6.7: Appendix I- Potts Model in One Dimension

Definition

The Potts model is defined by the Hamiltonian

$$H=-J\sum_{\langle ij\rangle}\delta_{\sigma_i,\sigma_j}-h\sum_i\delta_{\sigma_i,1}\,. \tag{6.7.1}$$

Here, the spin variables σ_i take values in the set $\{1,2,\ldots,q\}$on each site. The equivalent of an external magnetic field in the Ising case is a field h which prefers a particular value of σ ($\sigma=1$ in the above Hamiltonian). Once again, it is not possible to compute the partition function on general lattices, however in one dimension we may once again find Z using the transfer matrix method.

Transfer matrix

On a ring of N sites, we have

$$\begin{aligned}Z&=Tr\,e^{-\beta H}\\&=\sum_{\{\sigma_n\}}e^{\beta h\delta_{\sigma_1,1}}\,e^{\beta J\delta_{\sigma_1,\sigma_2}}\,\cdots\,e^{\beta h\delta_{\sigma_N,1}}\,e^{\beta J\delta_{\sigma_N,\sigma_1}}\\&=Tr\left(R^N\right)\,,\end{aligned}$$

where the $q\times q$ transfer matrix R is given by

$$R_{\sigma\sigma'}=e^{\beta J\delta_{\sigma\sigma'}}\,e^{\frac{1}{2\beta h\delta_{\sigma,1}}}\,e^{\frac{1}{2\beta h\delta_{\sigma',1}}}=\begin{cases}e^{\beta(J+h)} & \text{if } \sigma=\sigma'=1\\ e^{\beta J} & \text{if } \sigma=\sigma'\neq 1\\ e^{\beta h/2} & \text{if } \sigma=1 \text{ and } \sigma'\neq 1\\ e^{\beta h/2} & \text{if } \sigma\neq 1 \text{ and } \sigma'=1\\ 1 & \text{if } \sigma\neq 1 \text{ and } \sigma'\neq 1 \text{ and } \sigma\neq\sigma'\,.\end{cases} \tag{6.7.2}$$

In matrix form,

$$R=\begin{pmatrix}e^{\beta(J+h)} & e^{\beta h/2} & e^{\beta h/2} & \cdots & & e^{\beta h/2}\\ e^{\beta h/2} & e^{\beta J} & 1 & \cdots & & 1\\ e^{\beta h/2} & 1 & e^{\beta J} & \cdots & & 1\\ \vdots & \vdots & \vdots & \ddots & & \vdots\\ e^{\beta h/2} & 1 & 1 & \cdots & e^{\beta J} & 1\\ e^{\beta h/2} & 1 & 1 & \cdots & 1 & e^{\beta J}\end{pmatrix} \tag{6.7.3}$$

The matrix R has q eigenvalues λ_j, with $j=1,\ldots,q$. The partition function for the Potts chain is then

$$Z=\sum_{j=1}^q\lambda_j^N\,. \tag{6.7.4}$$

We can actually find the eigenvalues of R analytically. To this end, consider the vectors

$$\phi=\begin{pmatrix}1\\0\\\vdots\\0\end{pmatrix}\qquad,\qquad\psi=\left(q-1+e^{\beta h}\right)^{-1/2}\begin{pmatrix}e^{\beta h/2}\\1\\\vdots\\1\end{pmatrix}\,. \tag{6.7.5}$$

Then R may be written as

$$R=\left(e^{\beta J}-1\right)\mathbb{I}+\left(q-1+e^{\beta h}\right)|\,\psi\,\rangle\langle\,\psi\,|+\left(e^{\beta J}-1\right)\left(e^{\beta h}-1\right)|\,\phi\,\rangle\langle\,\phi\,|\,, \tag{6.7.6}$$

where $\mathbb{I}$ is the $q\times q$ identity matrix. When $h=0$, we have a simpler form,

$$R=\big(e^{\beta J}-1\big)\,\mathbb{I}+q\,|\,\psi\,\rangle\langle\,\psi\,|\,. \tag{6.7.7}$$

From this we can read off the eigenvalues:

$$\begin{aligned}\lambda_1&=e^{\beta J}+q-1\\ \lambda_j&=e^{\beta J}-1\quad,\quad j\in\{2,\ldots,q\}\,,\end{aligned}$$

since $|\,\psi\,\rangle$ is an eigenvector with eigenvalue $\lambda=e^{\beta J}+q-1$, and any vector orthogonal to $|\,\psi\,\rangle$ has eigenvalue $\lambda=e^{\beta J}-1$. The partition function is then

$$Z=\big(e^{\beta J}+q-1\big)^N+(q-1)\big(e^{\beta J}-1\big)^N\,. \tag{6.7.8}$$

In the thermodynamic limit $N\to\infty$, only the λ_1 eigenvalue contributes, and we have

$$F(T,N,h=0)=-Nk_BT\ln\big(e^{J/k_BT}+q-1\big)\qquad\text{for } N\to\infty\,. \tag{6.7.9}$$

When h is nonzero, the calculation becomes somewhat more tedious, but still relatively easy. The problem is that $|\,\psi\,\rangle$ and $|\,\phi\,\rangle$ are not orthogonal, so we define

$$|\,\chi\,\rangle=\frac{|\,\phi\,\rangle-|\,\psi\,\rangle\langle\,\psi\,|\,\phi\,\rangle}{\sqrt{1-\langle\,\phi\,|\,\psi\,\rangle^2}}\,, \tag{6.7.10}$$

where

$$x\equiv\langle\,\phi\,|\,\psi\,\rangle=\bigg(\frac{e^{\beta h}}{q-1+e^{\beta h}}\bigg)^{1/2}\,. \tag{6.7.11}$$

Now we have $\langle\,\chi\,|\,\psi\,\rangle=0$, with $\langle\,\chi\,|\,\chi\,\rangle=1$ and $\langle\,\psi\,|\,\psi\,\rangle=1$, with

$$|\,\phi\,\rangle=\sqrt{1-x^2}\;|\,\chi\,\rangle+x\,|\,\psi\,\rangle\,. \tag{6.7.12}$$

and the transfer matrix is then

$$\begin{aligned}R&=\big(e^{\beta J}-1\big)\,\mathbb{I}+\big(q-1+e^{\beta h}\big)\,|\,\psi\,\rangle\langle\,\psi\,|\\ &\qquad+\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)\Big[(1-x^2)\,|\,\chi\,\rangle\langle\,\chi\,|+x^2\,|\,\psi\,\rangle\langle\,\psi\,|+x\,\sqrt{1-x^2}\,\Big(|\,\chi\,\rangle\langle\,\psi\,|+|\,\psi\,\rangle\langle\,\chi\,|\Big)\Big]\\ &=\big(e^{\beta J}-1\big)\,\mathbb{I}+\Bigg[\big(q-1+e^{\beta h}\big)+\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)\bigg(\frac{e^{\beta h}}{q-1+e^{\beta h}}\bigg)\Bigg]\,|\,\psi\,\rangle\langle\,\psi\,|\\ &\qquad+\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)\bigg(\frac{q-1}{q-1+e^{\beta h}}\bigg)\,|\,\chi\,\rangle\langle\,\chi\,|\\ &\qquad+\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)\bigg(\frac{(q-1)\,e^{\beta h}}{q-1+e^{\beta h}}\bigg)^{1/2}\Big(|\,\chi\,\rangle\langle\,\psi\,|+|\,\psi\,\rangle\langle\,\chi\,|\Big)\,,\end{aligned}$$

which in the two-dimensional subspace spanned by $|\,\chi\,\rangle$ and $|\,\psi\,\rangle$ is of the form

$$R=\begin{pmatrix}a&c\\c&b\end{pmatrix}\,. \tag{6.7.13}$$

Recall that for any 2×2 Hermitian matrix,

$$\begin{aligned}M&=a_0\,\mathbb{I}+\mathbf{a}\cdot\tau\\ &=\begin{pmatrix}a_0+a_3&a_1-ia_2\\a_1+ia_2&a_0-a_3\end{pmatrix}\,,\end{aligned}$$

the characteristic polynomial is

$$P(\lambda)=det\,\big(\lambda\,\mathbb{I}-M\big)=(\lambda-a_0)^2-a_1^2-a_2^2-a_3^2\,, \tag{6.7.14}$$

and hence the eigenvalues are

$$\lambda_{\pm}=a_0 \pm \sqrt{a_1^2+a_2^2+a_3^2}\ . \tag{6.7.15}$$

For the transfer matrix of Equation ???, we obtain, after a little work,

$$\begin{aligned}\lambda_{1,2} &= e^{\beta J}-1+\frac{1}{2}\Big[q-1+e^{\beta h}+\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)\Big] \\ &\pm \frac{1}{2}\sqrt{\Big[q-1+e^{\beta h}+\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)\Big]^2-4(q-1)\big(e^{\beta J}-1\big)\big(e^{\beta h}-1\big)}\ .\end{aligned}$$

There are $q-2$ other eigenvalues, however, associated with the $(q-2)$-dimensional subspace orthogonal to $|\,\chi\,\rangle$ and $|\,\psi\,\rangle$. Clearly all these eigenvalues are given by

$$\lambda_j=e^{\beta J}-1 \qquad , \quad j\in\{3\,,\,\ldots\,,\,q\}\ . \tag{6.7.16}$$

The partition function is then

$$Z=\lambda_1^N+\lambda_2^N+(q-2)\,\lambda_3^N\ , \tag{6.7.17}$$

and in the thermodynamic limit $N\to\infty$ the maximum eigenvalue λ_1 dominates. Note that we recover the correct limit as $h\to 0$.