B33: Gauss's Law

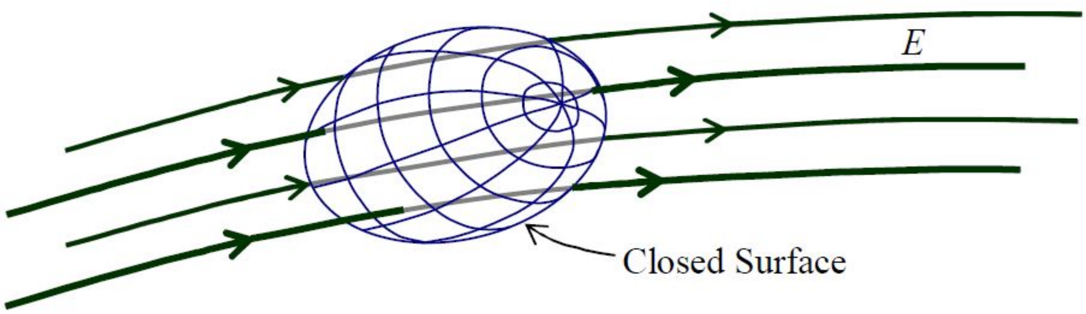
> *When asked to find the electric flux through a closed surface due to a specified non-trivial charge distribution, folks all too often try the immensely complicated approach of finding the electric field everywhere on the surface and doing the integral of $\vec{E}$ dot $\overrightarrow{dA}$ over the surface instead of just dividing the total charge that the surface encloses by ϵ_o.*

Conceptually speaking, Gauss's Law states that the number of electric field lines poking outward through an imaginary closed surface is proportional to the charge enclosed by the surface.

A closed surface is one that divides the universe up into two parts: inside the surface, and, outside the surface. An example would be a soap bubble for which the soap film itself is of negligible thickness. I'm talking about a spheroidal soap bubble floating in air. Imagine one in the shape of a tin can, a closed jar with its lid on, or a closed box. These would also be closed surfaces. To be closed, a surface has to encompass a volume of empty space. A surface in the shape of a flat sheet of paper would not be a closed surface. In the context of Gauss's law, an imaginary closed surface is often referred to as a Gaussian surface.

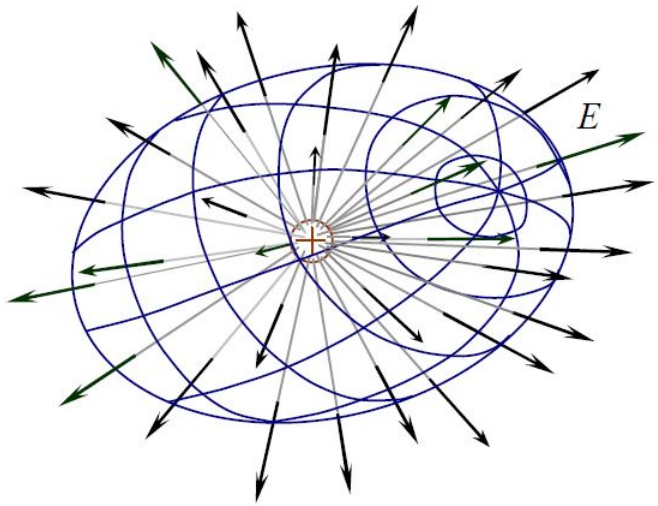
In conceptual terms, if you use Gauss's Law to determine how much charge is in some imaginary closed surface by counting the number of electric field lines poking outward through the surface, you have to consider inward-poking electric field lines as negative outward-poking field lines. Also, if a given electric field line pokes through the surface at more than one location, you have to count each and every penetration of the surface as another field line poking through the surface, adding $+1$ to the tally if it pokes outward through the surface, and -1 to the tally if it pokes inward through the surface.

So for instance, in a situation like:



we have 4 electric field lines poking inward through the surface which, together, count as -4 outward field lines, plus, we have 4 electric field lines poking outward through the surface which together count as $+4$ outward field lines for a total of 0 outward-poking electric field lines through the closed surface. By Gauss's Law, that means that the net charge inside the Gaussian surface is zero.

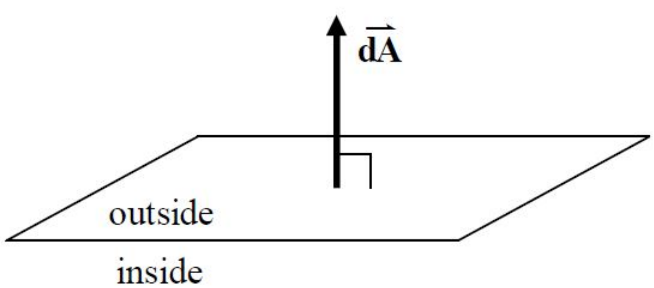
The following diagram might make our conceptual statement of Gauss's Law seem like plain old common sense to you:



The closed surface has the shape of an egg shell. There are 32 electric field lines poking outward through the Gaussian surface (and zero poking inward through it) meaning there must (according to Gauss's Law) be a net positive charge inside the closed surface. Indeed, from your understanding that electric field lines begin, either at positive charges or infinity, and end, either at negative charges or infinity, you could probably deduce our conceptual form of Gauss's Law. If the net number of electric field lines poking out through a closed surface is greater than zero, then you must have more lines beginning inside the surface than you have ending inside the surface, and, since field lines begin at positive charge, that must mean that there is more positive charge inside the surface than there is negative charge.

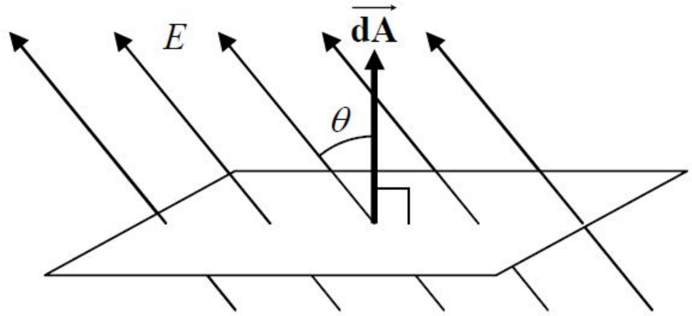
Our conceptual idea of the net number of electric field lines poking outward through a Gaussian surface corresponds to the net outward electric flux Φ_E through the surface.

To write an expression for the infinitesimal amount of outward flux $d\Phi_E$ through an infinitesimal area element dA, we first define an area element vector $\overrightarrow{dA}$ whose magnitude is, of course, just the area dA of the element; and; whose direction is perpendicular to the area element, and, outward. (Recall that a closed surface separates the universe into two parts, an inside part and an outside part. Thus, at any point on the surface, that is to say at the location of any infinitesimal area element on the surface, the direction outward, away from the inside part, is unambiguous.)



In terms of that area element, and, the electric field $\vec{E}$ at the location of the area element, we can write the infinitesimal amount of electric flux $d\Phi_E$ through the area element as:

$$d\Phi_E = \vec{E} \cdot \overrightarrow{dA}$$



Recall that the dot product $\vec{E}\cdot\overrightarrow{dA}$ can be expressed as $EdA\cos\theta$. For a given E and a given amount of area, this yields a maximum value for the case of $\theta=0^\circ$ (when $\vec{E}$ is parallel to $\overrightarrow{dA}$ meaning that $\vec{E}$ is perpendicular to the surface); zero when $\theta=90^\circ$ (when $\vec{E}$ is perpendicular to $\overrightarrow{dA}$ meaning that $\vec{E}$ is parallel to the surface); and; a negative value when θ is greater than 90° (with 180° being the greatest value of θ possible, the angle at which $\vec{E}$ is again perpendicular to the surface, but, in this case, into the surface.)

Now, the flux is the quantity that we can think of conceptually as the number of field lines. So, in terms of the flux, Gauss's Law states that the net outward flux through a closed surface is proportional to the amount of charge enclosed by that surface. Indeed, the constant of proportionality has been established to be $\frac{1}{\epsilon_0}$ where ϵ_0 (epsilon zero) is the universal constant known as the electric permittivity of free space. (You've seen ϵ_0 before. At the time, we stated that the Coulomb constant k is often expressed as $\frac{1}{4\pi\epsilon_0}$. Indeed, the identity $k=\frac{1}{4\pi\epsilon_0}$ appears on your formula sheet.) In equation form, Gauss's Law reads:

$$\oint\vec{E}\cdot\overrightarrow{dA}=\frac{Q_{\text{ENCLOSED}}}{\epsilon_0}\tag{B33.1}$$

The circle on the integral sign, combined with the fact that the infinitesimal in the integrand is an area element, means that the integral is over a closed surface. The quantity on the left is the sum of the product $\vec{E}\cdot\overrightarrow{dA}$ for each and every area element dA making up the closed surface. It is the total outward electric flux through the surface.

$$\Phi_E=\oint\vec{E}\cdot\overrightarrow{dA}\tag{B33.2}$$

Using this definition in Gauss's Law allows us to write Gauss's Law in the form:

$$\Phi_E=\frac{Q_{\text{ENCLOSED}}}{\epsilon_0}\tag{B33.3}$$

How You Will be Using Gauss's Law

Gauss's Law is an integral equation. Such an integral equation can also be expressed as a differential equation. We won't be using the differential form, but, because of its existence, the Gauss's Law equation

$$\oint\vec{E}\cdot\overrightarrow{dA}=\frac{Q_{\text{ENCLOSED}}}{\epsilon_0}$$

is referred to as the integral form of Gauss's Law. The integral form of Gauss's Law can be used for several different purposes. In the course for which this book is written, you will be using it in a limited manner consistent with the mathematical prerequisites and co-requisites for the course. Here's how:

1. Gauss's Law in the form $\Phi_E=\frac{Q_{\text{ENCLOSED}}}{\epsilon_0}$ makes it easy to calculate the net outward flux through a closed surface that encloses a known amount of charge Q_{ENCLOSED}. Just divide the amount of charge Q_{ENCLOSED} by ϵ_0 (given on your formula sheet as $\epsilon_0=8.85\times10^{-12}\frac{C^2}{N\cdot m^2}$ and you have the flux through the closed surface.

2. Given the electric field at all points on a closed surface, one can use the integral form of Gauss's Law to calculate the charge inside the closed surface. This can be used as a check for a case in which the electric field due to a given distribution of charge has been calculated by a means other than Gauss's Law. You will only be expected to do this in cases in which one can treat the closed surface as being made of one or more finite (not vanishingly small) surface pieces on which the electric field is constant over the entire surface piece so that the flux can be calculated algebraically as EA or $EA\cos\theta$. After doing so for each of the finite surface pieces making up the closed surface, you add the results and you have the flux

$$\Phi_E = \oint \vec{E} \cdot \overrightarrow{dA}$$

through the surface. To get the charge enclosed by the surface, you just plug that into
3. In cases involving a symmetric charge distribution, Gauss's Law can be used to calculate the electric field due to the charge distribution. In such cases, the right choice of the Gaussian surface makes E a constant at all points on each of several surface pieces, and in some cases, zero on other surface pieces. In such cases the flux can be expressed as EA and one can simply solve $EA = \frac{Q_{\text{enclosed}}}{\epsilon_o}$ for E and use one's conceptual understanding of the electric field to get the direction of $\vec{E}$. The remainder of this chapter and all of the next will be used to provide examples of the kinds of charge distributions to which you will be expected to be able to apply this method.

Using Gauss's Law to Calculate the Electric Field in the Case of a Charge Distribution Having Spherical Symmetry

A spherically-symmetric charge distribution has a well-defined center. Furthermore, if you rotate a spherically-symmetric charge distribution through any angle, about any axis that passes through the center, you wind up with the exact same charge distribution. A uniform ball of charge is an example of a spherically-symmetric charge distribution. Before we consider that one, however, let's take up the case of the simplest charge distribution of them all, a point charge.

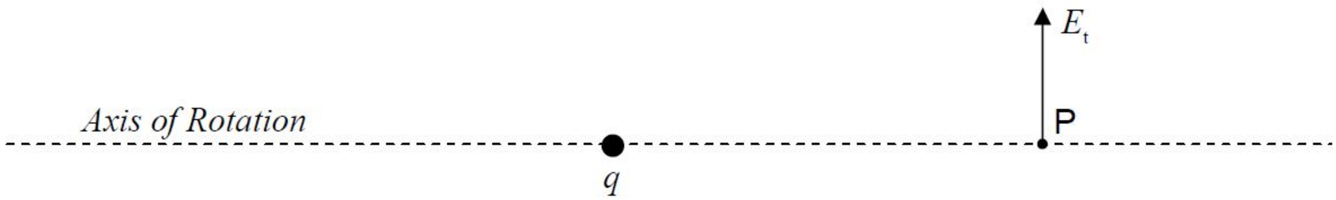
We use the symmetry of the charge distribution to find out as much as we can about the electric field and then we use Gauss's Law to do the rest. Now, when we rotate the charge distribution, we rotate the electric field with it. And, if a rotation of the charge distribution leaves you with the same exact charge distribution, then, it must also leave you with the same electric field.

We first prove that the electric field due to a point charge can have no tangential component by assuming that it does have a tangential component and showing that this leads to a contradiction.

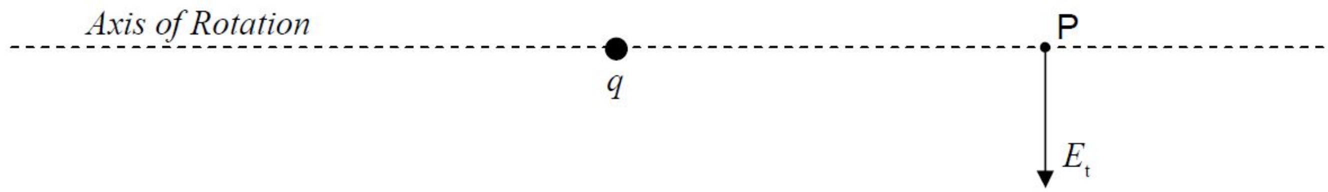
Here's our point charge q, and an assumed tangential component of the electric field at a point P which, from our perspective is to the right of the point charge.

(Note that a radial direction is any direction away from the point charge, and, a tangential direction is perpendicular to the radial direction.)

Now let's decide on a rotation axis for testing whether the electric field is symmetric with respect to rotation. Almost any will do. I choose one that passes through both the point charge, and, point P.



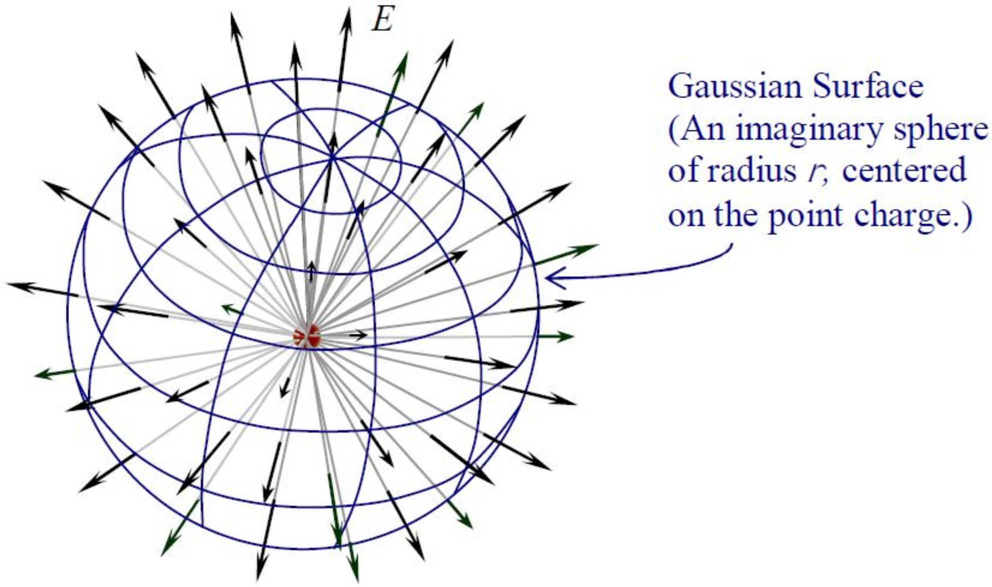
Now, if I rotate the charge, and its associated electric field, through an angle of 180° about that axis, I get:



This is different from the electric field that we started with. It is downward instead of upward. Hence the electric field cannot have the tangential component depicted at point P. Note that the argument does not depend on how far point P is from the point charge; indeed, I never specified the distance. So, no point to the right of our point charge can have an upward component to its electric field. In fact, if I assume the electric field at any point P' in space other than the point at which the charge is, to have a tangential component, then, I can adopt a viewpoint from which point P' appears to be to the right of the charge, and, the electric field appears to be upward. From that viewpoint, I can make the same rotation argument presented above to prove that the tangential component cannot exist. Thus, based on the spherical symmetry of the charge distribution, the electric field due to a point charge has to be strictly radial. Thus, at each point in space, the electric field must be either directly toward the point charge or directly away from it. Furthermore, again from symmetry, if the electric field is directly away from the point charge at one point in space, then it has to be directly away from the point charge at every point in space. Likewise, for the case in which it is directly toward the point charge at one point in space, the electric field has to be directly toward the point charge at every point in space.

We've boiled it down to a 50/50 choice. Let's assume that the electric field is directed away from the point charge at every point in space and use Gauss's Law to calculate the magnitude of the electric field. If the magnitude is positive, then the electric field is indeed directed away from the point charge. If the magnitude turns out to be negative, then the electric field is actually directed toward the point charge.

At this point we need to choose a Gaussian surface. To further exploit the symmetry of the charge distribution, we choose a Gaussian surface with spherical symmetry. More specifically, we choose a spherical shell of radius r, centered on the point charge.



At every point on the shell, the electric field, being radial, has to be perpendicular to the spherical shell. This means that for every area element, the electric field is parallel to our outward-directed area element vector $\overrightarrow{dA}$. This means that the $\vec{E} \cdot \overrightarrow{dA}$ in Gauss's Law,

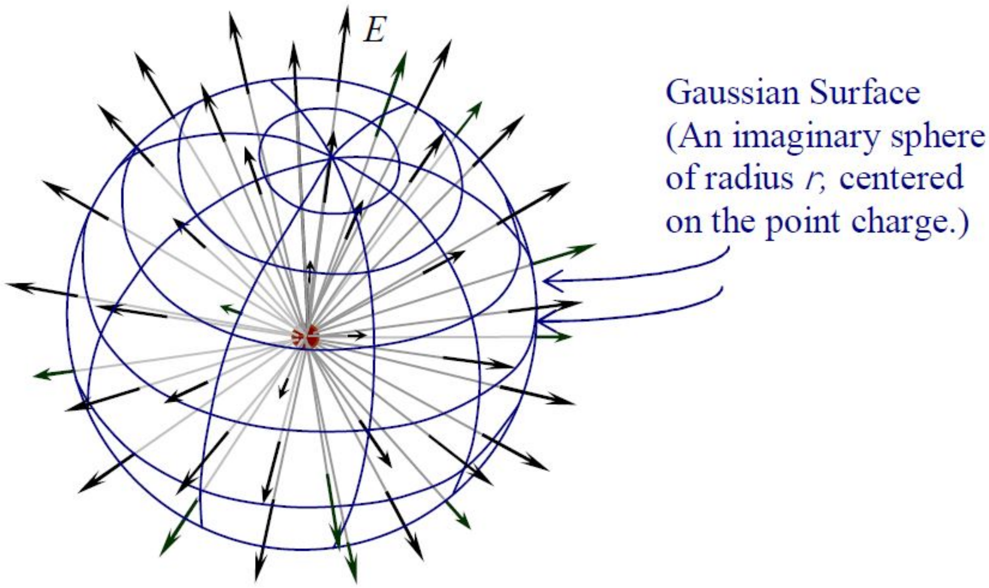
$$\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

evaluates to $E\, dA$. So, for the case at hand, Gauss's Law takes on the form:

$$\oint E dA = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

Furthermore, the magnitude of the electric field has to have the same value at every point on the shell. If it were different at a point P' on the spherical shell than it is at a point P on the spherical shell, then we could rotate the charge distribution about an axis through the point charge in such a manner as to bring the original electric field at point P' to position P. But this would represent a change in the electric field at point P, due to the rotation, in violation of the fact that a point charge has spherical symmetry. Hence, the electric field at any point P' on the Gaussian surface must have the same magnitude as the electric field at point P, which is what I set out to prove. The fact that E is a constant, in the integral, means that we can factor it out of the integral. So, for the case at hand, Gauss's Law takes on the form:

$$E \oint dA = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$



On the preceding page we arrived at $E\oint dA = \frac{Q_{\text{enclosed}}}{\epsilon_o}$.

Now $\oint dA$, the integral of dA over the Gaussian surface is the sum of all the area elements making up the Gaussian surface. That means that it is just the total area of the Gaussian surface. The Gaussian surface, being a sphere of radius r, has area $4\pi r^2$. So now, Gauss's Law for the case at hand looks like:

$$E4\pi r^2 = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

Okay, we've left that right side alone for long enough. We're talking about a point charge q and our Gaussian surface is a sphere centered on that point charge q, so, the charge enclosed, Q_{enclosed} is obviously q. This yields:

$$E4\pi r^2 = \frac{q}{\epsilon_o}$$

Solving for E gives us:

$$E = \frac{1}{4\pi\epsilon_o}\frac{q}{r^2}$$

This is positive when the charge q is positive, meaning that the electric field is directed outward, as per our assumption. It is negative when q is negative. So, when the charge q is negative, the electric field is directed inward, toward the charged particle. This expression is, of course, just Coulomb's Law for the electric field. It may look more familiar to you if we write it in terms of the Coulomb constant $k = \frac{1}{4\pi\epsilon_o}$ in which case our result for the outward electric field appears as:

$$E = \frac{kq}{r^2}$$

It's clear that, by means of our first example of Gauss's Law, we have derived something that you already know, the electric field due to a point charge.