4.1: Repackaging Newton's Second Law

Definition of Impulse

In chapter 2, we made a point of emphasizing that force is not possessed by objects – it is an interaction between them. One way we know this is that if the same force is exerted on identical objects that start at rest, the two objects are not necessarily moving the same afterward. There is an important ingredient missing here – the *duration* that the force acts. Since the force causes an acceleration, the longer it acts, the more the velocity is affected. So multiplying the force by the amount of time it acts may provide us with a useful quantity. The force may be changing magnitude or direction while it acts, but over a very short time this product is:

$$d\overrightarrow{J} = \overrightarrow{F}\,dt \tag{4.1.1}$$

If we want to know the totality of this quantity over a finite time interval, we need to add up all these little contributions. We give this quantity the name *impulse*.

Definition: Impulse

$$\overrightarrow{J}_{tot}\left(t_A \rightarrow t_B\right) \equiv \int\limits_{t_A}^{t_B} \overrightarrow{F}_{net}\,dt$$

This quantity is the sum of the product of the forces and the times over which those forces act. This certainly sounds very similar to work, which takes a product of forces and displacements. Also, impulse will have an impact on the motion of the object, as work did. But there are also many differences between these two quantities.

The first difference between impulse and work is that they obviously represent different physical quantities, because they have different units. While work has units of energy which we measure in Joules (or Newton-meters), impulse has units of force-times-time, measured in Newton-seconds. A second difference is that the impulse integral (mercifully) is not a line integral – there is no "path" to concern ourselves with when computing impulse. And third, because there is no dot product involved with the impulse integral, the result is a vector, in contrast to work, which is a scalar.

Definition of Momentum

The definition of impulse is not the end of the story, any more than the definition of work was. It needs to be related to the effect it has on the motion of the object. In the case of work, this relationship was expressed as the work-energy theorem:

$$\begin{array}{rcl} \text{actions of pushes and pulls} & = & W_{tot}\left(A \rightarrow B\right) \\ & = & \int\limits_A^B \overrightarrow{F}_{net} \cdot \overrightarrow{dl} \\ & = & \frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 \\ & = & \Delta KE \\ & = & \text{effect of pushes and pulls} \end{array} \tag{4.1.2}$$

For the case of impulse, we find this relationship again by coming back to Newton's second law, and noting that the integral of acceleration is velocity:

$$\begin{array}{rcl}\text{actions of pushes and pulls} & = & \overrightarrow{J}_{tot}\left(t_A \rightarrow t_B\right) \\ & = & \int\limits_{t_A}^{t_B} \overrightarrow{F}_{net} dt \\ & = & \int\limits_{t_A}^{t_B} \left[m\overrightarrow{a}_{cm}\right] dt \\ & = & \left[m\overrightarrow{v}_{cm}\right]_A^B \\ & = & \Delta\left(m\overrightarrow{v}_{cm}\right) \equiv \Delta\overrightarrow{p}_{cm} \\ & = & \text{effect of pushes and pulls}\end{array} \tag{4.1.3}$$

We call the quantity $\overrightarrow{p}$ *momentum* (which for a single object is defined as its mass multiplied by its velocity vector).

Definition: Momentum

$$\overrightarrow{p} \equiv m\overrightarrow{v}$$

Equation 4.1.3 is known as the *impulse-momentum theorem*. Like kinetic energy, momentum is related to the motion of the object (and the mass), but besides being a different function of mass and velocity than kinetic energy, it is also different in that it is a vector. This means that the total impulse can lead to a change in the magnitude or direction (or both) of the momentum vector.

The astute reader will undoubtedly realize that all we have really done here is to introduce a new vector, and use it to repackage Newton's second law. Indeed, we can rewrite the second law thus:

$$\overrightarrow{F}_{net} = m\overrightarrow{a}_{cm} = m\frac{d}{dt}\overrightarrow{v}_{cm} = \frac{d}{dt}\left(m\overrightarrow{v}_{cm}\right) = \frac{d}{dt}\overrightarrow{p}_{cm} \tag{4.1.4}$$

Given we are making comparisons between momentum and kinetic energy, it is useful to point out a direct mathematical relationship, which not only points out the difference between the two quantities, but will also be quite useful later on:

$$KE = \frac{1}{2}mv^2 = \frac{(mv)^2}{2m} = \frac{p^2}{2m} = \frac{\overrightarrow{p}\cdot\overrightarrow{p}}{2m} \tag{4.1.5}$$

Systems and Momentum Conservation

Let's continue following the trajectory from our discussion of work-energy, by returning to the idea of defining a system. As before, we define a system as an arbitrarily-grouped collection of objects, which can experience forces between themselves, or from outside the system. Previously we said that forces between objects within the system were responsible for internal work and forces exerted on objects within the system from outside provide external work. We will now similarly define internal impulse as coming from forces between objects within the system, and external impulse as coming from objects outside the system.

When it comes to forces between objects within our defined system, we know that the work done on one object does not cancel the work done on the other object. If the internal force is conservative, the non-zero total work done between the objects equals the energy converted between kinetic and potential energy. If the internal force is non-conservative, then the non-zero total work done between the objects equals the energy converted from mechanical to thermal energy. Is there an analogous process for impulse?

To answer this question, we need to determine whether impulses internal to a system don't cancel out, as in the case of work. We again start with Newton's third law, which ensures that the two forces involved in creating the pair of impulses are equal-and-opposite. Impulse vectors have the same directions as their associated force vectors, so the third law pair of forces results in a pair of impulses that are in opposite directions. But what about the magnitudes? Well, the force magnitudes are equal thanks to the third law, so all that remains is the time interval. There is never a moment when a force is acting that its third law pair isn't also acting, so the time intervals are the same. This leads to the following very important result: *All of the impulses internal to a system cancel each other out*. This means that there is no momentum analog to potential or thermal energy within a system. There is only momentum, and if the system experiences no external impulses, then momentum is conserved for the system. Comparing to what we got for energy, it looks like this:

$$\begin{array}{cc} \textbf{work-energy} & \textbf{impulse-momentum} \\ W_{ext} = \Delta KE \underbrace{- W_{cons} - W_{non-cons}}_{from\;internal\;forces} & \overrightarrow{J}_{ext} = \Delta \overrightarrow{p}_{cm} \underbrace{- \overrightarrow{J}_{cons} - \overrightarrow{J}_{non-cons}}_{from\;internal\;forces} \\ W_{ext} = \Delta KE + \Delta PE + \Delta E_{thermal} & \overrightarrow{J}_{ext} = \Delta \overrightarrow{p}_{cm} + 0 \quad + 0 \end{array} \tag{4.1.6}$$

There are two important features of this result:

1. ***It doesn't matter what forces are acting internally***. The result we obtained made no mention of whether the internal force was conservative or non-conservative – all forces satisfy Newton's third law, and the pairs act for equal periods of time, so the impulses cancel regardless of the nature of the force.
2. ***The quantity (momentum) that is conserved within the closed system is a vector***. This means that adding up all of the momentum vectors of a system at one point in time, then doing so again at another point in time, will give the same total vector in both cases, if the system is isolated from external impulses. This means that the total magnitude and direction don't change, or equivalently that the components measured in a given coordinate system don't change.

Example 4.1.1

Two blocks of different masses are attached to identical springs that are horizontal to the frictionless surface on which the block rests. If the springs are stretched the same distance and the blocks are released from rest, how do the following quantities compare for the two blocks when they reach the equilibrium point?

a. *kinetic energy*
b. *momentum*
c. *velocity*

Solution

a. The springs are stretched an equal amount, which means they both store the same potential energy. That means that when they get to the equilibrium point where they both have zero potential energy, they must have the same kinetic energy, since the mechanical energy is conserved.

b. We can determine the difference in momenta for the two blocks in two ways. First, we can consider the impulse given to each block by the spring. In the case of the more massive block, the spring force will accelerate it less, which means it will take longer to get to the equilibrium point. At every point during their journeys, the two blocks experience the same amount of force, but since the time interval for the heavier block is longer, it must experience the greater impulse. Therefore the heavier block gains more momentum, and since both blocks started with zero momentum, the heavier block must have more momentum at the equilibrium point. The second solution is much simpler: We already know that the two blocks end with the same KE, so since $KE = p^2/2m$, the block with the greater mass must have more momentum.

c. With the same kinetic energy, using $KE = \frac{1}{2}mv^2$, we see that the block with the greater mass must have the lower velocity.

The moral of this story: Although we tend to use kinetic energy, momentum, and velocity as proxies for motion, they are all quite different quantities.

Partial Momentum Conservation

We have to give some extra thought to what we mean by a conserved vector. Since a vector has both magnitude and direction, then to be conserved, both of those properties must remain unchanged. An equivalent way of saying this is that for the vector to be conserved, every component of that vector must be individually conserved. If the full momentum vector is not conserved, it is still possible for one or two of its components to be conserved, if the components of the external impulse in those directions is zero. So for example, a projectile (with no air resistance) conserves momentum in the two horizontal directions, but not in the vertical direction. This allows us to use momentum conservation to solve a much broader range of problems than if we can only consider complete momentum conservation.