4.5: Collisions

One-Dimensional Collisions

We know that in the case of a collision, the force acting between the two objects is irrelevant to momentum conservation, but is very important to determining the amount of energy converted to thermal energy. For example, if two blocks collide with a spring between them, then all the kinetic energy they come in with, they will also go out with, since the internal spring force involved in the collision is conservative, and there is no spring potential energy before or after. On the other hand, if an internal non-conservative force is present between the colliding objects, then some of the incoming kinetic energy is converted into thermal energy. The former sort of collision (where kinetic energy is conserved) we call *elastic*, and the second type of collision we call *inelastic*.

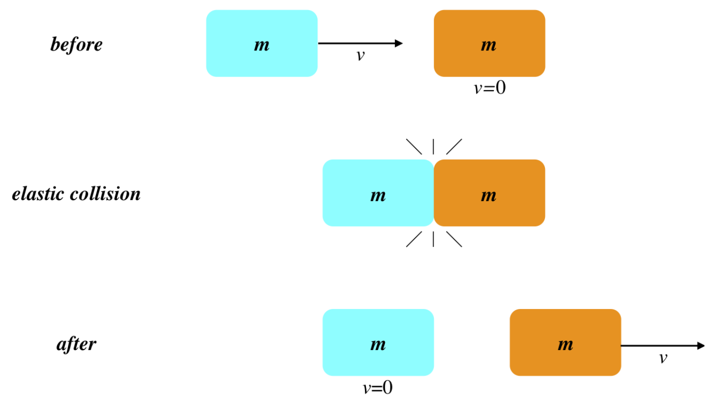
From our discussion in Section 4.4, it's clear that what determines the inelasticity of a collision is the deformation of the colliding objects. When a colliding object deforms, it's because the particles directly involved in the contact are accelerated more than other particles in the same object, thus introducing internal energy, and reducing the amount of mechanical energy available to go back into the motions of the objects.

A collision where the objects continue together with the same velocity after the collision (i.e. they remain stuck together), is often referred to as *totally* or *perfectly inelastic*. This of course does not mean that all of the kinetic energy is lost (the objects do continue moving at the end in most such collisions), only that they don't bounce off each other. From the perspective of the center of mass frame, we can see that such a collision maximizes the amount of internal energy that the collision can create: In this frame, the objects stop entirely after the collision, so all of the mechanical energy becomes internal. Changing frames doesn't change the amount of internal energy created (it only changes the mechanical energy we see), so having the objects stick together results in the largest possible creation of internal energy.

Elastic Collisions

If we are told that a given collision is elastic (or at least can be approximated as such), then that gives us an additional condition that we can use to solve the problem. Let’s see a couple of examples. in each case, the diagram will show the experimental result, which we will then show mathematically using the combination of momentum and kinetic energy conservation.

Figure 4.5.1 – Elastic Collision of Equal Masses, Target Stationary

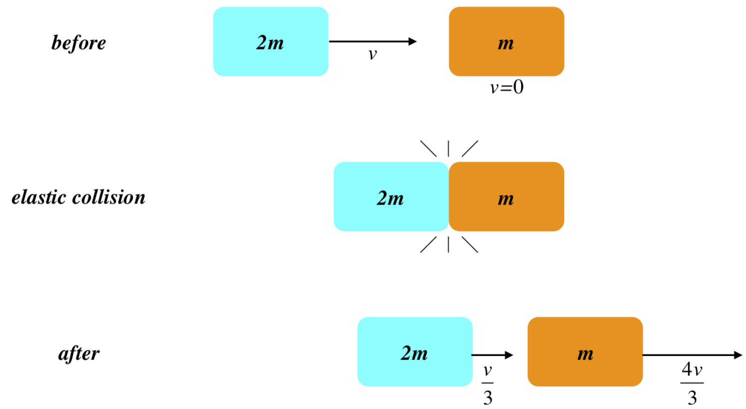


We see that the incoming cart stops completely and the target cart moves off with the same velocity as the original cart (note that the center of mass continues moving at a constant speed, as it should). We now show this mathematically... Dropping the vector arrows, since the motion is in one dimension, and choosing to the right as the (+) direction, we have:

$$\left.\begin{array}{lll} momentum\;conservation: & mv+0=mv_1+mv_2 & \Rightarrow \;\; v=v_1+v_2 \\ elastic\;collision & \frac{1}{2}mv^2+0=\frac{1}{2}mv_1^2+\frac{1}{2}mv_2^2 & \Rightarrow \;\; v^2=v_1^2+v_2^2 \end{array}\right\} \Rightarrow \;\; v_1=0,\;\; v_2=v \;\;\; or \;\;\; v_1=v,\;\; v_2=0 \tag{4.5.1}$$

Wait, why do we get two solutions? That is, why can *either* velocity equal zero? Well, if the incoming cart were to *miss the target cart*, then that too is an elastic “collision,” inasmuch as the momentum and kinetic are both conserved, so the math takes into account that as a possibility.

Figure 4.5.2 – Elastic Collision of Unequal Masses, Target Lighter and Stationary



The algebra is only a little tougher this time:

$$\left.\begin{array}{lll} momentum\ conservation: & 2mv+0=2mv_1+mv_2 & \Rightarrow \quad 2v=2v_1+v_2 \\ elastic\ collision & \frac{1}{2}2mv^2+0=\frac{1}{2}2mv_1^2+\frac{1}{2}mv_2^2 & \Rightarrow \quad 2v^2=2v_1^2+v_2^2 \end{array}\right\} \Rightarrow \quad 4v_1=v_2 \quad \Rightarrow \quad v_1=\frac{v}{3}, \quad v_2=\frac{4v}{3} \tag{4.5.2}$$

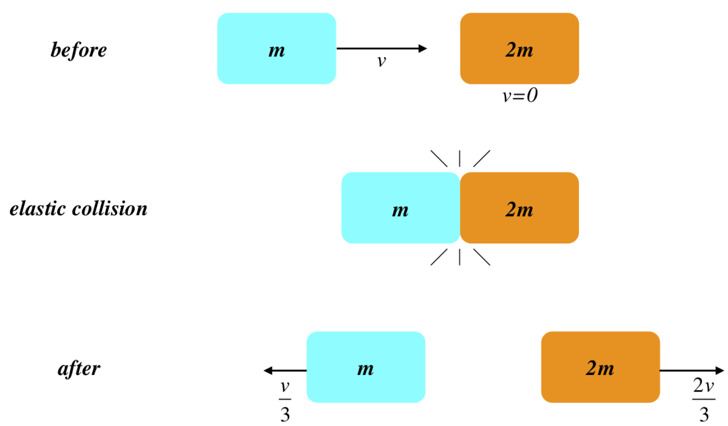
Both carts continue forward, the lighter one at 4 times the speed of the heavier one. Note that once again $v_1=v, \quad v_2=0$ is a solution (the incoming cart misses the target).

A clear application of this principle comes in bowling. Clearly we want the bowling ball to have more mass than a pin, so that it can carry through to the pins behind the front pin(s). If we consider collisions in two dimensions (which we will do later), we will find that the angular deflection of the ball when it doesn't strike the pin head-on will be less when the ball is heavier, which is one reason heavier bowling balls are more effective than lighter ones.

As a second example of this, suppose we are passengers in one of two vehicles involved in a head-on collision. Which vehicle would we rather be in, the lighter one or the heavier one? Intuitively we know we would rather be in the heavier vehicle, but why? Well, we would want to experience as little force as possible (force is what breaks bones). The force that our dashboard or steering column exerts on us is going to equal our mass times our acceleration (as it constitutes our net horizontal force), and we are constrained to experience the same acceleration as our car. So compare the accelerations of the two carts here. The heavier cart goes from a speed v down to a speed of v/3, for a change of 2v/3. The lighter cart's velocity changes from 0 to 4v/3 in the same period of time, which means it experiences twice the acceleration. More acceleration for our car means more acceleration for us, which means more force on us, which is bad.

Lastly, we look at the lighter object bouncing off the heavier one:

Figure 4.5.3 – Elastic Collision of Unequal Masses, Target Heavier and Stationary



The math:

$$\left.\begin{array}{lll} momentum\ conservation: & mv+0=mv_1+2mv_2 & \Rightarrow \quad v=v_1+2v_2 \\ elastic\ collision & \frac{1}{2}mv^2+0=\frac{1}{2}mv_1^2+\frac{1}{2}2mv_2^2 & \Rightarrow \quad v^2=v_1^2+2v_2^2 \end{array}\right\} \Rightarrow \quad 2v_1=-v_2 \quad \Rightarrow \quad v_1=-\frac{v}{3}, \quad v_2=\frac{2v}{3} \tag{4.5.3}$$

The lighter cart bounces off the heavier one at half the speed that the heavier one continues forward (or the incoming cart misses the target). There is actually a clever way we could have solved this case more quickly by using the solution of the previous case and what we know about relative motion. If we move along with the incoming block and declare ourselves to be "stationary," then we see the heavier mass coming toward us at a speed v, which is exactly the same physical situation as we had above. After the collision, we will see the heavier mass continuing in the same direction at a speed of $v/3$, while the target block moves in the same direction at a speed of $4v/3$. That is what *we see*. Going back to the original frame, these two speeds change by v, which means the heavy object is not going left at $v/3$ – it is going *right* at $v - v/3 = 2v/3$, while the smaller block is moving left at a speed of $4v/3 - v = v/3$.

Inelastic Collisions

Okay, so let’s look at some inelastic collisions. As we said above, these can include some component of an elastic collision, where the objects bounce off each other or they can be totally inelastic, where they stick together and continue as one mass. In both cases, some of the kinetic energy contained in the system is converted to thermal energy. [*Note: From this point on, "kinetic energy" will refer to mechanical kinetic energy, and all internal energy will be called thermal. We will no longer talk about "internal kinetic energy."*] This can be expressed in a couple of different ways:

$$KE_{after} = KE_{before} - \Delta E_{thermal} \,, \quad or \quad KE_{after} = (x\%)\, KE_{before} \tag{4.5.4}$$

Of the two types of inelastic collisions, the totally inelastic case is easier to handle, because we we don't require the information of what fraction of the internal force was conservative and what fraction was non-conservative. We can see the simplicity of the totally inelastic case comes from the fact that all of the information about the resulting motion is contained in just the momentum conservation equation, because both objects have the same final velocity:

$$m_1 v_1 + m_2 v_2 = (m_1 + m_2)\, v_f \quad \Rightarrow \quad v_f = \frac{m_1 v_1 + m_2 v_2}{m_1 + m_2} \tag{4.5.5}$$

Note that (naturally) the final speed of the combined masses is the speed of the center of mass, since everything is moving together. The kinetic energy converted to thermal energy can be computed from this result in terms of the incoming velocities:

$$-\Delta E_{thermal} = KE_f - KE_o = \frac{1}{2}(m_1 + m_2)\, v_f^2 - \left(\frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2\right) , \tag{4.5.6}$$

where v_f is given in terms of v_1 and v_2 in Equation 4.5.5.

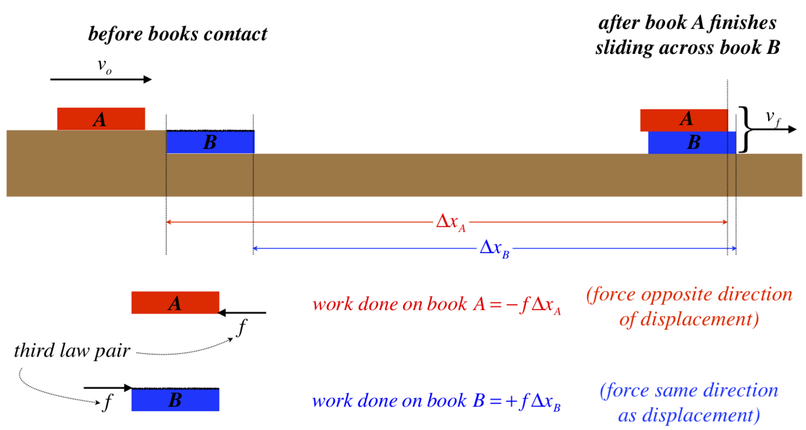
If the perfectly inelastic head-on collision involves an incoming object with mass m_1 and a stationary target with mass m_2, it is easy to use momentum conservation and Equation 4.1.5 to derive a simple relationship between the starting and ending kinetic energy of the system:

$$\frac{KE_{after}}{KE_{before}} = \frac{\dfrac{\cancel{p^2_{after}}}{2\,(m_1 + m_2)}}{\dfrac{\cancel{p^2_{before}}}{2m_1}} = \frac{m_1}{m_1 + m_2} \tag{4.5.7}$$

This certainly makes sense from the perspective of dropping a pile of mashed potatoes on the floor. The earth is the stationary target with a very large mass, and after the perfectly inelastic collision, pretty much all of the kinetic energy in the mashed potatoes becomes thermal. Of course, you may prefer to warm your mashed potatoes in a microwave oven instead.

Notice that nowhere here do we mention the details of the force acting between the two masses. This is strange, because we have seen that the specifics of the force determine the work done, which in turn determines the amount of energy converted to thermal. Let’s return to our two book example one more time:

Figure 4.5.4 – Work Done in Perfectly Inelastic Collision



Suppose we change the friction force by changing the coefficient of kinetic friction. In fact, let's assume we double that constant – what will happen? Well, the friction force will double, since the normal force is the same as before. From the reference frame of book B, this means the acceleration of book A has doubled. With double the acceleration and the same starting speed, in book B's reference frame it will take half as long for book A to come to rest. Therefore the time that both books experience double the acceleration is half as long as before. Putting this into the equations of motion for the two books gives:

$$\left.\begin{array}{l} book\;A: \;\; \Delta x_A = v_o t - \frac{1}{2}at^2 \\ book\;B: \;\; \Delta x_B = \frac{1}{2}at^2 \end{array}\right\} t \rightarrow \frac{t}{2},\; a \rightarrow 2a \Rightarrow \Delta x_A \rightarrow \frac{\Delta x_A}{2},\; \Delta x_B \rightarrow \frac{\Delta x_B}{2} \tag{4.5.8}$$

So both books travel half as far when the coefficient of friction is doubled. The friction force is twice as great, so the work done on each is the same as before! This shows that the same amount of energy is lost in both cases, and in general it will turn out to be the same no matter what sort of non-conservative force is acting, so long as the collision is perfectly inelastic.

Another way to see this is to use our favorite trick of going into the center of mass reference frame. In this frame, the two books are moving toward each other, and after they slide across each other, they come to a stop. This means that all of the initial kinetic energy becomes thermal. But notice that this is true no matter what force was acting between them! Different forces (like changing the coefficient of friction) will lead to different *rates* at which the kinetic energy becomes thermal, but all of it converts in the end.

Example 4.5.1

A large sled of mass 12.0kg is at rest on a horizontal, frictionless sheet of ice, when a heavy rock with a mass of 7.50kg is thrown onto it from behind. The rock is moving purely horizontally at a speed of 2.40m/s when it comes into contact with the sled, and it skids across the rough top surface of the sled until it and the sled are moving forward together at the same speed. The scrape marks on the sled indicate that the rock skidded across it a distance of 1.60m.

a. *Find the final speed of the sled.*
b. *Find the magnitude of the friction force between the sled and the rock.*

Solution

a. The only horizontal force present is the friction force between the rock and the sled, which is internal to the rock+sled system, so the system's horizontal momentum is conserved. We therefore have:

$$\underbrace{m_{rock}v_{rock} + m_{sled}v_{sled}}_{p_{before}} = \underbrace{\left(m_{rock} + m_{sled}\right)V}_{p_{after}} \quad \Rightarrow \quad V = \frac{m_{rock}v_{rock}}{m_{rock} + m_{sled}} = \boxed{0.92\frac{m}{s}}$$

b. The work done by friction equals the energy converted to thermal, which is the energy lost from kinetic. We therefore compute the kinetic energy lost first. We can do this the long way, but because this is the special case of a stationary target and a perfectly inelastic collision, we can use a shortcut to get the energy lost, using Equation 4.5.7:

$$\frac{KE_{after}}{KE_{before}} = \frac{m_1}{m_1 + m_2} = 0.385 \quad \Rightarrow \quad \Delta E_{thermal} = -\Delta KE = KE_{before} - KE_{after} = 0.615KE_{before} = 0.615\left(\frac{1}{2}m_{rock}v_{rock}^2\right) = 13.3J$$

The distance the rock scrapes across the sled is given, so since the work done during this scraping is the energy converted to thermal, we can compute the friction force:

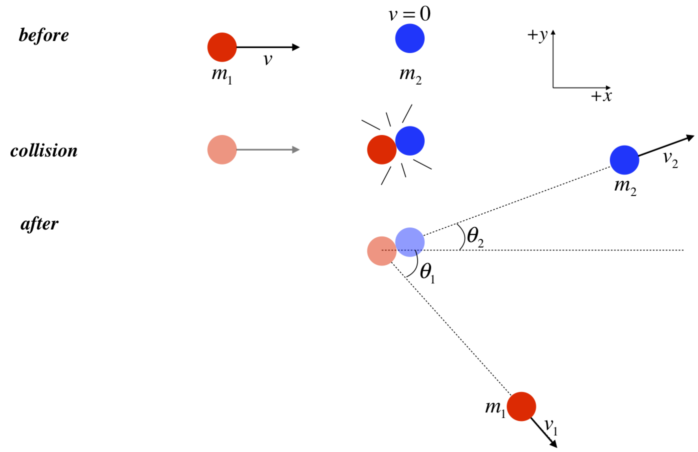
$$\Delta E_{thermal} = f \cdot \Delta x \quad \Rightarrow \quad f = \boxed{8.1N}$$

General Two-Dimensional Collisions

We have been saying for awhile now that one of the big differences between momentum conservation and energy conservation is the fact that momentum is a vector while energy is not. This means that there are actually three momentum quantities that are equal before and after (if the full momentum vector is conserved). Here we will look at what this entails.

Let's look at a standard two-dimensional collision. In this example, we will have a stationary ball struck by another. The two balls have different masses, and they collide off-center, so that they emerge from the collision in directions angled off the original direction of motion. We'll set up the geometry and label all the known and unknown variables with a diagram, and then do the physics:

Figure 4.5.5 – General Two-Dimensional Collision in the Target Frame



Now we need to apply momentum conservation. Since momentum is a conserved vector, each of its components are individually conserved, which means that momentum conservation provides us two separate equations to work with. In the "before" case, we have an x-component of momentum that is simply the incoming mass times the incoming velocity (m_1v), while the y-component of momentum is zero. in the "after" case, we need to resolve the momenta into components. Setting before equal to after gives:

$$\begin{array}{ll} x-direction: & m_1v = m_1v_1\cos\theta_1 + m_2v_2\cos\theta_2 \\ y-direction: & 0 = -m_1v_1\sin\theta_1 + m_2v_2\sin\theta_2 \end{array} \tag{4.5.9}$$

You'll note the minus sign for the component in the $-y$-direction. This is not strictly necessary, as this negative sign could be absorbed into θ_1, but it is generally less confusing to put the signs in explicitly, and let all the angle values be positive.

Let's consider what would be required to solve a problem that looks like this. We have two equations, and seven distinct variables. If this is all we know about the collision, then to completely unravel this physical situation, we need to know five of these quantities. So for example, we could be given the two masses, the incoming speed, and the outgoing speed and direction of one of the balls, and we can solve for the outgoing speed and direction of the other ball. If we also provided the target ball a starting velocity, or a y-component to the incoming ball's velocity, then there would be even more unknowns. But we can quickly reduce this problem back to the one above, by first rotating our coordinate system so that the incoming velocity is once again in the x direction, and then changing the reference frame to the rest frame of the target ball. It is also sometimes useful to change to the center of mass reference frame.

Notice that once such a problem is solved, one can then check to see if the collision is elastic, by comparing the kinetic energy before and after the collision:

$$KE_{before} = \frac{1}{2}m_1v^2 \qquad KE_{after} = \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \tag{4.5.10}$$

This comparison could be a difference (determining how much kinetic energy is lost), or a fraction (determining the percentage of kinetic energy remaining or the percentage lost). Note that a collision *can* result in an increase of kinetic energy, but this can only happen if there is some potential energy stored within the colliding objects that is unleashed by the collision. This is such an uncommon occurrence (the circumstances need to be quite contrived), that it is safe to assume that a collision is either elastic (conserves kinetic energy) or is inelastic such that kinetic energy is lost.

Not all problems are posed with five of the seven variables given. The energy condition can be given instead, which provides a third equation, requiring only four of the seven variables in the statement of the problem. Needless to say, these problems can require a lot of tedious algebra, but getting the equations set up using momentum conservation and the fate of the system's kinetic energy is where the physics is.

Elastic Two-Dimensional Collisions

As daunting as the full-blown problem shown above can be, there are cases where shortcuts or simplifications exist. We look first at the case of elastic collisions. If we want to know all the information shown above, we have no choice but to go through the algebra involved. But we can achieve an interesting result without recourse to the coordinate system at all. Namely, it turns out that the ratios of the masses of the colliding objects and their outgoing speeds completely determine the angle *between* the outgoing velocity vectors, $\theta_1+\theta_2$. To get this result, we will use Equation 4.1.5 extensively...

Let's call the incoming momentum $\overrightarrow{p}$ and the mass of the incoming object m_1. Then the kinetic energy of the system (in the frame where the target is stationary) is:

$$KE_{before} = \frac{1}{2m_1}\overrightarrow{p}\cdot\overrightarrow{p} \tag{4.5.11}$$

Now let's define the outgoing momenta of the two objects as $\overrightarrow{p}_1$ and $\overrightarrow{p}_2$, with the latter being for the target object after collision. The kinetic energy after the collision is therefore:

$$KE_{after} = \frac{1}{2m_1}\vec{p}_1 \cdot \vec{p}_1 + \frac{1}{2m_2}\vec{p}_2 \cdot \vec{p}_2 \tag{4.5.12}$$

Now we apply momentum conservation:

$$\vec{p} = \vec{p}_1 + \vec{p}_2 \quad \Rightarrow \quad KE_{before} = \frac{1}{2m_1}\left(\vec{p}_1 + \vec{p}_2\right) \cdot \left(\vec{p}_1 + \vec{p}_2\right) = \frac{1}{2m_1}\vec{p}_1 \cdot \vec{p}_1 + \frac{1}{2m_1}\vec{p}_2 \cdot \vec{p}_2 + \frac{1}{m_1}\vec{p}_1 \cdot \vec{p}_2 \tag{4.5.13}$$

Applying kinetic energy conservation (remember, we are assuming an elastic collision):

$$KE_{before} = KE_{after} \quad \Rightarrow \quad \cancel{\frac{1}{2m_1}\vec{p}_1 \cdot \vec{p}_1} + \frac{1}{2m_1}\vec{p}_2 \cdot \vec{p}_2 + \frac{1}{m_1}\vec{p}_1 \cdot \vec{p}_2 = \cancel{\frac{1}{2m_1}\vec{p}_1 \cdot \vec{p}_1} + \frac{1}{2m_2}\vec{p}_2 \cdot \vec{p}_2 \tag{4.5.14}$$

Now multiply through by m_1 and rearrange things a bit to get:

$$\vec{p}_1 \cdot \vec{p}_2 = \frac{1}{2}\left(\frac{m_1}{m_2} - 1\right)\vec{p}_2 \cdot \vec{p}_2 \tag{4.5.15}$$

Now write the dot products in terms of the magnitudes of the vectors and the angles between them:

$$p_1 p_2 \cos\theta = \frac{1}{2}\left(\frac{m_1}{m_2} - 1\right) p_2^2 \tag{4.5.16}$$

The angle θ is of course the angle between the two outgoing velocity vectors (which point the same direction as the momentum vectors). The $p_2 = 0$ solution to this corresponds to the case of the incoming object missing the target entirely (because the target remains stationary), so assuming the target is not missed, we can divide both sides by p_2 and if we also plug in $p_1 = m_1 v_1$ and $p_2 = m_2 v_2$, we get the promised relationship of the *scattering angle* in terms of the masses and outgoing speeds:

$$\theta = \cos^{-1}\left[\frac{1}{2}\left(1 - \frac{m_2}{m_1}\right)\frac{v_2}{v_1}\right] \tag{4.5.17}$$

We can extract some interesting information from this result:

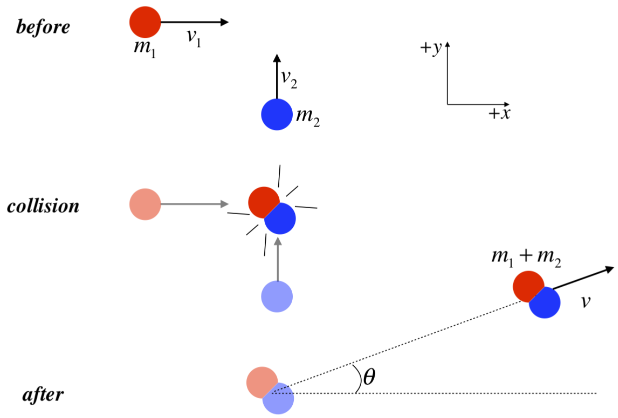
- We see that if the masses are equal, then the scattering angle is precisely 90^o, since the cosine of this angle vanishes. In this case, the scattering angle doesn't depend at all on how off-center the collision is (except that a direct head-on hit naturally leads to an angle of 0^o or 180^o). The degree of how off-center the collision is (which is measured by a quantity known as the *impact parameter*) *does* effect the angles θ_1 and θ_2 in Figure 4.5.5, but not the sum of those angles. If the masses are not equal, then the impact parameter does play a role in the scattering angle, because it has a say in the ratio of the outgoing speeds.
- If $m_2 > m_1$, the argument of the inverse cosine is negative, so the angle must be greater than 90^o. This makes sense, because if the target mass is greater than the incoming mass, the incoming mass "bounces back," rather than "plowing through" (a result we found for the one-dimensional elastic collisions we examined above), and since the target mass has a forward component to its final velocity, the angle is greater than 90^o.
- The argument of the inverse cosine can never be larger than +1 or smaller than –1, which places limits on the outgoing speeds given the masses. For example, if the incoming mass m_1 is twice the target mass m_2, then the largest possible ratio of the two outgoing velocities is 4. This ratio occurs when $\theta = 0$, and indeed we have seen this result already above (Equation 4.5.2).

It should be noted that this result could also be achieved using the formulas resulting from Figure 4.5.5, but it would require an unnatural desire to slog through trigonometric identities.

Perfectly Inelastic Two-Dimensional Collisions

As much as we were able to do with elastic collisions, perfectly inelastic collisions are even easier to handle. This is because the outgoing motions of the two objects are constrained to be the same (i.e. they stick together and have the same final speed and direction). This constraint means that if we are given all of the incoming conditions (the masses of the two objects, and their incoming velocity vectors), we can determine the result completely. That is, the amount of energy lost in the collision does not need to be given – it is unique and can in fact be computed. Figure 4.5.6 is a diagram for an example of a perfectly inelastic collision. [*This is somewhat simplified by having the incoming objects approach each other at right angles, but not as simple as the case of looking at it from the target frame, which makes the collision one-dimensional!*]

Figure 4.5.6 – A Perfectly Inelastic Two-Dimensional Collision



We follow the same procedure as we did for Figure 4.5.5, this time with the simplification that we have a single outgoing momentum:

$$\begin{aligned} x-direction: \quad & m_1 v_1 = (m_1+m_2)\,v\cos\theta \\ y-direction: \quad & m_2 v_2 = (m_1+m_2)\,v\sin\theta \end{aligned} \tag{4.5.18}$$

The amount of energy converted to thermal from this collision equals the loss of kinetic energy from the system, and as we saw in the one-dimensional case, this amount doesn't depend upon the details of the internal non-conservative force. It only matters that eventually (after the two objects end their tumultuous collision) settle into moving off together with a common velocity. The amount of energy converted is:

$$\Delta E_{thermal} = KE_{before} - KE_{after} = \left[\frac{1}{2}m_1 v_1^2 + \frac{1}{2}m_2 v_2^2\right] - \left[\frac{1}{2}(m_1+m_2)\,v^2\right] \tag{4.5.19}$$

For the case above where the two incoming objects have velocities are right angles to each other, we can turn this into an equation that includes only the masses and incoming speeds. Sparing the reader the algebra, the result is:

$$E_{thermal} = \frac{1}{2}\left(\frac{m_1 m_2}{m_1+m_2}\right)\left(v_1^2+v_2^2\right) \tag{4.5.20}$$

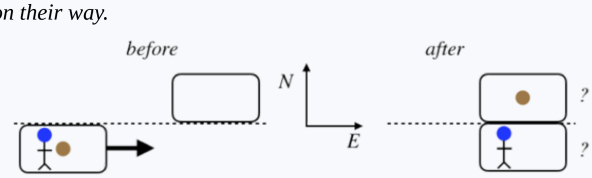
Notice that since the two velocities are perpendicular, the sum of their squares is actually the square of their *relative* velocity. This is not a surprising result, and in fact will translate into collisions at any angle (though the equation will look different), because we would not expect the post-collision blob to be any hotter when the collision is viewed in one frame as opposed to another. As mentioned above, we can always view this collision from the target frame, making the collision one-dimensional, and the total kinetic energy of the system before the collision is a function of the relative velocity. In that case, we can use Equation 4.5.7 to compute the energy converted to thermal:

$$\Delta E_{thermal} = KE_{before} - KE_{after} = KE_{before} - \left(\frac{m_1}{m_1+m_2}\right)KE_{before} = \left(\frac{m_2}{m_1+m_2}\right)KE_{before} \tag{4.5.21}$$

So suppose we drop a ball of clay to the ground. Viewing this from the earth's rest frame, the earth becomes the stationary target with mass m_2, and essentially all of the clay's incoming kinetic energy is converted to thermal (because $m_2 \approx m_1+m_2$), and the clay's (and earth's) temperature goes up a bit. If we view it from the clay's rest frame, then the kinetic energy of the earth is enormous (same relative speed, much larger mass), and after the collision we might therefore expect the temperatures to go up a lot, but making the clay the stationary target now makes the target mass m_2 very small compared to m_1+m_2 , which makes the fraction multiplying the earth's kinetic energy very small – exactly small enough to give the same energy change as before.

Example 4.5.1

A cart slides along a frictionless surface in an easterly direction at a speed of 2.40m/s. The cart has a mass of 100kg, and it contains a person who has a mass of 60.0kg, as well as a medicine ball that has a mass of 15.0kg. The cart slides past an identical (but empty) stationary cart, also on the frictionless surface. When the carts are side-by-side, the person throws the medicine ball into the other cart by pushing the ball in the north direction. At the moment of release, the person sees the ball is moving away from their hands at a speed of 3.20m/s. The ball comes to rest inside the other cart, and both carts continue on their way.



Find the speed and direction of both carts after the medicine ball has been exchanged. Express the directions as angles that are north or south (indicate which) of east.

Solution

Treating the two carts, the person, and the medicine ball as a single system, we know that all the forces on them are internal. We have some additional information to use. First, we are told that the medicine ball is pushed north. This will affect the north-south component of the momentum of the ball and the cart it lands in. The equal-and-opposite push of the ball on the person will affect the north-south component of momentum of the person and the cart they are in. But the east-west components of momentum are unaffected. We therefore know that the person + cart continues with the same eastward component of velocity as before, and the eastward momentum of the ball gets transferred to the ball + cart system, giving the final eastward component of the ball + cart:

$$\begin{array}{cccccc} & before & & after & & \\ east-west: & m_bv_o+m_c\left(0\right) & = & \left(m_b+m_c\right)v_{bc(east)} & \Rightarrow & v_{bc(east)}=\dfrac{m_bv_o}{m_b+m_c}=0.313\,\frac{m}{s}\end{array}$$

Now for the north-south components. Initially nothing has a north-south component of momentum, so the final total north-south component of momentum must also be zero for the whole system. We are given the <u>*relative*</u> *velocity of the ball and the person as it leaves their hands, but we are working in the earth frame, so we have to be careful. Calling the northward component of the ball's velocity relative to the earth v_b, and the southward component of the person + cart after the ball is released v_{pc} (which has a negative value), we can relate these through their relative motion by:*

$$v_{b(north)}=3.20\frac{m}{s}+v_{pc(south)}$$

Now apply momentum conservation (in the earth frame) along the north-south direction to get the north-south component of the velocity of the person + cart:

$$\begin{array}{cccccc} & before & & after & & \\ north-south: & 0 & = & m_bv_{b(north)}+\left(m_p+m_c\right)v_{pc(south)} & = & m_b\left(3.20\frac{m}{s}+v_{pc(south)}\right)+\left(m_p+m_c\right)v_{pc(south)}\end{array}$$

$$\Rightarrow \quad v_{pc(south)}=-\frac{m_b\left(3.20\frac{m}{s}\right)}{m_b+m_p+m_c}=-0.274\frac{m}{s}$$

This southward velocity of the person + cart gives us its final magnitude and direction of motion:

$$v_{pc}=\sqrt{v_o^2+v_{pc(south)}^2}=\boxed{2.42\frac{m}{s}}$$

$$\theta_{pc}=\tan^{-1}\left(\frac{v_{pc(south)}}{v_o}\right)=\boxed{6.51^o\ \ south\ of\ east}$$

The northward component of the momentum of the ball + cart is the same as the southward momentum of the person + cart, since the whole system's north-south momentum remains zero. We therefore have for the ball + cart:

$$0=\left(m_b+m_c\right)v_{bc(north)}+\left(m_p+m_c\right)v_{pc(south)} \quad \Rightarrow \quad v_{bc(north)}=-\left(\frac{m_p+m_c}{m_b+m_c}\right)v_{pc(south)}=0.381\frac{m}{s}$$

Now with both components of the velocity of the ball + cart, we complete the solution:

$$v_{bc}=\sqrt{v_{bc(east)}^2+v_{bc(north)}^2}=\boxed{0.493\frac{m}{s}}$$

$$\theta_{bc}=\tan^{-1}\left(\frac{v_{bc(north)}}{v_{bc(east)}}\right)=\boxed{50.1^o\ \ north\ of\ east}$$