

## 5.7: The Ideal Fermi Gas

### General formulation for noninteracting systems

Recall that the grand partition function for noninteracting bosons is given by

$$\Xi = \prod_{\alpha} \left( \sum_{n_{\alpha}=0}^{\infty} e^{\beta(\mu - \varepsilon_{\alpha})n_{\alpha}} \right) = \prod_{\alpha} \left( 1 - e^{\beta(\mu - \varepsilon_{\alpha})} \right)^{-1}, \quad (5.7.1)$$

In order for the sum to converge to the RHS above, we must have  $\mu < \varepsilon_{\alpha}$  for all single-particle states  $|\alpha\rangle$ . The density of particles is then

$$n(T, \mu) = -\frac{1}{V} \left( \frac{\partial \Omega}{\partial \mu} \right)_{T,V} = \frac{1}{V} \sum_{\alpha} \frac{1}{e^{\beta(\varepsilon_{\alpha} - \mu)} - 1} = \int_{\varepsilon_0}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{\beta(\varepsilon - \mu)} - 1}, \quad (5.7.2)$$

where  $g(\varepsilon) = \frac{1}{V} \sum_{\alpha} \delta(\varepsilon - \varepsilon_{\alpha})$  is the density of single particle states per unit volume. We assume that  $g(\varepsilon) = 0$  for  $\varepsilon < \varepsilon_0$ ; typically  $\varepsilon_0 = 0$ , as is the case for any dispersion of the form  $\varepsilon(\mathbf{k}) = A|\mathbf{k}|^r$ , for example. However, in the presence of a magnetic field, we could have  $\varepsilon(\mathbf{k}, \sigma) = A|\mathbf{k}|^r - g\mu_0 H\sigma$ , in which case  $\varepsilon_0 = -g\mu_0 |H|$ .

Clearly  $n(T, \mu)$  is an increasing function of both  $T$  and  $\mu$ . At fixed  $T$ , the maximum possible value for  $n(T, \mu)$ , called the *critical density*  $n_c(T)$ , is achieved for  $\mu = \varepsilon_0$ ,

$$n_c(T) = \int_{\varepsilon_0}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{\beta(\varepsilon - \varepsilon_0)} - 1}. \quad (5.7.3)$$

The above integral converges provided  $g(\varepsilon_0) = 0$ , assuming  $g(\varepsilon)$  is continuous<sup>5</sup>. If  $g(\varepsilon_0) > 0$ , the integral diverges, and  $n_c(T) = \infty$ . In this latter case, one can always invert the equation for  $n(T, \mu)$  to obtain the chemical potential  $\mu(T, n)$ . In the former case, where the  $n_c(T)$  is finite, we have a problem – what happens if  $n > n_c(T)$ ?

In the former case, where  $n_c(T)$  is finite, we can equivalently restate the problem in terms of a *critical temperature*  $T_c(n)$ , defined by the equation  $n_c(T_c) = n$ . For  $T < T_c$ , we apparently can no longer invert to obtain  $\mu(T, n)$ , so clearly something has gone wrong. The remedy is to recognize that the single particle energy levels are discrete, and separate out the contribution from the lowest energy state  $\varepsilon_0$ . we write  $[n(T, \mu)] = \overbrace{[n(T, \mu)]}^{n_0} + \overbrace{[n'(T, \mu)]}^{n'}$  where  $g_0$  is the degeneracy of the single particle state with energy  $\varepsilon_0$ . We assume that  $n_0$  is finite, which means that  $N_0 = Vn_0$  is extensive. We say that the particles have *condensed* into the state with energy  $\varepsilon_0$ . The quantity  $n_0$  is the *condensate density*. The remaining particles, with density  $n'$ , are said to comprise the *overcondensate*. With the total density  $n$  fixed, we have  $n = n_0 + n'$ . Note that  $n_0$  finite means that  $\mu$  is infinitesimally close to  $\varepsilon_0$ :

$$\mu = \varepsilon_0 - k_B T \ln \left( 1 + \frac{g_0}{Vn_0} \right) \approx \varepsilon_0 - \frac{g_0 k_B T}{Vn_0}. \quad (5.7.4)$$

Note also that if  $\varepsilon_0 - \mu$  is finite, then  $n_0 \propto V^{-1}$  is infinitesimal.

Thus, for  $T < T_c(n)$ , we have  $\mu = \varepsilon_0$  with  $n_0 > 0$ , and

$$n(T, n_0) = n_0 + \int_{\varepsilon_0}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{(\varepsilon - \varepsilon_0)/k_B T} - 1}. \quad (5.7.5)$$

For  $T > T_c(n)$ , we have  $n_0 = 0$  and

$$n(T, \mu) = \int_{\varepsilon_0}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{(\varepsilon - \mu)/k_B T} - 1}. \quad (5.7.6)$$

The equation for  $T_c(n)$  is

$$n = \int_{\varepsilon_0}^{\infty} d\varepsilon \frac{g(\varepsilon)}{e^{(\varepsilon - \varepsilon_0)/k_B T_c} - 1} . \quad (5.7.7)$$

For another take on ideal Bose gas condensation see the appendix in §10.

## Ballistic dispersion

We already derived, in §3.3, expressions for  $n(T, z)$  and  $p(T, z)$  for the ideal Bose gas (IBG) with ballistic dispersion  $\varepsilon(\mathbf{p}) = \mathbf{p}^2/2m$ . We found  $n(T, z) = \frac{1}{\lambda^d} \text{Li}_{-d}(\lambda^d z)$

and  $p(T, z) = \frac{k_B T}{\lambda^d} \text{Li}_{-d+1}(\lambda^d z)$  where  $g$  is the internal (spin) degeneracy of each single particle energy level. Here  $z = e^{\mu/k_B T}$  is the fugacity and

$$\text{Li}_s(z) = \sum_{m=1}^{\infty} \frac{z^m}{m^s} \quad (5.7.8)$$

is the polylogarithm function. For bosons with a spectrum bounded below by  $\varepsilon_0 = 0$ , the fugacity takes values on the interval  $z \in [0, 1]$ .

Clearly  $n(T, z)$  is an increasing function of  $z$  for fixed  $T$ . In Figure [zeta] we plot the function  $\text{Li}_s(z)$  versus  $z$  for three different values of  $s$ .

We note that the maximum value  $\text{Li}_s(z=1)$  is finite if  $s > 1$ . Thus, for  $d > 2$ , there is a maximum density  $n_{\text{max}}(T) = \frac{1}{\lambda^d} \text{Li}_{-d}(1)$  which is an increasing function of temperature  $T$ . Put another way, if we fix the density  $n$ , then there is a critical temperature  $T_c$  below which there is no solution to the equation  $n = n(T, z)$ . The critical temperature  $T_c(n)$  is then determined by the relation

$$n = \frac{1}{\lambda^d} \zeta\left(\frac{d}{2}\right) \left(\frac{mk_B T_c}{2\pi\hbar^2}\right)^{d/2} \implies k_B T_c = \frac{2\pi\hbar^2}{m} \left(\frac{n}{\zeta\left(\frac{d}{2}\right)}\right)^{2/d} . \quad (5.7.9)$$

What happens for  $T < T_c$ ?

[zeta] The polylogarithm function  $\text{Li}_s(z)$  versus  $z$  for  $s = \frac{1}{2}$ ,  $s = \frac{3}{2}$ , and  $s = \frac{5}{2}$ . Note that  $\text{Li}_s(1) = \zeta(s)$  diverges for  $s \leq 1$ .

[zeta] The polylogarithm function  $\text{Li}_s(z)$  versus  $z$  for  $s = \frac{1}{2}$ ,  $s = \frac{3}{2}$ , and  $s = \frac{5}{2}$ . Note that  $\text{Li}_s(1) = \zeta(s)$  diverges for  $s \leq 1$ .

As shown above in §7, we must separate out the contribution from the lowest energy single particle mode, which for ballistic dispersion lies at  $\varepsilon_0 = 0$ . Thus writing

$$n = \frac{1}{V} \frac{1}{z^{-1} - 1} + \frac{1}{V} \sum_{\substack{\alpha \\ (\varepsilon_\alpha > 0)}} \frac{1}{z^{-1} e^{\varepsilon_\alpha/k_B T} - 1} , \quad (5.7.10)$$

where we have taken  $g = 1$ . Now  $V^{-1}$  is of course very small, since  $V$  is thermodynamically large, but if  $\mu \rightarrow 0$  then  $z^{-1} - 1$  is also very small and their ratio can be finite, as we have seen. Indeed, if the density of  $\mathbf{k} = 0$  bosons  $n_0$  is finite, then their total number  $N_0$  satisfies

$$N_0 = V n_0 = \frac{1}{z^{-1} - 1} \implies z = \frac{1}{1 + N_0^{-1}} . \quad (5.7.11)$$

The chemical potential is then

$$\mu = k_B T \ln z = -k_B T \ln(1 + N_0^{-1}) \approx -\frac{k_B T}{N_0} \rightarrow 0^- . \quad (5.7.12)$$

In other words, the chemical potential is infinitesimally negative, because  $N_0$  is assumed to be thermodynamically large.

According to Equation [Oqsm], the contribution to the pressure from the  $\mathbf{k} = 0$  states is

$$p_0 = -\frac{k_B T}{V} \ln(1 - z) = \frac{k_B T}{V} \ln(1 + N_0^{-1}) \rightarrow 0^+ . \quad (5.7.13)$$

So the  $\mathbf{k} = 0$  bosons, which we identify as the condensate, contribute nothing to the pressure.

Having separated out the  $\mathbf{k} = 0$  mode, we can now replace the remaining sum over  $\alpha$  by the usual integral over  $\mathbf{k}$ . We then have

$$T < T_c \quad : \quad n = n_0 + g \zeta\left(\frac{d}{2}\right) \lambda_T^{-d}$$

$$p = g \zeta\left(\frac{d}{2}+1\right) k_B T \lambda_T^{-d}$$

and  $\left[ \begin{array}{l} T > T_c \\ T < T_c \end{array} \right] \quad n = \left[ \begin{array}{l} n_0 + g \zeta\left(\frac{d}{2}\right) \lambda_T^{-d} \\ g \zeta\left(\frac{d}{2}+1\right) \lambda_T^{-d} \end{array} \right]$

The condensate fraction  $n_0/n$  is unity at  $T = 0$ , when all particles are in the condensate with  $\mathbf{k} = 0$ , and decreases with increasing  $T$  until  $T = T_c$ , at which point it vanishes identically. Explicitly, we have

$$\frac{n_0(T)}{n} = 1 - \frac{g \zeta\left(\frac{d}{2}\right)}{n \lambda_T^{-d}} = 1 - \left( \frac{T}{T_c(n)} \right)^{d/2}. \quad (5.7.14)$$

Let us compute the internal energy  $E$  for the ideal Bose gas. We have


$$\frac{\partial}{\partial \beta} (\beta \Omega) = \Omega + \beta \frac{\partial \Omega}{\partial \beta} = \Omega - T \frac{\partial \Omega}{\partial T} = \Omega + TS \quad (5.7.15)$$

and therefore

$$\begin{aligned} E = \Omega + TS + \mu N &= \mu N + \frac{\partial}{\partial \beta} (\beta \Omega) \\ &= V \left( \mu n - \frac{\partial}{\partial \beta} (\beta p) \right) \\ &= \frac{1}{2} d g V k_B T \lambda_T^{-d} Li_{\frac{d}{2}+1}(z). \end{aligned}$$

This expression is valid at all temperatures, both above and below  $T_c$ . Note that the condensate particles do not contribute to  $E$ , because the  $\mathbf{k} = 0$  condensate particles carry no energy.

We now investigate the heat capacity  $C_{V,N} = \left( \frac{\partial E}{\partial T} \right)_{V,N}$ . Since we have been working in the GCE, it is very important to note that  $N$  is held constant when computing  $C_{V,N}$ . We'll also restrict our attention to the case  $d = 3$  since the ideal Bose gas does not condense at finite  $T$  for  $d \leq 2$  and  $d > 3$  is unphysical. While we're at it, we'll also set  $g = 1$ .

 Molar heat capacity of the ideal Bose gas (units of  $R$ ). Note the cusp at  $T = T_c$ .

[ibgc] Molar heat capacity of the ideal Bose gas (units of  $R$ ). Note the cusp at  $T = T_c$ .

The number of particles is

$$N = \begin{cases} N_0 + \zeta\left(\frac{3}{2}\right) V \lambda_T^{-3} & (T < T_c) \\ V \lambda_T^{-3} Li_{3/2}(z) & (T > T_c) \end{cases}, \quad (5.7.16)$$

and the energy is

$$E = \frac{3}{2} k_B T \frac{V}{\lambda_T^3} Li_{5/2}(z). \quad (5.7.17)$$

For  $T < T_c$ , we have  $z = 1$  and

$$C_{V,N} = \left( \frac{\partial E}{\partial T} \right)_{V,N} = \frac{15}{4} \zeta\left(\frac{5}{2}\right) k_B \frac{V}{\lambda_T^3}. \quad (5.7.18)$$

The molar heat capacity is therefore

$$c_{V,N}(T, n) = N_A \cdot \frac{C_{V,N}}{N} = \frac{15}{4} \zeta\left(\frac{5}{2}\right) R \cdot (n \lambda_T^3)^{-1}. \quad (5.7.19)$$

For  $T > T_c$ , we have

$$dE|_V = \frac{15}{4} k_B T Li_{5/2}(z) \frac{V}{\lambda_T^3} \cdot \frac{dT}{T} + \frac{3}{2} k_B T Li_{3/2}(z) \frac{V}{\lambda_T^3} \cdot \frac{dz}{z}, \quad (5.7.20)$$

where we have invoked Equation [zetarec]. Taking the differential of  $N$ , we have

$$dN|_V = \frac{3}{2} Li_{3/2}(z) \frac{V}{\lambda_T^3} \cdot \frac{dT}{T} + Li_{1/2}(z) \frac{V}{\lambda_T^3} \cdot \frac{dz}{z}. \quad (5.7.21)$$

We set  $dN = 0$ , which fixes  $dz$  in terms of  $dT$ , resulting in

$$c_{V,N}(T, z) = \frac{3}{2} R \cdot \left[ \frac{\frac{5}{2} Li_{5/2}(z)}{Li_{3/2}(z)} - \frac{\frac{3}{2} Li_{3/2}(z)}{Li_{1/2}(z)} \right]. \quad (5.7.22)$$

To obtain  $c_{V,N}(T, n)$ , we must invert the relation

$$n(T, z) = \lambda_T^{-3} Li_{3/2}(z) \quad (5.7.23)$$

in order to obtain  $z(T, n)$ , and then insert this into Equation [ibgcg]. The results are shown in Figure [ibgcv]. There are several noteworthy features of this plot. First of all, by dimensional analysis the function  $c_{V,N}(T, n)$  is  $R$  times a function of the dimensionless ratio  $T/T_c(n) \propto T n^{-2/3}$ . Second, the high temperature limit is  $\frac{3}{2}R$ , which is the classical value. Finally, there is a cusp at  $T = T_c(n)$ .

For another example, see §11.

## Isotherms for the ideal Bose gas

Let  $a$  be some length scale and define

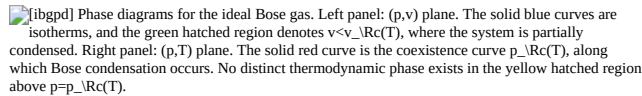
$$v_a = a^3, \quad p_a = \frac{2\pi\hbar^2}{ma^5}, \quad T_a = \frac{2\pi\hbar^2}{ma^2 k_B} \quad (5.7.24)$$

Then we have

$$\frac{v_a}{v} = \left( \frac{T}{T_a} \right)^{3/2} Li_{3/2}(z) + v_a n_0$$

$$\frac{p}{p_a} = \left( \frac{T}{T_a} \right)^{5/2} Li_{5/2}(z),$$

where  $v = V/N$  is the volume per particle<sup>7</sup> and  $n_0$  is the condensate number density;  $n_0$  vanishes for  $T \geq T_c$ , where  $z = 1$ . One identifies a critical volume  $v_c(T)$  by setting  $z = 1$  and  $n_0 = 0$ , leading to  $v_c(T) = v_a (T/T_a)^{3/2}$ . For  $v < v_c(T)$ , we set  $z = 1$  in Equation [BG1a] to find a relation between  $v$ ,  $T$ , and  $n_0$ . For  $v > v_c(T)$ , we set  $n_0 = 0$  in Equation [BG1a] to relate  $v$ ,  $T$ , and  $z$ . Note that the pressure is independent of volume for  $T < T_c$ . The isotherms in the  $(p, v)$  plane are then flat for  $v < v_c$ . This resembles the coexistence region familiar from our study of the thermodynamics of the liquid-gas transition. The situation is depicted in Fig. [ibgpd]. In the  $(T, p)$  plane, we identify  $p_c(T) = p_a (T/T_a)^{5/2}$  as the critical temperature at which condensation starts to occur.



[ibgpd] Phase diagrams for the ideal Bose gas. Left panel:  $(p, v)$  plane. The solid blue curves are isotherms, and the green hatched region denotes  $v < v_c(T)$ , where the system is partially condensed. Right panel:  $(p, T)$  plane. The solid red curve is the coexistence curve  $p_c(T)$ , along which Bose condensation occurs. No distinct thermodynamic phase exists in the yellow hatched region above  $p = p_c(T)$ .

Recall the Gibbs-Duhem equation,

$$d\mu = -s dT + v dp. \quad (5.7.25)$$

Along a coexistence curve, we have the Clausius-Clapeyron relation,

$$\left(\frac{dp}{dT}\right)_{\text{coex}} = \frac{s_2 - s_1}{v_2 - v_1} = \frac{\ell}{T \Delta v}, \quad (5.7.26)$$

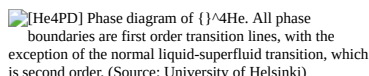
where  $\ell = T(s_2 - s_1)$  is the latent heat per mole, and  $\Delta v = v_2 - v_1$ . For ideal gas Bose condensation, the coexistence curve resembles the red curve in the right hand panel of Figure [ibgpd]. There is no meaning to the shaded region where  $p > p_c(T)$ . Nevertheless, it is tempting to associate the curve  $p = p_c(T)$  with the coexistence of the  $\mathbf{k} = 0$  condensate and the remaining uncondensed ( $\mathbf{k} \neq 0$ ) bosons<sup>3</sup>.

The entropy in the coexistence region is given by

$$s = -\frac{1}{N} \left( \frac{\partial \Omega}{\partial T} \right)_v = \frac{5}{2} \zeta\left(\frac{5}{2}\right) k_B v \lambda_T^{-3} = \frac{\frac{5}{2} \zeta\left(\frac{5}{2}\right)}{\zeta\left(\frac{3}{2}\right)} k_B \left(1 - \frac{n_0}{n}\right). \quad (5.7.27)$$

All the entropy is thus carried by the uncondensed bosons, and the condensate carries zero entropy. The Clausius-Clapeyron relation can then be interpreted as describing a phase equilibrium between the condensate, for which  $s_0 = v_0 = 0$ , and the uncondensed bosons, for which  $s' = s(T)$  and  $v' = v_c(T)$ . So this identification forces us to conclude that the specific volume of the condensate is zero. This is certainly false in an interacting Bose gas!

While one can identify, by analogy, a 'latent heat'  $\ell = T \Delta s = T s$  in the Clapeyron equation, it is important to understand that there is no distinct thermodynamic phase associated with the region  $p > p_c(T)$ . Ideal Bose gas condensation is a second order transition, and not a first order transition.

 [He4PD] Phase diagram of  ${}^4\text{He}$ . All phase boundaries are first order transition lines, with the exception of the normal liquid-superfluid transition, which is second order. (Source: University of Helsinki)

[He4PD] Phase diagram of  ${}^4\text{He}$ . All phase boundaries are first order transition lines, with the exception of the normal liquid-superfluid transition, which is second order. (Source: University of Helsinki)

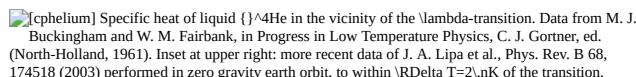
## The $\lambda$ -transition in Liquid ${}^4\text{He}$

Helium has two stable isotopes.  ${}^4\text{He}$  is a boson, consisting of two protons, two neutrons, and two electrons (hence an even number of fermions).  ${}^3\text{He}$  is a fermion, with one less neutron than  ${}^4\text{He}$ . Each  ${}^4\text{He}$  atom can be regarded as a tiny hard sphere of mass  $m = 6.65 \times 10^{-24}$  g and diameter  $a = 2.65$  Å. A sketch of the phase diagram is shown in Figure [He4PD]. At atmospheric pressure, Helium liquefies at  $T_l = 4.2$  K. The gas-liquid transition is first order, as usual. However, as one continues to cool, a second transition sets in at  $T = T_\lambda = 2.17$  K (at  $p = 1$  atm). The  $\lambda$ -transition, so named for the  $\lambda$ -shaped anomaly in the specific heat in the vicinity of the transition, as shown in Figure [cphelium], is continuous (second order).

If we pretend that  ${}^4\text{He}$  is a noninteracting Bose gas, then from the density of the liquid  $n = 2.2 \times 10^{22} \text{ cm}^{-3}$ , we obtain a Bose-Einstein condensation temperature  $T_c = \frac{2\pi\hbar^2}{m} \left(n / \zeta\left(\frac{3}{2}\right)\right)^{2/3} = 3.16$  K, which is in the right ballpark. The specific heat  $C_p(T)$  is found to be singular at  $T = T_\lambda$ , with

$$C_p(T) = A |T - T_\lambda(p)|^{-\alpha}. \quad (5.7.28)$$

$\alpha$  is an example of a *critical exponent*. We shall study the physics of critical phenomena later on in this course. For now, note that a cusp singularity of the type found in Figure [ibgcgv] corresponds to  $\alpha = -1$ . The behavior of  $C_p(T)$  in  ${}^4\text{He}$  is very nearly logarithmic in  $|T - T_\lambda|$ . In fact, both theory (renormalization group on the  $O(2)$  model) and experiment concur that  $\alpha$  is almost zero but in fact slightly negative, with  $\alpha = -0.0127 \pm 0.0003$  in the best experiments (Lipa, 2003). The  $\lambda$  transition is most definitely *not* an ideal Bose gas condensation. Theoretically, in the parlance of critical phenomena, IBG condensation and the  $\lambda$ -transition in  ${}^4\text{He}$  lie in different *universality classes*<sup>9</sup>. Unlike the IBG, the condensed phase in  ${}^4\text{He}$  is a distinct thermodynamic phase, known as a *superfluid*.

 [cphelium] Specific heat of liquid  ${}^4\text{He}$  in the vicinity of the  $\lambda$ -transition. Data from M. J. Buckingham and W. M. Fairbank, in Progress in Low Temperature Physics, C. J. Gortner, ed. (North-Holland, 1961). Inset at upper right: more recent data of J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003) performed in zero gravity earth orbit, to within  $\Delta T = 2$  nK of the transition.

[cphelium] Specific heat of liquid  ${}^4\text{He}$  in the vicinity of the  $\lambda$ -transition. Data from M. J. Buckingham and W. M. Fairbank, in Progress in Low Temperature Physics, C. J. Gortner, ed. (North-Holland, 1961). Inset at upper right: more recent data of J. A. Lipa et al., Phys. Rev. B 68, 174518 (2003) performed in zero gravity earth orbit, to within  $\Delta T = 2$  nK of the transition.


Note that  $C_p(T < T_c)$  for the IBG is not even defined, since for  $T < T_c$  we have  $p = p(T)$  and therefore  $dp = 0$  requires  $dT = 0$ .

## Fountain effect in superfluid $^4\text{He}$

At temperatures  $T < T_\lambda$ , liquid  $^4\text{He}$  has a superfluid component which is a type of Bose condensate. In fact, there is an important difference between condensate fraction  $N_{\mathbf{k}=0}/N$  and superfluid density, which is denoted by the symbol  $\rho_s$ . In  $^4\text{He}$ , for example, at  $T = 0$  the condensate fraction is only about 8%, while the superfluid fraction  $\rho_s/\rho = 1$ . The distinction between  $N_0$  and  $\rho_s$  is very interesting but lies beyond the scope of this course.

One aspect of the superfluid state is its complete absence of viscosity. For this reason, superfluids can flow through tiny cracks called *microleaks* that will not pass normal fluid. Consider then a porous plug which permits the passage of superfluid but not of normal fluid. The key feature of the superfluid component is that it has zero energy density. Therefore even though there is a transfer of particles across the plug, there is no energy exchange, and therefore a temperature gradient across the plug can be maintained<sup>10</sup>.

The elementary excitations in the superfluid state are sound waves called *phonons*. They are compressional waves, just like longitudinal phonons in a solid, but here in a liquid. Their dispersion is acoustic, given by  $\omega(k) = ck$  where  $c = 238 \text{ m/s}$ .<sup>11</sup> They have no internal degrees of freedom, hence  $g = 1$ . Like phonons in a solid, the phonons in liquid helium are not conserved. Hence their chemical potential vanishes and these excitations are described by photon statistics. We can now compute the height difference  $\Delta h$  in a U-tube experiment.

 [fountain] The fountain effect. In each case, a temperature gradient is maintained across a porous plug through which only superfluid can flow. This results in a pressure gradient which can result in a fountain or an elevated column in a U-tube.

[fountain] The fountain effect. In each case, a temperature gradient is maintained across a porous plug through which only superfluid can flow. This results in a pressure gradient which can result in a fountain or an elevated column in a U-tube.

Clearly  $\Delta h = \Delta p / \rho g$ , so we must find  $p(T)$  for the helium. In the grand canonical ensemble, we have

$$\begin{aligned} p &= -\Omega/V = -k_B T \int \frac{d^3k}{(2\pi)^3} \ln(1 - e^{-\hbar ck/k_B T}) \\ &= -\frac{(k_B T)^4}{(\hbar c)^3} \frac{4\pi}{8\pi^3} \int_0^\infty du u^2 \ln(1 - e^{-u}) = \frac{\pi^2}{90} \frac{(k_B T)^4}{(\hbar c)^3}. \end{aligned}$$

Let's assume  $T = 1 \text{ K}$ . We'll need the density of liquid helium,  $\rho = 148 \text{ kg/m}^3$ .

$$\begin{aligned} \frac{dh}{dT} &= \frac{2\pi^2}{45} \left( \frac{k_B T}{\hbar c} \right)^3 \frac{k_B}{\rho g} \\ &= \frac{2\pi^2}{45} \left( \frac{(1.38 \times 10^{-23} \text{ J/K})(1 \text{ K})}{(1.055 \times 10^{-34} \text{ J} \cdot \text{s})(238 \text{ m/s})} \right)^3 \times \frac{(1.38 \times 10^{-23} \text{ J/K})}{(148 \text{ kg/m}^3)(9.8 \text{ m/s}^2)} \simeq 32 \text{ cm/K}, \end{aligned}$$

a very noticeable effect!

## Bose condensation in optical traps

The 2001 Nobel Prize in Physics was awarded to Weiman, Cornell, and Ketterle for the experimental observation of Bose condensation in dilute atomic gases. The experimental techniques required to trap and cool such systems are a true *tour de force*, and we shall not enter into a discussion of the details here<sup>12</sup>.

The optical trapping of neutral bosonic atoms, such as  $^{87}\text{Rb}$ , results in a confining potential  $V(\mathbf{r})$  which is quadratic in the atomic positions. Thus, the single particle Hamiltonian for a given atom is written

$$\hat{H} = -\frac{\hbar^2}{2m} \nabla^2 + \frac{1}{2} m (\omega_1^2 x^2 + \omega_2^2 y^2 + \omega_3^2 z^2), \quad (5.7.29)$$

where  $\omega_{1,2,3}$  are the angular frequencies of the trap. This is an anisotropic three-dimensional harmonic oscillator, the solution of which is separable into a product of one-dimensional harmonic oscillator wavefunctions. The eigenspectrum is then given by a sum of one-dimensional spectra, viz.

$$E_{n_1, n_2, n_3} = \left(n_1 + \frac{1}{2}\right) \hbar \omega_1 + \left(n_2 + \frac{1}{2}\right) \hbar \omega_2 + \left(n_3 + \frac{1}{2}\right) \hbar \omega_3. \quad (5.7.30)$$

According to Equation [Ntot], the number of particles in the system is

$$N = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{n_3=0}^{\infty} \left[ y^{-1} e^{n_1 \hbar \omega_1 / k_B T} e^{n_2 \hbar \omega_2 / k_B T} e^{n_3 \hbar \omega_3 / k_B T} - 1 \right]^{-1} \\ = \sum_{k=1}^{\infty} y^k \left( \frac{1}{1 - e^{-k \hbar \omega_1 / k_B T}} \right) \left( \frac{1}{1 - e^{-k \hbar \omega_2 / k_B T}} \right) \left( \frac{1}{1 - e^{-k \hbar \omega_3 / k_B T}} \right),$$

where we've defined

$$y \equiv e^{\mu / k_B T} e^{-\hbar \omega_1 / 2 k_B T} e^{-\hbar \omega_2 / 2 k_B T} e^{-\hbar \omega_3 / 2 k_B T}. \quad (5.7.31)$$

Note that  $y \in [0, 1]$ .

Let's assume that the trap is approximately anisotropic, which entails that the frequency ratios  $\omega_1 / \omega_2$  are all numbers on the order of one. Let us further assume that  $k_B T \gg \hbar \omega_{1,2,3}$ . Then

$$\frac{1}{1 - e^{-k \hbar \omega_j / k_B T}} \approx \begin{cases} \frac{k_B T}{k \hbar \omega_j} & k \propto k^*(T) \\ 1 & k > k^*(T) \end{cases} \quad (5.7.32)$$

where  $k^*(T) = k_B T / \hbar \bar{\omega} \gg 1$ , with

$$\bar{\omega} = (\omega_1 \omega_2 \omega_3)^{1/3}. \quad (5.7.33)$$

We then have

$$N(T, y) \approx \frac{y^{k^*+1}}{1-y} + \left( \frac{k_B T}{\hbar \bar{\omega}} \right)^3 \sum_{k=1}^{k^*} \frac{y^k}{k^3}, \quad (5.7.34)$$

where the first term on the RHS is due to  $k > k^*$  and the second term from  $k \leq k^*$  in the previous sum. Since  $k^* \gg 1$  and since the sum of inverse cubes is convergent, we may safely extend the limit on the above sum to infinity. To help make more sense of the first term, write  $N_0 = (y^{-1} - 1)^{-1}$  for the number of particles in the  $(n_1, n_2, n_3) = (0, 0, 0)$  state. Then

$$y = \frac{N_0}{N_0 + 1}. \quad (5.7.35)$$

This is true always. The issue *vis-a-vis* Bose-Einstein condensation is whether  $N_0 \gg 1$ . At any rate, we now see that we can write

$$N \approx N_0 (1 + N_0^{-1})^{-k^*} + \left( \frac{k_B T}{\hbar \bar{\omega}} \right)^3 Li_3(y). \quad (5.7.36)$$

As for the first term, we have

$$N_0 (1 + N_0^{-1})^{-k^*} = \begin{cases} 0 & N_0 \ll k^* \\ N_0 & N_0 \gg k^* \end{cases} \quad (5.7.37)$$

Thus, as in the case of IBG condensation of ballistic particles, we identify the critical temperature by the condition  $y = N_0 / (N_0 + 1) \approx 1$ , and we have  $\bar{\nu} = \bar{\omega} / 2\pi$ . We see that  $k_B T_c \gg \hbar \bar{\omega}$  if the number of particles in the trap is large:  $N \gg 1$ . In this regime, we have

$$T < T_c : \quad N = N_0 + \zeta(3) \left( \frac{k_B T}{\hbar \bar{\omega}} \right)^3 \\ T > T_c : \quad N = \left( \frac{k_B T}{\hbar \bar{\omega}} \right)^3 Li_3(y).$$

It is interesting to note that BEC can also occur in two-dimensional traps, which is to say traps which are very anisotropic, with oblate equipotential surfaces  $V(\mathbf{r}) = V_0$ . This happens when  $\hbar \omega_3 \gg k_B T \gg \omega_{1,2}$ . We then have

$$T_c^{(d=2)} = \frac{\hbar \bar{\omega}}{k_B} \cdot \left( \frac{6N}{\pi^2} \right)^{1/2} \quad (5.7.38)$$

with  $\bar{\omega} = (\omega_1 \omega_2)^{1/2}$ . The particle number then obeys a set of equations like those in eqns. [trapab], *mutatis mutandis*<sup>13</sup>.

For extremely prolate traps, with  $\omega_3 \ll \omega_{1,2}$ , the situation is different because  $Li_1(y)$  diverges for  $y = 1$ . We then have

$$N = N_0 + \frac{k_B T}{\hbar \omega_3} \ln(1 + N_0) . \quad (5.7.39)$$

Here we have simply replaced  $y$  by the equivalent expression  $N_0/(N_0 + 1)$ . If our criterion for condensation is that  $N_0 = \alpha N$ , where  $\alpha$  is some fractional value, then we have

$$T_c(\alpha) = (1 - \alpha) \frac{\hbar \omega_3}{k_B} \cdot \frac{N}{\ln N} . \quad (5.7.40)$$

---

This page titled [5.7: The Ideal Fermi Gas](#) is shared under a [CC BY-NC-SA](#) license and was authored, remixed, and/or curated by [Daniel Arovas](#).