8.9: Stochastic Processes

A *stochastic process* is one which is partially random, it is not wholly deterministic. Typically the randomness is due to phenomena at the microscale, such as the effect of fluid molecules on a small particle, such as a piece of dust in the air. The resulting motion (called *Brownian motion* in the case of particles moving in a fluid) can be described only in a statistical sense. That is, the full motion of the system is a *functional* of one or more independent random variables. The motion is then described by its averages with respect to the various random distributions.

Langevin equation and Brownian motion

Consider a particle of mass M subjected to dissipative and random forcing. We'll examine this system in one dimension to gain an understanding of the essential physics. We write

$$\dot{p}+\gamma p=F+\eta(t)\ . \tag{8.9.1}$$

Here, γ is the damping rate due to friction, F is a constant external force, and $\eta(t)$ is a *stochastic random force*. This equation, known as the Langevin equation, describes a ballistic particle being buffeted by random forcing events. Think of a particle of dust as it moves in the atmosphere; F would then represent the external force due to gravity and $\eta(t)$ the random forcing due to interaction with the air molecules. For a sphere of radius a moving with velocity $\mathbf{v}$ in a fluid, the *Stokes drag* is given by $\mathbf{F}_{drag}=-6\pi\eta a\mathbf{v}$, where a is the radius. Thus,

$$\gamma_{Stokes}=\frac{6\pi\eta a}{M}\ , \tag{8.9.2}$$

where M is the mass of the particle. It is illustrative to compute γ in some setting. Consider a micron sized droplet ($a=10^{-4}$ cm) of some liquid of density $\rho\sim 1.0\,g/cm^3$ moving in air at $T=20^\circ\,C$. The viscosity of air is $\eta=1.8\times 10^{-4}\ g/cm\cdot s$ at this temperature[16]. If the droplet density is constant, then $\gamma=9\eta/2\rho a^2=8.1\times 10^4\ s^{-1}$, hence the time scale for viscous relaxation of the particle is $\tau=\gamma^{-1}=12\ \mu s$. We should stress that the viscous damping on the particle is of course due to the fluid molecules, in some average 'coarse-grained' sense. The random component to the force $\eta(t)$ would then represent the fluctuations with respect to this average.

We can easily integrate this equation:

$$\begin{aligned}\frac{d}{dt}\left(p\,e^{\gamma t}\right)&=F\,e^{\gamma t}+\eta(t)\,e^{\gamma t}\\ p(t)&=p(0)\,e^{-\gamma t}+\frac{F}{\gamma}\left(1-e^{-\gamma t}\right)+\int_0^t\!ds\,\eta(s)\,e^{\gamma(s-t)}\end{aligned}$$

Note that $p(t)$ is indeed a functional of the random function $\eta(t)$. We can therefore only compute averages in order to describe the motion of the system.

The first average we will compute is that of p itself. In so doing, we assume that $\eta(t)$ has zero mean: $\langle\eta(t)\rangle=0$. Then

$$\langle p(t)\rangle=p(0)\,e^{-\gamma t}+\frac{F}{\gamma}\left(1-e^{-\gamma t}\right)\ . \tag{8.9.3}$$

On the time scale γ^{-1}, the initial conditions $p(0)$ are effectively forgotten, and asymptotically for $t\gg\gamma^{-1}$ we have $\langle p(t)\rangle\to F/\gamma$, which is the terminal momentum.

Next, consider

$$\langle p^2(t)\rangle=\langle p(t)\rangle^2+\int_0^t\!ds_1\int_0^t\!ds_2\ e^{\gamma(s_1-t)}\,e^{\gamma(s_2-t)}\,\langle\eta(s_1)\,\eta(s_2)\rangle\ .$$

We now need to know the two-time correlator $\langle\eta(s_1)\,\eta(s_2)\rangle$. We assume that the correlator is a function only of the time difference $\Delta s=s_1-s_2$, so that the random force $\eta(s)$ satisfies

$$\begin{aligned}\langle\eta(s)\rangle&=0\\ \langle\eta(s_1)\,\eta(s_2)\rangle&=\phi(s_1-s_2)\ .\end{aligned}$$

The function $\phi(s)$ is the *autocorrelation function* of the random force. A macroscopic object moving in a fluid is constantly buffeted by fluid particles over its entire perimeter. These different fluid particles are almost completely uncorrelated, hence $\phi(s)$ is basically nonzero except on a very small time scale τ_ϕ, which is the time a single fluid particle spends interacting with the object. We can take $\tau_\phi \to 0$ and approximate

$$\phi(s) \approx \Gamma\,\delta(s)\ . \tag{8.9.4}$$

We shall determine the value of Γ from equilibrium thermodynamic considerations below.

With this form for $\phi(s)$, we can easily calculate the equal time momentum autocorrelation:

$$\begin{aligned}\big\langle p^2(t)\big\rangle &= \big\langle p(t)\big\rangle^2 + \Gamma\int\limits_0^t\!ds\; e^{2\gamma(s-t)}\\ &= \big\langle p(t)\big\rangle^2 + \frac{\Gamma}{2\gamma}\left(1-e^{-2\gamma t}\right)\ .\end{aligned}$$

Consider the case where $F=0$ and the limit $t \gg \gamma^{-1}$. We demand that the object thermalize at temperature T. Thus, we impose the condition

$$\left\langle \frac{p^2(t)}{2M}\right\rangle = \frac{1}{2}k_BT \qquad \Longrightarrow \qquad \Gamma = 2\gamma M k_BT \quad , \tag{8.9.5}$$

where M is the particle's mass. This determines the value of Γ.

We can now compute the general momentum autocorrelator:

$$\begin{aligned}\big\langle p(t)\,p(t')\big\rangle - \big\langle p(t)\big\rangle\big\langle p(t')\big\rangle &= \int\limits_0^t\!ds\int\limits_0^{t'}\!ds'\; e^{\gamma(s-t)}\,e^{\gamma(s'-t')}\,\big\langle\eta(s)\,\eta(s')\big\rangle\\ &= Mk_BT\,e^{-\gamma|t-t'|} \qquad (t,t'\to\infty\,,\ |t-t'|\ finite)\ .\end{aligned}$$

The full expressions for this and subsequent expressions, including subleading terms, are contained in an appendix, §14.

Let's now compute the position $x(t)$. We find

$$x(t) = \big\langle x(t)\big\rangle + \frac{1}{M}\int\limits_0^t\!ds\int\limits_0^s\!ds_1\;\eta(s_1)\,e^{\gamma(s_1-s)}\ , \tag{8.9.6}$$

where

$$\big\langle x(t)\big\rangle = x(0) + \frac{Ft}{\gamma M} + \frac{1}{\gamma}\left(v(0)-\frac{F}{\gamma M}\right)\left(1-e^{-\gamma t}\right)\ . \tag{8.9.7}$$

Note that for $\gamma t \ll 1$ we have $\big\langle x(t)\big\rangle = x(0) + v(0)\,t + \frac{1}{2}M^{-1}Ft^2 + \mathcal{O}(t^3)$, as is appropriate for ballistic particles moving under the influence of a constant force. This long time limit of course agrees with our earlier evaluation for the terminal velocity, $v_\infty = \big\langle p(\infty)\big\rangle/M = F/\gamma M$. We next compute the position autocorrelation:

$$\begin{aligned}\big\langle x(t)\,x(t')\big\rangle - \big\langle x(t)\big\rangle\big\langle x(t')\big\rangle &= \frac{1}{M^2}\int\limits_0^t\!ds\int\limits_0^{t'}\!ds'\;e^{-\gamma(s+s')}\int\limits_0^s\!ds_1\int\limits_0^{s'}\!ds_1'\;e^{\gamma(s_1+s_2)}\,\big\langle\eta(s_1)\,\eta(s_2)\big\rangle\\ &= \frac{2k_BT}{\gamma M}\,\mathsf{min}(t,t') + \mathcal{O}(1)\ .\end{aligned}$$

In particular, the equal time autocorrelator is

$$\big\langle x^2(t)\big\rangle - \big\langle x(t)\big\rangle^2 = \frac{2k_BT\,t}{\gamma M} \equiv 2D\,t\ , \tag{8.9.8}$$

at long times, up to terms of order unity. Here,

$$D = \frac{k_B T}{\gamma M} \tag{8.9.9}$$

is the *diffusion constant*. For a liquid droplet of radius $a = 1\,\mu\Rm$ moving in air at $T = 293\,K$, for which $\eta = 1.8 \times 10^{-4}\,P$, we have

$$D = \frac{k_B T}{6\pi\eta a} = \frac{(1.38 \times 10^{-16}\,erg/K)\,(293\,K)}{6\pi\,(1.8 \times 10^{-4}\,P)\,(10^{-4}\,cm)} = 1.19 \times 10^{-7}\,cm^2/s\,. \tag{8.9.10}$$

This result presumes that the droplet is large enough compared to the intermolecular distance in the fluid that one can adopt a continuum approach and use the Navier-Stokes equations, and then assuming a laminar flow.

If we consider molecular diffusion, the situation is quite a bit different. As we shall derive below in §10.3, the molecular diffusion constant is $D = \ell^2/2\tau$, where ℓ is the mean free path and τ is the collision time. As we found in Equation [nutaueqn], the mean free path ℓ, collision time τ, number density n, and total scattering cross section σ are related by

$$\ell = \bar{v}\tau = \frac{1}{\sqrt{2}\,n\sigma}\,, \tag{8.9.11}$$

where $\bar{v} = \sqrt{8k_BT/\pi m}$ is the average particle speed. Approximating the particles as hard spheres, we have $\sigma = 4\pi a^2$, where a is the hard sphere radius. At $T = 293\,K$, and $p = 1\,atm$, we have $n = p/k_BT = 2.51 \times 10^{19}\,cm^{-3}$. Since air is predominantly composed of N_2 molecules, we take $a = 1.90 \times 10^{-8}$ cm and $m = 28.0\,amu = 4.65 \times 10^{-23}\,g$, which are appropriate for N_2. We find an average speed of $\bar{v} = 471\,\Rm/s$ and a mean free path of $\ell = 6.21 \times 10^{-6}$ cm. Thus, $D = \frac{1}{2}\ell\bar{v} = 0.146\,cm^2/s$. Though much larger than the diffusion constant for large droplets, this is still too small to explain common experiences. Suppose we set the characteristic distance scale at $d = 10\,cm$ and we ask how much time a point source would take to diffuse out to this radius. The answer is $\Delta t = d^2/2D = 343\,s$, which is between five and six minutes. Yet if someone in the next seat emits a foul odor, your sense the offending emission in on the order of a second. What this tells us is that diffusion isn't the only transport process involved in these and like phenomena. More important are *convection* currents which distribute the scent much more rapidly.

Langevin equation for a particle in a harmonic well

Consider next the equation

$$M\ddot{X} + \gamma M\dot{X} + M\omega_0^2 X = F_0 + \eta(t)\,, \tag{8.9.12}$$

where F_0 is a constant force. We write $X = \frac{F_0}{M\omega_0^2} + x$ and measure x relative to the potential minimum, yielding

$$\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x = \frac{1}{M}\,\eta(t)\,. \tag{8.9.13}$$

At this point there are several ways to proceed.

Perhaps the most straightforward is by use of the Laplace transform. Recall:

$$\begin{aligned} \hat{x}(\nu) &= \int_0^\infty \!dt\; e^{-\nu t}\,\eta(\nu) \\ x(t) &= \int_{\mathcal{C}} \frac{d\nu}{2\pi i}\,e^{+\nu t}\,\hat{x}(\nu)\,, \end{aligned}$$

where the contour $\mathcal{C}$ proceeds from $a - i\infty$ to $a + i\infty$ such that all poles of the integrand lie to the left of $\mathcal{C}$. We then have

$$\begin{aligned} \frac{1}{M}\int_0^\infty \!dt\; e^{-\nu t}\,\eta(t) &= \frac{1}{M}\int_0^\infty \!dt\; e^{-\nu t}\,\Big(\ddot{x} + \gamma\,\dot{x} + \omega_0^2\,x\Big) \\ &= -(\nu + \gamma)\,x(0) - \dot{x}(0) + \big(\nu^2 + \gamma\nu + \omega_0^2\big)\,\hat{x}(\nu)\,. \end{aligned}$$

Thus, we have

$$\hat{x}(\nu)=\frac{(\nu+\gamma)\,x(0)+\dot{x}(0)}{\nu^2+\gamma\nu+\omega_0^2}+\frac{1}{M}\cdot\frac{1}{\nu^2+\gamma\nu+\omega_0^2}\int\limits_0^\infty\!dt\;e^{-\nu t}\,\eta(t)\,. \tag{8.9.14}$$

Now we may write

$$\nu^2+\gamma\nu+\omega_0^2=(\nu-\nu_+)(\nu-\nu_-)\,, \tag{8.9.15}$$

where

$$\nu_\pm=-\frac{1}{2}\gamma\pm\sqrt{\frac{1}{4}\gamma^2-\omega_0^2}\;. \tag{8.9.16}$$

Note that $\mathsf{Re}\,(\nu_\pm)\le 0$ and that $\gamma+\nu_\pm=-\nu_\mp$.

Performing the inverse Laplace transform, we obtain

$$\begin{aligned} x(t)\;&=\frac{x(0)}{\nu_+-\nu_-}\Big(\nu_+\,e^{\nu_- t}-\nu_-\,e^{\nu_+ t}\Big)+\frac{\dot{x}(0)}{\nu_+-\nu_-}\Big(e^{\nu_+ t}-e^{\nu_- t}\Big)\\ &\qquad\qquad+\int\limits_0^\infty\!ds\;K(t-s)\,\eta(s)\,, \end{aligned}$$

where

$$K(t-s)=\frac{\Theta(t-s)}{M\,(\nu_+-\nu_-)}\,\Big(e^{\nu_+(t-s)}-e^{\nu_-(t-s)}\Big) \tag{8.9.17}$$

is the *response kernel* and $\Theta(t-s)$ is the step function which is unity for $t>s$ and zero otherwise. The response is *causal*, $x(t)$ depends on $\eta(s)$ for all previous times $s<t$, but not for future times $s>t$. Note that $K(\tau)$ decays exponentially for $\tau\to\infty$, if $\mathsf{Re}(\nu_\pm)<0$. The marginal case where $\omega_0=0$ and $\nu_+=0$ corresponds to the diffusion calculation we performed in the previous section.

Discrete random walk

Consider an object moving on a one-dimensional lattice in such a way that every time step it moves either one unit to the right or left, at random. If the lattice spacing is ℓ, then after n time steps the position will be

$$x_n=\ell\sum_{j=1}^n\sigma_j\,, \tag{8.9.18}$$

where

$$\sigma_j=\begin{cases}+1 & \text{if motion is one unit to right at time step } j\\ -1 & \text{if motion is one unit to left at time step } j\backslash\,.\end{cases} \tag{8.9.19}$$

Clearly $\langle\sigma_j\rangle=0$, so $\langle x_n\rangle=0$. Now let us compute

$$\big\langle x_n^2\big\rangle=\ell^2\sum_{j=1}^n\sum_{j'=1}^n\langle\sigma_j\sigma_{j'}\rangle=n\ell^2\,, \tag{8.9.20}$$

where we invoke

$$\big\langle\sigma_j\sigma_{j'}\big\rangle=\delta_{jj'}\,. \tag{8.9.21}$$

If the length of each time step is τ, then we have, with $t=n\tau$,

$$\big\langle x^2(t)\big\rangle=\frac{\ell^2}{\tau}\,t\,, \tag{8.9.22}$$

and we identify the diffusion constant

$$D=\frac{\ell^2}{2\tau}\ . \tag{8.9.23}$$

Suppose, however, the random walk is *biased*, so that the probability for each independent step is given by

$$P(\sigma)=p\,\delta_{\sigma,1}+q\,\delta_{\sigma,-1}\ , \tag{8.9.24}$$

where $p+q=1$. Then

$$\langle\sigma_j\rangle=p-q=2p-1 \tag{8.9.25}$$

and

$$\begin{aligned}\langle\sigma_j\sigma_{j'}\rangle&=(p-q)^2\left(1-\delta_{jj'}\right)+\delta_{jj'}\\&=(2p-1)^2+4\,p\,(1-p)\,\delta_{jj'}\ .\end{aligned}$$

Then

$$\begin{aligned}\langle x_n\rangle&=(2p-1)\,\ell n\\\langle x_n^2\rangle-\langle x_n\rangle^2&=4\,p\,(1-p)\,\ell^2 n\ .\end{aligned}$$

Fokker-Planck equation

Suppose $x(t)$ is a stochastic variable. We define the quantity

$$\delta x(t)\equiv x(t+\delta t)-x(t)\ , \tag{8.9.26}$$

and we assume

$$\begin{aligned}\big\langle\delta x(t)\big\rangle&=F_1\big(x(t)\big)\,\delta t\\\big\langle\big[\delta x(t)\big]^2\big\rangle&=F_2\big(x(t)\big)\,\delta t\end{aligned}$$

but $\big\langle\big[\delta x(t)\big]^n\big\rangle=\mathcal{O}\big((\delta t)^2\big)$ for $n>2$. The $n=1$ term is due to *drift* and the $n=2$ term is due to *diffusion*. Now consider the conditional probability density, $P(x,t\,|\,x_0,t_0)$, defined to be the probability distribution for $x\equiv x(t)$ given that $x(t_0)=x_0$. The conditional probability density satisfies the composition rule,

$$P(x_2,t_2\,|\,x_0,t_0)=\int\limits_{-\infty}^{\infty}\!\!dx_1\,P(x_2,t_2\,|\,x_1,t_1)\,P(x_1,t_1\,|\,x_0,t_0)\ , \tag{8.9.27}$$

for any value of t_1. This is also known as the *Chapman-Kolmogorov equation*. In words, what it says is that the probability density for a particle being at x_2 at time t_2, given that it was at x_0 at time t_0, is given by the product of the probability density for being at x_2 at time t_2 given that it was at x_1 at t_1, multiplied by that for being at x_1 at t_1 given it was at x_0 at t_0, integrated over x_1. This should be intuitively obvious, since if we pick any time $t_1\in[t_0,t_2]$, then the particle had to be *somewhere* at that time. Indeed, one wonders how Chapman and Kolmogorov got their names attached to a result that is so obvious. At any rate, a picture is worth a thousand words: see Figure [FChaKol].

[FChaKol] Interpretive sketch of the mathematics behind the Chapman-Kolmogorov equation.

Proceeding, we may write

$$P(x,t+\delta t\,|\,x_0,t_0)=\int\limits_{-\infty}^{\infty}\!dx'\,P(x,t+\delta t\,|\,x',t)\,P(x',t\,|\,x_0,t_0)\ . \tag{8.9.28}$$

Now

$$\begin{aligned}P(x,t+\delta t\,|\,x',t)&=\big\langle\delta\big(x-\delta x(t)-x'\big)\big\rangle\\&=\bigg\{1+\big\langle\delta x(t)\big\rangle\,\frac{d}{dx'}+\frac{1}{2}\big\langle\big[\delta x(t)\big]^2\big\rangle\,\frac{d^2}{dx'^2}+\ldots\bigg\}\,\delta(x-x')\\&=\delta(x-x')+F_1(x')\,\frac{d\,\delta(x-x')}{dx'}\,\delta t+\frac{1}{2}F_2(x')\,\frac{d^2\delta(x-x')}{dx'^2}\,\delta t+\mathcal{O}\big((\delta t)^2\big)\ ,\end{aligned}$$

where the average is over the random variables. We now insert this result into Equation [CGEFPE], integrate by parts, divide by δt, and then take the limit $\delta t\to 0$. The result is the Fokker-Planck equation,

$$\frac{\partial P}{\partial t}=-\frac{\partial}{\partial x}\big[F_1(x)\,P(x,t)\big]+\frac{1}{2}\,\frac{\partial^2}{\partial x^2}\big[F_2(x)\,P(x,t)\big]\ . \tag{8.9.29}$$

Brownian motion redux

Let’s apply our Fokker-Planck equation to a description of Brownian motion. From our earlier results, we have

$$F_1(x)=\frac{F}{\gamma M}\qquad ,\qquad F_2(x)=2D\ . \tag{8.9.30}$$

A formal proof of these results is left as an exercise for the reader. The Fokker-Planck equation is then

$$\frac{\partial P}{\partial t}=-u\,\frac{\partial P}{\partial x}+D\,\frac{\partial^2 P}{\partial x^2}\ , \tag{8.9.31}$$

where $u=F/\gamma M$ is the average terminal velocity. If we make a Galilean transformation and define

$$y=x-ut\qquad ,\qquad s=t \tag{8.9.32}$$

then our Fokker-Planck equation takes the form

$$\frac{\partial P}{\partial s}=D\,\frac{\partial^2 P}{\partial y^2}\ . \tag{8.9.33}$$

This is known as the *diffusion equation*. Equation [FPEBM] is also a diffusion equation, rendered in a moving frame.

While the Galilean transformation is illuminating, we can easily solve Equation [FPEBM] without it. Let’s take a look at this equation after Fourier transforming from x to q:

$$\begin{aligned}P(x,t)&=\int\limits_{-\infty}^{\infty}\!\frac{dq}{2\pi}\,e^{iqx}\,\hat{P}(q,t)\\\hat{P}(q,t)&=\int\limits_{-\infty}^{\infty}\!dx\;e^{-iqx}\,P(x,t)\ .\end{aligned}$$

Then as should be well known to you by now, we can replace the operator $\frac{\partial}{\partial x}$ with multiplication by iq, resulting in

$$\frac{\partial}{\partial t}\,\hat{P}(q,t)=-(Dq^2+iqu)\,\hat{P}(q,t)\ , \tag{8.9.34}$$

with solution

$$\hat{P}(q,t)=e^{-Dq^2t}\,e^{-iqut}\,\hat{P}(q,0)\ . \tag{8.9.35}$$

We now apply the inverse transform to get back to x-space:

$$\begin{aligned} P(x,t) &= \int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\, e^{iqx}\, e^{-Dq^2 t}\, e^{-iqut}\int\limits_{-\infty}^{\infty} dx'\, e^{-iqx'}\, P(x',0) \\ &= \int\limits_{-\infty}^{\infty} dx'\, P(x',0)\int\limits_{-\infty}^{\infty}\frac{dq}{2\pi}\, e^{-Dq^2 t}\, e^{iq(x-ut-x')} \\ &= \int\limits_{-\infty}^{\infty} dx'\, K(x-x',t)\, P(x',0)\ , \end{aligned}$$

where

$$K(x,t) = \frac{1}{\sqrt{4\pi Dt}}\, e^{-(x-ut)^2/4Dt} \tag{8.9.36}$$

is the *diffusion kernel*. We now have a recipe for obtaining $P(x,t)$ given the initial conditions $P(x,0)$. If $P(x,0)=\delta(x)$, describing a particle confined to an infinitesimal region about the origin, then $P(x,t)=K(x,t)$ is the probability distribution for finding the particle at x at time t. There are two aspects to $K(x,t)$ which merit comment. The first is that the center of the distribution moves with velocity u. This is due to the presence of the external force. The second is that the standard deviation $\sigma=\sqrt{2Dt}$ is increasing in time, so the distribution is not only shifting its center but it is also getting broader as time evolves. This movement of the center and broadening are what we have called *drift* and *diffusion*, respectively.

Master Equation

Another way to model stochastic processes is via the *master equation*, which was discussed in chapter 3. Recall that if $P_i(t)$ is the probability for a system to be in state $|\, i\,\rangle$ at time t and W_{ij} is the transition rate from state $|\, j\,\rangle$ to state $|\, i\,\rangle$, then

$$\frac{dP_i}{dt} = \sum_j \big(W_{ij}P_j - W_{ji}P_i\big)\ . \tag{8.9.37}$$

Consider a birth-death process in which the states $|\, n\,\rangle$ are labeled by nonnegative integers. Let α_n denote the rate of transitions from $|\, n\,\rangle\to|\, n+1\,\rangle$ and let β_n denote the rate of transitions from $|\, n\,\rangle\to|\, n-1\,\rangle$. The master equation then takes the form[17]

$$\frac{dP_n}{dt} = \alpha_{n-1}P_{n-1} + \beta_{n+1}P_{n+1} - \big(\alpha_n+\beta_n\big)P_n\ . \tag{8.9.38}$$

Let us assume we can write $\alpha_n = K\bar{\alpha}(n/K)$ and $\beta_n = K\bar{\beta}(n/K)$, where $K\gg 1$. We assume the distribution $P_n(t)$ has a time-dependent maximum at $n=K\phi(t)$ and a width proportional to $\sqrt{K}$. We expand relative to this maximum, writing $n\equiv K\phi(t)+\sqrt{K}\,\xi$ and we define $P_n(t)\equiv\Pi(\xi,t)$. We now rewrite the master equation in Equation [MEPab] in terms of $\Pi(\xi,t)$. Since n is an independent variable, we set

$$dn = K\dot{\phi}\, dt + \sqrt{K}\, d\xi \quad\Rightarrow\quad d\xi\big|_n = -\sqrt{K}\,\dot{\phi}\, dt\ . \tag{8.9.39}$$

Therefore

$$\frac{dP_n}{dt} = -\sqrt{K}\,\dot{\phi}\,\frac{\partial\Pi}{\partial\xi} + \frac{\partial\Pi}{\partial t}\ . \tag{8.9.40}$$

Next, we write, for any function f_n,

$$\begin{aligned} f_n &= Kf\big(\phi+K^{-1/2}\xi\big) \\ &= Kf(\phi) + K^{1/2}\,\xi\, f'(\phi) + \frac{1}{2}\,\xi^2\, f''(\phi)+\ldots\ . \end{aligned}$$

Similarly,

$$\begin{aligned} f_{n\pm 1} &= Kf\big(\phi+K^{-1/2}\xi\pm K^{-1}\big) \\ &= Kf(\phi) + K^{1/2}\,\xi\, f'(\phi) \pm f'(\phi) + \frac{1}{2}\,\xi^2\, f''(\phi)+\ldots\ . \end{aligned}$$

Dividing both sides of Equation [MEPab] by $\sqrt{K}$, we have

$$-\frac{\partial \Pi}{\partial \xi}\,\dot{\phi}+K^{-1/2}\,\frac{\partial \Pi}{\partial t}=(\bar{\beta}-\bar{\alpha})\,\frac{\partial \Pi}{\partial \xi}+K^{-1/2}\bigg\{(\bar{\beta}'-\bar{\alpha}')\,\xi\,\frac{\partial \Pi}{\partial \xi}+\frac{1}{2}(\bar{\alpha}+\bar{\beta})\,\frac{\partial^2 \Pi}{\partial \xi^2}+(\bar{\beta}'-\bar{\alpha}')\Pi\bigg\}+\ldots\,. \tag{8.9.41}$$

Equating terms of order K^0 yields the equation

$$\dot{\phi}=f(\phi)\equiv\bar{\alpha}(\phi)-\bar{\beta}(\phi)\,. \tag{8.9.42}$$

Equating terms of order $K^{-1/2}$ yields the Fokker-Planck equation,

$$\frac{\partial \Pi}{\partial t}=-f'\big(\phi(t)\big)\,\frac{\partial}{\partial \xi}\,\big(\xi\,\Pi\big)+\frac{1}{2}g\big(\phi(t)\big)\,\frac{\partial^2 \Phi}{\partial \xi^2}\,, \tag{8.9.43}$$

where $g(\phi)\equiv\bar{\alpha}(\phi)+\bar{\beta}(\phi)$. If in the limit $t\to\infty$, Equation [Dphieqn] evolves to a stable fixed point ϕ^*, then the stationary solution of the Fokker-Planck Equation [FPEPi], $\Pi_{eq}(\xi)=\Pi(\xi,t=\infty)$ must satisfy

$$-f'(\phi^*)\,\frac{\partial}{\partial \xi}\,\big(\xi\,\Pi_{eq}\big)+\frac{1}{2}\,g(\phi^*)\,\frac{\partial^2 \Pi_{eq}}{\partial \xi^2}=0 \quad\Rightarrow\quad \Pi_{eq}(\xi)=\frac{1}{\sqrt{2\pi\sigma^2}}\,e^{-\xi^2/2\sigma^2}\,, \tag{8.9.44}$$

where

$$\sigma^2=-\frac{g(\phi^*)}{2\,f'(\phi^*)}\,. \tag{8.9.45}$$

Now both α and β are rates, hence both are positive and thus $g(\phi)>0$. We see that the condition $\sigma^2>0$, which is necessary for a normalizable equilibrium distribution, requires $f'(\phi^*)<0$, which is saying that the fixed point in Equation [Dphieqn] is stable.