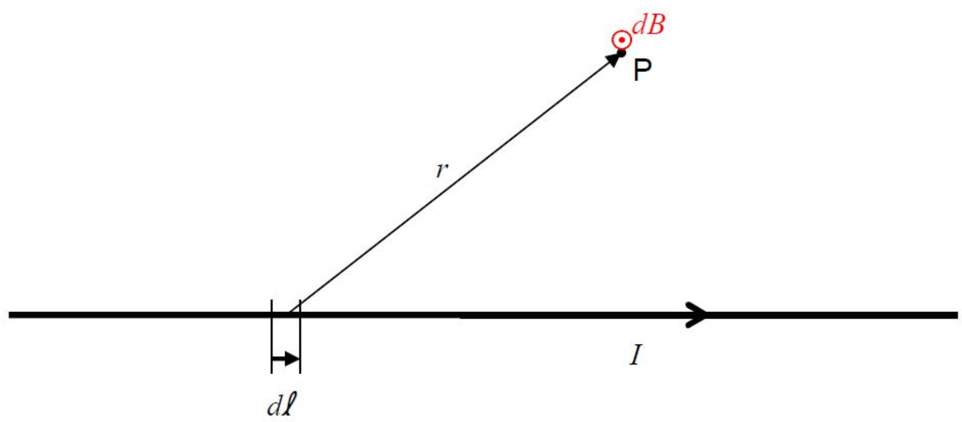
B36: The Biot-Savart Law

The Biot-Savart Law provides us with a way to find the magnetic field at an empty point in space, let's call it point P, due to current in wire. The idea behind the Biot-Savart Law is that each infinitesimal element of the current-carrying wire makes an infinitesimal contribution to the magnetic field at the empty point in space. Once you find each contribution, all you have to do is add them all up. Of course, there are an infinite number of contributions to the magnetic field at point P and each one is a vector, so, we are talking about an infinite sum of vectors. This business should seem familiar to you. You did this kind of thing when you were calculating the electric field back in *Chapter 30 The Electric Field Due to a Continuous Distribution of Charge on a Line*. The idea is similar, but here, of course, we are talking about magnetism.

The Biot-Savart Law gives the infinitesimal contribution to the magnetic field at point P due to an infinitesimal element of the current-carrying wire. The following diagram helps to illustrate just what the Biot-Savart Law tells us.



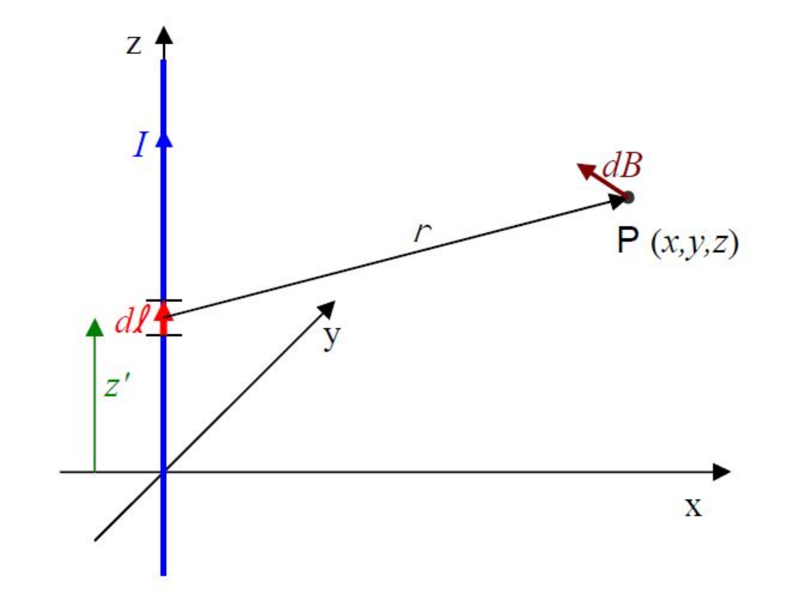
The Biot-Savart Law states that:

$$\overrightarrow{dB} = \frac{\mu_o}{4\pi} \frac{I\,\overrightarrow{dl} \times \vec{r}}{r^3}$$

The Biot-Savart Law represents a powerful straightforward method of calculating the magnetic field due to a current distribution.

Calculate the magnetic field due to a long straight wire carrying a current I along the z axis in the positive z direction. Treat the wire as extending to infinity in both directions.

Solution



Each infinitesimal element of the current-carrying conductor makes a contribution $\overrightarrow{dB}$ to the total magnetic field at point P.

The $\vec{r}$ vector extends from the infinitesimal element at $(0,0,z')$ to point P at (x,y,z).

$$\vec{r}=(x\hat{i}+y\hat{j}+z\hat{k})-z'\hat{k}$$

$$\vec{r}=x\hat{i}+y\hat{j}+(z-z')\hat{k}$$

The magnitude of $\vec{r}$ is thus:

$$r=\sqrt{x^2+y^2+(z-z')^2}$$

The $\overrightarrow{dl}$ vector points in the +z direction so it can be expressed as $\overrightarrow{dl}=dz'\hat{k}$

With these expressions for $\vec{r}$, r, and $\overrightarrow{dl}$ substituted into the Biot-Savart Law,

$$\overrightarrow{dB}=\frac{\mu_o}{4\pi}\frac{I\overrightarrow{dl}\times\vec{r}}{r^3}$$

we obtain:

$$\overrightarrow{dB}=\frac{\mu_o I}{4\pi}\frac{dz'\hat{k}\times\left(x\hat{i}+y\hat{j}+(z-z')\hat{k}\right)}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$$

$$\overrightarrow{dB}=\frac{\mu_o I}{4\pi}\frac{\left(x\hat{k}\times\hat{i}+y\hat{k}\times\hat{j}+(z-z')\hat{k}\times\hat{k}\right)}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$$

$$\overrightarrow{dB}=\frac{\mu_o I}{4\pi}\frac{dz'(x\hat{j}-y\hat{i})}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$$

$$\overrightarrow{dB}=-\frac{\mu_o I}{4\pi}y\frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}\hat{i}+\frac{\mu_o I}{4\pi}x\frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}\hat{j}$$

Let's work on this a component at a time. For the x component, we have:

$$dB_x = -\frac{\mu_o I}{4\pi} y \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$$

Integrating over z' from $-\infty$ to ∞ yields:

$$B_x = -\frac{\mu_o I}{4\pi} y \int_{-\infty}^{\infty} \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$$

I'm going to go with the following variable substitution:

$$u = z - z'$$

$$du = -dz', \text{ so, } dz' = du$$

Upper Limit: Evaluating $u = z - z'$ at $z' = \infty$ yields $-\infty$ for the upper limit of integration.

Lower Limit: Evaluating $u = z - z'$ at $z' = -\infty$ yields ∞ for the upper limit of integration.

So, our integral becomes:

$$B_x = -\frac{\mu_o I}{4\pi} y \int_{-\infty}^{\infty} \frac{-du}{(x^2+y^2+u^2)^{3/2}}$$

I choose to use one of the minus signs to interchange the limits of integration:

$$B_x = -\frac{\mu_o I}{4\pi} y \int_{-\infty}^{\infty} \frac{du}{(x^2+y^2+u^2)^{3/2}}$$

Using $\int \frac{dx}{(x^2+a^2)^{3/2}} = \frac{1}{a^2} \frac{x}{\sqrt{x^2+a^2}}$ from your formula sheet; and; identifying x^2+y^2 as a^2, and, u as the x appearing on the formula sheet, we obtain:

$$B_x = -\frac{\mu_o I}{4\pi} y \frac{1}{x^2+y^2} \frac{u}{\sqrt{u^2+x^2+y^2}} \Big|_{-\infty}^{\infty}$$

Now, I need to take the limit of that expression as u goes to ∞ and again as u goes to $-\infty$. To facilitate that, I want to factor a u out of the square root in the denominator. But, I have to be careful. The expression $\sqrt{u^2+x^2+y^2}$, which is equivalent to $\sqrt{(z-z')^2+x^2+y^2}$ is a distance. That means it is inherently positive, whether u (or z' for that matter) is positive or negative. So, when I factor u out of the square root, I'm going to have to use absolute value signs. For the denominator: $\sqrt{u^2+x^2+y^2} = \sqrt{u^2\left(1+\frac{x^2}{u^2}+\frac{y^2}{u^2}\right)} = |u|\sqrt{1+\frac{x^2}{u^2}+\frac{y^2}{u^2}}$, so,

$$B_x = -\frac{\mu_o I}{4\pi} y \frac{1}{x^2+y^2} \frac{u}{|u|} \frac{1}{\sqrt{1+\frac{x^2}{u^2}+\frac{y^2}{u^2}}} \Big|_{-\infty}^{\infty}$$

$$B_x = -\frac{\mu_o I}{4\pi} y \frac{1}{x^2+y^2} \left(1\frac{1}{\sqrt{1+0+0}} - -1\frac{1}{\sqrt{1+0+0}}\right)$$

$$B_x = -\frac{\mu_o I}{4\pi} y \frac{1}{x^2+y^2}(2)$$

$$B_x = -\frac{\mu_o I}{2\pi} y \frac{1}{x^2+y^2}$$

Now for the y component. Recall that we had:

$$\overrightarrow{dB} = -\frac{\mu_o I}{4\pi} y \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}} \hat{i} + \frac{\mu_o I}{4\pi} y \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}} \hat{j}$$

so,

$$dB_y = \frac{\mu_o I}{4\pi} x \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}}$$

But, except for the replacement of $-y$ by x, this is the same expression that we had for dB_x. And those, (the $-y$ in the expression for dB_x and the x in the expression for dB_y), are, as far as the integration over z' goes, constants, out front. They don't affect the integration, they just "go along for the ride." So, we can use our B_x result for B_y if we just replace the $-y$, in our expression for B_x, with x. In other words, without having to go through the entire integration process again, we have:

$$B_y = \frac{\mu_o I}{2\pi} x \frac{1}{x^2+y^2}$$

Since we have no z component in our expression

$$\overrightarrow{dB} = -\frac{\mu_o I}{4\pi} y \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}} \hat{i} + \frac{\mu_o I}{4\pi} y \frac{dz'}{\left[x^2+y^2+(z-z')^2\right]^{3/2}} \hat{j}$$

$\vec{B}$ itself must have no z component.

Substituting our results for $B_x, B_y, \text{ and } B_z$ into $\hat{i}, \hat{j}, \hat{k}$ expression for $\vec{B}$, (Namely, $\vec{B} = B_x\hat{i} + B_y\hat{j} + B_z\hat{k}$), we have:

$$\vec{B} = -\frac{\mu_o I}{2\pi} y \frac{1}{x^2+y^2} \hat{i} + \frac{\mu_o I}{2\pi} x \frac{1}{x^2+y^2} \hat{j} + 0\hat{k}$$

$$\vec{B} = \frac{\mu_o I}{2\pi} \frac{1}{x^2+y^2} (-y\hat{i} + x\hat{j})$$

The quantity x^2+y^2 is just r^2, the square of the distance that point P is from the current carrying wire (recall that we are finding the magnetic field due to a wire, with a current I, that extends along the z axis from $-\infty$ to ∞)

$$\vec{B} = \frac{\mu_o I}{2\pi} \frac{1}{r^2} (-y\hat{i} + x\hat{j})$$

Furthermore, the vector $(-y\hat{i} + x\hat{j})$ has magnitude $\sqrt{(-y)^2+x^2} = \sqrt{x^2+y^2} = r$. Hence, the unit vector $\hat{u}_B$ in the same direction as $(-y\hat{i} + x\hat{j})$ is given by

$$\hat{u}_B = \frac{-y\hat{i} + x\hat{j}}{r} = -\frac{y}{r}\hat{i} + \frac{x}{r}\hat{j}$$

and, expressed as its magnitude times the unit vector in its direction, the vector $(-y\hat{i} + x\hat{j})$ can be written as:

$$(-y\hat{i} + x\hat{j}) = r\hat{u}_B$$

Substituting $(-y\hat{i} + x\hat{j}) = r\hat{u}_B$ into our expression $\vec{B} = \frac{\mu_o I}{2\pi} \frac{1}{r^2} (-y\hat{i} + x\hat{j})$ yields:

$$\vec{B} = \frac{\mu_o I}{2\pi} \frac{1}{r^2} r\hat{u}_B$$

$$\vec{B} = \frac{\mu_o I}{2\pi} \frac{1}{r} \hat{u}_B$$

Note that the magnitude of $\vec{B}$ obtained here, namely $B = \frac{\mu_o I}{2\pi} \frac{1}{r}$, is identical to the magnitude obtained using the integral form of Ampere's Law. The direction $\hat{u}_B = -\frac{y}{r}\hat{i} + \frac{x}{r}\hat{j}$ for the magnetic field at any point P having coordinates (x, y, z), is also the same as, "the magnetic field extends in circles about that wire, in that sense of rotation (counterclockwise or clockwise) which is consistent with the right hand rule for something curly something straight with the something straight being the current and the something curly being the magnetic field."
