7.7: Global Symmetries

Symmetries and symmetry groups

Interacting systems can be broadly classified according to their *global symmetry group*. Consider the following five examples:

$$\begin{aligned}
\hat{H}_{Ising} &= -\sum_{i<j} J_{ij}\,\sigma_i\,\sigma_j & \sigma_i &\in \{-1,+1\} \\
\hat{H}_{p-clock} &= -\sum_{i<j} J_{ij}\,\cos\left(\frac{2\pi(n_i-n_j)}{p}\right) & n_i &\in \{1,2,\ldots,p\} \\
\hat{H}_{q-Potts} &= -\sum_{i<j} J_{ij}\,\delta_{\sigma_i,\sigma_j} & \sigma_i &\in \{1,2,\ldots,q\} \\
\hat{H}_{XY} &= -\sum_{i<j} J_{ij}\,\cos(\phi_i-\phi_j) & \phi_i &\in [0,2\pi] \\
\hat{H}_{\mathsf{O}(n)} &= -\sum_{i<j} J_{ij}\,\hat{\mathbf{\Omega}}_i\cdot\hat{\mathbf{\Omega}}_j & \hat{\mathbf{\Omega}}_i &\in S^{n-1}\,.
\end{aligned}$$

The Ising Hamiltonian is left invariant by the global symmetry group $\mathbb{Z}_2$, which has two elements, $\mathbb{I}$ and η, with

$$\eta\,\sigma_i = -\sigma_i\,. \tag{7.7.1}$$

$\mathbb{I}$ is the identity, and $\eta^2=\mathbb{I}$. By simultaneously reversing *all* the spins $\sigma_i\to-\sigma_i$, the interactions remain invariant.

The degrees of freedom of the *p-state clock model* are integer variables n_i each of which ranges from 1 to p. The Hamiltonian is invariant under the discrete group $\mathbb{Z}_p$, whose p elements are generated by the single operation η, where

$$\eta\,n_i = \begin{cases} n_i+1 & if \quad n_i\in\{1,2,\ldots,p-1\} \\ 1 & if \quad n_i=p\,. \end{cases} \tag{7.7.2}$$

Think of a clock with one hand and p 'hour' markings consecutively spaced by an angle $2\pi/p$. In each site i, a hand points to one of the p hour marks; this determines n_i. The operation η simply advances *all* the hours by one tick, with hour p advancing to hour 1, just as 23:00 military time is followed one hour later by 00:00. The interaction $\cos\left(2\pi(n_i-n_j)/p\right)$ is invariant under such an operation. The p elements of the group $\mathbb{Z}_p$ are then

$$\mathbb{I}\,,\ \eta\,,\ \eta^2\,,\ \ldots\,,\ \eta^{p-1}\,. \tag{7.7.3}$$

We've already met up with the q-state Potts model, where each site supports a 'spin' σ_i which can be in any of q possible states, which we may label by integers $\{1\,,\ \ldots\,,\ q\}$. The energy of two interacting sites i and j is $-J_{ij}$ if $\sigma_i=\sigma_j$ and zero otherwise. This energy function is invariant under global operations of the *symmetric group on q* characters, S_q, which is the group of permutations of the sequence $\{1\,,\,2\,,\,3\,,\,\ldots\,,\,q\}$. The group S_q has $q!$ elements. Note the difference between a $\mathbb{Z}_q$ symmetry and an S_q symmetry. In the former case, the Hamiltonian is invariant only under the q-element cyclic permutations,

$$\eta \equiv \begin{pmatrix} 1 & 2 & \cdots & q-1 & q \\ 2 & 3 & \cdots & q & 1 \end{pmatrix} \tag{7.7.4}$$

and its powers η^l with $l=0,\ldots,q-1$.

All these models – the Ising, p-state clock, and q-state Potts models – possess a global symmetry group which is *discrete*. That is, each of the symmetry groups $\mathbb{Z}_2$, $\mathbb{Z}_p$, S_q is a discrete group, with a finite number of elements. The XY Hamiltonian $\hat{H}_{XY}$ on the other hand is invariant under a *continuous* group of transformations $\phi_i\to\phi_i+\alpha$, where ϕ_i is the angle variable on site i. More to the point, we could write the interaction term $\cos(\phi_i-\phi_j)$ as $\frac{1}{2}\left(z_i^*z_j+z_iz_j^*\right)$, where $z_i=e^{i\phi_i}$ is a phase which lives on the unit circle, and z_i^* is the complex conjugate of z_i. The model is then invariant under the global transformation $z_i\to e^{i\alpha}z_i$. The phases $e^{i\alpha}$ form a group under multiplication, called $\mathsf{U}(1)$, which is the same as $\mathsf{O}(2)$. Equivalently, we could write the interaction as $\hat{\mathbf{\Omega}}_i\cdot\hat{\mathbf{\Omega}}_j$, where $\hat{\mathbf{\Omega}}_i=(\cos\phi_i\,,\,\sin\phi_i)$, which explains the $\mathsf{O}(2)$, symmetry, since the symmetry operations are global rotations in the plane, which is to say the two-dimensional orthogonal group. This last representation generalizes nicely to unit vectors in n dimensions, where

$$\hat{\mathbf{\Omega}} = (\Omega^1\ ,\ \Omega^2\ ,\ \ldots\ ,\ \Omega^n) \tag{7.7.5}$$

with $\hat{\mathbf{\Omega}}^2 = 1$. The dot product $\hat{\mathbf{\Omega}}_i \cdot \hat{\mathbf{\Omega}}_j$ is then invariant under global rotations in this n-dimensional space, which is the group $\mathsf{O}(n)$.

[DWonedim] A domain wall in a one-dimensional Ising model.

Lower critical dimension

Depending on whether the global symmetry group of a model is discrete or continuous, there exists a *lower critical dimension* d_ℓ at or below which no phase transition may take place at finite temperature. That is, for $d \le d_\ell$, the critical temperature is $T_c = 0$. Owing to its neglect of fluctuations, mean field theory generally *overestimates* the value of T_c because it overestimates the stability of the ordered phase. Indeed, there are many examples where mean field theory predicts a finite T_c when the actual critical temperature is $T_c = 0$. This happens whenever $d \le d_\ell$.

Let's test the stability of the ordered (ferromagnetic) state of the one-dimensional Ising model at low temperatures. We consider order-destroying *domain wall* excitations which interpolate between regions of degenerate, symmetry-related ordered phase, $\uparrow\uparrow\uparrow\uparrow\uparrow$ and $\downarrow\downarrow\downarrow\downarrow\downarrow$. For a system with a discrete symmetry at low temperatures, the domain wall is abrupt, on the scale of a single lattice spacing. If the exchange energy is J, then the energy of a single domain wall is $2J$, since a link of energy $-J$ is replaced with one of energy $+J$. However, there are N possible locations for the domain wall, hence its entropy is $k_B \ln N$. For a system with M domain walls, the free energy is

$$\begin{aligned} F &= 2MJ - k_B T \ln \binom{N}{M} \\ &= N \cdot \Big\{ 2Jx + k_B T \Big[x \ln x + (1-x) \ln(1-x) \Big] \Big\}\ , \end{aligned}$$

where $x = M/N$ is the density of domain walls, and where we have used Stirling's approximation for $k!$ when k is large. Extremizing with respect to x, we find

$$\frac{x}{1-x} = e^{-2J/k_B T} \qquad \Longrightarrow \qquad x = \frac{1}{e^{2J/k_B T} + 1}\ . \tag{7.7.6}$$

The average distance between domain walls is x^{-1}, which is finite for finite T. Thus, the thermodynamic state of the system is *disordered*, with no net average magnetization.

[DWIsing] Domain walls in the two-dimensional (left) and three-dimensional (right) Ising model.

Consider next an Ising domain wall in d dimensions. Let the linear dimension of the system be $L \cdot a$, where L is a real number and a is the lattice constant. Then the energy of a single domain wall which partitions the entire system is $2J \cdot L^{d-1}$. The domain wall

entropy is difficult to compute, because the wall can fluctuate significantly, but for a single domain wall we have $S \approx k_B \ln L$. Thus, the free energy $F = 2JL^{d-1} - k_B T \ln L$ is dominated by the energy term if $d > 1$, suggesting that the system *may* be ordered. We can do a slightly better job in $d = 2$ by writing

$$Z \approx \exp\left(L^d \sum_P N_P \, e^{-2PJ/k_B T} \right) , \tag{7.7.7}$$

where the sum is over all closd loops of perimeter P, and N_P is the number of such loops. An example of such a loop circumscribing a domain is depicted in the left panel of Figure [DWIsing]. It turns out that

$$N_P \simeq \kappa^P P^{-\theta} \cdot \Big\{ 1 + \mathcal{O}(P^{-1}) \Big\} , \tag{7.7.8}$$

where $\kappa = z - 1$ with z the lattice coordination number, and θ is some exponent. We can understand the κ^P factor in the following way. At each step along the perimeter of the loop, there are $\kappa = z-1$ possible directions to go (since one doesn't backtrack). The fact that the loop must avoid overlapping itself and must return to its original position to be closed leads to the power law term $P^{-\theta}$, which is subleading since $\kappa^P P^{-\theta} = \exp(P \ln \kappa - \theta \ln P)$ and $P \gg \ln P$ for $P \gg 1$. Thus,

$$F \approx -\frac{1}{\beta} L^d \sum_P P^{-\theta} \, e^{(\ln \kappa - 2\beta J) P} , \tag{7.7.9}$$

which diverges if $\ln \kappa > 2\beta J$, if $T > 2J/k_B \ln(z-1)$. We identify this singularity with the phase transition. The high temperature phase involves a proliferation of such loops. The excluded volume effects between the loops, which we have not taken into account, then enter in an essential way so that the sum converges. Thus, we have the following picture:

$$\begin{aligned} &\ln \kappa < 2\beta J \ : \ \text{large loops suppressed ; ordered phase} \\ &\ln \kappa > 2\beta J \ : \ \text{large loops proliferate ; disordered phase} . \end{aligned}$$

On the square lattice, we obtain

$$\begin{aligned} k_B T_c^{approx} &= \frac{2J}{\ln 3} = 1.82 \, J \\ k_B T_c^{exact} &= \frac{2J}{\sinh^{-1}(1)} = 2.27 \, J . \end{aligned}$$

The agreement is better than we should reasonably expect from such a crude argument.

Nota bene : Beware of arguments which allegedly prove the existence of an ordered phase. Generally speaking, any approximation will *underestimate* the entropy, and thus will overestimate the stability of the putative ordered phase.

Continuous symmetries

When the global symmetry group is continuous, the domain walls interpolate smoothly between ordered phases. The energy generally involves a stiffness term,

$$E = \frac{1}{2} \rho_s \int d^d r \, (\boldsymbol{\nabla} \theta)^2 , \tag{7.7.10}$$

where $\theta(\mathbf{r})$ is the angle of a local rotation about a single axis and where ρ_s is the *spin stiffness*. Of course, in $\mathsf{O}(n)$ models, the rotations can be with respect to several different axes simultaneously.

[XYdomainwall] A domain wall in an XY ferromagnet.

In the ordered phase, we have $\theta(\mathbf{r}) = \theta_0$, a constant. Now imagine a domain wall in which $\theta(\mathbf{r})$ rotates by 2π across the width of the sample. We write $\theta(\mathbf{r}) = 2\pi n x / L$, where L is the linear size of the sample (here with dimensions of length) and n is an integer telling us how many complete twists the order parameter field makes. The domain wall then resembles that in Figure [XYdomainwall]. The gradient energy is

$$E = \frac{1}{2}\rho_s \, L^{d-1} \int_0^L dx \left(\frac{2\pi n}{L}\right)^2 = 2\pi^2 n^2 \rho_s \, L^{d-2} \ . \tag{7.7.11}$$

Recall that in the case of discrete symmetry, the domain wall energy scaled as $E \propto L^{d-1}$. Thus, with $S \approx k_B \ln L$ for a single wall, we see that the entropy term dominates if $d \leq 2$, in which case there is no finite temperature phase transition. Thus, the lower critical dimension d_ℓ depends on whether the global symmetry is discrete or continuous, with

$$\begin{aligned} \text{discrete global symmetry} \quad &\Longrightarrow \quad d_\ell = 1 \\ \text{continuous global symmetry} \quad &\Longrightarrow \quad d_\ell = 2 \ . \end{aligned}$$

Note that all along we have assumed local, *short-ranged* interactions. Long-ranged interactions can enhance order and thereby suppress d_ℓ.

Thus, we expect that for models with discrete symmetries, $d_\ell = 1$ and there is no finite temperature phase transition for $d \leq 1$. For models with continuous symmetries, $d_\ell = 2$, and we expect $T_c = 0$ for $d \leq 2$. In this context we should emphasize that the two-dimensional XY model *does* exhibit a phase transition at finite temperature, called the *Kosterlitz-Thouless* transition. However, this phase transition is *not* associated with the breaking of the continuous global $\mathsf{O}(2)$ symmetry and rather has to do with the unbinding of vortices and antivortices. So there is still no true long-ranged order below the critical temperature T\ns_\ssr{KT}, even though there is a phase transition!

Random systems : Imry-Ma argument

Oftentimes, particularly in condensed matter systems, intrinsic randomness exists due to quenched impurities, grain boundaries, immobile vacancies, How does this quenched randomness affect a system's attempt to order at $T = 0$? This question was taken up in a beautiful and brief paper by J. Imry and S.-K. Ma, *Phys. Rev. Lett.* **35**, 1399 (1975). Imry and Ma considered models in which there are short-ranged interactions and a random local field coupling to the local order parameter:

$$\begin{aligned} \hat{H}_{RFI} &= -J \sum_{\langle ij \rangle} \sigma_i \, \sigma_j - \sum_i H_i \, \sigma_i \\ \hat{H}_{RF\mathsf{O}(n)} &= -J \sum_{\langle ij \rangle} \hat{\mathbf{\Omega}}_i \cdot \hat{\mathbf{\Omega}}_j - \sum_i H_i^\alpha \, \Omega_i^\alpha \ , \end{aligned}$$

where

$$\langle\!\langle \, H_i^\alpha \, \rangle\!\rangle = 0 \qquad , \qquad \langle\!\langle \, H_i^\alpha \, H_j^\beta \, \rangle\!\rangle = \Gamma \, \delta^{\alpha\beta} \, \delta_{ij} \ , \tag{7.7.12}$$

where $\langle\!\langle \, \cdot \, \rangle\!\rangle$ denotes a configurational average over the disorder. Imry and Ma reasoned that a system could try to lower its free energy by forming *domains* in which the order parameter takes advantage of local fluctuations in the random field. The size of these domains is assumed to be L_d, a length scale to be determined. See the sketch in the left panel of Figure [ImryMa].

[ImryMa] Left panel : Imry-Ma domains for an $\mathsf{O}(2)$ model. The arrows point in the direction of the local order parameter field $\langle \hat{\mathbf{\Omega}}(\mathbf{r}) \rangle$. Right panel : free energy density as a function of domain size L_d. Keep in mind that the minimum possible value for L_d is the lattice spacing a.

There are two contributions to the energy of a given domain: bulk and surface terms. The bulk energy is

$$E_{bulk} = -H_{rms}\,(L_d/a)^{d/2}\,, \tag{7.7.13}$$

where a is the lattice spacing. This is because when we add together $(L_d/a)^d$ random fields, the magnitude of the result is proportional to the square root of the number of terms, to $(L_d/a)^{d/2}$. The quantity $H_{rms} = \sqrt{\Gamma}$ is the root-mean-square fluctuation in the random field at a given site. The surface energy is

$$E_{surface} \propto \begin{cases} J\,(L_d/a)^{d-1} & (\text{discrete symmetry}) \\ J\,(L_d/a)^{d-2} & (\text{continuous symmetry})\,. \end{cases} \tag{7.7.14}$$

We compute the critical dimension d_c by balancing the bulk and surface energies,

$$\begin{aligned} d-1 &= \frac{1}{2}d \qquad \Longrightarrow \qquad d_c = 2 \qquad (discrete) \\ d-2 &= \frac{1}{2}d \qquad \Longrightarrow \qquad d_c = 4 \qquad (continuous)\,. \end{aligned}$$

The total free energy is $F = (V/L_d^d)\cdot\Delta E$, where $\Delta E = E_{bulk} + E_{surf}$. Thus, the free energy per unit cell is

$$f = \frac{F}{V/a^d} \approx J\left(\frac{a}{L_d}\right)^{\frac{1}{2}d_c} - H_{rms}\left(\frac{a}{L_d}\right)^{\frac{1}{2}d}\,. \tag{7.7.15}$$

If $d < d_c$, the surface term dominates for small L_d and the bulk term dominates for large L_d There is global minimum at

$$\frac{L_d}{a} = \left(\frac{d_c}{d}\cdot\frac{J}{H_{rms}}\right)^{\frac{2}{d_c-d}}\,. \tag{7.7.16}$$

For $d > d_c$, the relative dominance of the bulk and surface terms is reversed, and there is a global maximum at this value of L_d.

Sketches of the free energy $f(L_d)$ in both cases are provided in the right panel of Figure [ImryMa]. We must keep in mind that the domain size L_d cannot become smaller than the lattice spacing a. Hence we should draw a vertical line on the graph at $L_d = a$ and discard the portion $L_d < a$ as unphysical. For $d < d_c$, we see that the state with $L_d = \infty$, the ordered state, is never the state of lowest free energy. *In dimensions* $d < d_c$, the ordered state is always unstable to domain formation in the presence of a random field.

For $d > d_c$, there are two possibilities, depending on the relative size of J and H_{rms}. We can see this by evaluating $f(L_d = a) = J - H_{rms}$ and $f(L_d = \infty) = 0$. Thus, if $J > H_{rms}$, the minimum energy state occurs for $L_d = \infty$. In this case, the system has an ordered ground state, and we expect a finite temperature transition to a disordered state at some critical temperature $T_c > 0$. If, on the other hand, $J < H_{rms}$, then the fluctuations in H overwhelm the exchange energy at $T = 0$, and the ground state is disordered down to the very smallest length scale (the lattice spacing a).

Please read the essay, “Memories of Shang-Keng Ma,” at sip.clarku.edu/skma.html.
