9.4: Scaling Variables

We've seen how an RG transformation acts on the (infinite) set of couplings which define the Hamiltonian of a system. We found $\mathbf{K}' = \boldmath{\mathcal{R}}_b(\mathbf{K})$. If $\xi(\mathbf{K})$ is the correlation length in units of the lattice spacing, then since each RG step involves a rescaling by a factor b, we must have

$$\xi(\mathbf{K}) = b\,\xi(\mathbf{K}') = b^2\,\xi(\mathbf{K}'') = \cdots \ . \tag{9.4.1}$$

A *fixed point* of the transformation $\boldmath{\mathcal{R}}_b$ is a set of couplings $\mathbf{K}^*$ such that

$$\boldmath{\mathcal{R}}_b(\mathbf{K}^*) = \mathbf{K}^* \ . \tag{9.4.2}$$

Linearizing $\boldmath{\mathcal{R}}_b(\mathbf{K})$ about the fixed point, we have

$$K'_\alpha - K^*_\alpha = \sum_\beta Q_{\alpha\beta}\,(K_\beta - K^*_\beta) \qquad , \qquad Q_{\alpha\beta} = \frac{\partial K'_\alpha}{\partial K_\beta}\bigg|_{\mathbf{K}^*} \ . \tag{9.4.3}$$

The matrix $Q_{\alpha\beta}$ is real but not necessarily symmetric. We define the left eigenvectors of Q, $\phi^{(i)}_\alpha$, such that

$$\sum_\alpha \phi^{(i)}_\alpha\,Q_{\alpha\beta} = \lambda_i\,\phi^{(i)}_\beta \ . \tag{9.4.4}$$

The *scaling variable* u_i is then defined as

$$u_i \equiv \sum_\alpha \phi^{(i)}_\alpha\,(K_\alpha - K^*_\alpha) \ . \tag{9.4.5}$$

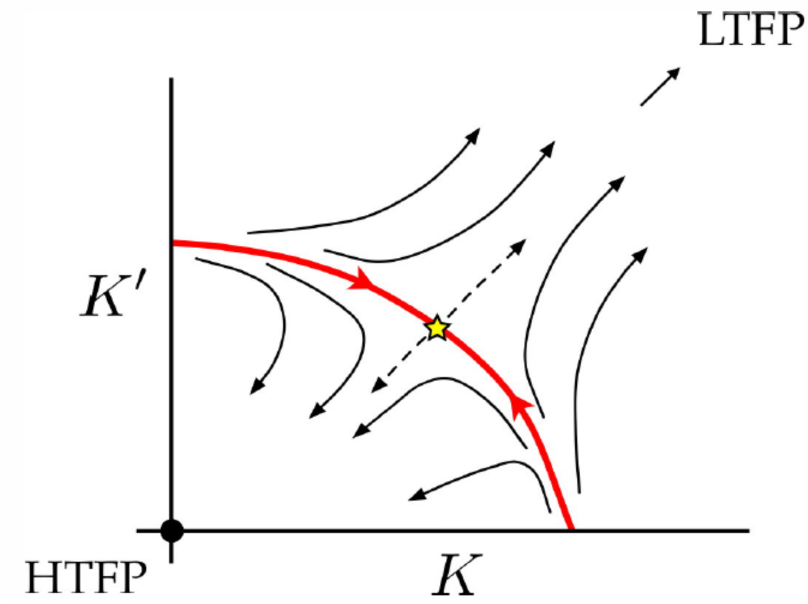
It should now be apparent that under an RG transformation, we have

$$u'_i = \sum_\alpha \phi^{(i)}_\alpha\,(K'_\alpha - K^*_\alpha) = \sum_{\alpha,\beta} \phi^{(i)}_\alpha\,,Q_{\alpha\beta}(K_\beta - K^*_\beta) = \lambda_i \sum_\alpha \phi^{(i)}_\alpha\,(K_\alpha - K^*_\alpha) = \lambda_i u_i \ . \tag{9.4.6}$$

We say that

$$u_i\ is\ \begin{cases} \text{\it relevant} & if\ \lambda_i > 1 \\ \text{\it irrelevant} & if\ \lambda_i < 1 \\ \text{\it marginal} & if\ \lambda_i = 1 \ . \end{cases} \tag{9.4.7}$$

Under renormalization, relevant scaling variables flow away from the fixed point, while irrelevant scaling variables flow toward the fixed point. For marginal variables, one must go to higher order, beyond the above linearization, to determine whether the flow is away from (marginally relevant) or toward (marginally irrelevant) the fixed point.


LTFP
K'
HTFP
K