8.10: Appendix I- Boltzmann Equation and Collisional Invariants

Problem : The linearized Boltzmann operator $L\psi$ is a complicated functional. Suppose we replace L by $\mathcal{L}$, where

$$\begin{aligned}\mathcal{L}\psi &= -\gamma\,\psi(\mathbf{v},t) + \gamma\bigg(\frac{m}{2\pi k_B T}\bigg)^{3/2}\!\int\! d^3\!u\;\exp\bigg(-\frac{m\mathbf{u}^2}{2k_B T}\bigg) \\ &\qquad\times\bigg\{1 + \frac{m}{k_B T}\,\mathbf{u}\cdot\mathbf{v} + \frac{2}{3}\bigg(\frac{m\mathbf{u}^2}{2k_B T}-\frac{3}{2}\bigg)\bigg(\frac{m\mathbf{v}^2}{2k_B T}-\frac{3}{2}\bigg)\bigg\}\,\psi(\mathbf{u},t)\,.\end{aligned}$$

Show that $\mathcal{L}$ shares all the important properties of L. What is the meaning of γ? Expand $\psi(\mathbf{v},t)$ in spherical harmonics and Sonine polynomials,

$$\psi(\mathbf{v},t) = \sum_{r\ell m} a_{r\ell m}(t)\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y^\ell_m(\hat{\mathbf{n}}), \tag{8.10.1}$$

with $x = mv^2/2k_B T$, and thus express the action of the linearized Boltzmann operator algebraically on the expansion coefficients $a_{r\ell m}(t)$.

The Sonine polynomials $S^n_\alpha(x)$ are a complete, orthogonal set which are convenient to use in the calculation of transport coefficients. They are defined as

$$S^n_\alpha(x) = \sum_{m=0}^{n} \frac{\Gamma(\alpha+n+1)\,(-x)^m}{\Gamma(\alpha+m+1)\,(n-m)!\,m!}\ , \tag{8.10.2}$$

and satisfy the generalized orthogonality relation

$$\int\limits_0^\infty\! dx\; e^{-x}\, x^\alpha\, S^n_\alpha(x)\, S^{n'}_\alpha(x) = \frac{\Gamma(\alpha+n+1)}{n!}\,\delta_{nn'}\ . \tag{8.10.3}$$

Solution : The ‘important properties’ of L are that it annihilate the five collisional invariants, 1, $\mathbf{v}$, and v^2, and that all other eigenvalues are *negative*. That this is true for $\mathcal{L}$ can be verified by an explicit calculation.

Plugging the conveniently parameterized form of $\psi(\mathbf{v},t)$ into $\mathcal{L}$, we have

$$\begin{aligned}\mathcal{L}\psi &= -\gamma\sum_{r\ell m} a_{r\ell m}(t)\, S^r_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y^\ell_m(\hat{\mathbf{n}}) \;+\; \frac{\gamma}{2\pi^{3/2}}\sum_{r\ell m} a_{r\ell m}(t)\!\int\limits_0^\infty\! dx_1\, x_1^{1/2}\, e^{-x_1} \\ &\qquad\times\int\! d\hat{\mathbf{n}}_1\Big[1 + 2\,x^{1/2}x_1^{1/2}\,\hat{\mathbf{n}}\!\cdot\!\hat{\mathbf{n}}_1 + \frac{2}{3}\big(x-\frac{3}{2}\big)\big(x_1-\frac{3}{2}\big)\Big]\, S^r_{\ell+\frac{1}{2}}(x_1)\, x_1^{\ell/2}\, Y^\ell_m(\hat{\mathbf{n}}_1)\,,\end{aligned}$$

where we’ve used

$$u = \sqrt{\frac{2k_B T}{m}}\; x_1^{1/2} \qquad , \qquad du = \sqrt{\frac{k_B T}{2m}}\; x_1^{-1/2}\, dx_1\ . \tag{8.10.4}$$

Now recall $Y^0_0(\hat{\mathbf{n}}) = \frac{1}{\sqrt{4\pi}}$ and

$$\begin{aligned}&Y^1_1(\hat{\mathbf{n}}) = -\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{i\varphi} \qquad && Y^1_0(\hat{\mathbf{n}}) = \sqrt{\frac{3}{4\pi}}\,\cos\theta \qquad && Y^1_{-1}(\hat{\mathbf{n}}) = +\sqrt{\frac{3}{8\pi}}\,\sin\theta\, e^{-i\varphi} \\ &S^0_{1/2}(x) = 1 && S^0_{3/2}(x) = 1 && S^1_{1/2}(x) = \frac{3}{2} - x\,,\end{aligned}$$

which allows us to write

$$\begin{aligned}1 &= 4\pi\, Y^0_0(\hat{\mathbf{n}})\, Y^{0\,*}_0(\hat{\mathbf{n}}_1) \\ \hat{\mathbf{n}}\!\cdot\!\hat{\mathbf{n}}_1 &= \frac{4\pi}{3}\Big[\, Y^1_0(\hat{\mathbf{n}})\, Y^{1\,*}_0(\hat{\mathbf{n}}_1) + Y^1_1(\hat{\mathbf{n}})\, Y^{1\,*}_1(\hat{\mathbf{n}}_1) + Y^1_{-1}(\hat{\mathbf{n}})\, Y^{1\,*}_{-1}(\hat{\mathbf{n}}_1)\Big]\,.\end{aligned}$$

We can do the integrals by appealing to the orthogonality relations for the spherical harmonics and Sonine polynomials:

$$\int d\hat{\mathbf{n}}\, Y^{\ell}_{m}(\hat{\mathbf{n}})\, Y^{l'\,*}_{m'}(\hat{\mathbf{n}}) = \delta_{ll'}\,\delta_{mm'}$$

$$\int_0^\infty dx\, e^{-x}\, x^{\alpha}\, S^{n}_{\alpha}(x)\, S^{n'}_{\alpha}(x) = \frac{\Gamma(n+\alpha+1)}{\Gamma(n+1)}\,\delta_{nn'}\,.$$

Integrating first over the direction vector $\hat{\mathbf{n}}_1$,

$$\begin{aligned}\mathcal{L}\psi = -\gamma\sum_{r\ell m} a_{r\ell m}(t)\, S^{r}_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y^{\ell}_{m}(\hat{\mathbf{n}}) \\ +\frac{2\gamma}{\sqrt{\pi}}\sum_{r\ell m} a_{r\ell m}(t)\int_0^\infty dx_1\, x_1^{1/2}\, e^{-x_1}\int d\hat{\mathbf{n}}_1\,\Big[Y^0_0(\hat{\mathbf{n}})\,Y^{0*}_0(\hat{\mathbf{n}}_1)\,S^0_{1/2}(x)\,S^0_{1/2}(x_1) \\ +\frac{2}{3}\,x^{1/2}x_1^{1/2}\sum_{m'=-1}^{1} Y^1_{m'}(\hat{\mathbf{n}})\,Y^{1\,*}_{m'}(\hat{\mathbf{n}}_1)\,S^0_{3/2}(x)\,S^0_{3/2}(x_1) \\ +\frac{2}{3}\,Y^0_0(\hat{\mathbf{n}})\,Y^{0*}_0(\hat{\mathbf{n}}_1)\,S^1_{1/2}(x)\,S^1_{1/2}(x_1)\Big]\,S^{r}_{\ell+\frac{1}{2}}(x_1)\,x_1^{\ell/2}\,Y^{\ell}_{m}(\hat{\mathbf{n}}_1)\,,\end{aligned}$$

we obtain the intermediate result

$$\begin{aligned}\mathcal{L}\psi = -\gamma\sum_{r\ell m} a_{r\ell m}(t)\, S^{r}_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y^{\ell}_{m}(\hat{\mathbf{n}}) \\ +\frac{2\gamma}{\sqrt{\pi}}\sum_{r\ell m} a_{r\ell m}(t)\int_0^\infty dx_1\, x_1^{1/2}\, e^{-x_1}\Big[Y^0_0(\hat{\mathbf{n}})\,\delta_{l0}\,\delta_{m0}\,S^0_{1/2}(x)\,S^0_{1/2}(x_1) \\ +\frac{2}{3}\,x^{1/2}x_1^{1/2}\sum_{m'=-1}^{1} Y^1_{m'}(\hat{\mathbf{n}})\,\delta_{l1}\,\delta_{mm'}\,S^0_{3/2}(x)\,S^0_{3/2}(x_1) \\ +\frac{2}{3}\,Y^0_0(\hat{\mathbf{n}})\,\delta_{l0}\,\delta_{m0}\,S^1_{1/2}(x)\,S^1_{1/2}(x_1)\Big]\,S^{r}_{\ell+\frac{1}{2}}(x_1)\,x_1^{1/2}.\end{aligned}$$

Appealing now to the orthogonality of the Sonine polynomials, and recalling that

$$\Gamma(\tfrac{1}{2}) = \sqrt{\pi}\qquad,\qquad \Gamma(1)=1\qquad,\qquad \Gamma(z+1)=z\,\Gamma(z)\,, \tag{8.10.5}$$

we integrate over x_1. For the first term in brackets, we invoke the orthogonality relation with $n=0$ and $\alpha=\frac{1}{2}$, giving $\Gamma(\frac{3}{2})=\frac{1}{2}\sqrt{\pi}$. For the second bracketed term, we have $n=0$ but $\alpha=\frac{3}{2}$, and we obtain $\Gamma(\frac{5}{2})=\frac{3}{2}\,\Gamma(\frac{3}{2})$, while the third bracketed term involves leads to $n=1$ and $\alpha=\frac{1}{2}$, also yielding $\Gamma(\frac{5}{2})=\frac{3}{2}\,\Gamma(\frac{3}{2})$. Thus, we obtain the simple and pleasing result

$$\mathcal{L}\psi = -\gamma{\sum_{r\ell m}}' a_{r\ell m}(t)\, S^{r}_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y^{\ell}_{m}(\hat{\mathbf{n}}) \tag{8.10.6}$$

where the prime on the sum indicates that the set

$$CI = \Big\{(0,0,0)\,,\quad (1,0,0)\,,\quad (0,1,1)\,,\quad (0,1,0)\,,\quad (0,1,-1)\Big\} \tag{8.10.7}$$

are to be excluded from the sum. But these are just the functions which correspond to the five collisional invariants! Thus, we learn that

$$\psi_{r\ell m}(\mathbf{v}) = \mathcal{N}_{r\ell m}\, S^{r}_{\ell+\frac{1}{2}}(x)\, x^{\ell/2}\, Y^{\ell}_{m}(\hat{\mathbf{n}}), \tag{8.10.8}$$

is an eigenfunction of $\mathcal{L}$ with eigenvalue $-\gamma$ if (r,ℓ,m) does not correspond to one of the five collisional invariants. In the latter case, the eigenvalue is zero. Thus, the algebraic action of $\mathcal{L}$ on the coefficients $a_{r\ell m}$ is

$$(\mathcal{L}a)_{r\ell m} = \begin{cases} -\gamma\, a_{r\ell m} & \text{if } (r,\ell,m) \notin CI \\ = 0 & \text{if } (r,\ell,m) \in CI \end{cases} \tag{8.10.9}$$

The quantity $\tau = \gamma^{-1}$ is the relaxation time.

It is pretty obvious that $\mathcal{L}$ is self-adjoint, since

$$\begin{aligned} \langle\, \phi \,|\, \mathcal{L}\psi \,\rangle &\equiv \int\! d^3\!v\, f^0(\mathbf{v})\, \phi(\mathbf{v})\, \mathcal{L}[\psi(\mathbf{v})] \\ &= -\gamma\, n \bigg(\frac{m}{2\pi k_B T}\bigg)^{3/2}\!\int\! d^3\!v\, \exp\bigg(-\frac{m\mathbf{v}^2}{2k_B T}\bigg)\, \phi(\mathbf{v})\, \psi(\mathbf{v}) \\ &\qquad + \gamma\, n \bigg(\frac{m}{2\pi k_B T}\bigg)^{3}\!\int\! d^3\!v\!\int\! d^3\!u\, \exp\bigg(-\frac{m\mathbf{u}^2}{2k_B T}\bigg)\, \exp\bigg(-\frac{m\mathbf{v}^2}{2k_B T}\bigg) \\ &\qquad\qquad \times \phi(\mathbf{v}) \left[1 + \frac{m}{k_B T}\, \mathbf{u}\cdot\mathbf{v} + \frac{2}{3}\bigg(\frac{m\mathbf{u}^2}{2k_B T} - \frac{3}{2}\bigg)\bigg(\frac{m\mathbf{v}^2}{2k_B T} - \frac{3}{2}\bigg)\right] \psi(\mathbf{u}) \\ &= \langle\, \mathcal{L}\phi \,|\, \psi \,\rangle\ , \end{aligned}$$

where n is the bulk number density and $f^0(\mathbf{v})$ is the Maxwellian velocity distribution.