5.S: Summary

References

- F. Reif, *Fundamentals of Statistical and Thermal Physics* (McGraw-Hill, 1987) This has been perhaps the most popular undergraduate text since it first appeared in 1967, and with good reason.
- A. H. Carter, *Classical and Statistical Thermodynamics* (Benjamin Cummings, 2000) A very relaxed treatment appropriate for undergraduate physics majors.
- D. V. Schroeder, *An Introduction to Thermal Physics* (Addison-Wesley, 2000) This is the best undergraduate thermodynamics book I've come across, but only 40% of the book treats statistical mechanics.
- C. Kittel, *Elementary Statistical Physics* (Dover, 2004) Remarkably crisp, though dated, this text is organized as a series of brief discussions of key concepts and examples. Published by Dover, so you can't beat the price.
- R. K. Pathria, *Statistical Mechanics* (2^{nd} edition, Butterworth-Heinemann, 1996) This popular graduate level text contains many detailed derivations which are helpful for the student.
- M. Plischke and B. Bergersen, *Equilibrium Statistical Physics* (3^{rd} edition, World Scientific, 2006) An excellent graduate level text. Less insightful than Kardar but still a good modern treatment of the subject. Good discussion of mean field theory.
- E. M. Lifshitz and L. P. Pitaevskii, *Statistical Physics* (part I, 3^{rd} edition, Pergamon, 1980) This is volume 5 in the famous Landau and Lifshitz *Course of Theoretical Physics*. Though dated, it still contains a wealth of information and physical insight.

Summary

\def\tpar{t\ns_\parallel} \def\mhat{\hat\Bm} \parindent=0pt \renewcommand*\rmdefault{ppl}\normalfont\upshape \physgreek \font\seventeenbf=cmbx10 scaled \magstep3 \setcounter{section}{4} \section{Quantum Statistics : Summary} $\bullet$ {\it Second-quantized Hamiltonians\/}: A noninteracting quantum system is described by a Hamiltonian $\HH=\sum_\alpha\ve\ns_\alpha\,\Hn\ns_\alpha$, where $\ve\ns_\alpha$ is the energy eigenvalue for the single particle state $\psi\ns_\alpha$ (possibly degenerate), and $\Hn\ns_\alpha$ is the number operator. Many-body eigenstates $\tket{\Vn}$ are labeled by the set of occupancies $\Vn=\{n\ns_\alpha\}$, with $\Hn\ns_\alpha\,\tket{\Vn}=n\ns_\alpha\tket{\Vn}$. Thus, $\HH\,\tket{\Vn}=E\ns_\Vn\>\tket{\Vn}$, where $E\ns_\Vn=\sum_\alpha n\ns_\alpha\,\ve\ns_\alpha$. $\bullet$ {\it Bosons and fermions\/}: The allowed values for $n\ns_\alpha$ are $n\ns_\alpha\in\{0,1,2,\ldots,\infty\}$ for bosons and $n\ns_\alpha\in\{0,1\}$ for fermions. $\bullet$ {\it Grand canonical ensemble\/}: Because of the constraint $\sum_\alpha n\ns_\alpha=N$, the ordinary canonical ensemble is inconvenient. Rather, we use the grand canonical ensemble, in which case

$$\Omega(T,V,\mu)=\pm k_{B}T\sum_{\alpha}\ln\Big(1\mp e^{-(\varepsilon_\alpha-\mu)/k_BT}\Big)\ ,$$

where the upper sign corresponds to bosons and the lower sign to fermions. The average number of particles occupying the single particle state $\psi\ns_\alpha$ is then

$$\langle\hat{n}_\alpha\rangle=\frac{\partial\Omega}{\partial\varepsilon_\alpha}=\frac{1}{e^{(\varepsilon_\alpha-\mu)/k_BT}\mp 1}\ .$$

In the Maxwell-Boltzmann limit, $\mu\ll -\kT$ and $\langle n\ns_\alpha\rangle = z\,e^{-\ve\ns_\alpha/\kT}$, where $z=e^{\mu/\kT}$ is the fugacity. Note that this low-density limit is common to both bosons and fermions. $\bullet$ {\it Single particle density of states\/}: The single particle density of states per unit volume is defined to be

$$g(\varepsilon)=\frac{1}{V}\,\mathsf{Tr}\ \delta(\varepsilon-\hat{h})=\frac{1}{V}\sum_{\alpha}\delta(\varepsilon-\varepsilon_\alpha)\ ,$$

where $\Hh$ is the one-body Hamiltonian. If $\Hh$ is isotropic, then $\ve=\ve(k)$, where $k=|\Bk|$ is the magnitude of the wavevector, and

$$g(\varepsilon)=\frac{\mathsf{g}\,\Omega_d}{(2\pi)^d}\,\frac{k^{d-1}}{d\varepsilon/dk}\ ,$$

where $\Sg$ is the degeneracy of each single particle energy state (due to spin, for example). $\bullet$ {\it Quantum virial expansion\/}: From $\Omega=-pV$, we have

$$n(T,z)=\int\limits_{-\infty}^{\infty}\!d\varepsilon\;\frac{g(\varepsilon)}{z^{-1}\,e^{\varepsilon/k_BT}\mp 1}=\sum_{j=1}^{\infty}(\pm 1)^{j-1}\,z^j\,C_j(T)$$

$$\frac{p(T,z)}{k_BT}=\mp\int\limits_{-\infty}^{\infty}\!d\varepsilon\;g(\varepsilon)\,\ln\!\big(1\mp z\,e^{-\varepsilon/k_BT}\big)=\sum_{j=1}^{\infty}(\pm 1)^{j-1}\,\frac{z^j}{j}\,C_j(T)\ ,$$

where

$$C_j(T)=\int\limits_{-\infty}^{\infty}\!d\varepsilon\;g(\varepsilon)\,e^{-j\varepsilon/k_BT}\ .$$

One now inverts \$n=n(T,z)\$ to obtain \$z=z(T,n)\$, then substitutes this into \$p=p(T,z)\$ to obtain a series expansion for the equation of state,

$$p(T,n)=nk_BT\Big(1+B_2(T)\,n+B_3(T)\,n^2+\ldots\Big)\ .$$

The coefficients \$B\ns_j(T)\$ are the {\it virial coefficients\/}. One finds

$$B_2=\mp\frac{C_2}{2C_1^2}\qquad,\qquad B_3=\frac{C_2^2}{C_1^4}-\frac{2\,C_3}{2\,C_1^3}\ .$$

\$\bullet\$ {\it Photon statistics\/}: Photons are bosonic excitations whose number is not conserved, hence \$\mu=0\$. The number distribution for photon statistics is then \$n(\ve)=1/(e^{\beta\ve}-1)\$. Examples of particles obeying photon statistics include phonons (lattice vibrations), magnons (spin waves), and of course photons themselves, for which \$\ve(k)=\hbar c k\$ with \$\Sg=2\$. The pressure and number density for the photon gas obey \$p(T) = A\ns_d\,T^{d+1}\$ and \$n(T)=A'_d\,T^d\$, where \$d\$ is the dimension of space and \$A\ns_d\$ and \$A'_d\$ are constants. \$\bullet\$ {\it Blackbody radiation\/}: The energy density per unit frequency of a three-dimensional blackbody is given{P by

$$\varepsilon(\nu,T)=\frac{8\pi h}{c^3}\cdot\frac{\nu^3}{e^{h\nu/k_BT}-1}\ .$$

The total power emitted per unit area of a blackbody is \${dP\over dA}=\sigma T^4\$, where \$\sigma=\pi^2 k_\ssr{B}^4/60\hbar^3 c^2 =5.67\times 10^{-8}\,\RW/\Rm^2\,\RK^4\$ is Stefan's constant. \$\bullet\$ {\it Ideal Bose gas\/}: For Bose systems, we must have \$\ve\ns_\alpha > \mu\$ for all single particle states. The number density is

$$n(T,\mu)=\int\limits_{-\infty}^{\infty}\!d\varepsilon\;\frac{g(\varepsilon)}{e^{\beta(\varepsilon-\mu)}-1}\ .$$

This is an increasing function of \$\mu\$ and an increasing function of \$T\$. For fixed \$T\$, the largest value \$n(T,\mu)\$ can attain is \$n(T,\ve\ns_0)\$, where \$\ve\ns_0\$ is the lowest possible single particle energy, for which \$g(\ve)=0\$ for \$\ve < \ve\ns_0\$. If \$n\ns_\Rc(T)\equiv n(T,\ve\ns_0) < \infty\$, this establishes a {\it critical density\/} above which there is {\it Bose condensation\/} into the energy \$\ve\ns_0\$ state. Conversely, for a given density \$n\$ there is a {\it critical temperature\/} \$T\ns_\Rc(n)\$ such that \$n\ns_0\$ is finite for \$T<t\ns_\rc\$\,.>T\ns_\Rc\$, \$n(T,\mu)\$ is given by the integral formula above, with \$n\ns_0=0\$. For a ballistic dispersion \$\ve(\Bk)=\hbar^2\Bk^2/2m\$, one finds \$n\lambda_{T\ns_\Rc}^d=\Sg\,\zeta(d/2)\$, \ie\ \$\kB T\ns_\Rc= {2\pi\hbar^2\over m} \left(n\big/\Sg\,\zeta(d/2)\right)^{2/d}\$. For \$T<t\ns_\rc(n)\$,>T\ns_\Rc(n)\$, one has \$n=\Sg\,{Li}\ns_{d\over 2}(z)\,\lambda_T^{-d}\$ and \$p=\Sg\,{Li}\ns_{{d\over 2}+1}(z)\,\kT\,\lambda_T^{-d}\$, where

$$Li_q(z)\equiv\sum_{n=1}^{\infty}\frac{z^n}{n^q}.$$

\$\bullet\$ {\it Ideal Fermi gas\/}: The Fermi distribution is \$n(\ve)=f(\ve-\mu)=1\big/\!\left(e^{(\ve-\mu)/\kT}+1\right)\$. At \$T=0\$, this is a step function: \$n(\ve)=\RTheta(\mu-\ve)\$, and \$n=\int\limits_{-\infty}^\mu\!\! d\ve\>g(\ve)\$. The chemical potential at \$T=0\$ is called the {\it Fermi energy\/}: \$\mu(T=0,n)=\veF(n)\$. If the dispersion is \$\ve(\Bk)\$, the locus of \$\Bk\$ values satisfying \$\ve(\Bk)=\veF\$ is called the {\it Fermi surface\/}. For an isotropic and monotonic dispersion \$\ve(k)\$, the Fermi surface is a sphere of radius $\kF$, the {\it Fermi wavevector\/}. For isotropic three-dimensional systems, $\kF=(6\pi^2 n/\Sg)^{1/3}$. $\bullet$ {\it Sommerfeld expansion\/}: Let $\phi(\ve)={d\Phi\over d\ve}$. Then

$$\int\limits_{-\infty}^{\infty} d\varepsilon \; f(\varepsilon-\mu)\,\phi(\varepsilon) = \pi D \csc(\pi D)\,\Phi(\mu)$$

$$= \bigg\{ 1 + \frac{\pi^2}{6}\,(k_{B}T)^2\,\frac{d^2}{d\mu^2} + \frac{7\pi^4}{360}\,(k_{B}T)^4\,\frac{d^4}{d\mu^4} + \ldots \bigg\}\,\Phi(\mu)\,,$$

where $D=\kT\,{d\over d\mu}$. One then finds, for example, $C\ns_V=\gamma V T$ with $\gamma=\third \pi^2 k_\ssr{B}^2\,g(\veF)$.