

8.14: Appendix V- Kramers-Krönig Relations

Suppose $\hat{\chi}(\omega) \equiv \hat{G}(\omega)$ is analytic in the UHP¹⁹. Then for all ν , we must have

$$\int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{\hat{\chi}(\nu)}{\nu - \omega + i\epsilon} = 0, \quad (8.14.1)$$

where ϵ is a positive infinitesimal. The reason is simple: just close the contour in the UHP, assuming $\hat{\chi}(\omega)$ vanishes sufficiently rapidly that Jordan's lemma can be applied. Clearly this is an extremely weak restriction on $\hat{\chi}(\omega)$, given the fact that the denominator already causes the integrand to vanish as $|\omega|^{-1}$.

Let us examine the function

$$\frac{1}{\nu - \omega + i\epsilon} = \frac{\nu - \omega}{(\nu - \omega)^2 + \epsilon^2} - \frac{i\epsilon}{(\nu - \omega)^2 + \epsilon^2}. \quad (8.14.2)$$

which we have separated into real and imaginary parts. Under an integral sign, the first term, in the limit $\epsilon \rightarrow 0$, is equivalent to taking a *principal part* of the integral. That is, for any function $F(\nu)$ which is regular at $\nu = \omega$,

$$\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{\nu - \omega}{(\nu - \omega)^2 + \epsilon^2} F(\nu) \equiv \wp \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} \frac{F(\nu)}{\nu - \omega}. \quad (8.14.3)$$

The *principal part* symbol \wp means that the singularity at $\nu = \omega$ is elided, either by smoothing out the function $1/(\nu - \epsilon)$ as above, or by simply cutting out a region of integration of width ϵ on either side of $\nu = \omega$.

The imaginary part is more interesting. Let us write

$$h(u) \equiv \frac{\epsilon}{u^2 + \epsilon^2}. \quad (8.14.4)$$

For $|u| \gg \epsilon$, $h(u) \simeq \epsilon/u^2$, which vanishes as $\epsilon \rightarrow 0$. For $u = 0$, $h(0) = 1/\epsilon$ which diverges as $\epsilon \rightarrow 0$. Thus, $h(u)$ has a huge peak at $u = 0$ and rapidly decays to 0 as one moves off the peak in either direction a distance greater than ϵ . Finally, note that

$$\int_{-\infty}^{\infty} du h(u) = \pi, \quad (8.14.5)$$

a result which itself is easy to show using contour integration. Putting it all together, this tells us that

$$\lim_{\epsilon \rightarrow 0} \frac{\epsilon}{u^2 + \epsilon^2} = \pi \delta(u). \quad (8.14.6)$$

Thus, for positive infinitesimal ϵ ,

$$\frac{1}{u \pm i\epsilon} = \frac{\wp}{u} \mp i\pi \delta(u), \quad (8.14.7)$$

a most useful result.

We now return to our initial result [kka], and we separate $\hat{\chi}(\omega)$ into real and imaginary parts:

$$\hat{\chi}(\omega) = \hat{\chi}'(\omega) + i\hat{\chi}''(\omega). \quad (8.14.8)$$

(In this equation, the primes do not indicate differentiation with respect to argument.) We therefore have, for every real value of ω ,

$$0 = \int_{-\infty}^{\infty} \frac{d\nu}{2\pi} [\chi'(\nu) + i\chi''(\nu)] \left[\frac{\wp}{\nu - \omega} - i\pi \delta(\nu - \omega) \right]. \quad (8.14.9)$$

Taking the real and imaginary parts of this equation, we derive the *Kramers-Krönig relations*:

$$\begin{aligned}\chi'(\omega) &= +\wp \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \frac{\hat{\chi}''(\nu)}{\nu - \omega} \\ \chi''(\omega) &= -\wp \int_{-\infty}^{\infty} \frac{d\nu}{\pi} \frac{\hat{\chi}'(\nu)}{\nu - \omega} .\end{aligned}$$

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