9.2: Real Space Renormalization

As alluded to previously, there are two different classes of renormalization. One class, called *real space renormalization group* (RSRG), eliminates local lattice-based degrees of freedom at each step in the RG process. The second class, called *momentum space renormalization group* (MSRG), is implemented by systematically lowering the cutoff Λ in the wavevector integrals. It turns out that the RSRG process, for reasons we shall see, is uncontrolled, and for 'professional' results one resorts to MSRG. Nevertheless RSRG provides us with perhaps the most vivid and intuitive understanding of what renormalization is all about, so we shall begin there.

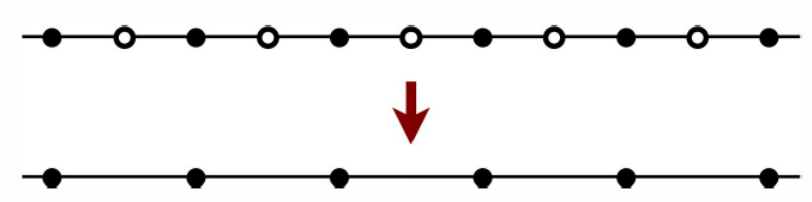

Figure 9.2.1: Real space renormalization of a one-dimensional lattice by 'integrating out' the degrees of freedom on half the lattice sites.

RSRG for the Ising chain

Consider a $d=1$ Ising model with Hamiltonian

$$\hat{H} = -J \sum_n \sigma_n \, \sigma_{n+1} \,. \tag{9.2.1}$$

Our goal is to compute the partition function $Z = \mathsf{Tr}\; e^{-\beta \hat{H}}$. We do this by first tracing over the degrees of freedom on all the odd index sites. We have

$$\begin{aligned}
\sum_{\sigma_{2n+1}} e^{\beta J \sigma_{2n} \sigma_{2n+1}} \, e^{\beta J \sigma_{2n+1} \sigma_{2n+2}} &= e^{\beta J (\sigma_{2n} + \sigma_{2n+1})} + e^{-\beta J (\sigma_{2n} + \sigma_{2n+1})} \\
&= \begin{cases} 2\cosh(\beta J) & \text{if } |\, \sigma_{2n} \, \sigma_{2n+2} \,\rangle = |\!\uparrow\uparrow\rangle \text{ or } |\!\downarrow\downarrow\rangle \\ 2 & \text{if } |\, \sigma_{2n} \, \sigma_{2n+2} \,\rangle = |\!\uparrow\downarrow\rangle \text{ or } |\!\downarrow\uparrow\rangle \end{cases} \\
&\equiv e^{\beta J' \sigma_{2n} \, \sigma_{2n+2}} \, e^{\beta \Delta\varepsilon} \,,
\end{aligned}$$

where

$$\begin{aligned}
e^{\beta J'} \, e^{\beta\Delta\varepsilon} &= 2\, \cosh(2\beta J) \\
e^{-\beta J'} \, e^{\beta\Delta\varepsilon} &= 2 \,,
\end{aligned}$$

from which we obtain

$$\begin{aligned}
e^{2\beta J'} &= \cosh(2\beta J) \\
e^{\beta\Delta\varepsilon} &= 2 \sqrt{\cosh(2\beta J)} \,.
\end{aligned}$$

Thus, if we write our original Hamiltonian as

$$\frac{\hat{H}}{k_{B}T} = \sum_n \big(c - K \sigma_n \, \sigma_{n+1} \big) \,, \tag{9.2.2}$$

where $K = \beta J$, then the RSRG transformation in which we trace out over every other site results in

$$\begin{aligned}
c' &= c - \ln 2 - \frac{1}{2} \ln\cosh(2K) \\
K' &= \frac{1}{2} \ln\cosh(2K) \\
a' &= 2a \,,
\end{aligned}$$

where the last equation describes the change in the lattice constant. The second of these equations may be written

$$\tanh K' = \tanh^2 K \,. \tag{9.2.3}$$

Suppose we perform this procedure n times. We then have, with $\ell_0 \equiv a$,

$$\ell_n = 2^n\, \ell_0 \qquad , \qquad \ln \tanh K_n = 2^n \ln \tanh K_0 \,. \tag{9.2.4}$$

At this point, we can imagine ℓ to be a continuous variable. We can now write down the behavior of the coupling constant K as a function of the microscopic length scale ℓ:

$$\tanh K(\ell) = (\tanh K_0)^{\ell/\ell_0} \,. \tag{9.2.5}$$

Let's define $g \equiv \tanh K$. We then have the *RG flow equation*

$$\beta(g) \equiv \frac{\partial \ln g}{\partial \ln b} = b\, \ln \tanh K_0 = \ln g < 0 \,, \tag{9.2.6}$$

where $b = \ell/\ell_0$. Thus, as ℓ increases, $\ln g$ flows to increasingly negative values, meaning $g \to 0$, which entails $K \to 0$. So as ℓ flows to larger and larger values, the coupling $K(\ell)$ gets smaller and smaller.

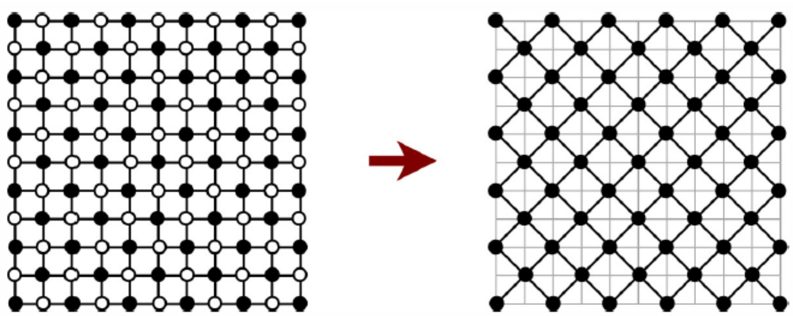

Figure 9.2.2: Real space renormalization of a two-dimensional square lattice.

Two-dimensional square lattice

Consider next a RSRG transformation of the two-dimensional square lattice Ising model. As depicted in Figure 9.2.2, the square lattice is bipartite, consisting of two interpenetrating $\sqrt{2} \times \sqrt{2}$ square sublattices. Let's try to do the same as for the one-dimensional Ising model and trace out over the degrees of freedom of one of the sublattices. To this end, let us trace out over a single site, which has four neighbors on the square lattice, as shown in Figure 9.2.3. We have[2]

$$\begin{aligned}\sum_{\sigma_0} e^{K\sigma_0(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} &= e^{K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} + e^{-K(\sigma_1+\sigma_2+\sigma_3+\sigma_4)} \\ &\equiv e^{K'(\sigma_1\sigma_2+\sigma_2\sigma_3+\sigma_3\sigma_4+\sigma_4\sigma_1)}\, e^{\widetilde{K}'(\sigma_1\sigma_3+\sigma_2\sigma_4)}\, e^{L'\sigma_1\sigma_2\sigma_3\sigma_4}\, e^{\alpha'} \,.\end{aligned}$$

It should be clear that $\widetilde{K}' = K'$, because the spin σ_0 couples to the sum $(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4)$ so there can be no distinction between induced nearest neighbor interactions ($K'\sigma_1\sigma_2$) and induced next-nearest neighbor interactions ($\widetilde{K}'\sigma_1\sigma_3$) at this stage. We setting $|\,\sigma_1\,\sigma_2\,\sigma_3\,\sigma_4\,\rangle$ to $|\!\uparrow\uparrow\uparrow\uparrow\rangle$, $|\,\uparrow\uparrow\uparrow\downarrow\rangle$, and $|\!\uparrow\uparrow\downarrow\downarrow\rangle$, respectively, we obtain the relations

$$\begin{aligned}2\cosh(4K) &= e^{6K'+L'+\alpha'} \\ 2\cosh(2K) &= e^{-L'+\alpha'} \\ 2 &= e^{-2K'+L'+\alpha'} \,.\end{aligned}$$

The solution is

$$\begin{aligned}K' &= \frac{1}{8}\ln\cosh(4K) \\ L' &= \frac{1}{8}\ln\cosh(4K) - \frac{1}{2}\ln\cosh(2K) \\ \alpha' &= 2\cosh^{1/8}(4K)\,\cosh^{1/2}(2K) \,.\end{aligned}$$

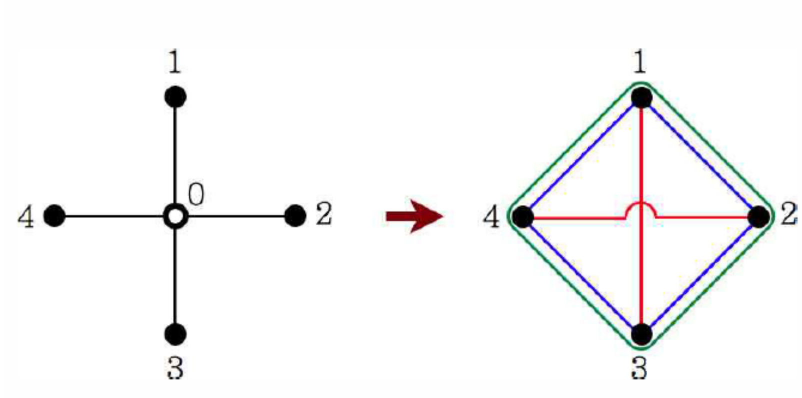

Figure 9.2.3: Tracing out over the central site results in three different interactions: nearest neighbor (blue), next-nearest neighbor (red), and a four-site plaquette term (green).

Note that *new couplings* have been generated at this very first step of the RSRG procedure. It now becomes very difficult to iterate this transformation a second time, since the presence of second neighbor and plaquette couplings $\tilde{K}$ and L means that we cannot exactly integrate out one of the sublattices as before. Still we could imagine iterating this RSRG procedure, if only perturbatively in certain couplings. We see, though, that rather than considering the effect of $\mathcal{R}$ on a single coupling K or the pair (K, α), we should instead consider, if only formally, the iteration of an *infinite set* of *all possible couplings*, $\{K_\alpha\}$. Writing this as a vector $\mathbf{K}$, we can write the RSRG transformation in the form

$$\mathbf{K}' = \boldmath{\mathcal{R}}_b(\mathbf{K}) \,. \tag{9.2.7}$$

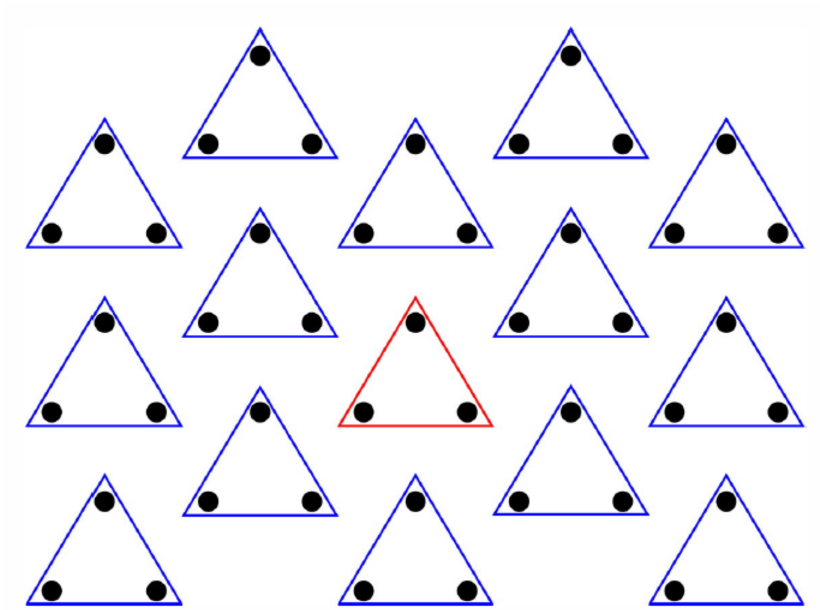

Figure 9.2.4: Spin blocking on the triangular lattice.