8.6: Linearized Boltzmann Equation

Linearizing the collision integral

We now return to the classical Boltzmann equation and consider a more formal treatment of the collision term in the linear approximation. We will assume time-reversal symmetry, in which case

$$\left(\frac{\partial f}{\partial t}\right)_{coll}=\int\!d^3\!p_1\!\int\!d^3\!p'\!\int\!d^3\!p'_1\;w(\mathbf{p}',\mathbf{p}'_1\,|\,\mathbf{p},\mathbf{p}_1)\,\Big\{f(\mathbf{p}')\,f(\mathbf{p}'_1)-f(\mathbf{p})\,f(\mathbf{p}_1)\Big\}\,. \tag{8.6.1}$$

The collision integral is nonlinear in the distribution f. We *linearize* by writing

$$f(\mathbf{p})=f^0(\mathbf{p})+f^0(\mathbf{p})\,\psi(\mathbf{p})\ , \tag{8.6.2}$$

where we assume $\psi(\mathbf{p})$ is small. We then have, to first order in ψ,

$$\left(\frac{\partial f}{\partial t}\right)_{coll}=f^0(\mathbf{p})\,\hat{L}\psi+\mathcal{O}(\psi^2)\ , \tag{8.6.3}$$

where the action of the *linearized collision operator* is given by

$$\begin{aligned}\hat{L}\psi\;&=\int\!d^3\!p_1\!\int\!d^3\!p'\!\int\!d^3\!p'_1\;w(\mathbf{p}',\mathbf{p}'_1\,|\,\mathbf{p},\mathbf{p}_1)\;f^0(\mathbf{p}_1)\,\Big\{\psi(\mathbf{p}')+\psi(\mathbf{p}'_1)-\psi(\mathbf{p})-\psi(\mathbf{p}_1)\Big\}\\ &=\int\!d^3\!p_1\!\int\!d\Omega\;|\mathbf{v}-\mathbf{v}_1|\;\frac{\partial\sigma}{\partial\Omega}\;f^0(\mathbf{p}_1)\,\Big\{\psi(\mathbf{p}')+\psi(\mathbf{p}'_1)-\psi(\mathbf{p})-\psi(\mathbf{p}_1)\Big\}\ ,\end{aligned}$$

where we have invoked Equation [BEsig] to write the RHS in terms of the differential scattering cross section. In deriving the above result, we have made use of the detailed balance relation,

$$f^0(\mathbf{p})\,f^0(\mathbf{p}_1)=f^0(\mathbf{p}')\,f^0(\mathbf{p}'_1)\ . \tag{8.6.4}$$

We have also suppressed the $\mathbf{r}$ dependence in writing $f(\mathbf{p})$, $f^0(\mathbf{p})$, and $\psi(\mathbf{p})$.

From Equation [bwig], we then have the linearized equation

$$\left(\hat{L}-\frac{\partial}{\partial t}\right)\psi=Y, \tag{8.6.5}$$

where, for point particles,

$$Y=\frac{1}{k_BT}\left\{\frac{\varepsilon(\mathbf{p})-c_pT}{T}\;\mathbf{v}\cdot\boldsymbol{\nabla}T+m\,v_\alpha v_\beta\,\mathcal{Q}_{\alpha\beta}-\frac{k_B\,\varepsilon(\mathbf{p})}{c_V}\;\boldsymbol{\nabla}\!\cdot\!\mathbf{V}-\mathbf{F}\cdot\mathbf{v}\right\}. \tag{8.6.6}$$

Equation [LBE] is an inhomogeneous linear equation, which can be solved by inverting the operator $\hat{L}-\frac{\partial}{\partial t}$.

Linear algebraic properties of $\hat{L}$

Although $\hat{L}$ is an integral operator, it shares many properties with other linear operators with which you are familiar, such as matrices and differential operators. We can define an *inner product*[9],

$$\langle\,\psi_1\,|\,\psi_2\,\rangle\equiv\int\!d^3\!p\;f^0(\mathbf{p})\,\psi_1(\mathbf{p})\,\psi_2(\mathbf{p})\ . \tag{8.6.7}$$

Note that this is not the usual Hilbert space inner product from quantum mechanics, since the factor $f^0(\mathbf{p})$ is included in the metric. This is necessary in order that $\hat{L}$ be *self-adjoint*:

$$\langle\,\psi_1\,|\,\hat{L}\psi_2\,\rangle=\langle\,\hat{L}\psi_1\,|\,\psi_2\,\rangle\ . \tag{8.6.8}$$

We can now define the spectrum of *normalized eigenfunctions* of $\hat{L}$, which we write as $\phi_n(\mathbf{p})$. The eigenfunctions satisfy the eigenvalue equation,

$$\hat{L}\phi_n=-\lambda_n\,\phi_n\ , \tag{8.6.9}$$

and may be chosen to be orthonormal,

$$\langle\, \phi_m \,|\, \phi_n \,\rangle = \delta_{mn} \ . \tag{8.6.10}$$

Of course, in order to obtain the eigenfunctions ϕ_n we must have detailed knowledge of the function $w(\mathbf{p}', \mathbf{p}_1' \,|\, \mathbf{p}, \mathbf{p}_1)$.

Recall that there are five collisional invariants, which are the particle number, the three components of the total particle momentum, and the particle energy. To each collisional invariant, there is an associated eigenfunction ϕ_n with eigenvalue $\lambda_n = 0$. One can check that these normalized eigenfunctions are

$$\begin{aligned} \phi_n(\mathbf{p}) &= \frac{1}{\sqrt{n}} \\ \phi_{p_\alpha}(\mathbf{p}) &= \frac{p_\alpha}{\sqrt{nmk_B T}} \\ \phi_\varepsilon(\mathbf{p}) &= \sqrt{\frac{2}{3n}} \left(\frac{\varepsilon(\mathbf{p})}{k_B T} - \frac{3}{2} \right) . \end{aligned}$$

If there are no temperature, chemical potential, or bulk velocity gradients, and there are no external forces, then $Y = 0$ and the only changes to the distribution are from collisions. The linearized Boltzmann equation becomes

$$\frac{\partial \psi}{\partial t} = \hat{L} \psi \ . \tag{8.6.11}$$

We can therefore write the most general solution in the form

$$\psi(\mathbf{p}, t) = \sum_n{}' C_n \, \phi_n(\mathbf{p}) \, e^{-\lambda_n t} \ , \tag{8.6.12}$$

where the prime on the sum reminds us that collisional invariants are to be excluded. All the eigenvalues λ_n, aside from the five zero eigenvalues for the collisional invariants, must be positive. Any negative eigenvalue would cause $\psi(\mathbf{p}, t)$ to increase without bound, and an initial nonequilibrium distribution would not relax to the equilibrium $f^0(\mathbf{p})$, which we regard as unphysical. Henceforth we will drop the prime on the sum but remember that $C_n = 0$ for the five collisional invariants.

Recall also the particle, energy, and thermal (heat) currents,

$$\begin{aligned} \mathbf{j} &= \int \! d^3\!p \; \mathbf{v} \, f(\mathbf{p}) = \int \! d^3\!p \; f^0(\mathbf{p}) \, \mathbf{v} \, \psi(\mathbf{p}) = \langle\, \mathbf{v} \,|\, \psi \,\rangle \\ \mathbf{j}_\varepsilon &= \int \! d^3\!p \; \mathbf{v} \, \varepsilon \, f(\mathbf{p}) = \int \! d^3\!p \; f^0(\mathbf{p}) \, \mathbf{v} \, \varepsilon \, \psi(\mathbf{p}) = \langle\, \mathbf{v} \, \varepsilon \,|\, \psi \,\rangle \\ \mathbf{j}_q &= \int \! d^3\!p \; \mathbf{v} \, (\varepsilon - \mu) \, f(\mathbf{p}) = \int \! d^3\!p \; f^0(\mathbf{p}) \, \mathbf{v} \, (\varepsilon - \mu) \, \psi(\mathbf{p}) = \langle\, \mathbf{v} \, (\varepsilon - \mu) \,|\, \psi \,\rangle \ . \end{aligned}$$

Note $\mathbf{j}_q = \mathbf{j}_\varepsilon - \mu \mathbf{j}$.

Steady state solution to the linearized Boltzmann equation

Under steady state conditions, there is no time dependence, and the linearized Boltzmann equation takes the form

$$\hat{L} \psi = Y \ . \tag{8.6.13}$$

We may expand ψ in the eigenfunctions ϕ_n and write $\psi = \sum_n C_n \, \phi_n$. Applying $\hat{L}$ and taking the inner product with ϕ_j, we have

$$C_j = -\frac{1}{\lambda_j} \, \langle\, \phi_j \,|\, Y \,\rangle \ . \tag{8.6.14}$$

Thus, the formal solution to the linearized Boltzmann equation is

$$\psi(\mathbf{p}) = -\sum_n \frac{1}{\lambda_n} \, \langle\, \phi_n \,|\, Y \,\rangle \, \phi_n(\mathbf{p}) \ . \tag{8.6.15}$$

This solution is applicable provided $|\, Y \,\rangle$ is orthogonal to the five collisional invariants.

Thermal conductivity

For the thermal conductivity, we take $\boldsymbol{\nabla} T=\partial_z T\,\hat{\mathbf{x}}$, and

$$Y=\frac{1}{k_B T^2}\,\frac{\partial T}{\partial x}\cdot X_\kappa\ , \tag{8.6.16}$$

where $X_\kappa\equiv(\varepsilon-c_p T)\,v_x$. Under the conditions of no particle flow ($\mathbf{j}=0$), we have $\mathbf{j}_q=-\kappa\,\partial_x T\,\hat{\mathbf{x}}$. Then we have

$$\langle\,X_\kappa\,|\,\psi\,\rangle=-\kappa\,\frac{\partial T}{\partial x}\ . \tag{8.6.17}$$

Viscosity

For the viscosity, we take

$$Y=\frac{m}{k_B T}\,\frac{\partial V_x}{\partial y}\cdot X_\eta\ , \tag{8.6.18}$$

with $X_\eta=v_x\,v_y$. We then

$$\Pi_{xy}=\langle\,m\,v_x\,v_y\,|\,\psi\,\rangle=-\eta\,\frac{\partial V_x}{\partial y}\ . \tag{8.6.19}$$

Thus,

$$\langle\,X_\eta\,|\,\psi\,\rangle=-\frac{\eta}{m}\,\frac{\partial V_x}{\partial y}\ . \tag{8.6.20}$$

Variational approach

Following the treatment in chapter 1 of Smith and Jensen, define $\hat{H}\equiv-\hat{L}$. We have that $\hat{H}$ is a positive semidefinite operator, whose only zero eigenvalues correspond to the collisional invariants. We then have the Schwarz inequality,

$$\langle\,\psi\,|\,\hat{H}\,|\,\psi\,\rangle\cdot\langle\,\phi\,|\,\hat{H}\,|\,\phi\,\rangle\geq\langle\,\phi\,|\,\hat{H}\,|\,\psi\,\rangle^2\ , \tag{8.6.21}$$

for any two Hilbert space vectors $|\,\psi\,\rangle$ and $|\,\phi\,\rangle$. Consider now the above calculation of the thermal conductivity. We have

$$\hat{H}\psi=-\frac{1}{k_B T^2}\,\frac{\partial T}{\partial x}\,X_\kappa \tag{8.6.22}$$

and therefore

$$\kappa=\frac{k_B T^2}{(\partial T/\partial x)^2}\,\langle\,\psi\,|\,\hat{H}\,|\,\psi\,\rangle\geq\frac{1}{k_B T^2}\,\frac{\langle\,\phi\,|\,X_\kappa\,\rangle^2}{\langle\,\phi\,|\,\hat{H}\,|\,\phi\,\rangle}\ . \tag{8.6.23}$$

Similarly, for the viscosity, we have

$$\hat{H}\psi=-\frac{m}{k_B T}\,\frac{\partial V_x}{\partial y}\,X_\eta\ , \tag{8.6.24}$$

from which we derive

$$\eta=\frac{k_B T}{(\partial V_x/\partial y)^2}\,\langle\,\psi\,|\,\hat{H}\,|\,\psi\,\rangle\geq\frac{m^2}{k_B T}\,\frac{\langle\,\phi\,|\,X_\eta\,\rangle^2}{\langle\,\phi\,|\,\hat{H}\,|\,\phi\,\rangle}\ . \tag{8.6.25}$$

In order to get a good lower bound, we want ϕ in each case to have a good overlap with $X_{\kappa,\eta}$. One approach then is to take $\phi=X_{\kappa,\eta}$, which guarantees that the overlap will be finite (and not zero due to symmetry, for example). We illustrate this method with the viscosity calculation. We have

$$\eta\geq\frac{m^2}{k_B T}\,\frac{\langle\,v_x v_y\,|\,v_x v_y\,\rangle^2}{\langle\,v_x v_y\,|\,\hat{H}\,|\,v_x v_y\,\rangle}\ . \tag{8.6.26}$$

Now the linearized collision operator $\hat{L}$ acts as

$$\langle\,\phi\,|\,\hat{L}\,|\,\psi\,\rangle=\int\!d^3\!p\,g^0(\mathbf{p})\,\phi(\mathbf{p})\!\int\!d^3\!p_1\!\int\!d\Omega\,\frac{\partial\sigma}{\partial\Omega}\,|\mathbf{v}-\mathbf{v}_1|\,f^0(\mathbf{p}_1)\Big\{\psi(\mathbf{p})+\psi(\mathbf{p}_1)-\psi(\mathbf{p}')-\psi(\mathbf{p}'_1)\Big\}\,. \tag{8.6.27}$$

Here the kinematics of the collision guarantee total energy and momentum conservation, so $\mathbf{p}'$ and $\mathbf{p}'_1$ are determined as in Equation [finalps].

Now we have

$$d\Omega=\sin\chi\,d\chi\,d\varphi\;, \tag{8.6.28}$$

where χ is the scattering angle depicted in Fig. [scat_impact] and φ is the azimuthal angle of the scattering. The differential scattering cross section is obtained by elementary mechanics and is known to be

$$\frac{\partial\sigma}{\partial\Omega}=\bigg|\frac{d(b^2/2)}{d\sin\chi}\bigg|\;, \tag{8.6.29}$$

where b is the *impact parameter*. The scattering angle is

$$\chi(b,u)=\pi-2\int\limits_{r_p}^{\infty}\!dr\,\frac{b}{\sqrt{r^4-b^2r^2-\frac{2U(r)r^4}{\tilde{m}u^2}}}\;, \tag{8.6.30}$$

where $\tilde{m}=\frac{1}{2}m$ is the reduced mass, and r_p is the relative coordinate separation at periapsis, the distance of closest approach, which occurs when $\dot{r}=0$,

$$\frac{1}{2}\tilde{m}u^2=\frac{\ell^2}{2\tilde{m}r_p^2}+U(r_p)\;, \tag{8.6.31}$$

where $\ell=\tilde{m}ub$ is the relative coordinate angular momentum.

[scat_impact] Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and \chi is the scattering angle. \phi_0 is the angle through which the relative coordinate moves between periapsis and infinity.

[scat_impact] Scattering in the CM frame. O is the force center and P is the point of periapsis. The impact parameter is b, and χ is the scattering angle. ϕ_0 is the angle through which the relative coordinate moves between periapsis and infinity.

We work in center-of-mass coordinates, so the velocities are

$$\begin{aligned}\mathbf{v}&=\mathbf{V}+\frac{1}{2}\mathbf{u} & \mathbf{v}'&=\mathbf{V}+\frac{1}{2}\mathbf{u}'\\ \mathbf{v}_1&=\mathbf{V}-\frac{1}{2}\mathbf{u} & \mathbf{v}'_1&=\mathbf{V}-\frac{1}{2}\mathbf{u}'\end{aligned}\qquad,$$

with $|\mathbf{u}|=|\mathbf{u}'|$ and $\hat{\mathbf{u}}\cdot\hat{\mathbf{u}}'=\cos\chi$. Then if $\psi(\mathbf{p})=v_xv_y$, we have

$$\Delta(\psi)\equiv\psi(\mathbf{p})+\psi(\mathbf{p}_1)-\psi(\mathbf{p}')-\psi(\mathbf{p}'_1)=\frac{1}{2}\big(u_xu_y-u'_xu'_y\big)\;. \tag{8.6.32}$$

We may write

$$\mathbf{u}'=u\,\Big(\sin\chi\cos\varphi\,\hat{\mathbf{e}}_1+\sin\chi\sin\varphi\,\hat{\mathbf{e}}_2+\cos\chi\,\hat{\mathbf{e}}_3\Big)\;, \tag{8.6.33}$$

where $\hat{\mathbf{e}}_3=\hat{\mathbf{u}}$. With this parameterization, we have

$$\int\limits_0^{2\pi}\!d\varphi\;\frac{1}{2}\big(u_\alpha u_\beta-u'_\alpha u'_\beta\big)=-\pi\sin^2\!\chi\,\big(u^2\,\delta_{\alpha\beta}-3u_\alpha u_\beta\big)\;. \tag{8.6.34}$$

Note that we have used here the relation

$$e_{1\alpha}\,e_{1\beta}+e_{2\alpha}\,e_{2\beta}+e_{3\alpha}\,e_{3\beta}=\delta_{\alpha\beta}\;, \tag{8.6.35}$$

which holds since the LHS is a projector $\sum_{i=1}^3|\hat{\mathbf{e}}_i\rangle\langle\hat{\mathbf{e}}_i|$.

It is convenient to define the following integral:

$$R(u) \equiv \int_0^\infty db\, b\, \sin^2\!\chi(b,u)\ . \tag{8.6.36}$$

Since the Jacobian

$$\left| \det \frac{(\partial \mathbf{v}, \partial \mathbf{v}_1)}{(\partial \mathbf{V}, \partial \mathbf{u})} \right| = 1\ , \tag{8.6.37}$$

we have

$$\langle\, v_x v_y \,|\, \hat{L} \,|\, v_x v_y \,\rangle = n^2 \left(\frac{m}{2\pi k_B T} \right)^3 \int d^3V \int d^3u\ e^{-m\mathbf{V}^2/k_BT}\ e^{-m\mathbf{u}^2/4k_BT} \cdot u \cdot \frac{3\pi}{2} u_x u_y \cdot R(u) \cdot v_x v_y\ . \tag{8.6.38}$$

This yields

$$\langle\, v_x v_y \,|\, \hat{L} \,|\, v_x v_y \,\rangle = \frac{\pi}{40}\, n^2 \, \big\langle u^5\, R(u) \big\rangle\ , \tag{8.6.39}$$

where

$$\big\langle F(u) \big\rangle \equiv \int_0^\infty du\, u^2\, e^{-mu^2/4k_BT}\, F(u) \bigg/ \int_0^\infty du\, u^2\, e^{-mu^2/4k_BT}\ . \tag{8.6.40}$$

It is easy to compute the term in the numerator of Equation [varvisc]:

$$\langle\, v_x v_y \,|\, v_x v_y \,\rangle = n \left(\frac{m}{2\pi k_B T} \right)^{3/2} \int d^3v\, e^{-m\mathbf{v}^2/2k_BT}\ v_x^2\, v_y^2 = n \left(\frac{k_B T}{m} \right)^2 . \tag{8.6.41}$$

Putting it all together, we find

$$\eta \ge \frac{40\, (k_B T)^3}{\pi\, m^2} \bigg/ \big\langle u^5\, R(u) \big\rangle\ . \tag{8.6.42}$$

The computation for κ is a bit more tedious. One has $\psi(\mathbf{p}) = (\varepsilon - c_p T)\, v_x$, in which case

$$\Delta(\psi) = \frac{1}{2} m \Big[(\mathbf{V} \cdot \mathbf{u})\, u_x - (\mathbf{V} \cdot \mathbf{u}')\, u_x' \Big]\ . \tag{8.6.43}$$

Ultimately, one obtains the lower bound

$$\kappa \ge \frac{150\, k_B\, (k_B T)^3}{\pi\, m^3} \bigg/ \big\langle u^5\, R(u) \big\rangle\ . \tag{8.6.44}$$

Thus, independent of the potential, this variational calculation yields a Prandtl number of

$$Pr = \frac{\nu}{a} = \frac{\eta\, c_p}{m\, \kappa} = \frac{2}{3}\ , \tag{8.6.45}$$

which is very close to what is observed in dilute monatomic gases (see Tab. [Prandtl]).

While the variational expressions for η and κ are complicated functions of the potential, for hard sphere scattering the calculation is simple, because $b = d \sin\phi_0 = d\cos(\frac{1}{2}\chi)$, where d is the hard sphere diameter. Thus, the impact parameter b is independent of the relative speed u, and one finds $R(u) = \frac{1}{3} d^3$. Then

$$\big\langle u^5\, R(u) \big\rangle = \frac{1}{3} d^3 \big\langle u^5 \big\rangle = \frac{128}{\sqrt{\pi}} \left(\frac{k_B T}{m} \right)^{5/2} d^2 \tag{8.6.46}$$

and one finds

$$\eta \ge \frac{5\, (m k_B T)^{1/2}}{16 \sqrt{\pi}\, d^2} \qquad , \qquad \kappa \ge \frac{75\, k_B}{64 \sqrt{\pi}\, d^2} \left(\frac{k_B T}{m} \right)^{1/2} . \tag{8.6.47}$$