17.5: Summary

We can define the **flux** of a uniform and constant vector field, $\vec{E}$, through a flat surface, as:

$$\Phi_E = \vec{E} \cdot \vec{A} = EA\cos\theta$$

where, $\vec{A}$, is a vector that is perpendicular to the surface with a magnitude equal to the area of that surface, and, θ, is the angle between $\vec{A}$ and $\vec{E}$. The flux of a field through a surface is proportional to the number of field lines that cross that surface. If the surface is parallel to the field ($\vec{A}$ and $\vec{E}$ are thus perpendicular), the flux through that surface is zero (no field lines cross the surface, the scalar product is zero).

If $\vec{E}$ and $\vec{A}$ change over the surface ($\vec{E}$ and/or $\vec{A}$ change magnitude and/or direction relative to each other along the surface), then we treat the surface as being made of infinitesimal surface elements over which the two vectors are constant. We define a vector $d\vec{A}$ to be perpendicular to the surface element with an infinitesimal area, dA. The total flux is then obtained by summing the fluxes through each surface element:

$$\Phi_E = \int \vec{E} \cdot d\vec{A} = \int EdA\cos\theta$$

Note that the direction of the vector $d\vec{A}$ (or $\vec{A}$) is ambiguous, as one can choose either of two directions perpendicular to a surface. Usually, one chooses the direction of $\vec{A}$ so that the flux is positive (i.e. $\vec{A}$ has a component parallel to $\vec{E}$). However, if the surface is "closed" (that is, it defines a volume), then we always choose the direction of $d\vec{A}$ so that it points outwards from the surface (since the surface encloses a volume, one can define an "inside" and an "outside").

In the case of the electric field, Gauss' Law relates the flux of the electric field from a closed surface to the amount of charge, Q^{enc}, contained in the volume enclosed by that surface:

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q^{enc}}{\epsilon_0}$$

Physically, Gauss' Law is a statement that field lines must begin or end on a charge (electric field lines originate on positive charges and terminate on negative charges). If there is a net number of lines coming out of a closed surface (a positive flux), that surface must enclose a positive charge from where those field lines originate. Similarly, if there are the same number of field lines entering a closed surface as there are lines exiting that surface (a flux of zero), then the surface encloses no charge. Gauss' Law states that the number of field lines exiting a closed surface is proportional to the amount of charge enclosed by that surface.

Gauss' Law is useful to determine the electric field. However, this can only be done analytically for charge distributions with a very high degree of symmetry. This is because the flux integral is not usually easy to evaluate unless:

1. **The electric field makes a constant angle with the surface**. When this is the case, the scalar product can be written in terms of the cosine of the angle between $\vec{E}$ and $d\vec{A}$, which can be taken out of the integral if it is constant:

$$\oint \vec{E} \cdot d\vec{A} = \oint E\cos\theta dA = \cos\theta \oint EdA$$

2. **The electric field is constant in magnitude along the surface**. When this is the case, the integral can be simplified further by factor out E, and simply becomes an integral over dA (which corresponds to the total area of the surface, A):

$$\oint \vec{E} \cdot d\vec{A} = \cos\theta \oint EdA = E\cos\theta \oint dA = EA\cos\theta$$

Note that Gauss' Law does not specify a closed surface over which to calculate the flux; it holds for any surface. We can thus choose a surface that will make the flux integral easy to evaluate - we call this choice a "gaussian surface" (not because it has some special property, but because we chose that surface to apply Gauss' Law). A procedure for applying Gauss' Law to determine the electric field at some point in space can be written as:

1. Make a diagram showing the charge distribution.
2. Use symmetry arguments to determine in which way the electric field vector points.

3. Choose a gaussian surface that goes through the point for which you want to know the electric field. Ideally, the surface is such that the electric field is constant in magnitude and always makes the same angle with the surface, so that the flux integral is straightforward to evaluate.
4. Calculate the flux, $\oint \vec{E} \cdot d\vec{A}$.
5. Calculate the amount of charge in the volume enclosed by the surface, Q^{enc}.
6. Apply Gauss' Law, $\oint \vec{E} \cdot d\vec{A} = \frac{Q^{enc}}{\epsilon_0}$.

We showed how Gauss' Law can be used to understand and quantify how charges arrange themselves on a conductor, in such a way that the electric field is zero everywhere in the conductor. Finally, we briefly introduced a more modern version of Gauss' Law that uses divergence instead of flux:

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

This last version has the advantage that it relates a local property of the field (divergence) to a local property of charge (charge density at some position in space).

Important Equations

Gauss' Law:

$$\Phi = \frac{Q_{enc}}{\epsilon_0}$$

$$\Phi = \int \vec{E} \cdot d\vec{A}$$

Important Definitions

Definition

Electric flux: A measure of the number of electric field lines crossing a surface. SI units: $[\mathrm{Vm}]$. Common variable(s): Φ_E.