# 5.5: Photon Statistics

## Thermodynamics of the photon gas

There exists a certain class of particles, including photons and certain elementary excitations in solids such as phonons ( lattice vibrations) and magnons ( spin waves) which obey bosonic statistics but with zero chemical potential. This is because their overall number is not conserved (under typical conditions) – photons can be emitted and absorbed by the atoms in the wall of a container, phonon and magnon number is also not conserved due to various processes, In such cases, the free energy attains its minimum value with respect to particle number when

$$\mu = \left(\frac{\partial F}{\partial N}\right)_{T.V} = 0\ . \tag{5.5.1}$$

The number distribution, from Equation ???, is then

$$n(\varepsilon) = \frac{1}{e^{\beta\varepsilon} - 1}\ . \tag{5.5.2}$$

The grand partition function for a system of particles with $\mu = 0$ is

$$\Omega(T,V) = V k_B T \int\limits_{-\infty}^{\infty} \! d\varepsilon\, g(\varepsilon)\, \ln\big(1 - e^{-\varepsilon/k_B T}\big)\ , \tag{5.5.3}$$

where $g(\varepsilon)$ is the density of states per unit volume.

Suppose the particle dispersion is $\varepsilon(\mathbf{p}) = A|\mathbf{p}|^\sigma$. We can compute the density of states $g(\varepsilon)$:

$$\begin{aligned} g(\varepsilon) &= \mathsf{g} \int \! \frac{d^d p}{h^d}\, \delta\big(\varepsilon - A|\mathbf{p}|^\sigma\big) = \frac{\mathsf{g}\Omega_d}{h^d} \int\limits_0^\infty \! dp\, p^{d-1}\, \delta(\varepsilon - A p^\sigma) \\ &= \frac{\mathsf{g}\Omega_d}{\sigma h^d}\, A^{-\frac{d}{\sigma}} \int\limits_0^\infty \! dx\, x^{\frac{d}{\sigma} - 1}\, \delta(\varepsilon - x) = \frac{2\,\mathsf{g}}{\sigma\,\Gamma(d/2)} \left(\frac{\sqrt{\pi}}{hA^{1/\sigma}}\right)^{\!d} \varepsilon^{\frac{d}{\sigma} - 1}\, \Theta(\varepsilon) \quad , \end{aligned}$$

where $\mathsf{g}$ is the internal degeneracy, due, for example, to different polarization states of the photon. We have used the result $\Omega_d = 2\pi^{d/2}\big/\Gamma(d/2)$ for the solid angle in $d$ dimensions. The step function $\Theta(\varepsilon)$ is perhaps overly formal, but it reminds us that the energy spectrum is bounded from below by $\varepsilon = 0$, there are no negative energy states.

For the photon, we have $\varepsilon(\mathbf{p}) = cp$, hence $\sigma = 1$ and

$$g(\varepsilon) = \frac{2\mathsf{g}\,\pi^{d/2}}{\Gamma(d/2)}\, \frac{\varepsilon^{d-1}}{(hc)^d}\, \Theta(\varepsilon)\ . \tag{5.5.4}$$

In $d = 3$ dimensions the degeneracy is $\mathsf{g} = 2$, the number of independent polarization states. The pressure $p(T)$ is then obtained using $\Omega = -pV$. We have

$$\begin{aligned} p(T) &= -k_B T \int\limits_{-\infty}^{\infty} \! d\varepsilon\, g(\varepsilon)\, \ln\big(1 - e^{-\varepsilon/k_B T}\big) \\ &= -\frac{2\,\mathsf{g}\,\pi^{d/2}}{\Gamma(d/2)}\, (hc)^{-d}\, k_B T \int\limits_0^\infty \! d\varepsilon\, \varepsilon^{d-1}\, \ln\big(1 - e^{-\varepsilon/k_B T}\big) \\ &= -\frac{2\,\mathsf{g}\,\pi^{d/2}}{\Gamma(d/2)}\, \frac{(k_B T)^{d+1}}{(hc)^d} \int\limits_0^\infty \! dt\, t^{d-1} \ln\big(1 - e^{-t}\big)\ . \end{aligned}$$

We can make some progress with the dimensionless integral:

$$\begin{aligned}\mathcal{I}_d &\equiv -\int_0^\infty dt\, t^{d-1} \ln\left(1-e^{-t}\right) \\ &= \sum_{n=1}^\infty \frac{1}{n} \int_0^\infty dt\, t^{d-1}\, e^{-nt} \\ &= \Gamma(d) \sum_{n=1}^\infty \frac{1}{n^{d+1}} = \Gamma(d)\, \zeta(d+1)\, .\end{aligned}$$

Finally, we invoke a result from the mathematics of the gamma function known as the *doubling formula*,

$$\Gamma(z) = \frac{2^{z-1}}{\sqrt{\pi}}\, \Gamma\big(\frac{z}{2}\big)\, \Gamma\big(\frac{z+1}{2}\big)\, . \tag{5.5.5}$$

Putting it all together, we find

$$p(T) = \mathsf{g}\, \pi^{-\frac{1}{2}(d+1)}\, \Gamma\big(\frac{d+1}{2}\big)\, \zeta(d+1)\, \frac{(k_B T)^{d+1}}{(\hbar c)^d}\, . \tag{5.5.6}$$

The number density is found to be

$$\begin{aligned}n(T) &= \int_{-\infty}^{\infty} d\varepsilon\, \frac{g(\varepsilon)}{e^{\varepsilon/k_B T}-1} \\ &= \mathsf{g}\, \pi^{-\frac{1}{2}(d+1)}\, \Gamma\big(\frac{d+1}{2}\big)\, \zeta(d) \left(\frac{k_B T}{\hbar c}\right)^d .\end{aligned}$$

For photons in $d=3$ dimensions, we have $\mathsf{g}=2$ and thus

$$n(T) = \frac{2\,\zeta(3)}{\pi^2} \left(\frac{k_B T}{\hbar c}\right)^3 \qquad , \qquad p(T) = \frac{2\,\zeta(4)}{\pi^2}\, \frac{(k_B T)^4}{(\hbar c)^3}\, . \tag{5.5.7}$$

It turns out that $\zeta(4) = \frac{\pi^4}{90}$.

Note that $\hbar c/k_B = 0.22855\, cm \cdot K$, so

$$\frac{k_B T}{\hbar c} = 4.3755\, T[K]\, cm^{-1} \quad \Longrightarrow \quad n(T) = 20.405 \times T^3 [K^3]\, cm^{-3}\, . \tag{5.5.8}$$

To find the entropy, we use Gibbs-Duhem:

$$d\mu = 0 = -s\, dT + v\, dp \quad \Longrightarrow \quad s = v\, \frac{dp}{dT}\, , \tag{5.5.9}$$

where $s$ is the entropy per particle and $v = n^{-1}$ is the volume per particle. We then find

$$s(T) = (d{+}1)\, \frac{\zeta(d{+}1)}{\zeta(d)}\, k_B\, . \tag{5.5.10}$$

The entropy per particle is constant. The internal energy is

$$E = -\frac{\partial \ln \Xi}{\partial \beta} = -\frac{\partial}{\partial \beta}\big(\beta p V\big) = d \cdot p\, V\, , \tag{5.5.11}$$

and hence the energy per particle is

$$\varepsilon = \frac{E}{N} = d \cdot pv = \frac{d \cdot \zeta(d{+}1)}{\zeta(d)}\, k_B T\, . \tag{5.5.12}$$

## Classical arguments for the photon gas

A number of thermodynamic properties of the photon gas can be determined from purely classical arguments. Here we recapitulate a few important ones.

- Suppose our photon gas is confined to a rectangular box of dimensions $L_x \times L_y \times L_z$ . Suppose further that the dimensions are all expanded by a factor $\lambda^{1/3}$, the volume is isotropically expanded by a factor of $\lambda$. The cavity modes of the electromagnetic radiation have quantized wavevectors, even within classical electromagnetic theory, given by

$$\mathbf{k} = \bigg(\frac{2\pi n_x}{L_x}\, , \, \frac{2\pi n_y}{L_y}\, , \, \frac{2\pi n_z}{L_z}\bigg)\ . \tag{5.5.13}$$

  Since the energy for a given mode is $\varepsilon(\mathbf{k}) = \hbar c|\mathbf{k}|$ , we see that the energy changes by a factor $\lambda^{-1/3}$ under an adiabatic volume expansion $V \to \lambda V$ , where the distribution of different electromagnetic mode occupancies remains fixed. Thus,

$$V\bigg(\frac{\partial E}{\partial V}\bigg)_{\!S} = \lambda\bigg(\frac{\partial E}{\partial \lambda}\bigg)_{\!S} = -\frac{1}{3}E\ . \tag{5.5.14}$$

  Thus,

$$p = -\bigg(\frac{\partial E}{\partial V}\bigg)_{\!S} = \frac{E}{3V}\ , \tag{5.5.15}$$

  as we found in Equation [photE]. Since $E = E(T,V)$ is extensive, we must have $p = p(T)$ alone.
- Since $p = p(T)$ alone, we have

$$\begin{aligned}\bigg(\frac{\partial E}{\partial V}\bigg)_{\!T} &= \bigg(\frac{\partial E}{\partial V}\bigg)_{\!p} = 3p \\ &= T\bigg(\frac{\partial p}{\partial T}\bigg)_{\!V} - p\ ,\end{aligned}$$

  where the second line follows the Maxwell relation $\big(\frac{\partial S}{\partial V}\big)_p = \big(\frac{\partial p}{\partial T}\big)_V$, after invoking the First Law $dE = TdS - p\,dV$ . Thus,

$$T\,\frac{dp}{dT} = 4p \qquad \Longrightarrow \qquad p(T) = A\,T^4\ , \tag{5.5.16}$$

  where $A$ is a constant. Thus, we recover the temperature dependence found microscopically in Equation [photp].
- Given an energy density $E/V$, the differential energy flux emitted in a direction $\theta$ relative to a surface normal is

$$dj_\varepsilon = c\cdot\frac{E}{V}\cdot\cos\theta\cdot\frac{d\Omega}{4\pi}\ , \tag{5.5.17}$$

  where $d\Omega$ is the differential solid angle. Thus, the power emitted per unit area is

$$\frac{dP}{dA} = \frac{cE}{4\pi V}\int\limits_0^{\pi/2}\!\! d\theta\!\int\limits_0^{2\pi}\!\! d\phi\,\sin\theta\cdot\cos\theta = \frac{cE}{4V} = \frac{3}{4}\,c\,p(T) \equiv \sigma\,T^4\ , \tag{5.5.18}$$

  where $\sigma = \frac{3}{4}cA$, with $p(T) = A\,T^4$ as we found above. From quantum statistical mechanical considerations, we have

  \sigma={\pi^2 k_\ssr{B}^4\over 60\,c^2\,\hbar^3}=5.67\times 10^{-8}\,{\RW\over\Rm^2\,\RK^4} \label{stefan}

  is *Stefan's constant*.

## Surface temperature of the earth

We derived the result $P = \sigma T^4\cdot A$ where $\sigma = 5.67\times10^{-8}\ W/\text{\Rm}^2\ K^4$ for the power emitted by an electromagnetic 'black body'. Let's apply this result to the earth-sun system. We'll need three lengths: the radius of the sun $R_\odot = 6.96\times10^8$ \Rm, the radius of the earth $R_e = 6.38\times10^6$ \Rm, and the radius of the earth's orbit $a_e = 1.50\times10^{11}$ \Rm. Let's assume that the earth has achieved a steady state temperature of $T_e$. We balance the total power incident upon the earth with the power radiated by the earth. The power incident upon the earth is

$$P_{incident} = \frac{\pi R_e^2}{4\pi a_e^2} \cdot \sigma T_\odot^4 \cdot 4\pi R_\odot^2 = \frac{R_e^2\, R_\odot^2}{a_e^2} \cdot \pi\sigma T_\odot^4 \, . \tag{5.5.19}$$

The power radiated by the earth is

$$P_{radiated} = \sigma T_e^4 \cdot 4\pi R_e^2 \, . \tag{5.5.20}$$

Setting $P_{incident} = P_{radiated}$ , we obtain

$$T_e = \left(\frac{R_\odot}{2\,a_e}\right)^{1/2} T_\odot \, . \tag{5.5.21}$$

Thus, we find $T_e = 0.04817\, T_\odot$, and with $T_\odot = 5780\,K$, we obtain $T_e = 278.4\,K$. The mean surface temperature of the earth is $\bar{T}_e = 287\,K$, which is only about $10\,K$ higher. The difference is due to the fact that the earth is not a perfect blackbody, an object which absorbs all incident radiation upon it and emits radiation according to Stefan's law. As you know, the earth's atmosphere retraps a fraction of the emitted radiation – a phenomenon known as the *greenhouse effect*.

[planck] Spectral density \rho_\ve(\nu,T) for blackbody radiation at three temperatures.

[planck] Spectral density $\rho_\varepsilon(\nu,T)$ for blackbody radiation at three temperatures.

## Distribution of blackbody radiation

Recall that the frequency of an electromagnetic wave of wavevector $\mathbf{k}$ is $\nu = c/\lambda = ck/2\pi$. Therefore the number of photons $\mathcal{N}_T(\nu,T)$ per unit frequency in thermodynamic equilibrium is (recall there are two polarization states)

$$\mathcal{N}(\nu,T)\,d\nu = \frac{2\,V}{8\pi^3} \cdot \frac{d^3k}{e^{\hbar ck/k_BT}-1} = \frac{V}{\pi^2} \cdot \frac{k^2\,dk}{e^{\hbar ck/k_BT}-1} \, . \tag{5.5.22}$$

We therefore have

$$\mathcal{N}(\nu,T) = \frac{8\pi V}{c^3} \cdot \frac{\nu^2}{e^{h\nu/k_BT}-1} \, . \tag{5.5.23}$$

Since a photon of frequency $\nu$ carries energy $h\nu$, the energy per unit frequency $\mathcal{E}(\nu)$ is

$$\mathcal{E}(\nu,T) = \frac{8\pi hV}{c^3} \cdot \frac{\nu^3}{e^{h\nu/k_BT}-1} \, . \tag{5.5.24}$$

Note what happens if Planck's constant $h$ vanishes, as it does in the classical limit. The denominator can then be written

$$e^{h\nu/k_BT} - 1 = \frac{h\nu}{k_BT} + \mathcal{O}(h^2) \tag{5.5.25}$$

and

\CE\ns_\ssr{CL}(\nu,T)=\lim_{h\to 0} \CE(\nu)=V\cdot{8\pi\kT\over c^3}\,\nu^2\ .

In classical electromagnetic theory, then, the total energy integrated over all frequencies *diverges*. This is known as the *ultraviolet catastrophe*, since the divergence comes from the large $\nu$ part of the integral, which in the optical spectrum is the ultraviolet portion. With quantization, the Bose-Einstein factor imposes an effective ultraviolet cutoff $k_BT/h$ on the frequency integral, and the total energy, as we found above, is finite:

$$E(T) = \int_0^\infty d\nu\; \mathcal{E}(\nu) = 3pV = V \cdot \frac{\pi^2}{15}\, \frac{(k_BT)^4}{(\hbar c)^3} \, . \tag{5.5.26}$$

We can define the spectral density $\rho_\varepsilon(\nu)$ of the radiation as

$$\rho_\varepsilon(\nu,T) \equiv \frac{\mathcal{E}(\nu,T)}{E(T)} = \frac{15}{\pi^4}\,\frac{h}{k_BT}\,\frac{(h\nu/k_BT)^3}{e^{h\nu/k_BT}-1} \tag{5.5.27}$$

so that $\rho_\varepsilon(\nu,T)\,d\nu$ is the fraction of the electromagnetic energy, under equilibrium conditions, between frequencies $\nu$ and $\nu+d\nu$, $\int\limits_0^\infty d\nu\,\rho_\varepsilon(\nu,T)=1$. In Figure [planck] we plot this in Figure [planck] for three different temperatures. The maximum occurs when $s\equiv h\nu/k_BT$ satisfies

$$\frac{d}{ds}\bigg(\frac{s^3}{e^s-1}\bigg)=0 \qquad\Longrightarrow\qquad \frac{s}{1-e^{-s}}=3 \qquad\Longrightarrow\qquad s=2.82144\ . \tag{5.5.28}$$

## What if the sun emitted ferromagnetic spin waves?

We saw in Equation [jephoton] that the power emitted per unit surface area by a blackbody is $\sigma T^4$. The power law here follows from the ultrarelativistic dispersion $\varepsilon=\hbar ck$ of the photons. Suppose that we replace this dispersion with the general form $\varepsilon=\varepsilon({\bf k})$. Now consider a large box in equilibrium at temperature $T$. The *energy current* incident on a differential area $dA$ of surface normal to $\hat{\bf z}$ is

$$dP=dA\cdot\!\int\!\frac{d^3\!k}{(2\pi)^3}\,\Theta(\cos\theta)\cdot\varepsilon({\bf k})\cdot\frac{1}{\hbar}\,\frac{\partial\varepsilon({\bf k})}{\partial k_z}\cdot\frac{1}{e^{\varepsilon({\bf k})/k_BT}-1}\ . \tag{5.5.29}$$

Let us assume an isotropic power law dispersion of the form $\varepsilon({\bf k})=Ck^\alpha$. Then after a straightforward calculation we obtain

$$\frac{dP}{dA}=\sigma\,T^{2+\frac{2}{\alpha}}\ , \tag{5.5.30}$$

where

$$\sigma=\zeta\big(2+\frac{2}{\alpha}\big)\,\Gamma\big(2+\frac{2}{\alpha}\big)\cdot\frac{{\sf g}\,k_B^{2+\frac{2}{\alpha}}\,C^{-\frac{2}{\alpha}}}{8\pi^2\hbar}\ . \tag{5.5.31}$$

One can check that for ${\sf g}=2$, $C=\hbar c$, and $\alpha=1$ that this result reduces to that of Equation [stefan].