7.8: Ginzburg-Landau Theory

Ginzburg-Landau free energy

Including gradient terms in the free energy, we write

$$F\big[m(\mathbf{x})\,,\,h(\mathbf{x})\big] = \int\!\! d^d\!x\,\bigg\{ f_0 + \frac{1}{2}a\,m^2 + \frac{1}{4}b\,m^4 + \frac{1}{6}c\,m^6\,-h\,m + \frac{1}{2}\kappa\,(\boldsymbol{\nabla} m)^2 + \ldots\bigg\}\,. \tag{7.8.1}$$

In principle, any term which does not violate the appropriate global symmetry will turn up in such an expansion of the free energy, with some coefficient. Examples include hm^3 (both m and h are odd under time reversal), $m^2(\boldsymbol{\nabla} m)^2$, We now ask: what function $m(\mathbf{x})$ extremizes the free energy functional $F\big[m(\mathbf{x})\,,\,h(\mathbf{x})\big]$? The answer is that $m(\mathbf{x})$ must satisfy the corresponding Euler-Lagrange equation, which for the above functional is

$$a\,m + b\,m^3 + c\,m^5 - h - \kappa\,\nabla^2 m = 0\,. \tag{7.8.2}$$

If $a>0$ and h is small (we assume $b>0$ and $c>0$), we may neglect the m^3 and m^5 terms and write

$$\big(a-\kappa\,\nabla^2\big)\,m = h\,, \tag{7.8.3}$$

whose solution is obtained by Fourier transform as

$$\hat{m}(\mathbf{q}) = \frac{\hat{h}(\mathbf{q})}{a+\kappa\mathbf{q}^2}\,, \tag{7.8.4}$$

which, with $h(\mathbf{x})$ appropriately defined, recapitulates the result in Equation [mhqeqn]. Thus, we conclude that

$$\hat{\chi}(\mathbf{q}) = \frac{1}{a+\kappa\mathbf{q}^2}\,, \tag{7.8.5}$$

which should be compared with Equation [xhiheqn]. For continuous functions, we have

$$\begin{aligned}\hat{m}(\mathbf{q}) &= \int\!\! d^d\!x\; m(\mathbf{x})\,e^{-i\mathbf{q}\cdot\mathbf{x}}\\ m(\mathbf{x}) &= \int\!\!\frac{d^d\!q}{(2\pi)^d}\;\hat{m}(\mathbf{q})\,e^{i\mathbf{q}\cdot\mathbf{x}}\,.\end{aligned}$$

We can then derive the result

$$m(\mathbf{x}) = \int\!\! d^d\!x'\;\chi(\mathbf{x}-\mathbf{x}')\,h(\mathbf{x}')\,, \tag{7.8.6}$$

where

$$\chi(\mathbf{x}-\mathbf{x}') = \frac{1}{\kappa}\int\!\!\frac{d^d\!q}{(2\pi)^d}\;\frac{e^{i\mathbf{q}\cdot(\mathbf{x}-\mathbf{x}')}}{\mathbf{q}^2+\xi^{-2}}\,, \tag{7.8.7}$$

where the correlation length is $\xi=\sqrt{\kappa/a}\propto(T-T_c)^{-1/2}$, as before.

If $a<0$ then there is a spontaneous magnetization and we write $m(\mathbf{x})=m_0+\delta m(\mathbf{x})$. Assuming h is weak, we then have two equations

$$\begin{aligned}a + b\,m_0^2 + c\,m_0^4 &= 0\\ (a+3b\,m_0^2+5c\,m_0^4-\kappa\,\nabla^2)\,\delta m &= h\,.\end{aligned}$$

If $-a>0$ is small, we have $m_0^2=-a/3b$ and

$$\delta\hat{m}(\mathbf{q}) = \frac{\hat{h}(\mathbf{q})}{-2a+\kappa\mathbf{q}^2}\,, \tag{7.8.8}$$

Domain wall profile

A particularly interesting application of Ginzburg-Landau theory is its application toward modeling the spatial profile of defects such as vortices and domain walls. Consider, for example, the case of Ising ($\mathbb{Z}_2$) symmetry with $h=0$. We expand the free energy density to order m^4:

$$F\big[m(\mathbf{x})\big]=\int\!d^dx\,\left\{f_0+\frac{1}{2}am^2+\frac{1}{4}bm^4+\frac{1}{2}\kappa\,(\boldsymbol{\nabla}m)^2\right\}. \tag{7.8.9}$$

We assume $a<0$, corresponding to $T<T_c$. Consider now a domain wall, where $m(x\to-\infty)=-m_0$ and $m(x\to+\infty)=+m_0$, where m_0 is the equilibrium magnetization, which we obtain from the Euler-Lagrange equation,

$$a\,m+b\,m^3-\kappa\,\nabla^2 m=0\ , \tag{7.8.10}$$

assuming a uniform solution where $\boldsymbol{\nabla}m=0$. This gives $m_0=\sqrt{|a|/b}$. It is useful to scale $m(\mathbf{x})$ by m_0, writing $m(\mathbf{x})=m_0\,\phi(\mathbf{x})$. The scaled order parameter function $\phi(\mathbf{x})$ interpolates between $\phi(-\infty)=-1$ and $\phi(+\infty)=1$.

It also proves useful to rescale position, writing $\mathbf{x}=\big(2\kappa/|a|\big)^{1/2}\zeta$. Then we obtain

$$\frac{1}{2}\boldsymbol{\nabla}^2\phi=-\phi+\phi^3\ . \tag{7.8.11}$$

We assume $\phi(\zeta)=\phi(\zeta)$ is only a function of one coordinate, $\zeta\equiv\zeta^1$. Then the Euler-Lagrange equation becomes

$$\frac{d^2\phi}{d\zeta^2}=-2\phi+2\phi^3\equiv-\frac{\partial U}{\partial\phi}\ , \tag{7.8.12}$$

where

$$U(\phi)=-\frac{1}{2}\big(\phi^2-1\big)^2\ . \tag{7.8.13}$$

The 'potential' $U(\phi)$ is an inverted double well, with maxima at $\phi=\pm1$. The equation $\ddot\phi=-U'(\phi)$, where dot denotes differentiation with respect to ζ, is simply Newton's second law with time replaced by space. In order to have a stationary solution at $\zeta\to\pm\infty$ where $\phi=\pm1$, the total energy must be $E=U(\phi=\pm1)=0$, where $E=\frac{1}{2}\dot\phi^2+U(\phi)$. This leads to the first order differential equation

$$\frac{d\phi}{d\zeta}=1-\phi^2\ , \tag{7.8.14}$$

with solution

$$\phi(\zeta)=\tanh(\zeta)\ . \tag{7.8.15}$$

Restoring the dimensionful constants,

$$m(x)=m_0\,\tanh\!\bigg(\frac{x}{\sqrt{2}\,\xi}\bigg)\ , \tag{7.8.16}$$

where the coherence length $\xi\equiv\big(\kappa/|a|\big)^{1/2}$ diverges at the Ising transition $a=0$.

Derivation of Ginzburg-Landau free energy

We can make some progress in systematically deriving the Ginzburg-Landau free energy. Consider the Ising model,

$$\frac{\hat H}{k_BT}=-\frac{1}{2}\sum_{i,j}K_{ij}\,\sigma_i\,\sigma_j-\sum_i h_i\,\sigma_i+\frac{1}{2}\sum_i K_{ii}\ , \tag{7.8.17}$$

where now $K_{ij}=J_{ij}/k_BT$ and $h_i=H_i/k_BT$ are the interaction energies and local magnetic fields in units of k_BT. The last term on the RHS above cancels out any contribution from diagonal elements of K_{ij}. Our derivation makes use of a generalization of the Gaussian integral,

$$\int\limits_{-\infty}^{\infty}\!dx\, e^{-\frac{1}{2}ax^2-bx}=\left(\frac{2\pi}{a}\right)^{1/2} e^{b^2/2a}\ . \tag{7.8.18}$$

The generalization is \[\int\limits_{-\infty}^\infty\!\!\!dx\ns_1\cdots\!\!\!\int\limits_{-\infty}^\infty\!\!\!dx\ns_N\> e^{-{1\over 2} A\ns_{ij} x\ns_ix\ns_j - b\ns_i x\ns_i}={(2\pi)^{N/2}\over \sqrt

\ e^\Tra \bigg[e^\!\!\int\limits_{-\infty}^\infty\!\!\!d\phi\ns_1 \cdots\!\!\!\int\limits_{-\infty}^\infty\!\!\!d\phi\ns_N\> e^{-{1\over 2} K^{-1}_{ij}\phi\ns_i\phi\ns_j}\,\Tra e^{(\phi\ns_i+h\ns_i)\sigma\ns_i}\\ &={det}^{-1/2}(2\pi K)\> e^{-{1\over 2} K\ns_{ii}}\!\!\int\limits_{-\infty}^\infty\!\!\!d\phi\ns_1 \cdots\!\!\!\int\limits_{-\infty}^\infty\!\!\!d\phi\ns_N\> e^{-{1\over 2} K^{-1}_{ij}\phi\ns_i\phi\ns_j}\,e^{\sum_i \ln\left[2\cosh(\phi\ns_i+h\ns_i)\right]}\\ &\equiv \int\limits_{-\infty}^\infty\!\!\!d\phi\ns_1\cdots\!\!\!\int\limits_{-\infty}^\infty\!\!\!d\phi\ns_N\> e^{-\RPhi(\phi\ns_1,\ldots,\phi\ns_N)}\ , \end{split}\] where

$$\Phi=\frac{1}{2}\sum_{i,j} K^{-1}_{ij}\,\phi_i\,\phi_j-\sum_i \ln\cosh(\phi_i+h_i)+\frac{1}{2}\ln\mathsf{det}\,(2\pi K)+\frac{1}{2}\,\mathsf{Tr}\,K-N\ln 2\ . \tag{7.8.19}$$

We assume the model is defined on a Bravais lattice, in which case we can write $\phi_i=\phi_{\mathbf{R}_i}$. We can then define the Fourier transforms,

$$\begin{aligned}\phi_{\mathbf{R}}&=\frac{1}{\sqrt{N}}\sum_{\mathbf{q}}\hat{\phi}_{\mathbf{q}}\,e^{i\mathbf{q}\cdot\mathbf{R}}\\ \hat{\phi}_{\mathbf{q}}&=\frac{1}{\sqrt{N}}\sum_{\mathbf{R}}\phi_{\mathbf{R}}\,e^{-i\mathbf{q}\cdot\mathbf{R}}\end{aligned}$$

and

$$\hat{K}(\mathbf{q})=\sum_{\mathbf{R}} K(\mathbf{R})\,e^{-i\mathbf{q}\cdot\mathbf{R}}\ . \tag{7.8.20}$$

A few remarks about the lattice structure and periodic boundary conditions are in order. For a Bravais lattice, we can write each direct lattice vector $\mathbf{R}$ as a sum over d basis vectors with integer coefficients, *viz.*

$$\mathbf{R}=\sum_{\mu=1}^{d} n_\mu\,\mathbf{a}_\mu\ , \tag{7.8.21}$$

where d is the dimension of space. The reciprocal lattice vectors $\mathbf{b}_\mu$ satisfy

$$\mathbf{a}_\mu\cdot\mathbf{b}_\nu=2\pi\,\delta_{\mu\nu}\ , \tag{7.8.22}$$

and any wavevector $\mathbf{q}$ may be expressed as

$$\mathbf{q}=\frac{1}{2\pi}\sum_{\mu=1}^{d}\theta_\mu\,\mathbf{b}_\mu\ . \tag{7.8.23}$$

We can impose periodic boundary conditions on a system of size $M_1\times M_2\times\cdots\times M_d$ by requiring

$$\phi_{\mathbf{R}+\sum_{\mu=1}^{d} l_\mu M_\mu \mathbf{a}_\mu}=\phi_{\mathbf{R}}\ . \tag{7.8.24}$$

This leads to the *quantization* of the wavevectors, which must then satisfy

$$e^{iM_\mu\,\mathbf{q}\cdot\mathbf{a}_\mu}=e^{iM_\mu\theta_\mu}=1\ , \tag{7.8.25}$$

and therefore $\theta_\mu=2\pi m_\mu/M_\mu$, where m_μ is an integer. There are then $M_1M_2\cdots M_d=N$ independent values of $\mathbf{q}$, which can be taken to be those corresponding to $m_\mu\in\{1,\ldots,M_\mu\}$.

Let's now expand the function $\Phi\big(\backslash\text{Vphi}\big)$ in powers of the ϕ_i, and to first order in the external fields h_i. We obtain

$$\begin{aligned}\Phi &= \frac{1}{2}\sum_{\mathbf{q}}\Big(\hat{K}^{-1}(\mathbf{q})-1\Big)\,|\hat{\phi}_{\mathbf{q}}|^2+\frac{1}{12}\sum_{\mathbf{R}}\phi^4_{\mathbf{R}}-\sum_{\mathbf{R}}h_{\mathbf{R}}\,\phi_{\mathbf{R}}+\mathcal{O}(\phi^6,h^2)\\ &\quad+\frac{1}{2}\,\mathsf{Tr}\;K+\frac{1}{2}\,\mathsf{Tr}\;\ln(2\pi K)-N\ln 2\end{aligned}$$

On a d-dimensional lattice, for a model with nearest neighbor interactions K_1 only, we have $\hat{K}(\mathbf{q})=K_1\sum_{\delta}e^{i\mathbf{q}\cdot\delta}$, where δ is a nearest neighbor separation vector. These are the eigenvalues of the matrix K_{ij}. We note that K_{ij} is then not positive definite, since there are negative eigenvalues[19]. To fix this, we can add a term K_0 everywhere along the diagonal. We then have

$$\hat{K}(\mathbf{q})=K_0+K_1\sum_{\delta}\cos(\mathbf{q}\cdot\delta)\ . \tag{7.8.26}$$

Here we have used the inversion symmetry of the Bravais lattice to eliminate the imaginary term. The eigenvalues are all positive so long as $K_0>zK_1$, where z is the lattice coordination number. We can therefore write $\hat{K}(\mathbf{q})=\hat{K}(0)-\alpha\,\mathbf{q}^2$ for small $\mathbf{q}$, with $\alpha>0$. Thus, we can write

$$\hat{K}^{-1}(\mathbf{q})-1=a+\kappa\,\mathbf{q}^2+\ldots\ . \tag{7.8.27}$$

To lowest order in $\mathbf{q}$ the RHS is isotropic if the lattice has cubic symmetry, but anisotropy will enter in higher order terms. We'll assume isotropy at this level. This is not necessary but it makes the discussion somewhat less involved. We can now write down our Ginzburg-Landau free energy density:

$$\mathcal{F}=a\,\phi^2+\frac{1}{2}\kappa\,|\boldsymbol{\nabla}\phi|^2+\frac{1}{12}\,\phi^4-h\,\phi\ , \tag{7.8.28}$$

valid to lowest nontrivial order in derivatives, and to sixth order in ϕ.

One might wonder what we have gained over the inhomogeneous variational density matrix treatment, where we found

$$\begin{aligned}F &= -\frac{1}{2}\sum_{\mathbf{q}}\hat{J}\,(\mathbf{q})\,|\hat{m}(\mathbf{q})|^2-\sum_{\mathbf{q}}\hat{H}(-\mathbf{q})\,\hat{m}(\mathbf{q})\\ &\quad+k_BT\sum_i\left\{\left(\frac{1+m_i}{2}\right)\ln\!\left(\frac{1+m_i}{2}\right)+\left(\frac{1-m_i}{2}\right)\ln\!\left(\frac{1-m_i}{2}\right)\right\}.\end{aligned}$$

Surely we could expand $\hat{J}\,(\mathbf{q})=\hat{J}\,(0)-\frac{1}{2}a\mathbf{q}^2+\ldots$ and obtain a similar expression for $\mathcal{F}$. However, such a derivation using the variational density matrix is only approximate. The method outlined in this section is exact.

Let's return to our complete expression for Φ:

$$\Phi\big(\textcolor{red}{\backslash\text{Vphi}}\big)=\Phi_0\big(\textcolor{red}{\backslash\text{Vphi}}\big)+\sum_{\mathbf{R}}v(\phi_{\mathbf{R}})\ , \tag{7.8.29}$$

where

$$\Phi_0\big(\textcolor{red}{\backslash\text{Vphi}}\big)=\frac{1}{2}\sum_{\mathbf{q}}G^{-1}(\mathbf{q})\,|\hat{\phi}(\mathbf{q})|^2+\frac{1}{2}\,\mathsf{Tr}\,\left(\frac{1}{1+G^{-1}}\right)+\frac{1}{2}\,\mathsf{Tr}\,\ln\left(\frac{2\pi}{1+G^{-1}}\right)-N\ln 2\ . \tag{7.8.30}$$

Here we have defined

$$\begin{aligned}v(\phi) &= \frac{1}{2}\phi^2-\ln\cosh\phi\\ &= \frac{1}{12}\,\phi^4-\frac{1}{45}\,\phi^6+\frac{17}{2520}\,\phi^8+\ldots\end{aligned}$$

and

$$G(\mathbf{q})=\frac{\hat{K}(\mathbf{q})}{1-\hat{K}(\mathbf{q})}\ . \tag{7.8.31}$$

We now want to compute

$$Z=\int\!D\Vphi\; e^{-\Phi_0(\Vphi)}\;e^{-\sum_{\mathbf{R}} v(\phi_{\mathbf{R}})} \tag{7.8.32}$$

where

$$D\Vphi\equiv d\phi_1\,d\phi_2\cdots d\phi_N\ . \tag{7.8.33}$$

We expand the second exponential factor in a Taylor series, allowing us to write

$$Z=Z_0\,\bigg(1-\sum_{\mathbf{R}}\big\langle v(\phi_{\mathbf{R}})\big\rangle+\frac{1}{2}\sum_{\mathbf{R}}\sum_{\mathbf{R}'}\big\langle v(\phi_{\mathbf{R}})\,v(\phi_{\mathbf{R}'})\big\rangle+\ldots\bigg)\ , \tag{7.8.34}$$

where

$$\begin{aligned} Z_0&=\int\!D\Vphi\; e^{-\Phi_0(\Vphi)} \\ \ln Z_0 &=\frac{1}{2}\,\textsf{Tr}\,\bigg[\ln(1+G)-\frac{G}{1+G}\bigg]+N\ln 2 \end{aligned}$$

and

$$\big\langle F\big(\Vphi\big)\big\rangle=\frac{\int\!D\Vphi\;F\,e^{-\Phi_0}}{\int\!D\Vphi\;e^{-\Phi_0}}\ . \tag{7.8.35}$$

To evaluate the various terms in the expansion of Equation [ZZZ], we invoke Wick’s theorem, which says

$$\begin{aligned} \big\langle x_{i_1}\,x_{i_2}\cdots x_{i_{2L}}\big\rangle &=\int\limits_{-\infty}^{\infty}\!\!dx_1\cdots\!\!\int\limits_{-\infty}^{\infty}\!\!dx_N\;e^{-\frac{1}{2}\,\mathcal{G}^{-1}_{ij}x_i x_j}\,x_{i_1}\,x_{i_2}\cdots x_{i_{2L}}\bigg/\int\limits_{-\infty}^{\infty}\!\!dx_1\cdots\!\!\int\limits_{-\infty}^{\infty}\!\!dx_N\;e^{-\frac{1}{2}\,\mathcal{G}^{-1}_{ij}x_i x_j} \\ &=\sum_{\substack{\textit{all distinct}\\ \textit{pairings}}}\mathcal{G}_{j_1 j_2}\,\mathcal{G}_{j_3 j_4}\cdots\mathcal{G}_{j_{2L-1}j_{2L}}\ , \end{aligned}$$

where the sets $\{j_1,\ldots,j_{2L}\}$ are all permutations of the set $\{i_1,\ldots,i_{2L}\}$. In particular, we have

$$\big\langle x_i^4\big\rangle=3\big(\mathcal{G}_{ii}\big)^2\ . \tag{7.8.36}$$

In our case, we have

$$\big\langle\phi_{\mathbf{R}}^4\big\rangle=3\,\bigg(\frac{1}{N}\sum_{\mathbf{q}}G(\mathbf{q})\bigg)^2\ . \tag{7.8.37}$$

Thus, if we write $v(\phi)\approx\frac{1}{12}\,\phi^4$ and retain only the quartic term in $v(\phi)$, we obtain

$$\begin{aligned} \frac{F}{k_BT}=-\ln Z_0 &=\frac{1}{2}\,\textsf{Tr}\,\bigg[\frac{G}{1+G}-\ln(1+G)\bigg]+\frac{1}{4N}\big(\textsf{Tr}\;G\big)^2-N\ln 2 \\ &=-N\ln 2+\frac{1}{4N}\big(\textsf{Tr}\;G\big)^2-\frac{1}{4}\,\textsf{Tr}\;\big(G^2\big)+\mathcal{O}\big(G^3\big)\ . \end{aligned}$$

Note that if we set K_{ij} to be diagonal, then $\hat{K}(\mathbf{q})$ and hence $G(\mathbf{q})$ are constant functions of $\mathbf{q}$. The $\mathcal{O}\big(G^2\big)$ term then vanishes, which is required since the free energy cannot depend on the diagonal elements of K_{ij}.

Ginzburg criterion

Let us define $A(T,H,V,N)$ to be the usual (thermodynamic) Helmholtz free energy. Then

$$e^{-\beta A}=\int\!Dm\;e^{-\beta F[m(\mathbf{x})]}\ , \tag{7.8.38}$$

where the functional $F[m(\mathbf{x})]$ is of the Ginzburg-Landau form, given in Equation [DWFE]. The integral above is a *functional integral*. We can give it a more precise meaning by defining its measure in the case of periodic functions $m(\mathbf{x})$ confined to a rectangular box. Then we can expand

$$m(\mathbf{x})=\frac{1}{\sqrt{V}}\sum_{\mathbf{q}}\hat{m}_{\mathbf{q}}\,e^{i\mathbf{q}\cdot\mathbf{x}}\ , \tag{7.8.39}$$

and we define the measure

$$Dm\equiv dm_0\prod_{\substack{\mathbf{q}\\ q_x>0}} d\,\mathsf{Re}\ \hat{m}_{\mathbf{q}}\ d\,\mathsf{Im}\ \hat{m}_{\mathbf{q}}\ . \tag{7.8.40}$$

Note that the fact that $m(\mathbf{x})\in\mathbb{R}$ means that $\hat{m}_{-\mathbf{q}}=\hat{m}^*_{\mathbf{q}}$. We'll assume $T>T_c$ and $H=0$ and we'll explore limit $T\to T_c^+$ from above to analyze the properties of the critical region close to T_c. In this limit we can ignore all but the quadratic terms in m, and we have

$$\begin{aligned}e^{-\beta A}\ &=\int\!Dm\ \exp\bigg(-\frac{1}{2}\beta\sum_{\mathbf{q}}(a+\kappa\,\mathbf{q}^2)\,|\hat{m}_{\mathbf{q}}|^2\bigg)\\ &=\prod_{\mathbf{q}}\bigg(\frac{\pi k_B T}{a+\kappa\,\mathbf{q}^2}\bigg)^{1/2}\ .\end{aligned}$$

Thus,

$$A=\frac{1}{2}k_B T\sum_{\mathbf{q}}\ln\!\bigg(\frac{a+\kappa\,\mathbf{q}^2}{\pi k_B T}\bigg)\ . \tag{7.8.41}$$

We now assume that $a(T)=\alpha t$, where t is the dimensionless quantity

$$t=\frac{T-T_c}{T_c}\ , \tag{7.8.42}$$

known as the *reduced temperature*.

We now compute the heat capacity $C_V=-T\,\frac{\partial^2 A}{\partial T^2}$. We are really only interested in the singular contributions to C_V, which means that we're only interested in differentiating with respect to T as it appears in $a(T)$. We divide by $N_{\mathsf{S}}k_B$ where N_{S} is the number of unit cells of our system, which we presume is a lattice-based model. Note $N_{\mathsf{S}}\sim V/\mathsf{a}^d$ where V is the volume and a the lattice constant. The dimensionless heat capacity per lattice site is then

$$c\equiv\frac{C_V}{N_{\mathsf{S}}}=\frac{\alpha^2\mathsf{a}^d}{2\kappa^2}\int^{\Lambda}\!\frac{d^dq}{(2\pi)^d}\ \frac{1}{(\xi^{-2}+\mathbf{q}^2)^2}\ , \tag{7.8.43}$$

where $\xi=(\kappa/\alpha t)^{1/2}\propto|t|^{-1/2}$ is the correlation length, and where $\Lambda\sim\mathsf{a}^{-1}$ is an ultraviolet cutoff. We define $R_*\equiv(\kappa/\alpha)^{1/2}$, in which case

$$c=R_*^{-4}\,\mathsf{a}^d\,\xi^{4-d}\cdot\frac{1}{2}\int^{\Lambda\xi}\!\frac{d^d\bar{q}}{(2\pi)^d}\ \frac{1}{(1+\bar{q}^2)^2}\ , \tag{7.8.44}$$

where $\bar{\mathbf{q}}\equiv\mathbf{q}\xi$. Thus,

$$c(t)\sim\begin{cases}const. & \text{if } d>4\\ -\ln t & \text{if } d=4\\ t^{\frac{d}{2}-2} & \text{if } d<4\ .\end{cases} \tag{7.8.45}$$

For $d>4$, mean field theory is qualitatively accurate, with finite corrections. In dimensions $d\le 4$, the mean field result is overwhelmed by fluctuation contributions as $t\to 0^+$ (as $T\to T_c^+$). We see that MFT is sensible provided the fluctuation contributions are small, provided

$$R_*^{-4}\,\mathsf{a}^d\,\xi^{4-d}\ll 1\ , \tag{7.8.46}$$

which entails t\gg t\ns_\ssr{G}, where

t\ns_\ssr{G}=\bigg({\Sa\over R\ns_*}\bigg)^{\!{2d\over 4-d}} is the *Ginzburg reduced temperature*. The criterion for the sufficiency of mean field theory, namely t\gg t\ns_\ssr{G}, is known as the *Ginzburg criterion*. The region |t|<t\ns_\ssr{G} is known as the *critical region*.

In a lattice ferromagnet, as we have seen, $R_* \sim \mathsf{a}$ is on the scale of the lattice spacing itself, hence t\ns_\ssr{G}\sim 1 and the critical regime is very large. Mean field theory then fails quickly as $T \to T_c$. In a (conventional) three-dimensional superconductor, R_* is on the order of the Cooper pair size, and $R_*/\mathsf{a} \sim 10^2 - 10^3$, hence t\ns_\ssr{G}=(a/R\ns_*)^6\sim 10^{-18} - 10^{-12} is negligibly narrow. The mean field theory of the superconducting transition – BCS theory – is then valid essentially all the way to $T = T_c$.
