11.8: Summary

Key Takeaways

We can describe the kinematics of rotational motion using vectors to indicate both an axis of rotation and the direction of rotation about that axis. If a particle with velocity vector, $\vec{v}$, is rotating in an circle about an axis, then its angular velocity vector, $\vec{\omega}$, relative to that axis is defined as:

$$\vec{\omega}=\frac{1}{r^{2}}\vec{r}\times\vec{v}$$

where $\vec{r}$ is a vector from the axis of rotation to the particle. The particle rotates in a circle that lies in the plane defined by $\vec{r}$ and $\vec{v}$, perpendicular to the axis of rotation. The direction of the angular velocity vector is co-linear with the axis of rotation and the direction of rotation is given by the right-hand rule for axial vectors.

One can define the angular velocity of a particle relative to a point of rotation, even if the particle is not moving in a circle. In that case, the angular velocity corresponds to the angular velocity of the particle as if it were instantaneously moving about a circle.

If a particle moving around a circle has a tangential acceleration, $\vec{a}_s$, then its angular acceleration vector is defined as:

$$\vec{\alpha}=\frac{1}{r^{2}}\vec{r}\times\vec{a}_{s}$$

The torque from a force, $\vec{F}$, exerted at a position $\vec{r}$, relative to an axis (or point) of rotation is defined as:

$$\vec{\tau}=\vec{r}\times\vec{F}$$

Torque is analogous to force in that it is used to model the causes of motion. Torques are only ever defined relative to an axis or point of rotation. The torque vector will be co-linear with the axis about which the object on which the force is exerted would rotate as a result of that force.

The magnitude of the torque can be written using either the component of the force, $F_{\perp}$ perpendicular to the vector $\vec{r}$, or the lever arm, $r_{\perp}$, of the force relative to the axis of rotation:

$$\begin{aligned}\tau &= rF\sin\phi\\ &= rF_{\perp}\\ &= r_{\perp}F\end{aligned}$$

where ϕ is the angle between the vectors $\vec{r}$ and $\vec{F}$ when these are placed “tail to tail”.

Using rotational/angular quantities, we can modify Newton’s Second Law to describe rotational dynamics about a given axis (or point) of rotation. For a point particle, this gives:

$$\vec{\tau}^{net}=mr^{2}\vec{\alpha}$$

where $\vec{\tau}^{net}$ is the net torque on the particle (the sum of the torques from each force exerted on the particle) about the axis, and $\vec{\alpha}$ is the resulting angular acceleration about that axis.

For an object (either continuous or made of point particles), the rotational version of Newton’s Second Law for rotation about a specific axis is given by:

$$\vec{\tau}^{net}=I\vec{\alpha}$$

where I is the moment of inertia of the object about that axis.

The moment of inertia of an object about an axis of rotation is given by

$$I=\sum_{i}m_{i}r_{i}^{2}$$

if the object is modeled as a system of point particles of mass m_i each a distance r_i from the axis of rotation. For a continuous object, the moment of inertia is given by:

$$I = \int r^2 dm$$

where dm is a small mass element a distance r from the axis of rotation and the integral is over the dimension of the object. Generally, one can set up the integral by expressing dm in terms of r using the density of the object, and then integrating r over the dimension of the object.

If the moment of inertia of an object of mass M about an axis that goes through the center of mass is given by I_{CM}, then the moment of inertia, I_h, of the object through an axis that is parallel and a distance h from the center of mass is given by the parallel axis theorem:

$$I_h = I_{CM} + Mh^2 \quad \text{Parallel axis theorem}$$

Objects are in equilibrium if they are not rotating when viewed in their center of mass frame of reference. Thus, for an object to be in equilibrium, the sum of the torques on the object, in the center of mass reference frame, must be zero.

An object is in static equilibrium if the center of mass is not accelerating, and thus the sum of the external forces on the object is zero. To model the torques on an object in static equilibrium, one can choose the axis about which to calculate the torques. A good choice is to choose an axis that is perpendicular to the plane in which the forces on the object are exerted (if such a plane exists), and to choose the axis to go through a point where at least one force is exerted (so that torques exerted at that point are identically zero).

An object is in dynamic equilibrium if the center of mass is accelerating, but the object does not rotate when viewed in the frame of reference of its center of mass. In dynamic equilibrium, if one models the torques exerted on the object about an axis that does not go through the center of mass, then one must remember to include an inertial force exerted at the center of mass.

Important Equations

Angular quantities:

$$\begin{aligned}
\vec\omega &= \frac{1}{r^2}\vec r \times \vec v \\
\vec\alpha &= \frac{1}{r^2}\vec r \times \vec a_\perp \\
\vec v_s &= \vec\omega \times \vec r \\
\vec a_s &= \vec\alpha \times \vec r
\end{aligned}$$

Torque from a force:

$$\begin{aligned}
\vec\tau &= \vec r \times \vec F \\
\tau &= rF\sin\phi \\
&= rF_\perp \\
&= r_\perp F
\end{aligned}$$

Newton's Second Law for a point particle about a given axis of rotation:

$$\vec\tau^{net} = mr^2\vec\alpha$$

Newton's Second Law for rotation about an axis:

$$\vec\tau^{net} = I\vec\alpha$$

Moment of Inertia:

$$I = \sum_i m_i r_i^2$$

$$I = \int r^2 dm$$

Parallel Axis Theorem:

$$I_h = I_{CM} + Mh^2$$

Important Definitions

Definition

Torque: A rotational equivalent of force which occurs when a force is applied at a distance r from the axis of rotation of a rigid body or particle. SI units: $[\mathrm{J}]$. Common variable(s): τ.

Definition

Moment of inertia: A property of matter which describes an object's resistance to rotational motion. SI units: $[\mathrm{kgm^2}]$. Common variable(s): I.

Definition

Linear mass density: The mass per unit length of an object. SI units: $[\mathrm{kgm^{-1}}]$. Common variable(s): λ.