8.12: Appendix III- General Linear Autonomous Inhomogeneous ODEs

We can also solve general autonomous linear inhomogeneous ODEs of the form

$$\frac{d^n x}{dt^n} + a_{n-1}\,\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1\,\frac{dx}{dt} + a_0\,x = \xi(t)\ . \tag{8.12.1}$$

We can write this as

$$\mathcal{L}_t\,x(t) = \xi(t)\ , \tag{8.12.2}$$

where $\mathcal{L}_t$ is the $\boxed{\text{n^\ssr{th}}}$ order differential operator

$$\mathcal{L}_t = \frac{d^n}{dt^n} + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\,\frac{d}{dt} + a_0\ . \tag{8.12.3}$$

The general solution to the inhomogeneous equation is given by

$$x(t) = x_h(t) + \int\limits_{-\infty}^{\infty}\!\!dt'\;G(t,t')\,\xi(t')\ , \tag{8.12.4}$$

where $G(t,t')$ is the Green's function. Note that $\mathcal{L}_t\,x_h(t)=0$. Thus, in order for eqns. [Leqn] and [time] to be true, we must have \[{\cal L}\nd_t\, x(t)=\stackrel

{\overbrace +\impi dt'\>{\cal L}\nd_t\,G(t,t')\,\xi(t')=\xi(t)\ ,\] which means that

$$\mathcal{L}_t\,G(t,t') = \delta(t-t')\ , \tag{8.12.5}$$

where $\delta(t-t')$ is the Dirac δ-function.

If the differential equation $\mathcal{L}_t\,x(t)=\xi(t)$ is defined over some finite or semi-infinite t interval with prescribed boundary conditions on $x(t)$ at the endpoints, then $G(t,t')$ will depend on t and t' separately. For the case we are now considering, let the interval be the entire real line $t\in(-\infty,\infty)$. Then $G(t,t')=G(t-t')$ is a function of the single variable $t-t'$.

Note that $\mathcal{L}_t=\mathcal{L}\big(\frac{d}{dt}\big)$ may be considered a function of the differential operator $\frac{d}{dt}$. If we now Fourier transform the equation $\mathcal{L}_t\,x(t)=\xi(t)$, we obtain

$$\begin{aligned}\int\limits_{-\infty}^{\infty}\!\!dt\;e^{i\omega t}\,\xi(t) &= \int\limits_{-\infty}^{\infty}\!\!dt\;e^{i\omega t}\,\bigg\{\frac{d^n}{dt^n} + a_{n-1}\,\frac{d^{n-1}}{dt^{n-1}} + \ldots + a_1\,\frac{d}{dt} + a_0\bigg\}\,x(t)\\ &= \int\limits_{-\infty}^{\infty}\!\!dt\;e^{i\omega t}\,\bigg\{(-i\omega)^n + a_{n-1}\,(-i\omega)^{n-1} + \ldots + a_1\,(-i\omega) + a_0\bigg\}\,x(t)\ .\end{aligned}$$

Thus, if we define

$$\hat{\mathcal{L}}(\omega) = \sum_{k=0}^n a_k\,(-i\omega)^k\ , \tag{8.12.6}$$

then we have

$$\hat{\mathcal{L}}(\omega)\,\hat{x}(\omega) = \hat{\xi}\,(\omega)\ , \tag{8.12.7}$$

where $a_n\equiv 1$. According to the Fundamental Theorem of Algebra, the $\boxed{\text{n^\ssr{th}}}$ degree polynomial $\hat{\mathcal{L}}(\omega)$ may be uniquely factored over the complex ω plane into a product over n roots:

$$\hat{\mathcal{L}}(\omega) = (-i)^n\,(\omega-\omega_1)(\omega-\omega_2)\cdots(\omega-\omega_n)\ . \tag{8.12.8}$$

If the $\{a_k\}$ are all real, then $\big[\hat{\mathcal{L}}(\omega)\big]^* = \hat{\mathcal{L}}(-\omega^*)$, hence if Ω is a root then so is $-\Omega^*$. Thus, the roots appear in pairs which are symmetric about the imaginary axis. if $\Omega=a+ib$ is a root, then so is $-\Omega^*=-a+ib$.

The general solution to the homogeneous equation is

$$x_h(t) = \sum_{\sigma=1}^n A_\sigma\, e^{-i\omega_\sigma t}\ , \tag{8.12.9}$$

which involves n arbitrary complex constants A_i. The susceptibility, or Green's function in Fourier space, $\hat{G}(\omega)$ is then

$$\hat{G}(\omega) = \frac{1}{\hat{\mathcal{L}}(\omega)} = \frac{i^n}{(\omega-\omega_1)(\omega-\omega_2)\cdots(\omega-\omega_n)}\ , \tag{8.12.10}$$

Note that $\big[\hat{G}(\omega)\big]^* = \hat{G}(-\omega)$, which is equivalent to the statement that $G(t-t')$ is a real function of its argument. The general solution to the inhomogeneous equation is then

$$x(t) = x_h(t) + \int\limits_{-\infty}^{\infty}\!\!dt'\, G(t-t')\, \xi(t')\ , \tag{8.12.11}$$

where $x_h(t)$ is the solution to the homogeneous equation, with zero forcing, and where

$$\begin{aligned}
G(t-t') &= \int\limits_{-\infty}^{\infty}\!\!\frac{d\omega}{2\pi}\, e^{-i\omega(t-t')}\, \hat{G}(\omega) \\
&= i^n \int\limits_{-\infty}^{\infty}\!\!\frac{d\omega}{2\pi}\, \frac{e^{-i\omega(t-t')}}{(\omega-\omega_1)(\omega-\omega_2)\cdots(\omega-\omega_n)} \\
&= \sum_{\sigma=1}^n \frac{e^{-i\omega_\sigma(t-t')}}{i\, \mathcal{L}'(\omega_\sigma)}\, \Theta(t-t')\ ,
\end{aligned}$$

where we assume that $Im\,\omega_\sigma < 0$ for all σ. This guarantees *causality* – the *response* $x(t)$ to the *influence* $\xi(t')$ is nonzero only for $t > t'$.

As an example, consider the familiar case

$$\begin{aligned}
\hat{\mathcal{L}}(\omega) &= -\omega^2 - i\gamma\omega + \omega_0^2 \\
&= -(\omega-\omega_+)\,(\omega-\omega_-)\ ,
\end{aligned}$$

with $\omega_\pm = -\frac{i}{2}\gamma \pm \beta$, and $\beta = \sqrt{\omega_0^2 - \frac{1}{4}\gamma^2}$. This yields

$$\mathcal{L}'(\omega_\pm) = \mp(\omega_+ - \omega_-) = \mp 2\beta\ . \tag{8.12.12}$$

Then according to equation [gfun],

$$\begin{aligned}
G(s) &= \bigg\{ \frac{e^{-i\omega_+ s}}{i\mathcal{L}'(\omega_+)} + \frac{e^{-i\omega_- s}}{i\mathcal{L}'(\omega_-)} \bigg\}\, \Theta(s) \\
&= \bigg\{ \frac{e^{-\gamma s/2}\, e^{-i\beta s}}{-2i\beta} + \frac{e^{-\gamma s/2}\, e^{i\beta s}}{2i\beta} \bigg\}\, \Theta(s) \\
&= \beta^{-1}\, e^{-\gamma s/2}\, \sin(\beta s)\, \Theta(s)\ .
\end{aligned}$$

Now let us evaluate the two-point correlation function $\big\langle x(t)\, x(t')\big\rangle$, assuming the noise is correlated according to $\big\langle \xi(s)\, \xi(s')\big\rangle = \phi(s-s')$. We assume $t, t' \to \infty$ so the transient contribution x_h is negligible. We then have

$$\begin{aligned}
\big\langle x(t)\, x(t')\big\rangle &= \int\limits_{-\infty}^{\infty}\!\!ds \int\limits_{-\infty}^{\infty}\!\!ds'\, G(t-s)\, G(t'-s')\, \big\langle \xi(s)\, \xi(s')\big\rangle \\
&= \int\limits_{-\infty}^{\infty}\!\!\frac{d\omega}{2\pi}\, \hat{\phi}(\omega)\, \big|\hat{G}(\omega)\big|^2\, e^{i\omega(t-t')}\ .
\end{aligned}$$

Higher order ODEs

Note that any n^\ssr{th} order ODE, of the general form

$$\frac{d^n x}{dt^n} = F\bigg(x\,,\, \frac{dx}{dt}\,,\,\ldots\,,\,\frac{d^{n-1} x}{dt^{n-1}}\bigg)\ , \tag{8.12.13}$$

may be represented by the first order system $\dot{\varphi} = \mathbf{V}(\varphi)$. To see this, define $\varphi_k = d^{k-1}x/dt^{k-1}$, with $k=1,\ldots,n$. Thus, for $k<n$ we have $\dot{\varphi}_k = \varphi_{k+1}$, and $\dot{\varphi}_n = F$. In other words, \[\stackrel

{\overbrace{ {d\over dt}

$$\begin{pmatrix} \varphi_1 \\ \vdots \\ \varphi_{n-1} \\ \varphi_n \end{pmatrix} \tag{8.12.14}$$

}}= \stackrel{\BV(\Bvphi)}{\overbrace{

$$\begin{pmatrix} \varphi_2 \\ \vdots \\ \varphi_n \\ F\big(\varphi_1,\ldots,\varphi_p\big) \end{pmatrix} \tag{8.12.15}$$

}}\ .\]

An inhomogeneous linear n^\ssr{th} order ODE,

$$\frac{d^n x}{dt^n} + a_{n-1}\,\frac{d^{n-1}x}{dt^{n-1}} + \ldots + a_1\,\frac{dx}{dt} + a_0\,x = \xi(t) \tag{8.12.16}$$

may be written in matrix form, as

$$\frac{d}{dt}\begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}}^{Q} \begin{pmatrix} \varphi_1 \\ \varphi_2 \\ \vdots \\ \varphi_n \end{pmatrix} + \overbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ \xi(t) \end{pmatrix}}^{\xi}\ . \tag{8.12.17}$$

Thus,

$$\dot{\varphi} = Q\,\varphi + \xi\ , \tag{8.12.18}$$

and if the coefficients c_k are time-independent, the ODE is *autonomous*.

For the homogeneous case where $\xi(t)=0$, the solution is obtained by exponentiating the constant matrix Qt:

$$\varphi(t) = \exp(Qt)\,\varphi(0)\ ; \tag{8.12.19}$$

the exponential of a matrix may be given meaning by its Taylor series expansion. If the ODE is not autonomous, then $Q=Q(t)$ is time-dependent, and the solution is given by the path-ordered exponential,

$$\varphi(t) = \mathsf{P}\,\exp\Bigg\{\int\limits_0^t\!dt'\,Q(t')\Bigg\}\,\varphi(0)\ , \tag{8.12.20}$$

where P is the *path ordering operator* which places earlier times to the right. As defined, the equation $\dot{\varphi} = \mathbf{V}(\varphi)$ is autonomous, since the t-advance mapping g_t depends only on t and on no other time variable. However, by extending the phase space $\mathbb{M} \ni \varphi$ from $\mathbb{M} \to \mathbb{M} \times \mathbb{R}$, which is of dimension $n+1$, one can describe arbitrary time-dependent ODEs.

In general, path ordered exponentials are difficult to compute analytically. We will henceforth consider the autonomous case where Q is a constant matrix in time. We will assume the matrix Q is real, but other than that it has no helpful symmetries. We can however decompose it into left and right eigenvectors:

$$Q_{ij} = \sum_{\sigma=1}^{n} \nu_\sigma \, R_{\sigma,i} \, L_{\sigma,j} \, . \tag{8.12.21}$$

Or, in bra-ket notation, $Q = \sum_\sigma \nu_\sigma \, |R_\sigma\rangle\langle L_\sigma|$. The normalization condition we use is

$$\big\langle \, L_\sigma \, \big| \, R_{\sigma'} \, \big\rangle = \delta_{\sigma\sigma'} \, , \tag{8.12.22}$$

where $\big\{ \nu_\sigma \big\}$ are the eigenvalues of Q. The eigenvalues may be real or imaginary. Since the characteristic polynomial $P(\nu) = \det\left(\nu \, \mathbb{I} - Q\right)$ has real coefficients, we know that the eigenvalues of Q are either real or come in complex conjugate pairs.

Consider, for example, the $n=2$ system we studied earlier. Then

$$Q = \begin{pmatrix} 0 & 1 \\ -\omega_0^2 & -\gamma \end{pmatrix} \, . \tag{8.12.23}$$

The eigenvalues are as before: $\nu_\pm = -\frac{1}{2}\gamma \pm \sqrt{\frac{1}{4}\gamma^2 - \omega_0^2}$. The left and right eigenvectors are

$$L_\pm = \frac{\pm 1}{\nu_+ - \nu_-} \, \begin{pmatrix} -\nu_\mp & 1 \end{pmatrix} \qquad , \qquad R_\pm = \begin{pmatrix} 1 \\ \nu_\pm \end{pmatrix} \, . \tag{8.12.24}$$

The utility of working in a left-right eigenbasis is apparent once we reflect upon the result

$$f(Q) = \sum_{\sigma=1}^{n} f(\nu_\sigma) \, \big| \, R_\sigma \, \big\rangle \big\langle \, L_\sigma \, \big| \tag{8.12.25}$$

for any function f. Thus, the solution to the general autonomous homogeneous case is

$$\begin{aligned} \big| \, \varphi(t) \, \big\rangle &= \sum_{\sigma=1}^{n} e^{\nu_\sigma t} \, \big| \, R_\sigma \, \big\rangle \big\langle \, L_\sigma \, \big| \, \varphi(0) \, \big\rangle \\ \varphi_i(t) &= \sum_{\sigma=1}^{n} e^{\nu_\sigma t} \, R_{\sigma,i} \, \sum_{j=1}^{n} L_{\sigma,j} \, \varphi_j(0) \, . \end{aligned}$$

If $\text{Re}\,(\nu_\sigma) \le 0$ for all σ, then the initial conditions $\varphi(0)$ are forgotten on time scales $\tau_\sigma = \nu_\sigma^{-1}$. Physicality demands that this is the case.

Now let's consider the inhomogeneous case where $\xi(t) \neq 0$. We begin by recasting Equation [phiQeqn] in the form

$$\frac{d}{dt}\big(e^{-Qt} \, \varphi\big) = e^{-Qt} \, \xi(t) \, . \tag{8.12.26}$$

We can integrate this directly:

$$\varphi(t) = e^{Qt} \, \varphi(0) + \int\limits_0^t \! ds \, e^{Q(t-s)} \, \xi(s) \, . \tag{8.12.27}$$

In component notation,

$$\varphi_i(t) = \sum_{\sigma=1}^{n} e^{\nu_\sigma t} \, R_{\sigma,i} \, \big\langle \, L_\sigma \, \big| \, \varphi(0) \, \big\rangle + \sum_{\sigma=1}^{n} R_{\sigma,i} \int\limits_0^t \! ds \, e^{\nu_\sigma (t-s)} \, \big\langle \, L_\sigma \, \big| \, \xi(s) \, \big\rangle . \tag{8.12.28}$$

Note that the first term on the RHS is the solution to the homogeneous equation, as must be the case when $\xi(s) = 0$.

The solution in Equation [CNsoln] holds for general Q and $\xi(s)$. For the particular form of Q and $\xi(s)$ in Equation [Qxieqn], we can proceed further. For starters, $\langle L_\sigma | \xi(s) \rangle = L_{\sigma,n} \, \xi(s)$. We can further exploit a special feature of the Q matrix to analytically determine all its left and right eigenvectors. Applying Q to the right eigenvector $|R_\sigma\rangle$, we obtain

$$R_{\sigma,j} = \nu_\sigma\, R_{\sigma,j-1} \qquad (j>1)\ . \tag{8.12.29}$$

We are free to choose $R_{\sigma,1} = 1$ for all σ and defer the issue of normalization to the derivation of the left eigenvectors. Thus, we obtain the pleasingly simple result,

$$R_{\sigma,k} = \nu_\sigma^{k-1}\ . \tag{8.12.30}$$

Applying Q to the left eigenvector $\langle L_\sigma|$, we obtain

$$\begin{aligned} -a_0\, L_{\sigma,n} &= \nu_\sigma\, L_{\sigma,1} \\ L_{\sigma,j-1} - a_{j-1}\, L_{\sigma,n} &= \nu_\sigma\, L_{\sigma,j} \qquad (j>1)\ . \end{aligned}$$

From these equations we may derive

$$L_{\sigma,k} = -\frac{L_{\sigma,n}}{\nu_\sigma} \sum_{j=0}^{k-1} a_j\, \nu_\sigma^{j-k-1} = \frac{L_{\sigma,n}}{\nu_\sigma} \sum_{j=k}^{n} a_j\, \nu_\sigma^{j-k-1}\ . \tag{8.12.31}$$

The equality in the above equation is derived using the result $P(\nu_\sigma) = \sum_{j=0}^n a_j\, \nu_\sigma^j = 0$. Recall also that $a_n \equiv 1$. We now impose the normalization condition,

$$\sum_{k=1}^n L_{\sigma,k}\, R_{\sigma,k} = 1\ . \tag{8.12.32}$$

This condition determines our last remaining unknown quantity (for a given σ), $L_{\sigma,p}$:

$$\big\langle\, L_\sigma \,\big|\, R_\sigma\,\big\rangle = L_{\sigma,n} \sum_{k=1}^n k\, a_k\, \nu_\sigma^{k-1} = P'(\nu_\sigma)\, L_{\sigma,n}\ , \tag{8.12.33}$$

where $P'(\nu)$ is the first derivative of the characteristic polynomial. Thus, we obtain another neat result,

$$L_{\sigma,n} = \frac{1}{P'(\nu_\sigma)}\ . \tag{8.12.34}$$

Now let us evaluate the general two-point correlation function,

$$C_{jj'}(t,t') \equiv \big\langle \varphi_j(t)\, \varphi_{j'}(t')\big\rangle - \big\langle\varphi_j(t)\big\rangle\, \big\langle\varphi_{j'}(t')\big\rangle\ . \tag{8.12.35}$$

We write

$$\big\langle \xi(s)\, \xi(s')\big\rangle = \phi(s-s') = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \hat\phi(\omega)\, e^{-i\omega(s-s')}\ . \tag{8.12.36}$$

When $\hat\phi(\omega)$ is constant, we have $\big\langle \xi(s)\, \xi(s')\big\rangle = \hat\phi(t)\, \delta(s-s')$. This is the case of so-called *white noise*, when all frequencies contribute equally. The more general case when $\hat\phi(\omega)$ is frequency-dependent is known as *colored noise*. Appealing to Equation [CNsoln], we have

$$\begin{aligned} C_{jj'}(t,t') &= \sum_{\sigma,\sigma'} \frac{\nu_\sigma^{j-1}}{P'(\nu_\sigma)}\, \frac{\nu_{\sigma'}^{j'-1}}{P'(\nu_{\sigma'})} \int_0^t\! ds\ e^{\nu_\sigma(t-s)} \int_0^{t'}\! ds'\ e^{\nu_{\sigma'}(t'-s')}\, \phi(s-s') \\ &= \sum_{\sigma,\sigma'} \frac{\nu_\sigma^{j-1}}{P'(\nu_\sigma)}\, \frac{\nu_{\sigma'}^{j'-1}}{P'(\nu_{\sigma'})} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\hat\phi(\omega)\, (e^{-i\omega t} - e^{\nu_\sigma t})(e^{i\omega t'} - e^{\nu_{\sigma'} t'})}{(\omega - i\nu_\sigma)(\omega + i\nu_{\sigma'})}\ . \end{aligned}$$

In the limit $t,t' \to \infty$, assuming $\mathsf{Re}\,(\nu_\sigma) < 0$ for all σ (no diffusion), the exponentials $e^{\nu_\sigma t}$ and $e^{\nu_{\sigma'} t'}$ may be neglected, and we then have

$$C_{jj'}(t,t') = \sum_{\sigma,\sigma'} \frac{\nu_\sigma^{j-1}}{P'(\nu_\sigma)}\, \frac{\nu_{\sigma'}^{j'-1}}{P'(\nu_{\sigma'})} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \frac{\hat\phi(\omega)\, e^{-i\omega(t-t')}}{(\omega - i\nu_\sigma)(\omega + i\nu_{\sigma'})}\ . \tag{8.12.37}$$