8.7: The Equations of Hydrodynamics

We now derive the equations governing fluid flow. The equations of mass and momentum balance are

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{V}) = 0$$
$$\frac{\partial(\rho\,V_\alpha)}{\partial t} + \frac{\partial \Pi_{\alpha\beta}}{\partial x^\beta} = 0\ ,$$

where \[\RPi\ns_{\alpha\beta}=\rho\, V\ns_\alpha V\ns_\beta + p\,\delta\ns_{\alpha\beta} \, - \, \stackrel

{\overbrace{\left\{\eta\Bigg({\pz V\ns_\alpha\over\pz x\ns_\beta} + {\pz V\ns_\beta\over\pz x\ns_\alpha}-\frac{2}
{3}\,\bnabla\!\cdot\!\BV\,\delta\ns_{\alpha\beta}\bigg) +\zeta\,\bnabla\!\cdot\!\BV\,\delta\ns_{\alpha\beta}\right\}}}\ .\]
Substituting the continuity equation into the momentum balance equation, one arrives at

$$\rho\,\frac{\partial \mathbf{V}}{\partial t} + \rho\,(\mathbf{V}\cdot\boldsymbol{\nabla})\mathbf{V} = -\boldsymbol{\nabla} p + \eta\,\nabla^2\mathbf{V} + (\zeta + \frac{1}{3}\eta)\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{V})\ , \tag{8.7.1}$$

which, together with continuity, are known as the *Navier-Stokes equations*. These equations are supplemented by an equation describing the conservation of energy,

$$T\,\frac{\partial s}{\partial T} + T\,\boldsymbol{\nabla}\cdot(s\mathbf{V}) = \tilde{\sigma}_{\alpha\beta}\,\frac{\partial V_\alpha}{\partial x^\beta} + \boldsymbol{\nabla}\cdot(\kappa\boldsymbol{\nabla}T)\ . \tag{8.7.2}$$

Note that the LHS of Equation [NSB] is $\rho\,D\mathbf{V}/Dt$, where D/Dt is the convective derivative. Multiplying by a differential volume, this gives the mass times the acceleration of a differential local fluid element. The RHS, multiplied by the same differential volume, gives the differential force on this fluid element in a frame instantaneously moving with constant velocity $\mathbf{V}$. Thus, this is Newton's Second Law for the fluid.

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