9.1: The Program of Renormalization

A statistical mechanical system is defined by a set of *degrees of freedom* and by a set of *coupling constants* $\{K_\alpha\}$. The degrees of freedom can be discrete, such as Ising spins σ_i, or continuous, such as a field $\phi(\mathbf{r})$. Additionally, each such system possesses a *microscopic length scale* ℓ. For discrete, lattice-based systems, this length scale is simply the lattice spacing: $\ell = a$. For continuous systems, we can define a microscopic length scale by imposing a *cutoff* Λ on the wavevectors we integrate over in all Fourier transforms. That is, we replace

$$\int \frac{d^dk}{(2\pi)^d}\, F(\mathbf{k}) \longrightarrow \int \frac{d^dk}{(2\pi)^d}\, F(\mathbf{k})\, g_\Lambda(\mathbf{k})\,, \tag{9.1.1}$$

where $F(\mathbf{k})$ is any function and $g_\Lambda(\mathbf{k})$ is the cutoff function. The simplest such case to imagine is a sharp cutoff which is isotropic in wavevector, $g_\Lambda(\mathbf{k}) = \Theta(\Lambda - |\mathbf{k}|)$. Other cutoff schemes, however, are possible, including 'soft cutoffs' where $g_\Lambda(\mathbf{k})$ is smooth. The microscopic length scale is then $\ell \sim \Lambda^{-1}$, which is the smallest distance in real space over which the system can independently fluctuate.

The idea behind renormalization is that we can successively winnow degrees of freedom from a system in some exact or approximate way, and in so doing we generate a new version of the system, at a different length scale $\ell' > \ell$, and with different couplings $\{K'_\alpha\}$. We then iterate this procedure. The result is a set of equations which tell us how the couplings behave under a change of the microscopic length scale. As we shall see, the fixed points of this procedure – where couplings do not change under a change of length scale – are critical points. Such a fixed point is defined by a set of couplings $\{K^*_\alpha\}$.

If we denote by $\mathcal{R}_b$ the renormalization procedure

$$\mathcal{R}_b\big(\ell\,,\{K_\alpha\}\big) = \big(\ell'\,,\{K'_\alpha\}\big)\,, \tag{9.1.2}$$

where $\ell' = b\,\ell$, then we have the composition law $\mathcal{R}_b\,\mathcal{R}_{b'} = \mathcal{R}_{b+b'}$. The set of transformations $\{\mathcal{R}_b\}$ is collectively referred to as the *renormalization group* (RG) because of this mathematical structure. It is somewhat of a misnomer, however, since the transformations are only defined for $b \geq 1$, which means that there is no inverse operation, and hence no true group structure[1]. Nevertheless we shall use the RG terminology because it has become universally accepted in the literature.