$C_p - C_V = T(\frac{\partial P}{\partial T})_V, N(\frac{\partial V}{\partial T})_{p,N} \tag{13.2.15}$$

where the partial derivatives are taken at: constant volume and constant number of particles, and at constant pressure and constant number of particles, respectively.

The heat capacity ratio or adiabatic index is the ratio of the heat capacity at constant pressure to heat capacity at constant volume. It is sometimes also known as the isentropic expansion factor:

$$\gamma = \frac{C_P}{C_V} = \frac{c_p}{c_v} \tag{13.2.16}$$

For an ideal gas, evaluating the partial derivatives above according to the equation of state, where R is the gas constant for an ideal gas yields:

$$pV = RT \tag{13.2.17}$$

$$C_p - C_V = T(\frac{\partial P}{\partial T})_V(\frac{\partial V}{\partial T})_p \tag{13.2.18}$$

$$C_p - C_V = -T(\frac{\partial P}{\partial V})_V(\frac{\partial V}{\partial T})_p^2 \tag{13.2.19}$$

$$P = \frac{RT}{V}n \rightarrow (\frac{\partial P}{\partial V})_T = \frac{-RT}{V^2} = \frac{-P}{V} \tag{13.2.20}$$

$$V = \frac{RT}{P}n \rightarrow (\frac{\partial V}{\partial T})_p^2 = \frac{R^2}{P^2} \tag{13.2.21}$$

substituting:

$$-T(\frac{\partial P}{\partial V})_V(\frac{\partial V}{\partial T})_p^2 = -T\frac{-P}{V}\frac{R^2}{P^2} = R \tag{13.2.22}$$

This equation reduces simply to what is known as Mayer’s relation:

Julius Robert Mayer: Julius Robert von Mayer (November 25, 1814 – March 20, 1878), a German physician and physicist, was one of the founders of thermodynamics. He is best known for his 1841 enunciation of one of the original statements of the conservation of energy (or what is now known as one of the first versions of the first law of thermodynamics): “Energy can be neither created nor destroyed. ” In 1842, Mayer described the vital chemical process now referred to as oxidation as the primary source of energy for any living creature. His achievements were overlooked and credit for the discovery of the mechanical equivalent of heat was attributed to James Joule in the following year. von Mayer also proposed that plants convert light into chemical energy.

$$\mathrm{C_P - C_V = R.} \tag{13.2.23}$$

It is a simple equation relating the heat capacities under constant temperature and under constant pressure.

Solving Problems with Calorimetry

Calorimetry is used to measure the amount of heat produced or consumed in a chemical reaction.

learning objectives

- Explain a bomb calorimeter is used to measure heat evolved in a combustion reaction

Calorimeters are designed to minimize energy exchange between the system being studied and its surroundings. They range from simple coffee cup calorimeters used by introductory chemistry students to sophisticated bomb calorimeters used to determine the energy content of food.

Calorimetry is used to measure amounts of heat transferred to or from a substance. To do so, the heat is exchanged with a calibrated object (calorimeter). The change in temperature of the measuring part of the calorimeter is converted into the amount of heat (since the previous calibration was used to establish its heat capacity). The measurement of heat transfer using this approach requires the definition of a system (the substance or substances undergoing the chemical or physical change) and its surroundings (the other components of the measurement apparatus that serve to either provide heat to the system or absorb heat from the system). Knowledge of the heat capacity of the surroundings, and careful measurements of the masses of the system and surroundings and their temperatures before and after the process allows one to calculate the heat transferred as described in this section.

A calorimeter is a device used to measure the amount of heat involved in a chemical or physical process. For example, when an exothermic reaction occurs in solution in a calorimeter, the heat produced by the reaction is absorbed by the solution, which increases its temperature. When an endothermic reaction occurs, the heat required is absorbed from the thermal energy of the solution, which decreases its temperature. The temperature change, along with the specific heat and mass of the solution, can then be used to calculate the amount of heat involved in either case.

Coffee-Cup Calorimeters

General chemistry students often use simple calorimeters constructed from polystyrene cups. These easy-to-use “coffee cup” calorimeters allow more heat exchange with their surroundings, and therefore produce less accurate energy values.

Structure of the Constant Volume (or “Bomb”) Calorimeter

Bomb Calorimeter: This is the picture of a typical setup of bomb calorimeter.

A different type of calorimeter that operates at constant volume, colloquially known as a bomb calorimeter, is used to measure the energy produced by reactions that yield large amounts of heat and gaseous products, such as combustion reactions. (The term “bomb” comes from the observation that these reactions can be vigorous enough to resemble explosions that would damage other calorimeters.) This type of calorimeter consists of a robust steel container (the “bomb”) that contains the reactants and is itself submerged in water. The sample is placed in the bomb, which is then filled with oxygen at high pressure. A small electrical spark is used to ignite the sample. The energy produced by the reaction is trapped in the steel bomb and the surrounding water. The temperature increase is measured and, along with the known heat capacity of the calorimeter, is used to calculate the energy produced by the reaction. Bomb calorimeters require calibration to determine the heat capacity of the calorimeter and ensure accurate results. The calibration is accomplished using a reaction with a known q, such as a measured quantity of benzoic acid ignited by a spark from a nickel fuse wire that is weighed before and after the reaction. The temperature change produced by the known reaction is used to determine the heat capacity of the calorimeter. The calibration is generally performed each time before the calorimeter is used to gather research data.

Example 13.2.1: Identifying a Metal by Measuring Specific Heat

A 59.7 g piece of metal that had been submerged in boiling water was quickly transferred into 60.0 mL of water initially at 22.0 °C. The final temperature is 28.5 °C. Use these data to determine the specific heat of the metal. Use this result to identify the metal.

Solution

Assuming perfect heat transfer, the heat given off by metal is the negative of the heat taken in by water, or:

$$\mathrm{q_{metal} = -q_{water}} \tag{13.2.24}$$

In expanded form, this is:

$$\mathrm{c_{metal} \times m_{metal} \times (T_{f,metal} - T_{i,metal}) = c_{water} \times m_{water} \times (T_{f,water} - T_{i,water})} \tag{13.2.25}$$

Noting that since the metal was submerged in boiling water, its initial temperature was 100.0 °C; and that for water, 60.0 mL = 60.0 g; we have:

$$\mathrm{(c_{metal})(59.7 g)(28.5^oC - 100.0^oC) = (4.18 J/g^oC)(60.0 g)(28.5 oC - 22.0 oC)} \tag{13.2.26}$$

Solving this:

$$\mathrm{c_{metal} = \frac{-(4.184 J/g^oC)(60.0 g)(6.5^oC)}{(59.7 g)(-71.5^oC)} = 0.38 J/g^oC} \tag{13.2.27}$$

Our experimental specific heat is closest to the value for copper (0.39 J/g °C), so we identify the metal as copper.

Key Points

- Heat capacity is the measurable physical quantity that characterizes the amount of heat required to change a substance’s temperature by a given amount. It is measured in joules per Kelvin and given by.
- The heat capacity is an extensive property, scaling with the size of the system.
- The heat capacity of most systems is not constant (though it can often be treated as such). It depends on the temperature, pressure, and volume of the system under consideration.
- Unlike the total heat capacity, the specific heat capacity is independent of mass or volume. It describes how much heat must be added to a unit of mass of a given substance to raise its temperature by one degree Celsius. The units of specific heat capacity are J/(kg °C) or equivalently J/(kg K).
- The heat capacity and the specific heat are related by $\mathrm{C = cm}$ or $\mathrm{c = \frac{C}{m}}$.
- The mass m, specific heat c, change in temperature ΔT, and heat added (or subtracted) Q are related by the equation: $\mathrm{Q = mc\Delta T}$.
- Values of specific heat are dependent on the properties and phase of a given substance. Since they cannot be calculated easily, they are empirically measured and available for reference in tables.
- A calorimeter is used to measure the heat generated (or absorbed) by a physical change or chemical reaction. The science of measuring these changes is known as calorimetry.
- In order to do calorimetry, it is crucial to know the specific heats of the substances being measured.

- Calorimetry can be performed under constant volume or constant pressure. The type of calculation done depends on the conditions of the experiment.
- The specific heat at constant volume for a gas is given as $(\frac{\partial U}{\partial T})_V = c_v$.
- The specific heat at constant pressure for an ideal gas is given as $(\frac{\partial H}{\partial T})_V = c_p = c_v + R$.
- The heat capacity ratio (or adiabatic index) is the ratio of the heat capacity at constant pressure to heat capacity at constant volume.
- Calorimetry is used to measure amounts of heat transferred to or from a substance.
- A calorimeter is a device used to measure the amount of heat involved in a chemical or physical process.
- This means that the amount of heat produced or consumed in the reaction equals the amount of heat absorbed or lost by the solution.

Key Terms

- **heat capacity**: The amount of heat energy needed to raise the temperature of an object or unit of matter by one degree Celsius; in units of joules per kelvin (J/K).
- **enthalpy**: the total amount of energy in a system, including both the internal energy and the energy needed to displace its environment
- **specific heat capacity**: The amount of heat that must be added (or removed) from a unit mass of a substance to change its temperature by one degree Celsius. It is an intensive property.
- **constant-pressure calorimeter**: An instrument used to measure the heat generated during changes that do not involve changes in pressure.
- **calorimeter**: An apparatus for measuring the heat generated or absorbed by either a chemical reaction, change of phase or some other physical change.
- **constant-volume calorimeter**: An instrument used to measure the heat generated during changes that do not involve changes in volume.
- **Fundamental Thermodynamic Relation**: In thermodynamics, the fundamental thermodynamic relation expresses an infinitesimal change in internal energy in terms of infinitesimal changes in entropy, and volume for a closed system in thermal equilibrium in the following way: dU=TdS-PdV. Here, U is internal energy, T is absolute temperature, S is entropy, P is pressure and V is volume.
- **adiabatic index**: The ratio of the heat capacity at constant pressure to heat capacity at constant volume.
- **specific heat**: The ratio of the amount of heat needed to raise the temperature of a unit mass of substance by a unit degree to the amount of heat needed to raise that of the same mass of water by the same amount.
- **heat of reaction**: The enthalpy change in a chemical reaction; the amount of heat that a systems gives up to its surroundings so it can return to its initial temperature.
- **combustion**: A process where two chemicals are combined to produce heat.

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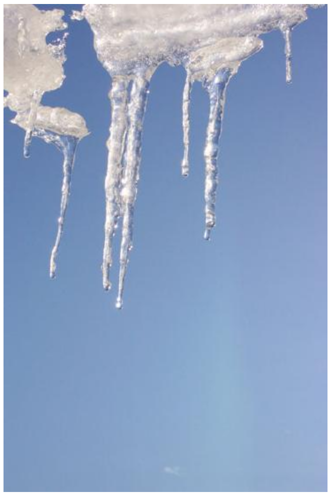
13.3: Phase Change and Latent Heat

learning objectives

- Describe the latent heat as a form of energy

Latent Heat

Previously, we have discussed temperature change due to heat transfer. No temperature change occurs from heat transfer if ice melts and becomes liquid water (i.e., during a phase change). For example, consider water dripping from icicles melting on a roof warmed by the Sun. Conversely, water freezes in an ice tray cooled by lower-temperature surroundings.



Melting Icicle: Heat from the air transfers to the ice causing it to melt.

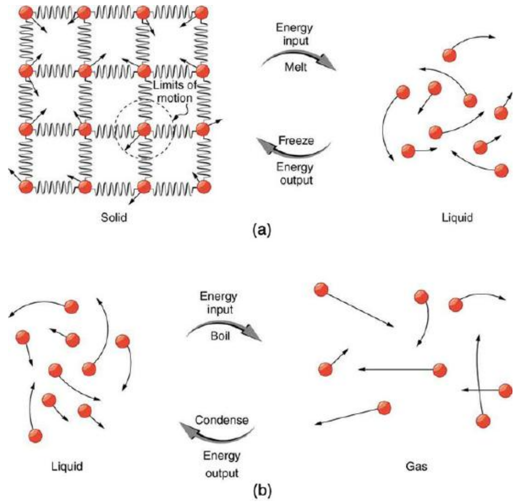
Energy is required to melt a solid because the cohesive bonds between the molecules in the solid must be broken apart so that the molecules can move around at comparable kinetic energies; thus, there is no rise in temperature. Similarly, energy is needed to vaporize a liquid, because molecules in a liquid interact with each other via attractive forces. There is no temperature change until a phase change is complete. The temperature of a glass of lemonade initially at 0 ºC stays at 0 ºC until all the ice has melted. Conversely, energy is released during freezing and condensation, usually in the form of thermal energy. Work is done by cohesive forces when molecules are brought together. The corresponding energy must be given off (dissipated) to allow them to stay together.

The energy involved in a phase change depends on two major factors: the number and strength of bonds or force pairs. The number of bonds is proportional to the number of molecules and thus to the mass of the sample. The strength of forces depends on the type of molecules. The heat Q required to change the phase of a sample of mass m is given by

$\mathrm{Q=mL_f}$ (melting or freezing)

$\mathrm{Q=mL_v}$ (evaporating or condensing)

where the latent heat of fusion, L_f, and latent heat of vaporization, L_v, are material constants that are determined experimentally.



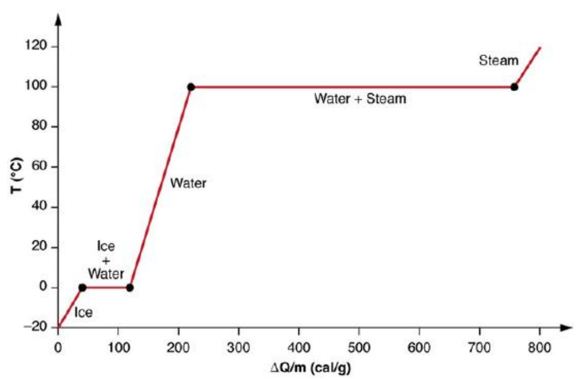
Phase Transitions: (a) Energy is required to partially overcome the attractive forces between molecules in a solid to form a liquid. That same energy must be removed for freezing to take place. (b) Molecules are separated by large distances when going from liquid to vapor, requiring significant energy to overcome molecular attraction. The same energy must be removed for condensation to take place. There is no temperature change until a phase change is complete.

Latent heat is an intensive property measured in units of J/kg. Both L_f and L_v depend on the substance, particularly on the strength of its molecular forces as noted earlier. L_f and L_v are collectively called latent heat coefficients. They are *latent*, or hidden, because in phase changes, energy enters or leaves a system without causing a temperature change in the system; so, in effect, the energy is hidden. Note that melting and vaporization are endothermic processes in that they absorb or require energy, while freezing and condensation are exothermic process as they release energy.

Heating Up Ice: Andrew Vanden Heuvel explores latent heat while trying to cool down his soda.

Significant amounts of energy are involved in phase changes. Let us look, for example, at how much energy is needed to melt a kilogram of ice at 0º C to produce a kilogram of water at 0°C. Using the equation for a change in temperature and the value for water (334 kJ/kg), we find that Q=mLf=(1.0kg)(334kJ/kg)=334kJ is the energy to melt a kilogram of ice. This is a lot of energy as it represents the same amount of energy needed to raise the temperature of 1 kg of liquid water from 0ºC to 79.8ºC. Even more energy is required to vaporize water; it would take 2256 kJ to change 1 kg of liquid water at the normal boiling point (100ºC at atmospheric pressure) to steam (water vapor). This example shows that the energy for a phase change is enormous compared to energy associated with temperature changes without a phase change.

Phase changes can have an enormous stabilizing effect (see figure below). Consider adding heat at a constant rate to a sample of ice initially at -20 ºC. Initially the temperature of the ice rises linearly, absorbing heat at a constant rate of 0.50 cal/g⋅C until it reaches 0 ºC. Once at this temperature, the ice begins to melt until all the entire sample has melted, absorbing a total of 79.8 cal/g of heat. The temperature remains constant at 0 ºC during this phase change. Once all the ice has melted, the temperature of the liquid water rises, absorbing heat at a new constant rate of 1.00 cal/g⋅C (remember that specific heats are dependent on phase). At 100ºC, the water begins to boil and the temperature again remains constant until the water absorbs 539 cal/g of heat to complete this phase change. When all the liquid has become steam, the temperature rises again, absorbing heat at a rate of 0.482 cal/g⋅C.



Heating and Phase Changes of Water: A graph of temperature versus energy added. The system is constructed so that no vapor evaporates while ice warms to become liquid water, and so that, when vaporization occurs, the vapor remains in of the system. The long stretches of constant temperature values at 0ºC and 100ºC reflect the large latent heat of melting and vaporization, respectively.

A phase change we have neglected to mention thus far is sublimation, the transition of solid directly into vapor. The opposite case, where vapor transitions directly into a solid, is called deposition. Sublimation has its own latent heat L*s* and can be used in the same way as $L_v$and L_f.

Key Points

- Energy is required to change the phase of a substance, such as the energy to break the bonds between molecules in a block of ice so it may melt.

- During a phase change energy my be added or subtracted from a system, but the temperature will not change. The temperature will change only when the phase change has completed.
- The heat Q required to change the phase of a sample of mass m is given by $Q = mL_f$ (melting or freezing) and $Q = mL_v$ (evaporating or condensing), where L_f and L_v are the latent heat of fusion and the latent heat of vaporization, respectively.

Key Terms

- **latent heat of fusion**: the energy required to transition one unit of a substance from solid to liquid; equivalently, the energy liberated when one unit of a substance transitions from liquid to solid.
- **latent heat of vaporization**: the energy required to transition one unit of a substance from liquid to vapor; equivalently, the energy liberated when one unit of a substance transitions from vapor to liquid.
- **sublimation**: the transition of a substance from the solid phase directly to the vapor state such that it does not pass through the intermediate, liquid phase

13.4: Methods of Heat Transfer

learning objectives

- Assess why particular characteristics are necessary for effective conduction

Conduction

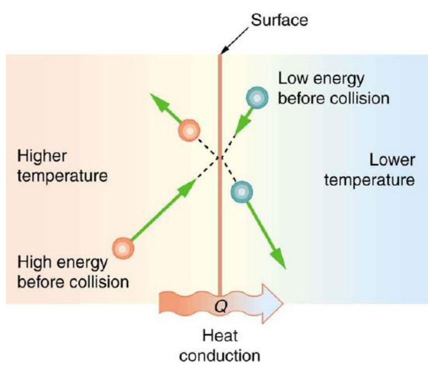
Conduction is the transfer of heat through stationary matter by physical contact. (The matter is stationary on a macroscopic scale—we know there is thermal motion of the atoms and molecules at any temperature above absolute zero.) Heat transferred from an electric stove to the bottom of a pot is an example of conduction.

Some materials conduct thermal energy faster than others. For example, the pillow in your room may the same temperature as the metal doorknob, but the doorknob feels cooler to the touch. In general, good conductors of electricity (metals like copper, aluminum, gold, and silver) are also good heat conductors, whereas insulators of electricity (wood, plastic, and rubber) are poor heat conductors.

Microscopic Description of Conduction

On a microscopic scale, conduction occurs as rapidly moving or vibrating atoms and molecules interact with neighboring particles, transferring some of their kinetic energy. Heat is transferred by conduction when adjacent atoms vibrate against one another, or as electrons move from one atom to another. Conduction is the most significant means of heat transfer within a solid or between solid objects in thermal contact. Conduction is greater in solids because the network of relatively close fixed spatial relationships between atoms helps to transfer energy between them by vibration.

Fluids and gases are less conductive than solids. This is due to the large distance between atoms in a fluid or (especially) a gas: fewer collisions between atoms means less conduction.

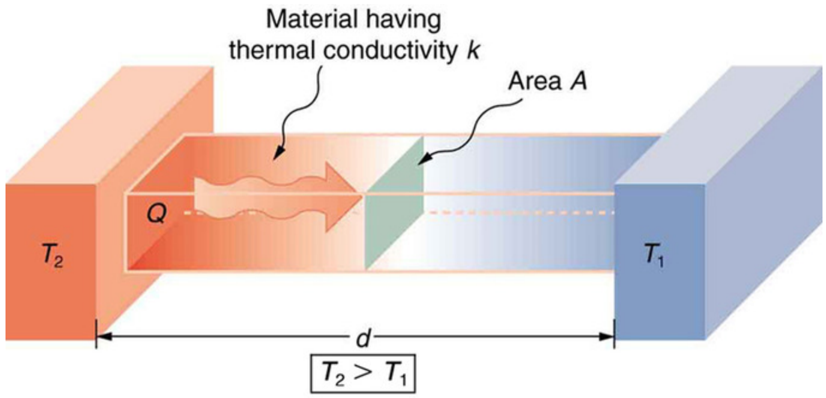


Microscopic Illustration of Conduction: The molecules in two bodies at different temperatures have different average kinetic energies. Collisions occurring at the contact surface tend to transfer energy from high-temperature regions to low-temperature regions. In this illustration, a molecule in the lower temperature region (right side) has low energy before collision, but its energy increases after colliding with the contact surface. In contrast, a molecule in the higher temperature region (left side) has high energy before collision, but its energy decreases after colliding with the contact surface.

The (average) kinetic energy of a molecule in the hot body is higher than in the colder body. If two molecules collide, an energy transfer from the hot to the cold molecule occurs (see the above figure). The cumulative effect from all collisions results in a net flux of heat from the hot body to the colder body. The heat flux thus depends on the temperature difference $\mathrm{T} = \mathrm{T_{hot}} - \mathrm{T_{cold}}$. Therefore, you will get a more severe burn from boiling water than from hot tap water. Conversely, if the temperatures are the same, the net heat transfer rate falls to zero, and equilibrium is achieved. Owing to the fact that the number of collisions increases with increasing area, heat conduction depends on the cross-sectional area. If you touch a cold wall with your palm, your hand cools faster than if you just touch it with your fingertip.

Factors Affecting the Rate of Heat Transfer Through Conduction

In addition to temperature and cross-sectional area, another factor affecting conduction is the thickness of the material through which the heat transfers. Heat transfer from the left side to the right side is accomplished by a series of molecular collisions. The thicker the material, the more time it takes to transfer the same amount of heat. If you get cold during the night, you may retrieve a thicker blanket to keep warm.



Effect of Thickness on Heat Conduction: Heat conduction occurs through any material, represented here by a rectangular bar. The temperature of the material is T_2 on the left and T_1 on the right, where T_2 is greater than T_1. The rate of heat transfer by conduction is directly proportional to the surface area A, the temperature difference $T_2 - T_1$, and the substance's conductivity kk. The rate of heat transfer is inversely proportional to the thickness d.

Lastly, the heat transfer rate depends on the material properties described by the coefficient of thermal conductivity. All four factors are included in a simple equation that was deduced from and is confirmed by experiments. The rate of conductive heat transfer through a slab of material, such as the one in the figure above is given by $\frac{Q}{t} = \frac{kA(T_2 - T_1)}{d}$ where $\frac{Q}{t}$ is the rate of heat transfer in Joules per second (Watts), k is the thermal conductivity of the material, A and d are its surface area and thickness, and $(T_2 - T_1)$ is the temperature difference across the slab.

Convection

Convection is the heat transfer by the macroscopic movement of a fluid, such as a car's engine kept cool by the water in the cooling system.

learning objectives

- Illustrate the mechanisms of convection with phase change

Example 13.4.1:

Calculating Heat Transfer by Convection: Convection of Air Through the Walls of a House.

Most houses are not airtight: air goes in and out around doors and windows, through cracks and crevices, following wiring to switches and outlets, and so on. The air in a typical house is completely replaced in less than an hour.

Suppose that a moderately-sized house has inside dimensions 12.0 m × 18.0 m × 3.00 m high, and that all air is replaced in 30.0 min. Calculate the heat transfer per unit time in watts needed to warm the incoming cold air by 10.0 ºC, thus replacing the heat transferred by convection alone.

Strategy:

Heat is used to raise the temperature of air so that $Q = mc\Delta T$. The rate of heat transfer is then $\frac{Q}{t}$, where t is the time for air turnover. We are given that ΔT is 10.0ºC, but we must still find values for the mass of air and its specific heat before we can calculate QQ. The specific heat of air is a weighted average of the specific heats of nitrogen and oxygen, which is $c = cp \cong 1000\ J/kg \cdot C$ (note that the specific heat at constant pressure must be used for this process).

Solution

(1) Determine the mass of air from its density and the given volume of the house. The density is given from the density ρ and the volume $m = \rho V = (1.29\ kg/m3)(12.0\ m \times 18.0\ m \times 3.00\ m) = 836\ kg$

(2) Calculate the heat transferred from the change in air temperature: $Q = mc\Delta T$ so that $Q = (836\ kg)(1000\ J/kg \cdot \circ C)(10 \circ C) = 8.36 \times 10^6\ J$

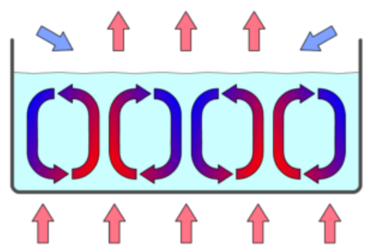
(3) Calculate the heat transfer from the heat Q and the turnover time t. Since air is turned over in $t = 0.500\ h = 1800\ s$, the heat transferred per unit time is $\frac{Q}{t} = \frac{8.36 \times 10^6\ J}{1800\ s} = 4.64\ kW$.

This rate of heat transfer is equal to the power consumed by about forty-six 100-W light bulbs.

Newly constructed homes are designed for a turnover time of 2 hours or more, rather than 30 minutes for the house of this example. Weather stripping, caulking, and improved window seals are commonly employed. More extreme measures are sometimes taken in very cold (or hot) climates to achieve a tight standard of more than 6 hours for one air turnover. Still longer turnover times are unhealthy, because a minimum amount of fresh air is necessary to supply oxygen for breathing and to dilute household pollutants. The term used for the process by which outside air leaks into the house from cracks around windows, doors, and the foundation is called "air infiltration."

Convection

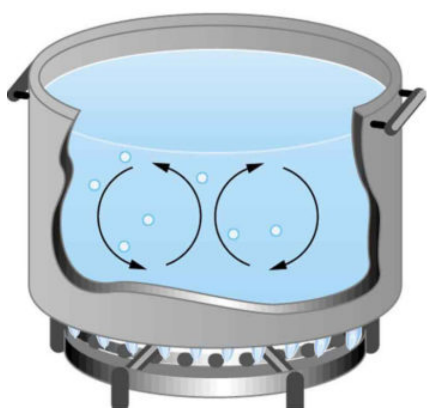
Convection (illustrated in) is the concerted, collective movement of ensembles of molecules within fluids (e.g., liquids, gases). Convection of mass cannot take place in solids, since neither bulk current flows nor significant diffusion can occur in solids. Instead heat diffusion in solids is called heat conduction, which we've just reviewed.



Convection Cells: Convection cells in a gravity field.

Convection is driven by large-scale flow of matter. In the case of Earth, the atmospheric circulation is caused by the flow of hot air from the tropics to the poles, and the flow of cold air from the poles toward the tropics. (Note that Earth's rotation causes changes in the direction of airflow depending on latitude.). An example of convection is a car engine kept cool by the flow of water in the cooling system, with the water pump maintaining a flow of cool water to the pistons.

While convection is usually more complicated than conduction, we can describe convection and perform some straightforward, realistic calculations of its effects. Natural convection is driven by buoyant forces: hot air rises because density decreases as temperature increases. This principle applies equally with any fluid. For example, the pot of water on the stove in is kept warm in this manner; ocean currents and large-scale atmospheric circulation transfer energy from one part of the globe to another.



Convection in a Pot of Water: Convection plays an important role in heat transfer inside this pot of water. Once conducted to the inside, heat transfer to other parts of the pot is mostly by convection. The hotter water expands, decreases in density, and rises to transfer heat to other regions of the water, while colder water sinks to the bottom. This process keeps repeating.

Convection and Insulation

Although air can transfer heat rapidly by convection, it is a poor conductor and thus a good insulator. The amount of available space for airflow determines whether air acts as an insulator or conductor. The space between the inside and outside walls of a house, for example, is about 9 cm (3.5 in)—large enough for convection to work effectively. The addition of wall insulation prevents airflow, so heat loss (or gain) is decreased. Similarly, the gap between the two panes of a double-paned window is about 1 cm, which prevents convection and takes advantage of air's low conductivity to prevent greater loss. Fur, fiber and fiberglass also take advantage of the low conductivity of air by trapping it in spaces too small to support convection. In animals, fur and feathers are lightweight and thus ideal for their protection.

Convection and Phase Changes

Some interesting phenomena happen when convection is accompanied by a phase change. It allows us to cool off by sweating, even if the temperature of the surrounding air exceeds body temperature. Heat from the skin is required in order for sweat to evaporate from the skin, but without air flow the air becomes saturated and evaporation stops. Air flow caused by convection replaces the saturated air by dry air and thus evaporation continues.

Another important example of the combination of phase change and convection occurs when water evaporates from the ocean. Heat is removed from the ocean when water evaporates. If the water vapor condenses in liquid droplets as clouds form, heat is released in the atmosphere (this heat release is latent heat) . Thus, an overall transfer of heat from the ocean to the atmosphere occurs. This process is the driving power behind thunderheads—great cumulus clouds that rise as much as 20.0 km into the stratosphere. Water vapor carried in by convection condenses, releasing tremendous amounts of energy, and this energy allows air to become more buoyant (warmer than its surroundings) and rise. As the air continues to rise, more condensation occurs, which in turn drives the cloud even higher. Such a mechanism is called positive feedback, since the process reinforces and accelerates itself. These systems sometimes produce violent storms with lightning and hail, and constitute the mechanism that drives hurricanes.

Cumulus Clouds: Cumulus clouds are caused by water vapor that rises because of convection. The rise of clouds is driven by a positive feedback mechanism.

Radiation

Radiation is the transfer of heat through electromagnetic energy

learning objectives

- Explain how the energy of electromagnetic radiation corresponds with wavelength

Radiation

You can feel heat transfer from a fire or the Sun. Yet the space between Earth and the Sun is largely empty, without any possibility of heat transfer by convection or conduction. Similarly, you can tell that an oven is hot without touching it or looking inside—it just warms you as you walk by.

In these examples, heat is transferred by radiation. The hot body emits electromagnetic waves that are absorbed by our skin, and no medium is required for them to propagate. We use different names for electromagnetic waves of different wavelengths: radio waves, microwaves, infrared radiation, visible light, ultraviolet radiation, X-rays, and gamma rays .

Radiation from a Fire: Most of the heat transfer from this fire to the observers is through infrared radiation. The visible light, although dramatic, transfers relatively little thermal energy. Convection transfers energy away from the observers as hot air rises, while conduction is negligibly slow here. Skin is very sensitive to infrared radiation so that you can sense the presence of a fire without looking at it directly.

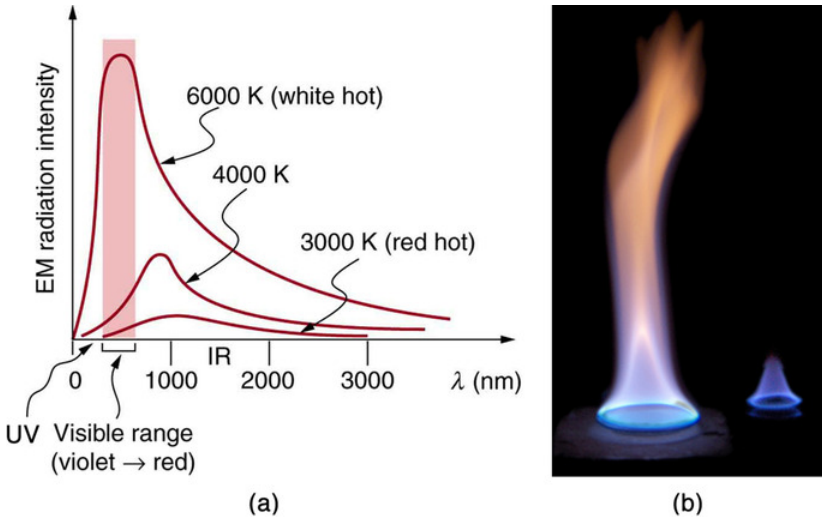
The energy of electromagnetic radiation depends on its wavelength (color) and varies over a wide range; a smaller wavelength (or higher frequency) corresponds to a higher energy. We can write this as:

$$\mathrm{E}=\mathrm{hf}=\frac{\mathrm{hc}}{\lambda} \tag{13.4.1}$$

where E is the energy, f is the frequency, λ is the wavelength, and h is a constant.

Because more heat is radiated at higher temperatures, a temperature change is accompanied by a color change. For example, an electrical element on a stove glows from red to orange, while the higher-temperature steel in a blast furnace glows from yellow to white. The radiation you feel is mostly infrared, which is lower in temperature still.

The radiated energy depends on its intensity, which is represented by the height of the distribution .



Radiation Spectrum: (a) A graph of the spectra of electromagnetic waves emitted from an ideal radiator at three different temperatures. The intensity or rate of radiation emission increases dramatically with temperature, and the spectrum shifts toward the visible and ultraviolet parts of the spectrum. The shaded portion denotes the visible part of the spectrum. It is apparent that the shift toward the ultraviolet with temperature makes the visible appearance shift from red to white to blue as temperature increases. (b) Note the variations in color corresponding to variations in flame temperature.

Heat Transfer

All objects absorb and emit electromagnetic radiation. The rate of heat transfer by radiation is largely determined by the color of the object. Black is the most effective, and white the least. People living in hot climates generally avoid wearing black clothing, for instance. Similarly, black asphalt in a parking lot will be hotter than the adjacent gray sidewalk on a summer day, because black absorbs better than gray. The reverse is also true—black radiates better than gray. Thus, on a clear summer night the asphalt will be colder than the gray sidewalk because black radiates energy more rapidly than gray.

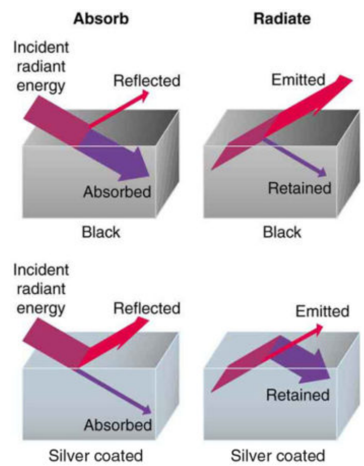
An ideal radiator, often called a blackbody, is the same color as an ideal absorber, and captures all the radiation that falls on it. In contrast, white is a poor absorber and also a poor radiator. A white object reflects all radiation, like a mirror. (A perfect, polished white surface is mirror-like in appearance, and a crushed mirror looks white.)

There is a clever relation between the temperature of an ideal radiator and the wavelength at which it emits the most radiation. It is called Wien's displacement law and is given by:

$$\mathrm{\lambda_max T = b} \tag{13.4.2}$$

where b is a constant equal to $2.9\times 10^{-3}\ \mathrm{m\cdot K}$.

Gray objects have a uniform ability to absorb all parts of the electromagnetic spectrum. Colored objects behave in similar but more complex ways, which gives them a particular color in the visible range and may make them special in other ranges of the nonvisible spectrum. Take, for example, the strong absorption of infrared radiation by the skin, which allows us to be very sensitive to it .



Good and Poor Radiators: A black object is a good absorber and a good radiator, while a white (or silver) object is a poor absorber and a poor radiator. It is as if radiation from the inside is reflected back into the silver object, whereas radiation from the inside of the black object is "absorbed" when it hits the surface and finds itself on the outside and is strongly emitted.

The rate of heat transfer by emitted radiation is determined by the Stefan-Boltzmann law of radiation:

$$\mathrm{\frac{Q}{t} = \sigma e A T^4} \tag{13.4.3}$$

where $\sigma = 5.67\times 10^{-8}\ \mathrm{\frac{J}{s\cdot m^2\cdot K^4}}$ is the Stefan-Boltzmann constant, A is the surface area of the object, and T is its absolute temperature in kelvin. The symbol e stands for the emissivity of the object, which is a measure of how well it radiates. An ideal jet-black (or blackbody) radiator has e=1e=1, whereas a perfect reflector has $\mathrm{e}=0$. Real objects fall between these two values. For example, tungsten light bulb filaments have an ee of about 0.5, and carbon black (a material used in printer toner), has the (greatest known) emissivity of about 0.99.

The radiation rate is directly proportional to the fourth power of the absolute temperature—a remarkably strong temperature dependence. Furthermore, the radiated heat is proportional to the surface area of the object. If you knock apart the coals of a fire, there is a noticeable increase in radiation due to an increase in radiating surface area.

Net Rate of Heat Transfer

The net rate of heat transfer by radiation (absorption minus emission) is related to both the temperature of the object and that of its surroundings. Assuming that an object with a temperature $\mathrm{T_1}$ is surrounded by an environment with uniform temperature $\mathrm{T_2}$, the net rate of heat transfer by radiation is:

\[\mathrm{\dfrac{Q_{net}}{t}=eAσ(T_2^4−T_1^4)}\)

where e is the emissivity of the object alone. In other words, it does not matter whether the surroundings are white, gray, or black; the balance of radiation into and out of the object depends on how well it emits and absorbs radiation. When $T_2 > T_1$, the quantity $\frac{Q_{net}}{t}$ is positive; that is, the net heat transfer is from hotter objects to colder objects.

Key Points

- On a microscopic scale, conduction occurs as rapidly moving or vibrating atoms and molecules interact with neighboring particles, transferring some of their kinetic energy.
- Conduction is the most significant form of heat transfer within a solid object or between solids in thermal contact.
- Conduction is most significant in solids, and less though in liquids and gases, due to the space between molecules.
- The rate of heat transfer by conduction is dependent on the temperature difference, the size of the area in contact, the thickness of the material, and the thermal properties of the material(s) in contact.
- Convection is driven by the large scale flow of matter in fluids. Solids cannot transport heat through convection.
- Natural convection is driven by buoyant forces: hot air rises because density decreases as temperature increases. This principle applies equally with any fluid.
- Convection can transport heat much more efficiently than conduction. Air is a poor conductor and a good insulator if the space is small enough to prevent convection.
- Convection often accompanies phase changes, such as when sweat evaporates from your body. This mass flow during convection allows humans to cool off even if the surrounding air's temperature exceeds the body temperature.
- The energy of electromagnetic radiation depends on the wavelength (color) and varies over a wide range: a smaller wavelength (or higher frequency) corresponds to a higher energy.
- All objects emit and absorb electromagnetic energy. The color of an object is related emissivity, or its efficiency of radiating away energy. Black is the most effective while white is the least effective ($e = 1$ and $e = 0$, respectively).
- An ideal radiator, often called a blackbody, is the same color as an ideal absorber and captures all the radiation that falls on it.
- The rate of heat transfer by emitted radiation is determined by the Stefan-Boltzmann law of radiation: $\frac{Q}{t} = \sigma A T^4$ where $\sigma = 5.67 \times 10^{-8} \ \frac{J}{s \cdot m^2 \cdot K^4}$ is the Stefan-Boltzmann constant, A is the surface area of the object, and T is its absolute temperature in kelvin.
- The net rate of heat transfer is related to the temperature of the object and the temperature of its surroundings. The larger the difference, the higher the net heat flux.
- The temperature of an object is very significant, because the radiation emitted is proportional to this quantity to the fourth power.

Key Terms

- **thermal conductivity**: the measure of a material's ability to conduct heat
- **natural convection**: A method for heat transport. A fluid surrounding a heat source receives heat, becomes less dense and rises. The surrounding, cooler fluid then moves to replace it. This cooler fluid is then heated and the process continues, forming a convection current.
- **positive feedback**: a feedback loop in which the output of a system is amplified with a net positive gain each cycle.
- **blackbody**: A theoretical body, approximated by a hole in a hollow black sphere, that absorbs all incident electromagnetic radiation and reflects none; it has a characteristic emission spectrum.
- **emissivity**: The energy-emitting propensity of a surface, usually measured at a specific wavelength.

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13.5: Global Warming

learning objectives

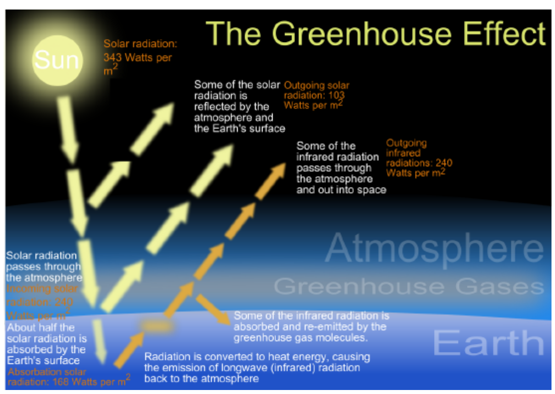
- Describe effect of greenhouse gases on the Earth’s equilibrium temperature

Radiation is a natural process of heat transfer; everything is constantly radiating heat. When a system is at equilibrium, we do not notice this exchange of energy because every object absorbs and emits precisely the amount of energy it receives, but this equilibrium is dynamic–the exchange is always taking place. In general, as the temperature of an object increases, it emits energy more rapidly by emitting photons at a faster rate, and by emitting photons of a greater average energy.

If we approximate the Earth as a perfect absorber and emitter of the radiation received from the sun (called a blackbody), we would expect the Earth to be at an average temperature of 5°C, rather than the 14°C which we observe. The extra temperature comes from the Earth and it’s atmosphere selectively absorbing certain wavelengths of radiation while reflecting other wavelengths. The 9°C discrepancy is due to the greenhouse effect. The gases of the atmosphere are “selective absorbers”; energy in the visible part of the electromagnetic spectrum passes through the atmosphere directly to the Earth’s surface (with some reflection occurring as well). The Earth absorbs this energy and then re-emits radiation in the infrared portion of the spectrum. The gases in the atmosphere, primarily CO_2 and water vapor are highly absorbent in the infrared part of the spectrum. The atmosphere absorbs the infrared radiation from the Earth, preventing it from escaping to space. Because all objects are continually emitting radiation, the atmosphere (having absorbed the Earth’s radiation) then emits radiation, some of which is then reabsorbed by the Earth’s surface. Thus the greenhouse effect is a continuous cycle of absorption and emission of energy between the Earth and atmosphere. This causes the Earth’s atmosphere and surface to be warmer than otherwise expected.

Radiative Transfer

The greenhouse effect is a phenomenon of radiative transfer, the process by which the energy of light waves is exchanged in matter. Radiative transfer dictates what energy is reflected, absorbed, and emitted.



The greenhouse effect: A summary of the heat transfer in the Earth’s atmosphere.

Atmospheric Absorbers

The radiative transfer properties of atmospheric chemicals depend on the energy of the radiation (both for absorption and emission), and those properties are unique to each chemical. In the context of global warming, we find that it is important to consider both how chemicals and particles reflect sunlight and how they absorb energy radiated from the Earth. Atmospheric reflecters, notably sulfates and nitrates, reflect and scatter light before it ever hits the surface of the Earth, effectively reducing the power that the Earth receives. On the other hand, greenhouse gases such as carbon dioxide (CO_2), methane (CH_4), and nitrous oxide (N_2O) are characteristically strong absorbers of the energy radiated by the Earth’s surface. They absorb so strongly because they typically exhibit resonant absorption behavior in the same energy range as the radiation emitted by the earth. These trap heat before it leaves the Earth, insulating the Earth and increasing the Earth’s equilibrium temperature.

Entropy and Solar Radiation

The Second Law of Thermodynamics implies that in order to produce energy sources, entropy must be produced. On a planetary scale, this production of entropy is primarily accounted for by the absorption and re-emisson of radiation. In general, the earth radiates the same energy that it receives. (If greenhouse gases increase, then temperature increases, and higher temperatures cause the Earth to radiate more, compensating for the greater energy absorbed in the atmosphere.) With light radiation, we find the entropy carried by the photons decreases as temperature increases, $S \propto 1T$$S \propto 1T$. Since the Earth is cooler than the Sun, as the Earth absorbs radiation from the Sun and re-emits radiation from the Earth's surface, there is a net production of entropy. This net production of entropy allows energy to be stored and reused, particularly in the form of chemical energy stored by cells through photosynthesis.

Key Points

- The atmosphere can both absorb and reflect radiation, either increasing or decreasing the Earth's average temperature.
- All bodies naturally emit radiation; warmer bodies emit more photons, with higher average energies.
- In absorbing sunlight and emitting infrared radiation, the Earth produces entropy.

Key Terms

- **selective absorber**: An object that will absorb radiation over a particular set of wavelengths but will not (is transparent) at other wavelengths.
- **greenhouse effect**: The process by which a planet is warmed by its atmosphere.
- **greenhouse gas**: Any gas, such as carbon dioxide or CFCs, that contributes to the greenhouse effect when released into the atmosphere.
- **radiative transfer**: The transfer of radiation (energy) leaving one object and being absorbed by another.
- **blackbody**: An object that is a perfect absorber and emitter of radiation.

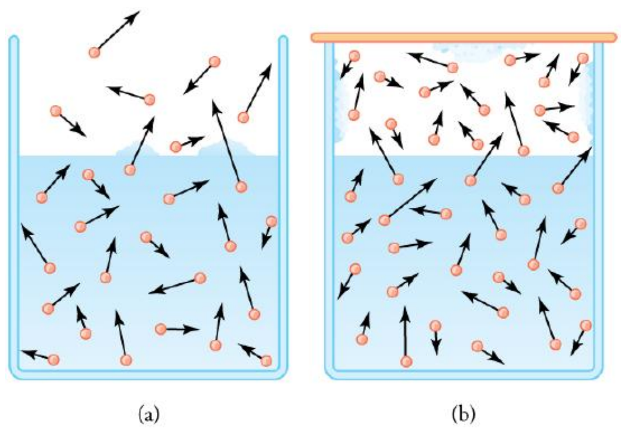
13.6: Phase Equilbrium

learning objectives

- Illustrate the causes for evaporation near the surface of a liquid

Evaporation is a type of vaporization of a liquid that only occurs on the liquid's surface. Usually, the molecules in a glass of water do not have enough heat energy to escape from the liquid. With sufficient heat, however, the liquid would quickly turn into vapor. When the molecules collide, they transfer energy to each other in varying degrees. Sometimes the transfer is so one-sided for a molecule near the surface that it achieves enough energy to escape the liquid.

Three key parts to evaporation are heat, atmospheric pressure (determines the percent humidity) and air movement. For molecules of a liquid to evaporate, they must be located near the surface, be moving in the proper direction, and have sufficient kinetic energy to overcome liquid- phase intermolecular forces. When only a small proportion of the molecules meet these criteria, the rate of evaporation is low. Since the kinetic energy of a molecule is proportional to its temperature, evaporation proceeds more quickly at higher temperatures. As the faster-moving molecules escape, the remaining molecules have lower average kinetic energy, and the temperature of the liquid decreases. This phenomenon is also called evaporative cooling. This is why evaporating sweat cools the human body. Evaporation also tends to proceed more quickly with higher flow rates between the gaseous and liquid phases and in liquids with higher vapor pressure. For example, laundry on a clothes line will dry (by evaporation) more rapidly on a windy day than on a still day.



Vapor Pressure Diagram: (a) Because of the distribution of speeds and kinetic energies, some water molecules can break away to the vapor phase even at temperatures below the ordinary boiling point. (b) If the container is sealed, evaporation will continue until there is enough vapor density for the condensation rate to equal the evaporation rate. This vapor density and the partial pressure it creates are the saturation values. They increase with temperature and are independent of the presence of other gases, such as air. They depend only on the vapor pressure of water.

Evaporation is an essential part of the water cycle. The sun (solar energy) drives evaporation of water from oceans, lakes, moisture in the soil, and other sources of water. In hydrology, evaporation and transpiration (which involves evaporation within plant stomata) are collectively termed evapotranspiration. Evaporation of water occurs when the surface of the liquid is exposed, allowing molecules to escape and form water vapor; this vapor can then rise and form clouds.

The Evaporating Atmosphere

At equilibrium, evaporation and condensation processes exactly balance and there is no net change in the volume of either phase.

learning objectives

- Explain how a substance can have multiple distinct phases in the same environment

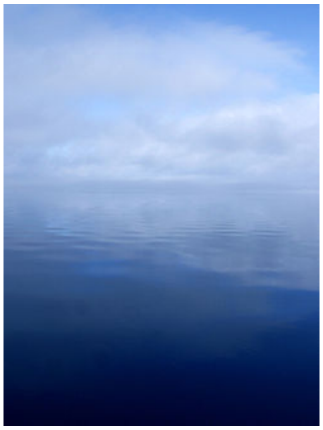
Phase Equilibrium

Left to equilibration, many compositions will form a uniform single phase, but depending on the temperature and pressure even a single substance may separate into two or more distinct phases. Within each phase, the properties are uniform but between the two phases properties differ.

Water in a closed jar with an air space over it forms a two phase system. Most of the water is in the liquid phase, where it is held by the mutual attraction of water molecules. Even at equilibrium, molecules are constantly in motion and, once in a while, a molecule in the liquid phase gains enough kinetic energy to break away from the liquid phase and enter the gas phase. Likewise, every once in a while a vapor molecule collides with the liquid surface and condenses into the liquid. At equilibrium, evaporation and condensation processes exactly balance and there is no net change in the volume of either phase.

At room temperature and pressure, the water jar reaches equilibrium when the air over the water has a humidity of about 3%. This percentage increases as the temperature goes up. At 100 °C and atmospheric pressure, equilibrium is not reached until the air is 100% water. If the liquid is heated a little over 100 °C, the transition from liquid to gas will occur not only at the surface, but throughout the liquid volume: the water boils.

The Earth's atmosphere is not unchanging. The water vapor in it changes phases. It is in a phase equilibrium. Collisions between water molecules in the atmosphere allows some to condense and some to remain in vapor. Similarly, several lighter gases can escape the gravitational field entirely.



Water Vapor in the Atmosphere: Water vapor condenses in the atmosphere

Key Points

- Evaporation turns liquids into gas.
- Evaporation can take place at temperatures below boiling point since the molecules in the liquid have different energies.
- As the molecules in a liquid collide, some achieve higher energies, allowing them to escape. This process lowers the energy of the remaining molecules and is the source of cooling in evaporating liquids.
- The atmosphere is made of gases in a phase equilibrium.
- As molecules in the atmosphere collide, they gain and lose energy.
- As water evaporates from the surface of the earth, water condenses in the atmosphere.

Key Terms

- **Vaporization**: a conversion of a solid or a liquid into a gas
- **Evaporation**: The process of a liquid converting to the gaseous state.
- **condensation**: The conversion of a gas to a liquid; the condensate so formed
- **equilibrium**: The state of a body at rest or in uniform motion, the resultant of all forces on which is zero.

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CHAPTER OVERVIEW

14: Thermodynamics

Topic hierarchy

14.1: Prelude to The First Law of Thermodynamics

Heat is the transfer of energy due to a temperature difference between two systems. Heat describes the process of converting from one form of energy into another. A car engine, for example, burns gasoline. Heat is produced when the burned fuel is chemically transformed into mostly CO_2 and H_2O, which are gases at the combustion temperature. These gases exert a force on a piston through a displacement, doing work and converting the piston's kinetic energy into a variety of other forms—into the car's kinetic energy; into electrical energy to run the spark plugs, radio, and lights; and back into stored energy in the car's battery.

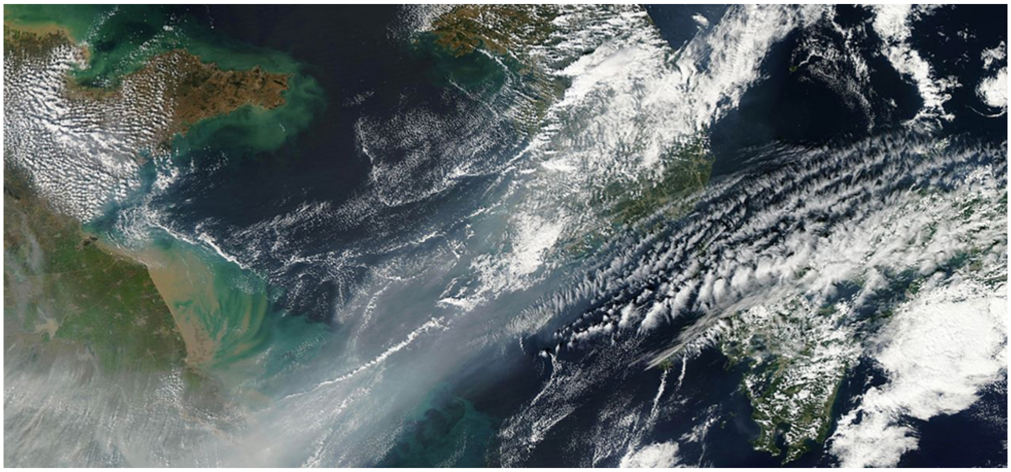


Figure 14.1.1: A weak cold front of air pushes all the smog in northeastern China into a giant smog blanket over the Yellow Sea, as captured by NASA's Terra satellite in 2012. To understand changes in weather and climate, such as the event shown here, you need a thorough knowledge of thermodynamics. (credit: modification of work by NASA)

Energy is conserved in all processes, including those associated with thermodynamic systems. The roles of heat transfer and internal energy change vary from process to process and affect how work is done by the system in that process. We will see that the first law of thermodynamics explains that a change in the internal energy of a system comes from changes in heat or work. Understanding the laws that govern thermodynamic processes and the relationship between the system and its surroundings is therefore paramount in gaining scientific knowledge of energy and energy consumption.

14.2: Thermodynamic Systems

Learning Objectives

By the end of this section, you will be able to:

- Define a thermodynamic system, its boundary, and its surroundings
- Explain the roles of all the components involved in thermodynamics
- Define thermal equilibrium and thermodynamic temperature
- Link an equation of state to a system

A **thermodynamic system** includes anything whose thermodynamic properties are of interest. It is embedded in its **surroundings** or **environment**; it can exchange heat with, and do work on, its environment through a **boundary**, which is the imagined wall that separates the system and the environment (Figure 14.2.1). In reality, the immediate surroundings of the system are interacting with it directly and therefore have a much stronger influence on its behavior and properties. For example, if we are studying a car engine, the burning gasoline inside the cylinder of the engine is the thermodynamic system; the piston, exhaust system, radiator, and air outside form the surroundings of the system. The boundary then consists of the inner surfaces of the cylinder and piston.

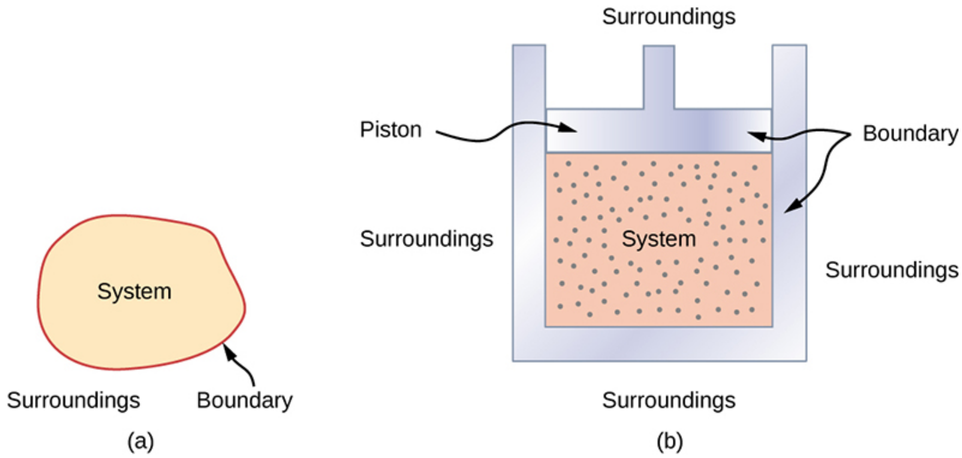


Figure 14.2.1: (a) A system, which can include any relevant process or value, is self-contained in an area. The surroundings may also have relevant information; however, the surroundings are important to study only if the situation is an open system. (b) The burning gasoline in the cylinder of a car engine is an example of a thermodynamic system.

Normally, a system must have some interactions with its surroundings. A system is called an isolated and **closed system** if it is completely separated from its environment—for example, a gas that is surrounded by immovable and thermally insulating walls. In reality, a closed system does not exist unless the entire universe is treated as the system, or it is used as a model for an actual system that has minimal interactions with its environment. Most systems are known as an **open system**, which can exchange energy and/or matter with its surroundings (Figure 14.2.2).

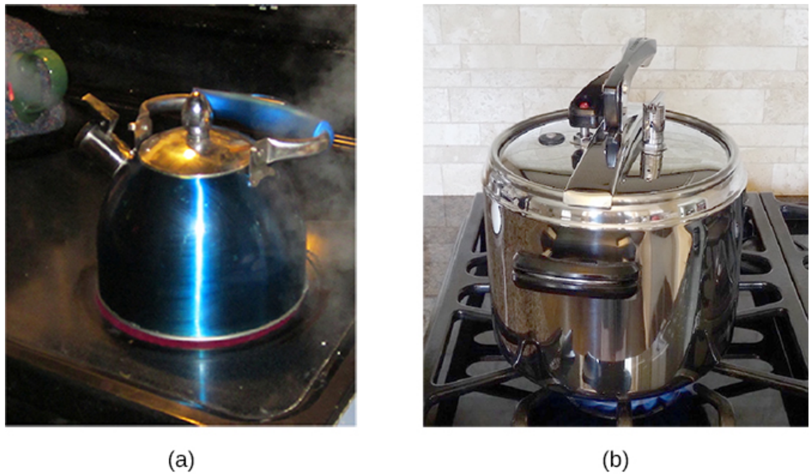


Figure 14.2.2: (a) This boiling tea kettle is an open thermodynamic system. It transfers heat and matter (steam) to its surroundings. (b) A pressure cooker is a good approximation to a closed system. A little steam escapes through the top valve to prevent explosion. (credit a: modification of work by Gina Hamilton)

When we examine a thermodynamic system, we ignore the difference in behavior from place to place inside the system for a given moment. In other words, we concentrate on the macroscopic properties of the system, which are the averages of the microscopic properties of all the molecules or entities in the system. Any thermodynamic system is therefore treated as a continuum that has the same behavior everywhere inside. We assume the system is in **equilibrium**. You could have, for example, a temperature gradient across the system. However, when we discuss a thermodynamic system in this chapter, we study those that have uniform properties throughout the system.

Before we can carry out any study on a thermodynamic system, we need a fundamental characterization of the system. When we studied a mechanical system, we focused on the forces and torques on the system, and their balances dictated the mechanical equilibrium of the system. In a similar way, we should examine the heat transfer between a thermodynamic system and its environment or between the different parts of the system, and its balance should dictate the thermal equilibrium of the system. Intuitively, such a balance is reached if the temperature becomes the same for different objects or parts of the system in thermal contact, and the net heat transfer over time becomes zero.

Thus, when we say two objects (a thermodynamic system and its environment, for example) are in **thermal equilibrium**, we mean that they are at the same temperature. Let us consider three objects at temperatures T_1, T_2, and T_3 respectively. How do we know whether they are in thermal equilibrium? The governing principle here is the **zeroth law of thermodynamics**:

Zeroth Law of Thermodyamics

If object 1 is in thermal equilibrium with objects 2 and 3, respectively, then objects 2 and 3 must also be in thermal equilibrium.

Mathematically, we can simply write the zeroth law of thermodynamics as

$$\text{If } T_1 = T_2 \text{ and } T_1 = T_3, \text{ then } T_2 = T_3.$$

This is the most fundamental way of defining temperature: Two objects must be at the same temperature thermodynamically if the net heat transfer between them is zero when they are put in thermal contact and have reached a thermal equilibrium.

The zeroth law of thermodynamics is equally applicable to the different parts of a closed system and requires that the temperature everywhere inside the system be the same if the system has reached a thermal equilibrium. To simplify our discussion, we assume the system is uniform with only one type of material—for example, water in a tank. The measurable properties of the system at least include its volume, pressure, and temperature. The range of specific relevant variables depends upon the system. For example, for a stretched rubber band, the relevant variables would be length, tension, and temperature. The relationship between these three basic properties of the system is called the **equation of state** of the system and is written symbolically **for a closed system** as

$$f(p, V, T) = 0,$$

where V, p, and T are the volume, pressure, and temperature of the system at a given condition.

In principle, this equation of state exists for any thermodynamic system but is not always readily available. The forms of $f(p,V,T)=0$ for many materials have been determined either experimentally or theoretically. In the preceding chapter, we saw an example of an equation of state for an ideal gas,

$$f(p,V,T)=pV-nRT=0.$$

We have so far introduced several physical properties that are relevant to the thermodynamics of a thermodynamic system, such as its volume, pressure, and temperature. We can separate these quantities into two generic categories. The quantity associated with an amount of matter is an **extensive variable**, such as the volume and the number of moles. The other properties of a system are **intensive variables**, such as the pressure and temperature. An extensive variable doubles its value if the amount of matter in the system doubles, provided all the intensive variables remain the same. For example, the volume or total energy of the system doubles if we double the amount of matter in the system while holding the temperature and pressure of the system unchanged.

14.3: Work, Heat, and Internal Energy

Learning Objectives

By the end of this section, you will be able to:

- Describe the work done by a system, heat transfer between objects, and internal energy change of a system
- Calculate the work, heat transfer, and internal energy change in a simple process

We discussed the concepts of work and energy earlier in mechanics. Examples and related issues of heat transfer between different objects have also been discussed in the preceding chapters. Here, we want to expand these concepts to a thermodynamic system and its environment. Specifically, we elaborated on the concepts of heat and heat transfer in the previous two chapters. Here, we want to understand how work is done by or to a thermodynamic system; how heat is transferred between a system and its environment; and how the total energy of the system changes under the influence of the work done and heat transfer.

Work Done by a System

A force created from any source can do work by moving an object through a displacement. Then how does a thermodynamic system do work? Figure $14.3.1$ shows a gas confined to a cylinder that has a movable piston at one end. If the gas expands against the piston, it exerts a force through a distance and does work on the piston. If the piston compresses the gas as it is moved inward, work is also done—in this case, on the gas.

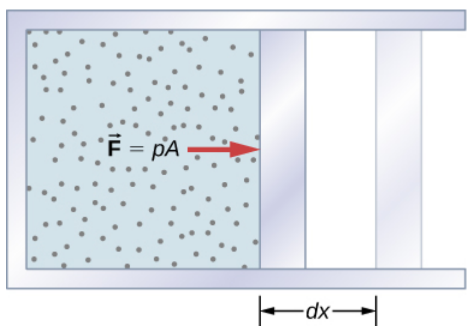


Figure 14.3.1: The work done by a confined gas in moving a piston a distance dx is given by $dW = Fdx = pdV$.

The work associated with such volume changes can be determined as follows: Let the gas pressure on the piston face be p. Then the force on the piston due to the gas is pA, where A is the area of the face. When the piston is pushed outward an infinitesimal distance dx, the magnitude of the work done by the gas is

$$dW = F\,dx = pA\,dx.$$

Since the change in volume of the gas is $dV = A\,dx$, this becomes

$$dW = pdV.$$

For a finite change in volume from V_1 to V_2, we can integrate this equation from V_1 to V_2 to find the net work:

$$W = \int_{V_1}^{V_2} p\,dV. \tag{14.3.1}$$

This integral is only meaningful for a **quasi-static process**, which means a process that takes place in infinitesimally small steps, keeping the system at thermal equilibrium. (We examine this idea in more detail later in this chapter.) Only then does a well-defined mathematical relationship (the equation of state) exist between the pressure and volume. This relationship can be plotted on a **pV** diagram of pressure versus volume, where the curve is the change of state. We can approximate such a process as one that occurs slowly, through a series of equilibrium states. The integral is interpreted graphically as the area under the **pV** curve (the shaded area of Figure $14.3.2$). Work done by the gas is positive for expansion and negative for compression.

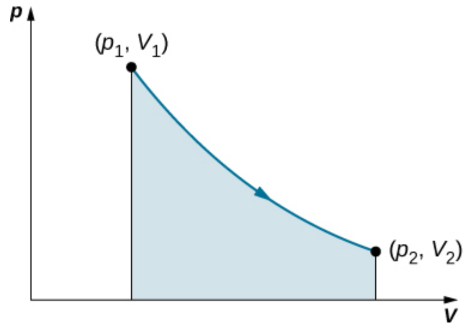


Figure 14.3.2: When a gas expands slowly from V_1 to V_2 the work done by the system is represented by the shaded area under the **pV** curve.

Consider the two processes involving an ideal gas that are represented by paths **AC** and **ABC** in Figure 14.3.3. The first process is an **isothermal expansion**, with the volume of the gas changing its volume from V_1 to V_2. This isothermal process is represented by the curve between points **A** and **C**. The gas is kept at a constant temperature T by keeping it in thermal equilibrium with a heat reservoir at that temperature. From Equation 14.3.1 and the ideal gas law,

$$W=\int_{V_1}^{V_2} p\,dV=\int_{V_1}^{V_2}\left(\frac{nRT}{V}\right)dV.$$

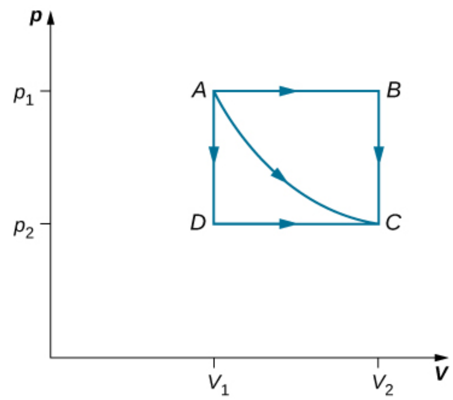


Figure 14.3.3: The paths **ABC**, **AC**, and **ADC** represent three different quasi-static transitions between the equilibrium states **A** and **C**.

The expansion is isothermal, so T remains constant over the entire process. Since n and R are also constant, the only variable in the integrand is V, so the work done by an ideal gas in an isothermal process is

$$W=nRT\int_{V_1}^{V_2}\frac{dV}{V}=nRT\ln\frac{V_2}{V_1}.$$

Notice that if $V_2>V_1$ (expansion), W is positive, as expected.

The straight lines from **A** to **B** and then from **B** to **C** represent a different process. Here, a gas at a pressure p_1 first expands isobarically (constant pressure) and quasi-statically from V_1 to V_2, after which it cools quasi-statically at the constant volume V_2 until its pressure drops to p_2. From **A** to **B**, the pressure is constant at p, so the work over this part of the path is

$$W=\int_{V_1}^{V_2} pdV=p_1\int_{V_1}^{V_2} dV=p_1(V_2-V_1).$$

From **B** to **C**, there is no change in volume and therefore no work is done. The net work over the path **ABC** is then

$$W=p_1(V_2-V_1)+0=p_1(V_2-V_1).$$

A comparison of the expressions for the work done by the gas in the two processes of Figure 14.3.3 shows that they are quite different. This illustrates a very important property of thermodynamic work: It is **path dependent**. We cannot determine the work done by a system as it goes from one equilibrium state to another unless we know its thermodynamic path. Different values of the work are associated with different paths.

✓ Isothermal Expansion of a van der Waals Gas

Studies of a van der Waals gas require an adjustment to the ideal gas law that takes into consideration that gas molecules have a definite volume (see The Kinetic Theory of Gases). One mole of a van der Waals gas has an equation of state

$$\left(p+\frac{a}{V_2}\right)(V-b)=RT,$$

where \(**a**\) and \(**b**\) are two parameters for a specific gas. Suppose the gas expands isothermally and quasi-statically from volume V_1 to volume V_2. How much work is done by the gas during the expansion?

Strategy

Because the equation of state is given, we can use Equation 14.3.1 to express the pressure in terms of V and T. Furthermore, temperature T is a constant under the isothermal condition, so V becomes the only changing variable under the integral.

Solution

To evaluate this integral, we must express p as a function of V. From the given equation of state, the gas pressure is

$$p=\frac{RT}{V-b}-\frac{a}{V^2}.$$

Because T is constant under the isothermal condition, the work done by 1 mol of a van der Waals gas in expanding from a volume V_1 to a volume V_2 is thus

$$\begin{aligned} W &= \int_{V_1}^{V_2}\left(\frac{RT}{V-b}-\frac{a}{V^2}\right)dV \\ &= \left[RT\ln(V-b)+\frac{a}{V}\right]_{V_1}^{V_2} \\ &= RT\ln\left(\frac{V_2-b}{V_1-b}\right)+a\left(\frac{1}{V_2}-\frac{1}{V_1}\right). \end{aligned}$$

Significance

By taking into account the volume of molecules, the expression for work is much more complex. If, however, we set $a=0$ and $b=0$ we see that the expression for work matches exactly the work done by an isothermal process for one mole of an ideal gas.

? Exercise 14.3.1

How much work is done by the gas, as given in Figure $14.3.3$, when it expands quasi-statically along the path **ADC**?

Answer

$p_2(V_2-V_1)$

Internal Energy

The **internal energy** E_{int} of a thermodynamic system is, by definition, the sum of the mechanical energies of all the molecules or entities in the system. If the kinetic and potential energies of molecule **i** are K_i and U_i respectively, then the internal energy of the system is the average of the total mechanical energy of all the entities:

$$E_{int}=\sum_i(\overline{K}_i+\overline{U}_i),$$

where the summation is over all the molecules of the system, and the bars over **K** and **U** indicate average values. The kinetic energy K_i of an individual molecule includes contributions due to its rotation and vibration, as well as its translational energy $m_iv_i^2/2$ where v_i is the molecule's speed measured relative to the center of mass of the system. The potential energy U_i is associated only with the interactions between molecule **i** and the other molecules of the system. In fact, neither the system's location nor its motion is of any consequence as far as the internal energy is concerned. The internal energy of the system is not affected by moving it from the basement to the roof of a 100-story building or by placing it on a moving train.

In an ideal monatomic gas, each molecule is a single atom. Consequently, there is no rotational or vibrational kinetic energy and $K_i = m_i v_i^2/2$. Furthermore, there are no interatomic interactions (collisions notwithstanding), so $U_i = constant$, which we set to zero. The internal energy is therefore due to translational kinetic energy only and

$$E_{int} = \sum_i \overline{K}_i = \sum_i \frac{1}{2} m_i \bar{v}_i^2.$$

From the discussion in the preceding chapter, we know that the average kinetic energy of a molecule in an ideal monatomic gas is

$$\frac{1}{2} m_i v_i^2 = \frac{3}{2} k_B T,$$

where **T** is the Kelvin temperature of the gas. Consequently, the average mechanical energy per molecule of an ideal monatomic gas is also $3k_B T/2$, that is

$$\overline{K_i + U_i} = \overline{K}_i = \frac{3}{2} k_B T.$$

The internal energy is just the number of molecules multiplied by the average mechanical energy per molecule. Thus for **n** moles of an ideal monatomic gas,

$$E_{int} = nN_A \left(\frac{3}{2} k_B T\right) = \frac{3}{2} nRT.$$

Notice that the internal energy of a given quantity of an ideal monatomic gas depends on just the temperature and is completely independent of the pressure and volume of the gas. For other systems, the internal energy cannot be expressed so simply. However, an increase in internal energy can often be associated with an increase in temperature.

We know from the zeroth law of thermodynamics that when two systems are placed in thermal contact, they eventually reach thermal equilibrium, at which point they are at the same temperature. As an example, suppose we mix two monatomic ideal gases. Now, the energy per molecule of an ideal monatomic gas is proportional to its temperature. Thus, when the two gases are mixed, the molecules of the hotter gas must lose energy and the molecules of the colder gas must gain energy. This continues until thermal equilibrium is reached, at which point, the temperature, and therefore the average translational kinetic energy per molecule, is the same for both gases. The approach to equilibrium for real systems is somewhat more complicated than for an ideal monatomic gas. Nevertheless, we can still say that energy is exchanged between the systems until their temperatures are the same.

14.4: First Law of Thermodynamics

Learning Objectives

By the end of this section, you will be able to:

- State the first law of thermodynamics and explain how it is applied
- Explain how heat transfer, work done, and internal energy change are related in any thermodynamic process

Now that we have seen how to calculate internal energy, heat, and work done for a thermodynamic system undergoing change during some process, we can see how these quantities interact to affect the amount of change that can occur. This interaction is given by the first law of thermodynamics. British scientist and novelist C. P. Snow (1905–1980) is credited with a joke about the four laws of thermodynamics. His humorous statement of the first law of thermodynamics is stated “you can’t win,” or in other words, you cannot get more energy out of a system than you put into it. We will see in this chapter how internal energy, heat, and work all play a role in the first law of thermodynamics.

Suppose Q represents the heat exchanged between a system and the environment, and W is the work done by or on the system. The first law states that the change in internal energy of that system is given by $Q-W$. Since added heat increases the internal energy of a system, Q is positive when it is added to the system and negative when it is removed from the system. When a gas expands, it does work and its internal energy decreases. Thus, W is positive when work is done by the system and negative when work is done on the system. This sign convention is summarized in Table $14.4.1$. The first law of thermodynamics is stated as follows:

First Law of Thermodynamics

Associated with every equilibrium state of a system is its internal energy E_{int}. The change in E_{int} for any transition between two equilibrium states is

$$\Delta E_{int} = Q - W \tag{14.4.1}$$

where Q and W represent, respectively, the heat exchanged by the system and the work done by the system.

Table 14.4.1: Thermodynamic Sign Conventions for Heat and Work

Process	Convention
Heat added to system	$Q > 0$
Heat removed from system	$Q < 0$
Work done by system	$W > 0$
Work done on system	$W < 0$

The first law (Equation 14.4.1) is a statement of **energy conservation** that tells us that a system can exchange energy with its surroundings by the transmission of heat and by the performance of work. The net energy exchanged is then equal to the change in the total mechanical energy of the molecules of the system (i.e., the system’s internal energy). Thus, if a system is isolated, its internal energy must remain constant.

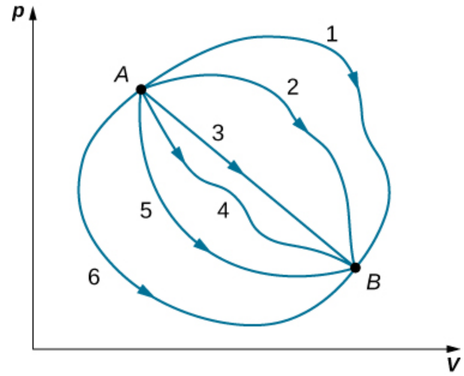


Figure 14.4.1: Different thermodynamic paths taken by a system in going from state A to state B. For all transitions, the change in the internal energy of the system $\Delta E_{int} = Q - W$ is the same.

Although Q and W both depend on the thermodynamic path taken between two equilibrium states, their difference $Q - W$ does not. Figure $14.4.1$ shows the **pV** diagram of a system that is making the transition from A to B repeatedly along different thermodynamic paths. Along path 1, the system absorbs heat Q_1 and does work W_1 along path 2, it absorbs heat Q_2 and does work W_2 and so on. The values of Q_i and W_i may vary from path to path, but we have

$$\underbrace{Q_1 - W_1}_{\text{path 1}} = \underbrace{Q_2 - W_2}_{\text{path 2}} = \cdots = \underbrace{Q_i - W_i}_{\text{path i}}$$

or

$$\underbrace{\Delta E_{\text{int 1}}}_{\text{path 1}} = \underbrace{\Delta E_{\text{int 2}}}_{\text{path 2}} = \cdots = \underbrace{\Delta E_{\text{int i}}}_{\text{path i}}$$

That is, the change in the internal energy of the system between A and B is path independent. In the chapter on potential energy and the conservation of energy, we encountered another path-independent quantity: the change in potential energy between two arbitrary points in space. This change represents the negative of the work done by a conservative force between the two points. The potential energy is a function of spatial coordinates, whereas the internal energy is a function of thermodynamic variables. For example, we might write $E_{int}(T, p)$ for the internal energy. Functions such as internal energy and potential energy are known as **state functions** because their values depend solely on the state of the system.

Often the first law is used in its differential form, which is

$$dE_{int} = dQ - dW.$$

Here dE_{int} is an infinitesimal change in internal energy when an infinitesimal amount of heat dQ is exchanged with the system and an infinitesimal amount of work **dW** is done by (positive in sign) or on (negative in sign) the system.

✓ Changes of State and the First Law

During a thermodynamic process, a system moves from state **A** to state **B**, it is supplied with 400 J of heat and does 100 J of work.

a. For this transition, what is the system’s change in internal energy?

b. If the system then moves from state **B** back to state **A**, what is its change in internal energy?
c. If in moving from **A** to **B** along a different path, $W'_{AB} = 400\ J$ of work is done on the system, how much heat does it absorb?

Strategy

The first law of thermodynamics relates the internal energy change, work done by the system, and the heat transferred to the system in a simple equation. The internal energy is a function of state and is therefore fixed at any given point regardless of how the system reaches the state.

Solution

1. From the first law, the change in the system's internal energy is

$$\begin{aligned}\Delta E_{intAB} &= Q_{AB} - W_{AB} \\ &= 400\ J - 100\ J = 300\ J.\end{aligned}$$

2. Consider a closed path that passes through the states **A** and **B**. Internal energy is a state function, so ΔE_{int} is zero for a closed path. Thus

$$\Delta E_{int} = \Delta E_{intAB} + \Delta E_{intBA} = 0,$$

and

$$\Delta E_{intAB} = -\Delta E_{intBA}.$$

This yields

$$\Delta E_{intBA} = -300\ J.$$

3. The change in internal energy is the same for any path, so

$$\begin{aligned}\Delta E_{intAB} &= \Delta E'_{intAB} \\ &= Q'_{AB} - W'_{AB} \\ 300\ J &= Q'_{AB} - (-400\ J),\end{aligned}$$

and the heat exchanged is

$$Q'_{AB} = -100\ J.$$

The negative sign indicates that the system loses heat in this transition.

Significance

When a closed cycle is considered for the first law of thermodynamics, the change in internal energy around the whole path is equal to zero. If friction were to play a role in this example, less work would result from this heat added. Example $14.4.1B$ takes into consideration what happens if friction plays a role.

Notice that in Example $14.4.1A$, we did not assume that the transitions were quasi-static. This is because the first law is not subject to such a restriction. It describes transitions between equilibrium states but is not concerned with the intermediate states. The system does not have to pass through only equilibrium states. For example, if a gas in a steel container at a well-defined temperature and pressure is made to explode by means of a spark, some of the gas may condense, different gas molecules may combine to form new compounds, and there may be all sorts of turbulence in the container—but eventually, the system will settle down to a new equilibrium state. This system is clearly not in equilibrium during its transition; however, its behavior is still governed by the first law because the process starts and ends with the system in equilibrium states.

✓ Polishing a Fitting

A machinist polishes a 0.50-kg copper fitting with a piece of emery cloth for 2.0 min. He moves the cloth across the fitting at a constant speed of 1.0 m/s by applying a force of 20 N, tangent to the surface of the fitting. (a) What is the total work done on the fitting by the machinist? (b) What is the increase in the internal energy of the fitting? Assume that the change in the internal energy of the cloth is negligible and that no heat is exchanged between the fitting and its environment. (c) What is the increase in the temperature of the fitting?

Strategy

The machinist's force over a distance that can be calculated from the speed and time given is the work done on the system. The work, in turn, increases the internal energy of the system. This energy can be interpreted as the heat that raises the temperature of the system via its heat capacity. Be careful with the sign of each quantity.

Solution

1. The power created by a force on an object or the rate at which the machinist does frictional work on the fitting is $\vec{F} \cdot \vec{v} = -Fv$. Thus, in an elapsed time Δt (2.0 min), the work done on the fitting is
$$\begin{aligned} W &= -Fv\Delta t \\ &= -(20\,N)(1.0\,m/s)(1.2 \times 10^2\ s) \\ &= -2.4 \times 10^3\ J. \end{aligned}$$
2. By assumption, no heat is exchanged between the fitting and its environment, so the first law gives for the change in the internal energy of the fitting:
$$\Delta E_{int} = -W = 2.4 \times 10^3\ J.$$
3. Since ΔE_{int} is path independent, the effect of the $2.4 \times 10^3\ J$ of work is the same as if it were supplied at atmospheric pressure by a transfer of heat. Thus,
4. $$\begin{aligned} 2.4 \times 10^3\ J &= mc\Delta T \\ &= (0.50\,kg)(3.9 \times 10^2\ J/kg \cdot\ ^oC)\Delta T, \end{aligned}$$
   and the increase in the temperature of the fitting is
$$\Delta T = 12^oC,$$
   where we have used the value for the specific heat of copper, $c = 3.9 \times 10^2\ J/kg \cdot\ ^oC$.

Significance

If heat were released, the change in internal energy would be less and cause less of a temperature change than what was calculated in the problem.

? Exercise 14.4.1

The quantities below represent four different transitions between the same initial and final state. Fill in the blanks.

Q (J)	W (J)	$\Delta E_{int}\,(J)$
–80	–120	
90		
	40	
	–40	

Solution

Line 1, $\Delta E_{int} = 40\ J$; line 2, $W = 50\ J$ and $\Delta E_{int} = 40\ J$; line 3, $Q = 80\ J$ and $\Delta E_{int} = 40\ J$; and line 4, $Q = 0$ and $\Delta E_{int} = 40\ J$

Example 14.4.2A: An Ideal Gas Making Transitions between Two States

Consider the quasi-static expansions of an ideal gas between the equilibrium states **A** and **C** of Figure 3.3.3. If 515 J of heat are added to the gas as it traverses the path **ABC**, how much heat is required for the transition along **ADC**? Assume that $p_1 = 2.10 \times 10^5\ N/m^2$, $p_2 = 1.05 \times 10^5\ N/m^2$, $V_1 = 2.25 \times 10^{-3} m^3$, and $V_2 = 4.50 \times 10^{-3} m^3$.

Strategy

The difference in work done between process **ABC** and process **ADC** is the area enclosed by **ABCD**. Because the change of the internal energy (a function of state) is the same for both processes, the difference in work is thus the same as the difference in heat transferred to the system.

Solution

For path **ABC**, the heat added is $Q_{ABC} = 515\ J$ and the work done by the gas is the area under the path on the **pV** diagram, which is

$$W_{ABC} = p_1(V_2 - V_1) = 373\ J.$$

Along **ADC**, the work done by the gas is again the area under the path:

$$W_{ADC} = p_2(V_2 - V_1) = 236\ J.$$

Then using the strategy we just described, we have

$$Q_{ADC} - Q_{ABC} = W_{ADC} - W_{ABC},$$

which leads to

$$Q_{ADC} = Q_{ABC} + W_{ADC} - W_{ABC} = (515 + 236 - 474)\ J = 278\ J.$$

Significance

The work calculations in this problem are made simple since no work is done along **AD** and **BC** and along **AB** and **DC**; the pressure is constant over the volume change, so the work done is simply $p\Delta V$. An isothermal line could also have been used, as we have derived the work for an isothermal process as $W = nRTln\frac{V_2}{V_1}$.

Example 14.4.2B: Isothermal Expansion of an Ideal Gas

Heat is added to 1 mol of an ideal monatomic gas confined to a cylinder with a movable piston at one end. The gas expands quasi-statically at a constant temperature of 300 K until its volume increases from **V** to 3**V**.

a. What is the change in internal energy of the gas?
b. How much work does the gas do?
c. How much heat is added to the gas?

Strategy

(a) Because the system is an ideal gas, the internal energy only changes when the temperature changes. (b) The heat added to the system is therefore purely used to do work that has been calculated in Work, Heat, and Internal Energy. (c) Lastly, the first law of thermodynamics can be used to calculate the heat added to the gas.

Solution

a. We saw in the preceding section that the internal energy of an ideal monatomic gas is a function only of temperature. Since $\Delta T = 0$, for this process, $\Delta E_{int} = 0$.
b. The quasi-static isothermal expansion of an ideal gas was considered in the preceding section and was found to be

$$W = nRTln\frac{V_2}{V_1} = nRtln\frac{3v}{V}$$

$$= (1.00\,mol)(8.314\ J/K \cdot mol)(300\ K)(ln3) = 2.74 \times 10^3\ J.$$

c. With the results of parts (a) and (b), we can use the first law to determine the heat added:

$$\Delta E_{int} = Q - W = 0,$$

which leads to

$$Q = W = 2.74 \times 10^3\ J.$$

Significance

An isothermal process has no change in the internal energy. Based on that, the first law of thermodynamics reduces to $Q = W$.

? Exercise 14.4.2

Why was it necessary to state that the process of Example $14.4.2B$ is quasi-static?

Solution

So that the process is represented by the curve $p = nRT/V$ on the **pV** plot for the evaluation of work.

✔ Example 14.4.3: Vaporizing Water

When 1.00 g of water at 100^oC changes from the liquid to the gas phase at atmospheric pressure, its change in volume is $1.67 \times 10^{-3} m^3$. (a) How much heat must be added to vaporize the water? (b) How much work is done by the water against the atmosphere in its expansion? (c) What is the change in the internal energy of the water?

Strategy

We can first figure out how much heat is needed from the latent heat of vaporization of the water. From the volume change, we can calculate the work done from $W = p\Delta V$ because the pressure is constant. Then, the first law of thermodynamics provides us with the change in the internal energy.

Solution

1. With L_v representing the latent heat of vaporization, the heat required to vaporize the water is

$$Q = mL_v = (1.00\ g)(2.26 \times 10^3\ J/g) = 2.26 \times 10^3\ J.$$

2. Since the pressure on the system is constant at $1\ atm = 1.01 \times 10^5 N/m^2$ the work done by the water as it is vaporized is

$$W = p\Delta V = (1.01 \times 10^5\ N/m^2)(1.67 \times 10^{-3} m^3) = 169\ J.$$

3. From the first law, the thermal energy of the water during its vaporization changes by

$$\Delta E_{int} = Q - W = 2.26 \times 10^3\ J - 169\ J = 2.09 \times 10^3\ J.$$

Significance

We note that in part (c), we see a change in internal energy, yet there is no change in temperature. Ideal gases that are not undergoing phase changes have the internal energy proportional to temperature. Internal energy in general is the sum of all energy in the system.

? Exercise 14.4.3

When 1.00 g of ammonia boils at atmospheric pressure and -33.0^oC, its volume changes from 1.47 to $1130\ cm^3$. Its heat of vaporization at this pressure is $1.37 \times 10^6\ J/kg$. What is the change in the internal energy of the ammonia when it vaporizes?

Solution

$1.26 \times 10^3\ J$.

Note

View this site to learn about how the first law of thermodynamics. First, pump some heavy species molecules into the chamber. Then, play around by doing work (pushing the wall to the right where the person is located) to see how the internal energy changes (as seen by temperature). Then, look at how heat added changes the internal energy. Finally, you can set a parameter constant such as temperature and see what happens when you do work to keep the temperature constant (**Note:** You might see a change in these variables initially if you are moving around quickly in the simulation, but ultimately, this value will return to its equilibrium value).

14.5: Thermodynamic Processes

Learning Objectives

By the end of this section, you will be able to:

- Define a thermodynamic process
- Distinguish between quasi-static and non-quasi-static processes
- Calculate physical quantities, such as the heat transferred, work done, and internal energy change for isothermal, adiabatic, and cyclical thermodynamic processes

In solving mechanics problems, we isolate the body under consideration, analyze the external forces acting on it, and then use Newton's laws to predict its behavior. In thermodynamics, we take a similar approach. We start by identifying the part of the universe we wish to study; it is also known as our system. (We defined a system at the beginning of this chapter as anything whose properties are of interest to us; it can be a single atom or the entire Earth.) Once our system is selected, we determine how the environment, or surroundings, interact with the system. Finally, with the interaction understood, we study the thermal behavior of the system with the help of the laws of thermodynamics.

The thermal behavior of a system is described in terms of **thermodynamic variables**. For an ideal gas, these variables are pressure, volume, temperature, and the number of molecules or moles of the gas. Different types of systems are generally characterized by different sets of variables. For example, the thermodynamic variables for a stretched rubber band are tension, length, temperature, and mass.

The state of a system can change as a result of its interaction with the environment. The change in a system can be fast or slow and large or small. The manner in which a state of a system can change from an initial state to a final state is called a **thermodynamic process**. For analytical purposes in thermodynamics, it is helpful to divide up processes as either **quasi-static** or **non-quasi-static**, as we now explain.

Quasi-static and Non-quasi-static Processes

A quasi-static process refers to an idealized or imagined process where the change in state is made infinitesimally slowly so that at each instant, the system can be assumed to be at a thermodynamic equilibrium with itself and with the environment. For instance, imagine heating 1 kg of water from a temperature 20^oC to 21^oC at a constant pressure of 1 atmosphere. To heat the water very slowly, we may imagine placing the container with water in a large bath that can be slowly heated such that the temperature of the bath can rise infinitesimally slowly from 20^oC to 21^oC. If we put 1 kg of water at 20^oC directly into a bath at 21^oC the temperature of the water will rise rapidly to 21^oC in a non-quasi-static way.

Quasi-static processes are done slowly enough that the system remains at thermodynamic equilibrium at each instant, despite the fact that the system changes over time. The thermodynamic equilibrium of the system is necessary for the system to have well-defined values of macroscopic properties such as the temperature and the pressure of the system at each instant of the process. Therefore, quasi-static processes can be shown as well-defined paths in state space of the system.

Since quasi-static processes cannot be completely realized for any finite change of the system, all processes in nature are non-quasi-static. Examples of quasi-static and **non-quasi-static processes** are shown in Figure $14.5.1$. Despite the fact that all finite changes must occur essentially non-quasi-statically at some stage of the change, we can imagine performing infinitely many quasi-static process corresponding to every quasi-static process. Since quasi-static processes can be analyzed analytically, we mostly study quasi-static processes in this book. We have already seen that in a quasi-static process the work by a gas is given by **pdV**.

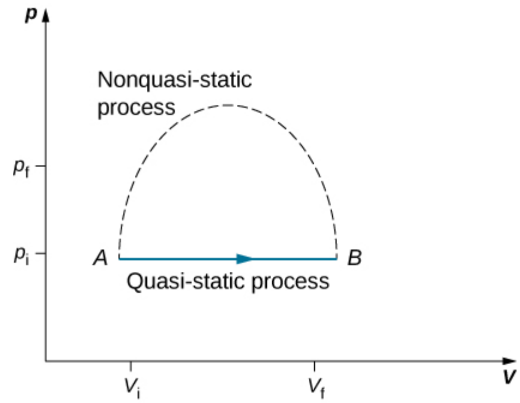


Figure 14.5.1: Quasi-static and non-quasi-static processes between states **A** and **B** of a gas. In a quasi-static process, the path of the process between **A** and **B** can be drawn in a state diagram since all the states that the system goes through are known. In a non-quasi-static process, the states between **A** and **B** are not known, and hence no path can be drawn. It may follow the dashed line as shown in the figure or take a very different path.

Isothermal Processes

An isothermal process is a change in the state of the system at a constant temperature. This process is accomplished by keeping the system in thermal equilibrium with a large heat bath during the process. Recall that a heat bath is an idealized "infinitely" large system whose temperature does not change. In practice, the temperature of a finite bath is controlled by either adding or removing a finite amount of energy as the case may be.

As an illustration of an isothermal process, consider a cylinder of gas with a movable piston immersed in a large water tank whose temperature is maintained constant. Since the piston is freely movable, the pressure inside P_{in} is balanced by the pressure outside P_{out} by some weights on the piston, as in Figure 14.5.2

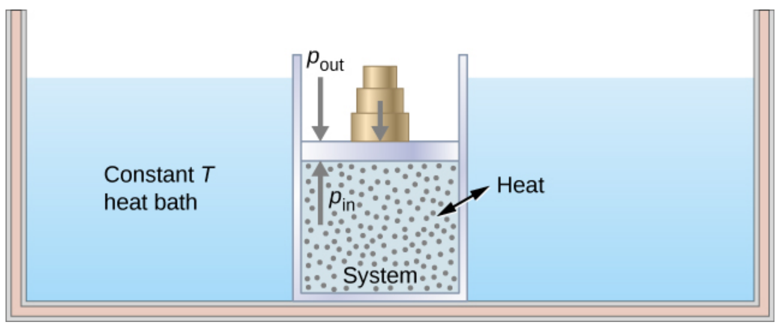


Figure 14.5.2: Expanding a system at a constant temperature. Removing weights on the piston leads to an imbalance of forces on the piston, which causes the piston to move up. As the piston moves up, the temperature is lowered momentarily, which causes heat to flow from the heat bath to the system. The energy to move the piston eventually comes from the heat bath.

As weights on the piston are removed, an imbalance of forces on the piston develops. The net nonzero force on the piston would cause the piston to accelerate, resulting in an increase in volume. The expansion of the gas cools the gas to a lower temperature, which makes it possible for the heat to enter from the heat bath into the system until the temperature of the gas is reset to the temperature of the heat bath. If weights are removed in infinitesimal steps, the pressure in the system decreases infinitesimally slowly. This way, an isothermal process can be conducted quasi-statically. An isothermal line on a (**p**, **V**) diagram is represented by a curved line from starting point **A** to finishing point **B**, as seen in Figure 14.5.3 For an ideal gas, an isothermal process is hyperbolic, since for an ideal gas at constant temperature, $p \propto \frac{1}{V}$.

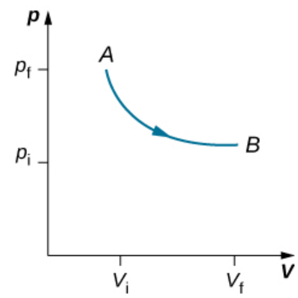


Figure 14.5.3: An isothermal expansion from a state labeled A to another state labeled B on a pV diagram. The curve represents the relation between pressure and volume in an ideal gas at constant temperature.

An isothermal process studied in this chapter is quasi-statically performed, since to be isothermal throughout the change of volume, you must be able to state the temperature of the system at each step, which is possible only if the system is in thermal equilibrium continuously. The system must go out of equilibrium for the state to change, but for quasi-static processes, we imagine that the process is conducted in infinitesimal steps such that these departures from equilibrium can be made as brief and as small as we like.

Other quasi-static processes of interest for gases are isobaric and isochoric processes. An **isobaric process** is a process where the pressure of the system does not change, whereas an **isochoric process** is a process where the volume of the system does not change.

Adiabatic Processes

In an adiabatic process, the system is insulated from its environment so that although the state of the system changes, no heat is allowed to enter or leave the system, as seen in Figure 14.5.3. An adiabatic process can be conducted either quasi-statically or non-quasi-statically. When a system expands adiabatically, it must do work against the outside world, and therefore its energy goes down, which is reflected in the lowering of the temperature of the system. An adiabatic expansion leads to a lowering of temperature, and an adiabatic compression leads to an increase of temperature. We discuss adiabatic expansion again in the section on Adiabatic Processes for an ideal Gas.

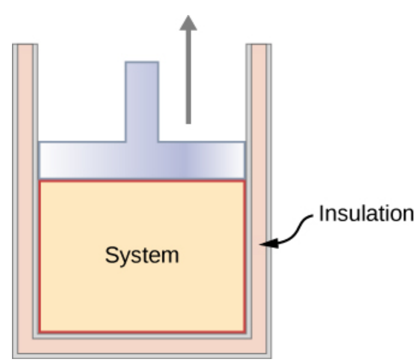


Figure 14.5.4: An insulated piston with a hot, compressed gas is released. The piston moves up, the volume expands, and the pressure and temperature decrease. The internal energy goes into work. If the expansion occurs within a time frame in which negligible heat can enter the system, then the process is called adiabatic. Ideally, during an adiabatic process no heat enters or exits the system.

Cyclic Processes

We say that a system goes through a cyclic process if the state of the system at the end is same as the state at the beginning. Therefore, state properties such as temperature, pressure, volume, and internal energy of the system do not change over a complete cycle:

$$\Delta E_{int} = 0.$$

When the first law of thermodynamics is applied to a cyclic process, we obtain a simple relation between heat into the system and the work done by the system over the cycle:

$$Q = W\ (cyclic\, process).$$

Thermodynamic processes are also distinguished by whether or not they are reversible. A reversible process is one that can be made to retrace its path by differential changes in the environment. Such a process must therefore also be quasi-static. Note, however, that a quasi-static process is not necessarily reversible, since there may be dissipative forces involved. For example, if friction occurred between the piston and the walls of the cylinder containing the gas, the energy lost to friction would prevent us from reproducing the original states of the system.

We considered several thermodynamic processes:

1. An isothermal process, during which the system's temperature remains constant
2. An adiabatic process, during which no heat is transferred to or from the system
3. An isobaric process, during which the system's pressure does not change
4. An isochoric process, during which the system's volume does not change

Many other processes also occur that do not fit into any of these four categories.

Note

View this site to set up your own process in a **pV** diagram. See if you can calculate the values predicted by the simulation for heat, work, and change in internal energy.

14.6: Heat Capacities of an Ideal Gas

Learning Objectives

By the end of this section, you will be able to:

- Define heat capacity of an ideal gas for a specific process
- Calculate the specific heat of an ideal gas for either an isobaric or isochoric process
- Explain the difference between the heat capacities of an ideal gas and a real gas
- Estimate the change in specific heat of a gas over temperature ranges

We learned about specific heat and molar heat capacity previously; however, we have not considered a process in which heat is added. We do that in this section. First, we examine a process where the system has a constant volume, then contrast it with a system at constant pressure and show how their specific heats are related.

Let's start with looking at Figure 14.6.1, which shows two vessels **A** and **B**, each containing 1 mol of the same type of ideal gas at a temperature **T** and a volume **V**. The only difference between the two vessels is that the piston at the top of **A** is fixed, whereas the one at the top of **B** is free to move against a constant external pressure **p**. We now consider what happens when the temperature of the gas in each vessel is slowly increased to $T+dT$ with the addition of heat.

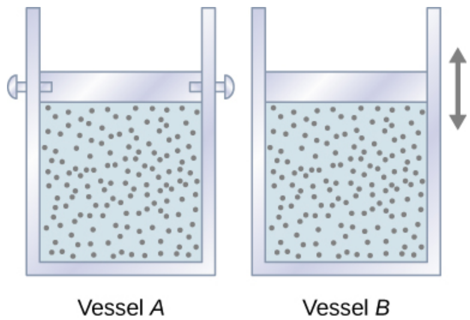


Figure 14.6.1: Two vessels are identical except that the piston at the top of A is fixed, whereas that atop B is free to move against a constant external pressure p.

Since the piston of vessel **A** is fixed, the volume of the enclosed gas does not change. Consequently, the gas does no work, and we have from the first law

$$dE_{int} = dQ - dW = dQ.$$

We represent the fact that the heat is exchanged at constant volume by writing

$$dQ = C_V n dT,$$

where C_V is the **molar heat capacity at constant volume** of the gas. In addition, since $dE_{int} = dQ$ for this particular process,

$$dE_{int} = C_V n dT. \tag{14.6.1}$$

We obtained this equation assuming the volume of the gas was fixed. However, internal energy is a state function that depends on only the temperature of an ideal gas. Therefore, $dE_{int} = C_V n dT$ gives the change in internal energy of an ideal gas for any process involving a temperature change **dT**.

When the gas in vessel **B** is heated, it expands against the movable piston and does work $dW = pdV$. In this case, the heat is added at constant pressure, and we write

$$dQ = C_p n dT,$$

where C_p is the **molar heat capacity at constant pressure** of the gas. Furthermore, since the ideal gas expands against a constant pressure,

$$d(pV) = d(RnT)$$

becomes

$$pdV = RndT.$$

Finally, inserting the expressions for **dQ** and **pdV** into the first law, we obtain

$$dE_{int} = dQ - pdV = (C_p n - Rn)dT.$$

We have found dE_{int} for both an isochoric and an isobaric process. Because the internal energy of an ideal gas depends only on the temperature, dE_{int} must be the same for both processes. Thus,

$$C_V ndT = (C_p n - Rn)dT,$$

and

$$C_p = C_V + R. \tag{14.6.2}$$

The derivation of Equation 14.6.2 was based only on the ideal gas law. Consequently, this relationship is approximately valid for all dilute gases, whether monatomic like He, diatomic like O_2, or polyatomic like CO_2 or NH_3.

In the preceding chapter, we found the molar heat capacity of an ideal gas under constant volume to be

$$C_V = \frac{d}{2}R,$$

where **d** is the number of degrees of freedom of a molecule in the system. Table 14.6.1 shows the molar heat capacities of some dilute ideal gases at room temperature. The heat capacities of real gases are somewhat higher than those predicted by the expressions of C_V and C_p given in Equation 14.6.2. This indicates that vibrational motion in polyatomic molecules is significant, even at room temperature. Nevertheless, the difference in the molar heat capacities, $C_p - C_V$, is very close to **R**, even for the polyatomic gases.

Table 14.6.1: Molar Heat Capacities of Dilute Ideal Gases at Room Temperature

		C_p	C_V	$C_p - C_V$
Type of Molecule	**Gas**	**(J/mol K)**	**(J/mol K)**	**(J/mol K)**
Monatomic	Ideal	$\frac{5}{2}R = 20.79$	$\frac{3}{2}R = 12.47$	$R = 8.31$
Diatomic	Ideal	$\frac{7}{2}R = 29.10$	$\frac{5}{2}R = 20.79$	$R = 8.31$
Polyatomic	Ideal	$4R = 33.26$	$3R = 24.04$	$R = 8.31$

Glossary

molar heat capacity at constant pressure

quantifies the ratio of the amount of heat added removed to the temperature while measuring at constant pressure

molar heat capacity at constant volume

quantifies the ratio of the amount of heat added removed to the temperature while measuring at constant volume

14.7: Adiabatic Processes for an Ideal Gas

Learning Objectives

By the end of this section, you will be able to:

- Define adiabatic expansion of an ideal gas
- Demonstrate the qualitative difference between adiabatic and isothermal expansions

When an ideal gas is compressed adiabatically $(Q = 0)$, work is done on it and its temperature increases; in an **adiabatic expansion**, the gas does work and its temperature drops. **Adiabatic compressions** actually occur in the cylinders of a car, where the compressions of the gas-air mixture take place so quickly that there is no time for the mixture to exchange heat with its environment. Nevertheless, because work is done on the mixture during the compression, its temperature does rise significantly. In fact, the temperature increases can be so large that the mixture can explode without the addition of a spark. Such explosions, since they are not timed, make a car run poorly—it usually "knocks." Because ignition temperature rises with the octane of gasoline, one way to overcome this problem is to use a higher-octane gasoline.

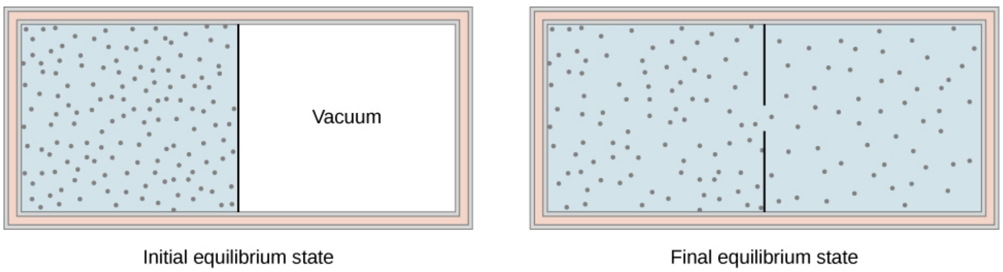


Figure 14.7.1: The gas in the left chamber expands freely into the right chamber when the membrane is punctured.

Another interesting adiabatic process is the free expansion of a gas. Figure 14.7.1 shows a gas confined by a membrane to one side of a two-compartment, thermally insulated container. When the membrane is punctured, gas rushes into the empty side of the container, thereby expanding freely. Because the gas expands "against a vacuum" $(p = 0)$, it does no work, and because the vessel is thermally insulated, the expansion is adiabatic. With $Q = 0$ and $W = 0$ in the first law, $\Delta E_{int} = 0$ so $E_{int\,i} = E_{int\,f}$ for free expansion.

If the gas is ideal, the internal energy depends only on the temperature. Therefore, when an ideal gas expands freely, its temperature does not change; this is also called a **Joule expansion**.

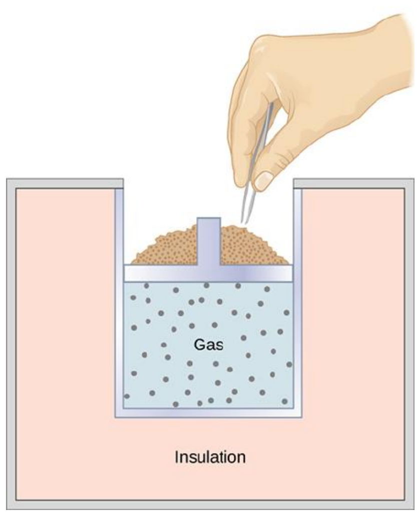


Figure 14.7.2: When sand is removed from the piston one grain at a time, the gas expands adiabatically and quasi-statically in the insulated vessel.

A quasi-static, adiabatic expansion of an ideal gas is represented in Figure 14.7.2, which shows an insulated cylinder that contains 1 mol of an ideal gas. The gas is made to expand quasi-statically by removing one grain of sand at a time from the top of the piston.

When the gas expands by dV, the change in its temperature is dT. The work done by the gas in the expansion is $dW = pdV$; $dQ = 0$ because the cylinder is insulated; and the change in the internal energy of the gas is

$$dE_{int} = C_V n dT.$$

Therefore, from the first law,

$$\begin{aligned} C_V n dT &= 0 - pdV \\ &= -pdV \end{aligned}$$

so

$$dT = -\frac{pdV}{C_V n}.$$

Also, for 1 mol of an ideal gas,

[d(pV) = d(RnT), \nonumber \]

so

$$pdV + V\,dp = RndT$$

and

$$dT = \frac{pdV + V\,dp}{Rn}.$$

We now have two equations for dT. Upon equating them, we find that

$$C_V nV\,dp + (C_V n + Rn)pdV = 0.$$

Now, we divide this equation by npV and use $C_p = C_V + R$. We are then left with

$$C_V \frac{dp}{p} + C_p \frac{dV}{V} = 0,$$

which becomes

$$\frac{dp}{p} + \gamma \frac{dV}{V} = 0,$$

where we define γ as the ratio of the molar heat capacities:

$$\gamma = \frac{C_p}{C_V}.$$

Thus

$$\int \frac{dp}{p} + \gamma \int \frac{dV}{V} = 0$$

and

$$\ln p + \gamma ln\, V = constant.$$

Finally, using $\ln(A^x) = x \ln A$ and $\ln\ AB = \ln\ A + \ln\ B$, we can write this in the form

$$pV^\gamma = constant. \tag{14.7.1}$$

This equation is the condition that must be obeyed by an ideal gas in a quasi-static adiabatic process. For example, if an ideal gas makes a quasi-static adiabatic transition from a state with pressure and volume p_1 and V_1 to a state with p_2 and V_2, then it must be true that $p_1 V_1^\gamma = p_2 V_2^\gamma$.

The adiabatic condition of Equation 14.7.1 can be written in terms of other pairs of thermodynamic variables by combining it with the ideal gas law. In doing this, we find that

$$p^{1-\gamma}T^{\gamma} = constant$$

and

$$TV^{\gamma-1} = constant.$$

A reversible adiabatic expansion of an ideal gas is represented on the **pV** diagram of Figure 14.7.1. The slope of the curve at any point is

$$\frac{dp}{dV} = \frac{d}{dV}\left(\frac{constant}{V^{\gamma}}\right) = -\gamma\frac{p}{V}.$$

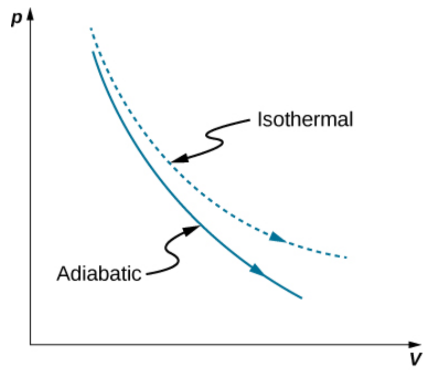


Figure 14.7.3: Quasi-static adiabatic and isothermal expansions of an ideal gas.

The dashed curve shown on this **pV** diagram represents an isothermal expansion where T (and therefore **pV**) is constant. The slope of this curve is useful when we consider the second law of thermodynamics in the next chapter. This slope is

$$\frac{dp}{dV} = \frac{d}{dV}\frac{nRT}{V} = -\frac{p}{V}.$$

Because $\gamma > 1$, the isothermal curve is not as steep as that for the adiabatic expansion.

Example 14.7.1: Compression of an Ideal Gas in an Automobile Engine

Gasoline vapor is injected into the cylinder of an automobile engine when the piston is in its expanded position. The temperature, pressure, and volume of the resulting gas-air mixture are 20^oC, $1.00 \times 10^5\ N/m^2$, and $240\ cm^3$, respectively. The mixture is then compressed adiabatically to a volume of $40\ cm^3$. Note that in the actual operation of an automobile engine, the compression is not quasi-static, although we are making that assumption here.

a. What are the pressure and temperature of the mixture after the compression?
b. How much work is done by the mixture during the compression?

Strategy

Because we are modeling the process as a quasi-static adiabatic compression of an ideal gas, we have $pV^{\gamma} = constant$ and $pV = nRT$. The work needed can then be evaluated with $W = \int_{V_1}^{V_2} pdV$.

Solution

a. For an adiabatic compression we have

$$p_2 = p_1\left(\frac{V_1}{V_2}\right)^{\gamma},$$

so after the compression, the pressure of the mixture is

$$p_2 = (1.00 \times 10^5\ N/m^2)\left(\frac{240 \times 10^{-6} m^3}{40 \times 10^{-6} m^3}\right)^{1.40} = 1.23 \times 10^6\ N/m^2.$$

From the ideal gas law, the temperature of the mixture after the compression is

$$\begin{aligned} T_2 &= \left(\frac{p_2V_2}{p_1V_1}\right)T_1 \\ &= \frac{(1.23\times10^6\ N/m^2)(40\times10^{-6}m^3)}{(1.00\times10^5\ N/m^2)(240\times10^{-6}m^3)}\cdot 293\ K \\ &= 600\ K = 328^oC. \end{aligned}$$

b. The work done by the mixture during the compression is

$$W=\int_{V_1}^{V_2} pdV.$$

With the adiabatic condition of Equation 14.7.1, we may write **p** as K/V^γ, where $K=p_1V_1^\gamma=p_2V_2^\gamma$. The work is therefore

$$\begin{aligned} W &= \int_{V_1}^{V_2}\frac{K}{V^\gamma}dV \\ &= \frac{K}{1-\gamma}\left(\frac{1}{V_2^{\gamma-1}}-\frac{1}{V_1^{\gamma-1}}\right) \\ &= \frac{1}{1-\gamma}\left(\frac{p_2V_2^\gamma}{V_2^{\gamma-1}}-\frac{p_1V_1^\gamma}{V_1^{\gamma-1}}\right) \\ &= \frac{1}{1-\gamma}(p_2V_2-p_1V_1) \\ &= \frac{1}{1-1.40}\left[(1.23\times10^6\ N/m^2)(40\times10^{-6}m^3)-(1.00\times10^5\ N/m^2)(240\times10^{-6}m^3)\right] \\ &= -63\ J. \end{aligned}$$

Significance

The negative sign on the work done indicates that the piston does work on the gas-air mixture. The engine would not work if the gas-air mixture did work on the piston.

14.8: The First Law of Thermodynamics (Exercise)

Conceptual Questions

3.2 Thermodynamic Systems

1. Consider these scenarios and state whether work is done by the system on the environment (SE) or by the environment on the system (ES):

(a) opening a carbonated beverage;

(b) filling a flat tire;

(c) a sealed empty gas can expands on a hot day, bowing out the walls.

3.3 Work, Heat, and Internal Energy

2. Is it possible to determine whether a change in internal energy is caused by heat transferred, by work performed, or by a combination of the two?

3. When a liquid is vaporized, its change in internal energy is not equal to the heat added. Why?

4. Why does a bicycle pump feel warm as you inflate your tire?

5. Is it possible for the temperature of a system to remain constant when heat flows into or out of it? If so, give examples.

3.4 First Law of Thermodynamics

6. What does the first law of thermodynamics tell us about the energy of the universe?

7. Does adding heat to a system always increase its internal energy?

8. A great deal of effort, time, and money has been spent in the quest for a so-called perpetual-motion machine, which is defined as a hypothetical machine that operates or produces useful work indefinitely and/or a hypothetical machine that produces more work or energy than it consumes. Explain, in terms of the first law of thermodynamics, why or why not such a machine is likely to be constructed.

3.5 Thermodynamic Processes

9. When a gas expands isothermally, it does work. What is the source of energy needed to do this work?

10. If the pressure and volume of a system are given, is the temperature always uniquely determined?

11. It is unlikely that a process can be isothermal unless it is a very slow process. Explain why. Is the same true for isobaric and isochoric processes? Explain your answer.

3.6 Heat Capacities of an Ideal Gas

12. How can an object transfer heat if the object does not possess a discrete quantity of heat?

13. Most materials expand when heated. One notable exception is water between **0°C** and **4°C**, which actually decreases in volume with the increase in temperature. Which is greater for water in this temperature region, C_p or C_V ?

14. Why are there two specific heats for gases C_p and C_V, yet only one given for solid?

3.7 Adiabatic Processes for an Ideal Gas

15. Is it possible for **γ** to be smaller than unity?

16. Would you expect **γ** to be larger for a gas or a solid? Explain.

17. There is no change in the internal energy of an ideal gas undergoing an isothermal process since the internal energy depends only on the temperature. Is it therefore correct to say that an isothermal process is the same as an adiabatic process for an ideal gas? Explain your answer.

18. Does a gas do any work when it expands adiabatically? If so, what is the source of the energy needed to do this work?

Problems

3.2 Thermodynamic Systems

19. A gas follows $pV = bp + c_T$ on an isothermal curve, where **p** is the pressure, **V** is the volume, **b** is a constant, and **c** is a function of temperature. Show that a temperature scale under an isochoric process can be established with this gas and is identical to that of an ideal gas.

20. A mole of gas has isobaric expansion coefficient $dV/dT = R/p$ and isochoric pressure-temperature coefficient $dp/dT = p/T$. Find the equation of state of the gas.

21. Find the equation of state of a solid that has an isobaric expansion coefficient $dV/dT = 2cT - bp$ and an isothermal pressure-volume coefficient $dV/dp = -bT$.

3.3 Work, Heat, and Internal Energy

22. A gas at a pressure of 2.00 atm undergoes a quasi-static isobaric expansion from 3.00 to 5.00 L. How much work is done by the gas?

23. It takes 500 J of work to compress quasi-statically 0.50 mol of an ideal gas to one-fifth its original volume. Calculate the temperature of the gas, assuming it remains constant during the compression.

24. It is found that, when a dilute gas expands quasi-statically from 0.50 to 4.0 L, it does 250 J of work. Assuming that the gas temperature remains constant at 300 K, how many moles of gas are present?

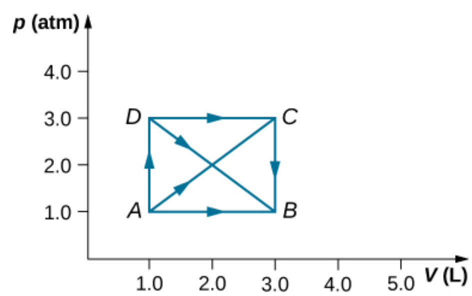
25. In a quasi-static isobaric expansion, 500 J of work are done by the gas. If the gas pressure is 0.80 atm, what is the fractional increase in the volume of the gas, assuming it was originally at 20.0 L?

26. When a gas undergoes a quasi-static isobaric change in volume from 10.0 to 2.0 L, 15 J of work from an external source are required. What is the pressure of the gas?

27. An ideal gas expands quasi-statically and isothermally from a state with pressure p and volume V to a state with volume 4V. Show that the work done by the gas in the expansion is pV(ln 4).

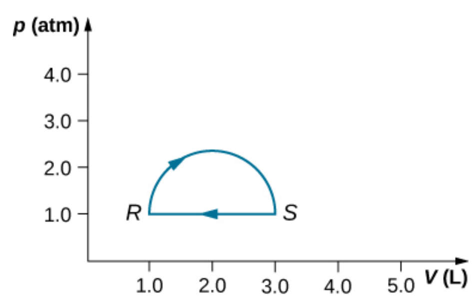
28. As shown below, calculate the work done by the gas in the quasi-static processes represented by the paths

(a) AB; (b) ADB; (c) ACB; and (d) ADCB



29. (a) Calculate the work done on the gas along the closed path shown below. The curved section between R and S is semicircular.

(b) If the process is carried out in the opposite direction, what is the work done on the gas?



30. An ideal gas expands quasi-statically to three times its original volume. Which process requires more work from the gas, an isothermal process or an isobaric one? Determine the ratio of the work done in these processes.

31. A dilute gas at a pressure of 2.0 atm and a volume of 4.0 L is taken through the following quasi-static steps: (a) an isobaric expansion to a volume of 10.0 L, (b) an isochoric change to a pressure of 0.50 atm, (c) an isobaric compression to a volume of 4.0 L, and (d) an isochoric change to a pressure of 2.0 atm. Show these steps on a pV diagram and determine from your graph the net work done by the gas.

32. What is the average mechanical energy of the atoms of an ideal monatomic gas at 300 K?

33. What is the internal energy of 6.00 mol of an ideal monatomic gas at **200°C**?

34. Calculate the internal energy of 15 mg of helium at a temperature of **0°C**.

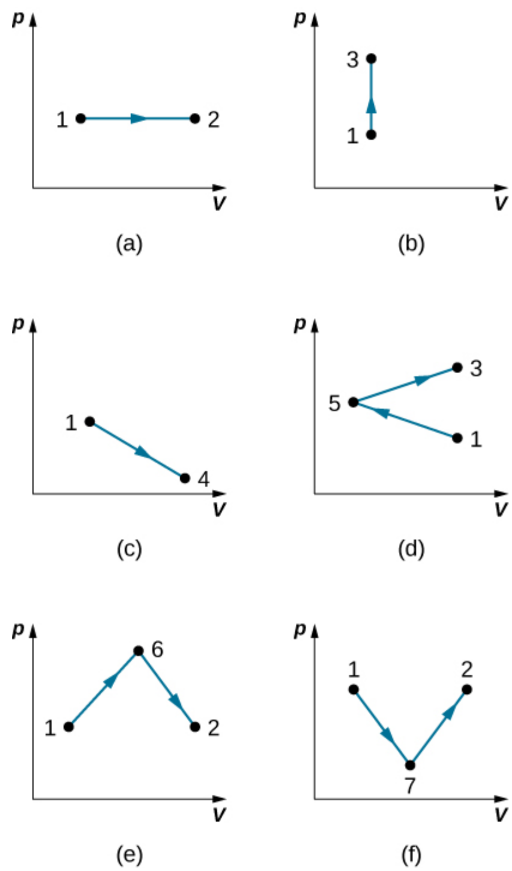
35. Two monatomic ideal gases A and B are at the same temperature. If 1.0 g of gas A has the same internal energy as 0.10 g of gas B, what are

(a) the ratio of the number of moles of each gas and

(b) the ration of the atomic masses of the two gases?

36. The van der Waals coefficients for oxygen are $a = 0.138 J \cdot m^3/mol^2$ and $b = 3.18 \times 10^{-5} m^3/mol$. Use these values to draw a van der Waals isotherm of oxygen at 100 K. On the same graph, draw isotherms of one mole of an ideal gas.

37. Find the work done in the quasi-static processes shown below. The states are given as (p, V) values for the points in the pV plane: 1 (3 atm, 4 L), 2 (3 atm, 6 L), 3 (5 atm, 4 L), 4 (2 atm, 6 L), 5 (4 atm, 2 L), 6 (5 atm, 5 L), and 7 (2 atm, 5 L).



3.4 First Law of Thermodynamics

38. When a dilute gas expands quasi-statically from 0.50 to 4.0 L, it does 250 J of work. Assuming that the gas temperature remains constant at 300 K,

(a) what is the change in the internal energy of the gas?

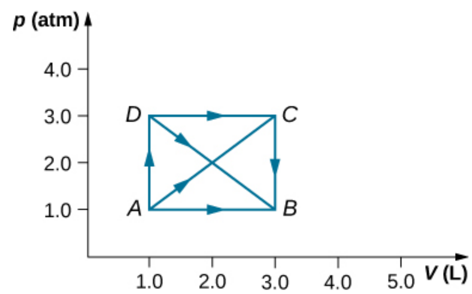
(b) How much heat is absorbed by the gas in this process?

39. In an expansion, 500 J of work are done by the gas. If the internal energy of the gas increased by 80 J in the expansion, how much heat does the gas absorb?

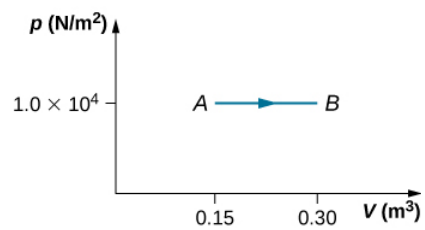
40. An ideal gas expands quasi-statically and isothermally from a state with pressure p and volume V to a state with volume 4V. How much heat is added to the expanding gas?

41. As shown below, if the heat absorbed by the gas along AB is 400 J, determine the quantities of heat absorbed along

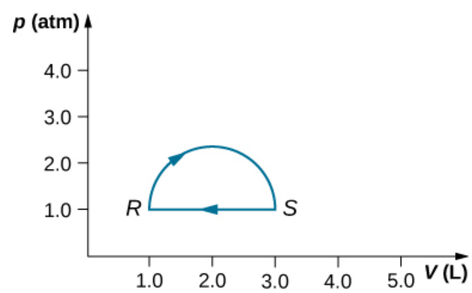
(a) ADB; (b) ACB; and (c) ADCB.



42. During the isobaric expansion from A to B represented below, 3,100 J of heat are added to the gas. What is the change in its internal energy?



43. (a) What is the change in internal energy for the process represented by the closed path shown below? (b) How much heat is exchanged? (c) If the path is traversed in the opposite direction, how much heat is exchanged?

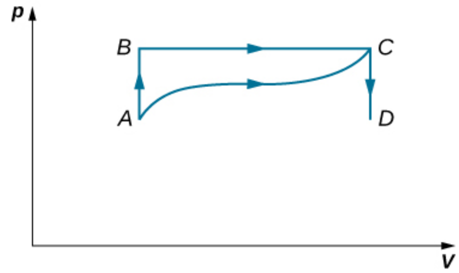


44. When a gas expands along path AC shown below, it does 400 J of work and absorbs either 200 or 400 J of heat.

(a) Suppose you are told that along path ABC, the gas absorbs either 200 or 400 J of heat. Which of these values is correct?

(b) Give the correct answer from part (a), how much work is done by the gas along ABC?

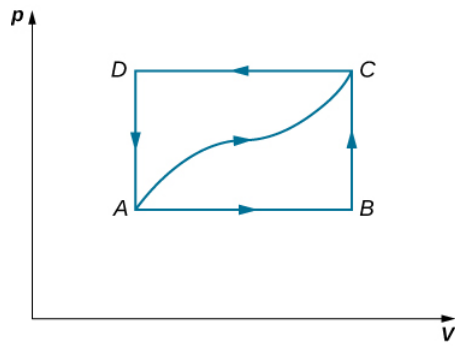
(c) Along CD, the internal energy of the gas decreases by 50 J. How much heat is exchanged by the gas along this path?



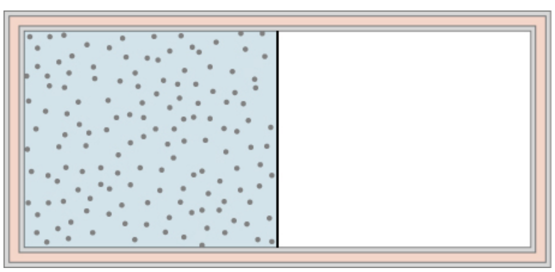
45. When a gas expands along AB (see below), it does 500 J of work and absorbs 250 J of heat. When the gas expands along AC, it does 400 J of work and absorbs 300 J of heat.

(a) How much heat does the gas exchange along BC?

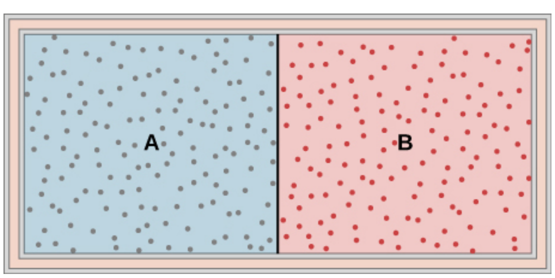
(b) When the gas makes the transition from C to A along CDA, 800 J of work are done on it from C to D. How much heat does it exchange along CDA?



46. A dilute gas is stored in the left chamber of a container whose walls are perfectly insulating (see below), and the right chamber is evacuated. When the partition is removed, the gas expands and fills the entire container. Calculate the work done by the gas. Does the internal energy of the gas change in this process?



47. Ideal gases A and B are stored in the left and right chambers of an insulated container, as shown below. The partition is removed and the gases mix. Is any work done in this process? If the temperatures of A and B are initially equal, what happens to their common temperature after they are mixed?



48. An ideal monatomic gas at a pressure of $.0 \times 10^5 N/m^2$ and a temperature of 300 K undergoes a quasi-static isobaric expansion from 2.0×10^3 to $4.0 \times 10^3 cm^3$.

(a) What is the work done by the gas?

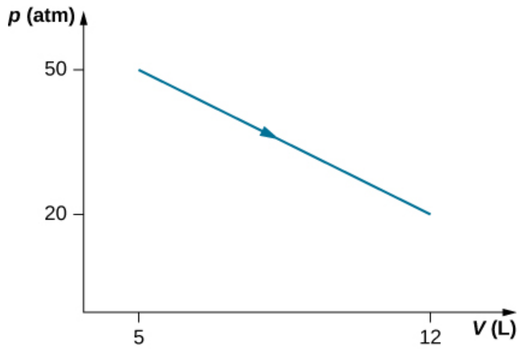
(b) What is the temperature of the gas after the expansion?

(c) How many moles of gas are there?

(d) What is the change in internal energy of the gas?

(e) How much heat is added to the gas?

49. Consider the process for steam in a cylinder shown below. Suppose the change in the internal energy in this process is 30 kJ. Find the heat entering the system.



50. The state of 30 moles of steam in a cylinder is changed in a cyclic manner from a-b-c-a, where the pressure and volume of the states are: a (30 atm, 20 L), b (50 atm, 20 L), and c (50 atm, 45 L). Assume each change takes place along the line connecting the initial and final states in the pV plane.

(a) Display the cycle in the pV plane.

(b) Find the net work done by the steam in one cycle.

(c) Find the net amount of heat flow in the steam over the course of one cycle.

51. A monatomic ideal gas undergoes a quasi-static process that is described by the function $p(V) = p_1 + 3(V - V_1)$, where the starting state is (p_1, V_1) and the final state (p_2, V_2). Assume the system consists of n moles of the gas in a container that can exchange heat with the environment and whose volume can change freely.

(a) Evaluate the work done by the gas during the change in the state.

(b) Find the change in internal energy of the gas.

(c) Find the heat input to the gas during the change.

(d) What are initial and final temperatures?

52. A metallic container of fixed volume of $2.5 \times 10^{-3} m^3$ immersed in a large tank of temperature **27°C** contains two compartments separated by a freely movable wall. Initially, the wall is kept in place by a stopper so that there are 0.02 mol of the nitrogen gas on one side and 0.03 mol of the oxygen gas on the other side, each occupying half the volume. When the stopper is removed, the wall moves and comes to a final position. The movement of the wall is controlled so that the wall moves in infinitesimal quasi-static steps.

(a) Find the final volumes of the two sides assuming the ideal gas behavior for the two gases.

(b) How much work does each gas do on the other?

(c) What is the change in the internal energy of each gas?

(d) Find the amount of heat that enters or leaves each gas.

53. A gas in a cylindrical closed container is adiabatically and quasi-statically expanded from a state A (3 MPa, 2 L) to a stateB with volume of 6 L along the path $1.8pV = constant$.

(a) Plot the path in the pV plane.

(b) Find the amount of work done by the gas and the change in the internal energy of the gas during the process.

54. Two moles of a monatomic ideal gas at (5 MPa, 5 L) is expanded isothermally until the volume is doubled (step 1). Then it is cooled isochorically until the pressure is 1 MPa (step 2). The temperature drops in this process. The gas is now compressed isothermally until its volume is back to 5 L, but its pressure is now 2 MPa (step 3). Finally, the gas is heated isochorically to return to the initial state (step 4).

(a) Draw the four processes in the pV plane.

(b) Find the total work done by the gas.

55. Consider a transformation from point A to B in a two-step process. First, the pressure is lowered from 3 MPa at point A to a pressure of 1 MPa, while keeping the volume at 2 L by cooling the system. The state reached is labeled C. Then the system is heated at a constant pressure to reach a volume of 6 L in the state B.

(a) Find the amount of work done on the ACB path.

(b) Find the amount of heat exchanged by the system when it goes from A to B on the ACB path.

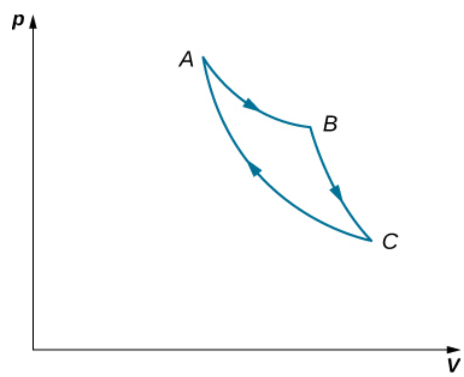
(c) Compare the change in the internal energy when the AB process occurs adiabatically with the AB change through the two-step process on the ACB path.

56. Consider a cylinder with a movable piston containing n moles of an ideal gas. The entire apparatus is immersed in a constant temperature bath of temperature T kelvin. The piston is then pushed slowly so that the pressure of the gas changes quasi-statically from p1p1 to p2p2 at constant temperature T. Find the work done by the gas in terms of **n**, **R**, **T**, p_1, and p_2.

57. An ideal gas expands isothermally along AB and does 700 J of work (see below).

(a) How much heat does the gas exchange along AB?

(b) The gas then expands adiabatically along BC and does 400 J of work. When the gas returns to A along CA, it exhausts 100 J of heat to its surroundings. How much work is done on the gas along this path?



58. Consider the processes shown below for a monatomic gas.

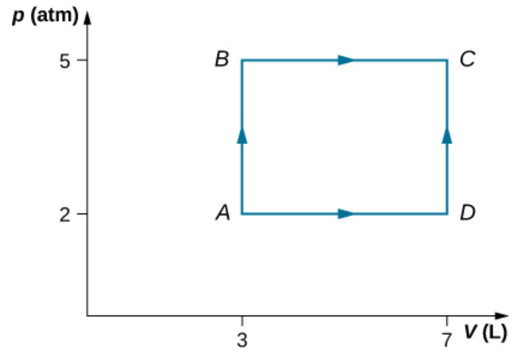
(a) Find the work done in each of the processes AB, BC, AD, and DC.

(b) Find the internal energy change in processes AB and BC.

(c) Find the internal energy difference between states C and A.

(d) Find the total heat added in the ADC process.

(e) From the information given, can you find the heat added in process AD? Why or why not?



59. Two moles of helium gas are placed in a cylindrical container with a piston. The gas is at room temperature **25°C** and under a pressure of $3.0 \times 10^5 Pa$. When the pressure from the outside is decreased while keeping the temperature the same as the room temperature, the volume of the gas doubles.

(a) Find the work the external agent does on the gas in the process.

(b) Find the heat exchanged by the gas and indicate whether the gas takes in or gives up heat. Assume ideal gas behavior.

60. An amount of n moles of a monatomic ideal gas in a conducting container with a movable piston is placed in a large thermal heat bath at temperature T_1 and the gas is allowed to come to equilibrium. After the equilibrium is reached, the pressure on the piston is lowered so that the gas expands at constant temperature. The process is continued quasi-statically until the final pressure is 4/3 of the initial pressure p_1.

(a) Find the change in the internal energy of the gas.

(b) Find the work done by the gas.

(c) Find the heat exchanged by the gas, and indicate, whether the gas takes in or gives up heat.

3.6 Heat Capacities of an Ideal Gas

61. The temperature of an ideal monatomic gas rises by 8.0 K. What is the change in the internal energy of 1 mol of the gas at constant volume?

62. For a temperature increase of **10°C** at constant volume, what is the heat absorbed by (a) 3.0 mol of a dilute monatomic gas; (b) 0.50 mol of a dilute diatomic gas; and (c) 15 mol of a dilute polyatomic gas?

63. If the gases of the preceding problem are initially at 300 K, what are their internal energies after they absorb the heat?

64. Consider 0.40 mol of dilute carbon dioxide at a pressure of 0.50 atm and a volume of 50 L. What is the internal energy of the gas?

65. When 400 J of heat are slowly added to 10 mol of an ideal monatomic gas, its temperature rises by **10°C**. What is the work done on the gas?

66. One mole of a dilute diatomic gas occupying a volume of 10.00 L expands against a constant pressure of 2.000 atm when it is slowly heated. If the temperature of the gas rises by 10.00 K and 400.0 J of heat are added in the process, what is its final volume?

3.7 Adiabatic Processes for an Ideal Gas

67. A monatomic ideal gas undergoes a quasi-static adiabatic expansion in which its volume is doubled. How is the pressure of the gas changed?

68. An ideal gas has a pressure of 0.50 atm and a volume of 10 L. It is compressed adiabatically and quasi-statically until its pressure is 3.0 atm and its volume is 2.8 L. Is the gas monatomic, diatomic, or polyatomic?

69. Pressure and volume measurements of a dilute gas undergoing a quasi-static adiabatic expansion are shown below. Plot ln p vs. V and determine γγ for this gas from your graph.

P (atm)	**V(L)**

20.0	1.0
17.0	1.1
14.0	1.3
11.0	1.5
8.0	2.0
5.0	2.6
2.0	5.2
1.0	8.4

70. An ideal monatomic gas at 300 K expands adiabatically and reversibly to twice its volume. What is its final temperature?

71. An ideal diatomic gas at 80 K is slowly compressed adiabatically and reversibly to half its volume. What is its final temperature?

72. An ideal diatomic gas at 80 K is slowly compressed adiabatically to one-third its original volume. What is its final temperature?

73. Compare the charge in internal energy of an ideal gas for a quasi-static adiabatic expansion with that for a quasi-static isothermal expansion. What happens to the temperature of an ideal gas in an adiabatic expansion?

74. The temperature of n moles of an ideal gas changes from T_1 to T_2 in a quasi-static adiabatic transition. Show that the work done by the gas is given by $W = \frac{nR}{\gamma - 1}(T_1 - T_2)$.

75. A dilute gas expands quasi-statically to three times its initial volume. Is the final gas pressure greater for an isothermal or an adiabatic expansion? Does your answer depend on whether the gas is monatomic, diatomic, or polyatomic?

76. (a) An ideal gas expands adiabatically from a volume of $2.0 \times 10^{-3} m^3$ to $2.5 \times 10^{-3} m^3$. If the initial pressure and temperature were $5.0 \times 10^5 Pa$ and 300 K, respectively, what are the final pressure and temperature of the gas? Use **γ=5/3** for the gas.

(b) In an isothermal process, an ideal gas expands from a volume of $2.0 \times 10^{-3} m^3$ to $2.5 \times 10^{-3} m^3$. If the initial pressure and temperature were $.0 \times 10^5 Pa$ and 300 K, respectively, what are the final pressure and temperature of the gas?

77. On an adiabatic process of an ideal gas pressure, volume and temperature change such that pV^γ is constant with **γ=5/3** for monatomic gas such as helium and **γ=7/5** for diatomic gas such as hydrogen at room temperature. Use numerical values to plot two isotherms of 1 mol of helium gas using ideal gas law and two adiabatic processes mediating between them. Use $T_1 = 500K, V_1 = 1L,$ and $T_2 = 300K$ for your plot.

78. Two moles of a monatomic ideal gas such as helium is compressed adiabatically and reversibly from a state (3 atm, 5 L) to a state with pressure 4 atm. (a) Find the volume and temperature of the final state. (b) Find the temperature of the initial state of the gas. (c) Find the work done by the gas in the process. (d) Find the change in internal energy of the gas in the process.

Additional Problems

79. Consider the process shown below. During steps AB and BC, 3600 J and 2400 J of heat, respectively, are added to the system.

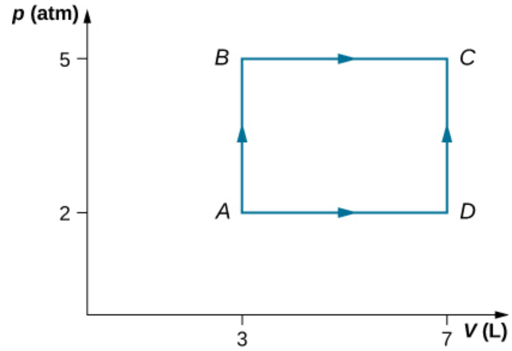
(a) Find the work done in each of the processes AB, BC, AD, and DC.

(b) Find the internal energy change in processes AB and BC.

(c) Find the internal energy difference between states C and A.

(d) Find the total heat added in the ADCprocess.

(e) From the information given, can you find the heat added in process AD? Why or why not?



80. A car tire contains $0.0380 m^3$ of air at a pressure of $2.20 \times 10^5 Pa$ (about 32 psi). How much more internal energy does this gas have than the same volume has at zero gauge pressure (which is equivalent to normal atmospheric pressure)?

81. A helium-filled toy balloon has a gauge pressure of 0.200 atm and a volume of 10.0 L. How much greater is the internal energy of the helium in the balloon than it would be at zero gauge pressure?

82. Steam to drive an old-fashioned steam locomotive is supplied at a constant gauge pressure of $1.75 \times 10^6 N/m^2$ (about 250 psi) to a piston with a 0.200-m radius.

(a) By calculating $p\Delta V$, find the work done by the steam when the piston moves 0.800 m. Note that this is the net work output, since gauge pressure is used.

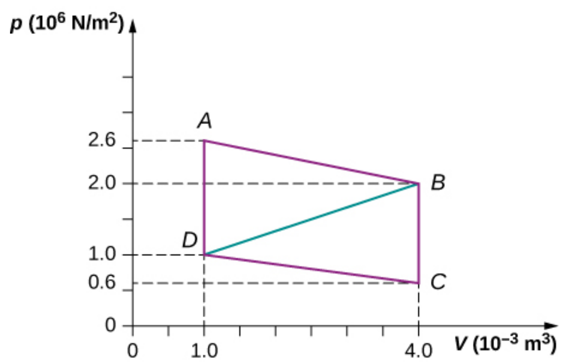
(b) Now find the amount of work by calculating the force exerted times the distance traveled. Is the answer the same as in part (a)?

83. A hand-driven tire pump has a piston with a 2.50-cm diameter and a maximum stroke of 30.0 cm.

(a) How much work do you do in one stroke if the average gauge pressure is $2.4 \times 10^5 N/m^2$ (about 35 psi)?

(b) What average force do you exert on the piston, neglecting friction and gravitational force?

84. Calculate the net work output of a heat engine following path ABCDA as shown below.



85. What is the net work output of a heat engine that follows path ABDA in the preceding problem with a straight line from B toD? Why is the work output less than for path ABCDA?

86. Five moles of a monatomic ideal gas in a cylinder at **27°C** is expanded isothermally from a volume of 5 L to 10 L.

(a) What is the change in internal energy?

(b) How much work was done on the gas in the process? (c) How much heat was transferred to the gas?

87. Four moles of a monatomic ideal gas in a cylinder at **27°C** is expanded at constant pressure equal to 1 atm until its volume doubles.

(a) What is the change in internal energy?

(b) How much work was done by the gas in the process?

(c) How much heat was transferred to the gas?

88. Helium gas is cooled from **20°C** to **10°C** by expanding from 40 atm to 1 atm. If there is 1.4 mol of helium,

(a) What is the final volume of helium?

(b) What is the change in internal energy?

89. In an adiabatic process, oxygen gas in a container is compressed along a path that can be described by the following pressure in atm as a function of volume V, with $V_0 = 1L : p = (3.0atm)(V/V_0)^{-1.2}$. The initial and final volumes during the process were 2 L and 1.5 L, respectively. Find the amount of work done on the gas.

90. A cylinder containing three moles of a monatomic ideal gas is heated at a constant pressure of 2 atm. The temperature of the gas changes from 300 K to 350 K as a result of the expansion. Find work done

(a) on the gas; and

(b) by the gas.

91. A cylinder containing three moles of nitrogen gas is heated at a constant pressure of 2 atm. The temperature of the gas changes from 300 K to 350 K as a result of the expansion. Find work done

(a) on the gas, and

(b) by the gas by using van der Waals equation of state instead of ideal gas law.

92. Two moles of a monatomic ideal gas such as helium is compressed adiabatically and reversibly from a state (3 atm, 5 L) to a state with a pressure of 4 atm.

(a) Find the volume and temperature of the final state.

(b) Find the temperature of the initial state.

(c) Find work done by the gas in the process.

(d) Find the change in internal energy in the process. Assume $C_V = 5R$ and $C_p = C_V + R$ for the diatomic ideal gas in the conditions given.

93. An insulated vessel contains 1.5 moles of argon at 2 atm. The gas initially occupies a volume of 5 L. As a result of the adiabatic expansion the pressure of the gas is reduced to 1 atm.

(a) Find the volume and temperature of the final state.

(b) Find the temperature of the gas in the initial state.

(c) Find the work done by the gas in the process.

(d) Find the change in the internal energy of the gas in the process.

Challenge Problems

94. One mole of an ideal monatomic gas occupies a volume of $1.0 \times 10^{-2} m^3$ at a pressure of $2.0 \times 10^5 N/m^2$.

(a) What is the temperature of the gas?

(b) The gas undergoes a quasi-static adiabatic compression until its volume is decreased to $5.0 \times 10^{-3} m^3$. What is the new gas temperature?

(c) How much work is done on the gas during the compression?

(d) What is the change in the internal energy of the gas?

95. One mole of an ideal gas is initially in a chamber of volume $1.0 \times 10^{-2} m^3$ and at a temperature of **27°C**.

(a) How much heat is absorbed by the gas when it slowly expands isothermally to twice its initial volume?

(b) Suppose the gas is slowly transformed to the same final state by first decreasing the pressure at constant volume and then expanding it isobarically. What is the heat transferred for this case?

(c) Calculate the heat transferred when the gas is transformed quasi-statically to the same final state by expanding it isobarically, then decreasing its pressure at constant volume.

96. A bullet of mass 10 g is traveling horizontally at 200 m/s when it strikes and embeds in a pendulum bob of mass 2.0 kg.

(a) How much mechanical energy is dissipated in the collision?

(b) Assuming that C_v for the bob plus bullet is 3R, calculate the temperature increase of the system due to the collision. Take the molecular mass of the system to be 200 g/mol.

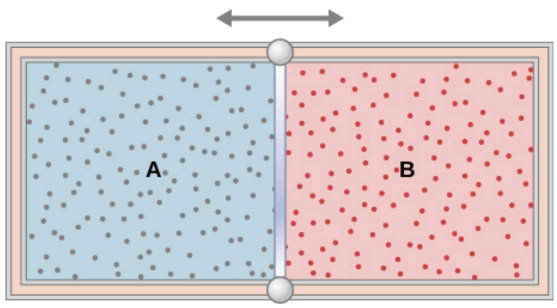
97. The insulated cylinder shown below is closed at both ends and contains an insulating piston that is free to move on frictionless bearings. The piston divides the chamber into two compartments containing gases A and B. Originally, each compartment has a volume of $5.0 \times 10^{-2} m^3$ and contains a monatomic ideal gas at a temperature of **0°C** and a pressure of 1.0 atm.

(a) How many moles of gas are in each compartment?

(b) Heat Q is slowly added to A so that it expands and B is compressed until the pressure of both gases is 3.0 atm. Use the fact that the compression of B is adiabatic to determine the final volume of both gases.

(c) What are their final temperatures?

(d) What is the value of Q?



98. In a diesel engine, the fuel is ignited without a spark plug. Instead, air in a cylinder is compressed adiabatically to a temperature above the ignition temperature of the fuel; at the point of maximum compression, the fuel is injected into the cylinder. Suppose that air at **20°C** is taken into the cylinder at a volume V_1 and then compressed adiabatically and quasi-statically to a temperature of **600°C** and a volume V_2. If **γ=1.4**, what is the ratio V_1/V_2? (Note: In an operating diesel engine, the compression is not quasi-static.)

14.9: The First Law of Thermodynamics (Answer)

Check Your Understanding

3.1. $p_2(V_2 - V_1)$

3.2. Line 1, $\Delta E_{int} = 40J$;

line 2, $W = 50J$ and $\Delta E_{int} = 40J$;

line 3, $Q = 80J$ and $\Delta E_{int} = 40J$; and

line 4, $Q = 0$ and $\Delta E_{int} = 40J$

3.3. So that the process is represented by the curve $p = nRT/V$ on the **pV** plot for the evaluation of work.

3.4. $1.26 \times 10 -^3 J$.

Conceptual Questions

1. a. SE; b. ES; c. ES

3. Some of the energy goes into changing the phase of the liquid to gas.

5. Yes, as long as the work done equals the heat added there will be no change in internal energy and thereby no change in temperature. When water freezes or when ice melts while removing or adding heat, respectively, the temperature remains constant.

7. If more work is done on the system than heat added, the internal energy of the system will actually decrease.

9. The system must be in contact with a heat source that allows heat to flow into the system.

11. Isothermal processes must be slow to make sure that as heat is transferred, the temperature does not change. Even for isobaric and isochoric processes, the system must be in thermal equilibrium with slow changes of thermodynamic variables.

13. Typically C_p is greater than C_V because when expansion occurs under constant pressure, it does work on the surroundings. Therefore, heat can go into internal energy and work. Under constant volume, all heat goes into internal energy. In this example, water contracts upon heating, so if we add heat at constant pressure, work is done on the water by surroundings and therefore, C_p is less than C_V.

15. No, it is always greater than 1.

17. An adiabatic process has a change in temperature but no heat flow. The isothermal process has no change in temperature but has heat flow.

Problems

19. $p(V - b) = -c_T$ is the temperature scale desired and mirrors the ideal gas if under constant volume.

21. $V - bpT + cT^2 = 0$

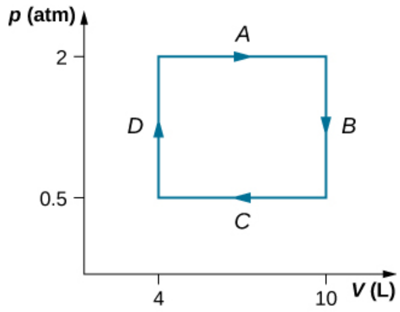
23. 74 K

25. 1.4 times

27. pVln(4)

29. a. 160 J; b. –160 J

31. $W = 900J$



33. $3.53 \times 10^{4} J$

35. a. 1:1;

b. 10:1

37. a. 600 J;

b. 0;

c. 500 J;

d. 200 J;

e. 800 J;

f. 500 J

39. 580 J

41. a. 600 J;

b. 600 J;

c. 800 J

43. a. 0;

b. 160 J;

c. –160 J

45. a. 150 J;

b. 700 J

47. No work is done and they reach the same common temperature.

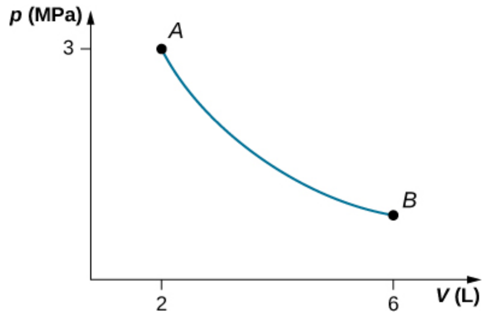
49. 54,500 J

51. a. $(p_1 + 3V_1^{\,2})(V_2 - V_1) - 3V_1(V_2^{\,2} - V_1^{\,2}) + (V_2^{\,3} - V_1^{\,3})$;

b. $\frac{3}{2}(p_2V_2 - p_1V_1)$;

c. the sum of parts (a) and (b); d. $T_1 = \frac{p_1V_1}{nR}$ and $T_2 = \frac{p_2V_2}{nR}$

53. a.



b. $W = 4.39kJ, \Delta E_{int} = -4.39kJ$

55. a. 1660 J;

b. −2730 J;

c. It does not depend on the process.

57. a. 700 J;

b. 500 J

59. a. −3 400 J;

b. 3400 J enters the gas

61. 100 J

63. a. 370 J;

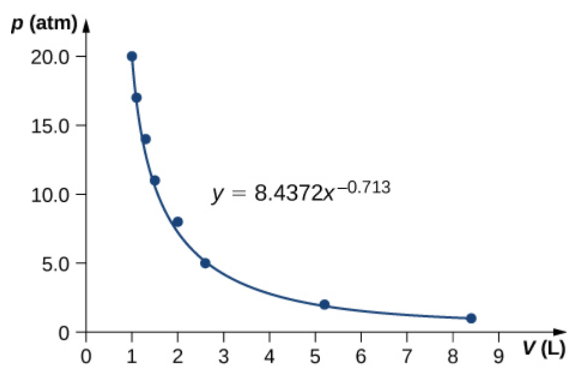
b. 100 J;

c. 500 J

65. 850 J

67. pressure decreased by 0.31 times the original pressure

69. $\gamma = 0.713$

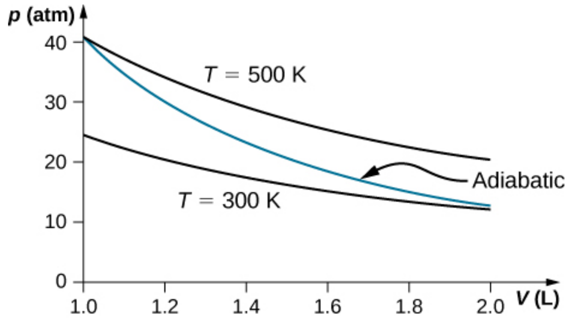


71. 84 K

73. An adiabatic expansion has less work done and no heat flow, thereby a lower internal energy comparing to an isothermal expansion which has both heat flow and work done. Temperature decreases during adiabatic expansion.

75. Isothermal has a greater final pressure and does not depend on the type of gas.

77.



Additional Problems

79. a. $W_{AB} = 0, W_{BC} = 2026J, W_{AD} = 810.4J, W_{DC} = 0;$

b. $\Delta E_{AB} = 3600J, \Delta E_{BC} = 374J;$

c. $\Delta E_{AC} = 3974J;$

d. $Q_{ADC} = 4784J;$

e. No, because heat was added for both parts **AD** and **DC**. There is not enough information to figure out how much is from each segment of the path.

81. 300 J

83. a. 59.5 J;

b. 170 N

85. $2.4 \times 10^3 J$

87. a. 15,000 J;

b. 10,000 J;

c. 25,000 J

89. 78 J

91. A cylinder containing three moles of nitrogen gas is heated at a constant pressure of 2 atm. a. −1220 J; b. +1220 J

93. a. 7.6 L, 61.6 K;

b. 81.3 K;

c. $3.63L \cdot atm = 367J;$

d. −367 J

Challenge Problems

95. a. 1700 J; b. 1200 J; c. 2400 J

97. a. 2.2 mol;

b. $V_A = 6.7 \times 10^{-2} m^3, V_B = 3.3 \times 10^{-2} m^3$;

c. $T_A = 2400K, T_B = 397K$; d. 26,000 J

14.10: The First Law of Thermodynamics (Summary)

Key Terms

adiabatic process	process during which no heat is transferred to or from the system
boundary	imagined walls that separate the system and its surroundings
closed system	system that is mechanically and thermally isolated from its environment
cyclic process	process in which the state of the system at the end is same as the state at the beginning
environment	outside of the system being studied
equation of state	describes properties of matter under given physical conditions
equilibrium	thermal balance established between two objects or parts within a system
extensive variable	variable that is proportional to the amount of matter in the system
first law of thermodynamics	the change in internal energy for any transition between two equilibrium states is $\Delta E_{int} = Q - W$
intensive variable	variable that is independent of the amount of matter in the system
internal energy	average of the total mechanical energy of all the molecules or entities in the system
isobaric process	process during which the system's pressure does not change
isochoric process	process during which the system's volume does not change
isothermal process	process during which the system's temperature remains constant
molar heat capacity at constant pressure	quantifies the ratio of the amount of heat added removed to the temperature while measuring at constant pressure
molar heat capacity at constant volume	quantifies the ratio of the amount of heat added removed to the temperature while measuring at constant volume
open system	system that can exchange energy and/or matter with its surroundings
quasi-static process	evolution of a system that goes so slowly that the system involved is always in thermodynamic equilibrium
reversible process	process that can be reverted to restore both the system and its environment back to their original states together
surroundings	environment that interacts with an open system
thermodynamic process	manner in which a state of a system can change from initial state to final state
thermodynamic system	object and focus of thermodynamic study

Key Equations

Equation of state for a closed system	$f(p, V, T) = 0$
Net work for a finite change in volume	$W = \int_{V_1}^{V_2} pdV$
Internal energy of a system (average total energy)	$E_{int} = \sum_i (\bar{K}_i + \bar{U}_i)$,

Internal energy of a monatomic ideal gas	$E_{int} = nN_A(\frac{3}{2}k_BT) = \frac{3}{2}nRT$
First law of thermodynamics	$\Delta E_{int} = Q - W$
Molar heat capacity at constant pressure	$C_p = C_V + R$
Ratio of molar heat capacities	$\gamma = C_p/C_V$
Condition for an ideal gas in a quasi-static adiabatic process	$pV^{\gamma} = constant$

Summary

3.2 Thermodynamic Systems

- A thermodynamic system, its boundary, and its surroundings must be defined with all the roles of the components fully explained before we can analyze a situation.
- Thermal equilibrium is reached with two objects if a third object is in thermal equilibrium with the other two separately.
- A general equation of state for a closed system has the form $f(p, V, T) = 0$, with an ideal gas as an illustrative example.

3.3 Work, Heat, and Internal Energy

- Positive (negative) work is done by a thermodynamic system when it expands (contracts) under an external pressure.
- Heat is the energy transferred between two objects (or two parts of a system) because of a temperature difference.
- Internal energy of a thermodynamic system is its total mechanical energy.

3.4 First Law of Thermodynamics

- The internal energy of a thermodynamic system is a function of state and thus is unique for every equilibrium state of the system.
- The increase in the internal energy of the thermodynamic system is given by the heat added to the system less the work done by the system in any thermodynamics process.

3.5 Thermodynamic Processes

- The thermal behavior of a system is described in terms of thermodynamic variables. For an ideal gas, these variables are pressure, volume, temperature, and number of molecules or moles of the gas.
- For systems in thermodynamic equilibrium, the thermodynamic variables are related by an equation of state.
- A heat reservoir is so large that when it exchanges heat with other systems, its temperature does not change.
- A quasi-static process takes place so slowly that the system involved is always in thermodynamic equilibrium.
- A reversible process is one that can be made to retrace its path and both the temperature and pressure are uniform throughout the system.
- There are several types of thermodynamic processes, including (a) isothermal, where the system's temperature is constant; (b) adiabatic, where no heat is exchanged by the system; (c) isobaric, where the system's pressure is constant; and (d) isochoric, where the system's volume is constant.
- As a consequence of the first law of thermodymanics, here is a summary of the thermodymaic processes:
  - (a) isothermal: $\Delta E_{int} = 0, Q = W$;
  - (b) adiabatic: $Q = 0, \Delta E_{int} = -W$;
  - (c) isobaric: $\Delta E_{int} = Q - W$; and
  - (d) isochoric: $W = 0, \Delta E_{int} = Q$.

3.6 Heat Capacities of an Ideal Gas

- For an ideal gas, the molar capacity at constant pressure C_p is given by $C_p = C_V + R = dR/2 + R$, where d is the number of degrees of freedom of each molecule/entity in the system.
- A real gas has a specific heat close to but a little bit higher than that of the corresponding ideal gas with $C_p \simeq C_V + R$.

3.7 Adiabatic Processes for an Ideal Gas

- A quasi-static adiabatic expansion of an ideal gas produces a steeper pV curve than that of the corresponding isotherm.
- A realistic expansion can be adiabatic but rarely quasi-static.

14.11: Prelude to The Second Law of Thermodynamics

According to the first law of thermodynamics, the only processes that can occur are those that conserve energy. But this cannot be the only restriction imposed by nature, because many seemingly possible thermodynamic processes that would conserve energy do not occur. For example, when two bodies are in thermal contact, heat never flows from the colder body to the warmer one, even though this is not forbidden by the first law. So some other thermodynamic principles must be controlling the behavior of physical systems.

One such principle is the **second law of thermodynamics**, which limits the use of energy within a source. Energy cannot arbitrarily pass from one object to another, just as we cannot transfer heat from a cold object to a hot one without doing any work. We cannot unmix cream from coffee without a chemical process that changes the physical characteristics of the system or its environment. We cannot use internal energy stored in the air to propel a car, or use the energy of the ocean to run a ship, without disturbing something around that object.

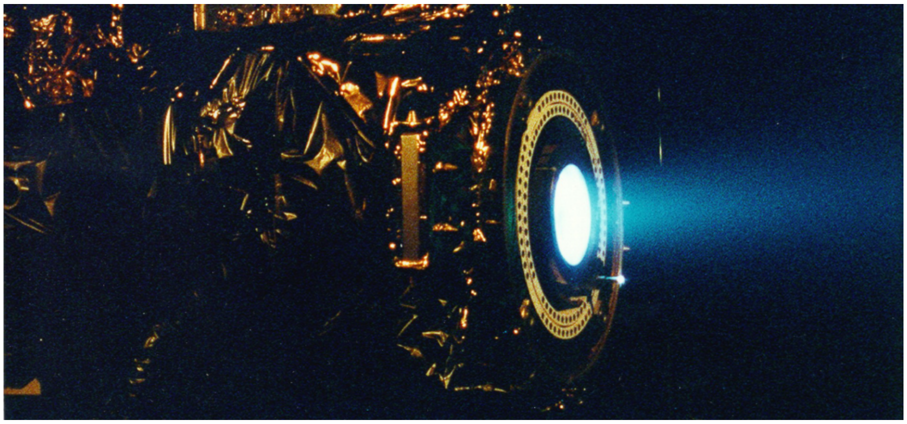


Figure 14.11.1: A xenon ion engine from the Jet Propulsion Laboratory shows the faint blue glow of charged atoms emitted from the engine. The ion propulsion engine is the first nonchemical propulsion to be used as the primary means of propelling a spacecraft.

In the chapter covering the first law of thermodynamics, we started our discussion with a joke by C. P. **Snow** stating that the first law means “you can’t win.” He paraphrased the second law as “you can’t break even, except on a very cold day.” Unless you are at zero kelvin, you cannot convert 100% of thermal energy into work. We start by discussing spontaneous processes and explain why some processes require work to occur even if energy would have been conserved.

14.12: Reversible and Irreversible Processes

Learning Objectives

By the end of this section, you will be able to:

- Define reversible and irreversible processes
- State the second law of thermodynamics via an irreversible process

Consider an ideal gas that is held in half of a thermally insulated container by a wall in the middle of the container. The other half of the container is under vacuum with no molecules inside. Now, if we remove the wall in the middle quickly, the gas expands and fills up the entire container immediately, as shown in Figure $14.12.1$.

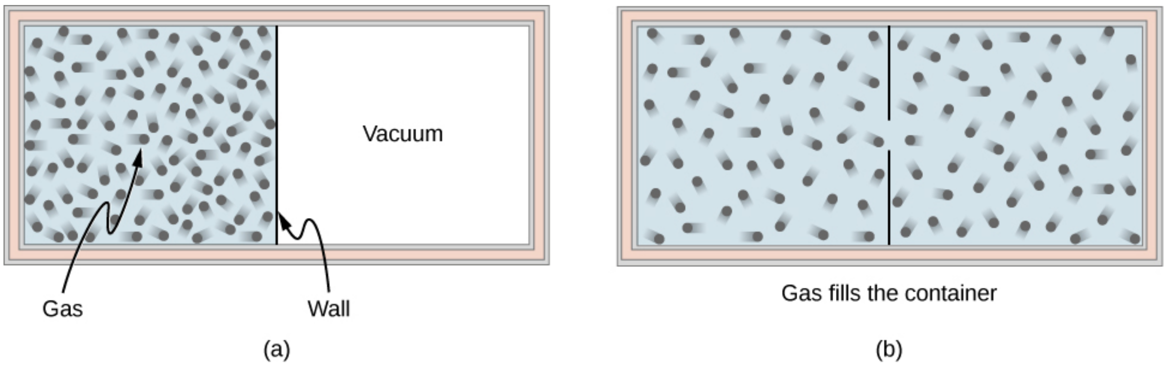


Figure 14.12.1: A gas expanding from half of a container to the entire container (a) before and (b) after the wall in the middle is removed.

Because half of the container is under vacuum before the gas expands there, we do not expect any work to be done by the system—that is, $W = 0$ - because no force from the vacuum is exerted on the gas during the expansion. If the container is thermally insulated from the rest of the environment, we do not expect any heat transfer to the system either, so $Q = 0$. Then the first law of thermodynamics leads to the change of the internal energy of the system,

$$\Delta E_{int} = Q - W = 0.$$

For an ideal gas, if the internal energy doesn't change, then the temperature stays the same. Thus, the equation of state of the ideal gas gives us the final pressure of the gas, $p = nRT/V = p_0/2$, where p_0 is the pressure of the gas before the expansion. The volume is doubled and the pressure is halved, but nothing else seems to have changed during the expansion.

All of this discussion is based on what we have learned so far and makes sense. Here is what puzzles us: Can all the molecules go backward to the original half of the container in some future time? Our intuition tells us that this is going to be very unlikely, even though nothing we have learned so far prevents such an event from happening, regardless of how small the probability is. What we are really asking is whether the expansion into the vacuum half of the container is **reversible**.

A **reversible process** is a process in which the system and environment can be restored to exactly the same initial states that they were in before the process occurred, if we go backward along the path of the process. The necessary condition for a reversible process is therefore the quasi-static requirement. Note that it is quite easy to restore a system to its original state; the hard part is to have its environment restored to its original state at the same time. For example, in the example of an ideal gas expanding into vacuum to twice its original volume, we can easily push it back with a piston and restore its temperature and pressure by removing some heat from the gas. The problem is that we cannot do it without changing something in its surroundings, such as dumping some heat there.

A reversible process is truly an ideal process that rarely happens. We can make certain processes close to reversible and therefore use the consequences of the corresponding reversible processes as a starting point or reference. In reality, almost all processes are irreversible, and some properties of the environment are altered when the properties of the system are restored. The expansion of an ideal gas, as we have just outlined, is irreversible because the process is not even quasi-static, that is, not in an equilibrium state at any moment of the expansion.

From the microscopic point of view, a particle described by Newton's second law can go backward if we flip the direction of time. But this is not the case, in practical terms, in a macroscopic system with more than 10^{23} particles or molecules, where numerous collisions between these molecules tend to erase any trace of memory of the initial trajectory of each of the particles. For example, we can actually estimate the chance for all the particles in the expanded gas to go back to the original half of the container, but the current age of the universe is still not long enough for it to happen even once.

An **irreversible process** is what we encounter in reality almost all the time. The system and its environment cannot be restored to their original states at the same time. Because this is what happens in nature, it is also called a natural process. The sign of an irreversible process comes from the finite gradient between the states occurring in the actual process. For example, when heat flows from one object to another, there is a finite temperature difference (gradient) between the two objects. More importantly, at any given moment of the process, the system most likely is not at equilibrium or in a well-defined state. This phenomenon is called **irreversibility**.

Let us see another example of irreversibility in thermal processes. Consider two objects in thermal contact: one at temperature T_1 and the other at temperature $T_2 > T_1$, as shown in Figure $14.12.2$

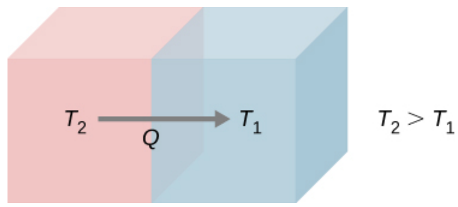


Figure 14.12.2: Spontaneous heat flow from an object at higher temperature T_2 to another at lower temperature T_1.

We know from common personal experience that heat flows from a hotter object to a colder one. For example, when we hold a few pieces of ice in our hands, we feel cold because heat has left our hands into the ice. The opposite is true when we hold one end of a metal rod while keeping the other end over a fire. Based on all of the experiments that have been done on spontaneous heat transfer, the following statement summarizes the governing principle:

Second Law of Thermodynamics (Clausius statement)

Heat never flows spontaneously from a colder object to a hotter object.

This statement turns out to be one of several different ways of stating the second law of thermodynamics. The form of this statement is credited to German physicist Rudolf **Clausius** (1822−1888) and is referred to as the **Clausius statement of the second law of thermodynamics**. The word "spontaneously" here means no other effort has been made by a third party, or one that is neither the hotter nor colder object. We will introduce some other major statements of the second law and show that they imply each other. In fact, all the different statements of the second law of thermodynamics can be shown to be equivalent, and all lead to the irreversibility of spontaneous heat flow between macroscopic objects of a very large number of molecules or particles.

Both isothermal and adiabatic processes sketched on a **pV** graph (discussed in The First Law of Thermodynamics) are reversible in principle because the system is always at an equilibrium state at any point of the processes and can go forward or backward along the given curves. Other idealized processes can be represented by **pV** curves; Table $14.12.1$ summarizes the most common reversible processes.

Table 14.12.1: Summary of Simple Thermodynamic Processes

Process	Constant Quantity and Resulting Fact
Isobaric	Constant pressure $W = p\Delta V$
Isochoric	Constant volume $W = 0$
Isothermal	Constant temperature $\Delta T = 0$
Adiabatic	No heat transfer $Q = 0$

- **4.2: Reversible and Irreversible Processes** by OpenStax is licensed CC BY 4.0. Original source: https://openstax.org/details/books/university-physics-volume-2.

14.13: Heat Engines

Learning Objectives

By the end of this section, you will be able to:

- Describe the function and components of a heat engine
- Explain the efficiency of an engine
- Calculate the efficiency of an engine for a given cycle of an ideal gas

A **heat engine** is a device used to extract heat from a source and then convert it into mechanical work that is used for all sorts of applications. For example, a steam engine on an old-style train can produce the work needed for driving the train. Several questions emerge from the construction and application of heat engines. For example, what is the maximum percentage of the heat extracted that can be used to do work? This turns out to be a question that can only be answered through the second law of thermodynamics.

The second law of thermodynamics can be formally stated in several ways. One statement presented so far is about the direction of spontaneous heat flow, known as the Clausius statement. A couple of other statements are based on heat engines. **Whenever we consider heat engines and associated devices such as refrigerators and heat pumps, we do not use the normal sign convention for heat and work**. For convenience, we assume that the symbols Q_h, Q_c, and **W** represent only the amounts of heat transferred and work delivered, regardless what the givers or receivers are. Whether heat is entering or leaving a system and work is done to or by a system are indicated by proper signs in front of the symbols and by the directions of arrows in diagrams.

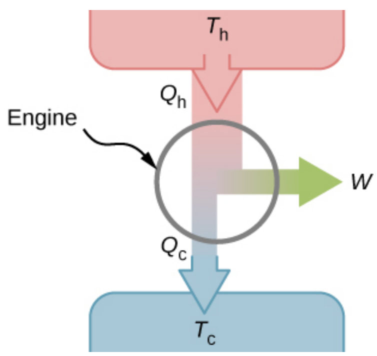


Figure 14.13.1: A schematic representation of a heat engine. Energy flows from the hot reservoir to the cold reservoir while doing work.

It turns out that we need more than one heat source/sink to construct a heat engine. We will come back to this point later in the chapter, when we compare different statements of the second law of thermodynamics. For the moment, we assume that a heat engine is constructed between a heat source (high-temperature reservoir or hot reservoir) and a heat sink (low-temperature reservoir or cold reservoir), represented schematically in Figure 14.13.1 The engine absorbs heat Q_h from a heat source (**hot reservoir**) of Kelvin temperature T_h, uses some of that energy to produce useful work **W**, and then discards the remaining energy as heat Q_c, into a heat sink (**cold reservoir**) of Kelvin temperature T_c. **Power plants** and internal combustion engines are examples of heat engines. Power plants use steam produced at high temperature to drive electric generators, while exhausting heat to the atmosphere or a nearby body of water in the role of the heat sink. In an **internal combustion engine**, a hot gas-air mixture is used to push a piston, and heat is exhausted to the nearby atmosphere in a similar manner.

Figure 14.13.2: The heat exhausted from a nuclear power plant goes to the cooling towers, where it is released into the atmosphere.

Actual heat engines have many different designs. Examples include internal combustion engines, such as those used in most cars today, and external combustion engines, such as the steam engines used in old steam-engine trains. Figure $14.13.2$shows a photo of a nuclear power plant in operation. The atmosphere around the reactors acts as the cold reservoir, and the heat generated from the nuclear reaction provides the heat from the hot reservoir.

Heat engines operate by carrying a **working substance** through a cycle. In a steam power plant, the working substance is water, which starts as a liquid, becomes vaporized, is then used to drive a turbine, and is finally condensed back into the liquid state. As is the case for all working substances in cyclic processes, once the water returns to its initial state, it repeats the same sequence.

For now, we assume that the cycles of heat engines are reversible, so there is no energy loss to friction or other irreversible effects. Suppose that the engine of Figure $14.13.1$goes through one complete cycle and that Q_h, Q_c, and **W** represent the heats exchanged and the work done for that cycle. Since the initial and final states of the system are the same, $\Delta E_{int} = 0$ for the cycle. We therefore have from the first law of thermodynamics,

$$W = Q - \Delta E_{int} \tag{14.13.1}$$

$$= (Q_h - Q_c) - 0, \tag{14.13.2}$$

so that

$$W = Q_h - Q_c. \tag{14.13.3}$$

The most important measure of a heat engine is its **efficiency (e)**, which is simply "what we get out" divided by "what we put in" during each cycle, as defined by

$$e = \frac{W_{out}}{Q_{in}}. \tag{14.13.4}$$

With a heat engine working between two heat reservoirs, we get out W and put in Q_h, so the efficiency of the engine is

$$e = \frac{W}{Q_h} \tag{14.13.5}$$

$$= 1 - \frac{Q_c}{Q_h}. \tag{14.13.6}$$

Here, we used Equation 14.13.3 in the final step of this expression for the efficiency.

Example 14.13.1: A Lawn Mower

A lawn mower is rated to have an efficiency of 25% and an average power of 3.00 kW. What are

a. the average work and
b. the minimum heat discharge into the air by the lawn mower in one minute of use?

Strategy

From the average power—that is, the rate of work production—we can figure out the work done in the given elapsed time. Then, from the efficiency given, we can figure out the minimum heat discharge $Q_c = Q_h(1-e)$ with $Q_h = Q_c + W$.

Solution

a. The average work delivered by the lawn mower is

$$W = P\Delta t \tag{14.13.7}$$

$$= 3.00 \times 10^3 \times 60 \times 1.00\, J \tag{14.13.8}$$

$$= 180\, kJ. \tag{14.13.9}$$

b. The minimum heat discharged into the air is given by

$$Q_c = Q_h(1-e) \tag{14.13.10}$$

$$= (Q_c + W)(1-e), \tag{14.13.11}$$

which leads to

$$Q_c = W(1/e - 1) \tag{14.13.12}$$

$$= 180 \times (1/0.25 - 1) kJ = 540\, kJ. \tag{14.13.13}$$

Significance

As the efficiency rises, the minimum heat discharged falls. This helps our environment and atmosphere by not having as much waste heat expelled.

14.14: Refrigerators and Heat Pumps

Learning Objectives

By the end of this section, you will be able to:

- Describe a refrigerator and a heat pump and list their differences
- Calculate the performance coefficients of simple refrigerators and heat pumps

The cycles we used to describe the engine in the preceding section are all reversible, so each sequence of steps can just as easily be performed in the opposite direction. In this case, the engine is known as a refrigerator or a heat pump, depending on what is the focus: the heat removed from the cold reservoir or the heat dumped to the hot reservoir. Either a refrigerator or a heat pump is an engine running in reverse.

- For a **refrigerator**, the focus is on removing heat from a specific area.
- For a **heat pump**, the focus is on dumping heat to a specific area.

We first consider a refrigerator (Figure $14.14.1$). The purpose of this engine is to remove heat from the cold reservoir, which is the space inside the refrigerator for an actual household refrigerator or the space inside a building for an air-conditioning unit.

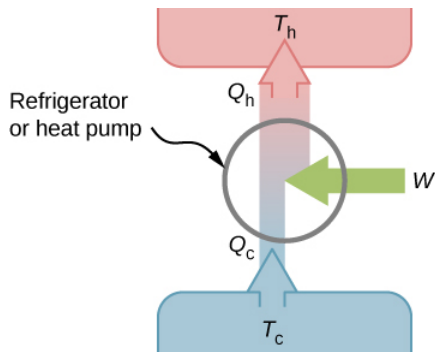


Figure 14.14.1: A schematic representation of a refrigerator (or a heat pump). The arrow next to work (W) indicates work being put into the system.

A refrigerator (or heat pump) absorbs heat Q_c from the cold reservoir at Kelvin temperature T_c and discards heat Q_h to the hot reservoir at Kelvin temperature T_h while work $\mathbf{W}$ is done on the engine's working substance, as shown by the arrow pointing toward the system in the figure. A household refrigerator removes heat from the food within it while exhausting heat to the surrounding air. The required work, for which we pay in our electricity bill, is performed by the motor that moves a coolant through the coils. A schematic sketch of a household refrigerator is given in Figure $14.14.2$

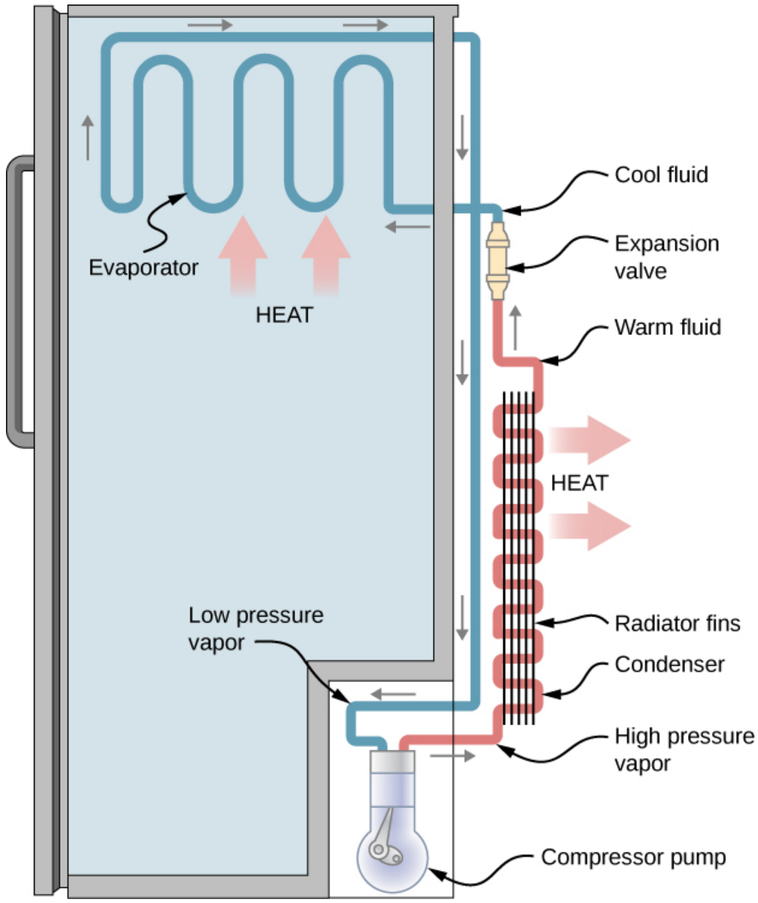


Figure 14.14.2: A schematic diagram of a household refrigerator. A coolant with a boiling temperature below the freezing point of water is sent through the cycle (clockwise in this diagram). The coolant extracts heat from the refrigerator at the evaporator, causing coolant to vaporize. It is then compressed and sent through the condenser, where it exhausts heat to the outside.

The effectiveness or **coefficient of performance** K_R of a refrigerator is measured by the heat removed from the cold reservoir divided by the work done by the working substance cycle by cycle:

$$K_R = \frac{Q_c}{W} = \frac{Q_c}{Q_h - Q_c}$$

Note that we have used the condition of energy conservation, $W = Q_h - Q_c$, in the final step of this expression.

The effectiveness or coefficient of performance K_P of a heat pump is measured by the heat dumped to the hot reservoir divided by the work done to the engine on the working substance cycle by cycle:

$$K_P = \frac{Q_h}{W} = \frac{Q_h}{Q_h - Q_c}.$$

Once again, we use the energy conservation condition $W = Q_h - Q_c$ to obtain the final step of this expression.

14.15: Statements of the Second Law of Thermodynamics

Learning Objectives

By the end of this section, you will be able to:

- Contrast the second law of thermodynamics statements according to Kelvin and Clausius formulations
- Interpret the second of thermodynamics via irreversibility

Earlier in this chapter, we introduced the Clausius statement of the second law of thermodynamics, which is based on the irreversibility of spontaneous heat flow. As we remarked then, the second law of thermodynamics can be stated in several different ways, and all of them can be shown to imply the others. In terms of heat engines, the second law of thermodynamics may be stated as follows:

Second Law of Thermodynamics (Kelvin statement)

It is impossible to convert the heat from a single source into work without any other effect.

This is known as the **Kelvin statement of the second law of thermodynamics**. This statement describes an unattainable "**perfect engine**," as represented schematically in Figure $14.15.1a$. Note that "without any other effect" is a very strong restriction. For example, an engine can absorb heat and turn it all into work, **but not if it completes a cycle**. Without completing a cycle, the substance in the engine is not in its original state and therefore an "other effect" has occurred. Another example is a chamber of gas that can absorb heat from a heat reservoir and do work isothermally against a piston as it expands. However, if the gas were returned to its initial state (that is, made to complete a cycle), it would have to be compressed and heat would have to be extracted from it.

The Kelvin statement is a manifestation of a well-known engineering problem. Despite advancing technology, we are not able to build a heat engine that is 100% efficient. The first law does not exclude the possibility of constructing a perfect engine, but the second law forbids it.

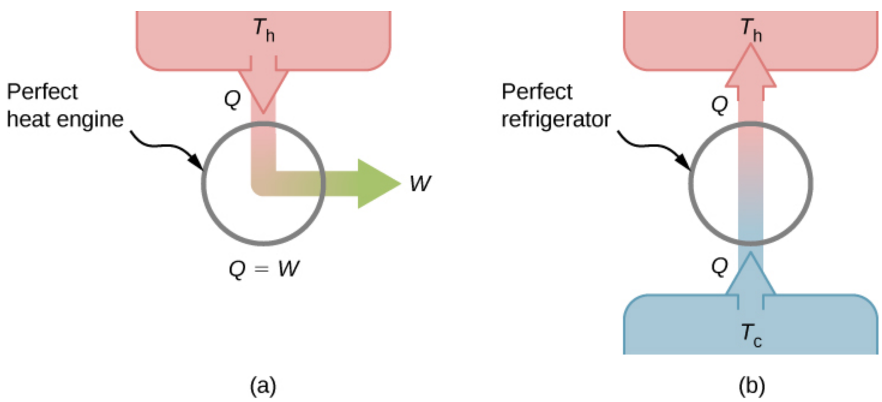


Figure 14.15.1: (a) A "perfect heat engine" converts all input heat into work. (b) A "perfect refrigerator" transports heat from a cold reservoir to a hot reservoir without work input. Neither of these devices is achievable in reality.

We can show that the Kelvin statement is equivalent to the Clausius statement if we view the two objects in the Clausius statement as a cold reservoir and a hot reservoir. Thus, the Clausius statement becomes: **It is impossible to construct a refrigerator that transfers heat from a cold reservoir to a hot reservoir without aid from an external source**. The Clausius statement is related to the everyday observation that heat never flows spontaneously from a cold object to a hot object. **Heat transfer in the direction of increasing temperature always requires some energy input**. A "perfect refrigerator," shown in Figure $14.15.1b$, which works without such external aid, is impossible to construct.

To prove the equivalence of the Kelvin and Clausius statements, we show that if one statement is false, it necessarily follows that the other statement is also false. Let us first assume that the Clausius statement is false, so that the perfect refrigerator of Figure $14.15.1b$ does exist. The refrigerator removes heat Q from a cold reservoir at a temperature T_c and transfers all of it to a hot reservoir at a temperature T_h. Now consider a real heat engine working in the same temperature range. It extracts heat $Q+\Delta Q$ from the hot reservoir, does work W, and discards heat Q to the cold reservoir. From the first law, these quantities are related by

$$W=(Q+\Delta Q)-Q=\Delta Q.$$

Suppose these two devices are combined as shown in Figure $14.15.2$ The net heat removed from the hot reservoir is ΔQ, no net heat transfer occurs to or from the cold reservoir, and work W is done on some external body. Since $W=\Delta Q$, the combination of a perfect refrigerator and a real heat engine is itself a perfect heat engine, thereby contradicting the Kelvin statement. Thus, if the Clausius statement is false, the Kelvin statement must also be false.

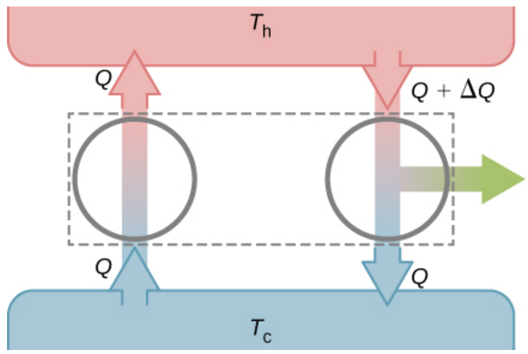


Figure 14.15.2: Combining a perfect refrigerator and a real heat engine yields a perfect heat engine because $W=\Delta Q$.

Using the second law of thermodynamics, we now prove two important properties of heat engines operating between two heat reservoirs. The first property is that **any reversible engine operating between two reservoirs has a greater efficiency than any irreversible engine operating between the same two reservoirs**.

The second property to be demonstrated is that **all reversible engines operating between the same two reservoirs have the same efficiency**. To show this, we start with the two engines D and E of Figure $14.15.3a$, which are operating between two common heat reservoirs at temperatures T_h and T_c. First, we assume that D is a reversible engine and that E is a hypothetical irreversible engine that has a higher efficiency than D. If both engines perform the same amount of work W per cycle, it follows from $e=\frac{W}{Q_h}=1-\frac{Q_C}{Q_h}$ that $Q_h>Q'_h$. It then follows from the first law that $Q_c>Q'_c$.

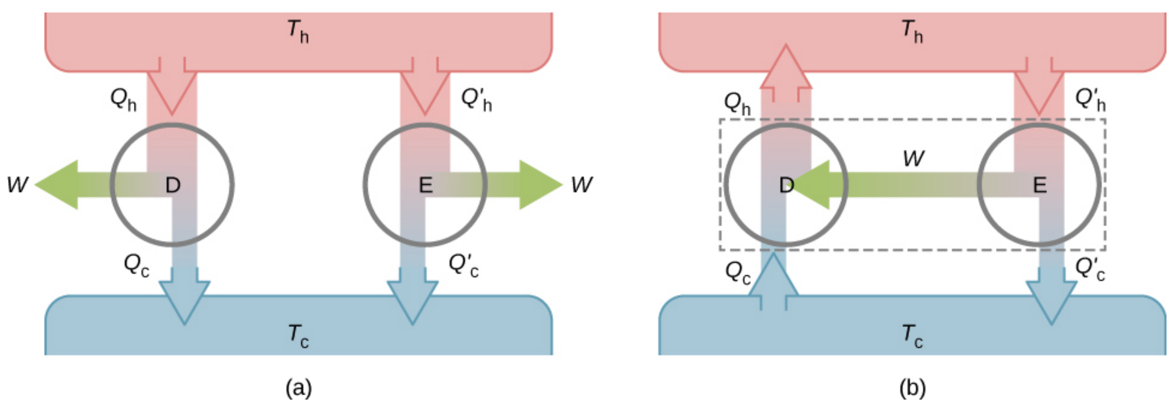


Figure 14.15.3: (a) Two uncoupled engines D and E working between the same reservoirs. (b) The coupled engines, with D working in reverse.

Suppose the cycle of D is reversed so that it operates as a refrigerator, and the two engines are coupled such that the work output of E is used to drive D, as shown in Figure $14.15.3b$ Since $Q_h>Q'_h$ and $Q_c>Q'_c$, the net result of each cycle is equivalent to a spontaneous transfer of heat from the cold reservoir to the hot reservoir, a process the second law does not allow. The original assumption must therefore be wrong, and it is impossible to construct an irreversible engine such that E is more efficient than the reversible engine D.

Now it is quite easy to demonstrate that the efficiencies of all reversible engines operating between the same reservoirs are equal. Suppose that D and E are both reversible engines. If they are coupled as shown in Figure $14.15.3b$, the efficiency of E cannot be greater than the efficiency of D, or the second law would be violated. If both engines are then reversed, the same reasoning implies that the efficiency of D cannot be greater than the efficiency of E. Combining these results leads to the conclusion that all reversible engines working between the same two reservoirs have the same efficiency.

Example 14.15.1

What is the efficiency of a perfect heat engine? What is the coefficient of performance of a perfect refrigerator?

Solution

A perfect heat engine would have $Q_c = 0$, which would lead to $e = 1 - Q_c/Q_h = 1$. A perfect refrigerator would need zero work, that is, $W = 0$, which leads to $K_R = Q_c/W \to \infty$.

Example 14.15.2

Show that $Q_h - Q'_h = Q_c - Q'_c$ for the hypothetical engine of Figure $14.15.1b$

Solution

From the engine on the right, we have $W = Q'_h - Q'_c$. From the refrigerator on the right, we have $Q_h = Q_c + W$. Thus $W = Q'_h - Q'_c = Q_h - Q_c$.

14.16: The Carnot Cycle

Learning Objectives

By the end of this section you will be able to:

- Describe the Carnot cycle with the roles of all four processes involved
- Outline the Carnot principle and its implications
- Demonstrate the equivalence of the Carnot principle and the second law of thermodynamics

In the early 1820s, Sadi **Carnot** (1786−1832), a French engineer, became interested in improving the efficiencies of practical heat engines. In 1824, his studies led him to propose a hypothetical working cycle with the highest possible efficiency between the same two reservoirs, known now as the **Carnot cycle**. An engine operating in this cycle is called a **Carnot engine**. The Carnot cycle is of special importance for a variety of reasons. At a practical level, this cycle represents a reversible model for the steam power plant and the refrigerator or heat pump. Yet, it is also very important theoretically, for it plays a major role in the development of another important statement of the second law of thermodynamics. Finally, because only two reservoirs are involved in its operation, it can be used along with the second law of thermodynamics to define an absolute temperature scale that is truly independent of any substance used for temperature measurement.

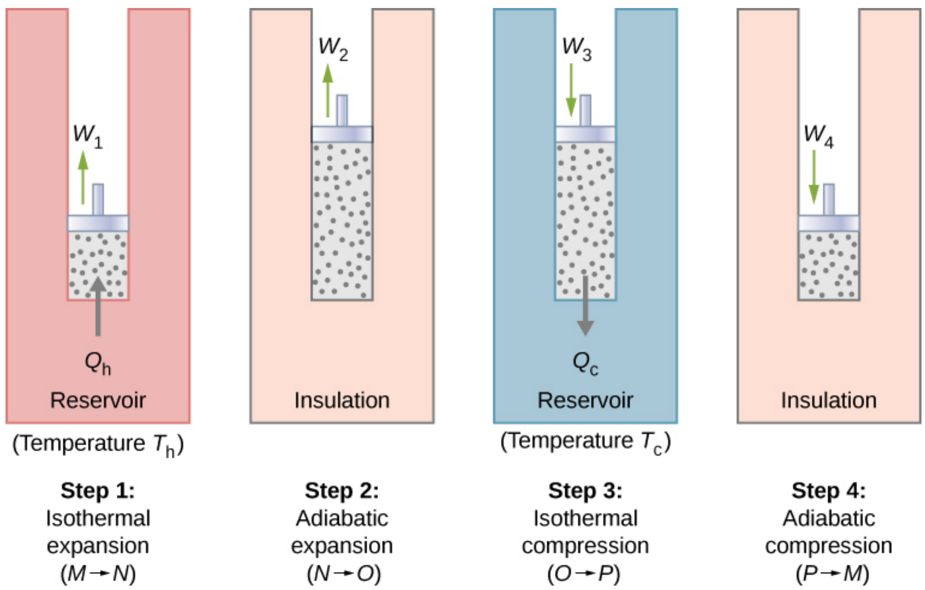


Figure 14.16.1: The four processes of the Carnot cycle. The working substance is assumed to be an ideal gas whose thermodynamic path *MNOP* is represented in Figure 14.16.2

With an ideal gas as the working substance, the steps of the Carnot cycle, as represented by Figure 14.16.1, are as follows.

1. **Isothermal expansion.** The gas is placed in thermal contact with a heat reservoir at a temperature T_h. The gas absorbs heat Q_h from the heat reservoir and is allowed to expand isothermally, doing work W_1. Because the internal energy E_{int} of an ideal gas is a function of the temperature only, the change of the internal energy is zero, that is, $\Delta E_{int} = 0$ during this isothermal expansion. With the first law of thermodynamics, $\Delta E_{int} = Q - W$, we find that the heat absorbed by the gas is

$$Q_h = W_1 = nRT_h \ln \frac{V_N}{V_M}.$$

2. **Adiabatic expansion**. The gas is thermally isolated and allowed to expand further, doing work W_2. Because this expansion is adiabatic, the temperature of the gas falls—in this case, from T_h to T_c. From $pV^{\gamma} = constant$ and the equation of state for an ideal gas, $pV = nRT$, we have

$$TV^{\gamma-1} = constant,$$

so that

$$T_h V_N^{\gamma-1} = T_c V_O^{\gamma-1}.$$

3. **Isothermal compression**. The gas is placed in thermal contact with a cold reservoir at temperature T_c and compressed isothermally. During this process, work W_3 is done on the gas and it gives up heat Q_c to the cold reservoir. The reasoning used in step 1 now yields

$$Q_c = nRT_c \ln \frac{V_O}{V_p},$$

where Q_c is the heat dumped to the cold reservoir by the gas.
4. **Adiabatic compression**. The gas is thermally isolated and returned to its initial state by compression. In this process, work W_4 is done on the gas. Because the compression is adiabatic, the temperature of the gas rises—from T_c to T_h in this particular case. The reasoning of step 2 now gives

$$T_c V_p^{\gamma-1} = T_h V_M^{\gamma-1}.$$

The total work done by the gas in the Carnot cycle is given by

$$W = W_1 + W_2 - W_3 - W_4.$$

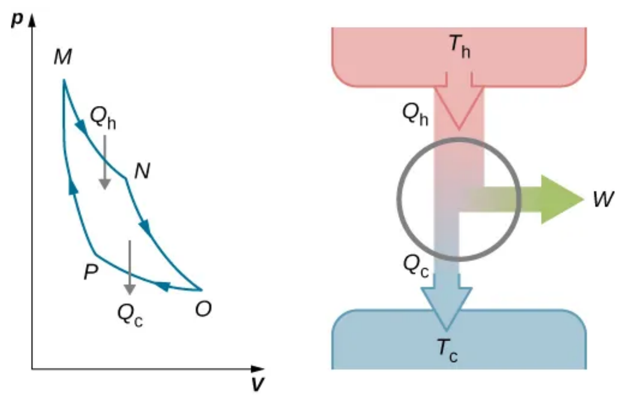


Figure 14.16.2: The total work done by the gas in the Carnot cycle is shown and given by the area enclosed by the loop MNOPM.

This work is equal to the area enclosed by the loop shown in the **pV** diagram of Figure 14.16.2 Because the initial and final states of the system are the same, the change of the internal energy of the gas in the cycle must be zero, that is, $\Delta E_{int} = 0$. The first law of thermodynamics then gives

$$W = Q - \Delta E_{int} = (Q_h - Q_c) - 0,$$

and

$$W = Q_h - Q_c$$

.

To find the efficiency of this engine, we first divide Q_c by Q_h:

$$\frac{Q_c}{Q_h} = \frac{T_c}{T_h} \frac{\ln V_O/V_P}{\ln V_N/V_M}.$$

When the adiabatic constant from step 2 is divided by that of step 4, we find

$$\frac{V_O}{V_P} = \frac{V_N}{V_M}.$$

Substituting this into the equation for Q_c/Q_h, we obtain

$$\frac{Q_c}{Q_h} = \frac{T_c}{T_h}.$$

Finally, with Equation 4.3.6, we find that the efficiency of this ideal gas Carnot engine is given by

$$e = 1 - \frac{T_c}{T_h}.$$

An engine does not necessarily have to follow a Carnot engine cycle. All engines, however, have the same **net** effect, namely the absorption of heat from a hot reservoir, the production of work, and the discarding of heat to a cold reservoir. This leads us to ask: Do all reversible cycles operating between the same two reservoirs have the same efficiency? The answer to this question comes from the second law of thermodynamics discussed earlier: **All reversible engine cycles produce exactly the same efficiency**. Also, as you might expect, all real engines operating between two reservoirs are less efficient than reversible engines operating between the same two reservoirs. This too is a consequence of the second law of thermodynamics shown earlier.

The cycle of an ideal gas Carnot refrigerator is represented by the **pV** diagram of Figure $14.16.3$ It is a Carnot engine operating in reverse. The refrigerator extracts heat Q_c from a cold-temperature reservoir at T_c when the ideal gas expands isothermally. The gas is then compressed adiabatically until its temperature reaches T_h, after which an isothermal compression of the gas results in heat Q_h being discarded to a high-temperature reservoir at T_h. Finally, the cycle is completed by an adiabatic expansion of the gas, causing its temperature to drop to T_c.

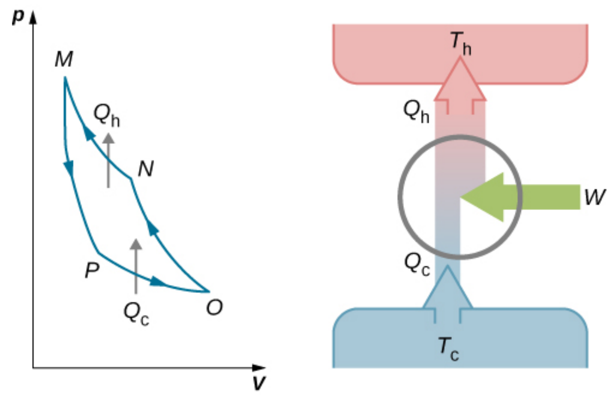


Figure 14.16.3. The work done on the gas in one cycle of the Carnot refrigerator is shown and given by the area enclosed by the loop **MPONM**.

The work done on the ideal gas is equal to the area enclosed by the path of the **pV** diagram. From the first law, this work is given by

$$W = Q_h - Q_c.$$

An analysis just like the analysis done for the Carnot engine gives

$$\frac{Q_c}{T_c} = \frac{Q_h}{T_h}.$$

When combined with Equation 4.4.1, this yields

$$K_R = \frac{T_c}{T_h - T_c}$$

for the coefficient of performance of the ideal-gas Carnot refrigerator. Similarly, we can work out the coefficient of performance for a Carnot heat pump as

$$K_P = \frac{Q_h}{Q_h - Q_c} = \frac{T_h}{T_h - T_c}.$$

We have just found equations representing the efficiency of a Carnot engine and the coefficient of performance of a Carnot refrigerator or a Carnot heat pump, assuming an ideal gas for the working substance in both devices. However, these equations are more general than their derivations imply. We will soon show that they are both valid no matter what the working substance is.

Carnot summarized his study of the Carnot engine and Carnot cycle into what is now known as **Carnot's principle**:

Carnot's Principle

No engine working between two reservoirs at constant temperatures can have a greater efficiency than a reversible engine.

This principle can be viewed as another statement of the second law of thermodynamics and can be shown to be equivalent to the Kelvin statement and the Clausius statement.

Example 14.16.1: The Carnot Engine

A Carnot engine has an efficiency of 0.60 and the temperature of its cold reservoir is 300 K. (a) What is the temperature of the hot reservoir? (b) If the engine does 300 J of work per cycle, how much heat is removed from the high-temperature reservoir per cycle? (c) How much heat is exhausted to the low-temperature reservoir per cycle?

Strategy

From the temperature dependence of the thermal efficiency of the Carnot engine, we can find the temperature of the hot reservoir. Then, from the definition of the efficiency, we can find the heat removed when the work done by the engine is given. Finally, energy conservation will lead to how much heat must be dumped to the cold reservoir.

Solution

1. From $e = 1 - T_c/T_h$ we have

$$0.60 = 1 - \frac{300\,K}{T_h},$$

so that the temperature of the hot reservoir is

$$T_h = \frac{300\,K}{1 - 0.60} = 750\,K.$$

2. By definition, the efficiency of the engine is $e = W/Q$, so that the heat removed from the high-temperature reservoir per cycle is

$$Q_h = \frac{W}{e} = \frac{300\,J}{0.60} = 500\,J.$$

3. From the first law, the heat exhausted to the low-temperature reservoir per cycle by the engine is

$$Q_c = Q_h - W = 500\,J - 300\,J = 200\,J.$$

Significance

A Carnot engine has the maximum possible efficiency of converting heat into work between two reservoirs, but this does not necessarily mean it is 100% efficient. As the difference in temperatures of the hot and cold reservoir increases, the efficiency of a Carnot engine increases.

Example 14.16.2: A Carnot Heat Pump

Imagine a Carnot heat pump operates between an outside temperature of 0^oC and an inside temperature of 20.0^oC. What is the work needed if the heat delivered to the inside of the house is 30.0 kJ?

Strategy

Because the heat pump is assumed to be a Carnot pump, its performance coefficient is given by $K_P = Q_h/W = T_h/(T_h - T_c)$. Thus, we can find the work **W** from the heat delivered Q_h.

Solution

The work needed is obtained from

$$W = Q_h/K_P = Q_h(T_h - T_c)/T_h = 30\,kJ \times (293\,K - 273\,K)/293\,K = 2\,kJ.$$

Significance

We note that this work depends not only on the heat delivered to the house but also on the temperatures outside and inside. The dependence on the temperature outside makes them impractical to use in areas where the temperature is much colder outside than room temperature.

In terms of energy costs, the **heat pump** is a very economical means for heating buildings (Figure 14.16.4). Contrast this method with turning electrical energy directly into heat with resistive heating elements. In this case, one unit of electrical energy furnishes at most only one unit of heat. Unfortunately, heat pumps have problems that do limit their usefullness. They are quite expensive to purchase compared to resistive heating elements, and, as the performance coefficient for a Carnot heat pump shows, they become less effective as the outside temperature decreases. In fact, below about -10^oC, the heat they furnish is less than the energy used to operate them.

Figure 14.16.4. A photograph of a heat pump (large box) located outside a house. This heat pump is located in a warm climate area, like the southern United States, since it would be far too inefficient located in the northern half of the United States. (credit: modification of work by Peter Stevens).

Exercise 14.16.1

A Carnot engine operates between reservoirs at 400^oC and 30^oC.

a. What is the efficiency of the engine?
b. If the engine does 5.0 J of work per cycle, how much heat per cycle does it absorb from the high-temperature reservoir?
c. How much heat per cycle does it exhaust to the cold-temperature reservoir?
d. What temperatures at the cold reservoir would give the minimum and maximum efficiency?

Answer a

$e = 1 - T_c/T_h = 0.55$

Answer b

$Q_h = eW = 9.1\ J$

Answer c

$Q_c = Q_h - W = 4.1\ J$

Answer d

-273^oC and 400^oC

Exercise 14.16.2

A Carnot refrigerator operates between two heat reservoirs whose temperatures are 0^oC and 25^oC.

a. What is the coefficient of performance of the refrigerator?
b. If 200 J of work are done on the working substance per cycle, how much heat per cycle is extracted from the cold reservoir?

c. How much heat per cycle is discarded to the hot reservoir?

Answer a

$K_R = T_c/(T_h - T_c) = 10.9$

Answer b

$Q_c = K_R W = 2.18\,kJ$

Answer c

$Q_h = Q_c + W = 2.38\,kJ$

14.17: Entropy

Learning Objectives

By the end of this section you will be able to:

- Describe the meaning of entropy
- Calculate the change of entropy for some simple processes

The second law of thermodynamics is best expressed in terms of a **change** in the thermodynamic variable known as **entropy**, which is represented by the symbol S. Entropy, like internal energy, is a state function. This means that when a system makes a transition from one state into another, the change in entropy ΔS is independent of path and depends only on the thermodynamic variables of the two states.

We first consider ΔS for a system undergoing a reversible process at a constant temperature. In this case, the change in entropy of the system is given by

$$\Delta S = \frac{Q}{T}, \tag{14.17.1}$$

where Q is the heat exchanged by the system kept at a temperature T (in kelvin). If the system absorbs heat—that is, with $Q > 0$ - the entropy of the system increases. As an example, suppose a gas is kept at a constant temperature of 300 K while it absorbs 10 J of heat in a reversible process. Then from Equation 14.17.1, the entropy change of the gas is

$$\Delta S = \frac{10\,J}{300\,K} = 0.033\,J/K.$$

Similarly, if the gas loses 5.0 J of heat; that is, $Q = -5.0\,J$, at temperature $T = 200\,K$, we have the entropy change of the system given by

$$\Delta S = \frac{-5.0\,J}{200\,K} = -0.025\,J/K.$$

Example 14.17.1: Entropy Change of Melting Ice

Heat is slowly added to a 50-g chunk of ice at 0^oC until it completely melts into water at the same temperature. What is the entropy change of the ice?

Strategy

Because the process is slow, we can approximate it as a reversible process. The temperature is a constant, and we can therefore use Equation 14.17.1in the calculation.

Solution

The ice is melted by the addition of heat:

$$Q = mL_f = 50\,g \times 335\,J/g = 16.8\,kJ.$$

In this reversible process, the temperature of the ice-water mixture is fixed at 0^oC or 273 K. Now from $\Delta S = Q/T$, the entropy change of the ice is

$$\Delta S = \frac{16.8\,kJ}{273\,K} = 61.5\,J/K$$

when it melts to water at 0^oC.

Significance

During a phase change, the temperature is constant, allowing us to use Equation 14.17.1 to solve this problem. The same equation could also be used if we changed from a liquid to a gas phase, since the temperature does not change during that process either.

The change in entropy of a system for an arbitrary, reversible transition for which the temperature is not necessarily constant is defined by modifying $\Delta S = Q/T$. Imagine a system making a transition from state **A** to **B** in small, discrete steps. The temperatures associated with these states are T_A and T_B, respectively. During each step of the transition, the system exchanges heat ΔQ_i reversibly at a temperature T_i. This can be accomplished experimentally by placing the system in thermal contact with a large number of heat reservoirs of varying temperatures, T_i, as illustrated in Figure 14.17.1 The change in entropy for each step is $\Delta S_i = Q_i/T_i$. The net change in entropy of the system for the transition is

$$\Delta S = S_B - S_A = \sum_i \Delta S_i = \sum_i \frac{\Delta Q_i}{T_i}.$$

We now take the limit as $\Delta Q_i \to 0$, and the number of steps approaches infinity. Then, replacing the summation by an integral, we obtain

$$\Delta S = S_B - S_A = \int_A^B \frac{dQ}{T}, \tag{14.17.2}$$

where the integral is taken between the initial state **A** and the final state **B**. This equation is valid only if the transition from **A** to **B** is reversible.

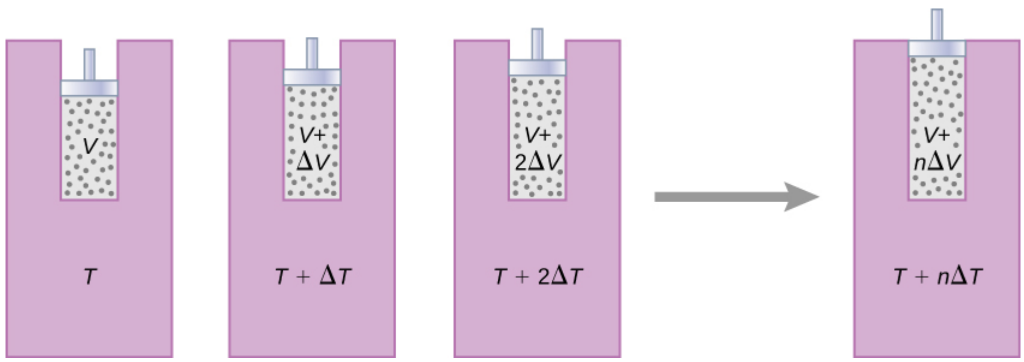


Figure 14.17.1: The gas expands at constant pressure as its temperature is increased in small steps through the use of a series of heat reservoirs.

As an example, let us determine the net entropy change of a reversible engine while it undergoes a single Carnot cycle. In the adiabatic steps 2 and 4 of the cycle shown in Figure 14.17.1, no heat exchange takes place, so $\Delta S_2 = \Delta S_4 = \int dQ/T = 0$. In step 1, the engine absorbs heat Q_h at a temperature T_h, so its entropy change is $\Delta S_1 = Q_h/T_h$. Similarly, in step 3, $\Delta S_3 = -Q_c/T_c$. The net entropy change of the engine in one cycle of operation is then

$$\Delta S_E = \Delta S_1 + \Delta S_2 + \Delta S_3 + \Delta S_4 = \frac{Q_h}{T_h} - \frac{Q_c}{T_c}.$$

However, we know that for a Carnot engine,

$$\frac{Q_h}{T_h} = \frac{Q_c}{T_c},$$

so

$$\Delta S_E = 0.$$

There is no net change in the entropy of the Carnot engine over a complete cycle. Although this result was obtained for a particular case, its validity can be shown to be far more general: There is no net change in the entropy of a system undergoing any complete reversible cyclic process. Mathematically, we write this statement as

$$\oint dS = \oint \frac{dQ}{T} = 0 \tag{14.17.3}$$

where $\oint$ represents the integral over a **closed reversible path**.

We can use Equation 14.17.3to show that the entropy change of a system undergoing a reversible process between two given states is **path independent**. An arbitrary, closed path for a reversible cycle that passes through the states **A** and **B** is shown in Figure

$14.17.2$ From Equation 14.17.3, we know that $\oint dS = 0$ for this closed path. We may split this integral into two segments, one along I, which leads from **A** to **B**, the other along II, which leads from **B** to **A**. Then

$$\left[\int_A^B dS\right]_I + \left[\int_B^A dS\right]_{II} = 0$$

Since the process is reversible,

$$\left[\int_A^B dS\right] = \left[\int_A^B dS\right]$$

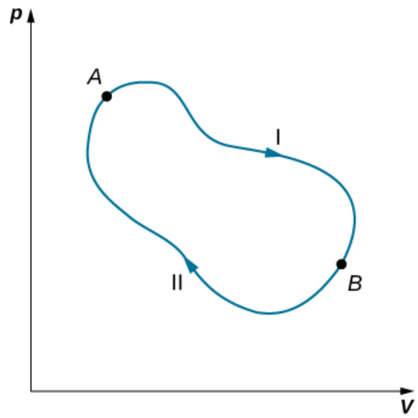


Figure 14.17.2: The closed loop passing through states A and B represents a reversible cycle.

Hence, the entropy change in going from **A** to **B** is the same for paths I and II. Since paths I and II are arbitrary, reversible paths, the entropy change in a transition between two equilibrium states is the same for all the reversible processes joining these states. Entropy, like internal energy, is therefore a state function.

What happens if the process is irreversible? When the process is irreversible, we expect the entropy of a closed system, or the system and its environment (the universe), to increase. Therefore we can rewrite this expression as

$$\Delta S \geq 0,$$

where S is the total entropy of the closed system or the entire universe, and the equal sign is for a reversible process. The fact is the entropy statement of the second law of thermodynamics:

Second Law of Thermodynamics (Entropy statement)

The entropy of a closed system and the entire universe **never** decreases.

We can show that this statement is consistent with the Kelvin statement, the Clausius statement, and the Carnot principle.

Example 14.17.2: Entropy Change of a System during an Isobaric Process

Determine the entropy change of an object of mass **m** and specific heat **c** that is cooled rapidly (and irreversibly) at constant pressure from T_h to T_c.

Strategy

The process is clearly stated as an irreversible process; therefore, we cannot simply calculate the entropy change from the actual process. However, because entropy of a system is a function of state, we can imagine a reversible process that starts from the same initial state and ends at the given final state. Then, the entropy change of the system is given by Equation 14.17.2, $\Delta S = \int_A^B dQ/T$.

Solution

To replace this rapid cooling with a process that proceeds reversibly, we imagine that the hot object is put into thermal contact with successively cooler heat reservoirs whose temperatures range from T_h to T_c. Throughout the substitute transition, the object loses infinitesimal amounts of heat dQ, so we have

$$\Delta S = \int_{T_h}^{T_c} \frac{dQ}{T}.$$

From the definition of heat capacity, an infinitesimal exchange dQ for the object is related to its temperature change dT by

$$dQ = m\,c\,dT.$$

Substituting this dQ into the expression for ΔS we obtain the entropy change of the object as it is cooled at constant pressure from T_h to T_c:

$$\Delta S = \int_{T_h}^{T_c} \frac{mcdT}{T} = mcln\frac{T_c}{T_h}.$$

Note that $\Delta S < 0$ here because $T_c > T_h$. In other words, the object has lost some entropy. But if we count whatever is used to remove the heat from the object, we would still end up with $S_{universe} > 0$ because the process is irreversible.

Significance

If the temperature changes during the heat flow, you must keep it inside the integral to solve for the change in entropy. If, however, the temperature is constant, you can simply calculate the entropy change as the heat flow divided by the temperature.

Example 14.17.3: Stirling Engine

The steps of a reversible **Stirling engine** are as follows. For this problem, we will use 0.0010 mol of a monatomic gas that starts at a temperature of 133^oC and a volume of $0.10m^3$ which will be called point **A**. Then it goes through the following steps:

1. Step **AB**: isothermal expansion at 33^oC from $0.20\,m^3$ to $0.10\,m^3$
2. Step **BC**: isochoric cooling to 33^oC
3. Step **CD**: isothermal compression at 33^oC from $0.20\,m^3$ to $0.10\,m^3$
4. Step **DA**: isochoric heating back to 133^oC and $0.10\,m^3$

(a) Draw the **pV** diagram for the Stirling engine with proper labels.

(b) Fill in the following table.

Step	**W** (J)	**Q** (J)	$\mathbf{\Delta S}$ **(J/K)**
Step **AB**			
Step **BC**			
Step **CD**			
Step **DA**			
Complete cycle			

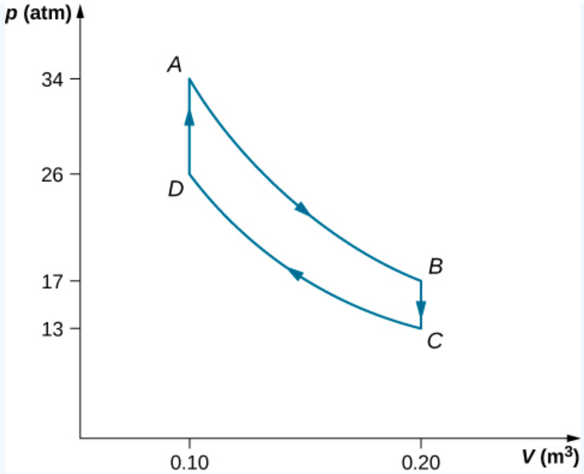
(c) How does the efficiency of the Stirling engine compare to the Carnot engine working within the same two heat reservoirs?

Strategy

Using the ideal gas law, calculate the pressure at each point so that they can be labeled on the **pV** diagram. Isothermal work is calculated using $W = nRT\,ln\left(\frac{V_2}{V_1}\right)$, and an isochoric process has no work done. The heat flow is calculated from the first law of thermodynamics, $Q = \Delta E_{int} - W$ where $\Delta E_{int} = \frac{3}{2}nR\Delta T$ for monatomic gasses. Isothermal steps have a change in entropy of **Q/T**, whereas isochoric steps have $\Delta S = \frac{3}{2}nR\,ln\left(\frac{T_2}{T_1}\right)$. The efficiency of a heat engine is calculated by using $e_{Stir} = W/Q_h$.

Solution

1. The graph is shown below.



2. The completed table is shown below.

Step	**W** (J)	**Q** (J)	**ΔS (J/K)**
Step **AB** Isotherm	2.3	2.3	0.0057
Step **BC** Isochoric	0	–1.2	0.0035
Step **CD** Isotherm	–1.8	–1.8	–0.0059
Step **DA** Isochoric	0	1.2	–0.0035
Complete cycle	0.5	0.5	~ 0

3. The efficiency of the Stirling heat engine is

$$e_{Stir} = W/Q_h = (Q_{AB} + Q_{CD})/(Q_{AB} + QDA) = 0.5/4.5 = 0.11.$$

If this were a Carnot engine operating between the same heat reservoirs, its efficiency would be

$$e_{Car} = 1 - \left(\frac{T_c}{T_h}\right) = 0.25$$

Therefore, the Carnot engine would have a greater efficiency than the Stirling engine.

Significance

In the early days of steam engines, accidents would occur due to the high pressure of the steam in the boiler. Robert Stirling developed an engine in 1816 that did not use steam and therefore was safer. The Stirling engine was commonly used in the nineteenth century, but developments in steam and internal combustion engines have made it difficult to broaden the use of the Stirling engine.

The Stirling engine uses compressed air as the working substance, which passes back and forth between two chambers with a porous plug, called the regenerator, which is made of material that does not conduct heat as well. In two of the steps, pistons in the two chambers move in phase.

14.18: Entropy on a Microscopic Scale

Learning Objectives

By the end of this section you will be able to:

- Interpret the meaning of entropy at a microscopic scale
- Calculate a change in entropy for an irreversible process of a system and contrast with the change in entropy of the universe
- Explain the third law of thermodynamics

We have seen how entropy is related to heat exchange at a particular temperature. In this section, we consider entropy from a statistical viewpoint. Although the details of the argument are beyond the scope of this textbook, it turns out that entropy can be related to how disordered or randomized a system is—the more it is disordered, the higher is its entropy. For example, a new deck of cards is very ordered, as the cards are arranged numerically by suit. In shuffling this new deck, we randomize the arrangement of the cards and therefore increase its entropy (Figure $14.18.1$). Thus, by picking one card off the top of the deck, there would be no indication of what the next selected card will be.

Figure 14.18.1: The entropy of a new deck of cards goes up after the dealer shuffles them. (credit: "Rommel SK"/YouTube)

The second law of thermodynamics requires that the entropy of the universe increase in any irreversible process. Thus, in terms of order, the second law may be stated as follows:

In any irreversible process, the universe becomes more disordered. For example, the irreversible free expansion of an ideal gas, results in a larger volume for the gas molecules to occupy. A larger volume means more possible arrangements for the same number of atoms, so disorder is also increased. As a result, the entropy of the gas has gone up. The gas in this case is a closed system, and the process is irreversible. Changes in phase also illustrate the connection between entropy and **disorder**.

Example 14.18.1: Entropy Change of the Universe

Suppose we place 50 g of ice at 0^oC in contact with a heat reservoir at 20^oC. Heat spontaneously flows from the reservoir to the ice, which melts and eventually reaches a temperature of 20^oC. Find the change in entropy of (a) the ice and (b) the universe.

Strategy

Because the entropy of a system is a function of its state, we can imagine two reversible processes for the ice: (1) ice is melted at $0^oC (T_A)$ and (2) melted ice (water) is warmed up from 9^oC to $20^oC (T_B)$ under constant pressure. Then, we add the change in entropy of the reservoir when we calculate the change in entropy of the universe.

Solution

1. From $\Delta S = S_B - S_A = \int_A^B \frac{dQ}{T}$, the increase in entropy of the ice is

$$\begin{aligned}\Delta S_{ice} &= \Delta S_1 + \Delta S_2 \\ &= \frac{mL_f}{T_A} + mc\int_A^B \frac{dT}{T} \\ &= \left(\frac{50\times 335}{273} + 50\times 4.19\times \ln\frac{293}{273}\right)\ J/K \\ &= 76.3\ J/K.\end{aligned}$$

2. During this transition, the reservoir gives the ice an amount of heat equal to

$$\begin{aligned}Q &= mL_f + mc(T_B - T_A) \\ &= 50\times(335 + 4.19\times 20)J \\ &= 2.10\times 10^4\ J.\end{aligned}$$

This leads to a change (decrease) in entropy of the reservoir:

$$\Delta S_{reservoir} = \frac{-Q}{T_B} = -71.7\ J/K.$$

The increase in entropy of the universe is therefore

$$\Delta S_{universe} = 76.3\ J/K - 71.7\ J/K = 4.6\ J/K > 0.$$

Significance

The entropy of the universe therefore is greater than zero since the ice gains more entropy than the reservoir loses. If we considered only the phase change of the ice into water and not the temperature increase, the entropy change of the ice and reservoir would be the same, resulting in the universe gaining no entropy.

This process also results in a more disordered universe. The ice changes from a solid with molecules located at specific sites to a liquid whose molecules are much freer to move. The molecular arrangement has therefore become more randomized. Although the change in average kinetic energy of the molecules of the heat reservoir is negligible, there is nevertheless a significant decrease in the entropy of the reservoir because it has many more molecules than the melted ice cube. However, the reservoir's decrease in entropy is still not as large as the increase in entropy of the ice. The increased disorder of the ice more than compensates for the increased order of the reservoir, and the entropy of the universe increases by 4.6 J/K.

You might suspect that the growth of different forms of life might be a net ordering process and therefore a violation of the second law. After all, a single cell gathers molecules and eventually becomes a highly structured organism, such as a human being. However, this ordering process is more than compensated for by the disordering of the rest of the universe. The net result is an increase in entropy and an increase in the disorder of the universe.

? Exercise 14.18.1

In Example 14.18.1, the spontaneous flow of heat from a hot object to a cold object results in a net increase in entropy of the universe. Discuss how this result can be related to an increase in disorder of the system.

Solution

When heat flows from the reservoir to the ice, the internal (mainly kinetic) energy of the ice goes up, resulting in a higher average speed and thus an average greater position variance of the molecules in the ice. The reservoir does become more ordered, but due to its much larger amount of molecules, it does not offset the change in entropy in the system.

The second law of thermodynamics makes clear that the entropy of the universe never decreases during any thermodynamic process. For any other thermodynamic system, when the process is reversible, the change of the entropy is given by $\Delta S = Q/T$. But what happens if the temperature goes to zero, $T \to 0$? It turns out this is not a question that can be answered by the second law.

A fundamental issue still remains: Is it possible to cool a system all the way down to zero kelvin? We understand that the system must be at its lowest energy state because lowering temperature reduces the kinetic energy of the constituents in the system. What happens to the entropy of a system at the absolute zero temperature? It turns out the absolute zero temperature is not reachable—at least, not though a finite number of cooling steps. This is a statement of the **third law of thermodynamics**, whose proof requires quantum mechanics that we do not present here. In actual experiments, physicists have continuously pushed that limit downward, with the lowest temperature achieved at about $1 \times 10^{-10} K$ in a low-temperature lab at the Helsinki University of Technology in 2008.

Like the second law of thermodynamics, the third law of thermodynamics can be stated in different ways. One of the common statements of the third law of thermodynamics is:

> The absolute zero temperature cannot be reached through any finite number of cooling steps.

In other words, the temperature of any given physical system must be finite, that is, $T > 0$. This produces a very interesting question in physics: Do we know how a system would behave if it were at the absolute zero temperature?

The reason a system is unable to reach 0 K is fundamental and requires quantum mechanics to fully understand its origin. But we can certainly ask what happens to the entropy of a system when we try to cool it down to 0 K. Because the amount of heat that can be removed from the system becomes vanishingly small, we expect that the change in entropy of the system along an isotherm approaches zero, that is

$$\lim_{T \to 0} (\Delta S)_T = 0$$

This can be viewed as another statement of the third law, with all the isotherms becoming **isentropic**, or into a reversible ideal adiabat. We can put this expression in words:

> A system becomes perfectly ordered when its temperature approaches absolute zero and its entropy approaches its absolute minimum.

The third law of thermodynamics puts another limit on what can be done when we look for energy resources. If there could be a reservoir at the absolute zero temperature, we could have engines with efficiency of 100%, which would, of course, violate the second law of thermodynamics.

Example 14.18.2: Entropy Change of an Ideal Gas in Free Expansion

An ideal gas occupies a partitioned volume V_1 inside a box whose walls are thermally insulating, as shown in Figure $14.18.2a$ When the partition is removed, the gas expands and fills the entire volume V_2 of the box, as shown in Figure $14.18.2b$ What is the entropy change of the universe (the system plus its environment)?

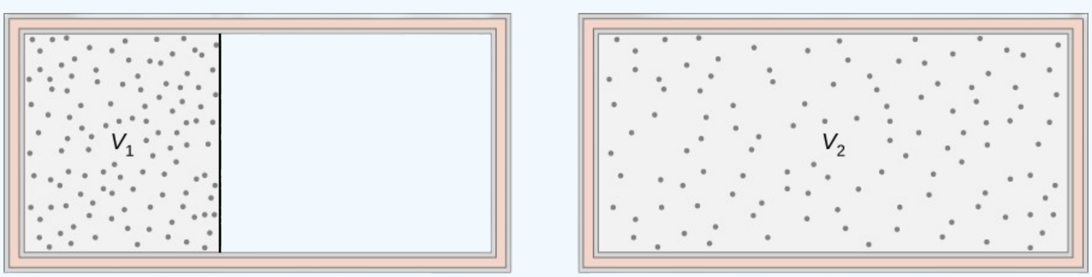


Figure 14.18.2: The adiabatic free expansion of an ideal gas from volume V_1 to volume V_2.

Strategy

The adiabatic free expansion of an ideal gas is an irreversible process. There is no change in the internal energy (and hence temperature) of the gas in such an expansion because no work or heat transfer has happened. Thus, a convenient reversible path connecting the same two equilibrium states is a slow, isothermal expansion from V_1 to V_2. In this process, the gas could be expanding against a piston while in thermal contact with a heat reservoir, as in step 1 of the Carnot cycle.

Solution

Since the temperature is constant, the entropy change is given by $\Delta S = Q/T$, where

$$Q = W = \int_{V_1}^{V_2} p\,dV$$

because $\Delta E_{int} = 0$. Now, with the help of the ideal gas law, we have

$$\Delta S = \frac{Q}{T} = nR\,\ln\frac{V_2}{V_1}.$$

Because $V_2 > V_1$, ΔS is positive, and the entropy of the gas has gone up during the free expansion.

Significance

What about the environment? The walls of the container are thermally insulating, so no heat exchange takes place between the gas and its surroundings. The entropy of the environment is therefore constant during the expansion. The net entropy change of the universe is then simply the entropy change of the gas. Since this is positive, the entropy of the universe increases in the free expansion of the gas.

Example 14.18.3: Entropy Change during Heat Transfer

Heat flows from a steel object of mass 4.00 kg whose temperature is 400 K to an identical object at 300 K. Assuming that the objects are thermally isolated from the environment, what is the net entropy change of the universe after thermal equilibrium has been reached?

Strategy

Since the objects are identical, their common temperature at equilibrium is 350 K. To calculate the entropy changes associated with their transitions, we substitute the irreversible process of the heat transfer by two isobaric, reversible processes, one for each of the two objects. The entropy change for each object is then given by $\Delta S = mc\,\ln(T_B/T_A)$.

Solution

Using $c = 450\ J/kg\cdot K$, the specific heat of steel, we have for the hotter object

$$\begin{aligned}\Delta S_h &= \int_{T_1}^{T_2}\frac{mc\,dT}{T} \\ &= mc\,\ln\left(\frac{T_2}{T_1}\right) \\ &= (4.00\,kg)(450\,J/kg\cdot K)\ln\frac{350\,K}{400\,K} = -240\,J/K.\end{aligned}$$

Similarly, the entropy change of the cooler object is

$$\Delta S_c = (4.00\,kg)(450\,J/kg\cdot K)\ln\frac{350\,K}{300\,K} = 277\,J/K.$$

The net entropy change of the two objects during the heat transfer is then

$$\Delta S_h + \Delta S_c = 37\,J/K.$$

Significance

The objects are thermally isolated from the environment, so its entropy must remain constant. Thus, the entropy of the universe also increases by 37 J/K.

Exercise 14.18.1

A quantity of heat **Q** is absorbed from a reservoir at a temperature T_h by a cooler reservoir at a temperature T_c. What is the entropy change of the hot reservoir, the cold reservoir, and the universe?

Answer

$-Q/T_h$; Q/T_c; and $Q(T_h - T_c)(T_hT_c)$

> **Exercise 14.18.2**
>
> A 50-g copper piece at a temperature of 20^oC is placed into a large insulated vat of water at 100^oC.
>
> a. What is the entropy change of the copper piece when it reaches thermal equilibrium with the water?
> b. What is the entropy change of the water?
> c. What is the entropy change of the universe?
>
> **Answer**
>
> a. 4.71 J/K; b. −4.18 J/K; c. 0.53 J/K

> View this site to learn about entropy and microstates. Start with a large barrier in the middle and 1000 molecules in only the left chamber. What is the total entropy of the system? Now remove the barrier and let the molecules travel from the left to the right hand side? What is the total entropy of the system now? Lastly, add heat and note what happens to the temperature. Did this increase entropy of the system?

14.19: The Second Law of Thermodynamics (Answer)

Check Your Understanding

4.1. A perfect heat engine would have $Q_c = 0$, which would lead to $e = 1 - Q_c/Q_h = 1$. A perfect refrigerator would need zero work, that is, $W = 0$, which leads to $K_R = Q_c/W \to \infty$.

4.2. From the engine on the right, we have $W = Q'_h - Q'_c$. From the refrigerator on the right, we have $Q_h = Q_c + W$. Thus, $W = Q'_h - Q'_c = Q_h - Q_c$.

4.3. a. $e = 1 - T_c/T_h = 0.55$;

b. $Q_h = eW = 9.1J$;

c. $Q_c = Q_h - W = 4.1J$;

d. $-273^\circ C$ and $400^\circ C$

4.4. a. $K_R = T_c/(T_h - T_c) = 10.9$;

b. $Q_c = K_R W = 2.18kJ$;

c. $Q_h = Q_c + W = 2.38kJ$

4.5. When heat flows from the reservoir to the ice, the internal (mainly kinetic) energy of the ice goes up, resulting in a higher average speed and thus an average greater position variance of the molecules in the ice. The reservoir does become more ordered, but due to its much larger amount of molecules, it does not offset the change in entropy in the system.

4.6. $-Q/T_h$; Q/T_c; and $Q(T_h - T_c)/(T_h T_c)$

4.7. a. 4.71 J/K;

b. −4.18 J/K;

c. 0.53 J/K

Conceptual Questions

1. Some possible solutions are frictionless movement; restrained compression or expansion; energy transfer as heat due to infinitesimal temperature nonuniformity; electric current flow through a zero resistance; restrained chemical reaction; and mixing of two samples of the same substance at the same state.

3. The temperature increases since the heat output behind the refrigerator is greater than the cooling from the inside of the refrigerator.

5. If we combine a perfect engine and a real refrigerator with the engine converting heat **Q** from the hot reservoir into work $W = Q$ to drive the refrigerator, then the heat dumped to the hot reservoir by the refrigerator will be $W + \Delta Q$, resulting in a perfect refrigerator transferring heat ΔQ from the cold reservoir to hot reservoir without any other effect.

7. Heat pumps can efficiently extract heat from the ground to heat on cooler days or pull heat out of the house on warmer days. The disadvantage of heat pumps are that they are more costly than alternatives, require maintenance, and will not work efficiently when temperature differences between the inside and outside are very large. Electric heating is much cheaper to purchase than a heat pump; however, it may be more costly to run depending on the electric rates and amount of usage.

9. A nuclear reactor needs to have a lower temperature to operate, so its efficiency will not be as great as a fossil-fuel plant. This argument does not take into consideration the amount of energy per reaction: Nuclear power has a far greater energy output than fossil fuels.

11. In order to increase the efficiency, the temperature of the hot reservoir should be raised, and the cold reservoir should be lowered as much as possible. This can be seen in Equation 4.3.

13. adiabatic and isothermal processes

15. Entropy will not change if it is a reversible transition but will change if the process is irreversible.

17. Entropy is a function of disorder, so all the answers apply here as well.

Problems

19. $11.0 \times 10^3 J$

21. $4.5 p V_0$

23. 0.667

25. a. 0.200;

b. 25.0 J

27. a. 0.67;

b. 75 J;

c. 25 J

29. a. 600 J;

b. 800 J

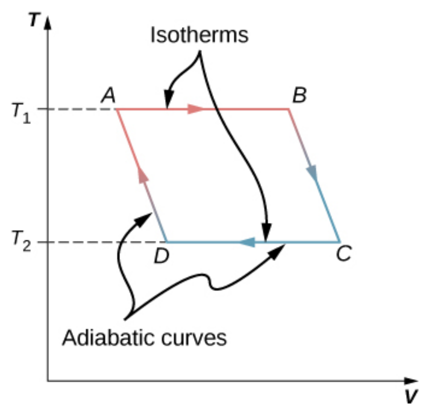
31. a. 69 J;

b. 11 J

33. 2.0

35. 50 J

37.



39. a. 381 J;

b. 619 J

41. a. 546 K;

b. 137 K

43. –1 J/K

45. –13 J(K mole)

47. $-\frac{Q}{T_h}, \frac{Q}{T_c}, Q(\frac{1}{T_c} - \frac{1}{T_h})$

49. a. –709 J/K;

b. 1300 J/K;

c. 591 J/K

51. a. $Q = nR\Delta T$;

b. $S = nRln(T_2/T_1)$

53. $3.78 \times 10^{-3} W/K$

55. 430 J/K

57. $80^\circ C, 80^\circ C, 6.70 \times 10^4 J$, 215 J/K, –190 J/K, 25 J/K

59. $\Delta S_{H2O} = 215 J/K, \Delta S_R = -208 J/K, \Delta S_U = 7 J/K$

61. a. 1200 J;

b. 600 J;

c. 600 J;

d. 0.50

63. $\Delta S = nC_V ln(\frac{T_2}{T_1}) + nC_p ln(\frac{T_3}{T_2})$

65. a. 0.33, 0.39;

b. 0.91

Additional Problems

67. $1.45 \times 10^7 J$

69. a. $V_B = 0.042 m^3, V_D = 0.018 m^3$;

b. 13,000 J;

c. 13,000 J;

d. –8,000 J;

e. –8,000 J;

f. 6200 J;

g. –6200 J;

h. **39%**; with temperatures efficiency is **40%**, which is off likely by rounding errors.

71. –670 J/K

73. a. –570 J/K;

b. 570 J/K

75. 82 J/K

77. a. 2000 J;

b. **40%**

79. 60%

81. 64.4%

Challenge Problems

83. derive

85.derive

87. 18 J/K

89. proof

91. $K_R = \frac{3(p_1 - p_2)V_1}{5p_2V_3 - 3p_1V_1 - p_2V_1}$

93. $W = 110,000 J$

14.20: The Second Law of Thermodynamics (Exercise)

Conceptual Questions

4.2 Reversible and Irreversible Processes

1. State an example of a process that occurs in nature that is as close to reversible as it can be.

4.3 Heat Engines

2. Explain in practical terms why efficiency is defined as W/Q_h.

4.4 Refrigerators and Heat Pumps

3. If the refrigerator door is left open, what happens to the temperature of the kitchen?

4. Is it possible for the efficiency of a reversible engine to be greater than 1.0? Is it possible for the coefficient of performance of a reversible refrigerator to be less than 1.0?

4.5 Statements of the Second Law of Thermodynamics

5. In the text, we showed that if the Clausius statement is false, the Kelvin statement must also be false. Now show the reverse, such that if the Kelvin statement is false, it follows that the Clausius statement is false.

6. Why don't we operate ocean liners by extracting heat from the ocean or operate airplanes by extracting heat from the atmosphere?

7. Discuss the practical advantages and disadvantages of heat pumps and electric heating.

8. The energy output of a heat pump is greater than the energy used to operate the pump. Why doesn't this statement violate the first law of thermodynamics?

9. Speculate as to why nuclear power plants are less efficient than fossil-fuel plants based on temperature arguments.

10. An ideal gas goes from state (p_i, V_i) to state (p_f, V_f) when it is allowed to expand freely. Is it possible to represent the actual process on a pV diagram? Explain.

4.6 The Carnot Cycle

11. To increase the efficiency of a Carnot engine, should the temperature of the hot reservoir be raised or lowered? What about the cold reservoir?

12. How could you design a Carnot engine with $\mathbf{100\%}$ efficiency?

13. What type of processes occur in a Carnot cycle?

4.7 Entropy

14. Does the entropy increase for a Carnot engine for each cycle?

15. Is it possible for a system to have an entropy change if it neither absorbs nor emits heat during a reversible transition? What happens if the process is irreversible?

4.8 Entropy on a Microscopic Scale

16. Are the entropy changes of the systems in the following processes positive or negative?

(a) water vapor that condenses on a cold surface;

(b) gas in a container that leaks into the surrounding atmosphere;

(c) an ice cube that melts in a glass of lukewarm water;

(d) the lukewarm water of part (c);

(e) a real heat engine performing a cycle; (f) food cooled in a refrigerator.

17. Discuss the entropy changes in the systems of Question 21.10 in terms of disorder.

Problems

4.2 Reversible and Irreversible Processes

18. A tank contains 111.0 g chlorine gas (Cl_2), which is at temperature **82.0°C** and absolute pressure $5.70 \times 10^5 Pa$. The temperature of the air outside the tank is **20.0°C**. The molar mass of Cl_2 is 70.9 g/mol.

(a) What is the volume of the tank?

(b) What is the internal energy of the gas?

(c) What is the work done by the gas if the temperature and pressure inside the tank drop to **31.0°C** and $3.80 \times 10^5 Pa$, respectively, due to a leak?

19. A mole of ideal monatomic gas at **0°C** and 1.00 atm is warmed up to expand isobarically to triple its volume. How much heat is transferred during the process?

20. A mole of an ideal gas at pressure 4.00 atm and temperature 298 K expands isothermally to double its volume. What is the work done by the gas?

21. After a free expansion to quadruple its volume, a mole of ideal diatomic gas is compressed back to its original volume adiabatically and then cooled down to its original temperature. What is the minimum heat removed from the gas in the final step to restoring its state?

4.3 Heat Engines

22. An engine is found to have an efficiency of 0.40. If it does 200 J of work per cycle, what are the corresponding quantities of heat absorbed and discharged?

23. In performing 100.0 J of work, an engine discharges 50.0 J of heat. What is the efficiency of the engine?

24. An engine with an efficiency of 0.30 absorbs 500 J of heat per cycle.

(a) How much work does it perform per cycle?

(b) How much heat does it discharge per cycle?

25. It is found that an engine discharges 100.0 J while absorbing 125.0 J each cycle of operation.

(a) What is the efficiency of the engine?

(b) How much work does it perform per cycle?

26. The temperature of the cold reservoir of the engine is 300 K. It has an efficiency of 0.30 and absorbs 500 J of heat per cycle.

(a) How much work does it perform per cycle?

(b) How much heat does it discharge per cycle?

27. An engine absorbs three times as much heat as it discharges. The work done by the engine per cycle is 50 J. Calculate

(a) the efficiency of the engine,

(b) the heat absorbed per cycle, and

(c) the heat discharged per cycle.

28. A coal power plant consumes 100,000 kg of coal per hour and produces 500 MW of power. If the heat of combustion of coal is 30 MJ/kg, what is the efficiency of the power plant?

4.4 Refrigerators and Heat Pumps

29. A refrigerator has a coefficient of performance of 3.0.

(a) If it requires 200 J of work per cycle, how much heat per cycle does it remove the cold reservoir?

(b) How much heat per cycle is discarded to the hot reservoir?

30. During one cycle, a refrigerator removes 500 J from a cold reservoir and discharges 800 J to its hot reservoir.

(a) What is its coefficient of performance?

(b) How much work per cycle does it require to operate?

31. If a refrigerator discards 80 J of heat per cycle and its coefficient of performance is 6.0, what are

(a) the quantity off heat it removes per cycle from a cold reservoir and

(b) the amount of work per cycle required for its operation?

32. A refrigerator has a coefficient of performance of 3.0.

(a) If it requires 200 J of work per cycle, how much heat per cycle does it remove the cold reservoir?

(b) How much heat per cycle is discarded to the hot reservoir?

4.6 The Carnot Cycle

33. The temperature of the cold and hot reservoirs between which a Carnot refrigerator operates are **−73°C** and **270°C**, respectively. Which is its coefficient of performance?

34. Suppose a Carnot refrigerator operates between T_c and T_h. Calculate the amount of work required to extract 1.0 J of heat from the cold reservoir if

(a) $T_c = 7^\circ, T_h = 27^\circ C$;

(b) $T_c = -73^\circ C, T_h = 27^\circ C$;

(c) $T_c = -173^\circ C, T_h = 27^\circ C$; and

(d) $T_c = -273^\circ C, T_h = 27^\circ C$.

35. A Carnot engine operates between reservoirs at 600 and 300 K. If the engine absorbs 100 J per cycle at the hot reservoir, what is its work output per cycle?

36. A 500-W motor operates a Carnot refrigerator between **−5°C** and **30°C**.

(a) What is the amount of heat per second extracted from the inside of the refrigerator?

(b) How much heat is exhausted to the outside air per second?

37. Sketch a Carnot cycle on a temperature-volume diagram.

38. A Carnot heat pump operates between **0°C** and **20°C**. How much heat is exhausted into the interior of a house for every 1.0 J of work done by the pump?

39. An engine operating between heat reservoirs at **20°C** and **200°C** extracts 1000 J per cycle from the hot reservoir.

(a) What is the maximum possible work that engine can do per cycle?

(b) For this maximum work, how much heat is exhausted to the cold reservoir per cycle?

40. Suppose a Carnot engine can be operated between two reservoirs as either a heat engine or a refrigerator. How is the coefficient of performance of the refrigerator related to the efficiency of the heat engine?

41. A Carnot engine is used to measure the temperature of a heat reservoir. The engine operates between the heat reservoir and a reservoir consisting of water at its triple point.

(a) If 400 J per cycle are removed from the heat reservoir while 200 J per cycle are deposited in the triple-point reservoir, what is the temperature of the heat reservoir?

(b) If 400 J per cycle are removed from the triple-point reservoir while 200 J per cycle are deposited in the heat reservoir, what is the temperature of the heat reservoir?

42. What is the minimum work required of a refrigerator if it is to extract 50 J per cycle from the inside of a freezer at **−10°C** and exhaust heat to the air at **25°C**?

4.7 Entropy

43. Two hundred joules of heat are removed from a heat reservoir at a temperature of 200 K. What is the entropy change of the reservoir?

44. In an isothermal reversible expansion at **27°C**, an ideal gas does 20 J of work. What is the entropy change of the gas?

45. An ideal gas at 300 K is compressed isothermally to one-fifth its original volume. Determine the entropy change per mole of the gas.

46. What is the entropy change of 10 g of steam at **100°C** when it condenses to water at the same temperature?

47. A metal rod is used to conduct heat between two reservoirs at temperatures T_h and T_c, respectively. When an amount of heat Q flows through the rod from the hot to the cold reservoir, what is the net entropy change of the rod, the hot reservoir, the cold reservoir, and the universe?

48. For the Carnot cycle of Figure 4.12, what is the entropy change of the hot reservoir, the cold reservoir, and the universe?

49. A 5.0-kg piece of lead at a temperature of **600°C** is placed in a lake whose temperature is **15°C**. Determine the entropy change of (a) the lead piece, (b) the lake, and (c) the universe.

50. One mole of an ideal gas doubles its volume in a reversible isothermal expansion.

(a) What is the change in entropy of the gas?

(b) If 1500 J of heat are added in this process, what is the temperature of the gas?

51. One mole of an ideal monatomic gas is confined to a rigid container. When heat is added reversibly to the gas, its temperature changes from T_1 to T_2.

(a) How much heat is added?

(b) What is the change in entropy of the gas?

52. (a) A 5.0-kg rock at a temperature of **20°C** is dropped into a shallow lake also at **20°C** from a height of $1.0 \times 10^3 m$. What is the resulting change in entropy of the universe?

(b) If the temperature of the rock is 100°C100°Cwhen it is dropped, what is the change of entropy of the universe? Assume that air friction is negligible (not a good assumption) and that **c=860J/kg·K** is the specific heat of the rock.

4.8 Entropy on a Microscopic Scale

53. A copper rod of cross-sectional area $5.0cm^2$ and length 5.0 m conducts heat from a heat reservoir at 373 K to one at 273 K. What is the time rate of change of the universe's entropy for this process?

54. Fifty grams of water at **20°C** is heated until it becomes vapor at **100°C**. Calculate the change in entropy of the water in this process.

55. Fifty grams of water at **0°C** are changed into vapor at **100°C**. What is the change in entropy of the water in this process?

56. In an isochoric process, heat is added to 10 mol of monoatomic ideal gas whose temperature increases from 273 to 373 K. What is the entropy change of the gas?

57. Two hundred grams of water at **0°C** is brought into contact with a heat reservoir at **80°C**. After thermal equilibrium is reached, what is the temperature of the water? Of the reservoir? How much heat has been transferred in the process? What is the entropy change of the water? Of the reservoir? What is the entropy change of the universe?

58. Suppose that the temperature of the water in the previous problem is raised by first bringing it to thermal equilibrium with a reservoir at a temperature of **40°C** and then with a reservoir at **80°C**. Calculate the entropy changes of (a) each reservoir, (b) of the water, and (c) of the universe.

59. Two hundred grams of water at **0°C** is brought into contact into thermal equilibrium successively with reservoirs at **20°C**, **40°C**, **60°C**, and **80°C**.

(a) What is the entropy change of the water?

(b) Of the reservoir?

(c) What is the entropy change of the universe?

60. (a) Ten grams of H_2O starts as ice at **0°C**. The ice absorbs heat from the air (just above **0°C**) until all of it melts. Calculate the entropy change of the H_2O, of the air, and of the universe.

(b) Suppose that the air in part (a) is at **20°C** rather than **0°C** and that the ice absorbs heat until it becomes water at **20°C**. Calculate the entropy change of the H_2O, of the air, and of the universe.

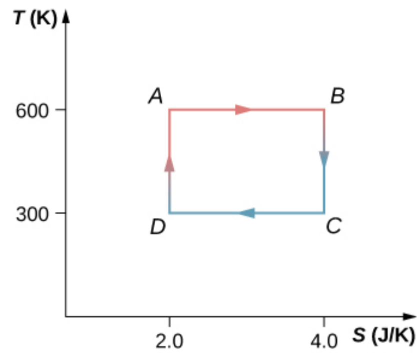
(c) Is either of these processes reversible?

61. The Carnot cycle is represented by the temperature-entropy diagram shown below.

(a) How much heat is absorbed per cycle at the high-temperature reservoir?

(b) How much heat is exhausted per cycle at the low-temperature reservoir?

(c) How much work is done per cycle by the engine? (d) What is the efficiency of the

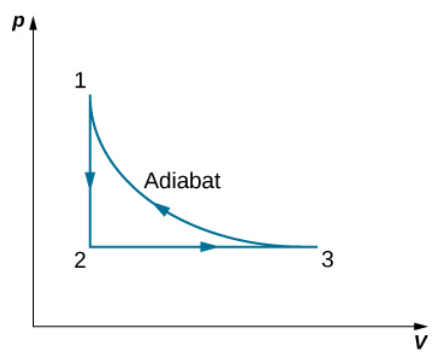


62. A Carnot engine operating between heat reservoirs at 500 and 300 K absorbs 1500 J per cycle at the high-temperature reservoir.

(a) Represent the engine's cycle on a temperature-entropy diagram.

(b) How much work per cycle is done by the engine?

63. A monoatomic ideal gas (n moles) goes through a cyclic process shown below. Find the change in entropy of the gas in each step and the total entropy change over the entire cycle.



64. A Carnot engine has an efficiency of 0.60. When the temperature of its cold reservoir changes, the efficiency drops to 0.55. If initially $T_c = 27^\circ C$, determine

(a) the constant value of T_h and

(b) the final value of T_c.

65. A Carnot engine performs 100 J of work while discharging 200 J of heat each cycle. After the temperature of the hot reservoir only is adjusted, it is found that the engine now does 130 J of work while discarding the same quantity of heat.

(a) What are the initial and final efficiencies of the engine?

(b) What is the fractional change in the temperature of the hot reservoir?

66. A Carnot refrigerator exhausts heat to the air, which is at a temperature of **25°C**. How much power is used by the refrigerator if it freezes 1.5 g of water per second? Assume the water is at **0°C.**

Additional Problems

67. A 300-W heat pump operates between the ground, whose temperature is **0°C**, and the interior of a house at **22°C**. What is the maximum amount of heat per hour that the heat pump can supply to the house?

68. An engineer must design a refrigerator that does 300 J of work per cycle to extract 2100 J of heat per cycle from a freezer whose temperature is **−10°C**. What is the maximum air temperature for which this condition can be met? Is this a reasonable condition to impose on the design?

69. A Carnot engine employs 1.5 mol of nitrogen gas as a working substance, which is considered as an ideal diatomic gas with **γ=7.5** at the working temperatures of the engine. The Carnot cycle goes in the cycle ABCDAwith AB being an isothermal expansion. The volume at points A and C of the cycle are $5.0 \times 10^{-3} m^3$ and 0.15 L, respectively. The engine operates between two thermal baths of temperature 500 K and 300 K.

(a) Find the values of volume at B and D.

(b) How much heat is absorbed by the gas in the AB isothermal expansion?

(c) How much work is done by the gas in the AB isothermal expansion?

(d) How much heat is given up by the gas in the CDisothermal expansion?

(e) How much work is done by the gas in the CD isothermal compression?

(f) How much work is done by the gas in the BC adiabatic expansion?

(g) How much work is done by the gas in the DA adiabatic compression?

(h) Find the value of efficiency of the engine based on the net work and heat input. Compare this value to the efficiency of a Carnot engine based on the temperatures of the two baths.

70. A 5.0-kg wood block starts with an initial speed of 8.0 m/s and slides across the floor until friction stops it. Estimate the resulting change in entropy of the universe. Assume that everything stays at a room temperature of **20°C**.

71. A system consisting of 20.0 mol of a monoatomic ideal gas is cooled at constant pressure from a volume of 50.0 L to 10.0 L. The initial temperature was 300 K. What is the change in entropy of the gas?

72. A glass beaker of mass 400 g contains 500 g of water at **27°C**. The beaker is heated reversibly so that the temperature of the beaker and water rise gradually to **57°C**. Find the change in entropy of the beaker and water together.

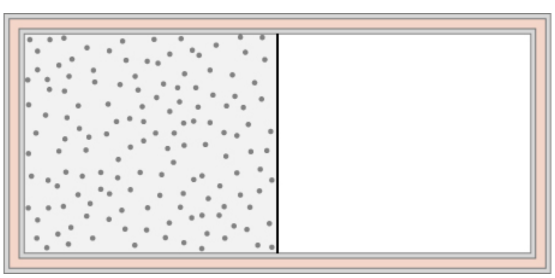
73. A Carnot engine operates between **550°C** and **20°C** baths and produces 300 kJ of energy in each cycle. Find the change in entropy of the (a) hot bath and (b) cold bath, in each Carnot cycle?

74. An ideal gas at temperature T is stored in the left half of an insulating container of volume V using a partition of negligible volume (see below). What is the entropy change per mole of the gas in each of the following cases?

(a) The partition is suddenly removed and the gas quickly fills the entire container.

(b) A tiny hole is punctured in the partition and after a long period, the gas reaches an equilibrium state such that there is no net flow through the hole.

(c) The partition is moved very slowly and adiabatically all the way to the right wall so that the gas finally fills the entire container.



75. A 0.50-kg piece of aluminum at **250°C** is dropped into 1.0 kg of water at **20°C**. After equilibrium is reached, what is the net entropy change of the system?

76. Suppose 20 g of ice at **0°C** is added to 300 g of water at **60°C**. What is the total change in entropy of the mixture after it reaches thermal equilibrium?

77. A heat engine operates between two temperatures such that the working substance of the engine absorbs 5000 J of heat from the high-temperature bath and discharges 3000 J to the low-temperature bath. The rest of the energy is converted into mechanical energy of the turbine. Find

(a) the amount of work produced by the engine and

(b) the efficiency of the engine.

78. A thermal engine produces 4 MJ of electrical energy while operating between two thermal baths of different temperatures. The working substance of the engine discharges 5 MJ of heat to the cold temperature bath. What is the efficiency of the engine?

79. A coal power plant consumes 100,000 kg of coal per hour and produces 500 MW of power. If the heat of combustion of coal is 30 MJ/kg, what is the efficiency of the power plant?

80. A Carnot engine operates in a Carnot cycle between a heat source at **550°C**550°C and a heat sink at **20°C**.20°C. Find the efficiency of the Carnot engine.

81. A Carnot engine working between two heat baths of temperatures 600 K and 273 K completes each cycle in 5 sec. In each cycle, the engine absorbs 10 kJ of heat. Find the power of the engine.

82. A Carnot cycle working between **100°C** and **30°C** is used to drive a refrigerator between **−10°C** and **30°C.**How much energy must the Carnot engine produce per second so that the refrigerator is able to discard 10 J of energy per second?

Challenge Problems

83. (a) An infinitesimal amount of heat is added reversibly to a system. By combining the first and second laws, show that $dU = TdS - dW$. (b) When heat is added to an ideal gas, its temperature and volume change from T_1 and V_1 to T_2 and V_2. Show that the entropy change of n moles of the gas is given by $\Delta S = nC_v \ln \frac{T_2}{T_1} + nR \ln \frac{V_2}{V_1}$.

84. Using the result of the preceding problem, show that for an ideal gas undergoing an adiabatic process, $TV^{\gamma-1}$is constant.

85. With the help of the two preceding problems, show that **ΔS** between states 1 and 2 of n moles an ideal gas is given by $\Delta S = nC_p \ln \frac{T_2}{T_1} - nRln \frac{p_2}{p_1}$.

86. A cylinder contains 500 g of helium at 120 atm and **20°C**. The valve is leaky, and all the gas slowly escapes isothermally into the atmosphere. Use the results of the preceding problem to determine the resulting change in entropy of the universe.

87. A diatomic ideal gas is brought from an initial equilibrium state at $p_1 = 0.50 atm$ and $T_1 = 300K$ to a final stage with $p_2 = 0.20 atm$ and $T_1 = 500K$. Use the results of the previous problem to determine the entropy change per mole of the gas.

88. The gasoline internal combustion engine operates in a cycle consisting of six parts. Four of these parts involve, among other things, friction, heat exchange through finite temperature differences, and accelerations of the piston; it is irreversible. Nevertheless, it is represented by the ideal reversible **Otto cycle**, which is illustrated below. The working substance of the cycle is assumed to be air. The six steps of the **Otto cycle** are as follows:

i. Isobaric intake stroke (**OA**). A mixture of gasoline and air is drawn into the combustion chamber at atmospheric pressure p_0 as the piston expands, increasing the volume of the cylinder from zero to V_A.

ii. Adiabatic compression stroke (**AB**). The temperature of the mixture rises as the piston compresses it adiabatically from a volume V_A to V_B.

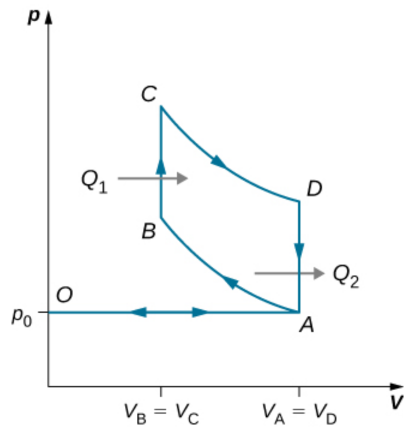
iii. Ignition at constant volume (**BC**). The mixture is ignited by a spark. The combustion happens so fast that there is essentially no motion of the piston. During this process, the added heat Q_1 causes the pressure to increase from p_B to p_CpBtopC at the constant volume $V_B (= V_C)$.

iv. Adiabatic expansion (**CD**). The heated mixture of gasoline and air expands against the piston, increasing the volume from V_C to V_D. This is called the power stroke, as it is the part of the cycle that delivers most of the power to the crankshaft.

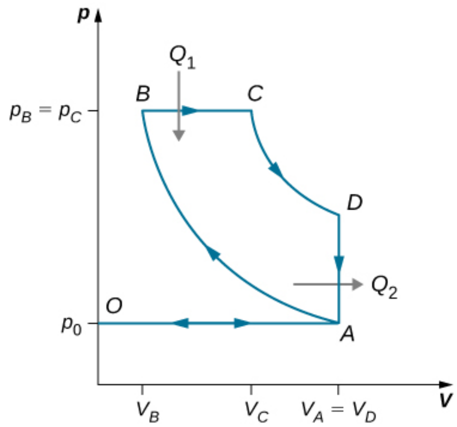
v. Constant-volume exhaust (**DA**). When the exhaust valve opens, some of the combustion products escape. There is almost no movement of the piston during this part of the cycle, so the volume remains constant at $V_A(=V_D)$. Most of the available energy is lost here, as represented by the heat exhaust Q_2.

vi. Isobaric compression (**AO**). The exhaust valve remains open, and the compression from V_A to zero drives out the remaining combustion products.

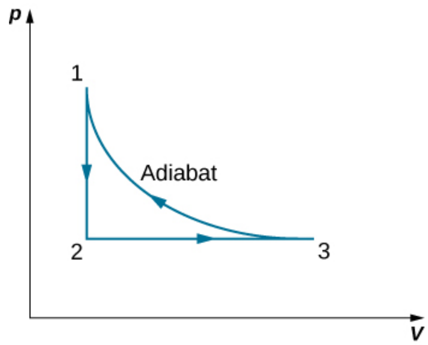
(a) Using (i) $e=W/Q_1$; (ii) $W=Q_1-Q_2$; and (iii) $Q_1=nC_v(T_C-T_B), Q_2=nC_v(T_D-T_A)$, show that $e=1-\frac{T_D-T_A}{T_C-T_B}$. (b) Use the fact that steps (ii) and (iv) are adiabatic to show that $e=1-\frac{1}{r^{\gamma-1}}$, where $r=V_A/V_B$. The quantity **r** is called the compression ratio of the engine. (c) In practice, **r** is kept less than around 7. For larger values, the gasoline-air mixture is compressed to temperatures so high that it explodes before the finely timed spark is delivered. This preignition causes engine knock and loss of power. Show that for **r=6** and **γ=1.4** (the value for air), **e=0.51**, or an efficiency of **51%**. Because of the many irreversible processes, an actual internal combustion engine has an efficiency much less than this ideal value. A typical efficiency for a tuned engine is about **25%** to **30%**.



89. An ideal **diesel cycle** is shown below. This cycle consists of five strokes. In this case, only air is drawn into the chamber during the intake stroke OA. The air is then compressed adiabatically from state **A** to state **B**, raising its temperature high enough so that when fuel is added during the power stroke **BC**, it ignites. After ignition ends at **C**, there is a further adiabatic power stroke **CD**. Finally, there is an exhaust at constant volume as the pressure drops from p_D to p_A, followed by a further exhaust when the piston compresses the chamber volume to zero. (a) Use $W=Q_1-Q_2, Q_1=nC_p(T_C-T_B)$, and $Q_2=nC_v(T_D-T_A)$ to show that $e=\frac{W}{Q_1}=1-\frac{T_D-T_A}{\gamma(T_C-T_B)}$. (b) Use the fact that **A→B** and **C→D** are adiabatic to show that $e=1-\frac{1}{\gamma}\frac{(\frac{V_C}{V_D})^{\gamma}-(\frac{V_B}{V_A})^{\gamma}}{(\frac{V_C}{V_D})-(\frac{V_B}{V_A})}$ (c) Since there is no preignition (remember, the chamber does not contain any fuel during the compression), the compression ratio can be larger than that for a gasoline engine. Typically, $V_A/V_B=15$ and $V_D/V_C=5$. For these values and **γ=1.4**,show that **ε=0.56**, or an efficiency of **56%**. Diesel engines actually operate at an efficiency of about **30%** to **35%** compared with **25%** to **30%** for gasoline engines.



91. Derive a formula for the coefficient of performance of a refrigerator using an ideal gas as a working substance operating in the cycle shown below in terms of the properties of the three states labeled 1, 2, and 3.



92. Two moles of nitrogen gas, with **γ=7/5** for ideal diatomic gases, occupies a volume of $10^{-2} m^3$ in an insulated cylinder at temperature 300 K. The gas is adiabatically and reversibly compressed to a volume of 5 L. The piston of the cylinder is locked in its place, and the insulation around the cylinder is removed. The heat-conducting cylinder is then placed in a 300-K bath. Heat from the compressed gas leaves the gas, and the temperature of the gas becomes 300 K again. The gas is then slowly expanded at the fixed temperature 300 K until the volume of the gas becomes $10^{-2} m^3$, thus making a complete cycle for the gas. For the entire cycle, calculate (a) the work done by the gas, (b) the heat into or out of the gas, (c) the change in the internal energy of the gas, and (d) the change in entropy of the gas.

93. A Carnot refrigerator, working between **0°C** and **30°C** is used to cool a bucket of water containing $10^{-2} m^3$ of water at **30°C** to **5°C** in 2 hours. Find the total amount of work needed.

14.21: The Second Law of Thermodynamics (Summary)

Key Terms

Carnot cycle	cycle that consists of two isotherms at the temperatures of two reservoirs and two adiabatic processes connecting the isotherms
Carnot engine	Carnot heat engine, refrigerator, or heat pump that operates on a Carnot cycle
Carnot principle	principle governing the efficiency or performance of a heat device operating on a Carnot cycle: any reversible heat device working between two reservoirs must have the same efficiency or performance coefficient, greater than that of an irreversible heat device operating between the same two reservoirs
Clausius statement of the second law of thermodynamics	heat never flows spontaneously from a colder object to a hotter object
coefficient of performance	measure of effectiveness of a refrigerator or heat pump
cold reservoir	sink of heat used by a heat engine
disorder	measure of order in a system; the greater the disorder is, the higher the entropy
efficiency (e)	output work from the engine over the input heat to the engine from the hot reservoir
entropy	state function of the system that changes when heat is transferred between the system and the environment
entropy statement of the second law of thermodynamics	entropy of a closed system or the entire universe never decreases
heat engine	device that converts heat into work
heat pump	device that delivers heat to a hot reservoir
hot reservoir	source of heat used by a heat engine
irreversibility	phenomenon associated with a natural process
irreversible process	process in which neither the system nor its environment can be restored to their original states at the same time
isentropic	reversible adiabatic process where the process is frictionless and no heat is transferred
Kelvin statement of the second law of thermodynamics	it is impossible to convert the heat from a single source into work without any other effect
perfect engine	engine that can convert heat into work with 100%100% efficiency
perfect refrigerator (heat pump)	refrigerator (heat pump) that can remove (dump) heat without any input of work
refrigerator	device that removes heat from a cold reservoir
reversible process	process in which both the system and the external environment theoretically can be returned to their original states
third law of thermodynamics	absolute zero temperature cannot be reached through any finite number of cooling steps

Key Equations

Result of energy conservation	$W = Q_h - Q_c$
Efficiency of a heat engine	$e = \frac{W}{Q_h} = 1 - \frac{Q_c}{Q_h}$
Coefficient of performance of a refrigerator	$K_R = \frac{Q_c}{W} = \frac{Q_c}{Q_h - Q}$
Coefficient of performance of a heat pump	$K_P = \frac{Q_h}{W} = \frac{Q_h}{Q_h - Q_c}$
Resulting efficiency of a Carnot cycle	$e = 1 - \frac{T_c}{T_h}$
Performance coefficient of a reversible refrigerator	$K_R = \frac{T_c}{T_h - T_c}$
Performance coefficient of a reversible heat pump	$K_P = \frac{T_h}{T_h - T_c}$
Entropy of a system undergoing a reversible process at a constant temperature	$\Delta S = \frac{Q}{T}$
Change of entropy of a system under a reversible process	$\Delta S = S_B - S_A = \int_A^B dQ/T$
Entropy of a system undergoing any complete reversible cyclic process	$\oint dS = \oint \frac{dQ}{T} = 0$
Change of entropy of a closed system under an irreversible process	$\Delta S \geq 0$
Change in entropy of the system along an isotherm	$\lim_{T \to 0} (\Delta S)_T = 0$

Summary

4.2 Reversible and Irreversible Processes

- A reversible process is one in which both the system and its environment can return to exactly the states they were in by following the reverse path.
- An irreversible process is one in which the system and its environment cannot return together to exactly the states that they were in.
- The irreversibility of any natural process results from the second law of thermodynamics.

4.3 Heat Engines

- The work done by a heat engine is the difference between the heat absorbed from the hot reservoir and the heat discharged to the cold reservoir, that is, $W = Q_h - Q_c$.
- The ratio of the work done by the engine and the heat absorbed from the hot reservoir provides the efficiency of the engine, that is, $e = W/Q_h = 1 - Q_c/Q_h$.

4.4 Refrigerators and Heat Pumps

- A refrigerator or a heat pump is a heat engine run in reverse.
- The focus of a refrigerator is on removing heat from the cold reservoir with a coefficient of performance K_R.
- The focus of a heat pump is on dumping heat to the hot reservoir with a coefficient of performance K_P.

4.5 Statements of the Second Law of Thermodynamics

- The Kelvin statement of the second law of thermodynamics: It is impossible to convert the heat from a single source into work without any other effect.
- The Kelvin statement and Clausius statement of the second law of thermodynamics are equivalent.

4.6 The Carnot Cycle

- The Carnot cycle is the most efficient engine for a reversible cycle designed between two reservoirs.
- The Carnot principle is another way of stating the second law of thermodynamics.

4.7 Entropy

- The change in entropy for a reversible process at constant temperature is equal to the heat divided by the temperature. The entropy change of a system under a reversible process is given by $\Delta S = \int_A^B dQ/T$.
- A system's change in entropy between two states is independent of the reversible thermodynamic path taken by the system when it makes a transition between the states.

4.8 Entropy on a Microscopic Scale

- Entropy can be related to how disordered a system is—the more it is disordered, the higher is its entropy. In any irreversible process, the universe becomes more disordered.
- According to the third law of thermodynamics, absolute zero temperature is unreachable.

14.22: Introduction

learning objectives

- Analyze the necessity to exclude energy transferred between system as heat from mechanical work

Work

In thermodynamics, work performed by a closed system is the energy transferred to another system that is measured by mechanical constraints on the system. Thermodynamic work encompasses mechanical work (gas expansion,) plus many other types of work, such as electrical. As such, thermodynamic work is a generalization of the concept of mechanical work in mechanics. It necessarily excludes energy transferred between systems as heat, which is modeled distinctly in thermodynamics. For closed systems, energy changes in a system other than as work transfer are as heat.

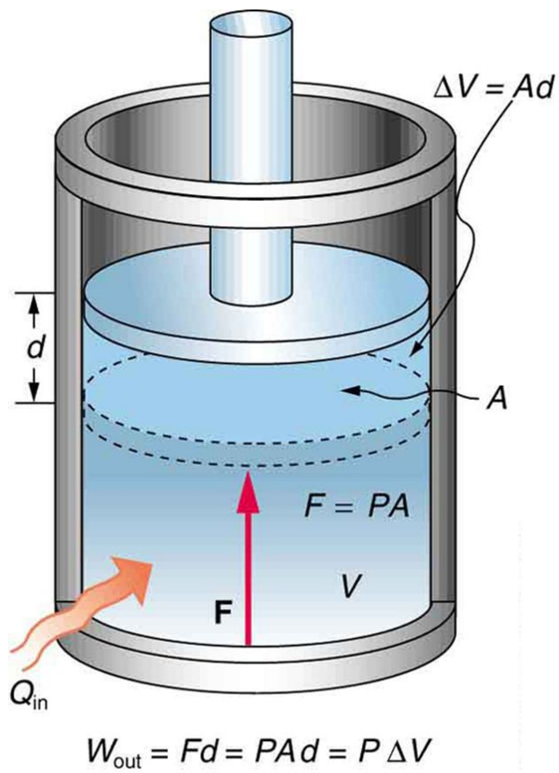


Fig 1: An isobaric expansion of a gas requires heat transfer during the expansion to keep the pressure constant. Since pressure is constant, the work done is PΔV.

Heat and Work

Heat transfer (often represented by Q) and doing work (W) are the two everyday means of bringing energy into or taking energy out of a system. The processes are quite different. Heat transfer, a less organized process, is driven by temperature differences. Work, a quite organized process(as in gas expansion), involves a macroscopic force exerted through a distance. Nevertheless, heat and work can produce identical results. Both heat and work can cause a temperature increase.

Heat transfer into a system, such as when the Sun warms the air in a bicycle tire, can increase its temperature, and so can work done on the system, as when the bicyclist pumps air into the tire. Once the temperature increase has occurred, it is impossible to tell whether it was caused by heat transfer or by doing work.

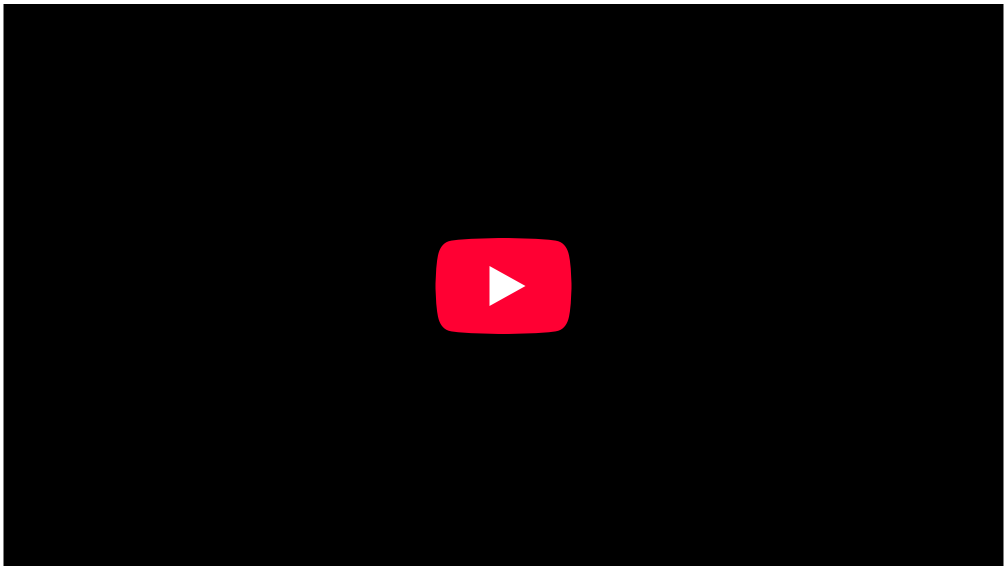
This uncertainty is an important point. Heat transfer and work are both energy in transit—neither is stored as such in a system. However, both can change the internal energy of a system. Internal energy is a form of energy completely different from either heat or work.

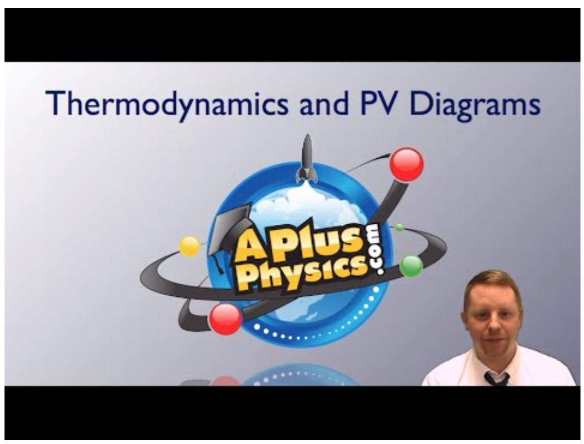
A Review of the Zeroth Law

Zeroth law justifies the use of thermodynamic temperature, defined as the shared temperature of three designated systems at equilibrium.

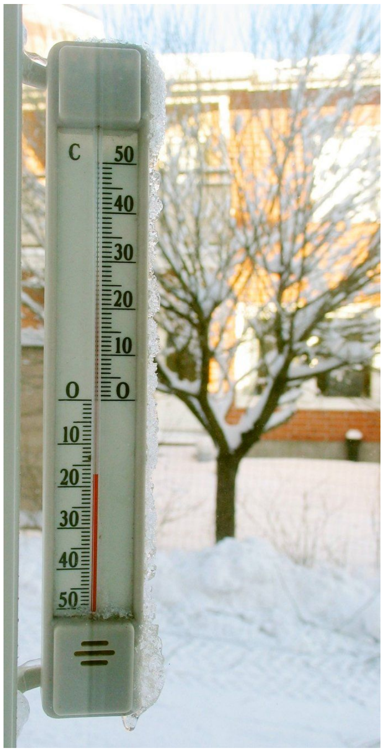
learning objectives

- Discuss how the Zeroth Law of Thermodynamics justifies the use of thermodynamic temperature





Thermodynamics and PV Diagrams: A brief introduction to the zeroth and 1st laws of thermodynamics as well as PV diagrams for students.



Thermometer: A thermometer calibrated in degrees Celsius

The Zeroth Law of Thermodynamics states: *If two systems, A and B, are in thermal equilibrium with each other, and B is in thermal equilibrium with a third system, C, then A is also in thermal equilibrium with C.*

This law was postulated in the 1930s, after the first and second laws of thermodynamics had been developed and named. It is called the “zeroth” law because it comes logically before the first and second laws (discussed in Atoms on the 1st and 2nd laws).

Two systems are in thermal equilibrium if they could transfer heat between each other, but don’t. Indeed, experiments have shown that if two systems, A and B, are in thermal equilibrium with each other, and B is in thermal equilibrium with a third system C, then A is also in thermal equilibrium with C. This conclusion may seem obvious, because all three have the same temperature, but zeroth law is basic to thermodynamics. Zeroth law justifies the use of thermodynamic temperature: the common “label” that the three systems in the definition above share is defined as the temperature of the systems.

Temperature

Thermometers actually take their own temperature, not the temperature of the object they are measuring. This raises the question of how we can be certain that a thermometer measures the temperature of the object with which it is in contact. The answer lies in the fact that any two systems placed in thermal contact (meaning heat transfer can occur between them) will reach the same temperature. That is, heat will flow from the hotter object to the cooler one until they reach exactly the same temperature. The objects are then in thermal equilibrium, and no further changes will occur. The systems interact and change because their temperatures differ, and the changes stop once their temperatures are the same. Thus, if enough time is allowed for this transfer of heat to run its course, the temperature a thermometer registers does represent the system with which it achieves thermal equilibrium.

Key Points

- Thermodynamic work is a generalization of the concept of mechanical work in mechanics.
- For closed systems, energy changes in a system other than as work transfer are as heat.
- Work in thermodynamics is a quite organized process (as in gas expansion), involving a macroscopic force exerted through a distance.
- The zeroth law of thermodynamics states that when systems, A and B, are in thermal equilibrium with each other, and B is in thermal equilibrium with a third system, C, then A is also in thermal equilibrium with C.
- Two systems are in thermal equilibrium if they could transfer heat between each other, but don’t.

- If enough time is allowed for heat transfer to occur between a thermometer and a system, the temperature the thermometer registers does represent the system with which it achieves thermal equilibrium.

Key Terms

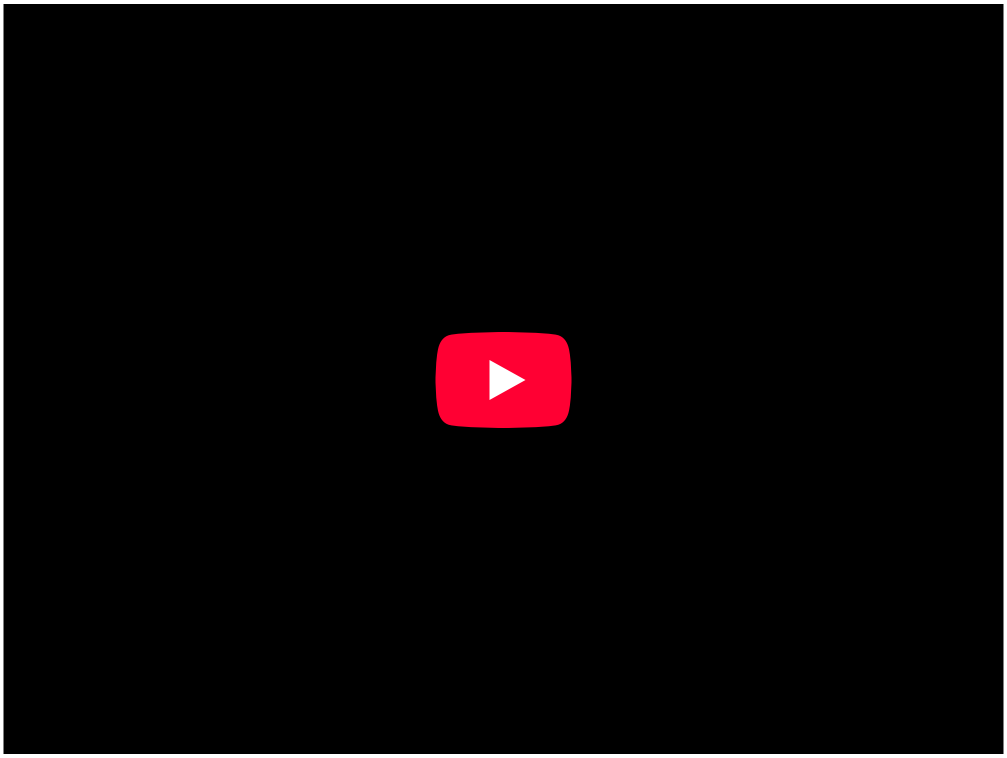
- **heat**: energy transferred from one body to another by thermal interactions
- **thermodynamics**: a branch of natural science concerned with heat and its relation to energy and work
- **internal energy**: The sum of all energy present in the system, including kinetic and potential energy; equivalently, the energy needed to create a system, excluding the energy necessary to displace its surroundings.
- **thermal equilibrium**: Two systems are in thermal equilibrium if they could transfer heat between each other, but don't.
- **thermodynamic temperature**: Temperature defined in terms of the laws of thermodynamics rather than the properties of a real material: expressed in kelvins

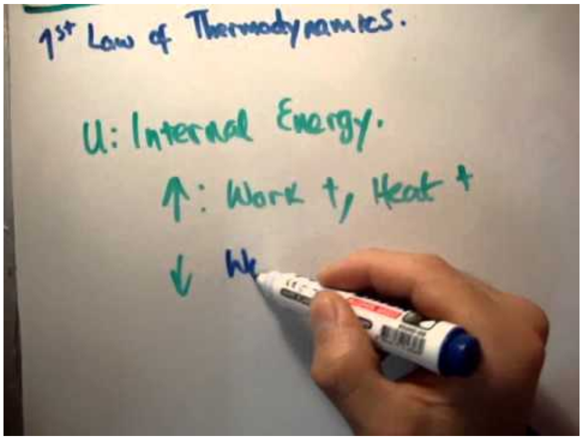
14.23: The First Law of Thermodynamics

learning objectives

- Explain how the net heat transferred and net work done in a system relate to the first law of thermodynamics

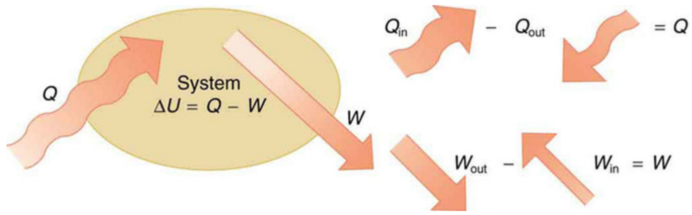
The first law of thermodynamics is a version of the law of conservation of energy specialized for thermodynamic systems. It is usually formulated by stating that the change in the internal energy of a closed system is equal to the amount of heat supplied to the system, minus the amount of work done by the system on its surroundings. The law of conservation of energy can be stated like this: The energy of an isolated system is constant.





First Law of Thermodynamics: In this video I continue with my series of tutorial videos on Thermal Physics and Thermodynamics. It's pitched at undergraduate level and while it is mainly aimed at physics majors, it should be useful to anybody taking a first course in thermodynamics such as engineers etc..

If we are interested in how heat transfer is converted into work, then the conservation of energy principle is important. The first law of thermodynamics applies the conservation of energy principle to systems where heat transfer and doing work are the methods of transferring energy into and out of the system. In equation form, the first law of thermodynamics is



Internal Energy: The first law of thermodynamics is the conservation-of-energy principle stated for a system where heat and work are the methods of transferring energy for a system in thermal equilibrium. Q represents the net heat transfer—it is the sum of all heat transfers into and out of the system. Q is positive for net heat transfer into the system. W is the total work done on and by the system. W is positive when more work is done by the system than on it. The change in the internal energy of the system, ΔU, is related to heat and work by the first law of thermodynamics, $\Delta U = Q - W$.

$$\Delta U = Q - W. \tag{14.23.1}$$

Here ΔU is the change in internal energy U of the system, Q is the net heat transferred into the system, and W is the net work done by the system. We use the following sign conventions: if Q is positive, then there is a net heat transfer into the system; if W is positive, then there is net work done by the system. So positive Q adds energy to the system and positive W takes energy from the system. Thus $\Delta U = Q - W$. Note also that if more heat transfer into the system occurs than work done, the difference is stored as internal energy. Heat engines are a good example of this—heat transfer into them takes place so that they can do work.

Constant Pressure and Volume

Isobaric process is one in which a gas does work at constant pressure, while an isochoric process is one in which volume is kept constant.

learning objectives

- Contrast isobaric and isochoric processes

According to the first law of thermodynamics, heat transferred to a system can be either converted to internal energy or used to do work to the environment. A process in which a gas does work on its environment at constant pressure is called an isobaric process, while one in which volume is kept constant is called an isochoric process.

Isobaric Process (Constant Pressure)

An isobaric process occurs at constant pressure. Since the pressure is constant, the force exerted is constant and the work done is given as PΔV. An example would be to have a movable piston in a cylinder, so that the pressure inside the cylinder is always at atmospheric pressure, although it is isolated from the atmosphere. In other words, the system is dynamically connected, by a movable boundary, to a constant-pressure reservoir. If a gas is to expand at a constant pressure, heat should be transferred into the system at a certain rate. This process is called an isobaric expansion.

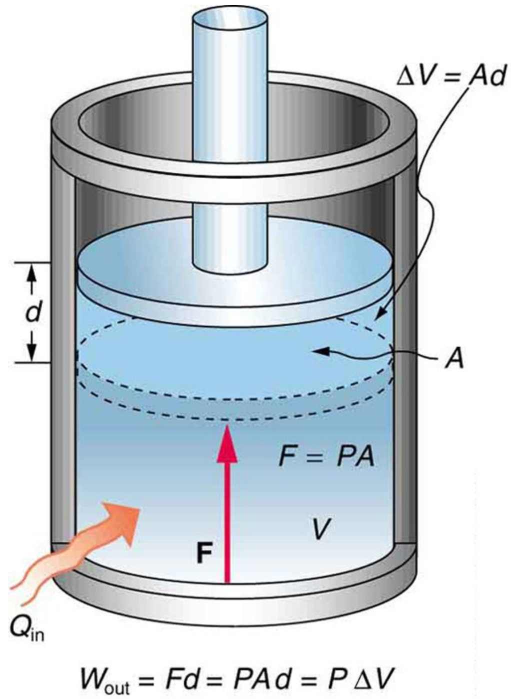


Fig 1: An isobaric expansion of a gas requires heat transfer during the expansion to keep the pressure constant. Since pressure is constant, the work done is PΔV.

Isochoric Process (Constant Volume)

An isochoric process is one in which the volume is held constant, meaning that the work done by the system will be zero. It follows that, for the simple system of two dimensions, any heat energy transferred to the system externally will be absorbed as internal energy. An isochoric process is also known as an isometric process or an isovolumetric process. An example would be to place a closed tin can containing only air into a fire. To a first approximation, the can will not expand, and the only change will be that the gas gains internal energy, as evidenced by its increase in temperature and pressure. Mathematically,

$$\Delta \mathrm{Q} = \Delta \mathrm{U}. \tag{14.23.2}$$

We may say that the system is dynamically insulated, by a rigid boundary, from the environment.

Isothermal Processes

An isothermal process is a change of a thermodynamic system, in which the temperature remains constant.

learning objectives

- Identify the typical systems in which an isothermal process occurs

An isothermal process is a change of a system, in which the temperature remains constant: $\Delta \mathrm{T} = 0$. This typically occurs when a system is in contact with an outside thermal reservoir (heat bath), and the change occurs slowly enough to allow the system to continually adjust to the temperature of the reservoir through heat exchange. In contrast, an adiabatic process is where a system exchanges no heat with its surroundings ($\mathrm{Q} = 0$). (See our atom on “Adiabatic Process. “) In other words, in an isothermal process, the value $\Delta \mathrm{T} = 0$ but $\mathrm{Q} \neq 0$, while in an adiabatic process, $\Delta \mathrm{T} \neq 0$ but $\mathrm{Q} = 0$.

Ideal Gas in an Isothermal Process

For an ideal, the product of pressure and volume (PV) is a constant if the gas is kept at isothermal conditions. (This is historically called Boyle’s law.) However, the cases where the product PV is an exponential term, does not comply. The value of the constant is nRT, where n is the number of moles of gas present and R is the ideal gas constant. In other words, the ideal gas law $\mathrm{PV} = \mathrm{nRT}$ applies. This means that

$$P = \frac{nRT}{V} = \frac{\text{constant}}{V} \tag{14.23.3}$$

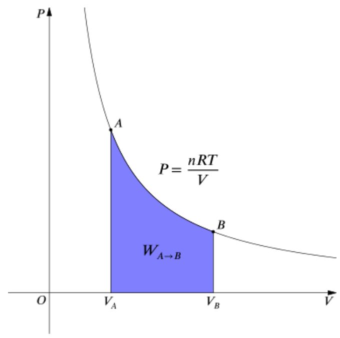
holds.

Calculation of Work

In thermodynamics, the work involved when a gas changes from state A to state B is simply

$$W_{A \rightarrow B} = \int_{V_A}^{V_B} P\, dV. \tag{14.23.4}$$

For an isothermal, reversible process, this integral equals the area under the relevant pressure-volume isotherm, and is indicated in blue in for an ideal gas. Again, $P = \frac{nRT}{V}$ applies and with T being constant (as this is an isothermal process), we have



Work Done by Gas During Expansion: The blue area represents "work" done by the gas during expansion for this isothermal change.

$$W_{A \rightarrow B} = nRT \int_{V_A}^{V_B} \frac{1}{V} dV = nRT \ln \frac{V_B}{V_A}. \tag{14.23.5}$$

It is also worth noting that, for many systems, if the temperature is held constant, the internal energy of the system also is constant, and so ΔU=0ΔU=0. From the first law of thermodynamics, it follows that Q=−WQ=−W for this same isothermal process.

Adiabatic Processes

An adiabatic process is any process occurring without gain or loss of heat within a system.

learning objectives

- Assess the environments in which isothermal processes typically occur

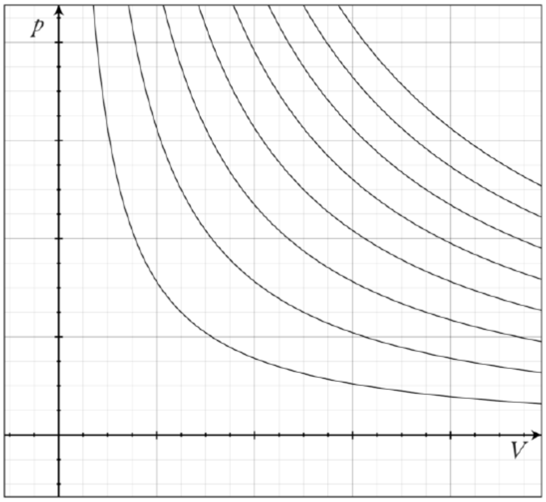
An isothermal process is a change of a system, in which the temperature remains constant: $\Delta T = 0$. This typically occurs when a system is in contact with an outside thermal reservoir (heat bath), and the change occurs slowly enough to allow the system to continually adjust to the temperature of the reservoir through heat exchange. In contrast, an adiabatic process is where a system exchanges no heat with its surroundings ($Q = 0$). (See our atom on "Adiabatic Process. ") In other words, in an isothermal process, the value $\Delta T = 0$ but $Q \neq 0$, while in an adiabatic process, $\Delta T \neq 0$ but $Q = 0$.

Ideal Gas in an Isothermal Process

For an ideal, the product of pressure and volume (PV) is a constant if the gas is kept at isothermal conditions. (This is historically called Boyle's law.) However, the cases where the product PV is an exponential term, does not comply. The value of the constant is nRT, where n is the number of moles of gas present and R is the ideal gas constant. In other words, the ideal gas law $PV = nRT$ applies. This means that

$$P = \frac{nRT}{V} = \frac{\text{constant}}{V} \tag{14.23.6}$$

holds. The family of curves generated by this equation is shown in. Each curve is called an isotherm.



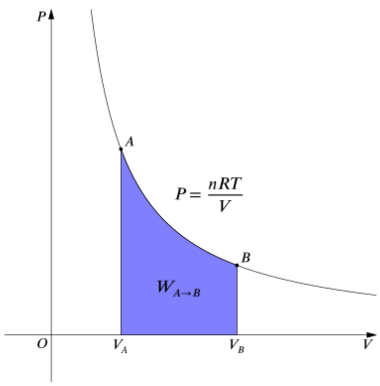
Isotherms of an Ideal Gas: Several isotherms of an ideal gas on a PV diagram.

Calculation of Work

In thermodynamics, the work involved when a gas changes from state A to state B is simply

$$W_{A\to B} = \int_{V_A}^{V_B} P\ dV. \tag{14.23.7}$$

For an isothermal, reversible process, this integral equals the area under the relevant pressure-volume isotherm, and is indicated in blue in for an ideal gas. Again, P = nRT / V applies and with T being constant (as this is an isothermal process), we have



Work Done by Gas During Expansion: The blue area represents "work" done by the gas during expansion for this isothermal change.

$$W_{A\to B} = nRT \int_{V_A}^{V_B} \frac{1}{V}\ dV = nRT \ln \frac{V_B}{V_A}. \tag{14.23.8}$$

It is also worth noting that, for many systems, if the temperature is held constant, the internal energy of the system also is constant, and so $\Delta U = 0$. From the first law of thermodynamics, it follows that $Q = -W$ for this same isothermal process.

Human Metabolism

The 1st law of thermodynamics explains human metabolism: the conversion of food into energy that is used by the body to perform activities.

learning objectives

- Contrast catabolism and anabolism in regards to energy

Metabolism in humans is the conversion of food into energy, which is then used by the body to perform activities. It is an example of the first law of thermodynamics in action. Considering the body as the system of interest, we can use the first law to examine heat transfer, doing work, and internal energy in activities ranging from sleep to heavy exercise. For example, one major factor in such activities is body temperature—normally kept constant by heat transfer to the surroundings, meaning that Q is negative (i.e., our body loses heat). Another factor is that the body usually does work on the outside world, meaning that W is positive. Thus, in such situations the body loses internal energy, since $\Delta U = Q - W$ is negative.

Eating

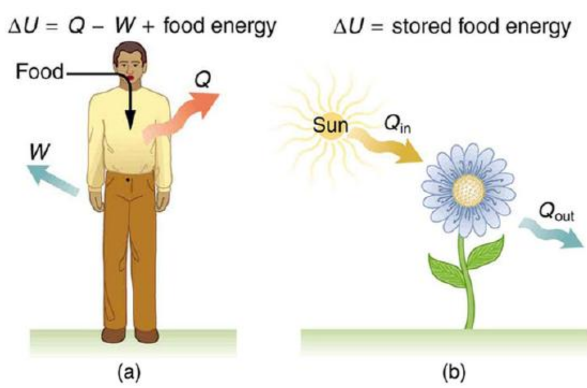
Now consider the effects of eating. The body metabolizes all the food we consume. Eating increases the internal energy of the body by adding chemical potential energy. In essence, metabolism uses an oxidation process in which the chemical potential energy of food is released. This implies that food input is in the form of work. Food energy is reported in a special unit, known as the Calorie. This energy is measured by burning food in a calorimeter, which is how the units are determined.

Catabolism and Anabolism

Catabolism is the pathway that breaks down molecules into smaller units and produces energy. Anabolism is the building up of molecules from smaller units. Anabolism uses up the energy produced by the catabolic break down of your food to create molecules more useful to your body.

Internal energy

Our body loses internal energy, and there are three places this internal energy can go—to heat transfer, to doing work, and to stored fat (a tiny fraction also goes to cell repair and growth). As shown in Fig 1 heat transfer and doing work take internal energy out of the body, and then food puts it back. If you eat just the right amount of food, then your average internal energy remains constant. Whatever you lose to heat transfer and doing work is replaced by food, so that, in the long run, $\Delta U=0$. If you overeat repeatedly, then ΔU is always positive, and your body stores this extra internal energy as fat. The reverse is true if you eat too little. If ΔU is negative for a few days, then the body metabolizes its own fat to maintain body temperature and do work that takes energy from the body. This process is how dieting produces weight loss.



Metabolism: (a) The first law of thermodynamics applied to metabolism. Heat transferred out of the body (Q) and work done by the body (W) remove internal energy, while food intake replaces it. (Food intake may be considered as work done on the body.) (b) Plants convert part of the radiant heat transfer in sunlight to stored chemical energy, a process called photosynthesis.

Metabolism

Life is not always this simple, as any dieter knows. The body stores fat or metabolizes it only if energy intake changes for a period of several days. Once you have been on a major diet, the next one is less successful because your body alters the way it responds to low energy intake. Your basal metabolic rate is the rate at which food is converted into heat transfer and work done while the body is at complete rest. The body adjusts its basal metabolic rate to compensate (partially) for over-eating or under-eating. The body will decrease the metabolic rate rather than eliminate its own fat to replace lost food intake. You will become more easily chilled and feel less energetic as a result of the lower metabolic rate, and you will not lose weight as fast as before. Exercise helps with weight loss because it produces both heat transfer from your body and work, and raises your metabolic rate even when you are at rest.

Irreversibility

The body provides us with an excellent indication that many thermodynamic processes are irreversible. An irreversible process can go in one direction but not the reverse, under a given set of conditions. For example, although body fat can be converted to do work and produce heat transfer, work done on the body and heat transfer into it cannot be converted to body fat. Otherwise, we could skip lunch by sunning ourselves or by walking down stairs. Another example of an irreversible thermodynamic process is photosynthesis. This process is the intake of one form of energy—light—by plants and its conversion to chemical potential energy. Both applications of the first law of thermodynamics are illustrated in. One great advantage of such conservation laws is that they accurately describe the beginning and ending points of complex processes (such as metabolism and photosynthesis) without regard to the complications in between.

Key Points

- The first law of thermodynamics is a version of the law of conservation of energy, specialized for thermodynamical systems.
- In equation form, the first law of thermodynamics is $\Delta U = Q - W$.
- Heat engines are a good example of the application of the 1st law; heat transfer into them takes place so that they can do work.
- An isobaric process occurs at constant pressure. Since the pressure is constant, the force exerted is constant and the work done is given as $P\Delta V$.
- An isobaric expansion of a gas requires heat transfer to keep the pressure constant.
- An isochoric process is one in which the volume is held constant, meaning that the work done by the system will be zero. The only change will be that a gas gains internal energy.
- For an ideal gas, the product of pressure and volume (PV) is a constant if the gas is kept at isothermal conditions.
- For an ideal gas, the work involved when a gas changes from state A to state B through an isothermal process is given as $W_{A\rightarrow B} = nRT \ln \frac{V_B}{V_A}$.
- For many systems, if the temperature is held constant, the internal energy of the system also is constant. It follows that $Q = -W$ in this case.
- Adiabatic processes can occur if the container of the system has thermally-insulated walls or the process happens in an extremely short time.
- For an adiabatically expanding ideal monatomic gas which does work on its environment (W is positive), internal energy of the gas should decrease.
- In a sense, isothermal process can be considered as the opposite extreme of adiabatic process. In isothermal processes, heat exchange is slow enough so that the system's temperature remains constant.
- Human metabolism is a complicated process. The 1st law of thermodynamics describes the beginning and ending points of these processes.
- Our body loses internal energy. There are three places this internal energy can go—to heat transfer, to doing work, and to stored fat.
- Our body provides a good example of irreversible processes. Although body fat can be converted to do work and produce heat transfer, work done on the body and heat transfer into it cannot be converted to body fat.

Key Terms

- **internal energy**: The sum of all energy present in the system, including kinetic and potential energy; equivalently, the energy needed to create a system, excluding the energy necessary to displace its surroundings.
- **heat**: energy transferred from one body to another by thermal interactions
- **law of conservation of energy**: The law stating that the total amount of energy in any isolated system remains constant, and cannot be created or destroyed, although it may change forms.
- **internal energy**: The sum of all energy present in the system, including kinetic and potential energy; equivalently, the energy needed to create a system, excluding the energy necessary to displace its surroundings.
- **reversible**: Capable of returning to the original state without consumption of free energy and increase of entropy.
- **ideal gas**: A hypothetical gas whose molecules exhibit no interaction and undergo elastic collision with each other and with the walls of the container.
- **Boyle's law**: The observation that the pressure of an ideal gas is inversely proportional to its volume at constant temperature.

- **ideal gas**: A hypothetical gas whose molecules exhibit no interaction and undergo elastic collision with each other and with the walls of the container.
- **metabolism**: The complete set of chemical reactions that occur in living cells.
- **oxidation**: A reaction in which the atoms of an element lose electrons and the valence of the element increases.
- **calorie**: The energy needed to increase the temperature of 1 kilogram of water by 1 kelvin. It is equivalent to 1,000 (small) calories.

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14.24: The Second Law of Thermodynamics

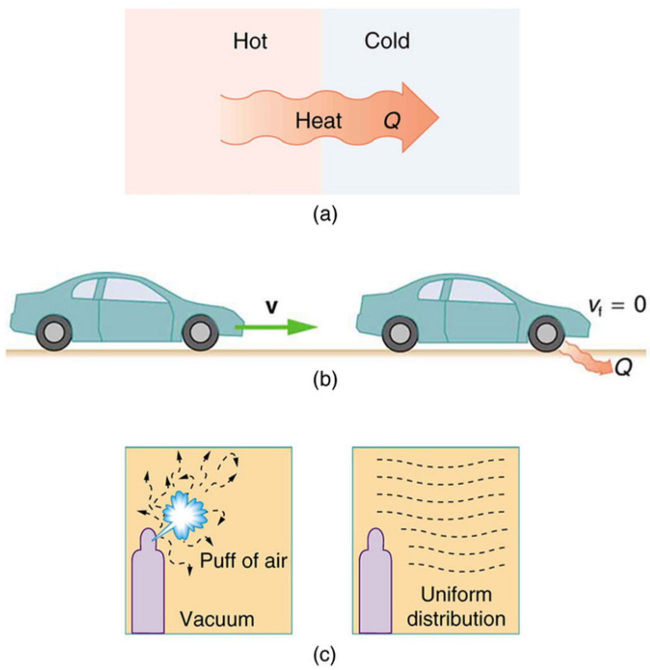
learning objectives

- Contrast the concept of irreversibility between the First and Second Laws of Thermodynamics

Irreversibility

The second law of thermodynamics deals with the direction taken by spontaneous processes. Many processes occur spontaneously in one direction only—that is, they areirreversible, under a given set of conditions. Although irreversibility is seen in day-to-day life—a broken glass does not resume its original state, for instance—complete irreversibility is a statistical statement that cannot be seen during the lifetime of the universe. More precisely, an irreversible process is one that depends on path. If the process can go in only one direction, then the reverse path differs fundamentally and the process cannot be reversible.

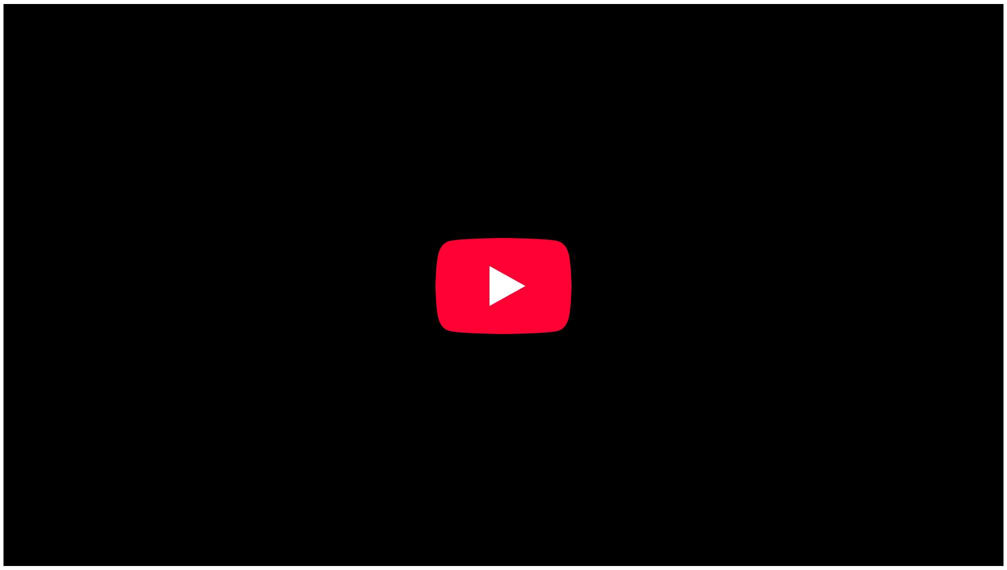
For example, heat involves the transfer of energy from higher to lower temperature. A cold object in contact with a hot one never gets colder, transferring heat to the hot object and making it hotter. Furthermore, mechanical energy, such as kinetic energy, can be completely converted to thermal energy by friction, but the reverse is impossible. A hot stationary object never spontaneously cools off and starts moving. Yet another example is the expansion of a puff of gas introduced into one corner of a vacuum chamber. The gas expands to fill the chamber, but it never regroups in the corner. The random motion of the gas molecules could take them all back to the corner, but this is never observed to happen.

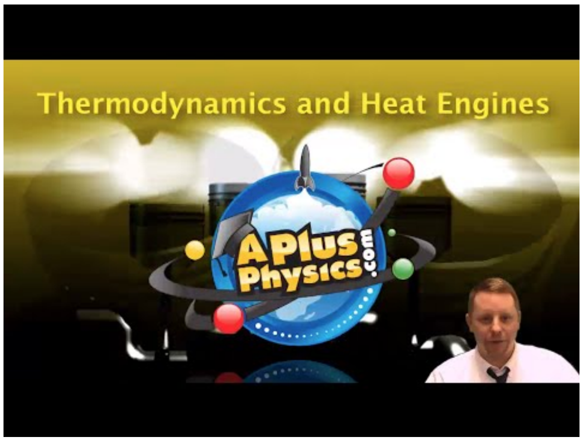


One-Way Processed in Nature: Examples of one-way processes in nature. (a) Heat transfer occurs spontaneously from hot to cold and not from cold to hot. (b) The brakes of this car convert its kinetic energy to heat transfer to the environment. The reverse process is impossible. (c) The burst of gas let into this vacuum chamber quickly expands to uniformly fill every part of the chamber. The random motions of the gas molecules will never return them to the corner.

Second Law of Thermodynamics

The fact that certain processes never occur suggests that there is a law forbidding them to occur. The first law of thermodynamics would allow them to occur—none of those processes violate conservation of energy. The law that forbids these processes is called the second law of thermodynamics. We shall see that the second law can be stated in many ways that may seem different, but these many ways are, in fact, equivalent. Like all natural laws, the second law of thermodynamics gives insights into nature, and its several statements imply that it is broadly applicable, fundamentally affecting many apparently disparate processes. The already familiar direction of heat transfer from hot to cold is the basis of our first version of the second law of thermodynamics.





Thermodynamics and Heat Engines: A brief introduction to heat engines and thermodynamic concepts such as the Carnot Engine for students.

The Second Law of Thermodynamics(first expression): *Heat transfer occurs spontaneously from higher- to lower-temperature bodies but never spontaneously in the reverse direction.*

The law states that it is impossible for any process to have as its sole result heat transfer from a cooler to a hotter object. We will express the law in other terms later on, most importantly in terms of entropy.

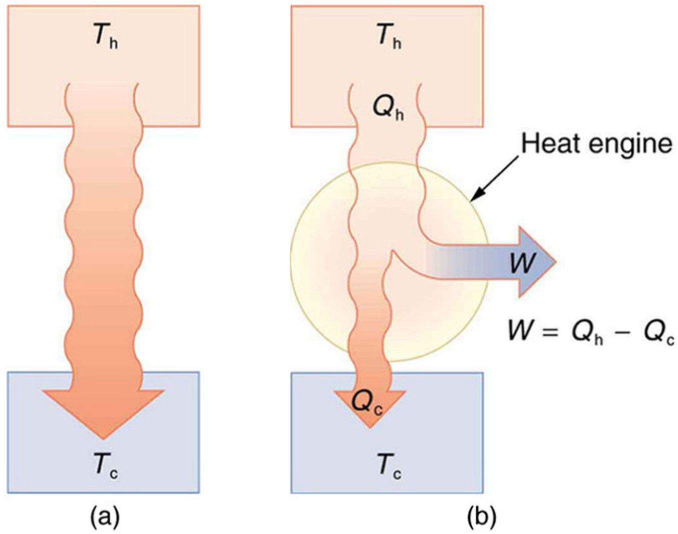
Heat Engines

In thermodynamics, a heat engine is a system that performs the conversion of heat or thermal energy to mechanical work.

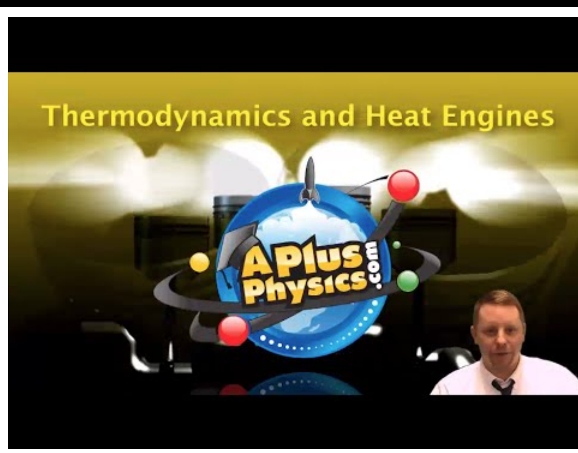
learning objectives

- Justify why efficiency is one of the most important parameters for any heat engine

In thermodynamics, a heat engine is a system that performs the conversion of heat or thermal energy to mechanical work. Gasoline and diesel engines, jet engines, and steam turbines are all heat engines that do work by using part of the heat transfer from some source. Heat transfer from the hot object (or hot reservoir) is denoted as Q_h, while heat transfer into the cold object (or cold reservoir) is Q_c, and the work done by the engine is W. The temperatures of the hot and cold reservoirs are T_h and T_c, respectively.



Heat Transfer: (a) Heat transfer occurs spontaneously from a hot object to a cold one, consistent with the second law of thermodynamics. (b) A heat engine, represented here by a circle, uses part of the heat transfer to do work. The hot and cold objects are called the hot and cold reservoirs. Qh is the heat transfer out of the hot reservoir, W is the work output, and Qc is the heat transfer into the cold reservoir.



Thermodynamics and Heat Engines: A brief introduction to heat engines and thermodynamic concepts such as the Carnot Engine for students.

Because the hot reservoir is heated externally, which is energy intensive, it is important that the work is done as efficiently as possible. In fact, we would like W to equal Q_h, and for there to be no heat transfer to the environment (Q_c=0). Unfortunately, this is impossible. The second law of thermodynamics (second expression) also states, with regard to using heat transfer to do work: *It is impossible in any system for heat transfer from a reservoir to completely convert to work in a cyclical process in which the system returns to its initial state.*

A cyclical process brings a system, such as the gas in a cylinder, back to its original state at the end of every cycle. Most heat engines, such as reciprocating piston engines and rotating turbines, use cyclical processes. The second law, in its second form, clearly states that such engines cannot have perfect conversion of heat transfer into work done.

Efficiency

A cyclical process brings the system back to its original condition at the end of every cycle. By definition, such a system's internal energy U is the same at the beginning and end of every cycle—that is, $\Delta U = 0$. The first law of thermodynamics states that $\Delta U = Q - W$, where Q is the net heat transfer during the cycle ($W = Q_h - Q_c$) and W is the net work done by the system. Since $\Delta U = 0$ for a complete cycle, we have $W = Q$. Thus the net work done by the system equals the net heat transfer into the system, or

$W = Q_h - Q_c$ (cyclical process),

just as shown schematically in (b).

Efficiency is one of the most important parameters for any heat engine. The problem is that in all processes, there is significant heat transfer Q_c lost to the environment. In the conversion of energy to work, we are always faced with the problem of getting less out than we put in. We define the efficiency of a heat engine (*Eff*) to be its net work output W divided by heat transfer to the engine Q_h:

$$E_{ff} = WQ_h. \tag{14.24.1}$$

Since $W = Q_h - Q_c$ in a cyclical process, we can also express this as

$Eff = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$ (for cyclical process),

making it clear that an efficiency of 1, or 100%, is possible only if there is no heat transfer to the environment (Q_c=0).

Carnot Cycles

The Carnot cycle is the most efficient cyclical process possible and uses only reversible processes through its cycle.

learning objectives

- Analyze why the Carnot engine is considered the perfect engine

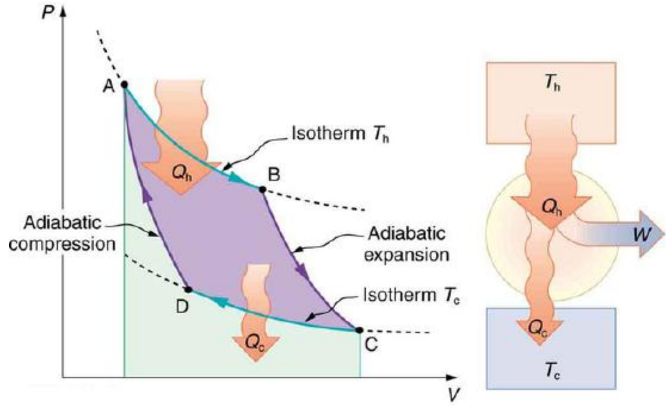
We know from the second law of thermodynamics that a heat engine cannot be 100 percent efficient, since there must always be some heat transfer Q_c to the environment. (See our atom on "Heat Engines. ") How efficient can a heat engine be then? This question was answered at a theoretical level in 1824 by a young French engineer, Sadi Carnot (1796-1832), in his study of the then-emerging heat engine technology crucial to the Industrial Revolution. He devised a theoretical cycle, now called the Carnot cycle, which is the most efficient cyclical process possible. The second law of thermodynamics can be restated in terms of the Carnot cycle, and so what Carnot actually discovered was this fundamental law. Any heat engine employing the Carnot cycle is called a Carnot engine.

What is crucial to the Carnot cycle is that only reversible processes are used. Irreversible processes involve dissipative factors, such as friction and turbulence. This increases heat transfer Q_c to the environment and reduces the efficiency of the engine. Obviously, then, reversible processes are superior.

The second law of thermodynamics (a third form): *A Carnot engine operating between two given temperatures has the greatest possible efficiency of any heat engine operating between these two temperatures. Furthermore, all engines employing only reversible processes have this same maximum efficiency when operating between the same given temperatures.*

Efficiency

The Carnot cycle comprises two isothermal and two adiabatic processes. Recall that both isothermal and adiabatic processes are, in principle, reversible.



PV Diagram for a Carnot Cycle: PV diagram for a Carnot cycle, employing only reversible isothermal and adiabatic processes. Heat transfer Qh occurs into the working substance during the isothermal path AB, which takes place at constant temperature Th. Heat transfer Qc occurs out of the working substance during the isothermal path CD, which takes place at constant temperature Tc. The net work output W equals the area inside the path ABCDA. Also shown is a schematic of a Carnot engine operating between hot and cold reservoirs at temperatures Th and Tc.

Carnot also determined the efficiency of a perfect heat engine—that is, a Carnot engine. It is always true that the efficiency of a cyclical heat engine is given by: $\mathrm{Eff} = \frac{Q_h - Q_c}{Q_h} = 1 - \frac{Q_c}{Q_h}$.

What Carnot found was that for a perfect heat engine, the ratio \mathrm{\frac{Q_c}{Q_h}} equals the ratio of the absolute temperatures of the heat reservoirs. That is, $\frac{Q_c}{Q_h} = \frac{T_c}{T_h}$ for a Carnot engine, so that the maximum or Carnot efficiency Eff_C is given by $\mathrm{Eff_c} = 1 - \frac{T_c}{T_h}$, where T_h and T_c are in kelvins. (Derivation of the formula is slightly beyond the scope of this atom.) No real heat engine can do as well as the Carnot efficiency—an actual efficiency of about 0.7 of this maximum is usually the best that can be accomplished.

Heat Pumps and Refrigerators

A heat pump is a device that transfers heat energy from a heat source to a heat sink against a temperature gradient.

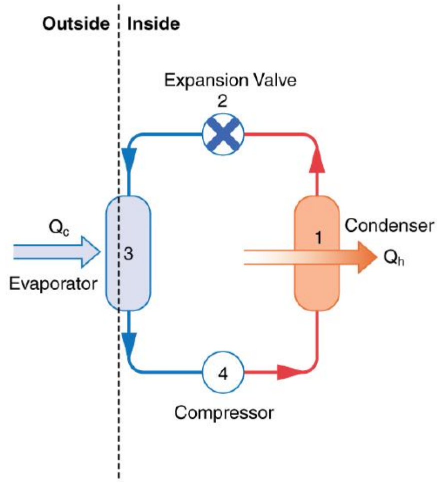
learning objectives

- Explain how the the components of a heat pump cause heat to transfer from a cold reservoir to a hot reservoir

Heat pumps, air conditioners, and refrigerators utilize heat transfer from cold to hot. Heat transfer (Q_c) occurs from a cold reservoir and into a hot one. This requires work input W, which is also converted to heat transfer. Thus the heat transfer to the hot reservoir is $Q_h = Q_c + W$. A heat pump's mission is for heat transfer Q_h to occur into a warm environment, such as a home in the winter. The mission of air conditioners and refrigerators is for heat transfer Q_c to occur from a cool environment, such as chilling a room or keeping food at lower temperatures than the environment. Actually, a heat pump can be used both to heat and cool a space. It is essentially an air conditioner and a heating unit all in one. In this section we will concentrate on its heating mode.

Heat Pumps

A working fluid such as a non-CFC refrigerant is used in a basic heat pump. The basic components of a heat pump in are a condenser, an expansion valve, an evaporator and a compressor. In the outdoor coils (the evaporator), heat transfer Q_c occurs to the working fluid from the cold outdoor air, turning it into a gas. The electrically driven compressor (work input W) raises the temperature and pressure of the gas and forces it into the condenser coils that are inside the heated space. Because the temperature of the gas is higher than the temperature inside the room, heat transfer to the room occurs and the gas condenses to a liquid. The liquid then flows back through a pressure-reducing valve to the outdoor evaporator coils, being cooled through expansion. (In a cooling cycle, the evaporator and condenser coils exchange roles and the flow direction of the fluid is reversed.)



Simple Heat Pump: A simple heat pump has four basic components: (1) condenser, (2) expansion valve, (3) evaporator, and (4) compressor.

Coefficient of Performance

The quality of a heat pump is judged by how much heat transfer $Q_h$occurs into the warm space compared with how much work input W is required. We define a heat pump's coefficient of performance (COP_{hp}) to be

$$\mathrm{COP_{hp}} = \frac{\mathrm{Q_h}}{\mathrm{W}}. \tag{14.24.2}$$

Since the efficiency of a heat engine is *Eff*=W/Q_h, we see that COP_{hp}=1/*Eff*. Since the efficiency of any heat engine is less than 1, it means that COP_{hp} is always greater than 1—that is, a heat pump always has more heat transfer Q_h than work put into it. Another interesting point is that heat pumps work best when temperature differences are small. The efficiency of a perfect engine (or Carnot engine) is

$$\mathrm{Eff_C} = 1\frac{\mathrm{T_c}}{\mathrm{T_h}}; \tag{14.24.3}$$

thus, the smaller the temperature difference, the smaller the efficiency and the greater the COP_{hp}.

Air Conditioners and Refrigerators

Air conditioners and refrigerators are designed to cool something down in a warm environment. As with heat pumps, work input is required for heat transfer from cold to hot. The quality of air conditioners and refrigerators is judged by how much heat transfer Q_c occurs from a cold environment compared with how much work input W is required. What is considered the benefit in a heat pump is considered waste heat in a refrigerator. We thus define the coefficient of performance (COP_{ref}) of an air conditioner or refrigerator to be

$$\mathrm{COP_{ref}} = \frac{\mathrm{Q_c}}{\mathrm{W}}. \tag{14.24.4}$$

Since $\mathrm{Q_h} = \mathrm{Q_c} + \mathrm{W}$ and $\mathrm{COP_{hp}} = \frac{\mathrm{Q_h}}{\mathrm{W}}$, we derive that

$$\mathrm{COP_{ref}} = \mathrm{COP_{hp}} - 1. \tag{14.24.5}$$

Also, from Q_h>Q_c, we see that an air conditioner will have a lower coefficient of performance than a heat pump.

Key Points

- Many thermodynamic phenomena, allowed to occur by the first law of thermodynamics, never occur in nature.
- Many processes occur spontaneously in one direction only, and the second law of thermodynamics deals with the direction taken by spontaneous processes.
- According to the second law of thermodynamics, it is impossible for any process to have heat transfer from a cooler to a hotter object as its sole result.

- A cyclical process brings a system, such as the gas in a cylinder, back to its original state at the end of every cycle. Most heat engines, such as reciprocating piston engines and rotating turbines, use cyclical processes.
- The second law of thermodynamics can be expressed as the following: It is impossible in any system for heat transfer from a reservoir to completely convert to work in a cyclical process in which the system returns to its initial state.
- The efficiency of a heat engine (Eff) is defined to be the engine's net work output W divided by heat transfer to the engine: Eff=WQh=1−QcQhEff=WQh=1−QcQh, where Q_c and Q_h denotes heat transfer to hot (engine) and cold (environment) reservoir.
- The second law of thermodynamics indicates that a Carnot engine operating between two given temperatures has the greatest possible efficiency of any heat engine operating between these two temperatures.
- Irreversible processes involve dissipative factors, which reduces the efficiency of the engine. Obviously, reversible processes are superior from the efficiency perspective.
- Carnot efficiency, the maximum achievable heat engine efficiency, is given as Effc=1−TcThEffc=1−TcTh.
- A heat pump 's mission is for heat transfer Qh to occur into a warm environment, such as a home in the winter.
- The mission of air conditioners and refrigerators is for heat transfer Qc to occur from a cool environment, such as chilling a room or keeping food at lower temperatures than the environment.
- A heat pump can be used both to heat and cool a space. It is essentially an air conditioner and a heating unit all in one. This is made possible by reversing the flow of its refrigerant, changing the direction net heat transfer.

Key Terms

- **entropy**: A measure of how evenly energy (or some analogous property) is distributed in a system.
- **the first law of thermodynamics**: A version of the law of conservation of energy, specialized for thermodynamical systems. Usually expressed as $\Delta U=Q-W$.
- **thermal energy**: The internal energy of a system in thermodynamic equilibrium due to its temperature.
- **internal energy**: The sum of all energy present in the system, including kinetic and potential energy; equivalently, the energy needed to create a system, excluding the energy necessary to displace its surroundings.
- **the second law of thermodynamics**: A law stating that states that the entropy of an isolated system never decreases, because isolated systems spontaneously evolve toward thermodynamic equilibrium—the state of maximum entropy. Equivalently, perpetual motion machines of the second kind are impossible.
- **heat engine**: Any device which converts heat energy into mechanical work.
- **CFC**: An organic compound that was commonly used as a refrigerant. Not commonly used anymore because of its ozone depletion effect.

14.25: Entropy

learning objectives

- Calculate the total change in entropy for a system in a reversible process

In this and following Atoms, we will study entropy. By examining it, we shall see that the directions associated with the second law — heat transfer from hot to cold, for example—are related to the tendency in nature for systems to become disordered and for less energy to be available for use as work. The entropy of a system can in fact be shown to be a measure of its disorder and of the unavailability of energy to do work.

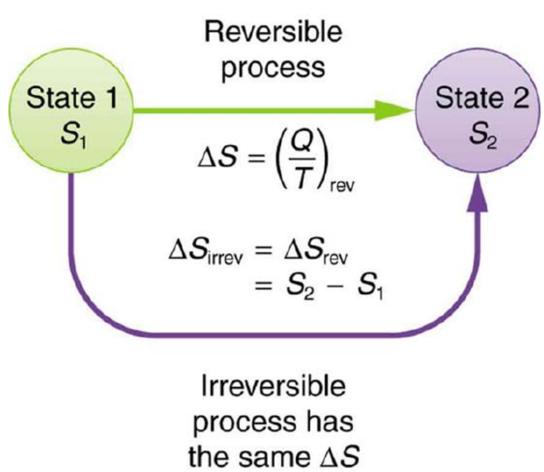
Definition of Entropy

We can see how entropy is defined by recalling our discussion of the Carnot engine. We noted that for a Carnot cycle, and hence for any reversible processes, $Q_c/Q_h=T_c/T_h$. Rearranging terms yields $\frac{Q_c}{T_c}=\frac{Q_h}{T_h}$ for any reversible process. Q_c and $Q_h$are absolute values of the heat transfer at temperatures T_c and T_h, respectively. This ratio of Q/T is defined to be the change in entropy ΔS for a reversible process,

$$\Delta S=(\frac{Q}{T})_{rev}, \tag{14.25.1}$$

where Q is the heat transfer, which is positive for heat transfer into and negative for heat transfer out of, and T is the absolute temperature at which the reversible process takes place. The SI unit for entropy is joules per kelvin (J/K). If temperature changes during the process, then it is usually a good approximation (for small changes in temperature) to take T to be the average temperature, avoiding the need to use integral calculus to find ΔS.

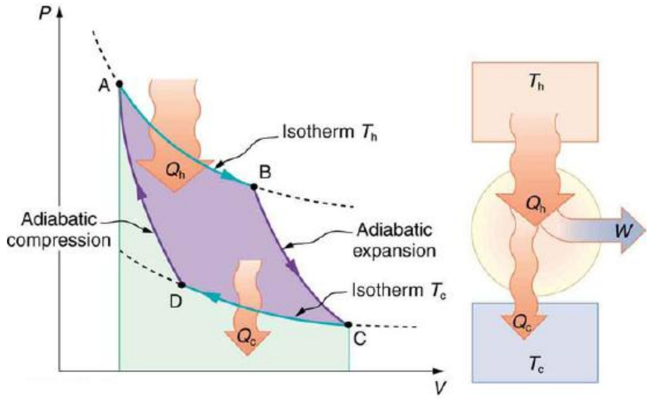
The definition of ΔS is strictly valid only for reversible processes, such as used in a Carnot engine. However, we can find ΔS precisely even for real, irreversible processes. The reason is that the entropy S of a system, like internal energy U, depends only on the state of the system and not how it reached that condition. Entropy is a property of state. Thus the change in entropy ΔS of a system between state one and state two is the same no matter how the change occurs. We just need to find or imagine a reversible process that takes us from state one to state two and calculate ΔS for that process. That will be the change in entropy for any process going from state one to state two.



Change in Entropy: When a system goes from state one to state two, its entropy changes by the same amount ΔS, whether a hypothetical reversible path is followed or a real irreversible path is taken.

Example

Now let us take a look at the change in entropy of a Carnot engine and its heat reservoirs for one full cycle. The hot reservoir has a loss of entropy $\Delta Sh=-Q_h/T_h$, because heat transfer occurs out of it (remember that when heat transfers out, then Q has a negative sign). The cold reservoir has a gain of entropy $\Delta S_c=Q_c/T_c$, because heat transfer occurs into it. (We assume the reservoirs are sufficiently large that their temperatures are constant.) So the total change in entropy is



PV Diagram for a Carnot Cycle: PV diagram for a Carnot cycle, employing only reversible isothermal and adiabatic processes. Heat transfer Qh occurs into the working substance during the isothermal path AB, which takes place at constant temperature Th. Heat transfer Qc occurs out of the working substance during the isothermal path CD, which takes place at constant temperature Tc. The net work output W equals the area inside the path ABCDA. Also shown is a schematic of a Carnot engine operating between hot and cold reservoirs at temperatures Th and Tc.

$$\Delta S_{tot} = \Delta S_h + \Delta S_c. \tag{14.25.2}$$

Thus, since we know that $Q_h/T_h = Q_c/T_c$ for a Carnot engine,

$$\Delta S_{tot} = -\frac{Q_h}{T_h} + \frac{Q_c}{T_c} = 0. \tag{14.25.3}$$

This result, which has general validity, means that the total change in entropy for a system in any reversible process is zero.

Stastical Interpretation of Entropy

According the second law of thermodynamics, disorder is vastly more likely than order.

learning objectives

- Calculate the number of microstates for simple configurations

The various ways of formulating the second law of thermodynamics tell what happens rather than why it happens. Why should heat transfer occur only from hot to cold? Why should the universe become increasingly disorderly? The answer is that it is a matter of overwhelming probability. Disorder is simply vastly more likely than order. To illustrate this fact, we will examine some random processes, starting with coin tosses.

Coin Tosses

What are the possible outcomes of tossing 5 coins? Each coin can land either heads or tails. On the large scale, we are concerned only with the total heads and tails and not with the order in which heads and tails appear. The following table shows all possibilities along with numbers of possible configurations (or microstate; a detailed description of every element of a system). For example, 4 heads and 1 tail instance may occur on 5 different configurations, with any one of the 5 coins showing tail and all the rest heads. (HHHHT, HHHTH, HHTHH, HTHHH, THHHH)

- 5 heads, 0 tails: 1 microstate
- 4 heads, 1 tail: 5 microstates
- 3 heads, 2 tails: 10 microstates
- 2 heads, 3 tails: 10 microstates
- 1 head, 4 tails: 5 microstates
- 0 head, 5 tails: 1 microstate

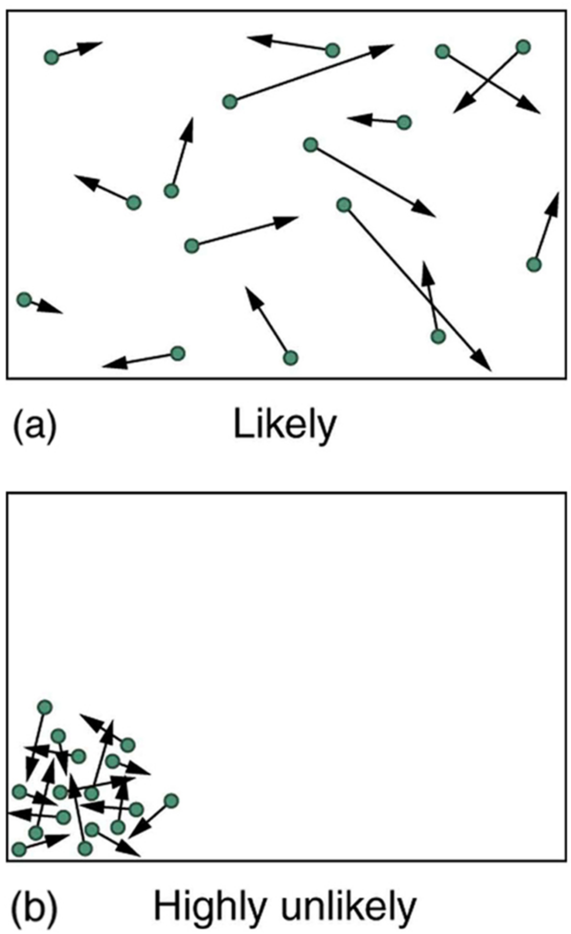
Note that all of these conclusions are based on the crucial assumption that each microstate is equally probable. Otherwise, the analysis will be erroneous.

The two most orderly possibilities are 5 heads or 5 tails. (They are more structured than the others.) They are also the least likely, only 2 out of 32 possibilities. The most disorderly possibilities are 3 heads and 2 tails and its reverse. (They are the least structured.) The most disorderly possibilities are also the most likely, with 20 out of 32 possibilities for the 3 heads and 2 tails and its reverse. If we start with an orderly array like 5 heads and toss the coins, it is very likely that we will get a less orderly array as a result, since 30 out of the 32 possibilities are less orderly. So even if you start with an orderly state, there is a strong tendency to go from order to disorder, from low entropy to high entropy. The reverse can happen, but it is unlikely.

This result becomes dramatic for larger systems. Consider what happens if you have 100 coins instead of just 5. The most orderly arrangements (most structured) are 100 heads or 100 tails. The least orderly (least structured) is that of 50 heads and 50 tails. There is only 1 way (1 microstate) to get the most orderly arrangement of 100 heads. The total number of different ways 100 coins can be tossed—is an impressively large 1.27×10^{30}. Now, if we start with an orderly macrostate like 100 heads and toss the coins, there is a virtual certainty that we will get a less orderly macrostate. If you tossed the coins once each second, you could expect to get either 100 heads or 100 tails once in 2×10^{22} years! In contrast, there is an 8% chance of getting 50 heads, a 73% chance of getting from 45 to 55 heads, and a 96% chance of getting from 40 to 60 heads. Disorder is highly likely.

Real Gas

The fantastic growth in the odds favoring disorder that we see in going from 5 to 100 coins continues as the number of entities in the system increases. In a volume of 1 m^3, roughly 10^{23} molecules (or the order of magnitude of Avogadro's number) are present in a gas. The most likely conditions (or macrostate) for the gas are those we see all the time—a random distribution of atoms in space with a Maxwell-Boltzmann distribution of speeds in random directions, as predicted by kinetic theory as shown in (a). This is the most disorderly and least structured condition we can imagine.



Kinetic Theory: (a) The ordinary state of gas in a container is a disorderly, random distribution of atoms or molecules with a Maxwell-Boltzmann distribution of speeds. It is so unlikely that these atoms or molecules would ever end up in one corner of the container that it might as well be impossible. (b) With energy transfer, the gas can be forced into one corner and its entropy greatly reduced. But left alone, it will spontaneously increase its entropy and return to the normal conditions, because they are immensely more likely.

In contrast, one type of very orderly and structured macrostate has all of the atoms in one corner of a container with identical velocities. There are very few ways to accomplish this (very few microstates corresponding to it), and so it is exceedingly unlikely ever to occur. (See (b).) Indeed, it is so unlikely that we have a law saying that it is impossible, which has never been observed to be violated—the second law of thermodynamics.

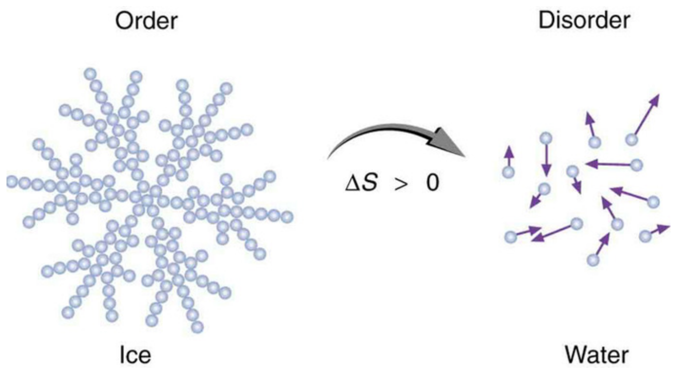
Order to Disorder

Entropy is a measure of disorder, so increased entropy means more disorder in the system.

learning objectives

- Discuss entropy and disorder within a system

Entropy is a measure of disorder. This notion was initially postulated by Ludwig Boltzmann in the 1800s. For example, melting a block of ice means taking a highly structured and orderly system of water molecules and converting it into a disorderly liquid in which molecules have no fixed positions. There is a large increase in entropy in the process.



Entropy of Ice: When ice melts, it becomes more disordered and less structured. The systematic arrangement of molecules in a crystal structure is replaced by a more random and less orderly movement of molecules without fixed locations or orientations. Its entropy increases because heat transfer occurs into it. Entropy is a measure of disorder.

Example $14.25.1$:

As an example, suppose we mix equal masses of water originally at two different temperatures, say 20.0º C and 40.0º C. The result is water at an intermediate temperature of 30.0º C. Three outcomes have resulted:

- Entropy has increased.
- Some energy has become unavailable to do work.
- The system has become less orderly.

Entropy, Energy, and Disorder

Let us think about each of the results. First, entropy has increased for the same reason that it did in the example above. Mixing the two bodies of water has the same effect as heat transfer from the hot one and the same heat transfer into the cold one. The mixing decreases the entropy of the hot water but increases the entropy of the cold water by a greater amount, producing an overall increase in entropy.

Second, once the two masses of water are mixed, there is only one temperature—you cannot run a heat engine with them. The energy that could have been used to run a heat engine is now unavailable to do work.

Third, the mixture is less orderly, or to use another term, less structured. Rather than having two masses at different temperatures and with different distributions of molecular speeds, we now have a single mass with a uniform temperature.

These three results—entropy, unavailability of energy, and disorder—are not only related but are in fact essentially equivalent.

Heat Death

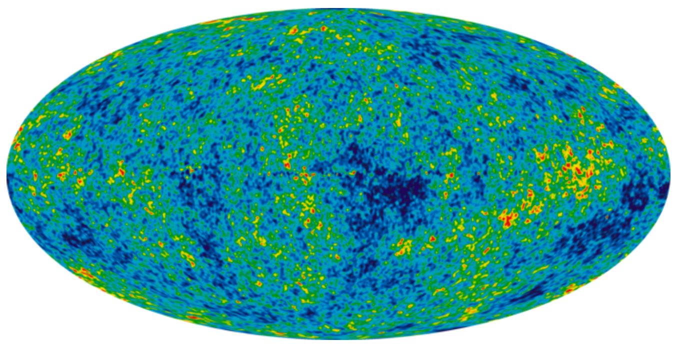
The entropy of the universe is constantly increasing and is destined for thermodynamic equilibrium, called the heat death of the universe.

learning objectives

- Describe processes that lead to the heat death of the universe

In the early, energetic universe, all matter and energy were easily interchangeable and identical in nature. Gravity played a vital role in the young universe. Although it may have seemed disorderly, there was enormous potential energy available to do work—all the future energy in the universe.

As the universe matured, temperature differences arose, which created more opportunity for work. Stars are hotter than planets, for example, which are warmer than icy asteroids, which are warmer still than the vacuum of the space between them. Most of these are cooling down from their usually violent births, at which time they were provided with energy of their own—nuclear energy in the case of stars, volcanic energy on Earth and other planets, and so on. Without additional energy input, however, their days are numbered.



Infant Universe: The image of an infant universe reveals temperature fluctuations (shown as color differences) that correspond to the seeds that grew to become the galaxies.

As entropy increases, less and less energy in the universe is available to do work. On Earth, we still have great stores of energy such as fossil and nuclear fuels; large-scale temperature differences, which can provide wind energy; geothermal energies due to differences in temperature in Earth's layers; and tidal energies owing to our abundance of liquid water. As these are used, a certain fraction of the energy they contain can never be converted into doing work. Eventually, all fuels will be exhausted, all temperatures will equalize, and it will be impossible for heat engines to function, or for work to be done.

Since the universe is a closed system, the entropy of the universe is constantly increasing, and so the availability of energy to do work is constantly decreasing. Eventually, when all stars have died, all forms of potential energy have been utilized, and all temperatures have equalized (depending on the mass of the universe, either at a very high temperature following a universal contraction, or a very low one, just before all activity ceases) there will be no possibility of doing work.

Either way, the universe is destined for thermodynamic equilibrium—maximum entropy. This is often called the heat death of the universe, and will mean the end of all activity. However, whether the universe contracts and heats up, or continues to expand and cools down, the end is not near. Calculations of black holes suggest that entropy can easily continue for at least 10^{100} years.

Living Systems and Evolution

It is possible for the entropy of one part of the universe to decrease, provided the total change in entropy of the universe increases.

learning objectives

- Formulate conditions that allow decrease of the entropy in one part of the universe

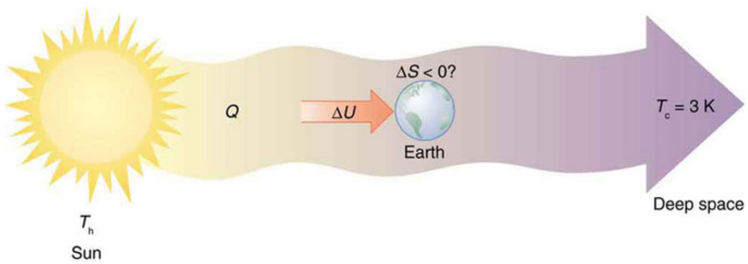
Some people misunderstand the second law of thermodynamics, stated in terms of entropy, to say that the process of the evolution of life violates this law. Over time, complex organisms evolved from much simpler ancestors, representing a large decrease in entropy of the Earth's biosphere. It is a fact that living organisms have evolved to be highly structured, and much lower in entropy than the substances from which they grow. But it is always possible for the entropy of one part of the universe to decrease, provided the total change in entropy of the universe increases. In equation form, we can write this as

$$\Delta S_{tot} = \Delta S_{sys} + \Delta S_{env} > 0. \tag{14.25.4}$$

Thus ΔS_{sys} can be negative as long as ΔS_{env} is positive and greater in magnitude.

How is it possible for a system to decrease its entropy? Energy transfer is necessary. If I gather iron ore from the ground and convert it into steel and build a bridge, my work (and used energy) has decreased the entropy of that system. Energy coming from the Sun can decrease the entropy of local systems on Earth—that is, ΔS_{sys} is negative. But the overall entropy of the rest of the universe increases by a greater amount—that is, ΔS_{env} is positive and greater in magnitude. Thus, $\Delta S_{tot}>0$, and the second law of thermodynamics is not violated.

Every time a plant stores some solar energy in the form of chemical potential energy, or an updraft of warm air lifts a soaring bird, the Earth can be viewed as a heat engine operating between a hot reservoir supplied by the Sun and a cold reservoir supplied by dark outer space—a heat engine of high complexity, causing local decreases in entropy as it uses part of the heat transfer from the Sun into deep space. However, there is a large total increase in entropy resulting from this massive heat transfer. A small part of this heat transfer is stored in structured systems on Earth, producing much smaller local decreases in entropy.



Earth's Entropy: Earth's entropy may decrease in the process of intercepting a small part of the heat transfer from the Sun into deep space. Entropy for the entire process increases greatly while Earth becomes more structured with living systems and stored energy in various forms.

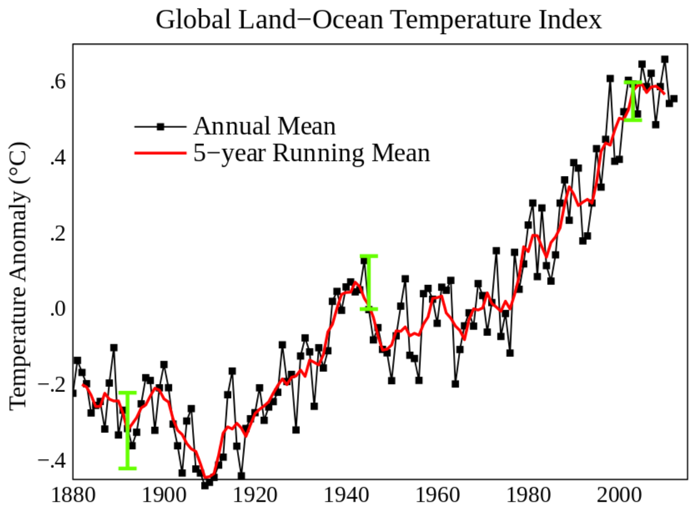
Global Warming Revisited

The Second Law of Thermodynamics may help provide explanation for the global warming over the last 250 years.

learning objectives

- Describe effect of the heat dumped into the environment on the Earth's atmospheric temperature

The Second Law of Thermodynamics may help provide explanation for why there have been increases in Earth's temperatures over the last 250 years (often called "Global Warming"), and many professionals are concerned that the entropy increase of the universe is a real threat to the environment.



Global Land-Ocean Temperature: Global mean land-ocean temperature change from 1880 – 2012, relative to the 1951 – 1980 mean. The black line is the annual mean and the red line is the five-year running mean. The green bars show uncertainty estimates.

As an engine operates, heat flows from a heat tank of greater temperature to a heat sink of lesser temperature. In between these states, the heat flow is turned into useful energy with the help of heat engines. As these engines operate, however, a great deal of heat is lost to the environment due to inefficiencies. In a Carnot engine, which is the most efficient theoretical engine based on Carnot cycle, the maximum efficiency is equal to one minus the temperature of the heat sink (T_c) divided by the temperature of the heat source (T_h).

$$(\mathrm{Eff_c} = 1 - \frac{\mathrm{T_c}}{\mathrm{T_h}}). \tag{14.25.5}$$

This ratio shows that for a greater efficiency to be achieved there needs to be the greatest difference in temperature available. This brings up two important points: optimized heat sinks are at absolute zero, and the longer engines dump heat into an isolated system the less efficient engines will become.

Unfortunately for engine efficiency, day-to-day life never operates in absolute zero. In an average car engine, only 14% to 26% of the fuel which is put in is actually used to make the car move forward. This means that 74% to 86% is lost heat or used to power accessories. According to the U.S. Department of Energy, 70% to 72% of heat produced by burning fuel is heat lost by the engine. The excess heat lost by the engine is then released into the heat sink, which in the case of many modern engines would be the Earth’s atmosphere. As more heat is dumped into the environment, Earth’s atmospheric (or heat sink) temperature will increase. With the entropy of the environment constantly increasing, searching for new, more efficient technologies and new non-heat engines has become a priority.

Thermal Pollution

Thermal pollution is the degradation of water quality by any process that changes ambient water temperature.

learning objectives

- Identify factors that lead to thermal pollution and its ecological effects

Thermal pollution is the degradation of water quality by any process that changes ambient water temperature. A common cause of thermal pollution is the use of water as a coolant, for example, by power plants and industrial manufacturers. When water used as a coolant is returned to the natural environment at a higher temperature, the change in temperature decreases oxygen supply, and affects ecosystem composition.

As we learned in our Atom on “Heat Engines”, all heat engines require heat transfer, achieved by providing (and maintaining) temperature difference between engine’s heat source and heat sink. Water, with its high heat capacity, works extremely well as a coolant. But this means that cooling water should be constantly replenished to maintain its cooling capacity.

Cooling Tower: This is a cooling tower at Gustav Knepper Power Station, Dortmund, Germany. Cooling water is circulated inside the tower.

Ecological Effects

Elevated water temperature typically decreases the level of dissolved oxygen of water. This can harm aquatic animals such as fish, amphibians, and other aquatic organisms. An increased metabolic rate may result in fewer resources; the more adapted organisms moving in may have an advantage over organisms that are not used to the warmer temperature. As a result, food chains of the old and new environments may be compromised. Some fish species will avoid stream segments or coastal areas adjacent to a thermal discharge. Biodiversity can decrease as a result. Many aquatic species will also fail to reproduce at elevated temperatures.

Some may assume that by cooling the heated water, we can possibly fix the issue of thermal pollution. However, as we noted in our previous Atom on “Heat Pumps and Refrigerators”, work required for the additional cooling leads to more heat exhaust into the environment. Therefore, it makes the situation even worse.

Key Points

- This ratio of Q/T is defined to be the change in entropy ΔS for a reversible process: $\Delta S=(\frac{Q}{T})_{rev}$.
- Entropy is a property of state. Therefore, the change in entropy ΔS of a system between two states is the same no matter how the change occurs.
- The total change in entropy for a system in any reversible process is zero.
- Each microstate is equally probable in the example of coin toss. However, as a macrostate, there is a strong tendency for the most disordered state to occur.
- When tossing 100 coins, if the coins are tossed once each second, you could expect to get either all 100 heads or all 100 tails once in 2×10^{22} years.
- Molecules in a gas follow the Maxwell-Boltzmann distribution of speeds in random directions, which is the most disorderly and least structured condition out of all the possibilities.
- Mixing two systems may decrease the entropy of one system, but increase the entropy of the other system by a greater amount, producing an overall increase in entropy.
- After mixing water at two different temperatures, the energy in the system that could have been used to run a heat engine is now unavailable to do work. Also, the process made the whole system more less structured.
- Entropy, unavailability of energy, and disorder are not only related but are in fact essentially equivalent.
- In the early, energetic universe, all matter and energy were easily interchangeable and identical in nature.
- As entropy increases, less and less energy in the universe is available to do work.
- The universe is destined for thermodynamic equilibrium —maximum entropy. This is often called the heat death of the universe, and will mean the end of all activity.
- Living organisms have evolved to be highly structured, and much lower in entropy than the substances from which they grow.
- It possible for a system to decrease its entropy provided the total change in entropy of the universe increases: $\Delta S_{tot}=\Delta S_{sys}+\Delta S_{env}>0$.
- The Earth can be viewed as a heat engine operating between a hot reservoir supplied by the Sun and a cold reservoir supplied by dark outer space.
- As heat engines operate, a great deal of heat is lost to the environment due to inefficiencies.

- Even in a Carnot engine, which is the most efficient theoretical engine, there is a heat loss determined by the ratio of temperature of the engine and its environment.
- As more heat is dumped into the environment, Earth's atmospheric temperature will increase.
- All heat engines require heat transfer, achieved by providing (and maintaining) temperature difference between engine's heat source and heat sink. Cooling water is typically used to maintain the temperature difference.
- Elevated water temperature typically decreases the level of dissolved oxygen of water, affecting ecosystem composition.
- Cooling heated water is not a solution for thermal pollution because extra work is required for the cooling, leading to more heat exhaust into the environment.

Key Terms

- **Carnot cycle**: A theoretical thermodynamic cycle. It is the most efficient cycle for converting a given amount of thermal energy into work.
- **reversible**: Capable of returning to the original state without consumption of free energy and increase of entropy.
- **disorder**: Absence of some symmetry or correlation in a many-particle system.
- **Maxwell-Boltzmann distribution**: A distribution describing particle speeds in gases, where the particles move freely without interacting with one another, except for very brief elastic collisions in which they may exchange momentum and kinetic energy.
- **entropy**: A measure of how evenly energy (or some analogous property) is distributed in a system.
- **geothermal**: Pertaining to heat energy extracted from reservoirs in the Earth's interior.
- **asteroid**: A naturally occurring solid object, which is smaller than a planet and is not a comet, that orbits a star.
- **Carnot cycle**: A theoretical thermodynamic cycle. It is the most efficient cycle for converting a given amount of thermal energy into work.
- **absolute zero**: The coldest possible temperature: zero on the Kelvin scale and approximately -273.15°C and -459.67°F. The total absence of heat; the temperature at which motion of all molecules would cease.
- **heat engine**: Any device which converts heat energy into mechanical work.
- **heat pump**: A device that transfers heat from something at a lower temperature to something at a higher temperature by doing work.

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14.26: The Third Law of Thermodynamics

learning objectives

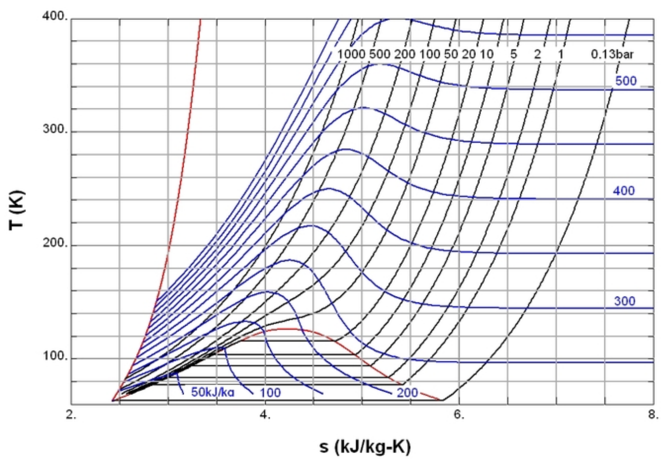
- Explain how absolute zero affects entropy

The third law of thermodynamics is sometimes stated as follows: *The entropy of a perfect crystal at absolute zero is exactly equal to zero.*

At zero kelvin the system must be in a state with the minimum possible energy, thus this statement of the third law holds true if the perfect crystal has only one minimum energy state. Entropy is related to the number of possible microstates, and with only one microstate available at zero kelvin the entropy is exactly zero.

The third law was developed by the chemist Walther Nernst during the years 1906-1912. It is often referred to as Nernst's theorem or Nernst's postulate. Nernst proposed that the entropy of a system at absolute zero would be a well-defined constant. Instead of being 0, entropy at absolute zero could be a nonzero constant, due to the fact that a system may have degeneracy (having several ground states at the same energy).

In simple terms, the third law states that the entropy of a perfect crystal approaches zero as the absolute temperature approaches zero. This law provides an absolute reference point for the determination of entropy. (diagrams the temperature entropy of nitrogen.) The entropy (S) determined relative to this point is the absolute entropy represented as follows:



Temperature Entropy of Nitrogen: Temperature–entropy diagram of nitrogen. The red curve at the left is the melting curve. Absolute value of entropy can be determined shown here, thanks to the third law of thermodynamics.

$$S = k_B \log W, \tag{14.26.1}$$

where k_B is the Bolzmann constant and W is the number of microstates. Provided that the ground state is unique (or $W = 1$), the entropy of a *perfect* crystal lattice as defined by Nernst's theorem is zero provided that its ground state is unique, because $\log(1) = 0$.

Adiabatic Processes

It is impossible to reduce the temperature of any system to zero temperature in a finite number of finite operations.

learning objectives

- Illustrate isentropic process, for example in terms of adiabtic demangetization

In our Atom on "Adiabatic Processes" (category: the First Law of Thermodynamics), we learned that an adiabatic process is any process occurring without gain or loss of heat within a system. We also learned a monatomic ideal gas expands adiabatically. In this

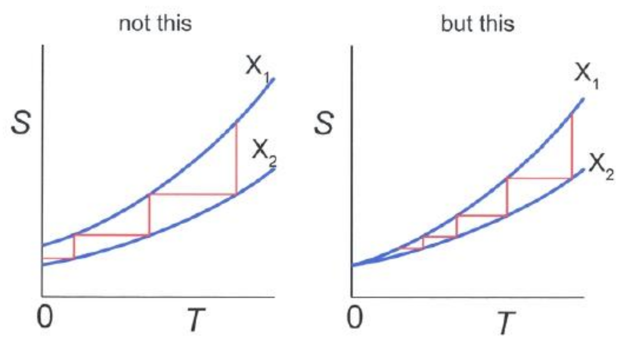
Atom, we discuss an adiabatic cooling process that can be used to cool a gas, as well as whether absolute zero can be obtained in real systems.

Absolute Zero?

Previously, we learned about the third law of thermodynamics, which states: *the entropy of a perfect crystal at absolute zero is exactly equal to zero.*

According to the third law, the reason that T=0 cannot be reached is explained as follows: Suppose the temperature of a substance can be reduced in an isentropic process by changing the parameter X from X_2 to X_1. As an example, one can think of a multistage adiabatic magnetization-demagnetization cycle setup where a magnetic field is switched on and off in a controlled way. (See below. The parameter X in this case would be the magnetization of the gas.)

Assuming an entropy difference at absolute zero, T=0 could be reached in a finite number of steps. However, going back to the third law, at T=0 there is no entropy difference, and therefore an infinite number of stepswould be needed for this process (illustrated in).



Can Absolute Zero be Reached?: Temperature-Entropy diagram. Horizontal lines represent isentropic processes, while vertical lines represent isothermal processes. Left side: Absolute zero can be reached in a finite number of steps if S(T=0,X1)≠S(T=0, X2). Right: An infinite number of steps is needed since S(0,X1)= S(0,X2).

Adiabatic Demagnetization Cooling

In simple terms, the cooling scheme mentioned above occurs by repeating the following steps:

1. A strong magnetic field is applied to adiabatically align magnetic moments of the particles in the gas.
2. The magnetic field is reduced adiabatically, and thermal energy of the gas causes "ordered" magnetic moments to become random again.

In the second step, thermal energy in the gas is used to cause disorder of magnetic moments. Therefore, the temperature of the gas is reduced (hence cooling works). This scheme is called *adiabatic demagnetization cooling*.

Key Points

- Entropy is related to the number of possible microstates, and with only one microstate available at zero kelvin, the entropy is exactly zero.
- The third law of thermodynamics provides an absolute reference point for the determination of entropy. The entropy determined relative to this point is the absolute entropy.
- Absolute entropy can be written as $S = k_B \backslash logW$, where W is the number of available microstates.
- Since at $T = 0$ there is no entropy difference, an infinite number of steps in a thermodynamic cooling process is required to reach T=0.
- In adiabatic demagnetization cooling, energy transfers from thermal entropy to magnetic entropy (or disorder of the magnetic moments).
- The fact that T=0 cannot be achieved in reality is a direct consequence of the third law of thermodynamics.

Key Terms

- **microstate**: The specific detailed microscopic configuration of a system.
- **absolute zero**: The coldest possible temperature: zero on the Kelvin scale and approximately -273.15°C and -459.67°F. The total absence of heat; the temperature at which motion of all molecules would cease.
- **degeneracy**: Two or more different quantum states are said to be degenerate if they are all at the same energy level.
- **isentropic**: Having a constant entropy.
- **demagnetization**: The process of removing the magnetic field from an object.

Index

T

U

V

Glossary

Sample Word 1 | Sample Definition 1

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    - 2.1: Basics of Kinematics - *Undeclared*
    - 2.2: Speed and Velocity - *Undeclared*
    - 2.3: Acceleration - *Undeclared*
    - 2.4: Problem-Solving for Basic Kinematics - *Undeclared*
    - 2.5: Free-Falling Objects - *Undeclared*
  - 3: Vectors - *CC BY 4.0*
    - 3.1: Prelude to Vectors - *CC BY 4.0*
    - 3.2: Scalars and Vectors (Part 1) - *CC BY 4.0*
    - 3.3: Scalars and Vectors (Part 2) - *CC BY 4.0*
    - 3.4: Coordinate Systems and Components of a Vector (Part 1) - *CC BY 4.0*
    - 3.5: Coordinate Systems and Components of a Vector (Part 2) - *CC BY 4.0*
    - 3.6: Algebra of Vectors - *CC BY 4.0*
    - 3.7: Algebra of Vectors Examples - *CC BY 4.0*
    - 3.8: Products of Vectors (Part 1) - *CC BY 4.0*
    - 3.9: Products of Vectors (Part 2) - *CC BY 4.0*
    - 3.10: Vectors (Answers) - *CC BY 4.0*
    - 3.11: Vectors (Exercises) - *CC BY 4.0*
    - 3.12: Vectors (Summary) - *CC BY 4.0*
  - 4: Motion Along a Straight Line - with Vectors - *CC BY 4.0*
    - 4.1: Prelude Motion Along a Straight Line - *CC BY 4.0*
    - 4.2: Position, Displacement, and Average Velocity - *CC BY 4.0*
    - 4.3: Instantaneous Velocity and Speed - *CC BY 4.0*
    - 4.4: Average and Instantaneous Acceleration - *CC BY 4.0*
    - 4.5: Motion with Constant Acceleration (Part 1) - *CC BY 4.0*
    - 4.6: Motion with Constant Acceleration (Part 2) - *CC BY 4.0*
    - 4.7: Free Fall - *CC BY 4.0*
    - 4.8: Finding Velocity and Displacement from Acceleration - *CC BY 4.0*
    - 4.9: Motion Along a Straight Line (Exercises) - *CC BY 4.0*
    - 4.10: Motion Along a Straight Line (Summary) - *CC BY 4.0*
  - 5: Two-Dimensional Kinematics - *Undeclared*
    - 5.1: Prelude to Motion in Two and Three Dimensions - *CC BY 4.0*
    - 5.2: Displacement and Velocity Vectors - *CC BY 4.0*
    - 5.3: Acceleration Vector - *CC BY 4.0*
    - 5.4: Projectile Motion - *CC BY 4.0*
    - 5.5: Uniform Circular Motion - *CC BY 4.0*
    - 5.6: Relative Motion in One and Two Dimensions - *CC BY 4.0*