4.2: Motion in three dimensions

The big challenge was to expand our description of motion from one dimension to two. Adding a third dimension ends up being trivial now that we know how to use vectors. In three dimensions, we describe the position of a point using three coordinates, so all of the vectors simply have three independent components, but are treated in exactly the same way as in the two dimensional case. The position of an object is now described by three independent functions, $x(t)$, $y(t)$, $z(t)$, that make up the three components of a position vector $\vec{r}(t)$:

$$\begin{aligned}\vec{r}(t) &= \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \\ \therefore \vec{r}(t) &= x(t)\hat{x} + y(t)\hat{y} + z(t)\hat{z}\end{aligned}$$

The velocity vector now has three components and is defined analogously to the 2D case:

$$\begin{aligned}\vec{v}(t) &= \frac{d\vec{r}}{dt} = \begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \\ \frac{dz}{dt} \end{pmatrix} = \begin{pmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{pmatrix} \\ \therefore \vec{v}(t) &= v_x(t)\hat{x} + v_y(t)\hat{y} + v_z(t)\hat{z}\end{aligned}$$

and the acceleration is defined in a similar way:

$$\begin{aligned}\vec{a}(t) &= \frac{d\vec{v}}{dt} = \begin{pmatrix} \frac{dv_x}{dt} \\ \frac{dv_y}{dt} \\ \frac{dv_z}{dt} \end{pmatrix} = \begin{pmatrix} a_x(t) \\ a_y(t) \\ a_z(t) \end{pmatrix} \\ \therefore \vec{a}(t) &= a_x(t)\hat{x} + a_y(t)\hat{y} + a_z(t)\hat{z}\end{aligned}$$

In particular, if an object has a constant acceleration, $\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$, and started at $t = 0$ with a position $\vec{r}_0$ and velocity $\vec{v}_0$, then its velocity vector is given by:

$$\vec{v}(t) = \vec{v}_0 + \vec{a}t = \begin{pmatrix} v_{0x} + a_x t \\ v_{0y} + a_y t \\ v_{0z} + a_z t \end{pmatrix}$$

and the position vector is given by:

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 t + \frac{1}{2}\vec{a}t^2 = \begin{pmatrix} x_0 + v_{0x}t + \frac{1}{2}a_x t^2 \\ y_0 + v_{0y}t + \frac{1}{2}a_y t^2 \\ z_0 + v_{0z}t + \frac{1}{2}a_z t^2 \end{pmatrix}$$

where again, we see how writing a single vector equation (e.g. $\vec{v}(t) = \vec{v}_0 + \vec{a}t$) is really just a way to write the three independent equations that are true for each component.