7.1: Universal Gravitation

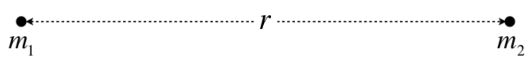
Newton Again

We return to a topic we have discussed only in the simplest of terms, but which has a great deal more depth. Of Newton's many achievements, one of his greatest has to have been the amazing realization that the gravity force is not simply a terrestrial phenomenon. Until he came along, people thought that objects "naturally" fall when they are near the Earth, and that heavenly bodies "naturally" do their little dance. To make the connection that the motions of planets could be explained using the very same paradigm that explains why things fall to Earth is truly a great achievement in human thought. Newton (apocryphally after seeing an apple fall from a tree) called this his *law of universal gravitation*, with emphasis on "universal," as it points out that the law applies both on Earth and in the heavens.

The key to Newton's idea is that the gravitational force actually exists between two objects and depends upon the masses of each and their separation in space. The Earth is no more special than the apple – both attract each other with equal force (which we know from the third law already), and the magnitude of that force depends upon their masses and their separations.

This actually does not fit well with our current understanding of the gravity force. In particular, we have been saying that the force equals mg, even as the height (distance from the Earth's surface) changes, so how is this dependent upon separation? First of all, it turns out that it is not the separation of the outer surfaces of the objects that matters, but rather their centers. In fact, it is even more complicated than that, so to simplify it, let's first just assume that the two gravitating objects are very small (point masses), so that their separation is well-defined:

Figure 7.1.1 – Two Point Masses Separated by *r*



In this case, the basis for Newton's law of universal gravitation can be described as follows:

- the force is exclusively attractive – experimentally, we only see gravity act as a "pull."
- the strength of the force grows linearly with the amount of each mass – experimentally, we find that the force doubles when we double either of the two masses involved, triples when either mass is tripled, and so on.
- the strength of the force varies in inverse proportion with the square of the separation – experimentally, we find that doubling the separation of the two objects reduces the force by a factor of four, tripling the separation reduces the force by a factor of nine, and so on.

Assuming these are the only factors that come into play for gravity (for example, the relative motions of the two objects doesn't affect the force), then we can write a proportionality for the magnitude of the gravity force between two point masses:

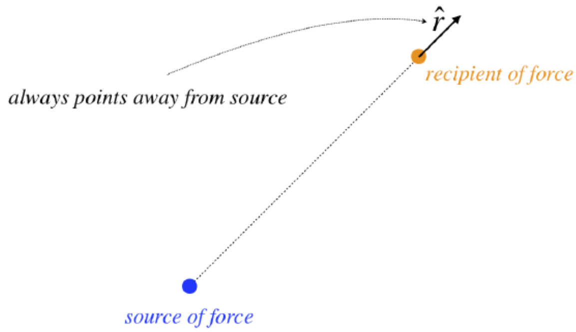
$$F_{gravity} \sim \frac{m_1 m_2}{r^2} \tag{7.1.1}$$

This satisfies all the criteria given above. All that remains is to turn it into an equality by inserting a multiplicative constant that turns it into units of force, with the correct observed magnitude:

$$F_{gravity} = \frac{G m_1 m_2}{r^2}, \qquad G \equiv 6.67 \times 10^{-11} \frac{Nm^2}{kg^2} \tag{7.1.2}$$

With the strength of the force, and the knowledge that it is attractive in nature, we have Newton's law of gravity. As usual, we would like to write this in a compact way that included the direction – as a vector equation. To do this, we temporarily discard the "equal partner" view, and treat one of the point masses as the source of the force (the object that the force is "by"), and the other as the recipient (the object the force is "on"). As an object's motion is determined by the forces on it, we treat the source of the force as the "origin," and define the position vector as pointing from the origin to the object on which the force acts. Therefore the unit vector of the position vector at the affected object always points away from the source of the gravity force.

Figure 7.1.2 – Defining Position Unit Vector for Gravity



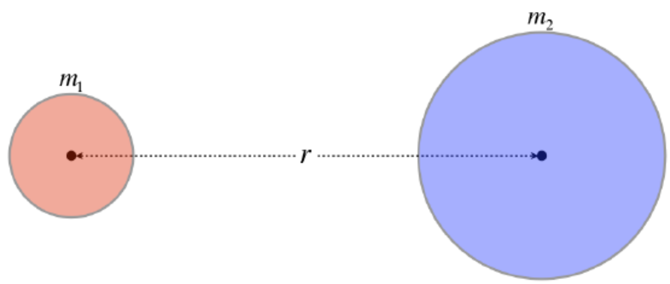
With this defined, we see that the attractive force on the recipient points in the $-\hat{r}$ direction, giving us a nice, compact vector equation for Newton's law of gravity:

$$\overrightarrow{F}_{gravity} = \frac{Gm_1m_2}{r^2}(-\hat{r}) \tag{7.1.3}$$

Spherical Bodies

Now of course we really aren't especially interested in gravity between point masses, when everything we see has some extension in space. So really what we have to do is treat two bodies as collections of point particles, all of which are attracting every other point particle. But this is quite cumbersome, and leads to all sorts of integral calculus. For our purposes, we will simply state the result that in the case of spherically-symmetric objects, they can be treated (in terms of gravitation) as if all of their mass were concentrated at their centers.

Figure 7.1.3 – Spheres Gravitate Like Point Masses at Their Centers



[*Note: This "spherical symmetry" does not require that the density of the spheres be uniform – the density can still vary radially. So the spheres can (for example) be more dense near their centers than near their surfaces, but the density cannot vary with the polar or azimuthal angles. That is, sampling the density throughout the sphere must reveal the same density everywhere that the distance from the sphere's center is held constant.*]

This turns out to be a convenient consequence of the inverse-square law, as you will no doubt examine in greater detail in more advanced math & physics classes. This result is something we will exploit greatly (at least as an approximation), since planets and stars are very close to being spherical.

Gravity at the Earth's Surface

Imagine a very small object (which can be effectively treated as a point object) was pulled toward a large spherical body, and stopped when it reached its surface. In that case, the gravitational force would be calculated using the radius of the large spherical object. Now we'll let that large spherical object be the Earth, and let the small object be a human (you).

Figure 7.1.4 – Universal Gravitation at the Earth's Surface

[*Note that the separation R_E points to the center of the Earth, not to a point on the Earth's surface just off the coast of Florida.]*

We can use now Newton's law of gravity to compute the force exerted on you. All we need is the radius of the Earth ($6.38\times10^6 m$), and the mass of the Earth ($5.98\times10^{24}kg$):

$$your\ weight=\frac{Gm_1m_2}{R_E^2}=\frac{\left(6.67\times10^{-11}\frac{Nm^2}{kg^2}\right)\left(5.98\times10^{24}kg\right)m}{\left(6.38\times10^6 m\right)^2}=\left(9.80\frac{m}{s^2}\right)m=mg \tag{7.1.4}$$

So now you know where our constant g comes from. Now you might be concerned that our projectile calculations have not been accurate, because g is only correct at the surface of the earth, and projectiles might go quite high. Let's look at an example – how much does the gravitational force decrease when we go high up in the sky in a commercial airline? Commercial flights typically fly at an altitude of about $10,000m$(about $33,000ft$), so making the adjustment to the gravitational force gives:

$$your\ weight\ in\ airplane=\frac{\left(6.67\times10^{-11}\frac{Nm^2}{kg^2}\right)\left(5.98\times10^{24}kg\right)m}{\left(6.38\times10^6 m+1\times10^4 m\right)^2}=\left(9.77\frac{m}{s^2}\right)m \tag{7.1.5}$$

It's hardly noticeable. If you weighed yourself on earth and were 150 lbs, then in the plane the scale would read 149.5lbs. Okay, so let's go to a place where we know the distance makes a big difference – all the way into outer space to the international space station (ISS). The altitude in this case is about $400,000m$

$$your\ weight\ in\ ISS=\frac{\left(6.67\times10^{-11}\frac{Nm^2}{kg^2}\right)\left(5.98\times10^{24}kg\right)m}{\left(6.38\times10^6 m+4\times10^5 m\right)^2}=\left(8.68\frac{m}{s^2}\right)m \tag{7.1.6}$$

Wait just a minute… How can those people be floating around their space station if they have only lost about 11% of their weight?

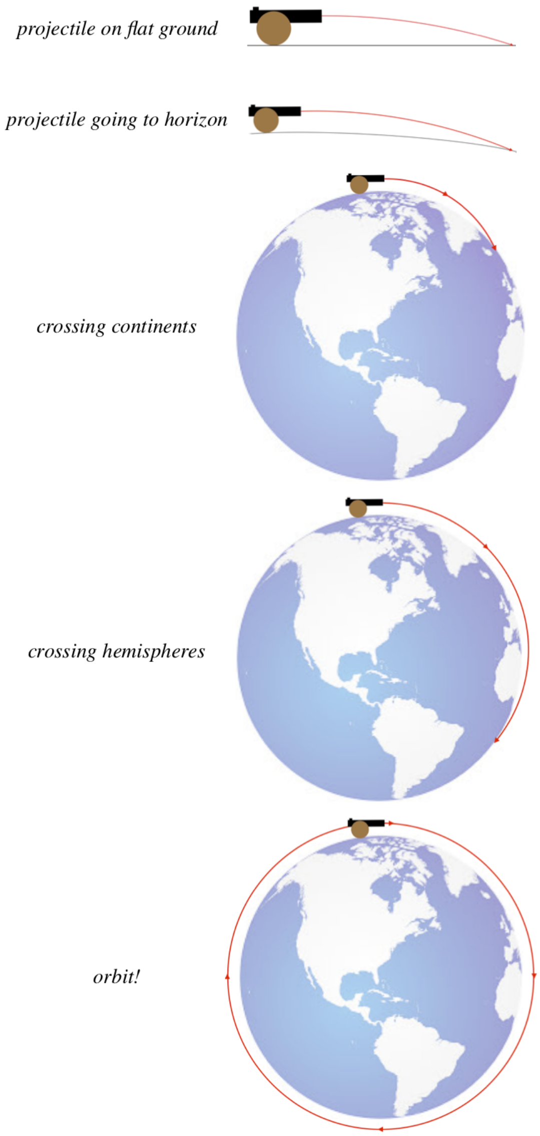
Free-Fall and Orbits

Suppose you are at the top floor of a skyscraper in an elevator when suddenly, tragically, the cable breaks. Assuming you could see past what I can only assume would be your abject terror, what would you see going on around you? The other screaming people around you, the hat on your head, and the penny that was on the floor would all be accelerating at the same rate, g. Since nothing is accelerating faster than anything else, if you hold out your pencil and release it, it doesn't drop to the floor of the elevator – it just "floats" there in front of you. In effect, the entire contents of the elevator is experiencing zero gravity.

Even conceding that being in a room in free-fall is equivalent to zero gravity, the space station is not plummeting to Earth, so how does it apply? Well, we know from our study of projectile motion that the horizontal motion of an object doesn't take away from the fact that it is in free-fall vertically. Indeed, there exist companies that fly planes in parabolic projectile trajectories so that the passengers can experience weightlessness for a couple minutes (before they have to pull out of the dive). So in fact if our elevator were a projectile, we would have the same zero gravity experience. Newton knew this, and came up with the following incredibly clever thought experiment:

If we fire a cannon horizontally, the cannonball follows the usual parabolic path, landing some distance away. If we increase the muzzle velocity, it goes farther before landing. Increase the muzzle velocity even more, and the landing point approaches the horizon. As we keep going this way, the projectile "falls over" the curvature of the Earth, and when the speed is finally fast enough, it never actually lands! Orbits are just the most extreme case of projectile motion.

Figure 7.1.5 – Newton's Cannon



So if being inside a container in free-fall is equivalent to being weightless, then a mouse inside a hollow cannonball fired by Newton's cannon would conclude that there is "no gravity," because the orbiting cannonball is a projectile in free-fall at all times. The astronauts on the space station experience weightlessness not because the Earth's gravitational influence is zero out there, but because that influence is the same on everything in the station, and everything is therefore in free-fall at the same rate. Indeed, if there was no gravity out there, then there would be no force to keep the space station moving in a circle, and it would fly away from Earth!

To understand how important this argument was in the context of his time, Newton used it to explain how a single phenomenon (gravity) could explain both terrestrial (projectile) motion and heavenly (orbital) motion at the same time. What is more, he backed it all up with mathematics! His law of universal gravitation predicted to very high precision the motions of the planets, even as it predicts the motions of cannonballs.