8.5: Diffusion and the Lorentz model

Failure of the relaxation time approximation

As we remarked above, the relaxation time approximation fails to conserve any of the collisional invariants. It is therefore unsuitable for describing hydrodynamic phenomena such as diffusion. To see this, let $f(\mathbf{r},\mathbf{v},t)$ be the distribution function, here written in terms of position, velocity, and time rather than position, momentum, and time as befor[7]. In the absence of external forces, the Boltzmann equation in the relaxation time approximation is

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}=-\frac{f-f^0}{\tau}\ . \tag{8.5.1}$$

The density of particles in velocity space is given by

$$\tilde{n}(\mathbf{v},t)=\int\! d^3\!r\, f(\mathbf{r},\mathbf{v},t)\ . \tag{8.5.2}$$

In equilibrium, this is the Maxwell distribution times the total number of particles: {\tilde n}\ns_0(\Bv)=N P\ns_\ssr{M}(\Bv). The number of particles as a function of time, $N(t)=\int\! d^3\!v\,\tilde{n}(\mathbf{v},t)$, should be a constant.

Integrating the Boltzmann equation one has

$$\frac{\partial\tilde{n}}{\partial t}=-\frac{\tilde{n}-\tilde{n}_0}{\tau}\ . \tag{8.5.3}$$

Thus, with $\delta\tilde{n}(\mathbf{v},t)=\tilde{n}(\mathbf{v},t)-\tilde{n}_0(\mathbf{v})$, we have

$$\delta\tilde{n}(\mathbf{v},t)=\delta\tilde{n}(\mathbf{v},0)\,e^{-t/\tau}\ . \tag{8.5.4}$$

Thus, $\tilde{n}(\mathbf{v},t)$ decays exponentially to zero with time constant τ, from which it follows that the total particle number exponentially relaxes to N_0. This is physically incorrect; local density perturbations can't just *vanish*. Rather, they *diffuse*.

Modified Boltzmann equation and its solution

To remedy this unphysical aspect, consider the modified Boltzmann equation,

$$\frac{\partial f}{\partial t}+\mathbf{v}\cdot\frac{\partial f}{\partial \mathbf{r}}=\frac{1}{\tau}\bigg[-f+\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,f\bigg]\equiv\frac{1}{\tau}\big(\mathbb{P}-1\big)f\ , \tag{8.5.5}$$

where $\mathbb{P}$ is a projector onto a space of isotropic functions of $\mathbf{v}$: $\mathbb{P}F=\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,F(\mathbf{v})$ for any function $F(\mathbf{v})$. Note that $\mathbb{P}F$ is a function of the speed $v=|\mathbf{v}|$. For this modified equation, known as the *Lorentz model*, one finds $\partial_t\tilde{n}=0$.

The model in Equation [Lormod] is known as the *Lorentz model*[8]. To solve it, we consider the Laplace transform,

$$\hat{f}(\mathbf{k},\mathbf{v},s)=\int\limits_0^\infty\! dt\,e^{-st}\!\int\! d^3\!r\,e^{-i\mathbf{k}\cdot\mathbf{r}}\,f(\mathbf{r},\mathbf{v},t)\ . \tag{8.5.6}$$

Taking the Laplace transform of Equation [Lormod], we find

$$\big(s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}\big)\,\hat{f}(\mathbf{k},\mathbf{v},s)=\tau^{-1}\,\mathbb{P}\,\hat{f}(\mathbf{k},\mathbf{v},s)+f(\mathbf{k},\mathbf{v},t=0)\ . \tag{8.5.7}$$

We now solve for $\mathbb{P}\,\hat{f}(\mathbf{k},\mathbf{v},s)$:

$$\hat{f}(\mathbf{k},\mathbf{v},s)=\frac{\tau^{-1}}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\,\mathbb{P}\,\hat{f}(\mathbf{k},\mathbf{v},s)+\frac{f(\mathbf{k},\mathbf{v},t=0)}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\ , \tag{8.5.8}$$

which entails

$$\mathbb{P}\,\hat{f}(\mathbf{k},\mathbf{v},s)=\bigg[\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,\frac{\tau^{-1}}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\bigg]\,\mathbb{P}\,\hat{f}(\mathbf{k},\mathbf{v},\mathbf{s})+\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,\frac{f(\mathbf{k},\mathbf{v},t=0)}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\ . \tag{8.5.9}$$

Now we have

$$\begin{aligned}\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,\frac{\tau^{-1}}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}&=\int\limits_{-1}^{1}\!dx\,\frac{\tau^{-1}}{s+ivkx+\tau^{-1}}\\&=\frac{1}{vk}\tan^{-1}\!\bigg(\frac{vk\tau}{1+\tau s}\bigg)\,.\end{aligned}$$

Thus,

$$\mathbb{P}\,f(\mathbf{k},\mathbf{v},s)=\Bigg[1-\frac{1}{vk\tau}\tan^{-1}\!\bigg(\frac{vk\tau}{1+\tau s}\bigg)\Bigg]^{-1}\!\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,\frac{f(\mathbf{k},\mathbf{v},t=0)}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\,. \tag{8.5.10}$$

We now have the solution to Lorentz's modified Boltzmann equation:

$$\begin{aligned}\hat{f}\,(\mathbf{k},\mathbf{v},s)\ &=\frac{\tau^{-1}}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\Bigg[1-\frac{1}{vk\tau}\tan^{-1}\!\bigg(\frac{vk\tau}{1+\tau s}\bigg)\Bigg]^{-1}\!\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,\frac{f(\mathbf{k},\mathbf{v},t=0)}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\\&\qquad+\frac{f(\mathbf{k},\mathbf{v},t=0)}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\,.\end{aligned}$$

Let us assume an initial distribution which is perfectly localized in both $\mathbf{r}$ and $\mathbf{v}$:

$$f(\mathbf{r},\mathbf{v},t=0)=\delta(\mathbf{v}-\mathbf{v}_0)\,. \tag{8.5.11}$$

For these initial conditions, we find

$$\int\!\frac{d\hat{\mathbf{v}}}{4\pi}\,\frac{f(\mathbf{k},\mathbf{v},t=0)}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}=\frac{1}{s+i\mathbf{v}_0\cdot\mathbf{k}+\tau^{-1}}\cdot\frac{\delta(v-v_0)}{4\pi v_0^2}\,. \tag{8.5.12}$$

We further have that

$$1-\frac{1}{vk\tau}\tan^{-1}\!\bigg(\frac{vk\tau}{1+\tau s}\bigg)=s\tau+\frac{1}{3}k^2v^2\tau^2+\ldots\,, \tag{8.5.13}$$

and therefore

$$\begin{aligned}\hat{f}\,(\mathbf{k},\mathbf{v},s)\ &=\frac{\tau^{-1}}{s+i\mathbf{v}\cdot\mathbf{k}+\tau^{-1}}\cdot\frac{\tau^{-1}}{s+i\mathbf{v}_0\cdot\mathbf{k}+\tau^{-1}}\cdot\frac{1}{s+\frac{1}{3}v_0^2\,k^2\,\tau+\ldots}\cdot\frac{\delta(v-v_0)}{4\pi v_0^2}\\&\qquad+\frac{\delta(\mathbf{v}-\mathbf{v}_0)}{s+i\mathbf{v}_0\cdot\mathbf{k}+\tau^{-1}}\,.\end{aligned}$$

We are interested in the long time limit $t\gg\tau$ for $f(\mathbf{r},\mathbf{v},t)$. This is dominated by $s\sim t^{-1}$, and we assume that τ^{-1} is dominant over s and $i\mathbf{v}\cdot\mathbf{k}$. We then have

$$\hat{f}\,(\mathbf{k},\mathbf{v},s)\approx\frac{1}{s+\frac{1}{3}v_0^2\,k^2\,\tau}\cdot\frac{\delta(v-v_0)}{4\pi v_0^2}\,. \tag{8.5.14}$$

Performing the inverse Laplace and Fourier transforms, we obtain

$$f(\mathbf{r},\mathbf{v},t)=(4\pi Dt)^{-3/2}\,e^{-r^2/4Dt}\cdot\frac{\delta(v-v_0)}{4\pi v_0^2}\,, \tag{8.5.15}$$

where the *diffusion constant* is

$$D=\frac{1}{3}v_0^2\,\tau\,. \tag{8.5.16}$$

The units are $[D]=L^2/T$. Integrating over velocities, we have the density

$$n(\mathbf{r},t)=\int\!d^3\!v\,f(\mathbf{r},\mathbf{v},t)=(4\pi Dt)^{-3/2}\,e^{-r^2/4Dt}\,. \tag{8.5.17}$$

Note that

$$\int d^3\!r\, n(\mathbf{r},t) = 1 \tag{8.5.18}$$

for all time. Total particle number is conserved!
