# 2.14: Appendix I- Integrating Factors

Suppose we have an inexact differential

$$\mathchar'26dW = A_i\, dx_i \ . \tag{2.14.1}$$

Here I am adopting the 'Einstein convention' where we sum over repeated indices unless otherwise explicitly stated; $A_i\, dx_i = \sum_i A_i\, dx_i$. An *integrating factor* $e^{L(\vec{x})}$ is a function which, when divided into $\mathchar'26dF$, yields an exact differential:

$$dU = e^{-L}\ \mathchar'26dW = \frac{\partial U}{\partial x_i}\, dx_i \ . \tag{2.14.2}$$

Clearly we must have

$$\frac{\partial^2 U}{\partial x_i\, \partial x_j} = \frac{\partial}{\partial x_i}\left(e^{-L}\, A_j\right) = \frac{\partial}{\partial x_j}\left(e^{-L}\, A_i\right) . \tag{2.14.3}$$

Applying the Leibniz rule and then multiplying by $e^L$ yields

$$\frac{\partial A_j}{\partial x_i} - A_j\, \frac{\partial L}{\partial x_i} = \frac{\partial A_i}{\partial x_j} - A_i\, \frac{\partial L}{\partial x_j} \ . \tag{2.14.4}$$

If there are $K$ independent variables $\{x_1, \ldots, x_K\}$, then there are $\frac{1}{2}K(K-1)$ independent equations of the above form – one for each distinct $(i,j)$ pair. These equations can be written compactly as

$$\Omega_{ijk}\, \frac{\partial L}{\partial x_k} = F_{ij} \ , \tag{2.14.5}$$

where

$$\Omega_{ijk} = A_j\, \delta_{ik} - A_i\, \delta_{jk}$$

$$F_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j} \ .$$

Note that $F_{ij}$ is antisymmetric, and resembles a field strength tensor, and that $\Omega_{ijk} = -\Omega_{jik}$ is antisymmetric in the first two indices (but is not totally antisymmetric in all three).

Can we solve these $\frac{1}{2}K(K-1)$ coupled equations to find an integrating factor $L$? In general the answer is *no*. However, when $K=2$ we can always find an integrating factor. To see why, let's call $x \equiv x_1$ and $y \equiv x_2$. Consider now the ODE

$$\frac{dy}{dx} = -\frac{A_x(x,y)}{A_y(x,y)} \ . \tag{2.14.6}$$

This equation can be integrated to yield a one-parameter set of integral curves, indexed by an initial condition. The equation for these curves may be written as $U_c(x,y) = 0$, where $c$ labels the curves. Then along each curve we have

$$\begin{aligned} 0 = \frac{dU_c}{dx} &= \frac{\partial U_x}{\partial x} + \frac{\partial U_c}{\partial y}\, \frac{dy}{dx} \\ &= \frac{\partial U_c}{\partial x} - \frac{A_x}{A_y}\, \frac{\partial U_c}{\partial y} \ . \end{aligned}$$

Thus,

$$\frac{\partial U_c}{\partial x}\, A_y = \frac{\partial U_c}{\partial y}\, A_x \equiv e^{-L} A_x\, A_y \ . \tag{2.14.7}$$

This equation defines the integrating factor $L$:

$$L = -\ln\left(\frac{1}{A_x}\,\frac{\partial U_c}{\partial x}\right) = -\ln\left(\frac{1}{A_y}\,\frac{\partial U_c}{\partial y}\right)\ . \tag{2.14.8}$$

We now have that

$$A_x = e^L\,\frac{\partial U_c}{\partial x} \qquad , \qquad A_y = e^L\,\frac{\partial U_c}{\partial y}\ , \tag{2.14.9}$$

and hence

$$e^{-L}\ \backslash\mathrm{mathchar}'26\!dW = \frac{\partial U_c}{\partial x}\,dx + \frac{\partial U_c}{\partial y}\,dy = dU_c\ . \tag{2.14.10}$$

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