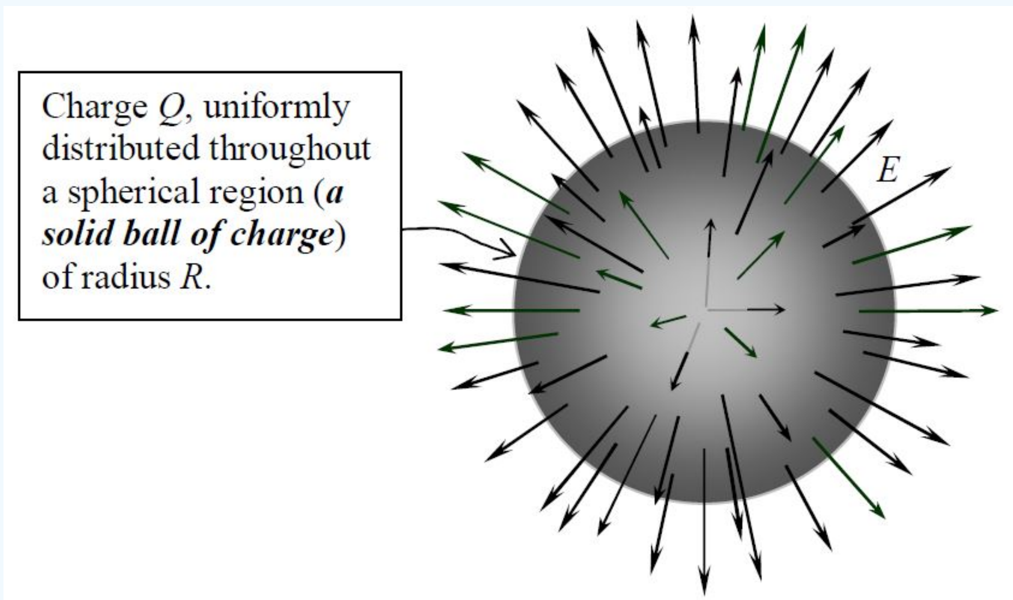
B34: Gauss's Law Example

We finished off the last chapter by using Gauss's Law to find the electric field due to a point charge. It was an example of a charge distribution having spherical symmetry. In this chapter we provide another example involving spherical symmetry.

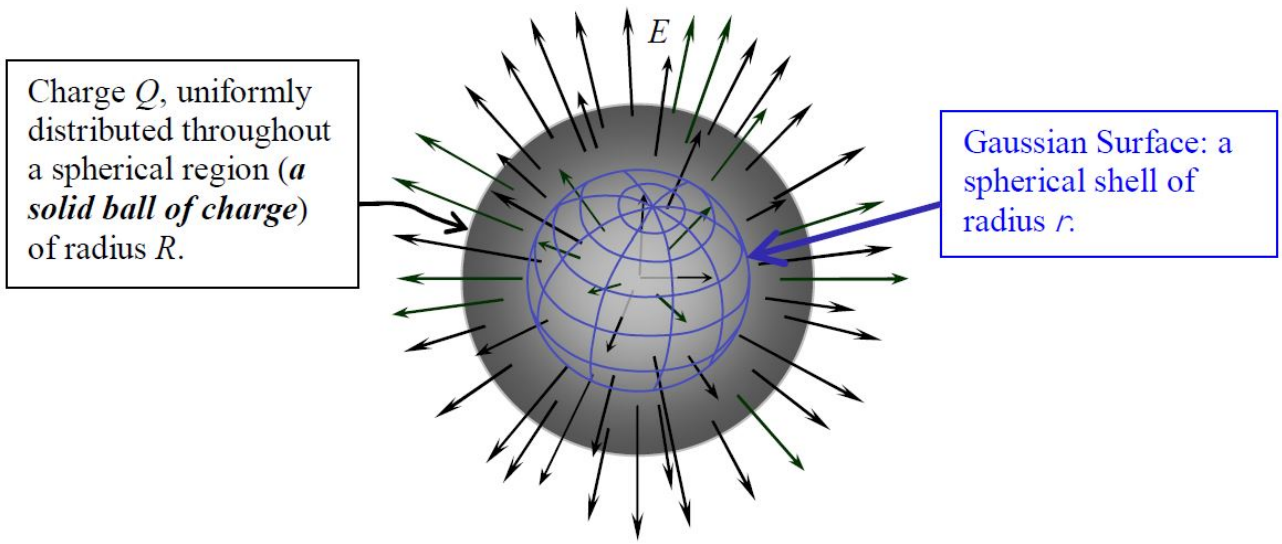
Find the electric field due to a uniform ball of charge of radius R and total charge Q. Express the electric field as a function of r, the distance from the center of the ball.

Solution

Again we have a charge distribution for which a rotation through any angle about any axis passing through the center of the charge distribution results in the exact same charge distribution. Thus, the same symmetry arguments used for the case of the point charge apply here with the result that, the electric field due to the ball of charge has to be strictly radially directed, and, the electric field has one and the same value at every point on any given spherical shell centered on the center of the ball of charge. Again, we assume the electric field to be outward-directed. If it turns out to be inward-directed, we'll simply get a negative value for the magnitude of the outward-directed electric field.



The appropriate Gaussian surface for any spherical charge distribution is a spherical shell centered on the center of the charge distribution.



Okay, let's go ahead and apply Gauss's Law.

$$\oint \vec{E} \cdot \overrightarrow{dA} = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

Since the electric field is radial, it is, at all points, perpendicular to the Gaussian Surface. In other words, it is parallel to the area element vector $\overrightarrow{dA}$. This means that the dot product $\vec{E} \cdot \overrightarrow{dA}$ is equal to the product of the magnitudes, EdA. This yields:

$$\oint EdA = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

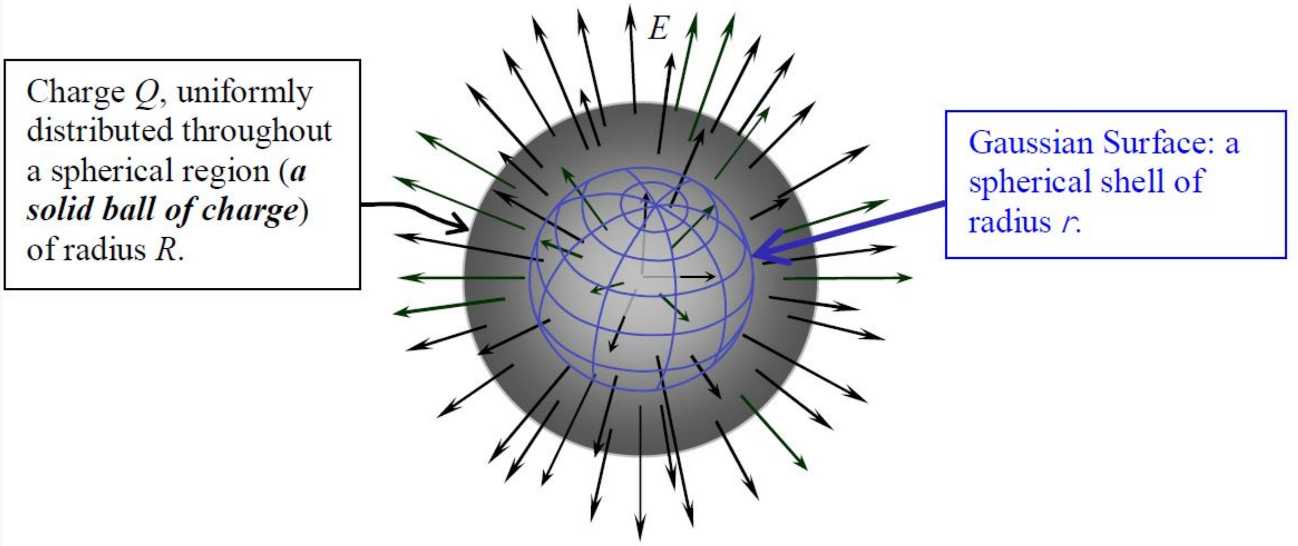
Again, since E has the same value at all points on the Gaussian surface of radius r, each dA in the infinite sum that the integral on the left is, is multiplied by the same value of E. Hence, we can factor the E out of the sum (integral). This yields

$$E \oint dA = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

The integral on the left is just the infinite sum of all the infinitesimal area elements making up the Gaussian surface, our spherical shell of radius r. The sum of all the area elements is, of course, the area of the spherical shell. The area of a sphere is $4\pi r^2$. So,

$$E 4\pi r^2 = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

Now the question is, how much charge is enclosed by our Gaussian surface of radius r?



There are two ways that we can get the value of the charge enclosed. Let's try it both ways and make sure we get one and the same result.

The first way: Because the charge is uniformly distributed throughout the volume, the amount of charge enclosed is directly proportional to the volume enclosed. So, the ratio of the amount of charge enclosed to the total charge, is equal to the ratio of the volume enclosed by the Gaussian surface to the total volume of the ball of charge:

$$\frac{Q_{\text{Enclosed}}}{Q}=\frac{\text{Volume of Gaussian Surface}}{\text{Volume of the Entire Ball of Charge}}$$

$$\frac{Q_{\text{Enclosed}}}{Q}=\frac{\frac{4}{3}\pi r^3}{\frac{4}{3}\pi R^3}$$

$$Q_{\text{Enclosed}}=\frac{r^3}{R^3}Q$$

The second way: The other way we can look at it is to recognize that for a uniform distribution of charge, the amount of charge enclosed by the Gaussian surface is just the volume charge density, that is, the charge-per-volume ρ, times the volume enclosed.

$$Q_{\text{Enclosed}}=\rho\,(\text{Volume of the Gaussian surface})$$

$$Q_{\text{enclosed}}=\rho\frac{4}{3}\pi r^3$$

In this second method, we again take advantage of the fact that we are dealing with a uniform charge distribution. In a uniform charge distribution, the charge density is just the total charge divided by the total volume. Thus:

$$\rho=\frac{Q}{\text{Volume of Ball of Charge}}$$

$$\rho=\frac{Q}{\frac{4}{3}\pi R^3}$$

Substituting this in to our expression $Q_{\text{enclosed}}=\rho\,4\pi r^2$ for the charge enclosed by the Gaussian surface yields:

$$Q_{\text{enclosed}}=\frac{Q}{\frac{4}{3}\pi R^3}\frac{4}{3}\pi r^3$$

$$Q_{\text{enclosed}}=\frac{r^3}{R^3}Q$$

which is indeed the same expression that we arrived at in solving for the charge enclosed the first way we talked about.

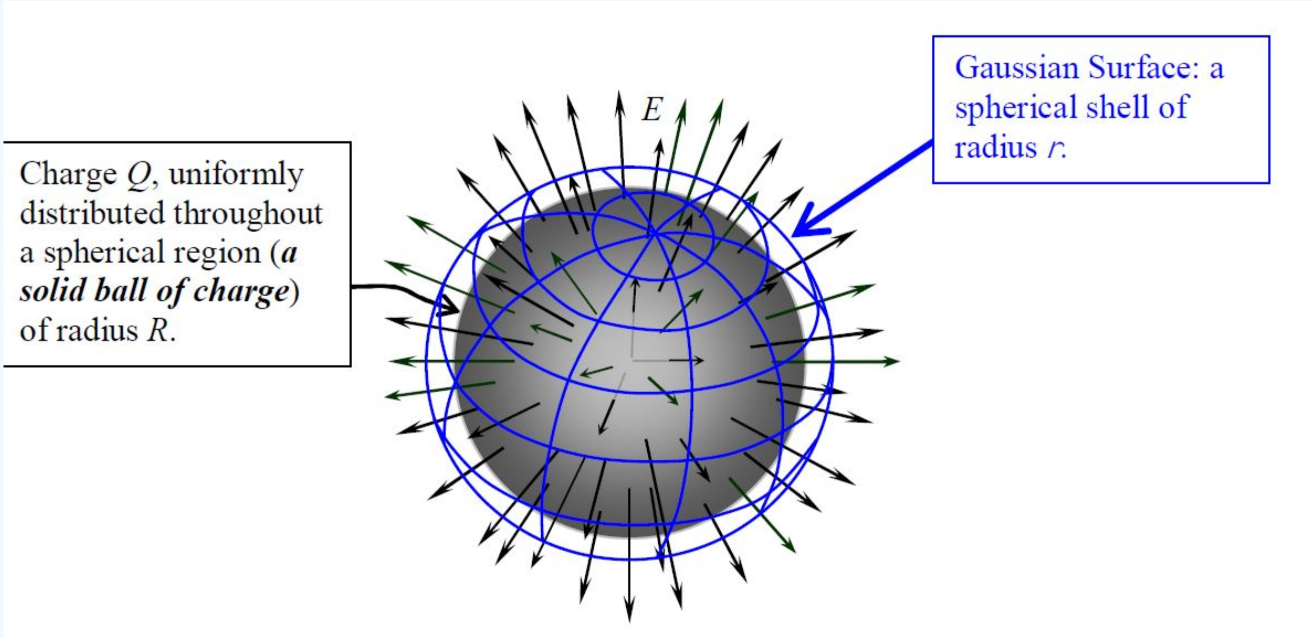
A couple of pages back we used Gauss's Law to arrive at the relation $E4\pi r^2 = \frac{Q_{\text{enclosed}}}{\epsilon_o}$ and now we have something to plug in for Q_{enclosed}. Doing so yields:

$$E4\pi r^2 = \frac{\left(\frac{r^3}{R^3}\right)Q}{\epsilon_o}$$

$$E = \frac{Q}{4\pi\epsilon_o R^3} r$$

This is our result for the magnitude of the electric field due to a uniform ball of charge at points inside the ball of charge $(r \leq R)$. E is directly proportional to the distance from the center of the charge distribution. E increases with increasing distance because, the farther a point is from the center of the charge distribution, the more charge there is inside the spherical shell that is centered on the charge distribution and upon which the point in question is situated. How about points for which $r \geq R$?

If $r \geq R$,



the analysis is identical to the preceding analysis up to and including the point where we determined that:

$$E4\pi r^2 = \frac{Q_{\text{enclosed}}}{\epsilon_o}$$

But as long as $r \geq R$, no matter by how much r exceeds R, all the charge in the spherical distribution of charge is enclosed by the Gaussian surface. "All the charge" is just Q the total amount of charge in the uniform ball of charge. So,

$$E4\pi r^2 = \frac{Q}{\epsilon_o}$$

$$E = \frac{1}{4\pi\epsilon_o}\frac{Q}{r^2}$$

The constant $\frac{1}{4\pi\epsilon_o}$ is just the Coulomb constant k so we can write our result as:

$$E = \frac{kQ}{r^2}$$

This result looks just like Coulomb's Law for a point charge. What we've proved here is that, at points outside a spherically-symmetric charge distribution, the electric field is the same as that due to a point charge at the center of the charge distribution.
