3.6: Force and Potential Energy

Force and the Potential Energy Function

We now have an alternative to the using the work-energy theorem when conservative forces are involved – it consists of computing potential energies and applying mechanical energy conservation. In essence we have developed the idea of potential energy starting from force. Specifically, we have, from Equation 3.4.4 and the definition of work, the following relationship between the potential energy difference between two points and the conservative force that does the work for which the use of potential energy is a shortcut:

$$U_B - U_A = -\int_A^B \overrightarrow{F} \cdot \overrightarrow{dl} \tag{3.6.1}$$

As we saw in Section 3.4, we can express the potential energy of a system as a function of position, so the question arises, "Is there some way to "reverse" Equation 3.6.1 so that we can obtain the functional form of the conservative force from the potential energy function?" We know that derivatives are the "opposite" of integrals, so it should not be too surprising that the reverse of Equation 3.6.1 takes the form of a derivative. To see how this works, let's consider only a very tiny change in potential energy due to a very small displacement. This changes the left hand side of Equation 3.6.1 to an infinitesimal, and the right hand side is no longer a sum of many pieces, but is instead only a single piece:

$$dU = -\overrightarrow{F} \cdot \overrightarrow{dl} \tag{3.6.2}$$

In three dimensions, the tiny displacement can be written as:

$$\overrightarrow{dl} = dx\ \hat{i} + dy\ \hat{j} + dz\ \hat{k} \tag{3.6.3}$$

This means that the dot product with the force vector is:

$$\overrightarrow{F} \cdot \overrightarrow{dl} = F_x dx + F_y dy + F_z dz \tag{3.6.4}$$

Suppose we make our tiny displacement only along the x-axis, so that dy and dz are zero. Then clearly all the work done by the force is given by the first term above, and we get that the small change in potential energy that occurs when the position changes a small amount in the x-direction is:

$$dU\left(x \rightarrow x + dx\right) = -F_x dx \quad \Rightarrow \quad F_x = -\frac{dU}{dx} \tag{3.6.5}$$

This is fine for a potential that changes only in the x-direction, but what happens if the potential energy is also a function of y and z? The answer is that we *treat y and z as though they are constants*, which means that $dy = dz = 0$, and our result above works. When we treat y and z as constants, we have to do something slightly different with our derivative. For example, if we take a derivative of the function $U\left(x, y\right) = xy$ with respect to x, we get, from the product rule:

$$\frac{dU}{dx} = \frac{d}{dx}(xy) = (1)\left(y\right) + \left(x\right)\left(\frac{dy}{dx}\right) \tag{3.6.6}$$

But if we we treat y and z as constants, the derivative of these variables are zero, making the second term above vanish. We call this "hold the other variables constant" derivative a *partial derivative*, and we even use a slightly different symbol to represent it:

$$partial\ derivative\ of\ function\ f\ with\ respect\ to\ x = \frac{\partial f}{\partial x}$$

So following the discussion above, we find that by holding two of the variables constant at a time (so that the displacement for the work is along only one axis), we can obtain all the components of the force from the potential function $U\left(x, y, z\right)$:

$$F_x = -\frac{\partial}{\partial x}U, \quad F_y = -\frac{\partial}{\partial y}U, \quad F_z = -\frac{\partial}{\partial z}U \tag{3.6.7}$$

Example 3.6.1

An object with a mass of 2.00kg moves through a region of space where it experiences only a conservative force whose potential energy function is given by:

$$U(x, y, z)=\beta x\left(y^2+z^2\right), \qquad \beta=-3.80 \frac{J}{m^3}$$

Find the magnitude of the acceleration of the object when it reaches the position $(x, y, z)=(1.50m, 3.00m, 4.00m)$

Solution

We know the mass of the object, so if we can determine the net force on it, we can get its acceleration from Newton's second law. The only force on this object is the conservative force with the given potential energy function, so that is the net force. We compute its components using the partial derivatives:

$$\begin{aligned}
F_x &= -\frac{\partial}{\partial x}\left[\beta x\left(y^2+z^2\right)\right]=-\beta x\left(y^2+z^2\right)=95.0N \\
F_y &= -\frac{\partial}{\partial y}\left[\beta x\left(y^2+z^2\right)\right]=-2\beta x y=34.2N \\
F_z &= -\frac{\partial}{\partial z}\left[\beta x\left(y^2+z^2\right)\right]=-2\beta x z=45.6N
\end{aligned}$$

And now for the magnitude of the acceleration:

$$a=\frac{\left|\overrightarrow{F}\right|}{m}=\frac{\sqrt{F_x^2+F_y^2+F_z^2}}{m}=\boxed{55.4\frac{m}{s^2}}$$

We can check to make sure that this method of deriving the force from the potential energy is consistent with the cases we have seen already:

Gravity: $U(x, y, z)=mgy+U_o$

$$\left.\begin{aligned}
F_x &= -\frac{\partial}{\partial x}U=-\frac{\partial}{\partial x}(mgy+U_o)=0 \\
F_y &= -\frac{\partial}{\partial y}U=-\frac{\partial}{\partial y}(mgy+U_o)=-mg \\
F_z &= -\frac{\partial}{\partial z}U=-\frac{\partial}{\partial z}(mgy+U_o)=0
\end{aligned}\right\} \Rightarrow \overrightarrow{F}_{gravity}=-mg\,\hat{j} \tag{3.6.8}$$

Elastic Force: $U(x, y, z)=\frac{1}{2}kx^2+U_o$

$$\left.\begin{aligned}
F_x &= -\frac{\partial}{\partial x}U=-\frac{\partial}{\partial x}\left(\tfrac{1}{2}kx^2+U_o\right)=-kx \\
F_y &= -\frac{\partial}{\partial y}U=-\frac{\partial}{\partial y}\left(\tfrac{1}{2}kx^2+U_o\right)=0 \\
F_z &= -\frac{\partial}{\partial z}U=-\frac{\partial}{\partial z}\left(\tfrac{1}{2}kx^2+U_o\right)=0
\end{aligned}\right\} \Rightarrow \overrightarrow{F}_{elastic}=-kx\,\hat{i} \tag{3.6.9}$$

Determining Conservative or Non-Conservative

We know that a potential energy can only be defined for a conservative force, and until now to show that a force is non-conservative we had to do two line integrals between the same two points and show that they yield different results, but this program for finding the force from the potential energy function gives us another less-onerous method for doing this. It goes something like this:

- Start with the force we want to know about, and integrate the x-component with respect to x to "undo" the negative partial derivative of the potential energy function with respect to x. Don't forget to leave an arbitrary constant added to the integration (this is an indefinite integral):

$$U(x,y,z) = -\int F_x dx + constant \tag{3.6.10}$$

- Because we have undone a partial derivative (which assumes the other variables are constant), even the variables y and z are fair game for the arbitrary constant of integration, so write the constant as an unknown function of those variables:

$$U(x,y,z) = -\int F_x dx + h(y,z) \tag{3.6.11}$$

This can readily be shown to be correct by taking the negative partial derivative with respect to x of both sides.

- Use this "candidate" potential energy function to get the other two components of the force vector. If this is possible, then the function $h(y,z)$ can be found (to within a numerical constant). If there is no way to get to the y and z components of the force vector, then it is non-conservative.

Example 3.6.2

Show that the force given in Example 3.2.1 (given again below) is not conservative, using the try-to-integrate-the-force method.

$$\overrightarrow{F}(y) = \alpha y \hat{i}$$

Solution

There is only an x-component of the force, so integrate that with respect to x:

$$U(x,y,z) = -\int F_x dx = -\alpha xy + h(y,z)$$

If we pick the function $h(y,z)$ equal to just zero, aren't we done? Haven't we shown that the force is conservative? After all, its derivative with respect to x gives us the x-component of the force, and that is the only component. Not so fast! The other components are zero, and we must be able to get those components from the partial derivatives as well. Here is where we run into trouble. Taking the partial derivative with respect to y and setting it equal to zero gives:

$$F_y = -\frac{\partial}{\partial y}U = -\frac{\partial}{\partial y}\left(-\alpha xy + h(y,z)\right) = \alpha x - \frac{\partial h}{\partial y}$$

This can only equal zero (and give the proper y component of the force) if $\frac{\partial h}{\partial y}$ equals αx. But how can this possibly be true, when the function h depends upon y and z? This is mathematically impossible, which means that this force is non-conservative.

While it's unlikely you have encountered it at this point unless you have taken more math courses than is typical at this point, you should be made aware of a shorthand notation that exists for this process of obtaining the force vector from the potential energy function. Rather than write three equations – one for each component of force – this relationship is often written as a vector equation that looks like this:

$$\overrightarrow{F} = -\overrightarrow{\nabla} U \tag{3.6.12}$$

The funny-looking triangle vector is called the *gradient operator*, or "del," and can be written like this:

$$\overrightarrow{\nabla} \equiv \hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}, \tag{3.6.13}$$

or, in column matrix notation:

$$\overrightarrow{\nabla} \equiv \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \tag{3.6.14}$$

Note that $\overrightarrow{\nabla}$ is not itself a vector – it has to "act upon" a function to create a vector. When it performs this function, the derivatives define vector components which are conveniently multiplied by the unit vectors.

Equipotential Surfaces

Consider the following potential energy function:

$$U(x,y,z) = -\alpha\left(x^2+y^2+z^2\right) \tag{3.6.15}$$

Notice that every point that is the same distance from the origin results in the same potential energy, since the potential energy function is proportional to the square of the radius of a sphere centered at the origin. This means that if an object moves between two points in space, where both points are the same distance from the origin, then (assuming this is the only force present) the object is moving the same speed at both points. This is because mechanical energy is conserved, and the potential energy hasn't changed, so the kinetic energy is also unchanged.

Every value available to the $U(x,y,z)$ above defines the surface of a sphere centered at the origin on which every point corresponds to the same potential energy. But the potential energy function above is not unique. Every such function defines surfaces of equal potential energy. We call these *equipotential surfaces*. A good example of these are represented by the dotted lines you see on topographical maps used by backpackers – each dotted line represents a fixed altitude, and therefore an equal gravitational potential.

Let's compute the force vector for the potential above:

$$\overrightarrow{F} = \hat{i}\left(-\frac{\partial U}{\partial x}\right)+\hat{j}\left(-\frac{\partial U}{\partial y}\right)+\hat{k}\left(-\frac{\partial U}{\partial z}\right) = 2\alpha\left(x\hat{i}+y\hat{j}+z\hat{k}\right) \tag{3.6.16}$$

Hopefully you recognize the part of this vector in parentheses. It is the position vector relative to the origin, Equation 1.6.1. This vector points directly to the point (x,y,z) from the origin, which means that it is *perpendicular to the sphere centered at the origin that contains that point*. It turns out to be a general property that *the conservative force associated with a potential is perpendicular to the equipotential surfaces everywhere in space*.

Notice that for the function $U(x,y,z)$ above, if $\alpha>0$, the potential energy gets *smaller* as one gets farther from the origin, and the force vector from this potential points away from the origin. This is also a general feature – *the conservative force associated with a potential points in the direction from greater potential to lower potential*. It should be clear on many fronts why this must be the case. If an object moves from a region of higher potential to one of lower potential, this decrease in *PE* must be balanced by an increase in *KE*, which means the object speeds up. Objects speed up when the net force on them points in the same direction that they are moving, so the force must point from where the PE is higher to where it is lower.
