21.9: Sample problems and solutions

Exercise 21.9.1

A cathode ray tube in a television accelerates an electron using a potential difference of $\Delta V = 500\text{V}$. The electron must be deflected upwards by a distance $h = 3\text{cm}$ using a uniform magnetic field, $\vec{B}$, before striking the phosphorescent screen, which is a distance $d = 5\text{cm}$ away. What direction and magnitude must the magnetic field have in order to steer the electron towards its destination?

Answer

First, we determine the velocity of the electron that were accelerated over a potential difference of $\Delta V = 500\text{V}$. Their kinetic energy is given by their charge times the potential difference::

$$\begin{aligned} K &= e\Delta V \\ \frac{1}{2}mv^2 &= e\Delta V \\ \therefore v &= \sqrt{\frac{2e\Delta V}{m}} = \sqrt{\frac{2(1.602\times 10^{-19}\text{C})(500\text{V})}{(9.109\times 10^{-31}\text{kg})}} \\ &= 1.326\times 10^{7}\text{ms}^{-1} \end{aligned}$$

Now that we have the velocity, we must determine the direction of the magnetic field. We know that the electron is moving directly towards the phosphorescent screen (which we will define as $\vec{x}$) and the electron must be deflected directly upwards (which we will define as $\vec{z}$). Knowing this, we can use the right hand rule to quickly determine that the magnetic force will be acting in the $-\vec{y}$ direction.

In the region with a magnetic field, the electron will undergo uniform circular motion with a radius give by the cyclotron radius, R:

$$R = \frac{mv}{qB}$$

We thus need to determine the radius of that circle for the electron to arrive that desired location on the screen. A section of the circle about which the electron moves is illustrated in Figure 21.9.1

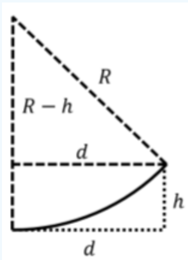

Figure 21.9.1: *Deflection of an electron noving in a uniform magnetic field.*

From geometry and Pythagoras' Theorem, we have:

$$\begin{aligned} R^2 &= (R-h)^2 + d^2 \\ R^2 &= R^2 - 2Rh + h^2 + d^2 \\ \therefore R &= \frac{h^2+d^2}{2h} = \frac{(3\text{cm})^2 + (5\text{cm})^2}{2(3\text{cm})} = 5.67\text{cm} \end{aligned}$$

The strength of the magnetic field is then given by:

$$B = \frac{mv}{qR} = \frac{(9.11\times 10^{-31}\text{kg})(1.326\times 10^{7}\text{ms}^{-1})}{(1.6\times 10^{-19}\text{C})(0.0567\text{m})} = 0.00135\text{T}$$

Exercise 21.9.2

A galvanometer has a square coil with a side length of $a = 2.5\text{cm}$ and $N = 70$ loops between two magnets which generate a radial magnetic field of $B = 8\text{mT}$. When a current runs through the coil, it generates a torque which is opposed by a spring with a torsional spring constant of $\kappa = 1.5 \times 10^{-8}\text{Nmrad}^{-1}$. If the deflection of the galvanometer's needle is 0.7, what is the current running through the coil?

Answer

First, we will determine the magnetic dipole moment of the square coil:

$$\mu = NIA$$

$$\mu = NIa^2$$

Now that we have the magnetic dipole moment, we can calculate the torque on the square coil that is produced by the magnetic field. Note that, in a galvanometer, the magnetic field is configured such that it is radial and always perpendicular to the magnetic dipole moment of the coil:

$$\tau_B = N\mu B sin(90^\circ) = NIa^2B$$

The deflection, θ, for a given current will occur when the torque produced by the wire is equal to the torque produced by the spring. The torque produced by the spring is given by:

$$\tau_s = \kappa\theta$$

where θ is measured in radians. The above equation is the rotational equivalent of Hooke's Law. Equating the torque from the spring and from the magnetic field, we can determine the current:

$$\begin{aligned} \tau_B &= \tau_S \\ NIa^2B &= \kappa\theta \\ I &= \frac{\kappa\theta}{Na^2B} = \frac{(1.5 \times 10^{-8}\text{Nm(rad)}^{-1})(0.7\text{rad})}{70(0.025\text{m})^2(8 \times 10^{-3}\text{T})} \\ &= 30\mu\text{A} \end{aligned}$$

Exercise 21.9.3

Integrate the equation $d\vec{F} = Id\vec{l} \times \vec{B}$ over a circular path to show that the torque exerted on a circular loop of radius, R, carrying current, I, immersed in a uniform magnetic field, $\vec{B}$, has a magnitude given by $\tau = \mu B$, where $\vec{\mu}$ is the magnetic dipole moment of the loop. You may simplify the problem by modeling the loop when its magnetic moment is perpendicular to the magnetic field.

Answer

Figure $21.9.2$illustrates a loop of radius, R, carrying current, I. The loop is in the $x-z$ plane, and there is a magnetic field, $\vec{B}$, in the negative x direction. By setting the loop up this way, it is easier to visualize some of the three-dimensional aspects.

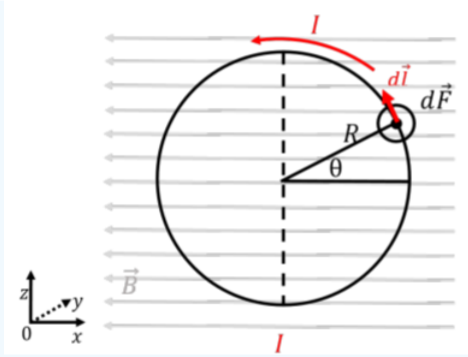

Figure 21.9.2 *A current-carrying loop in a magnetic field.*

Consider an infinitesimal section of the loop, with length, dl, located on the loop at a position labeled by the angle, θ, as illustrated. The vector, $d\vec{l}$, is given by:

$$d\vec{l} = dl(-\sin\theta\hat{x} + \cos\theta\hat{z})$$

The magnetic force on that element of the loop is given by:

$$\begin{aligned} d\vec{F} &= Id\vec{l} \times \vec{B} \\ &= Idl(-\sin\theta\hat{x} + \cos\theta\hat{z}) \times (-B\hat{x}) \\ &= -IBdl\cos\theta(\hat{z} \times \hat{x}) \\ &= -IBdl\cos\theta\hat{y} \end{aligned}$$

and the force on that element of wire is out of the page (negative y direction), as illustrated. That infinitesimal force will create an infinitesimal torque:

$$d\vec{\tau} = \vec{r} \times d\vec{F}$$

where $\vec{r}$ is the vector from the axis of rotation (through the center of the loop, parallel to the z axis) to the point where the force is exerted. The length of the vector, $\vec{r}$, is simply $r = R\cos\theta$, and the force is perpendicular to the vector $\vec{r}$. Thus, the torque on the infinitesimal element is given by:

$$\begin{aligned} d\vec{\tau} &= \vec{r} \times d\vec{F} = (R\cos\theta\hat{x}) \times (-IBdl\cos\theta\hat{y}) \\ &= -IBR\cos^2\theta dl(\hat{x} \times \hat{y}) = -IBR\cos^2\theta dl\hat{z} \end{aligned}$$

and the torque on that infinitesimal element is in the negative z direction, as anticipated from the direction of the force. Note that had we considered the loop to be oriented such that the magnetic field is not in the plane of the loop, the vector $\vec{r}$ in the torque would have a component in the y direction.

We can sum the torques on each element of the loop, from $\theta = 0$ to $\theta = 2\pi$. We can express the length, dl, using the infinitesimal angle, $d\theta$, that subtends the arc of length, dl, on the circle of radius, R:

$$dl = Rd\theta$$

The net torque is then given by:

$$\vec{\tau} = \int d\vec{\tau} = \int -IBR\cos^2\theta dl\hat{z} = (-IBR^2\hat{z})\int_0^{2\pi}\cos^2\theta d\theta = (-IBR^2\hat{z})\pi$$

The magnetic moment of the loop is:

$$\mu = IA = I\pi R^2$$

so that the torque is indeed given by $\tau = \mu B$. If we had rotated the loop so that the vector, $\vec{r}$, had a y component, then we would have found the general formula with a cross-product.