24.9: Summary

Key Takeaways

The Theory of Special Relativity is based on Einstein's two postulates:

1. The laws of physics are the same in all inertial reference frames. There is no experiment that can be performed to determine whether one is at rest or moving with constant velocity.
2. The speed of light propagating in vacuum is the same in all inertial reference frames. Any observer in an inertial frame of reference, regardless of their velocity, will measure that light has a speed of c, when it propagates in vacuum.

These postulates are required in order for the equations from electromagnetism to be valid in all inertial frames of reference. However, they lead to very counter-intuitive results. For example, if two events, A and B, are simultaneous in one frame of reference, an observer in a different frame of reference will observe event A to happen earlier/later than event B (earlier or later will depend on the direction of motion of the moving observer).

The Theory of Special Relativity allows us to relate observations made in one inertial frame of reference, S, to observations made in a different inertial frame of reference, S', that is moving with constant velocity, $\vec v$, relative to S. We always choose to define the x axis in the S and S' frames of reference so that they are both co-linear with the velocity of S', $\vec v$, which is defined to be in the positive x direction in frame, S. Furthermore, we assume that the origin of both frames of reference coincided at time $t=0$.

We define the gamma factor, γ, based on the speed, v, of S' relative to S:

$$\gamma=\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

The gamma factor is always greater or equal to 1.

If a time interval, Δt, is measured in frame, S, then a "dilated" time interval, $\Delta t'$, will be measured in frame S':

$$\Delta t'=\gamma\Delta t$$

since $\gamma\geq 1$. We call the time that is measured in a frame of reference that we consider "at rest" to be the "proper time" in that frame of reference. For example, a muon decays in $2.2\mu\text{s}$ when at rest. If a muon moves at high speed, in the frame of reference where the muon is moving, it will take *longer* (time dilation), for the muon to decay. The time $2.2\mu\text{s}$ is the "proper time" for the muon decay (since it is measured when the muon is at rest).

As a consequence of time dilation, observers in different frames of reference will measure different lengths due to "length contraction". If an object has a "proper length", L, in a frame of reference, S, that is at rest relative to the object, the object will have a contracted length, L', in a reference frame, S', moving with speed, v, relative to S:

$$L'=\frac{L}{\gamma}$$

Note that only the dimension of the object that is co-linear with the velocity vector, $\vec v$, is contracted.

We also noted that Special Relativity is intimately connected to electromagnetism. In particular, we described how what we model as a magnetic force in one frame of reference might be modeled as an electric force in a different frame of reference.

In order to describe the motion of objects, we found that we need to define a four-dimensional space-time, where positions in space-time are labeled by 4 "coordinates", (x,y,z,ct), instead of the usual 3 (space) position coordinates. This is a result of the fact that time is no longer absolute and depends on the frame of reference (e.g. time dilation).

In space-time, we think in terms of events that occur at specific locations in space and instants in time. We can visualize space-time using "space-time diagrams", where one axis corresponds to space (x), and the other axis corresponds to time (ct). The path of an object through space-time is called its "world line".

For a given event in space-time, we can define past and future "light cones". Only events in the past light-cone could have had a causal effect on the event. Similarly, only events in the future light-cone can ever be influenced by that event. Events that can be causally connected (within each other's light cones) are said to be "time-like". Events that are outside of each other's light cones are said to be “space-like”. If two events are time-like, all observers will agree on the order in which the events happened, preserving the notion of causality. Different observers can disagree on the order in which space-like events occurred.

The Lorentz transformations allow us to convert the coordinates of events in one frame of reference, S, to those in a frames, S', moving with constant speed, v, relative to S:

$$\begin{aligned} x' &= \gamma(x - vt) \\ y' &= y \\ z' &= z \\ t' &= \gamma\left(t - \frac{vx}{c^2}\right) \end{aligned}$$

and the inverse relations are easily found:

$$\begin{aligned} x &= \gamma(x' + vt') \\ y &= y' \\ z &= z' \\ t &= \gamma\left(t' + \frac{vx'}{c^2}\right) \end{aligned}$$

Certain quantities, which are measured to be the same in all frames of reference, are said to be “Lorentz invariant”. In particular, we can define the space-time interval, s, between two events in space-time as:

$$s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 \ - c^2 \Delta t^2$$

One can think of this as a sort of “distance” in space-time, that does not depend on the frame of reference.

If an object has a velocity vector, $\vec{u}$, as measured in frame of reference S, then its velocity, $\vec{u}'$, in a frame, S', moving with speed, v, relative to S, is given by:

$$\begin{aligned} u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\ u'_y &= \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \\ u'_z &= \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \end{aligned}$$

and the reverse transformations are given by:

$$\begin{aligned} u_x &= \frac{u'_x + v}{1 + \frac{vu_x}{c^2}} \\ u_y &= \frac{u'_y}{\gamma\left(1 + \frac{vu_x}{c^2}\right)} \\ u_z &= \frac{u'_z}{\gamma\left(1 + \frac{vu_x}{c^2}\right)} \end{aligned}$$

In order for momentum and energy to be conserved in Special Relativity, these need to be redefined. If a particles with rest mass, m_0, has a velocity, $\vec{u}$, in an inertial frame of reference, its relativistic momentum, $\vec{p}$, is defined to be:

$$\vec{p} = \gamma m_0 \vec{u}$$

where the gamma factor is evaluated using the speed, u:

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}}$$

This relativistic definition of momentum is equivalent to the classical definition when $u << c$. We can think of relativistic momentum in the same way as classical momentum, if we model the mass of the object as increasing with its speed:

$$m(u) = \gamma m_0$$
$$\therefore \vec p = m(u)\vec u$$

where m_0 is the mass of the object measured when the object is at rest (its "rest mass"). An object with a rest mass can never reach the speed of light, as this would correspond to it having infinite momentum (or infinite mass).

With the relativistic definition of momentum, one can still use Newton's Second Law in the form:

$$\vec F = \frac{d\vec p}{dt}$$

We define the total energy, E, of an object as:

$$E = K + m_0c^2$$

which has a contribution from its kinetic energy, K, and from its mass (the second term). The energy that an object has by virtue of having a mass is called "rest mass energy", which implies that mass and energy can really be thought of as the same thing; one can convert mass into energy and vice versa (as in a nuclear reactor).

The kinetic energy of an object moving with speed, u, is given by:

$$K = m_0c^2(\gamma - 1)$$

where the gamma factor is obtained using the speed, u. This relativistic definition of kinetic energy is equivalent to the classical definition when $u << c$. The total energy of a particle can also be written as:

$$E = \gamma m_0c^2$$

Since energy and mass are simply related by a constant, one can use units of energy to describe the mass of a particle. It is common in particle physics to express the mass of particles in units of MeV/c^2.

Finally, we saw that the relativistic momentum and energy of an object are related:

$$E^2 = p^2c^2 + m_0^2c^4$$

In particular, particles of light, which have no mass but have kinetic energy, have non-zero momentum:

$$p = \frac{E}{c}$$

Important Equations

Lorentz factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

Time dilation:

$$\Delta t' = \gamma \Delta t$$

Length contraction:

$$L' = \frac{L}{\gamma}$$

Lorentz transformations:

$$
\begin{aligned}
x' &= \gamma(x - vt) \\
y' &= y \\
z' &= z \\
t' &= \gamma\left(t - \frac{vx}{c^2}\right)
\end{aligned}
$$

Velocity addition:

$$
\begin{aligned}
u'_x &= \frac{u_x - v}{1 - \frac{vu_x}{c^2}} \\
u'_y &= \frac{u_y}{\gamma\left(1 - \frac{vu_x}{c^2}\right)} \\
u'_z &= \frac{u_z}{\gamma\left(1 - \frac{vu_x}{c^2}\right)}
\end{aligned}
$$

The spacetime interval:

$$s^2 = \Delta x^2 + \Delta y^2 + \Delta z^2 - c^2 \Delta t^t$$

Relativistic momentum:

$$\vec{p} = \gamma m_0 \vec{u}$$

Relativistic energy:

$$E = \gamma m_0 c^2 = K + m_0 c^2$$

Relativistic kinetic energy:

$$K = (\gamma - 1) m_0 c^2$$

Newton’s Second Law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$

Energy-momentum relation:

$$E^2 = p^2 c^2 + m_0^2 c^4$$

Important Definitions

Definition

Proper time: The time measured in a frame of reference considered at rest. SI units: $[\mathrm{s}]$. Common variable(s): Δt.

Definition

Proper length: The length of an object as measured at rest relative to the object. SI units: $[\mathrm{m}]$. Common variable(s): L.