8.3: Weakly Inhomogeneous Gas

Consider a gas which is only weakly out of equilibrium. We follow the treatment in Lifshitz and Pitaevskii, §6. As the gas is only slightly out of equilibrium, we seek a solution to the Boltzmann equation of the form $f = f^0 + \delta f$, where f^0 is describes a local equilibrium. Recall that such a distribution function is annihilated by the collision term in the Boltzmann equation but not by the streaming term, hence a correction δf must be added in order to obtain a solution.

The most general form of local equilibrium is described by the distribution

$$f^0(\mathbf{r},\Gamma) = C \exp\left(\frac{\mu - \varepsilon(\Gamma) + \mathbf{V}\cdot\mathbf{p}}{k_BT}\right) , \tag{8.3.1}$$

where $\mu = \mu(\mathbf{r},t)$, $T = T(\mathbf{r},t)$, and $\mathbf{V} = \mathbf{V}(\mathbf{r},t)$ vary in both space and time. Note that

$$\begin{aligned} df^0 &= \left(d\mu + \mathbf{p}\cdot d\mathbf{V} + (\varepsilon - \mu - \mathbf{V}\cdot\mathbf{p})\,\frac{dT}{T} - d\varepsilon\right)\left(-\frac{\partial f^0}{\partial\varepsilon}\right) \\ &= \left(\frac{1}{n}\,dp + \mathbf{p}\cdot d\mathbf{V} + (\varepsilon - h)\,\frac{dT}{T} - d\varepsilon\right)\left(-\frac{\partial f^0}{\partial\varepsilon}\right) \end{aligned}$$

where we have assumed $\mathbf{V} = 0$ on average, and used

$$\begin{aligned} d\mu &= \left(\frac{\partial\mu}{\partial T}\right)_{\!p} dT + \left(\frac{\partial\mu}{\partial p}\right)_{\!T} dp \\ &= -s\,dT + \frac{1}{n}\,dp\ , \end{aligned}$$

where s is the entropy per particle and n is the number density. We have further written $h = \mu + Ts$, which is the enthalpy per particle. Here, c_p is the heat capacity per particle at constant pressure[5]. Finally, note that when f^0 is the Maxwell-Boltzmann distribution, we have

$$-\frac{\partial f^0}{\partial\varepsilon} = \frac{f^0}{k_BT}\ . \tag{8.3.2}$$

The Boltzmann equation is written

$$\left(\frac{\partial}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial}{\partial\mathbf{r}} + \mathbf{F}\cdot\frac{\partial}{\partial\mathbf{p}}\right)\left(f^0 + \delta f\right) = \left(\frac{\partial f}{\partial t}\right)_{\!coll}\ . \tag{8.3.3}$$

The RHS of this equation must be of order δf because the local equilibrium distribution f^0 is annihilated by the collision integral. We therefore wish to evaluate one of the contributions to the LHS of this equation,

$$\begin{aligned} \frac{\partial f^0}{\partial t} + \frac{\mathbf{p}}{m}\cdot\frac{\partial f^0}{\partial\mathbf{r}} + \mathbf{F}\cdot\frac{\partial f^0}{\partial\mathbf{p}} &= \left(-\frac{\partial f^0}{\partial\varepsilon}\right)\left\{\frac{1}{n}\,\frac{\partial p}{\partial t} + \frac{\varepsilon - h}{T}\,\frac{\partial T}{\partial t} + m\mathbf{v}\cdot\Big[(\mathbf{v}\cdot\boldsymbol{\nabla})\,\mathbf{V}\Big]\right. \\ &\qquad \left. + \mathbf{v}\cdot\left(m\,\frac{\partial\mathbf{V}}{\partial t} + \frac{1}{n}\,\boldsymbol{\nabla} p\right) + \frac{\varepsilon - h}{T}\,\mathbf{v}\cdot\boldsymbol{\nabla} T - \mathbf{F}\cdot\mathbf{v}\right\} . \end{aligned}$$

To simplify this, first note that Newton's laws applied to an ideal fluid give $\rho\dot{\mathbf{V}} = -\boldsymbol{\nabla} p$, where $\rho = mn$ is the mass density. Corrections to this result, *e.g.* viscosity and nonlinearity in $\mathbf{V}$, are of higher order.

Next, continuity for particle number means $\dot{n} + \boldsymbol{\nabla}\cdot(n\mathbf{V}) = 0$. We assume $\mathbf{V}$ is zero on average and that all derivatives are small, hence $\boldsymbol{\nabla}\cdot(n\mathbf{V}) = \mathbf{V}\cdot\boldsymbol{\nabla} n + n\,\boldsymbol{\nabla}\cdot\mathbf{V} \approx n\,\boldsymbol{\nabla}\cdot\mathbf{V}$. Thus,

$$\frac{\partial\ln n}{\partial t} = \frac{\partial\ln p}{\partial t} - \frac{\partial\ln T}{\partial t} = -\boldsymbol{\nabla}\cdot\mathbf{V}\ , \tag{8.3.4}$$

where we have invoked the ideal gas law $n = p/k_BT$ above.

Next, we invoke conservation of entropy. If s is the entropy per particle, then ns is the entropy per unit volume, in which case we have the continuity equation

$$\frac{\partial(ns)}{\partial t}+\boldsymbol{\nabla}\cdot(ns\mathbf{V})=n\left(\frac{\partial s}{\partial t}+\mathbf{V}\cdot\boldsymbol{\nabla}s\right)+s\left(\frac{\partial n}{\partial t}+\boldsymbol{\nabla}\cdot(n\mathbf{V})\right)=0\,. \tag{8.3.5}$$

The second bracketed term on the RHS vanishes because of particle continuity, leaving us with $\dot{s}+\mathbf{V}\cdot\boldsymbol{\nabla}s\approx\dot{s}=0$ (since $\mathbf{V}=0$ on average, and any gradient is first order in smallness). Now thermodynamics says

$$\begin{aligned} ds &= \left(\frac{\partial s}{\partial T}\right)_{\!p} dT + \left(\frac{\partial s}{\partial p}\right)_{\!T} dp \\ &= \frac{c_p}{T}\,dT - \frac{k_B}{p}\,dp\,, \end{aligned}$$

since $T\big(\frac{\partial s}{\partial T}\big)_p=c_p$ and $\big(\frac{\partial s}{\partial p}\big)_T=\big(\frac{\partial v}{\partial T}\big)_p$, where $v=V/N$. Thus,

$$\frac{c_p}{k_B}\,\frac{\partial\ln T}{\partial t}-\frac{\partial\ln p}{\partial t}=0\,. \tag{8.3.6}$$

We now have in eqns. [ptea] and [pteb] two equations in the two unknowns $\frac{\partial\ln T}{\partial t}$ and $\frac{\partial\ln p}{\partial t}$, yielding

$$\begin{aligned} \frac{\partial\ln T}{\partial t} &= -\frac{k_B}{c_V}\,\boldsymbol{\nabla}\cdot\mathbf{V} \\ \frac{\partial\ln p}{\partial t} &= -\frac{c_p}{c_V}\,\boldsymbol{\nabla}\cdot\mathbf{V}\,. \end{aligned}$$

Thus Equation [LHSA] becomes

$$\begin{aligned} \frac{\partial f^0}{\partial t}+\frac{\mathbf{p}}{m}\cdot\frac{\partial f^0}{\partial\mathbf{r}}+\mathbf{F}\cdot\frac{\partial f^0}{\partial\mathbf{p}} &= \left(-\frac{\partial f^0}{\partial\varepsilon}\right)\bigg\{\frac{\varepsilon(\Gamma)-h}{T}\,\mathbf{v}\cdot\boldsymbol{\nabla}T+m\,v_\alpha v_\beta\,\mathcal{Q}_{\alpha\beta} \\ &\qquad +\frac{h-Tc_p-\varepsilon(\Gamma)}{c_V/k_B}\,\boldsymbol{\nabla}\cdot\mathbf{V}-\mathbf{F}\cdot\mathbf{v}\bigg\}\,, \end{aligned}$$

where

$$\mathcal{Q}_{\alpha\beta}=\frac{1}{2}\left(\frac{\partial V_\alpha}{\partial x_\beta}+\frac{\partial V_\beta}{\partial x_\alpha}\right)\,. \tag{8.3.7}$$

Therefore, the Boltzmann equation takes the form

$$\left\{\frac{\varepsilon(\Gamma)-h}{T}\,\mathbf{v}\cdot\boldsymbol{\nabla}T+m\,v_\alpha v_\beta\,\mathcal{Q}_{\alpha\beta}-\frac{\varepsilon(\Gamma)-h+Tc_p}{c_V/k_B}\,\boldsymbol{\nabla}\cdot\mathbf{V}-\mathbf{F}\cdot\mathbf{v}\right\}\frac{f^0}{k_BT}+\frac{\partial\,\delta\!f}{\partial t}=\left(\frac{\partial f}{\partial t}\right)_{\!coll}\,. \tag{8.3.8}$$

Notice we have dropped the terms $\mathbf{v}\cdot\frac{\partial\,\delta\!f}{\partial\mathbf{r}}$ and $\mathbf{F}\cdot\frac{\partial\,\delta\!f}{\partial\mathbf{p}}$, since $\delta\!f$ must already be first order in smallness, and both the $\frac{\partial}{\partial\mathbf{r}}$ operator as well as $\mathbf{F}$ add a second order of smallness, which is negligible. Typically $\frac{\partial\,\delta\!f}{\partial t}$ is nonzero if the applied force $\mathbf{F}(t)$ is time-dependent. We use the convention of summing over repeated indices. Note that $\delta_{\alpha\beta}\,\mathcal{Q}_{\alpha\beta}=\mathcal{Q}_{\alpha\alpha}=\boldsymbol{\nabla}\cdot\mathbf{V}$. For ideal gases in which only translational and rotational degrees of freedom are excited, $h=c_pT$.