12.4: Summary

Key Takeaways

If an object is rotating with angular speed, ω, about an axis that is fixed in an inertial frame of reference, the rotational kinetic energy of that object is given by:

$$K_{rot}=\frac{1}{2}I\omega^2$$

where I is the moment of inertia of that object about the axis of rotation.

The net work done by the net torque exerted on an object about a fixed axis or rotation in an inertial frame of reference is equal to object's change in rotational kinetic energy:

$$W=\int_{\theta_1}^{\theta_2}\vec{\tau}^{net}\cdot d\vec{\theta}=\frac{1}{2}I\omega_2^2-\frac{1}{2}I\omega_1^2$$

If a torque, $\vec{\tau}$, about a stationary axis is exerted on an object that is rotating with a constant angular velocity, $\vec{\omega}$, about that axis, then the torque does work at a rate:

$$P=\vec{\tau}\cdot\vec{\omega}$$

If an object of mass, M, is rotating about an axis through its center of mass, and the center of mass of is moving with speed, v_{CM}, relative to an inertial frame of reference, then the total kinetic energy of the object is given by:

$$K_{tot}=K_{rot}+K_{trans}=\frac{1}{2}I_{CM}\omega^2+\frac{1}{2}Mv_{CM}^2$$

where, ω, is the angular speed of the object about the center of mass, and, I_{CM}, is the moment of inertia of the object about the center of mass. The two terms in the kinetic energy come from the rotation about the center of mass (K_{rot}), and the translational motion of the center of mass (K_{trans}).

An object is said to be rolling without slipping on a surface if the point on the object that is in contact with the surface is instantaneously at rest relative to the surface. We can model an object that is rolling without slipping by superimposing rotational motion about the center of mass with translational motion of the center of mass. The angular speed, ω, and the angular acceleration, α, of the object about an axis through its center of mass are related to the speed, v_{CM}, and linear acceleration, a_{CM}, of the center of mass, respectively:

$$\begin{aligned}v_{CM}&=\omega R\\a_{CM}&=\alpha R\end{aligned}$$

These conditions are equivalent to stating that the object is rolling without slipping.

When an object is rolling without slipping, we can also model its motion as if it were instantaneously rotating about an axis that goes through the point of contact between the object and the ground (the instantaneous axis of rotation). The angular speed (and acceleration) about the instantaneous axis of rotation are the same as they are when the object is modeled as rotating about its (moving) center of mass.

An object can only be rolling without slipping if there is a force of static friction exerted by the surface on the object. Without this force, the object would slip along the surface.

We can define the angular momentum of a particle, $\vec{L}$, about a point in an inertial frame of reference as:

$$\vec{L}=\vec{r}\times\vec{p}$$

where, $\vec{r}$, is the vector from the point to the particle, and, $\vec{p}$, is the linear momentum of the particle. If the particle has an angular velocity, $\vec{\omega}$, relative to an axis of rotation its angular momentum about that axis can be written as:

$$\vec{L}=mr^2\vec{\omega}=I\vec{\omega}$$

where, r, is the distance between the particle and the axis of rotation, and $I = mr^2$, can be thought of as the moment of inertia of the particle about that axis.

We can write the equivalent of Newton's Second Law for the rotational dynamics of a particle using angular momentum:

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{net}$$

where, $\vec{\tau}^{net}$, is the net torque on the particle about the same point used to define angular momentum. That point must be in an inertial frame of reference.

The rate of change of the total angular momentum for a system of particles, $\vec{L} = \vec{L}_1 + \vec{L}_2 + \dots$, about a given point is given by:

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext}$$

where, $\vec{\tau}^{ext}$, is the net external torque on the system about the point of rotation. If the net external torque of the system is zero, then the total angular momentum of the system is constant (conserved). Again, the point of rotation must be in an inertial frame of reference[1].

For a solid object, in which all of the particles must move in unison, we can define the angular momentum of the object about a stationary axis to be:

$$\vec{L} = I\vec{\omega}$$

where, $\vec{\omega}$, is the angular velocity of the object about that axis, and, I, is the object's corresponding moment of inertia about that axis.

Many of the relations that exist between linear quantities have an analogue relation between the corresponding angular quantities, as summarized in the table below:

Name	Linear	Angular	Correspondence
Displacement	s	$\vec{\theta}$	$d\vec{\theta} = \frac{1}{r^2}\vec{r} \times d\vec{s}$
Velocity	$\vec{v}$	$\vec{\omega}$	$\vec{\omega} = \frac{1}{r^2}\vec{r} \times \vec{v}$, $v_s = \vec{\omega} \times \vec{r}$ [2]
Acceleration	$\vec{a}$	$\vec{\alpha}$	$\vec{\alpha} = \frac{1}{r^2}\vec{r} \times \vec{a}$, $a_s = \vec{\alpha} \times \vec{r}$ [3]
Inertia	m	I	$I = \sum_i m_i r_i^2$
Momentum	$\vec{p} = m\vec{v}$	$\vec{L} = I\vec{\omega}$	$\vec{L} = \vec{r} \times \vec{p}$
Newton's Second Law	$\vec{F}^{ext} = m\vec{a}_{CM}$	$\vec{\tau}^{ext} = I\vec{\alpha}$	$\vec{F} \to \vec{\tau}, m \to I, \vec{a} \to \vec{\alpha}$
Newton's Second Law	$\frac{d\vec{p}}{dt} = \vec{F}^{ext}$	$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext}$	$\vec{F} \to \vec{\tau}, \vec{p} \to \vec{L}$
Kinetic energy	$\frac{1}{2}mv^2$	$\frac{1}{2}I\omega^2$	$m \to I, v \to \omega$
Power	$\vec{F} \cdot \vec{v}$	$\vec{\tau} \cdot \vec{\omega}$	$\vec{F} \to \vec{\tau}, \vec{v} \to \vec{\omega}$

Table 12.4.1

Important Equations

Rotational kinetic energy of a rotating object:

$$K_{rot} = \frac{1}{2}I\omega^2$$

Total kinetic energy:

$$K_{tot} = K_{rot} + K_{trans} = \frac{1}{2} I_{CM} \omega^2 + \frac{1}{2} M v_{CM}^2$$

Work:

$$W = \int_{\theta_1}^{\theta_2} \vec{\tau}^{net} \cdot d\vec{\theta} = \frac{1}{2} I \omega_2^2 - \frac{1}{2} I \omega_1^2$$

Power:

$$P = \vec{\tau} \cdot \vec{\omega}$$

Angular momentum:

$$\vec{L} = \vec{r} \times \vec{p}$$

$$\vec{L} = m r^2 \vec{\omega} = I \vec{\omega}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{net}$$

$$\frac{d\vec{L}}{dt} = \vec{\tau}^{ext}$$

$$\vec{L} = I \vec{\omega}$$

Important Definitions

Definition

Angular momentum: The rotational equivalent of linear momentum. Angular momentum must be defined relative to an axis of rotation. SI units: $[\mathrm{kg} \cdot \mathrm{m}^2 \cdot \mathrm{s}^{-1}]$. Common variable(s): $\vec{L}$.

Definition

Rotational kinetic energy: The rotational equivalent of translation kinetic energy. Generally, an object can have both rotational and translational kinetic energy. SI units: $[J]$. Common variables: K_{rot}.

Footnotes

1. Technically, if the point is the center of mass, then this is valid even in an accelerating frame of reference.
2. This corresponds to the component of velocity perpendicular to $\vec{r}$.
3. This corresponds to the component of acceleration perpendicular to $\vec{r}$.