24.7: Relativistic momentum and energy

In this section, we show how to define momentum and energy in a way that is consistent with the postulates of Special Relativity. We expect that, since time and space depend on the frame of reference of the observer, so too will the momentum and the energy of an object. Consider an object of mass m_0, moving in a frame of reference, S, with velocity, $\vec{u}$ (we reserve $\vec{v}$ to represent the speed between two inertial frames of reference), in the x direction. At some time, t, the object will be at position, x, along the x axis. We define the relativistic momentum as:

$$p = m_0 \frac{dx}{dt'}$$

where t' is the time as measured in the rest frame of the object. By defining momentum in terms of the proper time of the object, all observers will agree on the value of t'. In the frame of reference, S, (with time t) this corresponds to:

$$p = m_0 \frac{dx}{dt'} = m_0 \frac{dt}{dt'}\frac{dx}{dt} = m_0 \frac{dt}{dt'} u$$

where u is the speed of the particle in frame, S. We can use time dilation to re-express the derivative:

$$\begin{aligned} \Delta t &= \gamma \Delta t' \\ \frac{\Delta t}{\Delta t'} &= \gamma \\ \therefore \frac{dt}{dt'} &= \gamma \end{aligned}$$

where in the last line, we simply took the limit of an infinitesimally short time interval. Therefore, the relativistic momentum of the particle, in frame, S, can be defined:

$$\vec{p} = m_0 \gamma \vec{u} = \frac{m_0 \vec{u}}{\sqrt{1 - \frac{u^2}{c^2}}} \tag{24.7.1}$$

where γ is calculated with the same speed, u, since that is the speed of the reference frame of the object relative to S. Note that as the speed, u, of the particle approaches the speed of light, the factor of γ approaches infinity. This means that an object with a mass can never reach the speed of light, as it would have an infinite momentum. In order to define momentum in a way that resembles the classic definition, one can think of the mass of the object as depending on the speed of the object. We define the rest-mass, m_0, of the object as the mass that is measured when the object is at rest. We can then model the mass of the object as increasing with its speed:

$$m(u) = \gamma m_0 = \frac{m_0}{\sqrt{1 - \frac{u^2}{c^2}}}$$

so that the relativistic momentum would be defined as:

$$\vec{p} = m(u)\vec{u}$$

In this case, we can think of the mass of the object as increasing with its speed. The object would acquire infinite mass if it were to reach the speed of light.

With the relativistic definition of momentum, Newton's Second Law can be written as:

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt} m_0 \gamma \vec{u}$$

Example 24.7.1

A constant force of $1 \times 10^{-22} \text{N}$ is applied to an electron (with mass $m_e = 9.11 \times 10^{-31} \text{kg}$) in order to accelerate it from rest to a speed of $u = 0.99c$. Compare the length of time over which the force must be applied using classical and relativistic dynamics.

Solution

In both cases, we can start with Newton's Second Law:

$$\vec{F} = \frac{d\vec{p}}{dt}$$
$$\therefore \int \vec{F}dt = \Delta\vec{p} = \vec{p}$$

where $\vec{p}$ is the final momentum of the electron (which is different depending on whether we use the classical or the relativistic definition of momentum). Since the force is constant:

$$\int \vec{F}dt = \vec{F}\Delta t = \vec{p}$$
$$\therefore \Delta t = \frac{p}{F}$$

where Δt is the length of time over which the force is applied. With the classical definition of momentum, the time is given by:

$$\Delta t = \frac{p}{F} = \frac{mu}{F} = \frac{(9.11\times 10^{-31}\text{kg})(0.99)(3\times 10^{8}\text{m/s})}{(1\times 10^{-22}\text{N})} = 2.71\text{s}$$

With the relativistic definition of momentum, we first need the gamma factor:

$$\gamma = \frac{1}{\sqrt{1-\frac{u^2}{c^2}}} = \frac{1}{\sqrt{1-(0.99)^2}} = 7.1$$

We can then calculate the time over which the force needs to be applied:

$$\Delta t = \frac{p}{F} = \frac{\gamma m_0 u}{F} = \gamma\frac{m_0 u}{F} = (7.1)(2.71\text{s}) = 19.2\text{s}$$

Discussion

When using the relativistic definition of momentum, we find that the time over which the force must be applied to reach a given speed is longer. This makes sense, since it will take infinitely long to reach the speed of light. Also, note that the time that is required using relativistic dynamics is just the time-dilated time that is required in classical dynamics.

Recall how we defined kinetic energy, in Section 7.2, by defining the change in kinetic energy of an object as the net work done on that object. We use the same formalism here to redefine kinetic energy using relativistic dynamics.

The work done by the net force, $\vec{F}$, on an object that goes from a position A to a position B, is given by

$$W = \int_A^B \vec{F}\cdot d\vec{l} = \int_0^t \left(\frac{d}{dt}m_0\gamma\vec{u}\right)\cdot(\vec{u}dt)$$

where we recognized that a infinitesimal segment $d\vec{l}$ along the path of the object is given by $d\vec{l} = \vec{u}dt$. The time infinitesimals, dt, cancel, and we are left with:

$$W = \int_0^t \left(\frac{d}{dt}m_0\gamma\vec{u}\right)\cdot(\vec{u}dt)$$
$$= \int d(m_0\gamma\vec{u})\cdot\vec{u}$$

which we can integrate by parts. We can integrate this over the speed, u, and we assume that the object started with a speed of $u = 0$ at the beginning of the path and has a speed, $u = U$, at the end of the path:

$$\begin{aligned}
W &= \int_0^U d(m_0\gamma\vec{u})\cdot\vec{u} = \left[\gamma m_0\vec{u}\cdot\vec{u}\right]_0^U - \int_0^U m_0\gamma u du \quad \text{(int. by parts)} \\
&= \gamma m_0 U^2 - m_0\int_0^U \frac{udu}{\sqrt{1-\frac{u^2}{c^2}}} \\
&= \gamma m_0 U^2 - m_0\left[c^2\sqrt{1-\frac{u^2}{c^2}}\right]_0^U \\
&= \gamma m_0 U^2 - m_0c^2 + m_0c^2\sqrt{1-\frac{U^2}{c^2}} \\
&= \gamma\left(m_0U^2 + m_0c^2\left(1-\frac{U^2}{c^2}\right)\right) - m_0c^2 \\
&= m_0c^2(\gamma-1)
\end{aligned}$$

Since the object started at rest (with a speed $u=0$) the above integral corresponds to what we would call the kinetic energy of the object, with a speed, u:

$$K = m_0c^2(\gamma-1) = m_0c^2\left(\frac{1}{\sqrt{1-\frac{u^2}{c^2}}}-1\right)$$

This form for the relativistic kinetic energy of the object is not at all similar to the form that we obtained in classical physics. As the speed of the object approaches the speed of light, the γ factor approaches infinity, as does the kinetic energy. Thus, it would take an infinite amount of work to accelerate an object to the speed of light, and again, we see that it is impossible for anything with mass to ever reach the speed of light. The formula above, however, should always be correct, even in the non-relativistic limit, when $v << c$. We can approximate the gamma factor using the binomial expansion for the case where $x << 1$:

$$(1+x)^n \sim 1+nx+\dots$$

So that, when $v<<c$ (and $v^2/c^2 << 1$), the gamma factor is approximated by:

$$\gamma = \left(1-\frac{u^2}{c^2}\right)^{-\frac{1}{2}} \sim 1+\frac{1}{2}\frac{u^2}{c^2}$$

In this limit, the relativistic kinetic energy reduces to:

$$\lim_{v<<c} K = \lim_{v<<c} m_0c^2(\gamma-1) \sim m_0c^2\left(1+\frac{1}{2}\frac{u^2}{c^2}-1\right) = \frac{1}{2}mu^2$$

which is the classical definition of kinetic energy. The kinetic energy is also zero when the speed is zero.

The kinetic energy has two terms in it:

$$K = m_0c^2\gamma - m_0c^2$$

The first term increases with speed and behaves as we would expect. The second term is constant, and depends only on the rest mass of the object (we call this term the rest mass energy). We can think of this in slightly different terms. Let us define the total energy, E, of the object as:

$$\begin{aligned}
E &= m_0c^2\gamma \\
\therefore E &= K + m_0c^2
\end{aligned} \tag{24.7.2}$$

so that the total energy is just the rest mass energy plus the kinetic energy. This highlights a key aspect of Special Relativity. An object will have energy, E, even when it is at rest. That energy, at rest, is called the rest mass energy, and corresponds to energy that an object has by virtue of having mass. This is, of course, Einstein's famous equation:

$$E = m_0c^2 \quad \text{(rest mass energy)} \tag{24.7.3}$$

This equation implies that mass can be thought of as a form of energy. Nuclear reactors function by converting a small amount of mass of uranium atoms into energy (in the form of heat), that is then used to produce high pressure steam to rotate a turbine.

Einstein's relation is often used to express the mass of subatomic particles in terms of energy. For example, an electron has a mass of $511\times 10^{3}\mathrm{eV/c^{2}}$ in these units.

Example 24.7.2

What is the mass of a proton, $m_p = 1.67\times 10^{-27}\mathrm{kg}$, in units of $\mathrm{MeV/c^2}$ (where the M stands for "Mega", and corresponds to $1\mathrm{MeV} = 1\times 10^{6}\mathrm{eV}$)?

Solution

We can first calculate the rest mass energy of the proton in Joules:

$$E = m_p c^2 = (1.67\times 10^{-27}\mathrm{kg})(3\times 10^{8}\mathrm{m/s})^2 = 1.503\times 10^{-10}\mathrm{J}$$

We can then convert from Joules to electron-volts:

$$\frac{(1.503\times 10^{-10}\mathrm{J})}{(1.6\times 10^{-19}\mathrm{J/eV})} = 939.4\times 10^{6}\mathrm{eV} = 939.4\mathrm{MeV}$$

The mass of the proton can then be expressed as $m_p = 939.4\mathrm{MeV/c^2}$.

Finally, it is interesting to examine the relationship between the momentum and the energy of a relativistic object. Consider the quantity c^2p^2:

$$\begin{aligned} c^2p^2 &= c^2(\gamma m_0 u)^2 = c^2\gamma^2 m_0^2 u^2 = c^4\gamma^2 m_0^2\frac{u^2}{c^2} = c^4\gamma^2 m_0^2\left(1-\frac{1}{\gamma^2}\right)\\ &= c^4\gamma^2 m_0^2 - c^4 m_0^2\\ &= E^2 - c^4 m_0^2 \end{aligned}$$

where we recognized that $c^4\gamma^2 m_0^2$ is simply the energy, E, squared. This is generally called the "energy-momentum" relation and written:

$$E^2 = p^2c^2 + m_0^2c^4 \tag{24.7.4}$$

An interesting consequence of this relationship is that particles with no mass will still have a momentum. For example, the photon, which is a particle of light and must thus have a mass of zero (or it could not move at the speed of light), will have a momentum given by:

$$p = \frac{E}{c}$$

Thus, one can use light to impart momentum to something. This is how a solar sail, a proposed propulsion mechanism for space travel, operates.

This page titled 24.7: Relativistic momentum and energy is shared under a CC BY-SA 4.0 license and was authored, remixed, and/or curated by Ryan D. Martin, Emma Neary, Joshua Rinaldo, and Olivia Woodman via source content that was edited to the style and standards of the LibreTexts platform.