25.5: Summary

Key Takeaways

Cartesian coordinate systems can be defined using an origin, and mutually perpendicular axes that specify a direction in which each corresponding coordinate increases. The position of a point is described by the coordinates of the point (one coordinate per axis). Polar, cylindrical and spherical coordinate systems can be defined relative to a Cartesian coordinate system and sometimes facilitate the description of situations with cylindrical (azimuthal) or spherical symmetry.

Vectors can be represented by arrows and are quantities that have both a magnitude and a direction, as opposed to “scalars”, which are simply numbers. Vectors are not fixed in space, so two vectors are equal if they have the same magnitude and direction, regardless of where they are drawn. We place a little arrow above a variable, $\vec d$, to indicate that it is a vector. There are several, equivalent, notations to indicate the components of a vector:

$$\begin{aligned}
\vec d &= (d_x, d_y, d_z) && \text{row vector}\\
&= \begin{pmatrix} d_x \\ d_y \\ d_z \end{pmatrix} && \text{column vector}\\
&= d_x\hat x + d_y\hat y + d_z\hat z && \text{using } \hat x,\ \hat y,\ \hat z\\
&= d_x\hat i + d_y\hat j + d_z\hat j && \text{using } \hat i,\ \hat j,\ \hat k
\end{aligned}$$

If we multiply (divide) a vector by a scalar, we multiply (divide) each component of the vector individually by that quantity. As a result, the magnitude of the vector will also be multiplied (divided) by that quantity:

$$a\vec d = \begin{pmatrix} ad_x \\ ad_y \\ ad_z \end{pmatrix}$$

In particular, we can define a unit vector, $\hat d$, to be a vector of length 1 in the same direction as $\vec d$, by simply dividing $\vec d$ by its magnitude, d:

$$\hat d = \frac{\vec d}{d}$$

where the magnitude of the vector, $||\vec d|| = d$, expressed in Cartesian coordinates, is given by:

$$||\vec d|| = d = \sqrt{d_x^2 + d_y^2 + d_z^2}$$

We can add two vectors by independently adding the individual components of the vectors:

$$\begin{aligned}
\vec c &= \vec a + \vec b\\
\therefore c_x &= a_x + b_x\\
\therefore c_y &= a_y + b_y\\
\therefore c_z &= a_z + b_z
\end{aligned}$$

Graphically, this corresponds to adding vectors “head to tail”. This also highlights that an equation written using vectors (as the first line above) really represents three independent equations, one for each coordinate of the vectors (or two in two dimensions). Subtraction of vectors is treated in the same way as addition (but using minus signs where appropriate).

One can define the scalar (or dot) product between two vectors, as a scalar quantity that is obtained from the two vectors:

$$\vec a \cdot \vec b = a_x b_x + a_y b_y + a_z b_z$$

The scalar product is also related to the angle, θ, between the two vectors when these are placed “tail to tail”:

$$\vec{a}\cdot\vec{b}=ab\cos\theta$$

In particular, the scalar product between two vectors is zero if the two vectors are perpendicular to each other ($\cos\theta=0$), and maximal when these are parallel to each other.

The vector (or cross) product between two vectors is a vector that is mutually perpendicular to both vectors and is defined as the following:

$$\vec{a}\times\vec{b}=\begin{pmatrix} a_yb_z-a_zb_y \\ a_zb_x-a_xb_z \\ a_xb_y-a_yb_x \end{pmatrix}$$

The vector product can only be defined in three dimensions, since it must be mutually perpendicular to the vectors. The magnitude of the vector product is given by:

$$||\vec{a}\times\vec{b}||=ab\sin\theta$$

where θ is the angle between the two vectors when these are placed tail to tail. In particular, the vector product between two vectors is zero if the two vectors are parallel to each other (and maximal when these are perpendicular). The direction of the vector product is given by the right-hand rule for the cross product.

An axial vector can be used to describe a quantity that is related to rotation. The direction of the axial vector is co-linear with the axis of rotation, its magnitude is given by the magnitude of the rotational quantity (e.g. angular speed), and its direction is defined using the right-hand rule for axial vectors.
