10A: Constant Acceleration Problems in Two Dimensions

In solving problems involving constant acceleration in two dimensions, the most common mistake is probably mixing the x and y motion. One should do an analysis of the x motion and a separate analysis of the y motion. The only variable common to both the x and the y motion is the time. Note that if the initial velocity is in a direction that is along neither axis, one must first break up the initial velocity into its components.

In the last few chapters we have considered the motion of a particle that moves along a straight line with constant acceleration. In such a case, the velocity and the acceleration are always directed along one and the same line, the line on which the particle moves. Here we continue to restrict ourselves to cases involving constant acceleration (constant in both magnitude and direction) but lift the restriction that the velocity and the acceleration be directed along one and the same line. If the velocity of the particle at time zero is not collinear with the acceleration, then the velocity will never be collinear with the acceleration and the particle will move along a curved path. The curved path will be confined to the plane that contains both the initial velocity vector and the acceleration vector, and in that plane, the trajectory will be a parabola. (The trajectory is just the path of the particle.)

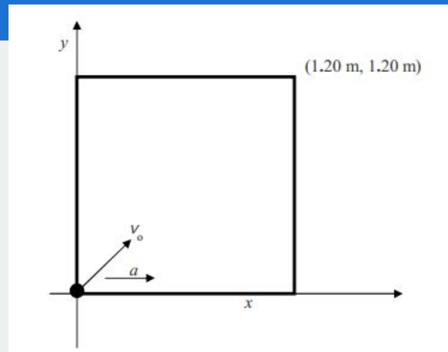
You are going to be responsible for dealing with two classes of problems involving constant acceleration in two dimensions:

1. Problems involving the motion of a single particle.
2. Collision Type II problems in two dimensions

We use sample problems to illustrate the concepts that you must understand in order to solve two-dimensional constant acceleration problems.

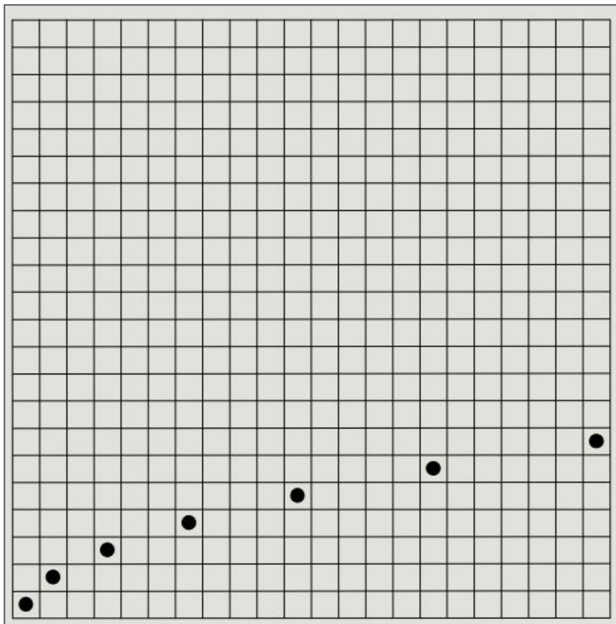
A horizontal square of edge length 1.20 m is situated on a Cartesian coordinate system such that one corner of the square is at the origin and the corner opposite that corner is at (1.20 m, 1.20 m). A particle is at the origin. The particle has an initial velocity of 2.20 m/s directed toward the corner of the square at (1.20 m, 1.20 m) and has a constant acceleration of $4.87 m/s^2$ in the $+x$ direction. Where does the particle hit the perimeter of the square?

Solution Let's start with a diagram.



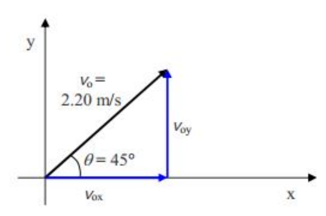
Now let's make some conceptual observations on the motion of the particle. Recall that the square is horizontal so we are looking down on it from above. It is clear that the particle hits the right side of the square because: It starts out with a velocity directed toward the far right corner. That initial velocity has an x component and a y component. The y component never changes because there is no acceleration in the y direction. The x component, however, continually increases. The particle is going rightward faster and faster. Thus, it will take less time to get to the right side of the square then it would without the acceleration and the particle will get to the right side of the square before it has time to get to the far side.

An important aside on the trajectory (path) of the particle: Consider an ordinary checker on a huge square checkerboard with squares of ordinary size (just a lot more of them then you find on a standard checkerboard). Suppose you start with the checker on the extreme left square of the end of the board nearest you (square 1) and every second, you move the checker right one square and forward one square. This would correspond to the checker moving toward the far right corner at constant velocity. Indeed you would be moving the checker along the diagonal. Now let's throw in some acceleration. Return the checker to square 1 and start moving it again. This time, each time you move the checker forward, you move it rightward one more square than you did on the previous move. So first you move it forward one square and rightward one square. Then you move it forward another square but rightward two more squares. Then forward one square and rightward three squares. And so on. With each passing second, the rightward move gets bigger. (That's what we mean when we say the rightward velocity is continually increasing.) So what would the path of the checker look like? Let's draw a picture.



As you can see, the checker moves on a curved path. Similarly, the path of the particle in the problem at hand is curved.

Now back to the problem at hand. The way to attack these two-dimensional constant acceleration problems is to treat the x motion and the y motion separately. The difficulty with that, in the case at hand, is that the initial velocity is neither along x nor along y but is indeed a mixture of both x motion and y motion. What we have to do is to separate it out into its x and y components. Let's proceed with that. Note that, by inspection, the angle that the velocity vector makes with the x axis is 45.0°.



$\cos\theta = \frac{v_{ox}}{v_o}$ $v_{ox} = v_o \cos\theta$ $v_{ox} = 2.20\frac{\text{m}}{\text{s}}\cos 45.0^\circ$ $v_{ox} = 1.556\frac{\text{m}}{\text{s}}$	By inspection (because the angle is 45.0°): $v_{oy} = v_{ox}$ So: $v_{oy} = 1.556\frac{\text{m}}{\text{s}}$

Now we are ready to attack the x motion and the y motion separately. Before we do, let's consider our plan of attack. We have established, by means of conceptual reasoning, that the particle will hit the right side of the square. This means that we already have the answer to half of the question "Where does the particle hit the perimeter of the square?" It hits it at $x = 1.20m$ and y = ? . All we have to do is to find out the value of y. We have established that it is the x motion that determines the time it takes for the particle to hit the perimeter of the square. It hits the perimeter of the square at that instant in time when x achieves the value of $1.20m$. So our plan of attack is to use one or more of the x-motion constant acceleration equations to determine the time at which the particle hits the perimeter of the square and to plug that time into the appropriate y-motion constant acceleration equation to get the value of y at which the particle hits the side of the square. Let's go for it.

x motion

We start with the equation that relates position and time:

$$x = \cancelto{0}{x_o} + v_{ox}t + \frac{1}{2}a_x t^2 \quad \text{(We need to find the time that makes } x = 1.20 \text{ m.)}$$

The x component of the acceleration is the total acceleration, that is $a_x = a$. Thus,

$$x = v_{0x}t + \frac{1}{2}at^2$$

Recognizing that we are dealing with a quadratic equation we get it in the standard form of the quadratic equation.

$$\frac{1}{2}at^2 + v_{0x}t - x = 0$$

Now we apply the quadratic formula:

$$t = \frac{-v_{0x} \pm \sqrt{v_{0x}^2 - 4(\frac{1}{2}a)(-x)}}{2(\frac{1}{2}a_x)}$$

$$t = \frac{-v_{0x} \pm \sqrt{v_{0x}^2 + 2ax}}{a_x}$$

substituting values with units (and, in this step, doing no evaluation) we obtain:

$$t = \frac{-1.556\frac{m}{s} \pm \sqrt{(1.556\frac{m}{s})^2 + 2(4.87\frac{m}{s^2})1.20m}}{4.87\frac{m}{s^2}}$$

Evaluating this expression yields:

$$t = 0.4518s$$

and

$$t = -1.091s.$$

We are solving for a future time so we eliminate the negative result on the grounds that it is a time in the past. We have found that the particle arrives at the right side of the square at time $t = 0.4518s$. Now the question is, "What is the value of y at that time?"

y motion

Again we turn to the constant acceleration equation relating position to time, this time writing it in terms of the y variables:

$$y = \cancelto{0}{y_o} + v_{oy}t + \frac{1}{2}\cancelto{0}{a_y}t^2$$

We note that y_0 is zero because the particle is at the origin at time 0 and ay is zero because the acceleration is in the $+x$ direction meaning it has no y component. Rewriting this:

$$y = V_{0y}t$$

Substituting values with units,

$$y = 1.556\frac{m}{s}(0.4518s)$$

evaluating, and rounding to three significant figures yields:

$$y = 0.703m$$

Thus, the particle hits the perimeter of the square at

$$(1.20m, 0.703m)$$

Next, let's consider a 2-D Collision Type II problem. Solving a typical 2-D Collision Type II problem involves finding the trajectory of one of the particles, finding when the other particle crosses that trajectory, and establishing where the first particle is when the second particle crosses that trajectory. If the first particle is at the point on its own trajectory where the second particle crosses that trajectory then there is a collision. In the case of objects rather than particles, one often has to do some further reasoning to solve a 2-D Collision Type II problem. Such reasoning is illustrated in the following example involving a rocket.

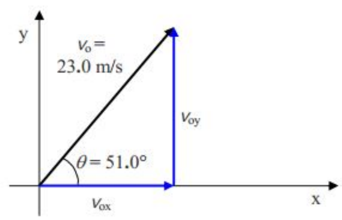
The positions of a particle and a thin (treat it as being as thin as a line) rocket of length $0.280m$ are specified by means of Cartesian coordinates. At time 0 the particle is at the origin and is moving on a horizontal surface at $23.0m/s$ at $51.0°$. It has a constant acceleration of $2.43m/s^2$ in the $+y$ direction. At time 0 the rocket is at rest and it extends from $(-.280m, 50.0m)$ to $(0, 50.0m)$, but it has a constant acceleration in the $+x$ direction. What must the acceleration of the rocket be in order for the particle to hit the rocket?

Solution

Based on the description of the motion, the rocket travels on the horizontal surface along the line $y = 50.0m$. Let's figure out where and when the particle crosses this line. Then we'll calculate the acceleration that the rocket must have in order for the nose of the rocket to be at that point at that time and repeat for the tail of the rocket. Finally, we'll quote our answer as being any acceleration in between those two values.

When and where does the particle cross the line $y = 50.0m$?

We need to treat the particle's x motion and the y motion separately. Let's start by breaking up the initial velocity of the particle into its x and y components.



$\cos\theta = \frac{V_{ox}}{V_o}$	$\sin\theta = \frac{V_{oy}}{V_o}$
$V_{ox} = V_o \cos\theta$	$V_{oy} = V_o \sin\theta$
$V_{ox} = 23.0\frac{m}{s}\cos 51.0°$	$V_{oy} = 23.0\frac{m}{s}\sin 51.0°$
$V_{ox} = 14.47\frac{m}{s}$	$V_{oy} = 17.87\frac{m}{s}$

Now in this case, it is the y motion that determines when the particle crosses the trajectory of the rocket because it does so when $y = 50.0m$. So let's address the y motion first.

y motion of the particle

$$y = \cancel{y_o}^{\,0} + V_{oy}\,t + \frac{1}{2}a_y t^2$$

Note that we can't just assume that we can cross out yo but in this case the time zero position of the particle was given as $(0, 0)$ meaning that y_o is indeed zero for the case at hand. Now we solve for t:

$$y = V_{0y}t + \frac{1}{2}a_y t^2 \quad \frac{1}{2}a_y t^2 + V_{0y}t - y = 0 \qquad t = \frac{-V_{0y} \pm \sqrt{V_{0y}^2 - 4(\frac{1}{2}a_y)(-y)}}{2(\frac{1}{2}a_y)}$$

$$t = \frac{-V_{0y} \pm \sqrt{V_{0y}^2 + 2a_y y}}{a_y} \qquad t = \frac{-17.87\frac{m}{s} \pm \sqrt{(17.87\frac{m}{s})^2 + 2(2.43\frac{m}{s^2})50.0m}}{2.43\frac{m}{s^2}}$$

$t = 2.405s$ and $t = -17.11s$

Again, we throw out the negative solution because it represents an instant in the past and we want a future instant. Now we turn to the x motion to determine where the particle crosses the trajectory of the rocket.

x motion of the particle

Again we turn to the constant acceleration equation relating position to time, this time writing it in terms of the x variables:

(because the particle starts at the origin)

(because the acceleration is in the y direction)

$$x=\cancelto{0}{x_o}+v_{ox}t+\frac{1}{2}\cancelto{0}{a_x}t^2$$

$$x=v_{ox}t$$

$$x=14.47\frac{\text{m}}{\text{s}}(2.405\ \text{s})$$

$$\text{x}=34.80\ \text{m.}$$

So the particle crosses the rocket’s path at $(34.80m, 50.0m)$ at time $t=2.450s$. Let’s calculate the acceleration that the rocket would have to have in order for the nose of the rocket to be there at that instant. The rocket has x motion only. It is always on the line $y=50.0m$.

Motion of the Nose of the Rocket

$$x'_n=\cancelto{0}{x'_{on}}+\cancelto{0}{v'_{oxn}}t+\frac{1}{2}a'_nt^2$$

where we use the subscript n for “nose” and a prime to indicate “rocket.” We have crossed out x'_{on} because the nose of the rocket is at $(0, 50.0m)$ at time zero, and we have crossed out v'_{oxn} because the rocket is at rest at time zero.

$$x'_n=\frac{1}{2}a'_nt^2$$

Solving for a'_n yields:

$$a'_n=\frac{2x'_n}{t^2}$$

Now we just have to evaluate this expression at $t=2.405s$, the instant when the particle crosses the trajectory of the rocket, and at $x'_n=x=34.80m$, the value of x at which the particle crosses the trajectory of the rocket.

$$a'_n=\frac{2(34.80m)}{(2.405s)^2}\quad a'_n=12.0\frac{m}{s^2}$$

It should be emphasized that the n for “nose” is not there to imply that the nose of the rocket has a different acceleration than the tail; rather; the whole rocket must have the acceleration $a'_n=12.0\frac{m}{s^2}$ in order for the particle to hit the rocket in the nose. Now let’s find the acceleration at’that the entire rocket must have in order for the particle to hit the rocket in the tail.

Motion of the Tail of the Rocket

$$x'_t=x'_{ot}+\cancelto{0}{v'_{oxt}}t+\frac{1}{2}a'_tt^2$$

where we use the subscript t for “tail” and a prime to indicate “rocket.” We have crossed out v'_{oxt} because the rocket is at rest at time zero, but x'_{ot} is not zero because the tail of the rocket is at ($-.280, 50.0m$) at time zero.

$$x'_t=x'_{ot}+\frac{1}{2}a'_tt^2$$

Solving for a'_t yields:

$$a'_t=\frac{2(x'_t-x'_{ot})}{t^2}$$

Evaluating at $t=2.405s$ and $x'_t=x=34.80m$ yields

$$a'_t=\frac{2(34.80m-(-0.280m))}{(2.405s)^2}\quad a'_t=12.1\frac{m}{s^2}$$

as the acceleration that the rocket must have in order for the particle to hit the tail of the rocket.

Thus: The acceleration of the rocket must be somewhere between $12.0\frac{m}{s^2}$ and $12.1\frac{m}{s^2}$, inclusive, in order for the rocket to be hit by the particle.