

8.5: The Lagrangian Formulation of Classical Physics

So far, we have seen that, based on Newton's Laws, one can formulate a description of motion that is based solely on the concept of energy. A lot of research was done in the eighteenth century to reformulate a theory of mechanics that would be equivalent to Newton's Theory but whose starting point is the concept of energy instead of the concept of force. This "modern" approach to classical mechanics is primarily based on the research by Lagrange and Hamilton.

Although it is beyond the scope of this text to go into the details of this formulation, it is worth taking a quick look in order to get a better sense of how physicists seek to generalize theories. It is also worth noting that the Lagrangian formulation is the method by which theories are developed for quantum mechanics and modern physics.

The Lagrangian description of a "system" is based on a quantity, L , called the "Lagrangian", which is defined as:

$$L = K - U \quad (8.5.1)$$

where K is the kinetic energy of the system, and U is its potential energy. A "system" can be a rather complex collection of objects, although we will illustrate how the Lagrangian formulation is implemented for a single object of mass m moving in one dimension under the influence of gravity. Let x be the direction of motion (which is vertical) such that the potential and kinetic energies of the object are given by:

$$\begin{aligned} U(x) &= mgx \\ K(v_x) &= \frac{1}{2}mv_x^2 \\ \therefore L(x, v_x) &= \frac{1}{2}mv_x^2 - mgx \end{aligned}$$

where we chose the potential energy to be zero at $x = 0$, and v_x is the velocity of the object.

In the modern formulation of classical mechanics, the motion of the system will be such that the following integral is minimized:

$$S = \int L dt$$

where L can depend on time explicitly or implicitly (through the fact that position and velocity, on which the Lagrangian depends, are themselves time-dependent). The requirement that the above integral be minimized is called the "Principle of Least Action"¹, and is thought to be the fundamental principle that describes all of the laws of physics. The condition for the action to be minimized is given by the Euler-Lagrange equation:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = 0 \quad (8.5.2)$$

Thus, in the Lagrangian formulation, one first writes down the Lagrangian for the system, and then uses the Euler-Lagrange equation to obtain the "equations of motion" for the system (i.e. equation that give the kinematic quantities, such as acceleration, for the system).

Given the Lagrangian that we found above for a particle moving in one dimension under the influence of gravity, we can determine each term in the Euler-Lagrange equation:

$$\begin{aligned} \frac{\partial L}{\partial v_x} &= \frac{\partial}{\partial v_x} \left(\frac{1}{2}mv_x^2 - mgx \right) = mv_x \\ \therefore \frac{d}{dt} \left(\frac{\partial L}{\partial v_x} \right) &= \frac{d}{dt} (mv_x) = ma_x \\ \frac{\partial L}{\partial x} &= \frac{\partial}{\partial x} \left(\frac{1}{2}mv_x^2 - mgx \right) = -mg \end{aligned}$$

Putting these into the Euler-Lagrange equation:

$$\begin{aligned}\frac{d}{dt} \left(\frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} &= 0 \\ (ma_x) - (-mg) &= 0 \\ ma_x &= -mg \\ \therefore a_x &= -g\end{aligned}$$

which is exactly equivalent to using Newton's Second Law (the second last step is equivalent to $F = ma$). In the Lagrangian formulation, we do not need the concept of force. Instead, we describe possible "interactions" by a potential energy function. That is why you may sometimes hear of physicists talking about the "Weak interaction" instead of the "Weak force" when they are talking about one of the four fundamental interactions (forces) of Nature. This is because, in the modern formulation of physics, one does not use the concept of force, and instead thinks of potential energy functions to model what we would call a force in the Newtonian approach.

Emmy Noether, a mathematician in the early twentieth century, proved a theorem that makes the Lagrangian formulation particularly aesthetic. Noether's theorem states that for any symmetry in the Lagrangian, there exists a quantity that is conserved. For example, if the Lagrangian does not depend explicitly on time, then a quantity, which we call energy, is conserved².

The Lagrangian that we had above for a particle moving under the influence of gravity did not depend on time explicitly, and thus energy is conserved (gravitational potential energy is converted into kinetic energy and there are no non-conservative forces). If the Lagrangian did not depend on position, then a quantity that we call "momentum"³ would be conserved. In this case, momentum in the x direction was not conserved because the Lagrangian depended on x through the potential energy.

Olivia's Thoughts

We saw in this chapter that describing systems in terms of energy is often easier than describing them in terms of forces. The Lagrangian gives us a way to get the same information we would get from Newton's laws (like the acceleration, etc.), but using energy as a starting point. The Lagrangian method is really useful when we are looking at motion in multiple dimensions, or when we are describing complicated systems. Using the Lagrangian is actually really simple, and just like with forces, you can pretty much approach every problem the same way. Here are the basic steps to follow:

1. Find two expressions for your system: one for the potential energy (U) and one for the kinetic energy (K). This often ends up being the hardest step.
2. Write down the Lagrangian, $L = K - U$, using the expressions you just found.
3. Pick a coordinate. (In one dimension, this is trivial, but it will be important once you start working in multiple dimensions). The Euler-Lagrange equation was given to you as:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_x} \right) - \frac{\partial L}{\partial x} = 0$$

because we are working in one dimension. You can actually pick whichever coordinate you are interested in. For example, if you were interested in the motion of your object in the y direction, you would pick y as your coordinate and write:

$$\frac{d}{dt} \left(\frac{\partial L}{\partial v_y} \right) - \frac{\partial L}{\partial y} = 0$$

4. Now you just have to do what the equation above tells you to do, which is to start with your Lagrangian (your $L = K - U$ equation) and take a bunch of derivatives. If you try to just plug L into the Euler-Lagrange equation and do all the derivatives at once, it can get confusing. I recommend finding the components separately. I like to start by taking the partial derivative with respect to velocity, $\frac{\partial L}{\partial v}$, then taking its derivative with respect to time. Next, I find $\frac{\partial L}{\partial y}$ and then put it all together.
5. That's it! When you've taken the derivatives (and simplified a bit), you'll have an "equation of motion" that gives you information about the motion of the object. You can then use this equation however you want!

Footnotes

1. The integral, S , is called the “action” of the system.
2. If the Lagrangian does not depend on time, then we can shift the system in time and the equations of motion would be unaffected. We say that the Lagrangian is symmetric, or unaffected, by changes in time.
3. See chapter 10

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