13A: Freefall, a.k.a. Projectile Motion

Note

The constant acceleration equations apply from the first instant in time after the projectile leaves the launcher to the last instant in time before the projectile hits something, such as the ground. Once the projectile makes contact with the ground, the ground exerts a huge force on the projectile causing a drastic change in the acceleration of the projectile over a very short period of time until, in the case of a projectile that doesn't bounce, both the acceleration and the velocity become zero. To take this zero value of velocity and plug it into constant acceleration equations that are devoid of post-ground-contact acceleration information is a big mistake. In fact, at that last instant in time during which the constant acceleration equations still apply, when the projectile is at ground level but has not yet made contact with the ground, (assuming that ground level is the lowest elevation achieved by the projectile) the magnitude of the velocity of the projectile is at its biggest value, as far from zero as it ever gets!

Consider an object in freefall with a non-zero initial velocity directed either horizontally forward; or both forward and vertically (either upward or downward). The object will move forward, and upward or downward—perhaps upward and then downward—while continuing to move forward. In all cases of freefall, the motion of the object (typically referred to as the projectile when freefall is under consideration) all takes place within a single vertical plane. We can define that plane to be the x-y plane by defining the forward direction to be the x direction and the upward direction to be the y direction.

One of the interesting things about projectile motion is that the horizontal motion is independent of the vertical motion. Recall that in freefall, an object continually experiences a downward acceleration of $9.80 \frac{m}{s^2}$ but has no horizontal acceleration. This means that if you fire a projectile so that it is approaching a wall at a certain speed, it will continue to get closer to the wall at that speed, independently of whether it is also moving upward and/or downward as it approaches the wall. An interesting consequence of the independence of the vertical and horizontal motion is the fact that, neglecting air resistance, if you fire a bullet horizontally from, say, shoulder height, over flat level ground, and at the instant the bullet emerges from the gun, you drop a second bullet from the same height, the two bullets will hit the ground at the same time. The forward motion of the fired bullet has no effect on its vertical motion.

> The most common mistake that folks make in solving projectile motion problems is combining the x and y motion in one standard constant-acceleration equation. Don't do that. Treat the x-motion and the y-motion separately.

In solving projectile motion problems, we take advantage of the independence of the horizontal (x) motion and the vertical (y) motion by treating them separately. The one thing that is common to both the x motion and the y motion is the time. The key to the solution of many projectile motion problems is finding the total time of "flight." For example, consider the following example.

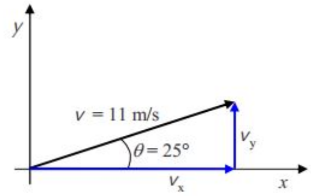
Example $13A.1$

A projectile is launched with a velocity of $11 m/s$ at an angle of 28° above the horizontal over flat level ground from a height of $2.0 m$ above ground level. How far forward does it go before hitting the ground? (Assume that air resistance is negligible.)

Solution

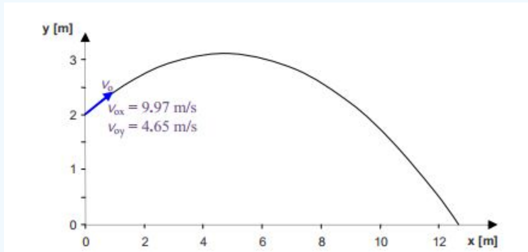
Before getting started, we better clearly establish what we are being asked to find. We define the forward direction as the x direction so what we are looking for is a value of x. More specifically, we are looking for the distance, measured along the ground, from that point on the ground directly below the point at which the projectile leaves the launcher, to the point on the ground where the projectile hits. This distance is known as the range of the projectile. It is also known as the range of the launcher for the given angle of launch and the downrange distance traveled by the projectile.

Okay, now that we know what we're solving for, let's get started. An initial velocity of $11 m/s$ at 28° above the horizontal, eh? Uh oh! We've got a dilemma. The key to solving projectile motion problems is to treat the x motion and the y motion separately. But we are given an initial velocity vo which is a mix of the two of them. We have no choice but to break up the initial velocity into its x and y components.



$\frac{v_x}{v} = \cos\theta$	$\frac{v_y}{v} = \sin\theta$
$v_x = v\cos\theta$	$v_y = v\sin\theta$
$v_x = 11\frac{\text{m}}{\text{s}}\cos 25^\circ$	$v_y = 11\frac{\text{m}}{\text{s}}\sin 25^\circ$
$v_x = 9.97\frac{\text{m}}{\text{s}}$	$v_y = 4.65\frac{\text{m}}{\text{s}}$

Now we're ready to get started. We'll begin with a sketch which defines our coordinate system, thus establishing the origin and the positive directions for x and y.



Recall that in projectile motion problems, we treat the x and y motion separately. Let's start with the x motion. It is the easier part because there is no acceleration.

x motion

$$x = \cancel{x_o}^{\,0} + v_{ox}\, t + \frac{1}{2}\cancel{a_x}^{\,0} t^2$$

$$x = v_{ox} t \tag{13A.1}$$

Note that for the x-motion, we start with the constant acceleration equation that gives the position as a function of time. (Imagine having started a stopwatch at the instant the projectile lost contact with the launcher. The time variable t represents the stopwatch reading.) As you can see, because the acceleration in the x direction is zero, the equation quickly simplifies to $x = V_{0x}t$. We are "stuck" here because we have two unknowns, x and t, and only one equation. It's time to turn to the y motion.

It should be evident that it is the y motion that yields the time, the projectile starts off at a known elevation $(y = 2.0m)$ and the projectile motion ends when the projectile reaches another known elevation, namely, $y = 0$.

y-motion

$$y = y_0 + V_{0y}t + \frac{1}{2}a_y t^2 \tag{13A.2}$$

This equation tells us that the y value at any time t is the initial y value plus some other terms that depend on t. It's valid for any time t, starting at the launch time $t = 0$, while the object is in projectile motion. In particular, it is applicable to that special time t, the last instant before the object makes contact with the ground, that instant that we are most interested in, the time when $y = 0$. What we can do, is to plug 0 in for y, and solve for that special time t that, when plugged into Equation 13A.2, makes y be 0. When we rewrite Equation 13A.2 with y set to 0, the symbol t takes on a new meaning. Instead of being a variable, it becomes a special time, the time that makes the y in the actual Equation 13A.2 ($y = y = y_0 + V_{0y}t + \frac{1}{2}a_y t^2$) zero.

$$0 = y_0 + V_{0y}t_* + \frac{1}{2}a_y t_*^2 \tag{13A.3}$$

To emphasize that the time in Equation 13A.3 is a particular instant in time rather than the variable time since launch, I have written it as t_* to be read "t star." Everything in Equation 13A.3 is a given except t_* so we can solve Equation 13A.3 for t_*. Recognizing that Equation 13A.3 is a quadratic equation in t_* we first rewrite it in the form of the standard quadratic equation $ax + bx^2 + c = 0$. This yields:

$$\frac{1}{2}a_y t_*^2 + V_{0y}t_* + y_0 = 0$$

Then we use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ which for the case at hand appears as:

$$t_* = \frac{-V_{0y} \pm \sqrt{V_{0y}^2 - 4(\frac{1}{2}a_y)y_0}}{2(\frac{1}{2}a_y)}$$

which simplifies to

$$t_* = \frac{-V_{0y} \pm \sqrt{V_{0y}^2 - 2a_y y_0}}{a_y}$$

Substituting values with units yields:

$$t_* = \frac{-4.65\frac{m}{s} \pm \sqrt{(-4.65\frac{m}{s})^2 - 2(-9.80\frac{m}{s^2})2.0m}}{-9.80\frac{m}{s^2}}$$

which evaluates to
$t_* = -0.321s$ and $t_* = 1.27s$

We discard the negative answer because we know that the projectile hits the ground after the launch, not before the launch.

Recall that t_* is the stopwatch reading when the projectile hits the ground. Note that the whole time it has been moving up and down, the projectile has been moving forward in accord with Equation 13A.1, $x = V_{0x}t$. At this point, all we have to do is plug $t_* = 1.27s$ into Equation 13A.1 and evaluate:

$$\begin{aligned} x &= V_{0x}t_* \\ &= 9.97\frac{m}{s}(1.27s) \\ &= 13m \end{aligned}$$

This is the answer. The projectile travels $13m$ forward before it hits the ground.