2.3: Mathematical Interlude - Exact and Inexact Differentials

The differential

$$dF = \sum_{i=1}^{k} A_i \, dx_i \tag{2.3.1}$$

is called *exact* if there is a function $F(x_1, \ldots, x_k)$ whose differential gives the right hand side of Equation 2.3.1. In this case, we have

$$A_i = \frac{\partial F}{\partial x_i} \qquad \Longleftrightarrow \qquad \frac{\partial A_i}{\partial x_j} = \frac{\partial A_j}{\partial x_i} \quad \forall \, i, j \, . \tag{2.3.2}$$

For exact differentials, the integral between fixed endpoints is path-independent:

$$\int_A^B dF = F(x_1^B, \ldots, x_k^B) - F(x_1^A, \ldots, x_k^A) \, , \tag{2.3.3}$$

from which it follows that the integral of dF around any closed path must vanish:

$$\oint dF = 0 \, . \tag{2.3.4}$$

When the cross derivatives are not identical, when $\partial A_i / \partial x_j \neq \partial A_j / \partial x_i$, the differential is *inexact*. In this case, the integral of dF is path dependent, and does not depend solely on the endpoints.

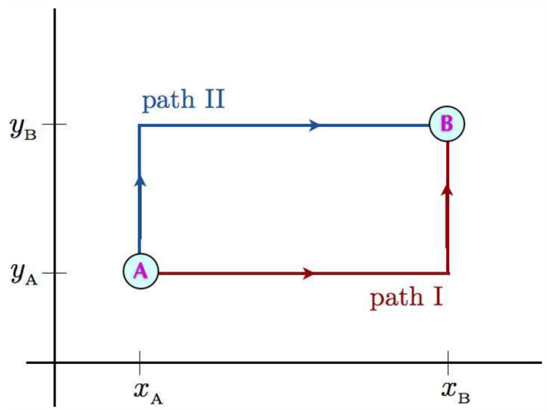

Figure [work_path] Two distinct paths with identical endpoints.

As an example, consider the differential

$$dF = K_1 \, y \, dx + K_2 \, x \, dy \, . \tag{2.3.5}$$

Let’s evaluate the integral of dF, which is the work done, along each of the two paths in Fig. [work_path]:

\begin{aligned} W^\ssr{(I)}&=K\ns_1\!\int\limits_{x\ns_\RA}^{x\nd_\RB}\!\!dx\>y\subA + K\ns_2\!\int\limits_{y\ns_\RA}^{y\nd_\RB}\!\!dy\>x\subB= K\ns_1\,y\subA \,(x\subB-x\subA) + K\ns_2\,x\subB\,(y\subB-y\sub

Note that in general W^\ssr{(I)}\ne W^\ssr{(II)}. Thus, if we start at point A, the kinetic energy at point B will depend on the path taken, since the work done is path-dependent.

The difference between the work done along the two paths is

W^\ssr{(I)}-W^\ssr{(II)}=\oint\!dF=(K\ns_2-K\ns_1)\,(x\subB-x\subA)\,(y\subB-y\subA)\ . \label{Wdiff}

Thus, we see that if $K_1 = K_2$, the work is the same for the two paths. In fact, if $K_1 = K_2$, the work would be path-independent, and would depend only on the endpoints. This is true for *any* path, and not just piecewise linear paths of the type depicted in Fig. [work_path]. Thus, if $K_1 = K_2$, we are justified in using the notation dF for the differential in Equation [dFe]; explicitly, we then have $F = K_1 \, xy$. However, if $K_1 \neq K_2$, the differential is inexact, and we will henceforth write \mathchar'26dF in such cases.