1.4: Kinematics

Equations of Motion

Okay, enough of the definitions. Let's see how these things all fit together, and how they can be used. What we will be looking at are called the *equations of motion*, and this topic is often referred to as *kinematics*. It is important to note that we are not yet dealing with causes for these motions, but only the motions themselves.

We will mostly only deal with constant accelerations (unless otherwise specified), and since instantaneous acceleration is the derivative of velocity, it is not difficult in the case of constant acceleration to integrate it to get the instantaneous velocity as a function of time:

$$\left.\begin{array}{l} a=\dfrac{dv}{dt} \quad \Rightarrow \quad v(t)=\int a\; dt=at+const \\ const=v(t=0)\equiv v_o \end{array}\right\} \quad v(t)=at+v_o \tag{1.4.1}$$

The constant of integration is found by plugging $t=0$ into Equation 1.4.1, which results in the velocity of the object at the starting time, which is typically designated as v_o.

We can play exactly the same game to obtain the equation of motion for position as a function of time, since we know how it relates to the instantaneous velocity:

$$\left.\begin{array}{l} v=\dfrac{dx}{dt} \quad \Rightarrow \quad x(t)=\int v dt=\int (at+v_o)\,dt=\frac{1}{2}at^2+v_o t+const \\ const=x(t=0)\equiv x_o \end{array}\right\} \quad x(t)=\frac{1}{2}at^2+v_o t+x_o \tag{1.4.2}$$

Notice that if we have all the details of this last equation, we can obtain the velocity equation above simply by taking a derivative. We cannot go in the opposite direction without also obtaining the starting position.

Example 1.4.1

The acceleration of a particle moving along the x-axis is given by the equation [note that it is **not** *constant!]:*

$$\alpha(t)=\beta+\lambda t, \qquad \beta=2.40\frac{m}{s^2}, \qquad \lambda=0.300\frac{m}{s^2}$$

The particle is at position $x=+4.60m$ and is moving in the $-x$ direction at a speed of $12.0\frac{m}{s}$ at time $t=0s$.

a. *Find the time at which the particle (briefly) comes to rest.*
b. *Find the position where the particle (briefly) comes to rest.*

Solution

a. We start by finding the equation for the velocity, as we are interested in the time at which this value goes to zero. The velocity function is the integral of the acceleration function:

$$v(t)=\int a(t)\,dt=\int [\beta+\lambda t]\,dt=\beta t+\frac{1}{2}\lambda t^2+const$$

We can determine the integration constant by plugging in what we know about the velocity at time $t=0$:

$$v(t=0)=\beta\cdot 0+\frac{1}{2}\lambda(0)^2+const \quad \Rightarrow \quad const=v_o=-12.0\frac{m}{s}$$

Plugging this back into our equation for velocity and setting the velocity equal to zero, we can calculate the time at which this occurs:

$$v(t=0)=\frac{1}{2}\lambda t^2+\beta t+v_o \quad \Rightarrow \quad t=\frac{-\beta\pm\sqrt{\beta^2-4\left(\frac{1}{2}\lambda\right)v_o}}{2\left(\frac{1}{2}\lambda\right)}=\boxed{4.00s}$$

b. We know the time at which it comes to rest, so we need the equation for the position as a function of time. We get this by integrating the velocity function:

$$x(t)=\int v(t)\,dt=\int\left[\frac{1}{2}\lambda t^2+\beta t+v_o\right]dt=\frac{1}{6}\lambda t^3+\frac{1}{2}\beta t^2+v_o t+const$$

As we did above, we find the constant of integration by putting in what we know about the position at $t=0$:

$$x\left(t=0\right)=\frac{1}{6}\lambda(0)^3+\frac{1}{2}\beta(0)^2+v_o\cdot 0+const \quad\Rightarrow\quad const=4.60m$$

And finally, plug this result back into the equation for $x\left(t\right)$ and put in the time we found above ($t=4s$) to get:

$$x\left(4\right)=\boxed{-21.0m}$$

Let's make an accounting of all the numbers we can encounter in a constant-acceleration situation:

- independent variable: t
- dependent variables: x, v
- constants of the motion: x_o, v_o, a (acceleration is constant by assumption)

With six numbers to work with, you can imagine there are many ways to set up a problem to solve for something unknown. Everything you need to solve any such problem is provided in the above equations. However, it is often easier to put those equations together to form a new equation, to cut down on the algebra needs for certain types of problems. The most common useful re-combining of these variables involves eliminating time from the two equations, since you may be given velocities and positions. The algebra is straightforward:

$$\left.\begin{array}{l} v_f=at+v_o \;\Rightarrow\; t=\dfrac{v_f-v_o}{a} \\ x_f-x_o=\frac{1}{2}at^2+v_ot \end{array}\right\} \quad x_f-x_o=\frac{1}{2}a\left(\frac{v_f-v_o}{a}\right)^2+v_o\left(\frac{v_f-v_o}{a}\right) \quad\Rightarrow\quad 2a\left(x_f-x_o\right)={v_f}^2-{v_o}^2 \tag{1.4.3}$$

You can think of this equation as the "before/after" equation, because it deals only with starting and ending positions and velocities, and has eliminated time as an input variable.

While we are accumulating useful (though unnecessary) equations for motion with constant acceleration, we should also include the two equations that involve average velocity. The first is just a rewriting of the definition of average velocity, with the "final" position occurring at time t:

$$v_{ave}=\frac{x_f-x_o}{t}=\frac{x\left(t\right)-x_o}{t} \quad\Rightarrow\quad x\left(t\right)=v_{ave}t+x_o \tag{1.4.4}$$

The second equation is quite useful, though it applies *only* to motion involving constant acceleration:

$$v_{ave}=\frac{x_f-x_o}{t}=\frac{\frac{1}{2}at^2+v_ot}{t}=\frac{1}{2}at+v_o=\frac{1}{2}\left(v_f-v_o\right)+v_o \quad\Rightarrow\quad v_{ave}=\frac{v_o+v_f}{2} \tag{1.4.5}$$

For constant acceleration, the average velocity simply equals the arithmetic average of the starting and ending velocities. We will better see why it comes out this way when we start discussing graphing shortly.

Free-Fall

There is one type of straight-line motion that involves constant acceleration that we are all familiar with: free-fall.

We will look more closely at how to explain this in terms of forces in a future section, but assuming air resistance has a small effect (remember, we are devising a simplified model here), then it turns out (as shown by Galileo dropping stones from the Tower of Pisa, and more dramatically in the demonstration) that objects all accelerate at the same constant rate as they fall to Earth. This rate of acceleration is commonly given the symbol g, and it has the value:

$$\text{acceleration due to gravity near the surface of the earth} = g = 9.8\,\frac{m}{s^2}$$

Note the units of distance-per-time-squared are the units of acceleration. This acceleration is of course always directed downward, and depending on our choice of coordinate system, this can be either positive or negative. Once the coordinate system is selected, the sign for g stays the same no matter which way the object is moving. If the positive direction is chosen to be upward, and the object is moving upward, then its velocity is positive and the negative value of g leads to a slowing of the object's motion. If it is moving down, then its velocity is negative, and the negative acceleration leads to the velocity becoming more negative (i.e. it is speeding up).

Example 1.4.2

A ball is thrown vertically upward at the same instant that a second ball is dropped from rest directly above it. The two balls are $12.0m$ *apart when they start their motion. Find the maximum speed at which the first ball can be thrown such that it doesn't collide with the second ball before it returns to its starting height. Treat the balls as being very small (i.e. ignore their diameters).*

Solution

Both balls are under the influence of the earth's gravity, and therefore both accelerate at a rate g downward. Treating up as the positive direction and the starting height of the ball thrown up as the origin, we have the following equations of motion for the two balls:

$$y_1(t) = -\frac{1}{2}gt^2 + v_o t + 0 \qquad y_2(t) = -\frac{1}{2}gt^2 + 0 \cdot t + y_o$$

We can use the equation for the dropped ball and its starting height to determine the time it takes to reach the origin, then plug that result into the equation for the thrown ball, to determine how high it will be when the dropped ball gets to the origin:

$$t = \sqrt{\frac{2y_o}{g}} \quad \Rightarrow \quad y_1 = -\frac{1}{2}g\left(\sqrt{\frac{2y_o}{g}}\right)^2 + v_o\left(\sqrt{\frac{2y_o}{g}}\right) = -y_o + \sqrt{\frac{2y_o}{g}}v_o$$

We insist that the the thrown ball has fallen back to at least its starting height by the time the dropped ball gets there, so we want y_1 to be no greater than zero, which gives us an inequality for v_o:

$$v_o \le \sqrt{\frac{gy_o}{2}} \quad \Rightarrow \quad v_o \le 7.67\,\frac{m}{s}$$