8.4: Energy diagrams and equilibria

We can write the mechanical energy of an object as:

$$E = K + U$$

which will be a constant if there are no non-conservative forces doing work on the object. This means that if the potential energy of the object increases, then its kinetic energy must decrease by the same amount, and vice-versa.

Consider a block that can slide on a frictionless horizontal surface and that is attached to a spring, as is shown in Figure $8.4.1$ (left side), where $x = 0$ is chosen as the position corresponding to the rest length of the spring. If you push on the block so as to compress the spring by a distance D and then release it, the block will initially accelerate because of the spring force in the positive x direction until the block reaches the rest position of the spring ($x = 0$ on the diagram). When it passes that point, the spring will exert a force in the opposite direction. The block will continue in the same direction and decelerate until it stops and turns around. It will then accelerate again towards the rest position of the spring, and then decelerate once the spring starts being compressed again, until the block stops and the motion repeats. We say that the block "oscillates" back and forth about the rest position of the spring.

We can describe the motion of the block in terms of its total mechanical energy, E. Its potential energy is given by:

$$U(x) = \frac{1}{2}kx^2$$

On the right of Figure $8.4.1$ is an "Energy Diagram" for the block, which allows us to examine how the total energy, E, of the block is divided between kinetic and potential energy depending on the position of the block. The vertical axis corresponds to energy and the horizontal axis corresponds to the position of the block.

The total mechanical energy, $E = 25\text{J}$, is shown by the horizontal red line. Also illustrated are the potential energy function ($U(x)$ in blue), and the kinetic energy, ($K = E - U(x)$, in dotted black).

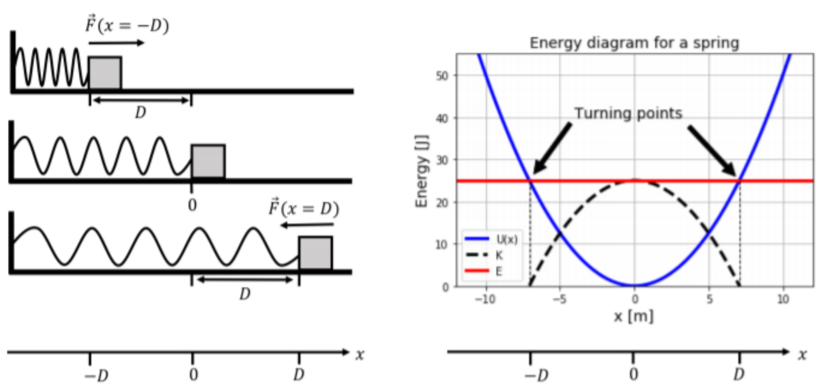

Figure $8.4.1$: Left: The block oscillates about the rest position of the spring, between $x = -D$ and $x = D$. Right: The energy diagram for the block. This diagram is for a spring with spring constant $k = 1\text{N/m}$.

The energy diagram allows us to describe the motion of the object attached to the spring in terms of energy. A few things to note:

1. At $x = \pm D$, the potential energy is equal to E, so the kinetic energy is zero. The block is thus instantaneously at rest at those positions.
2. At $x = 0$, the potential energy is zero, and the kinetic energy is maximal. This corresponds to where the block has the highest speed.
3. The kinetic energy of the block can never be negative[1], thus, the block cannot be located outside the range $[-D, +D]$, and we would say that the motion of the block is "bound". The points between which the motion is bound are called "turning points".

An analysis of the energy diagram tells us that the block is bound between the two turning points, which themselves are equidistant from the origin. When we initially compress the spring, we are "giving" the block "spring potential energy". As the block starts to move, the potential energy of the block is converted into kinetic energy as it accelerates and then back into potential energy as it decelerates.

? Exercise 8.4.1

Calculate the positions of the turning points for the situation shown in Figure 8.4.1. The total energy is $25\ \mathrm{J}$ and the spring constant is $k = 1\ \mathrm{N/m}$.

Answer

$7.1\ \mathrm{m}$

By looking at only the potential energy function, without knowing that it is related to a spring, we can come to the same conclusions; namely that the motion is bound as long as the total mechanical energy is not infinite. We call the point $x = 0$ a "stable equilibrium", because it is a local minimum of the potential energy function. If the object is displaced from the equilibrium point, it will want to move back towards that point. This can also be understood in terms of the force associated with the potential energy function:

$$F = -\frac{d}{dx}U(x)$$

The local minimum occurs where the derivative of the potential function is equal to zero. Thus, the **equilibrium point is given by the condition that the force associated with the potential is zero** ($x = 0$ in the case of the potential energy from a spring). The equilibrium is a stable equilibrium because the force associated with the potential energy function ($F(x) = -kx$ for the spring) points towards the equilibrium point.

The potential energy function for an object with total mechanical energy, E, can be thought of as a little "roller coaster", on which you place a marble and watch it "roll down" the potential energy function. You can think of placing a marble where $U(x) = E$ and releasing it. The marble would then roll down the potential energy function, just as an actual marble would roll down a real slope, mimicking the motion of the object along the x axis. This is illustrated in Figure 8.4.2 which shows an arbitrary potential energy function and a marble being placed at a location where the potential energy is equal to E.

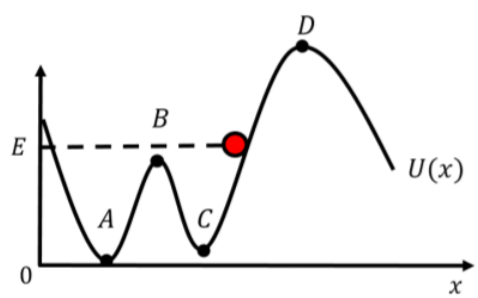

Figure 8.4.2: Arbitrary potential energy function and illustration of visualizing a marble rolling down the function by placing the marble on the potential energy function at a point where $U(x) = E$.

The motion of the marble will be bound between the two points where the potential energy function is equal to E. When the marble is placed as shown, it will roll towards the left, just as if it were a real marble on a track. Since the potential energy is increasing as a function of x at the point where we placed the marble, the force is in the negative x direction (remember, the force is the negative of the derivative of the potential energy function). With the given energy, the marble would never be able to make it to point D, as it does not have enough energy to "climb up the hill". It would roll down, through point C, up to point B, down to point A, and then turn around where $U(x) = E$ and return to where it started.

Locations A and C on the diagram are stable equilibria, because if a marble is placed in one of those locations and nudged slightly, it will come back to the equilibrium point (or oscillate about that point). Points B and D are "unstable equilibria", because if the marble is placed there and nudged, it will not immediately come back to those points. Note that if the marble were placed at point D and nudged towards the right, the motion of the marble would be unbound on the right, and it would keep going in that direction.

Now, say an object's potential energy is described by the function in Figure 8.4.2, and the object has total energy E. The object's motion along the x axis will be exactly the same as the projection of the marble's motion on the x axis.

Exercise 8.4.2

A force, $F(x)$, acts on an object. The potential energy function, $U(x)$, associated with the force is given by $U(x) = a(x-6)^2(x-1)(x-3)+20\text{ J}$, where a is a positive constant. $U(x)$ is plotted in Figure 8.4.3. Use the "marble" method to determine the direction of the force at $x=5$. Confirm your answer by finding the value of the force , $F(x)$, at $x=5$.

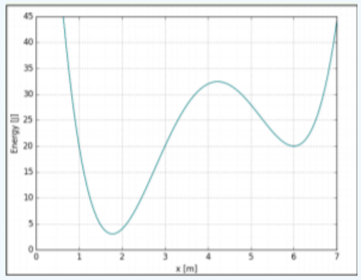

Figure 8.4.3: A potential energy function $U(x)$. The x-axis represents the x position and the y-axis represents the energy.

A. $F(x=5) = -10a$
B. $F(x=5) = 10a$
C. $F(x=5) = 20a$
D. $F(x=5) = -20a$

Answer

B.

Footnotes

1. Remember, the kinetic energy is given by $K = \frac{1}{2}mv^2$. Since neither mass nor the value of v^2 can be negative, the kinetic energy of an object can never be negative.