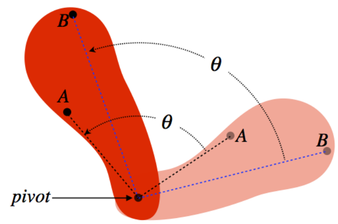
5.1: Rotational Kinematics

Our first foray into linear motion was with kinematics, and we start our discussion of rotation with the same topic.

Rigid Body Rotation

Whenever we talk about "rotation," there is something that is generally implied – we are not talking about a point mass or a collection of independently-moving point masses. Instead, we are generally referring to the rotation of a rigid object. A rigid object is nothing more than a collection of point objects that are confined to stay at specific positions relative to each other. When we talk about rotation, all these point objects follow different paths and travel different distances, but they all have something in common.

Figure 5.1.1 – Motion of Two Points on a Rotating Rigid Body



Drawing a straight line from the fixed point (called the *pivot*) to two different points on the object, we see that the angles through which these straight lines sweep are the same, and indeed this is true for *every* point on the object. So as we talk about rigid body rotation, our old language of linear motion (displacement, velocity, acceleration) that is based on units of distance and time, will have to give way to a new language for rotational motion, based on the units of radians (the most common unit of angular measure) and time. This language will be very similar to what we used for the linear case, usually with the word "angular" or "rotational" appended in front of the usual words.

Just because we are going to a new language, it doesn't mean we throw out the physical principles we have learned so far. But to apply them in our new area of study, we need to develop some way to translate between the two. Back in Section 1.7, in our discussion of circular motion, we came up with a translation between the arclength traveled by an object in circular motion and the angle is motion sweeps out. Certainly the points *A* and *B* in Figure 5.1.1 are following a circular path (they remain a fixed distance from the pivot), so this relation applies to them. If a given point on a rigid body is a distance r from the pivot, then the relationship between the distance it travels along the arclength and the angle measured in radians is given by Equation 1.7.2, and the relationship between its linear speed and the rate at which the angle is changing (in radians per second) is given by Equation 1.7.3, both of which we'll reiterate here:

$$s = R\theta, \qquad v = \frac{ds}{dt} = R\frac{d\theta}{dt} = R\omega \tag{5.1.1}$$

While s and v are different for every point on the rigid object, we see that θ and ω are common to all of them. We therefore embrace these as our *angular displacement* and *angular velocity* measurements, respectively, for the rigid body as a whole. We can similarly define an *angular acceleration* (α) in terms of the change of the linear speed of a spot on the rotating object:

$$a = \frac{dv}{dt} = r\frac{d\omega}{dt} = r\alpha \tag{5.1.2}$$

While each point mass comprising the rigid object may have its own linear velocity/acceleration, they all share a common angular velocity/acceleration. We therefore can simplify our discussion of rigid body rotation from tracking the many different motions of all of the individual parts of the object to one simple parameter common to all of them. We therefore (for the moment) step away from the translation between linear and angular motion – which we have already discussed in earlier sections – and instead focus on purely rotational motion, following exactly the same path as we did for linear motion. You'll note that as a rule the convention for rotational motion, we stick with Greek variables, in contrast to the Latin variables we used for linear motion.

Alert

Whenever the word "acceleration" is combined with circular motion, one naturally thinks of centripetal acceleration. Be careful not to make that association here! The link between linear acceleration and angular acceleration is through the component of acceleration responsible for speeding up the spot on the rigid object, not the acceleration responsible for changing its direction of motion (which is centripetal acceleration). So for example, an object rotating at a constant rate has no point on it that is *speeding up (and has zero angular acceleration), but every point on it (except at the pivot) experiencing a centripetal acceleration. Conversely, a rotating object that slows down, stops, and reverses its direction of motion is experiencing angular acceleration at all times, including the moment it stops, but the centripetal acceleration of points on the object is zero at the moment that it stops. And finally, the difference should be clear mathematically. For a point on the object, its acceleration has two components:*

$$\overrightarrow{a} = \overrightarrow{a}_{\perp} + \overrightarrow{a}_{\parallel}, \qquad where: \begin{cases} a_{\perp} = a_c = r\omega^2 \\ a_{\parallel} = r\alpha = r\dfrac{d\omega}{dt} \end{cases}$$

Rotational Equations of Motion

We define the following angular (rotational) versions of what we studied previously in kinematics:

$$\begin{aligned} position &: \theta(t) \\ displacement &: \Delta\theta = \theta_2 - \theta_1 \\ average\;velocity &: \omega_{ave} = \frac{\Delta\theta}{\Delta t} \\ instantaneous\;velocity &: \omega(t) = \frac{d\theta}{dt} \\ average\;acceleration &: \alpha_{ave} = \frac{\Delta\omega}{\Delta t} \\ instantaneous\;acceleration &: \alpha(t) = \frac{d\omega}{dt} \end{aligned} \tag{5.1.3}$$

The calculus that leads to the equations of motion works out exactly the same way as before (we have only changed the variable names), giving us:

$$\begin{aligned} \theta(t) = &\quad \theta_o + \omega_o t + \tfrac{1}{2}\alpha t^2 \\ \omega(t) = &\quad \omega_o + \alpha t \\ \omega_f^2 - \omega_o^2 = &\quad 2\alpha\left(\Delta\theta\right) \\ \omega_{ave} = &\quad \frac{\omega_o + \omega_f}{2} \quad (if\;\alpha = constant) \end{aligned} \tag{5.1.4}$$

Note that like the case of one-dimensional linear motion, we need to define at the outset a "positive" direction, but for rotation, this means choosing clockwise or counterclockwise from a specific perspective.

Example 5.1.1

A bug stands on the outer edge of a turntable as it begins to spin, accelerating rotationally in the horizontal plane from rest at a constant rate. Find the rate of angular acceleration of the turntable in terms of its radius and the coefficient of static friction if the bug slides off it just as the turntable completes its third full rotation.

Solution

The bug remains on the edge of the turntable thanks to the static friction force, which keeps it going in a circle. When the rotational speed becomes so great that the maximum static friction force is insufficient to maintain this centripetal acceleration, the bug will slide off. The maximum static friction force is the coefficient of static friction multiplied by the normal force, and since the turntable is horizontal and not accelerating vertically, the normal force equals the weight of the bug. We therefore have:

$$\left.\begin{aligned} f_{max} &= ma_c = mr\omega^2 \\ f_{max} &= \mu_s N \\ N &= mg \end{aligned}\right\} \quad \Rightarrow \quad \omega^2 = \frac{\mu_s g}{r}$$

The "no time" kinematics equation for rotation relates the angular acceleration (which we are looking for), the starting rotational speed (which is zero here, as the turntable starts from rest), the final speed (the speed that causes the bug to lose its grip), and the angle through which the object has rotated (which in this case is 6π – three full rotations):

$$\alpha=\frac{\omega_f^2-\cancel{\omega_o^2}}{2\Delta\theta}=\frac{\frac{\mu_s g}{r}}{2\left(6\pi\right)}=\boxed{\frac{\mu_s g}{12\pi r}}$$

Directions of Rotational Kinematics Vectors

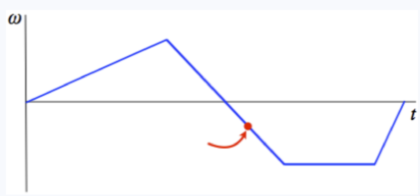
When we did all of this previously, we found it was easy to keep track of directions in one dimension, simply by checking the sign of the value, but when we extended to more dimensions, we needed to treat these quantities like vectors. How can we do this for rotational motion?

The answer comes from all the way back in Chapter 1 – the right hand rule! It goes like this: curl the fingers of your right hand (in their natural finger-curling manner) in the direction that the object is rotating, and your thumb points the direction of the vector. The direction is *perpendicular to the plane of rotation*.

This direction applies to all of the angular motion vectors – displacement, velocity, and acceleration. But be careful about the acceleration vector! Just as in the linear case, the acceleration vector points in the direction of the *changing* velocity vector, not the direction of the velocity vector itself. So if a rotating object is slowing down, the angular acceleration vector points in the opposite direction as the angular velocity vector.

Example 5.1.2

The graph below depicts the rotational velocity of a merry-go-round as a function of time, where the positive direction is defined to be downward (into the surface of the Earth). You are standing near the merry-go-round, watching children go by. At the point indicated in the graph, which of the following are you seeing?



a. *The kids closest to you are moving to the right and are speeding up.*
b. *The kids closest to you are moving to the right and are slowing down.*
c. *The kids closest to you are moving to the left and are speeding up.*
d. *The kids closest to you are moving to the left and are slowing down.*
e. *The kids closest to you are moving to the left, but their speed is not changing.*

Solution

From the RHR, we determine that the positive rotational direction is clockwise as you look at the merry-go-round from above (the kids on the merry-go-round are wondering why you are apparently giving their ride a thumbs-down!). Looking at it from ground level, this means that rotation in a positive direction results in seeing the nearest kids go by from right-to-left. At the point in question, the sign of the rotational velocity is negative, which means the kids are going by left-to-right. A short time later, the rotational velocity will be more negative, which means they are speeding up. So the answer is (a).