4.6: Problem Solving

In this chapter we consider three classic problems in momentum and energy conservation.

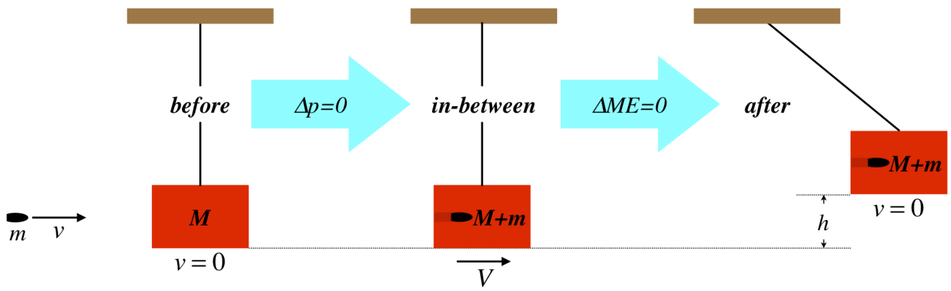
The Ballistic Pendulum

A bullet is fired into a block of wood that is hanging by a string from the ceiling. The mass of the bullet and the block are given, as is the incoming speed of the bullet. Find the height to which the block + bullet swings before stopping.

This problem involves both momentum and energy, but it requires great care that we don't invoke energy conservation when non-conservative forces are acting, and that we don't invoke momentum conservation when external forces are acting. Basically the trick is to dance around the problem only using the right tool for the job. The bullet colliding with the wood block involves a non-conservative force (the friction that slows the bullet within the block), so the mechanical energy of the bullet + block system is not conserved. But the force between the bullet and the block is internal to the bullet + block system, so momentum is conserved (we assume that the bullet digs-in and lodges so quickly that the block doesn't yet swing very far, so the external force from the string has no horizontal component during the collision). So we use momentum conservation to determine the speed of the block + bullet as they begin to swing. Then once it is swinging, the tension force from the string kills momentum conservation, but it does no work (it acts perpendicular to the direction of motion), so mechanical energy is conserved, and we use that to find the height.

Figure 4.6.1 – Ballistic Pendulum

Figure 4.5.1 – The Ballistic Pendulum



Applying momentum conservation for the first stage gives the following:

$$mv + 0 = (M+m)V \quad \Rightarrow \quad V = \frac{m}{M+m}v \tag{4.6.1}$$

Mechanical energy conservation for the second stage yields:

$$\frac{1}{2}\cancel{(M+m)}V^2 = \cancel{(M+m)}gh \quad \Rightarrow \quad h = \frac{V^2}{2g} \tag{4.6.2}$$

Plugging V from Equation 4.6.1 into Equation 4.6.2 gives an answer:

$$h = \frac{1}{2g}\left(\frac{m}{M+m}\right)^2 v^2 \tag{4.6.3}$$

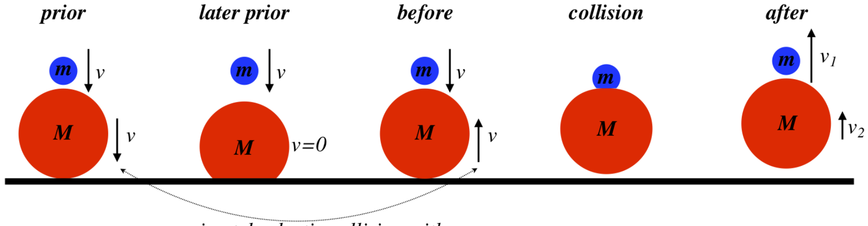
Notice that using conservation of energy from the beginning will not give the right answer – it *must* be broken into two parts, because we can only use the appropriate physical principles when they are applicable.

Stacked Balls

Two balls are dropped to the floor, with the lighter ball atop the heavier one. The balls collide approximately elastically with each other and with the floor. We see a reaction we are not used to – the small ball flies up to a height higher than it was dropped. Does this violate mechanical energy conservation?

Figure 4.6.2 – Stacked Balls

Figure 4.5.2 – The Stacked Ball Launch



As with the previous problem, we need to break this up into workable parts. If we assume an elastic collision with the floor (this is not necessary, but it makes it easier to compare the initial and final heights of the smaller ball), then the large ball leaves the floor at the same speed at which it struck the floor (which we are calling v). The balls were dropped together, so the small ball is moving at the same speed down as the large ball is moving up, and a collision occurs (depicted in the "before" portion of Figure 4.6.2). The momentum and kinetic energy conservation of this collision allows us to solve for the final velocities of the two balls:

$$\begin{array}{ll} \textit{momentum conservation}: & Mv - mv = mv_1 + Mv_2 \\ \textit{elastic collision}: & \frac{1}{2}mv^2 + \frac{1}{2}Mv^2 = \frac{1}{2}mv_1^2 + \frac{1}{2}Mv_2^2 \end{array} \tag{4.6.4}$$

We can now solve for v_1 and v_2 in terms of v, which we can determine from the height that the balls are dropped from. If v_1 comes out to be greater than v, then the small ball will rise to a height greater than that from which it is dropped. The algebra required to solve for v_1 and v_2 is rather daunting and too long to provide here, but here is the result:

$$v_1 = \left(\frac{3M-m}{M+m}\right)v\;, \qquad v_2 = \left(\frac{M-3m}{M+m}\right)v \tag{4.6.5}$$

We see that the small ball must rise to a height greater than that from which it was dropped, because the fraction in front of v is always greater than 1. Interestingly, if the larger ball is exactly three times the mass of the smaller one, the larger ball comes rest immediately after the collision with the smaller ball.

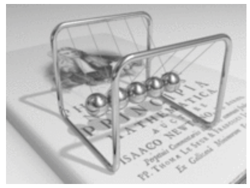
Perhaps you are worried about how we can assume a head-on collision of two balls going in opposite directions at the same speed, when they are in fact in contact with each other throughout? You are right to worry! This is one of those cases where we have *greatly* simplified the conditions in order to get a solution. Clearly if the balls were glued together for the collision with the floor, and then just as they leave contact with the floor the glue dissolves, then the balls would rise together at the same speed, staying in contact.

How do we explain this difference? When the balls remain in contact during the collision with the floor, the large ball is pushed from above by the smaller ball, and compressing the larger ball against the floor more than if the balls were not in contact. This greater compression leads to a greater impulse delivered by the floor. Indeed, we can compute this difference: With the balls in contact, they behave like a single ball with mass $M+m$, so the momentum change due to an elastic collision with the ground is $2\left(M+m\right)v$. If the balls lose contact, and the bottom ball bounces off the floor by itself, the momentum change is only $2Mv$. So the momentum of the system of two balls as they move up is greater when the large ball is in contact with the small ball during the bounce than when it bounces off the floor while not in contact with the small ball. All of the collisions are elastic, so regardless of whether or not the balls are in contact, the kinetic energy of the system as it leaves the floor is the same.

A system of two or more particles can of course have many different momenta for the same total kinetic energy. [*For example, a system of two identical particles moving at equal speeds toward each other has zero momentum, but if they are moving in the same direction at the same speed, the momentum is not zero, while the kinetic energy is the same in both cases.*] This is the case here – the two-ball system comes off the floor with the same kinetic energy in both cases, but the circumstances provide different momenta for those two cases, and this leads to different behaviors for the two objects in the system.

These are not the only two possibilities. If the balls are separated by a very small distance, so that the bottom ball does not fully reflect off the floor before colliding with the higher ball, then contact with the smaller ball will slow the larger ball's departure from the floor, increasing the impulse delivered by the floor. The increased impulse is not as great as if the balls were in contact the whole time, but it is more than the case where the larger ball bounces off the floor unimpeded. In this case, the momentum of the system is less than the case of the connected balls, and more than the case of the disconnected balls. The kinetic energy is still the same, and the result is that the small ball rises faster than the large ball, but not as fast as it does in the disconnected case. The case we calculated above is the *fastest* the smaller ball can be going at the end, and therefore the *highest* it can rise.

Newton's Cradle



Everyone is familiar with this desktop toy. As fun as it is to watch, it also comes with a puzzle: How does the other side of the line of balls know how many balls to send up? Obviously, if we send two balls down it will provide more momentum to come through the other side, but why can't that momentum come from a single ball that comes out twice as fast? Or four balls half as fast? The key is in the fact that the collisions are (nearly) elastic. If two balls go in and one ball comes out with that momentum, then it will have a different kinetic energy:

$$\begin{array}{ll} momentum\;in:\;2p & KE\;in:\;2\left(\frac{p^2}{2m}\right) \\ \underbrace{momentum\;out:2p}_{\substack{momentum\;conserved\\all\;forces\;internal}} & \underbrace{KE\;out:\left(\frac{(2p)^2}{2m}\right)}_{\substack{if\;one\;ball\;comes\;out,\;it\;does\\so\;with\;twice\;the\;momentum\\of\;a\;single\;ball\;coming\;in}} = 4\cancel{\frac{p^2}{2m}}\;(too\;much) \end{array} \tag{4.6.6}$$

We see a similar thing if too many balls come out (the total outgoing KE is too small). So the cradle "knows" how many balls to send out because it is the only way it can satisfy momentum conservation for the elastic collisions.