5.5: Torque

Rotational Newton's Second Law

As we saw for linear motion, we can only go so far with energy conservation. If we want to analyze aspects of motion such as elapsed time and direction of motion, we need more than mechanical energy conservation to work with. In the linear case, we found that this meant that we had to use Newton's Second Law. We now seek the rotational equivalent of that law.

The rotational equivalent of the Newton's Second Law must relate the reaction of the system (rotational acceleration) to an external influence (rotational force), with the degree of this effect being determined by an internal property of the system (rotational mass). That is, we need a rotational substitute for all of the participants of this formula:

$$\overrightarrow{a}_{cm} = \frac{\overrightarrow{F}_{net}}{m} \tag{5.5.1}$$

We already found a rotational version of acceleration in our discussion of rotational kinematics – it is the angular acceleration. We even defined a direction for this vector using the right-hand rule. The center of mass qualification in the case above is unneeded for the rotational case, because the angular acceleration is the same about every point on a rigid object.

We have also determined an appropriate candidate for the "rotational mass" – the rotational inertia. This is certainly a reasonable choice, for a couple of reasons. First, from our direct experience we know that it is easier to swing an object (e.g. a baseball bat) when holding the heavier end than when holding the lighter end, so the degree to which an extended object "resists" angular acceleration is determined by the distribution of mass. Second, if the physics is to remain consistent, why would the quantity that plays the role of mass in kinetic energy be different from the quantity that plays the role of mass for the second law?

With those two quantities established, we can now get a glimpse into what the "rotational force" is by examining the units:

$$[\alpha] = \frac{[rotational\ force]}{[I]} \quad \Rightarrow \quad [rotational\ force] = \left[\frac{rad}{s^2}\right]\left[kg \cdot m^2\right] = \frac{kg \cdot m^2}{s^2} \tag{5.5.2}$$

This is weird... These are units of energy! We'll need to chalk this up to coincidence, since clearly the vector quantity of rotational force cannot be a measure of energy. One way to see the difference is to remember the presence of radians in the numerator, even though they are not physical units. We will soon see the source of this coincidence, and it shouldn't take long before the apparent ambiguity between this quantity and energy fades away.

Alert

While the physical units are the same as energy, we ***never*** *refer to the SI units of this quantity as "joules." Using this term implies that we are talking about energy, which we are not. Generally we stick to "Newton-meters."*

We can't continue calling this vector "rotational force" forever, so we will henceforth refer to it by its proper name: *torque*. In keeping with our tradition of using greek variables for rotational quantities, we will represent torque with $\overrightarrow{\tau}$, giving as our rotational Newton's second law:

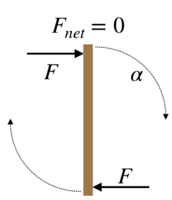
$$\overrightarrow{\alpha} = \frac{\overrightarrow{\tau}_{net}}{I} \tag{5.5.3}$$

Torque

In the cases of acceleration and inertia, we found a direct relationship between the linear and rotational quantities, so we would expect there to be a similar relationship between force and torque. Furthermore, since the linear/rotational bridge for acceleration and inertia both require a point of reference (the pivot), we would expect the same to be true for the bridge between force and torque.

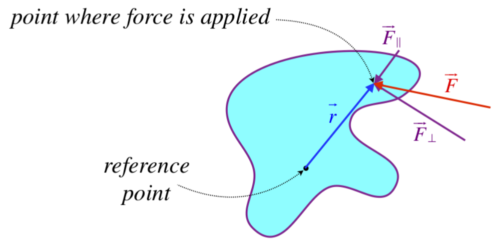
The first thing we notice is that an object can experience no net force and yet still experience a nonzero rotational acceleration:

Figure 5.5.1 – Zero Net Force Can Accelerate Rotationally



If the two forces shown in in Figure 5.5.1 are moved so that act at the same point on the object, then it's clear that they also cancel rotationally. So apparently the place *where* the force acts is important to computing torque. If we choose a reference point (we will refer to this as a "pivot" in cases when it happens to be a fixed point, but in general it does not), then the application point of a force can be described by a position vector $\overrightarrow{r}$ that points from the reference point to the point where the force is applied. But there is still more that we have to worry about here. If two forces with the same magnitudes as those in Figure 5.5.1 were applied at the same points on the bar, but were pointing vertically, then no angular acceleration would result. Let's put all this together...

Figure 5.5.2 – Parts of a Force that Cause Angular Acceleration



The force vector can be decomposed into two perpendicular vectors – one that is parallel to the position vector, and one perpendicular to it. When it comes to causing the object to accelerate its rotation around the pivot, it's clear that the part of the force that is parallel to the position vector $\overrightarrow{F}_{\parallel}$ will have no effect, while the perpendicular part of the force $\overrightarrow{F}_{\perp}$ will.

If we were to perform experiments to test the effects of various force magnitudes, we would find that the angular acceleration is proportional to the magnitude of the force – push twice as hard in the same direction as the same point on the object, and its angular acceleration is twice as great around the same pivot. If we were to perform further experiments to test the effects of applying the force at different positions, we would find that the angular acceleration is proportional to the magnitude of the position vector – extend the position vector in the same direction to twice its original length and apply the same force in the same direction, and the angular acceleration is once again twice as great around the same pivot. Mathematically, we express the results of these experiments this way:

$$\left|\overrightarrow{\tau}\right| \sim \left|\overrightarrow{r}\right| \left|\overrightarrow{F}\right| \tag{5.5.4}$$

Notice that the units of this product work out correctly, so all we need to do is incorporate the "only the perpendicular part of $\overrightarrow{F}$ has an effect" into the math. If we call the angle between the position vector and the force vector θ, then the perpendicular component is $F\sin\theta$. Assuming there are no other constants involved (and there aren't any), we get, for the magnitude of the torque:

$$\left|\overrightarrow{\tau}\right| = \left|\overrightarrow{r}\right| \left|\overrightarrow{F}\right| \sin\theta \tag{5.5.5}$$

This looks familiar – we actually saw something just like it, way back in Equation 1.2.9. Torque *is* a vector that is derived from the product of two other vectors. Is it possible that it is simply a cross-product of these two vectors? The magnitude works, but what about direction? In Figure 5.5.2, the force will accelerate the rotation counterclockwise, which means that according to the right-hand-rule, the acceleration vector points out of the page. If we perform a cross-product of the position vector (up to the right) and the force vector (up to the left), the right-hand-rule results in a vector that also points out of the page. We therefore write:

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F} \tag{5.5.6}$$

Example 5.5.1

A rigid object is pivoted around the origin. The force vector given below acts on this object at the position also indicated below. Find the torque vector exerted on the object due to this force.

$$\overrightarrow{F} = 1.5N\;\hat{i} + 0.80N\;\hat{j} - 2.4N\;\hat{k}, \qquad position:\;(x,\;y,\;z) = (3.0m,\;0.0m,\;-2.0m)$$

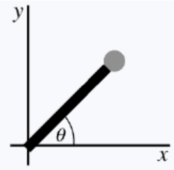
Solution

This is a straightforward calculation of a cross product:

$$\begin{aligned} \overrightarrow{\tau} &= \overrightarrow{r} \times \overrightarrow{F} \\ &= \left(3.0m\;\hat{i} + 0.0m\;\hat{j} - 2.0m\;\hat{k}\right) \times \left(1.5N\;\hat{i} + 0.80N\;\hat{j} - 2.4N\;\hat{k}\right) \\ &= \left[(3.0m)(0.80N) - (0.0m)(1.50N)\right]\hat{k} + \left[(0.0m)(-2.40N) - (-2.0m)(0.80N)\right]\hat{i} + \left[(-2.0m)(1.50N) - (3.0m)(-2.4N)\right]\hat{j} \\ &= \boxed{1.6Nm\;\hat{i} + 4.2Nm\;\hat{j} + 2.4Nm\;\hat{k}} \end{aligned}$$

Example 5.5.2

A small marble is attached to the end of a thin rigid rod of length L, whose other end is held fixed at the origin. The rod lies in the $x-y$ plane, and makes an angle θ up from the x-axis, as shown in the diagram.



The marble (but not the rod) is subjected to a force that gives rise to a potential energy field given by:

$$U(x,y) = \beta xy$$

Find the magnitude and direction of the torque exerted on the rod relative to the origin, in terms of L, θ, and β.

Solution

We can use the potential energy function to determine the force at every point in space:

$$\overrightarrow{F} = -\frac{\partial U}{\partial x}\,\hat{i} - \frac{\partial U}{\partial y}\,\hat{j} = -\beta\left(y\,\hat{i} + x\,\hat{j}\right)$$

The torque exerted relative to the origin at the point (x,y) is the cross-product of the position vector there and the force vector there:

$$\overrightarrow{\tau} = \overrightarrow{r} \times \overrightarrow{F} = \left(x\,\hat{i} + y\,\hat{j}\right) \times \left[-\beta\left(y\,\hat{i} + x\,\hat{j}\right)\right] = \beta\left(y^2 - x^2\right)\hat{k}$$

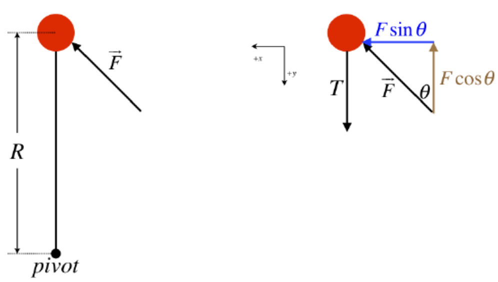
Now plug in for the coordinates of the marble in terms of L and θ:

$$\overrightarrow{\tau} = \beta L^2\left(\sin^2\theta - \cos^2\theta\right)\hat{k} = \boxed{-\beta L^2 \cos 2\theta\,\hat{k}}$$

Linking Rotational and Linear

Let's do a sanity check on our definition of torque and it role in the rotational second law. We can do it very simply by choosing a single point mass tied to a string whose other end is held as a fixed pivot (we'll leave gravity out of this). We'll start with the linear version of Newton's second law, and translate it into the rotational version.

Figure 5.5.3 – A Simple System Solved Two Ways



The forces in the x and y directions provide two equations through Newton's second law:

$$a_x = \frac{\sum F_x}{m} \quad \Rightarrow \quad a_{\parallel} = \frac{F\sin\theta}{m} \tag{5.5.7}$$

$$a_y = \frac{\sum F_y}{m} \quad \Rightarrow \quad a_{\perp} = \frac{T - F\cos\theta}{m} \tag{5.5.8}$$

Now we translate to rotational motion by first converting the parallel part of the acceleration into angular acceleration:

$$a_{\parallel} = R\alpha \tag{5.5.9}$$

Then convert mass into rotational inertia:

$$m = \frac{I}{R^2} \tag{5.5.10}$$

Plugging Equation 5.5.9 and Equation 5.5.10 into Equation 5.5.7 gives:

$$R\alpha = \frac{F\sin\theta}{\frac{I}{R^2}} \quad \Rightarrow \quad \alpha = \frac{FR\sin\theta}{I} = \frac{\tau}{I} \tag{5.5.11}$$

One important thing to note here is that while the torque and rotational inertia depend upon the pivot point (i.e. they are different values if we use a new reference point), the translation between the angular acceleration and linear acceleration exactly balances this difference. For example, if we replace the pivot defined above with a new one that is a distance $2R$ from the object, all of the math works out exactly the same. That is, the torque is twice as great and the rotational inertia is four times as great, resulting in a rotational acceleration that is half as large as before, but when it is multiplied by twice the radius to get the linear acceleration, the same result occurs, as it must.

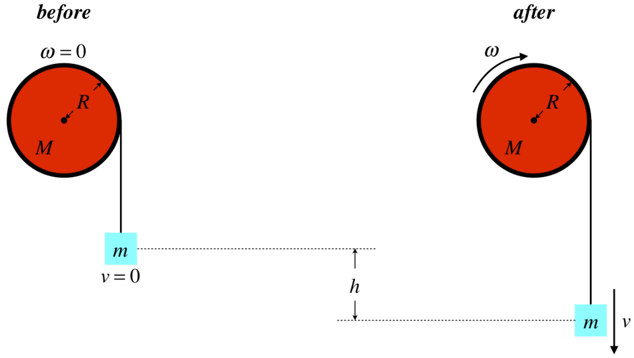
Solving Problems

Now we can do a whole set of problems involving torque causing rotational acceleration. There are many similarities with solving problems involving linear forces and accelerations, but here are some differences:

- Free-body diagrams now require that forces be placed appropriately on the objects, since torque depends upon force placement (no more using dots to represent the object).
- There usually is no need to resolve the torque vector into components. In fact, most problems can limit torque (and angular acceleration) to just "clockwise" and "counterclockwise" – the direction of the torque vector can be left until the end.
- One must either know or be able to calculate the rotational inertia of the object on which the torques acts.
- The perfect rolling condition is sometimes applied.

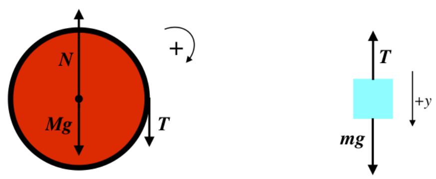
To get an idea of the process, we'll re-work the problem shown in Figure 5.5.4, this time using rotational second law instead of energy conservation:

Figure 5.5.4 – Falling Block Unwinds Spool (Redux)



Start with free-body diagrams:

Figure 5.4.5 – FBD's of Block and Spool



Next we need to right down the equations for Newton's second law for each object. The block is moving in a straight line, so we are already familiar with that one:

$$a_y = \frac{\sum F_y}{m} \quad \Rightarrow \quad a = \frac{mg - T}{m} = g - \frac{T}{m} \tag{5.5.12}$$

The spool is rotating, so we need to use the rotational version for it. Before we can sum the torques for the spool, we need to select a reference point, and its axle is a pretty obvious choice. The length of the position vector from this reference point to the where the gravity and normal forces act is zero, so those forces produce no torque around the axle (which makes sense – pushing on an axle should not cause something to spin around it). This leaves on the the tension force. It acts tangent to the spool, so this force is perpendicular to the position vector connecting the pivot to the point where the force acts, which makes the magnitude of torque it produces equal to simply the product of the tension and the radius of the spool. The direction of this torque is positive, since it causes a clockwise acceleration and our FBD defines that as the positive direction. As this is the only torque, it is the net torque, and we have:

$$\alpha = \frac{\tau_{net}}{I} \quad \Rightarrow \quad \alpha = \frac{T \cdot R}{I} \tag{5.5.13}$$

Now we have to incorporate our constraints (our "other information"). We know that the spool is a uniform solid disk with mass M, giving us its rotational inertia. Also, we know that the rate at which the string exits the spool is related to the rotation rate of the spool according to the usual "no slipping" condition, so we have an equation relating the block's linear acceleration a to the spool's angular acceleration α:

$$I = \frac{1}{2}MR^2, \qquad \alpha = \frac{a}{R} \tag{5.5.14}$$

Putting these constraints into Equation 5.4.13 and combining this with Equation 5.4.12 gives:

$$\left.\begin{array}{l} \frac{a}{R} = \frac{T \cdot R}{\frac{1}{2}MR^2} \quad \Rightarrow \quad T = \frac{Ma}{2} \\ a = g - \frac{T}{m} \end{array}\right\} \quad \Rightarrow \quad a = \frac{2m}{2m+M}g \tag{5.5.15}$$

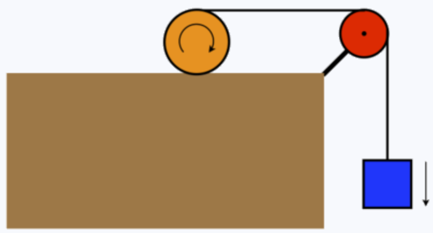
We see that the acceleration of the block is constant, so we can use a kinematics equation to determine the velocity after displacing a distance h from rest:

$$v_f^2 - v_o^2 = 2a\Delta y \quad \Rightarrow \quad v = \sqrt{\frac{4mgh}{2m+M}g} \tag{5.5.16}$$

This agrees with our previous answer.

Example 5.5.3

Let's take another look at Example 5.3.3 (see the diagram below). As the block falls, it pulls the uniform, solid cylinder (with a mass equal to that of the block), which rolls without slipping. If the horizontal surface was frictionless, then clearly this would be impossible. If it was very nearly frictionless, the disk would still slip. This means there must exist a minimum coefficient of static friction for which this physical situation can occur. Compute this minimum coefficient of static friction.



Solution

Let's get the constraints out of the way, as we already determined these in Example 5.3.3. The block moves twice as fast as the cylinder at all times, which means that it accelerates linearly at twice the rate of the cylinder as well:

$$v_{block} = 2v_{cylinder} \quad \Rightarrow \quad a_{block} = 2a_{cylinder} \equiv 2a$$

The rolling-without-slipping condition provided us with an equation relating the cylinder's linear velocity to its angular velocity. It naturally applies to the accelerations as well:

$$v_{cylinder} = R\omega \quad \Rightarrow \quad a_{cylinder} = R\alpha$$

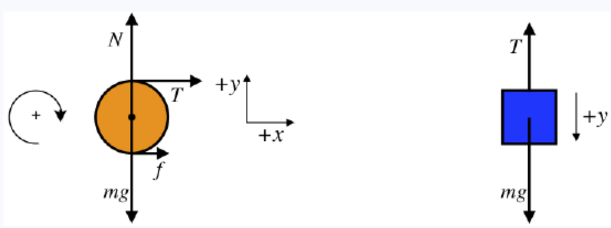
The cylinder is solid and uniform, so its rotational inertia is:

$$I = \frac{1}{2}mR^2$$

The only other constraint we have here is one we have encountered many times before – the cylinder will just barely not slip when the static friction force equals the coefficient of static friction multiplied by the normal force between the cylinder and the surface:

$$f = \mu_s N$$

Okay, now let's tackle the equations that come from Newton's second law. We of course start with force diagrams:



You might ask how we know that the friction force points in the direction indicated in the diagram. Technically, we don't yet know this, but we don't have to. If, in the course of our calculations, we find that the only way a solution can work out is if the value of f is negative, then the friction force must point the other way. We will see shortly that the direction on the diagram is in fact the only direction it can point.

There are three equations that come from Newton's second law for the cylinder (the horizontal and vertical linear net force equations, and the net torque equation), and there is one equation that comes out for the block:

$$\begin{array}{lll} & cylinder & block \\ x-direction: & a=\dfrac{T+f}{m} & \\ y-direction: & 0=N-mg & 2a=\dfrac{mg-T}{m} \\ torques: & \alpha=\dfrac{TR-fR}{I} & \end{array}$$

Plugging in for the rotational inertia and the angular acceleration gives:

$$\frac{a}{R}=\frac{TR-fR}{\frac{1}{2}mR^2} \quad \Rightarrow \quad \frac{a}{2}=\frac{T-f}{m}$$

Adding this equation to the x-direction equation for the cylinder gives:

$$\frac{3}{2}a=\frac{T+f}{m}+\frac{T-f}{m} \quad \Rightarrow \quad T=\frac{3}{4}ma$$

Now combine this result with the y-direction equation for the block to get:

$$2a=\frac{mg-\frac{3}{4}ma}{m} \quad \Rightarrow \quad a=\frac{4}{11}g$$

The y-direction equation for the cylinder and the friction/normal force constraint gives:

$$f=\mu_s mg$$

Solving for the friction force using the x-direction equation for the cylinder, and then plugging in what we found for the tension yields:

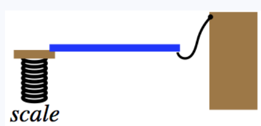
$$f=ma-T=ma-\frac{3}{4}ma=\frac{1}{4}ma$$

Combining these last three lines gives us our answer at last:

$$\mu_s mg=\frac{1}{4}ma=\frac{1}{4}m\left(\frac{4}{11}g\right) \quad \Rightarrow \quad \mu_s=\boxed{\frac{1}{11}}$$

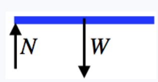
Example 5.5.4

A uniform rod weighing $32N$ is at rest horizontally, and is supported at one end by a scale. The support at the other end of the rod is suddenly removed, and the rod begins to fall & rotate. Find the force measured by the scale at the instant that the rod is released.



Solution

Start with a free-body diagram:



Naturally the normal force measured by the scale does not equal the weight of the rod, because the center of mass of the rod is accelerating. At the moment of release, the end of the rod on the scale is not free to move downward, so while this will not remain the case, at this moment that end of the bar is fixed, and the remainder of the bar experiences an angular acceleration around that point. We therefore have two sets of equations from Newton's second law – one for linear motion, and one for rotational motion. Setting downward and clockwise as the positive linear and rotational directions, and choosing the contact point with the scale as the reference, we have:

$$\begin{array}{llll} linear: & W-N=ma_{cm} & \Rightarrow & N=W-ma_{cm} \\ rotational: & W\left(\frac{1}{2}l\right)=I_{rod\ about\ end}\,\alpha=\frac{1}{3}ml^2\alpha & \Rightarrow & \alpha=\dfrac{3}{2ml}W \end{array}$$

The center of mass is half the length of the rod from the fixed point, so the linear acceleration of the center of mass is related to the angular acceleration according to:

$$a_{cm} = \frac{l}{2}\alpha$$

Plugging this back in above gives our answer:

$$N = W - m\left(\frac{l}{2}\alpha\right) = W - \frac{3}{4}W = \frac{1}{4}W = \boxed{8N}$$

Rotational Work

We have now discussed the rotational version of energy conservation and Newton's second law, so the link between these two topics – the work-energy theorem – should follow naturally. Rather than provide a derivation (which would really just resemble what we have done before for the linear case), we'll just write down the answer that makes sense from following our linear/rotational parallel.

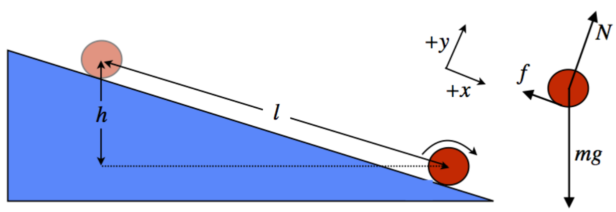
$$\begin{aligned} W_{A\rightarrow B}\,(linear) \quad &= \int_A^B \overrightarrow{F}\cdot\overrightarrow{dl} \quad = \Delta KE \quad = \tfrac{1}{2}mv_B^2 - \tfrac{1}{2}mv_A^2 \\ W_{A\rightarrow B}\,(rotational) \quad &= \int_A^B \overrightarrow{\tau}\cdot\overrightarrow{d\theta} \quad = \Delta KE \quad = \tfrac{1}{2}I\omega_B^2 - \tfrac{1}{2}I\omega_A^2 \end{aligned} \tag{5.5.17}$$

If we were so inclined, we could do the same unwinding-the-spool problem for a third time, this time with the rotational work-energy theorem. The approach looks slightly different, but when you actually sit down to do it, you see the same things come out of it as before. This time instead of relating the accelerations, we would relate the distance the mass drops to the angle the spool rotates.

Back when we discussed objects rolling down an inclined plane without slipping, we avoided talking about one potentially confusing point that we are now equipped to deal with. For a ball or cylinder to roll down, there has to be a friction force (otherwise it would merely slide). This friction force can only be static friction, because we are assuming there is no slipping, and we said that without any rubbing, the mechanical energy must be conserved. But this friction force acts *up the plane while the object moves down it*, which means that it does negative work on the object. This would seem to imply that mechanical energy should not be conserved, so how were we able to make the assumption that it is conserved?

The answer is, "Because the static friction force also does *positive* rotational work which adds energy to the object in rotational form, and this addition exactly balances the loss in linear form." This is not hard to prove. Start with a diagram and a FBD:

Figure 5.5.6 – Work Done on Cylinder by Static Friction as It Rolls Down Plane



Computing the work done by static friction for linear motion is very simple, since the friction force is constant and the motion is in a straight line:

$$W\left(x_1 \rightarrow x_2\right) = \int_{x_1}^{x_2} \overrightarrow{F}\cdot\overrightarrow{dx} = -f\cdot l \tag{5.5.18}$$

As expected, this work takes energy out of the cylinder system. Next we compute the rotational work done on the cylinder. The torque is a constant equal to fR, and is acting in the same direction as the rotational displacement, so

$$W\left(\theta_1 \rightarrow \theta_2\right) = \int_{\theta_1}^{\theta_2} \overrightarrow{\tau}\cdot\overrightarrow{d\theta} = +(fR)\cdot\theta \tag{5.5.19}$$

Putting these together gives us the total work done on the cylinder by the static friction force. Note that since it rolls without slipping, the linear distance it travels is related to the angle through which it rotates by the usual relation:

$$W_{static\ friction} = W\left(x_1 \rightarrow x_2\right) + W\left(\theta_1 \rightarrow \theta_2\right) = -f\cdot l + fR\theta = f\left(-l + R\theta\right) = 0 \tag{5.5.20}$$

So we see that in fact the work done by static friction here only serves to convert linear kinetic energy into rotational kinetic energy, and our understanding of how thermal energy is generated remains intact.

Rotational Power

We spoke before about how sometimes we are interested in the rate at which work is done, calling this value "power." Well, as with everything else we studied in linear motion, there is of course a rotational version:

$$P=\frac{dW}{dt}=\overrightarrow{\tau}\cdot\overrightarrow{\omega} \tag{5.5.21}$$

You sometimes hear the silly "debate" about torque vs. horsepower for car & truck engines. This should make it clear what the difference is. Power delivered to the wheels is directly related to torque exerted on them, but it is dependent upon how fast they are turning. Engines that can still produce a lot of torque at high speeds are powerful. To get an idea of why it might be hard to maintain torque at high speeds, imagine pedaling a bike downhill – when you get going fast enough, it's difficult to push hard on (provide torque to) your pedals. So generally the effectiveness of an engine is defined by torque at low speeds and power at high speeds. If you want fast acceleration off the line or the ability to pull a stump out of the ground, you want torque. If you want to go fast or tow a heavy trailer up a hill at a steady speed, you want power.