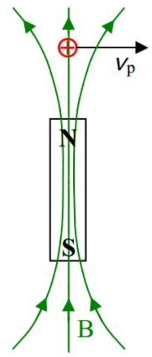
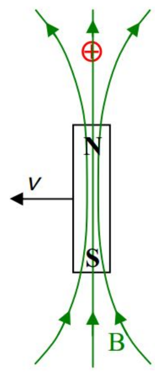
B20: Faraday's Law and Maxwell's Extension to Ampere's Law

Consider the case of a charged particle that is moving in the vicinity of a moving bar magnet as depicted in the following diagram:



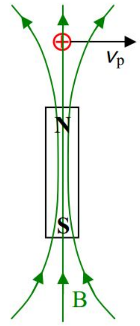
When we view the situation from the reference frame of the magnet, what we see (as depicted just above) is a charged particle moving in a stationary magnetic field. We have already studied the fact that a magnetic field exerts a force $\vec{F} = q\vec{v}_p \times \vec{B}$ on a charged particle moving in that magnetic field. Now let's look at the same phenomenon from the point of view of the charged particle:



(where $\vec{\mathbf{v}} = -\vec{\mathbf{v}}_p$).

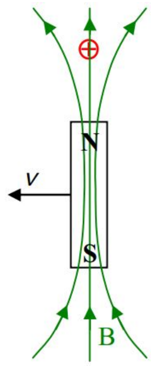
Surely we aren't going to change the force exerted on the charged particle by the magnetic field of the magnet just by looking at the situation from a different reference frame. In fact we've already addressed this issue. What I said was that it is the relative motion between the magnet and the charged particle that matters. Whether the charged particle is moving through magnetic field lines, or the magnetic field lines, due to their motion, are moving sideways through the particle, the particle experiences a force. Now here's the new viewpoint on this situation: What we say is, that the moving magnetic field doesn't really exert a force on the stationary charged particle, but rather, that by moving sideways through the point at which the particle is located, the magnetic field creates an electric field at that location, and it is the electric field that exerts the force on the charged particle. In this viewpoint, we have, at the location of the stationary charged particle, an electric field that is exerting a force on the particle, and a magnetic field that is exerting no force on the particle. At this stage it might seem that it would be necessary to designate the magnetic field as some special kind of magnetic field that doesn't exert a force on a charged particle despite the relative velocity between the charged particle and the magnetic field. Instead, what we actually do is to characterize the magnetic field as being at rest relative to the charged particle.

So, as viewed from the reference frame in which the magnet is at rest:



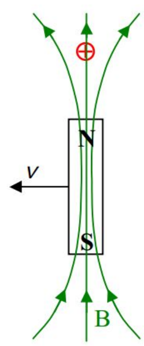
the particle experiences a force $\vec{F}$ directed out of the page in the diagram above due to its motion through the magnetic field.

And, as viewed from the reference frame in which the charged particle is at rest:



the particle finds itself in a stationary magnetic field but experiences the same force $\vec{F}$ because it also finds itself in an electric field directed out of the page.

So we have two models for explaining the force on the stationary charged particle in the case
depicted by:



In model 1 we simply say that in terms of the Lorentz Force $\vec{F} = q\vec{v} \times \vec{B}$, what matters is the relative velocity between the particle and the magnetic field and to calculate the force we identify the velocity $\vec{v}_p$ of the particle relative to the magnetic field as being rightward at magnitude $v_p = v$ in the diagram above so $\vec{F} = q\vec{v} \times \vec{B}$ (where q is the charge of the particle). In model 2 we say that the apparent motion of the magnetic field "causes" there to be an electric field and a stationary magnet field so the particle experiences a force $\vec{F} = q\vec{E}$. Of course we are using two different models to characterize the same force. In order for both models to give the same result we must have:

$$\vec{E} = \vec{v}_p \times \vec{B} \tag{B20.1}$$

where:

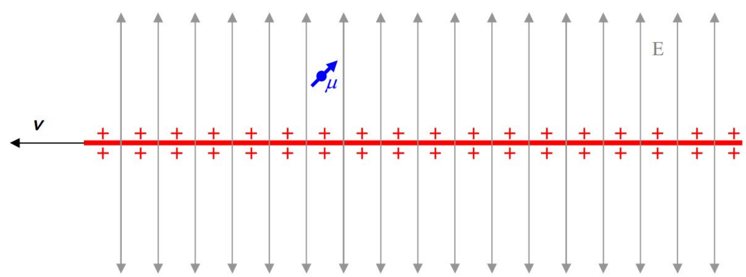
$\vec{E}$ is the electric field at an empty point in space due to the motion of that point relative to a magnetic field vector that exists at that point in space,

$\vec{v}_p$ is the velocity of the empty point in space relative to the magnetic field vector, and

$\vec{B}$ is the magnetic field vector.

Physicists have found model 2 to be more fruitful, especially when attempting to explain magnetic waves. The idea that a magnetic field in apparent sideways motion through a point in space "causes" there to be an electric field at that point in space, is referred to as Faraday's Law of Induction. Our mnemonic for Faraday's Law of Induction is: "A changing magnetic field causes an electric field."

The acceleration experienced by a charged particle in the vicinity of a magnet, when the charged particle is moving relative to the magnet represents an experimental result that we have characterized in terms of the model described in the preceding part of this chapter. The model is useful in that it can be used to predict the outcome of, and provide explanations regarding, related physical processes. Another experimental result is that a particle that has a magnetic dipole moment and is moving in an electric field with a velocity that is neither parallel nor antiparallel to the electric field, does (except for two special magnetic dipole moment directions) experience angular acceleration. We interpret this to mean that the particle experiences a torque. Recalling that a particle with a magnetic dipole moment that is at rest in an electric field experiences no torque, but one that is at rest in a magnetic field does indeed experience a torque (as long as the magnetic dipole moment and the magnetic field it is in are not parallel or antiparallel to each other), you might think that we can model the fact that a particle with a magnetic dipole moment experiences a torque when it is moving relative to an electric field, by defining a magnetic field "caused" by the apparent motion of the electric field relative to the particle. You would be right. To build such a model, we consider a charged particle that is moving in an electric field produced by a long line of charge that is uniformly distributed along the line. We start by depicting the situation in the reference frame in which the particle is at rest and the line of charge is moving:



Note that we have two different ways of accounting for the magnetic field due to the moving line of charge, at the location of the particle with a magnetic dipole moment. The moving line of charge is a current so we can think of the magnetic field as being caused by the current.

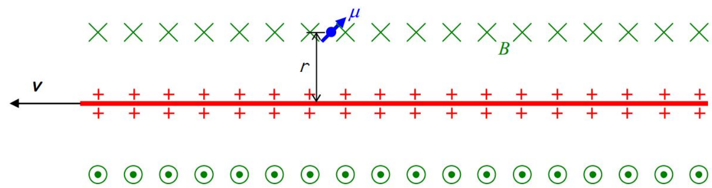
The other option is to view the magnetic field as being caused by the electric field lines moving sideways through the particle. There is, however, only one magnetic field, so, the two different ways of accounting for it must yield the same result. We are going to arrive at an expression for the magnetic field due to the motion of an electric field by forcing the two different ways of accounting for the magnetic field to be consistent with each other. First, we'll simply use Ampere's law to determine the magnetic field at the location of the particle. Let's define the linear charge density (the charge per length) of the line of charge to be λ and the distance that the particle is from the line of charge to be r. Suppose that in an amount of time dt the line of charge moves a distance dx. Then the amount of charge passing a fixed point on the line along which the charge is moving, in time dt, would be λdx. Dividing the latter by dt yields $\lambda dx/dt$ which can be expressed as λv and is just the rate at which charge is flowing past the fixed point, that is, it is the current I. In other words, the moving line of charge is a current $I = \lambda v$. Back in chapter 17 we gave the experimental result for the magnetic field due to a long straight wire carrying current I in the form of an equation that we called "Ampere's Law." It was equation 17-2; it read:

$$B = \frac{\mu_o}{2\pi}\frac{I}{r}$$

and it applies here. substituting $I=\lambda v$ into this expression for B yields

$$B=\frac{\mu_o}{2\pi}\frac{\lambda v}{r} \tag{B20.2}$$

By the right hand rule for something curly something straight we know that the magnetic field is directed into the page at the location of the particle that has a magnetic dipole moment, as depicted in the following diagram:



Now let's work on obtaining an expression for the same magnetic field from the viewpoint that it is the electric field moving sideways through the location of the particle that causes the magnetic field. First we need an expression for the electric field due to the line of charge, at the location of the particle, that is, at a distance r from the line of charge. The way to get that is to consider the line of charge as consisting of an infinite number of bits of charged material, each of which is a segment of infinitesimal length dx of the line of charge. Since the line of charge has a linear charge density λ, this means that each of the infinitesimal segments dx has charge λdx. To get the electric field at the location of the particle that has a magnetic dipole moment, all we have to do is to add up all the contributions to the electric field at the location of the particle, due to all the infinitesimal segments of charged material making up the line of charge. Each contribution is given by Coulomb's Law for the Electric Field. The difficulty is that there are an infinite number of contributions. You will be doing such calculations when you study chapter 30 of this textbook. At this stage, we simply provide the result for the electric field due to an infinitely long line of charge having a constant value of linear charge density λ:

$$E=\frac{\lambda}{2\pi r\epsilon_o}$$

Multiplying both sides by ϵ_o yields

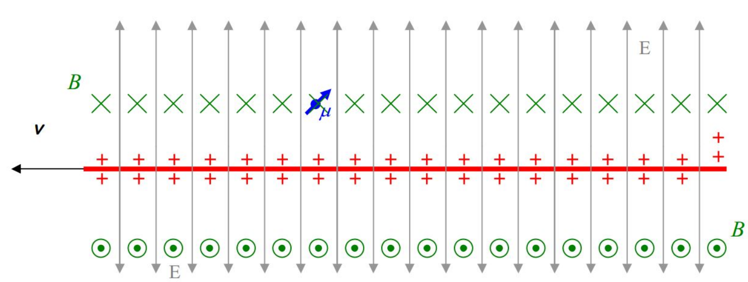
$$\epsilon_o E=\frac{\lambda}{2\pi r}$$

The expression on the right side of this equation appears in equation $ref20-2$, $B=\frac{\mu_o}{2\pi}\frac{\lambda v}{r}$. Substituting $\epsilon_o E$ for $\frac{\lambda}{2\pi r}$ where the latter appears in equation B20.2 yields:

$$B=\mu_o\epsilon_o E v$$

This represents the magnitude of the magnetic field that is experienced by a particle when it is moving with speed $v_p=v$ relative to an electric field $\vec{E}$ when the velocity is perpendicular to $\vec{E}$. Experimentally we find that a particle with a magnetic dipole moment experiences no torque (and hence no magnetic field) if its velocity is parallel or antiparallel to the electric field $\vec{E}$. As such, we can make our result more general (not only good for the case when the velocity is perpendicular to the electric field) if we write, $E_{\perp}$ in place of E.

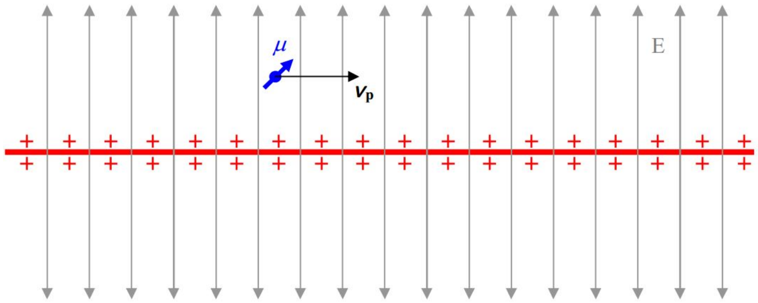
$$B=\mu_o\epsilon_o E_{\perp} v$$



Starting with the preceding equation, we can bundle both the magnitude and the direction (as determined from Ampere's Law and the right hand rule when we treat the moving line of charge as a current, and as depicted in the diagram above) of the magnetic field into one equation by writing:

$$\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$$

We can express $\vec{B}$ in terms of the velocity $\vec{v}_p$ of the particle relative to the line of charge



(instead of the velocity $\vec{v}$ of the line of charge relative to the particle) just by recognizing that

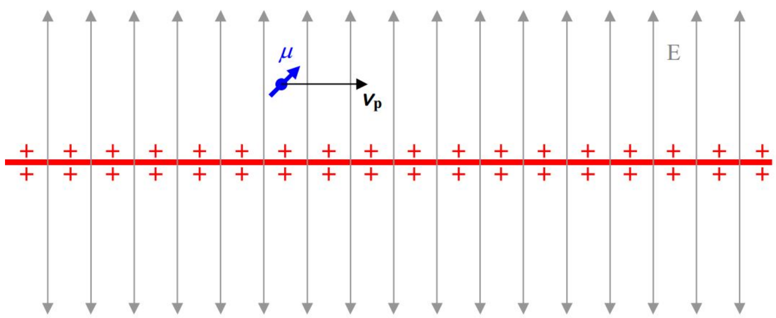
$$\vec{v}_p=-\vec{v}$$

Substituting this expression (\vec{v}_p=-\vec{v}) into our expression for the magnetic field ($\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$) yields:

$$\vec{B}=-\mu_o\epsilon_o\vec{v}_P\times\vec{E}$$

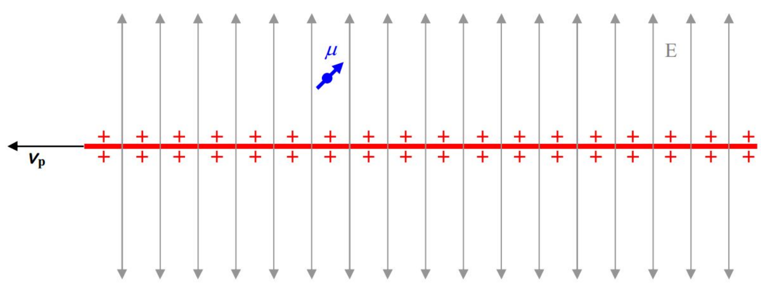
In this model, where we account for the torque experienced by a particle that has a magnetic dipole moment when that particle is moving in an electric field, by defining a magnetic field $\vec{B}=-\mu_o\epsilon_o\vec{v}_P\times\vec{E}$ which depends both on the velocity of the particle relative to the electric field and the electric field itself, the electric field itself is considered to exert no torque on the charged particle. At this stage it might seem that it would be necessary to designate the electric field as some special kind of electric field that doesn't exert a torque on a charged particle despite the relative velocity between the charged particle and the electric field. Instead, what we actually do is to characterize the electric field as being at rest relative to the charged particle.

So, as viewed from the reference frame in which the line of charge is at rest:



the particle that has a magnetic dipole moment experiences a torque due to its motion through the electric field.

And, as viewed from the reference frame in which the particle is at rest:

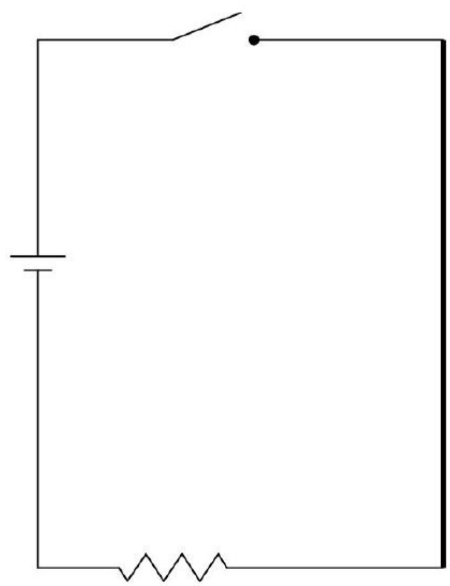


the particle that has a magnetic dipole moment finds itself in a stationary electric field but experiences the same torque because it also finds itself in a magnetic field directed, in the diagram above, into the page. One way of saying what is going on here is to say that, loosely speaking: A changing electric field “causes” a magnetic field. The phenomenon of a changing electric field “causing” a magnetic field is referred to as Maxwell’s Extension to Ampere’s Law.

So far, in this chapter we have addressed two major points: A magnetic field moving sideways through a point in space causes there to be an electric field at that point in space, and, an electric field moving sideways through a point in space causes there to be a magnetic field at that point in space. In the remainder of this chapter we find that putting these two facts together yields something interesting.

Expressing what we have found in terms of the point of view in which point P is fixed and the field is moving through point P with speed $\vec{v}=-\vec{v}_P$, we have: a magnetic field vector $\vec{B}$ moving with velocity $\vec{v}$ transversely through a point in space will “cause” an electric field $\vec{E}=-\vec{v}\times\vec{B}$ at that point in space; and; an electric field vector moving with velocity $\vec{v}$ transversely through a point in space will “cause” a magnetic field $\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$ at that point in space. The word “cause” is in quotes because there is never any time delay. A more precise way of putting it would be to say that whenever we have a magnetic field vector moving transversely through a point in space, there exists, simultaneously, an electric field $\vec{E}=-\vec{v}\times\vec{B}$ at that point in space, and whenever we have an electric field vector moving transversely through a point in space there exists, simultaneously, a magnetic field $\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$ at that point in space.

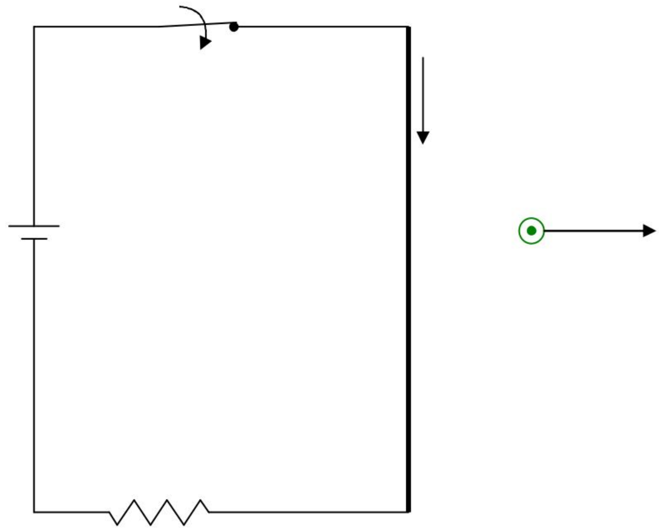
Consider the following circuit. Assume that we are looking down on the circuit from above, meaning that into the page is downward, and out of the page is upward.



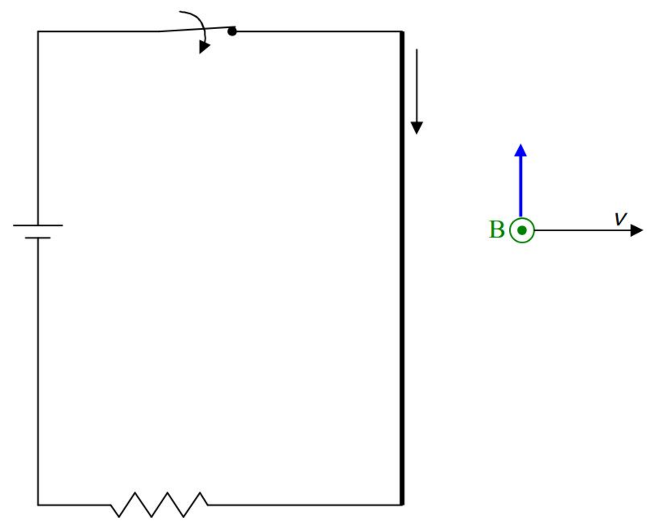
I want you to focus your attention on the rightmost wire of that circuit. As soon as someone closes that switch we are going to get a current through that wire and that current is going to produce a magnetic field. By means of the right-hand rule for something curly something straight, with the current being the something straight, and our knowledge that straight currents cause magnetic fields that make loops around the current, we can deduce that there will be an upward-directed (pointing out of the page) magnetic field at points to the right of the wire. In steady state, we understand that the upward-directed magnetic field vectors will be everywhere to the right of the wire with the magnitude of the magnetic field vector being smaller the greater the distance the point in question is from the wire. Now the question is, how long does it take for the magnetic field to become established at some point a specified distance to the right of the wire? Does the magnetic field appear instantly at every point to the right of the wire or does it take time? James Clerk Maxwell decided to explore the possibility that it takes time, in other words, that the magnetic field develops in the vicinity of the wire and moves outward with a finite velocity.

Here I want to talk about the leading edge of the magnetic field, the expanding boundary within which the magnetic field already exists, and outside of which, the magnetic field does not yet exist. With each passing infinitesimal time interval another infinitesimal layer is added to the region within which the magnetic field exists. While this is more a case of magnetic field vectors growing sideways through space, the effect of the motion of the leading edge through space is the same, at the growing boundary, as magnetic field vectors moving through space. As such, I am going to refer to this magnetic field growth as motion of the magnetic field through space.

To keep the drawing uncluttered I'm going to show just one of the infinite number of magnetic field vectors moving rightward at some unknown velocity (and it is this velocity that I am curious about) as the magnetic field due to the wire becomes established in the universe.



Again, what I'm saying is that, as the magnetic field builds up, what we have, are rightwardmoving upward (pointing out of the page, toward you) magnetic field lines due to the current that just began. Well, as a magnetic field vector moves through whatever location it is moving through, it "causes" an electric field $\vec{E}=-\vec{v}\times\vec{B}$.



At any point P through which the magnetic field vector passes, an electric field exists consistent with $\vec{E}=-\vec{v}$\time$\vec{B}$. What this amounts to is that we have both a magnetic field and an electric field moving rightward through space. But we said that an electric field moving transversely through space "causes" a magnetic field. More specifically we said that it is always accompanied by a magnetic field given by $\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$. Now we've argued around in a circle. The current "causes" the magnetic field and its movement through space "causes" an electric field whose movement through space "causes" the magnetic field. Again, the word "causes" here should really be interpreted as "exists simultaneously with." Still, we have two explanations for the existence of one and the same magnetic field and the two explanations must be consistent with each other. For that to be the case, if we take our expression for the magnetic field "caused" by the motion of the electric field,

$$\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$$

and substitute into it, our expression $\vec{E}=-\vec{v}\times\vec{B}$ for the electric field "caused" by the motion of the magnetic field, we must obtain the same $\vec{B}$ that, in this circular argument, is "causing" itself. Let's try it. Substituting $\vec{E}=-\vec{v}\times\vec{B}$ into $\vec{B}=\mu_o\epsilon_o\vec{v}\times\vec{E}$, we obtain:

$$\vec{B}=-\mu_o\epsilon_o\vec{v}\times(\vec{v}\times\vec{B})$$

All right. Noting that $\vec{v}$ is perpendicular to both $\vec{B}$ and $\vec{v}\times\vec{B}$, meaning that the magnitude of the cross product, in each case, is just the product of the magnitudes of the multiplicand vectors, we obtain:

$$\vec{B}=\mu_o\epsilon_o v^2\vec{B}$$

which I copy here for your convenience:

$$\vec{B}=\mu_o\epsilon_o v^2\vec{B}$$

Again, it is one and the same $\vec{B}$ on both sides, so, the only way this equation can be true is if $\mu_o\epsilon_o v^2$ is exactly equal to 1. Let's see where that leads us:

$$\mu_o\epsilon_o v^2=1$$

$$v^2=\frac{1}{\mu_o\epsilon_0}$$

$$v=\frac{1}{\sqrt{\mu_o\epsilon_o}}$$

$$v=\frac{1}{\sqrt{\left(4\pi\times10^{-7}\frac{T\cdot m}{A}\right)8.85\times10^{-12}\frac{C^2}{N\cdot m^2}}}$$

$$v=3.00\times10^8\frac{m}{s}$$

Wow! That's the speed of light! When James Clerk Maxwell found out that electric and magnetic fields propagate through space at the (already known) speed of light he realized that light is electromagnetic waves.