6A: One-Dimensional Motion (Motion Along a Line): Definitions and Mathematics

> A mistake that is often made in linear motion problems involving acceleration, is using the velocity at the end of a time interval as if it was valid for the entire time interval. The mistake crops up in constant acceleration problems when folks try to use the definition of average velocity $\bar{v}=\frac{\triangle x}{\triangle t}$ in the solution. Unless you are asked specifically about average velocity, you will never need to use this equation to solve a physics problem. Avoid using this equation—it will only get you into trouble. For constant acceleration problems, use the set of constant acceleration equations provided you.

Here we consider the motion of a particle along a straight line. The particle can speed up and slow down and it can move forward or backward but it does not leave the line. While the discussion is about a particle (a fictitious object which at any instant in time is at a point in space but has no extent in space—no width, height, length, or diameter) it also applies to a rigid body that moves along a straight line path without rotating, because in such a case, every particle of the body undergoes one and the same motion. This means that we can pick one particle on the body and when we have determined the motion of that particle, we have determined the motion of the entire rigid body.

So how do we characterize the motion of a particle? Let's start by defining some variables:

- t: How much time t has elapsed since some initial time. The initial time is often referred to as "the start of observations" and even more often assigned the value 0. We will refer to the amount of time t that has elapsed since time zero as the stopwatch reading. A time interval $\triangle t$ (to be read "delta t") can then be referred to as the difference between two stopwatch readings.
- x : Where the object is along the straight line. To specify the position of an object on a line, one has to define a reference position (the start line) and a forward direction. Having defined a forward direction, the backward direction is understood to be the opposite direction. It is conventional to use the symbol x to represent the position of a particle. The values that x can have, have units of length. The SI unit of length is the meter. (SI stands for "Systeme International," the international system of units.) The symbol for the meter is m. The physical quantity x can be positive or negative where it is understood that a particle which is said to be minus five meters forward of the start line (more concisely stated as $x=-5m$) is actually five meters behind the start line.
- v: How fast and which way the particle is going—the velocity of the object. Because we are considering an object that is moving only along a line, the "which way" part is either forward or backward. Since there are only two choices, we can use an algebraic sign ("+" or "−") to characterize the direction of the velocity. By convention, a positive value of velocity is used for an object that is moving forward, and a negative value is used for an object that is moving backward. Velocity has both magnitude and direction. The magnitude of a physical quantity that has direction is how big that quantity is, regardless of its direction. So the magnitude of the velocity of an object is how fast that object is going, regardless of which way it is going. Consider an object that has a velocity of $5m/s$. The magnitude of the velocity of that object is $5m/s$. Now consider an object that has a velocity of $-5m/s$. (It is going backward at $5m/s$.) The magnitude of its velocity is also $5m/s$. Another name for the magnitude of the velocity is the speed. In both of the cases just considered, the speed of the object is $5m/s$ despite the fact that in one case the velocity was $-5m/s$. To understand the "how fast" part, just imagine that the object whose motion is under study has a built-in speedometer. The magnitude of the velocity, a.k.a. the speed of the object, is simply the speedometer reading.
- a: Next we have the question of how fast and which way the velocity of the object is changing. We call this the acceleration of the object. Instrumentally, the acceleration of a car is indicated by how fast and which way the tip of the speedometer needle is moving. In a car, it is determined by how far down the gas pedal is pressed or, in the case of car that is slowing down, how hard the driver is pressing on the brake pedal. In the case of an object that is moving along a straight line, if the object has some acceleration, then the speed of the object is changing.

Okay, we've got the quantities used to characterize motion. Soon, we're going to develop some useful relations between those variables. While we're doing that, I want you to keep these four things in mind:

1. We're talking about an object moving along a line.
2. Being in motion means having your position change with time.
3. You already have an intuitive understanding of what instantaneous velocity is because you have ridden in a car. You know the difference between going 65 mph and 15 mph and you know very well that you neither have to go 65 miles nor travel for an hour to be going 65 mph. In fact, it is entirely possible for you to have a speed of 65 mph for just an instant (no time interval at all —it's how fast you are going (what your speedometer reading is) at that instant. To be sure, the speedometer needle may be

just “swinging through” that reading, perhaps because you are in the process of speeding up to 75 mph from some speed below 65 mph, but the 65 mph speed still has meaning and still applies to that instant when the speedometer reading is 65 mph. Take this speed concept with which you are so familiar, tack on some directional information, which for motion on a line just means, specify “forward” or “backward; and you have what is known as the instantaneous velocity of the object whose motion is under consideration.

A lot of people say that the speed of an object is how far that object travels in a certain amount of time. No! That’s a distance. Speed is a rate. Speed is never how far, it is how fast. So if you want to relate it to a distance you might say something like, “Speed is what you multiply by a certain amount of time to determine how far an object would go in that amount of time if the speed stayed the same for that entire amount of time.” For instance, for a car with a speed of 25 mph, you could say that 25 mph is what you multiply by an hour to determine how far that car would go in an hour if it maintained a constant speed of 25 mph for the entire hour. But why explain it in terms of position? It is a rate. It is how fast the position of the object is changing. If you are standing on a street corner and a car passes you going 35 mph, I bet that if I asked you to estimate the speed of the car that you would get it right within 5 mph one way or the other. But if we were looking over a landscape on a day with unlimited visibility and I asked you to judge the distance to a mountain that was 35 miles away just by looking at it, I think the odds would be very much against you getting it right to within 5 miles. In a case like that, you have a better feel for “how fast” than you do for “how far.” So why define speed in terms of distance when you can just say that the speed of an object is how fast it is going?

4. You already have an intuitive understanding of what acceleration is. You have been in a car when it was speeding up. You know what it feels like to speed up gradually (small acceleration) and you know what it feels like to speed up rapidly (big, “pedalto the-metal,” acceleration).

All right, here comes the analysis. We have a start line ($x = 0$) and a positive direction (meaning the other way is the negative direction).

Consider a moving particle that is at position x_1 when the clock reads t_1 and at position x_2 when the clock reads t_2.

The displacement of the particle is, by definition, the change in position $\Delta x = x_2 - x_1$ of the particle. The average velocity $barv$ is, by definition,

$$\bar{v} = \frac{\triangle x}{\triangle t}$$

where $\Delta t = t_2 - t_1$ is the change in clock reading. Now the average velocity is not something that one would expect you to have an intuitive understanding for, as you do in the case of instantaneous velocity. The average velocity is not something that you can read off the speedometer, and frankly, it’s typically not as interesting as the actual (instantaneous) velocity, but it is easy to calculate and we can assign a meaning to it (albeit a hypothetical meaning). It is the constant velocity at which the particle would have to travel if it was to undergo the same displacement $\Delta x = x_2 - x_1$ in the same time $\Delta t = t_2 - t_1$ at constant velocity. The importance of the average velocity in this discussion lies in the fact that it facilitates the calculation of the instantaneous velocity.

Calculating the instantaneous velocity in the case of a constant velocity is easy. Looking at what we mean by average velocity, it is obvious that if the velocity isn’t changing, the instantaneous velocity is the average velocity. So, in the case of a constant velocity, to calculate the instantaneous velocity, all we have to do is calculate the average velocity, using any displacement with its corresponding time interval, that we want. Suppose we have position vs. time data on, for instance, a car traveling a straight path at 24 m/s.

Here’s some idealized fictitious data for just such a case.

Data Reading Number	Times [second]	Position [meter]
0	0.00	0.0

Data Reading Number	Times [second]	Position [meter]
1	0.100	2.30
2	1.0	23.0
3	10.0	230
4	100.0	2300

Remember, the speedometer of the car is always reading 24 m/s. (It should be clear that the car was already moving as it crossed the start line at time zero—think of time zero as the instant a stopwatch was started and the times in the table as stopwatch readings.) The position is the distance forward of the start line.

Note that for this special case of constant velocity, you get the same average velocity, the known value of constant speed, no matter what time interval you choose. For instance, if you choose the time interval from 1.00 seconds to 10.0 seconds:

$$\bar{v}=\frac{\triangle x}{\triangle t} \text{ (\textbf{Average velocity})}$$

$$\bar{v}=\frac{x_3-x_2}{t_3-t_2}$$

$$\bar{v}=\frac{230m-23.0m}{10.0s-1.0s}$$

$$\bar{v}=23.0\frac{m}{s}$$

and if you choose the time interval 0.100 seconds to 100.0 seconds:

$$\bar{v}=\frac{\triangle x}{\triangle t}$$

$$\bar{v}=\frac{x_4-x_1}{t_4-t_1}$$

$$\bar{v}=\frac{2300m-2.30m}{100s-0.100s}$$

$$\bar{v}=23.0\frac{m}{s}$$

The points that need emphasizing here are that, if the velocity is constant then the calculation of the average speed yields the instantaneous speed (the speedometer reading, the speed we have an intuitive feel for), and when the velocity is constant, it doesn't matter what time interval you use to calculate the average velocity; in particular, a small time interval works just as well as a big time interval.

So how do we calculate the instantaneous velocity of an object at some instant when the instantaneous velocity is continually changing? Let's consider a case in which the velocity is continually increasing. Here we show some idealized fictitious data (consistent with the way an object really moves) for just such a case.

Data Reading Number	Time since object was at start time [s]	Position (distance ahead of start line) [m]	Velocity (This is what we are trying to calculate. Here are correct answers.) [m/s]
0	0	0	10
1	1	14	18
2	1.01	14.1804	18.08
3	1.1	15.84	18.8
4	2	36	26

Data Reading Number	Time since object was at start time [s]	Position (distance ahead of start line) [m]	Velocity (This is what we are trying to calculate. Here are correct answers.) [m/s]
5	5	150	50

What I want to do with this fictitious data is to calculate an average velocity during a time interval that begins with t = 1s and compare the result with the actual velocity at time t = 1s. The plan is to do this repeatedly, with each time interval used being smaller than the previous one.

Average velocity from t = 1s to t = 5s:

$$\bar{v} = \frac{\triangle x}{\triangle t}$$

$$\bar{v} = \frac{x_5 - x_1}{t_5 - t_1}$$

$$\bar{v} = \frac{150m - 14m}{5s - 1s}$$

$$\bar{v} = 34.0\frac{m}{s}$$

Note that this value is quite a bit larger than the correct value of the instantaneous velocity at t =1s (namely 18 m/s). It does fall between the instantaneous velocity of 18 m/s at t = 1s and the instantaneous velocity of 50 m/s at t = 5 seconds. That makes sense since, during the time interval, the velocity takes on various values which for 1s < t < 5s are all greater than 18 m/s but less than 50 m/s.

For the next two time intervals in decreasing time interval order (calculations not shown):

- Average velocity from t=1 to t =2s: 22 m/s
- Average velocity from t=1 to t =1.1s: 18.4 m/s

And for the last time interval, we do show the calculation

$$\bar{v} = \frac{\triangle x}{\triangle t}$$

$$\bar{v} = \frac{x_2 - x_1}{t_2 - t_1}$$

$$\bar{v} = \frac{14.1804m - 14m}{1.01s - 1s}$$

$$\bar{v} = 18.04\frac{m}{s}$$

Here I copy all the results so that you can see the trend:

- Average velocity from t = 1 to t = 5 s: 34 m/s
- Average velocity from t = 1 to t = 2 s: 22 m/s
- Average velocity from t = 1 to t = 1.1 s: 18.4 m/s
- Average velocity from t = 1 to t = 1.01 s: 18.04 m/s

Every answer is bigger than the instantaneous velocity at t =1s (namely 18 m/s). Why? Because the distance traveled in the time interval under consideration is greater than it would have been if the object moved with a constant velocity of 18 m/s. Why? Because the object is speeding up, so, for most of the time interval the object is moving faster than 18 m/s, so, the average value during the time interval must be greater than 18 m/s. But notice that as the time interval (that starts at t = 1s) gets smaller and smaller, the average velocity over the time interval gets closer and closer to the actual instantaneous velocity at t = 1s. By induction, we conclude that if we were to use even smaller time intervals, as the time interval we chose to use was made smaller and smaller, the average velocity over that tiny time interval would get closer and closer to the instantaneous velocity, so that when the time interval got to be so small as to be virtually indistinguishable from zero, the value of the average velocity would get to be indistinguishable from the value of the instantaneous velocity. We write that:

$$v=\lim_{\triangle x\to 0}\frac{\triangle x}{\triangle t}$$

(Note the absence of the bar over the v. This v is the instantaneous velocity.) This expression for v is, by definition, the derivative of x with respect to t. The derivative of x with respect to t is $\frac{dx}{dt}$ written as which means that

$$v=\frac{dx}{dt}$$

Note that, as mentioned, $\frac{dx}{dt}$ is the derivative of x with respect to t. It is not some variable d times x all divided by d times t. It is to be read “dee ex by dee tee” or, better yet, “the derivative of x with respect to t.” Conceptually what it means is, starting at that value of time t at which you wish to find the velocity, let t change by a very small amount. Find the (also very small) amount by which x changes as a result of the change in t and divide the tiny change in x by the tiny change in t. Fortunately, given a function that provides the position x for any time t, we don’t have to go through all of that to get v, because the branch of mathematics known as differential calculus gives us a much easier way of determining the derivative of a function that can be expressed in equation form. A function, in this context, is an equation involving two variables, one of which is completely alone on the left side of the equation, the other of which, is in a mathematical expression on the right. The variable on the left is said to be a function of the variable on the right. Since we are currently dealing with how the position of a particle depends on time, we use x and t as the variables in the functions discussed in the remainder of this chapter. In the example of a function that follows, we use the symbols x_o, v_o, and a to represent constants:

$$x=x_o+v_ot+\frac{1}{2}at^2$$

The symbol t represents the reading of a running stopwatch. That reading changes so t is a variable. For each different value of t, we have a different value of x, so x is also a variable. Some folks think that any symbol whose value is not specified is a variable. Not so. If you know that the value of a symbol is fixed, then that symbol is a constant. You don’t have to know the value of the symbol for it to be a constant; you just have to know that it is fixed. This is the case for x_o, v_o, and a in equation above.

Acceleration

At this point you know how to calculate the rate of change of something. Let’s apply that knowledge to acceleration. Acceleration is the rate of change of velocity. If you are speeding up, then your acceleration is how fast you are speeding up. To get an average value of acceleration over a time interval $\triangle t$, we determine how much the velocity changes during that time interval and divide the change in velocity by the change in stopwatch reading. Calling the velocity change $\triangle v$, we have

$$\bar{a}=\frac{\triangle v}{\triangle t}$$

To get the acceleration at a particular time t we start the time interval at that time t and make it an infinitesimal time interval. That is:

$$a=\lim_{\triangle t\to 0}\frac{\triangle v}{\triangle t}$$

The right side is, of course, just the derivative ofv with respect to t:

$$a=\frac{dv}{dt}$$