8.13: Appendix IV- Correlations in the Langevin formalism

As shown above, integrating the Langevin equation $\dot{p}+\gamma p=F+\eta(t)$ yields

$$p(t)=p(0)\,e^{-\gamma t}+\frac{F}{\gamma}\left(1-e^{-\gamma t}\right)+\int\limits_0^t\!ds\;\eta(s)\,e^{\gamma(s-t)}\ . \tag{8.13.1}$$

. Thus, the momentum autocorrelator is

$$\begin{aligned}\big\langle p(t)\,p(t')\big\rangle-\big\langle p(t)\big\rangle\big\langle p(t')\big\rangle &=\int\limits_0^t\!ds\!\int\limits_0^{t'}\!ds'\;e^{\gamma(s-t)}\,e^{\gamma(s'-t')}\,\big\langle\eta(s)\,\eta(s')\big\rangle\\ &=\Gamma\,e^{-\gamma(t+t')}\!\!\int\limits_0^{t_{min}}\!\!ds\;e^{2\gamma s}=Mk_{B}T\left(e^{-\gamma|t-t'|}-e^{-\gamma(t+t')}\right)\ ,\end{aligned}$$

where

$$t_{min}=min(t,t')=\begin{cases}t & if\ t<t'\\ t' & if\ t'<t\end{cases} \tag{8.13.2}$$

is the lesser of t and t'. Here we have used the result

$$\begin{aligned}\int\limits_0^t\!ds\!\int\limits_0^{t'}\!ds'\;e^{\gamma(s+s')}\,\delta(s-s') &=\int\limits_0^{t_{min}}\!\!ds\!\int\limits_0^{t_{min}}\!\!ds'\;e^{\gamma(s+s')}\,\delta(s-s')\\ &=\int\limits_0^{t_{min}}\!\!ds\;e^{2\gamma s}=\frac{1}{2\gamma}\Big(e^{2\gamma t_{min}}-1\Big)\ .\end{aligned}$$

One way to intuitively understand this result is as follows. The double integral over s and s' is over a rectangle of dimensions $t\times t'$. Since the δ-function can only be satisfied when $s=s'$, there can be no contribution to the integral from regions where $s>t'$ or $s'>t$. Thus, the only contributions can arise from integration over the square of dimensions $t_{min}\times t_{min}$. Note also

$$t+t'-2\,min(t,t')=|t-t'|\ . \tag{8.13.3}$$

[Fssprime] Regions for some of the double integrals encountered in the text.

Let's now compute the position $x(t)$. We have

$$\begin{aligned} x(t) &= x(0) + \frac{1}{M}\int\limits_0^t\!ds\, p(s) \\ &= x(0) + \int\limits_0^t\!ds \left[\left(v(0) - \frac{F}{\gamma M}\right) e^{-\gamma s} + \frac{F}{\gamma M}\right] + \frac{1}{M}\int\limits_0^t\!ds\!\int\limits_0^s\!ds_1\, \eta(s_1)\, e^{\gamma(s_1 - s)} \\ &= \langle x(t)\rangle + \frac{1}{M}\int\limits_0^t\!ds\!\int\limits_0^s\!ds_1\, \eta(s_1)\, e^{\gamma(s_1-s)}\ , \end{aligned}$$

with $v = p/M$. Since $\langle \eta(t)\rangle = 0$, we have

$$\begin{aligned} \langle x(t)\rangle &= x(0) + \int\limits_0^t\!ds \left[\left(v(0) - \frac{F}{\gamma M}\right) e^{-\gamma s} + \frac{F}{\gamma M}\right] \\ &= x(0) + \frac{Ft}{\gamma M} + \frac{1}{\gamma}\left(v(0) - \frac{F}{\gamma M}\right)\left(1 - e^{-\gamma t}\right) . \end{aligned}$$

Note that for $\gamma t \ll 1$ we have $\langle x(t)\rangle = x(0) + v(0)\, t + \frac{1}{2} M^{-1} F t^2 + \mathcal{O}(t^3)$, as is appropriate for ballistic particles moving under the influence of a constant force. This long time limit of course agrees with our earlier evaluation for the terminal velocity, $v_\infty = \langle p(\infty)\rangle/M = F/\gamma M$.

We next compute the position autocorrelation:

$$\begin{aligned} \langle x(t)\, x(t')\rangle - \langle x(t)\rangle\langle x(t')\rangle &= \frac{1}{M^2}\int\limits_0^t\!ds\!\int\limits_0^{t'}\!ds'\, e^{-\gamma(s+s')}\int\limits_0^s\!ds_1\!\int\limits_0^{s'}\!ds_1'\, e^{\gamma(s_1+s_2)}\, \langle \eta(s_1)\,\eta(s_2)\rangle \\ &= \frac{\Gamma}{2\gamma M^2}\int\limits_0^t\!ds\!\int\limits_0^{t'}\!ds'\left(e^{-\gamma|s-s'|} - e^{-\gamma(s+s')}\right) \end{aligned}$$

We have to be careful in computing the double integral of the first term in brackets on the RHS. We can assume, without loss of generality, that $t \ge t'$. Then

$$\begin{aligned} \int\limits_0^t\!ds\!\int\limits_0^{t'}\!ds'\, e^{-\gamma|s-s'|} &= \int\limits_0^{t'}\!ds'\, e^{\gamma s'}\int\limits_{s'}^t\!ds\ \, e^{-\gamma s} + \int\limits_0^{t'}\!ds'\, e^{-\gamma s'}\int\limits_0^{s'}\!ds\, e^{\gamma s} \\ &= 2\gamma^{-1} t' + \gamma^{-2}\left(e^{-\gamma t} + e^{-\gamma t'} - 1 - e^{-\gamma(t-t')}\right) . \end{aligned}$$

We then find, for $t > t'$,

$$\langle x(t)\, x(t')\rangle - \langle x(t)\rangle\langle x(t')\rangle = \frac{2k_BT}{\gamma M}\, t' + \frac{k_BT}{\gamma^2 M}\left(2e^{-\gamma t} + 2e^{-\gamma t'} - 2 - e^{-\gamma(t-t')} - e^{-\gamma(t+t')}\right) . \tag{8.13.4}$$

In particular, the equal time autocorrelator is

$$\langle x^2(t)\rangle - \langle x(t)\rangle^2 = \frac{2k_BT}{\gamma M}\, t + \frac{k_BT}{\gamma^2 M}\left(4e^{-\gamma t} - 3 - e^{-2\gamma t}\right) . \tag{8.13.5}$$

We see that for long times

$$\langle x^2(t)\rangle - \langle x(t)\rangle^2 \sim 2Dt\ , \tag{8.13.6}$$

where $D = k_BT/\gamma M$ is the diffusion constant.