5.6: Static Equilibrium

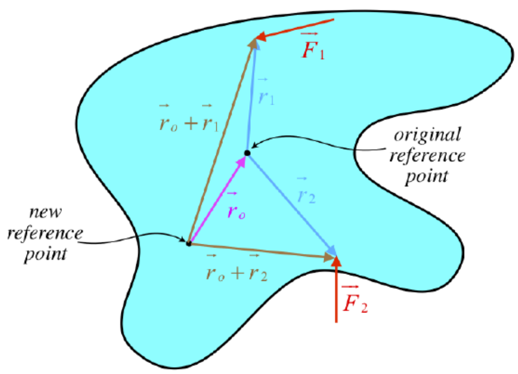
Pivots and Torque Reference Points

The definition of torque (Equation 5.5.6) includes the position vector $\overrightarrow{r}$, which points from a reference point to the point where the force is applied. When we are interested in how the torque is accelerating the object rotationally around a fixed point ("pivot"), it is convenient to choose the reference point to be that fixed point. This is because the forces applied at that fixed point (to keep it fixed) provide zero torque when referenced there, and those forces are generally not known. We explore here the effect of changing the reference point in the particular case when there is *no net force*, though perhaps there could be a net torque. The net torque around a given reference point is:

$$\overrightarrow{\tau}_{net}=\overrightarrow{r}_1\times\overrightarrow{F}_1+\overrightarrow{r}_2\times\overrightarrow{F}_2+\ldots \tag{5.6.1}$$

The reference point is located at the tails of the $\overrightarrow{r}_i$ vectors, but suppose we want to change that reference point. We can do this by simply adding the same constant vector $\overrightarrow{r}_o$ to every position vector. This has the effect of shifting the reference point from the point of $\overrightarrow{r}_o$ to its tail, as shown in Figure 5.6.1 [*Note: The figure shows only two of the many forces applied.*]

Figure 5.6.1 – Changing the Reference Point



The net torque around this new reference point is:

$$\begin{aligned}\overrightarrow{\tau}_{net}(new) &=\left(\overrightarrow{r}_o+\overrightarrow{r}_1\right)\times\overrightarrow{F}_1+\left(\overrightarrow{r}_o+\overrightarrow{r}_2\right)\times\overrightarrow{F}_2+\ldots\\ &=\left[\overrightarrow{r}_1\times\overrightarrow{F}_1+\overrightarrow{r}_2\times\overrightarrow{F}_2+\ldots\right]+\overrightarrow{r}_o\times\left[\overrightarrow{F}_1+\overrightarrow{F}_2+\ldots\right]\\ &=\overrightarrow{\tau}_{net}(original)+\overrightarrow{r}_o\times\overrightarrow{F}_{net}\end{aligned} \tag{5.6.2}$$

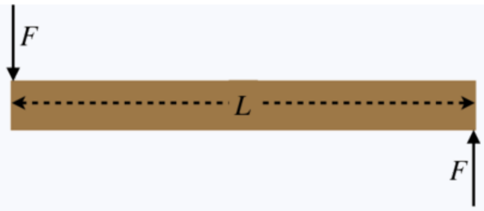
But we assumed that the net force was zero, so we get the remarkable result that the *net torque is the same around every reference point*!

Alert

As amazing as this result is, be careful not to mistake it for too general of a result. The net torque on an object by a collection of forces is only independent of the reference point if those forces result in zero net force.

Example 5.6.1

A non-uniform board of length L starts at rest and is pushed in opposite directions on both of its ends with forces of equal magnitude F at right angles to the board (see diagram). The forces continue to be applied at right angles with the same magnitude until the board has made a complete 360^o rotation, which occurs after a time T. Find the smallest possible rotational inertia for the board when it is rotated around an axis perpendicular to the page.



Solution

The smallest rotational inertia occurs when the axis of rotation is through the center of mass, so we are looking for I_{cm}, but we don't know where the center of mass is for this board, as it is non-uniform. However, in the physical situation given above, there is no net force on the board, so its center of mass, which started at rest, remains at rest, which means the board is rotating around its center of mass. If we can determine the torque about the center of mass and the angular acceleration of the board, then we can use the rotational second law to obtain the rotational inertia:

$$I = \frac{\tau_{net}}{\alpha}$$

We can get α from kinematics, since we know how far the board turns, how long it takes, and the fact that it started from rest:

$$\left.\begin{array}{l} \theta = \omega_o t + \frac{1}{2}\alpha t^2 \\ \theta = 2\pi \\ \omega_o = 0 \\ t = T \end{array}\right\} \quad \Rightarrow \quad \alpha = \frac{4\pi}{T^2}$$

Okay, so we know the acceleration of the board around its center of mass, and all we need is the net torque around that point, but we don't know where the center of mass is, so how are we supposed to continue? The net force is zero, which means we can choose any reference point, and it will give us the torque measured around every reference point, including the center of mass. Choosing the reference point to be one of the ends of the board, the torque due to the force on that end is zero, while the torque due to the force on the other end is simply FL, and the sum of these two torques is the net torque. Using this as the net torque around the center of mass gives us our answer:

$$\boxed{I = \frac{FLT^2}{4\pi}}$$

Note: One can pick a spot on the board and label it as the center of mass, calling the distance from one end x, which makes the distance from the other end equal to $L-x$. Then the torques can be written around the center of mass, and we'll find that the x's will cancel, giving the same result. But why go to all that trouble?

Static Equilibrium

We have spent a great deal of time studying motion in all its forms, but now we're going to step back and look at something called *static equilibrium*. Simply put, this means unmoving (static), and not about to move (equilibrium). This is a particularly important subject for engineers who aspire to build things that won't (easily) fall down. From Newton's laws for linear and rotational motion, we have two conditions for the equilibrium part of this condition:

- net force on object is zero
- net torque on object is zero

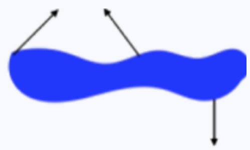
We are quite familiar with the net force part of this, but we need to do a bit of work on net torque. We know the formal definition of torque, but there is more we need to understand in order to apply this to static equilibrium problems. The first tool that we can immediately add to our toolbox for solving such problems is the result we got above. If the object is in static equilibrium, then it is experiencing zero net force, which means that no matter what reference point we choose, the net torque will be the same. But the net torque is *zero* for equilibrium, so we will have the following condition to work with:

For objects in static equilibrium, the net torque calculated around any reference point whatsoever is zero.

We will find the flexibility to choose any point we like as a reference to point to be very useful in what is to come.

Example 5.6.2

For the force diagram below, the force vectors are drawn in the proper locations on the object, and are pointing in the proper directions, but the lengths of the vectors are not to scale. Which of the following statements are true about the effects these forces can have on the motion of this object? Assume that none of the force magnitudes can be set to zero.



a. *The force magnitudes can be set so that the object will not accelerate rotationally, while at the same time its center of mass does not accelerate linearly.*
b. *There is no way to set the force magnitudes to prevent either linear or rotational acceleration.*
c. *The force magnitudes can be set so that either there is no linear acceleration of the object's center of mass, or there is no rotational acceleration of the object, but both cannot be achieved at the same time.*
d. *The force magnitudes can be set so that the object's center of mass will not accelerate linearly, but there is no way to prevent its rotational acceleration.*
e. *The force magnitudes can be set so that the object will not accelerate rotationally, but there is no way to prevent linear acceleration of its center of mass.*

Solution

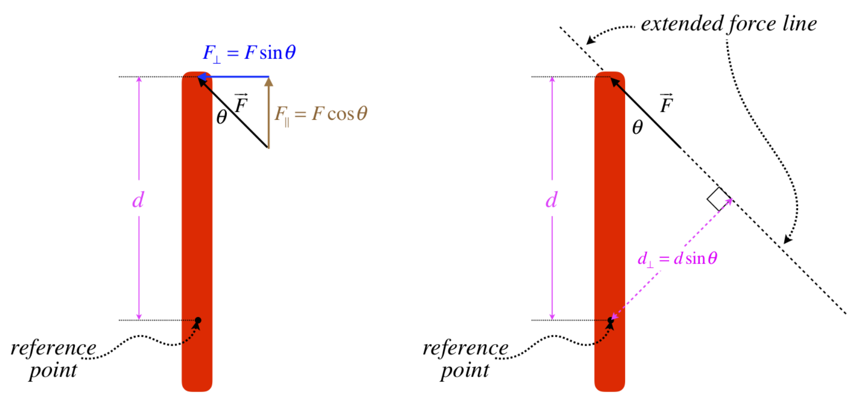
d

The two force vectors can be adjusted relative to each other so that their horizontal components cancel. Then both of their magnitudes can be adjusted in the same proportions so that the horizontal net force remains zero, while their combined vertical component of force cancels the other force vector. So zero net force is achievable. However, if we consider a reference point where the middle force acts on the object (giving that middle force zero contribution to torque), the torque of the other two forces will never cancel, no matter what adjustments are made to the force magnitudes. With no way to make the torque vanish, there is no way to prevent rotational acceleration.

Using Geometry to Determine Torque

Our definition of torque is all well-and-good, but in practice we rarely define a position vector and take a cross product. Instead, we tend to use the concept behind torque, and then some geometry. Figure 5.6.2 shows two ways to geometrically get to the same torque due to an applied force.

Figure 5.6.2 – Alternative Methods of Computing Torque



The left version consists of taking only the component of force that is perpendicular to the line joining the reference point and the point where the force is applied, giving the torque magnitude calculation:

$$\tau = F_\perp d = (F\sin\theta)\, d \tag{5.6.3}$$

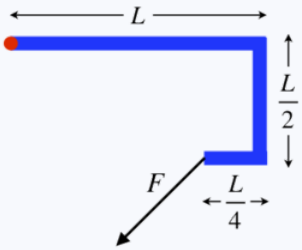
The right side of the figure shows another useful way to compute the same torque magnitude. Rather than finding the part of $\overrightarrow{F}$, it involves finding the perpendicular part of $\overrightarrow{r}$. This is done by extending the line of force and then geometrically determining the perpendiular distance from the reference point to that line. The result is the same as above:

$$\tau = F d_{\perp} = F\left(d\sin\theta\right) \tag{5.6.4}$$

The perpendicular distance from the reference point to the line of force is often referred to as the *moment-arm*, or *lever-arm*. We will find this to often be the method of choice of computing torques when it comes to solving problems.

Example 5.6.3

What can you say about the torque applied to the object due to the force F about the red pivot in the diagram?



a. *it equals $\frac{1}{2}FL$*
b. *it equals $\frac{1}{4}FL$*
c. *it is greater than $\frac{1}{2}FL$*
d. *it is less than $\frac{1}{4}FL$, but greater than zero*
e. *net torques always sum to zero*

Solution

c

There are a couple of ways to answer this. This first is to extend the force line along F and look at the perpendicular distance from the pivot to that line (this is the moment arm). It should be clear from the geometry that this moment arm exceeds $\frac{1}{2}L$, which means the torque must be greater than $\frac{1}{2}FL$. Another way to see it is to break F into two separate vectors, point pointing left and the other pointing down. Both of these forces produce clockwise torques, and the horizontal force has a moment arm of $\frac{1}{2}L$, while the vertical force has a moment arm of $\frac{3}{4}L$. Since the sum of these two force components exceeds the magnitude of the original force, and since one of them has a moment arm larger than $\frac{1}{2}L$, then the combined torques must exceed $\frac{1}{2}FL$.

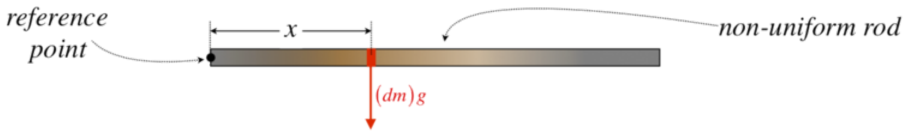
Center of Gravity

Up to now, whenever we have drawn a force diagram of an object, we have always placed the force vector for gravity at its center, while other forces are placed wherever they happen to act on the object. Gravity is somewhat special in that the force actually acts on every single atom in the object, but we can't draw all of those individual force vectors. Drawing it at the center of mass makes sense from the standpoint of Newton's second law, since if gravity is the only force, then it accelerates the object, and the part of the object that accelerates is the center of mass.

Wherever it happens to be appropriate to locate a single gravity force vector on a free-body diagram, it is called the object's *center of gravity*. We are currently dealing with torque, and the position at which a force acts has become quite important, so we need to examine more closely whether we can declare the center of mass of an object to be its center of gravity.

We choose as our test subject a horizontal non-uniform rod of length L, and select one of its ends as a reference point. The plan is to add up all of the infinitesimal torques that occur about this reference point due to gravity acting on every particle in the rod, and see if this total torque can be replaced by the entire gravity force acting at a single point (so that we can draw our free-body diagrams with only one gravity force vector!). An arbitrary piece of the rod will be a distance x from the reference point, and the torque exerted there will be the weight of that piece multiplied by x:

Figure 5.6.3 – Center of Gravity of a Non-Uniform Rod



$$d\tau = (dm\ g)\,x \quad \Rightarrow \quad \tau = \int_0^L dm\ gx = Mg\left[\frac{1}{M}\int_0^L dm\ x\right] = Mgx_{cm} \tag{5.6.5}$$

Sure enough, we get the same torque around the reference point if we put a single force vector with magnitude Mg (the object's full weight) acting at the object's center of mass.

Alert

It should be mentioned that there was a rather subtle assumption made in the above discussion – the gravity force is assumed to be the same at all points on the rod. If the gravity force can somehow vary from one end of the rod to the other, then the positions of these two centers will not coincide. If you are wondering how this can ever be the case, the answer is that the scale of distances must be very large, so that there are measurable differences in the gravity force from one point on the object to the other. This will not be an issue for our typically terrestrially-constrained studies, but can arise when talking about orbits of large bodies like moons.

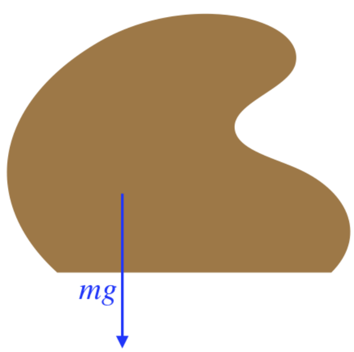
Note that like center of mass, the center of gravity of an object does not have to lie on the object. For example, a hoop's center of gravity is located in the empty space at its center. We now know how to locate the position of the gravity force on an object, and locating most other forces will be fairly intuitive (with one notable exception, which we will address next). This will enable us to use torque to analyze a whole range of real-world problems.

Placement of the Normal Force

Like the gravity force, the normal force can act at many places at once. When two surfaces come into contact, all of the particles at one surface repel all of the particles at the other. So once again we have the problem of where to draw a single force vector, this time for the normal force. The normal force is different from the gravity force, in one important way – it just *compensates* for other forces. That is, it adjusts according to other circumstances. Let's use what we know about static equilibrium to see how to place the normal force properly.

Consider the oddly-shaped object shown in Figure 5.6.4. We'll assume that the object sits at rest on a horizontal surface, the density of this object is not uniform, and that the center of gravity is at the position indicated in the diagram.

Figure 5.6.4 – Deducing the Normal Force Placement Balancing Only Gravity



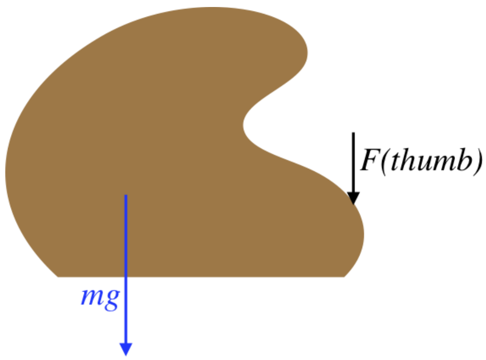
A (rather unsystematic) process for locating the position of the normal force goes like this:

1. Note that the object is in static equilibrium, which means that the normal force is equal to the weight (net force is zero), and that the net torque around any reference point we care to choose is also zero.
2. Try various positions for the normal force, and if we can prove that there must be a non-zero net torque around a reference point, then throw that position out.
3. Repeat step (2) for various positions until one is found that cannot be ruled-out.

For the object in Figure 5.6.4, we could try a normal force vector acting at the center of the base of the object. Then if we choose a reference point between the normal force vector and the weight vector, see see that those two forces must produce a non-zero counter-clockwise torque. We can similarly rule out any position to the right of the weight vector. If we try a position to the left of the weight vector, we get a similar result, this time with the torque being clockwise. We therefore conclude that for this case the normal force vector must be applied exactly where the weight vector intersects the base. No matter where we choose a reference point in that case, the two forces result in equal-and-opposite torques.

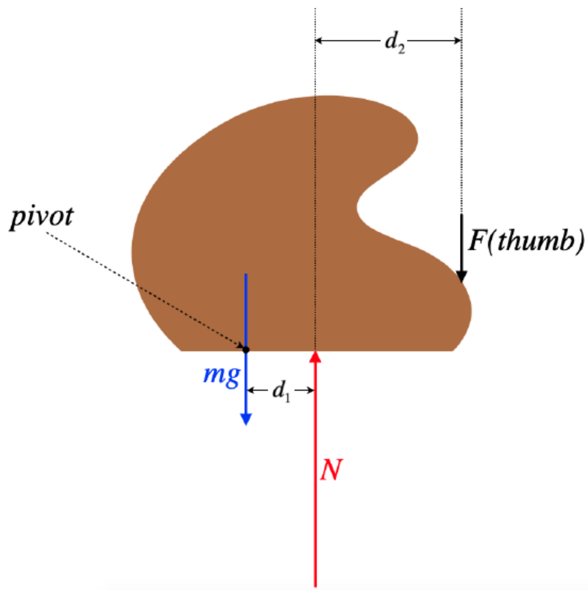
Let's complicate matters some by introducing a second force to our object – suppose we push down on the right side of the object with our thumb, as shown in Figure 5.6.5.

Figure 5.6.5a – Deducing the Normal Force Placement Balancing Two Forces



Let's try the same position for the normal force as before – in line with the gravity force. If we choose as a reference a point in line with these two forces, then they create no torque between the two of them, and the added force by the thumb creates a net clockwise torque. This isn't possible for an object in static equilibrium, so the normal force placement has moved from its original placement as a result of the added thumb force. It's easy to see that the normal force hasn't moved left, as placing the reference point at the normal force results in both the weight and the thumb force producing clockwise torques. So the normal force must move right, but how far? Perhaps it moves into line with the thumb force? No... We can choose the reference point to be in line with these two forces (so they both create zero torque), and the gravity force would yield a net counterclockwise torque.

Figure 5.6.6b – Deducing the Normal Force Placement Balancing Two Forces



So we conclude that the normal force must act somewhere between the gravity and thumb forces. If we know the magnitudes of these two forces, then we know the magnitude of the normal force (the net force is zero), and in fact we can also determine precisely where a single normal force is acting on the object. Calling the distance between the weight and normal force vector placements d_1 and the distance between the normal force and thumb force vector placements d_2, we can sum the torques around a reference point where the normal force acts (so it contributes no torque) to get:

$$0=\tau_{net}=-mgd_1+Fd_2 \quad \Rightarrow \quad \frac{d_1}{d_2}=\frac{F}{mg} \tag{5.6.6}$$

[*Note: In the torque sum, clockwise was chosen as the positive direction.*]

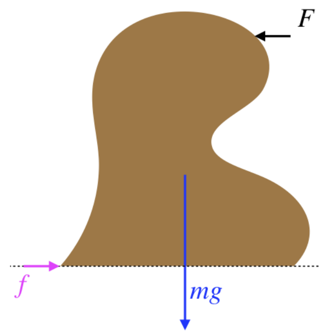
In the diagram the weight is shown to be greater than the thumb force, making the ratio less than 1, which means the placement of the normal force is closer to the placement of the weight vector than the thumb force vector. If the thumb pushes down more, then the normal force placement moves to the right. Note also that the same result arises if the reference point is chosen elsewhere. For example, if the reference point is chosen to be where the weight force acts, then the net torque equation gives zero contribution from the weight, and contributions from both the normal force (counterclockwise), and the thumb force (clockwise). The normal force can then be written in terms of the weight and thumb force (the net force is zero), giving:

$$0=\tau_{net}=-Nd_1+F\left(d_1+d_2\right) \quad \Rightarrow \quad 0=-\left(mg+\cancel{F}\right)d_1+\cancel{Fd_1}+Fd_2 \quad \Rightarrow \quad \frac{d_1}{d_2}=\frac{F}{mg} \tag{5.6.7}$$

Conditions for Tipping

Let's make a slight change to the situation just described. Suppose I push horizontally on the top of the object. What happens to the normal force position as the magnitude of the push increases? Assuming there is a static friction force to prevent the object from sliding, we have a free-body diagram (missing the normal force) that looks like this:

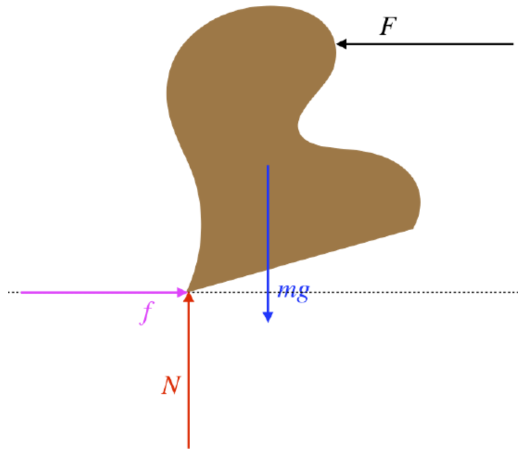
Figure 5.6.6c – Deducing the Normal Force Placement Balancing Two Forces



Choosing a pivot point at the intersection point of the gravity and friction forces, we see that the push force exerts a net counterclockwise torque. For the normal force to counteract this (and given that it must push straight up), we find that it must be placed to the left of the center of gravity.

Let's take a moment to consider the magnitudes of these forces. So long as the object doesn't slide, the static friction force must equal the push. The object doesn't accelerate up or down, so the normal force must have the same magnitude as the gravity force. Both of these conditions are important when we consider what happens when the push force is increased. The friction force also increases until it hits its maximum, at which point the object starts sliding. If we suppose the static friction force doesn't hit its maximum, how is the increased torque by the push compensated by the normal force, if it can't change magnitude? It must move left. But it can only move left for so long, and when it has gone as far as it can go, any added push results in angular acceleration – the object tips.

Figure 5.6.7 – Tipping



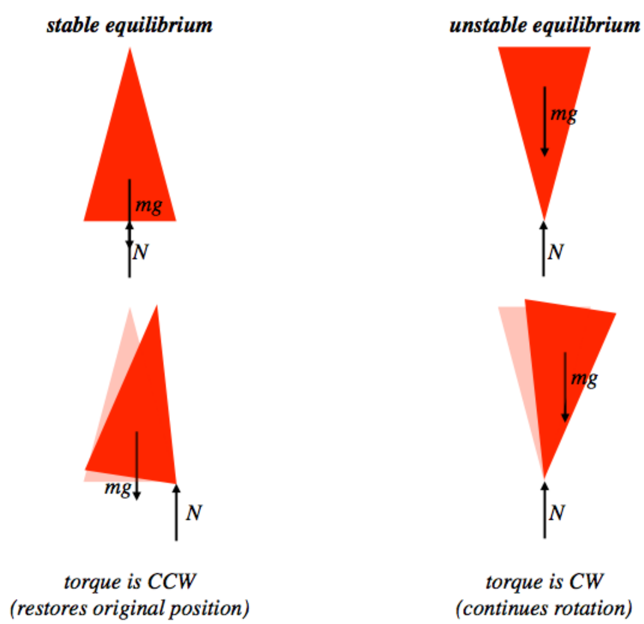
Suppose you want to push an object across the floor without tipping it over. To get it to slide, you have to push with a force at least equal to the static friction force, so to avoid tipping, this given amount of force needs to provide as little toque as possible – push at a point close to the bottom. With very little torque from the push force, the normal force can easily remain inside of the edge of the object, and the object won't tip before it slides.

Stable/Unstable Equilibrium

If the object is oddly-shaped and the only forces acting on it are gravity and the normal force, then this analysis gives us an answer as to whether the object falls over – if the normal force can be directly beneath the center of gravity, then it will stand up. By "can," of course we mean that some part of the base that is in contact with the surface (where the normal force acts) must be below the center of gravity.

In Section 3.7 we discussed stable and unstable equilibrium from the perspective of energy diagrams, and the concept of whether an equilibrium is stable or unstable was first addressed. The idea is that if the system is moved slightly from its equilibrium state, do the forces (or, in our current case, torques) act to return the system to equilibrium (stable), or to continue moving the system away from equilibrium (unstable). How these definitions apply to tipped objects is clear from the free-body diagrams, as shown in Figure 5.6.8. We can also define a *degree* of stability to a standing object. We define it as the angle through which we can rotate it such that if we let it go, restoring torques act to return it to its original position.

Figure 5.6.8 – Stable and Unstable Equilibrium



Note that this definition of stability matches with what we saw in energy diagrams. Recall that an equilibrium was a point where the potential energy function has zero slope, and the equilibrium is stable if the potential energy grows on both sides of the equilibrium, and is unstable if the potential energy falls off on both sides. Consider what happens to the gravitational potential energy of the object in both cases shown in Figure 5.6.8. In the stable case, tipping the object slightly *raises* the center of mass of the object (increasing its gravitational potential energy), while in the unstable case a slight tip *lowers* the center of gravity (decreasing its gravitational potential energy).

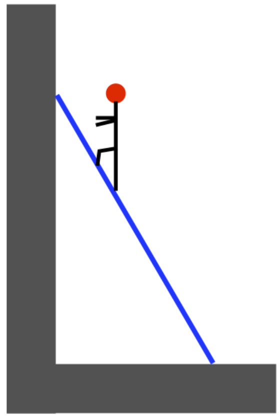
Problem-Solving

Problems involving static equilibrium can be approached in a very systematic way, the steps of which are outlined below:

1. Determine the object in static equilibrium you need to analyze and isolate it in a force diagram. This can sometimes be easier said than done. Sometimes the problem involves more than one extended object in contact with each other. In this case, determining the object you choose (or perhaps the combination of both objects) depends upon what you are asked to solve for (usually a force). You can't really go wrong here, though – if you choose an object that will not give you the answer you need, it should occur to you as you draw the force diagram. Also, you may find that a "wrong" choice of object may simply make your task a bit longer (more simultaneous equations) – annoying, but no real harm done.
2. Define a linear (x, y) coordinate system for force components, and a rotational coordinate system (positive rotation direction) for torques.
3. Extend each force vector with a dotted line as far as it will go on the page in both directions.
4. Choose a reference point. For now we won't worry about choosing a "good" one, choose any – but stick with it for the remaining steps. When you get better at these problems (which you can only achieve by doing them, especially if you do the same problem in multiple ways), you will get better at choosing convenient reference points. Please note that not all static equilibrium problems involve hinges or other "natural" pivots – The reference point doesn't need to be one of these!
5. Ignoring distractions like the object itself, use geometry to find the perpendicular distance from every force line to the pivot point (i.e. all the moment arms). Do not worry about what angle you use to find these (i.e. it doesn't have to be the angle from the torque equation $\tau = rf\sin\theta$) – just use geometry.
6. Multiply the moment arm by the magnitude of each force, and this is the magnitude of the torque due to each force.
7. Determine whether each torque is clockwise or counterclockwise, and give each the appropriate sign when summing the torques and setting that sum equal to zero for the equilibrium torque condition. Note that you could simply alternatively add up all the CW torques, place them on one side of the equation, and set them equal to the sum of the CCW torques on the other side of the equation. This is easier to implement, but loses the "flavor" of what equilibrium is (zero net torque), so I describe it both ways.
8. If you are lucky (or were clever at the outset), this equation may be all you need in order to find what you are looking for. If it isn't, you have two alternatives from here...
   - Write out the sum of the forces in the x and y directions (or just one of those directions, if that is all you need), and set the net forces equal to zero (another condition for equilibrium). These additional equations should be all you need to find what you are looking for.
   - Choose a new reference point and repeat the torque method described above. Recall that the torques should sum to zero around any point, so this is completely valid. The thing to keep in mind is that wherever you choose your reference point, if a force line goes through it, then that force won't appear in the torque equation because the moment arm for it is zero. Therefore you can choose your reference point at a spot through which lines for unknown forces pass, eliminating the need to eliminate them using simultaneous equations later. Whatever you do, don't choose a reference point that lies along a line of the force that you are actually looking for – it doesn't give you an equation that includes that force!

Let's look at some examples of this process in action with a couple of variants on a problem we'll call "Man Climbs Ladder"...

Figure 5.6.9 – Man Climbs Ladder



The following information is given:

- The uniform ladder makes a 60^o angle with the floor.
- The weight of the man is $700N$.
- The weight of the ladder is $150N$.
- The wall is frictionless, but the floor is not.

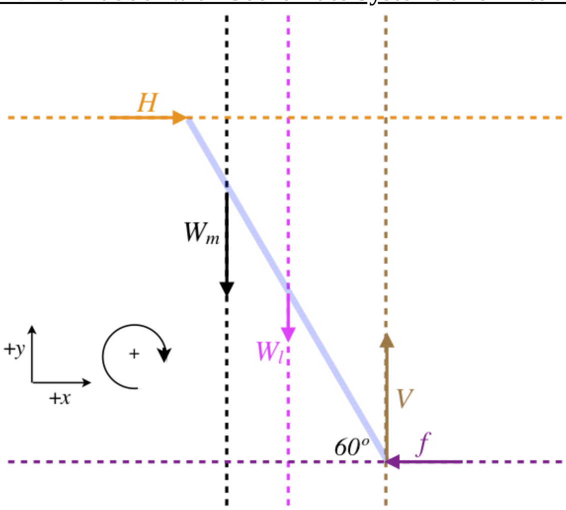
We will hold off for now on what is being asked, as this is sufficient for drawing a free-body diagram.

Alert

While it might seem unlikely, this is actually quite an effective approach to problem-solving in general, but it is especially true for static equilibrium problems: Start analyzing the situation without yet worrying about "where you are going." By removing the focus from a goal to fully understanding the situation, and getting the early steps out of the way, the solution tends to present itself more readily.

In this case, the object in equilibrium experiencing many different forces and torques is the ladder, so we isolate that. We will also include steps (2) and (3) given above (coordinate systems and extended force lines) for our diagram:

Figure 5.6.10 – FBD of Ladder with Coordinate Systems and Extended Force Lines



The vertical normal force by the floor has been labeled V, and the horizontal normal force by the floor is H. The wall is frictionless, so it provides no vertical component of force, and the friction force by the floor is called f. The weight of the ladder is W_l, and since the ladder is uniform, its center of gravity is its geometric center. The force of the man's foot on the ladder equals the weight of the man, and is called W_m. We don't yet know where this last force is placed on the ladder.

Version 1: Man is 75% of the Way Up, Find the Friction Force

We continue our problem solution with step (4) – picking a reference point. The point where the ladder meets the floor seems like a pretty natural place, so let's try that. We are not given the length of the ladder, so we will call it L. Now we need the moment arms (perpendicular distance from reference point to the line of force) for all the forces. Clearly it is zero for V and f. The right triangles formed by the force lines allow us to determine the moment arms for the other three forces easily:

$$\begin{array}{ll} \text{force} & \text{moment arm} \\ wall\;\;(H): & L\sin 60^o \\ man\;\;(W_m): & \frac{3}{4}L\cos 60^o \\ ladder\;weight\;\;(W_l): & \frac{1}{2}L\cos 60^o \end{array} \tag{5.6.8}$$

The force from the wall causes a clockwise (positive) torque, while the other two produce counterclockwise (negative) torques, so the torque equation for equilibrium is:

$$0=\tau_{net}=+H\left(L\sin 60^o\right)-W_m\left(\frac{3}{4}L\cos 60^o\right)-W_l\left(\frac{1}{2}L\cos 60^o\right)\;\;\;\Rightarrow\;\;\;H=346N \tag{5.6.9}$$

[*Note how the length of the ladder drops out of the equation.*]

This equation does not give us an answer for the friction force, but now that we have H, we can use the zero net force in the horizontal direction to get f immediately. The friction force is the only one opposing H, so it too equals 346N.

Before we move on to version #2, let's try using a different reference point, but this time give some thought to what reference point might save us a little bit of work. The overriding consideration is to pick a point that gives zero moment arm for the forces that we neither know, nor care to know. In this problem, those forces are H and V – every other force is either given (W_m and W_l) or desired (f). But those two force lines don't intersect on the ladder. ***It doesn't matter!*** The torques will sum to zero around any reference point whatsoever, and the intersection point of those two force lines works just fine. Calculating the non-zero moment arms from that point to the three force lines, we get:

$$\begin{array}{ll} \text{force} & \text{moment arm} \\ man\;\;(W_m): & \frac{3}{4}L\cos 60^o \\ ladder\;weight\;\;(W_l): & \frac{1}{2}L\cos 60^o \\ friction\;\;(f): & L\sin 60^o \end{array} \tag{5.6.10}$$

Putting these into the zero net torque equation gives us an equation with only f as an unknown, which gives us an instant solution:

$$0=\tau_{net}=-W_m\left(\frac{3}{4}L\cos 60^o\right)-W_l\left(\frac{1}{2}L\cos 60^o\right)+f\left(L\sin 60^o\right)\;\;\;\Rightarrow\;\;\;f=346N \tag{5.6.11}$$

Version 2: Coefficient of Static Friction with Floor is 0.4, Find Percentage of Ladder the Man Can Climb

This is a tougher version of the problem, because a constraint is thrown in. From Equation 5.6.11, it's clear that the higher the man climbs, the greater the static friction force needs to be to balance the torques. The ladder will start to slide when the friction force exceeds its maximum, so we are working with the constraint:

$$f_{max}=\mu_s V \tag{5.6.12}$$

The same free-body diagram applies, except that we now need to describe the distance up the ladder that the man has climbed when the maximum friction force is attained. We'll describe this distance in terms of the fraction of the total length of the ladder that the man is from the top, and we'll call the fraction β.

Okay, now we need to choose a reference point. There is no way to solve the whole problem with one equation, but we still don't care about the force from the wall (we are using both forces from the floor this time), so let's choose the point where the ladder touches the wall. The moment arms are:

$$\begin{array}{ll} \text{force} & \text{moment arm} \\ man\;\;(W_m): & \beta L\cos 60^o \\ ladder\;weight\;\;(W_l): & \frac{1}{2}L\cos 60^o \\ friction\;\;(f): & L\sin 60^o \\ floor\;\;(V): & L\cos 60^o \end{array} \tag{5.6.13}$$

The forces by the man, the weight of the ladder, and friction all cause positive torques, and the vertical force by the floor causes a negative torque, so the zero net torque equation is:

$$0=\tau_{net}=+W_m\left(\beta L\cos 60^o\right)+W_l\left(\frac{1}{2}L\cos 60^o\right)+f\left(L\sin 60^o\right)-V\left(L\cos 60^o\right) \tag{5.6.14}$$

The length of the ladder again cancels out, but still three unknowns remain in this equation. We can compute the vertical floor force from the zero net force in the y-direction:

$$0=-W_m-W_l+V \quad\Rightarrow\quad V=850N \tag{5.6.15}$$

We get the friction force from the constraint equation:

$$f_{max}=\mu_s V \quad\Rightarrow\quad f=340N \tag{5.6.16}$$

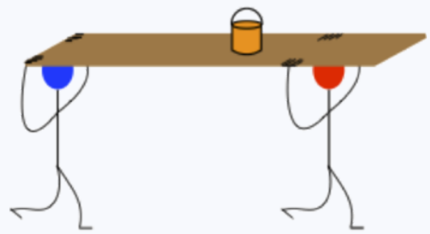
V and f can now be plugged into the torque equation to get our answer:

$$\beta=0.266 \tag{5.6.17}$$

This is the fraction of ladder from the top, so the man can climb 73.4% of the way up before the ladder starts to slide across the floor.

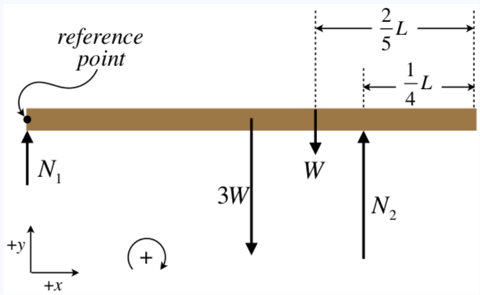
Example 5.6.4

Two painters carry a plank of plywood that they use for scaffolding over their heads on their way to the job site. The plank has a uniform mass distribution. On top it is a can of paint weighing one third as much as the plank. The painter in the rear is holding the plank at the very end and the painter in front is holding the plank one quarter of the the plank length from the front. The can of paint is two-fifths of the plank length from the front. Find the percentage of the total weight carried by the painter in front. Assume that the plank is horizontal as they carry it.



Solution

We start by identifying the object in equilibrium (the plank), and drawing a free-body diagram for it (we'll call the length of the plank L). We will choose the pivot to be the back of the plank, and will refer to the weights of the can of paint and plank as W and 3W, respectively. Also we have chosen an (x,y) coordinate system and the positive direction of rotation to be clockwise, as shown in the diagram.



Next apply the conditions of equilibrium. Clearly the x-direction forces are not meaningful, and the y-direction force equation and torque equations are:

$$\begin{array}{ll} vertical\ forces: & 0=N_1-3W-W+N_2 \\ torques: & 0=N_1(0)+3W\left(\frac{1}{2}L\right)+W\left(\frac{3}{5}L\right)-N_2\left(\frac{3}{4}L\right) \end{array}$$

The L's cancel out of the torque equation, resulting in a relation between the force exerted by the front painter and the weight of the can:

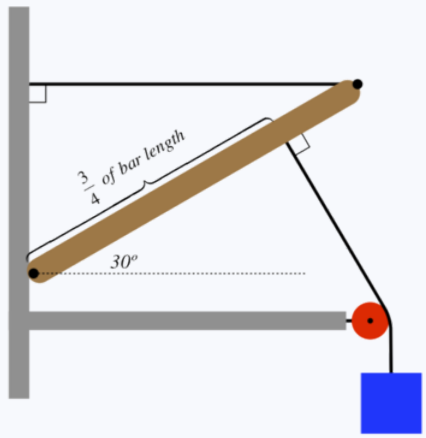
$$N_2 = \frac{14}{5}W$$

The total weight carried by the two painters is found from the force equation (or from common sense), and equals $4W$. So the percentage of the total weight carried by the front painter is:

$$\frac{N_2}{4W} = \boxed{70\%}$$

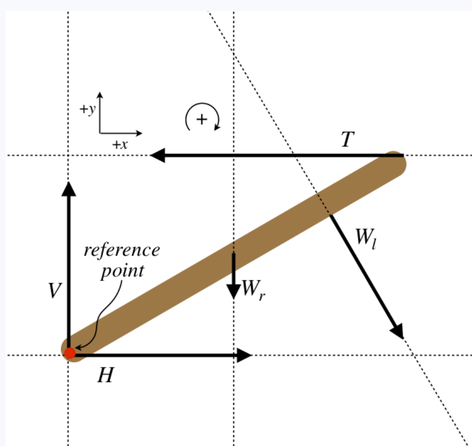
Example 5.6.5

A uniform rigid rod that weighs $400N$ is hinged on a vertical wall and connected to a support wire and a hanging load weighing $1250N$ as shown in the diagram below. Find the magnitude of the force exerted on the rod by the hinge.



Solution

We start with a free-body diagram of the rod, including a coordinate system, a positive direction of rotation, extended force lines, and a reference point:



Next we determine the moment arms. The hinge forces have zero moment arms, and in terms of the length of the rod (which we will call L), the other moment arms are:

force	moment arm
$weight\ of\ rod\ (W_r):$	$\frac{1}{2}L\cos 30^o$
$weight\ of\ load\ (W_l):$	$\frac{3}{4}L$
$tension\ (T):$	$L\sin 30^o$

Now for the zero-net-torque equation. The two weights cause clockwise torque, and the tension counterclockwise torque, so:

$$0=\tau_{net}=W_r\left(\frac{1}{2}L\cos 30^o\right)+W_l\left(\frac{3}{4}L\right)-T\left(L\sin 30^o\right)$$

The L cancels out of every term, and we can solve for the tension:

$$T=2221N$$

To get the horizontal and vertical forces by the hinge, we need the vertical and horizontal zero-net-force equations:

$$\begin{array}{ll} x-direction: & 0=H-T+W_l\sin 30^o \\ y-direction: & 0=V-W_r-W_l\cos 30^o \end{array}$$

Solve for H and V and combine them to get the magnitude of the force by the hinge:

$$\left.\begin{array}{l} H=1596N \\ V=1483N \end{array}\right\} \Rightarrow \quad F_{hinge}=\sqrt{H^2+V^2}=\boxed{2179N}$$