8.11: Appendix II- Distributions and Functionals

Let $x \in \mathbb{R}$ be a random variable, and $P(x)$ a probability distribution for x. The average of any function $\phi(x)$ is then

$$\big\langle \phi(x) \big\rangle = \int\limits_{-\infty}^{\infty} dx \; P(x) \, \phi(x) \bigg/ \int\limits_{-\infty}^{\infty} dx \; P(x) \, . \tag{8.11.1}$$

Let $\eta(t)$ be a random *function* of t, with $\eta(t) \in \mathbb{R}$, and let $P\big[\eta(t)\big]$ be the probability distribution *functional* for $\eta(t)$. Then if $\Phi\big[\eta(t)\big]$ is a functional of $\eta(t)$, the average of Φ is given by

$$\int\!\! D\eta \, P\big[\eta(t)\big] \; \Phi\big[\eta(t)\big] \bigg/ \int\!\! D\eta \, P\big[\eta(t)\big] \tag{8.11.2}$$

The expression $\int\! D\eta \, P[\eta] \, \Phi[\eta]$ is a *functional integral*. A functional integral is a continuum limit of a multivariable integral. Suppose $\eta(t)$ were defined on a set of t values $t_n = n\tau$. A functional of $\eta(t)$ becomes a multivariable function of the values $\eta_n \equiv \eta(t_n)$. The metric then becomes

$$D\eta \longrightarrow \prod_n d\eta_n \, . \tag{8.11.3}$$

In fact, for our purposes we will not need to know any details about the functional measure $D\eta$; we will finesse this delicate issue[18]. Consider the *generating functional*,

$$Z\big[J(t)\big] = \int\!\! D\eta \, P[\eta] \, \exp\!\bigg(\int\limits_{-\infty}^{\infty}\! dt \; J(t) \, \eta(t) \bigg) \, . \tag{8.11.4}$$

It is clear that

$$\frac{1}{Z[J]} \, \frac{\delta^n Z[J]}{\delta J(t_1) \cdots \delta J(t_n)} \bigg|_{J(t)=0} = \big\langle \eta(t_1) \cdots \eta(t_n) \big\rangle \, . \tag{8.11.5}$$

The function $J(t)$ is an arbitrary *source function*. We differentiate with respect to it in order to find the η-field correlators.

[Fdiscretize] Discretization of a continuous function $\eta(t)$. Upon discretization, a functional $\Phi\big[\eta(t)\big]$ becomes an ordinary multivariable function $\Phi(\{\eta_j\})$.

Let's compute the generating function for a class of distributions of the Gaussian form,

$$\begin{aligned} P[\eta] &= \exp\!\bigg(-\frac{1}{2\Gamma} \int\limits_{-\infty}^{\infty}\! dt \, \big(\tau^2 \, \dot\eta^2 + \eta^2 \big) \bigg) \\ &= \exp\!\bigg(-\frac{1}{2\Gamma} \int\limits_{-\infty}^{\infty}\! \frac{d\omega}{2\pi} \, \big(1 + \omega^2\tau^2\big) \, \big|\hat\eta(\omega)\big|^2 \bigg) \, . \end{aligned}$$

Then Fourier transforming the source function $J(t)$, it is easy to see that

$$Z[J]=Z[0]\cdot\exp\left(\frac{\Gamma}{2}\int\limits_{-\infty}^{\infty}\frac{d\omega}{2\pi}\,\frac{\left|\hat{J}\left(\omega\right)\right|^{2}}{1+\omega^{2}\tau^{2}}\right)\,. \tag{8.11.6}$$

Note that with $\eta(t)\in\mathbb{R}$ and $J(t)\in\mathbb{R}$ we have $\eta^{*}(\omega)=\eta(-\omega)$ and $J^{*}(\omega)=J(-\omega)$. Transforming back to real time, we have

$$Z[J]=Z[0]\cdot\exp\left(\frac{1}{2}\int\limits_{-\infty}^{\infty}\!dt\int\limits_{-\infty}^{\infty}\!dt'\;J(t)\,G(t-t')\,J(t')\right)\,, \tag{8.11.7}$$

where

$$G(s)=\frac{\Gamma}{2\tau}\,e^{-|s|/\tau}\qquad,\qquad\widehat{G}(\omega)=\frac{\Gamma}{1+\omega^{2}\tau^{2}} \tag{8.11.8}$$

is the *Green's function*, in real and Fourier space. Note that

$$\int\limits_{-\infty}^{\infty}\!ds\,G(s)=\widehat{G}(0)=\Gamma\,. \tag{8.11.9}$$

We can now compute

$$\begin{aligned}\big\langle\eta(t_1)\,\eta(t_2)\big\rangle&=G(t_1-t_2)\\ \big\langle\eta(t_1)\,\eta(t_2)\,\eta(t_3)\,\eta(t_4)\big\rangle&=G(t_1-t_2)\,G(t_3-t_4)+G(t_1-t_3)\,G(t_2-t_4)\\ &\qquad+G(t_1-t_4)\,G(t_2-t_3)\,.\end{aligned}$$

The generalization is now easy to prove, and is known as *Wick's theorem*:

$$\big\langle\eta(t_1)\cdots\eta(t_{2n})\big\rangle=\sum_{contractions}G(t_{i_1}-t_{i_2})\cdots G(t_{i_{2n-1}}-t_{i_{2n}})\,, \tag{8.11.10}$$

where the sum is over all distinct *contractions* of the sequence $1\text{-}2\cdots2n$ into products of pairs. How many terms are there? Some simple combinatorics answers this question. Choose the index 1. There are $(2n-1)$ other time indices with which it can be contracted. Now choose another index. There are $(2n-3)$ indices with which *that* index can be contracted. And so on. We thus obtain

$$C(n)\equiv\begin{matrix}\text{\# of contractions}\\ \text{of 1-2-3\,}\cdots 2n\end{matrix}=(2n-1)(2n-3)\cdots3\cdot1=\frac{(2n)!}{2^{n}\,n!}\,. \tag{8.11.11}$$

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