9.5: Summary

Key Takeaways

Kepler was the first to synthesize a large amount of data to quantitatively describe gravity with his three laws:

1. The path of a planet around the Sun is described by an ellipse with the Sun at once of its foci.
2. Planets move in such a way that the area swept by a line connecting the planet and the Sun in a given period of time is constant, independent of the location of the planet.
3. The ratio between the orbital periods, T, squared of two planets is equal to the ratio of the semi-major axes, s, of their orbits cubed:

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3$$

Newton described the attractive force of gravity exerted between two bodies of mass M_1 and M_2 (which must be point masses) as:

$$\vec{F}_{12} = -G\frac{M_1 M_2}{r^2}\hat{r}_{21}$$

where $\vec{F}_{12}$ is the force on body 1 from body 2, r is the distance between the two bodies, and $\vec{r}_{21}$ is the vector from body 2 to body 1. The motion of a body under the influence of only this force will satisfy all of Kepler's Laws, if the body is gravitationally bound.

The gravitational field, $\vec{g}(\vec{r})$, from a body of mass M, is defined as the gravitational force that another body would experience per unit mass:

$$\vec{g}(\vec{r}) = \frac{\vec{F}(\vec{r})}{m} = -G\frac{M}{r^2}\hat{r}$$

The field can be used to determine the corresponding gravitational force, $\vec{F}_g$, that a body of mass m would experience if located at a position $\vec{r}$ relative to the body of mass M:

$$F_g = m\vec{g}(\vec{r})$$

When describing the motion of objects near the surface of the Earth, it is thus more precise to refer to $g = 9.8\mathrm{N/kg}$ as the magnitude of the Earth's gravitational field at the surface of the Earth, then to refer to $g = 9.8\mathrm{m/s^2}$ as the acceleration due to Earth's gravity. The two are only equal if gravitational mass (the m in the above equation) and inertial mass (the m in Newton's Second Law) are the same.

Gauss' Law, which applies to all inverse-square force laws, can be used to determine the magnitude of the gravitational field from a body of mass M, even it if it not a point mass:

$$\oint \vec{g}(\vec{r}) \cdot d\vec{A} = 4\pi G M^{enc}$$

Since the force described by Newton's theory is conservative, we can define a potential energy function. The gravitational potential energy of a mass m located a distance r away from a mass M is:

$$U(r) = -G\frac{Mm}{r} + C$$

A convenient choice of the constant is $C = 0$, as this corresponds to the gravitational potential energy being equal to zero when m is infinitely far away from M.

The mechanical energy, E, of an object of mass m that is located at a distance r from an object of mass M, if gravity is the only conservative force exerted on m, is given by:

$$E = K + u = \frac{1}{2}mv^2 - G\frac{Mm}{r}$$

where we have explicitly chosen $C = 0$, and v is the speed of m relative M (considered to be at rest). Furthermore, if no non-conservative forces do work on the body of mass m, the mechanical energy, E, is constant.

If the mechanical energy of m is negative, it is gravitationally bound to M. Depending on the mechanical energy of m and its velocity at the point of closest approach to M, the orbit of m will be described by one of four conic sections (circle, ellipse, parabola, hyperbola).

Einstein's Theory of General Relativity describes gravitation as the bending of space and time caused by the presence of mass and energy. In Einstein's theory, objects follow straight (inertial) paths and do not feel a force of gravity. The curvature of space is what results in their apparent motion not being a straight line. Einstein's theory is based on the Equivalence Principle (inertial and gravitational mass are exactly equal) and the properties of how light propagates according to the Theory of Special Relativity.

Important Equations

Kepler's Third Law:

$$\left(\frac{T_1}{T_2}\right)^2 = \left(\frac{s_1}{s_2}\right)^3$$

Gravitational force and gravitational field:

$$\vec{F}_{12} = -G\frac{M_1 M_2}{r^2}\vec{r}_{21}$$

$$\vec{g}(\vec{r}) = -G\frac{M}{r^2}\hat{r}$$

$$F_g = m\vec{(\vec{r})}$$

Gauss's Law:

$$\oint \vec{g}(\vec{r}) \cdot d\vec{A} = 4\pi G M^{enc}$$

Gravitational potential energy and mechanical energy:

$$U(r) = -G\frac{Mm}{r} + C$$

$$E = K + U = \frac{1}{2}mv^2 - G\frac{Mm}{r}$$
